Abstract

Contact geometry is the study of odd-dimensional smooth manifolds equipped with contact structures, i.e. hyperplane distributions $\xi = \ker \alpha$ satisfying the contact condition

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0.$$

While they originally arise in the study of ODEs and in classical mechanics, the topological study of contact manifolds is a more recent and very active field of research.

A manifold can have multiple different contact structures, which can be either rigid (in which case one speaks of a "tight" manifold) or flexible (in the sense that they satisfy an h-principle). The latter contact manifolds are then called overtwisted. A foundational result of Eliashberg and Borman–Eliashberg–Murphy, roughly speaking, states that overtwisted contact manifolds exist in abundance, namely whenever the manifold admits the topological version of a contact structure (an *almost* contact structure), which is a first obvious obstruction. In dimension three, an almost contact structure is simply an oriented 2-plane field.

To illustrate this dichotomy, consider the sphere S^3. By a result of Eliashberg, it has precisely one tight contact structure. On the other hand, it has infinitely many overtwisted contact structures, corresponding to the infinitely many homotopy classes of 2-plane fields on the 3-sphere. There are other examples where there are infinitely many or no tight contact structures on a contact manifold.

A further interesting property of contact manifolds comes from the fact that contact geometry is the odd-dimensional counterpart to symplectic geometry. Often, it is possible to view a contact manifold as the boundary of a symplectic manifold. Manifolds that are in this sense "fillable" are always tight. The contrary, however, doesn't need to hold and one can ask the question under which conditions such tight, but non-fillable manifolds exist. The first examples of tight and non-fillable contact manifolds were constructed by Etnyre–Honda in dimension three, and by Massot–Niederkrueger–Wendl in higher dimensions.

More recently, Bowden–Gironella–Moreno–Zhou have shown that there exist homotopically standard, non-fillable but tight contact structures on all spheres S^{2n+1} with $n >= 2$. Starting with a specific open book decomposition of S^{2n-1}, one can construct a contact form on this manifold using a well-known construction by Thurston–Winkelnkemper. Then, according to Bourgeois, this contact structure can be extended to a tight contact structure on $S^{2n-1} \times T^2$. Applying subcritical surgery (preserving the tightness), one can kill the topology of the T^2-factor and obtain a tight contact structure on S^{2n+1}. Because of the special way of constructing it, one can show that it is non-fillable, but still homotopically standard.

The goal of my master thesis is to give a streamlined explanation of the results of Bowden–Gironella–Moreno–Zhou, including the necessary background needed to understand the main ideas.

Preliminaries

The Bourgeois construction

Definition 1. *An open book decomposition of a manifold M is a pair (B, p) where the binding B is a closed codim-2-submanifold of M and the map p is a smooth, locally trivial fibration* <u>*i.e. a fiber bundle?*</u>

$$p : M \setminus B \to S^1.$$

The fibres $p^{-1}(\varphi), \varphi \in S^1$ are called the pages. Moreover, it is required that the binding B has a trivial tubular neighborhood $B \times D^2$ in which p is given by the angular coordinate in the D^2-factor.

Definition 2 (another definition of open book)**.** *An open book decomposition of a manifold M is a pair (B, p), together with a defining map $\Phi : M \to \mathbb{R}^2$ so that each $z \in \mathrm{int}(D^2)$ is a regular value. Here, $B \subset M$ is a closed codimension-2 submanifold, $p : M \setminus B \to S^1$ is a fiber bundle, and Φ is such that $\Phi^{-1}(0) = B$ and $p = \Phi/|\Phi|$.*

Lemma 1. *Definition 1 and definition 2 are equivalent.*

Proof.

" $\Longrightarrow$ " First, we need to construct a defining map Φ. On the trivial tubular neighborhood take Φ to be the D^2-component. Outside this neighborhood, just set $\Phi = p$. Then, $\Phi^{-1}(0) = B, p = \Phi/|\Phi|$ and for all $z \in \mathrm{int}(D^2)$, there is always a small neighborhood s.t. Φ is just a projection map and hence, z is a regular value.

" $\Longleftarrow$ " We need to show the existence of the trivial tubular neighborhood of B. $\Phi^{-1}(D^2) \to D^2$ is a smooth submersion, as all points are regular points. According to the rank theorem, we can choose local coordinates for a neighborhood of any $x \in B$ s.t. Φ is just a projection to the D^2-component. Due to the compactness of B, finitely many such neighborhoods suffice to cover B. Moreover, the coordinate maps can be chosen in a compatible way and glued together. In total, we find a chart ψ of a neighborhood U of B s.t.

$$\psi(U) \subset \psi(B) \times D^2.$$

As there are only finitely many neighborhoods involved, we find an $\epsilon > 0$ s.t. $\Phi^{-1}(D^2_\epsilon) \subset \psi(U)$. Therefore, U contains a neighborhood of the form $B \times D^2$. This is the desired trivial tubular neighborhood.

□

Remark 1. *Let M be an oriented manifold with an open book decomposition (B, p). Choose a parametrization of S^1 (i.e. the direction of ∂_φ). Then, there are two equivalent ways of defining the orientation of B and the pages.*

- *Via the orientation of the pages: The orientation of M is the same as the orientation of the page together with ∂_φ.*
- *Via the orientation of the binding: $B \times D^2$ is an embedded submanifold of the same dimension as M. Therefore, it inherits the orientation. As $B \times D^2$ carries the product orientation and the orientation of D^2 is given by $r\mathrm{d}r \wedge \mathrm{d}\varphi$, we can deduce the orientation of B from the orientation of M.*

In both cases, the orientation of the pages and the binding have to agree in the sense that the induced orientation on the boundary of (the closure) of each page coincides with the orientation of B. To see that both options are in fact equivalent, choose a positive basis $\mathfrak{b}$ of the binding B. Then, $\mathfrak{b}, \partial_r, \partial_\varphi$ is a positive basis for M at that point according to the first option. As $-\partial_r$ is the outer normal vector of the page at this point, $-\partial_r, \mathfrak{b}$ is a positive basis of the page and so we get $-\partial_r, \mathfrak{b}, \partial_\varphi$ as a positive basis for M, which agrees with option 1 because B is odd-dimensional.

Definition 3. *Let (B, p) be an oriented open book decomposition of the oriented manifold M. A contact structure $\xi = \ker \alpha$ on M is said to be* ***supported*** *by the open book decomposition (B, p) of M if*

(i) the contact form α induces the positive orientation of M ($\alpha \wedge (\mathrm{d}\alpha)^n > 0$).

(ii) the 2-form $\mathrm{d}\alpha$ induces a symplectic form on each page, defining its positive orientation

(iii) the 1-form α induces a positive contact form on B, i.e.

$$\alpha|_{TB} \wedge (\mathrm{d}\alpha|_{TB})^{(n-2)} > 0.$$

Theorem 1. *Let $(M, \xi = \ker \alpha)$ be a closed contact manifold of dimension $2n-1, n \geq 2$. One can find an open book decomposition (B, p) of M supporting ξ. According to Bourgeois, ([Bou02]) there is a contact structure $\tilde{\xi}$ on $M \times T^2$ (where $\tilde{\xi}$ massively depends on the choice of open book).*

Proof. We follow the proof of [Gei08, Thm 7.3.6]. Wlog let M be connected. The existence of an open book decomposition for M is the theorem of Giroux-Mohsen as in [Gei08, Thm 7.3.5]. By definition of an open book, there exists a tubular neighborhood $B \times D^2$ with polar coordinates (r, p) on the D^2-part of the binding B, i.e. the angular coordinate is simply given by $p : M \setminus B \to S^1$ in that neighborhood. Now, we want to define smooth functions x_1, x_2 on M that coincide with the cartesian coordinate functions on D^2 close to the binding B. In order to do that, choose a smooth function $\rho(r)$ on $B \times D^2$, s.t.

- $\rho = r$ near the binding B,
- $\rho'(r) \geq 0$
- $\rho \equiv 1$ near $B \times \partial D^2$.

We extend this function to a smooth function $\rho : M \to [0, 1]$ by setting $\rho \equiv 1$ outside $B \times D^2$. Now, $x_1 := \rho \cos p$ and $x_2 := \rho \sin p$ are the desired smooth functions on M that coincide with the Cartesian coordinate functions on the D^2-factor near B. We compute

$$\begin{aligned} x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1 &= \rho^2 \cos^2 p \mathrm{d}p + \rho \cos p \sin p \mathrm{d}\rho + \rho^2 \sin^2 p \mathrm{d}p - \rho \cos p \sin p \mathrm{d}\rho \\ &= \rho^2 (\cos^2 p + \sin^2 p) \mathrm{d}p \\ &= \rho^2 \mathrm{d}p \end{aligned}$$

and, analogously,

$$\mathrm{d}x_1 \wedge \mathrm{d}x_2 = \rho \mathrm{d}\rho \wedge \mathrm{d}p.$$

On $M \times T^2$, choose coordinates (θ_1, θ_2) on the torus part of the manifold. Now we have all ingredients together to construct our contact form. Let

$$\tilde{\alpha} := x_1 \mathrm{d}\theta_1 - x_2 \mathrm{d}\theta_2 + \alpha.$$

This is a well-defined 1-form on $M \times T^2$ (α is extended to $M \times T^2$ as the pullback $\pi_1^* \alpha$) and we can compute the derivative

$$\mathrm{d}\tilde{\alpha} = \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 - \mathrm{d}x_2 \wedge \mathrm{d}\theta_2 + \mathrm{d}\alpha,$$

hence

$$\begin{aligned} (\mathrm{d}\tilde{\alpha})^n = &(n-1)(\mathrm{d}\alpha)^{n-1} \wedge (\mathrm{d}x_1 \wedge \mathrm{d}\theta_1 - \mathrm{d}x_2 \wedge \mathrm{d}\theta_2) \\ &- n(n-1)(\mathrm{d}\alpha)^{n-2} \wedge \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\theta_2. \end{aligned}$$

In order to verify the contact condition, we compute

$$\begin{aligned} \tilde{\alpha} \wedge (\mathrm{d}\tilde{\alpha})^n = &(x_1 \mathrm{d}\theta_1 - x_2 \mathrm{d}\theta_2 + \alpha) \wedge (n-1)(\mathrm{d}\alpha)^{n-1} \wedge (\mathrm{d}x_1 \wedge \mathrm{d}\theta_1 - \mathrm{d}x_2 \wedge \mathrm{d}\theta_2) \\ &- (x_1 \mathrm{d}\theta_1 - x_2 \mathrm{d}\theta_2 + \alpha) \wedge n(n-1)(\mathrm{d}\alpha)^{n-2} \wedge \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\theta_2 \\ = &(n-1)(\mathrm{d}\alpha)^{n-1} \wedge (x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1) \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \\ &+ \underbrace{\alpha \wedge (n-1)(\mathrm{d}\alpha)^{n-1} \wedge \mathrm{d}x_1}_{2n\text{-form on } M} \wedge \mathrm{d}\theta_1 - \underbrace{\alpha \wedge (n-1)(\mathrm{d}\alpha)^{n-1} \wedge \mathrm{d}x_2}_{2n\text{-form on } M} \wedge \mathrm{d}\theta_2 \\ &+ n(n-1)\alpha \wedge (\mathrm{d}\alpha)^{n-2} \wedge \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\theta_2 \end{aligned}$$

M has dimension $2n - 1$, i.e. the middle term is 0

$$\begin{aligned} = &(n-1)(\mathrm{d}\alpha)^{n-1} \wedge \rho^2 \mathrm{d}\phi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \\ &+ n(n-1)\alpha \wedge (\mathrm{d}\alpha)^{n-2} \wedge \rho \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \end{aligned}$$

By condition (ii) of definition 3, $(\mathrm{d}\alpha)^{n-1}$ must be a positive volume form on the pages. As explained in that definition, the orientation on M is given by ∂_ϕ and the orientation of the page. In particular, $(\mathrm{d}\alpha)^{n-1} \wedge \rho \mathrm{d}\phi$ is a positive volume form on M. Multiplied with a second ρ-factor, it vanishes along B. As $\theta_1 \wedge \theta_2$ is a positive volume form on T^2, the first term is non-negative everywhere and positive away from

$$\underbrace{B \times 0}_{\subset B \times D^2 \subset M} \times T^2.$$

Let $\mathfrak{b}$ be a basis of the binding B that is positively ordered. Then, $-\partial_r, \mathfrak{b}$ and (because the binding is odd-dimensional) $\mathfrak{b}, \partial_r$ are positive bases of the page. Clearly, then,

$$\mathfrak{a} := \mathfrak{b}, \partial_r, \partial_\phi, \partial_{\theta_1}, \partial_{\theta_2}$$

is an ordered basis of $M \times T^2$. Using $\rho'(r) \geq 0$ everywhere, we deduce that $\mathrm{d}\rho(\partial_r)$ is non-negative. Hence, plugging $\mathfrak{a}$ into the second term, we conclude

$$\begin{aligned} &\left(n(n-1)\alpha \wedge (\mathrm{d}\alpha)^{n-2} \wedge \rho \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2\right)(\mathfrak{a}) \\ =&n(n-1)\rho \cdot (\alpha \wedge (\mathrm{d}\alpha)^{n-2})(\mathfrak{b}) \cdot d\rho(\partial_r) \cdot \mathrm{d}\phi(\partial_\phi) \cdot \mathrm{d}\theta_1(\partial_{\theta_1}) \cdot \mathrm{d}\theta_2(\partial_{\theta_2}) \\ \geq& 0. \end{aligned}$$

By condition (iii) of definition 3, $\alpha \wedge (\mathrm{d}\alpha)^{n-2}$ is positive on B. Therefore, the second term is positive on $B \times 0 \times T^2$ (hence also on a neighborhood) and non-negative everywhere else. In total, we have checked the contact condition and $\tilde{\alpha}$ is indeed a contact form on $M \times T^2$. □

Definition 4. *An almost contact structure is a cooriented hyperplane field η (with an oriented trivial line bundle complementary to η defining the coorientation) and a complex bundle structure J on η. According to [Gei08, Prop 2.4.5], the space of complex bundle structures compatible to a symplectic form on η is non-empty and contractible. Thus, the almost contact structure can (up to homotopy) be defined by equipping η with a symplectic form.*

If we don't care about the coorientation, we can therefore talk of the almost contact structure (η, ω) where ω is a symplectic form on η.

Remark 2. *Let $\epsilon > 0$. The, $\tilde{\alpha}_\epsilon := \epsilon x_1 \mathrm{d}\theta_1 - \epsilon x_2 \mathrm{d}\theta_2 + \alpha$ is a contact form. By Gray stability, all these contact structures are isotopic, so the underlying almost contact structures are homotopic, too. For $\epsilon = 0$, we have the almost contact structure $\xi \oplus TT^2, \mathrm{d}\alpha \wedge \Omega$ where Ω is a volume form on T^2. This is homotopic to the almost contact structures for $\epsilon > 0$. Thus, up to homotopy, the almost contact structure of the Bourgeois form is given by*

$$\xi \oplus TT^2, \mathrm{d}\alpha \wedge \Omega.$$

The Thurston-Winkelnkemper construction

Definition 5 (mapping torus)**.** *Let Σ be a smooth manifold with boundary $\partial\Sigma$ and $\phi : \Sigma \to \Sigma$ a diffeomorphism that is equal to the identity close to $\partial\Sigma$. The mapping torus $\Sigma(\phi)$ is given by $\Sigma \times [0, 2\pi]/\sim$ where*

$$(x, 2\pi) \sim (\phi(x), 0).$$

The generalized mapping torus requires as additional data a smooth function $\overline{\varphi} : \Sigma \to \mathbb{R}^+$ *that is constant near* $\partial\Sigma$. *Then,*

$$\Sigma_{\overline{\varphi}}(\phi) \coloneqq \Sigma \times \mathbb{R}/\sim \quad \textit{where} \quad (x,\theta) \sim (\phi(x), \theta - \overline{\varphi}(x)).$$

Abstract open books Starting with a mapping torus $\Sigma(\phi)$, we can construct an abstract open book $M(\phi)$ with binding $\partial\Sigma$ (see fig. 1)

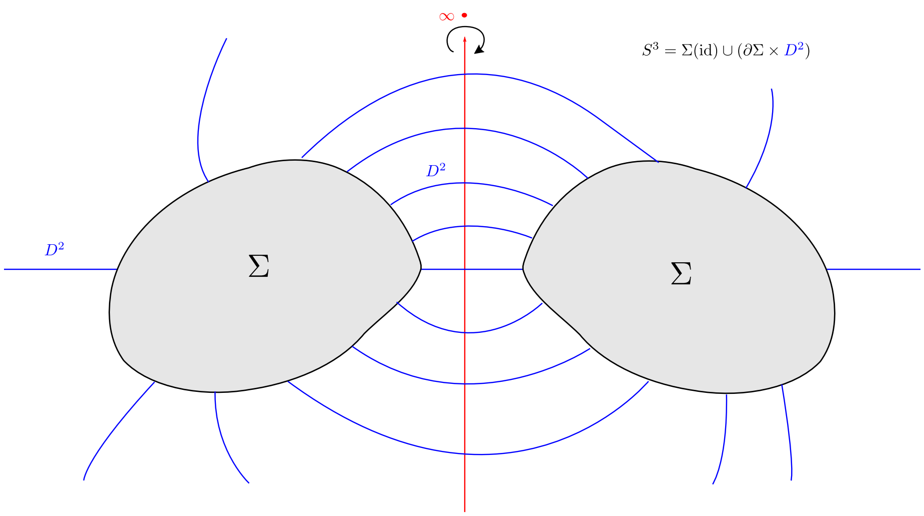

Figure 1: abstract open book

We define

$$M(\phi) \coloneqq \left(\Sigma(\phi) \cup \partial\Sigma \times D^2\right) / \sim$$

where we identify

$$[x \in \partial\Sigma, \theta] \sim (x, r = 1, \varphi = \theta)$$

The construction Let Σ^{2n} be a compact manifold admitting an exact symplectic form $\omega = \mathrm{d}\beta$ s.t. on the boundary $\partial\Sigma$, a contact form β_∂ is induced (this follows from the conditions requested in Geiges). Let the boundary be connected (i.e. the binding is also connected). Let the monodromy map ϕ be an exact symplectomorphism of (Σ, ω), equal to the identity near the boundary $\partial\Sigma$ (exactness is not necessary according to Geiges, as it can be obtained via a suitable isotopy of the symplectomorphism). An exact symplectomorphism ϕ of (Σ, ω) is such that

$$\phi^*(\beta) - \beta \eqqcolon \mathrm{d}\overline{\varphi}$$

is exact, i.e. there exists such a function $\overline{\varphi}$ on Σ (of course only defined up to adding a locally constant function. Choose it in such a way that it only takes positive values). The 1-form

$$\alpha := \beta + \mathrm{d}\varphi$$

is a contact form on $\Sigma \times \mathbb{R}$:

$$\alpha \wedge (\mathrm{d}\alpha)^n = (\beta + \mathrm{d}\varphi) \wedge \underbrace{(\mathrm{d}\beta)^n}_{=:\Omega} = \beta \wedge \Omega + \mathrm{d}\varphi \wedge \Omega = \mathrm{d}\varphi \wedge \Omega,$$

where Ω is a volume form on Σ (as β is a symplectic form). The $\beta \wedge \Omega$ term vanishes because both are forms on Σ, but Ω is already a top-level form. The resulting form is a wedge product of two volume forms on the product manifolds and therefore a volume form on $\Sigma \times \mathbb{R}$.

Now consider the transformation that induces the generalized mapping torus

$$F := (x, \varphi) \mapsto (\phi(x), \varphi - \overline{\varphi}(x)).$$

Remember that φ only takes positive values, i.e. the mapping torus is welldefined. The 1-form α is invariant under this transformation:

$$\begin{aligned} F^*(\alpha) &= F^*(\beta) + F^*(\mathrm{d}\varphi) && \mid \beta \text{ is independent of } \varphi \\ &= \phi^*(\beta) + \mathrm{d}F(\varphi) && \mid \text{definition of } \overline{\varphi}, F \\ &= \beta + \mathrm{d}\overline{\varphi} + \mathrm{d}\varphi - \mathrm{d}\overline{\varphi} \\ &= \alpha. \end{aligned}$$

It follows that α descends to a contact form on $\Sigma_{\overline{\varphi}}(\phi)$.

Now, we describe an adapted gluing construction for the abstract open book coming from a generalized mapping torus. Therefore, we construct a collar neighborhood on the generalized mapping torus s.t. on $[-1, 0] \times \partial\Sigma$, the symplectic form is given by $\mathrm{d}\left(e^s \beta_\partial\right)$ where s is the collar parameter, d.h. $\beta = e^s \beta_\partial$. Why does such a neighborhood exist?

Close to $\partial\Sigma$, ϕ is equal to the identity and therefore $\mathrm{d}\overline{\varphi}$ is locally constant (hence constant, as $\partial\Sigma$ is connected). Parametrize the neighborhood so that $\overline{\varphi}$ is constant on $[-1, 0] \times \partial\Sigma$.

Now, take a look at

$$M := \left(\Sigma_{\overline{\varphi}}(\phi) \ \dot\cup \ \left(\partial\Sigma \times D_2^2\right)\right) / \sim .$$

A simple linear reparametrization will make the notation a lot easier: As $\overline{\varphi}$ is constant on the neighborhood under consideration, we just pretend $\overline{\varphi} = 2\pi$. Furthermore, we parametrize the boundary $\partial\Sigma$ with $\theta \in S^1$. Then we identify

$$(s, \theta, \varphi) \in [-1, 0] \times \partial\Sigma \times S^1 \subset \Sigma_{\overline{\varphi}}(\phi)$$

with

$$(\theta, s = 1 - r, \varphi) \in \partial\Sigma \times D_2^2 =: \mathcal{N}$$

where (r, φ) are polar coordinates on D_2^2, i.e. we identify a collar neighborhood of Σ with an annulus in D_2^2. (See fig. 2)

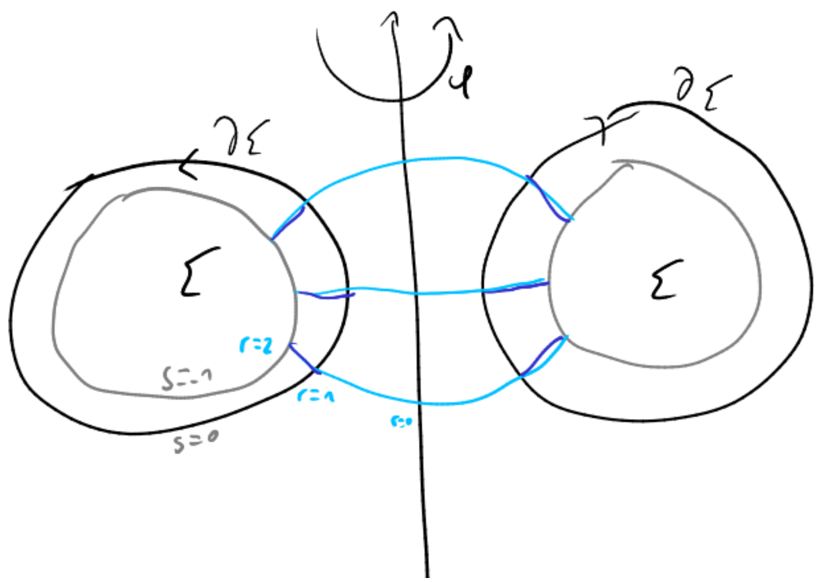

Figure 2: Detailled gluing process of the generalized abstract open book

Now we choose the ansatz

$$\alpha_{\text{ext}} \coloneqq h_1(r)\beta_\partial + h_2(r)\mathrm{d}\varphi.$$

for the extension of the contact form over $\mathcal{N}$. On the gluing area (i.e. $1 \leq r \leq 2$), α_{ext} has to agree with $\alpha = \beta + \mathrm{d}\varphi = e^s\beta_\partial + \mathrm{d}\varphi$, i.e.

$$h_1(r) = e^s = e^{1-r} \qquad h_2(r) = 1.$$

In order to ensure smoothness at $r = 0$, in a small neighborhood of $r = 0$ we set $h_1(r) = 2 - r^2$ and $h_2(r) = r^2$, obtaining

$$\alpha_{\text{ext}}|_0 = 2\beta_\partial$$

We compute

$$\mathrm{d}\alpha_{\text{ext}} = h_1'(r)\mathrm{d}r \wedge \beta_\partial + h_1(r)\mathrm{d}\beta_\partial + h_2'(r)\mathrm{d}r \wedge \mathrm{d}\varphi.$$

and

$$(\mathrm{d}\alpha_{\text{ext}})^n = n \cdot \mathrm{d}r \wedge (h_1'(r)\beta_\partial + h_2'(r)\mathrm{d}\varphi) \cdot h_1(r)^{n-1}(\mathrm{d}\beta_\partial)^{n-1} + \underbrace{h_1(r)^n(\mathrm{d}\beta_\partial)^n}_{=0},$$

where the second term vanishes because $(\mathrm{d}\beta_\partial)^n$ is a $2n$-form on $\partial\Sigma^{2n-1}$. Finally,

$$\begin{aligned}
\alpha_{\text{ext}} \wedge (\mathrm{d}\alpha_{\text{ext}})^n = h_1(r) n h_1(r)^{n-1} h_2'(r) \cdot \quad & \beta_\partial \wedge \mathrm{d}r \wedge \mathrm{d}\varphi \wedge (\mathrm{d}\beta_\partial)^{n-1} \\
+ h_2(r) n h_1(r)^{n-1} h_1'(r) \cdot \quad & \mathrm{d}\varphi \wedge \mathrm{d}r \wedge \beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1} \\
= n h_1(r)^{n-1} (h_1 h_2'(r) - h_2 h_1'(r)) \cdot \quad & \beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1} \wedge \mathrm{d}r \wedge \mathrm{d}\varphi \\
= n h_1(r)^{n-1} \det \begin{pmatrix} h_1(r) & h_2(r)/r \\ h_1'(r) & h_2'(r)/r \end{pmatrix} \cdot \quad & \beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1} \wedge r\mathrm{d}r \wedge \mathrm{d}\varphi
\end{aligned}$$

As β_∂ is a contact form on $\partial\Sigma$, $\beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1}$ is a positive volume form on $\partial\Sigma$. Furthermore, $r\mathrm{d}r \wedge \mathrm{d}\varphi$ is a positive volume form on the disk D_2^2. As a result, the right term of our result is a volume form on $\mathcal{N} = \partial\Sigma \times D_2^2$. The left term tells us that $h_1(r)$ musn't have any zeros for $r \in [0,2]$ and that $(h_1(r), h_2(r))$ must never be parallel to $(h_1'(r), h_2'(r))$, i.e.

$$h_1(r)^{n-1} \det \begin{pmatrix} h_1(r) & h_2(r)/r \\ h_1'(r) & h_2'(r)/r \end{pmatrix} > 0 \qquad \forall r \in [0,2].$$

(Close to zero, the determinant is given by $2 \cdot 2 - 0 \cdot 0 = 4 > 0$). Figure 4.7 in [Gei08] proves the existence of such a pair of functions h_1 and h_2.

In total, we obtain that α_{ext} induces the correct orientation on the extension and, as M is connected and orientable, on all of M. In particular, condition (i) of definition 3 holds and $\alpha \wedge (\mathrm{d}\alpha)^n = \mathrm{d}\varphi \wedge \Omega$ is a positive volume form on the mapping torus. As $\Omega = (\mathrm{d}\beta)^n = \omega^n$ for ω the symplectic form on Σ, it is a $2n$-form and we see that Ω is a positive volume form on Σ. Thus, on Σ, $\mathrm{d}\alpha = \mathrm{d}\beta = \omega$ is a symplectic form that induces the positive orientation of Σ. On $\mathcal{N}$, we need to check that the form induced by $\mathrm{d}\alpha_{\text{ext}}$ on the pages is symplectic with the right orientation. Inside $\mathcal{N}$, a page is given by the condition $\varphi = \text{const}$, i.e. $\mathrm{d}\varphi = 0$. We have

$$\begin{aligned}
(\mathrm{d}\alpha_{\text{ext}})^n &= n \cdot \mathrm{d}r \wedge (h_1'(r)\beta_\partial + h_2'(r)\mathrm{d}\varphi) \cdot h_1(r)^{n-1} (\mathrm{d}\beta_\partial)^{n-1} \\
&= n h_1'(r) h_1(r)^{n-1} \mathrm{d}r \wedge \beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1}
\end{aligned}$$

A positive volume form on Σ must be positive on $-\partial_r, \mathfrak{b}$ where $\mathfrak{b}$ is a positive basis of a point in $\partial\Sigma$. As $\beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1}$ is a positive volume form on $\partial\Sigma$, we obtain

$$\begin{aligned}
(\mathrm{d}\alpha_{\text{ext}})^n(-\partial_r, \mathfrak{b}) &= \underbrace{n h_1(r)^{n-1}}_{=:A>0} \cdot h_1'(r) \mathrm{d}r(-\partial_r) \wedge \underbrace{\left[\beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1}\right](\mathfrak{b})}_{=:B>0} \\
&= AB \cdot h_1'(r) \cdot -1 \\
&> 0,
\end{aligned}$$

where in the last line we've used $h_1'(r) = \frac{\mathrm{d}}{\mathrm{d}r}(2 - r^2) = -2r < 0$. We have thus verified condition (ii) inside $\mathcal{N}$ and outside $\mathcal{N}$, on Σ. As a result, is must hold on the whole page. Condition (iii) follows from the fact that on B, $\alpha_{\text{ext}} = 2\beta_\partial$ which is a positive contact form on $\partial\Sigma$ and therefore also on B.

Bourgeois & Thurston–Winkelnkemper

Applying the Thurston–Winkelnkemper construction yields a contact form α on

$$M = (\Sigma_{\overline{\varphi}}(\phi) \dot{\cup} \mathcal{N}) \,/ \sim .$$

Now, we apply the Bourgeois construction to it and obtain a contact form

$$\tilde{\alpha} = \alpha + x_1 \mathrm{d}\theta_1 - x_2 \mathrm{d}\theta_2$$

on $M \times T^2$ where θ_1, θ_2 are coordinates on T^2 and x_1, x_2 are coordinates as described in the section on the Bourgeois construction, i.e. there is a function $\rho : M \to [0,1]$ that agrees with r near the binding B and we define

$$x_1 \coloneqq \rho \cos(p); \qquad x_2 \coloneqq \rho \sin(p).$$

Inside $\mathcal{N}$, our projection map p is given by the angular coordinate φ. Therefore,

$$\alpha|_{\mathcal{N}} = \alpha_{\text{ext}} = h_1(r)\beta_\partial + h_2(r)\mathrm{d}\varphi = h_1(r)\beta_\partial + h_2(r)\mathrm{d}p.$$

In total, we obtain for the contact form on $\mathcal{U} \coloneqq \mathcal{N} \times T^2$

$$\tilde{\alpha} = h_1(r)\beta_\partial + h_2(r)\mathrm{d}p + \rho(r)(\cos(p)\mathrm{d}\theta_1 - \sin(p)\mathrm{d}\theta_2)$$

Outside $\mathcal{U}$, the form is given by

$$\tilde{\alpha} = \beta + \mathrm{d}p + \cos(p)\mathrm{d}\theta_1 - \sin(p)\mathrm{d}\theta_2,$$

as $\rho(r) = 1 \ \forall r \geq 1$. Collecting all the conditions on h_1 and h_2 from the last section, we require that

- $h_1(r) = e^s = e^{1-r}, h_2(r) = 1$ in the gluing area ($1 \leq r \leq 2$).
- Smoothness around the binding and contact condition on the binding: $h_1(r) = 2 - r^2$ and $h_2(r) = r^2$ around $r = 0$. The $-r^2$-part is only important to have $h_1'(r) < 0$ around 0 to satisfy the symplectic condition on the pages.
- Contact condition on the tubular neighborhood:

$$h_1(r)^{n-1} \det \begin{pmatrix} h_1(r) & h_2(r)/r \\ h_1'(r) & h_2'(r)/r \end{pmatrix} > 0 \qquad \forall r \in [0,2].$$

We now define two functions $\mu, \nu : M \to \mathbb{R}$ as follows:

$$\mu = \begin{cases} \frac{\rho'}{\rho' h_1 - \rho h_1'} & \text{inside } \mathcal{U} \\ 0 & \text{outside } \mathcal{U} \end{cases} \quad \text{and} \quad \nu = \begin{cases} \frac{-h_1'}{\rho' h_1 - \rho h_1'} & \text{inside } \mathcal{U} \\ 1 & \text{outside } \mathcal{U} \end{cases}.$$

Both μ and ν are smooth. First of all, we note that both are quotients of smooth functions. We have to make sure that $\rho' h_1 \neq \rho h_1'$. All of the involved functions are non-negative except for h_1', which is constantly negative (h_1 needs to be chosen in such a way that h_1' is nonzero). As a result,

$$\rho' h_1 > 0 > \rho h_1',$$

so the denominator is always nonzero. As ρ is constantly 1 for $r \geq 1$, we see that μ is 0 for $r \geq 1$. In particular, it is smooth everywhere. For ν, setting $\rho \equiv 1$ implies $\rho' = 0$ on $r \geq 1$. Therefore, the denominator simplifies to 1 and ν is smooth, too.

Lemma 2. *The Reeb vector field of the contact form $\tilde{\alpha}$ is given by*

$$R = \mu(r) R_B + \nu(r)[\cos(p)\partial_{\theta_1} - \sin(p)\partial_{\theta_2}],$$

where R_B is the Reeb vector field of β_∂, the contact form on $\partial\Sigma$.

Proof. Inside $\mathcal{U}$, we compute

$$\tilde{\alpha}(R) = h_1(r)\beta_\partial(R) + h_2(r)\mathrm{d}p(R) + \rho(r)(\cos(p)\mathrm{d}\theta_1 - \sin(p)\mathrm{d}\theta_2)(R)$$

β_∂ and $\mathrm{d}p$ are 0 on ∂_{θ_i}, p and θ_i are constant on R_B

$$\begin{aligned}
&= h_1\mu\beta_\partial(R_B) + \rho\nu(\cos(p)\mathrm{d}\theta_1 - \sin(p)\mathrm{d}\theta_2)(\cos(p)\partial_{\theta_1} - \sin(p)\partial_{\theta_2}) \\
&= h_1\mu + \rho\nu[\cos(p)\cos(p) + \sin(p)\sin(p)] \\
&= \frac{h_1\rho'}{\rho' h_1 - \rho h_1'} + \frac{-\rho h_1'}{\rho' h_1 - \rho h_1'} \\
&= 1.
\end{aligned}$$

Computing $\mathrm{d}\tilde{\alpha}$, we immediately drop the $\mathrm{d}r$-terms, as they evaluate to 0 on R.

$$\mathrm{d}\tilde{\alpha}(R, \cdot) = h_1(r)\mathrm{d}\beta_\partial(R, \cdot) - \rho(r)[\sin(p)\mathrm{d}p \wedge \mathrm{d}\theta_1 + \cos(p)\mathrm{d}p \wedge \mathrm{d}\theta_2](R, \cdot)$$

β_∂ and $\mathrm{d}p$ are 0 on ∂_{θ_i}, $\mathrm{d}\beta_\partial(R_B) = 0$, p is 0 on R_B

$$= \rho(r)\left[\sin(p)\mathrm{d}\theta_1(R)\mathrm{d}p + \cos(p)\theta_2(R)\mathrm{d}p\right]$$

$\mathrm{d}\theta_i$ is 0 on R_B

$$\begin{aligned}
&= \rho(r)\nu(r)\left[\sin(p)\cos(p) - \cos(p)\sin(p)\right]\mathrm{d}p \\
&= 0
\end{aligned}$$

Outside $\mathcal{U}$, we have

$$R = \cos(p)\partial_{\theta_1} - \sin(p)\partial_{\theta_2}.$$

Therefore,

$$\tilde{\alpha}(R) = \beta(R) + \mathrm{d}p(R) + \cos(p)\mathrm{d}\theta_1(R) - \sin(p)\mathrm{d}\theta_2(R)$$

β and dp are 0 on ∂_{θ_i}

$$
\begin{aligned}
&= [\cos(p)\mathrm{d}\theta_1 - \sin(p)\mathrm{d}\theta_2](\cos(p)\partial_{\theta_1} - \sin(p)\partial_{\theta_2}) \\
&= \cos(p)\cos(p) + \sin(p)\sin(p) \\
&= 1
\end{aligned}
$$

Also,

$$
\mathrm{d}\tilde{\alpha}(R) = \mathrm{d}\beta(R,\cdot) - \sin(p)\mathrm{d}p \wedge \mathrm{d}\theta_1(R,\cdot) - \cos(p)\mathrm{d}p \wedge \mathrm{d}\theta_2(R,\cdot)
$$

β and dp are 0 on ∂_{θ_i}

$$
\begin{aligned}
&= \sin(p)\mathrm{d}\theta_1(R)\mathrm{d}p + \cos(p)\mathrm{d}\theta_2(R)\mathrm{d}p \\
&= \sin(p)\cos(p)\mathrm{d}p - \cos(p)\sin(p)\mathrm{d}p \\
&= 0
\end{aligned}
$$

□

Claim: The computation in [Bou02] shows that, in order for $\tilde{\alpha}$ to be a contact form in the neighborhood $\mathcal{U}$ of $B \times T^2$ where $\alpha = h_1(r)\beta_\partial + h_2(r)\mathrm{d}p$, it is in fact enough that $h_1(h_1 - h_1') > 0$, a condition which only depends on h_1.

Proof. From our computations for the Bourgeois construction, we see that $(\mathrm{d}\alpha)^n$ must be a positive volume form on the pages and $\alpha \wedge (\mathrm{d}\alpha)^{n-1}$ must be a positive volume form on B for α to be a contact form. (The first condition is needed for the first term and the second condition for the second term in [Bou02, Equation (6)]). We have seen that

$$
\mathrm{d}\alpha = h_1'(r)\mathrm{d}r \wedge \beta_\partial + h_1(r)\mathrm{d}\beta_\partial + h_2'(r)\mathrm{d}r \wedge \mathrm{d}p.
$$

and

$$
(\mathrm{d}\alpha)^n = n \cdot \mathrm{d}r \wedge (h_1'(r)\beta_\partial + h_2'(r)\mathrm{d}\varphi) \cdot h_1(r)^{n-1}(\mathrm{d}\beta_\partial)^{n-1}
$$

In order for the latter to be a positive volume form on the pages, the ϕ coordinate doesn't matter. As a result, it is enough that $h_1'(r) \cdot h_1(r)^{n-1}$ is positive. Under the pullback to B, dr and dp-components vanish. Therefore,

$$
i^*\left(\alpha \wedge (\mathrm{d}\alpha)^{n-1}\right) = h_1(r)^n \beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1}.
$$

In order for this to be a positive volume form, we need $h_1(r)^n > 0$. Clearly, $h_1(r)^{2k}$ is always positive. Therefore, if n is odd, we need $h_1(r) > 0$ and $h_1'(r) > 0$. If n is even, we only need $h_1'(r) \cdot h_1(r) > 0$. In fact, these conditions only depend on h_1. However, it is not the same condition as claimed. □

In particular, one can homotope the pair (h_1, h_2) in a compactly supported way Why compactly supported? among pairs of functions satisfying this condition, and this will result in a homotopy of contact forms (hence isotopy by Gray's stability) on $M \times T^2$, independently of the fact that the resulting α on M might not be adapted to the open book or even a contact form. Moreover, the explicit formula in the last lemma still holds for the homotoped contact form, as the explicit computation does not use any specific property of the pair of functions (h_1, h_2) listed above. In particular, up to homotopy one can achieve the following form, which will be useful below: Where?

Lemma 3. *For $\delta > 0$ sufficiently small, up to a deformation among contact structures on $M \times T^2$ supported in the neighbourhood of radius 2δ of $B \times T^2$, one can assume that*

- $h_1(r) = 1$ *for* $r \leq \delta$
- $h_1(h_1 - h_1') > 0$ *everywhere*
- $h_2(r) = 0$ *for* $r \leq 3\delta/2$

Our description of the Reeb vector field also holds for the deformed contact form; in particular, it coincides with R_B for $r \leq \delta$.

The Milnor open book

The open book coming from an A_{k-1}-singularity Define $f \colon \mathbb{C}^n \to \mathbb{C}$ by

$$(z_0, \dots, z_{n-1}) \mapsto z_0^k + z_1^2 + \dots z_{n-1}^2.$$

Consider the sphere $S^{2n-1} \subset \mathbb{C}^n$. The intersection $f^{-1}(0) \cap S^{2n-1}$ is the so called Brieskorn sphere $B = \Sigma_{n-1}(k, 2, \dots, 2)$. On the complement $S^{2n-1} \setminus B$, the map

$$\pi_f \colon S^{2n-1} \setminus B \to S^1 \colon (z_0, \dots, z_{n-1}) \mapsto \frac{f(z_0, \dots, z_{n-1})}{|f(z_0, \dots, z_{n-1})|}$$

is a fibration over S^1, the Milnor fibration. According to definition 2, this is an open book where the fibers of the Milnor fibration (i.e. the Milnor fibers) form the pages. We only need to show that all points with $|f(x)| < 1$ are regular points. This is easy to see for all points except 0. At 0, however, I don't think it's true as $Tf_0 = 0$.

Alternative description: Positive stabilization of the standard open book $OB(D^{2n-2}, \mathrm{Id})$, i.e. $S^{2n-1} = OB(D^*S^{n-1}, \tau)$ where τ is the Dehn-Seidel-twist. Is this really the stabilization? Attach a $D^{n-1} \times D^{n-2}$ ($n-1$-handle) to the page. The 2-dimensional drawing suggests that at least the topology works out, i.e.

$$D^{2n-2} \cup h_{n-1} = D^*S^{n-1}.$$

Why (and in what sense) are the two descriptions equal? This is proven somewhere, see questions.md.

Summarizing the previous paragraph, we now have an open book decomposition of S^{2n-1}. We can apply the Thurston-Winkelnkemper construction to construct a supported contact structure and then, via the Bourgeois construction, obtain a contact structure on $S^{2n-1} \times T^2$.

Surgery

Our goal is to construct a contact structure on S^{2n+1}. Currently, we have a contact structure on $S^{2n-1} \times T^2$. Therefore, we perform multiple contact surgeries. First, we construct the necessary smooth surgeries. Then we will show that by some h-principle they can be realized as contact surgeries.

Smooth According to the paper by John Milnor, we can kill p-homotopy classes if $2n + 1 \geq 2p + 2$ (i.e. $n > p$) and there exists an embedding $f : S^p \times D^{2n+1-p} \to W$ representing the homotopy class we wish to kill, where W is the manifold under consideration. When does such an embedding exist? Let TW be the tangent bundle of W and $f_0 \colon S^p \to W$ a representative of the homotopy class λ. Consider the induced bundle $f_0^*(TW)$ over S^p.

Lemma 4. *Assume that $n \coloneqq \dim W \geq 2p + 1$. Then there exists an imbedding*

$$S^p \times D^{n-p} \to W$$

which represents λ if and only if the induced bundle $f_0^*(TW)$ is trivial.

If the induced bundle were nontrivial, we would have an embedding of a Mobius strip into the manifold (really?). Therefore, it has to be trivial. After killing the first generator of H_1, the manifold is still orientable, so that we can perform another 1-surgery to kill the second generator of H_1.

1-surgery doesn't affect H_2 (by general position?), so therefore we have $\pi_2 = H_2 = \mathbb{Z}$ (by the Hurewicz theorem?).

Lemma 5. *A simply connected homology sphere is homeomorphic to the sphere.*

Proof. Let M^n be a simply connected CW-complex that is a homology sphere, i.e. $H^1(M) = \mathbb{Z}$ and $H^n(M) = \mathbb{Z}$. As M is simply connected, i.e. $\pi_1(M) = 0$, we can apply the Hurewicz theorem, which in turn tells us that

$$\pi_k(M) = 0 \ \forall 1 < k < n \text{ and } \pi_n(M) = H^n(M) = \mathbb{Z}.$$

As a result, M is a homotopy sphere. Consider a generator $f : S^n \to M$ of $\pi_n(M)$. On π_0 level, we map one connected component to one connected component, so here the induced map is obviously bijective. On π_k level with $0 < k < n$, we just map 0 to 0 which is an isomorphism. On π_n level, we need to show that

$$f_* : \mathbb{Z} = \pi_n(S^n) \to \pi_n(M) = \mathbb{Z}$$

is an isomorphism. The identity map is a generator for $\pi_n(S^n)$. Now $f_*(\mathrm{id}) = f$, so we obtain a generator of $\pi_n(M)$. A map from $\mathbb{Z} \to \mathbb{Z}$ that sends 1 to 1 is a group isomorphism. These considerations show that f is a weak homotopy equivalence. As a smooth manifold, M is a CW-complex. By Whiteheads theorem we conclude that f is a homotopy equivalence, so together with the generalized Poincaré conjecture, we have shown that M is homeomorphic to the sphere. $\square$

Lemma 6. *A simply connected homology sphere that bounds a homology ball is diffeomorphic to the sphere.*

Proof. According to the last lemma, the homology sphere M is homotopy equivalent to a sphere.

As W is a zimply connected homology ball, all homotopy groups $\pi_{\geq 1}$ are 0 by the Hurewicz theorem. Taking any constant map on W therefore is a homotopy equivalence by Whiteheads theorem, i.e. W is contractible. Then, cut out a ball inside W, s.t. we obtain a cobordism W' from a sphere to M. Prove that W' is an h-cobordism. Thus, by the h-cobordism theorem (Ranicki, Theorem 1.9), we conclude that M is diffeomorphic to a sphere. $\square$

Proof of tightness

We now want to prove that the resulting contact structure is actually tight. We show algebraic tightness which implies tightness according to [BGMZ22, Proposition 3.2 (1)]. In example 2.10, we see that the Bourgeois contact structure on $S^{2n-1} \times T^2$ is 1-ADC, hence algebraically tight (ebd, (5)(c)). (explain that example?) As contact surgery gives us a Liouville cobordism from the original manifold to the surgered manifold (reference!), we can apply [BGMZ22, Proposition 3.2 (3)] to see that contact surgery preserves algebraic tightness.

The almost contact structure underlying ξ_{ex} is homotopic to the standard one: Why does it suffice to show that the almost complex structure is homotopic to the standard one? Isn't there more to an almost contact structure than just the almost complex structure? Are all hyperplane distributions homotopic?

Convex decomposition

Definition 6 (ideal Liouville domain)**.** *[Gir20, Definition 1] An ideal Liouville domain (F, ω) is a domain F endowed with an ideal Liouville structure ω. This ideal Liouville structure is an exact symplectic form on* $\operatorname{int} F$ *admitting a primitive λ such that: for some (and then any) function $u\colon F \to \mathbb{R}_{\geq 0}$ with regular level set $\partial F = \{u = 0\}$, the product $u\lambda$ extends to a smooth 1-form on F which induces a contact form on ∂F.*

Start with a Bourgeois manifold $\mathrm{BO}(\Sigma, \dots)$. Smoothly, we have

$$\mathrm{BO}(\Sigma, \dots) = \mathrm{OB}(\Sigma, \dots) \times T^2 = \left[\mathrm{OB}(\Sigma, \dots) \times S^1\right] \times S^1 =: V \times S^1$$

According to [DG12, Section 6] we obtain a convex decomposition of the first factor

$$V = V_+ \bigcup_{\Gamma} \overline{V}_-,$$

where $V_\pm$ are ideal Liouville domains and Γ is the dividing set. In [DG12, Section 5.3], the authors explicitly compute $V_\pm$ for the Bourgeois construction and obtain

$$V_+ = p^{-1}([0,\pi]) \times S^1, \qquad V_- = p^{-1}([\pi, 2\pi]) \times S^1,$$

i.e. topologically we get $V_\pm = \Sigma \times D^* S_1$.

We know that $V_\pm$ are ideal Liouville domains. But why are $V_\pm \times S^1$ with the induced contact structure round contactizations of $V_\pm$ and hence Giroux domains? -¿ see DG12, section 6 Is it important that they are Giroux domains?

The products $V_\pm \times S^1$ are Giroux domains, as they are round contactizations of ideal Liouville domains. My guess would be that a contactization works similar to a symplectization: One takes a symplectic manifold $\times \mathbb{R}$ and it becomes a contact manifold. A round contactization takes a manifold $\times S^1$.

What is the effect of the blowdown? The blowdown is applied to the boundary components of the Giroux domain.

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