Digital Signalbehandling 6

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1 Problem 1

1.1 a

Compute the sepctrum of $x[n]$ as given by (1)

$$x\,[n] = 0.9^n(u[n] - u[n - N_x]) \tag{1}$$

$$\begin{align}
x[n] &= 0.9^n u[n] - 0.9^n u[n - N_x] \tag{2}\\
&= 0.9^n u[n] - 0.9^{N_x} 0.9^{n-N_x} u[n - N_x] \tag{3}\\
0.9^n u[n] &\xrightarrow{\mathcal{F}} \frac{1}{1 - 0.9e^{-i\omega}} \tag{4}\\
&\text{shift theorem gives} \tag{5}\\
-0.9^{N_x} 0.9^{n-N_x} u[n - N_x] &\xrightarrow{\mathcal{F}} \frac{-0.9^{N_x} e^{-i\omega N_x}}{1 - 0.9e^{-i\omega}} \tag{6}\\
x[n] &\xrightarrow{\mathcal{F}} \frac{1 - (0.9e^{-i\omega})^{N_x}}{1 - 0.9e^{-i\omega}} = X(\omega) \tag{7}\\
& \tag{8}
\end{align}$$

N_x	f_1
7	0.1428
14	0.0.0714
28	0.0.03571
56	0.0.01785

Table 1: the freqency of the first bin.

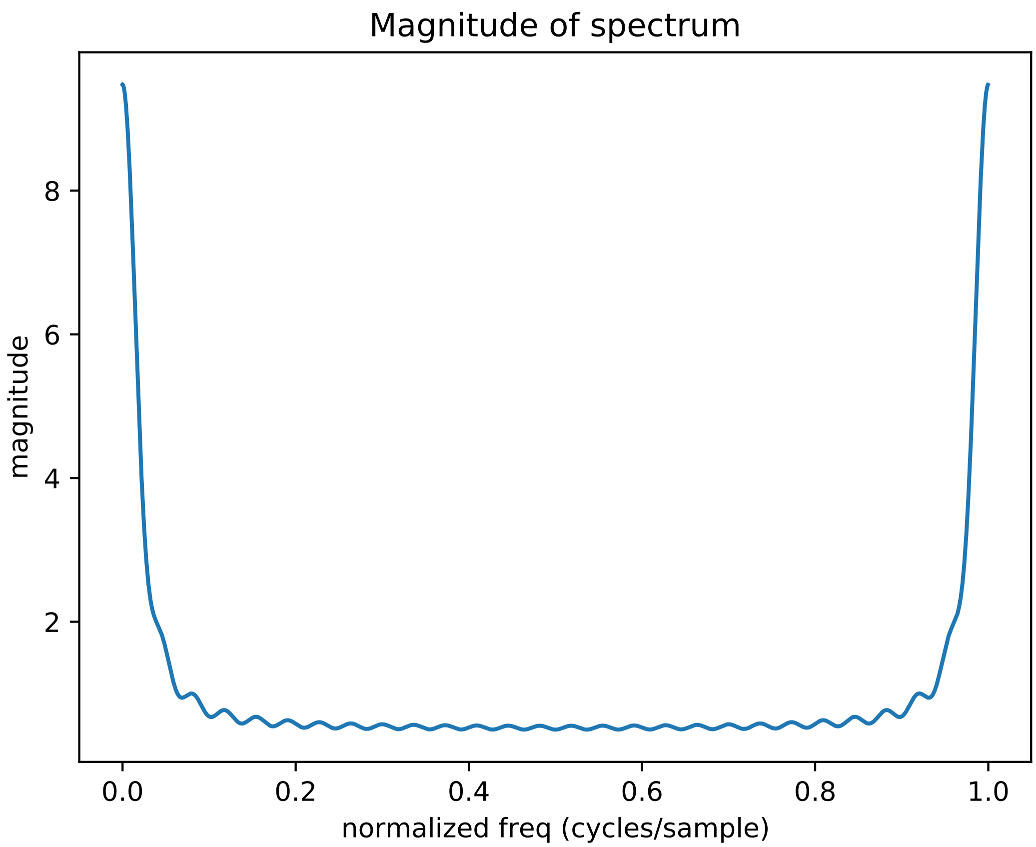


Figure 1: Magnitude spectrum for task1

1.2 b

1.3 c/e

The DFT is a sampling of the DTFT. so the kth index of the DFT corresponds to $f_k = k/2N$ if f_k, if $f$normalised to$f \in (0, 1)$. The reason for te factor 1/2 is that the freqencies above the nyquist frequency are mirror images of the lower freqs so it is not nessessay to compute them.

Note that python did not like to compute the full $f \in (0, 1)$ dft. but since the signal is real that isnt really an issue due to the symmetry properties of the DFT.

1.4 d

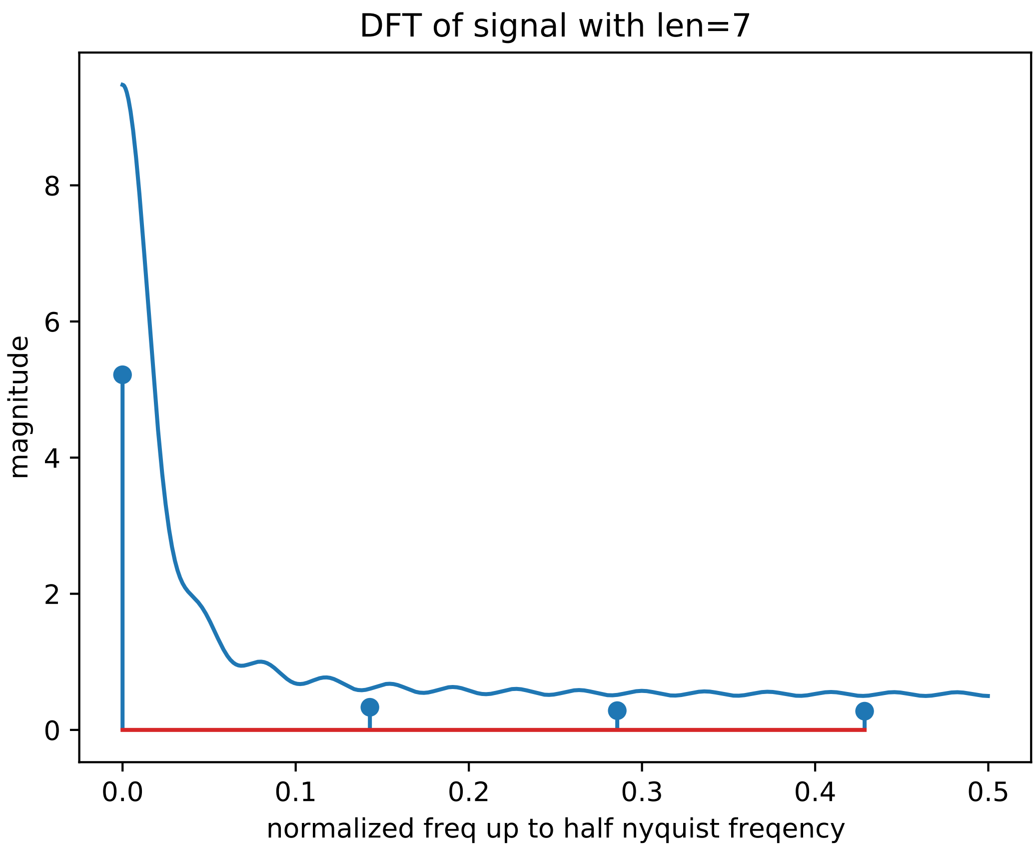


Figure 2: Magnitude spectrum for task1 with DFT at lenght 7

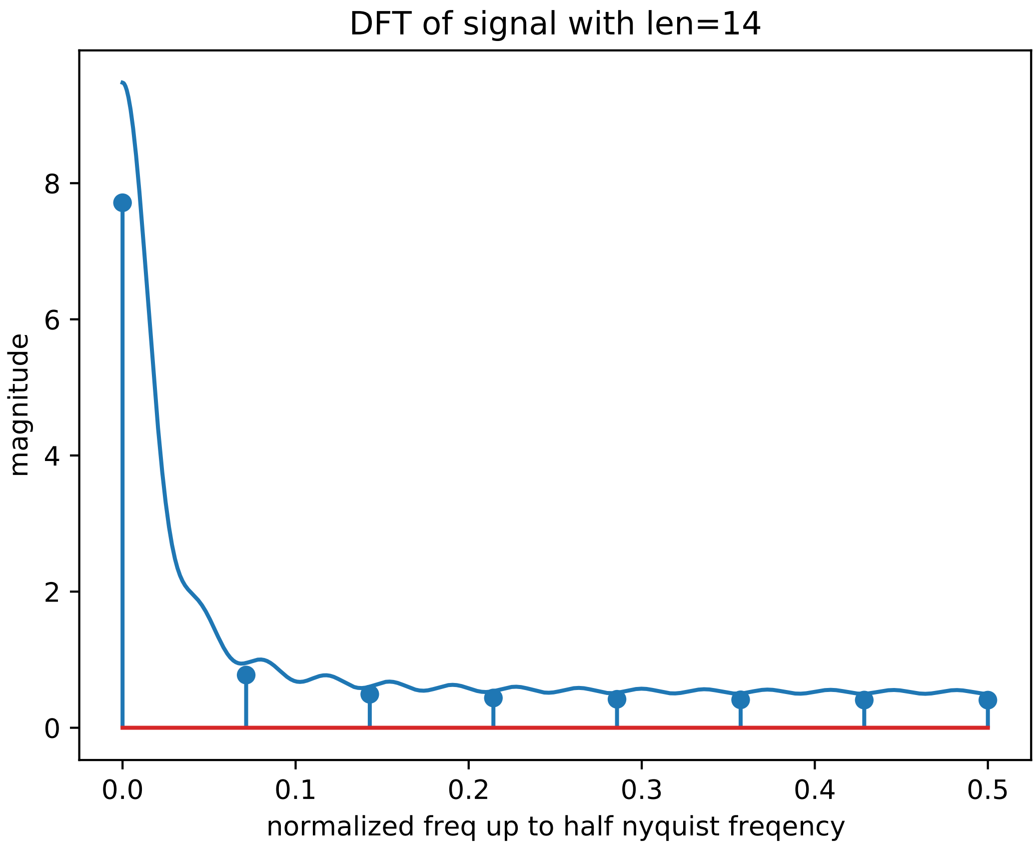


Figure 3: Magnitude spectrum for task1 with DFT at lenght 14

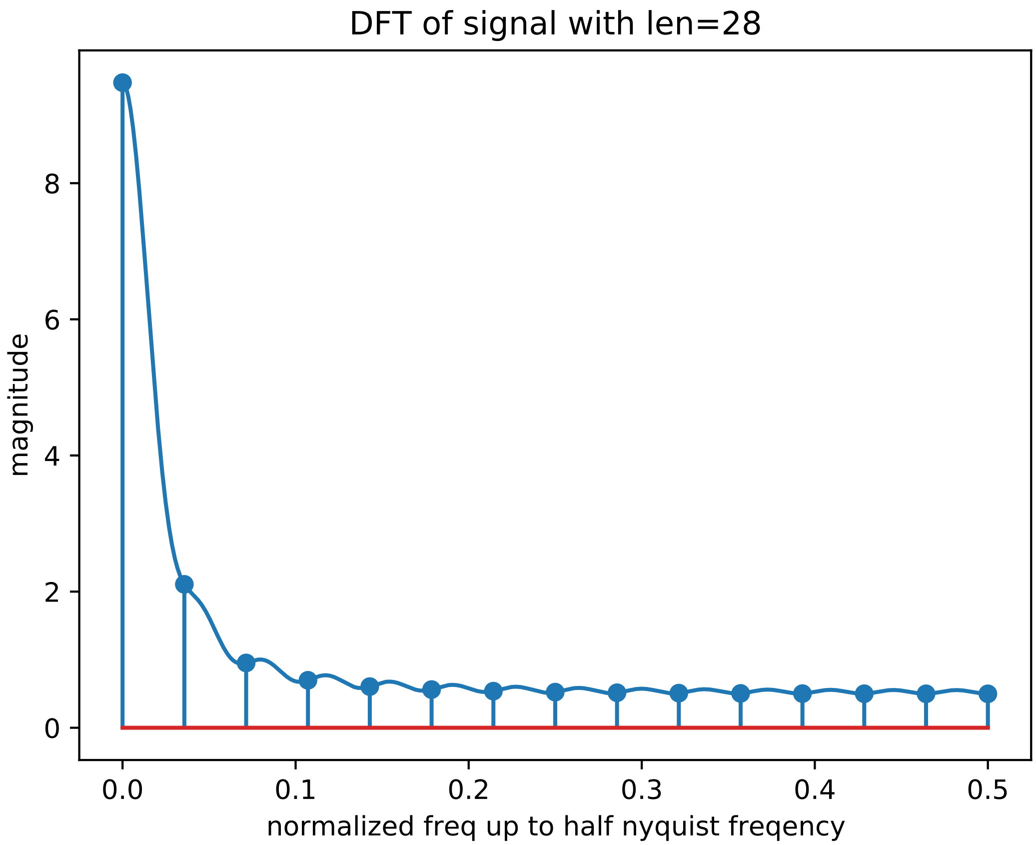


Figure 4: Magnitude spectrum for task1 with DFT at lenght 28

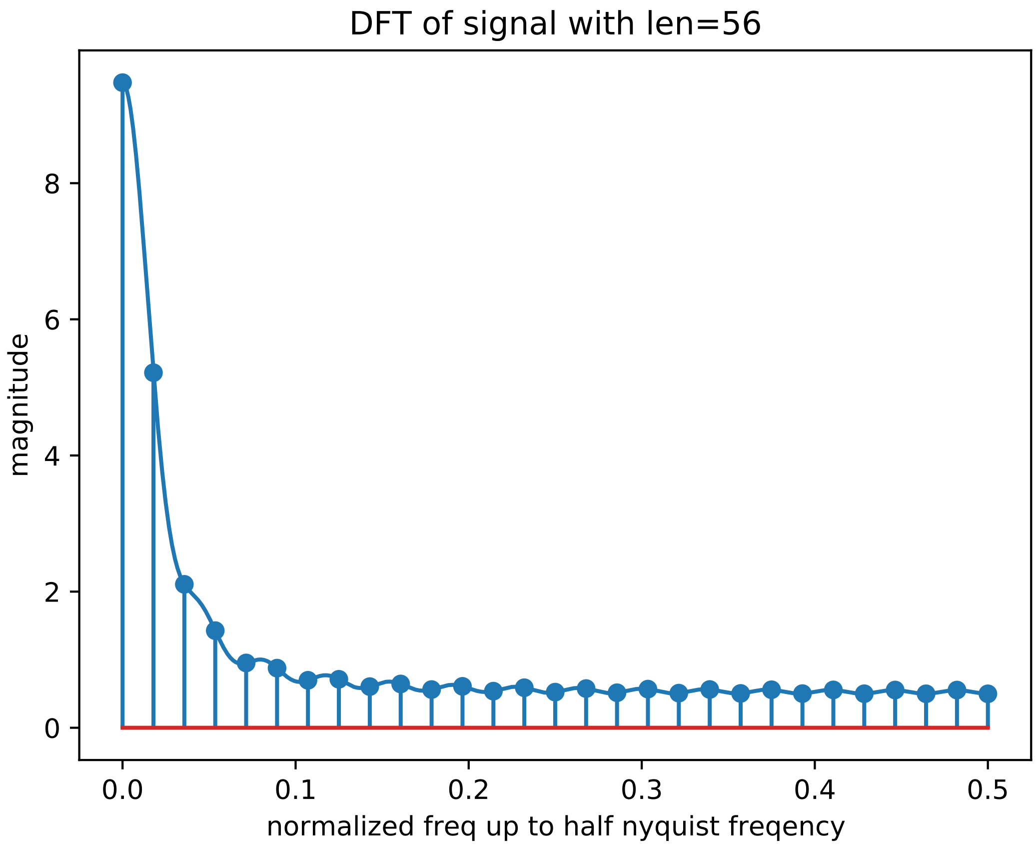


Figure 5: Magnitude spectrum for task1 with DFT at lenght 56

2 2

2.1 a

the length of the convolved sequence is 36, this corresponds with the length of a convolution between to sequences of lenght N and M being $N + M - 1$

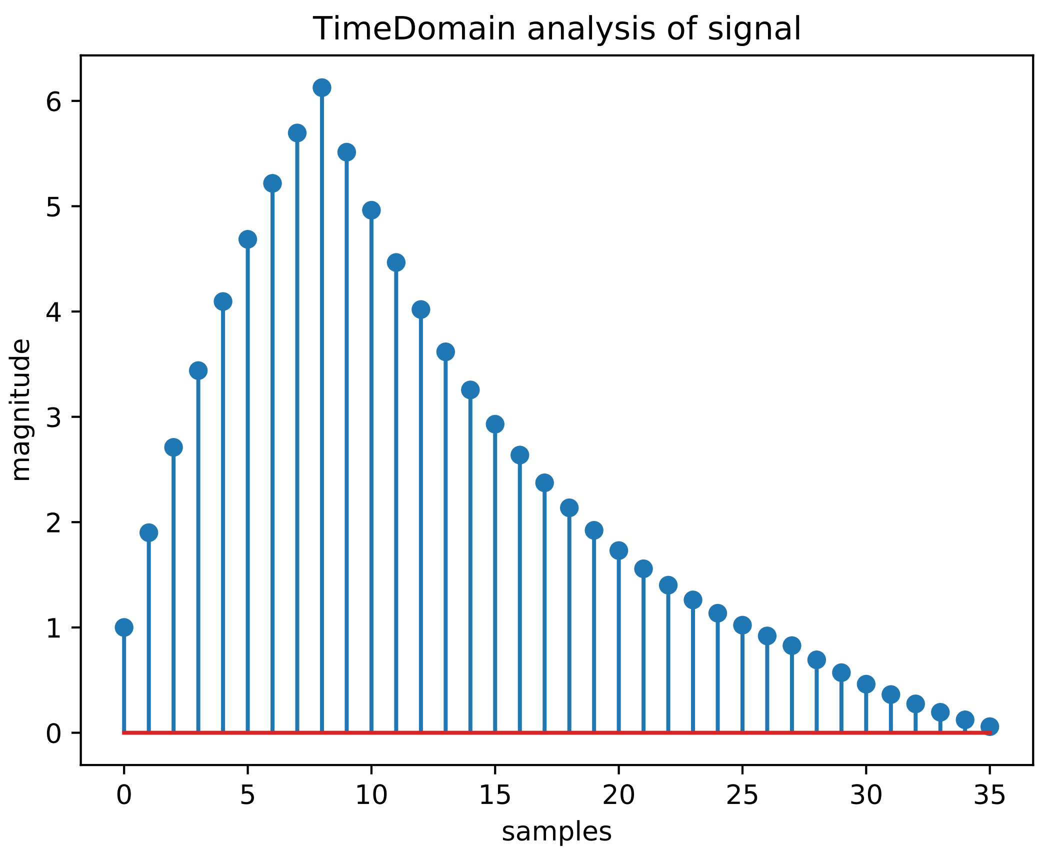


Figure 6: time domain convolution analysis for system in task 2

2.2 b

In order to recreate the signal exactly we need to use $N = 36$ for the lenght of the DFT and IDFT. This is to ensure that that the information in the signal is not lost or duplicated.

The filters that are shorter than 36 fail to represent the entire signal properly,there is information loss in the DFT conversions. At 36 samples the signal is reconstructed properly. The 72 sample DFT is zero padded due to the implementation in python. No information is strictly speaking lost, but the computation is unnessesary.

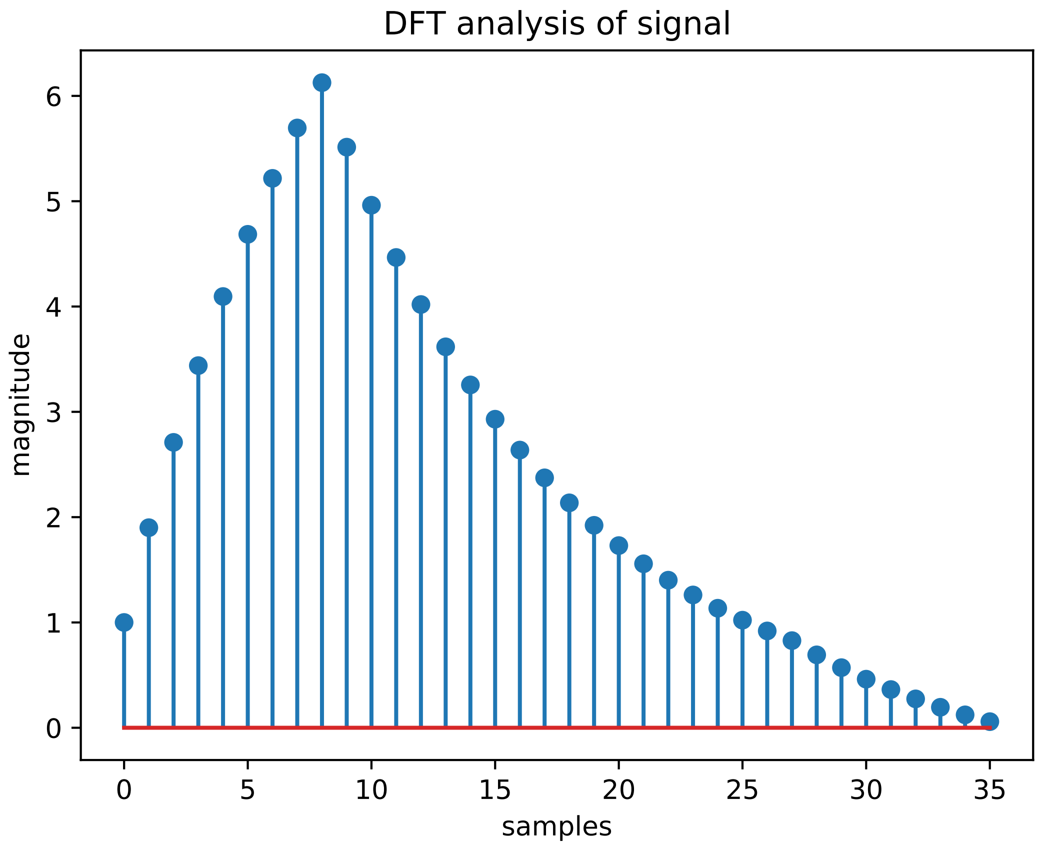


Figure 7: DFT analysis for system in task 2 length = 36

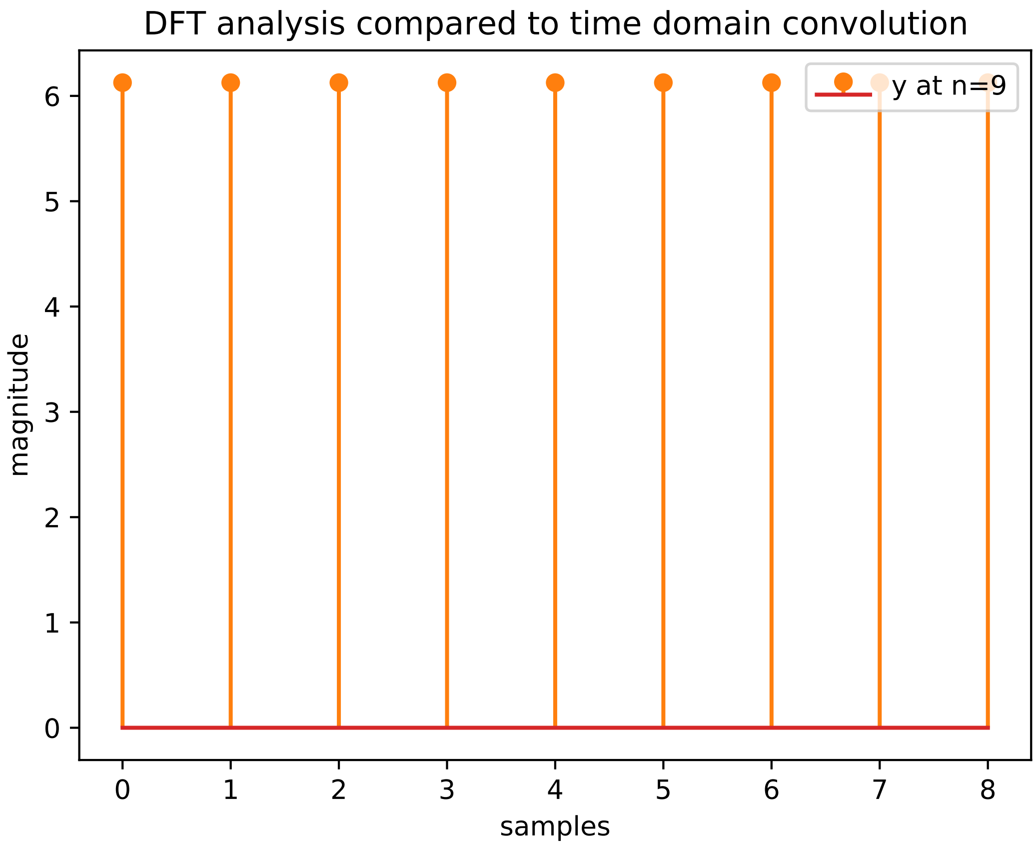


Figure 8: DFT analysis for system in task 2 length = 9

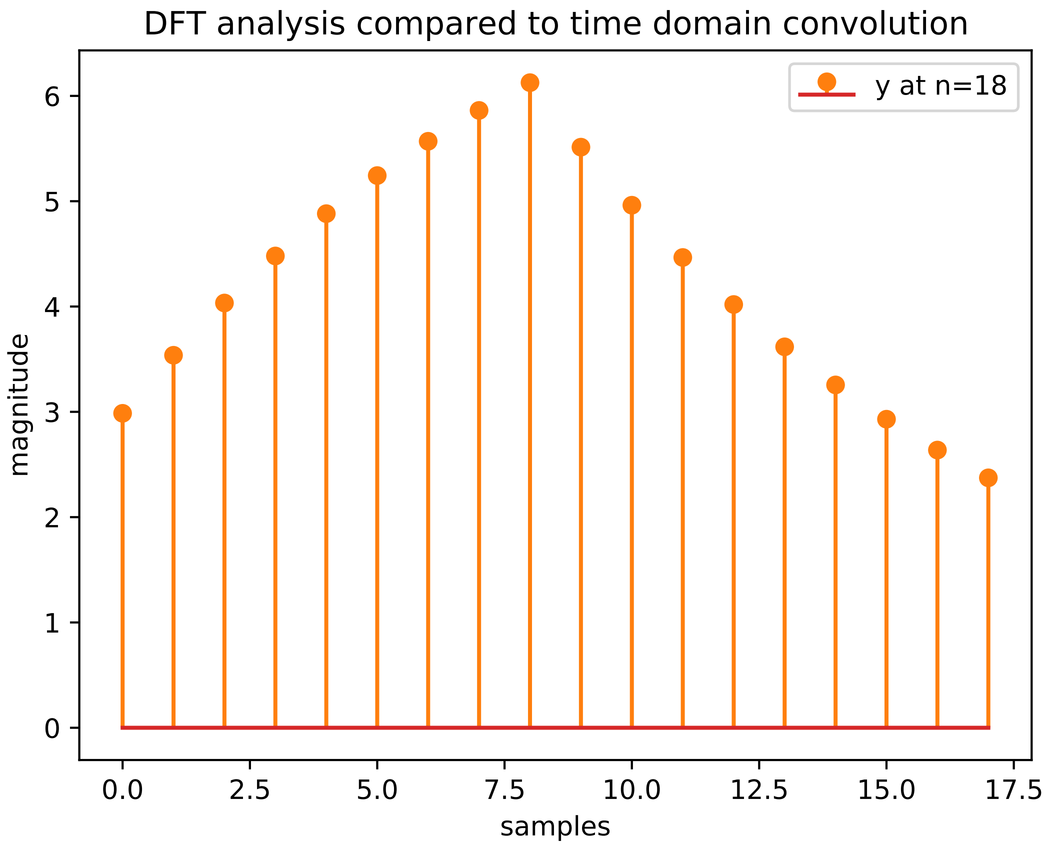


Figure 9: DFT analysis for system in task 2 length = 18

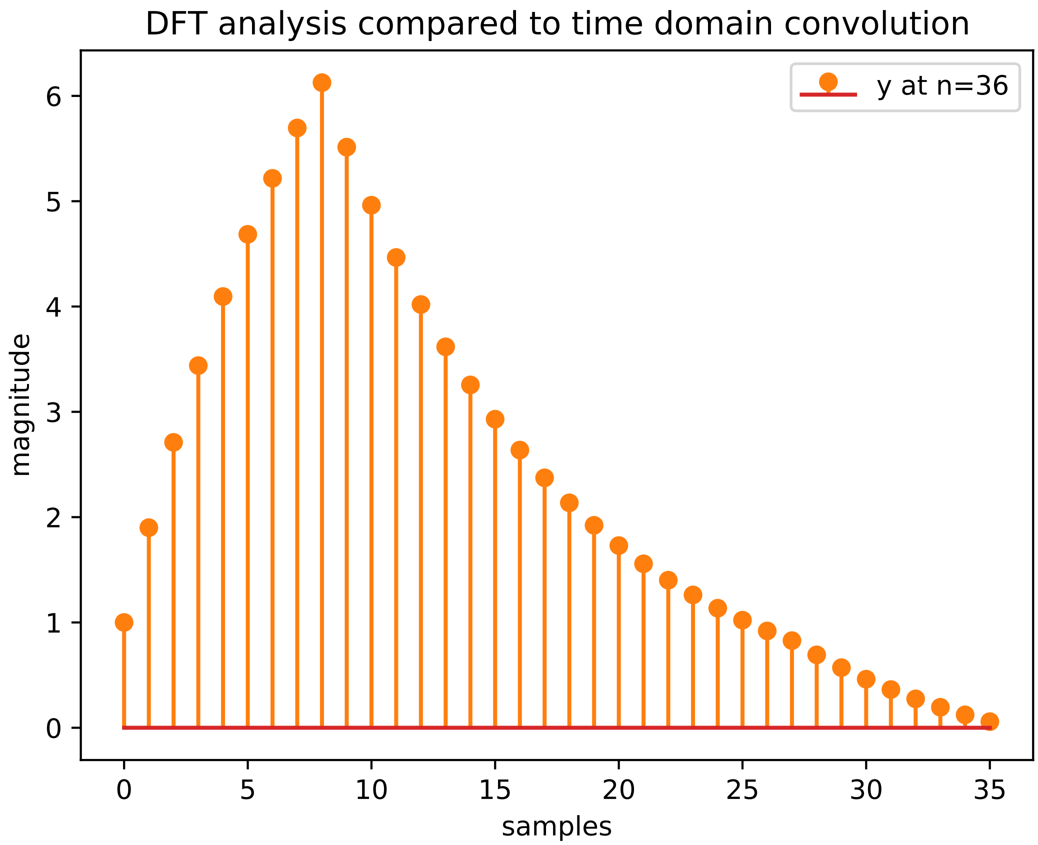


Figure 10: DFT analysis for system in task 2 length = 36

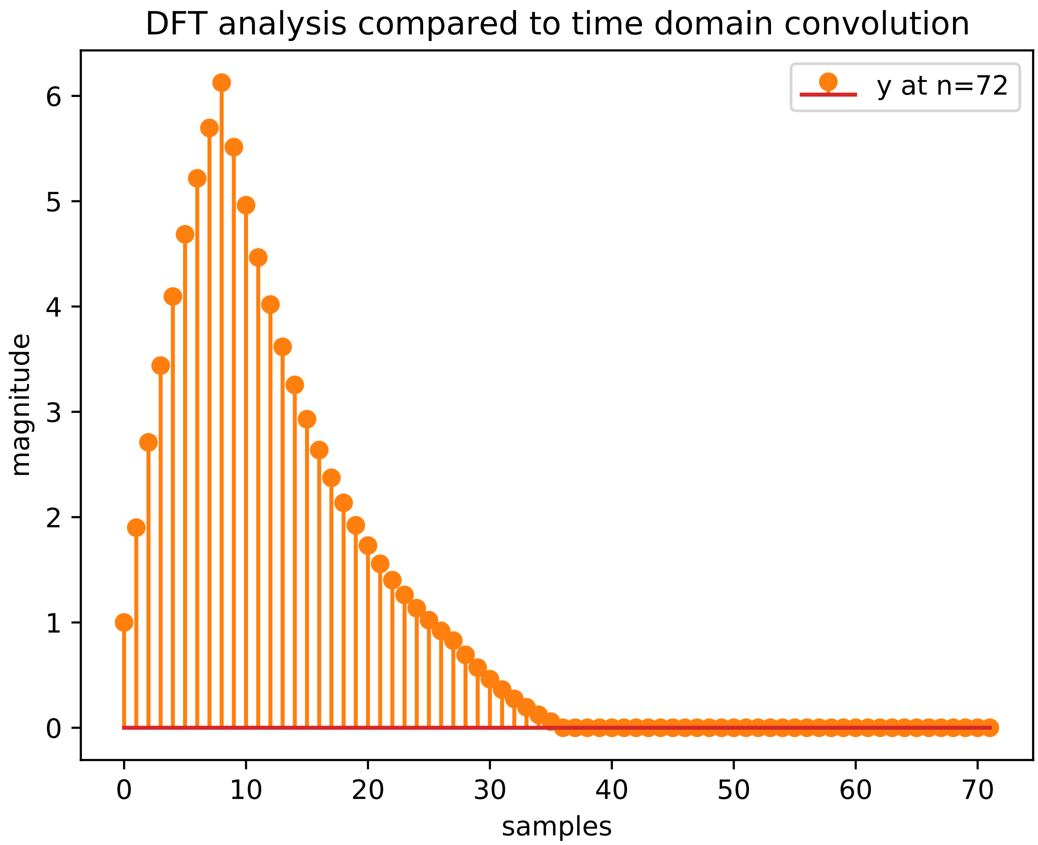


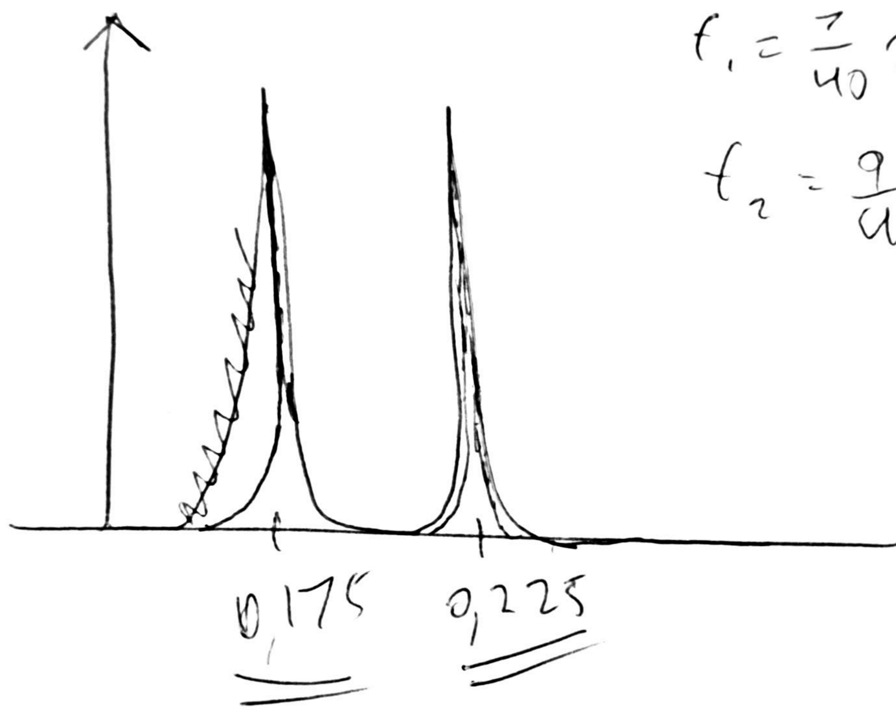
Figure 11: DFT analysis for system in task 2 length = 72

3 3

The estimated spectrums correspond quite well with the sketched spectrum. more bins seems to increase the smoothness of the plot, in the sense that it wont be as jagged in terms of sharp corners, while increasing singal length allows for a more precise measurement of the constituent frequecies.

$$f_1 = \frac{7}{40} \approx 0{,}175$$

$$f_2 = \frac{9}{40} \approx 0,$$



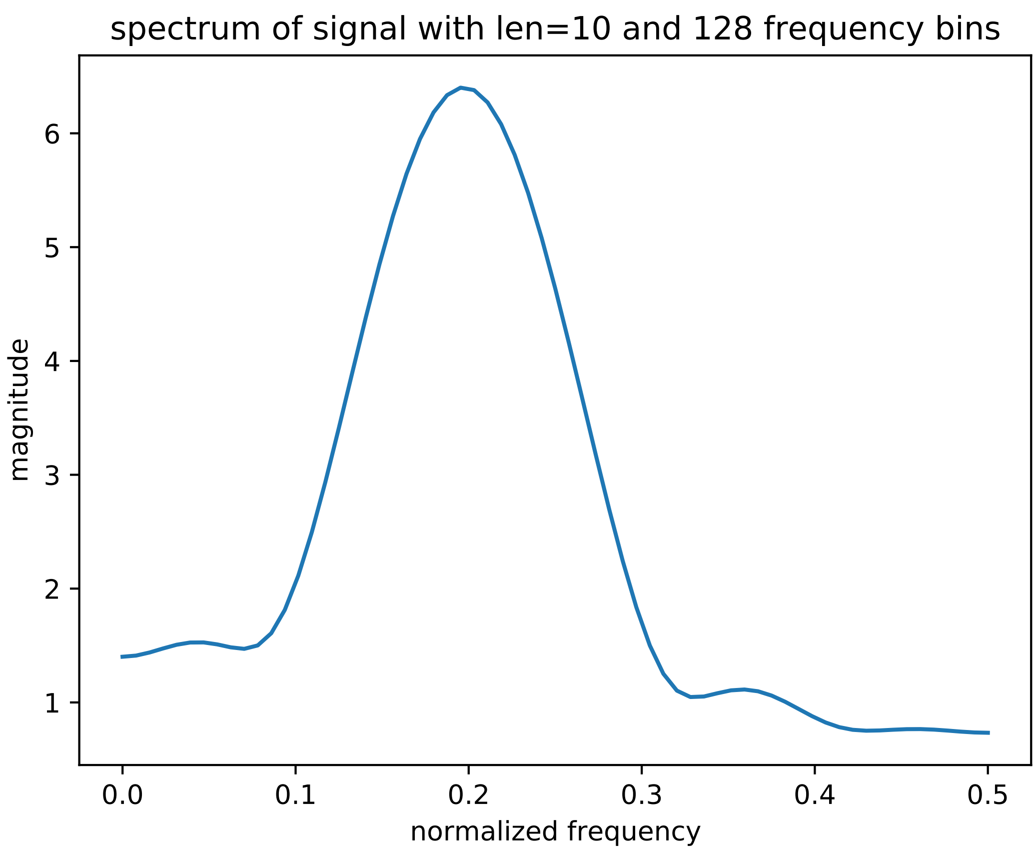


Figure 13: Spectrum Estimate signal lenght=10 DFT-length=128

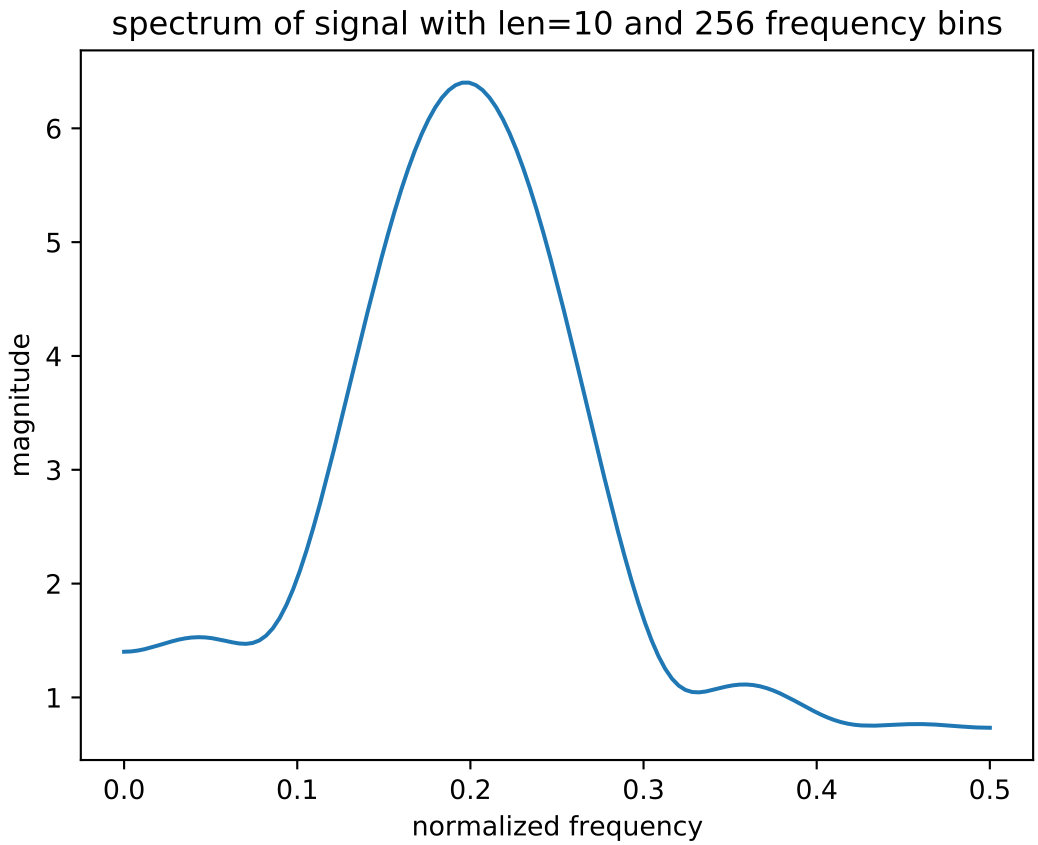


Figure 14: Spectrum Estimate signal lenght=10 DFT-length=256

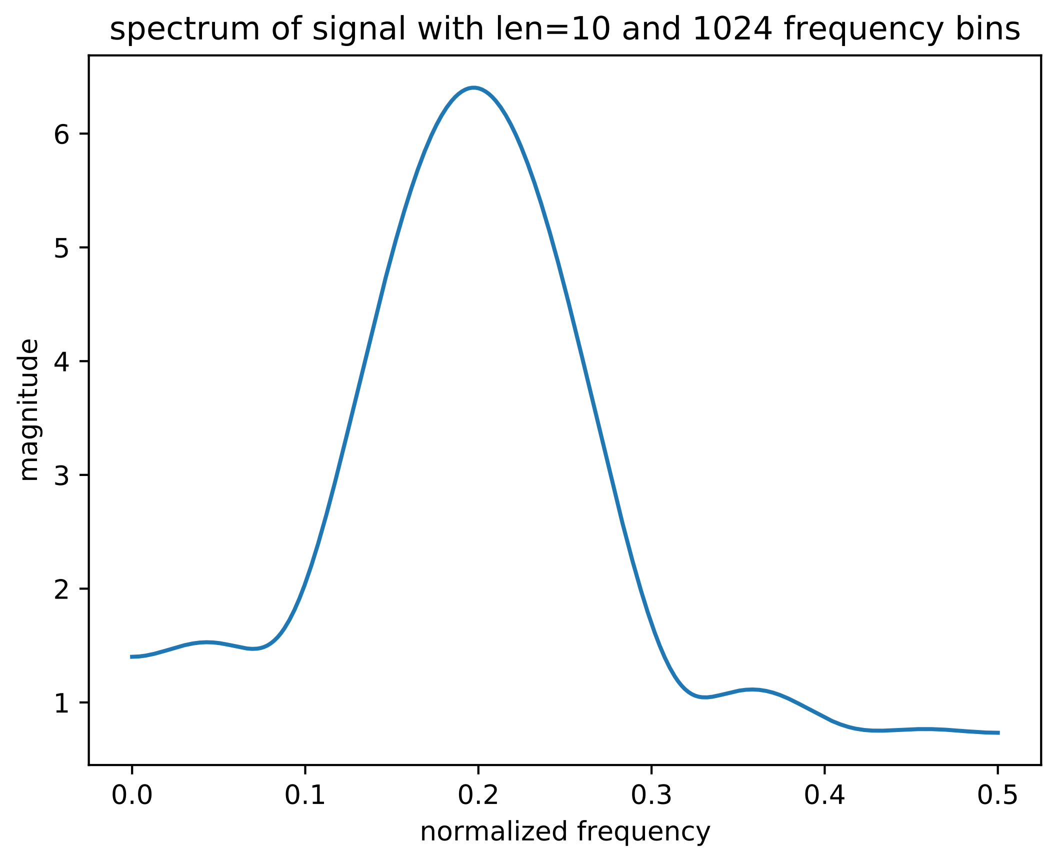


Figure 15: Spectrum Estimate signal lenght=10 DFT-length=1024

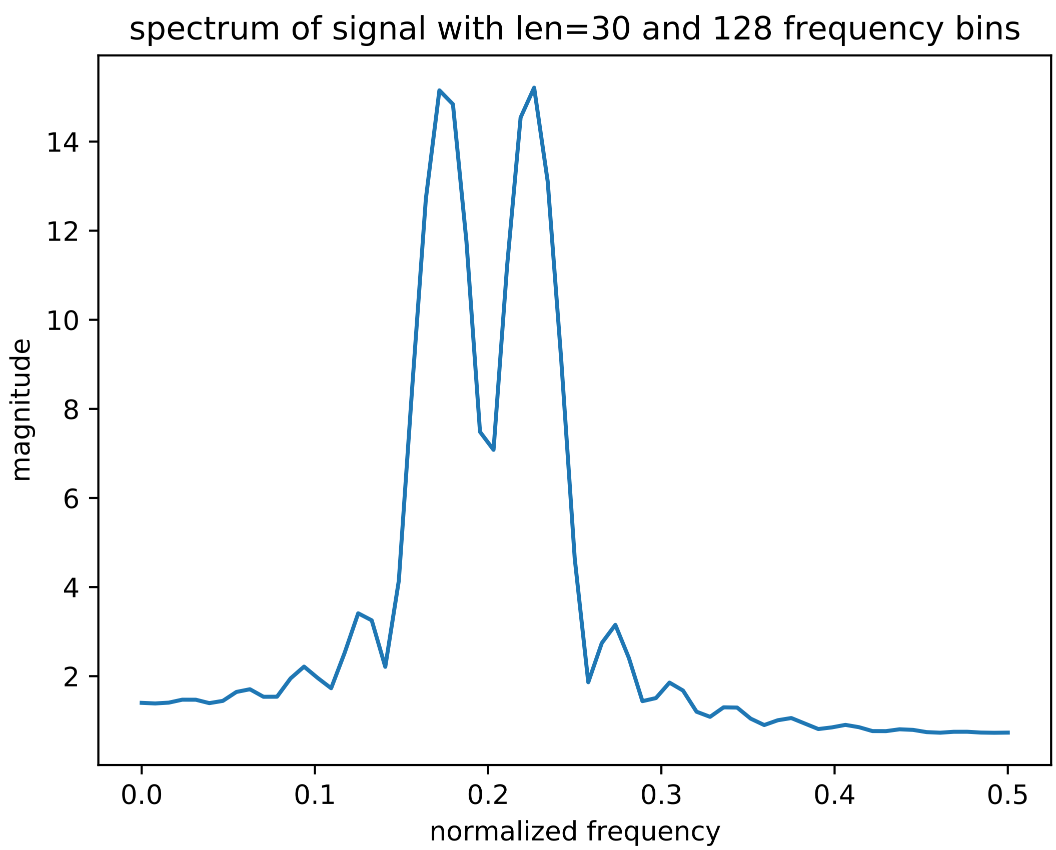


Figure 16: Spectrum Estimate signal lenght=30 DFT-length=128

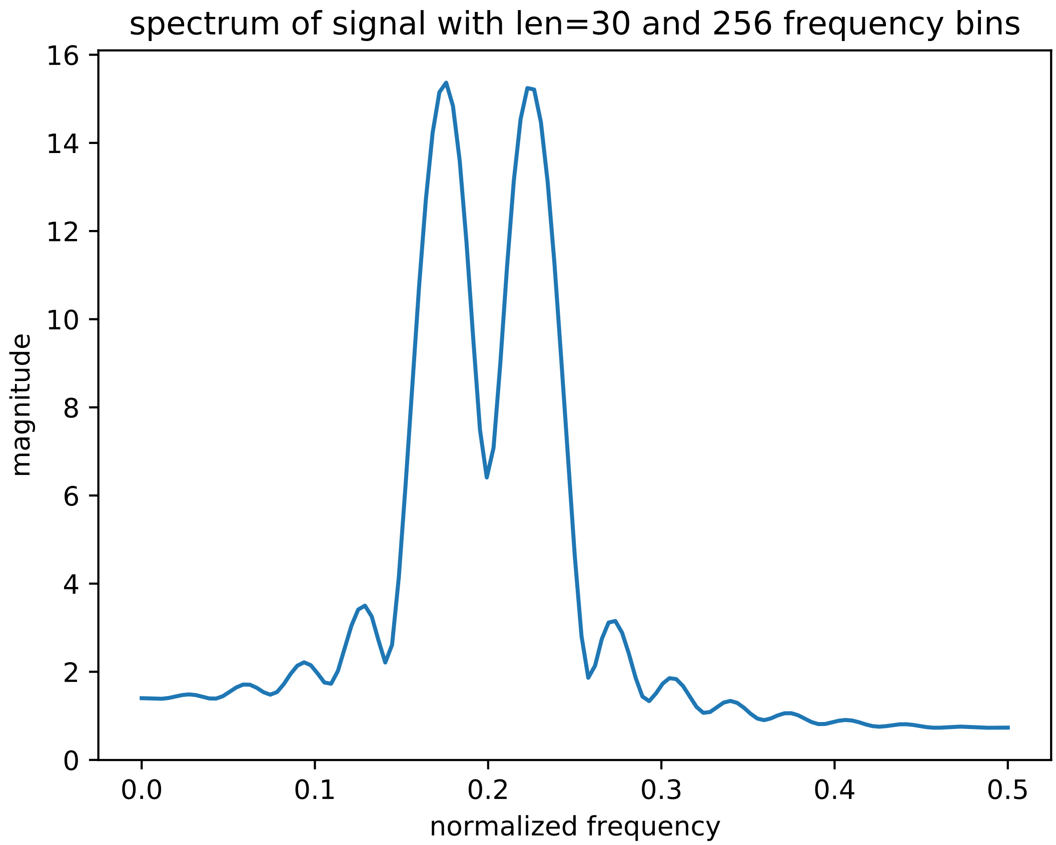


Figure 17: Spectrum Estimate signal lenght=10 DFT-length=256

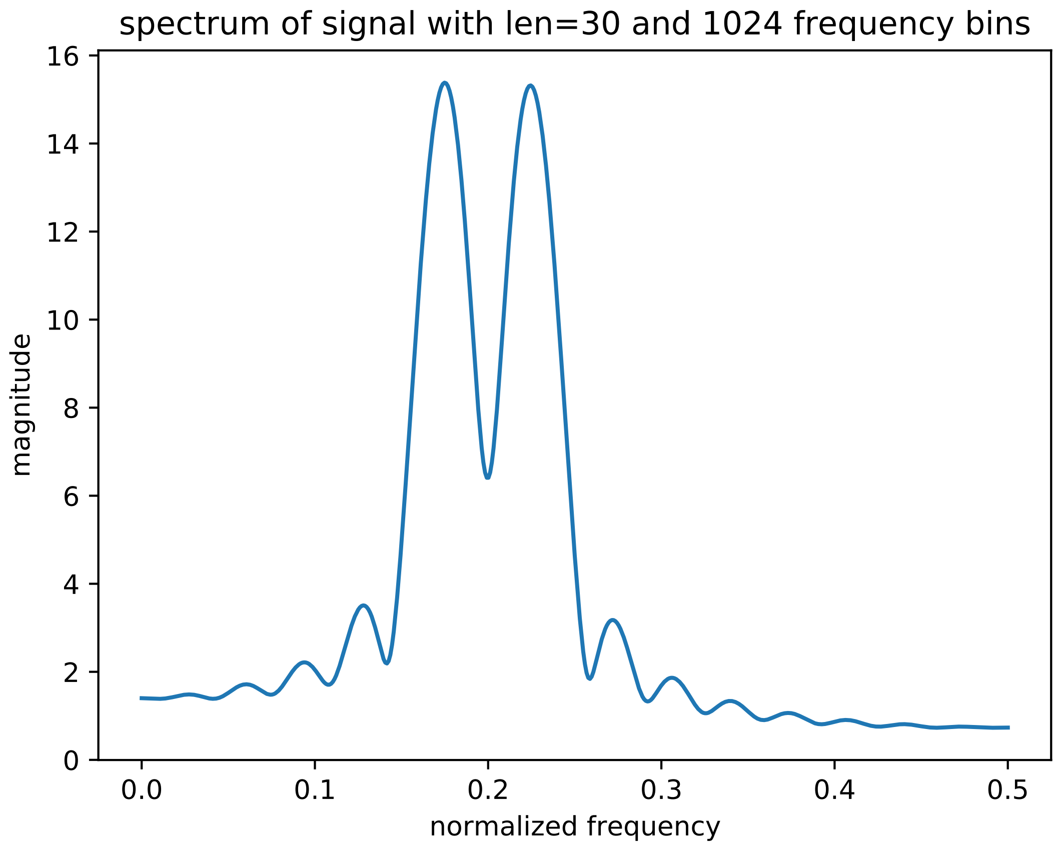


Figure 18: Spectrum Estimate signal lenght=30 DFT-length=1024

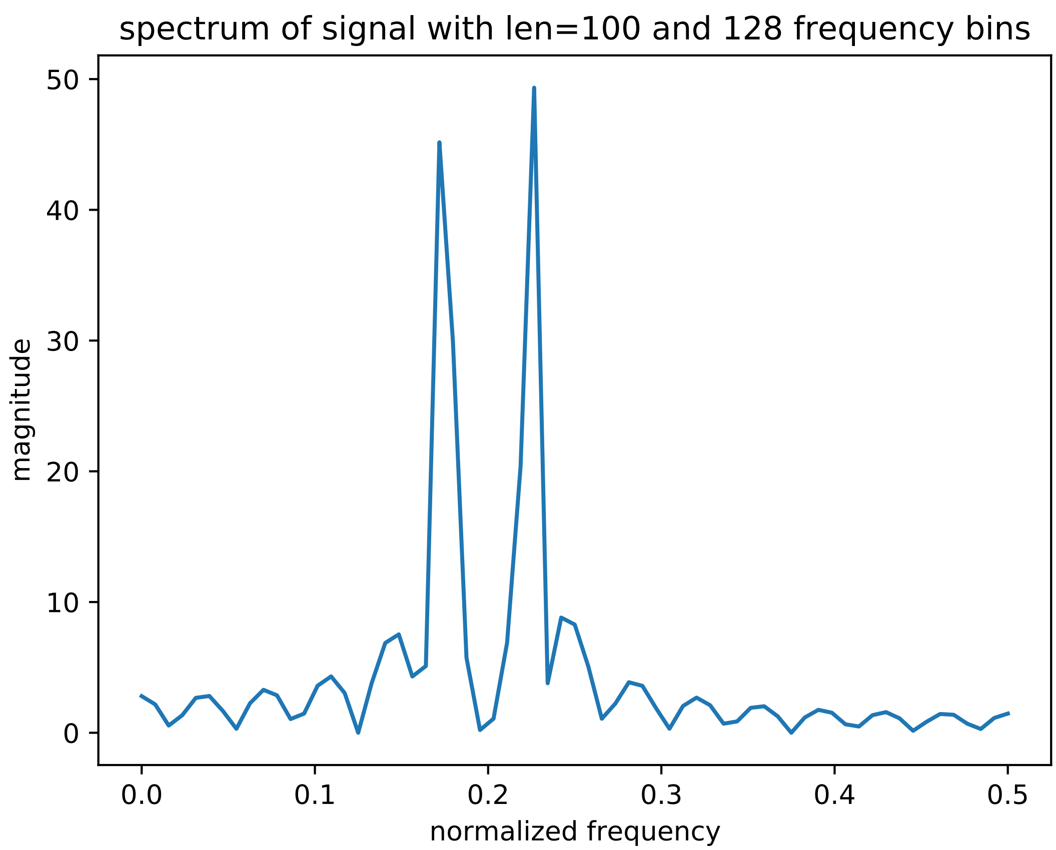


Figure 19: Spectrum Estimate signal lenght=100 DFT-length=128

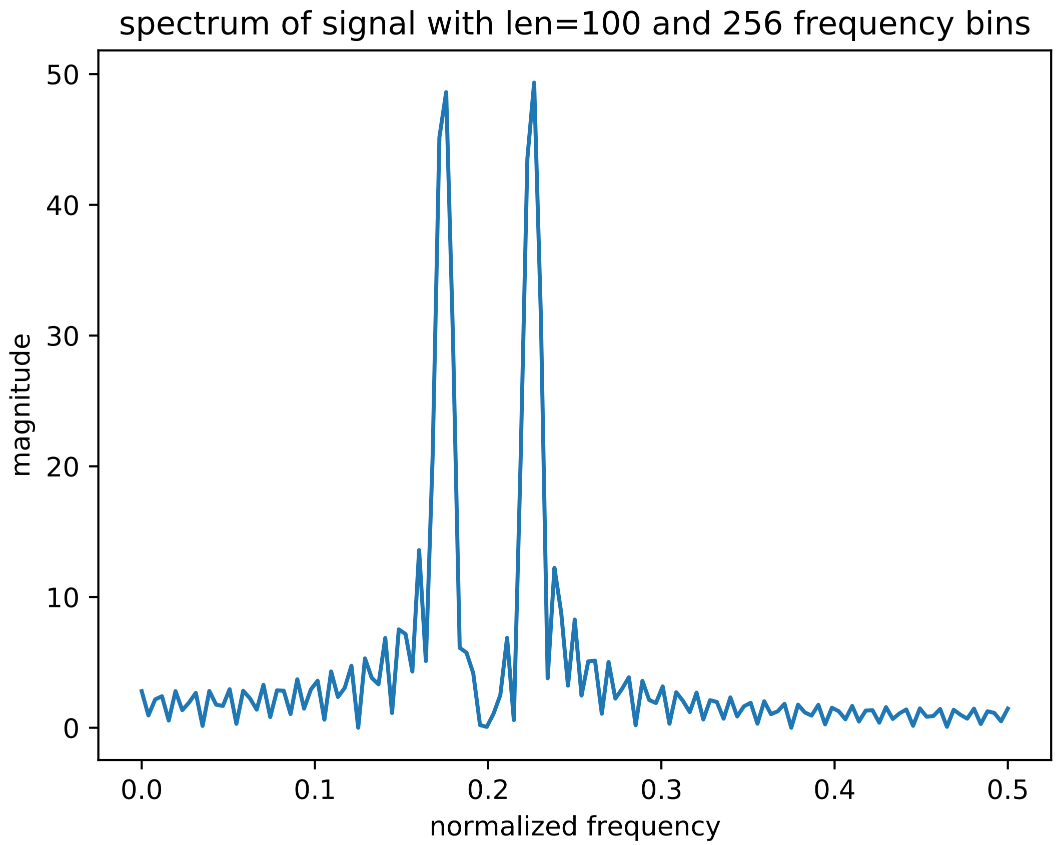


Figure 20: Spectrum Estimate signal lenght=100 DFT-length=256

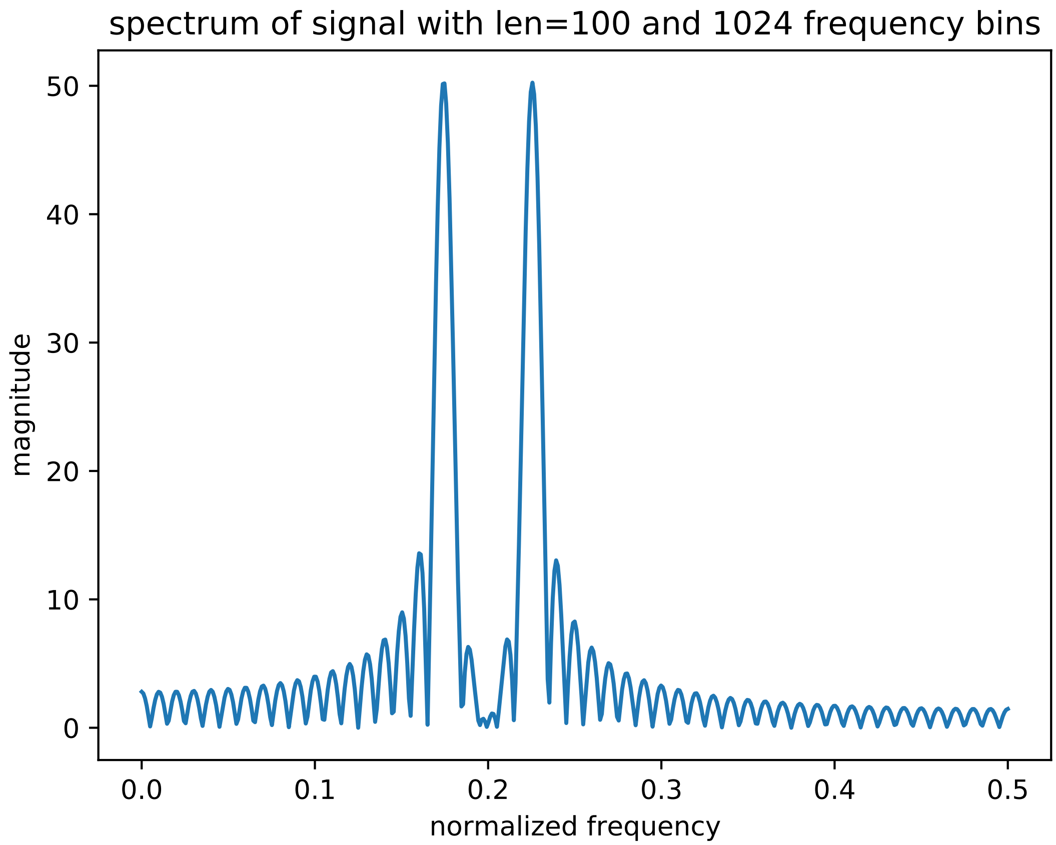


Figure 21: Spectrum Estimate signal lenght=100 DFT-length=1024

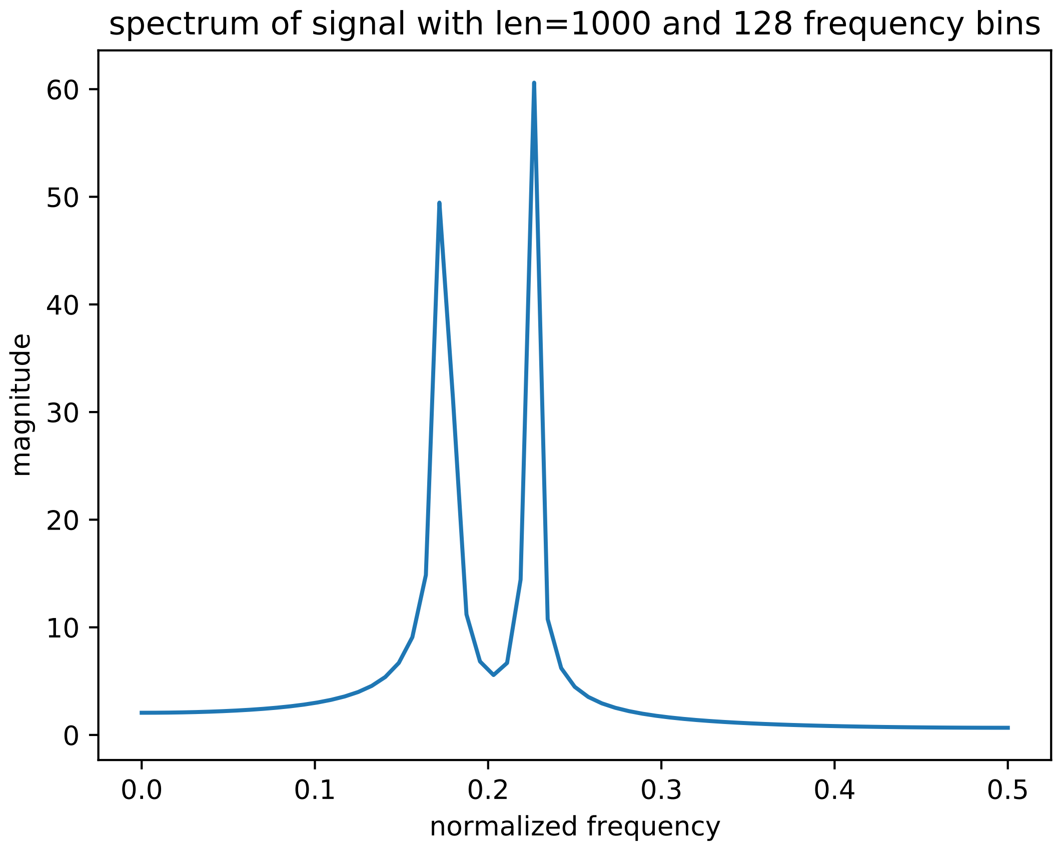


Figure 22: Spectrum Estimate signal lenght=1000 DFT-length=128

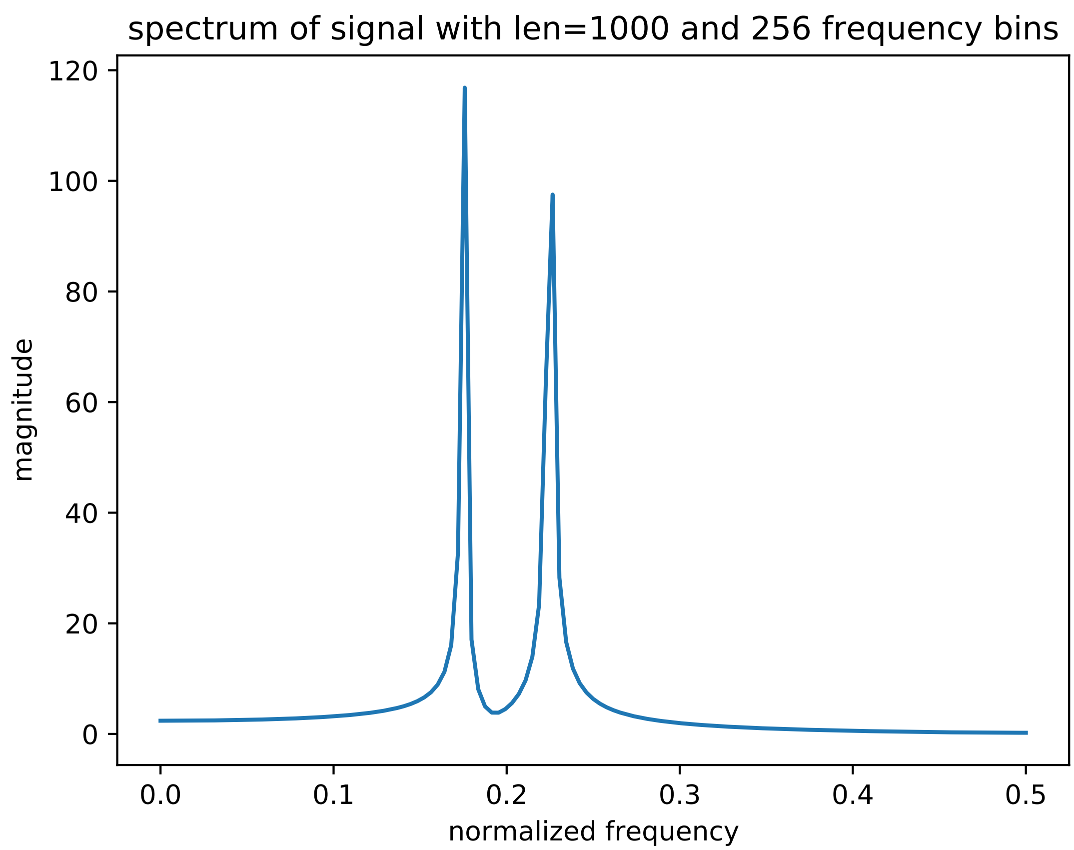


Figure 23: Spectrum Estimate signal lenght=1000 DFT-length=256

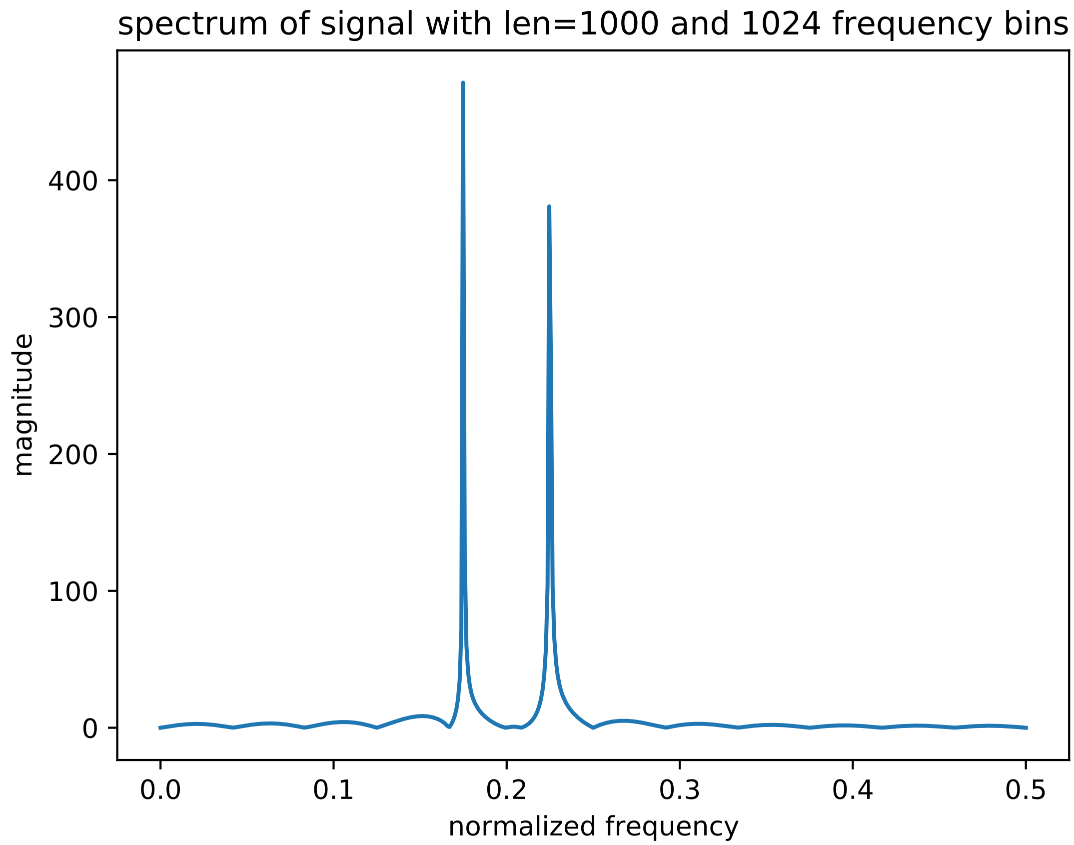


Figure 24: Spectrum Estimate signal lenght=1000 DFT-length=1024

4 4

4.1 a

The fast fourier transform (FFT) is an algorithm for computing the fourier transform in $O(N \log N)$ time complexity. it utilizes symetries in the fourier transform to achieve this.

4.2 b

The 2−radix FFT algorithm uses a divide and conquer approach to compute the DFT by recursively computing the odd and even indicies and combining them with a so called ”twindle factor”.

4.3 c

Obsever form DFT definition that at each frequency component we need N complex mulitplications and $N-1$ complex additions. To compute for the N frequency bins required for reconstruction, we then need N^2 multiplications and N^2+N additions.

4.4 d

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{\frac{-i2\pi nk}{N}} \tag{9}$$

$$\begin{aligned}
\sum_{n=0}^{N-1} x[n] e^{\frac{-i2\pi nk}{N}} &= \sum_{n=0}^{N/2-1} x[2n] e^{\frac{-i2\pi 2nk}{N}} + \sum_{n=1}^{N/2-1} x[2n+1] e^{\frac{-i2\pi(2n+1)k}{N}} && (10)\\
&= \sum_{n=0}^{N/2-1} x[2n] e^{\frac{-i2\pi nk}{N/2}} + e^{\frac{-i2\pi k}{N}} \sum_{n=1}^{N/2-1} x[2n+1] e^{\frac{-i2\pi nk}{N/2}} && (11)\\
&= F_1 + W_N^k F_2 && (12)
\end{aligned}$$

$$\begin{aligned}
F_1(k), F_2(k), \text{periodic in } M = N/2 && (13)\\
W_N^{k+N/2} = e^{\frac{-i2\pi k+N/2}{N}} = e^{\frac{-i2\pi k}{N}} e^{-i\pi} && (14)\\
= -e^{\frac{-i2\pi k}{N}} \text{thus} && (15)\\
X(k+M) = F_1(k) - W_N^k F_2(k) && (16)
\end{aligned}$$

4.5 e

At each k we need to compute M additions and M-1 additions. for each of the two separate transforms. we also need to add them together, thus we need $2\frac{N}{2}^2 + N = N^2/2 + N$ complex multiplications and $2\frac{N}{2}(\frac{N}{2}-1) + N = N^2/2$ complex additions.