6 Stability and initial conditions

Material: Chapters 6, 7, 10, and 16

6.1 Summary

- If we know the impulse response function $h(t)$ and can calculate its Laplace transform, and if we can calculate Laplace transform of the input signal $x(t)$, then we can find the output between $(0, \infty)$;
- If signal $x(t)$ exists at $t \to \infty$ then we can find the inverse Laplace transform without doing the integration in the complex plane;
- Poles and zeros are useful in describing the Laplace transform of the input or output signal. We can do inverse operation if the signal is defined for $t \to \infty$. If not we can always multiply it by a difference between two $\epsilon(t)$ on of which shifted $x(t) = \epsilon(t) - \epsilon(t-2)$ and $X(s) = \frac{1}{s}(1 + e^{-2s})$.

6.2 Combination of simple systems in the frequency-domain

We have seen previously that the time-domain response can be described as a convolution of the input signal and system response function, as

$$y(t) = \int_0^t x(\tau)h(t-\tau)\mathrm{d}\tau \,,$$

and using the convolution property of the Laplace transform,

$$Y(s) = H(s)X(s) \,.$$

We have also seen previously how to combine simple systems in the time-domain. We can use similarly simple relation-ships using the frequency domain description. For two system connected using:

Definition 15 serial coupling

$$\begin{aligned} Z(s) &= X(s)H_1(s) \\ Y(s) &= Z(s)H_2(s) \\ H(s) &= \frac{Y(s)}{X(s)} = H_1(s)H_2(s) = H_2(s)H_1(s) \,. \end{aligned}$$

The output signal of a system with system function $H(s) = H_1(s)H_2(s)$ is therefore identical to the output signal of the two systems $H_1(s)$ and $H_2(s)$ in series.

Definition 16 parallel coupling:
Because the system is linear

$$Y(s) = H_1(s)X(s) + H_2(s)X(s) = X(s)(H_1(s) + H_2(s)) = X(s)H(s)\,.$$

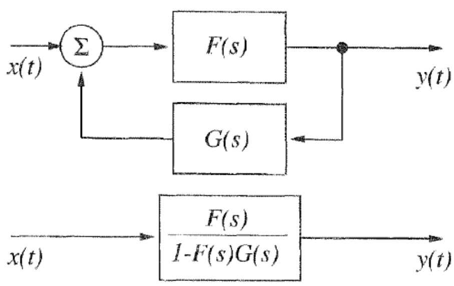
The output signal is therefore identified to the output signal of a systems with system function $H(s) = H_1(s) + H_2(s)$. We make use of this relation with partial fraction expansion. Expanding a system function into partial fractions is dividing it into simpler parts.

Definition 17 Feedback system:
Systems with feedback, like the one in Figure ??, are very important in control systems and have many useful applications. At the output of the feedback system we find the expression

$$Y(s) = F(s)\left[X(s) + Y(s)G(s)\right]$$

This yields the system function

$$H(s) = \frac{Y(s)}{X(s)} \frac{F(s)}{1 - F(s)G(s)}\,.$$



6.3 BIBO stability

There are several ways of defining the stability of a system. Among them, the BIBO stability is very useful for LTI-systems. A continuous LTI systems is stable if it reacts to a Bounded Input function $x(t)$ with a Bounded Output function $y(t)$ (BIBO). The BIBO establishes that for a bounded (well-behaved) input $x(t)$ the output of a BIBO stable system $y(t)$ is also bounded.

A function is said to be bound when its magnitude is less than a fixed limit for all time t,

$$|x(t)| < M < \infty, \quad \forall\, t\,.$$

Definition 18 BIBO stability From the general definition of BIBO-stability, we can derive the following condition for the stability of continous LTI-systems: a continuous LTI-system is stable if and only if its impulse response can be absolutely integrated, according to

$$\int_{-\infty}^{\infty} |h(t)| \mathrm{d}t \ < \ M_2 \ < \ \infty \,.$$

◇ **Example 24 (BIBO 1).** For a simple example we consider a system with impulse response

$$h(t) = e^{-at}\epsilon(t) \quad a \in \Re \,.$$

To determine the stability we investigate whether the impulse response can be absolutely integrated:

$$\int_{-\infty}^{\infty} |e^{-at}\epsilon(t)| \mathrm{d}t = \int_{0}^{\infty} e^{-at} \mathrm{d}t = \begin{cases} \frac{1}{a}, & a > 0 \\ \infty, & \text{otherwise} \end{cases} \,.$$

We can easily conclude that the system is stable for $a > 0$. ♣

◇ **Example 25 (BIBO 2).** Consider a positive feedback system created by a microphone close to a set of speakers that are putting out an amplified acoustic signal. The microphone picks up the input signal $x(t)$ as well as the amplified and delayed signal $\beta y(t-\tau), |\beta| \geq 1$. Determine if the system is BIBO stable or not - use $x(t) = \epsilon(t)$, $\beta = 2$, and $\tau = 1$ in doing so.
The input-output equation is

$$y(t) = x(t) + \beta y(t - \tau)$$

If we use the expression to obtain $y(t - \tau)$, we get

$$y(t - \tau) = x(t - \tau) + \beta y(t - 2\tau)$$

and replacing it above

$$y(t) = x(t) + \beta[x(t - \tau) + \beta y(t - 2\tau)] = x(t) + \beta x(t - \tau) + \beta^2 y(t - 2\tau)$$

Repeating the above scheme and replacing by the given data we obtain:

$$y(t) = \epsilon(t) + 2\epsilon(t - 1) + 4\epsilon(t - 2) + 8\epsilon(t - 3) + ...$$

which grows continuously as time increases. The output is not a bounded signal, although the input is bound. Thus, the system is unstable.

We can also show this by using LT and calculating $H(s)$:

$$\begin{aligned} y(t) &= x(t) + \beta y(t-\tau) \\ Y(s) &= X(s) + \beta Y(s)\mathrm{e}^{-s\tau} \\ Y(s)\left(1 - \beta\mathrm{e}^{-s\tau}\right) &= X(s) \\ H(s) &= \frac{1}{1-\beta\mathrm{e}^{-s\tau}} \end{aligned}$$

This will have a pole at:

$$\begin{aligned} 1 - \beta\mathrm{e}^{-s\tau} &= 0 \\ 1 = \beta\mathrm{e}^{-s\tau} &= 0 \\ 0 = \log(\beta\mathrm{e}^{-s\tau}) &= \log\beta - s\tau \\ \log\beta &= s\tau \\ s &= \frac{\log\beta}{\tau} \end{aligned}$$

The pole is in the right hand part of the s-plane and the system is not BIBO stable for $\beta > 1$. We can also get the impulse response function

$$\mathscr{L}^{-1}\left\{\frac{1}{1-\beta\mathrm{e}^{-s\tau}}\right\} = \mathscr{L}^{-1}\left\{\sum_{n=0}^{\infty}\left(\beta e^{-s\tau}\right)^n\right\} = \mathscr{L}^{-1}\left\{\sum_{n=0}^{\infty}\left(\beta^n e^{-sn\tau}\right)\right\} = \sum_{n=0}^{\infty}\beta^n\delta(t-n\tau)$$

That sum only converges for $|\beta| < 1$ ♣

6.4 Initial conditions

So far we have looked at systems that have a *forced response* associated with an input $x(t)$. If we have no input $x(t) = 0$ ($X(s) = 0$), then we get normally no output ($Y(s) = 0$). However, as will be seen, for *autonomous systems* ($x(t) = 0$) non-zero output is possible. The solutions are called *natural* or *unforced response*. So, these signals can be generated when the input signal is turned off and the system is disturbed in some way.

The poles of the transfer function appear at natural frequencies of the system with a general form of $e^{s_n t}$. Note, however, that if those frequencies are not 'transient', the system will be unstable.

The Laplace transform method for solving differential equations offers a clear separation between the *natural response* of the system to *initial conditions*, and the *forced response* of the system associated with the *input* $x(t)$.

From our "master" LDE with constant coefficients:

$$\sum_{i=0}^{N}\alpha_i\frac{\mathrm{d}^i y}{\mathrm{d}t^i} = \sum_{k=0}^{M}\beta_k\frac{\mathrm{d}^k x}{\mathrm{d}t^k} \tag{49}$$

if $x(t) = 0, \quad \forall\, t > 0$, then by definition $y(t) = 0$. So LDE only work for system for which initial conditions are zero and the input signal is casual (starts at $t = 0$). So how can we deal with cases when this is not the case?

6.4.1 First-order systems

Let us start with a simpler first-order system described by a differential equation as follows,

$$\alpha_1 \dot{y} + \alpha_0 y = \beta_1 \dot{x} + \beta_0 x \quad t > 0$$

$x(t)$ and $y(t)$ are only defined for $t > 0$, and the values $x(0)$ and $y(0)$, at the start of the observation, are known and can have any value. For a known input variable $x(t)$ it is then possible to calculate the systems response $y(t)$ for $t > 0$. How to analyze this resorting to Laplace transforms?

We start by setting some requirements: (i) $x(t)$ is a right-side signal, with $x(t) = 0$ for $t < 0$, (ii) $x(t)$ can be differentiated for $t > 0$, and (iii) $x(t)$ is of exponential order for $t \to \infty$. This guarantees that the Laplace transform of $x(t)$ exists. Let us also consider that there might be a step at $t = 0$,

$$\begin{aligned} \alpha_1 \dot{y} + \alpha_0 y &= \beta_1 \dot{x} + \beta_0 x \quad t > 0 \\ y(0) &= y_0 \end{aligned}$$

so, using the differentiation theorem (Eq. 45), leads to the algebraic equation,

$$\alpha_1 \left[sY(s) - y(0) \right] + \alpha_0 Y(s) = \beta_1 \left[sX(s) - x(0) \right] + \beta_0 X(s)$$

which can be solved for the Laplace transform of the output signal,

$$\begin{aligned} Y(s) &= \frac{\beta_1 s + \beta_0}{\alpha_1 s + \alpha_0} X(s) + \frac{\alpha_1 y(0) - \beta_1 x(0)}{\alpha_1 s + \alpha_0} && (50) \\ &= H(s)X(s) + \frac{\alpha_1 y(0) - \beta_1 x(0)}{\alpha_1 s + \alpha_0} \, . && (51) \end{aligned}$$

Definition 19 Non zero initial conditions for 1st order system

$$Y(s) = H(s)X(s) + \frac{\alpha_1 y(0) - \beta_1 x(0)}{\alpha_1 s + \alpha_0} \, .$$

The output signal consists then in two parts, the first contains the input signal for $t > 0$, weighted with the transfer function – *external part* (TB), *zero-state response* (TB2) or forced response to the system. The second part is given by the values of the input and the output signals at time $t = 0$ – *internal part* (TB), *zero-input response* (TB2), or natural response. The combination of the internal and external parts is schematically shown in Fig. 27.

The external part is calculated in the frequency domain with $Y_{\text{ext}} = H(s)X(s)$, and the internal part depends on a further transfer function $G(s)$ of the system state $t = 0$. At the input of $G(s)$, the initial state $z(0)$ appears which does not depend on the complex frequency.

Note: Do not confuse initial values with initial states. The initial value is the value of the input signal at time $t = 0$, while the initial state is the value of the internal states at time $t = 0$. It can be interpreted as the content of the energy stores (e.g., capacitor).

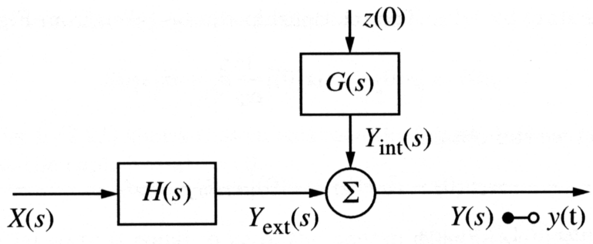


Figure 27: Combination of initial and external parts.

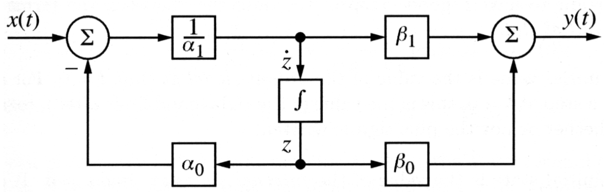


Figure 28: First-order systems in the direct form II.

Figure 28 shows the first-order system in a direct form II structure. With this representation we know the internal structure of the system. Let us express the output signal at $t = 0$ by the input signal $x(0)$ and the initial state $z(0)$. We can take the following relationship from Fig. 28,

$$\begin{aligned} y(0) &= \frac{\beta_1}{\alpha_1}\left[x(0) - \alpha_0 z(0)\right] + \beta_0 z(0) \\ \alpha_1 y(0) - \beta_1 x(0) &= z(0)\left[\alpha_1\beta_0 - \alpha_0\beta_1\right] . \end{aligned}$$

Replacing numeratof in the 2nd term in Eq. 51 with the left hand side of the equation above, we get

$$\begin{aligned} Y(s) &= \frac{\beta_1 s + \beta_0}{\alpha_1 s + \alpha_0} X(s) + z(0)\frac{[\alpha_1\beta_0 - \alpha_0\beta_1]}{\alpha_1 s + \alpha_0} \\ Y(s) &= H(s)X(s) + z(0)G(s) . \end{aligned}$$

If we use the nomenclature as in TB2, then,

$$Y(s) = \frac{B(s)}{A(s)}X(s) + \frac{1}{A(s)}I(s)$$
$$Y(s) = H(s)X(s) + H_1(s)I(s) .$$

Definition 20 zero-state and zero-input response With, as mentioned above, the *zero-state response* (forced response of the system) and *zero-input response* (natural response of the system) being defined by,

$$y_{zs}(t) = \mathscr{L}^{-1}\left\{H(s)X(s)\right\}$$
$$y_{zi}(t) = \mathscr{L}^{-1}\left\{H_1(s)I(s)\right\} .$$

In terms of convolution integrals, we can write

$$y(t) = \int_0^t x(\tau)h(t-\tau)\mathrm{d}\tau + \int_0^t i(\tau)h_1(t-\tau)\mathrm{d}\tau\,.$$

6.4.2 Second-order systems

If we have a second-order differential equation

$$\alpha_2\ddot{y} + \alpha_1\dot{y} + \alpha_0 y = \beta_2\ddot{x} + \beta_1\dot{x} + \beta_0 x\,,$$

and using the differentiation theorem for right-sided signals, for the first and second derivatives, we obtain

$$\begin{aligned}\mathscr{L}\{\dot{x}\} &= s\mathscr{L}\{x\} - x(0) = sX(s) - x(0)\\ \mathscr{L}\{\ddot{x}\} &= s\mathscr{L}\{\dot{x}\} - \dot{x}(0) = s^2X(s) - [sx(0) + \dot{x}(0)]\,.\end{aligned}$$

Thus, for a unique solution we require now two initial conditions, the value of the output signal $y(0)$ and its first derivative $\dot{y}(0)$. The corresponding values $x(0)$ and $\dot{x}(0)$ of the input signal are known. Solving as in the previous case we obtain the solution:

$$Y(s) = H(s)X(s) + \frac{s[\alpha_2 y(0) - \beta_2 x(0)] + [\alpha_1 y(0) + \alpha_2\dot{y}(0) - \beta_1 x(0) - \beta_2\dot{x}(0)]}{\alpha_2 s^2 + \alpha_1 s + \alpha_0}\,,$$

where the system function is,

$$H(s) = \frac{\beta_2 s^2 + \beta_1 + \beta_0}{\alpha_2 s^2 + \alpha_1 s + \alpha_0}\,.$$

Again, this can be expressed more simply, using the initial states, z_1 and z_2, according to,

$$Y(s) = H(s)X(s) + \frac{sz_1(0)z_2(0)}{\alpha_2 s^2 + \alpha_1 s + \alpha_0}\,.$$

◇ **Example 26 (natural response).** Find the natural response of a system which satisfies the following differential equation

$$\ddot{y}(t) \,+\, 2\dot{y}(t) \,+\, 4\,y(t) = 2\,\dot{x}(t) \,+\, x(t),$$

with the initial conditions $y(0) = 1$ and $\dot{y}(0) = 0$.

The natural response of a system is the response due to initial conditions with no external inputs, so we assume $x(t) = 0$.

• Replacing $y(t)$ and its derivatives with its Laplace transforms on both sides of the new differential equation (righthand side is zero) we get:

$$s^2Y(s) - sy(0) - \dot{y}(0) - 2(sY(s) - y(0)) + 4Y(s) = 0$$

and

$$s^2Y(s) - s + 2sY(s) - 2 + 4Y(s) = 0.$$

• Solving for $Y(s)$ we obtain

$$Y(s) = \frac{s+2}{s^2+2s+4}$$

- Poles are at

$$s^2 + 2s + 4 = 0,$$

allowing for

$$s_{1,2} = -1 \pm j\sqrt{3}$$

- Partial fraction expansion gives

$$\begin{aligned} Y(s) &= \frac{s+2}{s^2+2s+4} = \frac{s+2}{(s-s_1)\,(s-s_2)} = \frac{r_1}{s-s_1} + \frac{r_2}{s-s_2} \\ &= \frac{r_1}{s+1-j\sqrt{3}} + \frac{r_2}{s+1+j\sqrt{3}} \end{aligned}$$

where

$$\begin{aligned} r_1 &= \frac{s+2}{s+1+j\sqrt{3}}\Big|_{s=-1+j\sqrt{3}} = \frac{1+j\sqrt{3}}{2j\sqrt{3}} = \frac{j-\sqrt{3}}{-2\sqrt{3}} \\ &\Rightarrow\; r_1 = \frac{1}{2} - \frac{j}{2\sqrt{3}} \end{aligned}$$

Analogously

$$r_2 = \frac{1}{2} + \frac{j}{2\sqrt{3}}$$

Note that

$$r_1 = r_2^\star \quad \text{and} \quad s_1 = s_2^\star$$

- Taking the inverse Laplace transform we obtain the natural response of the system

$$y(t) = r_1\, e^{s_1 t} + r_2\, e^{s_2 t} = r_1\, e^{-(1-j\sqrt{3})t} + r_2\, e^{-(1+j\sqrt{3})t}$$

Using the Euler formula we get

$$y(t) = e^{-t}\left[r_1 \cos(\sqrt{3}t) + j\, r_1 \sin(\sqrt{3}t) + r_2 \cos(\sqrt{3}t) - j\, r_2 \sin(\sqrt{3}t)\right]$$

As $r_1 + r_2 = 1$ and $r_1 - r_2 = j/\sqrt{3}$ we obtain the (real) solution

$$y(t) = e^{-t}\left[\cos(\sqrt{3}t) + \frac{1}{\sqrt{3}} \sin(\sqrt{3}t)\right]$$

This expression can be written in the form

$$y(t) = A\, e^{-\sigma t} \cos(\omega t + \phi)$$

where the amplitude $A = 2/\sqrt{3}$, the phase $\phi = -85°$, $\sigma = 1$ and $\omega = \sqrt{3}$.

In summary:

— if $\sigma < 0$, $y(t)$ is an exponentially growing sinusoid; if $\sigma > 0$, $y(t)$ is an exponentially decaying sinusoid; if $\sigma = 0$, $y(t)$ is a sinusoid

— σ gives exponential rate of decay or growth; ω gives oscillation frequency

— amplitude A and phase ϕ determined by initial conditions

— $Ae^{\sigma t}$ is called the envelope of y

Conclusion:

The solution is transient (zero for large t). If the poles of the signal lay in the left hand part of the complex plane, the system is stable (type $e^{-\sigma t}$, where σ is real and positive). Otherwise the transient response will not decay (will be of a type $e^{\sigma t}$) and will completely overwhelm any response to a possible input signal and/or initial conditions (unstable system). ♣

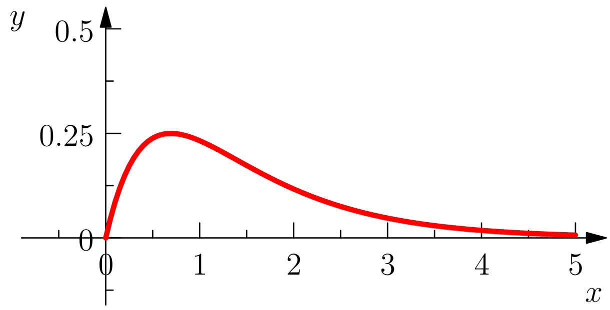


Figure 29: $h(t)$ of example 4 when the initial condition are zero.

◇ **Example 27.** Find the impulse response $h(t)$ and the unit step response $s(t)$ for a system described by:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$$

For the impulse response we set $x(t) = \delta(t)$ and $H(s) = Y(s)/X(s)$ ($\mathscr{L}\{\delta(t)\} = 1$),

$$Y(s)\left[s^2 + 3s + 2\right] = X(s)$$
$$H(s) = \frac{1}{s^2+3s+2} = \frac{A}{s+1} + \frac{B}{s+2}.$$

We can easily find that $A = 1$ and $B = -1$ so:

$$\begin{aligned} H(s) &= \frac{1}{s+1} - \frac{1}{s+2} \\ h(t) &= \left[e^{-t} - e^{-2t}\right]\epsilon(t)\,. \end{aligned}$$

We find the unit step response, by setting $x(t) = \epsilon(t)$,

$$\begin{aligned} S(s) &= H(s)\frac{1}{s} = \frac{1}{s(s^2+3s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \\ S(s) &= \frac{0.5}{s} - \frac{1}{s+1} + \frac{0.5}{s+2} \\ s(t) &= 0.5\epsilon(t) - e^{-t}\epsilon(t) + 0.5e^{-2t}\epsilon(t) \end{aligned}$$

So the system has a steady state response, $s(t) = 0.5$ when $t \to \infty$.

But what happens if initial conditions are not zero?
Find out the complete response $y(t)$ assuming $y(0) = 1$, $\dot{y}(0) = 0$, and $x(t) = \epsilon(t)$.
The Laplace transform of the differential equation gives

$$\begin{aligned} \left[s^2Y(s) - sy(0) - \dot{y}(0)\right] + 3\left[sY(s) - y(0)\right] + 2Y(s) &= X(s) \\ Y(s)\left(s^2+3s+2\right) - (s+3) &= X(s) \\ Y(s) = X(s)\frac{1}{(s+1)(s+2)} &+ \frac{s+3}{(s+1)(s+2)} \end{aligned}$$

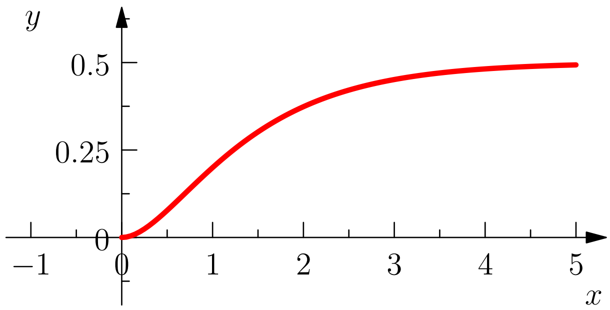


Figure 30: $s(t)$ of example 4 when the initial conditions are zero.

and since $X(s) = \mathscr{L}\{\epsilon(t)\} = \frac{1}{s}$, we have

$$\begin{aligned}
Y(s) &= \frac{1}{s(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)} \\
Y(s) &= \frac{1+s^2+3s}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \\
Y(s) &= \frac{0.5}{s} + \frac{1}{s+1} - \frac{0.5}{s+2} \\
y(t) &= 0.5\epsilon(t) + e^{-t}\epsilon(t) - 0.5e^{-2t}\epsilon(t)\,.
\end{aligned}$$

The steady-state response is 0.5 and the transient response is $[e^{-t} - 0.5e^{-2t}]\epsilon(t)$.

The complete solution, $y(t)$ is composed of the zero-state response, due to the input only, and the response due to the initial conditions only or the zero-input response. Thus, the system considers two different inputs: The $x(t) = \epsilon(t)$ and the initial conditions.

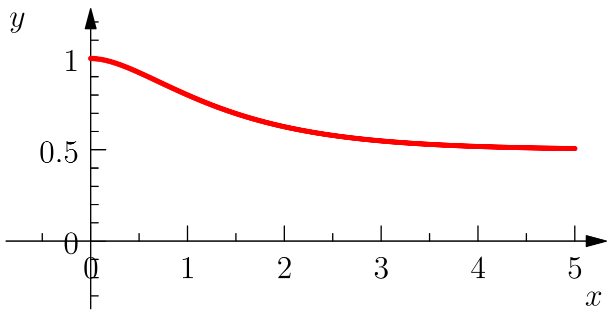


Figure 31: $s(t)$ for example 1; initial condition: $y(0) = 1$;

If we were able to find the transfer function $H(s) = Y(s)/X(S)$, its inverse would be $h(t)$. However, that is not possible when the initial conditions are non-zero. As seen above, in the case of non-zero initial conditions we get that the Laplace transform is

$$Y(s) = X(s)\frac{1}{(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)}\,,$$

and thus we cannot find the ratio $Y(s)/X(s)$. If we make the second term zero, we get $Y(s)/X(s) = H(s)$ and $h(t) = [e^{-t} - e^{-2t}]\epsilon(t)$.

◇ **Example 28 (Initial conditions applied to circuits).** If we consider a simple RL circuit shown below (Fig. 32).

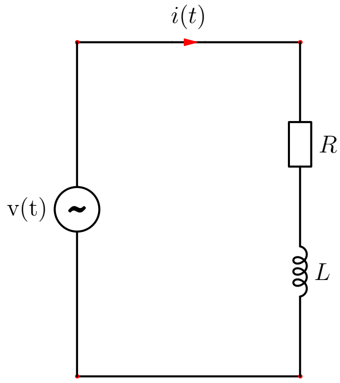


Figure 32: Simple RL circuit.

1. What will be the response to a step function $x(t) = V\epsilon(t)$?
2. What will be the natural response if we start with a initial current value of $i = V/R$?
♣

- For the step function response

$$\begin{aligned} L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri(t) &= V\epsilon(t) \\ L\left[sI(s) - i(0-)\right] + RI(s) &= V\frac{1}{s} \end{aligned}$$

There is no information on initial conditions, so we assume, for example, that the circuit was open, thus

$$i(0-) = 0$$

We get then,

$$\begin{aligned} LsI(s) + RI(s) &= V\frac{1}{s} \\ I(s)\left[sL + R\right] &= V\frac{1}{s} \\ I(s) &= \frac{V}{s}\frac{1}{(sL+R)} \\ I(s) = \frac{V}{L}\frac{1}{s(s+R/L)} &= \frac{a}{s} + \frac{b}{s+R/L} \\ a = L/R\,; \quad b &= -L/R \\ I(s) &= \frac{V}{R}\left(\frac{1}{s} - \frac{1}{s+R/L}\right) \\ i(t) &= \frac{V}{R}\left(\epsilon(t) - e^{-t\frac{R}{L}}\right) \end{aligned}$$

- For initial conditions not zero and input signal $x(t) = 0$ for $t > 0$

$$L\left[sI(s) - i(0-)\right] + RI(s) = V\frac{1}{s}$$
$$L\left[sI(s) - i_0\right] + RI(s) = 0$$
$$LsI(s) + RI(s) = Li_0 = \frac{LV}{R}$$
$$I(s) = \frac{V}{R}\left[\frac{L}{Ls+R}\right]$$
$$I(s) = \frac{V}{R}\left[\frac{1}{s+R/L}\right]$$
$$i(t) = \frac{V}{R}e^{-t\frac{R}{L}}$$

◇ **Example 29 (Initial conditions again).** For a simple serial RLC circuit, where the output signal is defined as current $i(t)$, and a input source voltage $v(t)$ we have a differential equation:

$$v(t) = Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\mathrm{d}\tau$$

Initial conditions are that there is no charge stored in the capacitor and if there is no input but the circuit is closed, there will be no current.

If we have some charge stored at $t = 0$, still $i(0) = 0$; as the circuit is open for $t < 0$. So we can not use the current through the resistor to describe the initial state. The only condition we have is

$$v_C(0) = \frac{1}{C}\int_{-\infty}^{0} i(\tau)\mathrm{d}\tau = Q$$

So we have to decribe the system using $v_C(t)$:

$$\begin{aligned}
v_C(t) &= \frac{1}{C}\int_{0}^{t} i(\tau)\mathrm{d}\tau + Q \\
v_C(0) &= V_0 = \frac{Q}{C} \\
\dot{v}_C(t) &= \frac{1}{C}i(t) \\
i(t) &= C\dot{v}_C(t)
\end{aligned}$$

Now

$$\begin{aligned}
v(t) &= RC\dot{v}_C(t) + LC\ddot{v}_C(t) + v_C(t) \\
V(s) &= RC(V_C(s)s - v_C(0)) + LC(V_C(s)s^2 - sv_C(0) - \dot{v}_C(0)) + V_C(s)
\end{aligned}$$

We have to solve for $V_C(s)$ in a usual way and then we can calculate the current.

$$\begin{aligned} v_C(0) &= v_C(0-) = Q/C \\ \dot{v}_C(0) &= \dot{v}_C(0-) = 0 \end{aligned}$$

$$I(s) = sCV_C(s)$$

$\diamondsuit$ **Example 30 (Analogous mechanical systems).** Consider the differential equation of the mass-spring system is

$$m\,\ddot{y}(t) \, + \, \beta\,\dot{y}(t) \, + \, k\,y(t) = F(t),$$

where the mass is $m = 1$ kg that is attached to a spring with constant $k = 5$ N/m. The medium offers a damping force with $\beta = 6$ N s/m.

(a) Determine the position of the mass with initial conditions $y(0) = 3$ m and $\dot{y}(0) = 1$ m/s. Consider no external force.

(b) Determine the position of the mass $y(t)$ if it is released with the initial conditions $y(0) = 0$ m and $\dot{y}(0) = 0$ m/s, and the system is driven by an external force $F(t) = 30\sin(2t)$ in Newton.

Verify that in (a) we obtain

$$y(t) = 4e^{-t} \, - \, e^{-5t}\,,$$

and in (b)

$$y(t) = 3e^{-t} \, - \, \frac{15}{29}\,e^{-5t} \, - \, \frac{72}{29}\cos(2t) \, + \, \frac{6}{29}\,\sin(2t)\,,$$

or, in other form,

$$y(t) = 3e^{-t} \, - \, \frac{15}{29}\,e^{-5t} + 2.49\,\sin(2t + 5^\circ)\,.$$

6.5 Bode plots

Bode plots depict the frequency response of LTI systems, where the frequency response is obtained by evaluating the transfer function $H(s)$ along the $j\omega$ axis. It usually consists on a Bode magnitude plot, showing the magnitude of the frequency response gain, and on a Bode phase plot, expressing the frequency response phase shift. Bode plots are used as a fast way of finding the approximate frequency response from the poles and zeros of the system.

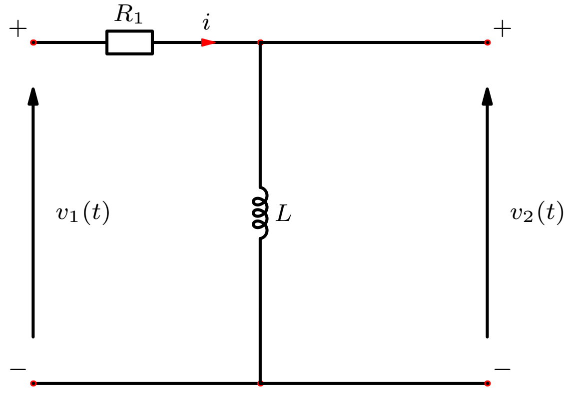


Figure 33:

6.5.1 System response function

$$
\begin{aligned}
v_1(t) &= iR + L\frac{\mathrm{d}i}{\mathrm{d}t} \\
V_1(s) &= I(s)R + LsI(s) \\
V_1(s) &= I(s)\left(R + Ls\right) \\
I(s) &= \frac{V(s)}{R + Ls} \\
v_2(t) &= L\frac{\mathrm{d}i}{\mathrm{d}t} \\
V_2(s) &= LsI(s) = Ls\frac{V(s)}{R + Ls} = V_1(s)\frac{sL}{R + sL} \\
\frac{V_2(s)}{V_1(s)} &= \left.\frac{sL}{R + sL}\right|_{s=j\omega} = \frac{j\omega L}{R + j\omega L}
\end{aligned}
$$

Here we do not include transient parts (complex exponential terms with $Re\{s\} \neq 0$), but would like to know the system response function in therms of $j\omega$ (frequency domain).

$$H(j\omega) = |H(j\omega)|\, e^{j\varphi(j\omega)} = |H(\omega)|\, e^{j\arg\{H(j\omega)\}}$$

6.5.2 Bode plots

As mentioned above the Bode plots consists in:

- The logarithm of the magnitude $|H(j\omega)|$ is plotted against the logarithm of the frequency, ω;
- The phase $\varphi(j\omega)$ is plotted linearly against the logarithm of the frequency.

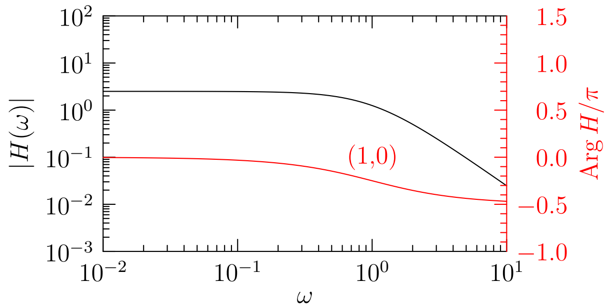


Figure 34: Example of a Bode Plot

$$
\begin{aligned}
H(\omega) &= \frac{j\omega L}{R+j\omega L} = \frac{j\omega L(R-j\omega L)}{(R+j\omega L)(R-j\omega L)} = \\
&= \frac{Rj\omega L+\omega^2L^2}{R^2+\omega^2L^2} = \frac{\omega^2L^2}{R^2+\omega^2L^2} + j\frac{R\omega L}{R^2+\omega^2L^2} \\
|H(\omega)| &= \left[Re\{H(s)\}^2 + Im\{H(s)\}^2\right]^{1/2} \\
|H(\omega)| &= \frac{\sqrt{(\omega^2L^2)^2+R^2\omega^2L^2}}{R^2+\omega^2L^2} \\
\tan(\varphi) &= \frac{R\omega L}{\omega^2L^2} = \frac{R}{\omega L}
\end{aligned}
$$

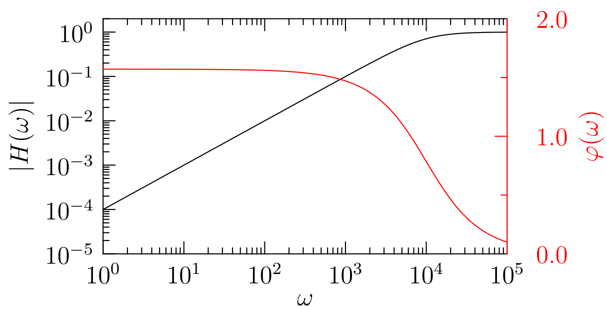


Figure 35: Example of a Bode Plot

The Bode plots are easily calculated using Matlab. To learn more about the characteristics of the Bode plots and how to plot them, only knowing the zeros and poles, refer to the text book.

6.6 Examples of Matlab code

◇ **Example 31 (A).** Imagine that the system is described by a simple equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(t)$$

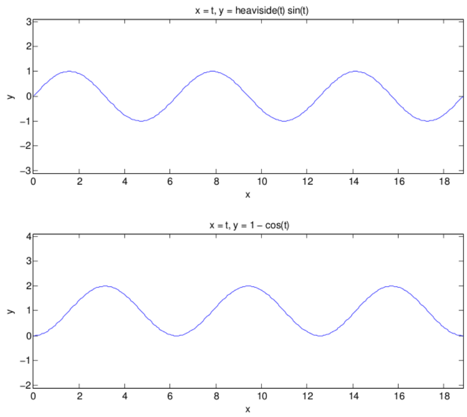
Can we find the output? What assumptions are necessary?

- $sY(s) = X(s)$
- This will only work if $y(0) = 0$

- Otherwise $sY(s) - y(0+) = X(s)$
- $H(s) = 1/s$
- If we have the input signal $x(t) = \sin(t)$; then $X(s) = \frac{1}{s^2+1}$

$$\begin{aligned} Y(s) &= \frac{1}{s}\frac{1}{s^2+1} = \frac{1}{s(s-j)(s+j)} = \frac{A}{s} + \frac{B}{s+j} + \frac{C}{s-j} \\ Y(s) &= \frac{1}{s} - \frac{1/2}{s+j} - \frac{1/2}{s-j} \\ y(t) &= \epsilon(t) - e^{jt} - e^{-jt} = 1 - \cos(t) \quad \forall t \geq 0 \end{aligned}$$

```
function Example1()
syms t s;
v=(sin(t))*heaviside(t);
V=laplace(v);
H=(1/s);
I=H*V; J=ilaplace(I);
subplot(2,1,1);
ezplot(t,v,[-0*pi(),6*pi()]);
subplot(2,1,2);
ezplot(t,J,[-0*pi(),6*pi()]);
end
```



◇ **Example 32 (B).** Lets look at the example above and calculate response for $x(t) = sin(10t)$. Initial conditions are zero.

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$$

What will be the output signal?

$$\begin{aligned} \mathscr{L}\{x(t)\} = \mathscr{L}\{\sin(10t)\} &= \frac{10}{s^2+10^2} \\ H(s) &= \frac{1}{s^2+3s+2} \\ Y(s) &= H(s)X(s) \end{aligned}$$

```
clear all; clf
syms s t w
w = 10; %angular frequency
%Laplace transform of the input signal:
%X(s) = w/(s^2+w^2)
%H(s) = 1/(s^2 + 3s+ 2)

num=[0 0 w ]; den= conv([1 3 2],[1 0 w^2]);
[r,p,k]=residue(num,den)
Y = r(1)/(s-p(1)) +  r(2)/(s-p(2)) + r(3)/(s-p(3))
      + r(4)/(s-p(4));
y=10*ilaplace(Y)
%Impulse response
%H(s) = 1/(s^2 + 3s+ 2)
%X(s) = 1
num=[0 0 1]; den= [1 3 2];
[r,p,k]=residue(num,den)
S0 = r(1)/(s-p(1)) +  r(2)/(s-p(2));
s0=ilaplace(S0);
ezplot(s0,[0,15]); hold;
ezplot(y,[0,15]); hold off;
axis([0 15 -0.5 1]);
grid;
end
```

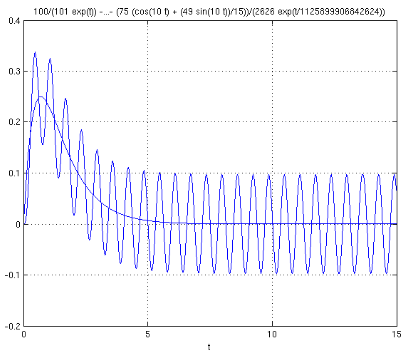


Figure 36: