4 Laplace Transform

Materials: TB - Ch4 and 6; TB2 - Ch3 (lots of examples)

4.1 Previously

We have seen previously that, in general, an output is a convolution of the input signal and the impulse response function:

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\mathrm{d}\tau \,.$$

It should be noted that if the input signal is a short impulse located at $t = 0$ ($\delta(\tau)$), the output can be written as

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\delta(\tau)\mathrm{d}\tau = h(t)$$

and this is the reason $h(t)$ is called impulse response function.

LTI systems based on a LDE can be described by an eigenvalue form, so that the output is

$$y(t) \quad = \quad Ce^{st} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau$$

with Ce^{st} the input and the eigenfunction of the system, and

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau$$

the eigenvalue, also called the Laplace transform of the impulse response.

We have also seen that we can use a combination of eigenfunctions and eigenvalues to get an output for an input composed of many signals;

$$y(t) = \sum_i C_i \, e^{s_i t} \, H(s_i) \,.$$

Thus, and considering that many signals and functions can be described by complex exponentials, it is often easier to work in the frequency domain. **Therefore we must master the process of:**

1. Decomposition of an input signal, $x(t)$, into e^{st} terms;
2. Superposition of e^{st} terms to get the signal $x(t)$ and determine the output, $y(t)$.

4.2 Laplace Transforms

Definition 9 Laplace Transform We define the Laplace transform of $x(t)$ so that,

$$\mathscr{L}\left\{x(t)\right\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\mathrm{d}t \qquad (41)$$

Definition 10 Inverse Laplace Transform The inverse operation, the inverse Laplace transform is defined as,

$$x(t) = \mathscr{L}^{-1}\left\{X(s)\right\} = \frac{1}{2\pi j}\int_{\Delta-j\infty}^{\Delta+j\infty} X(s)e^{st}ds \tag{42}$$

Where this is a path integral in the complex plane. We will not evaluate this integral directly but will resort to other methodologies to calculate $\mathscr{L}^{-1}\left\{x(t)\right\}$. We will see this later in the course, see also Chapter 5 in the TB.

Mathematically we integrate from $-\infty$ to $+\infty$, but in signal processing we are dealing with causal systems. The input (and the corresponding output) signal does not start until $t = 0$. So we can integrate from zero (unilateral Laplace transform, $\mathscr{L}_I\left\{x(t)\right\}$) or include the step function $\epsilon(t)$, such that $x'(t) = x(t)\cdot\epsilon(t)$. In this case:

$$\mathscr{L}\left\{x'(t)\right\} = \int_{-\infty}^{\infty} x(t)\epsilon(t)e^{-st}\mathrm{d}t = \int_{0}^{\infty} x(t)e^{-st}\mathrm{d}t$$

4.3 Important transform pairs for Laplace transforms

Find the Laplace transforms of the following functions:

- **Delta impulse**

$$\int_{-\infty}^{\infty}\delta(t-t_0)e^{-st}\mathrm{d}t = e^{-st_0}$$

$$\int_{-\infty}^{\infty}\delta(t)e^{-st}\mathrm{d}t = e^{-s0} = 1$$

- **Step function**

  We will denote the step function by both $\epsilon(t)$ and $u(t)$ since both are often used.

$$\begin{aligned}\epsilon(t) &= \begin{cases}1, & t \geq 0\\ 0, & t < 0\end{cases}\\ \int_{-\infty}^{\infty}\epsilon(t)e^{-st}\mathrm{d}t &= \int_{0}^{\infty}1e^{-st}\mathrm{d}t = -\frac{1}{s}e^{-st}\Big|_{0}^{+\infty} = 0+\frac{1}{s}; \quad \text{ROC} : \mathrm{Re}\{s\} > 0.\end{aligned}$$

  Due to the upper limit of the integral, $t \to \infty$, the integral converges only for $\mathrm{Re}\{s\} > 0$. We call the part of the complex plane where $\mathrm{Re}\{s\} > 0$ the region of convergence, ROC.

- **anti-causal step function**

both are often used.

$$\begin{aligned}
\epsilon'(t) &= \begin{cases} 0, & t > 0 \\ 1, & t \leq 0 \end{cases} \\
\int_{-\infty}^{\infty} \epsilon'(t) e^{-st} \mathrm{d}t &= \int_{-\infty}^{0} \mathrm{e}^{-st} = -\frac{1}{s} e^{-st} \Big|_{-\infty}^{0} = \\
&= -\frac{1}{s} + \lim_{t \to -\infty} \mathrm{e}^{-st} = -\frac{1}{s} \quad \text{ROC} : \mathrm{Re}\{s\} < 0.
\end{aligned}$$

Due to the lower limit of the integral, $t \to -\infty$, the integral converges only for $\mathrm{Re}\{s\} < 0$. We call the part of the complex plane where $\mathrm{Re}\{s\} < 0$ the region of convergence, ROC.

- **Exponential functions**

$$\mathscr{L}_I \left\{ e^{-at} \right\} = \mathscr{L} \left\{ e^{-at} \epsilon(t) \right\} = \int_0^{\infty} e^{-at} e^{-st} \mathrm{d}t = - \frac{1}{s+a} e^{-(s+a)t} \Big|_0^{\infty} =$$

$$= 0 + \frac{1}{s+a}; \quad \text{ROC} : \mathrm{Re}\{s\} > -a.$$

- **Combination of two exponentials functions**

Let us take,

$$\begin{aligned}
x(t) &= \epsilon(t) \left[e^{-t} + e^{-2t} \right] \\
\mathscr{L}\left\{ x(t) \right\} &= \mathscr{L}_I \left\{ \epsilon(t) \left[e^{-t} + e^{-2t} \right] \right\} = \int_0^{\infty} \left(e^{-(s+1)t} \right) \mathrm{d}t + \int_0^{\infty} \left(e^{-(s+2)t} \right) \mathrm{d}t \\
&= - \frac{1}{s+1} e^{-(s+1)t} \Big|_0^{\infty} - \frac{1}{s+2} e^{-(s+2)t} \Big|_0^{\infty} \\
&= \frac{1}{s+1} + \frac{1}{s+2} = \frac{(s+1)+(s+2)}{(s+1)(s+2)} = \frac{2s+3}{s^2+3s+2}; \quad \text{ROC} : \mathrm{Re}\{s\} > -1.
\end{aligned}$$

The Laplace transform has two poles at $s = -1$ and $s = -2$. As $X(s)$ only exists if both parts converge, the region of convergence is $Re\{s\} > -1$.

Let us now take

$$\begin{aligned}
x(t) &= \epsilon(t) e^{-2t} - \epsilon(-t) e^{-t} \\
\mathscr{L}\left\{ x(t) \right\} &= \int_{-\infty}^{\infty} \epsilon(t) e^{-2t} e^{-st} \mathrm{d}t - \int_{-\infty}^{\infty} \epsilon(-t) e^{-t} e^{-st} \mathrm{d}t = \int_0^{\infty} e^{-(s+2)t} \mathrm{d}t - \int_{-\infty}^{0} e^{-(s+1)t} \mathrm{d}t \\
&= - \frac{1}{s+2} e^{-(s+2)t} \Big|_0^{\infty} + \frac{1}{s+1} e^{-(s+1)t} \Big|_{-\infty}^{0} \\
&= \frac{1}{s+2} + \frac{1}{s+1} = \frac{2s+3}{s^2+3s+2}; \quad \text{ROC} : -2 < \mathrm{Re}\{s\} < -1.
\end{aligned}$$

- **Cos function**

$$
\begin{aligned}
\mathscr{L}_I\left\{\cos(bt)\right\} &= \mathscr{L}\left\{\cos(bt)\epsilon(t)\right\} \\
&= \mathscr{L}\left\{\frac{1}{2}\left(e^{jbt}+e^{-jbt}\right)\epsilon(t)\right\} = \\
&= \int_0^\infty \frac{1}{2}\left(e^{jbt}+e^{-jbt}\right)e^{-st}\mathrm{d}t = \frac{1}{2}\int_0^\infty \left(e^{-(s-jb)t}+e^{-(s+jb)t}\right)\mathrm{d}t \\
&= \frac{-1}{2(s-jb)}e^{-(s-jb)t}\Big|_0^\infty + \frac{-1}{2(s+jb)}e^{-(s+jb)t}\Big|_0^\infty = \frac{1}{2(s-jb)}+\frac{1}{2(s+jb)} \\
&= \frac{(s+jb)}{2(s+jb)(s-jb)}+\frac{(s-jb)}{2(s+jb)(s-jb)} = \frac{s}{s^2+b^2}; \quad \mathrm{Re}\{s\}>0\,.
\end{aligned}
$$

Poles: integral goes to infinity and those frequecies are not damped by the e^{-st} term in the transform, that is they are present in the signal.

◇ **Example 14 (LT from the definition, matlab code).** Here are two example of a simple matlab code where LT is calculated from the definition

```
clear all; close all;
t = [0 : 0.01 : 100]; % time vector
sigma = [0 : 0.05 : 2]; % real part of s
omega = [-2 : 0.05 : 2]; % imaginary part of s
H = zeros(length(sigma),length(omega)); % memory assignment for H
for m = 1 : length(sigma)
for n = 1 : length(omega)
s = sigma(m) + j*omega(n); % bulding 's = sigma + j*omega'
f = cos(-1*t).*exp(-s*t); % the function inside the integral
H(m,n) = trapz(t,f); % integral for the Laplace transform
end
end
figure;
surf(omega,sigma,log(abs(H))) % 3-D plot of H with respect to sigma and omega
ylabel('\sigma'); xlabel('\Omega'); zlabel('log|H(s)|');
box on; %grid on;
```

```
clear all; close all;
t = [0 : 0.01 : 100]; % time vector
sigma = [-1 : 0.05 : 2]; % real part of s
omega = [-2 : 0.05 : 2]; % imaginary part of s
H = zeros(length(sigma),length(omega)); % memory assignment for H
for m = 1 : length(sigma)
for n = 1 : length(omega)
s = sigma(m) + j*omega(n); % bulding 's = sigma + j*omega'
f = exp(-1*t).*exp(-s*t); % the function inside the integral
H(m,n) = trapz(t,f); % integral for the Laplace transform
end
end
figure;
surf(omega,sigma,log(abs(H))) % 3-D plot of H with respect to sigma and omega
ylabel('\sigma'); xlabel('\Omega'); zlabel('log|H(s)|');
box on; %grid on;
```

If the signal is not defined for all t ($0 < t < \infty$), then the LT integral will converge for all s. Still the value of the transform $\to \infty$ for some values of s. Those values are referred to as `poles`, and they can be used to guess the form of the signal leading to a particular Laplace transform. More about this later in the course. One can understand this in terms of weighing factors (which is the LT) connected to some complex exponential signal present in the $x(t)$.

4.4 Some properties of Laplace transforms

4.4.1 Multiplication by t

Let us consider $X(s) = \mathscr{L}\{x(t)\}$ the Laplace transform of time function $x(t)$. The theorem for the Laplace transform of $x(t)$ that has been multiplied by time variable t can be written as,

$$\mathscr{L}\{tx(t)\} = -\frac{\mathrm{d}X(s)}{\mathrm{d}s}; \quad s \in \mathrm{ROC}\{x\}\,. \tag{43}$$

and is also known as the differentiation in the frequency domain. Derivation:

$$\begin{aligned} X(s) &= \int\limits_{-\infty}^{\infty} e^{-st}\, x(t)\, \mathrm{d}t \\ \frac{\mathrm{d}X(s)}{\mathrm{d}s} &= \int\limits_{-\infty}^{\infty} -te^{-st}\, x(t)\, \mathrm{d}t \\ &= -\mathscr{L}\{tx(t)\}\ . \end{aligned}$$

4.4.2 Time shifting property

Let us consider $X(s) = \mathscr{L}\{x(t)\}$ the Laplace transform of time function $x(t)$. We can define the time shift theorem according to,

$$\mathscr{L}\{x(t-\tau)\} = e^{-s\tau}\, X(s); \quad s \in \mathrm{ROC}\{x\}\,. \tag{44}$$

Derivation:

$$\begin{aligned} \mathscr{L}\{x(t-\tau)\} &= \int\limits_{-\infty}^{\infty} e^{-st}\, x(t-\tau)\, \mathrm{d}t \\ &= \int\limits_{-\infty}^{\infty} e^{-s(t'+\tau)}\, x(t')\, \mathrm{d}t' \\ &= e^{-s\tau} \int\limits_{-\infty}^{\infty} e^{-st'}\, x(t')\, \mathrm{d}t' \\ &= e^{-s\tau}\, X(s)\,. \end{aligned}$$

◇ **Example 15 (Application of the time shifting property).** Find the Laplace transform of signal $x(t)$ represented in Fig. 20. This signal can be constructed with one line

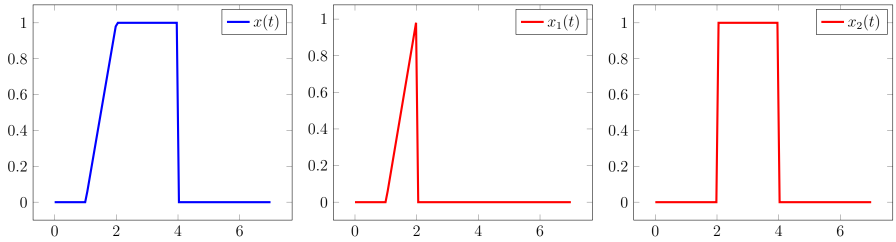

Figure 20: Signal $x(t)$ of example 1.

defined by $t-1$ ($x_1(t)$) and a rectangular pulse with unit amplitude starting at $t=2$ and ending at $t=4$ ($x_2(t)$):

$$\begin{aligned} x(t) &= x_1(t)+x_2(t) = (t-1)[\epsilon(t-1)-\epsilon(t-2)] + [\epsilon(t-2)-\epsilon(t-4)] \\ &= (t-1)\epsilon(t-1)-(t-1)\epsilon(t-2)+\epsilon(t-2)-\epsilon(t-4) \\ &= (t-1)\epsilon(t-1)-(t-2)\epsilon(t-2)-\epsilon(t-4) \end{aligned}$$

$$\begin{aligned} \epsilon(t) &\circ\!\!-\!\!\bullet \quad \frac{1}{s} \\ t\,\epsilon(t) &\circ\!\!-\!\!\bullet \quad -\frac{\mathrm{d}U(s)}{\mathrm{d}s} = \frac{1}{s^2} \end{aligned}$$

Applying the time shift property and taking each term individually:

$$\begin{aligned} (t-1)\epsilon(t-1) &\circ\!\!-\!\!\bullet \quad \frac{1}{s^2}\,e^{-s} \\ (t-2)\epsilon(t-2) &\circ\!\!-\!\!\bullet \quad \frac{1}{s^2}\,e^{-2s} \\ \epsilon(t-4) &\circ\!\!-\!\!\bullet \quad \frac{1}{s}\,e^{-4s}\,. \end{aligned}$$

Finally,

$$X(s) = \frac{1}{s^2}\,e^{-s} - \frac{1}{s^2}\,e^{-2s} - \frac{1}{s}\,e^{-4s}\,.$$

4.4.3 Time-differentiation and time-integration properties

It is necessary, when dealing with LDE, to know the Laplace transform of the derivation, and the integral of the time function. We start with the inverse Laplace transform,

$$x(t) = \mathscr{L}^{-1}\left\{X(s)\right\} = \frac{1}{2\pi j}\int_{\Delta-j\infty}^{\Delta+j\infty} X(s)e^{st}ds\,.$$

$X(s)$ is not time depend (only e^{st} is a function of time) and so, if differentiable, we obtain

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \frac{1}{2\pi j}\int_{\Delta-j\infty}^{\Delta+j\infty} sX(s)e^{st}ds \\ \mathscr{L}\left\{\frac{\mathrm{d}x}{\mathrm{d}t}\right\} &= sX(s)\,, \end{aligned}$$

Similarly, through the integration of $x(t)$ the *integration theorem* is obtained:

$$\mathscr{L}\left\{\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau\right\} = \frac{1}{s}X(s)\,.$$

Definition 11 Differentiation and Integration theorems

$$\begin{aligned}\mathscr{L}\left\{\frac{\mathrm{d}x}{\mathrm{d}t}\right\} &= sX(s)\,,\\ \mathscr{L}\left\{\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau\right\} &= \frac{1}{s}X(s)\,.\end{aligned}$$

and the differentiation theorem requires that $x(t)$ can be differentiated for all t (continuous in t)

The differentiation theorem is only valid for functions that can be differentiated for all t (continuous in t). This is not the case for causal signals, and we need a modification of that theorem that applies for causal signals. In order to use the theorem we represent the right-sided signal, $x(t)$, by a bilateral signal, $u(t)$ which can be differentiated, and the step function, according to (Fig. 21):

$$x(t) = u(t)\,\epsilon(t)\,.$$

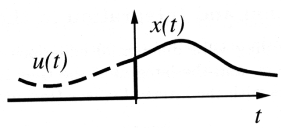

Figure 21: Signal with and without step at $t = 0$.

The Laplace transform of $x(t)$ is

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\mathrm{d}t = \int_{0}^{\infty} u(t)e^{-st}\mathrm{d}t$$

Partial integration of the second integral gives:

$$\begin{aligned}
\int_a^b f'(x)g(x)\mathrm{d}x &= f(x)g(x)|_a^b - \int_a^b f(x)g'(x)\mathrm{d}x \\
X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}\mathrm{d}t = \int_0^{\infty} u(t)e^{-st}\mathrm{d}t \\
X(s) &= -\frac{1}{s}\int_0^{\infty} -se^{-st}u(t)\mathrm{d}t = -\frac{1}{s}\left\{ e^{-st}\,u(t)|_0^{\infty} - \int_0^{\infty} e^{-st}\frac{\mathrm{d}u(t)}{\mathrm{d}t}\mathrm{d}t \right\} \\
X(s) &= \frac{1}{s}u(0) + \frac{1}{s}\int_0^{\infty} \frac{\mathrm{d}u(t)}{\mathrm{d}t}e^{-st}\mathrm{d}t \\
sX(s) &= u(0) + \mathscr{L}\{\dot{u}(t)\} \\
\mathscr{L}\{\dot{u}(t)\} &= sX(s) - u(0)
\end{aligned}$$

We can not simply replace $\dot{u}(t)$ with $\dot{x}(t)$ as the derivative at $t = 0$ does not exist. However if we introduce a new function

$$x^o(t) = \begin{cases} \dot{u}(t), & t > 0 \\ 0, & t < 0 \end{cases} = \begin{cases} \dot{x}(t), & t > 0 \\ 0, & t < 0 \end{cases} = \dot{x}(t) \quad \forall\, t \neq 0$$

Then

$$u(0) = x(0+)\,,$$

that is, at $t = 0$ the value of $u(0)$ is equal to the right-sided limit of $x(t)$ at $t \to 0$. We have now obtained the differentiation theorem of bilateral Laplace transform for right-sided signals:

$$\mathscr{L}\{x^o(t)\} = sX(s) - x(0+)$$

For signals which only exists for $t > 0$, unilateral $\mathscr{L}_I\{x(t)\}$ which integrates for $t > 0$ is often used.

Definition 12 Differentiation theorem for causal signals

$$\mathscr{L}_I\{\dot{x}(t)\} = sX(s) - x(0+) \tag{45}$$

it should be noticed that the function $x(t)$ and its derivative is only of interest for $t > 0$ and $x(0)$ is the right-handed limit.

Note: Signal called $u(t)$ in this example represents the bilateral version of the input $x(t)$. Not to be confused with the step or unit function, also using the same notation.

Similarly:

$$\mathscr{L}_I\left\{\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}\right\} = s^2X(s) - [sx(0+) + \dot{x}(0+)]$$

We can also can define integration theorem with the use of unilateral $\mathscr{L}_I\{x(t)\}$ (here $x(t)$ does not have to be differentiable):

$$\mathscr{L}_I\left\{\int_0^t x(\tau)\mathrm{d}\tau\right\} = \frac{1}{s}X(s)$$

WARNING: Often the unilateral $\mathscr{L}_I\{x(t)\}$ is defined with the left-handed limit, that is $x(0-)$ (for example in TB2). In principle this is equivalent to the treatment above, as long as we are consequent. If we use the convention with $x(0+)$, then, for example:

$$\mathscr{L}_I\{\delta(t)\} = \int_{0+}^{\infty}\delta(t)e^{-st}\mathrm{d}t = 0$$

If we use the convention with $x(0-)$, then

$$\mathscr{L}_I\{\delta(t)\} = \int_{0-}^{\infty}\delta(t)e^{-st}\mathrm{d}t = e^0 = 1$$

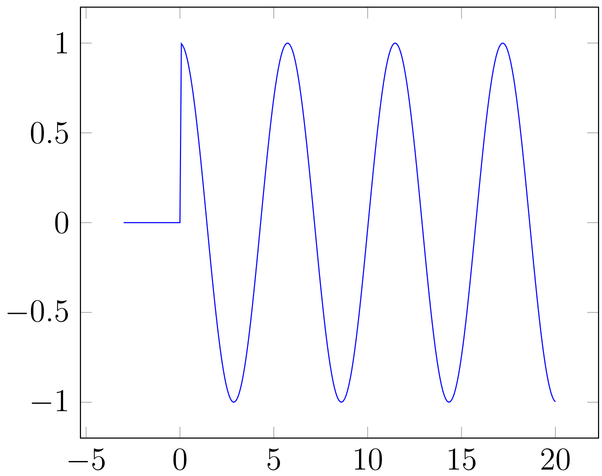

Figure 22: Signal $x(t)$ for Example 16

◇ **Example 16.** From Laplace transform of $x(t) = \cos(\Omega t)\epsilon(t)$ (Fig. 22) obtain the Laplace transform of $x(t) = \sin(\Omega t)\epsilon(t)$

Using $x(0-)$ limit where $x(0-) = 0$:

$$
\begin{aligned}
\mathscr{L}\left\{\cos(\Omega t)\epsilon(t)\right\} = X(s) &= \frac{s}{s^2+\Omega^2} \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= \epsilon(t)\frac{\mathrm{d}\cos(\Omega t)}{\mathrm{d}t} + \cos(\Omega t)\frac{\mathrm{d}\epsilon(t)}{\mathrm{d}t} \\
&= -\Omega\sin(\Omega t)\epsilon(t) + \cos(\Omega t)\delta(t) \\
&= -\Omega\sin(\Omega t)\epsilon(t) + \delta(t) \\
\mathscr{L}\left\{\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right\} &= sX(s) - x(0) = \mathscr{L}\left\{-\Omega\sin(\Omega t)\epsilon(t)\right\} + \mathscr{L}\left\{\delta(t)\right\} \\
\mathscr{L}\left\{-\sin(\Omega t)\epsilon(t)\right\} &= \frac{sX(s) - x(0) - 1}{\Omega} \\
\mathscr{L}\left\{\sin(\Omega t)\epsilon(t)\right\} &= \frac{1 - sX(s) + x(0)}{\Omega} = \frac{1 - sX(s)}{\Omega} \\
\mathscr{L}\left\{\sin(\Omega t)\epsilon(t)\right\} &= \frac{1}{\Omega}\frac{s^2+\Omega^2-s^2}{s^2+\Omega^2} \\
\mathscr{L}\left\{\sin(\Omega t)\epsilon(t)\right\} &= \frac{\Omega}{s^2+\Omega^2}
\end{aligned}
$$

For $x(0+)$ limit, $x(0+) = 1$:

$$
\begin{aligned}
\mathscr{L}\left\{\cos(\Omega t)\epsilon(t)\right\} = X(s) &= \frac{s}{s^2+\Omega^2} \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= \epsilon(t)\frac{\mathrm{d}\cos(\Omega t)}{\mathrm{d}t} + \cos(\Omega t)\frac{\mathrm{d}\epsilon(t)}{\mathrm{d}t} \\
&= -\Omega\sin(\Omega t)\epsilon(t) + \cos(\Omega t)\delta(t) \\
&= -\Omega\sin(\Omega t)\epsilon(t) + \delta(t) \\
\mathscr{L}\left\{\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right\} &= sX(s) - x(0) = \mathscr{L}\left\{-\Omega\sin(\Omega t)\epsilon(t)\right\} + \mathscr{L}\left\{\delta(t)\right\} \\
\mathscr{L}\left\{-\sin(\Omega t)\epsilon(t)\right\} &= \frac{sX(s) - 1 - 0}{\Omega} \\
\mathscr{L}\left\{\sin(\Omega t)\epsilon(t)\right\} &= \frac{1 - sX(s))}{\Omega} \\
\mathscr{L}\left\{\sin(\Omega t)\epsilon(t)\right\} &= \frac{1}{\Omega}\frac{s^2+\Omega^2-s^2}{s^2+\Omega^2} \\
\mathscr{L}\left\{\sin(\Omega t)\epsilon(t)\right\} &= \frac{\Omega}{s^2+\Omega^2}
\end{aligned}
$$

4.5 Laplace transforms for solving differential equations

Let us recall the definition of a linear differential equation:

$$
\sum_{i=0}^{N} \alpha_i \frac{\mathrm{d}^i y}{\mathrm{d}t^i} = \sum_{k=0}^{M} \beta_k \frac{\mathrm{d}^k x}{\mathrm{d}t^k} \tag{46}
$$

Or, in other form,

$$
\alpha_0 y(t) + \alpha_1 \dot{y}(t) + \alpha_2 \ddot{y}(t) + \ldots = \beta_0 x(t) + \beta_1 \dot{x}(t) + \beta_2 \ddot{x}(t) + \ldots
$$

Let us use the differential theorem assuming that $x(0+) = 0$, $\dot{x}(0+) = 0, \ldots$ and $y(0+) = 0$, $\dot{y}(0+) = 0, \ldots$ (Eq. 45),

$$\begin{aligned}
\alpha_0 Y(s) + \alpha_1 s Y(s) + \alpha_2 s^2 Y(s) + \ldots &= \beta_0 X(s) + \beta_1 s X(s) + \beta_2 s^2 X(s) + \ldots \\
(\alpha_0 + \alpha_1 s + \alpha_2 s^2 + \ldots) Y(s) &= (\beta_0 + \beta_1 s + \beta_2 s^2 + \ldots) X(s)
\end{aligned}$$

Finally,

$$\begin{aligned}
Y(s) &= \frac{(\beta_0 + \beta_1 s + \beta_2 s^2 + \ldots)}{(\alpha_0 + \alpha_1 s + \alpha_2 s^2 + \ldots)} X(s) \\
Y(s) &= H(s)\, X(s) \\
H(s) &= \frac{(\beta_0 + \beta_1 s + \beta_2 s^2 + \ldots)}{(\alpha_0 + \alpha_1 s + \alpha_2 s^2 + \ldots)}
\end{aligned}$$

where $H(s)$ is the system transfer function. So if we can find $X(s)$ and $H(s)$, and apply the inverse transform to get $y(t)$ from $Y(s)$. This way we get the response $y(t)$ of the LTI system described by a differential equation of the form given by Eq. 49 to the input signal $x(t)$.

◇ **Example 17.** Calculate the transfer function of the following differential equation, assuming that $y(0+) = \dot{y}(0+) = \ddot{y}(0+) = x(0+) = \dot{x}(0+) = 0$,

$$2\ddot{y} - 3\dot{y} + 5y = 10\dot{x} - 7x$$

using the differential theorem (Eq. 45), yields the algebraic equation

$$2s^2 Y(s) - 3sY(s) + 5Y(s) = 10sX(s) - 7X(s)$$

from which we can obtain the transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{10s - 7}{2s^2 - 3s + 5}\,.$$

◇ **Example 18.** Solve equation

$$m\ddot{y}(t) = x(t) - k_1 \dot{y}(t) - k_2 y(t)$$

`ode45` in matlab for $x(t) = \cos(100t)\mathrm{e}^{-t/100}$ and use the Laplace transform to determine transfer function of this system

$$p_1 = y(t)$$
$$p_2 = \dot{y}(t)$$

$$\begin{aligned}
\dot{p}_1 &= p_2 \\
\dot{p}_2 &= \frac{1}{m} x(t) - \frac{k_1}{m} p_2(t) - \frac{k_2}{m} p_1
\end{aligned}$$

$$
\begin{aligned}
ms^2Y(s) &= X(s) - k_1 sX(s) - k_2 Y(s) \\
Y(s)\left[ms^2 + k_1 s + k_2\right] &= X(s) \\
Y(s) &= \frac{1}{ms^2 + k_1 s + k_2} X(s) \\
H(s) &= \frac{1}{ms^2 + k_1 s + k_2} \\
H(j\omega) &= \left.\frac{1}{ms^2 + k_1 s + k_2}\right|_{s=j\omega}
\end{aligned}
$$

This will have poles at:

$$
s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-k_1 \pm \sqrt{k_1^2 - 4mk_2}}{2m}
$$

and for $k_2 \gg k_1$ the resonance frequency will be at

$$
\omega = \sqrt{\frac{k_2}{m}}
$$

---

5 Laplace Transform and LTI systems

5.1 Summary so far

- We can find the output of a system described by a LDE, provided we can find the Laplace transform of the input signal and the initial conditions are zero.
- The Laplace transform of the impulse response simplifies the description of elements such as differentiators and integrators.

5.2 Other important properties of Laplace transforms

- **Linearity**

  The Laplace transform is linear, so from the Laplace transform of a linear superposition of two functions of time, we can recover the Laplace transform of these functions. For any two complex constants a and b:

$$\mathscr{L}\{af(t) + bg(t)\} = a\mathscr{L}\{f(t)\} + b\mathscr{L}\{g(t)\} \ ,$$

  for all values on the complex frequency plane, as long as both $\mathscr{L}\{f(t)\}$ and $\mathscr{L}\{g(t)\}$ exist ($\mathrm{ROC}\{af + bg\} \supseteq \mathrm{ROC}\{f\} \cap \mathrm{ROC}\{g\}$).

  The region of convergence of the sum $\mathrm{ROC}\{af+bg\}$ is the intersection between the regions of convergence $\mathrm{ROC}\{f\}$ and $\mathrm{ROC}\{g\}$. These two regions of convergence must overlap for the Laplace transform of the sum to exist. However, singularities may be removed by the addition, so the region of convergence $\mathrm{ROC}\{af + bg\}$ can be larger than overlap of the individual regions of convergence.

- **Time-scaling property**

$$\mathscr{L}\{x(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right) \ ,$$

  The region of convergence is scaled the same way, i.e., $s \in \mathrm{ROC}\{x(at)\}$ if $\frac{s}{a} \in \mathrm{ROC}\{x(t)\}$.

- **Modulation or shifting in the frequency domain**

Definition 13 Modulation

$$\mathscr{L}\left\{e^{at}x(t)\right\} = X(s-a)\,, \quad s - \mathrm{Re}\{a\} \in \mathrm{ROC}\{s\}, \quad a \in C. \tag{47}$$

The ROC associated with $X(s-a)$ is that of $X(s)$ shifted by $\mathrm{Re}(a)$:

- **Convolution theorem**

  As mentioned previously, the time-domain response can be described as a convolution of the input signal and system response function.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau \, .$$

Using the Laplace transform, swapping the integrals and collecting together the terms we obtain:

$$\begin{aligned}
\mathscr{L}\{x(t) * h(t)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\mathrm{d}\tau \right] e^{-st}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)e^{-st}\mathrm{d}t \right] \mathrm{d}\tau \\
t' = t - \tau \Leftrightarrow t = t' + \tau & \\
&= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t')e^{-(t'+\tau)s}\mathrm{d}t' \right] \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} x(\tau)e^{-s\tau} \left[\int_{-\infty}^{\infty} h(t')e^{-st'}\mathrm{d}t' \right] \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} x(\tau)e^{-s\tau}\mathrm{d}\tau \int_{-\infty}^{\infty} h(t')e^{-st'}\mathrm{d}t' \\
&= \mathscr{L}\{x(t)\} \cdot \mathscr{L}\{h(t)\} \\
\mathscr{L}\{x(t) * h(t)\} &= Y(s) = H(s)X(s)\,.
\end{aligned}$$

Definition 14 Summarizing, the convolution theorem of the Laplace transform can be written as,

$$\mathscr{L}\{x(t) * y(t)\} = X(s)H(s)\,, \quad s \in \mathrm{ROC}\{x * h\} \supseteq \mathrm{ROC}\{x\} \cap \mathrm{ROC}\{h\}\,.$$

The theorem is applicable for all pairs of functions of time whose Laplace transform exists.

This property is one of the most important application of the Laplace transform to signal processing. The computation of the convolution integral is difficult even for simple signals. Not only the Laplace transform gives an efficient solution to the computation of the convolution integral but it also introduces an important representation of LTI systems, the *transfer function*. The transfer function provides information about the system, by inspection of the respective regions of convergence, zeros and poles, and it also indicates how the system reacts to changes in frequency. The design of filters, for example, depend on the transfer function.

5.3 Zeros and Poles

Usually, $X(s)$ is written in the following form:

$$X(s) = \frac{(\beta_0 + \beta_1 s + ... + \beta_m s^m)}{(\alpha_0 + \alpha_1 s + ... + \alpha_n s^n)}\,,$$

which can be written as,

$$X(s) = \frac{\beta_m}{\alpha_n}\frac{(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}$$

The roots of the numerator polynomial, z, are called the *zeros* of $X(s)$ because $X(s) = 0$ for those values of s. Similarly, the roots of the denominator polynomial, p, are called the poles of $X(s)$ because $X(s)$ is infinite for those values of s. Therefore, the poles of $X(s)$ lie outside the ROC since $X(s)$ does not converge at the poles, by definition. The zeros, on the other hand, may lie inside or outside the ROC. Except for a scale factor β_m/α_n, $X(s)$ can be completely specified by its zeros and poles. Thus, a very compact representation of $X(s)$ in the s-plane is to show the locations of poles and zeros in addition to the ROC. Traditionally, an $\times$ is used to indicate each pole location and an $\circ$ is used to indicate each zero.

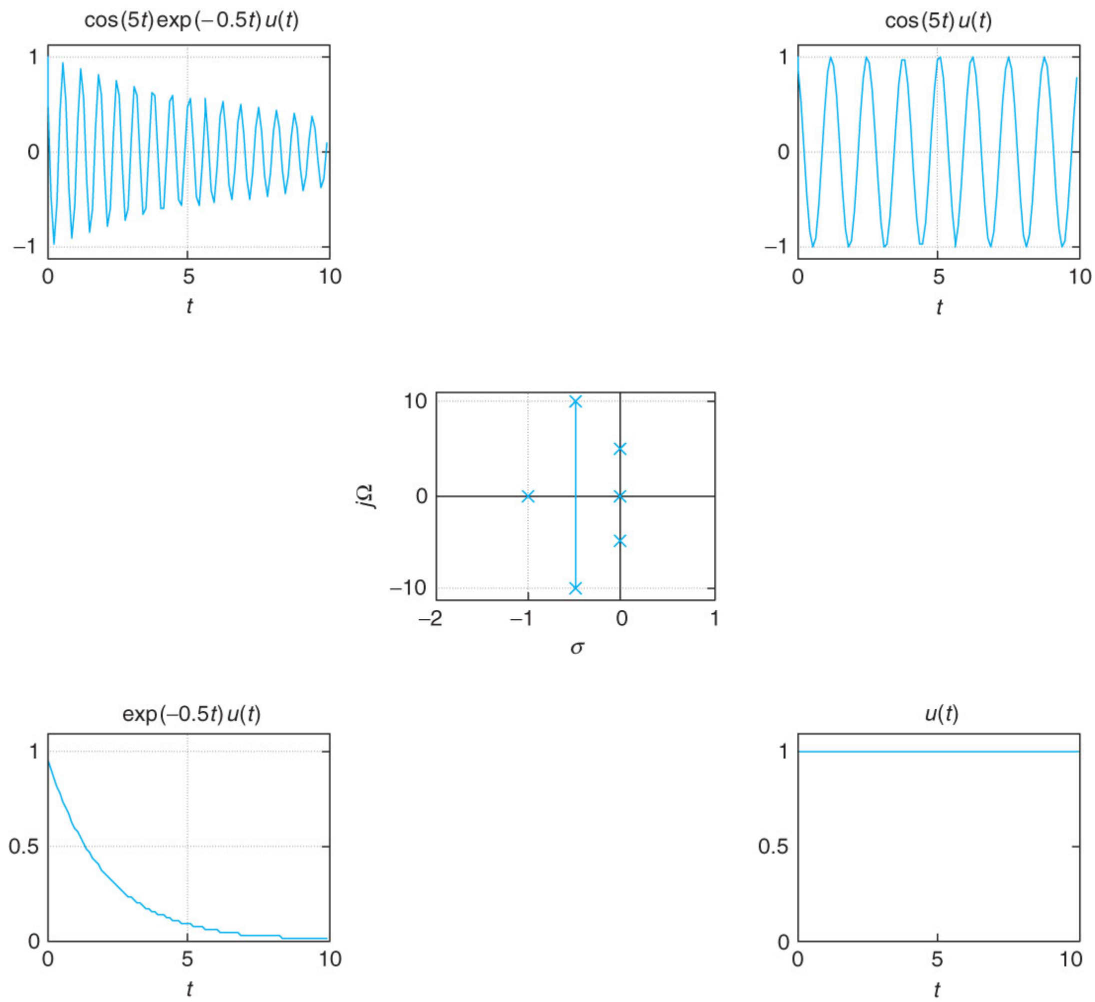

Figure 23: Poles of LT and the corresponding signals.

The location of the poles of the Laplace transform gives significant information about the signal. Let us take the Laplace transform of signal $x(t) = e^{-at}\epsilon(t)$

$$X(s) = \frac{1}{s+a} \quad Re\{s\} > -a$$

For any real a, $X(s)$ has a pole on the real axis of the s-plane at $Re\{s\} = -a$. The location of the pole is closely related to the signal characteristics. For example, if $a = 5$, $f(t) = e^{-5t}\epsilon(t)$ is a decaying exponential and the pole of $F(S)$ is at $s = -5$ (in the left-hand s-plane); if $a = -5$, we have a growing exponential and the pole is at $s = 5$. The larger the value of $|a|$, the faster the exponential decays (for $a > 0$) or increases (for $a < 0$). For $a = 0$, the pole at the origin $s = 0$ corresponds to the signal $f(t) = \epsilon(t)$, which is constant for $t \geq 0$, i.e., it does not decay. This transform has no zeros.

How about signal $x(t) = A\cos(\omega t)\epsilon(t)$?

$$\mathscr{L}\left\{A\cos(\omega t)\epsilon(t)\right\} = \frac{As}{s^2+\omega^2}$$

The transform of this signal has a zero at $s = 0$ and poles at $s^2 + \omega^2 = 0$. So $s = \pm j\omega$. The farther away from the origin of the $j\omega$ axis the poles are, the higher the frequency, and the closer the poles are to the origin, the lower the frequency. Location of the poles again describe the signal.

To conclude, signals are characterized by their damping and frequency and, as such, can be described by the poles of its Laplace transform. If we want to add the two signals considered here, the Laplace transform of the resulting signal is the sum of the Laplace transform of each of the signals (linear property) and the poles are the combination of the poles from each signal (if not canceled). When we want to find the inverse Laplace transform we need to do the opposite, that is, we isolate poles, or pairs of poles (when they are complex conjugate) to find the form of the signal.

Figure 23 shows the poles obtained for a number of signals.

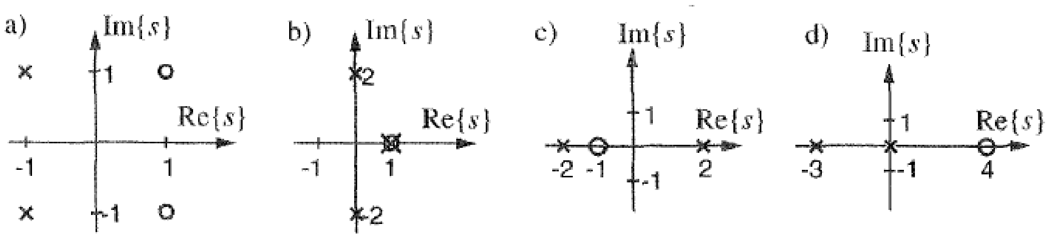

Figure 24: Pole-zero diagrams.

◇ **Example 19 (Exercise 6.3 TB).** Determine the transfer function that has the pole-zero diagrams shown in Fig. 24.

a)

$$H(s) = K\frac{(s-1+j)(s-1-j)}{(s+1+j)(s+1-j)} = K\frac{s^2-2s+2}{s^2+2s+2}$$

b)

$$H(s) = K\frac{1}{(s+2j)(s-2j)} = K\frac{1}{s^2+4}$$

c)

$$H(s) = K\frac{(s+1)}{(s+2)(s-2)} = K\frac{s+1}{s^2-4}$$

d)

$$H(s) = K\frac{(s-4)}{s(s+3)} = K\frac{s-4}{s^2+3s}$$

◇ **Example 20 (Exercise 4.5 TB).** $F(s) = \frac{2s+3}{s^2+3s+2}$ and $G(s) = \frac{3s+1}{s^2+4s+3}$ are the Laplace transforms of two right-sided signals.

1. Find the poles of $F(s)$ and give the region of convergence

$$\begin{aligned} s^2+3s+2=0 \quad &\Leftrightarrow \quad s=\frac{-3\pm\sqrt{9-8}}{2}=\frac{-3\pm 1}{2} \\ &\Leftrightarrow \quad s=-2 \vee s=-1 \\ \text{ROC} : \text{Re}\{s\} > -1. & \end{aligned}$$

2. Find the poles of $G(s)$ and give the region of convergence

$$\begin{aligned} s^2+4s+3=0 \quad &\Leftrightarrow \quad s=\frac{-4\pm\sqrt{16-12}}{2}=\frac{-4\pm 2}{2} \\ &\Leftrightarrow \quad s=-3 \vee s=-1 \\ \text{ROC} : \text{Re}\{s\} > -1. & \end{aligned}$$

3. Find the poles and zeros of $F(s)+G(s)$ and give the region of convergence.

$$F(s)+G(s) = \frac{2s+3}{(s+1)(s+2)}+\frac{3s+1}{(s+1)(s+3)}=\frac{5s^2+16s+11}{(s+1)(s+2)(s+3)}$$

Calculating the zeros:

$$\begin{aligned} 5s^2+16s+11=0 \quad &\Leftrightarrow \quad s=\frac{-16\pm\sqrt{256-220}}{10}=\frac{-16\pm 6}{10} \\ &\Leftrightarrow \quad s=-1 \vee s=-2.2 \end{aligned}$$

We can write,

$$\begin{aligned} F(s)+G(s) &= \frac{(s+1)(5s+11)}{(s+1)(s+2)(s+3)}=\frac{5s+11}{(s+2)(s+3)} \\ \text{ROC} : \text{Re}\{s\} > -2. & \end{aligned}$$

5.4 Inverse Laplace transforms

Inverting the Laplace transform consists in finding a function (either a signal or an impulse response of a system) that has the given transform with the given region of convergence. The inverse Laplace transform is defined as,

$$x(t)=\mathscr{L}^{-1}\{X(s)\}=\frac{1}{2\pi j}\int_{\Delta-j\infty}^{\Delta+j\infty} X(s)e^{st}ds \tag{48}$$

Where this is a path integral in the complex plane. We will not evaluate this integral directly but will resort to other methods to calculate $\mathscr{L}^{-1}\{x(t)\}$. The most common consists in using an appropriate combination of partial fraction expansions and the modulation theorem (Fig. 47), or other Laplace transform properties and pairs.

Some examples:

- **Simple real poles**

  If $X(s)$ is a proper rational function

$$X(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{\Pi_k(s-p_k)}$$

where the p_k are simple poles of $X(s)$, its partial fraction expansion and its inverse are given by

$$X(s) = \sum_k \frac{A_k}{s - p_k} \bullet\!\!-\!\!\circ\, x(t) = \sum_k A_k e^{p_k t},$$

where the expansion coefficients are computed as

$$A_k = X(s)(s - p_k)|_{s=p_k}.$$

5.4.1 Note on (not)Proper Rational Function

So what happens if X(s) in the example above is not a proper rational fraction, but the order of the polynomial N(s) is the same or higher then that of D(s). In that case partial fraction will have number of terms of type $A_n s^n$, with $n \geq 0$. So what is a inverse Laplace transform of that?

$$\mathscr{L}^{-1}\{s^n\} = ?$$

We can find it by looking at $\delta(t)$ and differentiation property of Laplace transform:

$$\begin{aligned} \mathscr{L}\{\delta(t)\} &= 1 \\ \mathscr{L}\{\dot{x}(t)\} &= sX(s) \\ x(t) = \delta(t) \quad X(s) &= 1 \\ \mathscr{L}\left\{\dot{\delta}(t)\right\} &= s \cdot 1 \\ \mathscr{L}\left\{\frac{\mathrm{d}^N}{\mathrm{d}t^N}\delta(t)\right\} &= s^N \end{aligned}$$

◇ **Example 21.** A general system is described by the differential equation:

$$2\dot{x} + 6x = \ddot{y} + 7\dot{y} + 6y$$

Find the output from a unit step.

1. Find $H(s)$:

$$\begin{aligned} (2s+6)X(s) &= (s^2+7s+6)Y(s) \\ H(s) = \frac{Y(s)}{X(s)} &= \frac{2s+6}{s^2+7s+6} \end{aligned}$$

2. Find $X(s)$:

$$X(s) = \mathscr{L}\{\epsilon(t)\} = \frac{1}{s}$$

3. Find the output $Y(s)$ and $y(t)$:

$$\begin{aligned} Y(s) &= \frac{2s+6}{s(s^2+7s+6)} \\ y(t) &= \frac{1}{2\pi j}\int_{\Delta-j\infty}^{\Delta+j\infty} Y(s)e^{st}ds = \frac{1}{2\pi j}\int_{\Delta-j\infty}^{\Delta+j\infty} \frac{2s+6}{s(s^2+7s+6)}e^{st}ds \end{aligned}$$

Recall:

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} &= \epsilon(t) \\ \mathscr{L}^{-1}\left\{\frac{1}{s+a}\right\} &= e^{-at}\epsilon(t) \end{aligned}$$

Partial fraction expansion:

$$\begin{aligned} Y(s) &= \frac{b(s)}{(s-p_1)(s-p_2)(s-p_3)} = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \frac{r_3}{s-p_3} \\ r_1 &= \frac{b(s)}{(s-p_1)(s-p_2)(s-p_3)}(s-p_1)\bigg|_{s=p_1} \\ r_2 &= \frac{b(s)}{(s-p_1)(s-p_2)(s-p_3)}(s-p_2)\bigg|_{s=p_2} \\ r_3 &= \frac{b(s)}{(s-p_1)(s-p_2)(s-p_3)}(s-p_3)\bigg|_{s=p_3} \end{aligned}$$

$$\begin{aligned} \frac{2s+6}{s(s^2+7s+6)} &= \frac{2s+6}{s(s+1)(s+6)} \\ r_1 = sY(s)|_{s=0} &= \frac{2s+6}{(s+1)(s+6)}\bigg|_{s=0} = 1 \\ r_2 = (s+1)Y(s)|_{s=-1} &= \frac{2s+6}{(s)(s+6)}\bigg|_{s=-1} = -\frac{4}{5} \\ r_3 = (s+6)Y(s)|_{s=-6} &= \frac{2s+6}{(s)(s+1)}\bigg|_{s=-6} = -\frac{1}{5} \end{aligned}$$

Finally,

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{0.8}{s+1} - \frac{0.2}{s+6} \\ y(t) &= [1 - 0.8e^{-t} - 0.2e^{-6t}]\epsilon(t) \end{aligned}$$

In this way we can find out where are the poles and describe the signal $y(t)$. ♣

- **Simple complex conjugate poles**

  The partial fraction expansion of a proper rational function

$$X(s) = \frac{N(s)}{(s+\alpha)^2+\Omega_0^2} = \frac{N(s)}{(s+\alpha-j\Omega_0)(s+\alpha+j\Omega_0)}$$

With complex conjugate poles $\{s_{1,2} = -\alpha \pm j\Omega_o\}$ is given by

$$X(s) = \frac{A}{s+\alpha-j\Omega_0} + \frac{A^*}{s+\alpha+j\Omega_0}\,,$$

where

$$A = X(s)(s+\alpha-j\Omega_0)|_{s=-\alpha+j\Omega_0} = |A|e^{j\theta}\,,$$

so that the inverse is the function

$$x(t) = 2|A|e^{-\alpha t}\cos(\Omega_0 t+\theta)\,.$$

An equivalent method consists in expressing the numerator $N(s)$ of $X(s)$, for constants A and B as $N(s) = A + B(s+\alpha)$, a first-order polynomial, so that,

$$X(s) = \frac{A+B(s+\alpha)}{(s+\alpha)^2+\Omega_0^2} = \frac{A}{\Omega_0}\frac{\Omega_0}{(s+\alpha)^2+\Omega_0^2} + B\frac{s+\alpha}{(s+\alpha)^2+\Omega_0^2}$$

The inverse Laplace is

$$x(t) = \left[\frac{A}{\Omega_0}\sin(\Omega_0 t) + B\cos(\Omega_0 t)\right]e^{-\alpha t}\,.$$

◇ **Example 22.** Consider the Laplace function,

$$X(s) = \frac{2s+3}{s^2+2s+4} = \frac{2s+3}{(s+1)^2+3}$$

Find the corresponding causal signal $x(t)$.

The poles are at $-1 \pm j\sqrt{3}$, so we expect that $x(t)$ is a decaying exponential with a damping factor of -1 (the real part of the poles) multiplied by a causal cosine of frequency $\sqrt{3}$. The partial fraction expansion is of the form

$$X(s) = \frac{2s+3}{s^2+2s+4} = \frac{a+b(s+1)}{(s+1)^2+3}$$

so that $3+2b = (a+bs)$, or $b=2$ and $a+b=3$ or $a=1$. Thus,

$$X(s) = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s+1)^2+3} + 2\frac{s+1}{(s+1)^2+3}\,,$$

which corresponds to

$$x(t) = \left[\frac{1}{\sqrt{3}}\sin(\sqrt{3}t) + 2\cos(\sqrt{3}t)\right]e^{-t}\epsilon(t)\,.$$

- **Double real poles**

  If $X(s)$ is a proper rational function

$$X(s) = \frac{N(s)}{(s+\alpha)^2} = \frac{a + b(s+\alpha)}{(s+\alpha)^2} = \frac{a}{(s+\alpha)^2} + \frac{b}{s+\alpha}$$

  then its inverse is

$$x(t) = ate^{-\alpha t} + be^{-\alpha t}\,,$$

  where a can be computed as

$$a = X(s)(s+\alpha)^2|_{s=-\alpha}$$

  after replacing it, b is found by computing $X(s_0)$ for a value $s_0 \neq \alpha$.

◇ **Example 23.** Find the inverse transform of,

$$X(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2}\,, \quad \text{ROC} : \text{Re}\{s\} > -3\,.$$

Lets us rewrite the function in the form:

$$X(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2} = \frac{a}{s+3} + \frac{b}{s+5} + \frac{c}{(s+5)^2}\,.$$

To calculate a, b, and c, let us realize that

$$a(s+5)^2 + b(s+3)(s+5) + c(s+3) = s^2 + 2s + 5\,,$$

which solved gives:

$$\begin{aligned} a &= 2 \\ b &= -1 \\ c &= -10\,. \end{aligned}$$

Replacing above we obtain

$$X(s) = \frac{2}{s+3} - \frac{1}{s+5} - \frac{10}{(s+5)^2},$$

and

$$x(t) = \left[2e^{-3t} - e^{-5t} - 10te^{-5t}\right]\epsilon(t)$$

- **Inverse functions containing $e^{-\rho s}$ terms**

  When $X(s)$ has exponential $e^{-\rho s}$ in the numerator or denominator, ignore this terms and perform partial fraction expansion on the rest, and at the end consider the exponentials to get the correct time shifting.

$$\mathscr{L}\left\{x(t-\tau)\right\} = e^{-s\tau}X(s)$$