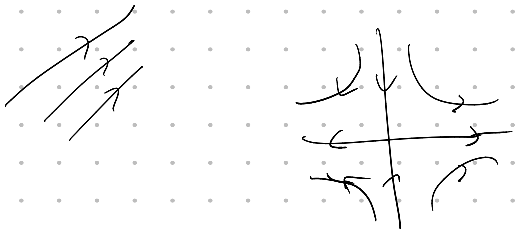
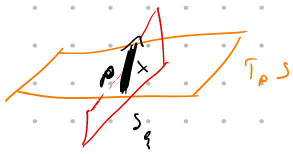
4. Surfaces in contact 3-mfds

Def A singular foliation on M is the equivalence class of a vector field X

s.t. $X \sim X' \iff \exists f: M \to \mathbb{R}_+$ s.t. $X = f X'$



Let $S \subset (M, \xi = \ker(\alpha))$ be an oriented surface. The characteristic foliation S_ξ of S is given by $TS \cap \xi$



If $T_pS = \xi_p \implies X(p) = 0$

Ex: $S = S^2 \subset (\mathbb{R}^3, \ker(x\,dy - y\,dx + dz))$

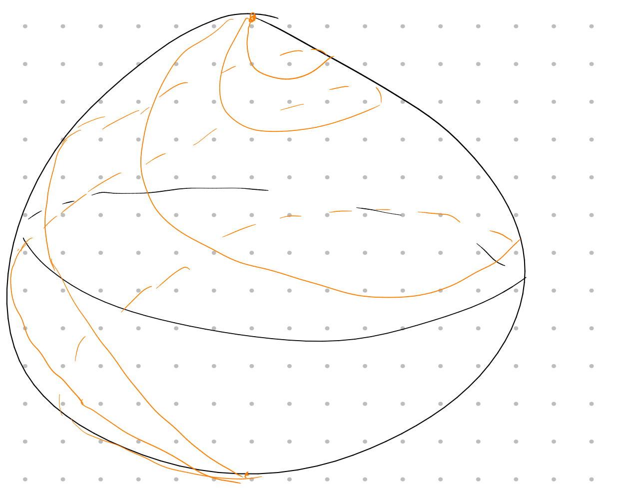
S_ξ is spanned by $X := (xz - y)\partial_x + (yz + x)\partial_y - (x^2+y^2)\partial_z$

$X \in TS^2$ & $X \in \xi$

$X \in TS^2$: $x^2z - yx + y^2z + xy - (x^2z + y^2z) = 0$ ✓

$X \in \xi$: $(xyz + x^2) - (yxz - y^2) - (x^2+y^2) = 0$ ✓

$X(x,y,z) = 0 \iff (x,y,z) = (0,0,\pm 1)$



Identify $\nu S \cong S \times \mathbb{R}$
$S \hookrightarrow S \times 0$

write $\alpha = \beta_z + u_z\,dz$

with $z \in \mathbb{R}$, β_z 1-forms on S

& $u_z: S \to \mathbb{R}$

$\implies d\alpha = d\beta_z - \dot\beta_z \wedge dz + du_z \wedge dz$

contact condition: $u_z\,d\beta_z + \beta_z \wedge (du_z - \dot\beta_z) > 0$

Let Ω be an area form on S

Lemma 1: S_ξ is given by a vf defined by

$$i_X \Omega = \beta_0 = \alpha|_{TS}$$

proof: $*$ $X(p)=0 \iff \beta_{0,p}=0 \iff T_pS = \xi_p$

$*$ $0 = L_X(i_X \Omega) = i_X \beta_0 \Rightarrow X \in \xi$ $\square$

Lemma 2: A vector field X on S defines a characteristic foliation of a contact structure ξ

$\iff$ $\forall p \in S$ with $X(p)=0 \Rightarrow \operatorname{div}_\Omega(X)(p) \neq 0$ $(*)$

Divergence: $\operatorname{Div}_\Omega(X)$ of a vector field X on S w.r.t. an area form Ω is defined by

$$\operatorname{Div}_\Omega(X) \cdot \Omega := L_X \Omega = d(i_X \Omega)$$

Exercise: Check that this gives the old defn in coordinates

proof: "$\Rightarrow$" If $X(p)=0 \Rightarrow \beta_{0,p}=0 \Rightarrow \alpha_p = u_0(p)\,dz \Rightarrow \xi_p = T_pS$

$$\Rightarrow d(i_X\Omega) = (d\beta_0)_p = d\alpha|_{\xi_p} \neq 0$$

"$\Leftarrow$" Let X with $(*)$. Let $\beta := i_X \Omega$ & $u: S \to \mathbb{R}$ def by $d\beta = u\Omega$ One-dimensional vector space $\Rightarrow \exists$ such a function

$(*) \Rightarrow$ if $\beta_p = 0 \Rightarrow u(p) \neq 0$

choose 1-form γ on S

$$\beta \wedge \gamma \geq 0 \geq (\beta\wedge\gamma)(p) > 0 \quad \text{if } \beta(p) \neq 0$$

$\beta_z := \beta + z(du - \gamma)$, $d\beta_0 = d\beta = u\Omega$ & $\dot\beta_0 = du - \gamma$;

$\alpha = \beta_z + u\,dz$ is C.F. near S. $[u\,d\beta_0 + \beta_0 \wedge (du - \dot\beta_0) = u^2\Omega + \beta\wedge\gamma > 0]$ $\square$

Ex: $S_1 = S^1 \times S^1 \subset (S^1 \times \mathbb{R}^2, \xi_n = \ker(\underbrace{\cos(n\theta) - \sin(n\theta) dy}_{\alpha_n}))$

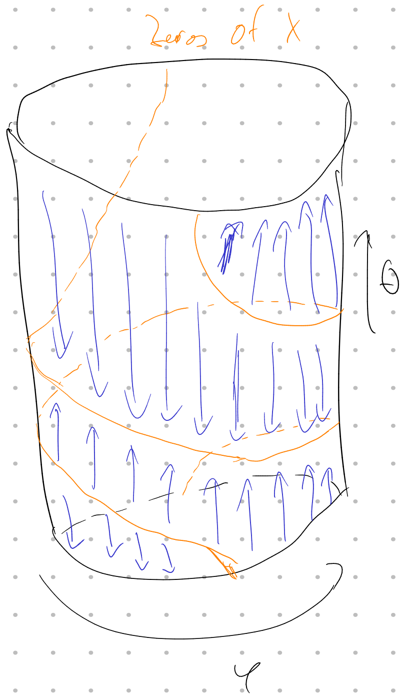
(r, φ) polar coordinates on $\mathbb{R}^2$

$\Omega := d\theta \wedge d\varphi$ defines the standard orientation on S

find X s.t. $\iota_X \Omega = \alpha_n|_{TS} = -(\cos(n\theta)\sin\varphi - \sin(n\theta)\cos\varphi)\, d\varphi$

Ansatz: $X(\theta, \varphi) = a(\theta,\varphi)\partial_\theta + b(\theta,\varphi)\partial_\varphi$

$\Rightarrow \boxed{X = -(x \sin(n\theta) + y\cos(n\theta))\partial_\theta}$



Thm 3: Let $S_i \subset (M_i, \xi_i)$, $i = 0,1$

& $\phi: S_0 \xrightarrow{\cong_{C^\infty}} S_1$ s.t.

$\phi(S_{0,\xi_0}) = S_{1,\xi_1}$ as oriented foliations

$\Rightarrow \exists$ tubular neighborhoods νS_0 & νS_1 in M_0 & M_1

& $\Phi: (\nu S_0, \xi_0) \xrightarrow{\cong} (\nu S_1, \xi_1)$ s.t. $\Phi|_{S_0} = \phi$

proof: the (Moser trick). ▨

(4.2) Singularities of S_ξ

$X(x,y) = a(x,y)\partial_x + b(x,y)\partial_y$ on A NBHD of $0 \in \mathbb{R}^2$
with isolated zero at 0 $\qquad \Omega := dx \wedge dy$

$d(\iota_X \Omega) = d(a\,dy - b\,dx) = (a_x + b_y)\, dx \wedge dy$

$\Rightarrow \operatorname{div}(X) = a_x + b_y$

characteristic foliation is given by flow lines φ_t of

$$(\dot{x}, \dot{y}) = X(x,y)$$

Linearized equation:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} a_x(0) & a_y(0) \\ b_x(0) & b_y(0) \end{pmatrix}}_{=:A} \begin{pmatrix} x \\ y \end{pmatrix}$$

A singularity is called <u>non-degenerate</u> $\Leftrightarrow$ every eigenvalue of A has real part $\neq 0$.

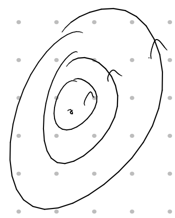
Remark [Hartman - Grobman thm]

If a singularity is non-deg. $\Rightarrow \exists\, C^1$-diffeo h on a NBHD of $0 \in \mathbb{R}^2$ s.t.

$$\varphi_t(h(x,y)) = h\left(e^{At}\cdot\begin{pmatrix}x\\y\end{pmatrix}\right)$$

Ex: $X = x\partial_y - y\partial_x$

$$\Rightarrow A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad EV = \pm i$$



degenerate singularity

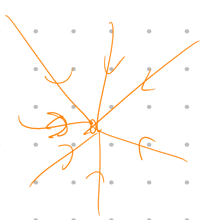
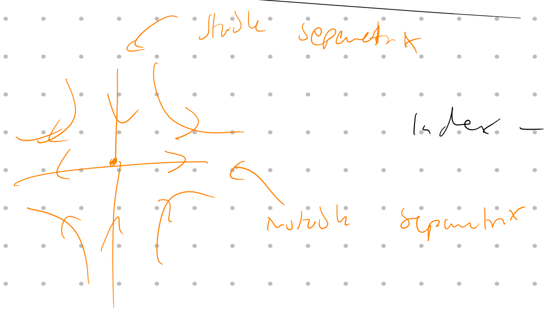
Div $= 0$

$\Rightarrow$ does <u>not</u> appear in characteristic foliation

If a singularity is non-degenerate we call it

* <u>elliptic</u> $:\Leftrightarrow$ $\exists$ only one eigenvalue or $\exists$ two eigenvalues with real parts of the same sign

* <u>hyperbolic</u> $:\Leftrightarrow$ $\exists$ two real eigenvalues of opposite signs.

<u>Elliptic</u>	<u>Hyperbolic</u>
 	
Index $+1$	Index -1
	Lin model: $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\lambda_1 > 0 > \lambda_2$

$$\text{Index}_X(p) := \deg\left(\begin{array}{ccc} S^1_q & \longrightarrow & S^1 \\ q & \longmapsto & \frac{X(q)}{|X(q)|} \end{array} \right)$$

Sign:	
$+1$ if source -1 if sink	sign of $(\lambda_1 + \lambda_2)$

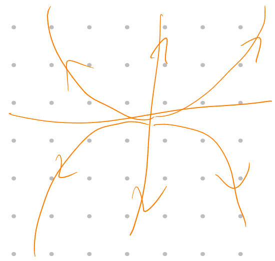
Example: Hyperbolic point

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \lambda_1 > 0 > \lambda_2$$

$$\Rightarrow X(x,y) = \lambda_1 x \partial_x + \lambda_2 y \partial_y$$

$\Rightarrow \beta = \iota_X(dx \wedge dy) = \lambda_1 x\, dy - \lambda_2 y\, dx$

$\Rightarrow \alpha = dz + \lambda_1 x\, dy - \lambda_2 y\, dx$ is a contact form $\Leftrightarrow \lambda_1 + \lambda_2 \neq 0$



Example: $X(x,y) = x\partial_x + y^3\partial_y \Rightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & 3y^2 \end{pmatrix}$

$\Rightarrow$ degenerated

But $\operatorname{div}(X)(0) = 1 \neq 0$

$\Rightarrow X$ defines the char. foliation of a contact str.

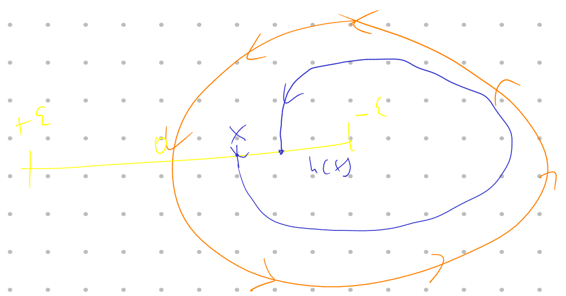
$(\alpha = -y^3\, dx + x\, dy + dz)$

By a C^∞-perturbation of X we get a non-deg elliptic point.

A vector field X on a closed surface S is called morse-smale $:\Leftrightarrow$

(i) $\exists$ only finitely many singularities & finitely many closed orbits, all non-deg

[a closed orbit is nondegenerate $:\Leftrightarrow$ Poincaré return map h satisfies $h'(0) \neq 1$

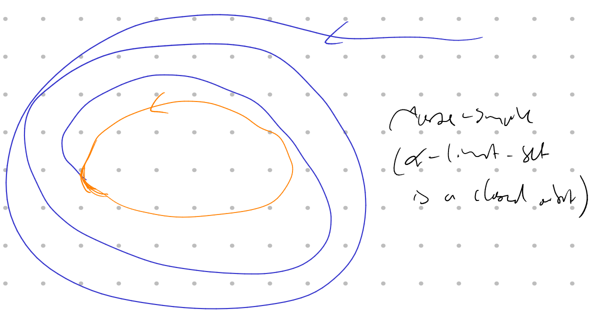


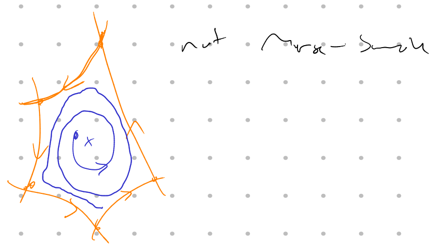
(ii) The α- & ω-limit sets of every flow-line are a single point or a closed orbit.

[φ_t flow of X

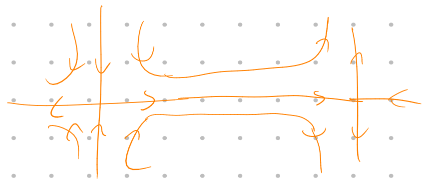
α-limit set of the orbit through $x_0 := \{\lim_{n\to\infty} \varphi_{t_n}(x_0) \mid t_n \nearrow \infty\}$

ω- " " $\{$ " $\mid t_n \searrow -\infty\}$





(iii) $\nexists$ flow line connecting hyperbolic points



not Morse–Smale

THM 4: After a C^∞-perturbation of the surface S we can assume that S_ξ is Morse–Smale

Proof: not easy $\longrightarrow$ see dynamical systems ▣

4.3 Convex surfaces (Giroux)

Def: $S \subset (M, \xi)$ is called <u>convex</u> $:\iff$

$\exists$ contact vector field Y near S s.t. $Y \pitchfork S$

Ex: $S^1 \times S^1 \subset (S^1 \times \mathbb{R}^2, \ker(\underbrace{\cos(n\theta)dx - \sin(n\theta)dy}_{\alpha_n}))$

$Y = x\partial x + y \partial y$ $\qquad Y \pitchfork S$

$L_Y \alpha_n = i_Y(d\alpha_n) + d(i_Y \alpha_n) = \alpha_n \implies Y$ is a contact ~~vector~~ field

Ex: unit sphere in $(\mathbb{R}^3, \xi_{ot})$.

Lemma 5: $S \subset (M, \xi)$ closed is convex

$\iff \exists \ \psi : S^1 \times \mathbb{R} \longrightarrow M$ s.t. $p \longmapsto \psi(p, 0)$ is the inclusion $S \hookrightarrow M$

& $\ker(\psi^* \alpha)$ is an $\mathbb{R}$-inv. contact structure on $S \times \mathbb{R}$

Proof:

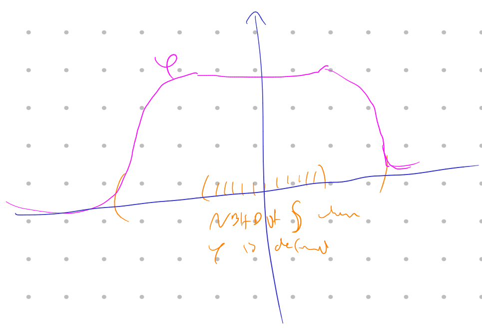
"$\Leftarrow$" $T\psi(\partial_t)$ is a contact vector field $\pitchfork S$

"$\Rightarrow$" Let Y be a contact vector field s.t. $Y \pitchfork S$

$H := \alpha(Y)$ defined near S

Let $\varphi : M \longrightarrow \mathbb{R}$ s.t. $\varphi \equiv 1$ near S

$\varphi \equiv 0$ on $M \setminus \nu S$



$\overline{Y}$ The contact vector field corresponding to $\varphi \cdot H$

$\psi_t :=$ flow of $\overline{Y}$

$\psi : S \times \mathbb{R} \longrightarrow M$

$(p, t) \longmapsto \psi_t(p)$

$\implies T\psi_{(p,t)}(\partial_t) = \dot{\psi}_t(p) = Y(\psi_t(p))$ near $S \implies \ker \psi^*\alpha$ is $\mathbb{R}$-inv. ▣