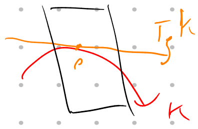
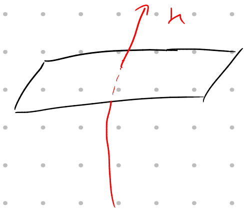
3. Knots in contact 3-manifolds

An embedding $K: S^1 \hookrightarrow (M^3, \xi)$ is called



- **Legendrian knot** $:\Leftrightarrow$ $TK \subset \xi$
- **Transverse knot** $:\Leftrightarrow$ $TK \oplus \xi = TM$



K_0 is isotopic to K_1 $:\Leftrightarrow$
$\exists K_t,\ t \in I,\ K_t$ is Legendrian (transverse) $\forall t$.

Notation: $K \subset (M, \xi)$ for the isotopy class

Examples: (0) $S^1 = \mathbb{R}/2\pi\mathbb{Z} \ni t \xmapsto{K} (3\sin(t)\cos(t), \cos(t), \sin^3(t)) \in (\mathbb{R}^3, \xi_{st} = \ker(x\,dy + dz))$

is a Legendrian unknot.

it's Legendrian: take the derivative, plug it into α

if it's $0 \Rightarrow$ it's Legendrian

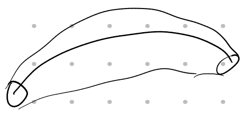
(1) $S^1 \ni \theta \longmapsto (\theta, 0, 0) \in (S^1 \times \mathbb{R}^2, \xi_n = \ker(\cos(n\theta)dx - \sin(n\theta)dy))$ is Legendrian.

(2) $S^1 \ni \theta \longmapsto (\theta, 0, 0) \in (S^1 \times \mathbb{R}^2, \xi = \ker(d\theta + r^2 d\varphi))$ is transverse.

3.1) Neighborhood & isotopy extension theorems

Thm 1: (1) Let $K \subset (M, \xi)$ be Legendrian $\Rightarrow$ $\exists$ tubular NBHD V_K of K in M and

$\forall r > 0\ \forall n \in \mathbb{Z} \setminus \{0\}$: $(V_K, \xi) \overset{cont}{\cong} (S^1 \times D^2_r, \xi_n)$



$$K \longmapsto S^1 \times 0$$

(2) Let $K \subset (M, \xi)$ be transverse. $\Rightarrow \exists$ tubular NBHD V_K of K in M

& $\exists\ \varepsilon > 0$ s.t. $(V_K, \xi) \overset{cont}{\cong} (S^1 \times D^2_\varepsilon, \ker(d\theta + r^2 d\varphi))$

$$K \longmapsto S^1 \times 0$$

Proof: HW. Hint: Use Moser trick as in Darboux's theorem.

Thm 2: Let $K_t: S^1 \hookrightarrow (M^3, \xi)$ be an isotopy of Legendrian (transverse) knots. $\Rightarrow \exists$ isotopy of contactomorphisms

$$\psi_t: (M, \xi) \xrightarrow{\cong\ cont} (M, \xi) \quad \text{s.t.}$$

$$\psi_0 = id \quad \& \quad \psi_t \circ K_0 = K_t$$

Proof: Construct φ_t as flow of a contact vectorfield from a hamilton function. (HW) ∎

9.2 The front projection

Let $K \subset (S^3, \xi_{st})$ be a knot

Thm 2.4 $\Rightarrow$ we can see $K \subset (\mathbb{R}^3, \xi_{st})$

$(x,y,z) \longmapsto (y,z)$ front projection

$S^1 \ni t \longmapsto (x(t), y(t), z(t)) \in (\mathbb{R}^3, \xi_{st} = \ker(x\,dy + dz))$

is Legendrian $\Leftrightarrow$ $0 = \alpha(TK) = x(t) \cdot y'(t) + z'(t) \quad \forall t$

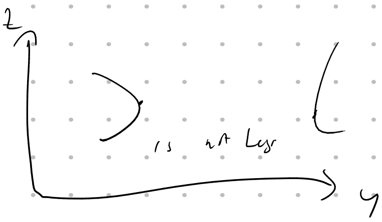
If K is generic (i.e. $y'(t) = 0$ only for finitely many t)

$\Rightarrow x(t) = -\frac{z'(t)}{y'(t)} = -\frac{dz}{dy}$ (if $y'(t) = 0 \Rightarrow z'(t) = 0$

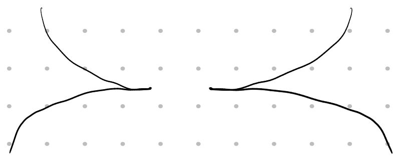
i.e. we can recover K from it's front projection via ⊛

Certain configurations do not appear as front projections of Legendrian knots.

⊛ if $y' = 0 \Rightarrow z' = 0 \Rightarrow$ there can't be vertical tangencies



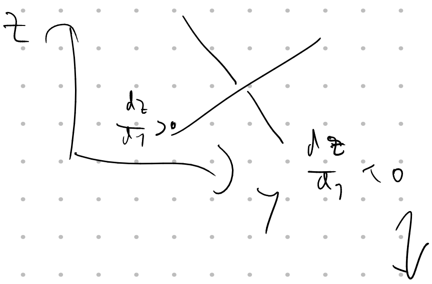
Instead: semicubical cusps:



$(x(t), y(t), z(t)) = (t, t^2, -\frac{2}{3}t^3)$

(every Legendrian knot has at least 2 of them)

⊛ No crossing as follows:

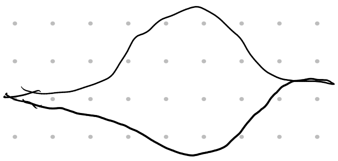


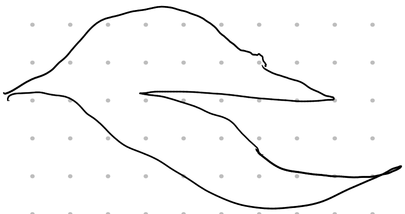
Instead:

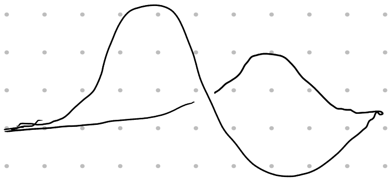
$\Rightarrow$ Any such front projection describes a unique Legendrian knot.

Convention: We draw Legendrian knots in the front projection

Examples:

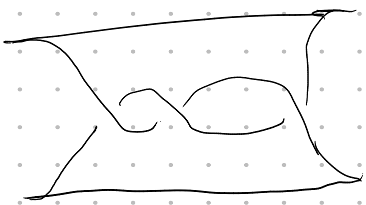






is isotopic to one of those on the left

trefoil



trefoil

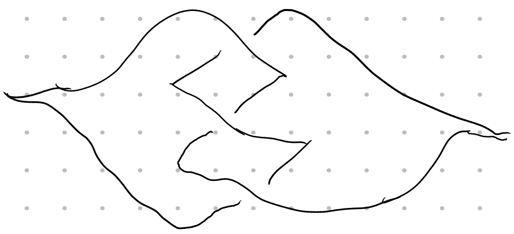


figure 8

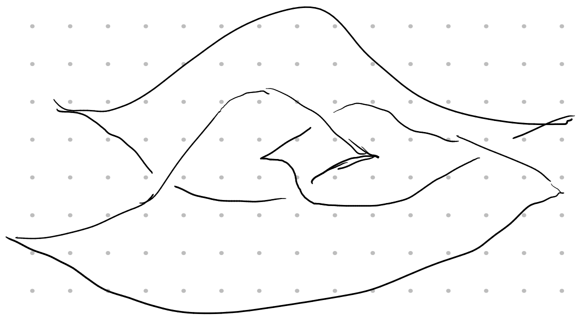
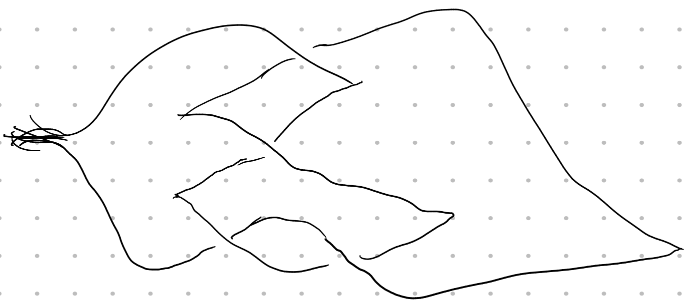
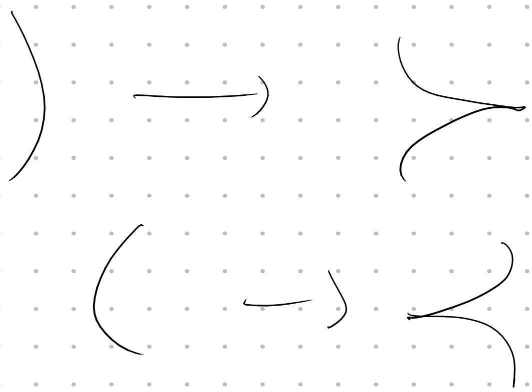


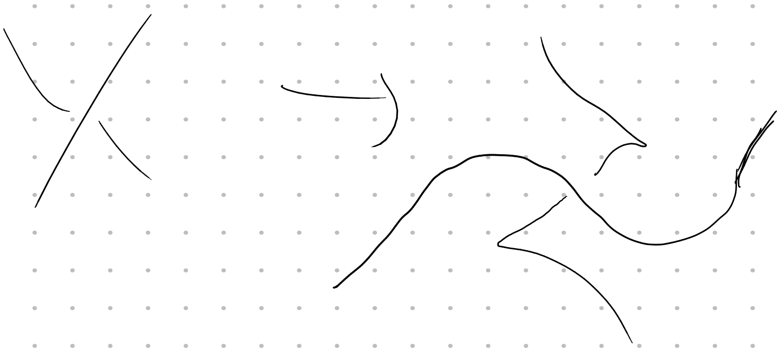
figure 8



Corollary 3: For every smooth knot $K \subset (\mathbb{R}^3, \xi_{st})$ $\exists$ isotopic knot $K' \subset (\mathbb{R}^3, \xi_{st})$ Leg.

Proof: front of K





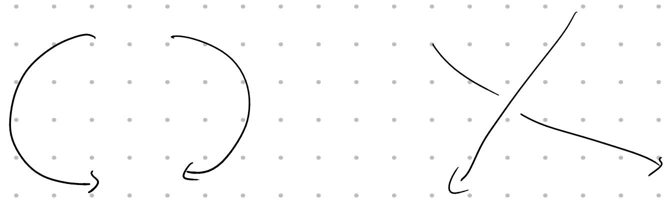
If h is positively transverse, i.e. $xy' + z' > 0$

Then: If $y' = 0 \Rightarrow z' > 0$

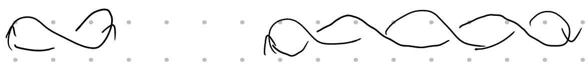
If $y' > 0 \Rightarrow x > -\frac{z'}{y'} = -\frac{dz}{dy}$

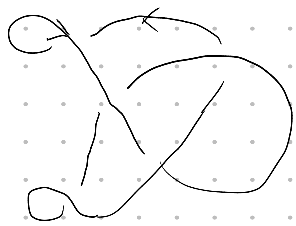
If $y' < 0 \Rightarrow x < -\frac{dz}{dy}$

$\Rightarrow$ The following configurations are excluded



All other configurations lift to transverse curves in $(\mathbb{R}^3, \xi_{st})$ unique up to shifts in x-direction





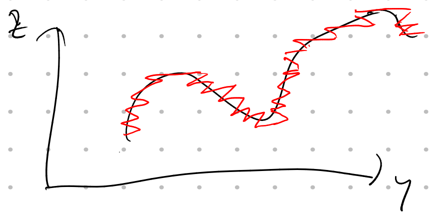
trefoil

Thm 4: Let $h: S^1 \hookrightarrow (M, \xi)$ a smooth knot

$\Rightarrow$ h can be C^0-close approximated by a legendrian (transverse) knot smoothly isotopic to h.

Proof: Legendrian case

Case 1: Let $\gamma: (a,b) \hookrightarrow (\mathbb{R}^3, \xi_{st})$ be an arc



Approximate the front of γ by a legendrian front projection s.t. $-\frac{dz}{dy}$ is close to the x-component of γ.

Case 2: If $\gamma: (a,b) \hookrightarrow (\mathbb{R}^3, \xi_{st})$ is legendrian near a & b

then we can choose the approximation near a & b to agree with γ.

Case 3: $h: S^1 \hookrightarrow (M, \xi)$ a knot. S^1 compact $\overset{\text{Lebesgue}}{\underset{\text{Lemma}}{\Rightarrow}}$ $\exists$ decomposition of S^1 into intervals I_i s.t. $h(I_i) \subset$ Darboux ball then use case 2.

Transverse case

Let $L : S^1 \hookrightarrow (M, \xi)$ be a Legendrian approximation of K.

Thm 1

$\Rightarrow$ wlog $L(t) = (\theta = t,\ x = 0,\ y = 0) \in (S^1 \times \mathbb{R}^2,\ \alpha = \cos\theta\, dx - \sin(\theta)\, dy)$

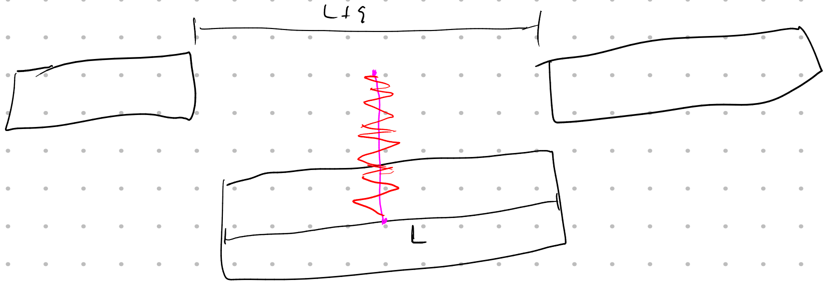
$L_\pm := (\theta = t,\ x = \pm\varepsilon \sin(t),\ y = \pm\varepsilon \cos(t))$ for $\varepsilon > 0$ small

$\Rightarrow$ $\dot{L}_\pm := (1, \pm\varepsilon\cos(t), \mp\varepsilon\sin(t))$

$\alpha(TL_\pm) = \pm\varepsilon(\cos^2(t) + \sin^2(t)) = \pm\varepsilon$

$L_\pm$ are called <u>transverse push-offs</u> of L

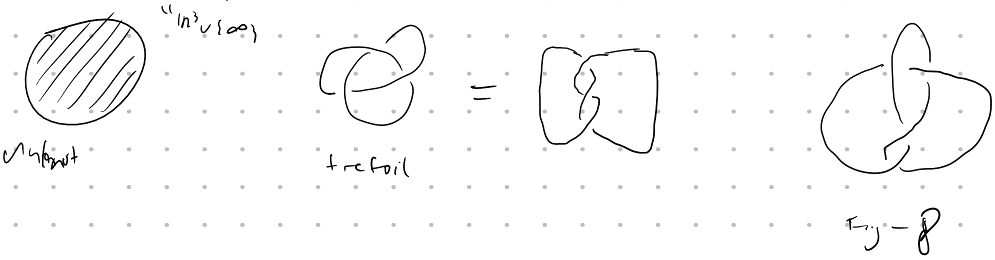
<u>real world application:</u>



3.4 Seifert surfaces & the Alexander polynomial

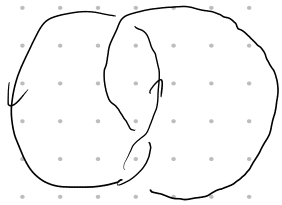
Let $K \in S^3$ ("$\mathbb{R}^3 \cup \{\infty\}$") a smooth knot

Ex:



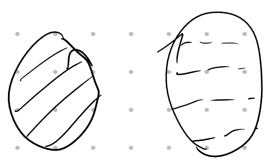
<u>Lemma 5:</u> $\forall K \subset S^3$ $\exists$ <u>Seifert surface</u>, i.e. $F^2 \xhookrightarrow{\text{smooth}} S^3$ compact, oriented, s.t. $\partial F = K$

<u>proof:</u> (1) Let D be a diagram of K



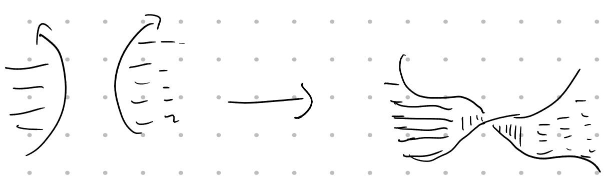
(2) Orient D

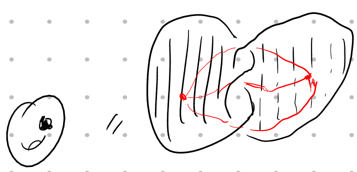
(3) resolve crossings following orientation



(4) get collection of circles in the plane

(5) Glue in a twisted band for every crossing

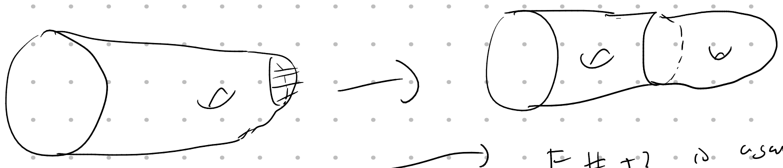




$\square$

Stabilization

F a Seifert surface of K



F

$F \# T^2$ is again a Seifert surface

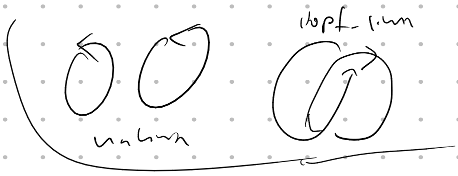
Thm 6 [Reidemeister – Singer] Any two Seifert surfaces of K have a common stabilization.

(Seifert)-genus: $g(K) := \min\{g(F) \mid F$ a Seifert surface of $K\}$

Ex: $g(K) = 0 \iff K = 0$

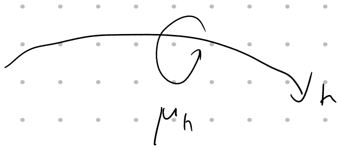
$g(\text{trefoil}) \leq 1$

Links: Let $K \subset M^3$ an oriented knot if K is nullhomologous i.e. $[K] = 0 \in H_1(M)$



$\Rightarrow H_1(M \setminus \nu K) = \mathbb{Z}_{\langle \mu_K \rangle} \oplus H_1(M)$

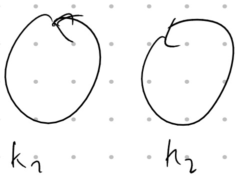
tubular nbhd of K

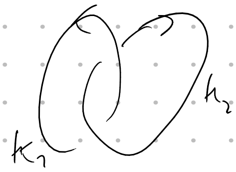


Let K_1, K_2 be oriented knots in M s.t. K_1 & K_2 are nullh.

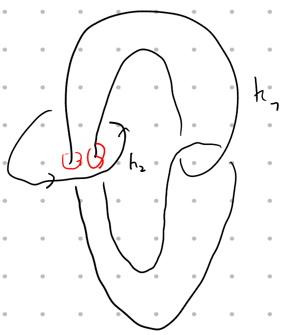
The linking number $lk(K_1, K_2) \in \mathbb{Z}$ is defined $[K_2] = lk(K_1,K_2)[\mu_{K_1}] \in H_1(M \setminus K_1) = H_1(M) \oplus \mathbb{Z}_{\langle \mu_{K_1} \rangle}$

$lk(K_1, -K_2) = lk(-K_1, K_2) = -lk(K_1, K_2)$





$[K_2] = -[\mu_{K_1}] \Rightarrow lk = -1$



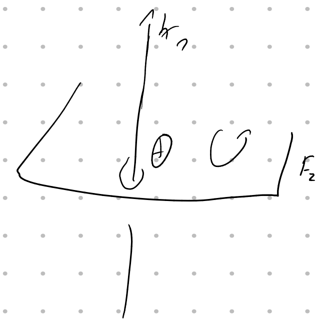
$[K_2] = -\mu_{K_2} + \mu_{K_2} = 0$

$lk = 0$

Lemma 7: (1) $K \subset S^3$ is nullhomologous $\Leftrightarrow$ K admits a Seifert surface

(2) $lk(K_1, K_2) = K_1 \cdot F_2$

with F_2 a Seifert surface of K_2

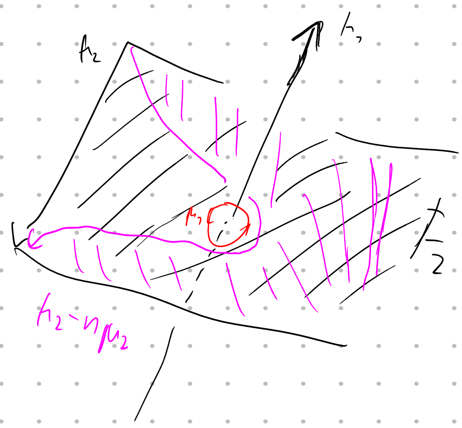


Proof: (1) "$\Leftarrow$" def of hom

"$\Rightarrow$" In S^3 Seifert alg.

general case $\rightarrow$ discussion session

(2) Let $K_1 \cdot F_2 = n > 0$ (algebraic intersection number)



$K_2 - n\mu_1$ bounds a surface that does not intersect K_1

$[K_2 - n\mu_1] = 0 \in H_1(\mathbb{R} \setminus K_1)$

$lk(K_1, K_2) = n = K_1 \cdot F_2$

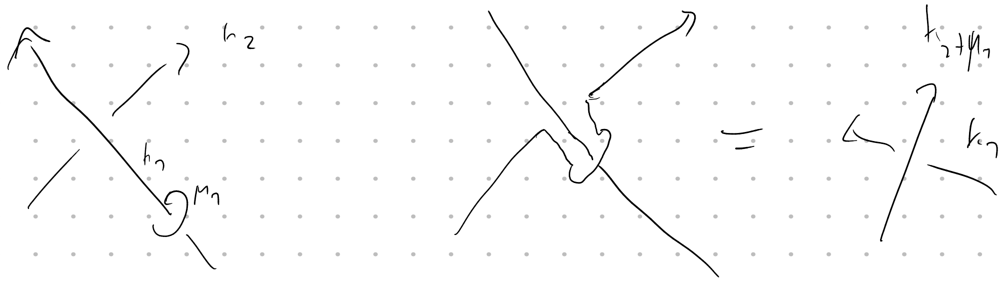
Lemma 8 Let $K_1, K_2 \subset S^3$ be oriented knots.

$\Rightarrow lk(K_1, K_2) = \#$ crossings of K_2 under K_1 with signs

Corollary 9: $lk(K_1, K_2) = lk(K_2, K_1)$ proof: "look at the diagram from behind the blackboard"

Proof (Lemma 8): $\circ$ $lk(K_1, \pm\mu_1) = \pm 1$

$\bullet$ $lk(K_1, K_2 \pm \mu_1) = lk(K_1, K_2) \pm 1$

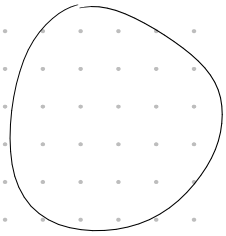


$n =$ # crossings of K_2 under K_1

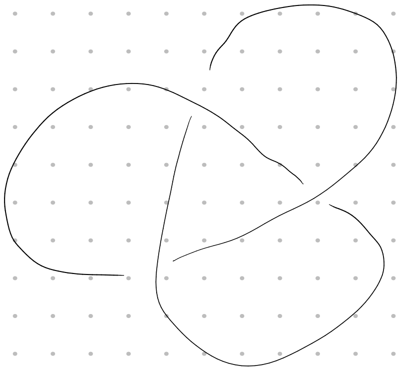
$\Rightarrow$ $K_2 - n\mu_1$ has no undercrossings with K_1

$\Rightarrow$ $lk(K_1, K_2 - n\mu_1) = 0$ $\Rightarrow$ $lk(K_1, K_2) = n$ $\square$

w.t.s.:



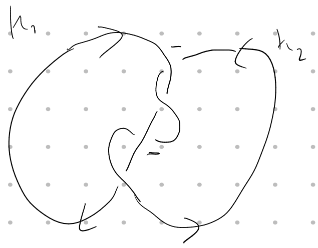
$\overset{\text{isotopic}}{\not\cong}$



$g(K) := \min\{g(F) \mid F \text{ is a seifert surface of } K\}$

$$[K_1] = \underset{\substack{\| \text{ Lemma 8}}}{lk(K_1, K_2)}[\mu_{K_2}] \in H_1(S^3 \setminus K_2) \cong \mathbb{Z}\langle \mu_{K_2} \rangle$$

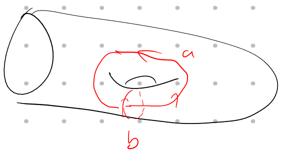
crossings of K_2 over K_1 counted with signs



$lk(K_1, K_2) = -2$

Alexander polynomial:

seifert form: Let F be a seifert surface of an oriented knot K.



$$S: H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$$
$$(a, b) \longmapsto lk(a, b^+)$$

push-off of b in positive normal dir. of F

Alexander polynomial: $\Delta_K(t) := \det\left(t^{-1/2} S - t^{1/2} S^T\right)$

Corollary 10: $\Delta_K(t)$ is a knot invariant.

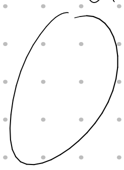
proof sketch: * $\Delta_K(t)$ is independent of the chosen matrix: $S' = PSP^T$ with $\det P = \pm 1$

* F' is a different Seifert surface $\Rightarrow$ F' & F have a common stabilization $\bar{F}$

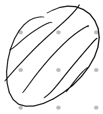
* $\bar{F}$ is a stabilization of F $\Rightarrow$ $\bar{S} = \begin{pmatrix} S & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\Rightarrow$ $\det\left(t^{-1/2}\bar{S} - t^{1/2}\bar{S}^T\right) = \det\left(t^{-1/2}S - t^{1/2}S^T\right)$

Examples:

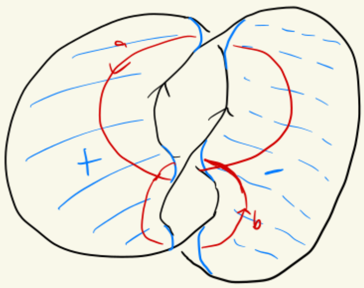
unknot



F =



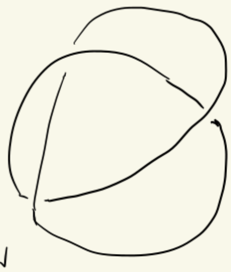
$H_1(F) = 0 \Rightarrow S = 0 \Rightarrow \Delta_K(t) = 1$



$\cong$

$\uparrow$

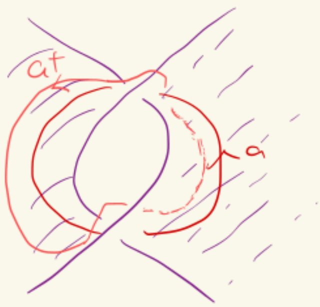
Reidemeister moves



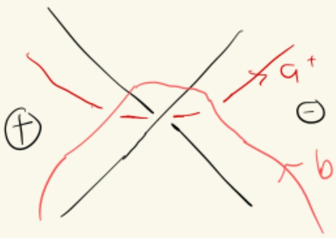
$$F \cong T^2 \setminus \mathring{D}^2$$

$$\Rightarrow H_1(F) \cong \mathbb{Z}^2_{\langle a,b \rangle}$$

$$S = \begin{pmatrix} lk(a,a^+) & lk(b,a^+) \\ lk(a,b^+) & lk(b,b^+) \end{pmatrix}$$



$$\Rightarrow S = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$



$$\Rightarrow \Delta_{\text{trefoil}}(t) = \det\left[\begin{pmatrix} -t^{-1/2} & 0 \\ t^{-1/2} & -t^{-1/2} \end{pmatrix} - \begin{pmatrix} -t^{1/2} & t^{1/2} \\ 0 & -t^{1/2} \end{pmatrix}\right]$$

$$= \det \begin{pmatrix} t^{1/2} - t^{-1/2} & -t^{1/2} \\ t^{1/2} & t^{1/2} - t^{-1/2} \end{pmatrix}$$

$$= t + t^{-1} - 2 + 1 = t - 1 + t^{-1} \neq 1$$

$$= \Delta_{\bigcirc}$$

HW: $\Delta_{\text{fig 8}}(t) = t - 3 - t^{-1}$

Corollary: $\deg(\Delta_K) \leq g(K)$ $\quad \left(S \in M_{2g,2g}(\mathbb{Z}) \right)$

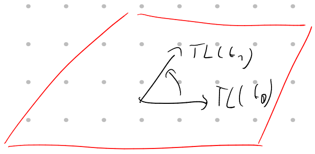
HW: $\forall g \in \mathbb{N}_0 : \exists K_g : g(K_g) = g$ (maybe via torus knots?)

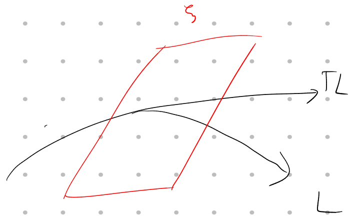
3.5 Classical invariants

Let $K \subset (\mathbb{R}^3, \xi)$ be a Legendrian knot.

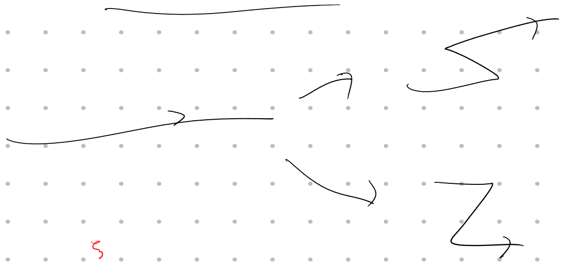
Idea: $tb(K) :=$ # twists of ξ around L

$rot(K) :=$ # twists of TL in ξ

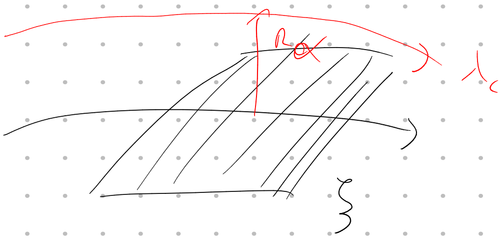




Stabilization:



Contact longitude of K: λ_c = push-off of K in the Reeb-direction



Thurston-Bennequin invariant: $tb(K) := lk(K, \lambda_c)$, K nullhomologous

Ex:

$tb(K) = lk(K, \lambda_c) = -1$

$R_\alpha = \partial z$

$tb(K) = lk(K, \lambda_c) = -2$

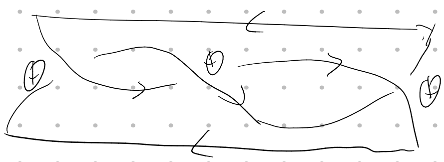
Lemma 12: Let $K \subset (\mathbb{R}^3, \xi_{st})$ be a Legendrian knot presented in front proj. Then $tb(K) = -\frac{1}{2}C + w$,

$C :=$ # cusps in front

$w :=$ # positive crossings − # negative crossings = writhe of front

Proof: Ex. □

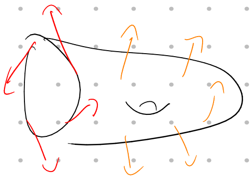
Ex



$tb = -\frac{1}{2}C + w = -\frac{1}{2}\cdot 4 + 3 = 1$

Let $K \subset (M, \xi)$ be an oriented, nullhomologous legendrian knot with Seifert surface Σ.

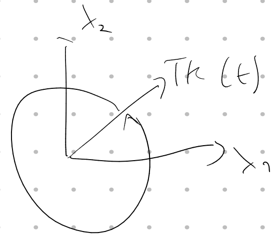
$\mathrm{rot}(K, [\Sigma]) := \langle e(\xi, K), [\Sigma] \rangle = pd(e(\xi, K)) \cdot [\Sigma]$



$= \#$ zeros of extension of TK over ξ (rather Σ?)

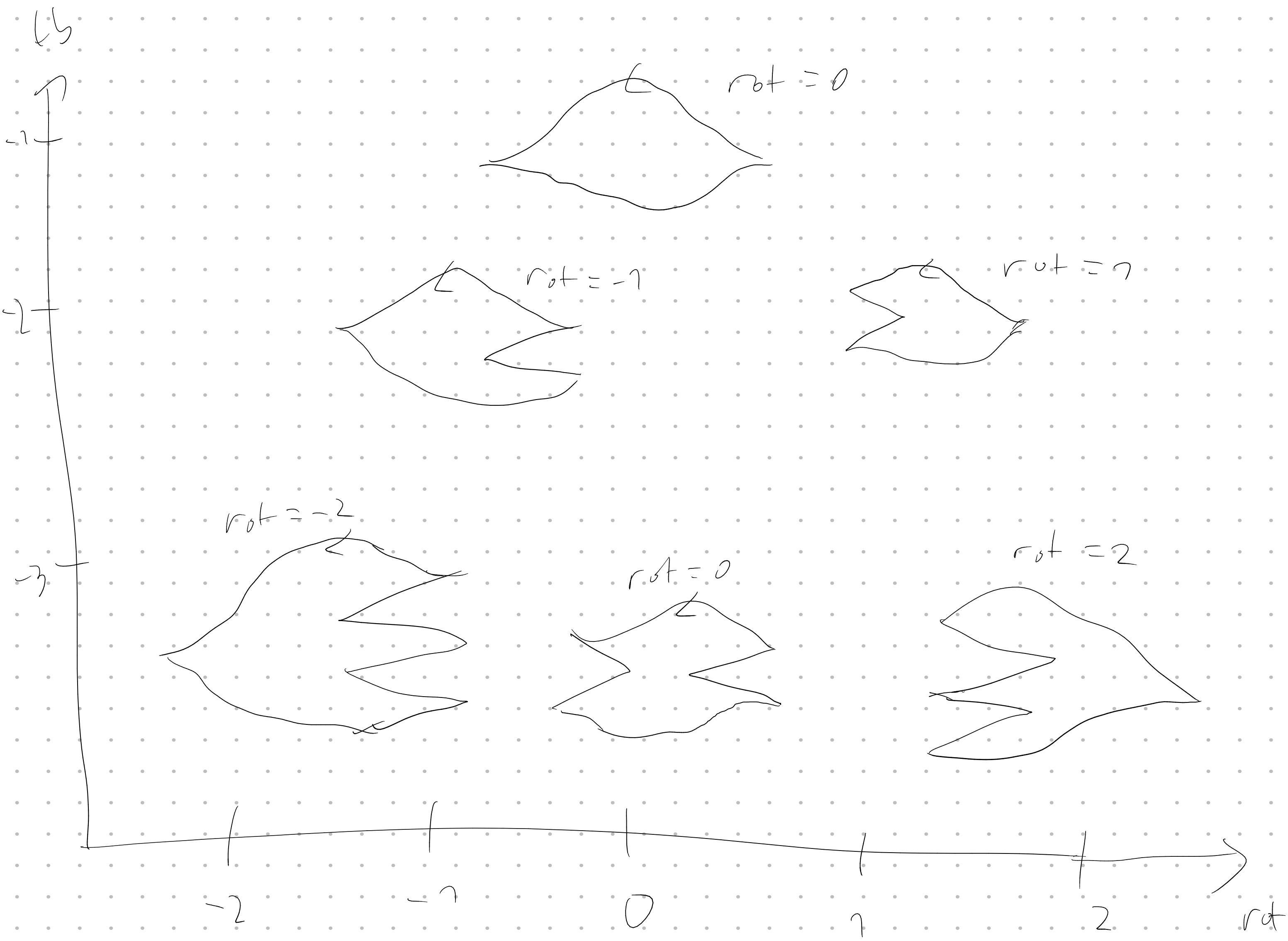
$= \#$ rotations of TK relative to a trivialization of ξ over Σ.

If ξ is trivializable, i.e. $\exists X_1, X_2$ v.f. st. $\xi = \langle X_1, X_2 \rangle$,
then $\mathrm{rot}(K) = \#$ rotations of TK relative to this trivialization.



Ex: $\xi_{ot} = \ker(x\,dy + dz) \implies X_1 = \partial_x,\ X_2 = \partial_y - x\partial_z$ span ξ_{ot}.

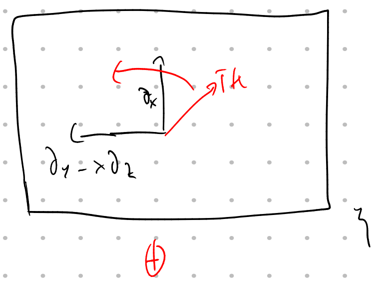
Remark: $\mathrm{rot}(-K) = -\mathrm{rot}(K)$

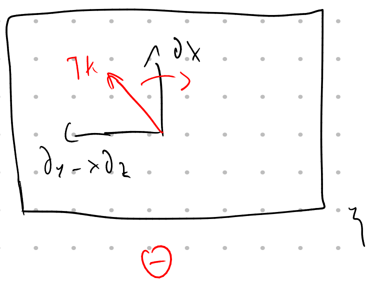


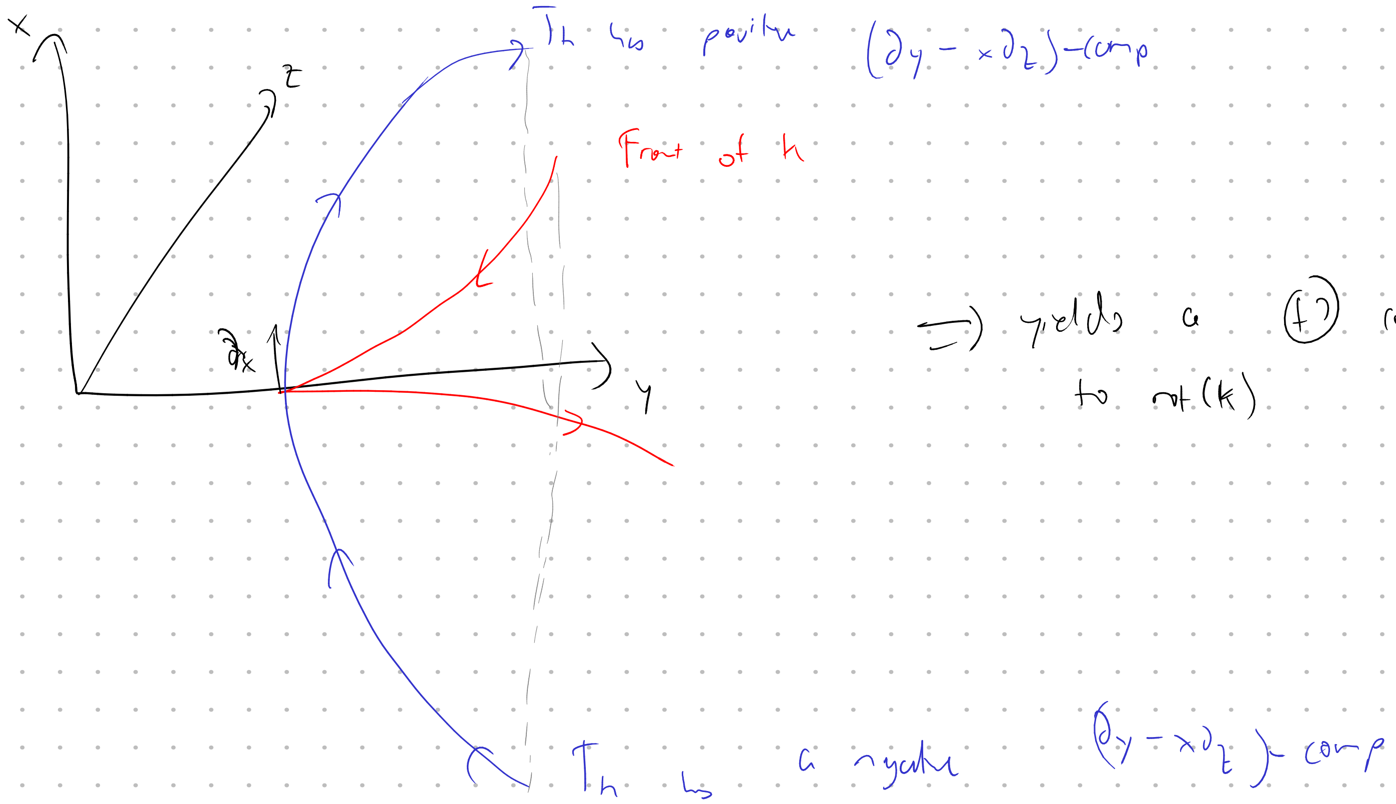
Lemma 73: $\mathrm{rot}(h) = \frac{1}{2}(C_- - C_+)$

[if ξ is trivializable (i.e. $\exists x_1, x_2$ s.t.

proof: Trivialization of $\xi_{st} = \langle \partial_x, \partial_y - x\partial_z \rangle$

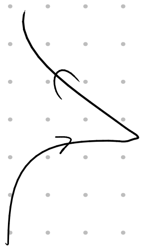






$\Rightarrow$ yields a $(+1)$ contribution to $\mathrm{rot}(k)$

Similar



yields a (-1) contribution

$\Rightarrow \mathrm{rot}(k) = \#$ left down cusps $- \#$ right up cusps

If we count w.r.t. $-\partial_x$

$\Rightarrow \mathrm{rot}(k) = \#$ right down cusps $- \#$ left up cusps

$\Rightarrow \mathrm{rot}(h) = \frac{1}{2}(C_- - C_+)$

Thm 14: [Bennequin]

If $K \subset (\mathbb{R}^3, \xi_{st})$ is a Legendrian knot $\Rightarrow$ $\underbrace{tb(K) \pm rot(K)}_{\text{Contact geometry}} \leq \underbrace{2g(K)-1}_{\text{smooth topology}}$

Proof: in section 4/5 □

Corollary 15: $(\mathbb{R}^3, \xi_{st}) \overset{\text{cont}}{\ncong} (\mathbb{R}^3, \xi_{OT})$

proof: $\exists$ Legendrian unknot K in $(\mathbb{R}^3, \xi_{OT})$ s.t. $TB(K) = 0$

$\Rightarrow TB(K) \pm Rot(K) = \pm Rot(K) > 0$ for one orientation on K

but $2g(K) - 1 = -1$

If (M, ξ) contains a Legendrian unknot with $tb = 0$ then (M, ξ) is called overtwisted. If not then it is called tight.

Thm 16 [Eliashberg]

(1) If ξ_1 & ξ_2 are OT contact structures on a closed 3-mfd M, then $\xi_1 \overset{\text{isotopic}}{\cong} \xi_2 \overset{\Rightarrow \text{ is clear}}{\Longleftrightarrow}$ ξ_1 is homotopic to ξ_2 as tangential 2-Plane fields.

(contact geom) (alg. top.)

(2) (M, ξ) is tight $\Leftrightarrow$ $\forall K \subset (M, \xi)$ nullhomologous & Leg:

$TB(K) \pm Rot(K) \leq 2g(K) - 1$

(3) If (M, ξ) admits a symplectic filling

$\Rightarrow M$ is tight.

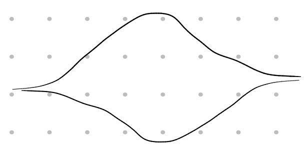
(4) $S^3, \mathbb{R}^3, S^1 \times S^2$ have unique tight contact structures

(5) T^3 has infinitely many contact structures [Giroux, Honda]

(6) $\exists M^3$ without tight contact structures (Etnyre - Honda)

proof: section 4/5 ? □

Thm 17 [Eliashberg - Fraser]

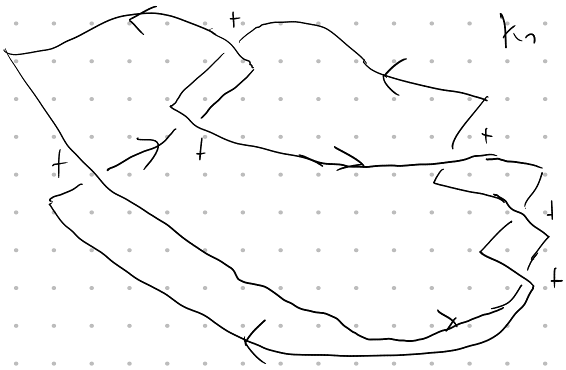
Every Legendrian unknot in $(\mathbb{R}^3, \xi_{st})$ is a stabilization of 

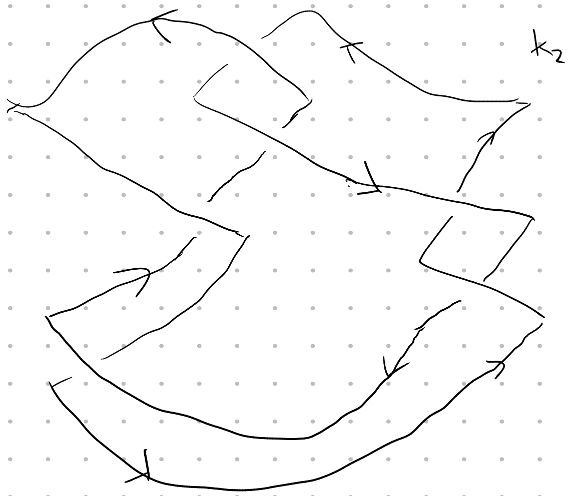
proof: in section 4 □

Etnyre - Honda: Similar results for trefoil & figure-8 knots

Thm 18 [CHEKANOV]

$\exists$ smoothly isotopic Legendrian knots





$TB = -5 + 6 = 1 \qquad rot = 0$

$tb = 1 \qquad rot = 0$

with same tb & rot, but not isotopic as Legendrian knots.

[Distinguished by their contact homologies].

However K_1 & K_2 are isotopic after one stabilization (Hw) ∎

Transverse Knots

Let $K \subset (M, \xi)$ be transverse and nullhomologous & choose a Seifert surface for K. Then we can define the

self-linking number

$$sl(K, [\Sigma]) := lk(K, K')$$

$K' :=$ push-off of K in direction of a non-vanishing vector field in $\xi|_\Sigma$

If ξ is trivializable

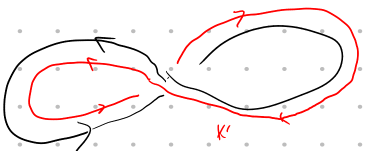
$K' :=$ push-off of K in direction of any non-vanishing vector field in ξ.

Lemma 195 $K \subset (\mathbb{R}^3, \xi_{st})$ is transverse & presented in the front

$\Rightarrow sl(K) = w$

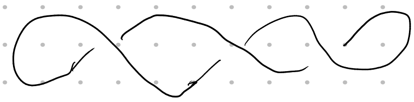
proof: $0 \neq \partial_x \in \xi_{st}$ $\quad K' =$ push-off of K in x-dir $\Rightarrow lk(K, K') = w$ ☐

Ex:

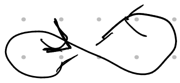


$SL(K) = lk(K, K') = -1$

Stabilization



$SL = -3$

Thm 20: [Eliashberg] Every positive transverse unknot in $(\mathbb{R}^3, \xi_{st})$ is a stabilization of  ∎