6. Symplectic fillings

6.1 Symplectic manifolds (dim = 4)

Let W be a 4-mfd. A 2-form ω on W is called symplectic

$:\Leftrightarrow$ $d\omega = 0$ and $\omega \wedge \omega \neq 0$

Ex: • $(\mathbb{R}^4, \omega_{st} := dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$

• $(\Sigma_g \times \Sigma_h, \Omega_{\Sigma_g} + \Omega_{\Sigma_h})$

• $\mathbb{C}P^2$ carries a (natural) symplectic structure (HW)

• S^4 doesn't carry a symplectic structure (HW)

• Let $(M, \xi = \ker\alpha)$ be a contact mfd.

$\Rightarrow$ $(W := \mathbb{R}_t \times M, \omega := d(e^t\alpha))$ is symplectic and is called symplectization

$[\omega \wedge \omega = (e^t dt \wedge \alpha + e^t d\alpha) \wedge (\ldots) = 2e^{2t} dt \wedge \alpha \wedge d\alpha \neq 0]$

Def: A diffeo $f: (W_1, \omega_1) \longrightarrow (W_2, \omega_2)$ is called

• symplectomorphism $:\Leftrightarrow$ $f^*\omega_2 = \omega_1$

• (deformation) equivalence $:\Leftrightarrow$ $f^*\omega_2$ is isotopic to ω_1

Theorem 1 [Darboux]: $\forall p \in (W, \omega)$ $\exists$ NBHD U of p s.t. $(U, \omega) \overset{\text{symp.}}{\cong} (\mathbb{R}^4, \omega_{st})$

proof: Moser trick (HW) ∎

6.2 Fillable contact manifolds

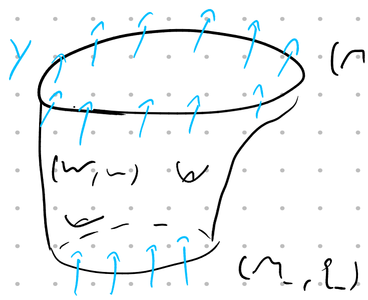
Let (M_+, ξ_+) and (M_-, ξ_-) be oriented (orientation induced by contact form) contact mfds.

Def: A (strong) symplectic cobordism from (M_-, ξ_-) to (M_+, ξ_+) is an oriented compact symplectic manifolds (W, ω) s.t.

• $\partial W = M_+ \sqcup -M_-$

• in a NBHD of ∂W $\exists$ Liouville vector field Y $(L_Y\omega = \omega)$ transverse to ∂W, pointing out of W along M_+ and pointing into W along M_-

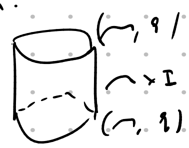
• $\ker(\iota_Y \omega|_{TM_\pm}) = \xi_\pm$



Ex: (M, ξ) is symplectically cobordant to (M, ξ).

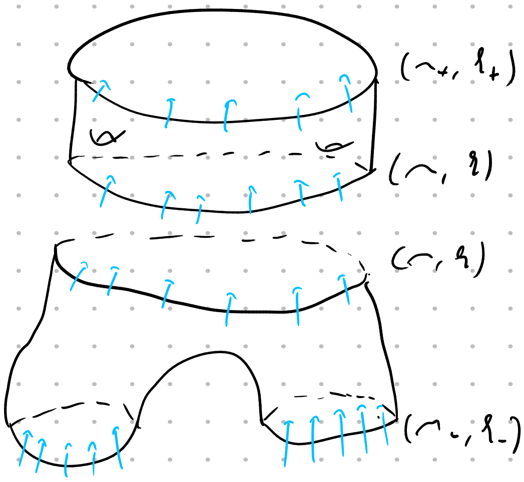
• $(I \times M, d(e^t\alpha))$ is a symplectic cobordism.

• $Y = \partial_t$ is a Liouville vector field.



Lemma 2 Let (W_+,ω_+) be a symplectic cobordism from (M,ξ) to (M_+,ξ_+)
& (W_-,ω_-) " " (M_-,ξ_-) to (M,ξ)
$\Rightarrow \exists$ symplectic cobordism (W,ω) from (M_-,ξ_-) to (M_+,ξ_+)

proof:



Glue $M\times I$ in between to smoothly transition Liouville v.f. into each other.

Def A (strong) symplectic filling of a contact mfd (M,ξ) is a symplectic cobordism from $\emptyset$ to (M,ξ).

Ex • (D^4,ω_{st}) is a filling of (S^3,ξ_{st})

• $(S^1\times S^2,\xi_{st})$ & (T^3,ξ_n) are also fillable (HW)

Theorem 3: [Eliashberg, Gromov] If (M,ξ) is fillable, then it is tight.

proof: In section 6.4.

Theorem 4: [Eliashberg] $\forall (M,\xi)$ $\exists$ symplectic cap, i.e. a cobordism from (M,ξ) to $\emptyset$.

proof: maybe in section 7

Corollary 5 (of Thm 3): (S^3,ξ_{st}), $(S^1\times S^2,\xi_{st})$, (T^3,ξ_n), $(\mathbb{R}^3,\xi_{st})$ & $(S^1\times D^2,\xi_n)$ are tight.

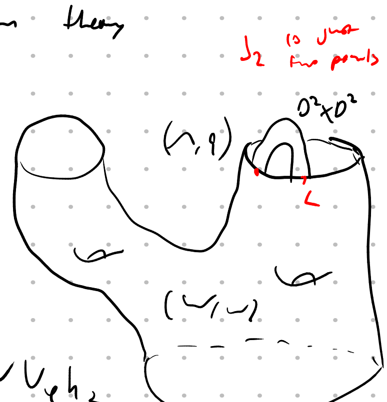
proof: • (S^3,ξ_{st}) is fillable $\Rightarrow$ tight.
• $(S^1\times S^2,\xi_{st})$ " "
• $(S^3\setminus\{pt\},\xi_{st})\cong(\mathbb{R}^3,\xi_{st})$ $\leadsto$ OT disk in RHS is also in LHS.
• universal cover of (T^3,ξ_n) & $(S^1\times D^2,\xi_n)$ is $(\mathbb{R}^3,\xi_{st})$ (sheet 9) $\square$

Remark: For every closed 3-mfd $\exists$ compact 4-mfd W st. $\partial W=M$ (see chapter 7).
$\leadsto$ cobordism theory

6.3 Weinstein handles and other types of fillings

Let $L\subset(M,\xi)$ be a Legendrian knot
& (W,ω) be a symplectic cobordism with $(M_+,\xi_+)\supset(M,\xi)$
& attach a 2-handle $h_2:=D^2\times D^2$ to W along L
via an embedding

$$\varphi:\partial D^2\times D^2\hookrightarrow\partial W$$
$$\partial D^2\times D\longmapsto L,\qquad W_L:=W\cup_\varphi h_2$$

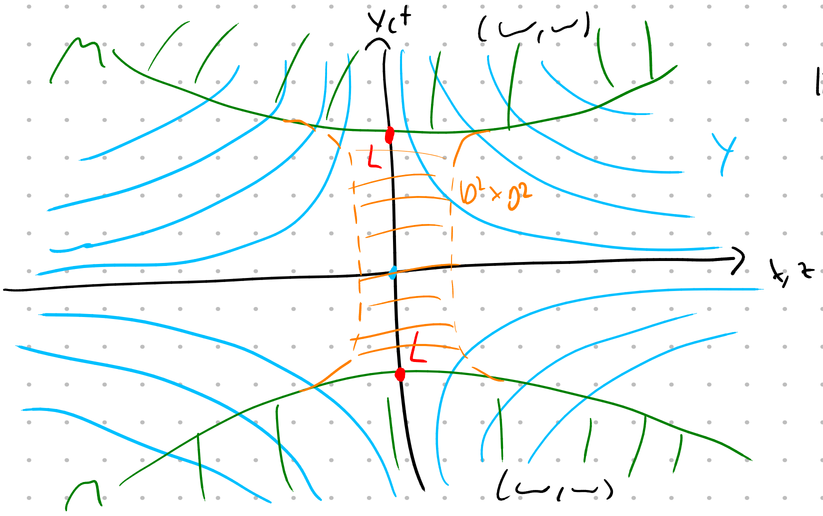


Thm 6: [Weinstein] $\forall L\ \exists \varphi$ s.t. $W_L = W \cup_\varphi h_2$ carries a symplectic structure extending ω over h_2 s.t. ω_L is a symplectic cobordism.

Remark: We can construct infinitely many contact mfds by starting with (D^4, ω_{st}) & attaching Weinstein two-handles along Legendrian links.

proof (Thm): $(\nu L, \xi) \overset{\text{cont.}}{\cong}$ std. model

$\Rightarrow$ wlog work in a local model & describe 2-handles there.



$\mathbb{R}^4,\ \omega = dx\wedge dy + dz\wedge dt$

$Y = 2x\partial_x - y\partial_y + 2z\partial_z - t\partial_t$

$M \cong \{x^2 - y^2 + z^2 - t^2 = -1\}$ contact $M \pitchfork Y$

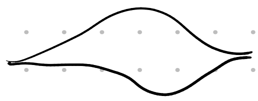
$L \cong \{x = z = 0 \mid y^2 + t^2 = 1\}$ Legendrian

[$\alpha := \iota_Y \omega = 2x\,dy + y\,dx + 2z\,dt + t\,dz = 0$ if you plug in TL]

Remark: If $(W,\omega) = (D^4, \omega_{st})$

$\partial W_L = (S^3 \setminus \nu L, \xi_{st}) \cup_\varphi (D^2 \times S^1, \xi')$

Surgery

Ex: $\partial(D^4 \cup h_2)$ attached along  $= (\mathbb{RP}^3, \xi_{st})$

$\Rightarrow (\mathbb{RP}^3, \xi_{st})$ is fillable (see chapter 7).