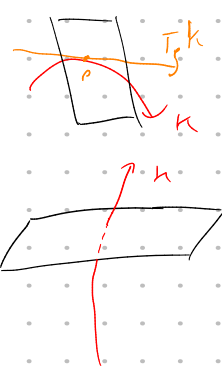


### 3. knots in contact 3-manifolds

An embedding  $K: S^1 \hookrightarrow (M^3, \eta)$  is called

\* Legendrian knot :  $(\subset) \quad T_k \subset \eta$

\* Transverse knot :  $(\subset) \quad T_k \cap \eta = \emptyset$



$h_0$  is isotopic to  $h_1 = \epsilon$

$\exists h_t, t \in I, h_t$  is Legendrian (transverse)  $\forall t$ .

Notation  $k \in (M, \eta)$  for the isotopy class

Example: (1)  $S^1 \hookrightarrow \mathbb{R}^3 / 2\pi \mathbb{Z} \ni t \mapsto (\cos(t), \sin(t), t) \in (\mathbb{R}^3, \eta_{std} = \ker(x dy - y dx))$   
 is a Legendrian unknot.

it's Legendrian: take the formula, plug it into  $\alpha$

if it's 0  $\Rightarrow$  it's Legendrian

(1)  $S^1 \ni \theta \mapsto (\theta, 0, 0) \in (S^1 \times \mathbb{R}^2, \eta = \ker(\cos \theta dx - \sin \theta dy))$  is Legendrian.

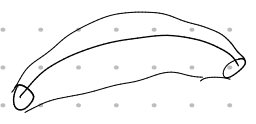
(2)  $S^1 \ni \theta \mapsto (\theta, 0, 0) \in (S^1 \times \mathbb{R}^2, \eta = \ker(d\theta + r^2 dy))$  is transverse.

### 3.1) Neighborhood & isotopy extension theorems

Thm 1: (1) Let  $k \in (M, \eta)$  be Legendrian.  $\Rightarrow \exists$  tubular NBHD  $V_k$  of  $k$  in  $M$  s.t.

$$\forall r > 0 \quad \exists \eta \in \mathcal{D}(\{0\}) : (V_k, \eta) \stackrel{\text{cont}}{\cong} (S^1 \times D_r^2, \eta_r)$$

$$k \mapsto S^1 \times \{0\}$$



(2) Let  $k \in (M, \eta)$  be transverse.  $\Rightarrow \exists$  tubular NBHD  $V_k$  of  $k$  in  $M$

$$\& \exists \eta^{cont} \text{ s.t. } (V_k, \eta) \stackrel{\text{cont}}{\cong} (S^1 \times D_r^2, \ker(d\theta + r^2 dy))$$

$$k \mapsto S^1 \times \{0\}$$

Proof:  $\# \checkmark$   $\Rightarrow$  Use Moser trick as in Darboux theorem

(2)

Thm 2: Let  $k_t: S^1 \hookrightarrow (M^3, \eta)$  be an isotopy of Legendrian (transverse) knots.

$\Rightarrow \exists$  isotopy of contactomorphisms

$$\psi_t: (M, \eta) \xrightarrow{\cong \text{ (cont) }} (M, \eta) \text{ s.t.}$$

$$\psi_0 = id$$

$$\& \psi_t \circ k_0 = k_t$$

Proof: Construct  $\Psi_t$  as flow of a contact vectorfield from a Hamilton function.  
(4-)

## 9.2 The front projection

Let  $K \subset (S^3, \gamma_{\text{std}})$  be a knot

Thm 2.4

$\Rightarrow$  We can see  $K \subset (\mathbb{R}^3, \gamma_{\text{std}})$

$(x(y, z) \mapsto (y, z))$  front projection

$S^1 \ni t \mapsto (x(t), y(t), z(t)) \in (\mathbb{R}^3, \gamma_{\text{std}} = \ker(x dy + y dz))$

is Legendrian  $(\Leftrightarrow) 0 = \alpha(\dot{\gamma}(t)) = x(t) \cdot y'(t) + y(t) \cdot z'(t) \quad \forall t$

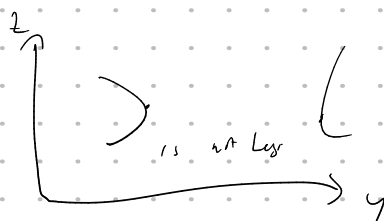
If  $K$  is generic (i.e.  $y'(t) \neq 0$  only for finitely many  $t$ )

$$\Rightarrow x(t) = -\frac{z'(t)}{y'(t)} = -\frac{dz}{dy} \quad (\text{if } y'(t) \neq 0 \Rightarrow z'(t) = 0)$$

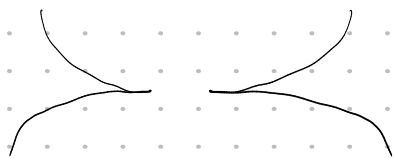
i.e. we can recover  $K$  from its front projection on  $\mathbb{R}^2$

Certain configurations do not appear as front projections of Legendrian knots.

\* if  $y' \geq 0 \Rightarrow z' \geq 0 \Rightarrow$  there can't be vertical tangencies



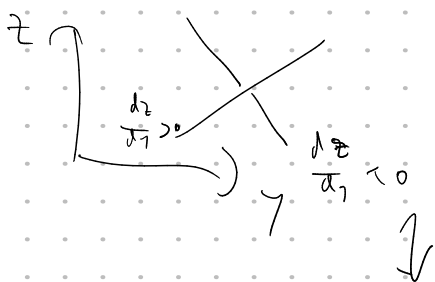
Important: semicircular cusps?



$$(x(t), y(t), z(t)) = (t, t^2, -\frac{2}{3}t^3)$$

(every Legendrian knot has at least 2 of them)

\* no crossing as follows:



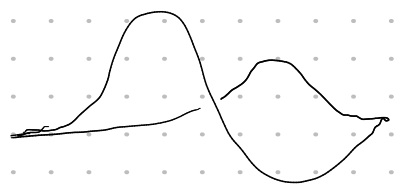
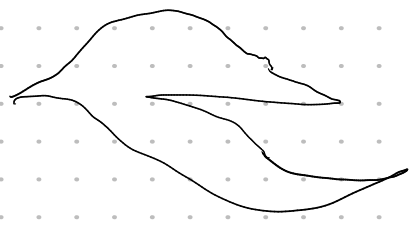
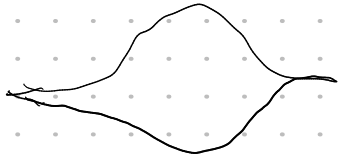
Legendrian:



$\Rightarrow$  Any such front projection describes a unique Legendrian knot.

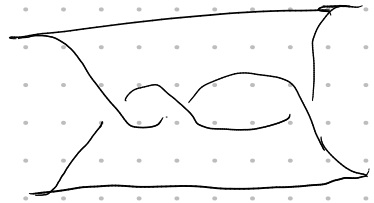
Legendre's: We have Legendre knots in the front projection

Examples:



is oblique to one of those on the left

helix



helix

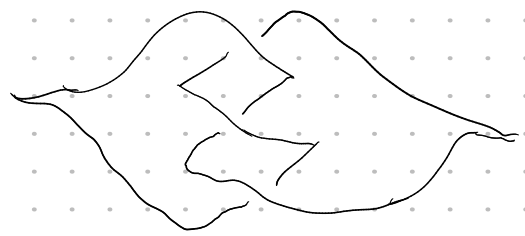


Figure 8

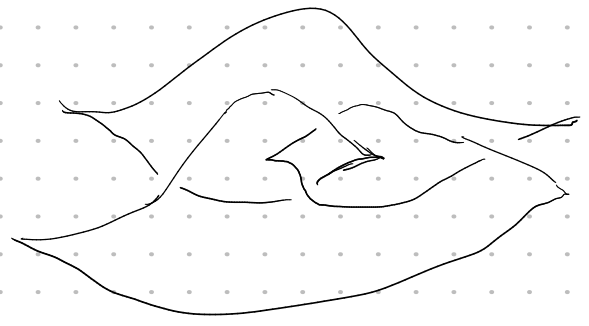
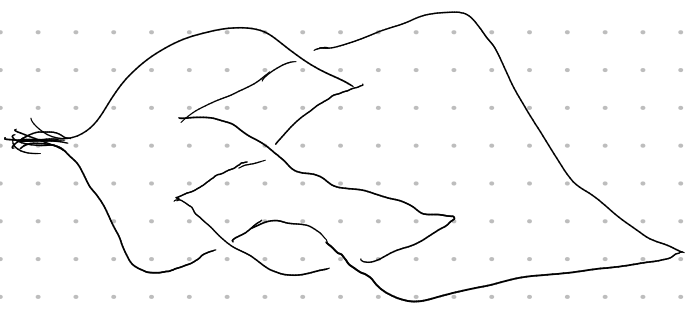
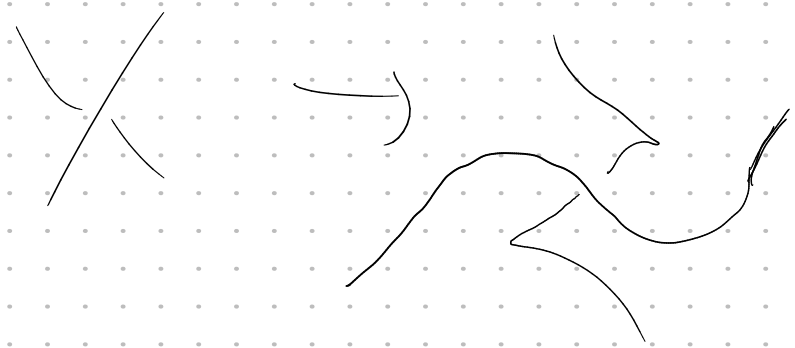
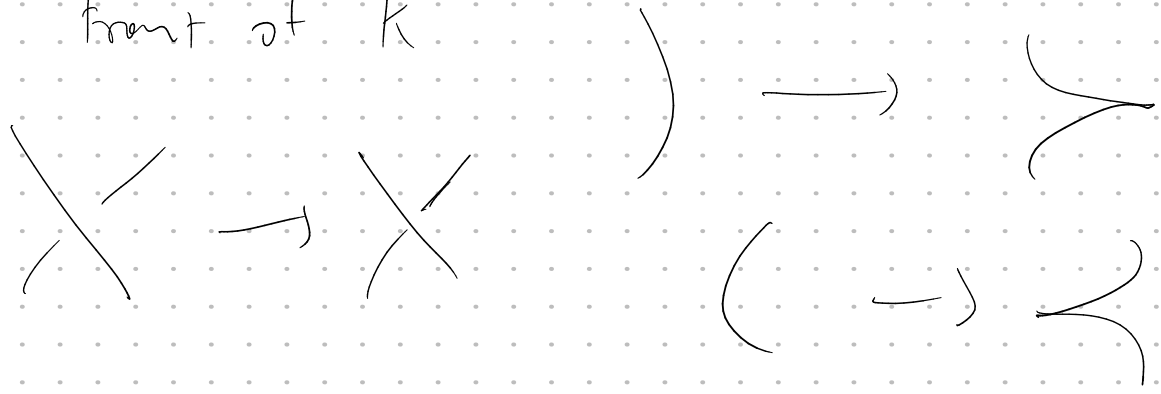


Figure 8



Corollary 3: For every smooth knot  $K \in C(n^3, \epsilon_0)$   
 $\exists$  isotopic knot  $K' \in C(n^3, \epsilon_0)$  Leg.

Proof: front of  $K$



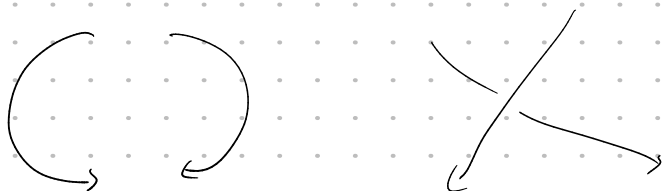
If  $h$  is positively transverse, i.e.  $x_{y'} + z' > 0$

Then: If  $y' = 0 \Rightarrow z' > 0$

$$\text{If } y' > 0 \Rightarrow x > -\frac{z'}{y'} = -\frac{dz}{dy}$$

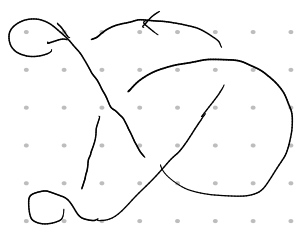
$$\text{If } y' < 0 \Rightarrow x < -\frac{dz}{dy}$$

$\Rightarrow$  The following configurations are excluded



All other configurations lift to transverse curves in  $(\mathbb{R}^3, \zeta_{st})$  unique up to

shifts in  $x$ -direction



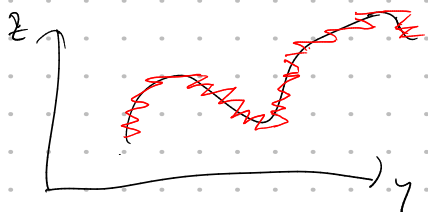
trivial

thm 4: Let  $h: S^1 \hookrightarrow (\mathbb{R}^3, \zeta)$  a smooth knot

$\Rightarrow h$  can be  $C^0$ -close approximated by a Legendrian (transverse) knot smoothly isotopic to  $h$ .

proof: Legendrian case

case 1: Let  $\gamma: (a, b) \hookrightarrow (\mathbb{R}^3, \zeta_{st})$  be an arc



Approximate the front of  $\gamma$  by a Legendrian front projection s.t.  $-\frac{dz}{dy}$  is close to the  $x$ -component of  $\gamma$ .

case 2: If  $\gamma: (a, b) \hookrightarrow (\mathbb{R}^3, \zeta_{st})$  is Legendrian near  $a$  &  $b$

then we can choose the approximation near  $a$  &  $b$  to agree with  $\gamma$ .

case 3:  $h: S^1 \hookrightarrow (\mathbb{R}^3, \zeta)$  a knot.  $S^1$  compact  $\xrightarrow{\text{Lebesgue}} \exists$  decomposition of  $S^1$  into intervals  $I_i$  s.t.  $h(I_i) \subset$  Darboux ball then use case 2.

## Transverse case

Let  $L : S^1 \longrightarrow (\mathbb{R}^3, g)$  be a Legendrian approximation of  $k$ .

Then  $\omega \log L(t) = (\theta = t, x=0, y=0) \in (S^1 \times \mathbb{R}^2, \alpha = (\cos \theta dx - \sin \theta) dy$

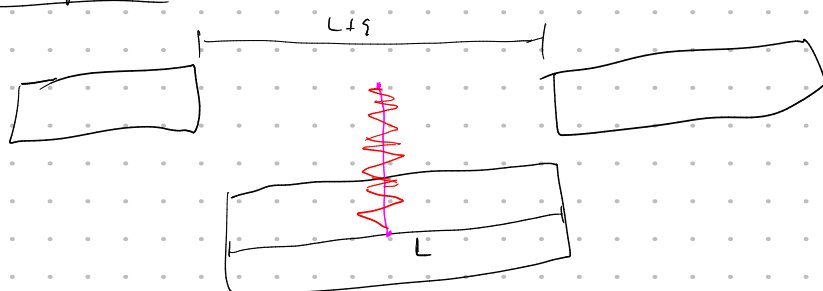
$$L_{\pm} := (\theta = t, x = \pm \epsilon \sin(t), y = \pm \epsilon \cos(t)) \quad \text{for } \epsilon > 0 \text{ small}$$

$$\Rightarrow L'_{\pm} := (1, \pm \epsilon (\cos(t), -\sin(t)))$$

$$\alpha(TL_{\pm}) = \pm \epsilon (\cos^2(t) + \sin^2(t)) = \pm \epsilon$$

$L_{\pm}$  are called transverse push-offs of  $L$

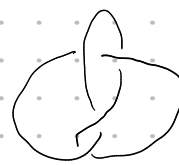
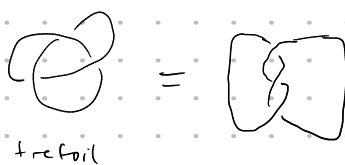
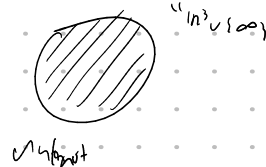
real world application:



## 3.4 Seifert surfaces & the Alexander polynomial

Let  $k \in S^3$  A smooth knot

Ex:



$F_2 - 8$

Lemma 5:  $\forall k \subset S^3 \exists$  Seifert surface, i.e.  $F^2 \xrightarrow{\text{smooth}} S^3$  compact, oriented, s.t.  $\partial F = k$

proof: (1) Let  $D$  be a diagram of  $k$

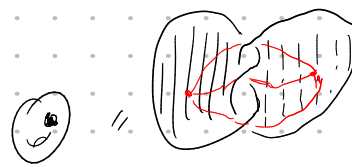
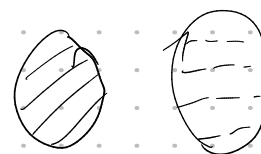
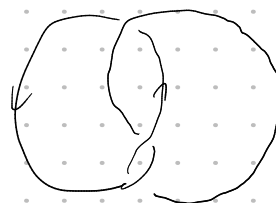
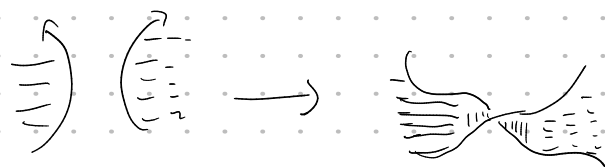
(2) orient  $D$

(3) resolve crossings following orientation



(4) get collection of circles in the plane

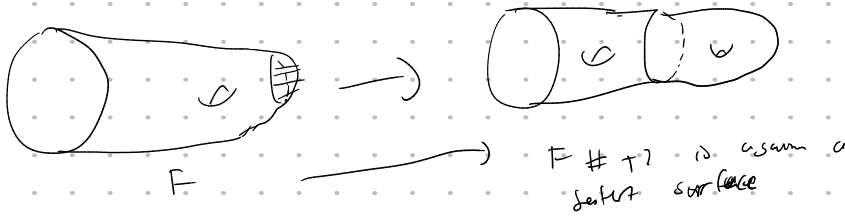
(5) glue in a twisted band for every crossing



$\square$

# Stabilization

$F$  a Seifert surface of  $K$



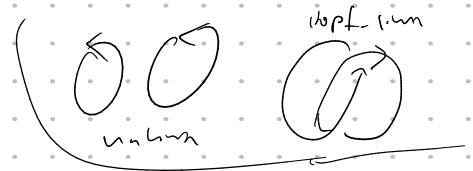
Thm 6 [Reidemeister - Singer] Any two Seifert surfaces of  $K$  have a common stabilization.

(Seifert)-genus:  $g(K) := \min \{g(F) \mid F \text{ a Seifert surface of } K\}$

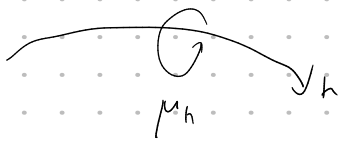
Ex:  $g(K) = 0 \iff K = 0$

$g(\bigcirc) \leq 1$

Links: Let  $K \subset M^3$  an oriented knot if  $K$  is nullhomologous  
i.e.  $[K] = 0 \in H_1(M)$

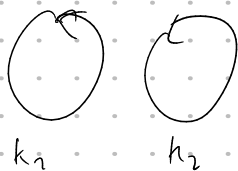


$\Rightarrow H_1(M \setminus \nu K) = \bigoplus_{\mu \in \mu(K)} H_1(M)$   
linker modulo of  $K$

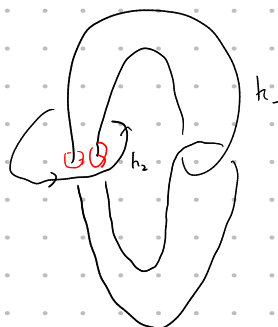


Let  $K_1, K_2$  be oriented knots in  $M$  s.t.  $K_1$  &  $K_2$  are nullh.  
 the linking number  $lk(K_1, K_2) \in \mathbb{Z}$  is defined  $[K_2] = lk(K_1, K_2) [K_1] \in H_1(M \setminus \nu K_1)$

$lk(K_1, -K_2) = lk(-K_1, K_2) = -lk(K_1, K_2)$



$[K_2] = -[K_1] \Rightarrow lk = -1$



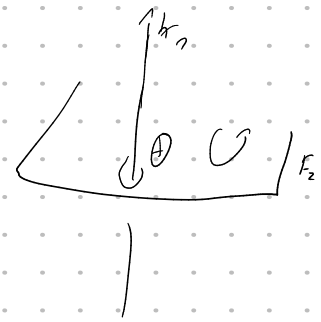
$[K_2] = -[K_1] + [K_1] = 0$

$lk = 0$

Lemma 7: (1)  $h \subset S^3$  is nullhomologous  $\Leftrightarrow h$  bounds a Seifert surface

(2)  $lk(h_1, h_2) = h_1 \cdot F_2$

— i.e.  $F_2$  a Seifert surface of  $h_2$



Proof: (1) "if" part of lemma

" $\Rightarrow$ " in  $S^3$  Seifert alg.

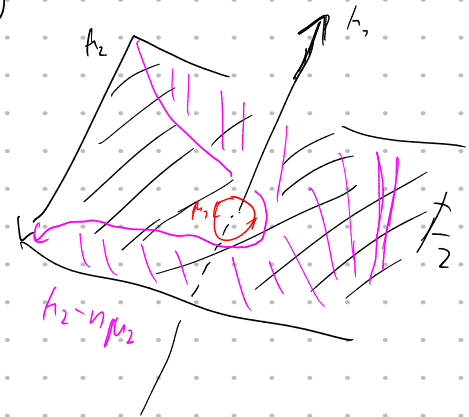
general case in discussion session

(2) Let  $h_1 \cdot F_2 = n > 0$  (algebraic intersection number)

$h_2 - n \mu_1$  bounds a surface but does not intersect  $h_1$

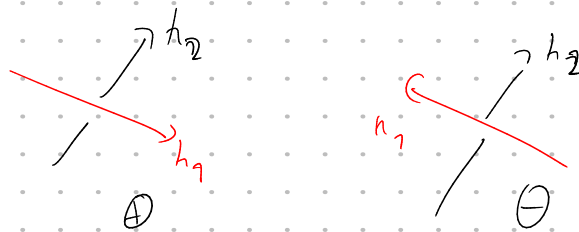
$lk(h_2 - n \mu_1, h_1) = 0 = lk(h_2, h_1)$

$lk(h_1, h_2) = n = h_1 \cdot F_2$



Lemma 8 Let  $h_1, h_2 \subset S^3$  be oriented knots

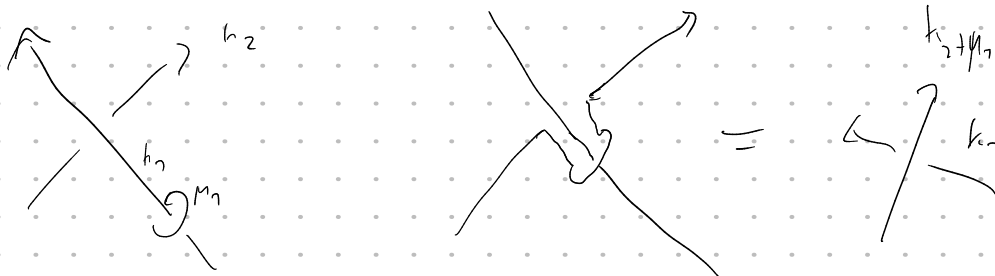
$\Rightarrow lk(h_1, h_2) = \#$  crossings of  $h_2$  under  $h_1$  with signs



Corollary 9:  $lk(h_1, h_2) = lk(h_2, h_1)$  proof: "look at the diagrams from behind the blackboard"

Proof (Lemma 8):  $lk(h_1, h_2 \pm \mu_1) = \pm 2$

$lk(h_1, h_2 \pm \mu_1) = lk(h_1, h_2) \pm 1$



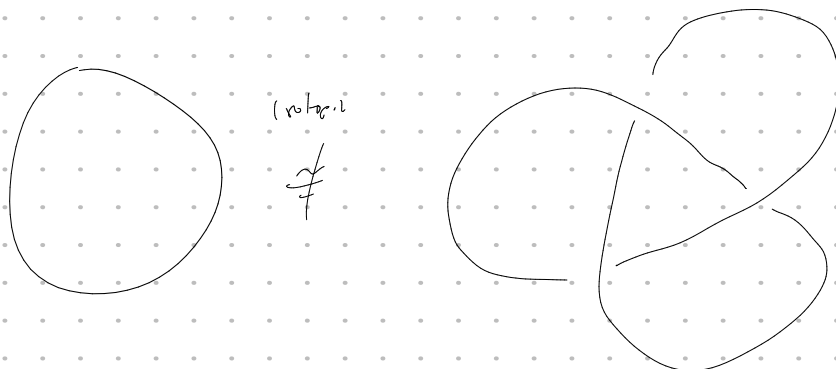
$n = \# \text{ crossings of } k_2 \text{ under } k_1$

$\Rightarrow k_2 - n \mu_1$  has no undercrossings with  $k_1$

$\Rightarrow \ell k(k_1, k_2 - n \mu_1) = 0 \Rightarrow \ell k(k_1, k_2) = n$



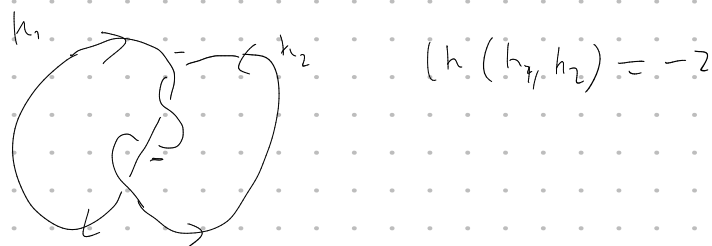
w.l.s :



$g(k) := \min \{g(F) \mid F \text{ is a Seifert surface of } k\}$

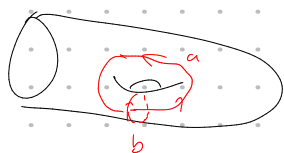
$[k_1] = \ell k(k_1, k_2) [\mu_{k_2}] \in H_1(S^3 \setminus k_2) \cong \mathbb{Z} \langle \mu_{k_2} \rangle$   
 // Lemma 8

# crossings of  $k_2$  over  $k_1$  counted with signs



Alexander polynomial:

seifert form Let  $F$  be a Seifert surface of an oriented knot  $k$ .



$$S: H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$$

$$(a, b) \longmapsto \ell k(a, b^+)$$

$\uparrow$   
 push-off of  $b$  in  
 positive normal dir. of  $F$

Alexander polynomial:  $\Delta_k(F) := \det(t^{-1/2} S - t^{1/2} S^T)$


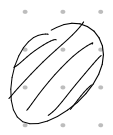


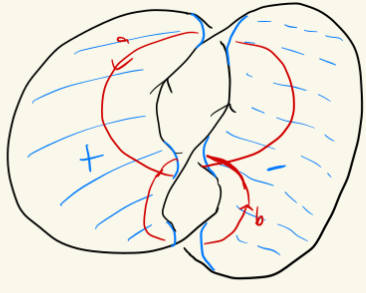
Corollary 10:  $\Delta_h(t)$  is a knot invariant.

proof sketch:  $\Delta_h(t)$  is independent of the Loren matrix:  $S = \rho S \rho^T$  with  $\det \rho = \pm 1$

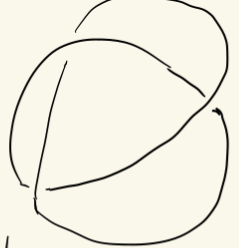
$\ast F'$  is a different defect surface  $\Rightarrow F'$  &  $F$  have a common stabilization  $\bar{F}$

$\ast \bar{F}$  is a stabilization of  $F \Rightarrow \bar{S} = \begin{pmatrix} S & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \det(t^{n/2} \bar{S} - t^{-n/2} \bar{S}^T) = \det(t^{n/2} S - t^{-n/2} S^T)$

Examples:   $F =$    $H_1(F) = 0 \Rightarrow S = 0 \Rightarrow \Delta_h(t) = 1$



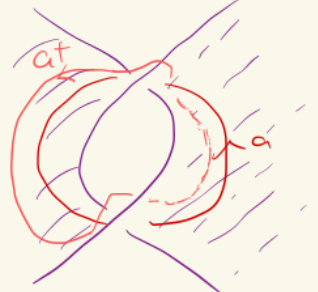
$\approx$   
 $\uparrow$   
Perd-  
mister  
moves



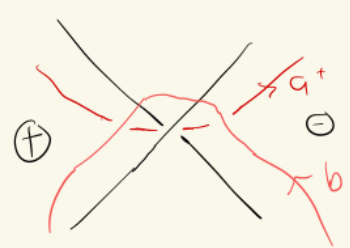
$F \equiv T^2 \setminus \dot{D}^2 = \mathbb{D}^2$

$\Rightarrow H_1(F) \cong \mathbb{Z}_{\langle a, b \rangle}^2$

$S = \begin{pmatrix} \text{ll}(a, a^+) & \text{ll}(b, a^+) \\ \text{ll}(a, b^+) & \text{ll}(b, b^+) \end{pmatrix}$



$\Rightarrow S = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$



$\Rightarrow \Delta_{\mathbb{Q}}(t) = \det \begin{pmatrix} -t^{n_1} & 0 \\ t^{n_1} & -t^{n_1} \end{pmatrix} - \begin{pmatrix} -t^{n_1} & t^{n_2} \\ 0 & -t^{n_2} \end{pmatrix}$

$= \det \begin{pmatrix} t^{n_2} - t^{n_1} & -t^{n_1} \\ t^{n_1} & t^{n_2} - t^{n_1} \end{pmatrix}$

$= t + t^{-2} - 2 + 1 = t - 1 + t^{-1} \neq 1$

$= \Delta_0$

Hw:  $\Delta_{fgr}(t) = t - 3 - t^{-1}$

Corollary:  $\text{dy}(\Delta_h) \leq g(h) \quad \left( S \in \mathcal{M}_{2g, 2g}(\mathbb{Z}) \right)$

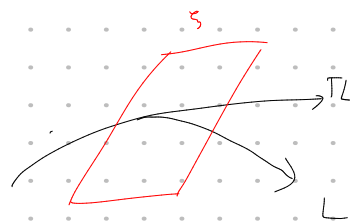
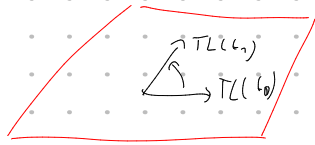
Hw:  $\forall g \in M_0 : \exists h_g : g(h_g) = g$  (maybe via torus knots?)

### 3.5 Classical Invariants

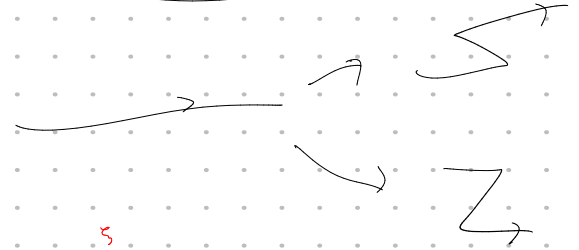
Let  $K \subset (\mathbb{R}^3, \xi)$  be a Legendrian knot.

(Def:  $tb(K) := \# \text{ twists of } \xi \text{ around } L$

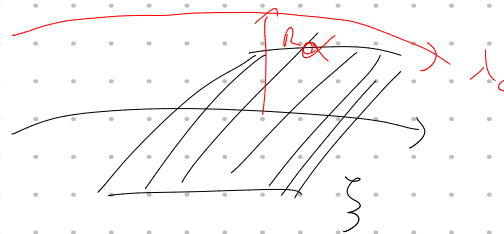
$rot(K) := \# \text{ twists of } TL \text{ in } \xi$



Stabilization:

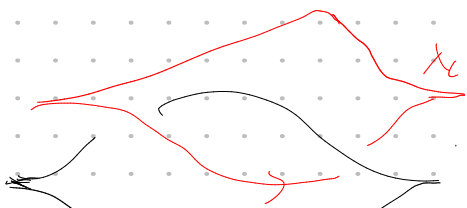


Contact longitude of  $K$ :  $\lambda_c = \text{push-out of } K \text{ in the Reeb-direction}$



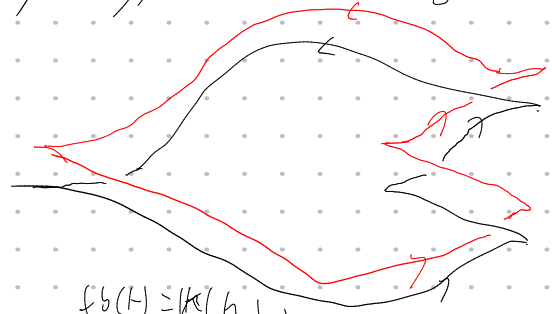
Thurston-Bennequin invariant:  $tb(K) := lk(K, \lambda_c)$ ,  $K$  nullhomologous

Ex:



$$tb(K) = lk(K, \lambda_c) = -7$$

$$Ra = \partial z$$



$$tb(K) = lk(K, \lambda_c) = -2$$

Lemma 12: Let  $K \subset (\mathbb{R}^3, \xi_{st})$  be a Legendrian knot presented in front proj.

then  $tb(K) = -\frac{1}{2}C + W$

$C := \# \text{ cusps in front}$

$W := \# \text{ positive crossings} - \# \text{ negative crossings}$   
 $= \underline{\text{writhe of front}}$

Proof:  $\square$

Ex



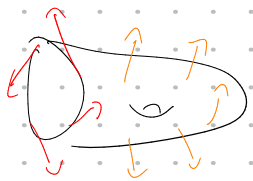
$$tb = -\frac{1}{2}C + W = -\frac{1}{2}4 + 3 = 1$$

Let  $K \subset (\gamma, \xi)$  be an oriented, nullhomologous legendrian knot with Seifert surface  $\Sigma$ .

$$\text{rot}(K, [\xi]) := \langle e(\xi, K), [\Sigma] \rangle = \text{pd}(e(\xi, K) \cdot [\Sigma])$$

$\cong \# \text{ zeros of extension of } TK \text{ over } \xi \text{ (rather } \Sigma?)$

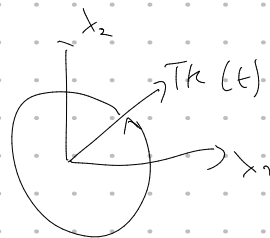
$\cong \# \text{ rotations of } TK \text{ relative to a trivialization of } \xi \text{ over } \Sigma$ .



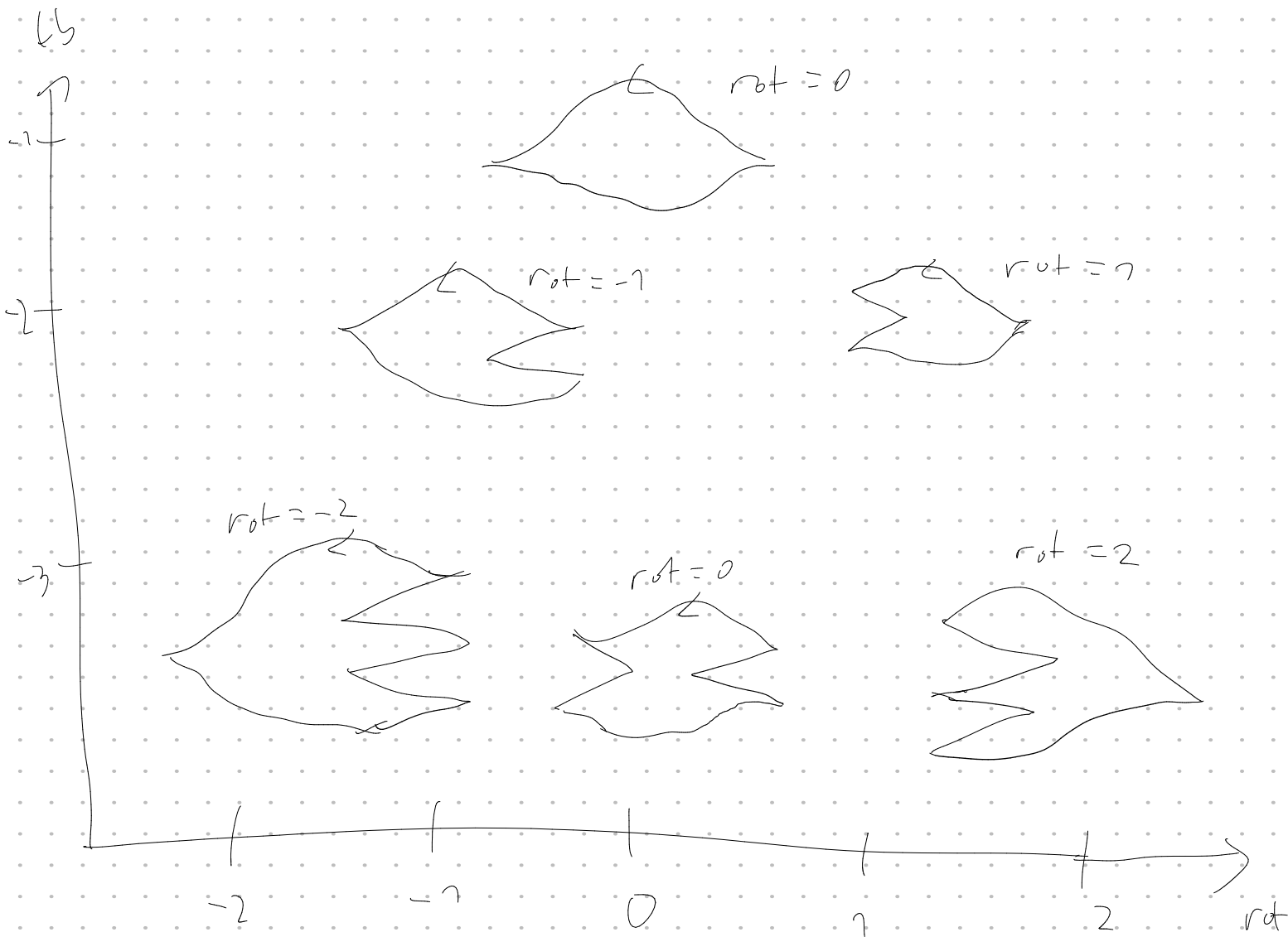
If  $\xi$  is trivializable, i.e.  $\exists x_1, x_2$  v.f. st.  $\xi = \langle x_1, x_2 \rangle$ ,

then  $\text{rot}(K) \cong \# \text{ rotations of } TK \text{ relative to this trivialization.}$

Ex:  $\xi_{\text{std}} = \text{Ker}(x dy + dz) \Rightarrow x_1 = \partial_x, x_2 = \partial_y - x \partial_z$  span  $\xi_{\text{std}}$ .



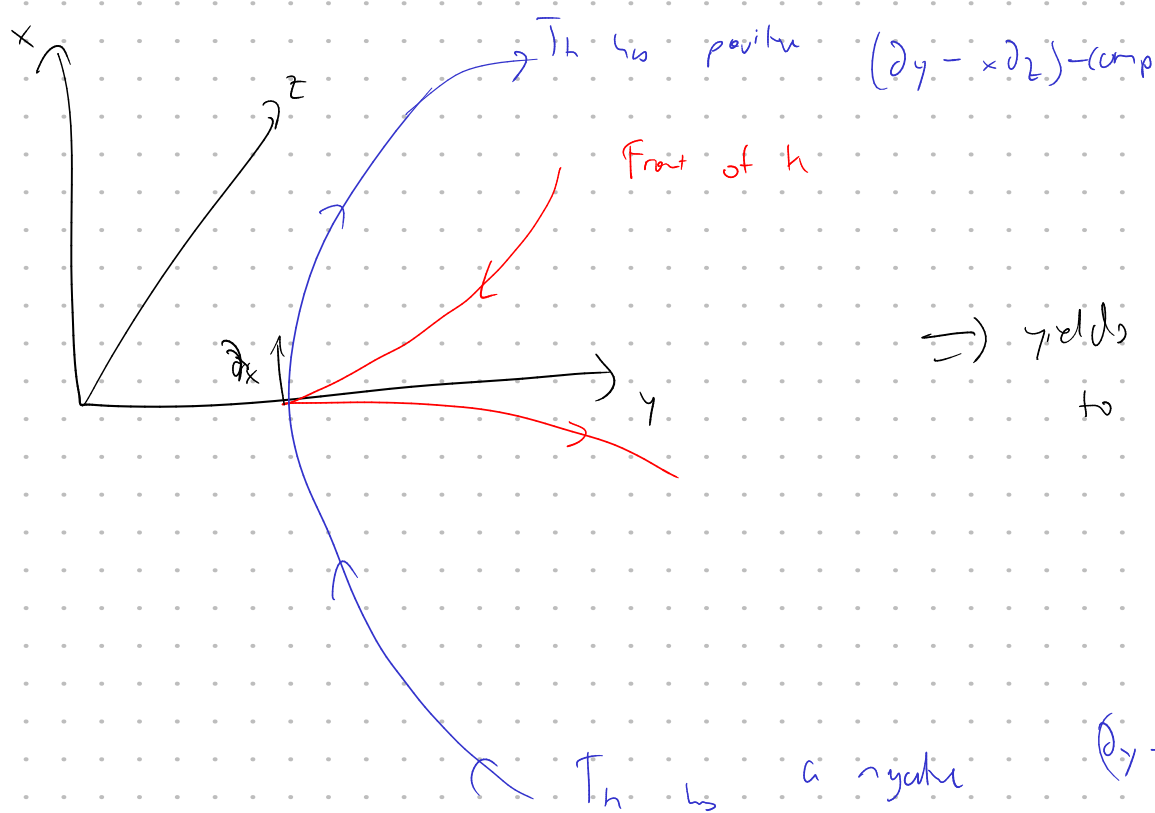
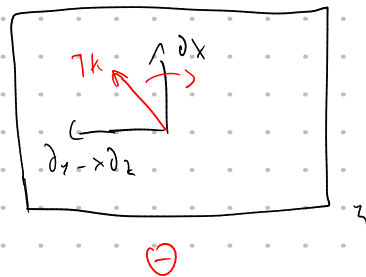
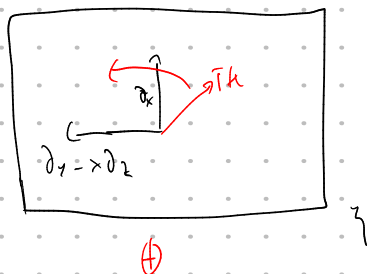
Remark:  $\text{rot}(-K) = -\text{rot}(K)$



Lemma 7.3.1:  $\text{rot}(h) = \frac{1}{2} (C_- - C_+)$

If  $q$  is initialized (i.e.  $\exists x_1, x_2$  s.t.

prob: Initialization of  $\vec{t}_{st} = \langle \partial_x, \partial_y - x\partial_z \rangle$



$\Rightarrow$  yields a  $(+)$  contribution to  $\text{rot}(k)$

smaller



Yields a  $(-)$  contribution

$$\Rightarrow \text{rot}(k) = \# \text{left down cusps} - \# \text{right up cusps}$$

If we count w.r.t.  $-\partial_x$

$$\Rightarrow \text{rot}(k) = \# \text{right down cusps} - \# \text{left up cusps}$$

$$\Rightarrow \text{rot}(h) = \frac{1}{2} (C_- - C_+)$$

# Thm 14: [Bennequin]

If  $K \subset (M^3, \xi_{st})$  is a Legendrian knot  $\Rightarrow \underbrace{tb(K) \pm rot(K)}_{\text{Contact geometry}} \leq \underbrace{2g(L) - 1}_{\text{Smooth topology}}$

Proof: in section 4/5

□

Corollary 15:  $(M^3, \xi_{st}) \stackrel{\text{cont.}}{\neq} (M^3, \xi_{OT})$

Proof:  $\exists$  Legendrian knot  $K$  in  $(M^3, \xi_{OT})$  s.t.  $TB(K) = 0$

$\Rightarrow TB(K) \pm rot(K) = \pm rot(K) > 0$  for one orientation on  $K$

but  $2g(K) - 1 = -1$

If  $(M, \xi)$  contains a Legendrian knot with  $tb = 0$  then  $(M, \xi) \neq \emptyset$  called overtwisted. If not then it is called tight.

## Thm 16 [Eliashberg]

(1) If  $\xi_1$  &  $\xi_2$  are OT contact structures on a closed 3-manifold  $M$ , then

$\xi_1 \stackrel{\text{isotopic}}{=} \xi_2 \Leftrightarrow \xi_1$  is homotopic to  $\xi_2$  as tangent 2-plane fields.

(Contact geom.)

(Top. top.)

(2)  $(M, \xi)$  is tight  $\Leftrightarrow \forall K \subset (M, \xi)$  null-homologous  $\wedge$  (s):

$$TB(K) \pm rot(K) \leq 2g(K) - 1$$

(3) If  $(M, \xi)$  admits a symplectic filling

$\Rightarrow M$  is tight.

(4)  $S^3, M^3, S^2 \times S^2$  have unique tight contact structures

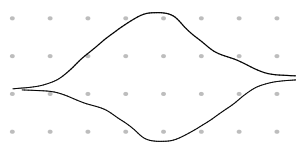
(5)  $T^2$  has infinitely many contact structures [Anany, Honda]

(6)  $\exists M^3$  without tight contact structures (Etnyre - Honda)

Proof: section 4/5? □

## Thm 17 [Eliashberg - Thurston]

Every Legendrian unknot in  $(M^3, \xi_{st})$  is a stabilization of

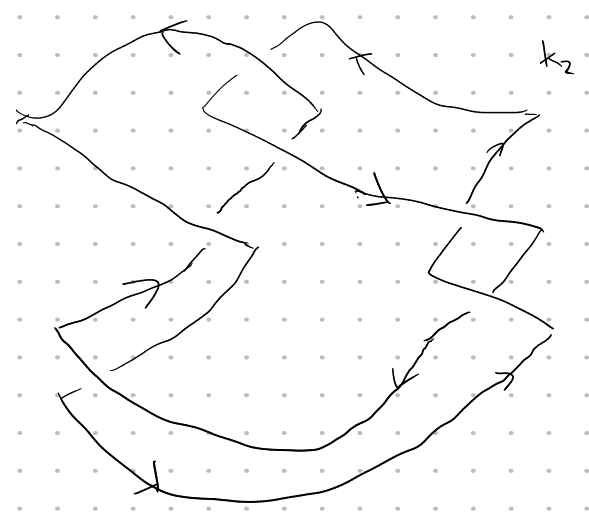
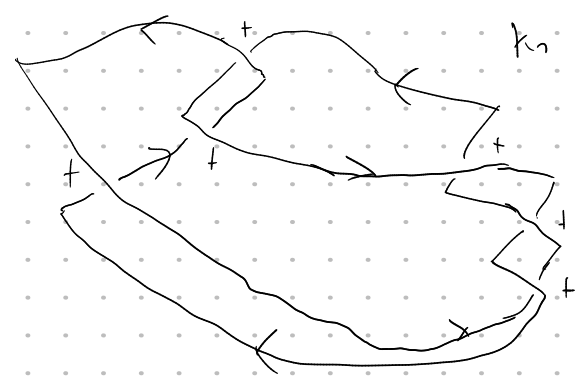


Proof: in section 4 □

Etnyre - Honda: similar results for pretails & figure-8 knots

Thm 18 [CHERKALOV]

Smooth isotopic Legendrian knots



$$\nabla\beta = -5 + 6 = 1 \quad \text{rot} = 0$$

$$\nabla\beta = 1 \quad \text{rot} = 0$$

with same  $\nabla\beta$  & rot, but not isotopic as Legendrian knots.

[Distinguished by their contact homologies].

However  $K_1$  &  $K_2$  are isotopic after one stabilization (Hart)

### Transverse Knots

Let  $K \subset (M, \gamma)$  be transverse and nullhomologous & choose a section surface for  $K$ . Then we can define the

### Self-linking Number

$$SL(K, [\Sigma]) := lk(K, K')$$

$K' :=$  push-off of  $K$  in direction of a non-vanishing vector field on  $\Sigma|_K$

If  $\eta$  is trivializable

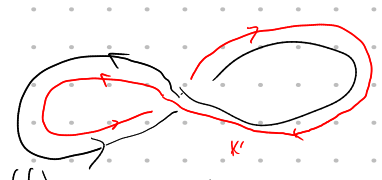
$K' :=$  push-off of  $K$  in direction of any non-vanishing vector field on  $\Sigma$ .

Lemma 19.5  $K \subset (M^3, \gamma_{st})$  is transverse & presented in the front

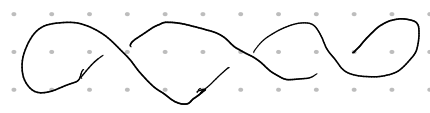
$$\Rightarrow SL(K) = w$$

proof:  $0 \neq \partial_x \in \gamma_{st} \quad K' = \text{push-off of } K \text{ in } x\text{-dir} \Rightarrow lk(K, K') = w \quad \square$

Ex:



Stabilization



$$SL(K) = lk(K, K') = -7$$

$$SL = -3$$

Thm 20: [B&S 5.2]

is a stabilization of

Every positive integer

has

least

n

$(n^3, \varphi_n)$

