Abstract

Contact geometry is the study of odd-dimensional smooth manifolds equipped with contact structures, i.e. hyperplane distributions $\xi = \ker \alpha$ satisfying the contact condition

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0.$$

While they originally arise in the study of ODEs and in classical mechanics, the topological study of contact manifolds is a more recent and very active field of research.

A manifold can have multiple different contact structures, which can be either rigid (in which case one speaks of a "tight" manifold) or flexible (in the sense that they satisfy an h-principle). The latter contact manifolds are then called overtwisted. A foundational result of Eliashberg and Borman–Eliashberg–Murphy, roughly speaking, states that overtwisted contact manifolds exist in abundance, namely whenever the manifold admits the topological version of a contact structure (an *almost* contact structure), which is a first obvious obstruction. In dimension three, an almost contact structure is simply an oriented 2-plane field.

To illustrate this dichotomy, consider the sphere S^3. By a result of Eliashberg, it has precisely one tight contact structure. On the other hand, it has infinitely many overtwisted contact structures, corresponding to the infinitely many homotopy classes of 2-plane fields on the 3-sphere. There are other examples where there are infinitely many or no tight contact structures on a contact manifold.

A further interesting property of contact manifolds comes from the fact that contact geometry is the odd-dimensional counterpart to symplectic geometry. Often, it is possible to view a contact manifold as the boundary of a symplectic manifold. Manifolds that are in this sense "fillable" are always tight. The contrary, however, doesn't need to hold and one can ask the question under which conditions such tight, but non-fillable manifolds exist. The first examples of tight and non-fillable contact manifolds were constructed by Etnyre–Honda in dimension three, and by Massot–Niederkrueger–Wendl in higher dimensions.

More recently, Bowden–Gironella–Moreno–Zhou have shown that there exist homotopically standard, non-fillable but tight contact structures on all spheres S^{2n+1} with $n >= 2$. Starting with a specific open book decomposition of S^{2n-1}, one can construct a contact form on this manifold using a well-known construction by Thurston–Winkelnkemper. Then, according to Bourgeois, this contact structure can be extended to a tight contact structure on $S^{2n-1} \times T^2$. Applying subcritical surgery (preserving the tightness), one can kill the topology of the T^2-factor and obtain a tight contact structure on S^{2n+1}. Because of the special way of constructing it, one can show that it is non-fillable, but still homotopically standard.

The goal of my master thesis is to give a streamlined explanation of the results of Bowden–Gironella–Moreno–Zhou, including the necessary background needed to understand the main ideas.

Preliminaries

The Bourgeois construction

Definition 1. *Let M be an oriented manifold with an open book decomposition (B, p) with oriented binding B. The pages are oriented by the requirement that the induced orientation on the boundary of (the closure) of each page coincides with the orientation of B.*

Question: I don't fully understand Geiges remark there (in Def 4.4.7). See Agustins remarks in workflowy notes.

A contact structure $\xi = \ker \alpha$ on M is said to be ***supported*** *by the open book decomposition (B, p) of M if*

(i) the contact form α induces the positive orientation of M $(\alpha \wedge (\mathrm{d}\alpha)^n > 0)$.

(ii) the 2-form $\mathrm{d}\alpha$ induces a symplectic form on each page, defining its positive orientation

(iii) the 1-form α induces a positive contact form on B, i.e.

$$\alpha|_{TB} \wedge (\mathrm{d}\alpha|_{TB})^{(n-2)} > 0.$$

Theorem 1. *Let $(M, \xi = \ker \alpha)$ be a closed contact manifold of dimension $2n - 1, n \geq 2$. One con find an open book decomposition (B, p) of M supporting ξ. According to Bourgeois, ([Bou02]) there is a contact structure $\tilde{\xi}$ on $M \times T^2$ (where $\tilde{\xi}$ massively depends on the choice of open book).*

Proof. We follow the proof of [Gei08, Thm 7.3.6]. Wlog let M be connected. The existence of an open book decomposition for M is the theorem of Giroux-Mohsen as in [Gei08, Thm 7.3.5]. By definition of an open book, there exists a tubular neighborhood $B \times D^2$ with polar coordinates (r, ϕ) on the D^2-part of the binding B s.t. $p : M \setminus B \to S^1$ is givenby ϕ in that neighborhood. Now, we want to define smooth functions x_1, x_2 on M that coincide with the cartesian coordinate functions on D^2 close to the binding B. In order to do that, choose a smooth function $\rho(r)$ on $B \times D^2$, s.t.

- $\rho = r$ near the binding B,
- $\rho'(r) \geq 0$
- $\rho \equiv 1$ near $B \times \partial D^2$.

We extend this function to a smooth function $\rho : M \to [0, 1]$ by setting $\rho \equiv 1$ outside $B \times D^2$. Now, $x_1 := \rho \cos \phi$ and $x_2 := \rho \sin \phi$ are the desired smooth functions on M that coincide with the Cartesian coordinate functions on the D^2-factor near B. We compute

$$\begin{aligned} x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1 &= \rho^2 \cos^2 \phi \mathrm{d}\phi + \rho \cos \phi \sin \phi \mathrm{d}\rho + \rho^2 \sin^2 \phi \mathrm{d}\phi - \rho \cos \phi \sin \phi \mathrm{d}\rho \\ &= \rho^2 (\cos^2 \phi + \sin^2 \phi) \mathrm{d}\phi \\ &= \rho^2 \mathrm{d}\phi \end{aligned}$$

and, analogously,

$$\mathrm{d}x_1 \wedge \mathrm{d}x_2 = \rho \mathrm{d}\rho \wedge \mathrm{d}\phi.$$

On $M \times T^2$, choose coordinates (θ_1, θ_2) on the torus part of the manifold. Now we have all ingredients together to construct our contact form. Let

$$\tilde{\alpha} \coloneqq x_1 \mathrm{d}\theta_1 - x_2 \mathrm{d}\theta_2 + \alpha.$$

This is a well-defined 1-form on $M \times T^2$ (α is extended to $M \times T^2$ in the obvious way) and we can compute the derivative

$$\mathrm{d}\tilde{\alpha} = \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 - \mathrm{d}x_2 \wedge \mathrm{d}\theta_2 + \mathrm{d}\alpha,$$

hence

$$\begin{aligned}(\mathrm{d}\tilde{\alpha})^n =&(n-1)(\mathrm{d}\alpha)^{n-1} \wedge (\mathrm{d}x_1 \wedge \mathrm{d}\theta_1 - \mathrm{d}x_2 \wedge \mathrm{d}\theta_2)\\ &- n(n-1)(\mathrm{d}\alpha)^{n-2} \wedge \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\theta_2.\end{aligned}$$

In order to verify the contact condition, we compute

$$\begin{aligned}\tilde{\alpha} \wedge (\mathrm{d}\tilde{\alpha})^n =&(x_1 \mathrm{d}\theta_1 - x_2 \mathrm{d}\theta_2 + \alpha) \wedge (n-1)(\mathrm{d}\alpha)^{n-1} \wedge (\mathrm{d}x_1 \wedge \mathrm{d}\theta_1 - \mathrm{d}x_2 \wedge \mathrm{d}\theta_2)\\ &- (x_1 \mathrm{d}\theta_1 - x_2 \mathrm{d}\theta_2 + \alpha) \wedge n(n-1)(\mathrm{d}\alpha)^{n-2} \wedge \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\theta_2\\ =&(n-1)(\mathrm{d}\alpha)^{n-1} \wedge (x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1) \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2\\ &+ \underbrace{\alpha \wedge (n-1)(\mathrm{d}\alpha)^{n-1} \wedge \mathrm{d}x_1}_{2n-\text{form on } M} \wedge \mathrm{d}\theta_1 - \underbrace{\alpha \wedge (n-1)(\mathrm{d}\alpha)^{n-1} \wedge \mathrm{d}x_2}_{2n-\text{form on } M} \wedge \mathrm{d}\theta_2\\ &+ n(n-1)\alpha \wedge (\mathrm{d}\alpha)^{n-2} \wedge \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\theta_2\end{aligned}$$

M has dimension $2n-1$, i.e. the middle term is 0

$$\begin{aligned}=&(n-1)(\mathrm{d}\alpha)^{n-1} \wedge \rho^2 \mathrm{d}\phi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2\\ &+ n(n-1)\alpha \wedge (\mathrm{d}\alpha)^{n-2} \wedge \rho \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2\end{aligned}$$

By condition (ii) of definition 1, $(\mathrm{d}\alpha)^{n-1}$ must be a positive volume form on the pages. As explained in that definition, the orientation on M is given by ∂_ϕ and the orientation of the page. In particular, $(\mathrm{d}\alpha)^{n-1} \wedge \rho \mathrm{d}\phi$ is a positive volume form on M. Multiplied with a second ρ-factor, it vanishes along B. As $\theta_1 \wedge \theta_2$ is a positive volume form on T^2, the first term is non-negative everywhere and positive away from

$$\underbrace{B \times 0}_{\subset B \times D^2 \subset M} \times T^2.$$

Let $\mathfrak{b}$ be a basis of the binding B that is positively ordered. Then, $-\partial_r, \mathfrak{b}$ and (because the binding is odd-dimensional) $\mathfrak{b}, \partial_r$ are positive bases of the page. Clearly, then,

$$\mathfrak{a} \coloneqq \mathfrak{b}, \partial_r, \partial_\phi, \partial_{\theta_1}, \partial_{\theta_2}$$

is an ordered basis of $M \times T^2$. Using $\rho'(r) \geq 0$ everywhere, we deduce that $\mathrm{d}\rho(\partial_r)$ is non-negative. Hence, plugging $\mathfrak{a}$ into the second term, we conclude

$$\begin{aligned}&\left(n(n-1)\alpha \wedge (\mathrm{d}\alpha)^{n-2} \wedge \rho \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2\right)(\mathfrak{a})\\ =&n(n-1)\rho \cdot (\alpha \wedge (\mathrm{d}\alpha)^{n-2})(\mathfrak{b}) \cdot d\rho(\partial_r) \cdot \mathrm{d}\phi(\partial_\phi) \cdot \mathrm{d}\theta_1(\partial_{\theta_1}) \cdot \mathrm{d}\theta_2(\partial_{\theta_2})\\ \geq&0.\end{aligned}$$

By condition (iii) of definition 1, $\alpha \wedge (\mathrm{d}\alpha)^{n-2}$ is positive on B. Therefore, the second term is positive on $B \times 0 \times T^2$ (hence also on a neighborhood) and non-negative everywhere else. In total, we have checked the contact condition and $\tilde{\alpha}$ is indeed a contact form on $M \times T^2$. $\square$

The Thurston-Winkelnkemper construction

Definition 2 (mapping torus)**.** *Let Σ be a smooth manifold with boundary $\partial\Sigma$ and $\phi : \Sigma \to \Sigma$ a diffeomorphism that is equal to the identity close to $\partial\Sigma$. The mapping torus $\Sigma(\phi)$ is given by $\Sigma \times [0, 2\pi]/\sim$ where*

$$(x, 2\pi) \sim (\phi(x), 0).$$

The generalized mapping torus requires as additional data a smooth function $\overline{\varphi} : \Sigma \to \mathbb{R}^+$ that is constant near $\partial\Sigma$. Then,

$$\Sigma_{\overline{\varphi}}(\phi) := \Sigma \times \mathbb{R}/\sim \quad \textit{where} \quad (x, \theta) \sim (\phi(x), \theta - \overline{\varphi}(x)).$$

Abstract open books Starting with a mapping torus $\Sigma(\phi)$, we can construct an abstract open book $M(\phi)$ with binding $\partial\Sigma$ (see fig. 1)

We define

$$M(\phi) := \left(\Sigma(\phi) \cup \partial\Sigma \times D^2\right) / \sim$$

where we identify

$$[x \in \partial\Sigma, \theta] \sim (x, r = 1, \varphi = \theta)$$

The construction Roadmap:

- Let 1-form on Σ descend to mapping torus and add $+\mathrm{d}\varphi$
- Extend this 1-form over the binding neighborhood of the open book
- Check that this actually gives a contact form with the desired properties

Creating a contact form α on the mapping torus Let Σ^{2n} be a compact manifold admitting an exact symplectic form $\omega = \mathrm{d}\beta$ s.t. on the boundary $\partial\Sigma$, a contact form β_∂ is induced (this follows from the conditions requested in Geiges). Let the boundary be connected (i.e. the binding is also connected). Let the monodromy map ϕ be an exact symplectomorphism of (Σ, ω), equal to the identity near the boundary $\partial\Sigma$ (exactness is not necessary according to

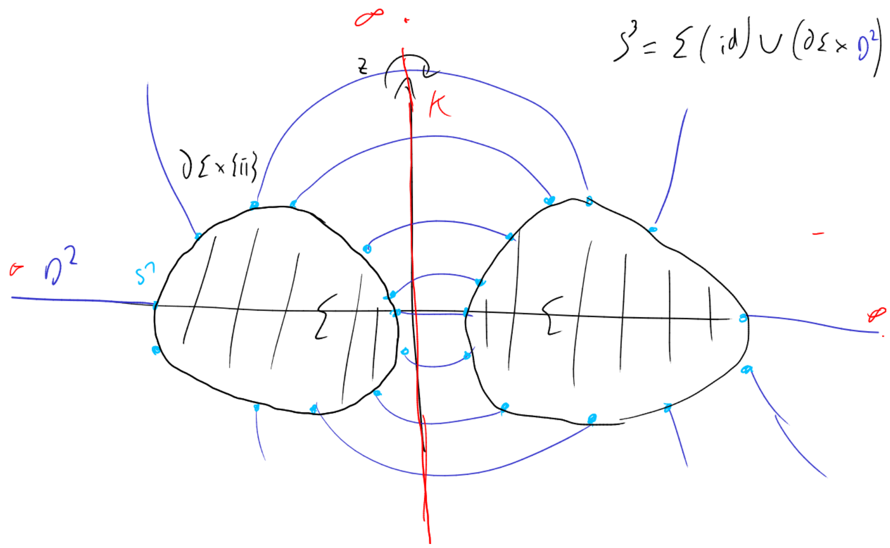

Figure 1: highly professional drawing of an abstract open book

Geiges, as it can be obtained via a suitable isotopy of the symplectomorphism). An exact symplectomorphism ϕ of (Σ, ω) is such that

$$\phi^*(\beta) - \beta =: \mathrm{d}\overline{\varphi}$$

is exact, i.e. there exists such a function $\overline{\varphi}$ on Σ (of course only defined up to adding a locally constant function. Choose it in such a way that it only takes positive values). The 1-form

$$\alpha \coloneqq \beta + \mathrm{d}\varphi$$

is a contact form on $\Sigma \times \mathbb{R}$:

$$\alpha \wedge (\mathrm{d}\alpha)^n = (\beta + \mathrm{d}\varphi) \wedge \underbrace{(\mathrm{d}\beta)^n}_{=:\Omega} = \beta \wedge \Omega + \mathrm{d}\varphi \wedge \Omega = \mathrm{d}\varphi \wedge \Omega,$$

where Ω is a volume form on Σ (as β is a symplectic form). The $\beta \wedge \Omega$ term vanishes because both are forms on Σ, but Ω is already a top-level form. The resulting form is a wedge product of two volume forms on the product manifolds and therefore a volume form on $\Sigma \times \mathbb{R}$.

Now consider the transformation that induces the generalized mapping torus

$$F \coloneqq (x, \varphi) \mapsto (\phi(x), \varphi - \overline{\varphi}(x)).$$

Remember that φ only takes positive values, i.e. the mapping torus is wellde-

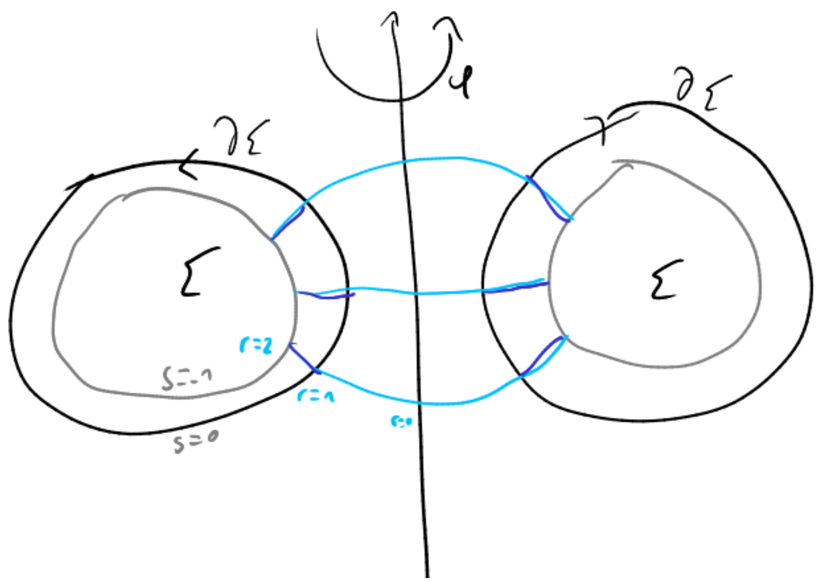

Figure 2: Detailled glueing process of the generalized abstract open book

fined. The 1-form α is invariant under this transformation:

$$\begin{aligned}
F^*(\alpha) &= F^*(\beta) + F^*(\mathrm{d}\varphi) && \mid \beta \text{ is independent of } \varphi \\
&= \phi^*(\beta) + \mathrm{d}F(\varphi) && \mid \text{definition of } \overline{\varphi}, F \\
&= \beta + \mathrm{d}\overline{\varphi} + \mathrm{d}\varphi - \mathrm{d}\overline{\varphi} \\
&= \alpha.
\end{aligned}$$

It follows that α descends to a contact form on $\Sigma_{\overline{\varphi}}(\phi)$.

Extending α to the whole abstract open book First, we describe a gluing construction for the abstract open book. Therefore, we construct a collar neighborhood on the generalized mapping torus s.t. on $[-1,0] \times \partial\Sigma$, the symplectic form is given by $\mathrm{d}\left(e^s \beta_\partial\right)$ where s is the collar parameter, d.h. $\beta = e^s \beta_\partial$. Why does such a neighborhood exist?

Close to $\partial\Sigma$, ϕ is equal to the identity and therefore $\mathrm{d}\overline{\varphi}$ is locally constant (hence constant, as $\partial\Sigma$ is connected). Parametrize the neighborhood so that $\overline{\varphi}$ is constant on $[-1,0] \times \partial\Sigma$.

Now, take a look at

$$\left(\Sigma_{\overline{\varphi}}(\phi) \,\dot\cup\, \left(\partial\Sigma \times D_2^2\right)\right) / \sim .$$

A simple linear reparametrization will make the notation a lot easier: As $\overline{\varphi}$ is constant on the neighborhood under consideration, we just pretend $\overline{\varphi} = 2\pi$. Furthermore, we parametrize the boundary $\partial\Sigma$ with $\theta \in S^1$. Then we identify

$$(s, \theta, \varphi) \in [-1,0] \times \partial\Sigma \times S^1 \subset \Sigma_{\overline{\varphi}}(\phi)$$

with

$$(\theta, s = 1 - r, \varphi) \in \partial\Sigma \times D^2_2$$

where (r, φ) are polar coordinates on D^2_2, i.e. we identify a collar neighborhood of Σ with an annulus in D^2_2. (See fig. 2)

Now we choose the ansatz

$$\alpha_{\text{ext}} \coloneqq h_1(r)\beta_\partial + h_2(r)\mathrm{d}\varphi.$$

for the extension of the contact form over $\partial\Sigma \times D^2$. On the gluing area (i.e. $1 \le r \le 2$), α_{ext} has to agree with $\alpha = \beta + \mathrm{d}\varphi = e^s\beta_\partial + \mathrm{d}\varphi$, d.h.

$$h_1(r) = e^s = e^{1-r} \qquad h_2(r) = 1.$$

In order to ensure smoothness at $r = 0$, in a small neighborhood of $r = 0$ we set $h_1(r) = 2$ and $h_2(r) = r^2$, obtaining

$$\alpha_{\text{ext}} = 2\beta_\partial + r^2\mathrm{d}\varphi.$$

We compute

$$\mathrm{d}\alpha_{\text{ext}} = h_1'(r)\mathrm{d}r \wedge \beta_\partial + h_1(r)\mathrm{d}\beta_\partial + h_2'(r)\mathrm{d}r \wedge \mathrm{d}\varphi.$$

and

$$(\mathrm{d}\alpha_{\text{ext}})^n = n \cdot \mathrm{d}r \wedge (h_1'(r)\beta_\partial + h_2'(r)\mathrm{d}\varphi) \cdot h_1(r)^{n-1}(\mathrm{d}\beta_\partial)^{n-1} + \underbrace{h_1(r)^n(\mathrm{d}\beta_\partial)^n}_{=0},$$

where the second term vanishes because $(\mathrm{d}\beta_\partial)^n$ is a $2n$-form on $\partial\Sigma^{2n-1}$. Finally,

$$\begin{aligned}
\alpha_{\text{ext}} \wedge (\mathrm{d}\alpha_{\text{ext}})^n = & h_1(r) n h_1(r)^{n-1} h_2'(r) \cdot & \beta_\partial \wedge \mathrm{d}r \wedge \mathrm{d}\varphi \wedge (\mathrm{d}\beta_\partial)^{n-1} \\
& + h_2(r) n h_1(r)^{n-1} h_1'(r) \cdot & \mathrm{d}\varphi \wedge \mathrm{d}r \wedge \beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1} \\
= & n h_1(r)^{n-1}(h_1 h_2'(r) - h_2 h_1'(r)) \cdot & \beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1} \wedge \mathrm{d}r \wedge \mathrm{d}\varphi \\
= & n h_1(r)^{n-1} \det \begin{pmatrix} h_1(r) & h_2(r)/r \\ h_1'(r) & h_2'(r)/r \end{pmatrix} \cdot & \beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1} \wedge r\mathrm{d}r \wedge \mathrm{d}\varphi
\end{aligned}$$

As β_∂ is a contact form on $\partial\Sigma$, $\beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1}$ is a positive volume form on $\partial\Sigma$. Furthermore, $r\mathrm{d}r \wedge \mathrm{d}\varphi$ is a positive volume form on the disk D^2_2. As a result, the right term of our result is a volume form on $\partial\Sigma \times D^2_2$. The left term tells us that $h_1(r)$ musn't have any zeros for $r \in [0, 2]$ and that $(h_1(r), h_2(r))$ must never be parallel to $(h_1'(r), h_2'(r))$. Figure 4.7 in [Gei08] proves the existence of such a pair of functions h_1 and h_2 such that

$$h_1(r)^{n-1} \det \begin{pmatrix} h_1(r) & h_2(r)/r \\ h_1'(r) & h_2'(r)/r \end{pmatrix} > 0 \qquad \forall r \in [0, 2].$$

(Close to zero, the determinant is given by $2 \cdot 2 - 0 \cdot 0 = 4 > 0$). In total, we obtain that α_{ext} induces the correct orientation on the extension. Does the orientation on the mapping torus agree with the orientation on the extension? How is the orientation on the mapping torus defined?

References

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[Gei08] Hansjörg Geiges. *An Introduction to Contact Topology*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, March 2008.