2. Contact Manifolds

17.10.23

2.1 Hyperplane fields

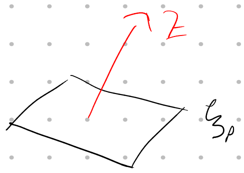
M^n = smooth n-MFD

$\left(p \longmapsto \xi_p^{n-1} \subset T_pM^n\right)$

- $\xi \in TM$ is called *hyperplane field* $:\Leftrightarrow$ $\forall p \in M$ $\exists$ NBHD U of p and linearly independent vector fields $X_1, \dots, X_{n-1}$ on U s.t. $\forall q \in U$:

$$\xi_q := \xi \cap T_qM = \langle X_1(q), \dots, X_{n-1}(q) \rangle$$

- ξ is called *co-orientable* $:\Leftrightarrow$ TM/ξ is orientable

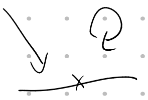


$p \mapsto T_pM/\xi_p = V_p$

Def vector bundle over M^n

$$E \supset p^{-1}(U) \xrightarrow[\text{Fib iso}]{\cong^{C^\infty}} U \times \mathbb{R}^k$$

$p: E \to M^n \ni p$, $\mathrm{pr}: U\times\mathbb{R}^k \to U$



Why is TM/ξ a bundle?

$\xi_q = \langle X_1(q), \dots, X_{n-1}(q) \rangle$

$N_q = \xi_q^\perp$

$p^{-1}(U) \cong U \times \langle N \rangle$

$p^{-1}(U) \cong U \times V_p$

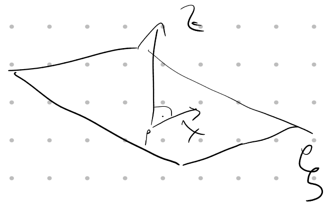
Lemma 1: $\xi \in TM$ is coorientable $\Leftrightarrow$ $\exists$ 1-form α on M s.t. $\xi = \ker(\alpha)$ (i.e. $\alpha_p : T_pM \to \mathbb{R}$)

18.10.23

proof: Let g be a riemannian metric.

"$\Rightarrow$" Choose a vector field Z on M with

$$\begin{cases} g(Z, X) = 0 & \forall X \in \xi \\ g(Z,Z) = 1 \end{cases}$$



$\alpha := g(Z, \cdot)$

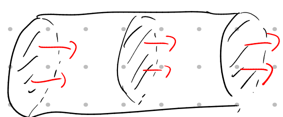
"$\Leftarrow$" A vector field Z with $\begin{cases} g(Z,Z) = 1 \\ \alpha(Z) \neq 0 \end{cases}$ defines a coorientation of ξ

$\square$

Convention: we restrict to coorientable hyperplane fields (often this is secretly assumed)

Example: $M^n := S^1 \times N^{n-1}$

$\forall p = (\theta, x) \in M \Rightarrow \xi_p := T_xN \subset T_pM$



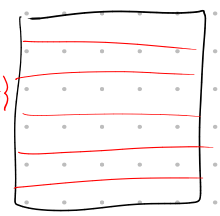
glue left and right side together

$\xi = \bigcup_{p\in M} \xi_p$ is a hyperplane field

$\xi = \ker(d\theta)$ codim-1-foliation

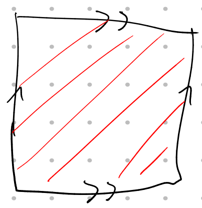
Foliation: $M^n = \bigsqcup_{\theta \in \Lambda} F_\theta$ "leafs" is called k-dim foliation

$:\Leftrightarrow$ $\exists$ Atlas of M (U_w, h_w) s.t. $\forall \theta\ \forall w:$ $h_w(F_\theta \cap U_w) = \emptyset$ or $\mathbb{R}^k \times p \cap h_w(U_w)$



Ex: Flow lines of a non-vanishing vector field are 1-dimensional foliations

$M = T^2$



If slope $\in \mathbb{Q}$ $\Rightarrow$ Leaves are compact

If slope $\in \mathbb{R}\setminus\mathbb{Q}$ $\Rightarrow$ " noncompact

If $k = n-1$ $\Rightarrow$ $\xi = \bigcup_{p \in M} T_p F(p)$ is a hyperplane field

"every foliation of codim 1 induces a hyperplane field"

Q: "when does a hyperplane come from a foliation?"

Theorem 2 [Frobenius]

$\ker(\alpha) = \xi \subset TM$ is induced by a foliation $\Leftrightarrow$ $\alpha \wedge d\alpha = 0$

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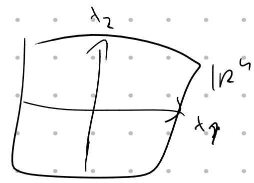
digression: $\alpha_p : T_pM \xrightarrow{1\text{-form}} \mathbb{R}$ lin

$\alpha_p \in (T_pM)^*$

Let $x_1, \dots, x_n$ be coordinates on M



$\Rightarrow$ coordinate vector fields $\partial x_1, \dots, \partial x_n$ form a basis of T_pM

Dual basis $dx_1, \dots, dx_n$, i.e. $dx_i(\partial x_j) = \delta_{ij}$

$\Rightarrow$ $\alpha_p = \sum c_i(p)\, dx_i$

k-form β: $\beta_p : \underbrace{(T_pM) \times \dots \times (T_pM)}_{k \text{ times}} \longrightarrow \mathbb{R}$ multilinear, alternating

$\wedge$-product:

1-forms $\alpha = \sum c_i\, dx_i$

$\beta = \sum b_j\, dx_j$.

$\alpha \wedge \beta =$ def. to be linear

$dx_i \wedge dx_j = -dx_j \wedge dx_i$

$dx_i \wedge dx_i = 0$

Differential

$d:$ k-form $\longrightarrow$ $(k+1)$-forms

$\sum c_i(p)\, dx_i \longmapsto d\alpha$

$\sum_{i,j} \frac{\partial c_i}{\partial x_j} dx_j \wedge dx_i$

Ex: $M = \mathbb{R}^3_{(x,y,z)}$ $\alpha = dx$ $\Rightarrow$ $\alpha \wedge d\alpha$ comes from a foliation

$d\alpha = ddx = 0$

(2) $\alpha = x\,dy + dz$

$$d\alpha = d(x\,dy + dz) = d(x\,dy) + \cancel{d(dz)} = \underbrace{\frac{\partial x}{\partial x}}_{=1} dx \wedge dy + \cancel{\frac{\partial x}{\partial y} dy \wedge dy}$$

$$\alpha \wedge d\alpha = (x\,dy + dz) \wedge (dx \wedge dy) = x\,dy \wedge (dx \wedge dy) + dz \wedge dx \wedge dy$$

$$= dx \wedge dy \wedge dz \neq 0$$

2.2 Contact structures

Let M^{2n+1}

* $\xi^{2n} = \ker(\alpha)$ is a contact structure $\iff$ $\alpha \wedge (d\alpha)^n$ is a volume form $\left\{ \begin{array}{l} \text{i.e. } \alpha \wedge (d\alpha)^n \neq 0 \\ \text{at every point} \end{array} \right\}$

$\alpha \wedge (d\alpha)^n = \alpha \wedge d\alpha \wedge \ldots \wedge d\alpha$

* α is called contact form

* The reeb vector field R_α of α is defined

$$\begin{cases} d\alpha(R_\alpha, \cdot) \equiv 0 \\ \alpha(R_\alpha) \equiv 1 \end{cases}$$

remark: $\alpha \wedge (d\alpha)^n$ is a vol form $\Rightarrow$ M orientable

* $\xi = \ker(\alpha) = \ker(\tilde{\alpha}) \Rightarrow \tilde{\alpha} = f\alpha$ for $f: M \to \mathbb{R}\setminus\{0\}$

$$\Rightarrow \tilde{\alpha} \wedge (d\tilde{\alpha})^n = (f\alpha) \wedge (d(f\alpha))^n = f\alpha \wedge (f\,d\alpha + df \wedge \alpha)^n$$

$\alpha \wedge \alpha = 0$

$$= \underbrace{f^{n+1}}_{\neq 0} \underbrace{\alpha \wedge (d\alpha)^n}_{\neq 0}$$

* R_α is well-defined:

$\alpha \wedge (d\alpha)^n \neq 0 \Rightarrow (d\alpha)^n|_\xi \neq 0$ $\Rightarrow$ $d\alpha$ has rank $2n$

ξ 2n-dimensional bundle $\Rightarrow$ $\ker(d\alpha)$ is 1-dim & $\alpha \neq 0$ on $\ker(d\alpha)$

Weinstein conj: M closed $\overset{?}{\Rightarrow}$ $\forall$ contact form α on M: R_α has a periodic orbit

Ex (1) Consider $\mathbb{R}^{2n+1}_{(x_1, y_1, \dots, x_n, y_n, z)}$

$\xi_{st} = \ker(\alpha_{st})$ standard contact structure

$$\alpha_{st} = \left(\sum_{i=1}^n x_i dy_i\right) + dz$$

$$\alpha_{st} \wedge (d\alpha_{st})^n = \left(\sum x_i dy_i + dz\right) \wedge \left(d\left(\sum x_i dy_i + dz\right)\right)^n = \left(\sum x_i dy_i + dz\right) \wedge \left(\sum dx_i \wedge dy_i\right)^n$$

$$= \left(\sum x_i dy_i + dz\right) \wedge \underbrace{\left(\sum dx_i \wedge dy_i\right) \wedge \left(\sum dx_i \wedge dy_i\right)} \wedge \left(\sum dx_i \wedge dy_i\right)^{n-2}$$

$$= \left(\sum x_i dy_i + dz\right) \wedge \sum_{i \neq j} dx_i \wedge dy_i \wedge dx_j \wedge dy_j \wedge \left(\sum dx_i \wedge dy_i\right)^{n-2}$$

$$= \left(\sum x_i dy_i + dz\right) \wedge n! \left(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \dots \wedge dx_n \wedge dy_n\right)$$

$$= dz \wedge n! \left(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \dots \wedge dx_n \wedge dy_n\right)$$

$$= n! \, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \dots \wedge dx_n \wedge dy_n \wedge dz \neq 0$$

Compute $R_{\alpha_{st}} = \left(\sum A_i \partial x_i + B_i \partial y_i\right) + C \partial_z$

$$d\alpha_{st}(R_{st}, \cdot) = \sum A_i \, dy_i - B_i \, dx_i = 0$$

$$\alpha_{st}(R_{st}) = C + \sum x_i B_i = 1$$

One solution: $R_{\alpha_{st}} = \partial_z$

(2) $\xi_{sym} = \ker\Big(\underbrace{\sum (x_i dy_i - y_i dx_i) + dz}_{\alpha}\Big)$

Homework:
- This is a contact structure
- $R_\alpha = \partial_z$
- for $n=1$: draw this
- read Milnor: Topology from a differential viewpoint
  Jeffrey Lee

Recall: $(M^{2n+1}, \ker(\alpha) = \xi^{2n} \subset TM)$ Contact manifold (19.10.23)

$\Leftrightarrow$ $\alpha \wedge (d\alpha)^n$ is a volume form (i.e. $\neq 0$)

$\mathbb{R}^{2n+1}$ $(x_1, y_1, \dots, x_n, y_n, z)$

(1) $\alpha_{st} = \left(\sum_{i=1}^{n} x_i dy_i\right) + dz \Rightarrow \alpha_{st} \wedge (d\alpha_{st})^n \neq 0$

(2) $\alpha_{sym} = \left(\sum_{i=1}^{n} x_i dy_i - y_i dx_i\right) + dz \Rightarrow \alpha_{sym} \wedge (d\alpha_{sym})^n \neq 0$

see homework
week 1

(3) $\mathbb{R}^3$ with cylindrical coordinates (θ, r, z)

$\xi_{OT} = \ker(\alpha_{OT}) = \ker(\cos(r)dz + r\sin(r)\,d\theta)$

$$d\alpha_{OT} = -\sin r\, dr \wedge dz + \underbrace{\frac{\partial \cos(r)}{\partial \theta}}_{=0} d\theta \wedge dz + (\sin(r) + r\cos(r))\, dr \wedge d\theta$$

$$\alpha_{OT} \wedge d\alpha_{OT} = \left[(r\sin^2 r) - \sin r \cos r - r\cos^2 r\right] d\theta \wedge dr \wedge dz$$

$$= -\Big[\underbrace{1 + \frac{\sin(r)\cos(r)}{r}}_{\neq 0}\Big] \underbrace{r\, d\theta \wedge dr \wedge dz}_{\text{volume form}} \neq 0$$

Exercises: $\cdot$ $f: (r, \theta, z) \longmapsto (x, y, z)$ If you pull that back, you get the volume form

$\cdot$ Look at the volume form on a sphere S^2

When are two contact structures equivalent?

Definition: [Contactomorphism]

$\cdot$ $f: (M_1, \xi_1) \longrightarrow (M_2, \xi_2)$ is called Contactomorphism

$:\Leftrightarrow$ f is a diffeomorphism s.t. $Tf(\xi_1) = \xi_2$

(i.e. $f^*(\alpha_2) = g \cdot \alpha_1$ for some $g: M_1 \to \mathbb{R} \setminus \{0\}$)

Pullback: $f: M_1 \longrightarrow M$

α_2 a 1-form on M_2

$\alpha_2 : TM \longrightarrow \mathbb{R}$

$(f^* \alpha_2)_p (X_p) := (\alpha_2)_{f(p)} (T_p f(X_p))$

SHORT! $f^* \alpha_2(X) = \alpha_2(Tf(X))$

Example: $f : (\mathbb{R}^{2n+1}, \xi_{st}) \longrightarrow (\mathbb{R}^{2n+1}, \xi_{sym})$

$(x, y, z) \longmapsto \left(\frac{x+y}{2}, \frac{y-x}{2}, z + \frac{\langle x, y \rangle}{2}\right)$

is a contactomorphism.

proof:

$$f^* \alpha_{sym} = \left(\sum_{i=1}^n \frac{x_i+y_i}{2} d\left(\frac{y_i - x_i}{2}\right) - \left(\frac{y_i-x_i}{2}\right) d\left(\frac{x_i+y_i}{2}\right)\right) + d\left(z + \frac{\langle x,y\rangle}{2}\right)$$

$$= \sum_{i=1}^n \frac{x_i+y_i}{2} \frac{dy_i - dx_i}{2} - \left(\frac{y_i-x_i}{2}\right) \frac{dx_i + dy_i}{2} + dz$$

$$+ \sum_{i=1}^n \frac{x_i dy_i}{2} + \sum_{i=1}^n \frac{y_i dx_i}{2}$$

$$= dz + \sum_{i=1}^n \left(\frac{x_i dy_i}{2} - \frac{y_i dx_i}{2} + \frac{x_i dy_i}{2} + \frac{y_i dx_i}{2}\right)$$

$$= dz + \sum_{i=1}^n x_i dy_i$$

If $f^* \alpha_2 = \alpha_1$ then f is called strict

Thm 3 [Bennequin]: $(\mathbb{R}^3, \xi_{st}) \overset{\text{cont}}{\not\cong} (\mathbb{R}^3, \xi_{OT})$ proof maybe in section 4

$\rightarrow$ goal: distinguish contact manifolds

Example: $S^{2n-1} \subset \mathbb{C}^n$

Standard contact structure on the sphere:

$\xi_{st} := TS^{2n-1} \cap (i TS^{2n-1})$ HW sheet 1

Thm 4: $(S^{2n-1} \setminus \{pt\}, \xi_{st}) \overset{\text{cont}}{\cong} (\mathbb{R}^{2n-1}, \xi_{st})$

proof: HW, sheet 2

Let W^{2n} be a $(2n)$-manifold

Def: A <u>symplectic form</u> is a 2-form ω s.t.

$d\omega = 0$ & ω^n is a volume form

Ex: $\left(\mathbb{R}^{2n}, \omega_{st} := \sum_{i=1}^{n} dx_i \wedge dy_i\right)$

Def: a <u>Liouville vector field</u> Y on (W, ω) is a vector field s.t. $d(\omega(Y, \cdot)) =: d(\iota_Y \omega) = \omega$ $\quad \iota :=$ plug in

Ex: $Y = \frac{1}{2} r \partial_r = \frac{1}{2}\left(\sum x_i \partial_{x_i} + y_i \partial_{y_i}\right)$ is Liouville on $(\mathbb{R}^{2n}, \omega_{st})$

$\iota_Y \omega_{st} = \frac{1}{2}\left(\sum x_i\, dy_i - y_i\, dx_i\right) \Rightarrow d(\iota_Y \omega_{st}) = \omega_{st}$

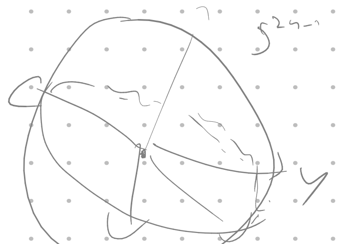
Lemma 5: Let Y be Liouville on (W^{2n}, ω)

$\Rightarrow$ $\alpha := \iota_Y \omega$ is a contact form on every hypersurface $M^{2n-1} \subset W$ transverse to Y.

Ex: $S^{2n-1} \subset (\mathbb{R}^{2n}, \omega_{st})$ is transverse to $Y = \frac{1}{2} r \partial_r$

$\Rightarrow$ $\alpha = \iota_Y \omega_{st}$ is a contact form

$= \sum x_i\, dy_i - y_i\, dx_i$



Homework: · check directly that α is a contact form on S^{2n-1}

· sheet 1: $\ker(\alpha) = \xi_{st}$

proof (Lemma 5):

$$\alpha \wedge (d\alpha)^{n} = \iota_Y \omega \wedge (d(\iota_Y \omega))^{n-1}$$

$$= \iota_Y \omega \wedge \omega^{n-1}$$

$$= \frac{1}{n} \iota_Y(\omega^n)$$

write down coordinates and compute it explicitly

$\omega^n \neq 0 \Rightarrow \alpha \wedge (d\alpha)^{n-1} \neq 0$ on Λ transverse to Y $\square$

(for me to understand $\alpha \wedge (d\alpha)^{n-1})(Y, \dots) = \frac{1}{n}\omega^n(Y, Y) = 0$)

Given a manifold B^n the space of contact elements:

$$\left\{ (b, V_b) \mid b \in B \ \&\ V_b^{n-1} \subset T_b B \text{ oriented \& co-oriented hyperplane} \right\}$$

Lemma 6: Space of contact elements $\cong S^*B$ (unit cotangent bundle)

proof: $(b, V_b) \overset{1:1}{\longmapsto} U_b^{V_b} : T_b B \longrightarrow \mathbb{R}$ linearly with $\ker(U_b^{V_b}) = V_b$

V_b oriented & cooriented

$U_b^{V_b}$ is unique up to scaling $\square$