

Tight and non-Fillable Contact Structures on the Sphere

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Background

Contact topology

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Fillability: *fillable* contact mflds are boundaries of symplectic mflds.

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Eliashberg, Borman–Eliashberg–Murphy:

Dichotomy: Rigidity vs. Flexibility.

- **tight** (*rigid/geometric*);
- **overtwisted** (*flexible/topological*).

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Theorem (Eliashberg–Gromov)

Fillable contact manifolds are tight.

Converse is false (Etnyre–Honda, Massot–Niederkrueger–Wendl).

Existence and classification

Topological obstruction: *almost* contact structure, i.e. reduction of structure group to $U(n) \times \mathbb{1}$.

Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))

Almost contact manifolds are contact, where the contact structure is overtwisted.

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Tight manifolds

How can **tight** contact manifolds be understood?

Contact structures on spheres

Standard contact structure

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Theorem (Eliashberg, '91)

On S^3 , it is the unique tight contact structure.

In particular, no tight and non-fillable contact structures on S^3 .

Tight and non-fillable structures in $\dim \geq 5$

Theorem (Bowden–Gironella–Moreno–Zhou '22-'24)

For every $n \geq 2$, the sphere \mathbb{S}^{2n+1} admits a tight, non-fillable contact structure that is homotopically standard.

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In $\dim \geq 7$, if M admits a tight structure, it also admits a tight and non strongly-fillable structure, in the same almost contact class.

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In $\dim = 5$, the same holds, if the first Chern class vanishes.

Tight and non-fillable spheres

Giroux correspondence

Giroux: Contact structures are *supported* by open books.

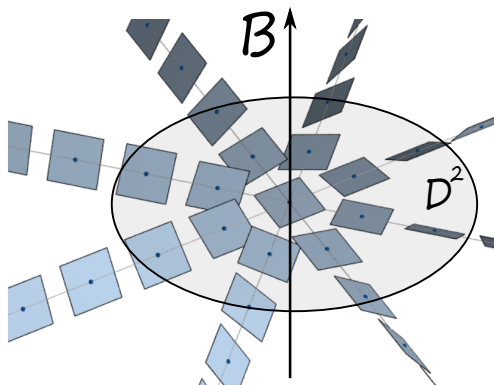


Figure: Supported contact structure.

Bourgeois contact structures

Theorem (Bourgeois '02)

Open book supporting $(M, \xi) \rightsquigarrow$ contact structure on $M \times \mathbb{T}^2$.

These are \mathbb{T}^2 -equivariant.

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- Milnor open book on \mathbb{S}^{2n-1} $\xrightarrow{\text{Bourgeois}}$ contact structure on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$
 \rightsquigarrow two 1-surgeries = $\mathbb{S}^{2n-1} \times \mathbb{S}^2 \rightsquigarrow$ one 2-surgery = \mathbb{S}^{2n+1} .

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Claim: $(\mathbb{S}^{2n+1}, \xi_{ex})$ is tight and non-fillable.

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 - Algebraic tightness is vanishing of a certain contact homology algebra.
- 2 Algebraic tightness is preserved under surgeries.
- 3 Algebraically tight \implies tight.

Milnor open book $\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex})$ is *tight*.

Fillability

- Pseudoholomorphic curves
- clever geometric construction
- homological obstruction

Thank you!

Fillability

Observation: Bourgeois manifolds have convex decomposition

$$\text{OB}(\Sigma, \phi) \times \mathbb{T}^2 = (\text{OB}(\Sigma, \phi) \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_{\phi} \overline{V}_- \times \mathbb{S}^1,$$

with $V_{\pm} = \Sigma \times D^*\mathbb{S}^1$, Σ = page of the open book, ϕ = monodromy.

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Theorem (Bowden–Gironella–Moreno)

$M = V \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_{\phi} \overline{V_-} \times \mathbb{S}^1$ with convex decomposition, $N = \partial V_{\pm}$ dividing set. If W is a symplectic filling of M , then

$$H_*(N) \rightarrow H_*(V_{\pm}) \rightarrow H_*(W),$$

induced by inclusion. Then second map is injective on image of the first.

Namely, if a homology class in N survives in V_{\pm} , then it survives in the filling.

Proof: W filling of $(\mathbb{S}^{2n+1}, \xi_{ex})$:

$$0 \neq H_n(N) \xrightarrow{\text{nontrivial}} H_n(W).$$

However, this factors as

$$0 \neq H_n(N) \rightarrow H_n(\partial W = \mathbb{S}^{2n+1}) = 0 \rightarrow H_n(W),$$

contradiction.