Tight and non-Fillable Contact Structures on the Sphere

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results by Bowden[1], Gironella[2], Moreno[3], Zhou[4]

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Background

Contact topology

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Fillability: *fillable* contact mflds are boundaries of symplectic mflds.

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Eliashberg, Borman–Eliashberg–Murphy:

Dichotomy: Rigidity vs. Flexibility.

- **tight** *(rigid/geometric)*;
- **overtwisted** *(flexible/topological)*.

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Theorem (Eliashberg–Gromov)

Fillable contact manifolds are tight.

Converse is false (Etnyre–Honda, Massot–Niederkrueger–Wendl).

Existence and classification

Topological obstruction: *almost* contact structure, i.e. reduction of structure group to $U(n) \times \mathbb{1}$.

Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))

Almost contact manifolds are contact, where the contact structure is overtwisted.

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Tight manifolds

How can **tight** contact manifolds be understood?

Contact structures on spheres

Standard contact structure

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Theorem (Eliashberg, '91)

On S^3, it is the unique tight contact structure.

In particular, no tight and non-fillable contact structures on S^3.

Tight and non-fillable structures in $\dim \geqslant 5$

Theorem (Bowden–Gironella–Moreno–Zhou '22-'24)

For every $n \geqslant 2$, the sphere $\mathbb{S}^{2n+1}$ admits a tight, non-fillable contact structure that is homotopically standard.

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In $\dim = 5$, the same holds, if the first Chern class vanishes.

Tight and non-fillable spheres

Giroux correspondence

Giroux: Contact structures are *supported* by open books.

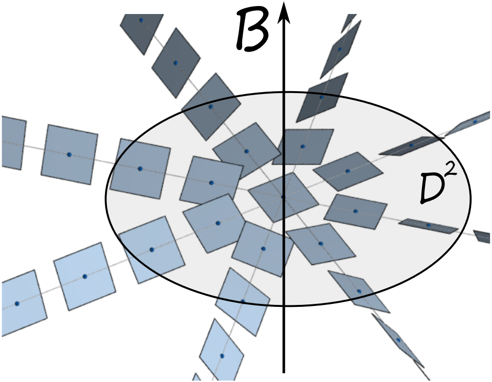

Figure: Supported contact structure.

Bourgeois contact structures

Theorem (Bourgeois '02)

Open book supporting $(M, \xi) \rightsquigarrow$ *contact structure on* $M \times \mathbb{T}^2$*.*

These are $\mathbb{T}^2$-equivariant.

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- Milnor open book on $\mathbb{S}^{2n-1} \overset{\text{Bourgeois}}{\rightsquigarrow}$ contact structure on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ $\rightsquigarrow$ two 1-surgeries = $\mathbb{S}^{2n-1} \times \mathbb{S}^2 \rightsquigarrow$ one 2-surgery = $\mathbb{S}^{2n+1}$.

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Claim: $(\mathbb{S}^{2n+1}, \xi_{ex})$ is tight and non-fillable.

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Milnor open book $\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex})$ is *tight*.

Fillability

- Pseudoholomorphic curves
- clever geometric construction
- homological obstruction

Thank you!

Fillability

Observation: Bourgeois manifolds have convex decomposition

$$\mathrm{OB}(\Sigma, \phi) \times \mathbb{T}^2 = (\mathrm{OB}(\Sigma, \phi) \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V}_- \times \mathbb{S}^1,$$

with $V_\pm = \Sigma \times D^*\mathbb{S}^1$, $\Sigma =$ page of the open book, $\phi =$ monodromy.

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Theorem (Bowden–Gironella–Moreno)

$M = V \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V_-} \times \mathbb{S}^1$ with convex decomposition, $N = \partial V_\pm$ dividing set. If W is a symplectic filling of M, then

$$H_*(N) \to H_*(V_\pm) \to H_*(W),$$

induced by inclusion. Then second map is injective on image of the first.

Namely, if a homology class in N survives in $V_\pm$, then it survives in the filling.

Proof: W filling of $(\mathbb{S}^{2n+1}, \xi_{ex})$:

$$0 \neq H_n(N) \xrightarrow{\text{nontrivial}} H_n(W).$$

However, this factors as

$$0 \neq H_n(N) \to H_n(\partial W = \mathbb{S}^{2n+1}) = 0 \to H_n(W),$$

contradiction.