14 Random signals

Previously all systems were described by **differential** or **difference equations**. The signals could be defined as mathematical functions, or superposition and combination of those (sin functions or series of delta impulses). In reality this is not always the case as random and chaotic behaviour can occur (noise). So we need to find out how to use our non-random or deterministic models with random signals.

- what an LTI system will do to a deterministic signal with noise
- Can we somehow use properties of the noise (expected values) to remove it?
- what if it is our LTI system which is introducing noise (for example during amplification, transmission, storage, etc)

Definition 35 Deterministic signal is a signal that has a known unambiguous value at every point in time. Those can be represented by mathematical functions, Fourier series, etc.

Non-deterministic, stochastic or random signal is a signal that are not known in any predictable way.

◇ **Example 51.** Amplification of '0' input signal. In that case we will not attempt to analyse individual noise patterns (see Figure 88) but the process that produces the noise. The output produced by an amplifier will not be the same for different amplifiers of the same type/kind. The same amplifier will also never produce the same nose pattern again. But for example given **noise power** is a deterministic property and can be calculated in a normal way. It should be the same (with in manufacturers tolerance) for all amplifiers of a particular model. ♣

Definition 36 Random process

A process that produces random signal will be called **random process**. It will produce an **ensemble** of random signals (all possible signals). Individual random signals $x_1(t)$, $x_2(t)$, etc are called sample functions or realisations of a random process. To characterise a random signal we use statistical averages. They can be classified into characteristics of random processes that which hold for a whole ensemble of random signals (expected value) and time averages which are found by averaging one sample function along time-axis.

By expected value (also ensemble mean) we define the mean value that is obtained at the same time from all sample functions of the same process:

$$E\{x(t_1)\} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i(t_1)$$

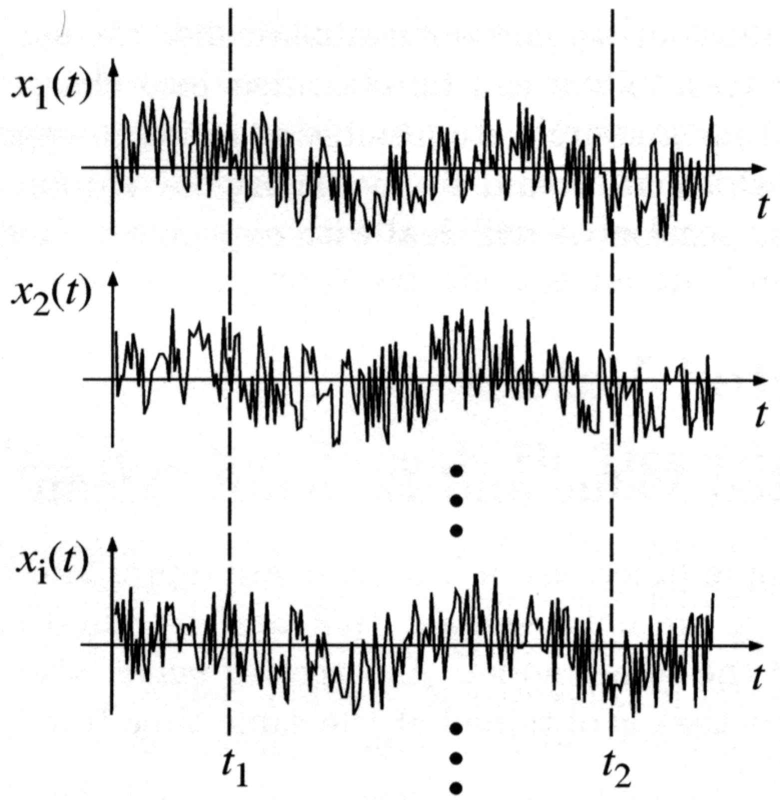


Figure 17.1: Example of sample functions $x_i(t)$

Figure 88: An example of a Non-deterministic signal

As we can obtain different means at different times, the expected value is in general time-dependent:

$$E\{x(t_1)\} \neq E\{x(t_2)\}$$

Expected value is an average across the process. Time-average is an average along the process. The definition of the expected value is a formal one and not a method for its calculations. It says it should be determined from all sample functions of a process which is in practice impossible. We can however get the expected value from

1. from precise knowledge of the random process without averaging the sample functions.

2. from averaging a finite number of sample functions, which can give an approximation of the expected value.

3. under certain conditions the ensemble average can also be expressed by the time-avarage for a sample function.

If we throw a die, one can use all three methods to get ensemble average. From method 1 and the definition

$$E\{x(t)\} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2}$$

Throwing many dice or a single one many times should give approximately the same result.

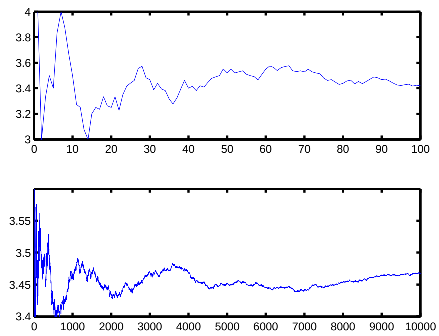


Figure 89: die throwing example: average as a function of number of throws for first 100 and 10^5 throws.

The expected value tells us what value to expect on average from a random process but it does not fully characterize the process. Signals in Figure 90 are quite different, still they have the same sample and time averages.

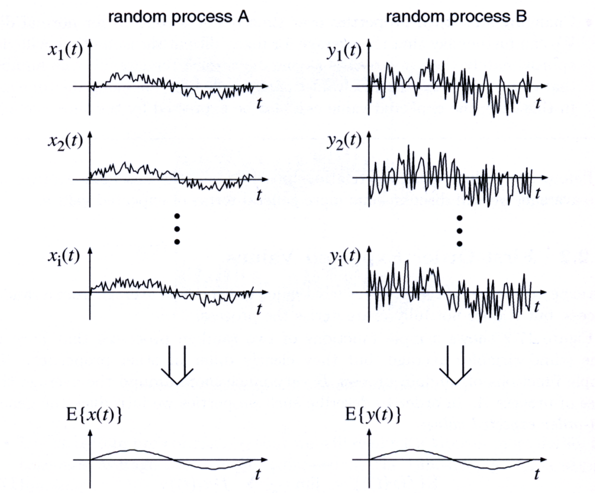


Figure 90:

Definition 37 First order expected value
To characterize such signals we introduce general first order expected value:

$$E\{f(x(t))\} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(x_i(t)) \tag{83}$$

By changing $f(x)$ different first order averages are obtained. First order refers to using amplitude of the signal $x(t)$ at only one time point. Higher order expected values will combine amplitudes at more then one time point. For $f(x) = x^2$ we get quadratic average:

$$E\{x^2(t)\} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i^2(t)$$

For $f(x) = x$ we get linear average $\mu_x(t)$:

$$E\{x(t)\} = \mu_x(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i(t)$$

variance:

$$\begin{aligned} E\{(x(t) - \mu_x(t))^2\} &= \sigma_x^2(t) \\ E\{x^2(t)\} - \mu_x^2 &= \sigma_x^2(t) \end{aligned}$$

Expected value $E\{\}$ is a linear operator and expected value of a deterministic signal is equal to that signal:

$$E\{ax_1(t) + bx_2(t)\} = aE\{x_1(t)\} + bE\{x_2(t)\}$$

where a and b can also be deterministic functions of time $a(t)$ and $b(t)$!

$$E\{d(t)\} = d(t)$$

First-order expected values hold for a certain point in time, and therefore they can not register the statistical dependencies that exist between different points in a signal. The second-order expected values link the signals at two different points.

Definition 38 Second-order expected values

$$E\{f(x(t_1), x(t_2))\} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(x(t_1), x(t_2)) \tag{84}$$

Definition 39 Auto-correlation function (ACF):

$$\varphi_{xx}(t_1, t_2) = E\{x(t_1)x(t_2)\} \tag{85}$$

For $t_1 = t_2 = t$:

$$\varphi_{xx}(t, t) = E\{x(t)x(t)\} = E\{x^2(t)\}$$

$$\varphi_{xx}(t_1, t_2) = E\{x(t_1)x(t_2)\}$$

and for $t_1 = t_2 = t$:

$$\varphi_{xx}(t, t) = E\{x(t)x(t)\} = E\{x^2(t)\}$$

Definition 40 Stationary process
A random process is stationary if its 2nd order expected values only depend on the difference τ between t_1 and t_2.

$$E\left\{f\left(x(t_1), x(t_2)\right)\right\} = E\left\{f\left(x(t_1 + \Delta t), x(t_2 + \Delta t)\right)\right\} \tag{86}$$

For a deterministic signal, this only can be a case for a signal which does not change with time (as for those expected value is equal to the signal itself). For a stationary process first order expected values will not depend on time and 2nd order ones will only depend on τ:

$$\varphi_{xx}(\tau) = E\left\{x(t)x(t+\tau)\right\}$$

Definition 41 Weak stationary process
A random process is weak stationary if its linear average and correlation properties contained in autocorrelation function do not depend on time (other 2nd order expected values might change with time).

14.1 Ergodic Random Processes

Definition 42 Ergodic random process
A stationary random process for which the time-averages of each function are the same as the ensemble averages is called ergodic random process.
If ergodicity conditions only hold for

$$f(x(t_1), x(t_2)) = x(t_1)x(t_2)$$
$$f(x(t_1), x(t_2)) = x(t_1)$$

then the process is called weak ergodic.

First order time average:

$$\overline{f(x_i(t))} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x_i(t))\mathrm{d}t$$

and the 2nd order time average

$$\overline{f(x_i(t), x_i(t-\tau))} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x_i(t), x_i(t-\tau))\mathrm{d}t$$

For an ergodic process this is the same for all sample functions.

14.2 Properties of ACF

It describes relationship between values of a random signal at times t_1 and t_2. Higher values of ACF indicates that $x(t)$ takes similar values at times t_1 and t_2. NOTE that in general $\varphi_{xx}(\tau) \neq \varphi_{xx}(t_1, t_2)$ and this is only true for stationary processes.

How do we calculate ACF?

Definition 43 ACF for an ergodic process

$$\varphi_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t-\tau)\mathrm{d}t \tag{87}$$

Basically, shift, multiply and sum!

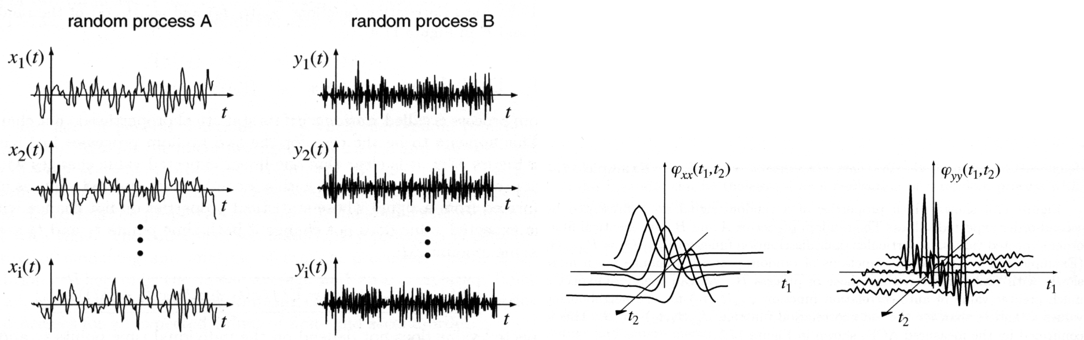


Figure 91: Two signals generated by random processes A and B and the corresponding $\varphi_{xx}(t_1, t_2)$ (which are 2D). One can notice that both processes are stationary

$$\varphi_{xx}(\tau) = E\left\{x(t)x(t+\tau)\right\}$$
$$\varphi_{xx}(0) = E\left\{x(t)x(t)\right\} = E\left\{x^2(t)\right\}$$

Now, if we consider:

$$E\left\{(x(t) - x(t \pm \tau))^2\right\} \geq 0$$
$$E\left\{(x^2(t)\right\} - 2E\left\{(x(t)x(t \pm \tau)\right\} + E\left\{(x^2(t \pm \tau)\right\} \geq 0$$

1st and last terms are equal and are first order expected values

$$E\left\{(x^2(t)\right\} - 2E\left\{(x(t)x(t \pm \tau)\right\} + E\left\{(x^2(t \pm \tau)\right\} \geq 0$$
$$2\varphi_{xx}(0) - 2\varphi_{xx}(\tau) \geq 0$$

So a maximum at $\tau = 0$ Considering $E\left\{(x(t) + x(t \pm \tau))^2\right\} \geq 0$ we get lower boundary as

$$2\varphi_{xx}(0) + 2\varphi_{xx}(\tau) \geq 0$$
$$\varphi_{xx}(\tau) \geq -\varphi_{xx}(0)$$

For signal with a non zero mean, we can get the lower and upper boundary as:

$$-\varphi_{xx}(0) + 2\mu_x^2 = -\sigma_x^2 + \mu_x^2 \leq \varphi_{xx}(\tau) \leq \varphi_{xx}(0) = \sigma_x^2 + \mu_x^2$$

where $\varphi_{xx}(0) = \sigma_x^2 + \mu_x^2$ comes from the fact that (see textbook p.410 and PC8):

$$E\left\{(x^2(t)\right\} = \sigma_x^2 + \mu_x^2$$

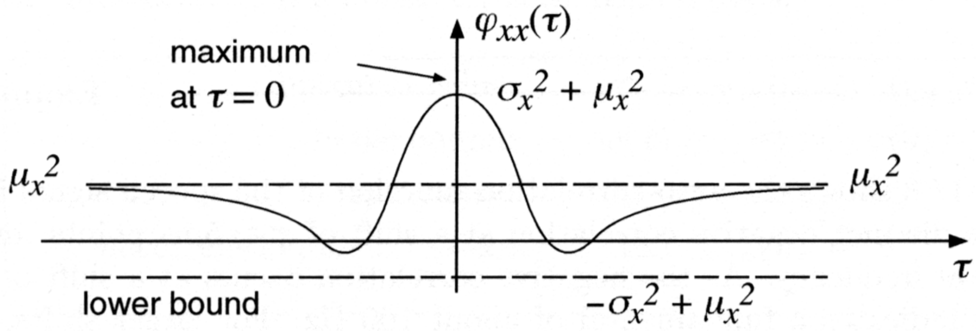


Figure 92: Properties of $\varphi_{xx}(\tau)$

◇ **Example 52.** We have to signal produced by two random processes

$$\begin{aligned} x_1(t) &= \sin(\omega_0 t + \phi_i) \\ x_2(t) &= A_i \sin(\omega_0 t + \phi_i) \end{aligned}$$

Are those processes stationary and/or ergodic?

$$\begin{aligned} \varphi_{xx}(t_1, t_2) &= E\{\sin(\omega_0 t_1 + \phi)\sin(\omega_0 t_2 + \phi)\} = \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sin(\omega_0 t_1 + \phi)\sin(\omega_0 t_2 + \phi)\mathrm{d}\phi = \\ &= \frac{1}{4\pi}\int_0^{2\pi} \left[\cos(\omega_0 t_1 - \omega_0 t_2) - \cos(\omega_0 t_1 + \omega_0 t_2 + 2\phi)\right]\mathrm{d}\phi \\ &= \frac{1}{2}\cos(\omega_0 t_1 - \omega t_2) + 0 = \frac{1}{2}\cos(\omega_0 \tau) \end{aligned}$$

So the process is stationary since $\varphi_{xx}(t_1, t_2) = \varphi_{xx}(\tau)$. The same is true for $x_2(t)$

Ergodicity:

Time average must be the same as the ensemble avarage. This will not be the case for $x_2(t)$ For $x_1(t)$

$$\begin{aligned}
\langle x_i(t)x_i(t-\tau)\rangle &= \lim_{T\mapsto\infty}\frac{1}{2T}\int\limits_{-T}^{T}\sin(\omega_0 t+\phi_i)\sin(\omega_0(t-\tau)+\phi_i)\mathrm{d}t = \\
&= \lim_{T\mapsto\infty}\frac{1}{2T}\int\limits_{-T}^{T}\frac{1}{2}\left[\cos(2\omega_0 t-\omega_0\tau+2\phi_i)+\cos(\omega_0\tau)\right]\mathrm{d}t = \\
&= \cos(\omega_0\tau)\lim_{T\mapsto\infty}\frac{1}{4T}\int\limits_{-T}^{T}\mathrm{d}t = \cos(\omega_0\tau)\frac{1}{4T}2T = \\
&= \frac{1}{2}\cos(\omega_0\tau)
\end{aligned}$$

Which is the same as calculated above for $x_1(t)$

14.3 ACF of periodic functions

For simple $A\sin(\omega_0 t)$, it will be similar to what we have shown above:

$$\begin{aligned}
\langle x_i(t)x_i(t-\tau)\rangle &= \lim_{T\mapsto\infty}\frac{A^2}{2T}\int\limits_{-T}^{T}\sin(\omega_0 t)\sin(\omega_0(t-\tau))\mathrm{d}t = \\
&= \lim_{T\mapsto\infty}\frac{A^2}{2T}\int\limits_{-T}^{T}\frac{1}{2}\left[\cos(2\omega_0 t-\omega_0\tau)+\cos(\omega_0\tau)\right]\mathrm{d}t = \\
&= \frac{A^2}{2}\cos(\omega_0\tau)
\end{aligned}$$

For a general periodic function

$$\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty}X_n e^{jn\omega t} \\
\langle x(t)x(t-\tau)\rangle &= \lim_{T\mapsto\infty}\frac{1}{2T}\int\limits_{-T}^{T}\sum_{n=-\infty}^{\infty}X_n e^{jn\omega t}\sum_{k=-\infty}^{\infty}X_k e^{jk\omega(t-\tau)} = \\
&= \sum_{n=-\infty}^{\infty}X_n\sum_{k=-\infty}^{\infty}X_k e^{-j\omega k\tau}\lim_{T\mapsto\infty}\frac{1}{2T}\int\limits_{-T}^{T}e^{j\omega(n+k)t} = \\
&= \sum_{n=-\infty}^{\infty}X_n X_{-n}e^{j\omega n\tau} = \sum_{n=-\infty}^{\infty}|X_n|^2 e^{j\omega n\tau} = \\
&= A_0^2+\frac{1}{2}\sum_{n=1}^{\infty}C_n^2\cos(n\omega\tau)
\end{aligned}$$

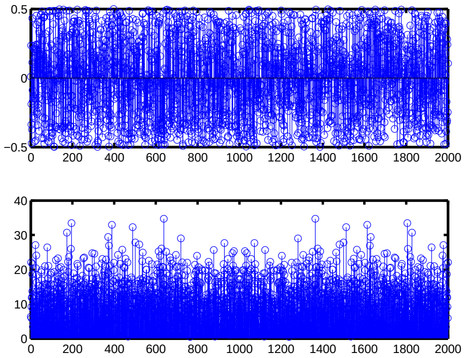


Figure 93: FFT of a white noise signal

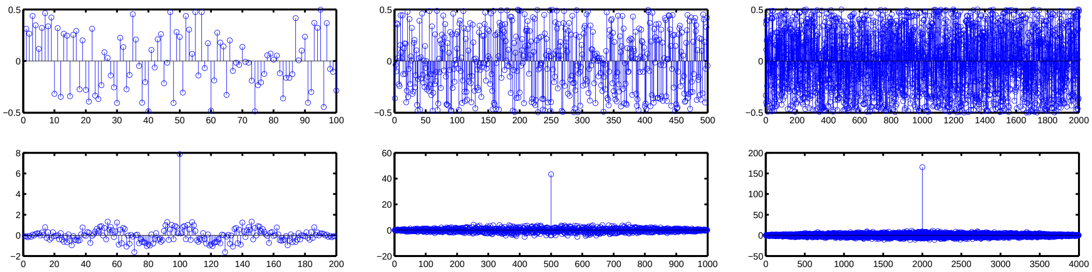


Figure 94: ACF for whit noise signals with different lengths

14.4 System analysis with noise

First we should investigate ACF for random (white) noise.

ACT for this random signal will look as shown below (see Figure 94) (calculate for different signal lengths)

If we calculate correlation between input and output from a system with a unknown transfer function then, the correlation output $y_0(t)$ is:

$$\begin{aligned} y_0(t) &= n(t) * h(t) \\ y_1(t) &= \varphi_{yn}(\tau) = y_0(t) * n(-t) \\ y_1(t) &= n(t) * h(t) * n(-t) = n(t) * n(-t) * h(t) = \\ &= \varphi_{nn}(\tau) * h(t) = \delta(t) * h(t) = h(t) \end{aligned}$$

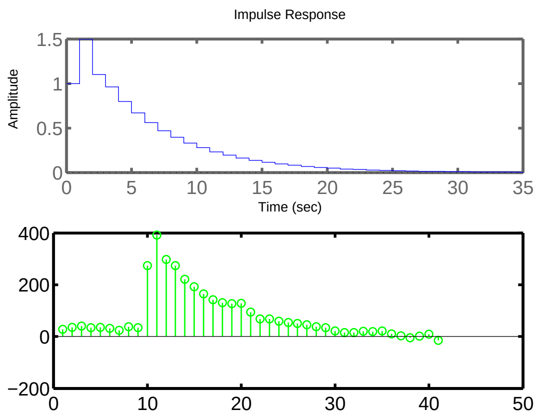
So, since $n(t) * n(-t)$ is an autocorrelation of random noise and it is approaching $\delta(t)$, we can measure impulse response!

Also,

$$\Phi_{y_0n}(j\omega) = H(j\omega)$$
$$\varphi_{yn}(t) = h(t)$$

Definition 44 cross-correlation We define cross-correlation as

$$\varphi_{xy}(\tau) = E\{x(t_1)y(t_2)\} \tag{88}$$




```
close all;
b =  [1 .9 0];
a = [1 -0.6 -0.2];
h = ldiv(b,a);
subplot(2,1,1);
dimpulse(b,a);
set(gca,'linewidth',2);
set(gca,'fontsize',14);
%or
subplot(2,1,2);
x = 0.5*sign(randn(1,10000));
y = conv(x,h);
A = xcorr(y,x);
stem(A(10010:10050),'LineWidth',1,'Color','green');
```

Figure 95:

For a complex random process autocorrelation and coross-corellation functions are given by:

$$\varphi_{xx}(\tau) = E\left\{x(t+\tau)x^*(t)\right\}$$
$$\varphi_{xy}(\tau) = E\left\{x(t+\tau)y^*(t)\right\}$$

Cross- and auto covariance functions are obtained by converting our signal to zero mean:

$$\tilde{x}(t) = x(t) - \mu_x$$
$$\tilde{y}(t) = y(t) - \mu_y$$

14.5 time delay from autocorrelation

Signal recorded with sampling rate 44100 Hz; If the signal is composed from the main signal and additional echo's then the ACF should have a maxima for $\tau = t_e$. With a bit of luck we could use that to for example measure the distance echo's of the original wave are travelling before reaching the micorphone (radar). We can also use a simple filter to remove the echo. Output (signal recorded by the microphone) is composed from input signal and time-delayed and amplitude reduced input signal:

$$\begin{aligned}
y[n] &= x[n] + Ax[n-N] \\
Y(z) &= X(z) + z^{-N}AX(z) \\
Y(z) &= X(z)\left(1 + Az^{-N}\right) \\
H(z) &= \frac{Y(z)}{X(x)} = 1 + Az^{-N} \\
X(z) &= Y(z)\frac{1}{H(z)}
\end{aligned}$$

We need to filter measured $Y(z)$ through filter $H(z)^{-1}$ to recover original signal without the echo.

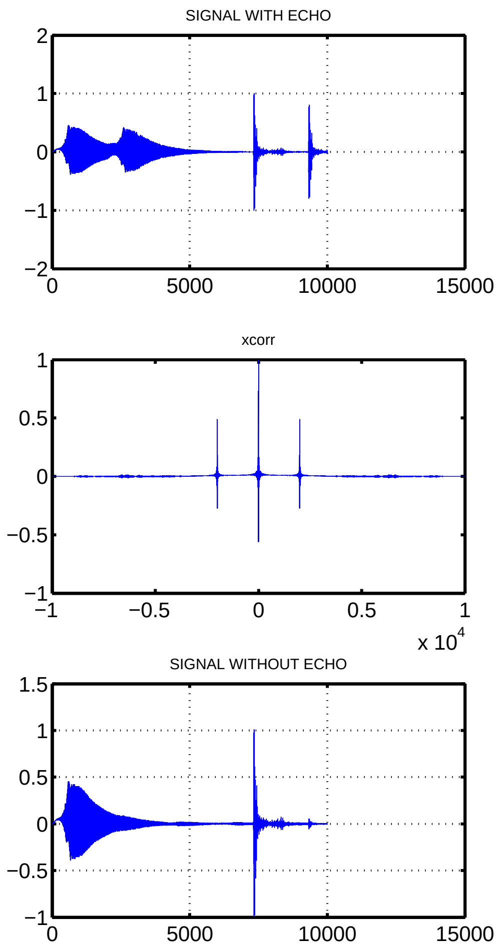


Figure 96: Signal with echo, ACF and signal with removed echo.

14.6 Power density spectrum and complex signals

Random signal: properties of random signals can be described by expected values (1st and second order)which are deterministic quantities (no fluctuations as the signal itself, do not depend on exact time-point where they are determined). In the past we have described “non-random” signals in the frequency domain, which often resulted in elegant description of the LTI system. Can we do the same for random signal? Due to the fact that stationary signals can not be integrated absolutely (which is a requirement for existence of Fourier transform), the Fourier transform exists only in special cases.

$$\int_{-\infty}^{\infty} |x_i(t)| \mathrm{d}t < \infty$$

Instead we can form expected value in the time domain and then transform into frequency domain.

Definition 45 Power density spectrum of a random process Fourier transform of autocorrelation or cross-correlation function is called *power density spectrum of a*

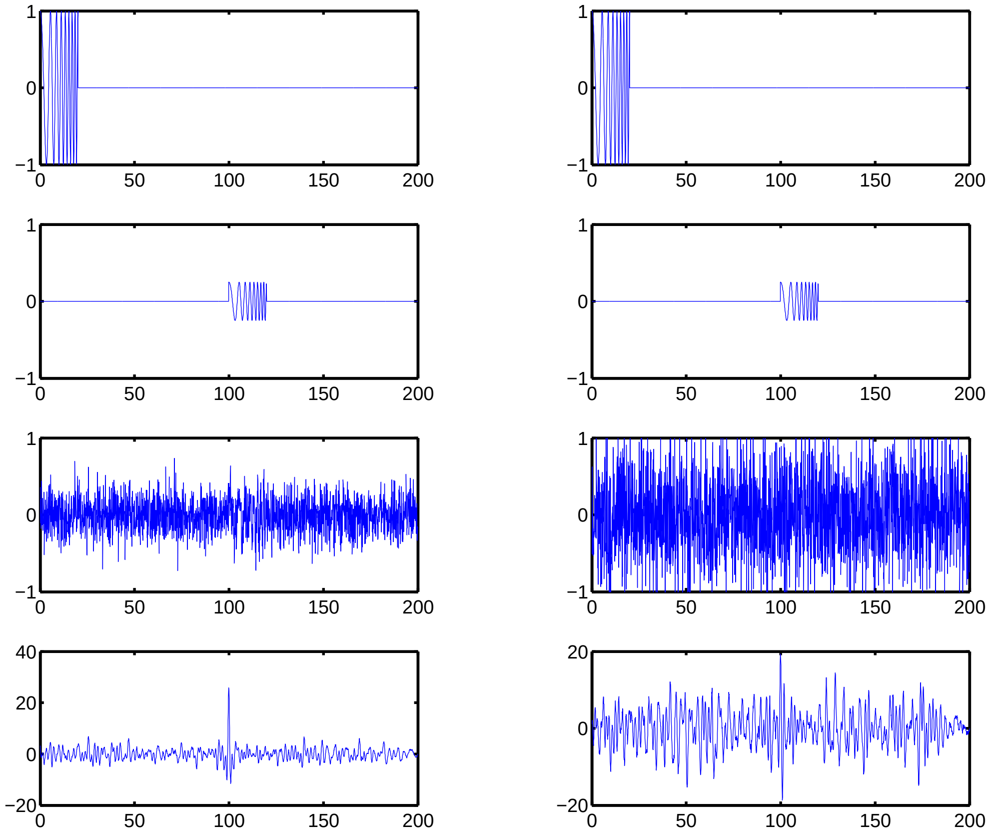


Figure 97: recovery of the time delay through cross-correlation; signal (echo amplitude) to noise (added random noise) is 1:1 (left) and 1:2 (right). Delay og 100s is identified in both cases. Chirp impulse is used, since a periodic impulse would created more uncertainty in the determined time delay (see figure below).

random process

$$\Phi_{xx}(j\omega) = \mathcal{F}\left\{\varphi_{xx}(\tau)\right\} \tag{89}$$

Also:

$$E\{|x(t)|^2\} = \varphi_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{xx}(j\omega)e^{j\omega\tau}\mathrm{d}\omega\bigg|_{\tau=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{xx}(j\omega)\mathrm{d}\omega$$

And therefore $\Phi_{xx}(j\omega)$ is the Power density spectrum

---

◇ **Example 53.**

$$x_i(t) = \sin(\omega_0 t + \varphi_i)$$

where φ_i is a random variable. Now

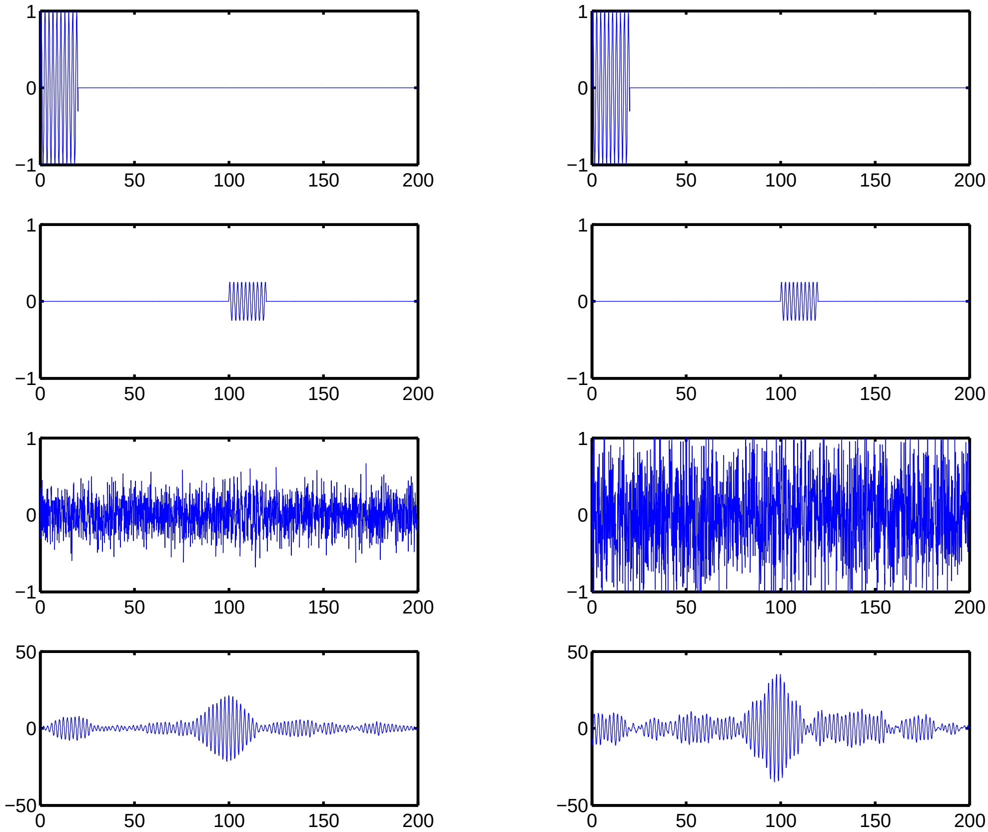


Figure 98: recovery of the time delay for sin impulse (parameters the same as above).

$$X(j\omega) = \left[e^{j\varphi_i}\pi\delta(\omega-\omega_0) - e^{-j\varphi_i}\pi\delta(\omega+\omega_0)\right]$$
$$E\{X(j\omega)\} = 0$$
$$E\{|X(j\omega)|\} = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$

For given random process

$$\begin{aligned}\varphi_{xx}(\tau) &= \frac{1}{2}\cos\omega_0 t\\ \Phi_{xx}(j\omega) &= \frac{\pi}{2}\delta(\omega-\omega_0) + \frac{\pi}{2}\delta(\omega+\omega_0)\end{aligned}$$

This is similar to $E\{|X(j\omega)|\}$ and indicates that the random signal only contains frequencies $\pm\omega_0$

$$\varphi_{xx}(\tau) = \mathcal{F}^{-1}\{\Phi_{xx}(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{xx}(j\omega)e^{j\omega\tau}\mathrm{d}\omega$$

$$\varphi_{xx}(0) = E\left\{|x(t)|^2\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{xx}(j\omega)\mathrm{d}\omega$$

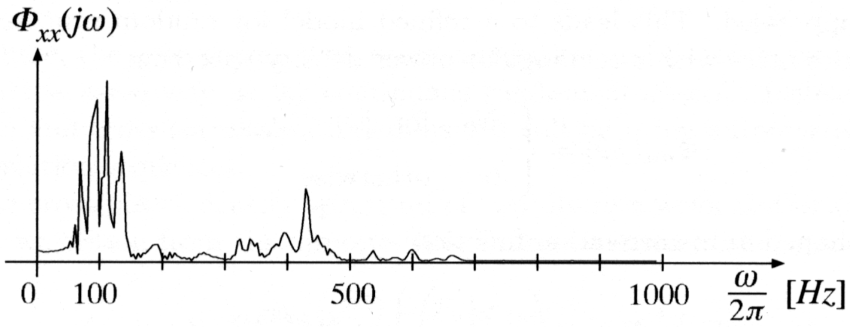


Figure 99: Power density spectra for a speech signal

Now we can return to white noise and show that $\varphi_{nn}(\tau) = N_0\delta(\tau)$

$$\varphi_{nn}(\tau) = \mathcal{F}^{-1}\left\{\Phi_{nn}(j\omega)\right\} = \mathcal{F}^{-1}\left\{N_0\right\} = N_0\delta(\tau)$$

$$\varphi_{nn}(0) = \frac{N_0}{2\pi}\int_{-\infty}^{-\infty} \mathrm{d}\omega \longrightarrow \infty$$

If the noise is band limited (we have a maximum frequency ω_max)

$$\Phi_{nn} = \begin{cases} N_0 & \text{for} |\omega| < \omega_{max} \\ 0 & \text{otherwise} \end{cases}$$

This will give $sin(x)/x$ shaped ACF.

$$\varphi_{nn}(\tau) = \mathcal{F}^{-1}\left\{\Phi_{nn}(j\omega)\right\} = \mathcal{F}^{-1}\left\{N_0 \mathrm{rect}\left(\frac{\omega}{2\omega_{max}}\right)\right\} = N_0\frac{\omega_{max}}{\pi} si(\tau\omega_{max})$$

15 Random signals and LTI-Systems

15.1 Combining random signals

Multiplication by an arbitrary complex factor K.

$$\begin{aligned} y(t) &= Kx(t) \\ \varphi_{yy}(\tau) &= E\{y(t+\tau)y^*(t)\} = E\{Kx(t+\tau)K^*x^*(t)\} = |K|^2\varphi_{xx}(\tau) \\ \varphi_{xy}(\tau) &= E\{x(t+\tau)y^*(t)\} = E\{x(t+\tau)K^*x^*(t)\} = K^*\varphi_{xx}(\tau) \\ \Phi_{yy}(j\omega) &= |K|^2\Phi_{xx}(j\omega) \end{aligned}$$

Adding of Random Signals

$$\begin{aligned} y(t) &= f(t) + g(t) \\ \varphi_{yy}(\tau) &= E\{y(t+\tau)y^*(t)\} = \varphi_{ff}(\tau) + \varphi_{fg}(\tau) + \varphi_{gf}(\tau) + \varphi_{gg}(\tau) \end{aligned}$$

If processes are uncorrelated and one of them has zero mean. Then

$$\begin{aligned} \varphi_{gf}(\tau) &= \varphi_{fg}(\tau) = 0 \\ \varphi_{yy}(\tau) &= \varphi_{ff}(\tau) + \varphi_{gg}(\tau) \\ \Phi_{yy}(j\omega) &= \Phi_{ff}(j\omega) + \Phi_{gg}(j\omega) \end{aligned}$$

15.2 Response of LTI-System to random signals

The fact that our system is time invariant means that stationary/ergodicity characterizing input signal will also be preserved for the output. If we have a random signal $x(t)$ as an input, we would like to describe expected values of the output depending on the expected values of the input $x(t)$ and system characteristics (for example $h(t)$, $H(s)$, differential equation, etc). To get linear mean of at the output of an LTI system we define output as a convolution between input and impulse response function.

$$\begin{aligned} y(t) &= x(t) * h(t) \\ E\{y(t)\} &= \mu_y(t) \\ \mu_y(t) &= E\{x(t) * h(t)\} = E\{x(t)\} * h(t) = \mu_x(t) * h(t) \end{aligned}$$

The relationship between expected values has the same form as between input and output. If the input is stationary, then $E\{x(t)\} = \mu_x$, then

$$\begin{aligned} \mu_y(t) &= E\{x(t)\} * h(t) = \mu_x * h(t) = \int_{-\infty}^{\infty} \mu_x h(\tau)\mathrm{d}\tau = \mu_x H(0) \\ H(0) &= \mathcal{F}\left\{h(t)\right\}\big|_{\omega=0} \end{aligned}$$

NOTE that value of the integral will not depend on the direction of integration, as well as shift by a constant value t.

$$\int_{-\infty}^{\infty} h(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} h(\tau-t)\mathrm{d}\tau = \int_{-\infty}^{\infty} h(\tau)\mathrm{d}\tau$$

So, the mean is transformed as DC ($\omega = 0$) component of the input signal.

Now we have to find out how autocorrelation function of the output corresponds to autocorrelation of the input. This will allow us to find power density of the output which we can compare with the input. Proof in the textbook p 443-445.

$$\begin{aligned} \varphi_{yy}(\tau) &= E\{y(t+\tau)y^*(t)\} \\ \varphi_{yy}(\tau) &= \varphi_{hh}(\tau) * \varphi_{xx}(\tau) \\ \varphi_{hh}(\tau) &= h(\tau) * h(-\tau) \end{aligned}$$

$\varphi_{hh}(\tau)$ is often called filter autocorrelation function. Since $h(\tau)$ is deterministic, $\varphi_{hh}(\tau)$ do not represent a expected value of a random process.

ACF between input and output

$$\begin{aligned} \varphi_{xy}(\tau) &= h^*(-\tau) * \varphi_{xx}(\tau) \\ \varphi_{yx}(\tau) &= h(\tau) * \varphi_{xx}(\tau) \end{aligned}$$

For example, if we have a system which introduces time delayed t_0

$$\varphi_{yx}(\tau) = \delta(\tau - t_0) * \varphi_{xx}(\tau) = \varphi_{xx}(\tau - t_0)$$

Since $\varphi_{xx}(\tau - t_0)$ will have a maximum at $\tau = t_0$, we can use this property to find t_0 (as we already did above).

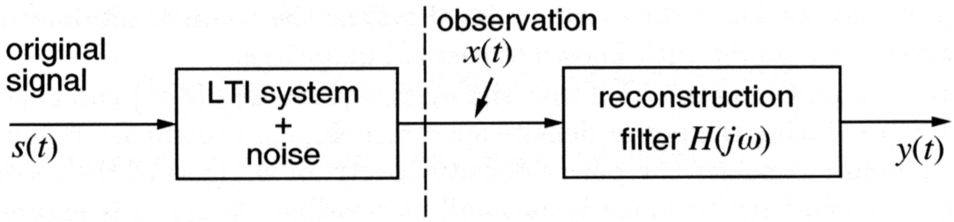


Figure 100: Signal $s(t)$ is a subject to noise and distortion by LTI system. Wiener filter will reconstruct signal $y(t)$ from $x(t)$ which is as similar as possible to $s(t)$.

At the same time

$$\begin{aligned}
\varphi_{hh}(\tau) &= h(\tau) * h(-\tau) = \delta(\tau - t_0) * \delta(-\tau + t_0) = \delta(\tau - t_0) * \delta(\tau - t_0) = \\
&= \int_{-\infty}^{\infty} \delta(t - t_0)\delta(\tau - t + t_0)\mathrm{d}t = \delta(\tau) \\
\varphi_{yy}(\tau) &= \delta(\tau) * \varphi_{xx}(\tau) = \varphi_{xx}(\tau)
\end{aligned}$$

So the autocorrelation function is not changed by the delay circuit. Power density spectrum:

$$\begin{aligned}
\Phi_{yy}(j\omega) &= \Phi_{xx}(j\omega)|H(j\omega)|^2 && (90) \\
\Phi_{yx}(j\omega) &= \Phi_{xx}(j\omega)H(j\omega) && (91) \\
\Phi_{xy}(j\omega) &= \Phi_{xx}(j\omega)H^*(j\omega) && (92)
\end{aligned}$$

We can now show use a more complete interpretation of the power density spectrum. If we define a ideal band pass filter, for which $H(j\omega) \neq 0$ for $\omega_0 < \omega < \omega_0 + \Delta\omega$, then

$$E\{|y(t)^2|\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(j\omega)|H(j\omega)|^2 \mathrm{d}\omega = \frac{1}{2\pi}\Phi_{xx}(j\omega_0)\Delta\omega$$

15.3 Signal estimation using the Wiener filter (18.3)

We have signal $s(t)$ which passes through LTI system with unknown characteristics and is subject to interfering noise. Those two results in signal $x(t)$ which we would like to pass through a filter which will results in a output signal, as close to $s(t)$ as possible. This process is important in communication, data storage, measurement technology, etc.

$$S\{s(t) + n(t)\} = x(t)$$
$$S_F\{x(t)\} = y(t) \approx s(t)$$

We define error signal

$$e(t) = y(t) - s(t)$$
$$E\{|e(t)|^2\} = E\{|y(t) - s(t)|^2\}$$

Now:

$$\Phi_{ee}(j\omega) = \Phi_{yy}(j\omega) - \Phi_{ys}(j\omega) - \Phi_{sy}(j\omega) + \Phi_{ss}(j\omega)$$

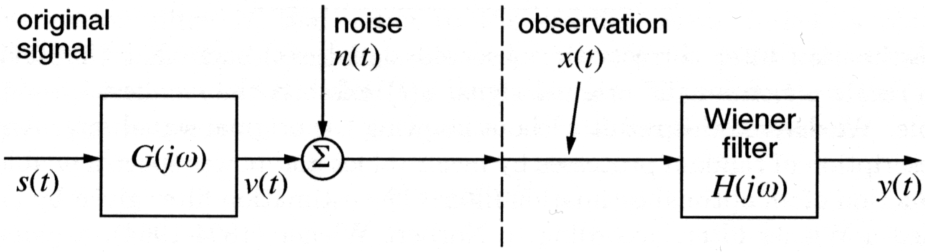


Figure 101: LTI system with frequency response $G(j\omega)$ and added noise $n(t)$

We need to find how Φ_{ee} depends on $H(i\omega)$ to get smallest $\Phi_{ee}(j\omega)$.

$$\begin{aligned}
\Phi_{yy}(j\omega) &= \Phi_{xx}(j\omega)HH^* \\
\Phi_{ee}(j\omega) &= \Phi_{xx}(j\omega)HH^* - \Phi_{xs}(j\omega)H - \Phi_{sx}(j\omega)H^* + \Phi_{ss}(j\omega)
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}\Phi_{ee}}{\mathrm{d}|H|} &= 0 \\
\frac{\mathrm{d}\Phi_{ee}}{\mathrm{d}\phi} &= 0 \\
H &= |H|e^{j\phi}
\end{aligned}$$

Now we can say something about H

$$\begin{aligned}
\frac{\mathrm{d}\Phi_{ee}}{\mathrm{d}|H|} &= 2|H|\Phi_{xx} - \Phi_{xs}\frac{\mathrm{d}|H|e^{j\phi}}{\mathrm{d}|H|} - \Phi_{xs}^*\frac{\mathrm{d}|H|e^{-j\phi}}{\mathrm{d}|H|} = |H|\Phi_{xx} - (\Phi_{xs}e^{j\phi} - \Phi_{xs}^*e^{-j\phi}) \\
\frac{\mathrm{d}\Phi_{ee}}{\mathrm{d}|H|} &= 2|H|\Phi_{xx} - 2Re\{\Phi_{xs}e^{j\phi}\} = 0 \\
2H\Phi_{xx} &= 2Re\{\Phi_{xs}e^{j\phi}\} \\
H &= \frac{Re\{\Phi_{xs}e^{j\phi}\}}{\Phi_{xx}}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}\Phi_{ee}}{\mathrm{d}\phi} &= -j|H|e^{j\phi}\Phi_{xs} + j|H|e^{-j\phi}\Phi_{xs}^* = 0 \\
-j|H|e^{j\phi}\Phi_{xs} + j|H|e^{-j\phi}\Phi_{xs}^* &= -j|H|e^{j\phi}|\Phi_{xs}|e^{jArg\{\Phi_{xs}\}} + j|H|e^{-j\phi}|\Phi_{xs}|e^{-jArg\{\Phi_{xs}\}} \\
\text{which is zero only for:} & \\
\phi &= -\mathrm{Arg}\{\Phi_{xs}\} \\
\text{as:} & \\
-j|H|e^{j\phi}|\Phi_{xs}|e^{-j\phi} + j|H|e^{-j\phi}|\Phi_{xs}|e^{j\phi} &= -j|H||\Phi_{xs}| + j|H||\Phi_{xs}| = 0
\end{aligned}$$

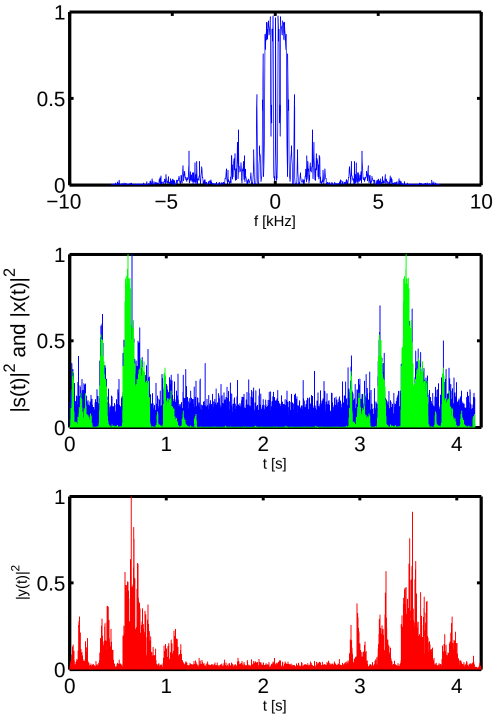
$$H = \frac{\Phi_{sx}(j\omega)}{\Phi_{xx}(j\omega)}$$

For case where $G(j\omega)$ and added noise $n(t)$ are separated (above we have discussed more general case), from Equation 91 and Equation 90:

$$\begin{aligned}
\Phi_{sx}(j\omega) &= \Phi_{ss}(j\omega)G(j\omega) \\
\Phi_{xx}(j\omega) &= \Phi_{ss}(j\omega)|G(j\omega)|^2 + \Phi_{nn}(j\omega) \\
H &= \frac{\Phi_{ss}(j\omega)G(j\omega)}{\Phi_{ss}(j\omega)|G(j\omega)|^2 + \Phi_{nn}(j\omega)}
\end{aligned}$$

In case where we only have noise ($G(j\omega) = 1$)

$$H = \frac{\Phi_{ss}(j\omega)}{\Phi_{ss}(j\omega) + \Phi_{nn}(j\omega)}$$




```
clear all; close all;
load signals_white.mat
s = speech;
n = 0.2*randn(size(s));
sn =  s + n;
S = s;
X = sn;
N = n;
t = 1/fs:1/fs:size(s,1)/fs;
SS = CPSD(S,S,[],512,1024, 'twosided');
NN = CPSD(N,N,[],512,1024, 'twosided');
XN = CPSD(X,N,[],512,1024, 'twosided');
XX = CPSD(X,X,[],512,1024, 'twosided');
H = SS./(SS+NN);
subplot(311)
f = (-512+(1:size(H,1)))/1024*fs/1000;
plot(f,real(fftshift(H)));
xlabel('f [kHz]')
set(gca,'linewidth',2);
set(gca,'fontsize',14);

h = ifft(H);
y = conv(X,h);
subplot(312)
plot(t,(X/max(abs(X))).^2)
hold on
plot(t,(S/max(abs(S))).^2,'-g')
hold off;
ylim([0 1])
xlabel('t [s]')
xlim([0 size(y,1)/fs])
set(gca,'linewidth',2);
set(gca,'fontsize',14);
t = 1/fs:1/fs:size(y,1)/fs;
ylabel('|s(t)|^2 and |x(t)|^2')
subplot(313)
plot(t,(y/max(abs(y))).^2,'-r')


ylim([0 1])
xlim([0 size(y,1)/fs])
xlabel('t [s]')
ylabel('|y(t)|^2')

%soundsc(y, fs)
%pause(5)
%soundsc(X, fs)
YY = CPSD(y,y,[],512,1024, 'twosided');
set(gca,'linewidth',2);
set(gca,'fontsize',14);

set(gcf,'PaperUnits','points');
set(gcf,'PaperSize',[400 700]);
set(gcf,'Position',[0 0 450 900]);
set(gcf,'PaperPositionMode','auto')
print('-dpdf','-r300','wiener')
```

Figure 102: Wiener filter for speech signal with added white noise. See code for explanation