7 Fourier Series and Transforms

Material: Chapter 9

7.1 Summary

We have used the Laplace transform to describe signals and systems in the frequency-domain. Consists in representing signals as complex exponentials and use the superposition principle to compose the output signal.

There are however some disadvantages to the Laplace transform approach:

- It only provides a simple system model for LTI-systems that can be described by ordinary differential equations with constant coefficients;
- The Laplace transform does not really exist for signals e^{st}, $\cos(t)$, $\sin(t)$, for example;
- From the system function, the properties of a system in the frequency domain cannot easily be seen.

As an alternative to the Laplace transform, let us consider the Fourier transform.

7.2 Fourier representation of signals and LTI systems

The study of signals and systems using sinusoidal representations is termed Fourier analysis and is used in several branches of engineering and science. Fourier analysis consists in representing a signal as a weighted superposition of complex sinusoids. If such a signal is applied to an LTI system, then the output is a weighted superposition of the system response to each complex exponential. This does not only lead to a useful expression for the system output, but also provides an insightful characterization of signals and systems. For example, the sound that we hear from a choir contains bass, tenor, alto, and soprano parts, each of which contributes to a different frequency range in the overall sound. Here, the representation of signals can be viewed analogously: The weighted associated with a sinusoid of a given frequency represents the contribution of the sinusoid to the overall signal.

Why to use sinusoids? For example, when a sinusoidal signal goes though a LTI-systems (*e.g.*, RCL circuit), the system will only change the signal's amplitude and phase.

How can we use sinusoids to build periodic signals?

7.3 Fourier Series

If we let $f_p(t)$ be a periodic signal with a period T.

Fourier showed that any such periodic signal can be decomposed into a d.c. (direct current) term and sinusoidal (cos) terms with $2\pi\frac{1}{T}$, $2\pi\frac{2}{T}$, $2\pi\frac{3}{T}$,... frequencies components.

Definition 21 Any periodic signal can be decomposed into a d.c. (direct current) term and harmonic terms with $2\pi\frac{1}{T}$, $2\pi\frac{2}{T}$, $2\pi\frac{3}{T}$,... frequencies components with a weighing factor C_n and phase θ_n according to:

$$f_p(t) = c_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)\,, \tag{52}$$

where $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency.

We can reorganize Eq. 52 using

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$f_p(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \tag{53}$$

with

$$a_n = C_n \cos(\theta_n)\,; \quad b_n = -C_n \sin(\theta_n)$$

or, since

$$\cos(n\omega_0 t + \theta_n) = \frac{1}{2}\left(e^{j(n\omega_0 t+\theta_n)} + e^{-j(n\omega_0 t+\theta_n)}\right)\,,$$

Definition 22 The periodic function can be written in a more general form

$$f_p(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}\,, \tag{54}$$

where,

$$F_n = \frac{C_n}{2}e^{j\theta_n} \quad ; \quad F_{-n} = \frac{C_n}{2}e^{-j\theta_n}$$
$$F_{-n} = F_n^*$$

Eqs. 52, 53 and 54 are equivalent ways of representing Fourier series. A function and its Fourier series is completely described by its Fourier coefficients (C_n in Eq. 52, a_n and b_n in Eq. 53, and F_n in Eq. 54) and the fundamental frequency ω_0.

Let us consider the exponential form of the Fourier series. How to find F_n?

We multiply both members of this equation by the complex conjugate of a member of the base $e^{-ik\omega_0 t}$ and integrate over a period T,

$$\begin{aligned}\int_0^T e^{-jk\omega_0 t} f_p(t)\mathrm{d}t &= \int_0^T \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} e^{-jk\omega_0 t}\mathrm{d}t = \sum_{n=-\infty}^{\infty} F_n \int_0^T e^{j(n-k)\omega_0 t}\mathrm{d}t \\ &= \sum_{n=-\infty}^{\infty} F_n \int_0^T \cos\left((n-k)\,\omega_0 t\right) + j\sin\left((n-k)\,\omega_0 t\right)\mathrm{d}t \end{aligned} \tag{55}$$

The following relations are verified

$$\int_0^T \cos(m\,\omega_0\,t)\,\mathrm{d}t = \int_0^T \sin(m\,\omega_0\,t)\,\mathrm{d}t = 0, \quad m = 1, 2, 3, ...$$

As

$$\int_0^T \cos(0)\,\mathrm{d}t = T, \quad \text{and} \quad \int_0^T \sin(0)\,\mathrm{d}t = 0,$$

we can write Eq. 55 as

$$\int_0^T e^{-jk\omega_0 t} f_p(t)\mathrm{d}t = \sum_{n=-\infty}^{\infty} F_n\,\delta_{nk} \int_0^T \mathrm{d}t = F_k T.$$

So,

Definition 23 Coefficient F_k can be found by:

$$F_k = \frac{1}{T}\int_0^T e^{-jk\omega_0 t} f_p(t)\mathrm{d}t.$$

Each coefficient is associated with a frequency $k\omega_0$. So we can talk about frequency spectrum of the starting signal $f_p(t)$.

◇ **Example 33. Square wave function**

Find the Fourier coefficients associated with a square wave function.

$$f_p(t) = \begin{cases} -V & -\frac{T_0}{2} < t < 0 \\ +V & 0 < t < \frac{T_0}{2} \end{cases}$$

$$\begin{aligned}
F_k &= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{0} (-V)e^{-jk\omega_0 t}\mathrm{d}t + \frac{1}{T_0}\int_{0}^{\frac{T_0}{2}} (V)e^{-jk\omega_0 t}\mathrm{d}t = \\
&= \frac{1}{T_0}\frac{(-V)}{-jk\omega_0}\left\{\left[e^{-jk\omega_0 t}\right]_{-\frac{T_0}{2}}^{0} - \left[e^{-jk\omega_0 t}\right]_{0}^{\frac{T_0}{2}}\right\} = \\
&= \frac{1}{T_0}\frac{V}{jk\omega_0}\left\{e^0 - e^{\frac{jk\omega_0 T_0}{2}} - e^{\frac{-jk\omega_0 T_0}{2}} + e^0\right\} \\
\omega_0 &= \frac{2\pi}{T_0} \\
F_k &= \frac{1}{T_0}\frac{VT_0}{jk2\pi}\left[2 - \left(e^{jk\pi} + e^{-jk\pi}\right)\right] \\
&= \frac{V}{jk2\pi}\left[2 - \left(e^{jk\pi} + e^{-jk\pi}\right)\right] \\
&= \frac{V}{jk2\pi}\left[2 - 2\cos(k\pi)\right]
\end{aligned}$$

Let us calculate some coefficients:

$$\begin{aligned}
F_0 &= \frac{V}{0j2\pi}(2-2) = 0 \\
F_1 &= \frac{V}{j2\pi}(2-(-2)) = 4\frac{V}{j2\pi} \\
F_2 &= \frac{V}{2j2\pi}(2-2) = 0 \\
F_3 &= \frac{V}{3j2\pi}(2-(-2)) = 4\frac{V}{3j2\pi} \\
F_k &= 4\frac{V}{jk2\pi} \quad k = 2n+1
\end{aligned}$$

We can now evaluate $f(p)$,

$$\begin{aligned}
f_p(t) &= \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \\
F_1 e^{j\omega_0 t} &= 4\frac{V}{\pi}\frac{1}{2j}\left(e^{j\omega_0 t}\right) \\
F_{-1} e^{-j\omega_0 t} &= -4\frac{V}{\pi}\frac{1}{2j}\left(e^{-j\omega_0 t}\right) \\
f_p(t) &= \sum_{n=-1}^{1} F_n e^{jn\omega_0 t} = 4\frac{V}{\pi}\frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) = 4\frac{V}{\pi}\sin(\omega_0 t) \\
f_p(t) &= \sum_{n=-3}^{3} F_n e^{jn\omega_0 t} = \frac{4V}{\pi}\left[\sin(\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t)\right] \\
f_p(t) &= \sum_{n=-5}^{5} F_n e^{jn\omega_0 t} = \frac{4V}{\pi}\left[\sin(\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) + \frac{1}{5}\sin(5\omega_0 t)\right] \\
f_p(t) &= \frac{4V}{\pi}\left[\sin(\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) + \frac{1}{5}\sin(5\omega_0 t) + \frac{1}{7}\sin(7\omega_0 t) + ...\right]
\end{aligned}$$

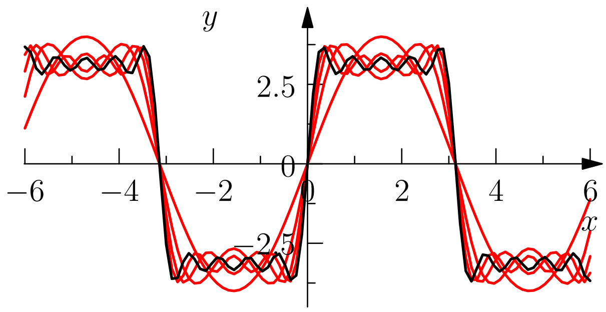

Figure 37: f_p constructed using up to n = 9 (black line) with $\omega_0 = 1$ and $V = \pi$

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7.4 Some properties of Fourier transforms

How can we use sinusoids to build more general non-periodic signals? A non-periodic function $f(t)$ can be thought of as a periodic signal with an infinite period. The Fourier transform is, then, a generalization of the complex Fourier series in the limit as $T_0 \to \infty$. Consequently the sum changes to an integral, and $\omega = \frac{2\pi}{T_0} = \mathrm{d}\omega$ in this limit.

Let $f(t)$ be an arbitrary function of time with Fourier transform $F(w)$, so the equations for the Fourier series become

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\mathrm{d}\omega \tag{56}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, and

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\mathrm{d}t \tag{57}$$

is the Fourier transform.

Let us look into some basic signals that are important for signal processing, such as $\cos(\omega_0 t)$; the short impulse $\delta(t-t_0)$ and the step function $\epsilon(t)$.

Note: In mathematical analysis Fubini's theorem gives conditions to compute a **double integral** using iterated integrals. As a consequence it allows to **change** the order of integration. This is going to be applied in the demonstrations below.

Let us start by finding the dirac δ function in time and Fourier domain.

7.4.1 Impulse function in the time domain, $\delta(t-t_0)$

We start with Eq. 56 where we substitute $F(\omega)$ by [Eq. 57] with $t \to t_0$. So, we have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[F(\omega)]e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t_0)e^{-j\omega t_0}dt_0\right]e^{j\omega t}d\omega$$

We can interchange the order of integration in this equation by writing

$$f(t) = \int_{-\infty}^{\infty} f(t_0)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t_0}e^{j\omega t}d\omega\right)\mathrm{d}t_0\,.$$

Let us recall the selective property of the impulse function:

$$f(t) = \int_{-\infty}^{\infty} f(t_0)\delta(t-t_0)\mathrm{d}t_0\,.$$

Comparing the two previous equations we can write

$$\delta(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t_0}e^{j\omega t}\mathrm{d}\omega$$

and further comparison with Eq. 56 allows to conclude that

$$\delta(t-t_0) = \mathcal{F}^{-1}\{e^{-j\omega t_0}\}$$

So, applying a Fourier transform to each side, we have for the **impulse function**:

$$\mathcal{F}\{A\delta(t-t_0)\} = Ae^{-j\omega t_0}\,.$$

For $t_0 = 0$, $\mathcal{F}\{\delta(t)\} = 1$. The magnitude is equal to the constant A and the phase is $\varphi = -\omega t_0$.

7.4.2 Impulse function in the frequency domain, $\delta(\omega-\omega_0)$

To obtain the impulse function in the **frequency domain**, $\delta(\omega-\omega_0)$, we can proceed as in the previous case. Now we start with Eq. 57 where we substitute $f(t)$ by Eq. 56 with $\omega \to \omega_0$. So, we have

$$F(\omega) = \int_{-\infty}^{\infty} [f(t)] e^{-j\omega t} \mathrm{d}t = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega_0) e^{j\omega_0 t} d\omega_0 \right] e^{-j\omega t} \mathrm{d}t$$

We can interchange the order of integration in this equation by writing

$$F(\omega) = \int_{-\infty}^{\infty} F(\omega_0) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} \mathrm{d}t \right) \mathrm{d}\omega_0$$

This result can be written in the form

$$F(\omega) = \int_{-\infty}^{\infty} F(\omega_0) \left(\delta(\omega-\omega_0) \right) \mathrm{d}\omega_0$$

by introducing

$$\delta(\omega-\omega_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} \mathrm{d}t$$

which comparison with Eq. 57 allows to conclude that

$$\delta(\omega-\omega_0) = \frac{1}{2\pi} \mathcal{F}\{e^{j\omega_0 t}\} \Rightarrow \mathcal{F}\{e^{j\omega_0 t}\} \ = 2\pi\delta(\omega-\omega_0)\,.$$

Alternatively, we can start with the calculation of the inverse Fourier transform of $\delta(\omega-\omega_0)$ using Eq. 56. In this way we obtain successively

$$\mathcal{F}^{-1}\{\delta(\omega-\omega_0)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

So, applying a Fourier transform to each side, we obtain the same result

$$\mathcal{F}\{e^{j\omega_0 t}\} \ = 2\pi\delta(\omega-\omega_0)$$

which shows the coherence of the treatment.

Note 1: The delta function satisfies to the condition $\delta(x-x') = \delta(x'-x)$. In fact, both forms gives the same result when applied to a general function $f(x)$:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x') dx = f(x')$$

and

$$\int_{-\infty}^{\infty} f(x)\,\delta(x'-x) dx = f(x')$$

Note 2: The convolution of two delta functions is easily calculated. For instance (see Exercise 5.1b)

$$\delta(\omega-10) * \delta(\omega+11) = \int_{-\infty}^{\infty} \delta(\omega_0-10)\,\delta(\omega-\omega_0+11) d\omega_0 = \delta(\omega+1)$$

In fact, we must have, for instance, $\omega_0 = \omega+11$ in order to have a non-zero integral.

7.4.3 Fourier transform of $\dot{\delta}(t)$

$$\mathcal{F}\{\dot{\delta}(t)\} = \int_{-\infty}^{\infty} \dot{\delta}(t) e^{-j\omega t} \mathrm{d}t.$$

We can start by realizing that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\delta(t)e^{-j\omega t}\right] = \dot{\delta}(t)e^{-j\omega t} - j\omega\, e^{-j\omega t}\,\delta(t)$$

Replacing $\dot{\delta}(t)e^{-j\omega t}$ in the definition above we get,

$$\begin{aligned}\mathcal{F}\{\dot{\delta}(t)\} &= \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\left[\delta(t)e^{-j\omega t}\right] \mathrm{d}t + j\omega \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\,[1] + j\omega = j\omega\end{aligned}$$

This conclusion could be obtained directly using the following property:

$$\int_{-\infty}^{\infty} f(t)\dot{\delta}(t-t_0)\,\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}t}f(t)\bigg|_{t=t_0}.$$

Other identities involving the derivative of the delta function are:

$$\int_{-\infty}^{\infty} \dot{\delta}(t)\mathrm{d}t = 0$$

$$\int_{-\infty}^{\infty} f(t)\ddot{\delta}(t-t_0)\,\mathrm{d}t = +\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)|_{t=t_0}$$

7.4.4 Fourier transform of sine and cosine functions

These transformations follow directly from the previous properties.

$$\begin{aligned}\mathcal{F}\left\{\cos(\omega_0 t)\right\} &= \mathcal{F}\left\{\frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right\} = \frac{1}{2}\left(2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right) \\ &= \pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)\end{aligned}$$

$$\mathcal{F}\left\{\sin(\omega_0 t)\right\} = \frac{\pi}{j}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$$

7.4.5 Fourier transform of the step function, $\epsilon(t)$

The step function can be decomposed in an even and an odd component and written as the sum of the two functions according to,

$$\epsilon(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)$$

Using the linearity property (Section 7.5.1) and the Fourier transforms of the unit and sgn(t) shown in Fig. 38 we can write,

$$\mathcal{F}\left\{\epsilon(t)\right\} = \frac{1}{2}\mathcal{F}\left\{1\right\} + \frac{1}{2}\mathcal{F}\left\{\mathrm{sgn}(t)\right\} = \pi\delta(\omega) + \frac{1}{j\omega} \tag{58}$$

7.4.6 Fourier transform of a rectangular function, $\mathrm{rect}\left(\frac{\mathrm{t}}{\tau}\right)$

The function is defined as

$$\mathrm{rect}\left(\frac{\mathrm{t}}{\tau}\right) = \epsilon\left(\mathrm{t} + \frac{\tau}{2}\right) - \epsilon\left(\mathrm{t} - \frac{\tau}{2}\right)$$

So,

$$\mathrm{rect}\left(\frac{\mathrm{t}}{\tau}\right) = \begin{cases} 1, & \text{for } |t| \leq \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\mathcal{F}\left\{\mathrm{rect}\left(\frac{\mathrm{t}}{\tau}\right)\right\} = \int_{-\infty}^{\infty} e^{-j\omega\, t}\epsilon\left(t + \frac{\tau}{2}\right)\mathrm{d}t - \int_{-\infty}^{\infty} e^{-j\omega\, t}\epsilon\left(t - \frac{\tau}{2}\right)\mathrm{d}t$$

Performing the change of variable $t + \frac{\tau}{2} = t'$ in the first integral and $t - \frac{\tau}{2} = t'$ in the second one, we can write

$$\mathcal{F}\left\{\mathrm{rect}\left(\frac{\mathrm{t}}{\tau}\right)\right\} = e^{j\omega\,\frac{\tau}{2}}\int_{-\infty}^{\infty} e^{-j\omega\, t'}\epsilon(t')\mathrm{d}t' - e^{-j\omega\,\frac{\tau}{2}}\int_{-\infty}^{\infty} e^{-j\omega\, t'}\epsilon(t')\mathrm{d}t'$$

Now we can use the Eq. 58, which defines the Fourier transform of the step function, allowing for

$$\mathcal{F}\left\{\mathrm{rect}\left(\frac{\mathrm{t}}{\tau}\right)\right\} = e^{j\omega\,\frac{\tau}{2}}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) - e^{-j\omega\,\frac{\tau}{2}}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)$$

$$\mathcal{F}\left\{\mathrm{rect}\left(\frac{\mathrm{t}}{\tau}\right)\right\} = \pi\delta(\omega)\left(e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right) + \frac{1}{j\omega}\left(e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right)$$

With the properties of the delta function, $\delta(\omega-\omega_0)f(t) = \delta(\omega-\omega_0)f(\omega_0)$, and the Euler relations we obtain,

$$\begin{aligned} \mathcal{F}\left\{\mathrm{rect}\left(\frac{\mathrm{t}}{\tau}\right)\right\} &= \pi\delta(\omega)\left(e^{0} - e^{0}\right) + \frac{1}{j\omega}\left(e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right) \\ &= \frac{1}{j\omega}\left(e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right) \\ &= \frac{2}{\omega}\sin\left(\omega\frac{\tau}{2}\right) = \tau\frac{\sin\left(\omega\frac{\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right) \end{aligned}$$

7.4.7 Fourier transform of the gaussian function, $e^{-a^2t^2}$

$$\mathcal{F}\{e^{-a^2t^2}\} = \int_{-\infty}^{\infty} e^{-a^2t^2}\, e^{-j\omega t}\mathrm{d}t = \int_{-\infty}^{\infty} e^{-(a^2t^2 + j\omega t)}\mathrm{d}t$$

We can write

$$(a^2t^2 + j\omega\, t) = \left(a\,t + \frac{j\omega}{2a}\right)^2 + \frac{\omega^2}{4a^2}$$

So,

$$\mathcal{F}\{e^{-a^2t^2}\} = \int_{-\infty}^{\infty} e^{-(a\,t + \frac{j\omega}{2a})^2}\, e^{-\frac{\omega^2}{4a^2}}\,\mathrm{d}t = e^{-\frac{\omega^2}{4a^2}}\int_{-\infty}^{\infty} e^{-(a\,t + \frac{j\omega}{2a})^2}\,\mathrm{d}t$$

The change of variable: $a\,t + \frac{j\omega}{2a} = t' \Rightarrow a\,\mathrm{d}t = \mathrm{d}t'$, and the result

$$\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}$$

allows to obtain

$$\mathcal{F}\{e^{-a^2t^2}\} = \frac{\sqrt{\pi}}{a}\, e^{-\frac{\omega^2}{4a^2}}.$$

The Fourier transforms of these and other functions are summarised in Figure 38.

$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\dot{\delta}(t)$	$j\omega$
$\frac{1}{T}\text{Ш}\left(\frac{t}{T}\right)$	$\text{Ш}\left(\frac{\omega T}{2\pi}\right)$
$\varepsilon(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\text{rect}(at)$	$\frac{1}{\|a\|}\text{si}\left(\frac{\omega}{2a}\right)$
$\text{si}(at)$	$\frac{\pi}{\|a\|}\text{rect}\left(\frac{\omega}{2a}\right)$
$\frac{1}{t}$	$-j\pi\text{sign}(\omega)$
$\text{sign}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos(\omega_0 t)$	$\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$
$\sin(\omega_0 t)$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
$e^{-\alpha\|t\|}$, $\alpha > 0$	$\frac{2\alpha}{\alpha^2 + \omega^2}$
$e^{-a^2 t^2}$	$\frac{\sqrt{\pi}}{a} e^{-\frac{\omega^2}{4a^2}}$

Figure 38: Fourier transform pairs.

7.5 Fourier transform properties

Let us go through some important properties of the Fourier transforms. There are some that we will not cover now but can be found in Figure 39.

7.5.1 Linearity of the Fourier Transform

It follows directly from the linearity of the integration that the principle of superposition applies to the Fourier transform and its inverse:

$$\mathcal{F}\left\{af(t) + bg(t)\right\} = a\mathcal{F}\left\{f(t)\right\} + b\mathcal{F}\left\{g(t)\right\}$$

$$\mathcal{F}^{-1}\left\{cF(j\omega) + dG(j\omega)\right\} = c\mathcal{F}^{-1}\left\{F(j\omega)\right\} + d\mathcal{F}^{-1}\left\{G(j\omega)\right\} ,$$

where a, b, c, and d can be any real or complex constants.

Example 2. The Fourier transform of a delta impulse pair can be obtained as the sum of the transforms of two individual impulses that are each shifted by $\pm\tau$:

$$\delta(t+\tau) + \delta(t-\tau) \quad \circ\!\!-\!\!\bullet \quad e^{-j\omega\tau} + e^{j\omega\tau} = 2\cos\omega\tau\,.$$

	$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(j\omega) + BX_2(j\omega)$
Delay	$x(t-\tau)$	$e^{-j\omega\tau}X(j\omega)$
Modulation	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
'Multiplication by t' Differentiation in the frequency domain	$tx(t)$	$-\dfrac{dX(j\omega)}{d(j\omega)}$
Differentiation in the time domain	$\dfrac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$X(j\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$ $= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right), \quad a \in \mathbb{R}\backslash\{0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega)\cdot X_2(j\omega)$
Multiplication	$x_1(t)\cdot x_2(t)$	$\frac{1}{2\pi}X_1(j\omega) * X_2(j\omega)$
Duality	$x_1(t)$ $x_2(jt)$	$x_2(j\omega)$ $2\pi x_1(-\omega)$
Symmetry relations	$x(-t)$ $x^*(t)$ $x^*(-t)$	$X(-j\omega)$ $X^*(-j\omega)$ $X^*(j\omega)$
Parseval theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\, dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$

Figure 39: Properties of the Fourier transforms.

7.5.2 Convolution in the Fourier domain

Problem 1. Show that the following property is verified,

$$\mathcal{F}\{f * g\} \;=\; F(\omega)G(\omega)\,.$$

We start with the definition of convolution

$$f * g \;=\; \int_{-\infty}^{\infty} f(\tau)[g(t-\tau)]\mathrm{d}\tau$$

Applying Eq. 56 to $[g(t-\tau)]$ we can write

$$f * g \;=\; \int_{-\infty}^{\infty} f(\tau)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega(t-\tau)}\mathrm{d}\omega\right]\mathrm{d}\tau$$

We can interchange the order of integration in this equation by writing

$$f * g \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\left(\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}\mathrm{d}\tau\right)e^{j\omega t}\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\,F(\omega)e^{j\omega t}\mathrm{d}\omega$$

In this expression we have used Eq. 57 and we can introduce Eq. 56 in the last step. So,

$$f * g \;=\; \mathcal{F}^{-1}\{G(\omega)\,F(\omega)\}$$

So, applying a Fourier transform to each side, we have finally

$$\mathcal{F}\{f * g\} \;=\; F(\omega)\,G(\omega)$$

Problem 2. Show that the following properties is verified

$$\mathcal{F}\{f(t)g(t)\} \;=\; \frac{1}{2\pi}F(\omega) * G(\omega)$$

We start using Eq. 57,

$$\mathcal{F}\{f(t)g(t)\} \;=\; \int_{-\infty}^{\infty} f(t)g(t)e^{-j\omega\, t}\mathrm{d}t$$

Substituting $f(t)$ by using Eq. 56 we can write

$$\mathcal{F}\{f(t)g(t)\} \;=\; \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega_0)e^{j\omega_0\, t}\mathrm{d}\omega_0\right] g(t)e^{-j\omega\, t}\mathrm{d}t$$

By interchanging the order of integration in this equation we write

$$\mathcal{F}\{f(t)g(t)\} \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega_0)\left(\int_{-\infty}^{\infty} g(t)e^{-j(\omega-\omega_0)t}\mathrm{d}t\right)\mathrm{d}\omega_0$$

From Eq. 57 we can write

$$G(\omega-\omega_0) = \int_{-\infty}^{\infty} g(t)e^{-j(\omega-\omega_0)t}\mathrm{d}t$$

Consequently we have

$$\mathcal{F}\{f(t)g(t)\} \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega_0)G(\omega-\omega_0)d\omega_0 = \frac{1}{2\pi}F(\omega) * G(\omega)$$

where we have used the concept of convolution in the last step.

Note: There are other methods to prove this type of relations.

7.6 Integration theorem

Consider the convolution with $\epsilon(t)$:

$$f(t) * \epsilon(t) = \int_{-\infty}^{\infty} \epsilon(t)f(t-\tau)\mathrm{d}\tau = \int_{0}^{\infty} f(t-\tau)\mathrm{d}\tau$$

Here we consider the time-shift associated with the function f (again, convolution is commutative). This shows that convolution with the unit step function, $\epsilon(t)$, corresponds to an integration in the time-domain.

Applying a Fourier transform to each side we obtain,

$$\mathcal{F}\{f(t) * \epsilon(t)\} = F(\omega)E(\omega) = \mathcal{F}\left\{\int_{0}^{\infty} f(t-\tau)d\tau\right\}$$

Performing a change of variable $t-\tau = t'$ ($\mathrm{d}\tau = -\mathrm{d}t'$) in the integral we can write

$$\int_0^{\infty} f(t-\tau)\mathrm{d}\tau = \int_t^{-\infty} f(t')(-\mathrm{d}t') = \int_{-\infty}^{t} f(t')\mathrm{d}t' = \int_{-\infty}^{t} f(\tau)\mathrm{d}\tau$$

Using this result and Eq. 58 in the equation above we can write

$$F(\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] = \mathcal{F}\left\{\int_{-\infty}^{t} f(\tau)\mathrm{d}\tau\right\}$$

Consequently,

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(\tau)\mathrm{d}\tau\right\} = \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

7.6.1 Differentiation theorem

There is a differential theorem for both the time-domain and the frequency-domain. Differentiation in the time-domain corresponds to convolution with $\dot{\delta}(t)$:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t) * \dot{\delta}(t)\,.$$

With the transform pair $\mathcal{F}\left\{\dot{\delta}(t)\right\} = j\omega$ and the convolution property we obtain the differentiation theorem:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \quad \circ\!\!-\!\!\bullet \quad j\omega X(\omega)\,.$$

Note that it can be derived from the differentiation theorem of the bilateral Laplace transform by replacing s with $j\omega$. Again, this requires $x(t)$ to be differentiable.

The differentiation theorem in the frequency-domain corresponds to a multiplication of the time signal with $-t$:

$$-tx(t) \quad \circ\!\!-\!\!\bullet \quad \frac{\mathrm{d}X(\omega)}{\mathrm{d}(\omega)}\,.$$

It can be demonstrated by differentiating the equation that defines the Fourier transform (Eq. 57) with respect to $j\omega$. Note that it must be possible to differentiate $X(\omega)$.

7.6.2 Parseval's theorem and energy spectra

The Parseval relationships state that the energy in time-domain representation of a signal is equal to the energy in the frequency-domain representation, normalized by 2π; that is, for a signal $x(t)$ with finite energy and corresponding Fourier transform $X(\omega)$, the energy is conserved when going from the time to the frequency-domain - **Parseval's Energy Conservation Theorem**

$$
\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t &= \int_{-\infty}^{\infty} x(t)x^*(t)\mathrm{d}t = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\mathrm{d}\omega \right]^* \mathrm{d}t \\
&= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t}\mathrm{d}\omega \right] \mathrm{d}t \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t \right] \mathrm{d}\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)X(\omega)\mathrm{d}\omega \\
\int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, \mathrm{d}\omega
\end{aligned}
$$

The quantity $|X(\omega)|^2$ plotted against ω is termed the *energy spectrum* of the signal.

7.7 FT : short summary

Fourier Series

$$f_p(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_0^T f_p(t)e^{-jn\omega_0 t}\mathrm{d}t$$

Fourier Transform

$$f(t) = \mathcal{F}^{-1}\left\{F(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\mathrm{d}\omega$$

$$F(\omega) = \mathcal{F}\left\{f(t)\right\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\mathrm{d}t$$

For $T_0 \to \infty$ then $\frac{2\pi}{T_0} = \mathrm{d}\omega$

In the text book a Fourier transform if a function is written as $X(j\omega)$ and not $X(\omega)$. This is to highlight the relationship between FT and LT. FT of a function of time is the same as its Laplace transform on the imaginary axis $s = j\omega$ of the complex plane (this is true is the LT exists as well as FT). FT can exist without LT. FT only deals with "oscilating" parts of the signals and not with e^{st} in general.

1. Impulse function

$$\mathcal{F}\left\{A\delta(t-t_0)\right\} = \int_{-\infty}^{\infty} A\delta(t-t_0)e^{-j\omega t}\mathrm{d}t = Ae^{-j\omega t_0}$$

   So the magnitude is equal to the constant A and the phase is $\varphi = -\omega t_0$. For $\mathcal{F}\left\{\delta(t)\right\} = 1$, as $t_0 = 0$

2. Impulse function in the frequency domain $\delta(\omega - \omega_0)$

$$\mathcal{F}^{-1}\left\{\delta(\omega-\omega_0)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{j\omega t}\mathrm{d}\omega = \frac{1}{2\pi}e^{j\omega_0 t}$$

   So,

$$\mathcal{F}\left\{e^{j\omega_0 t}\right\} = 2\pi\delta(\omega-\omega_0)$$
$$e^{j\omega_0 t} \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega-\omega_0)$$

3. sine and cosine functions

$$\begin{aligned}\mathcal{F}\left\{\cos(\omega_0 t)\right\} &= \mathcal{F}\left\{\frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right\} = \frac{1}{2}\left(2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right) = \\ &= \pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)\end{aligned}$$

$$\mathcal{F}\left\{\sin(\omega_0 t)\right\} = \frac{\pi}{j}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$$

4. step function $\epsilon(t)$

$$\mathcal{F}\left\{\epsilon(t)\right\} = \pi\delta(\omega) + \frac{1}{j\omega}$$

5. window function $\mathrm{rect}\left(\frac{\mathrm{t}}{\tau}\right)$ Use the fact that time shift by γ gives a factor $e^{-j\omega\gamma}$ in the $\mathcal{F}\left\{\right\}$

$$\begin{aligned}
\mathrm{rect}\left(\frac{\mathrm{t}}{\tau}\right) &= \epsilon(\mathrm{t}+\frac{\tau}{2}) - \epsilon(\mathrm{t}-\frac{\tau}{2}) \\
\mathcal{F}\left\{\mathrm{rect}\left(\frac{\mathrm{t}}{\tau}\right)\right\} &= e^{j\omega\frac{\tau}{2}}\left(\pi\delta(\omega)+\frac{1}{j\omega}\right) - e^{-j\omega\frac{\tau}{2}}\left(\pi\delta(\omega)+\frac{1}{j\omega}\right) \\
&= \pi\delta(\omega)\left(e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right) + \frac{1}{j\omega}\left(e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right)
\end{aligned}$$

using the fact that $\delta(t-t_0)f(t) = \delta(t-t_0)f(t_0)$

$$\begin{aligned}
\mathcal{F}\left\{\mathrm{rect}\left(\frac{\mathrm{t}}{\tau}\right)\right\} = \pi\delta(\omega)\left(e^0 - e^0\right) + \frac{1}{j\omega}\left(e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right) &= \\
= \frac{1}{j\omega}\left(e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right) &= \\
= \frac{2}{\omega}\sin\left(\omega\frac{\tau}{2}\right) = \tau\frac{\sin\left(\omega\frac{\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} &= \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)
\end{aligned}$$

Differentiation theorem:

$$\begin{array}{ccc}
\frac{\mathrm{d}x(t)}{\mathrm{d}t} & = & x(t) * \dot{\delta}(t) \\
\dot{\delta}(t) & \circ\!\!-\!\!\bullet & j\omega \\
\frac{\mathrm{d}x(t)}{\mathrm{d}t} & \circ\!\!-\!\!\bullet & j\omega X(\omega) \\
-tx(t) & \circ\!\!-\!\!\bullet & \frac{\mathrm{d}X(\omega)}{\mathrm{d}(\omega)}
\end{array}$$

Integration theorem:

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(\tau)\mathrm{d}\tau\right\} = \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

7.8 Duality

$$\begin{array}{ccc}
f_1(t) & \circ\!\!-\!\!\bullet & f_2(\omega) \\
f_2(t) & \circ\!\!-\!\!\bullet & 2\pi f_1(-\omega)
\end{array}$$

For example

$$\begin{array}{ccc}
\mathrm{rect}(t) & \circ\!\!-\!\!\bullet & \mathrm{sinc}\left(\frac{\omega}{2}\right) \\
\mathrm{sinc}\left(\frac{\mathrm{t}}{2}\right) & \circ\!\!-\!\!\bullet & 2\pi\,\mathrm{rect}(-\omega)
\end{array}$$

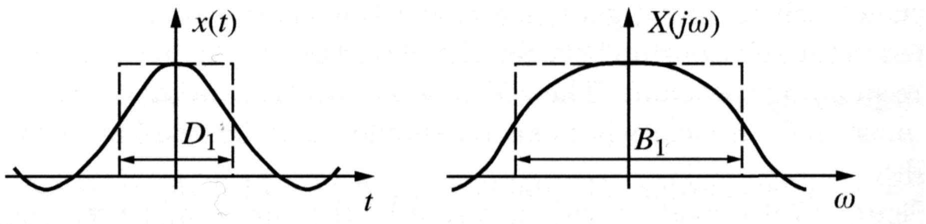

Figure 40:

7.9 Uncertainty Principle and time-bandwidth product

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)\mathrm{d}t &= D_1 x(0) \\
D_1 &= \frac{1}{x(0)}\int_{-\infty}^{\infty} x(t)\mathrm{d}t = \frac{1}{x(0)}X(0) \\
X(0) &= \mathcal{F}\left\{x(t)\right\}\big|_{\omega=0} = \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t\right]\Bigg|_{\omega=0} = \int_{-\infty}^{\infty} x(t)\mathrm{d}t \\
D_1 &= \frac{X(0)}{x(0)}
\end{aligned}$$

In the frequency domain:

$$\begin{aligned}
B_1 &= \frac{1}{X(0)}\int_{-\infty}^{\infty} X(\omega)\mathrm{d}\omega = 2\pi x(0) \\
x(0) &= \mathcal{F}^{-1}\left\{X(\omega)\right\}\big|_{t=0} = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\mathrm{d}\omega\right]\Bigg|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\mathrm{d}\omega \\
B_1 &= 2\pi\frac{x(0)}{X(0)}
\end{aligned}$$

Then

$$D_1 B_1 = 2\pi$$

This is true for the definition of the duration and band-width we have choosen here (equal area rectangles for all real and symmetrical time signals).

A different possibility to define duration and bandwidth is to use tolerance. See Figure 41. Here, outside the duration D_2 the magnitude of the signal $x(t)$ is always less then q. Similar is true for bandwidth B_2.

$$\begin{aligned}
|x(t)| \leq q \cdot max(x(t)) \quad \forall t \notin [t_0, t_0 + D_2] \\
|X(\omega)| \leq q \cdot max(X(\omega)) \quad \forall |\omega| > \frac{B_2}{2}
\end{aligned}$$

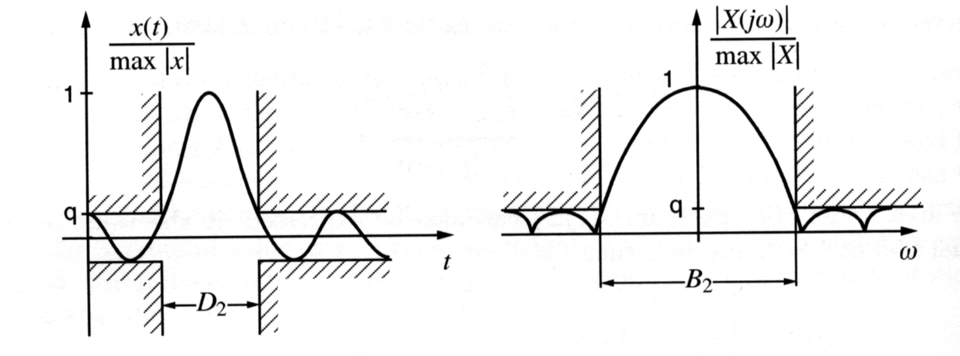

Figure 41: duration and bandwidth defined using tolerance

For a Gaussian impulse $x(t) = e^{-\alpha^2 t^2}$:

$$x(t) = e^{-\alpha^2 t^2}$$
$$\mathcal{F}\left\{e^{-\alpha^2 t^2}\right\} = \frac{\sqrt{\pi}}{\alpha} e^{-\frac{\omega^2}{4\alpha^2}}$$
$$e^{-\alpha^2 \frac{D_2^2}{2^2}} = q$$
$$D_2 = \frac{2}{\alpha}\sqrt{-\ln q}$$

and

$$\frac{\sqrt{\pi}}{\alpha} e^{-\frac{(B_2/2)^2}{4\alpha^2}} = q\frac{\sqrt{\pi}}{\alpha} B_2 = 4\alpha\sqrt{-\ln q}$$

Then

$$D_2 B_2 = -8\ln q$$

Which is again constant for defined q. For small q and given D, B_2 will be large.

The third method of defining define duration and bandwidth is based on the 2nd order moments for the magnitude square of the signal.

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t = \int_{-\infty}^{\infty} \frac{1}{2\pi} |F(\omega)|^2 \,\mathrm{d}\omega$$

$$\hat{t} = \frac{1}{E}\int_{-\infty}^{\infty} t\,|f(t)|^2 \,\mathrm{d}t$$
$$\hat{\omega} = \frac{1}{2\pi E}\int_{-\infty}^{\infty} \omega\,|F(\omega)|^2 \,\mathrm{d}\omega$$

$$(\Delta t)^2 = \frac{1}{E}\int_{-\infty}^{\infty} (t-\hat{t})^2\,|f(t)|^2 \,\mathrm{d}t$$
$$(\Delta \omega)^2 = \frac{1}{2\pi E}\int_{-\infty}^{\infty} (\omega-\hat{\omega})^2\,|F(\omega)|^2 \,\mathrm{d}\omega$$

$$\Delta t \Delta \omega \geq \frac{1}{2}$$

Or as defined win the textbook (p235)

$$D_3 B_3 \geq \sqrt{\frac{\pi}{2}}$$

where

$$D_3^2 = \frac{1}{E} \int_{-\infty}^{\infty} (t - \hat{t})^2 \left| f(t) \right|^2 \mathrm{d}t$$

$$B_3^2 = \frac{1}{E} \int_{-\infty}^{\infty} (\omega - \hat{\omega})^2 \left| F(\omega) \right|^2 \mathrm{d}\omega$$

It can also be shown that $\Delta t \Delta \omega = \min$ for a Gaussian signal.

8 Application of FT in filter design

Most important of application of LTI system is in filtering, when we want to remove part of the signal. If our signal consists of noise $\eta(t)$ and signal containing information $x(t)$, $y(t) = x(t) + n(t)$, and the spectral characteristics of both signals are know. The idea is then to design a LTI system that will get rid of as much noise as possible. This is achieved by finding a transfer function $F(s)$ of our filter system that will do the job, that is reduce $n(t)$ and effect $x(t)$ as little as possible.

$$H_{tot}(s) = H(s)F(s)$$

The filter can clean up frequency spectra of the input signal, or select one frequency from the output for further processing/measurements.

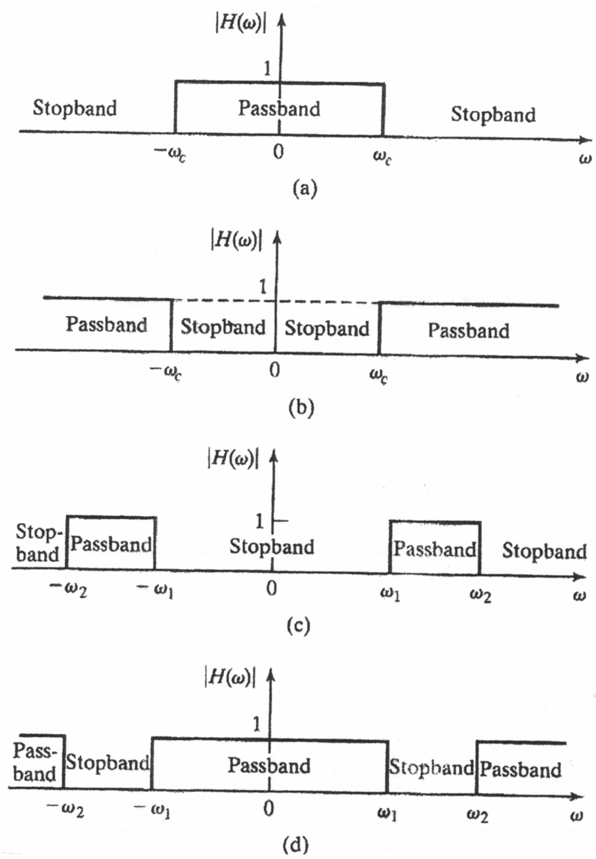

Figure 42: Ideal Filters

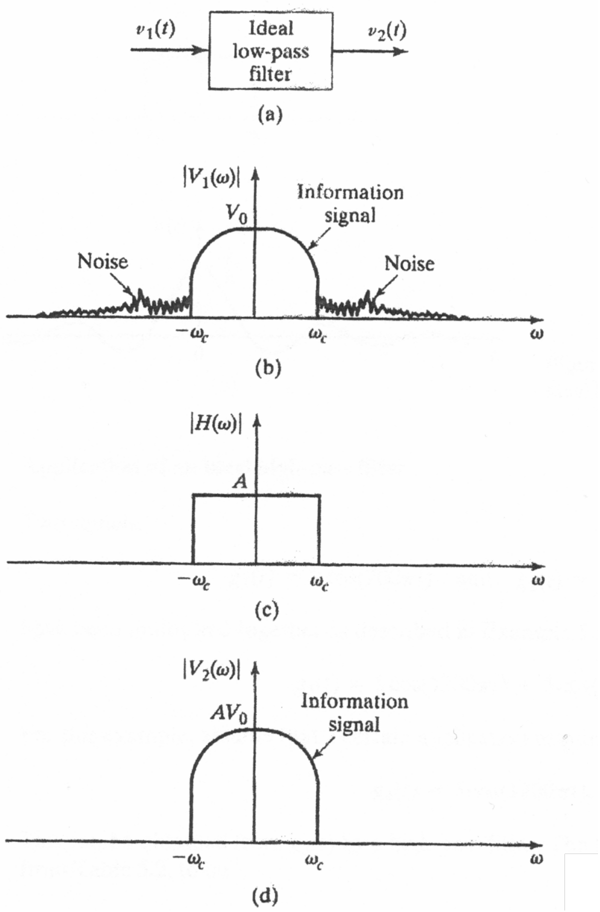

Figure 43:

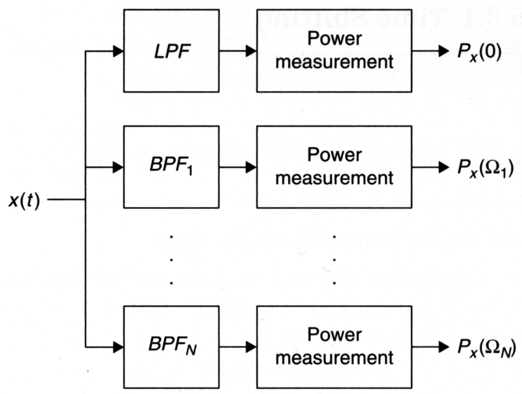

Figure 44:

8.1 How to make a filter

If we build a system with a transfer function with a general form:

$$F(s) = \frac{1}{1+s}$$

then

$$F(\omega) = \frac{1}{1+j\omega}$$

And this will be low pass filter, as $|F(\omega \to \infty)| = 0$ and $|F(0)| > 0$

Similarly, we can think of a high pass filter, where:

$$F(s) = \frac{1}{1+1/s}$$
$$F(\omega) = \frac{1}{1+1/j\omega}$$

How to implement this? If we consider a system shown below:

Then

$$v_i(t) = Ri(t) + \frac{1}{C}\int_0^t i(\tau)\mathrm{d}\tau$$
$$V_i(s) = RI(s) + \frac{1}{sC}I(s)$$
$$\frac{V_i(s)}{R+1/sC} = I(s)$$
$$v_o(t) = \frac{1}{C}\int_0^t i(\tau)\mathrm{d}\tau$$
$$V_o(s) = \frac{1}{sC}I(s) = \frac{1}{sC}\frac{V_i(s)}{R+1/sC}$$
$$V_o(s) = V_i(s)\frac{1}{1+sRC}$$
$$H(s) = \frac{V_o}{V_i} = \frac{1}{1+sRC}$$
$$H(\omega) = \frac{1}{1+j\omega RC}$$

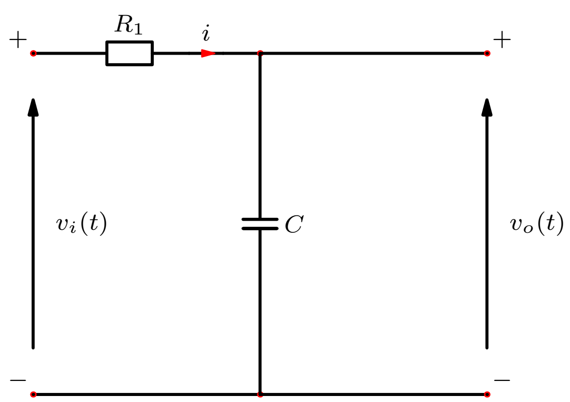

Figure 45:

Now we define ω_C

$$\begin{aligned}
\omega_C &= \frac{1}{RC} \\
H(\omega) &= \frac{1}{1 + j\frac{\omega}{\omega_C}} \\
H(\omega) = |H(\omega)|e^{j\varphi} &= \frac{1 - j\frac{\omega}{\omega_C}}{1 + \frac{\omega^2}{\omega_C^2}} \\
|H(\omega)| = \frac{\sqrt{1 + \frac{\omega^2}{\omega_C^2}}}{1 + \frac{\omega^2}{\omega_C^2}} &= \frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_C^2}}} \\
\tan\varphi &= -\frac{\omega}{\omega_C}
\end{aligned}$$

For $\omega = \omega_C$ we het cut off or half power frequency:

$$|H(\omega = \omega_C)| = \frac{1}{\sqrt{2}} = -3\text{dB}$$

A measure of attenuation is given by the loss function in decibels, defined as:

$$\alpha(\omega) = -10\log|H(j\omega)|^2 = -20\log_{10}|H(j\omega)|$$

10^5 difference corresponds to 100dB, difference of $1/\sqrt{2} = 0.71.$ corresponds to 3dB.

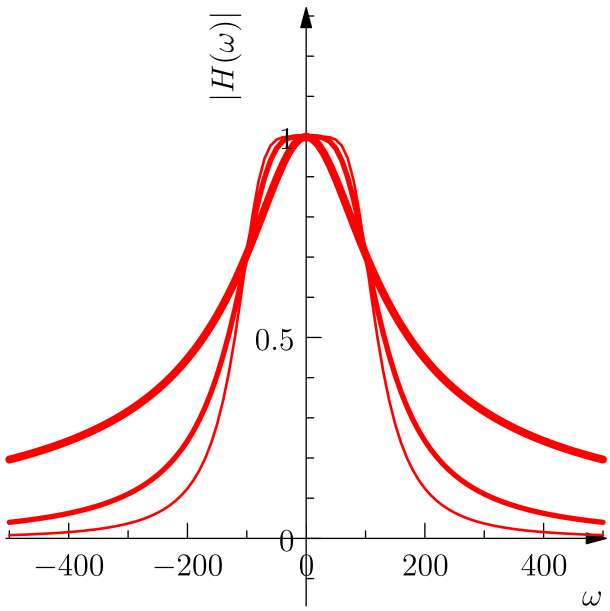

Figure 46: Low pass filter with $\omega_C = 100$

Can we make it better? 2nd order filter with $H(s) \propto s^{-2}$ could improve things. We have to use several energy storing compartments which will respond in a different way to different frequencies.

Now

$$
\begin{aligned}
V_i(s) &= RI(s) + sLI(s) + \frac{1}{sC}I(s) \\
I(s) &= \frac{V_i(s)}{R + sL + \frac{1}{sC}} \\
V_o(s) = \frac{1}{sC}I(s) &= \frac{V_i(s)}{R + sL + \frac{1}{sC}}\frac{1}{sC} \\
\frac{V_o}{V_i} = \frac{1}{R + sL + \frac{1}{sC}}\frac{1}{sC} &= \frac{1}{sCR + s^2LC + 1} \\
\frac{V_o}{V_i} &= \frac{\frac{1}{LC}}{s^2 + s\frac{R}{L} + \frac{1}{LC}}
\end{aligned}
$$

Now:

$$
\begin{aligned}
\omega_n &= \frac{1}{\sqrt{LC}} \\
H(s) &= \frac{\omega_n^2}{s^2 + R\sqrt{\frac{C}{L}}\omega_n s + \omega_n^2} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\end{aligned}
$$

We define the cut of frequency (half power frequency) ω_c and let :

$$
\begin{aligned}
L &= \frac{R}{\omega_c\sqrt{2}} \\
C &= \frac{\sqrt{2}}{\omega_c R}
\end{aligned}
$$

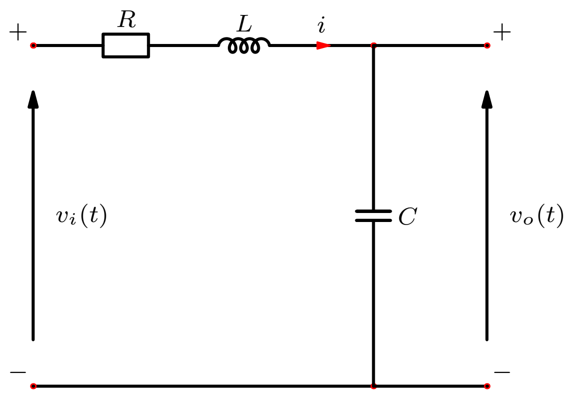

Figure 47:

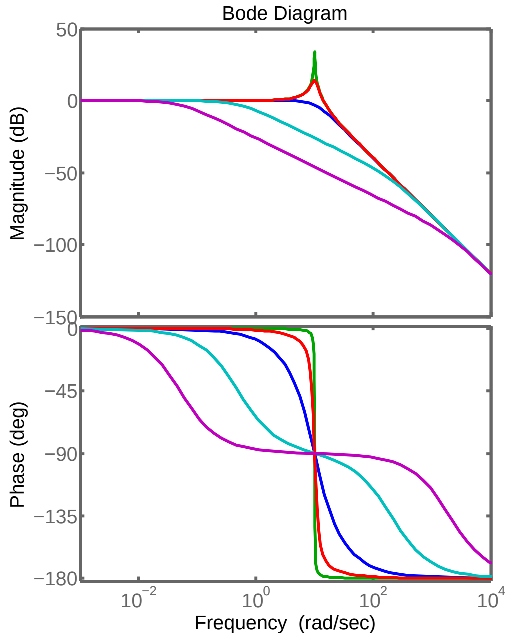

Figure 48: filter response as a function of ζ; $w_n = 10$. For blue curve, $\zeta = \frac{1}{\sqrt{2}}$ and $\omega_n = \omega_c$

Then

$$
\begin{aligned}
\omega_n &= \frac{1}{\sqrt{LC}} = \left(\frac{R}{\omega_c\sqrt{2}}\right)^{-0.5}\left(\frac{\sqrt{2}}{\omega_c R}\right)^{-0.5} = \omega_c \\
\zeta &= \frac{R}{2}\sqrt{\frac{C}{L}} \\
\frac{C}{L} &= \frac{\sqrt{2}}{\omega_c R}\frac{\omega_c\sqrt{2}}{R} = \frac{2}{R^2} \\
\zeta &= \frac{R}{2}\sqrt{\frac{2}{R^2}} = \frac{1}{\sqrt{2}}
\end{aligned}
$$

Now:

$$
H(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}
$$
$$
s = j\omega
$$

$$
H(\omega) = \frac{\omega_c^2}{-\omega^2 + \sqrt{2}j\omega_c\omega + \omega_c^2} = \frac{\omega_c^2}{\omega_c^2 - \omega^2 + \sqrt{2}j\omega_c\omega} =
$$
$$
= \frac{\omega_c^2}{\sqrt{(\omega_c^2 - \omega^2)^2 + 2\omega_c^2\omega^2}}e^{j\varphi(\omega)} = \frac{\omega_c^2}{\sqrt{\omega_c^4 + \omega^4 - 2\omega_c^2\omega^2 + 2\omega_c^2\omega^2}}e^{j\varphi(\omega)} =
$$
$$
= \frac{\omega_c^2}{\sqrt{\omega_c^4 + \omega^4}}e^{i\varphi(\omega)} = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^4}}e^{j\varphi(\omega)} = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2N}}}e^{j\varphi(\omega)}
$$

This is a 2nd order Butterworth filter. Above we have already described 1st order filter.

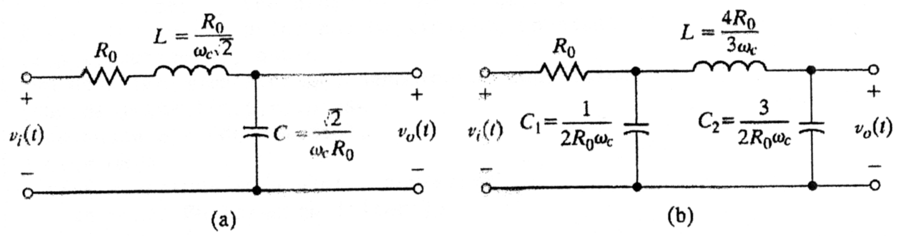

Figure 6.12 Butterworth filters.

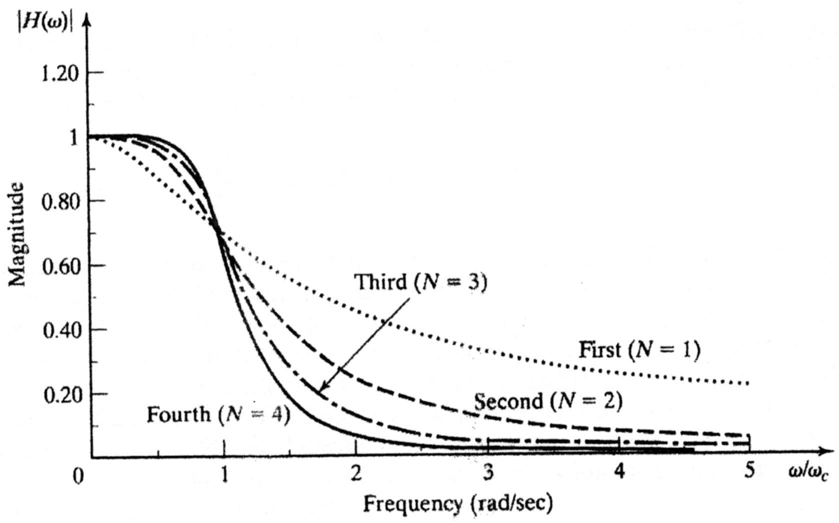

Figure 6.13 Frequency spectra of Butterworth filters.

8.2 High pass filter

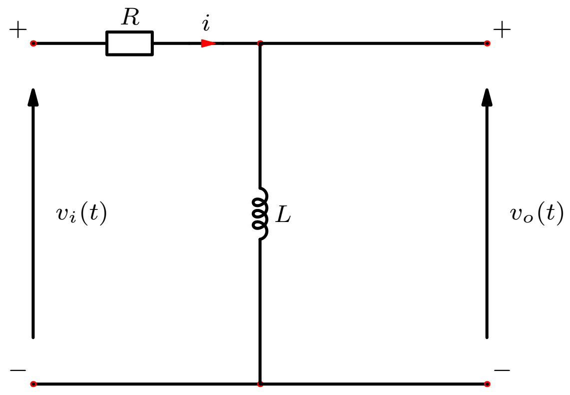

Figure 49: High Pass Filter

$$\frac{V_i}{V_o} = \frac{sL}{R+sL} = \frac{1}{1+\frac{R}{sL}} = \frac{1}{1+\frac{R}{j\omega L}}$$
$$\frac{R}{L} = \omega_L$$
$$H(\omega) = \frac{1}{1-j\frac{\omega_L}{\omega}}$$
$$H(\omega) = \frac{1+j\frac{\omega_L}{\omega}}{1+\left(\frac{\omega_L}{\omega}\right)^2}$$
$$|H(\omega)| = \sqrt{\frac{1}{\left(1+\left(\frac{\omega_L}{\omega}\right)^2\right)^2} + \frac{\frac{\omega_L^2}{\omega^2}}{\left(1+\left(\frac{\omega_L}{\omega}\right)^2\right)^2}} = $$
$$= \frac{1}{\sqrt{1+\left(\frac{\omega_L}{\omega}\right)^2}}$$

For $\omega = \omega_L$ we get $|H(\omega)| = \frac{1}{\sqrt{2}}$

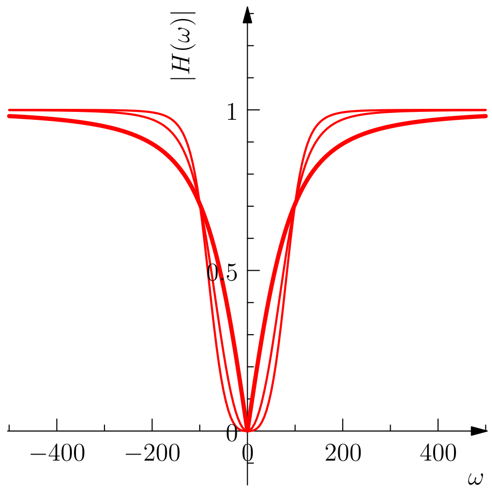

Figure 50: High pass filters with $\omega_L = 100$

Higher order filters are made by switching L and C in the higher order low pass Butterworth filters and finding matching R, L and C.

8.3 Band-pass filters

Band pass filter can be realised by defining the voltage drop on the resistor as out output signal. Current in the circuit will be largest close to the resonance frequency and this band will be attenuated less then low and high frequencies.

$$\frac{V_o}{V_i} = \frac{R}{R+sL+\frac{1}{sC}} = \frac{R/L}{R/L+s+\frac{1}{sCL}}$$

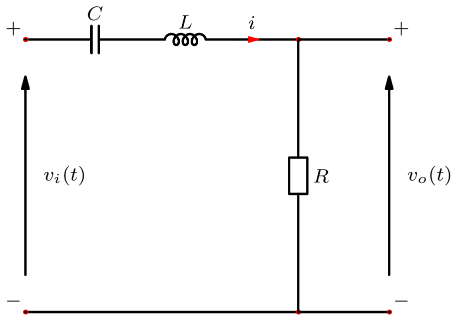

Figure 51: Band-pass filter

8.4 BIBO stability

$$\begin{aligned} H(s) &= \frac{1}{s-1} \\ \frac{Y(s)}{X(s)} &= \frac{1}{s-1} \\ Y(s)(1-s) &= X(s) \\ Y(s) - sY(s) &= X(s) \\ y(t) - \dot{y}(t) &= x(t) \end{aligned}$$

Impulse response:

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = e^t$$

This system will not be stable and will not produce "reasonable" output.
We can in general define BIBO (bound input bound output) stability:

$$\int_{-\infty}^{\infty} |h(t)| \mathrm{d}t < \infty$$

So, as long as for a causal system, H(s) contains poles in the *left* part of the complex plain, the system is stable.

8.5 Ideal and real filters

In general a zero phase low pass filter defined by:

$$H(\omega) = \epsilon(j\omega + j\omega_0) - \epsilon(j\omega - j\omega_0)$$

Will be non-causal as, the inverse Fourier transform will be a sinc function existing both for $t < 0$ and $t > 0$. A causal filter with a stable response should satisfy the following condition (Paley-Wiener integral condition):

$$\int_{-\infty}^{\infty} \frac{|\log(H(j\omega))|}{1+\omega^2} \mathrm{d}\omega < \infty$$

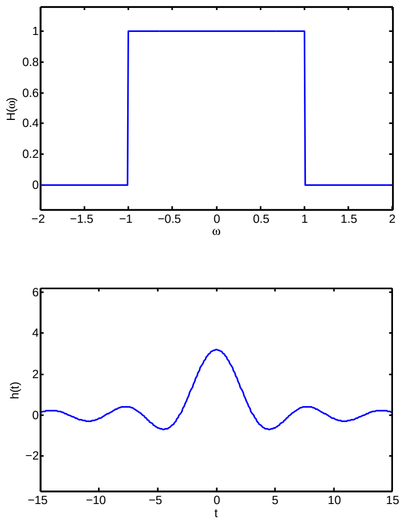

Figure 52:

To satisfy this condition $H(i\omega)$ can not be zero for any band of frequencies. So what is a good filter? A measure of attenuation is given by the loss function in decibels, defined as:

$$\alpha(\omega) = -10 \log |H(j\omega)|^2 = -20 \log_{10} |H(j\omega)|$$

10^5 difference corresponds to 100dB.