Module 9: Recommended Exercises

TMA4268 Statistical Learning V2020

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Problem 1

Work through the lab in Section 9.6.1 of the course book.

Problem 2 (Book Ex.2)

We have seen that in $p = 2$ dimensions, a linear decision boundary takes the form $\beta_0 + \beta_1 X_1 + \beta_2 X_2 = 0$. We now investigate a non-linear decision boundary.

a) Sketch the curve
$$(1 + X_1)^2 + (2 - X_2)^2 = 4.$$

b) On your sketch, indicate the set of points for which
$$(1 + X_1)^2 + (2 - X_2)^2 > 4,$$
as well as the set of points for which
$$(1 + X_1)^2 + (2 - X_2)^2 \leq 4.$$

c) Suppose that a classifier assigns an observation to the blue class if
$$(1 + X_1)^2 + (2 - X_2)^2 > 4,$$
and to the red class otherwise. To what class is the observation $(0, 0)$ classified? $(-1, 1)$? $(2, 2)$? $(3, 8)$?

d) Argue that while the decision boundary in (c) is not linear in terms of X_1 and X_2, it is linear in terms of X_1, X_1^2, X_2, and X_2^2.

Problem 3

The SVM is an extension of the support vector classifier, by enlarging the feature space using kernels. The polynomial kernel is a popular choice and has the following form

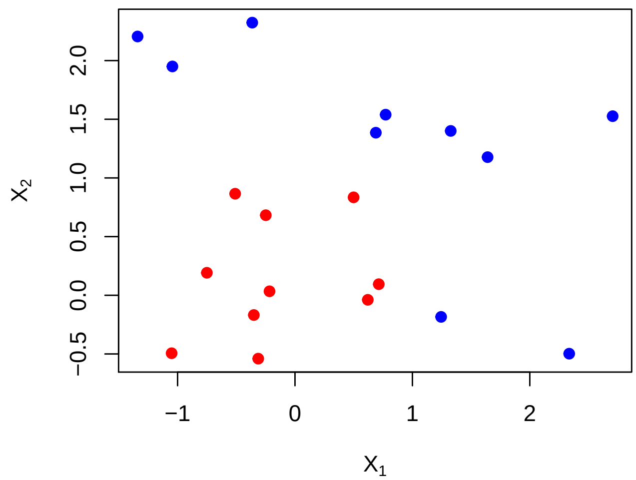
$$K(\boldsymbol{x}_i, \boldsymbol{x}_i') = (1 + \sum_{j=1}^{p} x_{ij} x_{i'j'})^d$$

Show that for a feature space with inputs X_1 and X_2 and for degree $d = 2$, the above kernel can be represented as the inner product
$$K(\boldsymbol{x}_i, \boldsymbol{x}_i') = \langle h(x), h(x') \rangle \ ,$$
where $h(x)$ is a 6-dimensional transformation function in an enlarged space.

Problem 4

This problem involves plotting of decision boundaries for different kernels and it's taken from Lab video.

```r
# code taken from video by Trevor Hastie
set.seed(10111)
x <- matrix(rnorm(40), 20, 2)
y <- rep(c(-1, 1), c(10, 10))
x[y == 1, ] <- x[y == 1, ] + 1
plot(x, col = y + 3, pch = 19, xlab = expression(X[1]), ylab = expression(X[2]))
```




```r
dat = data.frame(x, y = as.factor(y))
```

(a) Plot the decision boundary of the `svmfit` model by using the function `make.grid`. Hint: Use the predict function for the grid points and then plot the predicted values {-1,1} with different colors.

R-hints:

```r
library(e1071)
svmfit = svm(y ~ ..., ..., kernel = "...", cost = ..., scale = ...)
```

The following function may help you to generate a grid for plotting:

```r
make.grid = function(x, n = 75) {
    # takes as input the data matrix x and number of grid points n in
    # each direction the default value will generate a 75x75 grid
    grange = apply(x, 2, range)  # range for x1 and x2
    x1 = seq(from = grange[1, 1], to = grange[2, 1], length.out = n)  # sequence from the lowest to the
    x2 = seq(from = grange[1, 2], to = grange[2, 2], length.out = n)  # sequence from the lowest to the
    expand.grid(X1 = x1, X2 = x2)  #create a uniform grid according to x1 and x2 values
}
```

(b) On the same plot add the training points and indicate the support vectors.

(c) The solutions to the SVM optimization problem is given by

$$\hat{\beta} = \sum_{i \in \mathcal{S}} \hat{\alpha}_i y_i x_i \ ,$$

where S is the set of the support vectors. From the `svm()` function we cannot extract $\hat{\beta}$, but instead we have access to $\text{coef}_i = \hat{\alpha}_i y_i$, and $\hat{\beta}_0$ is given as *rho*. For more details see here.

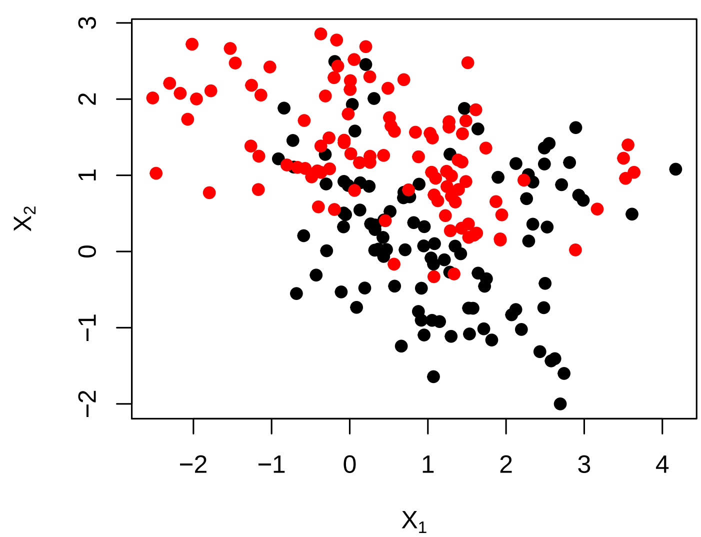
Calculate the coeficients $\hat{\beta}_0, \hat{\beta}_1$ and $\hat{\beta}_2$. Then add the decisision boundary and the margins using the function `abline()` on the plot from (b).

Problem 5

Now we fit an svm model with radial kernel to the following data taken from Hastie, Tibshirani, and Friedman (2009). Use cross-validation to find the best set of tuning parameters (cost C and γ). Using the same idea as in Problem 4a) plot the non-linear decision boundary, and add the training points. Furthermore if you want to create the decision boundary curve you can use the argument `decision.values=TRUE` in the function predict, and then you can plot it by using the `contour()` function.

R-hints:

```
load(url("https://web.stanford.edu/~hastie/ElemStatLearn/datasets/ESL.mixture.rda"))
# names(ESL.mixture)
rm(x, y)
attach(ESL.mixture)
plot(x, col = y + 1, pch = 19, xlab = expression(X[1]), ylab = expression(X[2]))
```




```
dat = data.frame(y = factor(y), x)
```

To run cross-validation over a grid for (C, γ), you can use a two-dimensional list of values in the `ranges` argument:

```
r.cv <- tune(svm, factor(y) ~ ., data = dat, kernel = "...", ranges = list(cost = c(...),
    gamma = c(...)))
```

For the plot:

```
xgrid = make.grid(x)
ygrid = predict(..., xgrid)
plot(xgrid, col = as.numeric(ygrid), pch = 20, cex = 0.2)
points(x, col = y + 1, pch = 19)

# decision boundary
func = predict(..., xgrid, decision.values = TRUE)
func = attributes(func)$decision
contour(unique(xgrid[, 1]), unique(xgrid[, 2]), matrix(func, 75, 75),
    level = 0, add = TRUE)  #svm boundary
```

Problem 6 - optional (Book Ex. 7)

This problem involves the OJ data set which is part of the ISLR package.

(a) Create a training set containing a random sample of 800 observations, and a test set containing the remaining observations.

(b) Fit a support vector classifier to the training data using `cost=0.01`, with Purchase as the response and the other variables as predictors. Use the `summary()` function to produce summary statistics, and describe the results obtained.

(c) What are the training and test error rates?

(d) Use the `tune()` function to select an optimal cost. Consider values in the range 0.01 to 10.

(e) Compute the training and test error rates using this new value for cost.

(f) Repeat parts (b) through (e) using a support vector machine with a radial kernel. Use the default value for `gamma`.

(g) Repeat parts (b) through (e) using a support vector machine with a polynomial kernel. Set `degree=2`.

(h) Overall, which approach seems to give the best results on this data?

Hastie, Trevor, Robert Tibshirani, and Jerome Friedman. 2009. *The Elements of Statistical Learning*. 2nd ed. Vol. 1. Springer series in statistics New York.