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# 3. Übungsblatt

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## 1. Lösung:

(a) Für $q = 1$ folgt aus (1)

$$\Delta\Phi = -4\pi\delta(\boldsymbol{x}) \tag{1}$$
$$\text{mit } \Phi = \frac{1}{r} = \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{2}$$

Damit gilt für die Komponenten von $\boldsymbol{A}(\boldsymbol{x})$ :

$$A_i(\boldsymbol{x}) = \frac{1}{4\pi}\int \mathrm{d}^3x'\, \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} w_i(\boldsymbol{x}') \tag{3}$$
$$\Delta A_i(\boldsymbol{x}) = \frac{1}{4\pi}\int \mathrm{d}^3x'\, \Delta\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} w_i(\boldsymbol{x}') \tag{4}$$
$$= \frac{1}{4\pi}\int \mathrm{d}^3x'\, -4\pi\delta^3(\boldsymbol{x}-\boldsymbol{x}') w_i(\boldsymbol{x}') \tag{5}$$
$$= -w_i(\boldsymbol{x}) \tag{6}$$
$$\rightarrow \Delta\boldsymbol{A}(\boldsymbol{x}) = -\boldsymbol{w}(\boldsymbol{x}) \tag{7}$$

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \tag{8}$$
$$= \frac{1}{4\pi}\int \mathrm{d}^3x'\, \boldsymbol{\nabla}\times(G(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{w}(\boldsymbol{x}')) \tag{9}$$
$$= \frac{1}{4\pi}\int \mathrm{d}^3x'\, G(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{\nabla}\times\boldsymbol{w}(\boldsymbol{x}') - \boldsymbol{w}(\boldsymbol{x}')\times\boldsymbol{\nabla}G(\boldsymbol{x},\boldsymbol{x}') \tag{10}$$
$$= -\frac{1}{4\pi}\int \mathrm{d}^3x'\, \boldsymbol{w}(\boldsymbol{x}')\times\boldsymbol{\nabla}G(\boldsymbol{x},\boldsymbol{x}') \tag{11}$$
$$= \frac{1}{4\pi}\int \mathrm{d}^3x'\, \frac{\boldsymbol{w}(\boldsymbol{x}')\times(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \tag{12}$$

(b)

$$\boldsymbol{\nabla} \boldsymbol{A} = \boldsymbol{\nabla} \left[ -\frac{1}{2} (\boldsymbol{r} \times \boldsymbol{B}) \right] \tag{13}$$
$$= -\frac{1}{2} \left[ \boldsymbol{b} (\boldsymbol{\nabla} \times \boldsymbol{r}) - \boldsymbol{r} (\boldsymbol{\nabla} \times \boldsymbol{B}) \right] \tag{14}$$
$$= -\boldsymbol{r} (\boldsymbol{\nabla} \times \boldsymbol{B}) \tag{15}$$
$$= 0 \qquad \leftarrow \text{ da } \boldsymbol{B} = B_0 \hat{\boldsymbol{e}}_i \text{ uniform ist.} \tag{16}$$

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times \left[ -\frac{1}{2}) \boldsymbol{r} \times \boldsymbol{B}) \right] \tag{17}$$
$$= \frac{1}{2} [(\boldsymbol{B} \boldsymbol{\nabla}) \boldsymbol{r} - (\boldsymbol{r} \boldsymbol{\nabla}) \boldsymbol{B} + \boldsymbol{r} (\boldsymbol{\nabla} \boldsymbol{B}) - \boldsymbol{B} (\boldsymbol{\nabla} \boldsymbol{r})] \tag{18}$$
$$= \frac{1}{2} [3 \boldsymbol{B} - \boldsymbol{B}] \tag{19}$$
$$= \boldsymbol{B} \tag{20}$$

Hier wurde wieder verwendet, dass $\boldsymbol{b}$ uniform ist sodass $\boldsymbol{\nabla} \boldsymbol{B} = 0$ und $(\boldsymbol{r} \boldsymbol{\nabla}) \boldsymbol{B} = 0$. Außerdem ist $\boldsymbol{\nabla} \boldsymbol{r} = 3$ und

$$(\boldsymbol{B} \boldsymbol{\nabla}) \boldsymbol{r} = B_i \partial_i r_j \hat{\boldsymbol{e}}_j \tag{21}$$
$$= B_i \hat{\boldsymbol{e}}_i \tag{22}$$
$$= \boldsymbol{B}. \tag{23}$$

Das Potential ist nicht eindeutig. $\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla} f$ löst ebenfalls die angegebenen Gleichungen, falls $\Delta f = 0$

**2. Lösung:**

(a)

$$\mathcal{F} [\alpha f(x) + \beta g(x); k] = \int \mathrm{d}x \, e^{-ikx} (\alpha f(x) + \beta g(x)) \tag{24}$$
$$= \alpha \int \mathrm{d}x \, e^{-ikx} f(x) + \beta \int \mathrm{d}x \, e^{-ikx} g(x) \tag{25}$$
$$= \alpha \mathcal{F} [f(x); k] + \beta \mathcal{F} [g(x); k] \tag{26}$$

(b)

$$\mathcal{F} [\alpha f(x-a); k] = \int \mathrm{d}x \, e^{-ikx} f(x-a) \tag{27}$$
$$= e^{-ikxa} \int \mathrm{d}x' \, e^{-ikx'} f(x') \tag{28}$$
$$= e^{-ikxa} \mathcal{F} [f(x); k] \tag{29}$$

(c)

$$\mathcal{F}\left[f(ax);k\right] = \int \mathrm{d}x\, e^{-ikx} f(ax) \tag{30}$$
$$= \int \frac{\mathrm{d}x'}{a}\, e^{-\dfrac{ikx'}{a}} f(x') \tag{31}$$
$$= \frac{1}{a}\mathcal{F}\left[f(x);k/a\right] \tag{32}$$

(d)

$$\mathcal{F}\left[f(-x);k\right] = \int \mathrm{d}x\, e^{-ikx} f(-x) \tag{33}$$
$$= \int \mathrm{d}x'\, e^{ikx'} f(x') \tag{34}$$
$$= \mathcal{F}\left[f(x);-k\right] \tag{35}$$

(e)

$$\mathcal{F}\left[f'(x);k\right] = \int \mathrm{d}x\, e^{-ikx} f'(x) \tag{36}$$
$$= -\int \mathrm{d}x\, \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-ikx}\right) f(x) \tag{37}$$
$$= ik \int \mathrm{d}x\, e^{-ikx} f(x) \tag{38}$$
$$= ik\mathcal{F}\left[f(x);k\right] \tag{39}$$

(f)

$$\mathcal{F}\left[xf(x);k\right] = \int \mathrm{d}x\, e^{-ikx} x f(x) \tag{40}$$
$$= i\frac{\mathrm{d}}{\mathrm{d}k} \int \mathrm{d}x\, e^{-ikx} f(x) \tag{41}$$
$$= i\frac{\mathrm{d}}{\mathrm{d}k}\mathcal{F}\left[f(x);k\right] \tag{42}$$

(g)

$$\mathcal{F}\left[f(x);-k\right] = \int \mathrm{d}x\, e^{ikx} f(x) \tag{43}$$
$$= \left[\int \mathrm{d}x\, e^{-ikx} f(x)\right]^* \tag{44}$$
$$= \mathcal{F}\left[f(x);k\right]^* \tag{45}$$

**3. Lösung:**

Die konsistente Definition der Fouriertransformation ist

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t}\, f(t) \tag{46}$$
$$\tilde{f}(\boldsymbol{k}) = \int_{-\infty}^{\infty} \mathrm{d}^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, f(\boldsymbol{x}) \tag{47}$$
$$\tilde{f}(\omega, \boldsymbol{k}) = \int_{-\infty}^{\infty} \mathrm{d}^3x\mathrm{d}t\, e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\, f(t, \boldsymbol{x}) \tag{48}$$

und die Rücktransformation ist

$$f(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, e^{-i\omega t}\tilde{f}(\omega) \tag{49}$$
$$f(\boldsymbol{x}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^3k}{(2\pi)^3}\, e^{-i\boldsymbol{k}\boldsymbol{x}}\tilde{f}(\boldsymbol{k}) \tag{50}$$
$$f(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^3k\mathrm{d}t}{(2\pi)^4}\, e^{-i\boldsymbol{k}\boldsymbol{x}-\omega t}\tilde{f}(\boldsymbol{k}, \omega) \tag{51}$$

(a) I

$$\nabla \boldsymbol{E}(\boldsymbol{x}, t) = 4\pi\rho(\boldsymbol{x}, t) \tag{52}$$

$$\partial_i \int \frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4} e^{i(\boldsymbol{k}\boldsymbol{x}-\omega t)}\tilde{\boldsymbol{E}}_i(\boldsymbol{k}, \omega) = 4\pi \int \frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4} e^{i(\boldsymbol{k}\boldsymbol{x}-\omega t)}\tilde{\rho}(\boldsymbol{k}, \omega) \tag{53}$$
$$\int \frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4} e^{i(\boldsymbol{k}\boldsymbol{x}-\omega t)} ik_i\tilde{\boldsymbol{E}}_i(\boldsymbol{k}, \omega) = 4\pi \int \frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4} e^{i(\boldsymbol{k}\boldsymbol{x}-\omega t)}\tilde{\rho}(\boldsymbol{k}, \omega) \tag{54}$$

Da die Fouriertransformation invertierbar ist, folgt:

$$\boldsymbol{k}\tilde{\boldsymbol{E}}(\boldsymbol{k}, \omega) = -i4\pi\tilde{\rho}(\boldsymbol{k}, \omega) \tag{55}$$

II

$$\nabla \times \boldsymbol{E}(\boldsymbol{x}, t) = -\frac{1}{c}\partial_t \boldsymbol{B}(\boldsymbol{x}, t) \tag{56}$$

$$\epsilon_{ijk}\partial_j \int \frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4} e^{i(\boldsymbol{k}\boldsymbol{x}-\omega t)}\tilde{\boldsymbol{E}}_i(\boldsymbol{k}, \omega) = -\frac{1}{c}\partial_t \int \frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4} e^{i(\boldsymbol{k}\boldsymbol{x}-\omega t)}\tilde{\boldsymbol{B}}_i(\boldsymbol{k}, \omega) \tag{57}$$
$$\int \frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4} e^{i(\boldsymbol{k}\boldsymbol{x}-\omega t)}\epsilon_{ijk} ik_j\tilde{\boldsymbol{E}}_i(\boldsymbol{k}, \omega) = \frac{i\omega}{c}\partial_t \int \frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4} e^{i(\boldsymbol{k}\boldsymbol{x}-\omega t)}\tilde{\boldsymbol{B}}_i(\boldsymbol{k}, \omega) \tag{58}$$

$$\boldsymbol{k} \times \tilde{\boldsymbol{E}}(\boldsymbol{k}, \omega) = \frac{\omega}{c}\tilde{\boldsymbol{B}}(\boldsymbol{k}, \omega) \tag{59}$$

III Es folgt analog:

$$\nabla \boldsymbol{B}(\boldsymbol{x}, t) = 0 \tag{60}$$
$$\boldsymbol{k}\tilde{\boldsymbol{B}}(\boldsymbol{k}, \omega) = 0 \tag{61}$$

IV

$$\nabla \times \boldsymbol{B}(\boldsymbol{x},t) = \frac{1}{c}\partial_t \boldsymbol{E}(\boldsymbol{x},t) + \frac{4\pi}{c}\boldsymbol{\jmath}(\boldsymbol{x},t) \tag{62}$$

$$\boldsymbol{k} \times \tilde{\boldsymbol{B}}(\boldsymbol{k},\omega) = -\frac{\omega}{c}\tilde{\boldsymbol{E}}(\boldsymbol{k},\omega) - i\frac{4\pi}{c}\tilde{\boldsymbol{\jmath}}(\boldsymbol{k},\omega) \tag{63}$$

(b) Ausgehend von IV ergibt sich:

$$\boldsymbol{k} \times \tilde{\boldsymbol{B}}(\boldsymbol{k},\omega) = -\frac{\omega}{c}\tilde{\boldsymbol{E}}(\boldsymbol{k},\omega) - i\frac{4\pi}{c}\tilde{\boldsymbol{\jmath}}(\boldsymbol{k},\omega) \tag{64}$$

$$\boldsymbol{k} \cdot (\boldsymbol{k} \times \tilde{\boldsymbol{B}}(\boldsymbol{k},\omega)) = -\frac{\omega}{c}\boldsymbol{k} \cdot \tilde{\boldsymbol{E}}(\boldsymbol{k},\omega) - i\frac{4\pi}{c}\boldsymbol{k} \cdot \tilde{\boldsymbol{\jmath}}(\boldsymbol{k},\omega) \tag{65}$$

$$(\boldsymbol{k} \times \boldsymbol{k}) \cdot \tilde{\boldsymbol{B}}(\boldsymbol{k},\omega) = -\frac{\omega}{c}\tilde{(}-i4\pi)\rho(\boldsymbol{k},\omega) - i\frac{4\pi}{c}\boldsymbol{k} \cdot \tilde{\boldsymbol{\jmath}}(\boldsymbol{k},\omega) \tag{66}$$

$$0 = -\omega\tilde{\rho}(\boldsymbol{k},\omega) + \boldsymbol{k} \cdot \tilde{\boldsymbol{\jmath}}(\boldsymbol{k},\omega) \tag{67}$$

$$\omega\tilde{\rho}(\boldsymbol{k},\omega) = \boldsymbol{k} \cdot \tilde{\boldsymbol{\jmath}}(\boldsymbol{k},\omega) \tag{68}$$