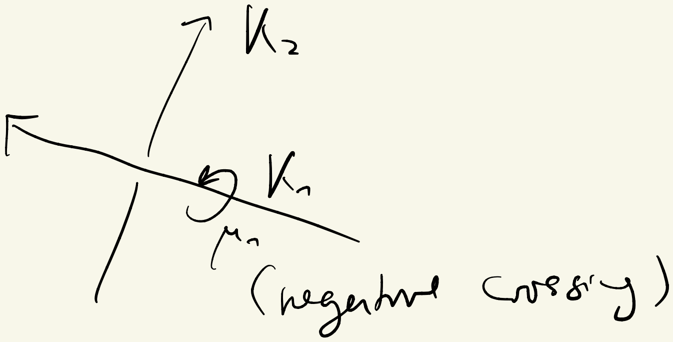
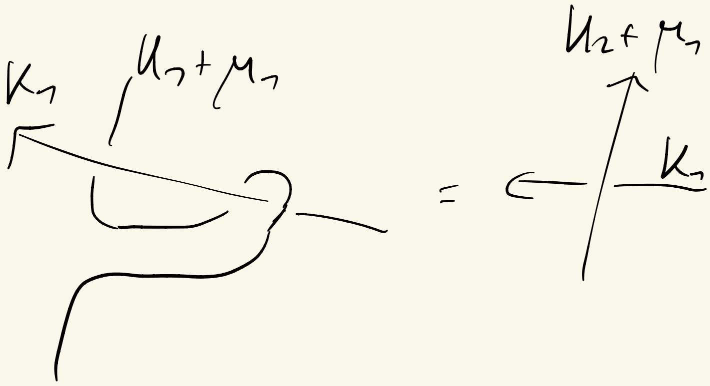
Pf of Lemma 8: *) $lk(K_1, \pm\mu_1) = \pm 1$

*) $lk(K_1, K_2 \pm \mu_1) = lk(K_1, K_2) \pm 1$




$n :=$ # crossings of $K_2$ under $K_1$

$\Rightarrow K_2 - n\mu_1$ has no undercrossing with $K_1$

$\Rightarrow lk(K_1, K_2 - n\mu_1) = 0$

$\Rightarrow lk(K_1, K_2) = n$ ∎

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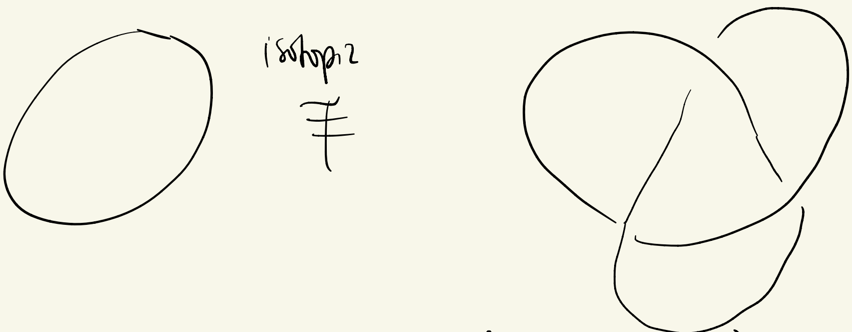
W.t.s.



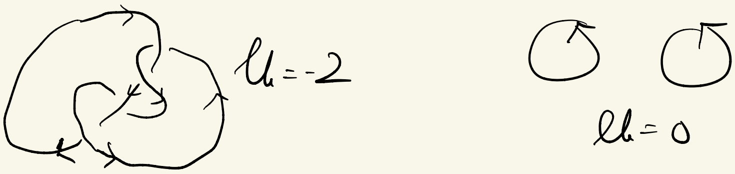

$g(K) := \min(g(F) \mid F$ is a Seifert srf of $K)$

If $K_1$ & $K_2$ are oriented disjoint knots in $S^3$

$[K_1] = lk(K_1, K_2)[\mu_{K_2}] \in H_1(S^3 \setminus K_2) \cong \mathbb{Z}\langle \mu_{K_2} \rangle$

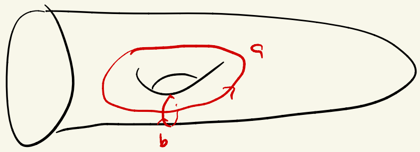
|| Lemma 8

# crossings of $K_2$ under $K_1$ counted with signs



# Alexander polynomial:

**Seifert form:** Let $F$ be a Seifert sf of an oriented knot $K$




$$S: H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$$
$$(a,b) \longmapsto lk(a, b^+)$$

$b^+$: push-off of $b$ in positive normal dir. of $F$.

**Alexander polynomial:** $\Delta_K(t) := \det\left(t^{-1/2} S - t^{1/2} S^T\right)$

**Corollary 10:** $\Delta_K(t)$ is a knot invariant.

**proof sketch:** * $\Delta_K$ is independent of the chosen matrix.

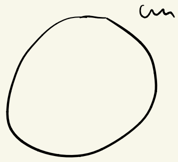
$S' = P S P^T$ with $\det(P) = \pm 1$

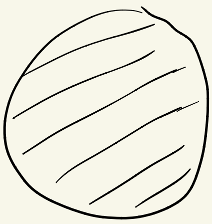
* $F'$ is a different Seifert surface $\Rightarrow$ $F'$ & $F$ have a common stabilization $\overline{\overline{F}}$

* $\overline{F}$ is a stabilization of $F$ $\Rightarrow$ $\overline{S} = \begin{pmatrix} S & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

details: Exercise

**Example:**

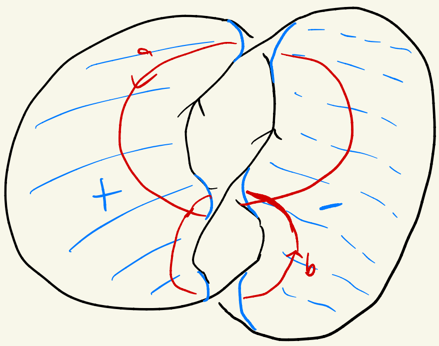



$F =$ 

$H_1(F) = 0$

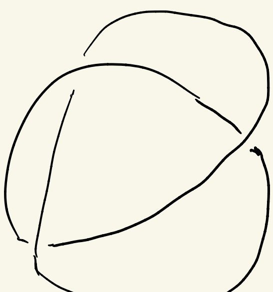
$\Rightarrow S = 0$

$\Rightarrow \Delta_K(t) = 1$




$\cong$

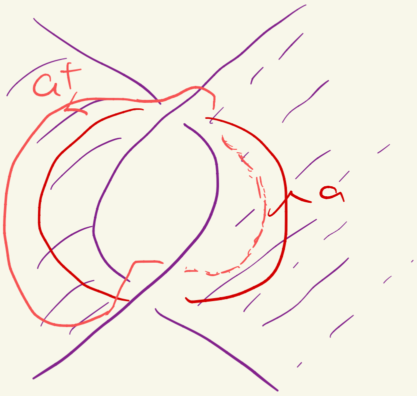
Reidemeister moves



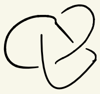
$F \cong T^2 \setminus \mathring{D}^2$

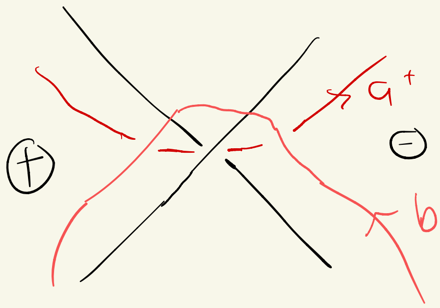
$\Rightarrow H_1(F) \cong \mathbb{Z}^2_{\langle a,b \rangle}$

$$S = \begin{pmatrix} lk(a,a^+) & lk(b,a^+) \\ lk(a,b^+) & lk(b,b^+) \end{pmatrix}$$




$$\Rightarrow S = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

$\Rightarrow \Delta$  $(t) = \det\left[\begin{pmatrix} -t^{1/2} & 0 \\ t^{1/2} & -t^{-1/2} \end{pmatrix} - \begin{pmatrix} -t^{1/2} & t^{1/2} \\ 0 & -t^{1/2} \end{pmatrix}\right]$




$$= \det \begin{pmatrix} t^{1/2} - t^{-1/2} & -t^{1/2} \\ t^{1/2} & t^{1/2} - t^{-1/2} \end{pmatrix}$$

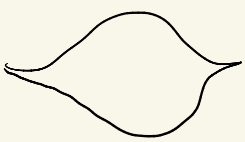
$$= t + t^{-2} - 2 + 1 = t - 1 + t^{-1} \neq 1$$

$$= \Delta_O$$

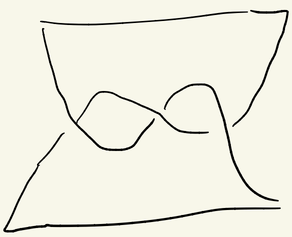
<u>HW</u>: $\Delta_{fig8}(t) = t - 3 + t^{-1}$

<u>Corollary</u>: $\deg(\Delta_K) \leq g(K)$ ☒

<u>HW</u>: $\forall g \in \mathbb{N}_0 : \exists K_g : g(K_g) = g$

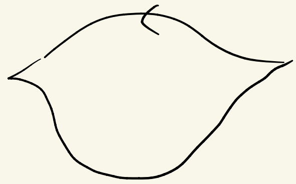


Leg iso
$\neq$

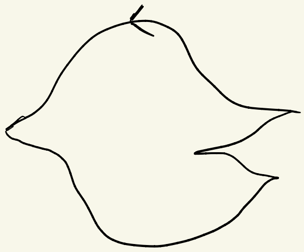


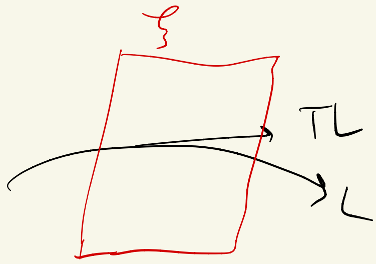
## 3.5 Classical invariants

Ex.:



**Stabilization:**

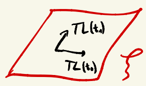


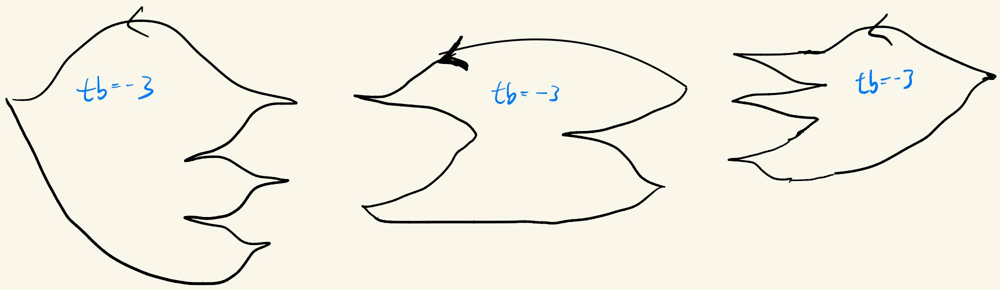



idea:

$tb(K) := \#$ twists of $\xi$ around $L$

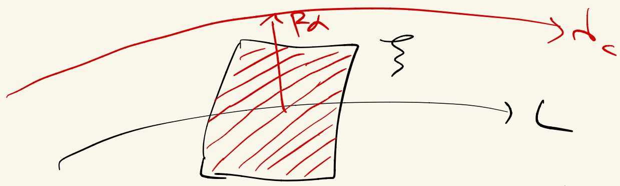
$rot(K) := \#$ twists of $TL$ in $\xi$



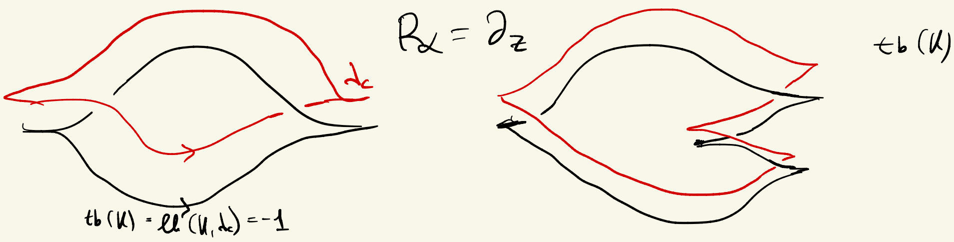

Let $K \subset (M^3, \xi)$ be a Legendrian knot

**Contact longitude:** of $K$ $\quad \lambda_c$ = push-off of $K$ in the Reeb-direction




**Thurston-Bennequin invariant:** $tb(K) := lk(K, \lambda_c)$, $K$ nullhomologous

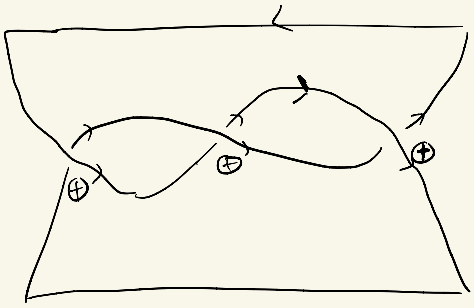
Ex.:




**Lemma 12:** Let $K \subset (\mathbb{R}^3, \xi_{st})$ be a Legendrian presented in a front proj.
Then $tb(K) = -\frac{1}{2}C + W$, $\quad C := \#$ cusps in front

$W := \#$ positive crossings $- \#$ negative crossings

$=$ **writhe of front**

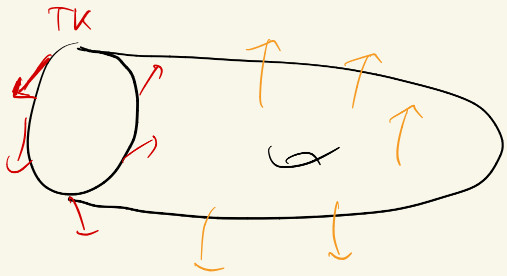


$tb = -\frac{1}{2} \cdot 4 + 3 = \underline{1}$

pf: HW $\boxed{\Sigma}$

Let $K \subset (M, \xi)$ be an oriented nullhomologous Leg. knot with Seifert sf $\Sigma$. $\mathrm{rot}(K, [\Sigma]) := \langle e(\xi, K), [\Sigma] \rangle$

$= pd(e(\xi, K)) \cdot [\Sigma]$

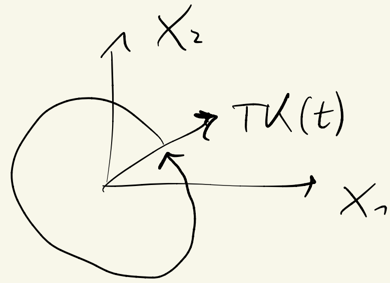



$= \#$ zeros of extension of $TK$ over $\xi$

$= \#$ rotations of $TK$ relative to a trivialization of $\xi$ over $\Sigma$

If $\xi$ is trivializable, i.e., $\exists X_1, X_2$ vf s.t. $\xi = \langle X_1, X_2 \rangle$

then $\mathrm{rot}(K) := \#$ rotations of $TK$ relative to this trivialization




Ex.: $\xi_{st} = \ker(x\,dy + dz)$

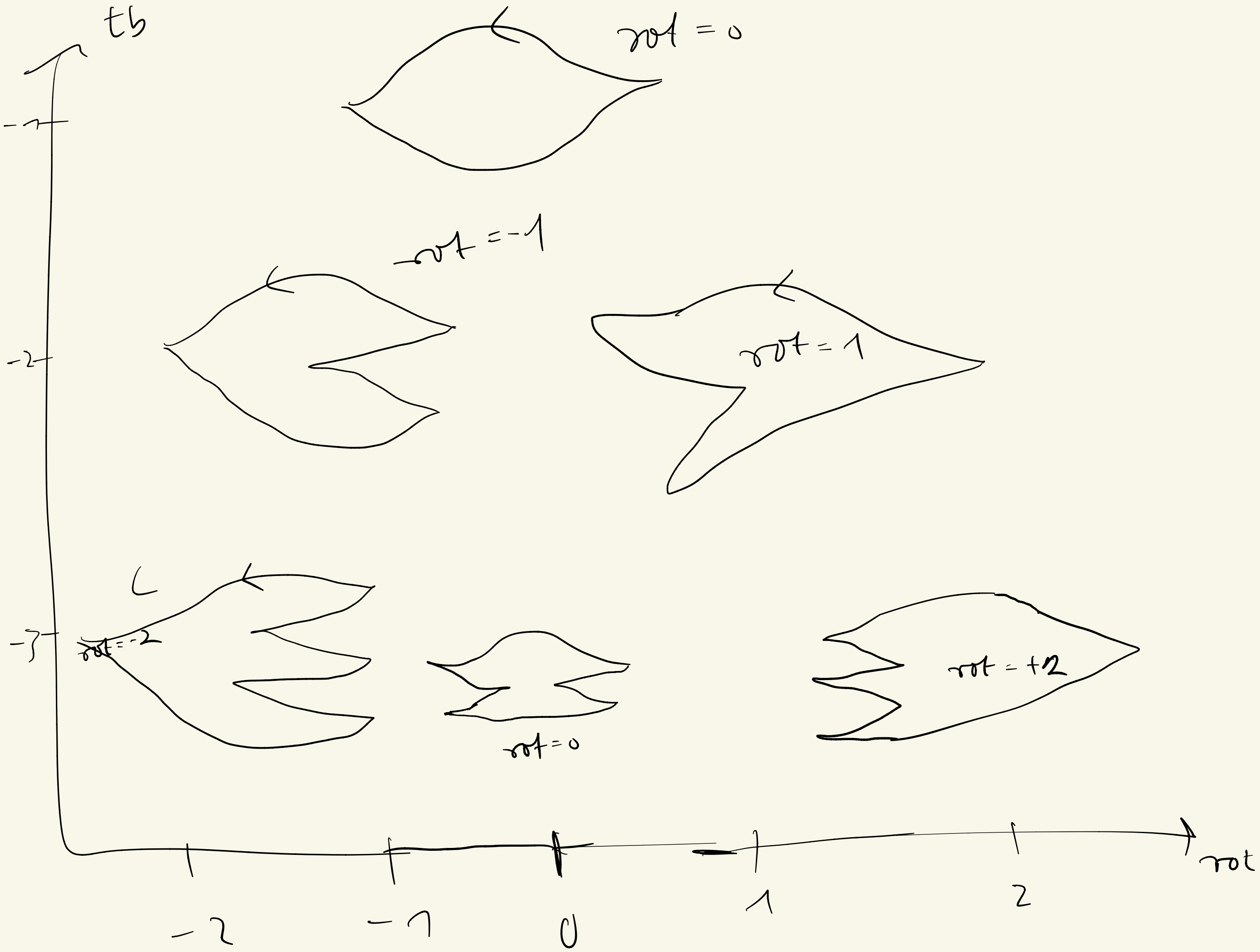
$\rightarrow X_1 = \partial_x, \; X_2 = \partial_y - x\partial_z$ span $\xi_{st}$

Rmk: $\mathrm{rot}(-K) = -\mathrm{rot}(K)$

Lemma 30: Let $K \subset (\mathbb{R}^3, \xi_{st})$ be an oriented Leg. knot $\underbrace{\text{. Then}}_{\text{presented in a front proj}}$ $\mathrm{rot}(K) = \frac{1}{2}(C_- - C_+)$,

$C_\pm := \#$ cusps oriented upwards/downwards

pf: next lecture.


tb
rot = 0
rot = -1
rot = 1
rot = -2
rot = 0
rot = +2
-1
-2
-3
-2
-1
0
1
2
rot