Module 7: Recommended Exercises

TMA4268 Statistical Learning V2020

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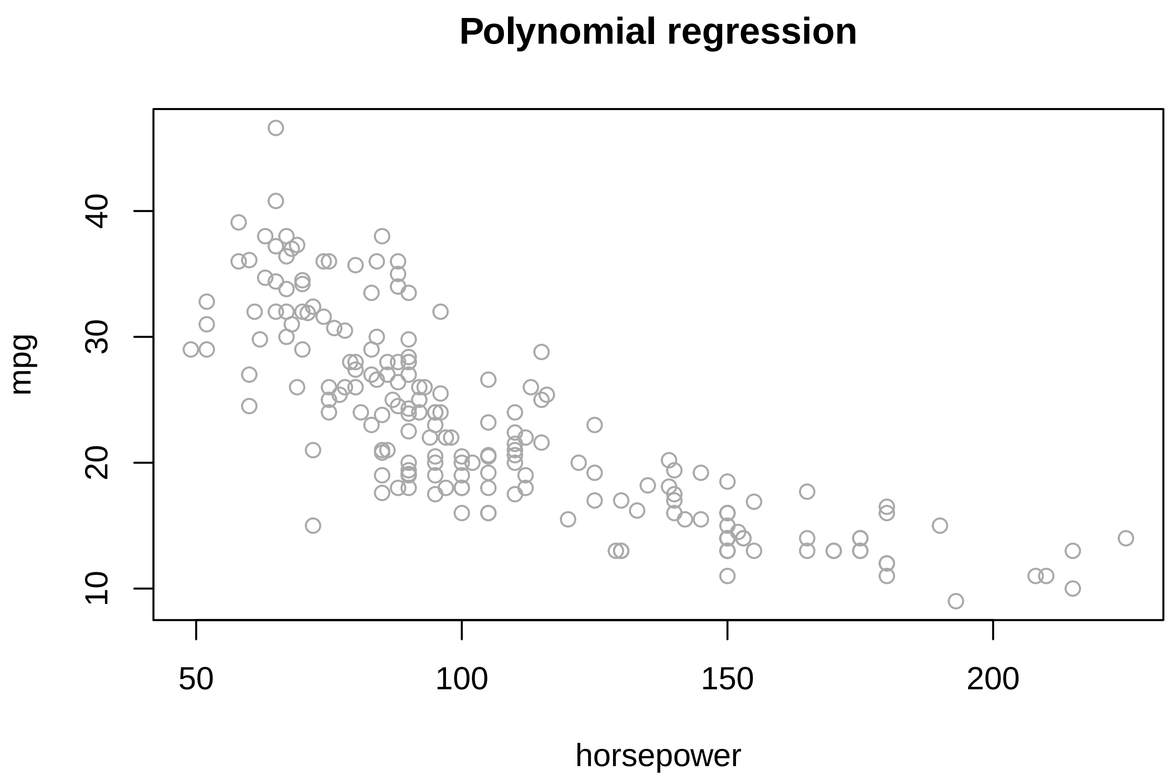
Problem 1

Let us take a look at the `Auto` data set. We want to model miles per gallon `mpg` by engine horsepower `horsepower`. Separate the observations into training and test. A training set is plotted below.

Perform polynomial regression of degree 1, 2, 3 and 4. Use `lines()` to add the fitted values to the plot below.

Also plot the test error by polynomial degree.

```r
library(ISLR)
# extract only the two variables from Auto
ds = Auto[c("horsepower", "mpg")]
n = nrow(ds)
# which degrees we will look at
deg = 1:4
set.seed(1)
# training ids for training set
tr = sample.int(n, n/2)
# plot of training data
plot(ds[tr, ], col = "darkgrey", main = "Polynomial regression")
```



Problem 2

We will continue working with the `Auto` data set. The variable `origin` is 1,2 or 3, corresponding to American, European or Japanese origin. Use `factor(origin)` for conversion to a factor variable. Predict `mpg` by origin with a linear model. Plot the fitted values and approximative 95 percent confidence intervals. Selecting `se = T` in `predict()` gives standard errors of the prediction.

- Hint: make a new dataframe of the three origins (as factors) and use this new data in your predict function.
- Hint: to plot the confidence intervals, you can add `geom_segment(aes(x=origin, y=lwr, xend = origin, yend=upr))` to your ggplot, where `origin`, `lwr` and `upr` comes from a dataframe with `lwr` as the lower bound and `upr` as the upper bound.

Problem 3

Now, let us look at the `Wage` data set. The section on Additive Models explains how we can create an AM by adding components together. One component we saw is a natural spline in `year` with one knot. Derive the expression for the design matrix $\mathbf{X}_2$ from the natural spline basis:

$$b_1(x_i) = x_i, \quad b_{k+2}(x_i) = d_k(x_i) - d_K(x_i),\ k = 0, \dots, K-1,$$

$$d_k(x_i) = \frac{(x_i - c_k)^3_+ - (x_i - c_{K+1})^3_+}{c_{K+1} - c_k}.$$

Problem 4

We will continue working with the same AM as in problem 3. The R call `model.matrix(~ bs(age,knots=c(40,60)) + ns(year,knots=2006) + education)` gives a design matrix for the AM. This matrix is what `gam()` uses. However, it does not equal our design matrix for AM $\mathbf{X} = (\mathbf{1}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$. The predicted responses will still be the same.

Write code that produces $\mathbf{X}$. The code below may be useful.

```r
# X_1
mybs = function(x, knots) {
    cbind(x, x^2, x^3, sapply(knots, function(y) pmax(0, x - y)^3))
}

d = function(c, cK, x) (pmax(0, x - c)^3 - pmax(0, x - cK)^3)/(cK - c)
# X_2
myns = function(x, knots) {
    kn = c(min(x), knots, max(x))
    K = length(kn)
    sub = d(kn[K - 1], kn[K], x)
    cbind(x, sapply(kn[1:(K - 2)], d, kn[K], x) - sub)
}
# X_3
myfactor = function(x) model.matrix(~x)[, -1]
```

If the code is valid, the predicted response $\hat{\mathbf{y}} = \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{y}$ should be the same as when using the built-in R function.

```r
# install.packages('gam'')
library(gam)
library(ISLR)
# your X-matrix
X = # fitted model with our X
myhat = lm(wage ~ X - 1)$fit
# fitted model with gam
yhat = gam(wage ~ bs(age, knots = c(40, 60)) + ns(year, knots = 2006) + education)$fit
# are they equal?
all.equal(myhat, yhat)
```

How can `myhat` equal `yhat` when the design matrices differ?

Problem 5

In this exercise we take a quick look at different non-linear regression methods. We continue using the Auto dataset from above, but with more variables.

Fit an additive model using the function `gam` from package `gam`. Call the result `gamobject`.

- `mpg` is the response,
- `displace` is a cubic spline (hint: `bs`) with one knot at 290,
- `horsepower` is a polynomial of degree 2 (hint: `poly`),
- `weight` is a linear function,
- `acceleration` is a smoothing spline with `df=3` (hint: `s`),
- `origin` is a step function (what we previously have called dummy variable coding).

Plot the resulting curves (hint: first set `par(mfrow=c(2,3)` and then `plot(gamobject,se=TRUE,col="blue")`). Comment on what you see.

Problem 6 (Advanced)

Back to Wage-data. In the part where we discussed smoothing splines there is a section explaining how to compute $\mathbf{S}$, where $\hat{\mathbf{y}} = \mathbf{S}\mathbf{y}$. This is implemented below, with $\mathbf{x}$ as unique observations of `age` and $\mathbf{y}$ the coresponding `wage`.

```r
K = function(x) {
    xi = sort(unique(x))
    n = length(xi)
    h = xi[-1] - xi[-n]
    i = seq.int(n - 2)
    D = diag(1/h[i], ncol = n)
    D[cbind(i, i + 1)] = -1/h[i] - 1/h[i + 1]
    D[cbind(i, i + 2)] = 1/h[i + 1]
    W = diag(h[i] + h[i + 1]/3)
    W[cbind(i[-1], i[-1] - 1)] = h[i[-1]]/6
    W[cbind(i[-1] - 1, i[-1])] = h[i[-1]]/6
    t(D) %*% solve(W) %*% D
}

x = sort(unique(age))
y = wage[order(age[!duplicated(age)])]
eig = eigen(K(x))
U = eig$vectors
d = eig$values
```

```r
lambda = 2000
S = U %*% diag(1/(1 + lambda * d)) %*% t(U)
```

Use **S** from this code to compute $\hat{\mathbf{y}}$. Also compute $\hat{\mathbf{y}}$ using `smooth.spline()` with the correct degrees of freedom. Finally, plot both sets of fitted values and observe that they are equal.

```r
myhat = yhat = smooth.spline(x, y, df = )$y

plot(x, y, main = "Comparison of fitted values")
co = c("blue", "red")
w = c(5, 2)
# add lines of fitted values to the plot
lines(x, myhat, lwd = w[1], col = co[1])  #lwd = linewidth
lines(x, yhat, lwd = w[2], col = co[2])
legend("topright", legend = c("myhat", "yhat"), col = co, lwd = w, inset = c(0,
    -0.19), xpd = T)
```