Homotopically Standard Tight Non-fillable Contact Structures on the Sphere

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results by Bowden, Gironella, Moreno and Zhou

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Background

Contact topology

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Fillability: *fillable* contact mflds are boundaries of symplectic mflds.

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Eliashberg, Borman–Eliashberg–Murphy:

Dichotomy: Rigidity vs. Flexibility.

- **tight** *(rigid/geometric)*;
- **overtwisted** *(flexible/topological)*.

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Dichotomy: Rigidity vs. Flexibility.

- **tight** *(rigid/geometric)*;
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Theorem (Eliashberg–Gromov)

Fillable contact manifolds are tight.

Converse is false (Etnyre–Honda, Massot–Niederkrueger–Wendl).

Existence and classification

Topological obstruction: *almost* contact structure, i.e. reduction of structure group to $U(n) \times \mathbb{1}$.

Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))

Almost contact manifolds are contact, where the contact structure is overtwisted.

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Tight manifolds

How can we understand **tight** contact manifolds?

Contact topology: fillability

Hierarchy of fillability:

$$\{Stein\} \overset{①}{=} \{Weinstein\} \overset{②}{\subsetneq} \{Liouville\} \overset{③}{\subsetneq} \{strong\}$$

$$\overset{④}{\subsetneq} \{weak\} \overset{⑤}{\subsetneq} \{tight\}$$

- $dim = 3$: ① Cieliebak–Eliashberg, ② Bowden, ③ Ghiggini, ④ Eliashberg, ⑤ Etnyre–Honda.
- $dim \geqslant 5$: ① Cieliebak–Eliashberg, ② Bowden–Crowley–Stipsicz, ③ Zhou, ④ Bowden–Gironella–Moreno, ⑤ Massot–Niederkrüger–Wendl.

Contact structures on spheres

First step in classification: contact structures on spheres.

Standard contact structure

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Theorem (Eliashberg, '91)

On S^3, it is the unique tight contact structure.

In particular, no tight and non-fillable contact structures on S^3.

Tight and non-fillable structures in $\dim \geqslant 5$

Theorem (Bowden–Gironella–Moreno–Zhou '22-'24)

In $\dim \geqslant 7$, if M admits a tight structure, it also admits a tight and non strongly-fillable structure, in the same almost contact class.

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In $\dim \geqslant 7$*, if* M *admits a tight structure, it also admits a tight and non strongly-fillable structure, in the same almost contact class.*

In $\dim = 5$*, the same holds, if the first Chern class vanishes.*

Case of spheres

The general theorem follows by connected sum with an “exotic” sphere:

Theorem (Bowden–Gironella–Moreno–Zhou ’22-’24)

For every $n \geqslant 2$, the sphere $\mathbb{S}^{2n+1}$ admits a tight, non-fillable contact structure that is homotopically standard.

General remarks

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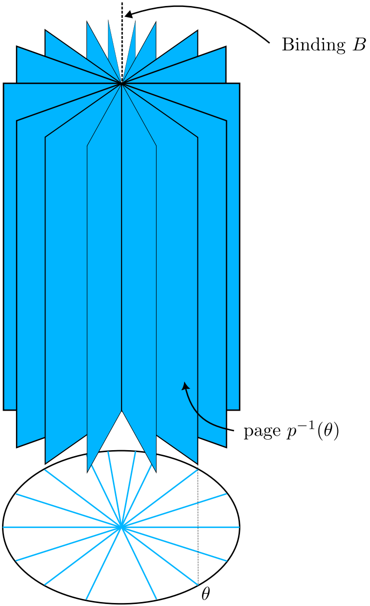
- Suggests that higher-dimensional contact phenomena should occur independently of underlying smooth topology.

Tight and non-fillable spheres

Open books



Giroux correspondence

Giroux: Contact structures are *supported* by open books.

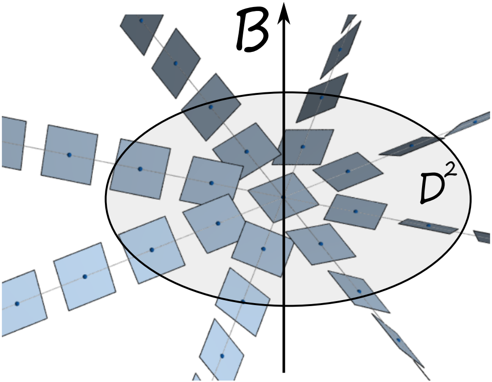


Figure: Supported contact structure.

Bourgeois contact structures

Theorem (Bourgeois '02)

Open book supporting $(M, \xi) \rightsquigarrow$ *contact structure on* $M \times \mathbb{T}^2$.

These are $\mathbb{T}^2$-equivariant.

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- Milnor open book on $\mathbb{S}^{2n-1}$ $\rightsquigarrow$ Bourgeois manifold on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ $\rightsquigarrow$ two 1-surgeries = $\mathbb{S}^{2n-1} \times \mathbb{S}^2$ $\rightsquigarrow$ one 2-surgery = $\mathbb{S}^{2n+1}$.

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Claim: $(\mathbb{S}^{2n+1}, \xi_{ex})$ is tight and non-fillable.

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Milnor open book $\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex})$ is *tight*.

Non-fillability

Non-fillability of $(\mathbb{S}^{2n+1}, \xi_{ex})$ can be proven via:

1. Homological obstruction and cobordisms as in [Bowden–Gironella–Moreno], building on [Massot–Niederkrüger–Wendl].
2. Symplectic cohomology computations as in [Zhou].

Homological obstructions

Observation: Bourgeois manifolds have convex decomposition

$$M \times \mathbb{T}^2 = (M \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V}_- \times \mathbb{S}^1,$$

with $V_\pm = \Sigma \times D^*\mathbb{S}^1$, $\Sigma =$ page of the open book, $\phi =$ monodromy.

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Theorem (Bowden–Gironella–Moreno)

$M \times T^2 = V \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V_-} \times \mathbb{S}^1$ with convex decomposition, $\Gamma = \partial V_\pm$ dividing set. If W is a symplectic filling of $M \times T^2$, then

$$H_*(\Gamma) \to H_*(V_\pm) \to H_*(W),$$

induced by inclusion. Then second map is injective on image of the first.

Namely, if a homology class in Γ survives in $V_\pm$, then it survives in the filling.

Idea of proof

- Capping cobordism from $M \times T^2$ to $\Gamma \times \mathbb{S}^2$, via handles $V_\pm \times D^2$ with co-core $C_\pm \simeq V_\pm$.

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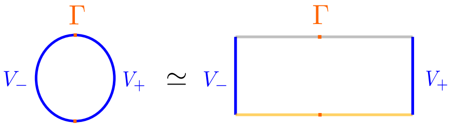

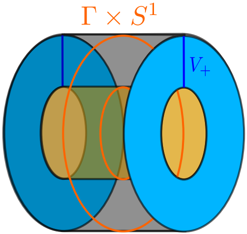

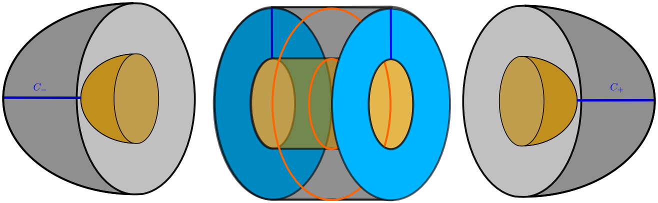

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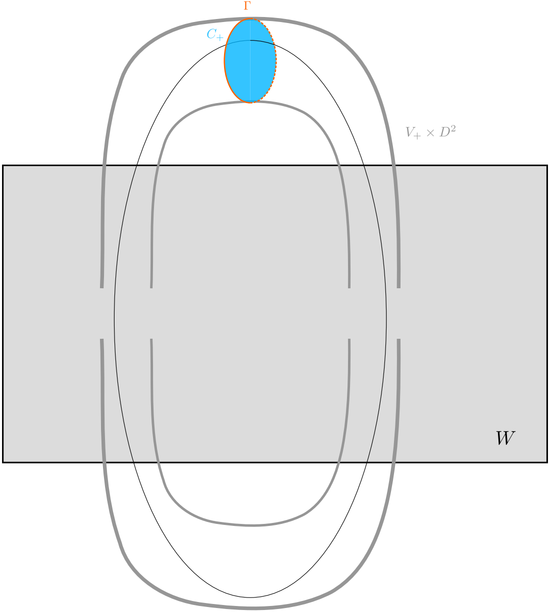

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- Second factor gives moduli space of spheres $\mathcal{M}_*$ with evaluation map $ev : \mathcal{M}_* \to W$.
- Spheres intersect $C_\pm$ precisely once $\rightsquigarrow$ intersection map $\mathcal{I}_\pm : \mathcal{M}_* \to C_\pm$.

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- Spheres intersect $C_\pm$ precisely once $\rightsquigarrow$ intersection map $\mathcal{I}_\pm : \mathcal{M}_* \to C_\pm$.
- If $\sigma \subset W$ satisfies $\partial\sigma = z$ with z cycle in Γ, then z also bounds $b = \mathcal{I}_\pm ev^{-1}(\sigma) \subset V_\pm$. $\square$

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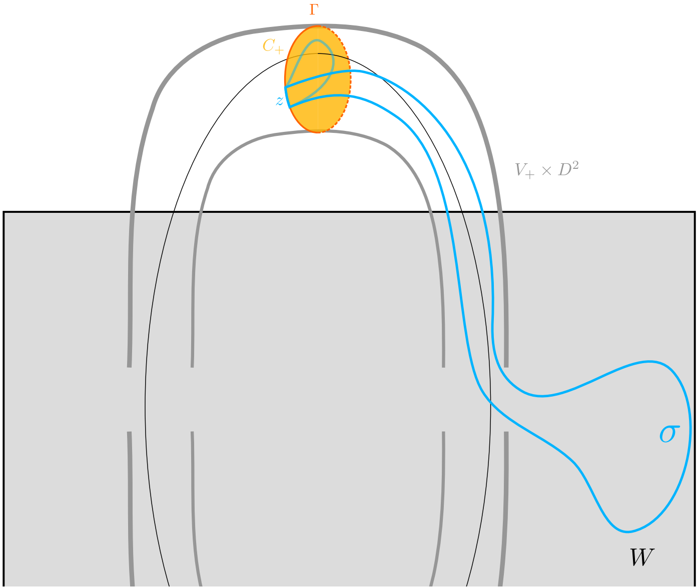

Homological obstructions

Fact:

1. If $\dim \geqslant 7$, subcritical surgeries on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ can be pushed away from dividing set to V_+.

$$\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex}) \text{ still has a dividing set } \Gamma,$$

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   $$\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex}) \text{ still has a dividing set } \Gamma,$$

   with $H_n(\Gamma) \neq 0$.
2. Homological obstruction theorem persists under surgery away from dividing set (capping cobordisms).

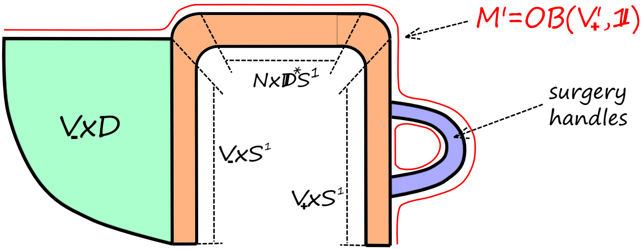

Figure: Capping cobordism.

End of the proof: W filling of $(\mathbb{S}^{2n+1}, \xi_{ex}) \Rightarrow$ Homological obstruction:

$$0 \neq H_n(N) \hookrightarrow H_n(W).$$

However, this factors as

$$0 \neq H_n(N) \to H_n(\mathbb{S}^{2n+1}) = 0 \to H_n(W),$$

contradiction.

Thank you!