CLASSIFYING CONTACT STRUCTURES ON THE SPHERE

Introduction

Contact geometry is the study of odd-dimensional smooth manifolds equipped with contact structures, i.e. hyperplane distributions $\xi = \ker \alpha$ satisfying the contact condition

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0.$$

While they originally arise in the study of ODEs and in classical mechanics, the topological study of contact manifolds is a more recent and very active field of research.

A manifold can have multiple different contact structures, which can be either rigid (in which case one speaks of a "tight" manifold) or flexible (in the sense that they satisfy an h-principle). The latter contact manifolds are then called overtwisted. A foundational result of Eliashberg [Eli89] and Borman–Eliashberg–Murphy [BEM15], roughly speaking, states that overtwisted contact manifolds exist in abundance, namely whenever the manifold admits the topological version of a contact structure (an *almost* contact structure), which is a first obvious obstruction. In dimension three, an almost contact structure is simply an oriented 2-plane field.

To illustrate this dichotomy, consider the sphere S^3. By a result of Eliashberg, it has precisely one tight contact structure. On the other hand, it has infinitely many overtwisted contact structures, corresponding to the infinitely many homotopy classes of 2-plane fields on the 3-sphere. There are other examples where there are infinitely many or no tight contact structures on a contact manifold.

A further interesting property of contact manifolds comes from the fact that contact geometry is the odd-dimensional counterpart to symplectic geometry. Often, it is possible to view a contact manifold as the boundary of a symplectic manifold. Manifolds that are in this sense "fillable" are always tight [Gro85, Eli91]. The contrary, however, doesn't need to hold and one can ask the question under which conditions such tight, but non-fillable manifolds exist. The first examples of tight and non-fillable contact manifolds were constructed by Etnyre–Honda [EH02] in dimension three, and by Massot–Niederkrueger–Wendl [MNW13] in higher dimensions.

In the following, I will start with a brief overview of the results and methods of my Master's thesis (for more details see the summary of the thesis). Then, I give a brief overview of the current research status in the area and finally state and explain some open questions that I plan to work on in the dissertation.

Tight non-fillable contact structures on the sphere

Recently, Bowden–Gironella–Moreno–Zhou [BGMZ22] have proved the existence of homotopically standard, non-fillable but tight contact structures on all spheres S^{2n+1} with $n >= 2$. One fundamental ingredient of the work is the Giroux correspondence [Gir02]. There is a one to one correspondence between oriented contact

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structures up to isotopy and open book decompositions up to positive stabilization. Starting with a specific open book decomposition of S^{2n-1}, one can, as an instance of the Giroux correspondence, construct a contact form on this manifold using a well-known construction by Thurston–Winkelnkemper. Then, according to Bourgeois, this contact structure can be extended to a tight contact structure on $S^{2n-1} \times T^2$ that is homotopically standard. Applying subcritical surgery (preserving the tightness), one can kill the topology of the T^2-factor and obtain a tight contact structure on S^{2n+1}. The symplectic fillability fails for homological reasons as can be exhibited by clever use of a certain strong cobordism.

Fillability hierarchy

As explained above, the notion of fillability is crucial in understanding tight contact structures. However, there are actually a lot of different notions of fillability in symplectic geometry.

Definition 1 (weak and strong filling)**.** *Let (M, ξ) be a contact manifold and (W, ω) a symplectic manifold s.t. $\partial W = M$.*

- *(W, ω) is a weak filling of (M, ξ) iff $\omega|_\xi > 0$*
- *(W, ω) is a strong filling of (M, ξ) iff $\omega = \mathrm{d}\alpha$ near the boundary.*

Definition 2 (Liouville filling)**.** *A Liouville cobordism is a compact symplectic manifold where the globally defined Liouville vector field X is negatively transverse along ∂W_- (concave side) and positively transverse along ∂W_+ (convex side). If $\partial W_- = \emptyset$, we call it a Liouville domain. A Liouville filling of a contact manifold $(M, \xi = ker\alpha)$ is a Liouville domain (W, λ) such that $(\partial W, ker\lambda|\partial W)$ is contactomorphic to $(M, ker\alpha)$.*

Definition 3 (Weinstein filling)**.** *A Weinstein filling is a Liouville filling such that the Liouville vector field X is gradient-like for a Morse function which is locally constant on the boundary.*

In general, we have the following sequence of proper inclusions (we abbreviate e.g. "Weinstein" for "Weinstein fillable contact manifold")

$$\{\text{Weinstein}\} \overset{1}{\subsetneq} \{\text{Liouville}\} \overset{2}{\subsetneq} \{\text{strong}\} \overset{3}{\subsetneq} \{\text{weak}\} \overset{4}{\subsetneq} \{\text{tight}\}$$

In dimension ≥ 5, 1 was proved by[BCS14, Theorem 1.5], 2 by [Zho21], 3 by [BGM22] and 4 by [MNW13] (in dimension 3, they have been known for a while). All of the above examples where such an inclusion is proper rely on special geometric constructions. Now, due to [BGMZ22], the last inclusion is known to be proper for all contact manifolds (of course only if there are tight contact structures at all). In the same spirit, they proof that for any $n \geq 3$, there exist homotopically standard Liouville fillable contact structures on S^{2n+1} that are not Weinstein fillable.

Open questions

- For being able to further classify contact structures on the sphere, it is certainly interesting to better understand the underlying almost contact structures. Using the fact that S^{2n+1} is stably trivial, we can show that an almost contact structure on S^{2n+1} corresponds to a map that associates an almost complex structure on R^{2n+2} to every point of the sphere. Almost complex structures on R^{2n+2} are given by $\mathrm{SO}(2n+2)/\,\mathrm{U}(n+1)$, so we have the following correspondence

$$\mathrm{Acont}\left(S^{2n+1}\right) = \pi_{2n+1}(\mathrm{SO}(2n+2)/\,\mathrm{U}(n+1)).$$

This fundamental group has been computed by Harris [Har63] and we obtain

$$\mathrm{Acont}\left(S^{2n+1}\right) = \pi_{2n+1}(\mathrm{SO}(2n+2)/\,\mathrm{U}(n+1)) = \begin{cases} \mathbb{Z}_{n!} & n = 0 \bmod 4 \\ \mathbb{Z} & n = 1 \bmod 4 \\ \mathbb{Z}_{n!/2} & n = 2 \bmod 4 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & n = 3 \bmod 4 \end{cases}.$$

It is possible to choose the first identification in such a way that ξ_{standard} corresponds to 0. Taking the negative stabilization of that open book, we obtain a contact structure ξ_{neg}. Which element does ξ_{neg} correspond to?

- Is there a Liouville but not Weinstein fillable contact structure on S^5?
- Is there a strong but not Liouville fillable structure on S^{2n+1}, $n \geq 2$?

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