

Homotopically Standard Tight Non-fillable Contact Structures on the Sphere

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results by Bowden, Gironella, Moreno and Zhou

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Background

Contact topology

Contact manifold:

- odd-dimensional smooth manifold
- codim-1 hyperplane distribution $\ker \alpha$
- contact condition:

$$\alpha \wedge d\alpha \neq 0$$

Contact topology

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Eliashberg, Borman–Eliashberg–Murphy:

Dichotomy: Rigidity vs. Flexibility.

- **tight** (*rigid/geometric*);
- **overtwisted** (*flexible/topological*).

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Theorem (Eliashberg–Gromov)

Fillable contact manifolds are tight.

Converse is false (Etnyre–Honda, Massot–Niederkrueger–Wendl).

Existence and classification

Topological obstruction: *almost* contact structure, i.e. reduction of structure group to $U(n) \times \mathbb{1}$.

Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))

Almost contact manifolds are contact, where the contact structure is overtwisted.

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Tight manifolds

How can we understand **tight** contact manifolds?

Contact topology: fillability

Hierarchy of fillability:

$$\begin{array}{ccccccc} \{Stein\} & \overset{\textcircled{1}}{=} & \{Weinstein\} & \overset{\textcircled{2}}{\subsetneq} & \{Liouville\} & \overset{\textcircled{3}}{\subsetneq} & \{strong\} \\ & & & & & & \\ & & \overset{\textcircled{4}}{\subsetneq} & \{weak\} & \overset{\textcircled{5}}{\subsetneq} & \{tight\} & \end{array}$$

- $dim = 3$: $\textcircled{1}$ Cieliebak–Eliashberg, $\textcircled{2}$ Bowden, $\textcircled{3}$ Ghiggini, $\textcircled{4}$ Eliashberg, $\textcircled{5}$ Etnyre–Honda.
- $dim \geq 5$: $\textcircled{1}$ Cieliebak–Eliashberg,
 $\textcircled{2}$ Bowden–Crowley–Stipsicz, $\textcircled{3}$ Zhou,
 $\textcircled{4}$ Bowden–Gironella–Moreno, $\textcircled{5}$ Massot–Niederkrüger–Wendl.

Contact structures on spheres

First step in classification: contact structures on spheres.

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Theorem (Eliashberg, '91)

On S^3 , it is the unique tight contact structure.

In particular, no tight and non-fillable contact structures on S^3 .

Exotic spheres

Theorem (Bowden–Gironella–Moreno–Zhou '22-'24)

For every $n \geq 2$, the sphere \mathbb{S}^{2n+1} admits a tight, non-fillable contact structure that is homotopically standard.

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In $\dim \geq 7$, if M admits a tight structure, it also admits a tight and non strongly-fillable structure, in the same almost contact class.

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In $\dim = 5$, the same holds, if the first Chern class vanishes.

General remarks

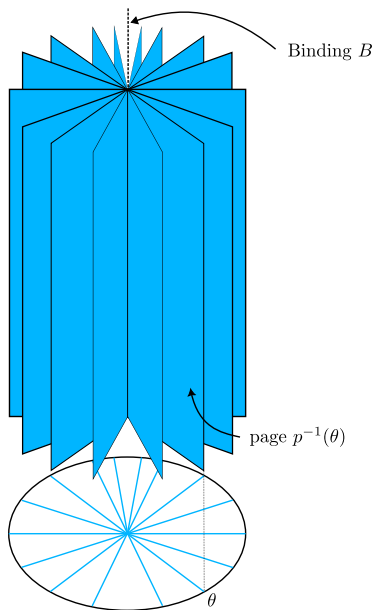
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- Suggests that higher-dimensional contact phenomena should occur independently of underlying smooth topology.

Tight and non-fillable spheres

Open books



Giroux correspondence

Giroux: Contact structures are *supported* by open books.

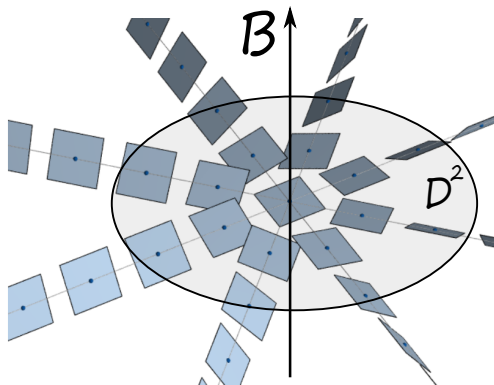


Figure: Supported contact structure.

Bourgeois contact structures

Theorem (Bourgeois '02)

Open book supporting $(M, \xi) \rightsquigarrow$ contact structure on $M \times \mathbb{T}^2$.

These are \mathbb{T}^2 -equivariant.

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Claim: $(\mathbb{S}^{2n+1}, \xi_{ex})$ is homotopically standard, tight and non-fillable.

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All almost complex structures on the ball are homotopic

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- 1 Milnor open book \Rightarrow algebraically tight Bourgeois manifold.
 - Algebraic tightness is nonvanishing of a certain contact homology algebra.
- 2 Algebraic tightness is preserved under surgeries.
- 3 Algebraically tight \implies tight.

Milnor open book $\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex})$ is *tight*.

Non-fillability

Non-fillability of $(\mathbb{S}^{2n+1}, \xi_{ex})$ can be proven via:

- 1 Homological obstruction and cobordisms as in [Bowden–Gironella–Moreno], building on [Massot–Niederkrüger–Wendl].
- 2 Symplectic cohomology computations as in [Zhou].

Convex Decomposition and Capping Cobordism

Observations:

- Bourgeois manifolds have convex decomposition

$$M \times \mathbb{T}^2 = (M \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_{\phi} \overline{V}_- \times \mathbb{S}^1,$$

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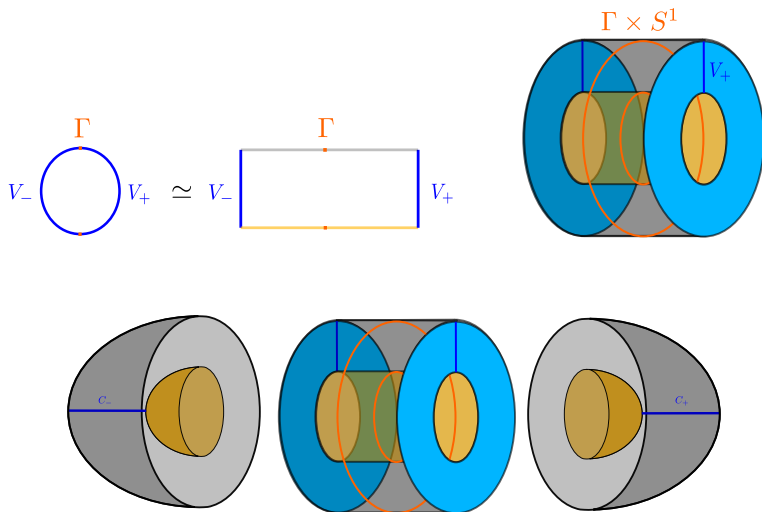
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- Capping cobordism from $M \times T^2$ to $\Gamma \times \mathbb{S}^2$, via handles $V_{\pm} \times D^2$ with co-core $C_{\pm} \simeq V_{\pm}$.

Capping Cobordism: Example



Homological obstructions

Theorem (Bowden–Gironella–Moreno)

$M \times T^2 = V \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V_-} \times \mathbb{S}^1$ with convex decomposition, $\Gamma = \partial V_\pm$ dividing set. If W is a symplectic filling of $M \times T^2$, then

$$H_*(\Gamma) \rightarrow H_*(V_\pm) \rightarrow H_*(W),$$

induced by inclusion. Then second map is injective on image of the first.

Namely, if a homology class in Γ survives in V_\pm , then it survives in the filling.

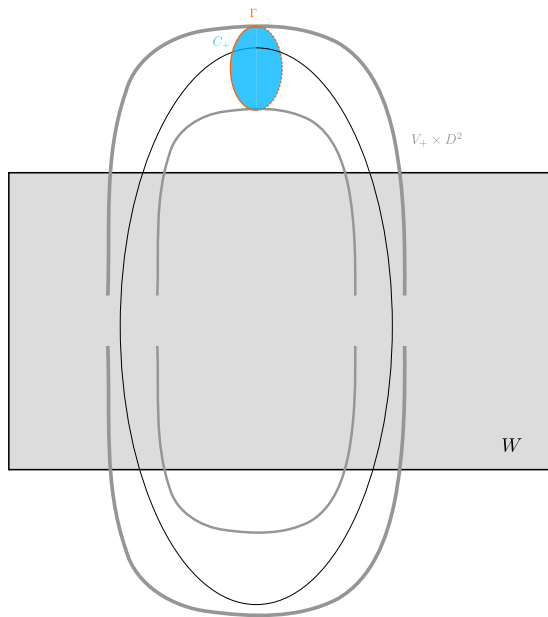
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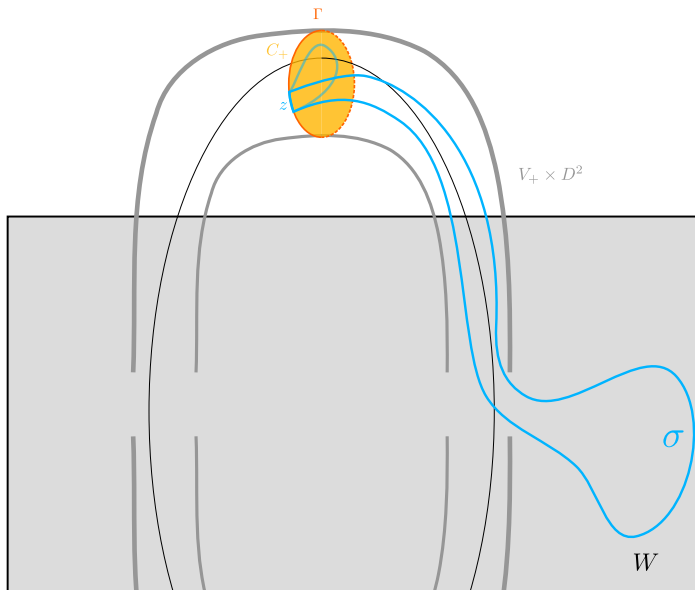
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- Spheres intersect C_{\pm} precisely once \rightsquigarrow intersection map $\mathcal{I}_{\pm} : \mathcal{M}_* \rightarrow C_{\pm}$.
- If $\sigma \subset W$ satisfies $\partial\sigma = z$ with z cycle in Γ , then z also bounds $b = \mathcal{I}_{\pm} ev^{-1}(\sigma) \subset V_{\pm}$. □

Homological obstructions

Fact:

- ① If $\dim \geq 7$, subcritical surgeries on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ can be pushed away from dividing set to V_+ .

$\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex})$ still has a dividing set Γ ,

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- 2 Homological obstruction theorem persists under surgery away from dividing set (capping cobordisms).

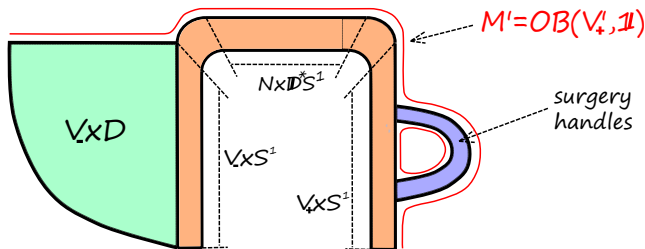


Figure: Capping cobordism.

End of the proof: Assume W filling of $(\mathbb{S}^{2n+1}, \xi_{ex})$.

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- \exists homology class $z \in \Gamma$ s.t. z survives in V_{\pm} .
- Homological obstruction theorem: $H_n(\Gamma) \rightarrow H_n(W) \neq 0$.
- However, this factors as

$$0 \neq H_n(\Gamma) \rightarrow H_n(\mathbb{S}^{2n+1}) = 0 \rightarrow H_n(W),$$

contradiction.

Thank you!