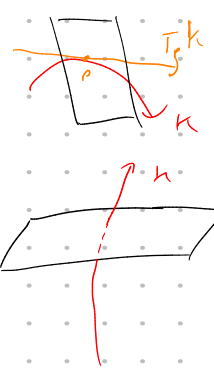


3. knots in contact 3-manifolds

An embedding $K: S^1 \hookrightarrow (M^3, \eta)$ is called

* Legendrian knot : $(\subset) \quad T_k \subset \eta$

* Transverse knot : $(\subset) \quad T_k \cap \eta = \emptyset$



h_0 is isotopic to $h_1 = \epsilon$

$\exists h_t, t \in I, h_t$ is Legendrian (transverse) $\forall t$.

Notation $h \in (\pi, \eta)$ for the isotopy class

Example: (1) $S^1 \hookrightarrow \mathbb{R}^3 / 2\pi \mathbb{Z} \ni t \mapsto (\cos(t), \sin(t), t) \in (\mathbb{R}^3, \eta_{\text{std}} = \ker(x dy - y dx))$
 is a Legendrian unknot.

It's Legendrian: take the framing, plug it into α

if it's 0 \Rightarrow it's Legendrian

(1) $S^1 \ni \theta \mapsto (\theta, 0, 0) \in (S^1 \times \mathbb{R}^2, \eta = \ker(\cos \theta dx - \sin \theta dy))$ is Legendrian.

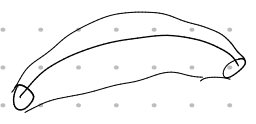
(2) $S^1 \ni \theta \mapsto (\theta, 0, 0) \in (S^1 \times \mathbb{R}^2, \eta = \ker(d\theta + r^2 dy))$ is transverse.

3.1) Neighborhood & isotopy extension theorems

Thm 1: (1) Let $K \subset (\pi, \eta)$ be Legendrian. $\Rightarrow \exists$ tubular NBHD V_h of h in π s.t.

$$\forall r > 0 \quad \exists \eta \in \mathcal{D}(\{0\}) : (V_h, \eta) \stackrel{\text{cont}}{\cong} (S^1 \times D_r^2, \eta_r)$$

$$h \mapsto S^1 \times 0$$



(2) Let $h \subset (\pi, \eta)$ be transverse. $\Rightarrow \exists$ tubular NBHD V_h of h in π

$$\& \exists \eta^{cont} \text{ s.t. } (V_h, \eta) \stackrel{\text{cont}}{\cong} (S^1 \times D_r^2, \ker(d\theta + r^2 dy))$$

$$h \mapsto S^1 \times 0$$

Proof: $\# \checkmark$ \Rightarrow Use Moser trick as in Darboux theorem

(2)

Thm 2: Let $K_t: S^1 \hookrightarrow (\pi, \eta)$ be an isotopy of Legendrian (transverse)

knots. $\Rightarrow \exists$ isotopy of contactomorphisms

$$\psi_t: (\pi, \eta) \xrightarrow{\cong \text{ (cont) }} (\pi, \eta) \text{ s.t.}$$

$$\psi_0 = id$$

$$\& \psi_t \circ K_0 = K_t$$

Proof: Construct Ψ_t as flow of a contact vectorfield from a Hamilton function.
(4-)

9.2 The front projection

Let $K \subset (S^3, \gamma_{\text{std}})$ be a knot

Thm 2.4

\Rightarrow We can see $K \subset (\mathbb{R}^3, \gamma_{\text{std}})$

$(x(y, z) \mapsto (y, z)$ front projection

$S^1 \ni t \mapsto (x(t), y(t), z(t)) \in (\mathbb{R}^3, \gamma_{\text{std}} = \ker(x dy + y dz))$

is Legendrian $(\Leftrightarrow) 0 = \alpha(\dot{\gamma}(t)) = x(t) \cdot y'(t) + y(t) \cdot z'(t) \quad \forall t$

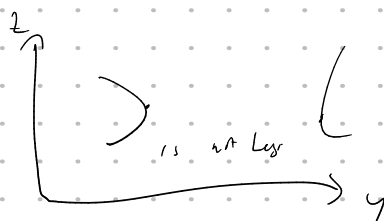
If K is generic (i.e. $y'(t) = 0$ only for finitely many t)

$\Rightarrow x(t) = -\frac{z'(t)}{y'(t)} = -\frac{dz}{dy}$ (if $y'(t) = 0 \Rightarrow z'(t) = 0$)

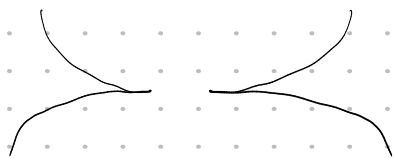
i.e. we can recover K from its front projection on \mathbb{R}^2

Certain configurations do not appear as front projections of Legendrian knots.

* if $y' \geq 0 \Rightarrow z' \geq 0 \Rightarrow$ there can't be vertical tangencies



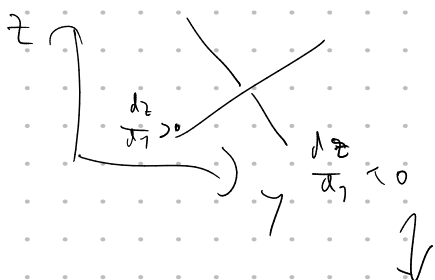
important: semicubical cusps?



$$(x(t), y(t), z(t)) = (t, t^2, -\frac{2}{3}t^3)$$

(every Legendrian knot has at least 2 of them)

* no crossing as follows:



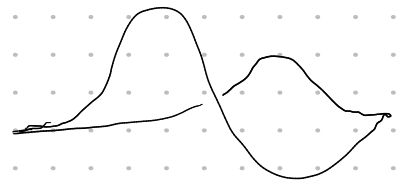
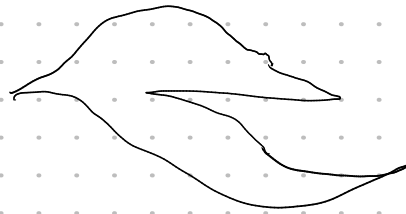
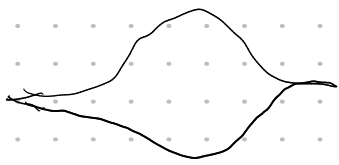
Legendrian:



\Rightarrow Any such front projection describes a unique Legendrian knot.

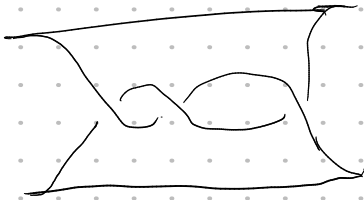
Legendre's: We have Legendre knots in the front projection

Examples:



is oblique to one of those on the left

helix



helix

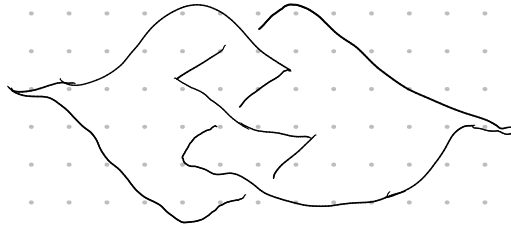


Figure 8

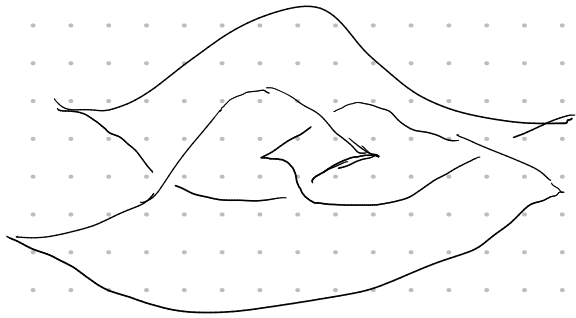
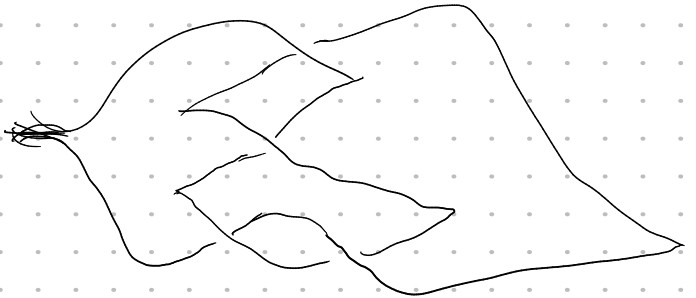
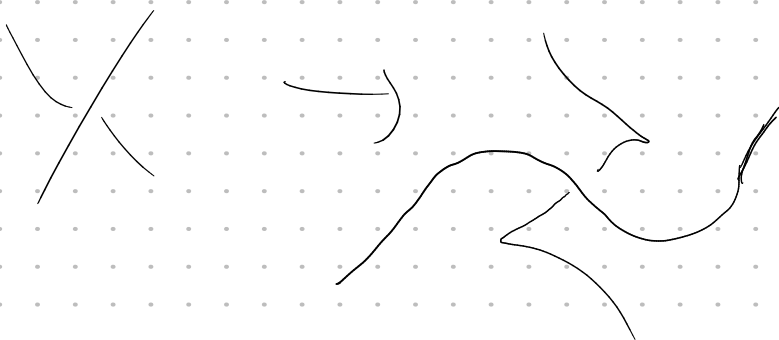
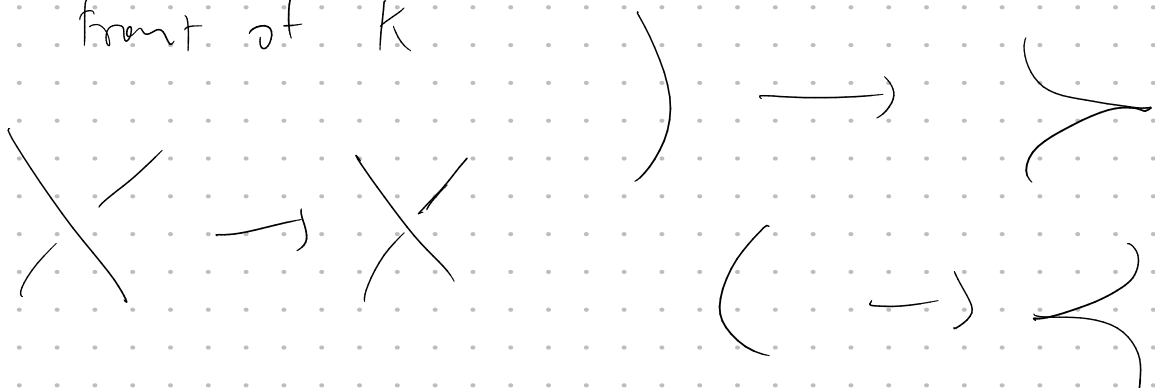


Figure 8



Corollary 3: For every smooth knot $K \in C(n^3, \epsilon_0)$
 \exists isotopic knot $K' \in C(n^3, \epsilon_0)$ Leg.

Proof: front of K



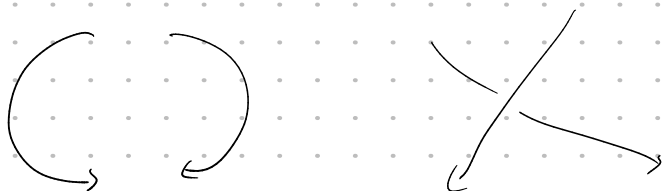
If h is positively transverse, i.e. $x_{y'} + z' > 0$

Then: If $y' = 0 \Rightarrow z' > 0$

$$\text{If } y' > 0 \Rightarrow x > -\frac{z'}{y'} = -\frac{dz}{dy}$$

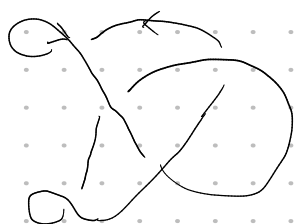
$$\text{If } y' < 0 \Rightarrow x < -\frac{dz}{dy}$$

\Rightarrow The following configurations are excluded



All other configurations lift to transverse curves in $(\mathbb{R}^3, \zeta_{st})$ unique up to

shifts in x -direction



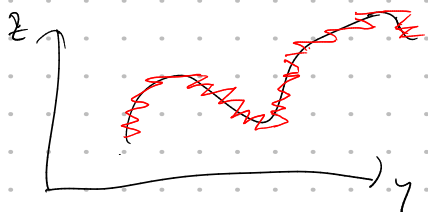
trivial

thm 4: Let $h: S^1 \hookrightarrow (\mathbb{R}^3, \zeta)$ a smooth knot

$\Rightarrow h$ can be C^0 -close approximated by a Legendrian (transverse) knot smoothly isotopic to h .

proof: Legendrian case

case 1: Let $\gamma: (a, b) \hookrightarrow (\mathbb{R}^3, \zeta_{st})$ be an arc



Approximate the front of γ by a Legendrian front projection s.t. $-\frac{dz}{dy}$ is close to the x -component of γ .

case 2: If $\gamma: (a, b) \hookrightarrow (\mathbb{R}^3, \zeta_{st})$ is Legendrian near a & b

then we can choose the approximation near a & b to agree with γ .

case 3: $h: S^1 \hookrightarrow (\mathbb{R}^3, \zeta)$ a knot. S^1 compact $\xrightarrow{\text{Lebesgue}} \exists$ decomposition of S^1 into intervals I_i s.t. $h(I_i) \subset$ Darboux ball then use case 2.

Transverse case

Let $L : S^1 \longrightarrow (\mathbb{R}^3, g)$ be a Legendrian approximation of k .

Then $\omega \log L(t) = (\theta=t, x=0, y=0) \in (S^1 \times \mathbb{R}^2, \alpha = (\cos \theta dx - \sin \theta dy))$

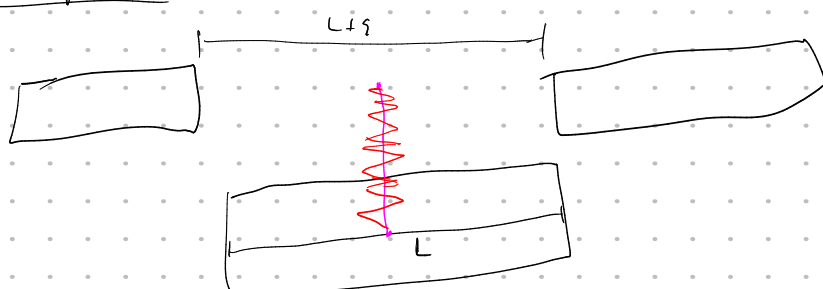
$$L_{\pm} := (\theta=t, x = \pm \epsilon \sin(t), y = \pm \epsilon \cos(t)) \quad \text{for } \epsilon > 0 \text{ small}$$

$$\Rightarrow L'_{\pm} := (1, \pm \epsilon (\cos(t), -\sin(t)))$$

$$\alpha(TL'_{\pm}) = \pm \epsilon (\cos^2(t) + \sin^2(t)) = \pm \epsilon$$

L_{\pm} are called transverse push-offs of L

real world application:



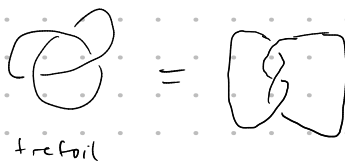
3.4 Seifert surfaces & the Alexander polynomial

Let $k \in S^3$ a smooth knot

Ex:



unknot



trefoil

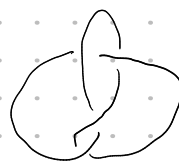


Fig-8

Lemma 5: $\forall k \subset S^3 \exists$ Seifert surface, i.e. $F^2 \xrightarrow{\text{surj}} S^3$ compact, oriented, s.t. $\partial F = k$

proof: (1) Let D be a diagram of k

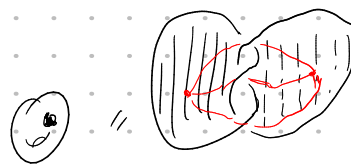
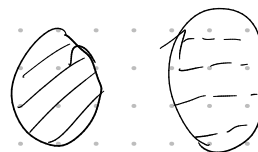
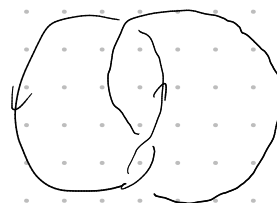
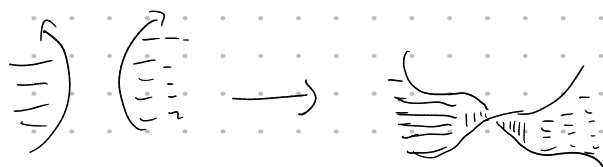
(2) orient D

(3) resolve crossings following orientation



(4) get collection of circles in the plane

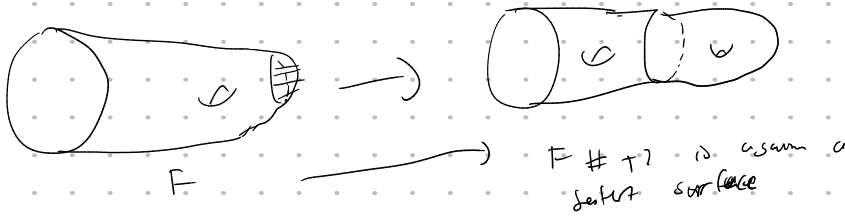
(5) glue in a twisted band for every crossing



D

Stabilization

F a Seifert surface of K



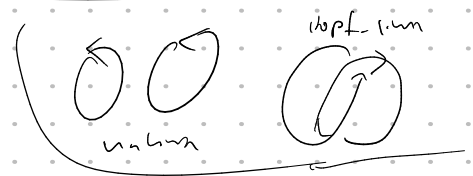
Thm 6 [Reidemeister - Singer] Any two Seifert surfaces of K have a common stabilization.

(Seifert)-genus: $g(K) := \min \{g(F) \mid F \text{ a Seifert surface of } K\}$

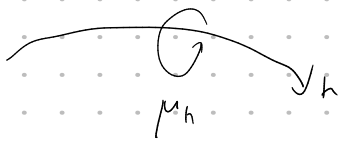
Ex: $g(K) = 0 \iff K = 0$

$g(\bigcirc) \leq 1$

Links: Let $K \subset M^3$ an oriented knot if K is nullhomologous
i.e. $[K] = 0 \in H_1(M)$

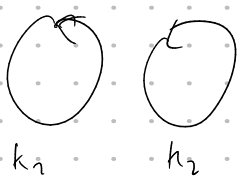


$\Rightarrow H_1(M \setminus \nu K) = \bigoplus_{\mu \in \mu(K)} H_1(M)$
linker modulo of K

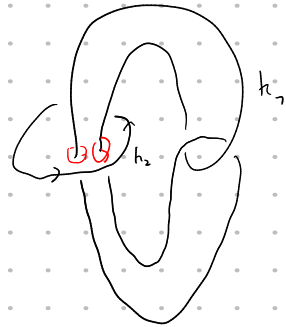


Let K_1, K_2 be oriented knots in M s.t. K_1 & K_2 are nullh.
 the linking number $lk(K_1, K_2) \in \mathbb{Z}$ is defined $[K_2] = lk(K_1, K_2) [K_1] \in H_1(M \setminus \nu K_1) \cong H_1(M) \oplus \mathbb{Z} \langle [K_1] \rangle$

$lk(K_1, -K_2) = lk(-K_1, K_2) = -lk(K_1, K_2)$



$[K_2] = -[K_1] \Rightarrow lk = -1$



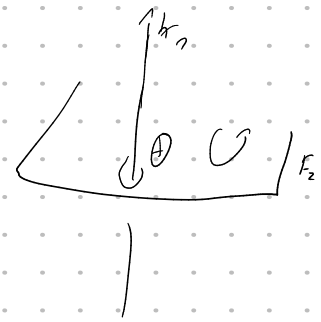
$[K_2] = -[K_1] + [K_1] = 0$

$lk = 0$

Lemma 7: (1) $h \subset S^3$ is nullhomologous $\Leftrightarrow h$ admits a Seifert surface

(2) $lk(h_1, h_2) = h_1 \cdot F_2$

— i.e. F_2 a Seifert surface of h_2



Proof: (1) " \Leftarrow " part of Lemma

" \Rightarrow " in S^3 Seifert alg.

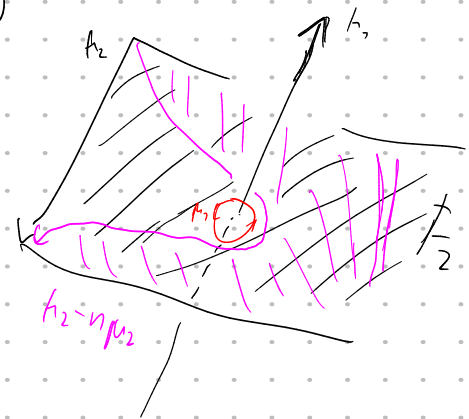
general case in discussion session

(2) Let $h_1 \cdot F_2 = n > 0$ (algebraic intersection number)

$h_2 - n \mu_1$ bounds a surface but does not intersect h_1

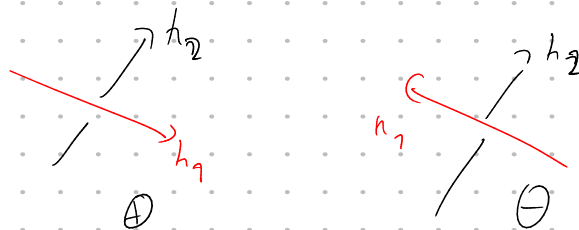
$lk(h_2 - n \mu_1, h_1) = 0 \in H_2(M \setminus h_1)$

$lk(h_1, h_2) = n = h_1 \cdot F_2$



Lemma 8 Let $h_1, h_2 \subset S^3$ be oriented knots

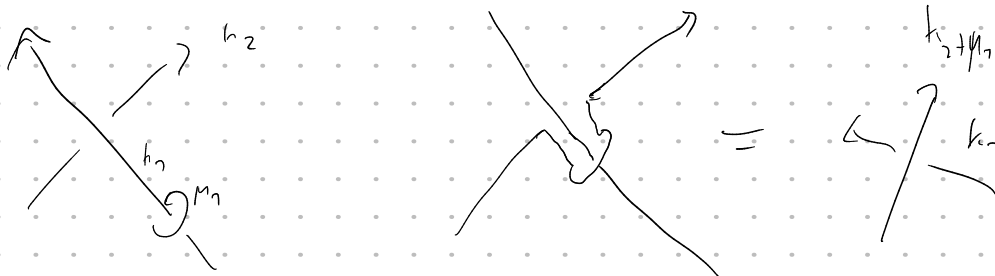
$\Rightarrow lk(h_1, h_2) = \#$ crossings of h_2 under h_1 with signs



Corollary 9: $lk(h_1, h_2) = lk(h_2, h_1)$ proof: "look at the diagrams from behind the blackboard"

Proof (Lemma 8): $lk(h_1, h_2 \pm \mu_1) = \pm 2$

$lk(h_1, h_2 \pm \mu_1) = lk(h_1, h_2) \pm 1$



$n = \# \text{ crossings of } k_2 \text{ under } k_1$

$\Rightarrow k_2 - n \mu_1$ has no undercrossings with k_1

$\Rightarrow \text{lk}(k_1, k_2 - n \mu_1) = 0 \quad \Rightarrow \quad \text{lk}(k_1, k_2) = n$

