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Exercise 8 - Solution

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Homework Problem 8.1 (Examples for Tangent-, Linearizing and Normal Cones) 5 Points

Consider the feasible set

$$F := \left\{x \in \mathbb{R}^n \,\middle|\, g_i(x) \le 0 \text{ for all } i = 1, \ldots, n_{\text{ineq}},\ h_j(x) = 0 \text{ for all } j = 1, \ldots, n_{\text{eq}}\right\} \tag{7.2}$$

without any equality restrictions h and with the inequality constraints $g\colon \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$g(x) = \begin{pmatrix} (x_1 - 1)^2 + x_2^2 - 1 \\ (x_1 - 3)^2 + x_2^2 - 1 \\ x_3 + 1 \\ -x_3 - 2 \end{pmatrix} \qquad \text{at} \qquad x^* = (2,\ 0,\ -1)^\intercal \in F.$$

Find the set of active indices $\mathcal{A}(x^*)$, an explicit representation of F, the tangent cone $\mathcal{T}_F(x^*)$, the **normal cone** $\mathcal{T}_F(x^*)^\circ$ and the linearizing cone $\mathcal{T}_F^{\text{lin}}(x^*)$ and sketch F and the cones.

Solution.

We have that

$$g(x^*) = \begin{pmatrix} (2 - 1)^2 - 1 \\ (2 - 3)^2 - 1 \\ -1 + 1 \\ -(-1) - 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \le 0$$

and therefore that in fact $x \in F$ with the set of active inequality constraint indices $\mathcal{A}(x^*) = \{1, 2, 3\}$.

The feasible set is the intersection of two cylinders of cross-section radius 1 with $(1, 0, x_3)$- and $(3, 0, x_3)$-axes, respectively, and two x_3-halfspaces, i. e., the line segment

$$F = \{(2, 0, x_3) \in \mathbb{R}^3 \mid x_3 \in [-2, -1]\}.$$

This makes it easy to compute the tangent cone, because all sequences in F are on a line, so the tangent cone ends up being the ray

$$\begin{aligned}\mathcal{T}_F(x^*) &:= \left\{d \in \mathbb{R}^n \,\middle|\, \text{there exist sequences } x^{(k)} \in F \text{ and } t^{(k)} \searrow 0 \text{ such that } d = \lim_{k\to\infty} \frac{x^{(k)} - x^*}{t^{(k)}}\right\} \\ &= \{d \in \mathbb{R}^3 \mid d_1 = d_2 = 0,\ d_3 \le 0\}.\end{aligned}$$

Accordingly the normal cone

$$\begin{aligned}\mathcal{T}_F(x^*)^\circ &:= \{s \in \mathbb{R}^n \mid s^\mathsf{T} x \le 0 \text{ for all } x \in \mathcal{T}_F(x^*)\} \\ &= \left\{s \in \mathbb{R}^3 \,\middle|\, s^\mathsf{T} \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} \le 0 \text{ for all } x_3 \le 0\right\} \\ &= \{s \in \mathbb{R}^3 \mid s_3 \ge 0\}\end{aligned}$$

is the closed halfspace at 0 defined by the normal vector to be any nonzero element from the tangent ray.

To find $\mathcal{T}_F^{\text{lin}}(x^*)$, we compute the derivatives of the active inequality constraints, which are

$$\begin{aligned} g_1'(x) &= \begin{pmatrix} 2(x_1 - 1) \\ 2x_2 \\ 0 \end{pmatrix} &&\Rightarrow & g_1'(x^*) &= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \\ g_2'(x) &= \begin{pmatrix} 2(x_1 - 3) \\ 2x_2 \\ 0 \end{pmatrix} &&\Rightarrow & g_2'(x^*) &= \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \\ g_3'(x) &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &&\Rightarrow & g_3'(x^*) &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{T}_F^{\text{lin}}(x^*) &:= \left\{d \in \mathbb{R}^n \,\middle|\, \begin{array}{ll} g_i'(x^*)\, d \le 0 & \text{for all } i \in \mathcal{A}(x^*) \\ h_j'(x^*)\, d = 0 & \text{for all } j = 1, \ldots, n_{\text{eq}} \end{array}\right\} \\ &= \{d \in \mathbb{R}^3 \mid d_1 = 0,\ d_3 \le 0\} \\ &\supsetneq \mathcal{T}_F(x^*)\end{aligned}$$

(5 Points)

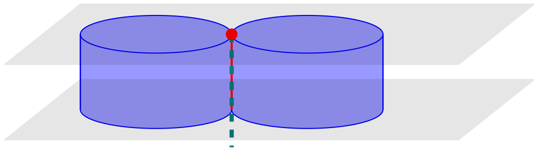

Figure 0.1: Constraining cylinders (blue), limiting hyperplanes of constraints on x_3 (gray), feasible set (red), point x^*, tangent cone (teal, dashed) shifted to x^* and limiting hyperplane of the half space normal cone (coincides with upper limiting hyperplane).

Homework Problem 8.2 (Examples and Properties of Polar Cones) 4 Points

(i) Prove Lemma 7.9 of the lecture notes, i. e., for arbitrary sets $M, M_1, M_2 \subseteq \mathbb{R}^n$ the statements

(a) M° is a closed convex cone.

(b) $M_1 \subseteq M_2$ implies $M_2^\circ \subseteq M_1^\circ$.

(ii) Verify the claimed forms of the polar cones in Example 7.10, i. e., the following:

(a) Suppose that A is an affine subspace of $\mathbb{R}^n$ of the form $A = U + \{\bar{x}\}$. Then $A^\circ = \{\bar{x}\}^\circ \cap U^\perp$.

(b) In the absence of inequality constraints, the polar of the linearizing cone $\mathcal{T}_F^{\text{lin}}(x)$ for $x \in F$ has the representation

$$\begin{aligned}\mathcal{T}_F^{\text{lin}}(x)^\circ &= \{s \in \mathbb{R}^n \mid s \text{ is some linear combination of } h_j'(x)^\intercal,\ j = 1, \ldots, n_{\text{eq}}\} \\ &= \operatorname{range} h'(x)^\intercal.\end{aligned}$$

(c) Let $N := (\mathbb{R}_{\geq 0})^n$ denote the non-negative orthant in $\mathbb{R}^n$. Then $N^\circ = (\mathbb{R}_{\leq 0})^n$ is the non-positive orthant.

Solution.

(i) (a) Closedness of M° is simply due the fact that the definition involves a non strict inequality. Convexity is due to the linearity of the defining condition.

(b) This is due to the fact, that having to check a condition for all elements of a larger set is more restrictive.

(ii) (a) Let $A = U + \{\bar{x}\}$. Clearly, for all $s \in \{\bar{x}\}^\circ \cap U^\perp$, and all $\bar{x} + u \in \{\bar{x}\} + U$,

$$s^\mathsf{T}(\bar{x} + u) = s^\mathsf{T}\bar{x} \leq 0,$$

so $\{\bar{x}\}^\circ \cap U^\perp \subseteq (\{\bar{x}\} + U)^\circ$.

Conversely, if $s \in (\{\bar{x}\} + U)^\circ$, then $s^\mathsf{T}\bar{x} = s^\mathsf{T}(\bar{x} + \underbrace{0}_{\in U}) \leq 0$, so $s \in \{\bar{x}\}^\circ$. Also, assuming there were a $u \in U$ with $s^\mathsf{T}u \neq 0$, we know that λu is in the linear subspace U for all $\lambda \in \mathbb{R}$, and therefore

$$s^\mathsf{T}(\bar{x} + \lambda u) = s^\mathsf{T}\bar{x} + \lambda s^\mathsf{T}u \xrightarrow{\lambda \to \operatorname{sgn}(s^\mathsf{T}u)\infty} \infty$$

which is a contradiction to $s \in (\{\bar{x}\} + U^\circ)$, showing that in fact $\{x\}^\circ \cap U^\perp \supseteq (\{\bar{x}\} + U)^\circ$.

(b) This is of course a consequence of Lemma 7.13, which we did not know anything about at the time the remark was made. Luckily, in this special case, we immediately obtain that

$$\mathcal{T}_F^{\text{lin}}(x) = \{d \in \mathbb{R}^n \mid h_i'(x)d = 0,\ i = 1, \ldots, n_{\text{eq}}\} = \{d \in \mathbb{R}^n \mid h'(x)d = 0\} = \ker h'(x),$$

which is a linear subspace, hence

$$\ker h'(x)^\circ = \ker h'(x)^\perp = \operatorname{range} h'(x)^\mathsf{T}.$$

(c) Let s be in the the non-positive orthant $(\mathbb{R}_{\leq 0})^n$ and d in the non-negative orthant $(\mathbb{R}_{\geq 0})^n$, then

$$s^\mathsf{T}d = \sum_{i=1}^n \underbrace{s_i}_{\leq 0} \underbrace{d_i}_{\geq 0} \leq 0$$

showing that $(\mathbb{R}_{\leq 0})^n \subseteq (\mathbb{R}_{\geq 0})^{n\circ}$.

Assuming there were an $s \in (\mathbb{R}_{\geq 0})^{n\circ}$ and an index i such that $s_i > 0$, then

$$s\mathsf{T} \underbrace{e_i}_{\in \mathbb{R}^n_{\geq 0}} = s_i > 0,$$

which finalizes the proof.

(4 Points)

Homework Problem 8.3 (Lin. Cone and CQs Depend on Description of Feasible Set) 3 Points

The optimization problems

$$\begin{aligned} \text{Minimiere} \quad & f(x) \quad \text{über } x \in \mathbb{R} \\ \text{unter} \quad & x = 0 \end{aligned} \tag{P_1}$$

and

$$\begin{aligned} \text{Minimiere} \quad & f(x) \quad \text{über } x \in \mathbb{R} \\ \text{unter} \quad & x^2 = 0 \end{aligned} \tag{P_2}$$

for any $f \in C^1(\mathbb{R})$ have their obvious solution (because sole feasible point) at $x^* = 0$.

Show that the Abadie and Guignard constraint qualifications are satisfied at $x^* = 0$ for (P_1) but not (P_2).

Solution.

We are dealing with the two feasible sets

$$F^{(1)} = \{x \in \mathbb{R} \mid \underbrace{x}_{h^{(1)}(x)} = 0\}, \quad F^{(2)} = \{x \in \mathbb{R} \mid \underbrace{x^2}_{h^{(2)}(x)} = 0\}$$

Both coincide with the singleton $F^{(1)} = F^{(2)} = \{0\} =: F$.

Accordingly, the tangent cones coincide as well, as they don't depend on the description of the feasible set. Since the only sequence in the feasible sets is the constant zero sequence, we know that

$$\mathcal{T}_F(x) = \{0\} \text{ and therefore } \mathcal{T}_F(x)^\circ = \mathbb{R}.$$

For linearizing cones, we compute

$$\begin{aligned} h^{(1)\prime}(x) &= 1 \quad && \Rightarrow \quad h^{(1)\prime}(x^*) = 1 \\ h^{(2)\prime}(x) &= 2x \quad && \Rightarrow \quad h^{(2)\prime}(x^*) = 0 \end{aligned}$$

and therefore obtain that

$$\begin{aligned} &\mathcal{T}^{\text{lin}}_{F^{(1)}}(x^*) = \{0\} \quad && \text{and} \quad \mathcal{T}^{\text{lin}}_{F^{(2)}}(x^*)^\circ && = \mathbb{R} \quad && \Leftrightarrow \quad \text{ACQ/GCQ} \\ &\mathcal{T}^{\text{lin}}_{F^{(2)}}(x^*) = \mathbb{R} \quad && \text{and} \quad \mathcal{T}^{\text{lin}}_{F^{(2)}}(x^*)^\circ && = \{0\} \quad && \Leftrightarrow \text{no ACQ/GCQ}. \end{aligned}$$

Note: The message here is that unnecessarily increasing the order of a constraint can remove linearization information and kill CQs.

(3 Points)

Homework Problem 8.4 (ACQ for Problems with Affine Constraints) 6 Points

Consider

$$F := \left\{x \in \mathbb{R}^n \,\middle|\, g_i(x) \le 0 \text{ for all } i = 1, \ldots, n_{\text{ineq}},\ h_j(x) = 0 \text{ for all } j = 1, \ldots, n_{\text{eq}}\right\} \tag{7.2}$$

and

$$F^{\text{lin}}(x) = \left\{ y \in \mathbb{R}^n \,\middle|\, \begin{aligned} g_i(x) + g_i'(x)\,(y - x) &\le 0 && \text{for all } i = 1, \ldots, n_{\text{ineq}} \\ h_j(x) + h_j'(x)\,(y - x) &= 0 && \text{for all } j = 1, \ldots, n_{\text{eq}} \end{aligned} \right\}$$

for $x \in F$.

(i) Show that $\mathcal{T}_F^{\text{lin}}(x) = \mathcal{T}_{F^{\text{lin}}(x)}(x)$ for $x \in F$. (Remark 7.6 Statement (i))

(ii) Show that $\mathcal{T}_F^{\text{lin}}(x)$ is a closed convex cone. (Remark 7.6 Statement (ii))

(iii) Prove Theorem 8.9 by showing that the Abadie CQ holds at any feasible point of problems of the form

$$\left.\begin{aligned} \text{Minimize} \quad & f(x) && \text{where } x \in \mathbb{R}^n \\ \text{subject to} \quad & A_{\text{ineq}}\, x \le b_{\text{ineq}} \\ \text{and} \quad & A_{\text{eq}}\, x = b_{\text{eq}} \end{aligned}\right\} \tag{8.10}$$

Solution.

(i) Let $x \in F$.

To show that $\mathcal{T}_F^{\text{lin}}(x) \subseteq \mathcal{T}_{F^{\text{lin}}(x)}(x)$, let $d \in \mathcal{T}_F^{\text{lin}}(x)$. By definition,

$$\begin{aligned} g_i'(x)\, d &\le 0 && \text{for all } i \in \mathcal{A}(x) \\ h_j'(x)\, d &= 0 && \text{for all } j = 1, \ldots, n_{\text{eq}} \end{aligned}. \tag{0.1}$$

For all $t > 0$ with

$$t < \inf_{i \in \mathcal{I}(x), g_i'(x)d > 0} \frac{g_i(x)}{g_i'(x)d}$$

we have that

$$\begin{aligned} g_i(x) + g_i'(x)\,((x+td)-x) &= g_i(x) + g_i'(x)\,td && \leq 0 && \text{for all } i = 1,\dots,n_{\text{ineq}} \\ h_j(x) + h_j'(x)\,((x+td)-x) &= h_j(x) + h_j'(x)\,td = h_j(x) && = 0 && \text{for all } j = 1,\dots,n_{\text{eq}} \end{aligned}$$

so $x + td \in F^{\text{lin}}(x)$ for all of those t, so choosing $y^{(k)} = x + t^{(k)}d$ for any sufficiently small positive nullsequence $t^{(k)}$ yields a sequence in $F^{\text{lin}}(x)$ such that

$$\frac{y^{(k)} - x}{t^{(k)}} = d \to d$$

so $d \in \mathcal{T}_{F^{\text{lin}}(x)}(x)$.

To show that $\mathcal{T}_F^{\text{lin}}(x) \supseteq \mathcal{T}_{F^{\text{lin}}(x)}(x)$, let $d \in \mathcal{T}_{F^{\text{lin}}(x)}(x)$. Then there exists a sequence $y^{(k)}$ in $F^{\text{lin}}(x)$ and a positive nullsequence $t^{(k)}$ such that $d^{(k)} := \frac{y^{(k)}-x}{t^{(k)}} \to d$. The definition of $F^{\text{lin}}(x)$ and the feasibility of x immediately implies (0.1) for $d^{(k)}$ instead of d, so the same holds for d due to continuity.

(ii) Closedness is an immediate consequence of the continuous differentiability of g and h and convexity is clear because both conditions defining the linearizing cone are – by design – linear.

(iii) The feasible set is of the form

$$\begin{aligned} & g(x) := A_{\text{ineq}}\,x - b_{\text{ineq}} \leq 0 \\ \text{and} \quad & h(x) := \quad A_{\text{eq}}\,x - b_{\text{eq}} = 0 \end{aligned}$$

with affine linear constraints g and h, where $g' \equiv A_{\text{ineq}}$ and $h' \equiv A_{\text{eq}}$.

Accordingly, for any feasible $x \in F$, we have that

$$\begin{aligned} g_i(x) + g_i'(x)\,(y-x) &= A_{\text{ineq}}\,x - b_{\text{ineq}} + A_{\text{ineq}}(y-x) && = A_{\text{ineq}}\,y - b_{\text{ineq}} && = g(y) && \text{for all } i = 1,\dots,n_{\text{ineq}} \\ h_j(x) + h_j'(x)\,(y-x) &= A_{\text{eq}}\,x - b_{\text{eq}} + A_{\text{ineq}}(y-x) && = A_{\text{eq}}\,y - b_{\text{eq}} && = h(y) && \text{for all } j = 1,\dots,n_{\text{eq}} \end{aligned},$$

which means that $F^{\text{lin}}(x) = F$ for every $x \in F$ and therefore

$$\mathcal{T}_F^{\text{lin}}(x) = \mathcal{T}_{F^{\text{lin}}(x)}(x) = \mathcal{T}_F(x)$$

which is exactly the ACQ.

(6 Points)

Please submit your solutions as a single pdf and an archive of programs via moodle.