Homotopically Standard Tight Non-fillable Contact Structures on the Sphere

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results by Bowden, Gironella, Moreno and Zhou

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Background

Contact topology

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Fillability: *fillable* contact mflds are boundaries of symplectic mflds.

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Eliashberg, Borman–Eliashberg–Murphy:

Dichotomy: Rigidity vs. Flexibility.

- **tight** *(rigid/geometric)*;
- **overtwisted** *(flexible/topological)*.

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Dichotomy: Rigidity vs. Flexibility.

- **tight** *(rigid/geometric)*;
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Theorem (Eliashberg–Gromov)

Fillable contact manifolds are tight.

Converse is false (Etnyre–Honda, Massot–Niederkrueger–Wendl).

Existence and classification

Topological obstruction: *almost* contact structure, i.e. reduction of structure group to $U(n) \times \mathbb{1}$.

Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))

Almost contact manifolds are contact, where the contact structure is overtwisted.

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Tight manifolds

How can we understand **tight** contact manifolds?

Contact topology: fillability

Hierarchy of fillability:

$$\{Stein\} \overset{①}{=} \{Weinstein\} \overset{②}{\subsetneq} \{Liouville\} \overset{③}{\subsetneq} \{strong\}$$

$$\overset{④}{\subsetneq} \{weak\} \overset{⑤}{\subsetneq} \{tight\}$$

- $dim = 3$: ① Cieliebak–Eliashberg, ② Bowden, ③ Ghiggini, ④ Eliashberg, ⑤ Etnyre–Honda.
- $dim \geqslant 5$: ① Cieliebak–Eliashberg, ② Bowden–Crowley–Stipsicz, ③ Zhou, ④ Bowden–Gironella–M., ⑤ Massot–Niederkrüger–Wendl.

Contact structures on spheres

First step in classification: contact structures on spheres.

Standard contact structure

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Theorem (Eliashberg, '91)

On S^3, it is the unique tight contact structure.

In particular, no tight and non-fillable contact structures on S^3.

Tight and non-fillable structures in $\dim \geqslant 5$

Theorem (Bowden–Gironella–M.–Zhou '22-'24)

In $\dim \geqslant 7$, if M admits a tight structure, it also admits a tight and non strongly-fillable structure, in the same almost contact class.

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In $\dim \geqslant 7$*, if* M *admits a tight structure, it also admits a tight and non strongly-fillable structure, in the same almost contact class.*

In $\dim = 5$*, the same holds, if the first Chern class vanishes.*

Case of spheres

The general theorem follows by connected sum with an “exotic” sphere:

Theorem (Bowden–Gironella–M.–Zhou ’22-’24)

For every $n \geqslant 2$, the sphere $\mathbb{S}^{2n+1}$ admits a tight, non-fillable contact structure that is homotopically standard.

General remarks

- This is a novel and strictly higher-dimensional phenomenon (false in dim 3).

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- Suggests that higher-dimensional contact phenomena should occur independently of underlying smooth topology.

Tight and non-fillable spheres

Giroux correspondence

Giroux: Contact structures are *supported* by open books.

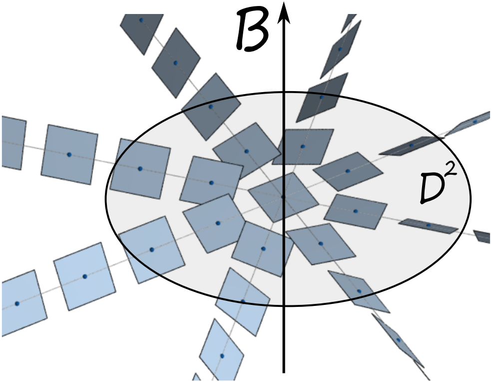

Figure: Supported contact structure.

Bourgeois contact structures

Theorem (Bourgeois '02)

Open book supporting $(M, \xi) \rightsquigarrow$ *contact structure on* $M \times \mathbb{T}^2$.

These are $\mathbb{T}^2$-equivariant.

Geometric construction

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- Milnor open book on $\mathbb{S}^{2n-1}$ $\rightsquigarrow$ Bourgeois manifold on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ $\rightsquigarrow$ two 1-surgeries = $\mathbb{S}^{2n-1} \times \mathbb{S}^2$ $\rightsquigarrow$ one 2-surgery = $\mathbb{S}^{2n+1}$.

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Claim: $(\mathbb{S}^{2n+1}, \xi_{ex})$ is tight and non-fillable.

Tightness and fillability from algebraic perspective

Contact homology algebra $CHA(Y)$ (homology well-defined by Pardon).

Definition

1. Y is *algebraically* tight if $CHA(Y) \neq 0$.
2. Y is *algebraically* fillable if there is a DGA augmentation of $CHA(Y)$ at the chain level.

Similarly for *algebraically* overtwisted/non-fillable.

Note: This definition is well-defined, due to functoriality of the DGA, even though homotopy type of chain level is not.

Formal algebraic properties

Lemma

1. *Algebraically tight $\Rightarrow$ tight.*
2. *Algebraically fillable $\Rightarrow$ algebraically tight.*
3. *Algebraically non-fillable $\Rightarrow$ non-fillable.*
4. 1*-ADC $\Rightarrow$ algebraically tight.*

1-ADC is an *index-positivity* condition (Lazarev, Zhou).

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Facts:

1. (Advek '22) tight contact manifolds can be algebraically overtwisted, in dim 3.
2. Algebraic tightness is preserved under surgeries. Tightness is also, in dim 3 (Wand '14).
3. 1-ADC binding of fillable open book $\Rightarrow$ 1-ADC algebraically fillable Bourgeois manifold $\Rightarrow$ algebraically tight.
4. E.g. Milnor A_k-singularity open book has 1-ADC binding.

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Note: Heuristically, there is a priori *many* choices of open book. This suggests *many* non-standard structures. However, distinguishing is subtle.

Non-fillability

Non-fillability of $(\mathbb{S}^{2n+1}, \xi_{ex})$ can be proven via:

1. Homological obstruction and cobordisms as in [Bowden–Gironella–M.], building on [Massot–Niederkrüger–Wendl].
2. Symplectic cohomology computations as in [Zhou].

Homological obstructions

Observation: Bourgeois manifolds have convex decomposition

$$M \times \mathbb{T}^2 = (M \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V}_- \times \mathbb{S}^1,$$

with $V_\pm = \Sigma \times D^*\mathbb{S}^1$, $\Sigma =$ page of the open book, $\phi =$ monodromy.

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Theorem (Bowden–Gironella–M.)

$M = V \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V_-} \times \mathbb{S}^1$ with convex decomposition, $N = \partial V_\pm$ dividing set. If W is a symplectic filling of M, then

$$H_*(N) \to H_*(V_\pm) \to H_*(W),$$

induced by inclusion. Then second map is injective on image of the first.

Namely, if a homology class in N survives in $V_\pm$, then it survives in the filling.

Idea of proof

- Capping cobordism from M to $N \times \mathbb{S}^2$ with a SHS, via handles $H_\pm$ with co-core $V_\pm$.

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- Spheres intersect $H_\pm$ precisely once $\rightsquigarrow$ intersection map $\mathcal{I}_\pm : \mathcal{M}_* \to V_\pm$.
- If $\sigma \subset W$ satisfies $\partial\sigma = c$ with c cycle in N, then $b = \mathcal{I}_\pm ev^{-1}(\sigma)$ bounds σ in $V_\pm$. $\square$

Homological obstructions

Fact:

1. If $\dim \geqslant 7$, subcritical surgeries on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ can be pushed away from dividing set to V_+.

$$\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex}) \text{ still has a dividing set } N,$$

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$$\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex}) \text{ still has a dividing set } N,$$
with $H_n(N) \neq 0$.
2. Homological obstruction theorem persists under surgery away from dividing set (capping cobordisms).

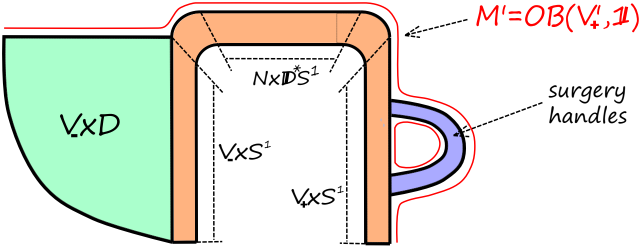

Figure: Capping cobordism.

End of the proof: W filling of $(\mathbb{S}^{2n+1}, \xi_{ex}) \Rightarrow$ Homological obstruction:

$$0 \neq H_n(N) \hookrightarrow H_n(W).$$

However, this factors as

$$0 \neq H_n(N) \to H_n(\mathbb{S}^{2n+1}) = 0 \to H_n(W),$$

contradiction.

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1. **Homotopically standard:** Fixed ξ, Bourgeois manifolds have same almost contact class $\xi \oplus T\mathbb{T}^2$, so suffices with trivial open book. *h*-cobordism theorem gives standard smooth topology on sphere.
2. **Dimension** 5**:** Needs careful *flexible* version of the homological obstruction theorem.
3. **Symplectic cohomology:** Capping cobordisms reach $\partial(V \times \mathbb{D}^2)$. Zhou's computations of $SH_+(\partial(V \times \mathbb{D}^2))$ and SH_+ computations of Brieskorn spheres as by [Kwon–van-Koert] can be used.

Thank you!