ON THE NUMBER OF OPTIMAL SEQUENCES TO SOLVE THE K-PEG TOWER OF HANOI PROBLEM

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Abstract. In this paper we only look at sequences that are performed according to the Frame-Stewart-Algorithm. There are several possibilities to choose the number of disks which are placed on peg one. If this number is chosen within certain boundaries, the resulting sequence needs the minimal number of moves required to solve the problem. By calculating those boundaries, the number of possibilities can be determined by applying a recursive formula. We then find an explicit formula for the number of optimal sequences and prove it.

1. Introduction

The 3-peg Tower of Hanoi is a well-known problem that was first mentioned in 1883 by Edouard Lucas [4]. The number of moves as well as the number of optimal sequences are already known. In 1908, Dudeney raised a more complex question. In his book *The Canterbury Puzzles* he published the "Reves Puzzle", asking for the minimal number of moves with 4 pegs. This question was answered in 2014 by Bousch in [2]. The k-peg Tower of Hanoi problem was mentioned in [3] for the first time in 1947. The question was a notorious open problem for several decades, until in 2019 [1] a proof was found. Nevertheless, until now nobody has answered the question, how many sequences there are to solve the problem only using the minimal number of moves. We do only take into account sequences that are created according to the Frame-Stewart-Algorithm.

Definition 1 (k-peg Tower of Hanoi)**.** There are k pegs and n disks with $k, n \in \mathbb{N}_0 \wedge k \geq 3$.

(1) A move consists of moving exactly one disk d ($d \in \{0, 1, \ldots, n-1\}$) from a peg a to a peg b ($a, b \in \{0, 1, \ldots, k-1\}$). That will be denoted as $(d; a \to b)$
(2) Only disks with no other disks on top of them can be moved
(3) A disk can only be moved to a peg where no smaller disk is laying

The game is solved when all disks are moved to peg $k-1$.

Strategy 1. In order to solve the Tower of Hanoi problem, move the biggest disk only once. We denote by **part-1-sequence** the part of the sequence until, but not including this move. The rest of the sequence after the move of disk $n-1$ is then called **part-2-sequence**

Definition 2. At the end of a part-1-sequence there is one peg with the biggest disk on it. Let this peg be the startpeg. Let the be the peg without any disks on it be the endpeg. All remaining pegs with the disks on them are called intermediate towers.

Definition 3. We call a part-2-sequence inverted to a part-1-sequence if the following criteria are fulfilled:

(1) Invert the order of the moves
(2) Any move $(d; a \to b)$ is replaced by $(d; b \to a)$
(3) Any move $(d; \text{startpeg} \to f)$ is replaced by $(d; \text{endpeg} \to f)$
(4) Any move $(d; \text{endpeg} \to f)$ is replaced by $(d; \text{startpeg} \to f)$
(5) Any move $(d; f \to \text{startpeg})$ is replaced by $(d; f \to \text{endpeg})$
(6) Any move $(d; f \to \text{endpeg})$ is replaced by $(d; f \to \text{startpeg})$

Strategy 2.
Choose $m_1,\ m_2,\ \ldots,\ m_{k-2}$ in a way that the number of moves of the following strategy is minimal.

step 1:	Move all disks $\leq m_1$ from peg 0 to peg 1, creating the 1-intermediate tower In order to perform these and the following moves, apply this strategy recursively (so that peg 1 is the endpeg).
step x:	Move all disks $\leq m_x$ from peg 0 to peg $x\ \lvert x \in \{2,3,\ldots,k-2\}$, creating the x-intermediate tower.
step $k-1$:	Move the biggest disk from peg 0 to the endpeg.
step k:	perform the moves of the inverted part-2-sequence

Strategy 3 (Frame-Stewart)**.**
Choose m_1 in a way that the number of moves of the following strategy is minimal.

step 1:	move all disks $\leq m_1$ from peg 0 to 1. In order to perform these and the following moves, apply this strategy recursively (so that peg 1 is the endpeg).
step 2:	move all remaining disks from peg 0 to the endpeg.
step 3:	move all disks $\leq m_1$ from peg 0 to the endpeg.

Fact 1.1. *Strategy 2 and strategy 3 are equivalent.*

When we talk about an optimal strategy or sequence, we mean a strategy or sequence that yields the minimum number of moves to solve the Tower of Hanoi game. That does not necessarily mean that this strategy or sequence is the only one with a minimum number of moves. As proved in [1], a sequence according to strategy 3 requires a minimal number of moves.

Definition 4. Let the minimal number of moves required to solve a Tower of Hanoi game with k pegs and n disks be $M(n,k)$.

Definition 5. We denote by $I(n,k)$ the increment $M(n,k) - M(n-1,k)$.

Definition 6. All numbers $M(n,k),\ M(n+1,k),\ldots,M(n+q,k)$, with

$$I(n,k) = I(n+1,k) = \cdots = I(n+q,k) = 2^t$$

are called a t_k**-block** with the length $L(t,k) = q$. If t and k are of any value it is also called an incrementblock.

Corollary 1.2. *The p-intermediate tower needs $M(x_p, k-p+1)$ moves to be constructed.*

Proof. When the 1-intermediate tower is constructed, the larger disks have no influence on its construction, so all k pegs can be used and we need $M(x_1,k)$ moves. As for the 2-intermediate tower, all its disks are bigger than the biggest disk of the 1-intermediate tower. Therefore, all pegs except the first peg can be used for its construction so that we need $M(x_2,k-1)$ moves. As the smallest disk of the p-intermediate tower is larger than the biggest disks of all p-intermediate towers that have been constructed so far, all pegs except the pegs $1,2,\ldots,p$ can be used for its construction. □

2. Formula for the minimal number

Theorem 1. *Consider a Tower of Hanoi game with k pegs and n disks. Then*

$$M(n,k) = \sum_{i=0}^{t} \left(2^i * \binom{i+k-3}{k-3} \right) + 2^t \left(n - \binom{t+k-2}{k-2} \right)$$

where

$$\binom{t+k-2}{k-2} \geq n > \binom{t+k-3}{k-2}$$

.

Lemma 2.1. *Any increment $I(n,k)$ can be expressed as 2^t with $t \in \mathbb{N}_0$.*

Proof. If the height of one of the intermediate towers is increased, that leads to a certain increment in the number of moves needed to construct it. As the part-2-sequence needs exactly the same number of moves as the part-1-sequence, the total increment is twice as big as the increment of this intermediate tower. The smallest increment is the increment from 0 disks to one disks, which is 1. Therefore any increment is either 1 or twice the increment of the enlarged intermediate tower. Therefore only powers of 2 can occur as increments. □

Lemma 2.2. *There is exactly one t_k-block. (see definition 6)*

Proof. We can prove this by showing that

(1) the increment never decreases (prove by induction on n) and
(2) there is no maximum increment for any number of pegs (prove by induction on n and k).

The increment increases up to two disks (one disk: 1, two disks: 2 as it requires three moves for any number of pegs). Let us assume, that the increment never decreases up to n disks. For the sake of a contradiction, let us now consider a configuration in which the increment is 2^t and in which we could achieve a increment $\leq 2^t$ for $n+1$ disks. As the number of disks has increased by one, we have to increase the height of one of the intermediate towers. Let the intermediate tower whose height has been increased by one for n disks be the p-intermediate tower. If now one of the other intermediate towers (not p) could be increased by one and yield a increment $< 2^{t-1}$, then this intermediate tower would have been increased for n disks instead of the p-intermediate tower. Therefore, the p-intermediate tower has to be increased once again with an increment $< 2^{t-1}$. But when he was increased for n disks, his increment was 2^{t-1}. Therefore his increment would have decreased. As the p-intermediate tower has less than n disks, this contradicts our assumption. As for the second part of our lemma, for three pegs there is no maximum increment. First, let us consider a configuration with 4 pegs. The 1-intermediate tower is constructed with 4 pegs, the 2-intermediate tower is constructed with 3 pegs. Let us assume there is a change of increment for n disks. Is it possible for the increment to stay always the same from then on? As the 2-intermediate tower can't be increased because that would immediately increase the increment, we have to increase the 1-intermediate tower. Once it reaches height n, the increment increases. Therefore, for 4 pegs there is no maximum increment. By induction on k this can be proved for any number of pegs. □

Lemma 2.3.

$$L(t,k) = \binom{t+k-3}{k-3}$$

Proof. The increment in an incrementblock is always the same, so the increment of the intermediate towers also has to be the same. Therefore, the length of an incrementblock is equal to the sum of the lengths of the incrementblocks of the intermediate towers. As there are $k-2$ intermediate towers, we get the following.

$$L(t,k) = \sum_{i=0}^{k-3} L(t-1,k-i) = \sum_{i=3}^{k} L(t-1,i)$$

In order to show the connection to the binomial coefficients, we introduce a new function $L_2(u,l)$ with

$$L(t,k) = L_2(t+k-3,k-3)$$

Transforming the last equation into an expression of this function, we get:

$$L_2(t+k-3,k-3) = \sum_{i=3}^{k} L_2(t-1+i-3,i-3)$$

$$L_2(t+k-3,k-3) = \sum_{i=0}^{k-3} L_2(t-1+i,i)$$

$$L_2(t+k,k) = \sum_{i=0}^{k} L_2(t-1+i,i) \tag{2.1}$$

In case of three pegs, the length of an incrementblock is always 1

$$L(t,3) = L_2(t,0) = 1 \tag{2.2}$$

For one disk, the increment also is always 1. As k is always ≥ 3, the following does not restrict the number of pegs

$$L(0,k+3) = L_2(k,k) = 1 \tag{2.3}$$

To show that $L_2(x,y) = \binom{x}{y}$, we consider the following three conditions. If these three conditions are true of a function $f(x,y)$, then it is equal to $\binom{x}{y}$

$$f(y+x,x) = \sum_{i=0}^{x} f(y-1+i,i) \tag{2.4}$$

$$f(x,0) = 1 \tag{2.5}$$

$$f(x,x) = 1 \tag{2.6}$$

As 2.1 is analogous to 2.4, 2.2 is analogous to 2.5 and 2.3 is analogous to 2.6, we prove our lemma.

$$L(t,k) = L_2(t+k-3, k-3) = \binom{t+k-3}{k-3}$$

□

Lemma 2.4. *The number of moves needed until the completion of the t_k-block is*

$$\sum_{i=0}^{t} \left(2^i * \binom{i+k-3}{k-3} \right)$$

.

Proof. Per definition, the increment of the t_k-block is 2^t. The number of moves needed in the t_k-block is its length multiplied with the increment. As the lowest increment is 2^0 and the highest increment is 2^k and the increment always doubles, we prove our lemma. □

Lemma 2.5. *The current increment 2^t can be calculated with the following inequality:*

$$\binom{t+k-2}{k-2} \geq n > \binom{t+k-3}{k-2}$$

Proof. The current increment is determined by the number of incrementblocks that have been filled by now. Therefore, we calculate the sum of t incrementblocks.

$$\sum_{i=0}^{t} \binom{i+k-3}{k-3} = \binom{t+k-2}{k-2}$$

The number of disks n has to be lower or equal to the sum of the lengths of t incrementblocks for increment 2^t, but it has to be bigger than the sum of $t-1$ incrementblocks.

$$\sum_{i=0}^{t-1} \binom{i+k-3}{k-3} = \binom{t-1+k-2}{k-2} = \binom{t+k-3}{k-2}$$

□

The additional number of moves (in comparison to the correct number of moves) until the completion of the current incrementblock equals the difference between $\binom{t+k-2}{k-2}$ and n multiplied with the increment in the last incrementblock 2^t, that is $2^t * \left(\binom{t+k-2}{k-2} - n \right)$. Adding up the number of moves until the completion of the current incrementblock and then subtracting the additional number of moves yields the formula proposed in Theorem 1.

3. Numbers of optimal sequences

Definition 7. We denote by the term configuration with a **filled** t-incrementblock a configuration that has the increment 2^t, but if one disk was added, the increment would increase to 2^{t+1}. As a result, the number of disks of a configuration with k pegs and a filled t-incrementblock is $\binom{t+k-2}{k-2}$.

Lemma 3.1. *Consider a configuration with k pegs and a filled t-incrementblock. Let $h(p;t;k)$ be the height of the p-intermediate tower. Then $h(p;t;k) = \binom{t+k-2-p}{t-1}$.*

Proof. As the whole configuration has a filled t_k-block, the p-intermediate tower has a filled $(t-1)_{k-p+1}$-incrementblock $\forall 0 < p < k-1$. The p-intermediate tower has $k-p+1$ pegs that can be used for construction. If the incrementblock of at least one intermediate tower was not filled, then its height could be increased without changing the current increment. That would be a contradiction. As we know that the p-intermediate tower has $k-p+1$ pegs that can be used for construction, we can calculate its height. The number of disks of a configuration with $k-p+1$ pegs and a filled $(t-1)$-incrementblock is $\binom{t-1+k-p+1-2}{k-p+1-2} = \binom{t+k-2-p}{t-1}$. □

Definition 8. $t(n,k) := log_2(I(n,k))$. Because of Lemma 2.5, $t(n,k)$ can be calculated with the following inequality:

$$\binom{t(n,k)+k-2}{k-2} \geq n > \binom{t(n,k)+k-3}{k-2}$$

As we use it very often in the following, we will write τ instead of $t(n,k)$.

Lemma 3.2. *Let the height of the p-intermediate tower be h_p. Then*

$$h(p;t;k) \geq h_p \geq h(p;t-1;k)$$

where $\forall q \quad \text{with} \quad 1 \leq q < k-1$

$$h(q;t;k) \geq h_q \geq h(q;t-1;k)$$

Proof. The number of moves can not be changed. Otherwise, either our strategy or the resulting sequence would not be optimal. Nevertheless, the height of the intermediate towers can be changed if their increment is not changed. For the sake of a contradiction, let us assume there exists a intermediate tower q with $h_q < h(q;t-1;k)$. Increasing the height of this intermediate tower would yield an increment of 2^{t-1}. The increment of the whole configuration is 2^t, so that the number of moves could be reduced by increasing the height of this intermediate tower and decreasing the height of an other one. As a result, not filling the $t-1_{k-p+1}$ leads to a higher number of moves. It is trivial that increasing the increment of a disk will increase the number of moves. □

Lemma 3.3. *We denote by $b(p;t;k)$ the number of different h_p where the increment of the p-intermediate tower is still 2^t and the number of pegs is k. Then,*

$$b(p;t;k) = \binom{t+k-3-p}{t-1}$$

Proof. The maximum number of disks with increment t is reached, when the t_{k-p+1}-block is filled, whereas the maximum number of disks with an increment lower than 2^t is reached, when the $t-1_{k-p+1}$-block is filled. Therefore,

$$\begin{aligned} b(p;t;k) &= h(p;t;k) - h(p;t-1;k) \\ &= \binom{t+k-2-p}{t-1} - \binom{t+k-3-p}{t-2} \\ &= \binom{t+k-3-p}{t-1} \end{aligned}$$

□

Definition 9.

$$\mathbf{min(a,b)} = \begin{cases} a & |a \leq b \\ b & |a > b \end{cases}$$

$$\mathbf{max(a,b)} = \begin{cases} a & |a \geq b \\ b & |a < b \end{cases}$$

Lemma 3.4. *The minimum height of the 1-intermediate tower in a end-configuration with n disks and k pegs is* $h(1;\tau-1;k) + \max\left(0;\ n - \binom{\tau+k-3}{\tau-1} - \binom{\tau+k-4}{\tau}\right)$.

Proof. Normally, the minimum height of the 1-intermediate tower is $h(1;\tau-1;k)$ as shown in Lemma 3.2. But there may be cases in which the number of disks is so big that choosing this height for the 1-intermediate tower enforces choosing a height bigger than $h(q;\tau;k)$ for the q-intermediate tower. The minimal number of disks that have to be placed on peg 1 in addition to the $h(1;\tau-1;k)$ disks to prevent this equals

$$
\begin{aligned}
&n - \text{the sum of the maximal heights of all intermediate towers except the first one plus one disk (the biggest)}\\
&\quad - h(1;\tau-1;k)\\
=&n - \left(\sum_{q=2}^{k-2} h(q;\tau;k) + 1\right) - h(1;\tau-1;k)\\
=&n - \left(\sum_{q=2}^{k-2} \binom{\tau+k-2-q}{\tau-1} + 1\right) - \binom{\tau-1-2+k-1}{\tau-2}\\
=&n - \left(\sum_{q=0}^{k-3} \binom{\tau-1+q}{\tau-1}\right) - \binom{\tau+k-4}{\tau-2}\\
=&n - \binom{\tau+k-3}{\tau} - \binom{\tau-1-2+k-1}{\tau-2}\\
=&n - \left(\binom{\tau+k-4}{\tau} + \binom{\tau+k-4}{\tau-1} + \binom{\tau+k-4}{\tau-2}\right)\\
=&n - \binom{\tau+k-3}{\tau-1} - \binom{\tau+k-4}{\tau}
\end{aligned}
$$

□

Lemma 3.5. *The maximum height of the 1-intermediate tower in an end-configuration with n disks and k pegs is* $h(1;\tau-1;k) + \min\left(n - \binom{\tau+k-3}{\tau-1}; \binom{\tau+k-4}{\tau-1}\right)$

Proof. Normally, the maximum height of the 1-intermediate tower is

$$h(1;\tau;k) = h(1;\tau-1;k) + b(1;\tau;k) = h(1;\tau-1;k) + \binom{\tau+k-4}{\tau-1}$$

.
But there may be cases in which the number of disks is so small that choosing this height for the 1-intermediate tower enforces choosing a height smaller than $h(q;\tau-1;k)$ for the q-intermediate tower. In order to prevent such an event, the maximum number of disks that can be placed on peg 1 equals

$$
\begin{aligned}
=&n - (\text{the sum of the minimal heights of all intermediate towers plus one (for the biggest)})\\
=&n - \left(\sum_{p=1}^{k-2} h(p;\tau-1;k) + 1\right)\\
=&n - \left(\sum_{p=1}^{k-2} \binom{\tau+k-3-p}{\tau-2} + 1\right)\\
=&n - \left(\sum_{p=1}^{k-2} \binom{\tau-2+p}{\tau-2} + \binom{\tau-2}{\tau-2}\right)\\
=&n - \binom{\tau+k-3}{\tau-1}
\end{aligned}
$$

□

Lemma 3.6. *Let h_1 be the height of the 1-intermediate tower. Then, $h_1 = h(1;\tau-1;k) + a$. We denote by $\Upsilon(n,k)$ the number of optimal sequences to solve a configuration with n disks and k pegs.*

Note that we do not take into account sequences that differ to other sequences only in changing pegs. It is also important to know that we only consider sequences with inverted part-2-sequences.

$$\begin{aligned}
\Upsilon(n,3) &= 1\\
\Upsilon(0,k) &= 1\\
\Upsilon(n,k) &= \sum_{a=\max\left(0;\ n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}\right)}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\Upsilon(h_1,k)\cdot\Upsilon(n-h_1,k-1)\right)
\end{aligned}$$

Proof. The number of optimal sequences for 3 pegs as well as for 0 disks is trivially 1. To get the number of optimal sequences for n disks and k pegs, we choose the height h_1 of the 1-intermediate tower. Then, for each height h_1 the number of optimal sequences can be calculated by multiplying the number of optimal sequences to build the 1-intermediate tower with the number of optimal sequences to build the rest of the end-configuration. As all k pegs can be used to build the 1-intermediate tower, the number of optimal sequences is just $\Upsilon(h_1,k)$. The rest of the end-configuration has 1 peg less to be constructed. Therefore, the number of optimal sequences to build it is $\Upsilon(n-h_1,k-1)$. Finally, we add up the number of optimal sequences for all heights of the 1-intermediate tower, where the height matches the criteria of Lemma 3.4 and Lemma 3.5. □

Lemma 3.7.

$$t(n-h_1,k-1)=\tau$$

Proof. $t(n-h_1,k-1)$ equals the increment of a configuration where peg 1 and the corresponding disks of the 1-intermediate tower have been removed. Therefore, the increment stays the same. □

Lemma 3.8. *Let h_1 be the height of the 1-intermediate tower. Then,*

$$t(h_1,k)=\begin{cases}\tau-2 & \big|a=0\\ \tau-1 & \Big|0<a\le\binom{\tau+k-4}{\tau-1}\end{cases}$$

where $h_1=h(1;\tau-1;k)+a$.

Proof.

$$\begin{aligned}
\binom{t(h_1,k)+k-2}{k-2} &\ge h_1 &&> \binom{t(h_1,k)+k-3}{k-2}\\
\binom{t(h_1,k)+k-2}{k-2} &\ge h(1;\tau-1;k)+a &&> \binom{t(h_1,k)+k-3}{k-2}\\
\binom{t(h_1,k)+k-2}{k-2} &\ge \binom{\tau+k-4}{k-2}+a &&> \binom{t(h_1,k)+k-3}{k-2}
\end{aligned}$$

If $t(h_1,k)=\tau-2$ the binomial coefficient in the middle is equal to that one on the left. Therefore, for $a=0$, $t(h_1,k)=\tau-2$. If $t(h_1,k)=\tau-1$ the binomial coefficient in the middle is equal to that one on the right. Therefore, for $a>0$, $t(h_1,k)=\tau-1$. Somehow, there also might be an a so that

$$\begin{aligned}
\binom{t(h_1,k)+k-2}{k-2} &< \binom{\tau+k-4}{k-2}+a \text{ for } t(h_1,k)=\tau-1\\
\binom{\tau+k-3}{k-2} &< \binom{\tau+k-4}{k-2}+a\\
\binom{\tau+k-4}{k-3}+\binom{\tau+k-4}{k-2} &< \binom{\tau+k-4}{k-2}+a\\
\binom{\tau+k-4}{\tau-1} &< a
\end{aligned}$$

As a result, for all $0<a\le\binom{\tau+k-4}{\tau-1}$, $t(h_1,k)=\tau-1$

□

Definition 10.

$$\Upsilon_2(n,k)=\binom{\binom{t(n,k)+k-3}{t(n,k)}}{\binom{t(n,k)+k-2}{t(n,k)}-n}$$

Lemma 3.9.

$$\sum_{a=\max\left(0;\ n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}\right)}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)}\left(\Upsilon_2\left(h_1;\ k\right)\cdot\Upsilon_2\left(n-h_1;\ k-1\right)\right)$$

$$=\sum_{a=0}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)}\left(\binom{\binom{\tau+k-4}{\tau-1}}{a}\cdot\binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a}\right)$$

Proof.

$$\sum_{a=\max\left(0;\ n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}\right)}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)}\left(\Upsilon_2\left(h_1;\ k\right)\cdot\Upsilon_2\left(n-h_1;\ k-1\right)\right)$$

$$=\sum_{a=\max\left(0;\ n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}\right)}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)}\left(\binom{\binom{t(h_1,k)+k-3}{t(h_1,k)}}{\binom{t(h_1,k)+k-2}{t(h_1,k)}-h_1}\cdot\binom{\binom{t((n-h_1),k-1)+k-1-3}{t((n-h_1),k-1)}}{\binom{t((n-h_1),k-1)+k-1-2}{t((n-h_1),k-1)}-(n-h_1)}\right)$$

We apply Lemma 3.7

$$=\sum_{a=\max\left(0;\ n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}\right)}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)}\left(\binom{\binom{t(h_1,k)+k-3}{t(h_1,k)}}{\binom{t(h_1,k)+k-2}{t(h_1,k)}-h_1}\cdot\binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau}-n+h_1}\right)$$

case distinction:

case 1 $\binom{\tau+k-3}{\tau-1}+\binom{\tau+k-4}{\tau}\geq n$

case 2 $\binom{\tau+k-3}{\tau-1}+\binom{\tau+k-4}{\tau}<n$

case 1.

$$\sum_{a=0}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)}\left(\binom{\binom{t(h_1,k)+k-3}{t(h_1,k)}}{\binom{t(h_1,k)+k-2}{t(h_1,k)}-h_1}\cdot\binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau}-n+h_1}\right)$$

$$=\sum_{a=0}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)}\left(\binom{\binom{t(h(1;\tau-1;k)+a,k)+k-3}{t(h(1;\tau-1;k)+a,k)}}{\binom{t(h(1;\tau-1;k)+a,k)+k-2}{t(h(1;\tau-1;k)+a,k)}-h(1;\tau-1;k)-a}\cdot\binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau}-n+h(1;\tau-1;k)+a}\right)$$

$$=\left(\binom{\binom{t(h(1;\tau-1;k),k)+k-3}{t(h(1;\tau-1;k),k)}}{\binom{t(h(1;\tau-1;k),k)+k-2}{t(h(1;\tau-1;k),k)}-h(1;\tau-1;k)}\cdot\binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau}-n+h(1;\tau-1;k)}\right)$$

$$+\sum_{a=1}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)}\left(\binom{\binom{t(h(1;\tau-1;k)+a,k)+k-3}{t(h(1;\tau-1;k)+a,k)}}{\binom{t(h(1;\tau-1;k)+a,k)+k-2}{t(h(1;\tau-1;k)+a,k)}-h(1;\tau-1;k)-a}\cdot\binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau}-n+h(1;\tau-1;k)+a}\right)$$

Lemma 3.8

$$= \left(\binom{\binom{\tau-2+k-3}{\tau-2}}{\binom{\tau-2+k-2}{\tau-2} - h(1;\tau-1;k)} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau} - n + h(1;\tau-1;k)} \right)$$
$$+ \sum_{a=1}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\binom{\binom{\tau-1+k-3}{\tau-1}}{\binom{\tau-1+k-2}{\tau-1} - h(1;\tau-1;k) - a} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau} - n + h(1;\tau-1;k) + a} \right)$$

We can replace now $h(1;\tau-1;k)$ with $\binom{\tau+k-4}{\tau-2}$

$$= \left(\binom{\binom{\tau+k-5}{\tau-2}}{\binom{\tau+k-4}{\tau-2} - \binom{\tau+k-4}{\tau-2}} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau} - n + \binom{\tau+k-4}{\tau-2}} \right)$$
$$+ \sum_{a=1}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\binom{\binom{\tau+k-4}{\tau-1}}{\binom{\tau+k-3}{\tau-1} - \binom{\tau+k-4}{\tau-2} - a} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau} - n + \binom{\tau+k-4}{\tau-2} + a} \right)$$

Using several identities for binomial coefficients yields

$$= \left(\binom{\binom{\tau+k-5}{\tau-2}}{0} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau} - n + \binom{\tau+k-4}{\tau-2}} \right)$$
$$+ \sum_{a=1}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\binom{\binom{\tau+k-4}{\tau-1}}{\binom{\tau+k-4}{\tau-1} - a} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau} - n + \binom{\tau+k-4}{\tau-2} + a} \right)$$

$$= \left(\binom{\binom{\tau+k-5}{\tau-2}}{0} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau} - n + \binom{\tau+k-4}{\tau-2}} \right)$$
$$+ \sum_{a=1}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\binom{\binom{\tau+k-4}{\tau-1}}{a} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau} - n + \binom{\tau+k-4}{\tau-2} + a} \right)$$

Setting $a = 0$ in the sum yields exactly the term before the sum. Therefore we can put it into the sum.

$$= \sum_{a=0}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\binom{\binom{\tau+k-4}{\tau-1}}{a} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau} - n + \binom{\tau+k-4}{\tau-2} + a} \right)$$

We use the symmetry of binomial coefficients to simplify the last term

$$= \sum_{a=\max\left(0;\ n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}\right)}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\binom{\binom{\tau+k-4}{\tau-1}}{a} \cdot \binom{\binom{\tau+k-4}{\tau}}{n - \binom{\tau+k-3}{\tau-1} - a} \right)$$

case 2.

$$\sum_{a=n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\binom{\binom{t(h_1,k)+k-3}{t(h_1,k)}}{\binom{t(h_1,k)+k-2}{t(h_1,k)}-h_1} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau}-n+h_1}\right)$$

$$= \sum_{a=n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\binom{\binom{t(h(1;\tau-1;k)+a,k)+k-3}{t(h(1;\tau-1;k)+a,k)}}{\binom{t(h(1;\tau-1;k)+a,k)+k-2}{t(h(1;\tau-1;k)+a,k)}-h(1;\tau-1;k)-a} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau}-n+h(1;\tau-1;k)+a}\right)$$

As $a \geq 1$, we can apply Lemma 3.8

$$= \sum_{a=n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\binom{\binom{\tau-1+k-3}{\tau-1}}{\binom{\tau-1+k-2}{\tau-1}-h(1;\tau-1;k)-a} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau}-n+h(1;\tau-1;k)+a}\right)$$

We can replace now $h(1;\tau-1;k)$ with $\binom{\tau+k-4}{\tau-2}$

$$= \sum_{a=n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\binom{\binom{\tau-1+k-3}{\tau-1}}{\binom{\tau-1+k-2}{\tau-1}-\binom{\tau+k-4}{\tau-2}-a} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau}-n+\binom{\tau+k-4}{\tau-2}+a}\right)$$

Using several identities for binomial coefficients yields

$$= \sum_{a=n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\binom{\binom{\tau+k-4}{\tau-1}}{\binom{\tau+k-4}{\tau-1}-a} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau}-n+\binom{\tau+k-4}{\tau-2}+a}\right)$$

$$= \sum_{a=n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\binom{\binom{\tau+k-4}{\tau-1}}{a} \cdot \binom{\binom{\tau+k-4}{\tau}}{\binom{\tau+k-3}{\tau}-n+\binom{\tau+k-4}{\tau-2}+a}\right)$$

$$\sum_{a=n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\binom{\binom{\tau+k-4}{\tau-1}}{a} \cdot \binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a}\right) \tag{3.1}$$

Now have a look at the following sum:

$$\sum_{a=0}^{n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}-1} \left(\binom{\binom{\tau+k-4}{\tau-1}}{a} \cdot \binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a}\right) \tag{3.2}$$

We prove that this sum is always 0 by showing that the lower part of the second binomial coefficient is always bigger than the upper part.

$$\binom{\tau+k-4}{\tau} - \left(n - \binom{\tau+k-3}{\tau-1} - a\right) < 0 \qquad \left|\, 0 \leq a \leq n - \binom{\tau+k-3}{\tau-1} - \binom{\tau+k-4}{\tau} - 1\right.$$

$$\binom{\tau+k-4}{\tau} + \binom{\tau+k-3}{\tau-1} - n < -a$$

$$n - \binom{\tau+k-4}{\tau} - \binom{\tau+k-3}{\tau-1} > a$$

If this inequality is true for the biggest a, it is true for all a.
Therefore, let $a = n - \binom{\tau+k-3}{\tau-1} - \binom{\tau+k-4}{\tau} - 1$

$$n - \binom{\tau+k-4}{\tau} - \binom{\tau+k-3}{\tau-1} > n - \binom{\tau+k-3}{\tau-1} - \binom{\tau+k-4}{\tau} - 1 \qquad \Leftrightarrow \qquad 1 > 0$$

As we proved that sum (3.2) is 0, we can just add it to (3.1). Thereby we get

$$\begin{aligned}
&\sum_{a=n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\left(\binom{\binom{\tau+k-4}{\tau-1}}{a}\right) \cdot \left(\binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a}\right)\right) \\
= &\sum_{a=0}^{n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}-1} \left(\left(\binom{\binom{\tau+k-4}{\tau-1}}{a}\right) \cdot \left(\binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a}\right)\right) \\
+ &\sum_{a=n-\binom{\tau+k-3}{\tau-1}-\binom{\tau+k-4}{\tau}}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\left(\binom{\binom{\tau+k-4}{\tau-1}}{a}\right) \cdot \left(\binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a}\right)\right) \\
= &\sum_{a=0}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\left(\binom{\binom{\tau+k-4}{\tau-1}}{a}\right) \cdot \left(\binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a}\right)\right)
\end{aligned}$$

As a result, case 1 and case 2 are equivalent. ☐

Lemma 3.10.

$$\sum_{a=0}^{\min\left(n-\binom{\tau+k-3}{\tau-1};\ \binom{\tau+k-4}{\tau-1}\right)} \left(\left(\binom{\binom{\tau+k-4}{\tau-1}}{a}\right) \cdot \left(\binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a}\right)\right) = \binom{\binom{\tau+k-3}{\tau}}{\binom{\tau+k-2}{\tau}-n}$$

Proof. case distinction:

case 1 $n - \binom{\tau+k-3}{\tau-1} > \binom{\tau+k-4}{\tau-1}$
case 2 $n - \binom{\tau+k-3}{\tau-1} \leq \binom{\tau+k-4}{\tau-1}$

case 1.

$$\sum_{a=0}^{\binom{\tau+k-4}{\tau-1}} \left(\left(\binom{\binom{\tau+k-4}{\tau-1}}{a}\right) \cdot \left(\binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a}\right)\right) \tag{3.3}$$

In order to prove the equivalence of case 1 and 2, we look at the following sum.

$$\sum_{a=\binom{\tau+k-4}{\tau-1}+1}^{n-\binom{\tau+k-3}{\tau-1}} \left(\left(\binom{\binom{\tau+k-4}{\tau-1}}{a}\right) \cdot \left(\binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a}\right)\right) \tag{3.4}$$

We prove that sum 3.4 is always 0 by showing that lower part of the first binomial coefficient is smaller than the upper part.

$$\binom{\tau+k-4}{\tau-1} - a < 0 \qquad \left| \binom{\tau+k-4}{\tau-1} + 1 \leq a \leq n - \binom{\tau+k-3}{\tau-1} \right.$$
$$\binom{\tau+k-4}{\tau-1} < a$$

If this inequality is true for the smallest a, then it is true for all a.
Therefore, let $a = \binom{\tau+k-4}{\tau-1} + 1$

$$\binom{\tau+k-4}{\tau-1} < \binom{\tau+k-4}{\tau-1} + 1 \qquad \Leftrightarrow \qquad 0 < 1$$

As we proved that sum (3.4) is 0, we can just add it to (3.3). Thereby we get

$$\begin{aligned}
&\sum_{a=0}^{\binom{\tau+k-4}{\tau-1}} \left(\binom{\binom{\tau+k-4}{\tau-1}}{a} \cdot \binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a} \right) \\
=&\sum_{a=0}^{\binom{\tau+k-4}{\tau-1}} \left(\binom{\binom{\tau+k-4}{\tau-1}}{a} \cdot \binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a} \right) \\
+&\sum_{a=\binom{\tau+k-4}{\tau-1}+1}^{n-\binom{\tau+k-3}{\tau-1}} \left(\binom{\binom{\tau+k-4}{\tau-1}}{a} \cdot \binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a} \right) \\
=&\sum_{a=0}^{n-\binom{\tau+k-3}{\tau-1}} \left(\binom{\binom{\tau+k-4}{\tau-1}}{a} \cdot \binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a} \right)
\end{aligned}$$

As a result, case 1 and case 2 are equivalent.

case 2.

We apply Vandermonde's identity

$$\sum_{a=0}^{n-\binom{\tau+k-3}{\tau-1}} \left(\binom{\binom{\tau+k-4}{\tau-1}}{a} \cdot \binom{\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}-a} \right) = \binom{\binom{\tau+k-4}{\tau-1}+\binom{\tau+k-4}{\tau}}{n-\binom{\tau+k-3}{\tau-1}}$$

Using the symmetry of binomial coefficients, we obtain

$$\binom{\binom{\tau+k-3}{\tau}}{n-\binom{\tau+k-3}{\tau-1}} = \binom{\binom{\tau+k-3}{\tau}}{\binom{\tau+k-2}{\tau}-n}$$

□

Theorem 2.

$$\Upsilon_2(n,k) = \Upsilon(n,k)$$

Proof. By Lemma 3.9 and Lemma 3.10. □

4. Discussion

We considerer it to be really astounding that the number of possibilities does show such beautiful patterns (see appendix). Although the question doesn't seem to have any connection to Pascals triangle at a first view, table 1 does show surprising similarities and symmetries. Theorem 2 is also solving a question of combinatorics: It calculates the number of possibilities to place n identical objects on m pegs, where for each peg there is a minimum and a maximum number of objects, if these minimum and maximum numbers have the form $h(p;t-1;k)$ and $h(p;t;k)$ for certain t. Of course, this is not spectacular, because it only applies to very specific conditions. However, there might be done some research based on this attempt to solve the problem for more general minimum and maximum numbers.

5. Appendix

Table 1. Number of possibilities to solve the n-disk k-peg Tower of Hanoi problem for $n, k \leq 15$

	number of disks														
$k \downarrow$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	1	2	1	3	3	1	4	6	4	1	5	10	10	5	1
5	1	3	3	1	6	15	20	15	6	1	10	45	120	210	252
6	1	4	6	4	1	10	45	120	210	252	210	120	45	10	1
7	1	5	10	10	5	1	15	105	455	1365	3003	5005	6435	6435	5005
8	1	6	15	20	15	6	1	21	210	1330	5985	20349	54264	116280	203490
9	1	7	21	35	35	21	7	1	28	378	3276	20475	98280	376740	1184040
10	1	8	28	56	70	56	28	8	1	36	630	7140	58905	376992	1947792
11	1	9	36	84	126	126	84	36	9	1	45	990	14190	148995	1221759
12	1	10	45	120	210	252	210	120	45	10	1	55	1485	26235	341055
13	1	11	55	165	330	462	462	330	165	55	11	1	66	2145	45760
14	1	12	66	220	495	792	924	792	495	220	66	12	1	78	3003
15	1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1	91

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