2.A Appendix: Fundamental Concepts of Linear Algebra

a **Matrices.** Matrix, more precisely an $n \times m$ matrix:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

Rows $i = 1, ..., n$, columns $j = 1, ..., m$. Elements a_{ij}.

Square Matrix: Equal number of rows and columns, $n = m$.

Symmetric Matrix: It holds that $a_{ij} = a_{ji}$.

Diagonal of a Square Matrix: The elements $[a_{11}, a_{22}, ..., a_{nn}]$.

Diagonal Matrix: Consists "only of the diagonal ", $d_{ij} = 0$ for $i \neq j$.

$$\boldsymbol{D} = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix}$$

b **Transposed Matrix**: If we switch the rows and columns of a matrix $\boldsymbol{A}$, we get the transposed matrix $\boldsymbol{A}^T$:

$$\boldsymbol{A}^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix}$$

Notes:

1. It is obvious that $(\boldsymbol{A}^T)^T = \boldsymbol{A}$ (like a mattress, turned twice).
2. For symmetric matrices, $\boldsymbol{A}^T = \boldsymbol{A}$.

c **Vectors.** A vector, more precisely a column vector: n numbers, written on top of one another.

$$\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Elements b_i.

d **Transposed Vectors**: Column vectors become row vectors if we transpose them:

$$\underline{b}^T = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}^T = [b_1, b_2, ..., b_n] \, .$$

Row vectors save more printing space than column vectors, and so column vectors are often written as transposed row vectors: $\underline{b} = [b_1, b_2, ..., b_n]^T$.

e **Simple Calculation Operations.** Addition and subtraction: This is possible only if the two matrices (vectors) have equal dimensions. We add or subtract the corresponding elements.

Multiplication by a number (a "scalar"): Each elements is multiplied. Division by a number: analogous.

Quite often in statistics and elsewhere we meet so called **linear combinations** of vectors. This is a name for expressions of the form

$$\lambda_1 \underline{b}_1 + \lambda_2 \underline{b}_2$$

plus possibly other such terms – we add multiples of the vectors involved.

f **Matrix Multiplication.** Matrices can only be multiplied if the dimensions fit: $\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B}$ is defined if the number of columns of $\boldsymbol{A}$ equals the number of rows of $\boldsymbol{B}$. Then

$$c_{ik} = \sum\nolimits_{j=1}^{m} a_{ij} b_{jk}$$

Example:

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + 1 \cdot 4 & 2 \cdot 1 + 1 \cdot (-2) \\ (-1) \cdot 3 + 0 \cdot 4 & (-1) \cdot 1 + 0 \cdot (-2) \\ 3 \cdot 3 + 1 \cdot 4 & 3 \cdot 1 + 1 \cdot (-2) \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ -3 & -1 \\ 13 & 1 \end{bmatrix}$$

Notes:

1. In the example $\boldsymbol{B} \cdot \boldsymbol{A}$ is not defined, since $\boldsymbol{B}$ has 2 columns, but $\boldsymbol{A}$ has 3 rows.
2. If $\boldsymbol{A} \cdot \boldsymbol{B}$ and $\boldsymbol{B} \cdot \boldsymbol{A}$ are both defined, they are, in general, different; $\boldsymbol{A} \cdot \boldsymbol{B} \neq \boldsymbol{B} \cdot \boldsymbol{A}$! Matrices in a multiplication are **not interchangeable**.
3. We can have $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{0}$ even if neither $\boldsymbol{A} = \boldsymbol{0}$ nor $\boldsymbol{B} = \boldsymbol{0}$.
4. The associative property holds: $(\boldsymbol{A} \cdot \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot (\boldsymbol{B} \cdot \boldsymbol{C})$
5. The distributive property holds: $\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$ and similarly $(\boldsymbol{A} + \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{C}$.
6. Transposition of a product:
$$(\boldsymbol{A} \cdot \boldsymbol{B})^T = \boldsymbol{B}^T \cdot \boldsymbol{A}^T$$
We must thus **reverse the order when we transpose!**
7. The product $\boldsymbol{A} \cdot \boldsymbol{A}^T$ is always symmetric.

g All this also holds for vectors: If $\underline{a}$ and $\underline{b}$ are column vectors, then

$$\underline{a} \cdot \underline{b}^T = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_m \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_m \\ \vdots & & & \vdots \\ a_n b_1 & a_n b_2 & \dots & a_n b_m \end{bmatrix}.$$

If they have the same length,

$$\underline{a}^T \cdot \underline{b} = \sum\nolimits_i a_i \cdot b_i .$$

"A matrix times a column vector " (if defined) gives a column vector: $\boldsymbol{A} \cdot \underline{b} = \underline{c}$.

h The **length of a vector** is the root of $\sum_i a_i^2$. We often denote this with $\|\underline{a}\|$. We can write

$$\|\underline{a}\|^2 = \underline{a}^T \cdot \underline{a} \,.$$

i The **identity matrix** (of dimension m) is defined as a diagonal with only ones:

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

It does not change matrices under multiplication: $\boldsymbol{I} \cdot \boldsymbol{A} = \boldsymbol{A}$, $\boldsymbol{A} \cdot \boldsymbol{I} = \boldsymbol{A}$.

j **Inverse Matrix.** If $\boldsymbol{A}$ is square and $\boldsymbol{B} \cdot \boldsymbol{A} = \boldsymbol{I}$, $\boldsymbol{B}$ is called the inverse matrix of $\boldsymbol{A}$; we write $\boldsymbol{B} = \boldsymbol{A}^{-1}$.

Notes:

1. It then also holds that $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{I}$. So, if $\boldsymbol{B} = \boldsymbol{A}^{-1}$, then also $\boldsymbol{A} = \boldsymbol{B}^{-1}$.
2. There does not exist an inverse for every square matrix $\boldsymbol{A}$. If it exists, we call $\boldsymbol{A}$ **regular**, and there exists only one inverse. If there is no inverse, we call $\boldsymbol{A}$ **singular**.
3. $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$.
4. Inverse of a matrix product: If $\boldsymbol{A}$ and $\boldsymbol{B}$ are square,

   $$(\boldsymbol{A} \cdot \boldsymbol{B})^{-1} = \boldsymbol{B}^{-1} \cdot \boldsymbol{A}^{-1}$$

   The order must thus be reversed when inverting a product!
5. We have $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$. We often write in short $\boldsymbol{A}^{-T}$.

k **Linear Equation System.** Summarized in short: The equation system

$$\begin{aligned} a_{11}\beta_1 + a_{12}\beta_2 + ... + a_{1m}\beta_m &= y_1 \\ a_{21}\beta_1 + a_{22}\beta_2 + ... + a_{2m}\beta_m &= y_2 \\ \dots & \quad .. \\ a_{m1}\beta_1 + a_{m2}\beta_2 + ... + a_{mm}\beta_m &= y_m \end{aligned}$$

(for the β_j) can be written as

$$\boldsymbol{A}\underline{\beta} = \underline{y}$$

(for $\underline{\beta}$). There is exactly one solution if $\boldsymbol{A}$ is regular, thus if the inverse $\boldsymbol{A}^{-1}$ exists. Then

$$\underline{\beta} = \boldsymbol{A}^{-1}\underline{y}$$

is this solution.

l If the **matrix $\boldsymbol{A}$ is singular**, then there exists a row $[a_{i1}, a_{i2}, ..., a_{im}]$ that can be written as a linear combination of the others. The corresponding equation leads either to an inconsistency (no solution) or is superfluous (infinitely many solutions). We speak of a **linear dependency** of the rows of the matrix or of the equations.

(If the matrix is singular, there also exists a column that can be written as a linear combination of the others. So, the column vectors are also linearly dependent.)