1 Linear and time-invariant systems

1.1 Basic definitions

Definition 1 (linear system) If the response of a system to a linear combination of input signals always consists of the corresponding combination of the output signals, then for this system superposition principle applies. Such system is called **Linear** system.

Definition 2 (time-invariant system) A system that responds to a delayed input signal with a corresponding delayed output signal is called a **time-invariant system**

Definition 3 (linear time-invariant system) A system that is both time-invariant and linear is termed an LTI-system **linear time-invariant system**

continuous	-	discrete
analoque	-	digital
real-valued	-	complex-valued
unidimensional	-	multidimensional
deterministic	-	stochastic
causal	-	noncausal
with memory	-	memory-loss
linear	-	nonlinear
time-invariant	-	time-variant
translation-invariant	-	translation-variant

Table 2: Classification of systems

Definition 4 (causal system) A system is causal if its response to the arrival of a signal does not begin before this arrival.

◇ **Example 1 (Exercise 1.4).** Which of the following system descriptions designate linear, time-invariant, memory or causal systems?

c)

$$y(t) = x(t-T)$$

Linearity:

$$\begin{aligned} S\left\{Ax_1(t) + Bx_2(t)\right\} &= Ax_1(t-T) + Bx_2(t-T) \\ &= AS_1\left\{x_1(t)\right\} + BS_2\left\{x_2(t)\right\} \end{aligned}$$

Time-invariance:

$$S\{x(t-\tau)\} = x(t-\tau-T) = y(t-\tau)$$

The systems is linear, time-invariant (the response to the input signal shifted by τ is the same as the output signal shifted by τ), causal (the input at time t does not depend on future input signals), and with memory (the delay requires that the signal is saved/stored).

e)

$$y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$$

Linearity:

$$\begin{aligned} S\left\{Ax_1(t) + Bx_2(t)\right\} &= \frac{\mathrm{d}}{\mathrm{d}t}(Ax_1(t) + Bx_2(t)) \\ &= A\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} + B\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} \\ &= AS_1\left\{x_1(t)\right\} + BS_2\left\{x_2(t)\right\} \end{aligned}$$

Time-invariance:

$$S\{x(t-\tau)\} = \frac{\mathrm{d}x(t-\tau)}{\mathrm{d}t}$$

Changing the variable from t to t':

$t' = t - \tau \iff t = t' + \tau$;

$\mathrm{d}t = \mathrm{d}t'$, since τ is a constant.

$$S\{x(t-\tau)\} = \frac{\mathrm{d}x(t')}{\mathrm{d}t'} = y(t') = y(t-\tau)$$

The system is linear, time-invariant, causal and with memory.

f)

$$y(t) = \frac{1}{T}\int\limits_{t-T}^{t} x(t')dt'$$

Linearity:

$$\begin{aligned} S\{Ax_1(t)+Bx_2(t)\} &= \frac{1}{T}\int\limits_{t-T}^{t}[Ax_1(t')+Bx_2(t')]dt' \\ &= \frac{A}{T}\int\limits_{t-T}^{t}x_1(t')dt' + \frac{B}{T}\int\limits_{t-T}^{t}x_2(t')dt' \\ &= AS_1\{x_1(t)\}+BS_2\{x_2(t)\} \end{aligned}$$

Time-invariance:

$$S\{x(t-\tau)\} = \frac{1}{T}\int\limits_{t-T}^{t}x(t'-\tau)\mathrm{d}t'$$

Changing the variable from t' to η:

$\eta = t'-\tau$;

$\mathrm{d}t' = \mathrm{d}\eta$, since τ is a constant;

integration limits, $t\to t-\tau$ and $t-T\to t-T-\tau$;

$$\begin{aligned} S\{x(t-\tau)\} &= \frac{1}{T}\int\limits_{t-T-\tau}^{t-\tau}x(\eta)\mathrm{d}\eta \\ &= y(t-\tau) \end{aligned}$$

The system is linear, time-invariant, causal, and with memory.

h)

$$y(t) = A(t)x(t)$$

We can first define a delayed input signal $x_d(t)$

$$x_d(t) = x(t-\tau)$$

For this signal, the output will be:

$$S\{x_d(t)\} = A(t)x_d(t) = A(t)x(t-\tau)$$

This is not the same as a delayed output calculated from the formula describing the output signal:

$$y(t-\Delta t) = A(t-\tau)x(t-\tau)$$

i)

$$y(t) = x(t-T(t)),\ \ T(t) \text{ arbitrary}$$

Linearity:

$$\begin{aligned} S\{Ax_1(t-T(t))+Bx_2(t-T(t))\} &= Ax_1(t-T(t))+Bx_2(t-T(t)) \\ &= AS_1\{x_1(t)\}+BS_2\{x_2(t)\} \end{aligned}$$

Time-invariance: As above, we can define a time-delayed signal $x_d(t) = x(t-\tau)$, response to that signal is:

$$S\{x(t-\tau)\} = S\{x_d(t)\} = x_d(t - T(t)) = x(t - \tau - T(t))$$

This is different from a time-delayed output $y(t-\tau)$ calculated from the definition of the output signal.

$$\begin{aligned} y(t-\tau) &= x(t-\tau-T(t-\tau)) \\ S\{x(t-\tau)\} &\neq y(t-\tau) \end{aligned}$$

The system is linear, time-variant, non-causal, and with memory ♣

◇ **Example 2 (Exercise 1.5).** Two systems S_1 and S_2 respond to the input signal $x(t)$ with the output signals:

$$\begin{aligned} y_1(t) &= S_1\{x(t)\} = mx(t)\cos(\omega_T t) \\ y_2(t) &= S_2\{x(t)\} = [1 + mx(t)]\cos(\omega_T t),\ m \in \Re \end{aligned}$$

Are the systems a) linear; b) time-invariant; c) real-valued d) memoryless?

a) System S is linear if $S\{Ax_a(t) + Bx_b(t)\} = AS\{x_a(t)\} + BS\{x_b(t)\}$

$$\begin{aligned} S_1\{Ax_a(t) + Bx_b(t)\} &= m[Ax_a(t) + Bx_b(t)]\cos(\omega_T t) = \\ &= mAx_a(t)\cos(\omega_T t) + mBx_b(t)\cos(\omega_T t) = \\ &= AS_1\{x_a(t)\} + BS_1\{x_b(t)\} \end{aligned}$$

$$\begin{aligned} S_2\{Ax_a(t) + Bx_b(t)\} &= \{1 + m[Ax_a(t) + Bx_b(t)]\}\cos(\omega_T t) = \\ &= \cos(\omega_T t) + mAx_a(t)\cos(\omega_T t) + mBx_b(t))\cos(\omega_T t) \\ AS_2\{x_a(t)\} + BS_2\{x_b(t)\} &= A[1 + mx_a(t)]\cos(\omega_T t) + B[1 + mx_b(t)]\cos(\omega_T t) \\ &= (A + B)\cos(\omega_T t) + mAx_a(t)\cos(\omega_T t) + mBx_b(t)\cos(\omega_T t) \\ S_2\{Ax_a(t) + Bx_b(t)\} &\neq AS_2\{x_a(t)\} + BS_2\{x_b(t)\} \end{aligned}$$

System S_1 is linear and system S_2 is not linear.

b) System S is time-invariant if $S\{x(t-\tau)\} = y(t-\tau)$

$$\begin{aligned} S_1\{x(t-\tau)\} &= m(x(t-\tau))\cos(\omega_T t) \\ y_1(t-\tau) &= m(x(t-\tau))\cos(\omega_T(t-\tau)) \\ S_1\{x(t-\tau)\} &\neq y_1(t-\tau) \end{aligned}$$

$$\begin{aligned} S_2\{x(t-\tau)\} &= [1 + m(x(t-\tau))]\cos(\omega_T t) \\ y_2(t-\tau) &= [1 + m[x(t-\tau))]\cos(\omega_T(t-\tau)) \\ S_2\{x(t-\tau)\} &\neq y_2(t-\tau) \end{aligned}$$

Systems S_1 and S_2 vary with time.

c) Systems S_1 and S_2 are real-valued as from $x(t) \in \Re$ it is obtained, $y_{1,2}(t) \in \Re$.

d) Systems S_1 and S_2 are memoryless as both input signals only depend on t, that is, the input signals are only used at time t. ♣