Tight and non-Fillable Contact Structures on the Sphere

Josua Kugler
results by Bowden[1], Gironella[2], Moreno[3], Zhou[4]

Heidelberg University

[1]University of Regensburg
[2]University of Nantes
[3]Heidelberg University
[4]Morningside Center of Mathematics, CAS

Background

Contact topology

Contact topology: The study of contact manifolds, up to isotopy.

Contact topology

Contact topology: The study of contact manifolds, up to isotopy.

Fillability: *fillable* contact mflds are boundaries of symplectic mflds.

Contact topology

Contact topology: The study of contact manifolds, up to isotopy.

Fillability: *fillable* contact mflds are boundaries of symplectic mflds.

Fillability question

Which contact manifolds are **fillable**?

Contact topology

Contact topology: The study of contact manifolds, up to isotopy.

Fillability: *fillable* contact mflds are boundaries of symplectic mflds.

Fillability question

Which contact manifolds are **fillable**?

Eliashberg, Borman–Eliashberg–Murphy:

Dichotomy: Rigidity vs. Flexibility.

- **tight** *(rigid/geometric)*;
- **overtwisted** *(flexible/topological)*.

Contact topology

Contact topology: The study of contact manifolds, up to isotopy.

Fillability: *fillable* contact mflds are boundaries of symplectic mflds.

Fillability question

Which contact manifolds are **fillable**?

Eliashberg, Borman–Eliashberg–Murphy:

Dichotomy: Rigidity vs. Flexibility.

- **tight** *(rigid/geometric)*;
- **overtwisted** *(flexible/topological)*.

Theorem (Eliashberg–Gromov)

Fillable contact manifolds are tight.

Converse is false (Etnyre–Honda, Massot–Niederkrueger–Wendl).

Existence and classification

Topological obstruction: *almost* contact structure, i.e. reduction of structure group to $U(n) \times \mathbb{1}$.

Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))

Almost contact manifolds are contact, where the contact structure is overtwisted.

Existence and classification

Topological obstruction: *almost* contact structure, i.e. reduction of structure group to $U(n) \times \mathbb{1}$.

Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))

Almost contact manifolds are contact, where the contact structure is overtwisted.

Tight manifolds

How can **tight** contact manifolds be understood?

Contact structures on spheres

Standard contact structure

The standard contact structure is $(S^{2n-1}, \xi) = \partial(B^{2n}, \omega_{std})$.

Contact structures on spheres

Standard contact structure

The standard contact structure is $(S^{2n-1}, \xi) = \partial(B^{2n}, \omega_{std})$.

Theorem (Eliashberg, '91)

On S^3, it is the unique tight contact structure.

In particular, no tight and non-fillable contact structures on S^3.

Tight and non-fillable structures in $\dim \geqslant 5$

Theorem (Bowden–Gironella–Moreno–Zhou '22-'24)

For every $n \geqslant 2$, the sphere $\mathbb{S}^{2n+1}$ admits a tight, non-fillable contact structure that is homotopically standard.

Tight and non-fillable structures in $\dim \geqslant 5$

Theorem (Bowden–Gironella–Moreno–Zhou '22-'24)

For every $n \geqslant 2$, the sphere $\mathbb{S}^{2n+1}$ admits a tight, non-fillable contact structure that is homotopically standard.

By connected sum with such an "exotic" sphere, it can be concluded

Theorem (Bowden–Gironella–Moreno–Zhou '22-'24)

In $\dim \geqslant 7$, if M admits a tight structure, it also admits a tight and non strongly-fillable structure, in the same almost contact class.

Tight and non-fillable structures in $\dim \geqslant 5$

Theorem (Bowden–Gironella–Moreno–Zhou '22-'24)

For every $n \geqslant 2$, the sphere $\mathbb{S}^{2n+1}$ admits a tight, non-fillable contact structure that is homotopically standard.

By connected sum with such an “exotic” sphere, it can be concluded

Theorem (Bowden–Gironella–Moreno–Zhou '22-'24)

In $\dim \geqslant 7$, if M admits a tight structure, it also admits a tight and non strongly-fillable structure, in the same almost contact class.
In $\dim = 5$, the same holds, if the first Chern class vanishes.

Tight and non-fillable spheres

Giroux correspondence

Giroux: Contact structures are *supported* by open books.

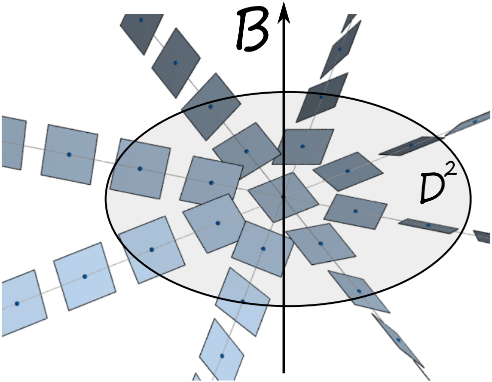

Figure: Supported contact structure.

Bourgeois contact structures

Theorem (Bourgeois '02)

Open book supporting $(M, \xi) \rightsquigarrow$ *contact structure on* $M \times \mathbb{T}^2$.

These are $\mathbb{T}^2$-equivariant.

Geometric construction

Geometric construction: Construct a tight and non-fillable contact structure on $\mathbb{S}^{2n+1}$:

Geometric construction

Geometric construction: Construct a tight and non-fillable contact structure on $\mathbb{S}^{2n+1}$:

- Milnor open book on $\mathbb{S}^{2n-1} \overset{\text{Bourgeois}}{\rightsquigarrow}$ contact structure on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ $\rightsquigarrow$ two 1-surgeries = $\mathbb{S}^{2n-1} \times \mathbb{S}^2$ $\rightsquigarrow$ one 2-surgery = $\mathbb{S}^{2n+1}$.

Geometric construction

Geometric construction: Construct a tight and non-fillable contact structure on $\mathbb{S}^{2n+1}$:

- Milnor open book on $\mathbb{S}^{2n-1} \overset{\text{Bourgeois}}{\rightsquigarrow}$ contact structure on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ $\rightsquigarrow$ two 1-surgeries = $\mathbb{S}^{2n-1} \times \mathbb{S}^2 \rightsquigarrow$ one 2-surgery = $\mathbb{S}^{2n+1}$.
- If $n \geqslant 3$, surgeries are *subcritical* $\rightsquigarrow$ by 'Eliashberg's' h-pple, Weinstein cobordism $\rightsquigarrow$ contact manifold $(\mathbb{S}^{2n+1}, \xi_{ex})$.

Geometric construction

Geometric construction: Construct a tight and non-fillable contact structure on $\mathbb{S}^{2n+1}$:

- Milnor open book on $\mathbb{S}^{2n-1} \overset{\text{Bourgeois}}{\rightsquigarrow}$ contact structure on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ $\rightsquigarrow$ two 1-surgeries = $\mathbb{S}^{2n-1} \times \mathbb{S}^2 \rightsquigarrow$ one 2-surgery = $\mathbb{S}^{2n+1}$.
- If $n \geqslant 3$, surgeries are *subcritical* $\rightsquigarrow$ by 'Eliashberg's' h-pple, Weinstein cobordism $\rightsquigarrow$ contact manifold $(\mathbb{S}^{2n+1}, \xi_{ex})$.

Claim: $(\mathbb{S}^{2n+1}, \xi_{ex})$ is tight and non-fillable.

Tightness

Facts:

1. Milnor open book $\Rightarrow$ algebraically tight Bourgeois manifold.

Tightness

Facts:

1. Milnor open book $\Rightarrow$ algebraically tight Bourgeois manifold.
    - Algebraic tightness is vanishing of a certain contact homology algebra.

Tightness

Facts:

1. Milnor open book $\Rightarrow$ algebraically tight Bourgeois manifold.
   - Algebraic tightness is vanishing of a certain contact homology algebra.
2. Algebraic tightness is preserved under surgeries.

Tightness

Facts:

1. Milnor open book $\Rightarrow$ algebraically tight Bourgeois manifold.
   - Algebraic tightness is vanishing of a certain contact homology algebra.
2. Algebraic tightness is preserved under surgeries.
3. Algebraically tight $\implies$ tight.

Tightness

Facts:

1. Milnor open book $\Rightarrow$ algebraically tight Bourgeois manifold.
   - Algebraic tightness is vanishing of a certain contact homology algebra.
2. Algebraic tightness is preserved under surgeries.
3. Algebraically tight $\implies$ tight.

> Milnor open book $\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex})$ is *tight*.

Thank you!

Fillability

Observation: Bourgeois manifolds have convex decomposition

$$\mathrm{OB}(\Sigma, \phi) \times \mathbb{T}^2 = (\mathrm{OB}(\Sigma, \phi) \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V}_- \times \mathbb{S}^1,$$

with $V_\pm = \Sigma \times D^*\mathbb{S}^1$, $\Sigma =$ page of the open book, $\phi =$ monodromy.

Fillability

Observation: Bourgeois manifolds have convex decomposition

$$\mathrm{OB}(\Sigma, \phi) \times \mathbb{T}^2 = (\mathrm{OB}(\Sigma, \phi) \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V_-} \times \mathbb{S}^1,$$

with $V_\pm = \Sigma \times D^*\mathbb{S}^1$, $\Sigma =$ page of the open book, $\phi =$ monodromy.

Theorem (Bowden–Gironella–Moreno)

$M = V \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V_-} \times \mathbb{S}^1$ with convex decomposition, $N = \partial V_\pm$ dividing set. If W is a symplectic filling of M, then

$$H_*(N) \to H_*(V_\pm) \to H_*(W),$$

induced by inclusion. Then second map is injective on image of the first.

Namely, if a homology class in N survives in $V_\pm$, then it survives in the filling.

Proof: W filling of $(\mathbb{S}^{2n+1}, \xi_{ex})$:

$$0 \neq H_n(N) \xrightarrow{\text{nontrivial}} H_n(W).$$

However, this factors as

$$0 \neq H_n(N) \to H_n(\partial W = \mathbb{S}^{2n+1}) = 0 \to H_n(W),$$

contradiction.