1. Minimizing $t^{2k} - \alpha C_{2k+2}(t)$ and the p-frame potential

This note is related to minimization of the two potentials $t^{2k} - \alpha C_{2k+2}(t)$ and the p-frame potential, $|t|^p$. Numerically there is evidence that minimizers as $\alpha \to 0$ of the first potential may be related to minimizers as $p \to 2k$ in the second problem. More precisely it may be that for

$$\mu^* = \lim_{p \to 2k} arg \min_{\mu} \iint |\langle x, y\rangle|^p d\mu(x) d\mu(y), \quad and$$

$$\nu^* = \lim_{\alpha \to 0} arg \min_{\nu} \iint \langle x, y\rangle^{2k} - \alpha C_{2k+2}(\langle x, y\rangle) d\nu(x) d\nu(y)$$

the measures $\mu^* = \nu^*$.

This equality holds for the unit circle. One reason why the equality may hold in general is that both problems may have (small) weighted designs as solutions assuming the limits above exist. The evidence for this in p-frame potential case is supported by numerics and a few cases (for tight designs) where the minimizers are known precisely. It is worth noting that there are a couple of related optimization problems that are of independent interest to the two problems above. These are the problems

$$\xi^* = arg \max_{\xi} \iint C_{2k+2}(\langle x, y\rangle) d\xi(x) d\xi(y) \text{ s.t. } \iint C_l(\langle x, y\rangle) d\xi(x) d\xi(y) = 0 \text{ for } 1 < l \leqslant 2k-1,$$
$$\int d\xi(x) = 1 \text{ and}$$

$$\eta^* = arg \min_{\eta} |supp(\eta)| \text{ s.t. } \iint C_l(\langle x, y\rangle) d\eta(x) d\eta(y) = 0 \text{ for } 1 < l \leqslant 2k-1, \text{ and } \int d\eta(x) = 1.$$

The third problem is expected to be equivalent to the second while the relation of the fourth to the first three is not clear (this raises the question of how often one of the first three problems and the fourth have the same solution). Below are simple computations on how Gegenbauer coefficients in the p-frame potential behave as $p \to 2k$ for the unit sphere (things are equally doable in the other dimensions). Note that the integral $\int_{-1}^{1} |t|^p C_{2k}^{(1/2)}(t) dt$ are the coefficients (up to a constant) in the expansion of $|t|^p$ into the Gegenbauer (Legendre) polynomials. The value of this integral can be checked (or check references) to be equal to (p greater than 2 is the only current area of interest)

$$\gamma_{2k} = \int_{-1}^{1} |t|^p C_{2k}^{(1/2)}(t) dt = \frac{\Gamma(\frac{p}{2}+1)\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2}-k+1)\Gamma(\frac{p+3}{2}+k)}.$$

This shows that the ratio of coefficients $\frac{\gamma_{2k+2}}{\gamma_{2k}}$ is of size $\frac{\epsilon}{4k+3}$ for p of size $2k-\epsilon$, so that γ_{2k+2} goes linearly to zero (along with the other larger coefficients but with smaller constants) as $p \to 2k$ linearly. For this reason minimizing the first potential as $\alpha \to 0$ corresponds to minimizing as $p \to 2k$ the truncated $2k+2$-term Gegenbauer expansion of the p-frame potential.

2. Minimizing $t^{2k} - \alpha C_{2k+2}(t)$ on the unit circle

For the two dimensional (circle) problem all the equally distributed measures of even size have Chebyshev polynomial averages of the form $1, 0, 0, ..., 0, 1, 0, 0, ..., 0, 1$. Thus if one constrains the set of measures that one wishes to minimize over to ones of this form, then minimizing the potentials $f(t) = \sum_j \alpha_j C_j(t) - \sum_k \beta_k C_k(t)$ just corresponds to finding the smallest sub-sum (that is the sum over coefficients in the expansion of $f(t)$ over all positive indices which are multiples of $2m$ for some integer m) of the sequence of coefficients given by ones potential.

Note that this new problem does not have unique solutions. For instance, letting the Chebyshev polynomials be denoted $T_k(x)$ so that $T_k(\cos x) = \cos kx$ then the function $f(t) = T_0(t) - T_2(t) + T_4(t) - 3T_6(t)$ attains its minimum value over the new constraint set for two distinct measures (one has support a non-trivial subset of the other), namely for two antipodal points for which the averages of the Chebyshev polynomials are $1, 0, 1, 0, 1, 0, ...$ giving the value -2 and the regular hexagon which has average coeff. sequence $1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, ...$ also taking the minimum value of -2 for $f(t)$.

Neither of these configurations seem to be optimal numerically for the unconstrained problem and instead the value ~ -2.059078 is attainable by a measure supported on two points with inner product $-\frac{\sqrt{19-\sqrt{73}}}{6}$, which is a root of the derivative of $f(t)$. Similar is the potential $g(t) = T_0(t) - T_2(t) + T_4(t) - T_6(t) + T_8(t) - 3T_{10}(t)$. Here there is also a 2 point configuration that appears as optimal with the angle between the two points $-.349755$ which is close to a root of the derivative of g. Similar for $h(t) = T_0(t) - T_2(t) + T_4(t) - 2T_8(t)$ (the inner product here is $.716874$, and is approximately a root of h). Lastly to answer questions about numerics when some coefficients are skipped, one may consider the function $p(x) = T_0(x) - T_4(x) + T_8(x) - 3T_{12}(x) = -2 + 192x^2 - 2368x^4 + 10496x^6 - 20608x^8 + 18432x^{10} - 6144x^{12}$. Numerics indicate that the support size of a minimizer for this problem (without antipodes) is of size four with inner products $\sim \pm.48014$ and $\sim \pm.87719$ appearing which may correspond to zeros $\pm\frac{1}{2}(\sqrt{2 - \frac{\sqrt{19-\sqrt{73}}}{3}})$ and $\pm\frac{1}{2}(\sqrt{2 + \frac{\sqrt{19-\sqrt{73}}}{3}})$ of $p'(x)$. It should be mentioned that the weights in the above configurations need not be equal. This leads to the following conjecture

Conjecture. *If $f(t) = \sum_j \alpha_j C_j(t) - \sum_k \beta_k C_k(t)$ does not have a unique minimal sub-sum then it has as minimizer a measure supported on a set of points which has inner products appearing in the zero set of the derivative of $f(t)$.*

The size of the support in these cases at first glance appears to depend on the smallest value of m for which a sub-sum on the multiples of $2m$ is minimal for $f(t)$. A reasonable conjecture is that the support of a minimizer in this case is of size $2m$. Another question remains whether this is the only case where the simple procedure noted above (constraining to equally distributed discrete measures) does not apply

Conjecture. *If $f(t) = \sum_j \alpha_j C_j(t) - \sum_k \beta_k C_k(t)$ has a unique minimal sub-sum then a min. measure is supported on a tight design of corresponding order to the minimal sub-sum.*

The second conjecture looks to be false after looking at random coeff. potentials.