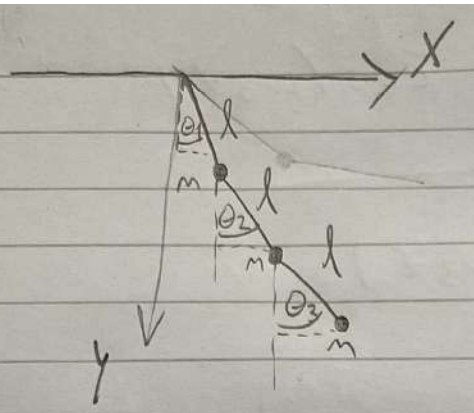

$$x_1 = l\sin\theta_1$$
$$x_2 = l\sin\theta_1 + l\sin\theta_2$$
$$x_3 = l\sin\theta_1 + l\sin\theta_2 + l\sin\theta_3$$
$$y_1 = l\cos(\theta_1)$$
$$y_2 = l\cos(\theta_1) + l\cos(\theta_2)$$
$$y_3 = l\cos(\theta_1) + l\cos(\theta_2) + l\cos(\theta_3)$$

$$\dot{x}_1 = l\cos\theta_1\,\dot\theta_1$$
$$\dot{x}_2 = l\cos\theta_1\,\dot\theta_1 + l\cos\theta_2\,\dot\theta_2$$
$$\dot{x}_3 = (l\cos\theta_1\,\dot\theta_1 + l\cos\theta_2\,\dot\theta_2) + l\cos\theta_3\,\dot\theta_3$$
$$\dot{y}_1 = -l\sin\theta_1\,\dot\theta_1$$
$$\dot{y}_2 = -l\sin\theta_1\,\dot\theta_1 - l\sin\theta_2\,\dot\theta_2$$
$$\dot{y}_3 = (-l\sin\theta_1\,\dot\theta_1 - l\sin\theta_2\,\dot\theta_2) - l\sin\theta_3\,\dot\theta_3$$

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2\right) \qquad \cos(x-y) = \cos x\cos y + \sin x\sin y$$

$$\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2 =$$

$$= l^2\cos^2\theta_1\,\dot\theta_1^2 + 2l^2\cos^2\theta_1\,\dot\theta_1^2 + 4l^2\dot\theta_1\dot\theta_2\cos\theta_1\cos\theta_2 + 2l^2\cos^2\theta_2\,\dot\theta_2^2$$
$$+2l^2\dot\theta_1\dot\theta_3\cos\theta_1\cos\theta_3 + 2l^2\dot\theta_2\dot\theta_3\cos\theta_2\cos\theta_3 + l^2\cos^2\theta_3\,\dot\theta_3^2$$
$$+l^2\sin^2\theta_1\,\dot\theta_1^2 + 2l^2\sin^2\theta_1\,\dot\theta_1^2 + 4l^2\dot\theta_1\dot\theta_2\sin\theta_1\sin\theta_2 + 2l^2\sin^2\theta_2\,\dot\theta_2^2$$
$$+2l^2\dot\theta_1\dot\theta_3\sin\theta_1\sin\theta_3 + 2l^2\dot\theta_2\dot\theta_3\sin\theta_2\sin\theta_3 + l^2\sin^2\theta_3\,\dot\theta_3^2$$

$$= l^2\dot\theta_1^2 + 2l^2\dot\theta_1^2 + 2l^2\dot\theta_2^2 + l^2\dot\theta_3^2 + 4l^2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)$$
$$+2l^2\dot\theta_1\dot\theta_3\cos(\theta_1-\theta_3) + 2l^2\dot\theta_2\dot\theta_3\cos(\theta_2-\theta_3)$$

$$T = \frac{ml^2}{2}\Big(3\dot\theta_1^2 + 2\dot\theta_2^2 + \dot\theta_3^2 + 4\dot\theta_1\dot\theta_2\underbrace{\cos(\theta_1-\theta_2)}_{\approx 1} + 2\dot\theta_1\dot\theta_3\underbrace{\cos(\theta_1-\theta_3)}_{\approx 1}$$
$$+ 2\dot\theta_2\dot\theta_3\underbrace{\cos(\theta_2-\theta_3)}_{\approx 1}\Big)$$

$U = -mgy_1 - mgy_2 - mgy_3$

$U = -3mgl\cos(\theta_1) - 2mgl\cos\theta_2 - mgl\cos\theta_3$

$\frac{dU}{d\theta_1} = \sin(\theta_1) <$ $\theta_1 = 0$ stabilna, $\theta_1 = \pi$ nestabilna $\quad \cos\theta_1 \approx 1 - \frac{\theta_1^2}{2}$

$\frac{dU}{d\theta_2} = \sin(\theta_2) <$ $\theta_2 = 0$ stabilna, $\theta_2 = \pi$ nestabilna $\quad \cos\theta_2 \approx 1 - \frac{\theta_2^2}{2}$

$\frac{dU}{d\theta_3} = \sin(\theta_3) <$ $\theta_3 = 0$ stabilna, $\theta_3 = \pi$ nestabilna $\quad \cos\theta_3 \approx 1 - \frac{\theta_3^2}{2}$

$$T \approx \frac{ml^2}{2}\left(3\dot\theta_1^2 + 2\dot\theta_2^2 + \dot\theta_3^2 + 4\dot\theta_1\dot\theta_2 + 2\dot\theta_1\dot\theta_3 + 2\dot\theta_2\dot\theta_3\right)$$

$$U \approx -3mgl + \frac{3mgl}{2}\theta_1^2 - 2mgl + mgl\theta_2^2 - mgl + \frac{mgl}{2}\theta_3^2$$

$$U \approx \underbrace{-6mgl}_{\text{do na konst.}} + \frac{3}{2}mgl\theta_1^2 + mgl\theta_2^2 + \frac{1}{2}mgl\theta_3^2$$

$L = T - U$

$$L = \frac{ml^2}{2}\left(3\dot\theta_1^2 + 2\dot\theta_2^2 + \dot\theta_3^2 + 4\dot\theta_1\dot\theta_2 + 2\dot\theta_1\dot\theta_3 + 2\dot\theta_2\dot\theta_3\right)$$
$$+ 6mgl - \frac{3}{2}mgl\theta_1^2 - mgl\theta_2^2 - \frac{1}{2}mgl\theta_3^2$$

$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot\theta_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \qquad \frac{ml^2}{2}\left(6\ddot\theta_1 + 4\ddot\theta_2 + 2\ddot\theta_3\right) + 3mgl\theta_1 = 0 \;/ : ml^2$

$$\boxed{\frac{g}{l} = \omega_0^2}$$

$$3\ddot\theta_1 + 2\ddot\theta_2 + \ddot\theta_3 + 3\omega_0^2\theta_1 = 0$$

$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot\theta_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 \qquad \frac{ml^2}{2}\left(4\ddot\theta_2 + 4\ddot\theta_1 + 2\ddot\theta_3\right) + 2mgl\theta_2 = 0$

$$2\ddot\theta_1 + 2\ddot\theta_2 + \ddot\theta_3 + 2\omega_0^2\theta_2 = 0$$

$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot\theta_3}\right) - \frac{\partial L}{\partial \theta_3} = 0 \qquad \frac{ml^2}{2}\left(2\ddot\theta_3 + 2\ddot\theta_1 + 2\ddot\theta_2\right) + mgl\theta_3 = 0$

$$\ddot\theta_1 + \ddot\theta_2 + \ddot\theta_3 + \omega_0^2\theta_3 = 0$$

$$\Theta_i = \sum_{k=1}^{3} C_k a_{ik} e^{i\omega_k t}$$

$$\sum_{k=1}^{3} C_k e^{i\omega_k t}\left[-3\omega_k^2 a_{1k} - 2\omega_k^2 a_{2k} - \omega_k^2 a_{3k} + 3\omega_0^2 a_{1k}\right] = 0$$

$$\sum_{k=1}^{3} C_k e^{i\omega_k t}\left[-2\omega_k^2 a_{1k} - 2\omega_k^2 a_{2k} - \omega_k^2 a_{3k} + 2\omega_0^2 a_{2k}\right] = 0$$

$$\sum_{k=1}^{3} C_k e^{i\omega_k t}\left[-\omega_k^2 a_{1k} - \omega_k^2 a_{2k} - \omega_k^2 a_{3k} + \omega_0^2 a_{3k}\right] = 0$$

$$\begin{vmatrix} -3\omega_k^2 + 3\omega_0^2 & -2\omega_k^2 & -\omega_k^2 \\ -2\omega_k^2 & -2\omega_k^2 + 2\omega_0^2 & -\omega_k^2 \\ -\omega_k^2 & -\omega_k^2 & -\omega_k^2 + \omega_0^2 \end{vmatrix} = 0$$

Uz Paython i $\omega_0^2 = 1 \Rightarrow \omega_1^2 = 0.4157746\,\omega_0^2$

$$\omega_2^2 = 2.2942804\,\omega_0^2$$

$$\omega_3^2 = 6.2899451\,\omega_0^2$$

$\underline{\omega_1^2}$:

$$-3\omega_1^2 a_{11} - 2\omega_1^2 a_{21} - \omega_1^2 a_{31} + 3\omega_0^2 a_{11} = 0$$

$$-2\omega_1^2 a_{11} - 2\omega_1^2 a_{21} - \omega_1^2 a_{31} + 2\omega_0^2 a_{21} = 0$$

$$-\omega_1^2 a_{11} + 3\omega_0^2 a_{11} - 2\omega_0^2 a_{21} = 0$$

$$a_{11}(3\omega_0^2 - \omega_1^2) = 2\omega_0^2 a_{21}$$

$$a_{11}\,\frac{3\omega_0^2 - \omega_1^2}{2\omega_0^2} = a_{21}$$

$$-3\omega_1^2 a_{11} - \frac{\omega_1^2(3\omega_0^2 - \omega_1^2)}{\omega_0^2} a_{11} - \omega_1^2 a_{31} + 3\omega_0^2 a_{11} = 0$$

$$-3\omega_1^2 a_{11} - 3\omega_1^2 a_{11} + \frac{\omega_1^4}{\omega_0^2} a_{11} - \omega_1^2 a_{31} + 3\omega_0^2 a_{11} = 0 \,/ : \omega_1^2$$

$$\left(-6 + \frac{\omega_1^2}{\omega_0^2} + 3\frac{\omega_0^2}{\omega_1^2}\right) a_{11} = a_{31}$$

$\underline{\omega_2^2:}$ $$a_{12}\frac{3\omega_0^2-\omega_2^2}{2\omega_0^2}=a_{22}$$

$$\left(-6+\frac{\omega_1^2}{\omega_0^2}+3\frac{\omega_0^2}{\omega_2^2}\right)a_{12}=a_{32}$$

$\underline{\omega_3^2:}$ $$a_{13}\frac{3\omega_0^2-\omega_3^2}{2\omega_0^2}=a_{23}$$

$$\left(-6+\frac{\omega_g^2}{\omega_0^2}+3\frac{\omega_0^2}{\omega_3^2}\right)a_{13}=a_{33}$$

$$\Theta_i=\sum_{k=1}^{3}a_{ik}\underbrace{C_k e^{i\omega_k t}}_{Q_k}$$

$$\Theta_1=a_{11}Q_1+a_{12}Q_2+a_{13}Q_3$$

<0

~ 1.2921127 ~ 0.35286 ~ -1.64497255

$$\Theta_2=\frac{3\omega_0^2-\omega_1^2}{2\omega_0^2}a_{11}Q_1+\frac{3\omega_0^2-\omega_2^2}{2\omega_0^2}a_{12}Q_2+\frac{3\omega_0^2-\omega_3^2}{2\omega_0^2}a_{13}Q_3$$

$$\Theta_3=\left(-6+\frac{\omega_1^2}{\omega_0^2}+3\frac{\omega_0^2}{\omega_1^2}\right)a_{11}Q_1+\left(-6+\frac{\omega_2^2}{\omega_0^2}+3\frac{\omega_0^2}{\omega_2^2}\right)a_{12}Q_2+\left(-6+\frac{\omega_3^2}{\omega_0^2}+3\frac{\omega_0^2}{\omega_3^2}\right)a_{13}Q_3$$

~ 1.631 ~ -2.39812 ~ 0.766897

<1

mod 1: - mase su u fazi, $A_{m^{(3)}}>A_{m^{(2)}}>A_{m^{(1)}}$

mod 2: - $m^{(1)}$ i $m^{(2)}$ su u fazi, $m^{(3)}$ im je u protufazi

, $A_{m^{(3)}}>A_{m^{(1)}}>A_{m^{(2)}}$ *

mod 3: $m^{(1)}$ i $m^{(3)}$ su u fazi, $m^{(2)}$ im je u protufazi

, $A_{m^{(2)}}>A_{m^{(1)}}>A_{m^{(3)}}$

$$\Theta_1(t) = A_1 \cos(\omega_1 t + \alpha_1) + A_2 \cos(\omega_2 t + \alpha_2) + A_3 \cos(\omega_3 t + \alpha_3)$$

$$\Theta_2(t) = \frac{3\omega_0^2 - \omega_1^2}{2\omega_0^2} A_1 \cos(\omega_1 t + \alpha_1) + \frac{3\omega_0^2 - \omega_2^2}{2\omega_0^2} A_2 \cos(\omega_2 t + \alpha_2) + \frac{3\omega_0^2 - \omega_3^2}{2\omega_0^2} A_3 \cos(\omega_3 t + \alpha_3)$$

$$\Theta_3(t) = \left(-6 + \frac{\omega_1^2}{\omega_0^2} + 3\frac{\omega_0^2}{\omega_1^2}\right) A_1 \cos(\omega_1 t + \alpha_1) + \left(-6 + \frac{\omega_2^2}{\omega_0^2} + 3\frac{\omega_0^2}{\omega_2^2}\right) A_2 \cos(\omega_2 t + \alpha_2)$$

$$+ \left(-6 + \frac{\omega_3^2}{\omega_0^2} + 3\frac{\omega_0^2}{\omega_3^2}\right) A_3 \cos(\omega_3 t + \alpha_3)$$

$$\boxed{\begin{aligned} y_1 &= \dot{\theta}_1 \\ y_2 &= \dot{\theta}_2 \\ y_3 &= \dot{\theta}_3 \end{aligned}}$$

$$3\dot{y}_1 + 2\dot{y}_2$$

<u>Optimiziranje za Python!</u>

$$\ddot{\theta}_3 = -3\ddot{\theta}_1 - 2\ddot{\theta}_2 - 3\omega_0^2\theta_1$$

$$2\ddot{\theta}_1 + 2\ddot{\theta}_2 - 3\ddot{\theta}_1 - 2\ddot{\theta}_2 - 3\omega_0^2\theta_1 + 2\omega_0^2\theta_2 = 0$$

$$-\ddot{\theta}_1 - 3\omega_0^2\theta_1 + 2\omega_0^2\theta_2 = 0$$

$$\boxed{\ddot{\theta}_1 = -3\omega_0^2\theta_1 + 2\omega_0^2\theta_2}$$

$$\ddot{\theta}_1 + \ddot{\theta}_2 - 3\ddot{\theta}_1 - 2\ddot{\theta}_2 - 3\omega_0^2\theta_1 + \omega_0^2\theta_3 = 0$$

$$-2\ddot{\theta}_1 - \ddot{\theta}_2 - 3\omega_0^2\theta_1 + \omega_0^2\theta_3 = 0$$

$$6\omega_0^2\theta_1 - 4\omega_0^2\theta_2 - 3\omega_0^2\theta_1 + \omega_0^2\theta_3 = \ddot{\theta}_2$$

$$\boxed{\ddot{\theta}_2 = 3\omega_0^2\theta_1 - 4\omega_0^2\theta_2 + \omega_0^2\theta_3}$$

$$\ddot{\theta}_3 = 9\omega_0^2\theta_1 - 6\omega_0^2\theta_2 - 6\omega_0^2\theta_1 + 8\omega_0^2\theta_2 - 2\omega_0^2\theta_3 - 3\omega_0^2\theta_1$$

$$\boxed{\ddot{\theta}_3 = 2\omega_0^2\theta_2 - 2\omega_0^2\theta_3}$$

$$\dot{y}_1 = -3\omega_0^2\theta_1 + 2\omega_0^2\theta_2$$

$$\dot{y}_2 = 3\omega_0^2\theta_1 - 4\omega_0^2\theta_2 + \omega_0^2\theta_3$$

$$\dot{y}_3 = 2\omega_0^2\theta_2 - 2\omega_0^2\theta_3$$