

## Preliminaries

### The Bourgeois construction

**Definition 1.** Let  $M$  be an oriented manifold with an open book decomposition  $(B, p)$  with oriented binding  $B$ . The pages are oriented by the requirement that the induced orientation on the boundary of (the closure) of each page coincides with the orientation of  $B$ .

*Question: I don't fully understand Geiges remark there.*

A contact structure  $\xi = \ker \alpha$  on  $M$  is said to be **supported** by the open book decomposition  $(B, p)$  of  $M$  if

- (i) the contact form  $\alpha$  induces the positive orientation of  $M$  ( $\alpha \wedge (d\alpha)^n > 0$ ).
- (ii) the 2-form  $d\alpha$  induces a symplectic form on each page, defining its positive orientation
- (iii) the 1-form  $\alpha$  induces a positive contact form on  $B$ , i.e.  $\alpha \wedge (d\alpha)^{(n-2)} > 0$  on  $B$ . *Question: Is that i.e.-part correct?*

**Theorem 1.** Let  $(M, \xi = \ker \alpha)$  be a closed contact manifold of dimension  $2n - 1, n \geq 2$ . One can find an open book decomposition  $(B, p)$  of  $M$  supporting  $\xi$ . According to Bourgeois, ([Bou02]) there is a contact structure  $\tilde{\xi}$  on  $M \times T^2$  (where  $\tilde{\xi}$  massively depends on the choice of open book).

*Proof.* We follow the proof of [Gei08, Thm 7.3.6]. Wlog let  $M$  be connected. The existence of an open book decomposition for  $M$  is the theorem of Giroux-Mohsen as in [Gei08, Thm 7.3.5]. By definition of an open book, there exists a tubular neighborhood  $B \times D^2$  with polar coordinates  $(r, \phi)$  on the  $D^2$ -part of the binding  $B$  s.t.  $p : M \setminus B \rightarrow S^1$  is given by  $\phi$  in that neighborhood. Now, we want to define smooth functions  $x_1, x_2$  on  $M$  that coincide with the cartesian coordinate functions on  $D^2$  close to the binding  $B$ . In order to do that, choose a smooth function  $\rho(r)$  on  $B \times D^2$ , s.t.

- $\rho = r$  near the binding  $B$ ,
- $\rho'(r) \geq 0$
- $\rho \equiv 1$  near  $B \times \partial D^2$ .

We extend this function to a smooth function  $\rho : M \rightarrow [0, 1]$  by setting  $\rho \equiv 1$  outside  $B \times D^2$ . Now,  $x_1 := \rho \cos \phi$  and  $x_2 := \rho \sin \phi$  are the desired smooth functions on  $M$  that coincide with the Cartesian coordinate functions on the  $D^2$ -factor near  $B$ . We compute

$$\begin{aligned} x_1 dx_2 - x_2 dx_1 &= \rho^2 \cos^2 \phi d\phi + \rho \cos \phi \sin \phi d\rho + \rho^2 \sin^2 \phi d\phi - \rho \cos \phi \sin \phi d\rho \\ &= \rho^2 (\cos^2 \phi + \sin^2 \phi) d\phi \\ &= \rho^2 d\phi \end{aligned}$$

and, analogously,

$$dx_1 \wedge dx_2 = \rho d\rho \wedge d\phi.$$

On  $M \times T^2$ , choose coordinates  $(\theta_1, \theta_2)$  on the torus part of the manifold. Now we have all ingredients together to construct our contact form. Let

$$\tilde{\alpha} := x_1 d\theta_1 - x_2 d\theta_2 + \alpha.$$

This is a well-defined 1-form on  $M \times T^2$  ( $\alpha$  is extended to  $M \times T^2$  in the obvious way) and we can compute the derivative

$$d\tilde{\alpha} = dx_1 \wedge d\theta_1 - dx_2 \wedge d\theta_2 + d\alpha,$$

hence

$$\begin{aligned} (d\tilde{\alpha})^n &= (n-1)(d\alpha)^{n-1} \wedge (dx_1 \wedge d\theta_1 - dx_2 \wedge d\theta_2) \\ &\quad - n(n-1)(d\alpha)^{n-2} \wedge dx_1 \wedge d\theta_1 \wedge dx_2 \wedge d\theta_2. \end{aligned}$$

In order to verify the contact condition, we compute

$$\begin{aligned} \tilde{\alpha} \wedge (d\tilde{\alpha})^n &= (x_1 d\theta_1 - x_2 d\theta_2 + \alpha) \wedge (n-1)(d\alpha)^{n-1} \wedge (dx_1 \wedge d\theta_1 - dx_2 \wedge d\theta_2) \\ &\quad - (x_1 d\theta_1 - x_2 d\theta_2 + \alpha) \wedge n(n-1)(d\alpha)^{n-2} \wedge dx_1 \wedge d\theta_1 \wedge dx_2 \wedge d\theta_2 \\ &= (n-1)(d\alpha)^{n-1} \wedge (x_1 dx_2 - x_2 dx_1) \wedge d\theta_1 \wedge d\theta_2 \\ &\quad + \underbrace{\alpha \wedge (n-1)(d\alpha)^{n-1} \wedge dx_1 \wedge d\theta_1}_{2n\text{-form on } M} - \underbrace{\alpha \wedge (n-1)(d\alpha)^{n-1} \wedge dx_2 \wedge d\theta_2}_{2n\text{-form on } M} \\ &\quad + n(n-1)\alpha \wedge (d\alpha)^{n-2} \wedge dx_1 \wedge d\theta_1 \wedge dx_2 \wedge d\theta_2 \end{aligned}$$

$M$  has dimension  $2n-1$ , i.e. the middle term is 0

$$\begin{aligned} &= (n-1)(d\alpha)^{n-1} \wedge \rho^2 d\phi \wedge d\theta_1 \wedge d\theta_2 \\ &\quad + n(n-1)\alpha \wedge (d\alpha)^{n-2} \wedge \rho d\rho \wedge d\phi \wedge d\theta_1 \wedge d\theta_2 \end{aligned}$$

By condition (ii) of definition 1,  $(d\alpha)^{n-1}$  must be a positive volume form on the pages. As explained in that definition, the orientation on  $M$  is given by  $\partial_\phi$  and the orientation of the page. In particular,  $(d\alpha)^{n-1} \wedge \rho d\phi$  is a positive volume form on  $M$ . Multiplied with a second  $\rho$ -factor, it vanishes along  $B$ . As  $\theta_1 \wedge \theta_2$  is a positive volume form on  $T^2$ , the first term is non-negative everywhere and positive away from

$$\underbrace{B \times 0}_{\subset B \times D^2 \subset M} \times T^2.$$

Let  $\mathfrak{b}$  be a basis of the binding  $B$  that is positively ordered. Then,  $-\partial_r, \mathfrak{b}$  and (because the binding is odd-dimensional)  $\mathfrak{b}, \partial_r$  are positive bases of the page. Clearly, then,

$$\mathfrak{a} := \mathfrak{b}, \partial_r, \partial_\phi, \partial_{\theta_1}, \partial_{\theta_2}$$

is an ordered basis of  $M \times T^2$ . Using  $\rho'(r) \geq 0$  everywhere, we deduce that  $d\rho(\partial_r)$  is non-negative. Hence, plugging  $\mathfrak{a}$  into the second term, we conclude

$$\begin{aligned} & (n(n-1)\alpha \wedge (d\alpha)^{n-2} \wedge \rho d\rho \wedge d\phi \wedge d\theta_1 \wedge d\theta_2)(\mathfrak{a}) \\ &= n(n-1)\rho \cdot (\alpha \wedge (d\alpha)^{n-2})(\mathfrak{b}) \cdot d\rho(\partial_r) \cdot d\phi(\partial_\phi) \cdot d\theta_1(\partial_{\theta_1}) \cdot d\theta_2(\partial_{\theta_2}) \\ &\geq 0. \end{aligned}$$

By condition (iii) of definition 1,  $\alpha \wedge (d\alpha)^{n-2}$  is positive on  $B$ . Therefore, the second term is positive on  $B \times 0 \times T^2$  (hence also on a neighborhood) and non-negative everywhere else. In total, we have checked the contact condition and  $\tilde{\alpha}$  is indeed a contact form on  $M \times T^2$ .  $\square$

## References

- [Bou02] Frédéric Bourgeois. Odd dimensional tori are contact manifolds. *International Mathematics Research Notices*, 2002(30):1571–1574, January 2002.
- [Gei08] Hansjörg Geiges. *An Introduction to Contact Topology*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, March 2008.