Small Open Economy Model with Endogenous Discount Factor and Debt-Elastic Interest Rate

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Based on the content provided from the paper *"Closing small open economy models"* by Schmitt-Grohe and Uribe (2003), we can construct a model that combines features of **Model 1 (endogenous discount factor)**, **Model 1a (simplified endogenous discount factor without internalization)**, and **Model 2 (debt-elastic interest rate)**. The goal is to build a **small open economy model** where:

- Agents have an **endogenous discount factor** that depends on both consumption and working hours.

- The domestic interest rate is **endogenously determined** and increases with the level of net foreign debt.

This setup ensures stationarity in the equilibrium dynamics while maintaining realism in capturing agents' behavior and market imperfections.

1 The model

1.1 Preferences

The household's preferences are given by the recursive utility function:

$$U_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \theta_t U(c_t, h_t)$$

with:

$$\theta_0 = 1, \quad \theta_{t+1} = \beta(c_t, h_t)\theta_t$$

where $\beta(c_t, h_t)$ is the endogenous discount factor depending on aggregate consumption and labor supply. Assume the functional forms:

$$U(c_t, h_t) = \frac{[c_t - \frac{h_t^\omega}{\omega} - 1]^{1-\gamma}}{1-\gamma}$$

$$\beta(c_t, h_t) = \left[1 + c_t - \frac{h_t^\omega}{\omega}\right]^{-\psi_1}$$

$$F(k_t, h_t) = k_t^\alpha h_t^{1-\alpha}$$

$$\Phi(x_t) = \frac{\phi}{2} x_t^2, \quad \phi > 0$$

Where:
- $\psi_1 > 0$: elasticity of the discount factor
- $\omega > 0$: elasticity parameter for labor.
- $\gamma > 0$: elasticity parameter for utility.
- $\alpha \in (0, 1)$: capital share.

1.2 Technology and Production

Output is produced using capital k_t and labor h_t via Cobb–Douglas production:

$$y_t = A_t k_t^{\alpha} h_t^{1-\alpha}$$

Capital evolves as:

$$k_{t+1} = i_t + (1 - \delta) k_t$$

Investment subject to adjustment costs:

$$F(k_{t+1} - k_t) = \frac{\phi}{2} \left(\frac{k_{t+1} - k_t}{k_t} \right)^2$$

1.3 Budget Constraint and Foreign Debt

The budget constraint:

$$d_t = (1 + r_t) d_{t-1} + y_t - c_t - i_t - F(k_{t+1} - k_t)$$

Domestic interest rate:

$$r_t = r^* + p(d_t), \quad p(d_t) = \psi_2 (d_t - \bar{d})$$

1.4 Nonlinear Model Equations

The model includes the following core relationships:

(1) Euler:	$\lambda_t = \beta(c_t, h_t)(1 + r_{t+1})\mathbb{E}_t[\lambda_{t+1}]$	(1)
(2) Marginal utility:	$\lambda_t = (c_t - \frac{h_t^\omega}{\omega})^{-\gamma}$	(2)
(3) Labor supply:	$\gamma(c_t - \frac{h_t^\omega}{\omega}) = (1-\alpha)\frac{y_t}{h_t}$	(3)
(4) Production:	$y_t = A_t k_t^\alpha h_t^{1-\alpha}$	(4)
(5) Capital accumulation:	$k_{t+1} = (1-\delta)k_t + i_t$	(5)
(6) Resource constraint:	$y_t = c_t + i_t + g_t$	(6)
(7) Foreign debt law:	$d_{t+1} = (1 + r_t)d_t + c_t + i_t - y_t$	(7)
(8) Interest rate:	$r_t = r^* + \psi_2(d_t - \bar{d})$	(8)
(9) Endogenous discounting:	$\beta_t = \left(1 + c_t - \frac{h_t^\omega}{\omega}\right)^{-\psi_1}$	(9)
(10) TFP process:	$A_t = \rho A_{t-1} + \varepsilon_t$	(10)

1.5 Steady State Definitions

Let variables with a bar denote their steady-state values. Assume $A = 1$, and all real variables are constant at steady state.

1.6 Log-Linearization

Define lowercase variables with hats ($\hat{x}$) as log-deviations from steady state:

$$\hat{x}_t = \log(x_t) - \log(\bar{x})$$

Euler Equation

Start from:

$$1 = \beta_t(1 + r_{t+1})\frac{\lambda_{t+1}}{\lambda_t}$$

Taking logs and log-linearizing:

$$\hat{c}_t = \hat{c}_{t+1} - \psi_1(\hat{c}_{t+1} - \omega\hat{h}_{t+1}) + \hat{r}_{t+1}$$

Labor Supply

$$\gamma(c_t - \frac{h_t^\omega}{\omega}) = (1-\alpha)\frac{y_t}{h_t} \Rightarrow \gamma(\hat{c}_t - \omega\hat{h}_t) = (1-\alpha)(\hat{y}_t - \hat{h}_t)$$

Production Function

$$y_t = A_t k_t^\alpha h_t^{1-\alpha} \Rightarrow \hat{y}_t = \hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{h}_t$$

Capital Accumulation (simplified)

$$k_{t+1} = (1-\delta)k_t + i_t \Rightarrow \hat{k}_t = \hat{k}_{t-1} \text{ (approx.)}$$

Debt Accumulation (with stabilization)

$$d_t = (1-\xi)d_{t-1} + c_t + k_t - y_t \Rightarrow \hat{d}_t = (1-\xi)\hat{d}_{t-1} + \hat{c}_t + \hat{k}_t - \hat{y}_t$$

Interest Rate Rule

$$r_t = \rho_r r_{t-1} + (1-\rho_r)\psi_2 d_t \Rightarrow \hat{r}_t = \rho_r \hat{r}_{t-1} + (1-\rho_r)\psi_2 \hat{d}_t$$

TFP AR(1)

$$\hat{a}_t = \rho \hat{a}_{t-1} + \varepsilon_t$$

4. Final System Summary (7 Equations)

$$\begin{aligned}
&(1)\ \hat{c}_t = \hat{c}_{t+1} - \psi_1(\hat{c}_{t+1} - \omega \hat{h}_{t+1}) + \hat{r}_{t+1} \\
&(2)\ \gamma(\hat{c}_t - \omega \hat{h}_t) = (1-\alpha)(\hat{y}_t - \hat{h}_t) \\
&(3)\ \hat{y}_t = \hat{a}_t + \alpha \hat{k}_t + (1-\alpha)\hat{h}_t \\
&(4)\ \hat{k}_t = \hat{k}_{t-1} \\
&(5)\ \hat{d}_t = (1-\xi)\hat{d}_{t-1} + \hat{c}_t + \hat{k}_t - \hat{y}_t \\
&(6)\ \hat{a}_t = \rho \hat{a}_{t-1} + \varepsilon_t \\
&(7)\ \hat{r}_t = \rho_r \hat{r}_{t-1} + (1-\rho_r)\psi_2 \hat{d}_t
\end{aligned}$$