5. Existence & classification of contact 3-mfds & leg. knots

5.1 big picture

OT = flexible:

$$\{\xi \text{ OT}\}/\text{Isotopy} \xleftrightarrow{1:1} \{\text{Tang } 2\text{-plane fields}\}/\text{Hom.}$$

$$\Big\updownarrow 1:1$$

Alg top

i.e. 'solved'

Tight = rigid

$\exists$ mfds with a unique tight c.s.
$\exists$ mfds with no ''
$\exists$ '' ∞ ''

5.2 classification of tight contact structures

Thm 1: $(\mathbb{R}^3, \xi_{st})$, (S^3, ξ_{st}), $(S^1\times\mathbb{R}, \xi_{st})$, $(S^1\times S^2, \xi_{st})$ are tight

proof: in sect 6

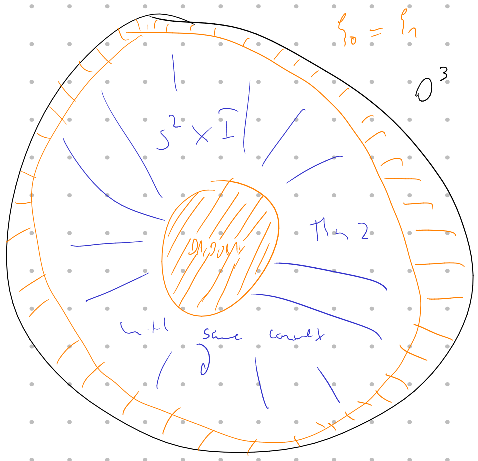
Thm 2: A tight contact structure ξ on $S^2\times[0,1]$ with convex boundary is determined (up to isotopy fixing ∂) by the dividing set of ∂

Thm 3 [Eliashberg]

Let ξ_0, ξ_1 be tight c.s. on D^3 s.t. ∂D^3 is convex and $\Gamma^0_{\partial D^3} = \Gamma^1_{\partial D^3}$

$\Rightarrow$ ξ_0 is isotopic to ξ_1 rel ∂D^3

proof:



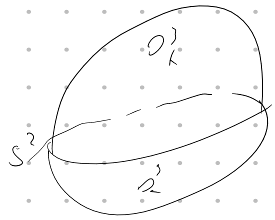
Thm 4 [Eliashberg]

$S^3, \mathbb{R}^3, S^1\times S^2$ admit unique tight c.s.

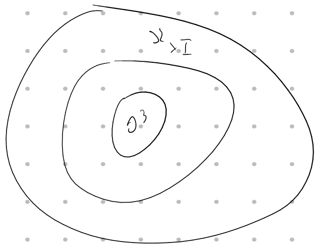
Proof: (1) $S^3 = D^3_+ \cup_{S^2} D^3_-$ wlog: S^2 convex

ξ tight $\Rightarrow$ Γ_{S^2} is connected

Thm 3 $\Rightarrow$ ξ_0 is isotopic to ξ_1



(2) $\mathbb{R}^3 = D^3(0) \cup S^2\times[1,2] \cup S^2\times[2,3] \cup \cdots$



wlog $S^2\times n$ convex $\forall n\in\mathbb{Z}$

ξ tight $\Rightarrow$ $\Gamma_{S^2\times n}$ is connected

Thm 2&3 $\Rightarrow$ ξ_0 is isotopic to ξ_1

(3) $S^1\times S^2 = [-1,1]\times S^2 \Big/ -1\times S^2 \sim 1\times S^2$

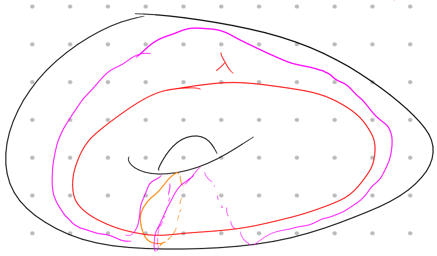
$\Rightarrow$ wlog $\pm 1\times S^2$ is convex with connected Γ

Let V be a solid torus with tight c.s. s.t. ∂V is convex

Fix: meridian μ: s.c.c. on ∂V non-trivial in ∂V but trivial in V.

Longitude λ: s.c.c. on ∂V nontrivial in ∂V s.t. $\mu \pitchfork \lambda = pt$

$\lambda' = \lambda + n\mu$

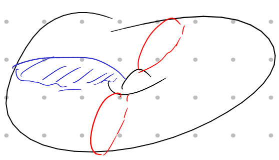


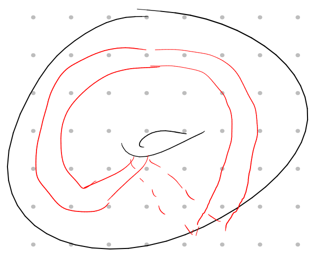
ξ tight $\Rightarrow$ $\Gamma_{\partial V}$ consist of linear curves i.e. parallel copies of $p\mu + q\lambda$

$\mathrm{Slope}(\partial V) = \mathrm{slope}(\Gamma_{\partial V})$

Ex: $\mathrm{Slope}(S^1\times D^2, \xi_n) = -\frac{1}{n}$

no slope 0:





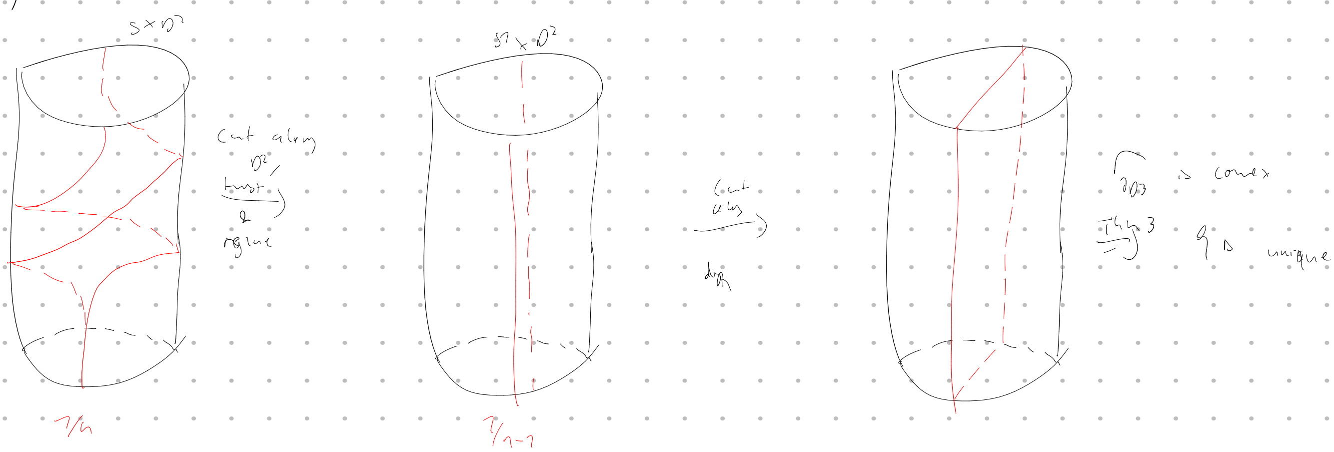
Thm 5: [Giroux, Honda]

(1) $\forall n \in \mathbb{Z}$ $\exists!$ tight c.s. on $S^1 \times D^2$ with convex ∂ with 2 dividing curves of slope $= \frac{1}{n}$

(2) $\forall r \in \mathbb{Q} \setminus \{0\}$: $\exists$ finitely many tight c.s. " " of slope $= r$

proof sketch:

(1)



(2)