Regla de Simpson de 3/8

Catalina Guatibonza y Jose Cañón

24 de septiembre de 2023

1. Demostrar que

$$\int_{x_i}^{x_i+3} f(x)\,dx \approx \frac{3h}{8}[f(x_i) + 3f(x_i+1) + 3f(x_i+2) + f(x+3)]$$

Primero usamos los siguientes reemplazos:

$$x_i = x_0 \tag{1}$$
$$x_i + 3 = x_3 \tag{2}$$

Reemplazando en la integral tenemos:

$$\int_{x_0}^{x_3} f(x)\,dx \approx \int_{x_0}^{x_3} P_3(x)\,dx$$

Donde P_3 corresponde a la ecuacion polinomial de Lagrange con grado 3:

$$\begin{aligned} P_3(x) = &\frac{y_o(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{y_1(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ &+ \frac{y_2(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{y_3(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \end{aligned} \tag{3}$$

Sabemos que $h = \frac{b-a}{n}$, donde a y b corresponden a la posicion en el eje x y n al numero de divisiones o rectangulos que se crean, entonces graficando nuestras x tendriamos las siguientes equivalencias como se muestra en la figura 1:

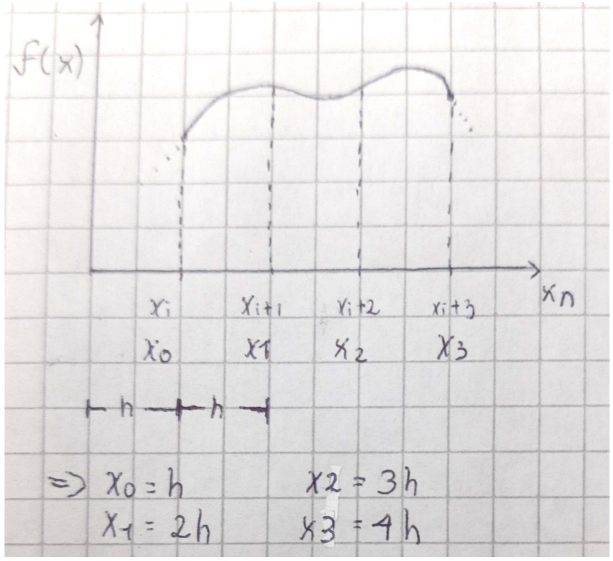

Figura 1: Valores de x_0, x_1, x_2, x_3

Reemplazando estos valores de x_n en (3) y operando, se obtiene:

$$\begin{aligned} P_3(x) = {} & \frac{y_o(x-x_1)(x-x_2)(x-x_3)}{(-h)(-2h)(-3h)} + \frac{y_1(x-x_0)(x-x_2)(x-x_3)}{(h)(-h)(-2h)} \\ & + \frac{y_2(x-x_0)(x-x_1)(x-x_3)}{(2h)(h)(-h)} + \frac{y_3(x-x_0)(x-x_1)(x-x_2)}{(3h)(2h)(h)} \end{aligned}$$

$$\begin{aligned} P_3(x) = {} & \frac{y_o(x-x_1)(x-x_2)(x-x_3)}{-6h^3} + \frac{y_1(x-x_0)(x-x_2)(x-x_3)}{2h^3} \\ & + \frac{y_2(x-x_0)(x-x_1)(x-x_3)}{-2h^3} + \frac{y_3(x-x_0)(x-x_1)(x-x_2)}{6h^3} \end{aligned} \tag{4}$$

Ahora repartimos la integral para cada fracción:

$$\begin{aligned} P_3(x) = {} & \int_{x_0}^{x_3} \frac{y_o(x-x_1)(x-x_2)(x-x_3)}{-6h^3}\,dx + \int_{x_0}^{x_3} \frac{y_1(x-x_0)(x-x_2)(x-x_3)}{2h^3} \\ & + \int_{x_0}^{x_3} \frac{y_2(x-x_0)(x-x_1)(x-x_3)}{-2h^3} + \int_{x_0}^{x_3} \frac{y_3(x-x_0)(x-x_1)(x-x_2)}{6h^3} \end{aligned} \tag{5}$$

Vemos que las integrales son de la forma:

$$
\begin{aligned}
f(x) &= \int (x-a)(x-b)(x-c)\,dx \\
&= \frac{(x-a)(x-b)(x-c)^2}{2} - \frac{(x-c)^4}{4} + (-c+\frac{a}{2}+\frac{b}{2})(\frac{(x-c)^3}{3})
\end{aligned}
\tag{6}
$$

Con esta propiedad se solucionara cada integral por separado, empezando por la primera:

$$
\begin{aligned}
I_1(x) &= \int_{x_0}^{x_3} \frac{y_o(x-x_1)(x-x_2)(x-x_3)}{-6h^3}\,dx \\
&= \frac{y_0}{-6h^3}\int_{x_0}^{x_3}(x-x_1)(x-x_2)(x-x_3)\,dx \\
&= \frac{y_0}{-6h^3}[\frac{(x-x_1)(x-x_2)(x-x_3)^2}{2} - \frac{(x-x_3)^4}{4} + (-x_3+\frac{x_1}{2}+\frac{x_2}{2})(\frac{(x-x_3)^3}{3})]
\end{aligned}
$$

Evaluamos los limites de la integral y se cancela todo el primer termino donde $x = x_3$

$$
\begin{aligned}
I_1(x) &= \frac{y_0}{-6h^3}[-[\frac{(x_0-x_1)(x_0-x_2)(x_0-x_3)^2}{2} - \frac{(x_0-x_3)^4}{4} + (-x_3+\frac{x_1}{2}+\frac{x_2}{2})(\frac{(x_0-x_3)^3}{3})]] \\
&= \frac{y_0}{6h^3}[\frac{(-h)(-2h)(-3h)^2}{2} - \frac{(-3h)^4}{4} + (\frac{-3h}{2})(\frac{(-3h)^3}{3})] \\
&= \frac{y_0}{6h^3}[9h^6 - \frac{81h^4}{4} + \frac{-27h^4}{2}] \\
&= \frac{y_0}{6h^3}[\frac{9h^4}{4} \\
&= \frac{3}{8}y_0h
\end{aligned}
$$

Continuamos con la segunda integral:

$$
\begin{aligned}
I_2(x) &= \int_{x_0}^{x_3} \frac{y_1(x-x_0)(x-x_2)(x-x_3)}{2h^3}\,dx \\
&= \frac{y_1}{2h^3}\int_{x_0}^{x_3}(x-x_0)(x-x_2)(x-x_3)\,dx \\
&= \frac{y_1}{2h^3}[\frac{(x-x_0)(x-x_2)(x-x_3)^2}{2} - \frac{(x-x_3)^4}{4} + (-x_3+\frac{x_0}{2}+\frac{x_2}{2})(\frac{(x-x_3)^3}{3})]
\end{aligned}
$$

Nuevamente evaluamos los limites de la integral y se cancela todo el primer termino donde $x = x_3$

$$
\begin{aligned}
I_2(x) &= \frac{y_1}{2h^3}[-[\frac{(x_0-x_0)(x_0-x_2)(_0x-x_3)^2}{2} - \frac{(x_0-x_3)^4}{4} + (-x_3 + \frac{x_0}{2} + \frac{x_2}{2})(\frac{(x_0-x_3)^3}{3})]] \\
&= \frac{y_1}{2h^3}[\frac{(-3h)^4}{4} - (-2h)(-9h^3)] \\
&= \frac{y_1}{2h^3}[\frac{(81h)^4}{4} - (18h^4)] \\
&= \frac{y_1}{2h^3}[\frac{9h^4}{4}] \\
&= \frac{9}{8}y_1 h
\end{aligned}
$$

Continuamos con la tercera integral:

$$
\begin{aligned}
I_3(x) &= \int_{x_0}^{x_3} \frac{y_2(x-x_0)(x-x_1)(x-x_3)}{-2h^3}\,dx \\
&= \frac{y_2}{-2h^3}\int_{x_0}^{x_3}(x-x_0)(x-x_1)(x-x_3)\,dx \\
&= \frac{y_2}{-2h^3}[\frac{(x-x_0)(x-x_1)(x-x_3)^2}{2} - \frac{(x-x_3)^4}{4} + (-x_3 + \frac{x_0}{2} + \frac{x_1}{2})(\frac{(x-x_3)^3}{3})]
\end{aligned}
$$

Nuevamente evaluamos los limites de la integral y se cancela todo el primer termino donde $x = x_3$

$$
\begin{aligned}
I_3(x) &= \frac{y_2}{-2h^3}[-[\frac{(x_0-x_0)(x-x_1)(x-x_3)^2}{2} - \frac{(x_0-x_3)^4}{4} + (-x_3 + \frac{x_0}{2} + \frac{x_1}{2})(\frac{(x_0-x_3)^3}{3})]] \\
&= \frac{y_2}{-2h^3}[-[-\frac{(-3h)^4}{4} + (\frac{-5h}{2})(-9h^3)]] \\
&= \frac{y_2}{-2h^3}[-(-\frac{-81h^4}{4} + \frac{45h^4}{2})] \\
&= \frac{y_2}{-2h^3}[-(\frac{9h^4}{4})] \\
&= \frac{9}{8}y_2 h
\end{aligned}
$$

Continuamos con la cuarta integral:

$$
\begin{aligned}
I_4(x) &= \int_{x_0}^{x_3} \frac{y_3(x-x_0)(x-x_1)(x-x_2)}{6h^3}\,dx \\
&= \frac{y_3}{6h^3}\int_{x_0}^{x_3}(x-x_0)(x-x_1)(x-x_2)\,dx \\
&= \frac{y_3}{6h^3}[\frac{(x-x_0)(x-x_1)(x-x_2)^2}{2} - \frac{(x-x_2)^4}{4} + (-x_2+\frac{x_0}{2}+\frac{x_1}{2})(\frac{(x-x_2)^3}{3})] \\
&= \frac{y_3}{6h^3}[[\frac{(x_3-x_0)(x_3-x_1)(x_3-x_2)^2}{2} - \frac{(x_3-x_2)^4}{4} + (-x_2+\frac{x_0}{2}+\frac{x_1}{2})(\frac{(x_3-x_2)^3}{3})] \\
&- [\frac{(x_0-x_0)(x_0-x_1)(x_0-x_2)^2}{2} - \frac{(x_0-x_2)^4}{4} + (-x_2+\frac{x_0}{2}+\frac{x_1}{2})(\frac{(x_0-x_2)^3}{3})]] \\
&= \frac{y_3}{6h^3}[[3h^4 - \frac{h^4}{4} + (\frac{-3h}{2})(\frac{h^3}{3})] - [-4h^4 + (\frac{-3h}{2})(\frac{-8h^3}{3})]] \\
&= \frac{y_3}{6h^3}[[3h^4 - \frac{h^4}{4} - \frac{h^4}{2}] - [-4h^4 + 4h^4]] \\
&= \frac{y_3}{6h^3}[\frac{9h^4}{4}] \\
&= \frac{3}{8}y_3h
\end{aligned}
$$

Reemplazando en (5) se obtiene:

$$
\begin{aligned}
P_3(x) &= I_1(x) + I_2(x) + I_3(x) + I_4(x) \\
&= \frac{3}{8}y_0h + \frac{9}{8}y_1h + \frac{9}{8}y_2h + \frac{3}{8}y_3h \\
&= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)
\end{aligned}
$$

Sabemos que $f(x_0) = y_0$ entonces:

$$
P_3(x) = \frac{3h}{8}(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)) \tag{7}
$$

Ahora revirtiendo la sustitucion de $x_0 = x_1$ y similares se obtiene:

$$
P_3(x) = \frac{3h}{8}(f(x_i) + 3f(x_i+1) + 3f(x_i+2) + f(x_i+3)) \tag{8}
$$

Lo cual demuestra la igualdad presentada al inicio.