2 LDE and block diagrams

2.1 Linear differential equations (LDE)

◇ **Example 3.** Consider simple RLC circuit shown in Figure 1.

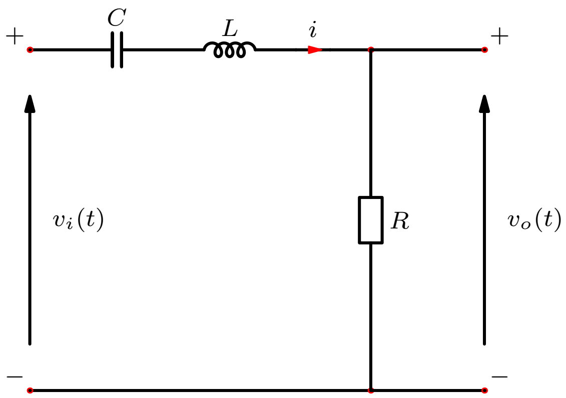


Figure 1: Simple RLC circuit

$$
\begin{aligned}
v_R(t) &= Ri(t) = v_o(t) \\
v_C(t) &= \frac{Q}{C} = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\mathrm{d}\tau \qquad (1)\\
v_L(t) &= L\frac{\mathrm{d}i(t)}{\mathrm{d}t} \\
v_i(t) &= v_R + v_C + v_L \\
v_i(t) &= Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\mathrm{d}\tau + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} \qquad (2)\\
\dot{v}_i(t) &= R\dot{i}(t) + \frac{1}{C}i(t) + L\ddot{i}(t) \qquad (3)\\
\dot{v}_i(t) &= \dot{v}_o(t) + \frac{1}{RC}v_o(t) + \frac{L}{R}\ddot{v}_o(t) \qquad (4)
\end{aligned}
$$

Figure 2: LDE of system in Fig. 1. In principle it is possible to arbitrary decide if $v_i(t)$ or $v_0(t)$ is the input signal.

To illustrate the relationship between input and output signals, we can do a backward calculation. If we know that th output is given by $v_o(t) = Re^{-t}t$ for $t > 0$ (zero otherwise, see Fig. 3) and $R = 2$, $L = 1$, $C = 1$ in appropriate units, what was the input $v_i(t)$ which resulted in this output signal? We can calculate directly from the differential equation in Eq. 3. Note that it would not be as straightforward to calculate output $v_o(t)$ for a given $v_i(t)$ signal, as multiple time derivatives of a unknown function $v_o(t)$ appear in the system equation.

$$
\begin{aligned}
\dot{v}_i(t) &= \dot{v}_o(t) + R^{-1}v_0(t) + R^{-1}\ddot{v}_o(t) = 2e^{-t} - 2e^{-t}t + e^{-t}t + \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t} - te^{-t}\right) = \\
&= 2e^{-t} - 2e^{-t}t + e^{-t}t - e^{-t} - e^{-t} + e^{-t}t = 2e^{-t} - 2e^{-t} - 2e^{-t}t + 2e^{-t}t = 0
\end{aligned}
$$

so $v(t) = v = \text{const}$. To get v we need to find $v_C(\infty)$. From Eq. 1

$$v_C = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\mathrm{d}\tau = \int_0^t e^{-\tau}\,\tau\,\mathrm{d}\tau$$

Partial integration of the integral yields

$$v_C(t) = -\tau\, e^{-\tau}|_0^t + \int_0^t \dot{\tau}\, e^{-\tau}\,\mathrm{d}\tau \;=\; -\,t\,e^{-t} \;-\; e^{-t} \;+\; 1$$

As $v_C(\infty) \;=\; 1$, we conclude that $v = 1$.

Let us verify that $i(\infty) = 0$. We find

$$i(t) = t\,e^{-t} = \frac{t}{e^t} \Rightarrow \lim_{t\to\infty}\frac{t}{e^t} = \frac{\infty}{\infty},$$

which is indeterminate. We solve the indeterminacy using L'Hopital rule

$$\lim_{t\to\infty}\frac{t}{e^t} = \lim_{t\to\infty}\frac{\dot{t}}{\dot{(e^t)}} = \lim_{t\to\infty}\frac{1}{e^t} = 0$$

As we know starting conditions $i(0) = 0$ and $\dot{i}(0) = 1$, we can also get v by using $v_C(0) = 0$. So,

$$v(0) = v_R(0) + v_C(0) + v_L(0) = 1.$$

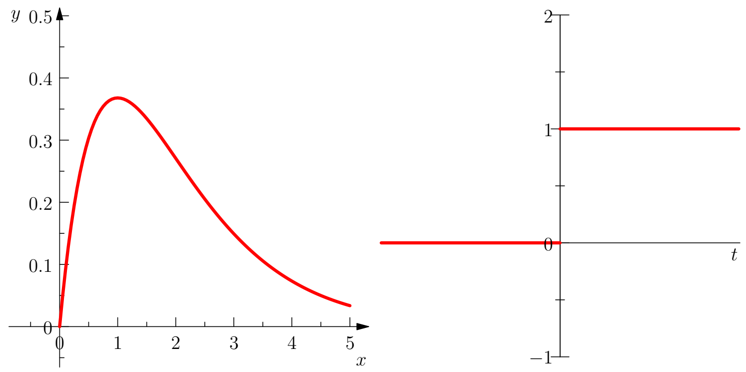


Figure 3: Plots of $i(t)$ and corresponding output signal $v(t)$.

As we have seen from Eq. 3, the circuit is described as a linear combination of derivatives of the input signal and the output signal with constant time-independent coefficients, that is, it is described by a linear different equation with constant coefficients. ♣

If we then assume a general LDE system with constant coefficients. We want to relate input $x(t)$ with the output $y(t)$.

Definition 5 Every system that can be modelled using a LDE with constant coefficients is a LTI-system.

$$\sum_{i=0}^{N}\alpha_i\frac{\mathrm{d}^i y(t)}{\mathrm{d}t^i} = \sum_{k=0}^{M}\beta_k\frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k} \tag{5}$$

For simplicity, lets consider $N = M$ in the following considerations:

- The order of the equation is given by the highest order derivative.
- For a given function $x(t)$ there are up to N different linearly independent solutions $y(t)$ to Eq. 31.
- For a particular solution we have to give N conditions.
- For initial condition problems, these should be N initial conditions $y(0)$, $\dot{y}(0)$, $\ddot{y}(0)$, ...
- The differential equation describes a continuous-time system.

2.2 Block diagrams

Let us consider a simple example:

$$\alpha_0 y + \alpha_1 \frac{\mathrm{d}y}{\mathrm{d}t} + \alpha_2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \beta_0 x + \beta_1 \frac{\mathrm{d}x}{\mathrm{d}t} + \beta_2 \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \tag{6}$$

$$\alpha_0 y + \alpha_1 \dot{y} + \alpha_2 \ddot{y} = \beta_0 x + \beta_1 \dot{x} + \beta_2 \ddot{x} \tag{7}$$

Note that $y(t) = 0$ if $x(t) = 0 \; \forall t$. This has to be the case for LTI systems.
Example of integration:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= x \\
\int_{-\infty}^{t} \frac{\mathrm{d}y}{\mathrm{d}\tau} \mathrm{d}\tau &= \int_{-\infty}^{t} x \mathrm{d}\tau \\
y(t) - y(-\infty) &= \int_{-\infty}^{t} x \mathrm{d}\tau \\
y(t) &= \int_{-\infty}^{t} x \mathrm{d}\tau
\end{aligned}$$

Eq. 7 can be solved by integrating both sides,

$$\begin{aligned}
\alpha_0 y + \alpha_1 \dot{y} + \alpha_2 \ddot{y} &= \beta_0 x + \beta_1 \dot{x} + \beta_2 \ddot{x} \\
\alpha_0 \int y + \alpha_1 y + \alpha_2 \dot{y} &= \beta_0 \int x + \beta_1 x + \beta_2 \dot{x} \\
\alpha_0 \iint y + \alpha_1 \int y + \alpha_2 y &= \beta_0 \iint x + \beta_1 \int x + \beta_2 x \\
\alpha_0 \int_{(2)} y + \alpha_1 \int_{(1)} y + \alpha_2 \int_{(0)} y &= \beta_0 \int_{(2)} x + \beta_1 \int_{(1)} x + \beta_2 \int_{(0)} x
\end{aligned}$$

Redefining the constants,

$$\begin{aligned}
a_i = \alpha_{N-i} \,&;\, b_k = \beta_{N-k} \\
a_0 &= \alpha_{2-0} = \alpha_2 \\
a_1 &= \alpha_{2-1} = \alpha_1 \\
a_2 &= \alpha_{2-2} = \alpha_0 \,,
\end{aligned}$$

we obtain

$$a_2 \int_{(2)} y + a_1 \int_{(1)} y + a_0 y = b_2 \int_{(2)} x + b_1 \int_{(1)} x + b_0 x \,. \tag{8}$$

From this we can take the output,

$$\boxed{y(t) = \frac{1}{a_0} \left[b_2 \int_{(2)} x + b_1 \int_{(1)} x + b_0 x - a_2 \int_{(2)} y - a_1 \int_{(1)} y \right]} \,.$$

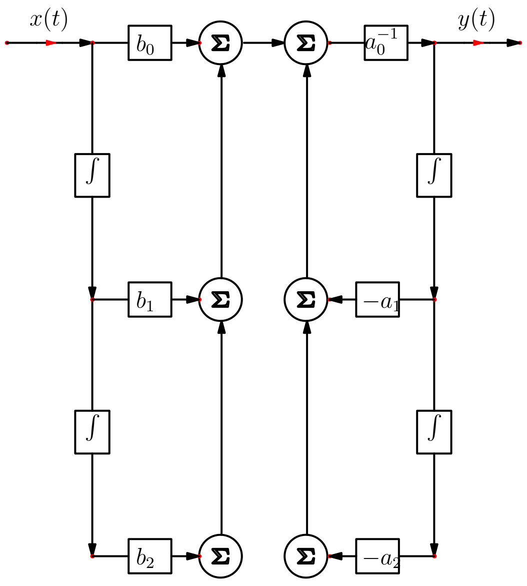


Figure 4: LTI systems represented as a block diagram in the direct form I.

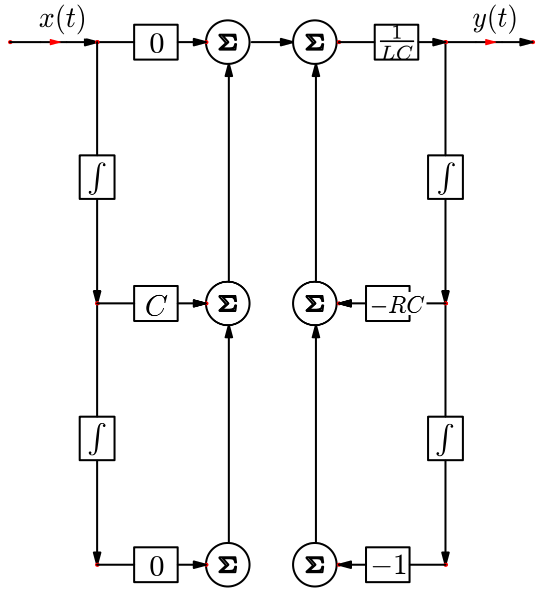


Figure 5: Direct form I for the RLC circuit in Figure 1

More generally:

$$\boxed{y(t) = \int_{(0)} y\mathrm{d}t = \frac{1}{a_0}\left[\sum_{k=0}^{N} b_k \int_{(k)} x\mathrm{d}t - \sum_{i=1}^{N} a_i \int_{(i)} y\mathrm{d}t\right]}.$$

The diagram shown in Fig. 10 is called *Direct Form I*. It works for all LTI systems described by LDE with constant coefficients.

◇ **Example 4.** We can look at the right side of the diagram in more details. If we call the signal between the left and right stage (containing only $x(t)$ and $y(t)$ respectivelly) for $w(t)$, then the signals in the right stage can be expressed in terms of $w(t)$:

$$\begin{aligned}
\frac{1}{a_0}\left[-a_1\int_{(1)} y(t) - a_2\int_{(2)} y(t) + ... + w(t)\right] &= y(t) \\
-a_1\int_{(1)} y(t) - a_2\int_{(2)} y(t) - ... + w(t) &= a_0 y(t) \\
w(t) = a_0 y(t) + a_1\int_{(1)} y(t) + a_2\int_{(2)} y(t) + ...
\end{aligned}$$

Could we make such a diagram for the circuit shown in Fig. 1? If we define $x(t) = v(t)$ and $y(t) = i(t)$, then from Eq. 3 we can write

$$C\dot{v}(t) = RC\dot{i}(t) + i(t) + LC\ddot{i}(t) \tag{9}$$

and

$$a_0 = LC\,;\; a_1 = RC\,;\; a_2 = 1 \tag{10}$$
$$b_0 = 0\,;\; b_1 = C\,;\; b_2 = 0\,, \tag{11}$$

as represented in Figure 5.

◇ **Example 5.** Let us now take the same circuit from Fig. 1 considering the same input, $x(t) = v(t)$, but the ouput as $y(t) = v_C(t)$. Starting by deriving Eq. 1, we obtain

$$\dot{y}(t) = \dot{v_C}(t) = \frac{1}{C} i(t) \tag{12}$$
$$C\dot{y}(t) = i(t)\,, \tag{13}$$

and replacing $i(t)$ from Eq. 13 in Eq. 3 we obtain,

$$\begin{aligned} \dot{x}(t) &= RC\ddot{y}(t) + \dot{y}(t) + LC\dddot{y}(t) \\ x(t) &= RC\dot{y}(t) + y(t) + LC\ddot{y}(t)\,. \end{aligned}$$

$$\begin{aligned} \int_{(2)} x(t) &= RC\int_{(1)} y(t) + \int_{(2)} y(t) + LC\int_{(0)} y(t) \\ \int_{(2)} y(t) + RC\int_{(1)} y(t) + LC\int_{(0)} y(t) &= \int_{(2)} x(t) + 0\int_{(1)} x(t) + 0\int_{(0)} x(t) \\ a_2\int_{(2)} y + a_1\int_{(1)} y + a_0 y &= b_2\int_{(2)} x + b_1\int_{(1)} x + b_0 x \\ a_0 = LC\,;\; a_1 = RC\,;\; a_2 = 1 \quad &; \quad b_0 = 0\,;\; b_1 = 0\,;\; b_2 = 1\,. \end{aligned}$$

Note the difference between the coefficients obtained in this example and in example 1a where a different output signal was chosen. ♣

2.3 Form II

Is this a unique/best representation of our system? For this scheme 2N integrations have to be performed. We can reduce that if we swap the 1st and 2nd stages in Form I. This is possible as cascaded LTI systems can be interchanged without affecting the overall transfer function. After the interchange, both cascade of integrators run parallel and so we can unit them, arriving to the direct form II, as shown in Figure 6.
From the block diagram we can write directly the relationships

$$y = \sum_{i=0}^{N} b_i z_i = b_0 z_0 + b_1 z_1 + b_2 z_2 = b_0 z_0 + b_1 \int_{(1)} z_0 + b_2 \int_{(2)} z_0 \tag{14}$$

$$z_0 = \frac{1}{a_0}\left[x - \sum_{i=1}^{N} a_i z_i\right] = \frac{1}{a_0}\left[x - a_1 z_1 - a_2 z_2\right]\,, \tag{15}$$

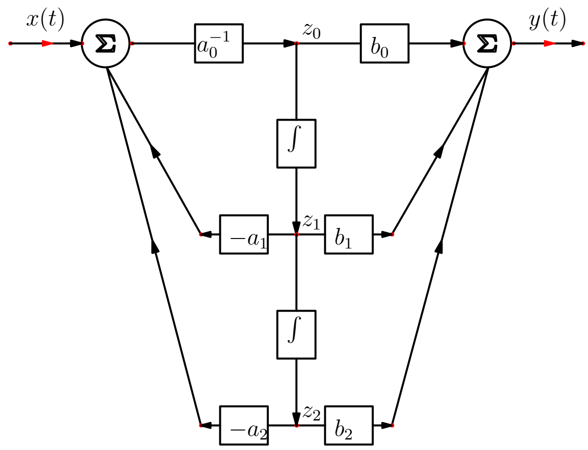


Figure 6: LTI systems in the direct form II.

Knowing that every state variable, z_i, (for $i = 1, 2, ..., N$), which describes the internal state of the system, can be obtained by integrating z_0 i times we write,

$$z_i = \int_{(i)} z_0 \mathrm{d}t \,. \tag{16}$$

Eq. 15 can be rewritten as

$$x = \sum_{i=0}^{N} a_i z_i = a_0 z_0 + a_1 z_1 + a_2 z_2 = a_0 z_0 + a_1 \int_{(1)} z_0 + a_2 \int_{(2)} z_0 \,. \tag{17}$$

We can check if this transformation is equivalent to the initial form by replacing Eqs. 14 and 17 in Eq. 8, according to,

$$a_2 \int_{(2)} (b_0 z_0 + b_1 z_1 + b_2 z_2) + a_1 \int_{(1)} (b_0 z_0 + b_1 z_1 + b_2 z_2) + a_0 (b_0 z_0 + b_1 z_1 + b_2 z_2) =$$

$$b_2 \int_{(2)} (a_0 z_0 + a_1 z_1 + a_2 z_2) + b_1 \int_{(1)} (a_0 z_0 + a_1 z_1 + a_2 z_2) + b_0 (a_0 z_0 + a_1 z_1 + a_2 z_2)$$

$$a_2 b_0 \int_{(2)} z_0 + a_2 b_1 \int_{(2)} z_1 + a_2 b_2 \int_{(2)} z_2 + a_1 b_0 \int_{(1)} z_0 + a_1 b_1 \int_{(1)} z_1 + a_1 b_2 \int_{(1)} z_2 + a_0 b_0 z_0 + a_0 b_1 z_1 + a_0 b_2 z_2 =$$

$$b_2 a_0 \int_{(2)} z_0 + b_2 a_1 \int_{(2)} z_1 + b_2 a_2 \int_{(2)} z_2 + b_1 a_0 \int_{(1)} z_0 + b_1 a_1 \int_{(1)} z_1 + b_1 a_2 \int_{(1)} z_2 + b_0 a_0 z_0 + b_0 a_1 z_1 + b_0 a_2 z_2$$

From Eq. 16 it follows that

$$a_2 b_0 z_2 + a_2 b_1 \int_{(1)} z_2 + a_2 b_2 \int_{(2)} z_2 + a_1 b_0 z_1 + a_1 b_1 z_2 + a_1 b_2 \int_{(1)} z_2 + a_0 b_0 z_0 + a_0 b_1 z_1 + a_0 b_2 z_2 =$$

$$b_2 a_0 z_2 + b_2 a_1 \int_{(1)} z_2 + b_2 a_2 \int_{(2)} z_2 + b_1 a_0 z_1 + b_1 a_1 z_2 + b_1 a_2 \int_{(1)} z_2 + b_0 a_0 z_0 + b_0 a_1 z_1 + b_0 a_2 z_2$$

Both sides of this equation are equal and so Eq. 8 is fulfilled after the transformation. In practical applications we have large freedom to describe the system by defining state variables, as long as for a given $x(t)$ we get the correct $y(t)$.

The circuit from examples 1a and 1b can be represented using block diagrams of type II, see Figs. 7 and 8.

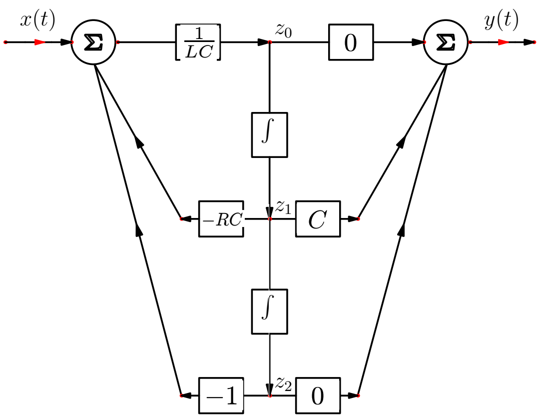


Figure 7: Block diagram of type II representation of the circuit in Fig. 1, considering $x(t) = v(t)$ and $y(t) = i(t)$.

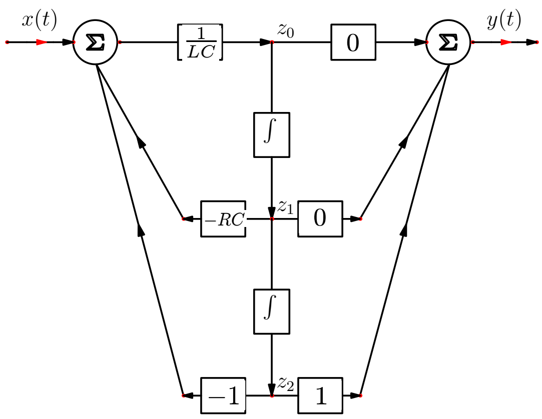


Figure 8: Block diagram of type II representation of the circuit in Fig. 1, considering $x(t) = v(t)$ and $y(t) = v_C(t)$.

2.4 State model

The state model gives the possibility of representing the internal behaviour of a system, as opposed to a differential equation. The differential equations is a single equation of order N, while the state model state model representation consists of a system of N 1st-order equations. Each differential equation is valid for one of the state variables. The output signal is obtained by a linear combination of the states.

◇ **Example 6 (Electrical circuit).** Consider an electrical circuit shown in Figure 9. For this circuit,

$$y(t) = -R_2 i_2(t) \tag{18}$$

$$u_1 = \frac{Q}{C} \tag{19}$$

$$Q = \int_{-\infty}^{\tau} i(\tau)\mathrm{d}\tau \tag{20}$$

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = \frac{\mathrm{d}Q/C}{\mathrm{d}t} = \frac{1}{C}i_1 + i_2 \tag{21}$$

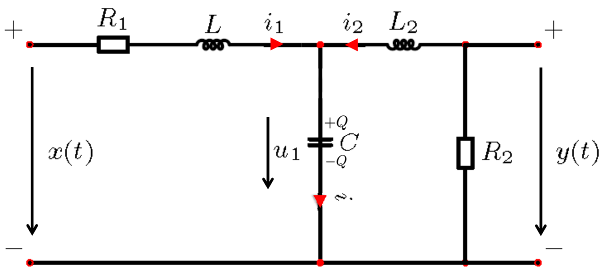


Figure 9: RCL circuit

Using Kirchhoff's rules,

$$\begin{aligned} i &= i_1 + i_2 && (22)\\ -x(t) &+ R_1 i_1(t) + L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + \frac{Q}{C} = 0 && (23)\\ \frac{Q}{C} &+ R_2 i_2(t) + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} = 0\,. && (24) \end{aligned}$$

It is also known that

$$i = \frac{\mathrm{d}Q}{\mathrm{d}t}\,, \tag{25}$$

so, from Eqs. 25 and 22 it follows that

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = i_1(t) + i_2(t)\,,$$

and

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = \frac{1}{C}\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{1}{C} i_1(t) + \frac{1}{C} i_2(t)\,. \tag{26}$$

Replacing Eq. 19 in Eqs. 23 and 24,

$$\begin{aligned} \frac{\mathrm{d}i_1}{\mathrm{d}t} &= -\frac{R_1}{L_1} i_1 - \frac{1}{L_1} u_1 + \frac{1}{L_1} x(t) && (27)\\ \frac{\mathrm{d}i_2}{\mathrm{d}t} &= -\frac{R_2}{L_2} i_2 - \frac{1}{L_2} u_1\,. && (28) \end{aligned}$$

♣

From the three first-order differential equations (Eqs. 26, 27, and 28) we can eliminate i_1, i_2, and u_1 and obtain a third-order differential equation in the form of Eq. 31. However, this would result in the loss of information that determine the behaviour of the system. Instead it is often useful to represent Eq. 26, 27, 28, and 18 in matrix form:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_1 \\ i_2 \\ u_1 \end{bmatrix} &= \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ u_1 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} x(t) \\ y(t) &= \begin{bmatrix} 0 & -R_2 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ u_1 \end{bmatrix}. \end{aligned}$$

We can generalize this for any LTI system described a block diagram in Form II or a system described by LDE. See textbook pages 32 and 33 for more details. Matrices A, B, C and D can be constructed based on a_i and b_i coefficients.

◇ **Example 7 (LDE for the same system).**

$$
\begin{aligned}
\frac{\mathrm{d}i_2}{\mathrm{d}t} &= -\frac{R_2}{L_2}i_2 - \frac{1}{L_2}u_1 \\
\ddot{i_2} &= -\frac{R_2}{L_2}\dot{i_2} - \frac{1}{L_2}\dot{u_1} \\
\frac{\mathrm{d}u_1}{\mathrm{d}t} &= \frac{1}{C}i_1 + \frac{1}{C}i_2 \\
\ddot{i_2} &= -\frac{R_2}{L_2}\dot{i_2} - \frac{1}{CL_2}(i_1 + i_2) \\
i_1 &= \left(\ddot{i_2} + \frac{R_2}{L_2}\dot{i_2} + \frac{1}{CL_2}i_2\right)CL_2 \\
\ddot{i_1} &= -\frac{R_1}{L_1}\dot{i_1} - \frac{1}{CL_1}(i_1 + i_2) + \frac{1}{L_1}x(t) \\
\ldots
\end{aligned}
$$

Definition 6 State model We can easily generalise the previous example and write the general form of the state model:

$$
\begin{aligned}
\dot{\mathbf{z}} &= \hat{A}\mathbf{z} + \hat{B}\mathbf{x} \qquad (29)\\
\mathbf{y} &= \hat{C}\mathbf{z} + \hat{D}\mathbf{x}\,, \qquad (30)
\end{aligned}
$$

where $\mathbf{z}$ is the state variable, Eq. 29 is the state equation, and Eq. 30 is the output equation. $\hat{A}$, the system matrix, describes how change in state vector $\dot{\mathbf{z}}$ depends on instantaneous $\mathbf{z}$; $\hat{B}$ describes how input changes state $\mathbf{z}$; $\hat{C}$ describes the effect of the state on the output; and $\hat{D}$ describes the direct influence of the input on the output.

If we know $z(t_0)$ at time t_0 and $x(t)$ between t_0 and t we can uniquely determine the state $z(t)$ and output $y(t)$.

◇ **Example 8 (Moving object).** We use the same method to illustrate motion of a object attached to a spring under viscous friction damping. Here

$$m\ddot{y}(t) = x(t) - k_1\dot{y}(t) - k_2 y(t)$$

where $y(t)$ is the position, $x(t)$ is the applied force. We can express this equation in terms of internal variables u_1 and u_2

$$
\begin{aligned}
u_1 &= y(t) \\
u_2 &= \dot{y}(t)
\end{aligned}
$$

$$
\begin{aligned}
m\ddot{y}(t) &= x(t) - k_1\dot{y}(t) - k_2 y(t) \\
\ddot{y}(t) &= \frac{1}{m}x(t) - \frac{k_1}{m}\dot{y}(t) - \frac{k_2}{m}y(t) \\
\dot{u_2}(t) &= \frac{1}{m}x(t) - \frac{k_2}{m}u_1(t) - \frac{k_1}{m}u_2(t) \\
\dot{u_1}(t) &= u_2(t)
\end{aligned}
$$

Now, we can write the last two equations in a matrix format:

$$
\begin{aligned}
\begin{bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{k_2}{m} & -\frac{k_1}{m} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} x(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}
\end{aligned}
$$

◇ **Example 9 (Moving object, matlab code).** Here is a example code to solve the equation above. Check matlab help for description of `ss, impulse, step, lsim`

```
close all; clear all;
%Example 7; moving object on a spring, in a viscous friction enviroment
k1 = 1;   %friction coefficient
k2 = 100; %spring constant; resonance at frequency (100/1)^0.5
m = 1;

A = [0 1; -k2/m -k1/m];
B = [0; 1/m];
C = [1, 0];
D = [];
sys = ss(A,B,C,D);
step(sys);
hold on
impulse(sys);
t = 0:.01:20;
u = sin(10*t);
hold off; figure;
lsim(sys,u,t)
hold off;
figure;
bode(sys);
```

Matlab/IPython Project 1

Familiarize yourself with IPython/Jupyter notebook concept and try to do the simulation in python. You will find many good examples on the net, here is one: Scipy Examples and More Examples. There are also many good books on the topics, including this one and this one from the university of Oslo.
Delivery: MATLAB: Working code and a short report documenting how to use it and showing that it solves the problem correctly. Python: ipython/Jupyter notebook with the code, code description and solution of the problem given below. Help will be available during problem class on Thursday 18.01.2018. Please bring a PC with you.
Delivery deadline: 26.01.2017 at 16:00. Delivery through `blackboard`.

Exercise

We would like to numerically solve following Linear Differential Equation (LED)

$$\sum_{i=0}^{N} \alpha_i \frac{\mathrm{d}^i y(t)}{\mathrm{d}t^i} = \sum_{k=0}^{M} \beta_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k} \tag{31}$$

by transforming it to an integral form

$$\boxed{y(t) = \int_{(0)} y\mathrm{d}t = \frac{1}{a_0}\left[\sum_{k=0}^{N} b_k \int_{(k)} x\mathrm{d}t - \sum_{i=1}^{N} a_i \int_{(i)} y\mathrm{d}t\right]}.$$

and then integrating it numerically accordingly to the flow diagram shown below (drawn for a system where N = 2). Please do not solve this using any other method, the point of this exercise is to program LDE solver based on the block diagram discussed in the course.

Note that integration for the output part can not be done exactly but needs a approximation that

$$\int_0^t y(\tau)\mathrm{d}\tau \approx \int_0^{t-\Delta t} y(\tau)\mathrm{d}\tau$$

Apply this method to calculate step response $s(t)$ for a system described by the following equations

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$$
$$y(0) = \dot{y}(0) = \ddot{y}(0) = 0$$

where

$$s(t) = S\{\epsilon(t)\}$$

and

$$\epsilon(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Show that the calculated output agrees with an analytical solution given below and which we will obtain using the Laplace transform later in the course:

$$s(t) = 0.5\epsilon(t) - e^{-t}\epsilon(t) + 0.5e^{-2t}\epsilon(t)$$

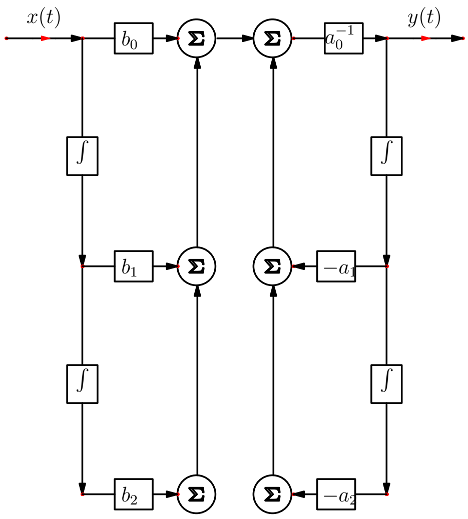


Figure 10: LTI systems represented as a block diagram in the direct form I.

3 Solving LDE

3.1 Summary so far

- We want to find the relation between the input $x(t)$ and the output $y(t)$ of LTI systems in the time domain:
- Use a differential equation as the mathematical representation of that relationship;
- Construct block diagrams as graphical representations of the differential equations;
- Represent the system as a state model, which are equivalent to block diagrams but allow more information about the internal energy stores.

3.2 Impulse response

Definition 7 By impuls response we define a response of the LTI system to a impuls input:

$$h(t) = S\{\delta(t)\}$$

If the input is a linear combination of impulses then the output will a linear combination of impulse responses (linearity of the system):

$$y(t) = S\left\{\sum_{i=N_1}^{N_2} \alpha_i \delta(t-t_i)\right\} = \sum_{i=N_1}^{N_2} \alpha_i S\left\{\delta(t-t_i)\right\} = \sum_{i=N_1}^{N_2} \alpha_i h(t-t_i)$$

We can use this interpretation to an arbitrary input using approximation of an impulse and relate the output to the input in terms of delayed impulse responses.

$$\delta(t-t_i) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \Pi\left(\frac{t-t_i}{\Delta t}\right) \tag{32}$$

$$x(t) \cong \sum_{i=N_1}^{N_2} \left[x(n\Delta t)\Delta t\right] \frac{1}{\Delta t} \Pi\left(\frac{t-n\Delta t}{\Delta t}\right) \tag{33}$$

a sum of $\Pi(t)$ functions, where

$$\Pi\left(\frac{t-t_0}{\tau}\right) = \begin{cases} 1, & |t-t_0| < \tau/2 \\ 0, & \text{otherwise} \end{cases}$$

Approximated signal $x(t)$ is defined for all t, for finite Δx is not a smooth function.

$$x(t) \quad = \quad \sum_{n=-\infty}^{\infty} \lim_{\Delta t \to 0} x(n\Delta t)\left[\frac{1}{\Delta t}\Pi\left(\frac{t-n\Delta t}{\Delta t}\right)\right]\Delta t = \int_{-\infty}^{\infty} x(\beta)\delta(t-\beta)\mathrm{d}\beta \tag{34}$$

where

$$n\Delta t = \beta$$

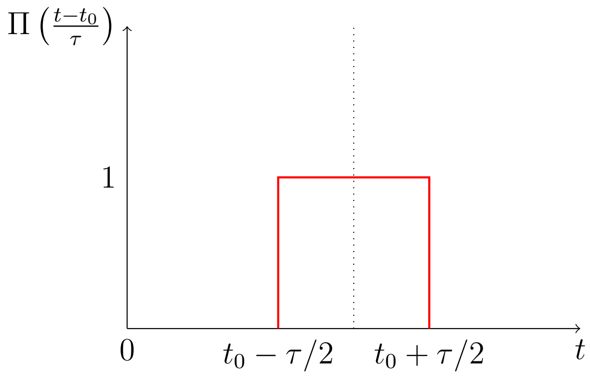


Figure 11: $\Pi\left(\frac{t-t_0}{\tau}\right)$. Note $\int\limits_{-\infty}^{\infty}\Pi\left(\frac{t-t_0}{\tau}\right)=\tau$

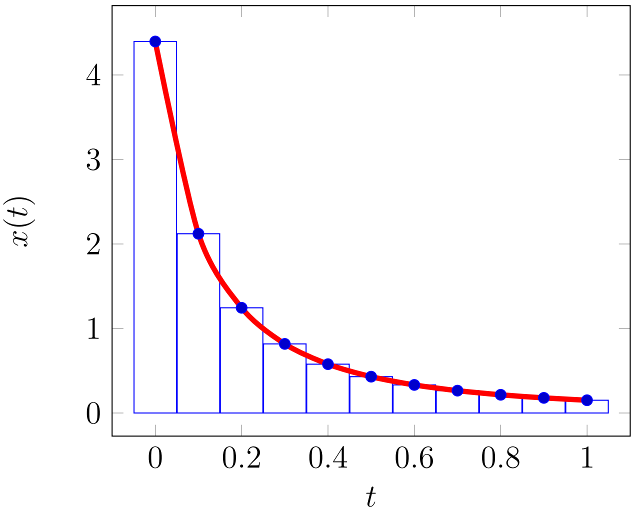


Figure 12: Plot of $x(t)$ (red) and $x(t)\Pi\left(\frac{t-\Delta t n}{\Delta t}\right)$. Note $\int\limits_{-\infty}^{\infty} x(t)\Pi\left(\frac{t-\Delta t n}{\Delta t}\right)\mathrm{d}t \approx \int\limits_{-\infty}^{\infty} x(t)\mathrm{d}t$

and

$$\Delta t = \mathrm{d}\beta$$

and β is a continuous variable in a limit where $\Delta t \to 0$. So, the input signal is given be a convolution between input signal and delta-function.

$$x(t) = \int\limits_{-\infty}^{\infty} x(\beta)\delta(t-\beta)\mathrm{d}\beta \tag{35}$$

An arbitraty signal x(t), continuous for all t, For a LTI system, the output will be a corresponding integral, where $\delta(t)$ is replaced by the impolse response $h(t)$

$$y(t) = \int\limits_{-\infty}^{\infty} x(\beta)h(t-\beta)\mathrm{d}\beta \tag{36}$$

Sine the output from a single $x(n\Delta t)\delta(t-n\Delta t)$ will be $x(n\Delta t)h(t-n\Delta t)$. This integral corresponds to a convolution between input signal and the impulse response:

Definition 8 (Output of LTI system) The output any linear, time-invariant system can be described as the convolution of its impulse response with an arbitrary input.

$$y(t) = x(t) * h(t) = \int\limits_{-\infty}^{\infty} x(\beta)h(x-\beta)\mathrm{d}\beta = \int\limits_{-\infty}^{\infty} h(\alpha)x(t-\alpha)\mathrm{d}\alpha = h(t) * x(t) \quad (37)$$

3.3 Convolution

The combination of two functions of time (time signals) $f(t)$ and $g(t)$ is a function $(f*g)$ called *convolution* and defined by the integral

$$(f*g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau\,,$$

for all signals $f(t)$, $g(t)$ defined on $\Re$. It is important to note that the operation of convolution is commutative, meaning that $f*g = g*f$ for all signals f and g. Thus, the convolution operation can also be stated using the equivalent definition

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$$

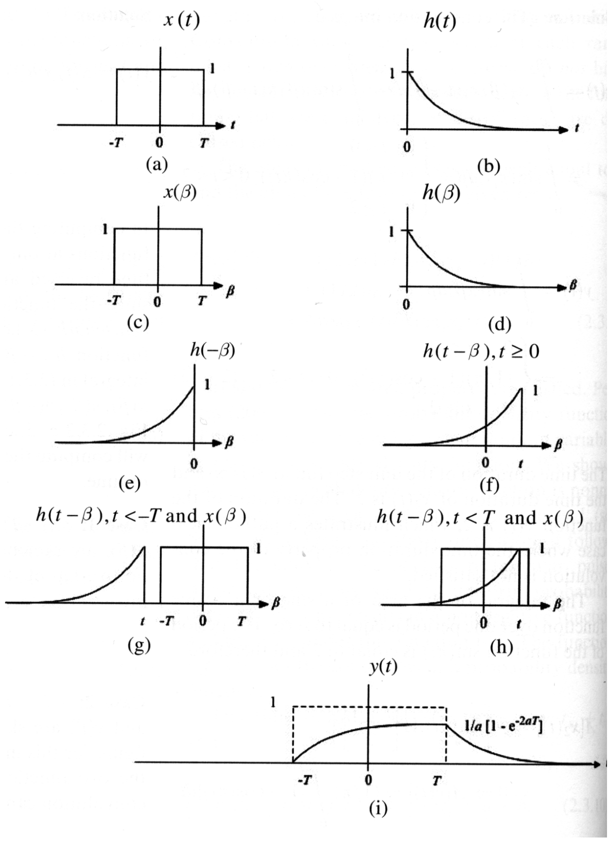
So, we can compute this operation by choosing the way that minimizes the algebraic complexity of the expression, that is, it is desirable to select the easier function for time shifting. Convolution has several other important properties.

$\diamondsuit$ **Example 10.** One can calculate convolution between two function using following steps:

- $x(t) \longmapsto x(\beta)$
- $h(t) \longmapsto h(\beta) \longmapsto h(-\beta)$
- 2nd signal is then shifted by a particular value t_0 and integral is calculated:

$$y(t_0) = \int\limits_{-\infty}^{\infty} x_1(\beta)x_2(t_0-\beta)\mathrm{d}\beta$$

- this needs repeating for all values of $-\infty < t_0 < \infty$



♣

◇ **Example 11 (Convolution again).** Consider convolution between $h(t) = \mathrm{e}^{-t}$ (red) and $x(t) = \Pi\left(\frac{t-2}{4}\right)$ (blue) shown in figure Figure 13:

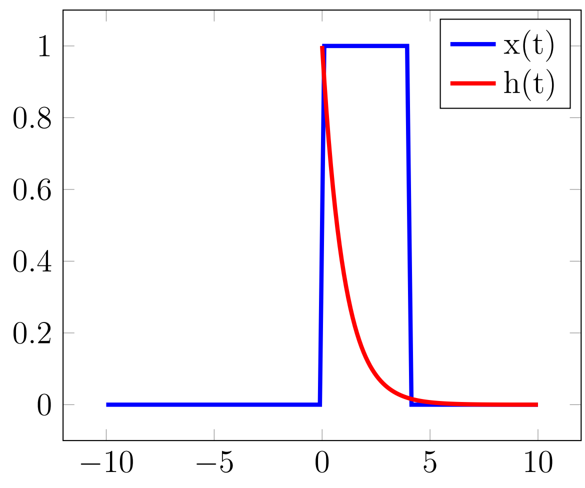


Figure 13:

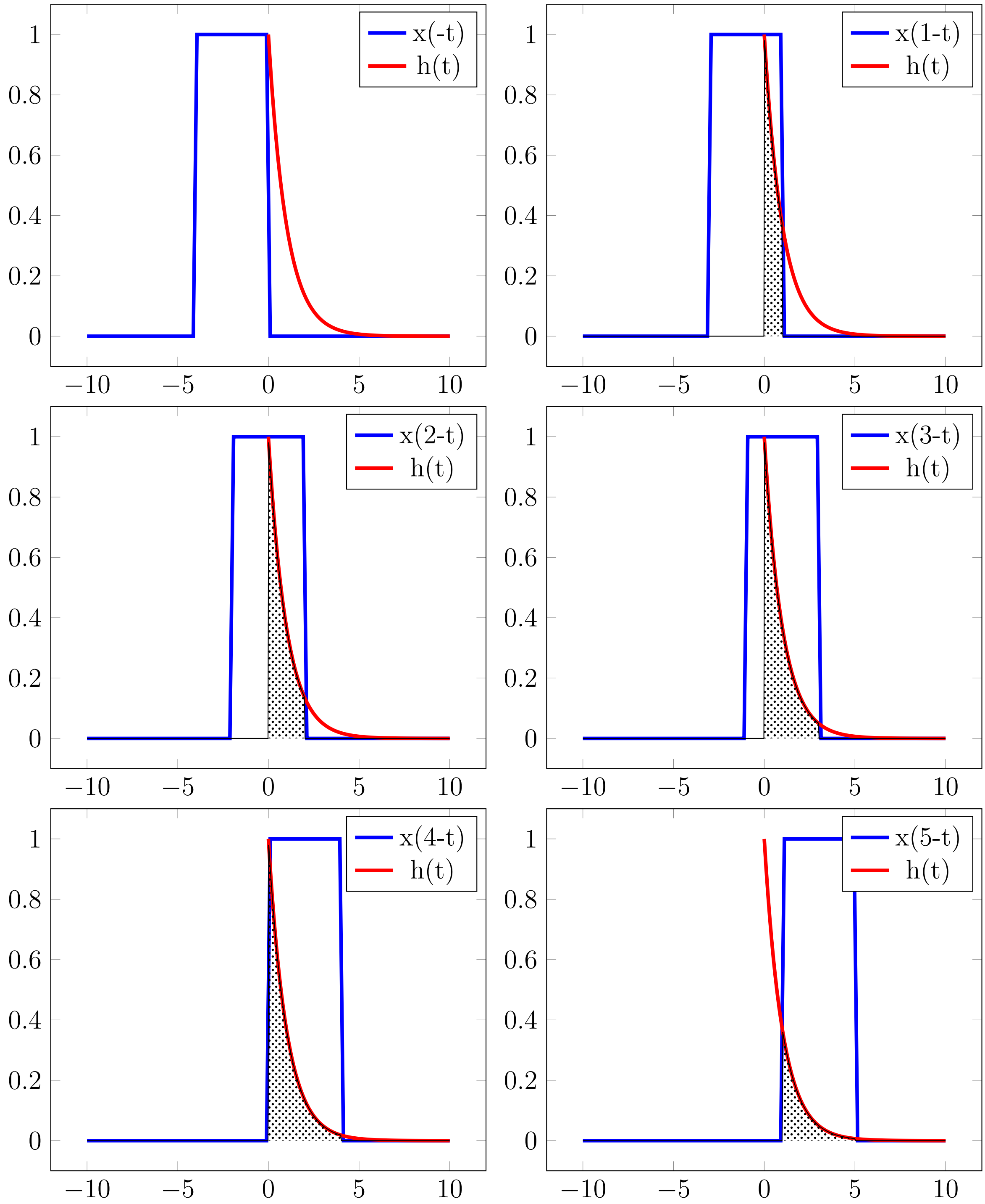


Figure 14: Plot of $x(t_i - \beta)$, $h(\beta)$ and $x(t_i - \beta)h(\beta)$ (shaded)for $t_i = 0, 1, .., 5$. Shaded area represents the convolution integral for given t_i.

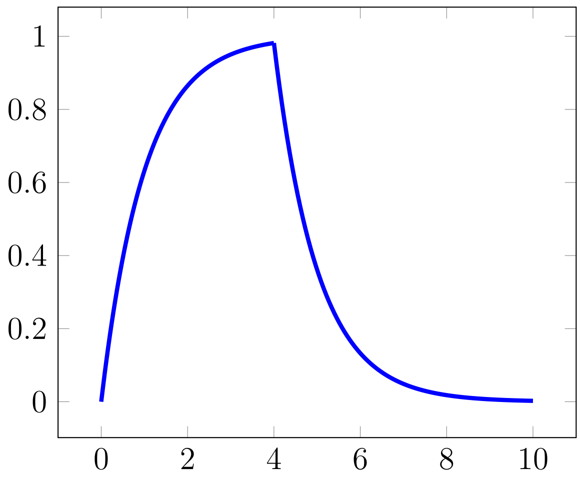


Figure 15: $y(t)$ calculated for the example above.

3.4 Exponential signals

Now we assume that our input signal $x(t)$ has the form of a complex exponent, $Ce^{s_0 t}$:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(t-\tau) C e^{s_0 \tau} \mathrm{d}\tau \\
t-\tau &= t' \leftrightarrow \tau = t - t' \\
\mathrm{d}\tau &= -\mathrm{d}t' \quad \text{and here } t \text{ is a constant} \\
\infty &\to t - \infty = -\infty \;;\; -\infty \to t - (-\infty) = \infty \\
y(t) &= \int_{\infty}^{-\infty} h(t') C e^{s_0(t-t')} (-\mathrm{d}t') \\
&= \int_{-\infty}^{\infty} h(t') C e^{s_0(t-t')} \mathrm{d}t' \\
y(t) &= C e^{s_0 t} \int_{-\infty}^{\infty} h(t') e^{-s_0 t'} \mathrm{d}t' \qquad (38) \\
H(s_0) &= \int_{-\infty}^{\infty} h(t') e^{-s_0 t'} \mathrm{d}t' = \mathscr{L}\{h(t)\} \qquad (39) \\
y(t) &= H(s_0) \cdot x(t) \\
H(s_0) e^{s_0 t} &= S\left\{e^{s_0 t}\right\} \qquad (40)
\end{aligned}$$

$H(s_0)$ is the **Laplace transform** of the impulse response function calculated at s_0. Since the same exponent $e^{s_0 t}$ appears on both the input and output, $e^{s_0 t}$ is called the **eigenfunction** of the system and $H(s_0)$ will be a corresponding **eigenvalue**.

3.5 Eigenfunctions

We can compare the previous conclusions with linear algebra and the notion of the eigenvectors of a matrix system. A linear time invariant (LTI) system S operates on a continuous input $x(t)$ to produce continuous time output $y(t)$:

$$S\{x(t)\} = y(t)$$

$$x(t) \rightarrow \boxed{S} \rightarrow y(t)$$

We can write a mathematical analogy to an $N \times N$ matrix $\hat{A}$ operating on a vector $\boldsymbol{b}$ to produce another vector $\boldsymbol{b}$:

$$\hat{A}\,\boldsymbol{a} = \boldsymbol{b}$$

$$\boldsymbol{a} \rightarrow \boxed{\hat{A}} \rightarrow \boldsymbol{b}$$

An **eigenvector** of $\hat{A}$ is a vector $\boldsymbol{v}$ such that $\hat{A}\,\boldsymbol{v} \;= \lambda\,\boldsymbol{v}$, with λ, the eigenvalue, $\in C$:

$$\boldsymbol{v} \rightarrow \boxed{\hat{A}} \rightarrow \lambda\boldsymbol{v}$$

We can define an **eigenfunction** (or eigensignal) of an LTI system S to be a signal $e(t)$ such that

$$S\{e(t)\} \;=\; \lambda\, e(t), \quad \vee\, \lambda \in C$$

and according to the diagram in Fig. 16.

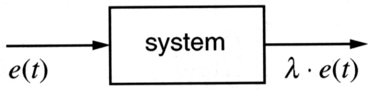


Figure 16: Driving a system with the eigenfunction $e(t)$.

Eigenfunctions are the simplest possible signals for S to operate on: to calculate the output, we simply multiply the input by a complex number λ. Let us show that this is true for any $e^{s_0 t}$ signal and LTI system.

3.6 Exponential function as eigenfunctions of LTI systems

Let us show that such exponential functions, e^{st} are eigenfunctions of LTI systems by showing that $y(t) = \lambda x(t)$. Let us consider $x(t) = e^{s_0 t}$, and look for the corresponding $y(t)$ signal,

$$y(t) = S\left\{x(t)\right\} = S\left\{e^{s_0 t}\right\}\,.$$

Let us start looking at the response to an input signal shifted in time,

$$x(t-\tau) = e^{s_0(t-\tau)}\,.$$

Because of time-invariance we can write

$$y(t-\tau) = S\left\{x(t-\tau)\right\} = S\left\{e^{s_0(t-\tau)}\right\} = S\left\{e^{-s_0\tau}e^{s_0 t}\right\}\,.$$

$e^{-s_0\tau}$ does not depend on time and so, because of linearity, we can write

$$\begin{aligned} y(t) &= S\left\{Ax(t)\right\} = AS\left\{x(t)\right\} \\ y(t-\tau) &= S\left\{e^{-s_0\tau}e^{s_0 t}\right\} = e^{-s_0\tau}S\left\{e^{s_0 t}\right\} = e^{-s_0\tau}y(t) = \lambda y(t)\,. \end{aligned}$$

The constant λ identifies the behaviour of S and it generally depends on the complex frequency s_0 ($\lambda = H(s_0)$), where $H(s_0)$ is the system function or transfer function.

This is not generally true for a signal for which $x(t) = 0$ for $t < 0$.

◇ **Example 12.** Show that the one-sided exponential function

$$x(t) = \begin{cases} e^{st}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

is not an eigenfunction of the LTI-system,

$$y(t) = \int_{-\infty}^{t} e^{s\tau} \mathrm{d}\tau \,.$$

For $t < 0$,

$$y(t) = 0 = H(s)\, x(t)\,.$$

For $t \geq 0$,

$$y(t) = \int_{0}^{t} e^{s\tau} \mathrm{d}\tau = \frac{1}{s} e^{s\tau} \Big|_0^t = \frac{1}{s}(e^{st} - 1)$$

$$y(t) \neq H(s) e^{s\,t}\,.$$

3.7 Response of LTI-system to sinusoidal input

Finally, due to linearity, we can use a combination of eigenfunction and eigenvalues to get output for a input composed of many signals:

$$y(t) = \sum_i C_i\, e^{s_i t}\, H(s_i)\,,$$

where the input signal is given by

$$x(t) = \sum_i C_i\, e^{s_i t}\,,$$

and we will have to learn how to decompose.

◇ **Example 13.** Given a causal LTI system with transfer function

$$H(s) = \frac{s^2 + 4}{(s+1)(s+2)(s+3)},$$

find the output signal $y(t)$ produced by the input signal $x(t) = \cos(2t)$.

We start by using the Euler formula to write: $\cos(2t) = \frac{1}{2}\left(e^{j2t} + e^{-j2t}\right)$

This indicates that the input signal is a linear combination of two (eigenfunction) signals: $x_1(t) = e^{s_1 t}$ and $x_2(t) = e^{s_2 t}$, with $s_1 = 2j$ and $s_2 = -2j$.

So, we can use a combination of eigenfunctions and eigenvalues to get the output for a input composed of many signals:

$$y(t) \;=\; \sum_{i=1}^{2} C_i\, e^{s_i t}\, H(s_i).$$

From this equation we can write

$$y(t) = \frac{1}{2} e^{s_1 t} H(s_1) \;+\; \frac{1}{2} e^{s_2 t} H(s_2).$$

$$\text{As}\quad H(s_i) = 0, i = 1, 2 \quad \Rightarrow \quad y(t) = 0.$$

3.7.1 Eigenfunctions and RTC circuit

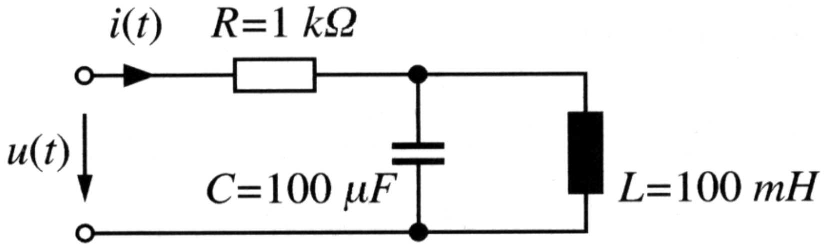


Figure 17: Parallel resonant RLC circuit

1. $i = i_1 + i_2$; $i - i_1 - i_2 = 0$
2. $u(t) = Ri + L\frac{\mathrm{d}i_2}{\mathrm{d}t}$
3. $u(t) = Ri + \frac{1}{C}\int\limits_{-\infty}^{t} i_1 \mathrm{d}\tau$
4. from (1): $i_2 = i - i_1$
5. from (4 and 2): $u = Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} - L\frac{\mathrm{d}i_1}{\mathrm{d}t}$
6. from (3): $\frac{\mathrm{d}u}{\mathrm{d}t} = R\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}i_1$
7. from (6): $i_1 = C\frac{\mathrm{d}u}{\mathrm{d}t} - CR\frac{\mathrm{d}i}{\mathrm{d}t}$

We can then combine (7 and 5):

$$\begin{aligned}
u(t) = Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} - L\frac{\mathrm{d}}{\mathrm{d}t}\left(C\frac{\mathrm{d}u(t)}{\mathrm{d}t} - CR\frac{\mathrm{d}i(t)}{\mathrm{d}t}\right) \\
u(t) = Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} - LC\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2} + LCR\frac{\mathrm{d}^2i(t)}{\mathrm{d}t^2} \\
u(t) + LC\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2} = Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + LCR\frac{\mathrm{d}^2i(t)}{\mathrm{d}t^2}
\end{aligned}$$

Now we apply some arbitrary voltage

$$\begin{aligned}
u(t) &= u_0\, e^{\sigma_0 t}\, \cos(\omega_0 t + \varphi_0) \\
&= u_0 e^{\sigma_0 t}\frac{1}{2}\left[e^{j(\omega_0 t+\varphi_0)} + e^{-j(\omega_0 t+\varphi_0)}\right] \\
&= \frac{u_0}{2}e^{j\varphi_0}e^{(\sigma_0+j\omega_0)t} + \frac{u_0}{2}e^{-j\varphi_0}e^{(\sigma_0-j\omega_0)t} \\
&= U_1 e^{s_1 t} + U_2 e^{s_2 t} \\
U_1 &= U_2^* = \frac{u_0}{2}e^{j\varphi_0} \\
s_1 &= s_2^* = (\sigma_0 + j\omega_0)
\end{aligned}$$

- The input signal is a sum of two exponential functions (complex).
- The system can be modelled by LDE with constant coefficients so, it is a LTI system. Consequently, it has complex exponential eigenfunctions.

- The output signal may be written as the sum of two exponential functions.

$$i(t) = I_1 e^{s_1 t} + I_2 e^{s_2 t}$$

I_1 and I_2 are unknown and must be determined from the model of the system. We can do this separately for two signals U_1 and U_2.

$$\begin{aligned}
u(t) + LC\frac{\mathrm{d}^2 u(t)}{\mathrm{d}t^2} &= Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + LCR\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} \\
U_1 e^{s_1 t} + LCU_1 s_1^2 e^{s_1 t} &= RI_1 e^{s_1 t} + LI_1 s_1 e^{s_1 t} + LCRs_1^2 I_1 e^{s_1 t} \\
U_2 e^{s_2 t} + LCU_2 s_2^2 e^{s_2 t} &= RI_2 e^{s_2 t} + LI_2 s_2 e^{s_2 t} + LCRs_2^2 I_2 e^{s_2 t}
\end{aligned}$$

As U_1 and U_2 are known we can determine I_1 and I_2:

$$\begin{aligned}
U_1(e^{s_1 t} + LCs_1^2 e^{s_1 t}) &= I_1(Re^{s_1 t} + Ls_1 e^{s_1 t} + LCRs_1^2 e^{s_1 t}) \\
U_2(e^{s_2 t} + LCs_2^2 e^{s_2 t}) &= I_2(Re^{s_2 t} + Ls_2 e^{s_2 t} + LCRs_2^2 e^{s_2 t})
\end{aligned}$$

$$\begin{aligned}
I_1 &= U_1 \frac{1 + LCs_1^2}{R + Ls_1 + LCRs_1^2} \\
I_2 &= U_2 \frac{1 + LCs_2^2}{R + Ls_2 + LCRs_2^2}
\end{aligned}$$

And

$$\begin{aligned}
i(t) &= I_1 e^{s_1 t} + I_2 e^{s_2 t} \\
&= I_1 e^{s_1 t} + I_1^* e^{s_1^* t} = 2Re\left\{I_1 e^{s_1 t}\right\} \\
i(t) &= 2Re\left\{u_0 \frac{1}{2} e^{j\varphi_0}\left[\frac{1 + LC(\sigma_0 + j\omega_0)^2}{R + L(\sigma_0 + j\omega_0) + LCR(\sigma_0 + j\omega_0)^2}\right] e^{(\sigma_0 + j\omega_0)t}\right\}
\end{aligned}$$

If $\sigma_0 = 0$ then we will have a steady state frequency response. By plotting the dispersion (ω_0) we see the amplitude response of the system. We will return to that later. For now we can just plot the amplitude response (see Fig. 18). As you can see the current is zero for $f = 50$Hz, so our circuit will work as a stop band filter removing 50Hz frequency from the input signal.

Let us summarise the general procedure:

- To obtain the desired output we have to determine the factors of the eigenfunctions, here I_1 and I_2;
- These factors can be expressed as algebraic equations;
- The LDE does not need to be solved as a whole, instead we only need to form the derivative (inductance) or integral (capacitance);
- The latter step results simply on the multiplication or division of the complex frequency, s.

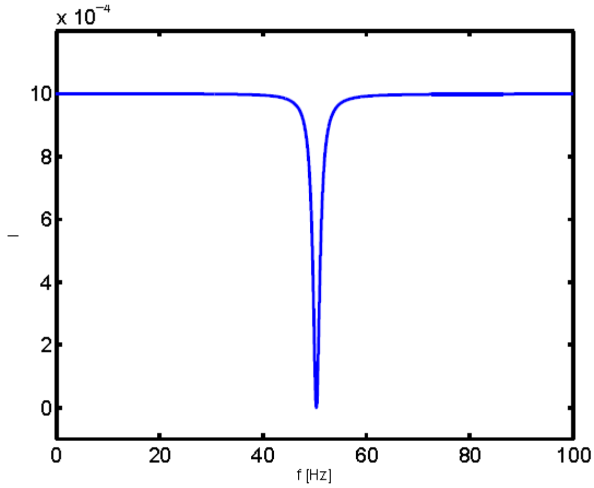


Figure 18: Current amplitude as a function of frequency for circuit shown in Fig. 17.

3.8 Impedance

Let us consider the system and corresponding LDE of example 3,

$$u(t) + LC\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = Ri(t) + L\frac{\mathrm{d}i}{\mathrm{d}t} + LCR\frac{\mathrm{d}^2i}{\mathrm{d}t^2}$$

Let us consider each of the components using Ohm's law in the impedance form,

$$U(s) = Z(s) \cdot I(s)$$

Resistor: $U(s) = RI(s); \;\; Z_R = R$;

Capacitance: $u(t) = \frac{1}{C}\int\limits_{-\infty}^{t} i(\tau)\mathrm{d}\tau$ and as a consequence $i(t) = C\frac{\mathrm{d}u}{\mathrm{d}t} = sC \cdot u(t)$;
So, $U(s) = \frac{1}{sC}I(s); \;\; Z_C = \frac{1}{sC}$;

Inductance: $u(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = sL \cdot i(t)$; $U(s) = sLI(s); \;\; Z_L = sL$.

We can redraw the circuit of example 3 using the impedances of the different elements, Fig. 19.

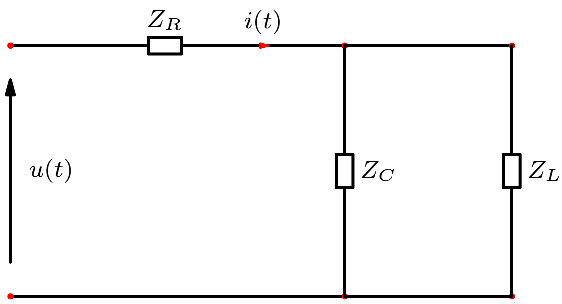


Figure 19: Electric circuit modelled as an impedance.

$$\begin{aligned}
Z_{tot} &= Z_R + \frac{Z_C Z_L}{Z_C + Z_L} \\
U &= Z_{tot} \cdot I \\
I &= \frac{1}{Z_{tot}} U \\
Z_{tot}^{-1} &= \frac{1}{Z_R + \frac{Z_C Z_L}{Z_C + Z_L}} = \frac{1}{R + \frac{\frac{1}{sC} sL}{\frac{1}{sC} + sL}} = \frac{1}{R + \frac{\frac{L}{C}}{\frac{1+s^2LC}{sC}}} = \frac{1}{R + \frac{sL}{1+s^2LC}} = \frac{1}{\frac{R(1+s^2LC)+sL}{1+s^2LC}} \\
Z_{tot}^{-1} &= \frac{1+s^2LC}{R+sL+s^2RLC} \\
I &= \frac{1+s^2LC}{R+sL+s^2RLC} U
\end{aligned}$$

$$i(t) = Z_{tot}^{-1}(s_1)U_1 e^{s_1 t} + Z_{tot}^{-1}(s_2)U_2 e^{s_2 t}$$

The impedance model only applies to LTI systems, as only those have exponential functions as eigenfunctions.