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5. Übungsblatt

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1. Lösung:

$$\Phi(\boldsymbol{r}) = \int d^3r' \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{1}$$

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{|\boldsymbol{r}|} + r'_i \frac{\partial}{\partial r'_i} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\bigg|_{\boldsymbol{r}'=0} + \frac{1}{2!} r'_i r'_j \frac{\partial}{\partial r'_i}\frac{\partial}{\partial r'_j} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\bigg|_{\boldsymbol{r}'=0} + \dots \tag{2}$$

$$= \frac{1}{|\boldsymbol{r}|} - r'_i \frac{\partial}{\partial r_i} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\bigg|_{\boldsymbol{r}'=0} + \frac{1}{2!} r'_i r'_j \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\bigg|_{\boldsymbol{r}'=0} + \dots \tag{3}$$

$$= \frac{1}{r} - r'_i \frac{\partial}{\partial r_i}\frac{1}{r} + \frac{1}{2!} r'_i r'_j \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\frac{1}{r} + \dots \tag{4}$$

$$\text{Erste Ordnung:} \tag{5}$$

$$-r'_i \frac{\partial}{\partial r_i}\frac{1}{r} = \frac{r'_i r_i}{r^3} \tag{6}$$

$$\text{Zweite Ordnung:} \tag{7}$$

$$r'_i r'_j \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\frac{1}{r} = -r'_i r'_j \frac{\partial}{\partial r_i}\frac{r_j}{r^3} \tag{8}$$

$$= -r'_i r'_j \left(\frac{\delta_{ij}}{r^3} - \frac{3r_j}{r^4}\frac{r_i}{r}\right) \tag{9}$$

$$= r'_i r'_j \left(\frac{3r_i r_j}{r^5} - \frac{\delta_{ij}}{r^3}\right) \tag{10}$$

Damit ergibt sich für das Potential:

$$\Phi(\boldsymbol{r}) = \frac{1}{r}\int d^3r' \rho(\boldsymbol{r}') \tag{11}$$

$$+ \frac{r_i}{r^3}\int d^3r' r'_i \rho(\boldsymbol{r}') \tag{12}$$

$$+ \frac{1}{2}\left(\frac{3r_i r_j}{r^5} - \frac{\delta_{ij}}{r^3}\right)\int d^3r' r'_i r'_j \rho(\boldsymbol{r}') \tag{13}$$

Die Ausdrücke für Monopol und Dipol können direkt ersetzt werden:

$$\Phi(\boldsymbol{r}) = \frac{1}{r}Q \tag{14}$$
$$+ \frac{\boldsymbol{r}}{r^3}\boldsymbol{P} \tag{15}$$
$$+ \frac{1}{2}\left(\frac{3r_ir_j}{r^5} - \frac{\delta_{ij}}{r^3}\right)\int d^3r' r'_i r'_j \rho(\boldsymbol{r}') \tag{16}$$
$$\tag{17}$$

Unter Verwendung der Einstein'schen Summenkonvention können wir zeigen, dass:

$$(3r_ir_j - \delta_{ij}r^2)\delta_{ij} = 3r_ir_i - \delta_{ii}r^2 \tag{18}$$
$$= 3r^2 - 3r^2 \tag{19}$$
$$= 0 \tag{20}$$

Somit folgt

$$\left(\frac{3r_ir_j - \delta_{ij}r^2}{r^5}\right) r'_i r'_j = \left(\frac{3r_ir_j - \delta_{ij}r^2}{r^5}\right)(r'_i r'_j - \frac{\delta_{ij}}{3}r'^2) \tag{21}$$
$$\tag{22}$$

und

$$\Phi(\boldsymbol{r}) = \frac{1}{r}Q + \frac{\boldsymbol{r}}{r^3}\boldsymbol{P} \tag{23}$$
$$+ \frac{1}{2}\left(\frac{3r_ir_j}{r^5} - \frac{\delta_{ij}}{r^3}\right)\int d^3r' \rho(\boldsymbol{r}') \tag{24}$$
$$= \frac{1}{r}Q + \frac{\boldsymbol{r}}{r^3}\boldsymbol{P} \tag{25}$$
$$+ \frac{1}{2}\left(\frac{3r_ir_j}{r^5} - \frac{\delta_{ij}}{r^3}\right)\int d^3r' r'_i r'_j - \frac{\delta_{ij}}{3}r'^2 \rho(\boldsymbol{r}') \tag{26}$$
$$= \frac{1}{r}Q + \frac{\boldsymbol{r}}{r^3}\boldsymbol{P} + \frac{1}{2}\left(\frac{3r_ir_j}{r^5} - \frac{\delta_{ij}}{r^3}\right)\boldsymbol{q}_{ij} \tag{27}$$

Transformationen:
Q ist ein Skalar und damit invariant unter Transformationen. Man kann dies auch explizit zeigen via:

$$\rho(\boldsymbol{x})\mathrm{d}^3x \to \rho'(\boldsymbol{x}')\mathrm{d}^3x' = \rho(\boldsymbol{x})\frac{1}{\det J}\mathrm{d}^3x' = \rho(\boldsymbol{x})\mathrm{d}^3x \tag{28}$$

(a)

$$\boldsymbol{r} \to \boldsymbol{r}' = \boldsymbol{r} + a \tag{29}$$

$$P_i' = \int d^3r'\ \rho'(\boldsymbol{r}')(r_i + a_i) \tag{30}$$

$$= P_i + a_i Q \tag{31}$$

$$\boldsymbol{q}_{ij}' = \int d^3r'\ \rho'(\boldsymbol{r}')(r_i r_j - \frac{1}{3}|\boldsymbol{r}|^2\delta_{ij}) \tag{32}$$

$$+ \int d^3r'\ \rho'(\boldsymbol{r}')(a_i a_j - \frac{1}{3}|\boldsymbol{a}|^2\delta_{ij}) \tag{33}$$

$$+ \int d^3r'\ \rho'(\boldsymbol{r}')(a_i r_j + r_i a_j - \frac{2}{3}\boldsymbol{a}\boldsymbol{r}\delta_{ij}) \tag{34}$$

$$= \boldsymbol{q}_{ij} + (a_i a_j - \frac{1}{3}|\boldsymbol{a}|^2\delta_{ij})Q \tag{35}$$

$$+ a_i P_j + a_j P_i - \frac{2}{3}\boldsymbol{a}\boldsymbol{P}\delta_{ij} \tag{36}$$

(b)

$$\boldsymbol{r} \to \boldsymbol{r}' = -\boldsymbol{r} \tag{37}$$

$$P_i' = \int d^3r'\ \rho'(\boldsymbol{r}')(-r_i) \tag{38}$$

$$= -P_i \tag{39}$$

$$\boldsymbol{q}_{ij}' = \int d^3r'\ \rho'(\boldsymbol{r}')((-r_i)(-r_j) - \frac{1}{3}|-\boldsymbol{r}|^2\delta_{ij}) \tag{40}$$

$$= \int d^3r'\ \rho'(\boldsymbol{r}')(r_i r_j - \frac{1}{3}|\boldsymbol{r}|^2\delta_{ij}) \tag{41}$$

$$= \boldsymbol{q}_{ij} \tag{42}$$

(c)

$$r_i \to r_i' = D_{ij} r_j \tag{43}$$

$$P_i' = \int d^3r'\ \rho'(\boldsymbol{r}') D_{ij} r_j \tag{44}$$

$$= D_{ij} P_j \tag{45}$$

$$\boldsymbol{q}_{ij}' = \int d^3r'\ \rho'(\boldsymbol{r}')(D_{ik} r_k D_{jl} r_l - \frac{1}{3}|\boldsymbol{r}|^2\delta_{ij}) \tag{46}$$

$$= \int d^3r'\ \rho'(\boldsymbol{r}')(D_{ik} r_k D_{jl} r_l - \frac{1}{3}|\boldsymbol{r}|^2 D_{ik}\delta_{kl}D_{jl}) \tag{47}$$

$$= D_{ik} D_{jl} \boldsymbol{q}_{kl} \tag{48}$$

2. Lösung:

$$\Phi_0(\boldsymbol{r}) = \frac{1}{r}\int_{-L}^{+L} \mathrm{d}x' \int_{-L}^{+L} \mathrm{d}y' \int_{-L}^{+L} \mathrm{d}z' \rho \tag{49}$$
$$= \rho\frac{8L^3}{r} \tag{50}$$
$$= \frac{Q}{r} \leftarrow \rho = \frac{Q}{8L^3} \tag{51}$$

$$\Phi_i(\boldsymbol{r}) = \frac{r_i}{r^3}\int_{-L}^{+L} \mathrm{d}x' \int_{-L}^{+L} \mathrm{d}y' \int_{-L}^{+L} \mathrm{d}z' r_i' \rho \tag{52}$$
$$= 0 \leftarrow r_i' \text{ ist eine ungerade Funktion} \tag{53}$$

$$\Phi_{ij}(\boldsymbol{r}) = \frac{1}{2}\left(\frac{3r_ir_j}{r^5} - \frac{\delta_{ij}}{r^3}\right)\int_{-L}^{+L} \mathrm{d}x' \int_{-L}^{+L} \mathrm{d}y' \int_{-L}^{+L} \mathrm{d}z' (3r_i'r_j' - r'^2\delta_{ij})\rho \tag{54}$$

$$\text{Für } i \neq j : \Phi_{ij}(\boldsymbol{r}) = \frac{1}{2}\left(\frac{3r_ir_j}{r^5} - \frac{\delta_{ij}}{r^3}\right)\int_{-L}^{+L} \mathrm{d}x' \int_{-L}^{+L} \mathrm{d}y' \int_{-L}^{+L} \mathrm{d}z' 3r_i'r_j'\rho \tag{55}$$
$$= 0 \tag{56}$$

$$\text{Für } i = j : \Phi_{ii}(\boldsymbol{r}) = \frac{1}{2}\left(\frac{3r_i^2}{r^5} - \frac{1}{r^3}\right)\int_{-L}^{+L} \mathrm{d}x' \int_{-L}^{+L} \mathrm{d}y' \int_{-L}^{+L} \mathrm{d}z' (3r_i'^2 - r'^2)\rho \tag{57}$$
$$i \neq k \neq l \rightarrow = \frac{1}{2}\left(\frac{3r_i^2}{r^5} - \frac{1}{r^3}\right)\int_{-L}^{+L} \mathrm{d}x' \int_{-L}^{+L} \mathrm{d}y' \int_{-L}^{+L} \mathrm{d}z' (2r_i'^2 - r_k'^2 - r_l'^2)\rho \tag{58}$$
$$= 0 \tag{59}$$

3. Lösung:

$$\boldsymbol{X} = \boldsymbol{E} + i\boldsymbol{B} \tag{60}$$

(a)

$$\boldsymbol{\nabla}\boldsymbol{X} = 4\pi\rho \tag{61}$$
$$\mathrm{Re}(\boldsymbol{\nabla}\boldsymbol{X}) = \boldsymbol{\nabla}\boldsymbol{E} = 4\pi\rho \tag{62}$$
$$\mathrm{Im}(\boldsymbol{\nabla}\times\boldsymbol{X}) = \boldsymbol{\nabla}\boldsymbol{B} = 0 \tag{63}$$

(b)

$$\boldsymbol{\nabla}\times\boldsymbol{X} = i\partial_{ct}\boldsymbol{X} + \boldsymbol{y} \tag{64}$$
$$\mathrm{Re}(\boldsymbol{\nabla}\times\boldsymbol{X}) = \boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_{ct}\boldsymbol{B} + \mathrm{Re}(\boldsymbol{y}) \tag{65}$$
$$\mathrm{Im}(\boldsymbol{\nabla}\times\boldsymbol{X}) = \boldsymbol{\nabla}\times\boldsymbol{B} = +\partial_{ct}\boldsymbol{E} + \mathrm{Im}(\boldsymbol{y}) \tag{66}$$
$$\rightarrow \boldsymbol{y} = \frac{4\pi}{c}i\boldsymbol{\jmath} \tag{67}$$

(c)

$$\boldsymbol{\nabla}\boldsymbol{\nabla}\times\boldsymbol{X} = 0 \tag{68}$$
$$= \boldsymbol{\nabla}(i\partial_{ct}\boldsymbol{X}) + \frac{4\pi}{c}i\boldsymbol{\nabla}\boldsymbol{\jmath} \tag{69}$$
$$= i(\partial_{ct}(4\pi\rho) + \frac{4\pi}{c}\boldsymbol{\nabla}\boldsymbol{\jmath}) \tag{70}$$
$$\rightarrow 0 = \partial_t\rho + \boldsymbol{\nabla}\boldsymbol{\jmath} \tag{71}$$

$$Q_v = \int_V \mathrm{d}^3r\ \rho \tag{72}$$

$$\frac{\mathrm{d}Q_v}{\mathrm{d}t} = \int \mathrm{d}^3x\ \partial_t\rho = -\int_V \mathrm{d}^3x\ \boldsymbol{\nabla}\cdot\boldsymbol{\jmath} = -\int_{\partial V}\mathrm{d}S\ \boldsymbol{\jmath}\cdot\boldsymbol{n} \tag{73}$$

Die Ladung ändert sich nur mit der Zeit, wenn ein Strom aus dem von ∂V umschlossenen Volumen abfließt. Andernfalls ($\boldsymbol{\jmath} = 0$) ist die Ladung in V erhalten.

(d)

$$\boldsymbol{\nabla}\boldsymbol{X} = \boldsymbol{\nabla}\boldsymbol{E} + i\boldsymbol{\nabla}\boldsymbol{B} \tag{74}$$
$$= -\boldsymbol{\nabla}\boldsymbol{\nabla}\Phi - \partial_{ct}\boldsymbol{\nabla}\boldsymbol{A} + i(\boldsymbol{\nabla}\boldsymbol{\nabla}\times\boldsymbol{A}) \tag{75}$$
$$= -\Delta\Phi + \partial_{ct}^2\Phi \qquad \leftarrow \text{Lorentz Eichung} \tag{76}$$
$$= 4\pi\rho \tag{77}$$

4. Lösung: Es gilt

$$\boldsymbol{V}(\boldsymbol{x}) = -\frac{1}{4\pi}\int \mathrm{d}^3x'\ \frac{\boldsymbol{F}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{78}$$
$$\Delta\boldsymbol{V}(\boldsymbol{x}) = -\frac{1}{4\pi}\Delta\int \mathrm{d}^3x'\ \frac{\boldsymbol{F}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{79}$$
$$= -\frac{1}{4\pi}\int \mathrm{d}^3x'\ \boldsymbol{F}(\boldsymbol{x}')\Delta\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{80}$$
$$= -\frac{1}{4\pi}\int \mathrm{d}^3x'\ \boldsymbol{F}(\boldsymbol{x}')(-4\pi)\delta^3(\boldsymbol{x}-\boldsymbol{x}') \tag{81}$$
$$= \boldsymbol{F}(\boldsymbol{x}) \tag{82}$$

Damit kann man schreiben

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{\nabla}\Phi(\boldsymbol{x}) + \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{x}) \tag{83}$$

mit

$$\Phi(\boldsymbol{x}) = -\boldsymbol{\nabla}\boldsymbol{V}(\boldsymbol{x}) \tag{84}$$
$$\boldsymbol{A}(\boldsymbol{x}) = -\boldsymbol{\nabla}\times\boldsymbol{V}(\boldsymbol{x}) \tag{85}$$

da

$$\Delta\boldsymbol{V}(\boldsymbol{x}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{V}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{V}) \tag{86}$$