2. Contact Manifolds

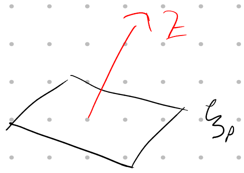
17.10.23

2.1 Hyperplane fields

M^n = smooth n-MFD $\quad \left(p \longmapsto \xi_p^{n-1} \subset T_pM^n\right)$

- $\xi \subset TM$ is called <u>hyperplane field</u> :⟺ $\forall p \in M$ ∃ NBHD U of p and linearly independent vector fields $X_1, \dots, X_{n-1}$ on U s.t. $\forall q \in U$:
$$\xi_q := \xi \cap T_qM = \langle X_1(q), \dots, X_{n-1}(q) \rangle$$

- ξ is called <u>co-orientable</u> :⟺ TM/ξ is orientable



$$p \mapsto T_pM/\xi_p = V_p$$

<u>Def</u> vector bundle over M^n

$$\begin{tikzcd}
E \supset p^{-1}(U) \arrow[r, "\cong^{C^\infty}", "\text{Fib iso}"'] \arrow[d, "p"] & U \times \mathbb{R}^k \arrow[dl, "pr"] \\
M^n \ni p & U
\end{tikzcd}$$

<u>Why is TM/ξ a bundle?</u>

$\xi_q = \langle X_1(q), \dots, X_{n-1}(q) \rangle$

$N_q = \xi_q^\perp$

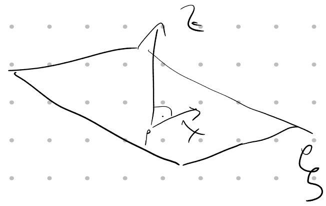
$p^{-1}(U) \cong U \times \langle N \rangle$

$p^{-1}(U) \cong U \times V_p$

<u>Lemma 1</u>: $\xi \subset TM$ is coorientable ⟺ ∃ 1-form α on M s.t. $\xi = \ker(\alpha)$ $\quad$ 18.10.23
(i.e. $\alpha_p : T_pM \to \mathbb{R}$)

<u>proof</u>: Let g be a riemannian metric.

"⟹" Choose a vector field Z on M with
$$\begin{cases} g(Z, X) = 0 \quad \forall X \in \xi \\ g(Z,Z) = 1 \end{cases}$$



$\alpha := g(Z, \cdot)$

"⟸" A vector field Z with $\begin{cases} g(Z,Z) = 1 \\ \alpha(Z) \neq 0 \end{cases}$ defines a coorientation of ξ

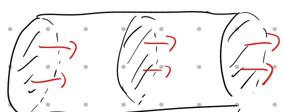
$\square$

<u>Convention</u>: we restrict to coorientable hyperplane fields (often this is secretly assumed)

<u>Example</u>:
$$M^n := S^1 \times N^{n-1}$$
$\forall p = (\theta, x) \in M \Rightarrow \xi_p := T_x N \subset T_pM$

$\xi = \bigcup_{p \in M} \xi_p$ is a hyperplane field

$\xi = \ker(d\theta)$ $\quad$ codim-1-foliation

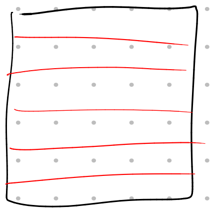


Foliation:

$M^n = \bigsqcup_{\theta\in\Lambda} F_\theta$ is called h-dim foliation ("leafs")

$:\Leftrightarrow$ $\exists$ Atlas of M (U_α, h_α) s.t. $\forall\theta\ \forall\alpha$: $h_\alpha(F_\theta \cap U_\theta) = \emptyset$ or $\mathbb{R}^h \times p \cap h_\alpha(U_\alpha)$

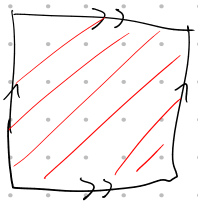
$(F_\theta \overset{h}{\cong})\ \mathbb{R}^h \times (*)$



$\mathbb{R}^n$ $(\overset{h_\alpha}{\subset} M)$

Ex: Flow lines of a non-vanishing vector field are 1-dimensional foliations

$M = T^2$



If slope $\in \mathbb{Q} \Rightarrow$ Leaves are compact

If slope $\in \mathbb{R}\setminus\mathbb{Q} \Rightarrow$ " noncompact

If $h = n-1 \Rightarrow \xi = \bigcup_{p\in M} T_pF(p)$ is a hyperplane field

"every foliation of codim 1 induces a hyperplane field"

Q: "When does a hyperplane come from a foliation?"

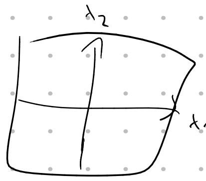
Theorem 2 (Frobenius)

$\ker(\alpha) = \xi \subset TM$ is induced by a foliation $\Leftrightarrow$ $\alpha \wedge d\alpha = 0$

digression: $\alpha_p : T_pM \xrightarrow{\text{1-form}} \mathbb{R}$ lin

$\alpha_p \in (T_pM)^*$

Let $x_1, \dots, x_n$ be coordinates on M



x_2, x_1, $\mathbb{R}^n$

$\Rightarrow$ coordinate vector fields $\partial x_1, \dots, \partial x_n$ form a basis of T_pM

Dual basis $dx_1, \dots, dx_n$, i.e. $dx_i(\partial x_j) = \delta_{ij}$

$\Rightarrow \alpha_p = \sum c_i(p)\, x_i$

k-form β: $\beta_p : \underbrace{(T_pM)^* \times \dots \times (T_pM)^*}_{k \text{ times}} \longrightarrow \mathbb{R}$ multilinear, alternating

$\wedge$-product:

1-forms $\alpha = \sum c_i\, dx_i$

$\beta = \sum b_j\, dx_j$.

$\alpha \wedge \beta =$ def. to be linear

$dx_i \wedge dx_j = -dx_j \wedge dx_i$

$dx_i \wedge dx_i = 0$

Differential

d: k-forms $\longrightarrow$ $(k+1)$-forms

$\sum c_i(p)\, dx_i \longmapsto d\alpha$

$\sum_{i,j} \frac{\partial c_i}{\partial x_j}\, dx_j \wedge dx_i$

Ex: $M = \mathbb{R}^3_{(x,y,z)}$ $\alpha = dx$ $\Rightarrow$ $\alpha \wedge d\alpha$ comes from a foliation

$d\alpha = ddx = 0$

(2) $\alpha = x\,dy + dz$

$$d\alpha = d(x\,dy + dz) = d(x\,dy) + \cancel{d(dz)} = \underbrace{\frac{\partial x}{\partial x}}_{=1} dx \wedge dy + \cancel{\frac{\partial x}{\partial y} dy \wedge dy}$$

$$\alpha \wedge d\alpha = (x\,dy + dz) \wedge (dx \wedge dy) = x\,dy \wedge (dx \wedge dy) + dz \wedge dx \wedge dy$$

$$= dx \wedge dy \wedge dz \neq 0$$

2.2 Contact structures

Let M^{2n+1}

* $\xi^{2n} = \ker(\alpha)$ is a contact structure $\Leftrightarrow$ $\alpha \wedge (d\alpha)^n$ is a volume form $\left\{\begin{array}{l}\text{i.e. } \alpha \wedge (d\alpha)^n \neq 0 \\ \text{at every point}\end{array}\right\}$

$\alpha \wedge (d\alpha)^n = \alpha \wedge d\alpha \wedge \dots \wedge d\alpha$

* α is called contact form

* the reeb vector field R_α of α is defined

$$\begin{cases} d\alpha(R_\alpha, \cdot) \equiv 0 \\ \alpha(R_\alpha) \equiv 1 \end{cases}$$

remark: $\alpha \wedge (d\alpha)^n$ is a vol form $\Rightarrow$ M orientable

* $\xi = \ker(\alpha) = \ker(\tilde{\alpha}) \Rightarrow \tilde{\alpha} = f\alpha$ for $f: M \to \mathbb{R}\setminus\{0\}$

$$\Rightarrow \tilde{\alpha} \wedge (d\tilde{\alpha})^n = (f\alpha) \wedge (d(f\alpha))^n = f\alpha \wedge (f\,d\alpha + df \wedge \alpha)^n$$

$\alpha \wedge \alpha = 0$

$$= \underbrace{f^{n+1}}_{\neq 0} \underbrace{\alpha \wedge (d\alpha)^n}_{\neq 0}$$

* R_α is well-defined:

$\alpha \wedge (d\alpha)^n \neq 0 \Rightarrow (d\alpha)^n\big|_\xi \neq 0 \Rightarrow d\alpha$ has rank $2n$

ξ: $2n$-dimensional bundle $\Rightarrow$ $\ker(d\alpha)$ is 1-dim

& $\alpha \neq 0$ on $\ker(d\alpha)$

Weinstein conj: M closed $\overset{?}{\Rightarrow}$ $\forall$ contact form α on M: R_α has a periodic orbit

Ex) (1) Consider $\mathbb{R}^{2n+1}_{(x_1, y_1, \dots, x_n, y_n, z)}$

$\xi_{st} = \ker(\alpha_{st})$ standard contact structure

$\alpha_{st} = \left(\sum_{i=1}^{n} x_i dy_i\right) + dz$

$$\alpha_{st} \wedge (d\alpha_{st})^n = \left(\sum x_i dy_i + dz\right) \wedge \left(d\left(\sum x_i dy_i + dz\right)\right)^n = \left(\sum x_i dy_i + dz\right) \wedge \left(\sum dx_i \wedge dy_i\right)^n$$

$$= \left(\sum x_i dy_i + dz\right) \wedge \underbrace{\left(\sum dx_i \wedge dy_i\right) \wedge \left(\sum dx_i \wedge dy_i\right)} \wedge \left(\sum dx_i \wedge dy_i\right)^{n-2}$$

$$= \left(\sum x_i dy_i + dz\right) \wedge \sum_{i \neq j} dx_i \wedge dy_i \wedge dx_j \wedge dy_j \wedge \left(\sum dx_i \wedge dy_i\right)^{n-2}$$

$$= \left(\sum x_i dy_i + dz\right) \wedge n! \left(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \dots \wedge dx_n \wedge dy_n\right)$$

$$= dz \wedge n! \left(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \dots \wedge dx_n \wedge dy_n\right)$$

$$= n!\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \dots \wedge dx_n \wedge dy_n \wedge dz \neq 0$$

Compute $R_{\alpha_{st}} = \left(\sum A_i \partial x_i + B_i \partial y_i\right) + C \partial z$

$$d\alpha_{st}(R_{st}, \cdot) = \sum A_i \, dy_i - B_i \, dx_i = 0$$

$$\alpha_{st}(R_{st}) = C + \sum x_i B_i = 1$$

One solution: $R_{\alpha_{st}} = \partial_z$

(2) $\xi_{syn} = \ker\big(\underbrace{\sum (x_i dy_i - y_i dx_i) + dz}_{\alpha}\big)$

Homework:
- This is a contact structure
- $R_\alpha = \partial_z$
- for $n=1$: draw this
- read Milnor: Topology from a differential viewpoint
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