# Tight and non-Fillable Contact Structures on the Sphere

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# Background

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> **Dichotomy:** Rigidity vs. Flexibility.
> - **tight** *(rigid/geometric)*;
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### Theorem (Eliashberg–Gromov)

*Fillable contact manifolds are tight.*

Converse is false (Etnyre–Honda, Massot–Niederkrueger–Wendl).

# Existence and classification

*Topological* obstruction: *almost* contact structure, i.e. reduction of structure group to $U(n) \times \mathbb{1}$.

> **Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))**
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### Tight manifolds

How can **tight** contact manifolds be understood?

# Contact structures on spheres

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## Theorem (Eliashberg, '91)

*On $S^3$, it is the unique tight contact structure.*

In particular, no tight and non-fillable contact structures on $S^3$.

# Tight and non-fillable structures in $\dim \geqslant 5$

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*In $\dim = 5$, the same holds, if the first Chern class vanishes.*

# Tight and non-fillable spheres

# Giroux correspondence

**Giroux:** Contact structures are *supported* by open books.
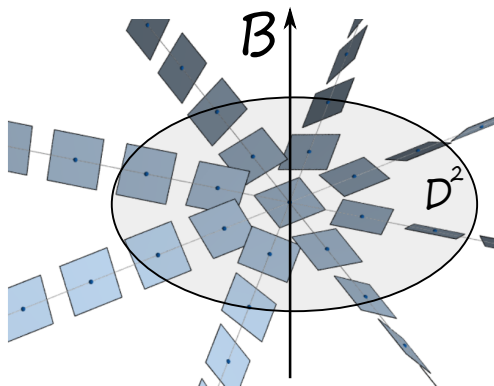


Figure: Supported contact structure.

# Bourgeois contact structures

### Theorem (Bourgeois '02)

*Open book supporting $(M, \xi) \rightsquigarrow$ contact structure on $M \times \mathbb{T}^2$.*

These are $\mathbb{T}^2$-equivariant.

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> **Claim:** $(\mathbb{S}^{2n+1}, \xi_{ex})$ is tight and non-fillable.

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Milnor open book $\Rightarrow$ $(\mathbb{S}^{2n+1}, \xi_{ex})$ is *tight*.

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# Fillability

- Pseudoholomorphic curves
- clever geometric construction
- homological obstruction

Thank you!

# Fillability

**Observation:** Bourgeois manifolds have convex decomposition

$$\mathrm{OB}(\Sigma, \phi) \times \mathbb{T}^2 = (\mathrm{OB}(\Sigma, \phi) \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V}_- \times \mathbb{S}^1,$$

with $V_\pm = \Sigma \times D^*\mathbb{S}^1$, $\Sigma$ = page of the open book, $\phi$ = monodromy.

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with $V_\pm = \Sigma \times D^*\mathbb{S}^1$, $\Sigma =$ page of the open book, $\phi =$ monodromy.

Theorem (Bowden–Gironella–Moreno)

$M = V \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V_-} \times \mathbb{S}^1$ *with convex decomposition,*
$N = \partial V_\pm$ *dividing set. If $W$ is a symplectic filling of $M$, then*

$$H_*(N) \to H_*(V_\pm) \to H_*(W),$$

*induced by inclusion. Then second map is injective on image of the first.*

Namely, if a homology class in $N$ survives in $V_\pm$, then it survives in the filling.

**Proof:** $W$ filling of $(\mathbb{S}^{2n+1}, \xi_{ex})$:

$$0 \neq H_n(N) \xrightarrow{\text{nontrivial}} H_n(W).$$

However, this factors as

$$0 \neq H_n(N) \to H_n(\partial W = \mathbb{S}^{2n+1}) = 0 \to H_n(W),$$

contradiction.