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Design and Analysis of Experiments

04 - Statistical Intervals

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BY NC SA

“*I attribute my success to this: I never gave or took an excuse.*”

“*I think one’s feelings waste themselves in words; they ought all to be distilled into actions, and into actions which bring results.*”

Florence Nightingale
1820-1910

English statistician and founder of modern nursing

Image: https://www.pinterest.com.au/pin/572238696374683644/

Statistical intervals are important in quantifying the uncertainty associated to a given estimate;

As an example, consider the situation: *a coaxial cable manufacturing operation produces cables with a target resistance of* 50Ω *and a known standard deviation of* 2Ω*. Assume that the resistance values can be well modeled by a normal distribution*.

Let us now suppose that a sample mean of $n = 25$ observations of resistance yields $\bar{x} = 48$. Given the sampling variability, it is very likely that this value is not exactly the true value of μ, but we are so far unable quantify how much uncertainty there is in this estimate.

Statistical intervals define regions that are likely to contain the true value of an estimated quantity.

Intervals are used to quantify the uncertainty associated with a given estimate, allowing the derivation of statements at quantifiable levels of confidence.

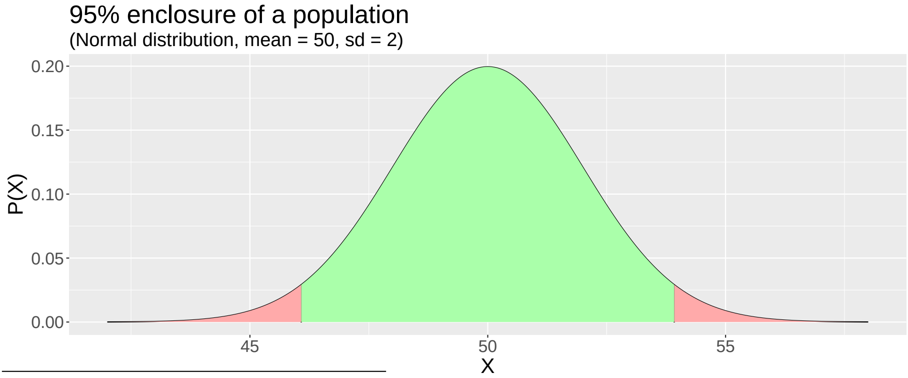
Three (plus one) of the most common types of interval are:

- Tolerance intervals;
- Prediction intervals;
- Highest density intervals;
- Confidence intervals.

Tolerance Intervals

Definition

"*A tolerance interval is an* ***enclosure*** *interval for a specified* ***proportion of the sampled population****, not its mean or standard deviation. For a specified confidence level, you may want to determine lower and upper bounds such that a given percent of the population is contained within them.*"[1].



[1] J.G. Ramírez: `https://git.io/v5ZFh`

The common practice in engineering of defining specification limits by adding $\pm 3\sigma$ to a given estimate of the mean arises from this definition - for a Normal population, $\approx 99.75\%$ of observations fall within $\mu \pm 3\sigma$.

Since σ^2 is usually unknown, we use s^2 and compensate for the additional uncertainty in this estimation.

For a Normal population, the *two-sided tolerance interval* for a given population proportion γ at the confidence level $1 - \alpha$ is given as:[2]

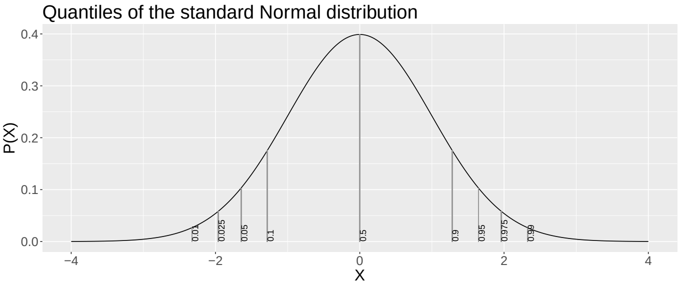
$$\bar{X} \pm S \sqrt{\frac{(n-1)}{n} \frac{\left(n + z^2_{(\alpha/2)}\right)}{\chi^{2\,(n-1)}_{(\gamma)}}}$$

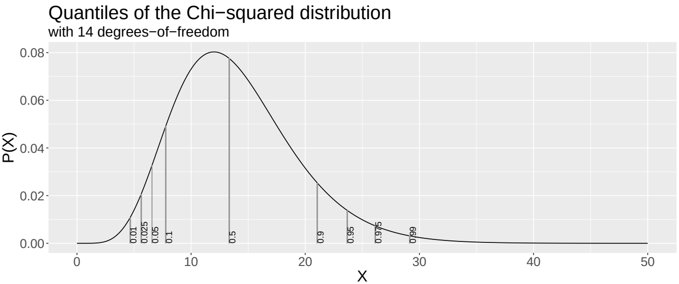
where $z_{(q)}$ is the q-quantile of the standard normal distribution; and $\chi^{2\,(n-1)}_{(q)}$ is the q-quantile of the χ^2 distribution with $n - 1$ degrees of freedom.

[2] NIST Engineering Statistics Handbook, `https://goo.gl/m6cxC6`

Tolerance Intervals

Quantiles





Confidence Intervals
Definition

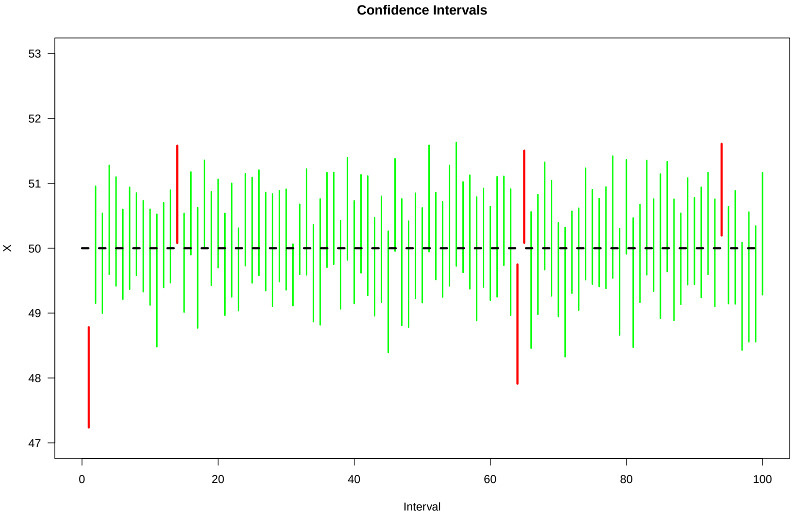
Confidence intervals quantify the degree of uncertainty associated with the **estimation of population parameters** such as the mean or the variance.

Can be defined as "*the interval that contains the true value of a given population parameter with a confidence level of* $100(1-\alpha)$";

Another useful definition is to think about confidence intervals in terms of confidence *in the method*: "The method used to derive the interval has a hit rate of 95%" - i.e., the interval generated has a 95% chance of 'capturing' the true population parameter."

Confidence Intervals

Example: 100 $CI_{.95}$ for a sample of 25 observations



For an interactive demonstration of the factors involved in the definition of a confidence interval, download the files from `https://git.io/vxXGj` and run on RStudio.

The two-sided $CI_{(1-\alpha)}$ for the mean of a normal population with known variance σ^2 is given by:

$$\bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$$

where $(1-\alpha)$ is the confidence level and z_x is the x-quantile of the standard normal distribution.

For the more usual case with an unknown variance,

$$\bar{x} + t_{\alpha/2}^{(n-1)}\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{1-\alpha/2}^{(n-1)}\frac{s}{\sqrt{n}}$$

where $t_x^{(n-1)}$ is the x-quantile of the t distribution with $n-1$ degrees of freedom.

A two-sided confidence interval on the variance of a normal variable can also be easily calculated:

$$\frac{(n-1)s^2}{\chi^{2\,(n-1)}_{1-\alpha/2}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^{2\,(n-1)}_{\alpha/2}}$$

For the standard deviation one simply needs to take the squared root of the confidence limits.

Statistical Intervals
Wrapping up

Statistical intervals quantify the uncertainty associated with different aspects of estimation;

Reporting intervals is always better than point estimates, as it provides the necessary information to quantify the location and uncertainty of your estimated values;

The correct interpretation is a little tricky (although not very difficult)[3], but it is essential in order to derive the correct conclusions based on the statistical interval of interest.

Intervals for other distributions (and even distribution-free intervals) can be obtained using analytical or resampling methods.

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[3] See the table at the end of `https://git.io/v5ZFh`

Bibliography

Required reading

1. J.G. Ramírez, *Statistical Intervals: Confidence, Prediction, Enclosure*: `https://git.io/v5ZFh`
2. D.C. Montgomery and G.C. Runger, *Applied Statistics and Probability for Engineers*, Chapter 8. 3rd Ed., Wiley 2005.
3. J. Orloff and J. Bloom, *Bootstrap confidence intervals*: `https://goo.gl/XrT1ao`

Recommended reading

1. Simply Statistics (blog) - `http://simplystatistics.org`
2. R. Dawkins, *Climbing Mount Improbable*, W.W.Norton&Co.,1997.

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