

# Homotopically Standard Tight Non-fillable Contact Structures on the Sphere

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results by Bowden, Gironella, Moreno and Zhou

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# Background

# Contact topology

## Contact manifold:

- odd-dimensional smooth manifold
- codim-1 hyperplane distribution  $\ker \alpha$
- contact condition:

$$\alpha \wedge d\alpha \neq 0$$

# Contact topology

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Eliashberg, Borman–Eliashberg–Murphy:

**Dichotomy:** Rigidity vs. Flexibility.

- **tight** (*rigid/geometric*);
- **overtwisted** (*flexible/topological*).

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## Theorem (Eliashberg–Gromov)

*Fillable contact manifolds are tight.*

Converse is false (Etnyre–Honda, Massot–Niederkrueger–Wendl).



# Existence and classification

*Topological* obstruction: *almost* contact structure, i.e. reduction of structure group to  $U(n) \times \mathbb{1}$ .

Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))

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## Tight manifolds

How can we understand **tight** contact manifolds?

# Contact topology: fillability

## Hierarchy of fillability:

$$\begin{array}{ccccccc} \{Stein\} & \overset{\textcircled{1}}{=} & \{Weinstein\} & \overset{\textcircled{2}}{\subsetneq} & \{Liouville\} & \overset{\textcircled{3}}{\subsetneq} & \{strong\} \\ & & & & & & \\ & & \overset{\textcircled{4}}{\subsetneq} & \{weak\} & \overset{\textcircled{5}}{\subsetneq} & \{tight\} & \end{array}$$

- $dim = 3$ :  $\textcircled{1}$  Cieliebak–Eliashberg,  $\textcircled{2}$  Bowden,  $\textcircled{3}$  Ghiggini,  $\textcircled{4}$  Eliashberg,  $\textcircled{5}$  Etnyre–Honda.
- $dim \geq 5$ :  $\textcircled{1}$  Cieliebak–Eliashberg,  
 $\textcircled{2}$  Bowden–Crowley–Stipsicz,  $\textcircled{3}$  Zhou,  
 $\textcircled{4}$  Bowden–Gironella–Moreno,  $\textcircled{5}$  Massot–Niederkrüger–Wendl.

# Contact structures on spheres

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## Standard contact structure

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## Theorem (Eliashberg, '91)

*On  $S^3$ , it is the unique tight contact structure.*

In particular, no tight and non-fillable contact structures on  $S^3$ .

# Exotic spheres

Theorem (Bowden–Gironella–Moreno–Zhou '22-'24)

*For every  $n \geq 2$ , the sphere  $\mathbb{S}^{2n+1}$  admits a tight, non-fillable contact structure that is homotopically standard.*

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*In  $\dim \geq 7$ , if  $M$  admits a tight structure, it also admits a tight and non strongly-fillable structure, in the same almost contact class.*



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*In  $\dim = 5$ , the same holds, if the first Chern class vanishes.*

# General remarks

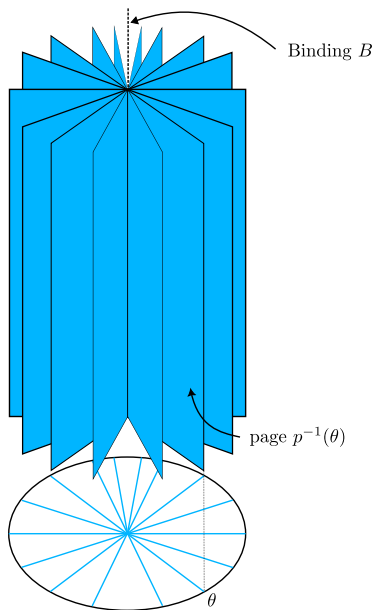
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- Suggests that higher-dimensional contact phenomena should occur independently of underlying smooth topology.

# **Tight and non-fillable spheres**

# Open books



# Giroux correspondence

**Giroux:** Contact structures are *supported* by open books.

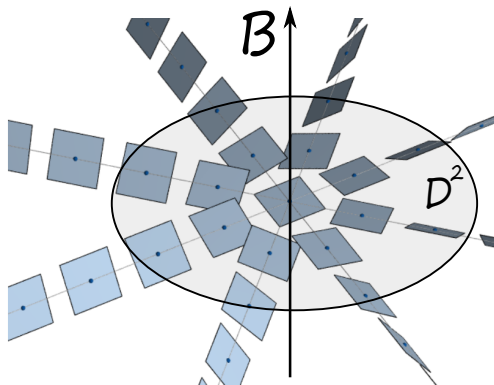


Figure: Supported contact structure.

# Bourgeois contact structures

## Theorem (Bourgeois '02)

*Open book supporting  $(M, \xi) \rightsquigarrow$  contact structure on  $M \times \mathbb{T}^2$ .*

These are  $\mathbb{T}^2$ -equivariant.

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**Claim:**  $(\mathbb{S}^{2n+1}, \xi_{ex})$  is homotopically standard, tight and non-fillable.

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All almost complex structures on the ball are homotopic

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- 2 Algebraic tightness is preserved under surgeries.
- 3 Algebraically tight  $\implies$  tight.

Milnor open book  $\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex})$  is *tight*.

# Non-fillability

**Non-fillability** of  $(\mathbb{S}^{2n+1}, \xi_{ex})$  can be proven via:

- 1 Homological obstruction and cobordisms as in [Bowden–Gironella–Moreno], building on [Massot–Niederkrüger–Wendl].
- 2 Symplectic cohomology computations as in [Zhou].

# Convex Decomposition and Capping Cobordism

## Observations:

- Bourgeois manifolds have convex decomposition

$$M \times \mathbb{T}^2 = (M \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_{\phi} \overline{V}_- \times \mathbb{S}^1,$$

with  $V_{\pm} = \Sigma \times D^*\mathbb{S}^1$ ,  $\Sigma$  = page of the open book,  $\phi$  = monodromy.



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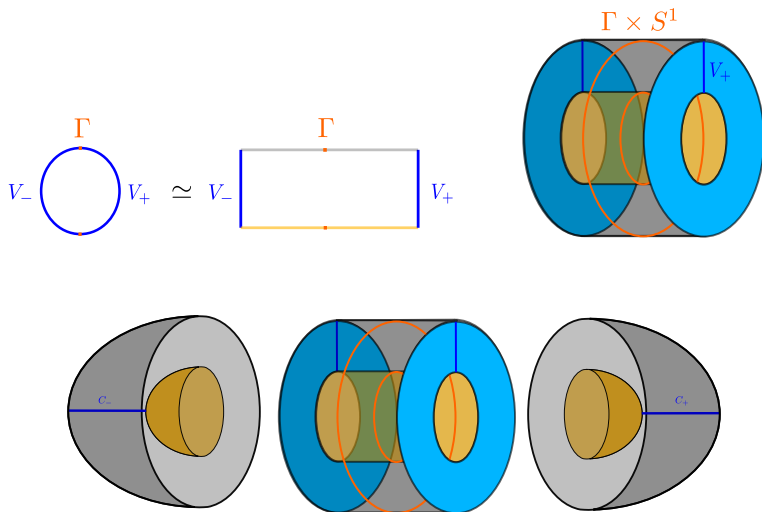
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- Capping cobordism from  $M \times T^2$  to  $\Gamma \times \mathbb{S}^2$ , via handles  $V_{\pm} \times D^2$  with co-core  $C_{\pm} \simeq V_{\pm}$ .

# Capping Cobordism: Example



# Homological obstructions

## Theorem (Bowden–Gironella–Moreno)

$M \times T^2 = V \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V_-} \times \mathbb{S}^1$  with convex decomposition,  $\Gamma = \partial V_\pm$  dividing set. If  $W$  is a symplectic filling of  $M \times T^2$ , then

$$H_*(\Gamma) \rightarrow H_*(V_\pm) \rightarrow H_*(W),$$

induced by inclusion. Then second map is injective on image of the first.

Namely, if a homology class in  $\Gamma$  survives in  $V_\pm$ , then it survives in the filling.

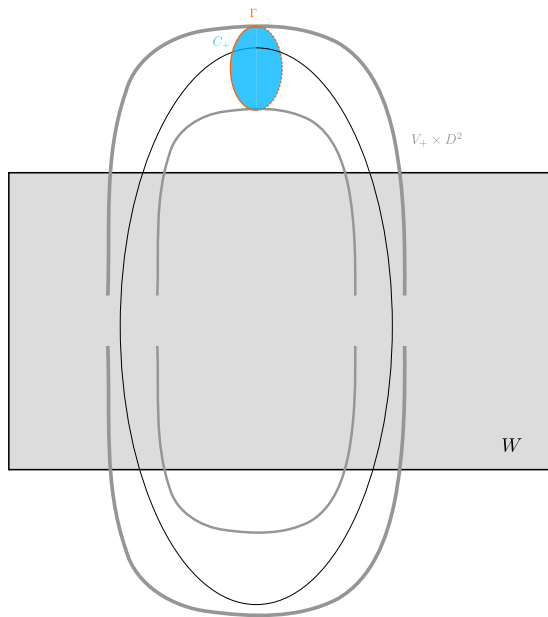
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- Second factor gives moduli space of spheres  $\mathcal{M}_*$  with evaluation map  $ev : \mathcal{M}_* \rightarrow W$ .

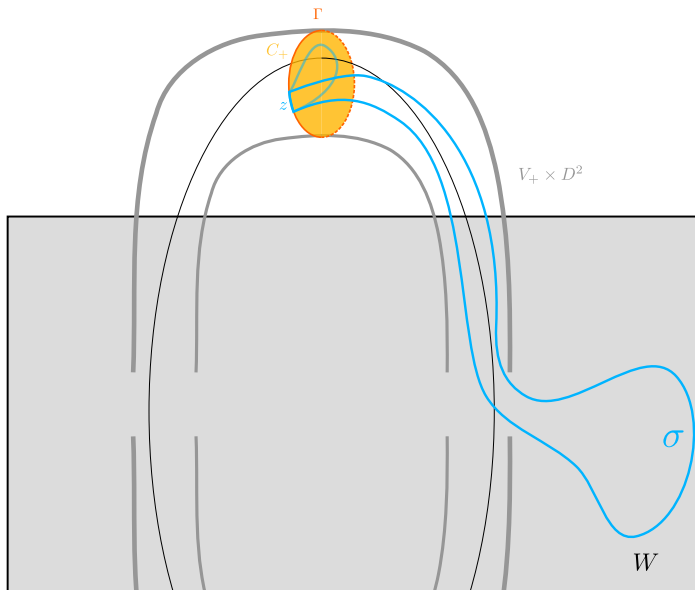
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- Spheres intersect  $C_{\pm}$  precisely once  $\rightsquigarrow$  intersection map  $\mathcal{I}_{\pm} : \mathcal{M}_* \rightarrow C_{\pm}$ .
- If  $\sigma \subset W$  satisfies  $\partial\sigma = z$  with  $z$  cycle in  $\Gamma$ , then  $z$  also bounds  $b = \mathcal{I}_{\pm} ev^{-1}(\sigma) \subset V_{\pm}$ . □

# Homological obstructions

## Fact:

- ① If  $\dim \geq 7$ , subcritical surgeries on  $\mathbb{S}^{2n-1} \times \mathbb{T}^2$  can be pushed away from dividing set to  $V_+$ .

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- 2 Homological obstruction theorem persists under surgery away from dividing set (capping cobordisms).

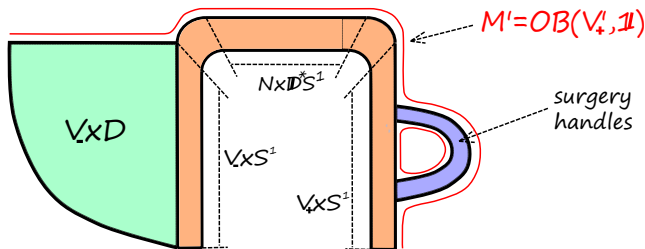


Figure: Capping cobordism.

**End of the proof:** Assume  $W$  filling of  $(\mathbb{S}^{2n+1}, \xi_{ex})$ .

- $\exists$  homology class  $z \subset \Gamma$  s.t.  $z$  survives in  $V_{\pm}$ .

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- However, this factors as

$$0 \neq H_n(\Gamma) \rightarrow H_n(\mathbb{S}^{2n+1}) = 0 \rightarrow H_n(W),$$

contradiction.

Thank you!