2 Contact Manifolds

17.10.23

2.1 Hyperplane fields

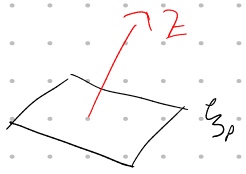
M^n = smooth manifold

$\left(p \longmapsto \xi_p^{n-1} \subset T_pM \right)$

- $\xi \subset TM$ is called hyperplane field $:\Leftrightarrow$ $\forall p \in M$ $\exists$ nbhd U of p and linearly independent vector fields $X_1, \dots, X_{n-1}$ on U s.t. $\forall q \in U$:

$$\xi_q := \xi \cap T_qM = \langle X_1(q), \dots, X_{n-1}(q) \rangle$$

- ξ is called co-orientable $:\Leftrightarrow$ TM/ξ is orientable



$p \mapsto T_pM/\xi_p = V_p$

Def vector bundle over M^n

$E \supset \pi^{-1}(U) \xrightarrow{\cong^{C^\infty}_{\text{fib. lsom}}} U \times \mathbb{R}^k$

$\pi \downarrow$ $\qquad$ $\searrow$ $\swarrow$ pr

$M^n \ni p$ $\qquad$ U

Why is TM/ξ a bundle?

$\xi_q = \langle X_1(q), \dots, X_{n-1}(q) \rangle$

$N_q = \xi_q^\perp$

$\pi^{-1}(U) \cong U \times \langle N \rangle$

$\pi^{-1}(U) \cong U \times V_p$

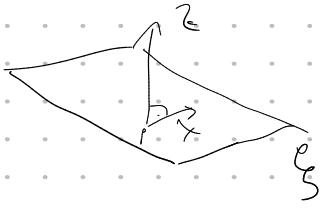
Lemma 1: $\xi \subset TM$ is coorientable $\Leftrightarrow$ $\exists$ 1-form α on M s.t. $\xi = \ker(\alpha)$ (ie. $\alpha_p : T_pM \to \mathbb{R}$) (18.10.23)

proof: Let g be a riemannian metric.

"$\Rightarrow$" Choose a vector field Z on M s.t.

$$\begin{cases} g(Z, X) = 0 & \forall X \in \xi \\ g(Z, Z) = 1 \end{cases}$$

$\alpha := g(Z, \cdot)$



"$\Leftarrow$" A vector field Z s.t. $\begin{cases} \alpha(Z) = 1 \\ d\alpha(Z, \cdot) \neq 0 \end{cases}$ defines a coorientation of ξ □

Convention: We restrict to coorientable hyperplane fields (often this is secretly assumed)

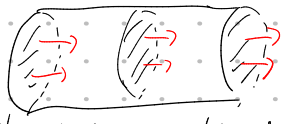
Example:

$M^n := S^1 \times N^{n-1}$

$\forall p = (\theta, x) \in M = \xi_p := T_xN \subset T_pM$

$\xi = \bigcup_{p \in M} \xi_p$ is a hyperplane field

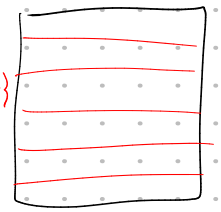
$\xi = \ker(d\theta)$ (odim − 1-foliation)


glue left and right side together

Foliations:

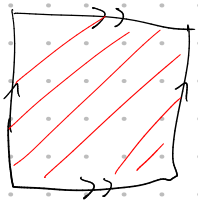
$\mathcal{F}^n = \bigsqcup_{\theta \in \Lambda} F_\theta$ (F_θ "leafs") is called n-dim foliation

$:\Leftrightarrow$ $\exists$ Atlas of $\mathcal{F}$ (U_α, h_α) s.t. $\forall \theta \forall \alpha$: $h_\alpha(F_\theta \cap U_\theta) = \emptyset$ or $\mathbb{R}^n \times p \cap h_\alpha(U_\alpha)$



Ex: Flow lines of a non-vanishing vector field are n-dimensional foliations

$\mathcal{F} = T^2$



If slope $\in \mathbb{Q} \Rightarrow$ Leaves are compact
If slope $\in \mathbb{R} \setminus \mathbb{Q} \Rightarrow$ " noncompact

If $k = n-1 \Rightarrow \xi = \bigcup_{p \in F} T_p F(p)$ is a hyperplane field

"every foliation of codim 1 induces a hyperplane field"

Q: "When does a hyperplane come from a foliation?"

Theorem 2 [Frobenius]

$\ker(\alpha) = \xi \subset T\mathcal{F}$ is induced by a foliation $\Leftrightarrow$ $\alpha \wedge d\alpha$

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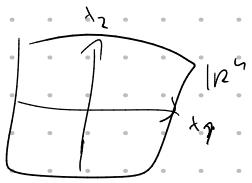
digression: $\alpha_p : T_p \mathcal{F} \xrightarrow{1\text{-form}} \mathbb{R}$ i.e.

$\alpha_p \in (T_p \mathcal{F})^*$



Let $x_1, \dots, x_n$ be coordinates on $\mathcal{F}$

$\Rightarrow$ coordinate vector fields $\partial_{x_1}, \dots, \partial_{x_n}$ form a basis of $T_p \mathcal{F}$

Dual basis $dx_1, \dots, dx_n$, i.e. $dx_i(\partial_{x_j}) = \delta_{ij}$

$\Rightarrow \alpha_p = \sum c_i(p) dx_i$

k-form β: $\beta_p : \underbrace{(T_p \mathcal{F})^* \times \dots \times (T_p \mathcal{F})^*}_{k \text{ times}} \to \mathbb{R}$ multilinear, alternating

$\wedge$-product:

1-forms $\alpha = \sum c_i dx_i$

$\beta = \sum b_j dx_j$.

$\alpha \wedge \beta =$ def. to be linear

$dx_i \wedge dx_j = -dx_j \wedge dx_i$

$dx_i \wedge dx_i = 0$

Differential

d: k-forms $\to$ $(k+1)$-forms

$\sum c_i(p) dx_i \longmapsto d\alpha$

$\sum_{ij} \frac{\partial c_i}{\partial x_j} dx_j \wedge dx_i$

Ex: $\Omega = \mathbb{R}^3_{(x,y,z)}$ $\alpha = dx$

$d\alpha = d\,dx = 0$ $\Rightarrow$ $\alpha \wedge d\alpha$ comes from a foliation

(2) $\alpha = x\,dy + dz$

$d\alpha = d(x\,dy + dz) = d(x\,dy) + \cancel{d(dz)} = \underbrace{\frac{\partial x}{\partial x}}_{=1} dx \wedge dy + \cancel{\frac{\partial x}{\partial y} dy \wedge dy}$

$\alpha \wedge d\alpha = (x\,dy + dz) \wedge (dx \wedge dy) = x\,dy \wedge (dx \wedge dy) + dz \wedge dx \wedge dy$

$= dx \wedge dy \wedge dz \neq 0$

2.2 Contact structures Let Ω^{2n+1}

* $\xi^{2n} = \ker(\alpha)$ is a contact structure $\Leftrightarrow$ $\alpha \wedge (d\alpha)^n$ is a volume form $\left\{ \text{i.e. } \alpha \wedge (d\alpha)^n \neq 0 \text{ at every point} \right\}$. $\alpha \wedge (d\alpha)^n = \alpha \wedge d\alpha \wedge \dots \wedge d\alpha$

* α is called contact form

* the reeb vector field R_α of α is defined

$$\begin{cases} d\alpha(R_\alpha, \cdot) \equiv 0 \\ \alpha(R_\alpha) \equiv 1 \end{cases}$$

Remark: $\alpha \wedge (d\alpha)^n$ is a vol form $\Rightarrow$ Ω orientable

* $\xi = \ker(\alpha) = \ker(\tilde\alpha) \Rightarrow \tilde\alpha = f\alpha$ for $f : \Omega \to \mathbb{R} \setminus \{0\}$

$\Rightarrow \tilde\alpha \wedge (d\tilde\alpha)^n = (f\alpha) \wedge (d(f\alpha))^n = f\alpha \wedge (f\,d\alpha + df \wedge \alpha)^n$

$\alpha \wedge d\alpha = 0$

$= \underbrace{f^{n+1}}_{\neq 0} \underbrace{\alpha \wedge (d\alpha)^n}_{\neq 0}$

* R_α is well-defined:

$\alpha \wedge (d\alpha)^n \neq 0 \Rightarrow (d\alpha)^n|_\xi \neq 0$ $\Rightarrow$ $d\alpha$ has rank $2n$

(ξ: 2n-dimensional subspace) $\Rightarrow$ $\ker(d\alpha)$ is 1-dim & $\alpha \neq 0$ on $\ker(d\alpha)$

Warnholm (say): α closed $\overset{?}{\Rightarrow}$ $\forall$ contact form α on $\mathbb{R}^{2n+1}$: R_α is a periodic orbit

Ex (1) Consider $\mathbb{R}^{2n+1}_{(x_1,y_1,\dots,x_n,y_n,z)}$

$\xi_{st} = \ker(\alpha_{st})$ Standard contact structure

$\alpha_{st} = \left(\sum_{i=1}^{n} x_i dy_i\right) + dz$

$$\alpha_{st} \wedge (d\alpha_{st})^n = \left(\sum x_i dy_i + dz\right) \wedge \left(d\left(\sum x_i dy_i + dz\right)\right)^n = \left(\sum x_i dy_i + dz\right) \wedge \left(\sum dx_i \wedge dy_i\right)^n$$

$$= \left(\sum x_i dy_i + dz\right) \wedge \underbrace{\left(\sum dx_i \wedge dy_i\right) \wedge \left(\sum dx_i \wedge dy_i\right)}_{\sum_{i\neq j} dx_i \wedge dy_i \wedge dx_j \wedge dy_j} \wedge \left(\sum dx_i \wedge dy_i\right)^{n-2}$$

$$= \left(\sum x_i dy_i + dz\right) \wedge n! \left(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \dots \wedge dx_n \wedge dy_n\right)$$

$$= dz \wedge n! \left(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \dots \wedge dx_n \wedge dy_n\right)$$

$$= n!\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \dots \wedge dx_n \wedge dy_n \wedge dz \neq 0$$

Compute $R_{\alpha_{st}} = \left(\sum A_i \partial_{x_i} + B_i \partial_{y_i}\right) + C\partial_z$

$$d\alpha_{st}\left(R_{st}, \cdot\right) = \sum A_i dy_i - B_i dx_i = 0$$

$$\alpha_{st}\left(R_{st}\right) = C + \sum x_i B_i = 1$$

One solution: $R_{\alpha_{st}} = \partial_z$

(2) $\xi_{Sym} = \ker\Big(\underbrace{\sum (x_i dy_i - y_i dx_i) + dz}_{\alpha}\Big)$

Harmonic: • This is a contact structure
• $R_\alpha = \partial_z$
• for $n=1$: draw this
• read Milnor: Topology from a differential viewpoint
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