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THE CLOTHOID COMPUTATION: A SIMPLE AND EFFICIENT NUMERICAL ALGORITHM

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ABSTRACT

The clothoid (also known as Cornu spiral or Euler spiral) is a curve that is characterized by its curvature being proportional to its length. This property makes it very useful as a transition curve when designing the layout of roads and railway tracks. In this paper, we analyse two methods for computing the clothoid: the classical method, which is based on the use of explicit formulas obtained from Taylor expansions of sine and cosine functions, and an alternative algorithm, which is based on the numerical solution of the initial value problems giving the clothoid parametrization. This alternative method is simple and efficient and its effectiveness is shown through its application to three classical problems of horizontal road alignment.

Keywords: Clothoid, transition curves, horizontal road alignment, Runge-Kutta methods

INTRODUCTION

In horizontal road alignments, *transition curves* are used to connect two stretches having different radii of curvature $R_1 \neq R_2$, so that there is a smooth change from R_1 to R_2. This smooth curvature change means a gradual increase/decrease of centrifugal force experienced by the vehicle, which, in addition to avoid the disturbance passengers, significantly reduce the risk of accident. Moreover, transition curves for highway vertical alignments have been introduced, and analysis of these curves was recently presented (Easa and Hassan 2000a; Easa and Hassan 2000b; Kobryn 2015). Recent works suggest different types of curves as transition curves (Tari and Baykal 2005; Kobryn 2011; Bosurgi and DAndrea 2012; Eliou and Kaliabetsos 2014; Kobryn 2014), but the transition curve that has been most commonly used in design of roads is the clothoid (Baass 1984; Kobryn 1993; Baykal et al. 1997; Dong et al. 2007).

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Although the clothoid (also called Euler spiral or Cornu spiral) has been known for hundreds of years, it is currently widely used as a transition curve in the design of roads and railways and continues to be subject of interesting mathematical studies (see, for example, Korkut et al. 2008; Narajan 2014). First works on this curve date from the end of the seventeenth century. These are framed in the study of an elasticity problem, introduced in 1694 by the Swiss mathematician Jakob Bernoulli (1654-1705), and successfully solved, in the middle of the eighteenth century, by another great Swiss scientist, Leonhard Euler (1701-1783). Some years later, the curve was rediscovered by two French physicists and used to study the diffraction of light: first, Augustin-Jean Fresnel (1788-1827) parametrized the curve in terms of integrals (the famous Fresnel integrals) and later, Marie Alfred Cornu (1841-1902) used this parameter to draw the curve accurately. Finally, its use as a transition curve for railways was introduced by the American engineer Arthur Newell Talbot (1857-1942) in the late nineteenth century (more details on the historical evolution of this curve can be found in Levien 2008).

In the next section of this paper, from the properties characterizing the clothoid, we recall how to obtain the initial value problems giving the arc length parametrization. Next, we compare two numerical methods to compute the clothoid: the classical method used in surveying engineering (based on the use of explicit formulas obtained from Taylor expansions of sine and cosine functions) and an alternative algorithm, based on the numerical solution of the initial value problems giving the clothoid parametrization. This alternative method is simple and efficient and its effectiveness its shown through its application to three problems of horizontal road alignment: connecting an oriented straight stretch with an arc of an oriented circle, connecting two oriented straight stretches with different directions, and connecting two oriented circles. Finally, in the last section, we present some brief and interesting conclusions.

ARC LENGTH PARAMETRIZATION

The clothoid is a curve in which, at every point, the product of the radius of curvature by the arch length is constant. This property is what makes it useful as transition curve and allows for easy determination of its arc length parametrization.

Let C be a smooth plane curve of length L, and denote by $r(s) = (x(s), y(s))$ its arc length parametrization, $s \in [0, L]$. Because $r'(s) = (x'(s), y'(s))$ is a unit vector, if $\Phi(s) \in [0, 2\pi)$ denotes the angle of the vector tangent to the curve C at the point $r(s)$ with the positive abscissa axis (OX^+) (see Fig. 1), it verifies that

$$\begin{cases} x'(s) = \cos\Phi(s), & s \in (0, L), \\ y'(s) = \sin\Phi(s), & s \in (0, L). \end{cases} \tag{1}$$

On the other hand, the curvature of C in the point $r(s)$ is defined as

$$\kappa(s) = ||r''(s)||.$$

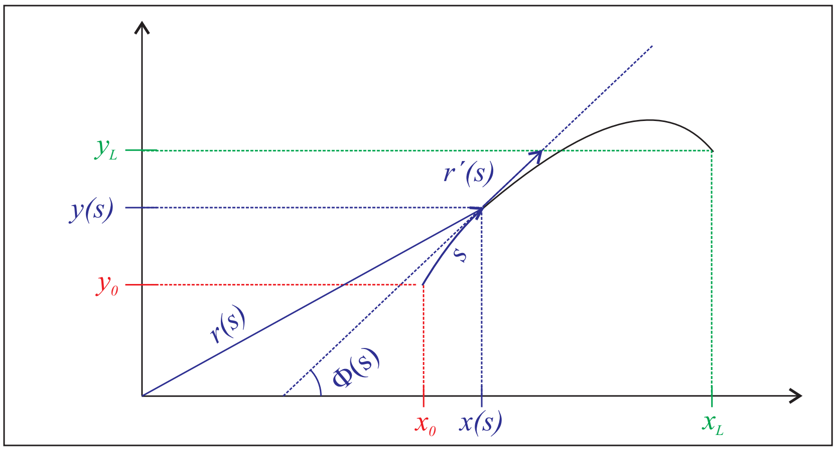

FIG. 1. Curve C: Arc length parametrization.

In this case,

$$r''(s) = (x''(s), y''(s)) = \Phi'(s)(-\sin\Phi(s), \cos\Phi(s)),$$

and consequently,

$$\kappa(s) = |\Phi'(s)|.$$

Similarly, the radius of curvature is defined as the inverse of the curvature, and therefore

$$r_c(s) = \frac{1}{|\Phi'(s)|}, \quad s \in (0, L).$$

Because C is a clothoid, the product of the radius of curvature by the arch length is constant, and there must be $A \in \mathbb{R}$ (called the clothoid parameter) verifying,

$$s\, r_c(s) = A^2, \quad s \in [0, L], \tag{2}$$

that is,

$$|\Phi'(s)| = \frac{s}{A^2}, \quad s \in (0, L).$$

Thus, if Φ_0 denotes the angle of the tangent vector to the curve C in the point $r(0) = (x_0, y_0)$ with OX^+, then

$$\Phi(s) = \lambda \frac{s^2}{2A^2} + \Phi_0, \quad s \in [0, L], \tag{3}$$

where $\lambda = 1$ if $\Phi(s)$ is increasing (if the clothoid is traversed in the counter-clockwise direction) and $\lambda = -1$ if $\Phi(s)$ is decreasing (if the clothoid is traversed in the clockwise direction), see Fig. 3.

Replacing the above expression in Eq. (1), the clothoid parametrization is given as the solution of the following initial value problems:

$$\begin{cases} x'(s) = \cos\left(\lambda \dfrac{s^2}{2A^2} + \Phi_0\right), & s \in (0, L), \\ x(0) = x_0. \end{cases} \tag{4}$$

$$\begin{cases} y'(s) = \sin\left(\lambda \dfrac{s^2}{2A^2} + \Phi_0\right), & s \in (0, L), \\ y(0) = y_0. \end{cases} \tag{5}$$

Remark 1. *In some applications, for example, to compute the clothoid linking a straight stretch with a given point of a circle arc (see Fig. 4), the initial point $r(0) = (x_0, y_0)$ is not known, but the end point $r(L) = (x_L, y_L)$ is known (it is the given point). In this case, we have to change in (4)-(5) the initial conditions $x(0) = x_0$ and $y(0) = y_0$ by the end conditions $x(L) = x_L$, $y(L) = y_L$, so that we have to solve two final value problems instead of two initial value problems.*

Remark 2. *The clothoid parameter A is completely determined if the radius of curvature is known at any point of the curve (for any value of s). For example, if we want to compute the clothoid with radius of curvature R at the end point (when $s = L$), Eq. (2) leads to $RL = A^2$ and, consequently, the clothoid will given by parameter $A = \sqrt{RL}$.*

Similarly, for any value of $A > 0$, the expression (2) implies

$$\lim_{s \to 0^+} r_c(s) = +\infty,$$

which means that the curvature of any clothoid at the initial point $r(0) = (x_0, y_0)$ is null. We say that two curves "link" at a point when the two curves have the same slope and the same curvature in that point. In this sense, all clothoids link at the starting point with the straight line passing through that point with slope given by $\Phi(0) = \Phi_0$.

NUMERICAL COMPUTATION

As we have seen, for a given parameter A, there exist two clothoids coming out of the point $r(0) = (x_0, y_0)$ with slope given by $\Phi(0) = \Phi_0$. Once we have decided which of the two clothoids we want to get (once the value of λ is fixed), we only have to solve problems (4)-(5) in order to compute it. The differential equations that govern these problems are trivial, and so

$$x(s) = \int_0^s \cos\left(\lambda \frac{\tau^2}{2A^2} + \Phi_0\right) d\tau + x_0, \qquad s \in [0, L], \tag{6}$$

$$y(s) = \int_0^s \sin\left(\lambda \frac{\tau^2}{2A^2} + \Phi_0\right) d\tau + y_0, \qquad s \in [0, L]. \tag{7}$$

From these expressions we should emphasize that:

1. The clothoid of parameter A starting at $r(0) = (0, 0)$, tangent to OX^+ ($\Phi(0) = 0$) and for $s > 0$ that is above the axis OX ($\lambda = 1$), is given by

$$\hat{x}(s) = \int_0^s \cos\left(\frac{\tau^2}{2A^2}\right) d\tau, \qquad s \in [0, L], \tag{8}$$

$$\hat{y}(s) = \int_0^s \sin\left(\frac{\tau^2}{2A^2}\right) d\tau, \qquad s \in [0, L]. \tag{9}$$

   Any other clothoid of parameter A can be obtained by a translation (moving the origin to the point (x_0, y_0)), a rotation (of angle Φ_0) and, eventually (when $\lambda = -1$), a symmetry with respect to the new axis (the axis OX translated and rotated). This property makes it necessary to compute that clothoid, call it *standard*, and then apply to it the required basic transformations. This approach is widely used in surveying engineering and in the next section (*The classical method*) we recall a widespread method for the computation of that clothoid.
2. The integrals appearing in Eqs. (6)-(7) have been thoroughly studied and, for example, for the case $\Phi_0 = 0$, with the introduction of a new dummy variable $\sigma = \tau/(\sqrt{\pi}A)$ in (8)-(9), these integrals are transformed into the known Fresnel integrals (see, for example, (Abramowitz and Stegun 1972) and the references therein). The direct computation of these integrals is not possible, and it is necessary to use some kind of numerical approximation. Thus, it can be more useful to employ a numerical method for directly solving the initial value problems (4)-(5), rather than integrals in Eqs. (6)-(7). This approach, equivalent to the other (see remark 3), has the advantage of, by applying the method in a previous step, allowing for more control over the model. The section discussing the *alternative method* presents an example of the algorithm obtained by applying the simplest numerical method to solve the problems (4)-(5).

The classical method

As we have said, any clothoid of parameter A can be obtained by basic transformations from the standard clothoid, given by Eqs. (8)-(9). To compute the integrals in Eqs. (8)-(9), the following algorithm is widely used:

1. Consider the McLaurin polynomials of a certain degree for $\cos x$ and $\sin x$.
2. Replace the values of the integrands in Eqs. (8)-(9) with these polynomials evaluated at point $\tau^2/(2A^2)$.
3. Compute the resulting integral with the use of Barrow's rule.

For example, if we consider the polynomial of degree ten for $\cos x$ and degree eleven for $\sin x$, we obtain the following approximations (see Abramowitz and

Stegun 1972):

$$\hat{x}(s) \approx s - \frac{s^5}{2!\,5(\sqrt{2}A)^4} + \frac{s^9}{4!\,9(\sqrt{2}A)^8} - \frac{s^{13}}{6!\,13(\sqrt{2}A)^{12}} + \frac{s^{17}}{8!\,17(\sqrt{2}A)^{16}} - \frac{s^{21}}{10!\,21(\sqrt{2}A)^{20}}, \qquad s \in [0, L], \tag{10}$$

$$\hat{y}(s) \approx \frac{s^3}{2!\,3A^2} - \frac{s^7}{3!\,7(\sqrt{2}A)^6} + \frac{s^{11}}{5!\,11(\sqrt{2}A)^{10}} - \frac{s^{15}}{7!\,15(\sqrt{2}A)^{14}} + \frac{s^{19}}{9!\,19(\sqrt{2}A)^{18}} - \frac{s^{23}}{11!\,23(\sqrt{2}A)^{22}}, \qquad s \in [0, L]. \tag{11}$$

It should be noted that the McLaurin series of $\cos x$ and $\sin x$ are approximations of these functions in the neighborhood of zero, so that these approaches are only good for small values of $\tau^2/(2A^2)$. However, the dummy variable τ takes values in the interval $[0, s]$ and, consequently, the previous approaches can present problems for large values of s. To illustrate this fact, in Fig. 2 we draw the standard clothoid of parameter $A = 17.32$ with a dashed line, and the approximation given by Eqs. (10)-(11) with a solid line. We observe that the approximation is very good for small values of s, but after a certain value, as expected, it becomes much less accurate.

Despite the limitations, it is essential to note that the approaches (10)-(11) are perfectly valid for their applications in surveying engineering. In effect, as mentioned in remark 2, the clothoid ending with radius of curvature R corresponds with a parameter $A = \sqrt{RL}$. Consequently, the approximations (10)-(11) are good if $s^2/(2LR)$ is not too large, and because of $s \in [0, L]$, this happens whenever $L/(2R)$ is not too large. The numerical experiments show that these approximations are good enough for values of $L/(2R)$ lower than three. After this value, the clothoid is distorted and stops being useful as transition curve.

An alternative method

An alternative way to proceed is to obtain the desired clothoid parametrization using a numerical method to solve the problems (4)-(5). There are many methods that can be used (see, eg, Atkinson et al. 2009) and each one of them leads to a different algorithm. As an example, we show the algorithm obtained by using the simple explicit Euler's method:

Let $L > 0$, we choose a natural number $N \in \mathbb{N}$, define $\Delta s = L/N$ and, for each $n = 0, 1, \ldots, N$, consider $s^n = n\Delta s$. Euler's method is based on the approximations

$$x'(s^n) = \frac{x(s^{n+1}) - x(s^n)}{\Delta s}, \qquad y'(s^n) = \frac{y(s^{n+1}) - y(s^n)}{\Delta s}.$$

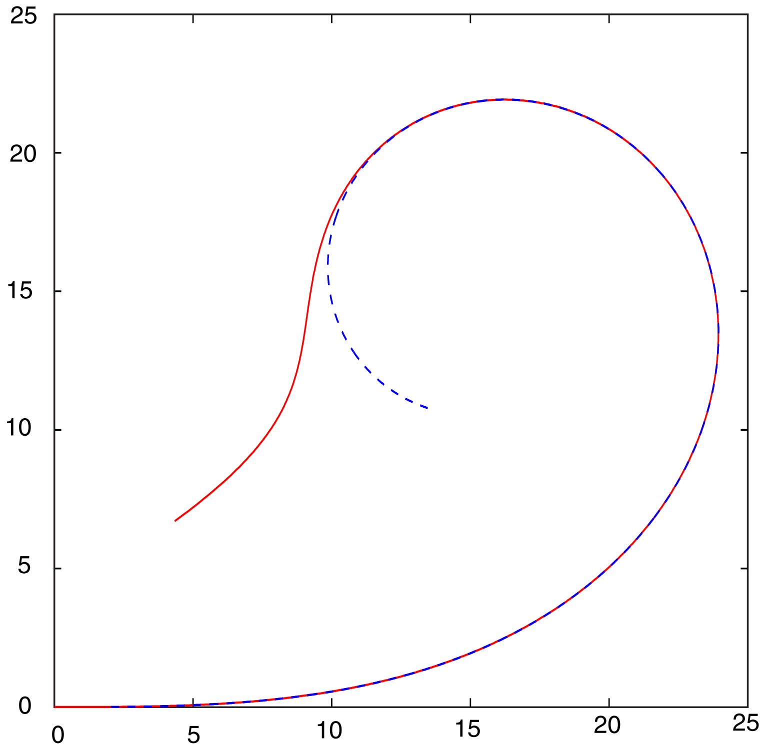

FIG. 2. Approximations of the clothoid of parameter $A = 17.32$, for an arc length $L = 60$; the solid line was computed with the classical method, Eqs. (10)-(11), and the dashed line with the alternative method, Eqs. (12).

Thus, we define $x^0 = x_0$, $y^0 = y_0$, and for all $n = 0, \ldots, N-1$ compute

$$\begin{aligned} x^{n+1} &= x^n + \Delta s \cos\left(\lambda \frac{(s^n)^2}{2A^2} + \Phi_0\right), \\ y^{n+1} &= y^n + \Delta s \sin\left(\lambda \frac{(s^n)^2}{2A^2} + \Phi_0\right), \end{aligned} \tag{12}$$

and accept the approximations $x(s^{n+1}) \approx x^{n+1}$, $y(s^{n+1}) \approx y^{n+1}$.

Remark 3. *The method just presented is the same as that obtained if we compute the integrals in Eqs. (6)-(7) with the left rectangles formula. Similarly, if we solve problems (4)-(5) with the modified Euler's method, the algorithm would be the same if we compute the integrals in Eqs. (6)-(7) with the trapezoidal rule. In general, using a numerical method to solve Eqs. (4)-(5) is equivalent to approximating the integrals in Eqs. (6)-(7) with a numerical quadrature formula.*

Despite its simplicity, the previous algorithm is effective and, if N is *large enough* (Δs is *small enought*), the values of x^n and y^n obtained are good approximations of $x(s^n)$ and $y(s^n)$. In fact, we have the following error estimates (see

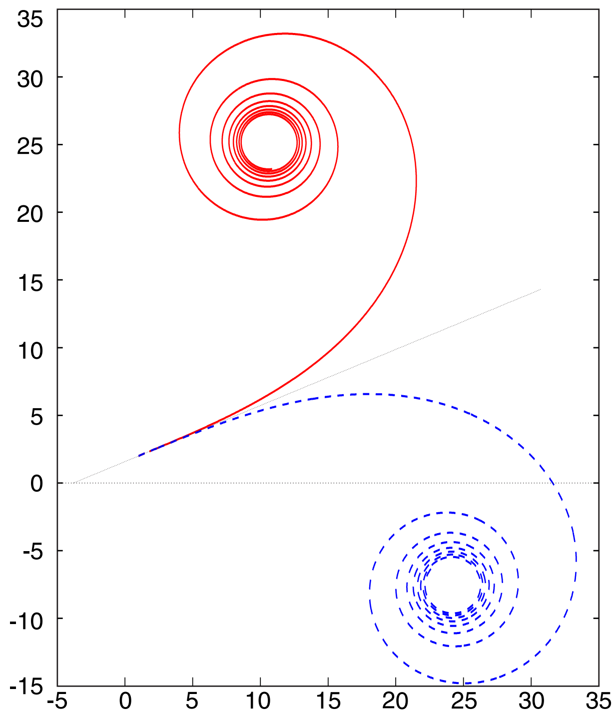

FIG. 3. Approximations of the two clothoids of parameter $A = 20$ that link at point $(x_0, y_0) = (1, 2)$ with the straight line of slope given by $\Phi_0 = \pi/8$. The solid line corresponds with $\lambda = 1$ and the dashed line with $\lambda = -1$.

Appendix 1) which guarantee previous affirmation:

$$|x^n - x(s^n)| \leq \frac{(\Delta s)^3 n(n+1)}{4A^2},$$
$$|y^n - y(s^n)| \leq \frac{(\Delta s)^3 n(n+1)}{4A^2}.$$

To ilustrate the accuracy of both methods (classical and alternative) we compute the standard clothoid of parameter $A = 17.32$ (see Fig. 2) with both algorithms. Table 1 shows the coordinates of some points obtained by both methods, the differences between them, and the error estimate for the alternative method. We can observe that the error estimate is always small (the alternative method had good accuracy). For small values of s both methods gave similar points, but for large values of s the difference were much larger than the error estimate, showing the poor performance of the classical method for large values of s (which is also reflected in Fig. 2). The good performance of the alternative method is illustrated in Fig. 3, which shows the approximations of the two clothoids of parameter $A = 20$ that link at point $(x_0, y_0) = (1, 2)$ with the straight line of slope given by $\Phi_0 = \pi/8$. Unlike with the result obtained with the use of the method described in the previous section, this algorithm directly computes the needed clothoid (without later translations, rotations and/or symmetries) and also its behavior is acceptable for large values of s.

TABLE 1. Comparision of classical method and alternative method for computing a standard clothoid of parameter $A = 17.32$

	Alternative		Classical		Differences		Error estimate
$s = s^n$	x^n	y^n	$\hat{x}(s^n)$	$\hat{y}(s^n)$	$\lvert x^n - \hat{x}(s^n)\rvert$	$\lvert y^n - \hat{y}(s^n)\rvert$	$\frac{(\Delta s)^3 n(n+1)}{4A^2}$
0	0	0	0	0	0	0	0
15	14.7908	1.8541	14.7904	1.8563	0.0004	0.0022	0.0023
30	23.9233	12.7493	23.9177	12.7553	0.0056	0.0060	0.0090
45	14.6790	21.7373	14.6594	21.7340	0.0196	0.0033	0.0203
55	10.1555	14.0968	9.0509	13.6903	1.1046	0.4065	0.0303
60	13.6087	10.7421	4.3401	6.7009	9.2685	4.0411	0.0360

APPLICATIONS

In this section we present three examples in which the clothoid is used as transition curve. First, the clothoid is used to connect a straight stretch oriented (given by a free vector $\mathbf{v}$) with a particular point in a given oriented circle. Second, it is used to design the layout of a road, with known principal directions (intersecting lines) and the permitted minimum radius of curvature. Finally, by combining the previous applications, we deal with the transition between two oriented circles.

Connecting an oriented straight stretch with an arc of an oriented circle

Suppose we headed for a straight stretch whose direction and sense are determined by a known vector $\mathbf{v}$ (see Fig. 4). We want to connect with a point $F = (x_F, y_F)$ of a circle centered on $C = (x_C, y_C)$, to be traveled with an orientation given by the indicator λ_C ($\lambda_C = -1$ clockwise, $\lambda_C = 1$ counterclockwise). We want to obtain the shortest clothoid providing such connection. Then we observe:

1. Obviously, the sign of the clothoid (λ) is determined by the circle orientation, such that $\lambda = \lambda_C$.
2. The end point of the clothoid must be linked to point F. To ensure that (as already proposed in remark 1) we must replace, in Eqs. (4)-(5), the initial conditions for the final terms conditions

$$x(L) = x_F, \quad y(L) = y_F.$$

3. When the clothoid is linked with the circle at point F, both curves must have the same slope and the same radius of curvature. This property determines the length of the clothoid, as follows:

   i. First, we compute the radius of curvature at the link point as the norm of the vector $\mathbf{CF} = \mathbf{OF} - \mathbf{OC}$. Obviously,

$$\mathbf{CF} = (x_F - x_C, y_F - y_C)$$

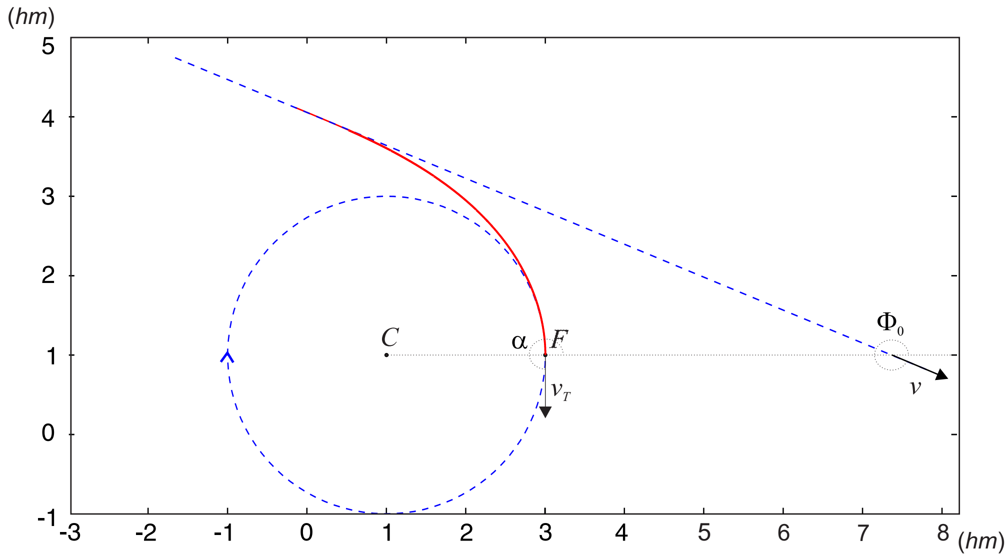

FIG. 4. Clothoid connecting a straight stretch -with direction and sense given by $v = (\cos(\frac{15\pi}{8}), \sin(\frac{15\pi}{8}))$-, with the point $F = (3, 1)$ of the circle centered on $C = (1, 1)$ and clockwise oriented.

and

$$R = \sqrt{(x_F - x_C)^2 + (y_F - y_C)^2}.$$

ii. From vector **CF**, and taking into account the orientation of the circle, we compute the tangent vector at point F, which is given by

$$\mathbf{v_T} = \lambda_C(y_C - y_F, x_F - x_C).$$

iii. We denote by $\Phi_0 \in [0, 2\pi)$ and $\alpha \in [0, 2\pi)$ the angles of vectors **v** and $\mathbf{v_T}$, respectively, with OX^+ (see Fig. 4).

iv. For the clothoid and circle to have the same slope at the link point, it must be verified that $\Phi(L) = \alpha$. Then, using Eq. (3) and taking into account that $A = \sqrt{RL}$, we obtain

$$\lambda \frac{L}{2R} + \Phi_0 = \alpha. \tag{13}$$

If we denote by $\tau \in [0, 2\pi)$ the angle that must be rotated (*rotation angle*), we can write

$$\tau = \begin{cases} \lambda(\alpha - \Phi_0) & \text{if } \lambda(\alpha - \Phi_0) \geq 0, \\ 2\pi + \lambda(\alpha - \Phi_0) & \text{if } \lambda(\alpha - \Phi_0) < 0, \end{cases}$$

and, from Eq. (13) we obtain that $L = 2R\tau$.

Taking these calculations into account, the clothoid is now completely determined: we have the end point, the minimum radius R, its length L (consequently its parameter $A = \sqrt{RL}$) and also the value of λ and Φ_0. The alternative method described in previous section gives the clothoid as follows:

Define $x^N = x_F$, $y^N = y_F$; for all $n = N-1, \dots, 0$ compute

$$\begin{aligned} x^n &= x^{n+1} - \Delta s \cos\left(\lambda \frac{(s^n)^2}{2A^2} + \Phi_0\right), \\ y^n &= y^{n+1} - \Delta s \sin\left(\lambda \frac{(s^n)^2}{2A^2} + \Phi_0\right), \end{aligned} \tag{14}$$

and approach $r(s^n) = (x(s^n), y(s^n)) \approx (x^n, y^n)$.

As an example, Fig. 4 shows the clothoid connecting a straight stretch, with direction and sense given by $\mathbf{v} = (\cos(\frac{15\pi}{8}), \sin(\frac{15\pi}{8}))$, with the point $F = (3, 1)$ of the circle centered on $C = (1, 1)$ and clockwise oriented. As indicated earlier, it should be noted that with these data the clothoid is uniquely determined. The straight line of linkage (the line linking with the clothoid) must not be a data point, but it is obtained a posteriori, from the *director vector* $\mathbf{v}$, once the initial point of the clothoid, $r(0) \approx (x^0, y^0)$, is computed.

Connecting two oriented straight stretches

Suppose that we want to connect two straight stretches intersecting at a point V. To set directions (and senses) we assume known two points P_1 and P_2 (see Fig. 5). We want to connect the half-line from P_1 to V (r_1) with the other one from V to P_2 (r_2). Once we know the minimum radius of curvature R that can have the road in this stretch, we seek two clothoids (one that begins tangent to r_1, with the sense of vector $\mathbf{P_1V} = \mathbf{OV} - \mathbf{OP_1}$, and another that begins tangent to r_2, with the sense of vector $\mathbf{P_2V} = \mathbf{OV} - \mathbf{OP_2}$). These clothoids must be connected by an arc of circle of radius R and given angle $\omega \geq 0$ (see Fig. 5). Next, we detail the algorithm to compute these clothoids and thereby design the layout of the road (hereinafter, the superscript 1 refers the clothoid that departs from r_1, and 2 refers the one that departs from r_2; matching parameters in both clothoids do not carry superscript):

1. From points P_1, P_2 and V, we compute:

   i. The angle $\Phi_0^1 \in [0, 2\pi)$ between $\mathbf{P_1V}$ and OX$^+$,
   ii. The angle $\Phi_0^2 \in [0, 2\pi)$ between $\mathbf{P_2V}$ and OX$^+$,
   iii. The angle $\beta \in (0, \pi)$ between the two straight lines,
   iv. The signs of clothoids λ^1 and λ^2 (obviously $\lambda^2 = -\lambda^1$).

2. From angle β, we compute the angle $\tau \in (0, \pi/2)$ (the angle that forms the tangent vector to the curve at point F^1 with the vector $\mathbf{P_1V}$-). Fig. 5 shows that $\tau + \omega/2$ is the complementary of $\beta/2$, and thus

$$\tau = \frac{\pi - (\omega + \beta)}{2}.$$

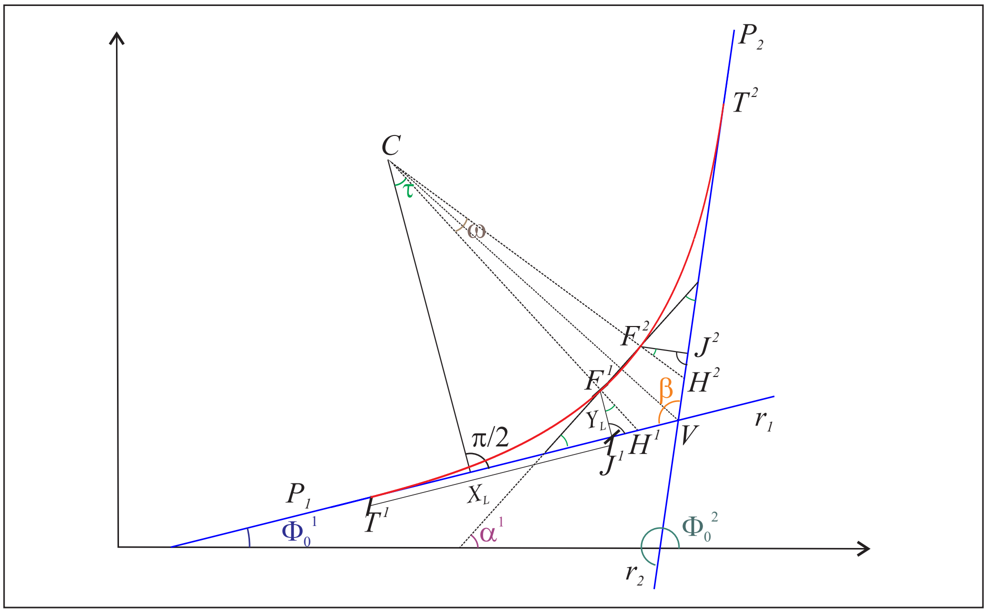

FIG. 5. Scheme of the curve clothoid-circle-clothoid, used to connect two straight stretches.

3. From radius R and angle τ (Fig. 5 shows that $\tau = \alpha^1 - \Phi_0^1$ is the rotation angle), we compute the length of the clothoids. As given in the previous example, $L = 2R\tau$, and the parameter of the clothoids is $A = \sqrt{RL}$.
4. From the parameter A of the clothoids, we compute the points of tangency at r_1 and r_2 (points T^1 and T^2, respectively). We proceed as follows:

   i. We compute the end point (X_L, Y_L) of the standard clothoid of parameter A. For example, using the alternative method, we obtain that

$$X_L = \Delta s \sum_{k=0}^{N-1} \left(\cos \left(\frac{\left(s^k\right)^2}{2A^2} \right) \right), \quad Y_L = \Delta s \sum_{k=0}^{N-1} \left(\sin \left(\frac{\left(s^k\right)^2}{2A^2} \right) \right).$$

   ii. From X_L and Y_L we compute the proper distances of the clothoid (see Fig. 5):

$$\begin{aligned}
dist(T^1, J^1) &= dist(T^2, J^2) &&= X_L, \\
dist(J^1, H^1) &= dist(J^2, H^2) &&= Y_L \tan\tau, \\
dist(H^1, V) &= dist(H^2, V) &&= \left(R + \frac{Y_L}{\cos\tau} \right) \left(\frac{\sin(\omega/2)}{\sin(\beta/2)} \right), \\
dist(T^1, V) &= dist(T^2, V) &&= dist(T^1, J^1) + dist(J^1, H^1) + \\
& && \quad dist(H^1, V).
\end{aligned}$$

iii. From the previous distances, compute the points of tangency T_1 and T_2 given by

$$\mathbf{OT^1} = \mathbf{OV} - dist(T^1, V)\frac{\mathbf{P_1V}}{||\mathbf{P_1V}||},$$
$$\mathbf{OT^2} = \mathbf{OV} - dist(T^2, V)\frac{\mathbf{P_2V}}{||\mathbf{P_2V}||}.$$

5. Taking point T^1 (respectively T^2) as initial point, we use the alternative method given by Eqs. (12) to compute the clothoid of parameter A, initial angle Φ_0^1 (respectively Φ_0^2) and sign λ^1 (respectively λ^2).

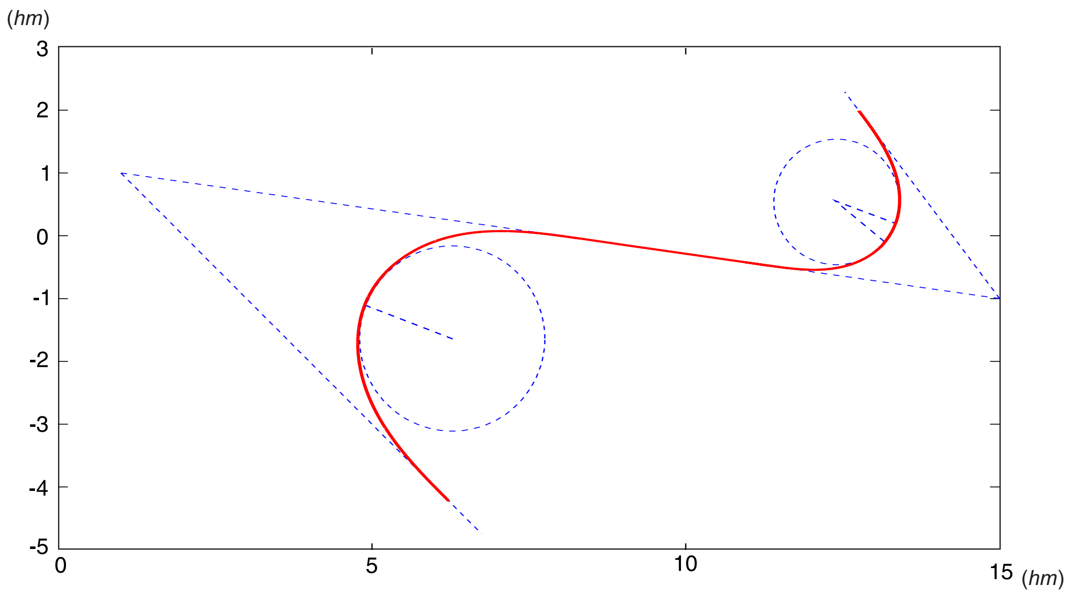

FIG. 6. Layout of a road connecting three straight stretches determined by the points $P_1 = (0, 2)$**,** $V_1 = (1, 1)$**,** $P_2 = V_2 = (15, -1)$ **and** $P_3 = (12, 3)$**.**

The previous algorithm is described for a fixed system of reference and then can be used to connect several different straight stretches with curves of different radius. This can be very useful in the design (layout) of a road. As an example, Fig. 6 shows the path of a road that links three straight stretches, the first two with a clothoid-clothoid curve of minimum radius $R = 1.5\,hm$ and the next two with a clothoid-circle-clothoid curve of radius $R = 1\,hm$ and angle $\omega = \frac{\pi}{8}$. To use the algorithm, the straight lines are determined by the following points: $P_1 = (0, 2)$, $V_1 = (1, 1)$, $P_2 = V_2 = (15, -1)$ and $P_3 = (12, 3)$.

Connecting two oriented circles

Combining previous applications allows for the easy connection of two oriented circles. In this case we suppose that we have two circles centered on $C_1 = (x_{C_1}, y_{C_1})$ and $C_2 = (x_{C_2}, y_{C_2})$, to be traveled with orientations given, respectively, by indicators λ_{C_1} and λ_{C_2}. We want to connect a point of the first circle $F_1 = (x_{F_1}, y_{F_1})$ with another point of the second circle $F_2 = (x_{F_2}, y_{F_2})$, with the use of a transition curve that combines the clothoids (possibly connected by straight stretchs and/or arc circles). The process is as follows:

- *Step 1:* Choose two straight directions (angles Φ_0^1, $\Phi_0^2 \in [0, 2\pi)$) determining the slope of the clothoids at the initial points (Fig. 7). Choose the minimum radius permitted, $R > 0$, and the angle $\omega \geq 0$ determining the arc circle used to combined the clothoids (obviously, if $\omega = 0$ the arc circle is not used). All of these parameters should be choosen by the engineers according to their own preferences or needs (e.g. to avoid obstacles, mimimize the road lenght, maximize safety).
- *Step 2:* As described previously, compute the clothoids connecting the straight stretches with slopes given by Φ_0^1 and Φ_0^2, with the oriented circles at points F_1 and F_2.
- *Step 3:* As described previously, compute the transition curve (clothoid-circle-clothoid or clothoid-clothoid) connecting the two oriented straight stretches completely determined in Step 2.

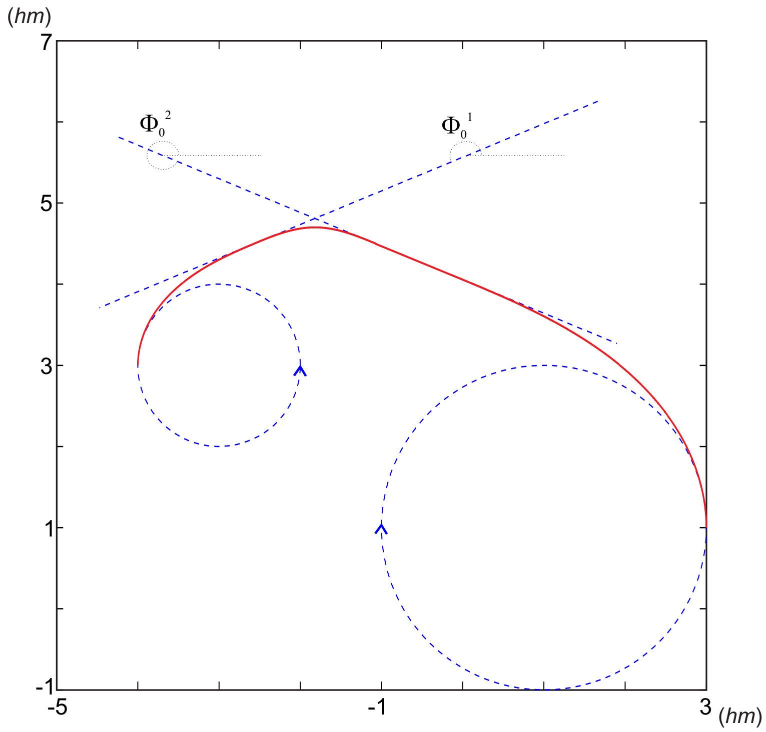

FIG. 7. Transition curve connecting two given points of two oriented cicles

As an example, Fig. 7 shows the transition curve obtained with the previous method to connect, at points $F_1 = (-4, 3)$ and $F_2 = (3, 1)$, the circles centered, respectively, in $C_1 = (-3, 3)$ and $C_2 = (1, 1)$. We assume that the circles have opposite orientations ($\lambda_{C_1} = 1$, $\lambda_{C_2} = -1$), and choose $\Phi_0^1 = \frac{9}{8}\pi$, $\Phi_0^2 = \frac{15}{8}\pi$, $R = 1\,hm$, $\omega = 0$.

Note that, with $\Phi_0^2 = \Phi_0^1 + \pi$, the previous method can be used to connect two circles covered along the same direction (*egg-shaped transition*), and also covered along opposite direction (*reversing circular curves*) (the same sign of λ_{C_1} and λ_{C_2} corresponds to reversing circular curves, and the opposite to egg-shaped transition). Fig. 8 shows both connections for two circles with radii $1\,hm$ and $2.1\,hm$. In this example, $\Phi_0^1 = \frac{7}{8}\pi$ was used for the egg-shaped transition (Fig. 8(a)) and $\Phi_0^1 = \frac{11}{8}\pi$ for the reversing circular curves (Fig. 8(b)).

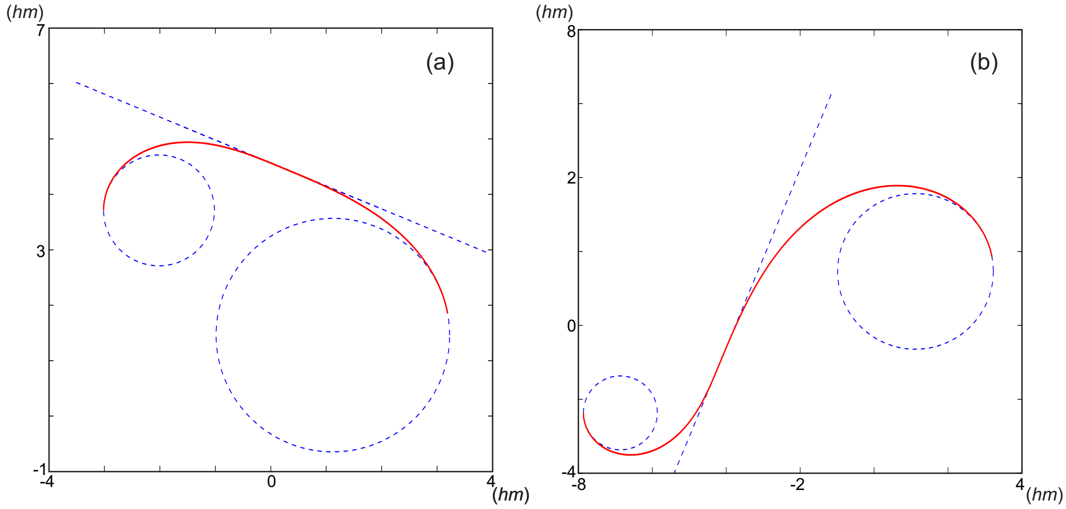

FIG. 8. (a) Egg-shaped transition; (b) transition between two reversing circular curves.

CONCLUSIONS

In this paper we compared two different methods for computing clothoids: a classical method based on the use of explicit formulas obtained from Taylor expansions, and an alternative method based on the numerical solution of the initial value problems appearing in their initial parametrization. This second method does not use complex formulas, and therefore it is conceptually very simple and easy to apply. It also provides good approximations, even when the classical method fails (when the ratio between length and radius of curvature is large), and needs no rotations, translations or subsequent symmetries. It can be easily included in any model (computer application) for horizontal road design, and thus it can be a useful tool for optimizing the horizontal alignment of roads (see Mondal et al. 2015).

APPENDIX 1: ERROR ESTIMATES

In this appendix we detail the proof of the following result, which gives useful error estimates for the alternative method.

Theorem 1. *For all $n = 0, 1, \ldots,$ the approximations given by the expression (12) verify*

$$|x^n - x(s^n)| \leq \frac{(\Delta s)^3 n(n+1)}{4A^2}, \tag{15a}$$

$$|y^n - y(s^n)| \leq \frac{(\Delta s)^3 n(n+1)}{4A^2}, \tag{15b}$$

and consequently

$$|x^n - x(s^n)| \leq \frac{L}{4R}\Delta s + \frac{1}{4R}(\Delta s)^2, \tag{16a}$$

$$|y^n - y(s^n)| \leq \frac{L}{4R}\Delta s + \frac{1}{4R}(\Delta s)^2. \tag{16b}$$

To proof this result, we'll use the following error estimate for the left rectangles formula (see Atkinson et al. 2009):

Lemma 1. *If $f(x)$ is a continuously differentiable function in $[a, b]$, then*

$$\left| \int_a^b f(x)dx - f(a)(b-a) \right| \leq M \frac{(b-a)^2}{2},$$

where $M = \max_{x \in [a,b]} |f'(x)|$.

With a direct application of this lemma, the following result is obtained

Corollary 1.

$$\left| \int_{s^k}^{s^{k+1}} \cos\left(\lambda \frac{\tau^2}{2A^2} + \Phi_0\right) d\tau - \Delta s \cos\left(\lambda \frac{(s^k)^2}{2A^2} + \Phi_0\right) \right| \leq \frac{(k+1)(\Delta s)^3}{2A^2}, \tag{17a}$$

$$\left| \int_{s^k}^{s^{k+1}} \sin\left(\lambda \frac{\tau^2}{2A^2} + \Phi_0\right) d\tau - \Delta s \sin\left(\lambda \frac{(s^k)^2}{2A^2} + \Phi_0\right) \right| \leq \frac{(k+1)(\Delta s)^3}{2A^2}. \tag{17b}$$

Proof. Taking $f(\tau) = \cos\left(\lambda \frac{\tau^2}{2A^2} + \Phi_0\right)$ in $[s^k, s^{k+1}]$, we obtain $|f'(\tau)| \leq \frac{\tau}{A^2}$ and consecuently

$$M = \max_{\tau \in [s^k, s^{k+1}]} |f'(\tau)| \leq \frac{s^{k+1}}{A^2} = \frac{(k+1)\Delta s}{A^2}.$$

The inequality (17a) is now obtained as a direct application of the previous lemma, and (17b) is obtained in a similar way. □

Now, Theorem 1 can be proved:

Proof of Theorem 1. We are going to obtain (15a) and (16a), inequalities (15b) and (16b) can be obtained in a similar way.

The first equality of (12) is equivalent to

$$x^{n+1} = x_0 + \sum_{k=0}^{n} \Delta s \cos\left(\lambda \frac{(s^k)^2}{2A^2} + \Phi_0\right). \tag{18}$$

Eqs. (6) and (18) result in

$$|x^n - x(s^n)| = \left| \sum_{k=0}^{n-1} \Delta s \cos\left(\lambda \frac{(s^k)^2}{2A^2} + \Phi_0\right) - \int_0^{s^n} \cos\left(\lambda \frac{\tau^2}{2A^2} + \Phi_0\right) d\tau \right|.$$

Obviously,

$$\int_0^{s^n} \cos\left(\lambda \frac{\tau^2}{2A^2} + \Phi_0\right) d\tau = \sum_{k=0}^{n-1} \int_{s^k}^{s^{k+1}} \cos\left(\lambda \frac{\tau^2}{2A^2} + \Phi_0\right) d\tau.$$

Thus Eq. (17a) gives

$$\begin{aligned} |x^n - x(s^n)| &\leq \sum_{k=0}^{n-1} \left| \int_{s^k}^{s^{k+1}} \cos\left(\lambda \frac{\tau^2}{2A^2} + \Phi_0\right) d\tau - \Delta s \cos\left(\lambda \frac{(s^k)^2}{2A^2} + \Phi_0\right) \right| \\ &\leq \sum_{k=0}^{n-1} \frac{(k+1)(\Delta s)^3}{2A^2} = \frac{(\Delta s)^3}{2A^2} \sum_{k=1}^{n} k = \frac{(\Delta s)^3 n(n+1)}{4A^2}. \end{aligned}$$

Finally, taking into account that $\Delta s = \frac{L}{N}$, $A = \sqrt{RL}$ and $n \leq N$, we obtain

$$|x^n - x(s^n)| \leq \frac{(\Delta s)^3 N(N+1)}{4RL} = \frac{(\Delta s)^2}{4R}\left(\frac{L}{\Delta s} + 1\right) = \frac{L}{4R}\Delta s + \frac{1}{4R}(\Delta s)^2.$$

□

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REFERENCES

Abramowitz, M. and Stegun, I. (1972). *Handbook of mathematical functions : with formulas, graphs, and mathematical tables*. Dover, New York.

Atkinson, K., Han, W., and Stewart, D. (2009). *Numerical Solution of Ordinary Differential Equations*. John Wiley & Sons, Inc., New Jersey.

Baass, K. (1984). “Use of clothoid templates in highway design.” *Transportation Forum*, 1(3), 47–52.

Baykal, O., Tari, E., Coskun, Z., and Sahin, M. (1997). "New transition curve joining two straight lines." *Journal of Transportation Engineering*, 123(5), 337–345.

Bosurgi, G. and DAndrea, A. (2012). "A polynomial parametric curve (ppc-curve) for the design of horizontal geometry of highways." *Computer-Aided Civil and Infrastructure Engineering*, 27(4), 304–a312.

Dong, H., Easa, S. M. and Li, J. (2007). "Approximate extraction of spiralled horizontal curves from satellite imagery." *Journal of Surveying Engineering*, 133(1), 36–40.

Easa, S. M. and Hassan, Y. (2000a). "Development of transitioned vertical curve. I properties." *Transportation Research Part A: Policy and Practice*, 34(6), 481 – 496.

Easa, S. M. and Hassan, Y. (2000b). "Development of transitioned vertical curve. II sight distance." *Transportation Research Part A: Policy and Practice*, 34(7), 565 – 584.

Eliou, N. and Kaliabetsos, G. (2014). "A new, simple and accurate transition curve type, for use in road and railway alignment design." *European Transport Research Review*, 6(2), 171–179.

Kobryn, A. (1993). "General mathematical transition curves for alignment between two rectilinear road sections." *Zeitschrift fur Vermessungswesen*, 5, 227–242.

Kobryn, A. (2011). "Polynomial solutions of transition curves." *Journal of Surveying Engineering*, 137(3), 71–80.

Kobryn, A. (2014). "New solutions for general transition curves." *Journal of Surveying Engineering*, 140(1), 12–21.

Kobryn, A. (2015). "Vertical arcs design using polynomial transition curves." *KSCE Journal of Civil Engineering*, doi: 10.1007/s12205-015-0492-z.

Korkut, L., Vlah, D., Zubrinic, D., and Zupanovic, V. (2008). "Generalized fresnel integrals and fractal properties of related spirals." *Applied Mathematics and Computation*, 206, 236–244.

Levien, R. (2008). *The Euler spiral: a mathematical history, Technical Report UCB/EECS-2008*, Vol. 111. EECS Department, University of California, Berkeley.

Mondal, S., Lucet, Y., and Hare, W. (2015). "Optimizing horizontal alignment of roads in a specified corridor." *Computers & Operations Research*, 64, 130 – 138.

Narajan, S. (2014). "Approximating cornu spirals by arc splines." *Journal of Computational and Applied Mathematics*, 256, 789–804.

Tari, E. and Baykal, O. (2005). "A new transition curve with enhanced properties." *Canadian Journal of Civil Engineering*, 32(5), 913–923.