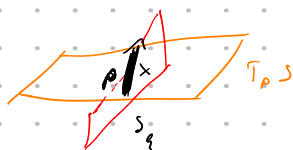


4. Surfaces in contact 3-manifolds

Def A singular foliation on M is the equivalence class of a vector field X s.t.

$$X \sim X' \iff \exists f: M \rightarrow \mathbb{R}^+ \text{ s.t. } X = f X'$$


Let $S \subset (M, \xi = \ker(\alpha))$ be an oriented surface. the characteristic foliation S_ξ of S is given by $TS \cap \xi$



$$(X \in T_p S = \xi_p \implies X(p) = 0)$$

Ex: $S = S^2 \subset (\mathbb{R}^3, \ker(X dy - y dx + dz))$

$$S_\xi \text{ is spanned by } X := (xz - y)\partial_x + (yz + x)\partial_y - (x^2 + y^2)\partial_z$$

$$X \in TS^2 \quad \& \quad X \in \xi$$

$$X \in TS^2: x^2 z - yx + y^2 z + xy - (x^2 + y^2) = 0 \quad \checkmark$$

$$X \in \xi: (xyz + x^2) - (yz + y^2) - (x^2 + y^2) = 0 \quad \checkmark$$

$$X(x, y, z) = 0 \iff (-y, z) = (0, \pm 1)$$

Identify $\sqrt{S} = S \times \mathbb{R}$
 $S \hookrightarrow S \times 0$

Write $\alpha = \beta_z + u_z dz$

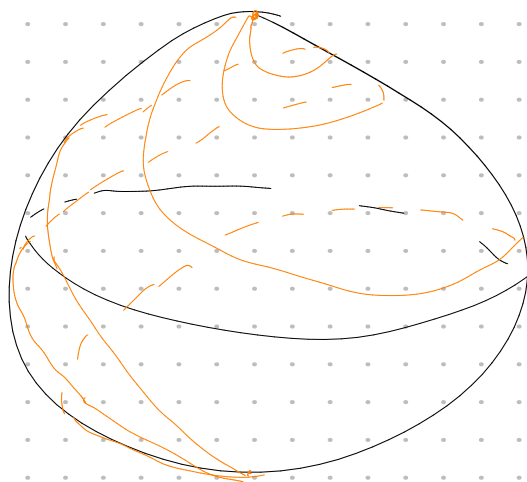
with $z \in \mathbb{R}$, β_z 1-forms on S

& $u_z: S \rightarrow \mathbb{R}$

$$\implies d\alpha = d\beta_z - \beta_z \wedge dz + du_z \wedge dz$$

contact condition: $u_z d\beta_z + \beta_z \wedge (du_z - \beta_z) \geq 0$

Let \mathcal{U} be an area form on S



Lemma 1: S_g is given by a v.f. defined by

$$i_X \Omega = \beta_0 = \alpha|_{T_S}$$

proof: $X(p) \neq 0 \Leftrightarrow \beta_{0,p} = 0 \Leftrightarrow T_p S = \xi_p$

$$* \quad 0 = L_X(i_X \Omega) = L_X \beta_0 \Rightarrow X \in \xi \quad \square$$

Lemma 2: A vector field X on S defines a characteristic foliation of a contact structure ξ

$$(\Leftrightarrow) \quad \forall p \in S \text{ with } X(p) \neq 0 \Rightarrow d\iota_{X_p}(\xi_p) \neq 0 \quad (*)$$

Definition: $d\iota_{X_p}(\xi_p)$ of a vector field X on S is an area form Ω is defined by

$$d\iota_{X_p}(\xi_p) \cdot \Omega := L_X \Omega = d(L_X \Omega)$$

Exercise: Check that this only depends on α

proof: (\Leftarrow) If $X(p) \neq 0 \Rightarrow \beta_{0,p} = 0 \Rightarrow \alpha_p = u_0(p) dz \Rightarrow \xi_p = T_p S$

$$\Rightarrow d(L_X \Omega) = (d\beta_0)_p = d\alpha|_{\xi_p} \neq 0$$

if \Leftarrow Let X with $(*)$. Let $\beta := L_X \Omega$ & $u: S \rightarrow \mathbb{R}$ def by $\Rightarrow \exists$ some u s.t. $d\beta = u \Omega$

$$(*) \Rightarrow \text{if } \beta_p = 0 \Rightarrow u(p) \neq 0$$

choose γ from ξ on S

$$\beta \wedge \gamma \geq 0 \geq (\beta \wedge \gamma)(p) > 0 \quad \text{if } \beta(p) \neq 0$$

$$\beta_2 := \beta + 2(d\alpha - \gamma), \quad d\beta_0 = d\beta = u \Omega \quad \& \quad \beta_0 = d\alpha - \gamma;$$

$$\alpha = \beta_2 + u dz \quad \text{is C.F. near } S. \quad [u d\beta_0 + \beta_0 \wedge (d\alpha - \beta_0) = u^2 \Omega + \beta \wedge \gamma > 0] \quad \square$$

Ex: $S := S^1 \times S^1 \subset (S^1 \times \mathbb{R}^2, \frac{g}{h} = \underbrace{(\cos(u)\theta) dx - \sin(u)\theta dy}_{du})$

(r, φ) polar coordinates on \mathbb{R}^2

$\omega := d\theta \wedge d\varphi$ defines the standard orientation on S

$x = \cos \varphi \Rightarrow dx = -\sin \varphi d\varphi$
 $y = \sin \varphi \Rightarrow dy = \cos \varphi d\varphi$

Find X s.t. $L_X \omega = \alpha|_{TS} = -(\cos(u)\sin \varphi - \sin(u)\cos \varphi) d\varphi$

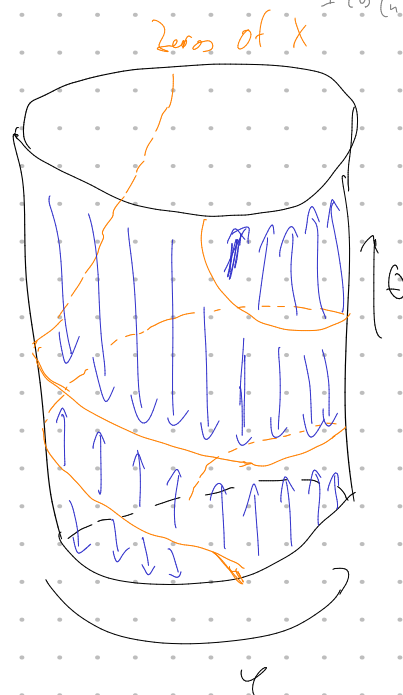
$L_X \omega = -\sin(u)\theta \cos \varphi + \cos(u)\theta \sin \varphi$

Ansatz: $X(\theta, \varphi) = a(\theta, \varphi) \partial_\theta + b(\theta, \varphi) \partial_\varphi$

$\Rightarrow X = -(\sin(u)\cos \varphi + \cos(u)\sin \varphi) \partial_\theta$

$\varphi = -\theta + \pi$
 $\varphi = -\theta$

$= -(\sin(u)\cos \varphi + \cos(u)\sin \varphi) \partial_\theta \downarrow \alpha|_{TS}?$



Thm 3: Let $S_i \subset (\mathcal{M}_i, g_i)$, $i=0,1$

& $\phi: S_0 \xrightarrow{\cong} S_1$ s.t.

$\phi(S_{0,p_0}) = S_{1,p_1}$ as oriented foliations

$\Rightarrow \exists$ tubular neighborhoods νS_0 & νS_1 in \mathcal{M}_0 & \mathcal{M}_1

& $\Phi: (\nu S_0, g_0) \xrightarrow{\cong} (\nu S_1, g_1)$ s.t. $\Phi|_{S_0} = \phi$

proof: the (non-trivial). □

4.2) Singularities of S_g

$X(x, y) = a(x, y) \partial_x + b(x, y) \partial_y$

on a nbhd of $0 \in \mathbb{R}^2$

w/ a isolated zero at 0

$\omega := dx \wedge dy$

$d(L_X \omega) = d(a dy - b dx) = (a_x + b_y) dx \wedge dy$

$\Rightarrow \text{div}(X) = a_x + b_y$

Characteristic foliation is given by the lines ℓ_x of

$(\dot{x}, \dot{y}) = X(x, y)$

Linearized equation:

$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} a_x(0) & a_y(0) \\ b_x(0) & b_y(0) \end{pmatrix}}_{=: A} \begin{pmatrix} x \\ y \end{pmatrix}$

A singularity is called non-degenerate \Leftrightarrow any eigenvalue of A has real part $\neq 0$.

Remark [Jacobian = cross product then]

If a singularity is non-deg. $\Rightarrow \exists C^1$ -diffeo h on a nbhd of $0 \in \mathbb{R}^2$ s.t.

$$\varphi_t(h(y)) = h(e^{At} \cdot \begin{pmatrix} x \\ y \end{pmatrix})$$

Ex:

$$X = x \partial_y - y \partial_x$$

$$\Rightarrow A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{Eig} = \pm i$$



degenerate singularity

$$\det = 0$$

\Rightarrow does not appear in characteristic foliation

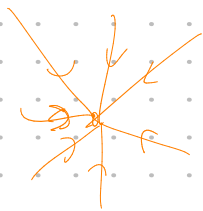
If a singularity is non-degenerate we call it

* elliptic : $\Leftrightarrow \exists$ only one eigenvalue

or \exists two eigenvalues with real parts of the same sign

* hyperbolic : $\Leftrightarrow \exists$ two real eigenvalues of opposite signs.

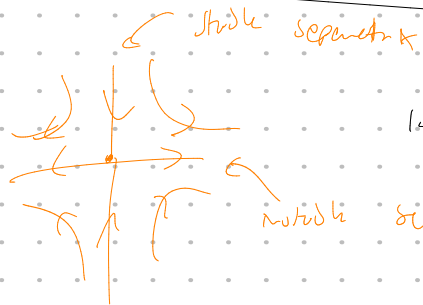
elliptic



Index +1



hyperbolic



Index -1

stable manifold

unstable manifold

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \lambda_1 > 0 > \lambda_2$$

$$\text{Index}_X(p) := \deg \left(\begin{array}{ccc} S_q^1 & \longrightarrow & S^1 \\ q & \longmapsto & \frac{X(q)}{|X(q)|} \end{array} \right)$$

sign: +1 if source
-1 if sink

$$\text{sign} = (\lambda_1 + \lambda_2)$$

Example: Hyperbolic point

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \lambda_1 > 0 > \lambda_2$$

$$\Rightarrow X(x, y) = \lambda_1 x \partial_x + \lambda_2 y \partial_y$$

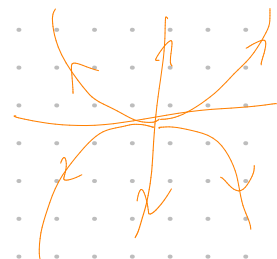
$$\Rightarrow \beta = \langle X, \alpha \rangle = \lambda_1 x \cdot dy - \lambda_2 y \cdot dx$$

$$\Rightarrow \alpha = dz + \lambda_1 x dy - \lambda_2 y dx \quad \text{is a contact form} \Leftrightarrow \lambda_1 + \lambda_2 \neq 0$$

Example: $X(y) = x \partial_x + y^3 \partial_y \rightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & 3y^2 \end{pmatrix}$

\Rightarrow degenerated.

but $\text{div}(X)(0) = 1 \neq 0$



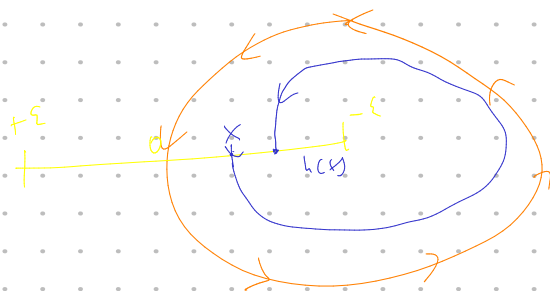
$\Rightarrow X$ defines the char foliation of a contact str.

$$(\alpha = -y^3 dx + x dy + dz)$$

By a C^∞ -perturbation of X we get a non-deg elliptic point.

A vector field X on a closed surface S is called morse-smale \Leftrightarrow

- (i) \Rightarrow only finitely many singularities & finitely many closed orbits, all non-deg
 Γ a closed orbit is non-degenerate \Leftrightarrow Poincaré return map h satisfies $h'(0) \neq 1$



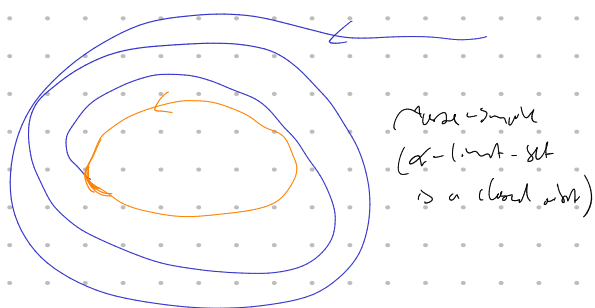
(ii) The α - & ω -limit sets of every flow-line are a single point or a closed orbit.

Γ φ_t flow of X

α -limit set of the orbit through $x_0 := \left\{ \lim_{h \rightarrow \infty} \varphi_{-h}(x_0) \mid t_h \nearrow \infty \right\}$

$\omega = 1$

" $\left\{ \quad \quad \quad \mid t_h \searrow -\infty \right\}$



not morse-smale

not Morse-smooth

not horse-shoe

Proof: not clear \rightarrow See dynamical system



Def 1 $S \subset (\gamma, \eta)$ is called convex \Leftrightarrow

Ex: $S^1 \times S^1 \subset (S^1 \times \mathbb{R}^2, \underbrace{\ker(\cos(u)\,dx - \sin(u)\,dy)}_{\alpha_n})$

$$Y = x \cdot \partial_x + y \cdot \partial_y$$

YHS

$$L_Y \alpha_n = i_Y(\text{d}\alpha_n) + d(i_Y \alpha_n) = \alpha_n \quad \Leftrightarrow \quad Y \text{ is a contact vector field}$$

Lemma 5: $SC(n, 1)$ closed is convex

(\Rightarrow) $\exists \psi: S \times \mathbb{R} \rightarrow M$ s.t. $p \mapsto \psi(p, 0)$ is the inclusion $S \hookrightarrow M$
 & $\mu(\psi^* \alpha)$ is an \mathbb{R} -val. contact structure on $S \times \mathbb{R}$

Proofs " \subseteq " $T\psi(\partial_t)$ is a contact vector field ∇S

" \Rightarrow " Let γ be a contact vector field s.t. $\gamma \perp S$

$h := \alpha(y)$ defined near S

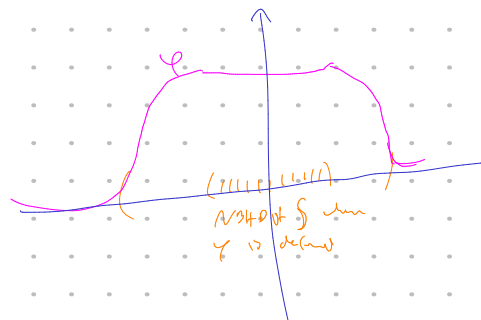
Let $\varphi: \Lambda \rightarrow \mathbb{R}$ s.t.

$\varphi \equiv 1$ near S
 $\varphi \equiv 0$ on $\Lambda \setminus \nu S$

✓ The contact vector field corresponding to $\gamma \cdot H$

$$y_k := \text{flow at } \overline{y}$$

$$\begin{array}{ccc} \varphi: S \times \mathbb{R} & \longrightarrow & \cap \\ (p, t) & \longmapsto & \varphi_t(p) \end{array}$$

$$\Rightarrow T\psi_{(p,t)}(x) = \psi_t(p) = \psi(\psi_t(p)) \text{ near } \Rightarrow \text{ker } \psi^* \alpha \text{ is } \mathbb{R}\text{-inv}$$


write $\alpha = \beta + u dt$

$$\Rightarrow \alpha \wedge d\alpha = (\beta + u dt) \wedge (d\beta + du \wedge dt) \\ = (u d\beta + \beta \wedge du) \wedge dt$$

contact condition: $\boxed{u d\beta + \beta \wedge du > 0}$

$$\neq d\beta = d(l_X \Omega) = \text{Div}_\Omega(X) \Omega$$

$$\ast du \wedge \Omega = 0 \quad (\text{3-form on a surface})$$

$$\Rightarrow 0 = l_X(du \wedge \Omega) = X(u) \Omega - du \wedge l_X \Omega = X(u) \Omega + \beta \wedge du$$

$$\Rightarrow \text{contact condition} \quad u \text{ Div}_\Omega(X) - X(u) > 0$$

Ex: $S^3 \subset \mathbb{C}^2 \quad \alpha = r_1^2 dy_1 + r_2^2 dy_2$

$$S := \{r_1^2 = c, r_2^2 = 1-c\} \cong \mathbb{T}^2 \quad \text{for } c \in (0,1)$$

$$\beta = \alpha|_S = c dy_1 + (1-c) dy_2$$

$$\Omega = dy_1 \wedge dy_2$$

$$\Rightarrow X = (1-c) \partial_{y_1} - c \partial_{y_2}$$

$$l_X \Omega = (1-c) dy_2 + c dy_1 = \beta$$

$$d\beta = 0 \Rightarrow \text{Div}_\Omega(X) = 0$$

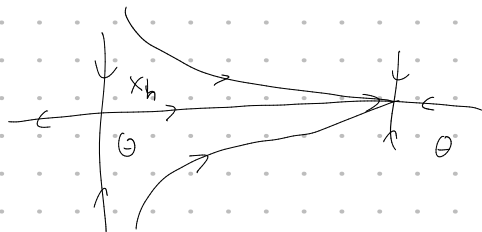
$$\Rightarrow S \text{ is not convex since } -X(u) > 0 \text{ admits no solution } u \text{ on } S.$$

6.4 The elimination lemma

An elliptic point X_e & a hyperbolic point X_h are in

elimination position $\Leftrightarrow \text{sign}(x_e) = \text{sign}(x_h)$ & \exists a separatrix of x_h connecting

x_h & x_e



(dea: replace by



Lemma 6: Let X_e & X_h be in elim pos $\Rightarrow \exists$ an annulus $A \subset S$ s.t.

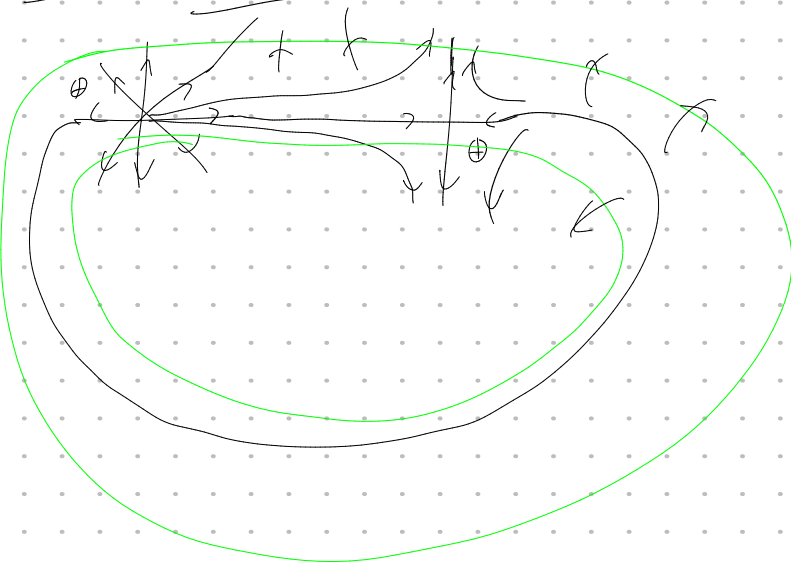
$\ast X_e$ & X_h are the only singularities in A

$\ast A$ has no closed orbit

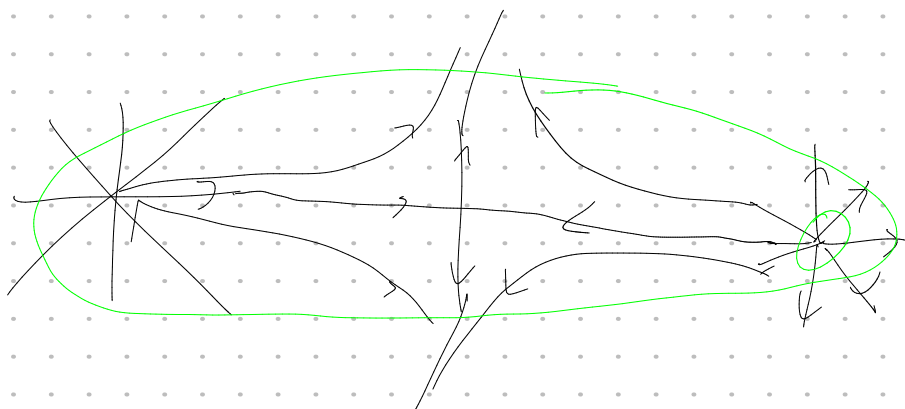
$\ast A$ is transverse to ∂A

Proof:

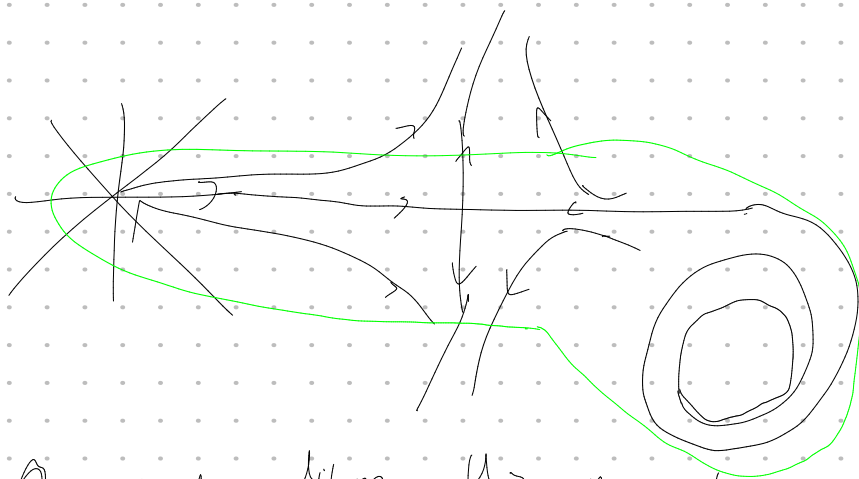
Case 1:



Case 2:



Case 3:



By the Poincaré-Bendixon condition this one also has a limit cycle (it can't go into a hyperbolic point)

Lemma 7: $\exists f: S \rightarrow \mathbb{R}^2$ s.t. $\operatorname{div}(f_x) > 0$ on A

Proof: $\operatorname{div}(f \circ x) \lrcorner = d(i_{f_x} \lrcorner) = d(f \lrcorner x \lrcorner)$
 $= df \lrcorner \lrcorner_x \lrcorner + f \lrcorner (\lrcorner_x \lrcorner) = df \lrcorner \lrcorner_x \lrcorner + f \operatorname{div}(x) \lrcorner$

$df \lrcorner \lrcorner_x \lrcorner \Rightarrow \Rightarrow 0 = \lrcorner_x (df \lrcorner \lrcorner) = x(f) \lrcorner - df \lrcorner \lrcorner_x \lrcorner$

$\Rightarrow \operatorname{div}(f_x) = x(f) + f \operatorname{div}(x)$

choose f constant on a nbhd of x_c^+ & x_h^+ s.t.

$$X(t) + f \operatorname{div}(X) > 0 \quad \text{on } A$$

For $(x) \geq -c$ on A (as a continuous function on the compact set A)
 $X(t) > cf$ admits a solution f
 simple differential inequality, can always be solved. □

Then §: [Evolution Lemma, Growth]

Let $S \subset (\overline{\Omega}, q = \operatorname{tr}(\alpha))$ & $x_c, x_h \in S$ in div. position

$\Rightarrow \exists$ a small C^0 isotopy $\psi_t : S \rightarrow \overline{\Omega}$, $t \in [0, 1]$ s.t.

* $\psi_0 = \text{inclusion } S \hookrightarrow \overline{\Omega}$

* $\psi_t = \text{id}$ on $S \setminus A$

* $(\psi_t(A))_t$ admits no singularities

proof: replace α by $f \cdot \alpha \Rightarrow \beta = d\alpha|_S$ & X d.t. by $(X)^\omega = \beta$
 satisfies $\operatorname{div}(X) > 0$ on A

$\Rightarrow \int \beta > 0$ on A

$\Rightarrow \beta + dz$ is a contact form on $A \times \mathbb{R}$ (i.e. A is convex)

$\Rightarrow \exists$ C^0 -small nbhd of A in $(\overline{\Omega}, q)$ is contactomorphic to $(A \times \mathbb{R}, \beta + dz)$

replace $A = A \times 0 \hookrightarrow A \times \mathbb{R}$ by the graph of a function

$g : A \rightarrow \mathbb{R}$ s.t. $g \geq 0$ near ∂A

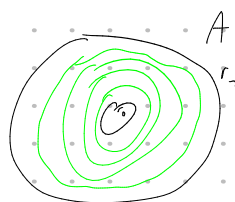
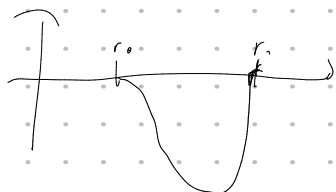
\exists isotopy $\psi_t(p) = (p, t g(p))$

$$(\psi_t^* \alpha = \beta + dg)$$

\downarrow

$\Rightarrow (\psi_t(A))_t$ is given by $X + X_g$ with $(x, t)_t \Omega = \beta + dg$

choose $g : A \rightarrow [-\infty, 0]$ as follows



$$(X_g)^\omega = dg \Rightarrow dg(X_g) = \omega(X_g, X_g) = 0 \Rightarrow X_g \text{ is tangent to } S^1 \times \{t\} \subset S^1 \times \mathbb{R} = A$$

\Rightarrow $X + X_g \neq 0$ near ∂A on $A \setminus V(\partial A)$ choose d_g so large that $\beta + d_g \neq 0$

\Rightarrow $X + X_g \neq 0$ on $A \setminus V(\partial A)$

\Rightarrow no singularities

Ex 1: The dividing set

Def: the dividing set Γ_S of a convex surface $S \subset (M, g)$ is

$$\Gamma_S := \{p \in S \mid \exists \gamma \in \mathcal{F} \text{ (w.r.t. to a contact v.f. } \gamma)\}$$

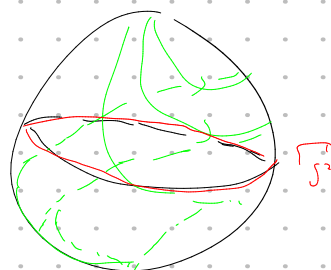
Ex 1 $S^2 \subset (\mathbb{R}^3, \ker(\alpha - x dx + dy + dz))$

$$\gamma = x \partial_x + y \partial_y + z \partial_z \quad \gamma \nmid S^2$$

$$L_\gamma \alpha = L_\gamma dx + d(L_\gamma x) = z \alpha$$

$\Rightarrow S^2$ is convex

$$\alpha(\gamma) = z \neq 0 \quad \Rightarrow \quad \Gamma_{S^2} = \{z = 0\} \subset S^2$$



Ex 2 $S^1 \times S^1 \subset (S^1 \times \mathbb{R}^2, \ker(\cos(n\theta) dx - \sin(n\theta) dy))$

$$\gamma = x \partial_x + y \partial_y \text{ is a contact v.f. } \nmid S^1 \times S^1$$

$$\alpha_n(\gamma) = x \cos(n\theta) - y \sin(n\theta)$$

$$\Rightarrow \Gamma_{S^1 \times S^1} = \{(\theta, \pm \sin(n\theta), \pm \cos(n\theta))\}$$

Def: Let X be a v.f. repr. a singular foliation on a surface S with volume form ω

$$\Gamma \subset S \text{ dividing } X \Leftrightarrow$$

$$* \Gamma \subset S$$

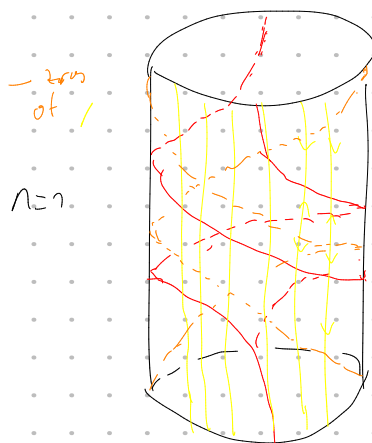
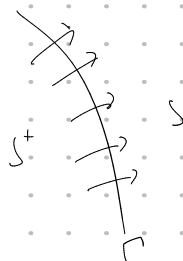
$$* X \nmid \Gamma$$

$$* \Gamma \neq \emptyset$$

$$* L_X \omega \neq 0 \text{ on } S \setminus \Gamma$$

$$* \text{ for } S_\pm = \{p \in S \mid \pm \alpha_X \omega(p) > 0\}$$

then X points out of S_+ along Γ



Corollary 9: Γ_S of a convex surface S divides S_η

Proof: $\Gamma_S = \{u(p)=0\}$ contact condition $du \neq 0$ on Γ_S .

$\Rightarrow \Gamma_S$ is a η -mf

* If $p \in \Gamma_S = u^{-1}(0)$ st. $x \in T_p \Gamma_S$

$\Rightarrow T_p \Gamma_S \subset \ker(\alpha)$ (because $x \in \ker(\beta)$)

$\Rightarrow \forall v \in T_p \Gamma_S = \ker(\alpha dp + \beta \wedge d\eta) = 0$ \downarrow contact condition

* If $\Gamma = \emptyset \Rightarrow u \neq 0 \Rightarrow \beta/u + dz$ is a contact form

$\Rightarrow d(\beta/u)$ is an exact area form on S \downarrow

$$\Gamma \neq \emptyset \Rightarrow \text{area}(S) = \int_S d(\beta/u) \stackrel{\text{Stokes}}{=} \int_{\partial S} \beta/u = 0$$

* on $S_\pm \times \mathbb{R}$ we can write $\alpha = \ker(\underbrace{\beta/|u|}_{=: \tilde{\beta}} \pm dz)$

\Rightarrow contact condition $= \pm d\tilde{\beta} > 0$ on S_\pm

$\Rightarrow \pm d\tilde{\beta}$ is an area form ν_\pm on S_\pm

$\Rightarrow d(L_x \nu_\pm) = d\tilde{\beta} = \pm \nu_\pm$

$\Rightarrow \text{div}_{\nu_\pm}(X) = \pm 1$ on S

$L_X \nu_\pm = f \nu_\pm$ on S_\pm for $f: S_\pm \rightarrow \mathbb{R}$

$\stackrel{\text{continuity}}{\Rightarrow} L_X \nu \neq 0$ on $S \setminus \Gamma$ & X points out of S_\pm along Γ



Thm 20: (a) $S \subset (\mathbb{R}^3, \eta)$ is convex $\Leftrightarrow S_\eta$ is divided by a η -mf $\Gamma \subset S$

(b) Γ is (up to topology) determined by S_η

Proof: (a) " \Rightarrow ": $\Gamma = \Gamma_S$ by Corollary 9

" \Leftarrow ": Let u be a value function & x representative of S_η

$$\beta := L_x u \quad \& \quad \alpha := \beta + u dz \quad \text{on } S \times \mathbb{R}$$

α is contact $\Leftrightarrow u \text{div}_{\nu_\pm}(X) - X(u) > 0$

Γ divides $S_\eta \Rightarrow \exists$ suitable choice of u s.t. $u \text{div}_{\nu_\pm}(X) - X(u) > 0$

$\Rightarrow \ker(\alpha)$ is an \mathbb{R}^2 -invariant contact structure on $S \times \mathbb{R}$ & $S_\eta = S_{\ker \alpha}$

thm 3

$\Rightarrow \exists k$ for α are contactomorphic near S

\downarrow convex w.r.t. $\ker \alpha$

$\Rightarrow S$ convex w.r.t. η

(5) Let γ_0, γ_1 be contract vector fields tangent to S

$$\Gamma_i := \{ \gamma_i(p) \in \mathbb{R}_p \}$$

write $\gamma_i = \exp(B + u_i \cdot dz)$ (u_i corresponding to Γ_i)

$$\Rightarrow \Gamma_i = \{ u_i(p) = 0 \}$$

consider $\alpha_\epsilon = B + (1-\epsilon)u_0 + \epsilon u_1 \cdot dz$

Apply the Moser trick to α_ϵ to get a contact isotopy ψ_ϵ

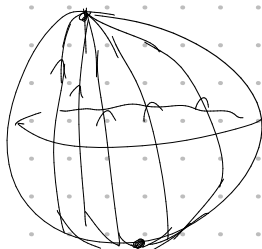
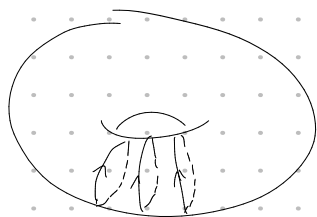
$$\text{s.t. } \psi_\epsilon(\Gamma_0) = \Gamma_1$$

□

THM 11: Poincaré-Hopf

X a vector field with isolated singularities on a surface S

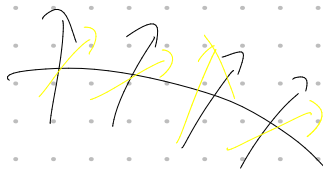
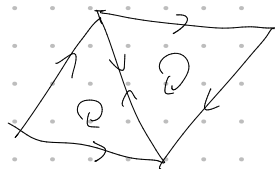
$$\text{s.t. } \int_{\partial S} \langle X, \nu \rangle ds = \sum_{p \in S} \text{Ind}_p(X)$$



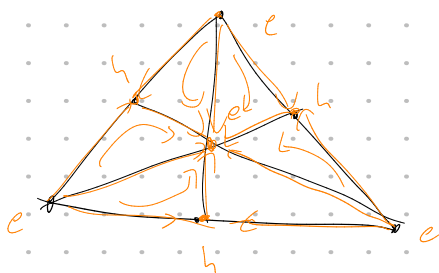
Proof: (i) Let X & Y be such a v.f.

Choose a triangulation T of S s.t. no zero of X or Y lies on an edge or vertex of T & every triangle contains at most one zero

$$\Rightarrow \sum_{p \in S} (\text{Ind}_p(X) - \text{Ind}_p(Y)) = \sum_{\Delta \in T} \frac{1}{2\pi} \left(\text{Change of angle between } X \& Y \text{ along } \partial \Delta \right) = 0$$



(ii) Let T be a triangle. $\Rightarrow \exists X$ s.t. $\sum_{p \in S} \text{Ind}_p(X) = V - E + F = \chi(S)$



$$\begin{array}{lcl} V & \xrightarrow{\quad} & \text{zero of index} = 1 \\ E & \xrightarrow{\quad} & \text{"} = -1 \\ F & \xrightarrow{\quad} & \text{"} = 1 \end{array}$$

□

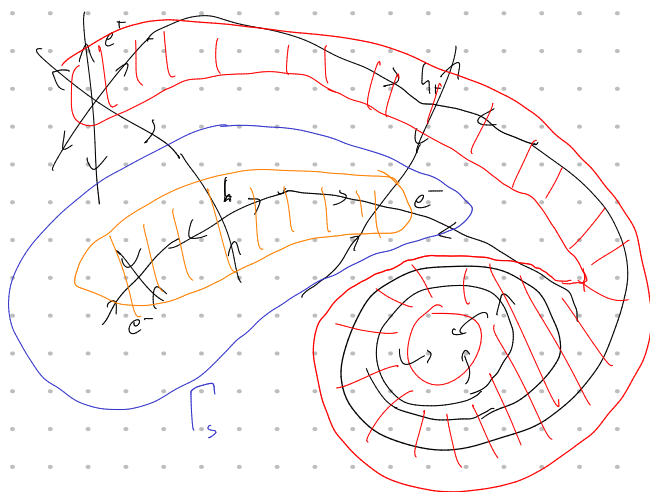
Thm 22 If S_g is non-singular $\Rightarrow S$ is convex

proof: $S_{\pm}^0 :=$ Disk around \pm elliptic points

\cup annuli and repelling/attracting periodic orbits

\cup bands around stable/unstable separatrices of \pm hyp. point

s.t. $S_+^0 \cap S_-^0 = \emptyset$ & $X \nmid \partial S_{\pm}^0$



rescale $g|_V$ with $g: S \rightarrow \mathbb{R}^4$ s.t. $\pm \det_{g|_V}(x) > 0$ on S_{\pm}^0
On $A := S \setminus (S_+^0 \cup S_-^0)$ we have A_g is non-singular

principle
proof $X \nmid \partial S \Rightarrow A = \bigcup \text{Annuli} \Rightarrow \Gamma := \text{Spines of } A \text{ divide } S_g$

Corollary: Every $SC(\mathcal{M}, \mathcal{F})$ is C^∞ -close to a convex surface

proof: Thm 4 + Thm 22

Thm 24: [Gromov flexibility]

Let $SC(\mathcal{M}, \mathcal{F})$ be convex. For a singular foliation on S divided by Γ_S

& vS a NBD of S in \mathcal{M}

$\Rightarrow \exists$ isotopy $\psi_t: S \rightarrow vS$ of embeddings s.t.

(i) $\psi_0 =$ inclusion $SC \hookrightarrow \mathcal{M}$

(ii) $\psi_t(S)$ is convex with boundary set $\psi_t(\Gamma_S)$

(iii) $(\psi_t(S))_t = \psi_t(F)$

remark: Γ_S determines S_g up to perturbation & S_g determines \mathcal{F} near S

proof: HW

Idea: (1) Use proof of Thm 12 to find vectorfields X_0 & X_1 defining S_0 & F s.t.
 $\alpha_t = (X_0)_t dt + (X_1)_t dt$ are \mathbb{R}^2 -equiv
 for $X_t := (1-t)X_0 + tX_1 \Rightarrow f = S_{g_1} \& S_{g_0} \subset S_g$

(2) Apply Poincaré trick to $\alpha_t \Rightarrow$ get isotopy ψ_t with (i) - (iii) \square

6.6. The Legendrian realization principle & Giroux's Criterion

Thm 15 [Legendrian realization principle Karshon-Kurka] \square

Let $S(K, g)$ be convex & $G \subset S$ be a non-rotating graph
 (i.e. $G \not\cap \Gamma_S$ & Γ_S intersects every component of $S \setminus G$)

$\Rightarrow \exists$ isotopy of S s.t. G is contained in char. foliation (i.e. Legendrian) \square

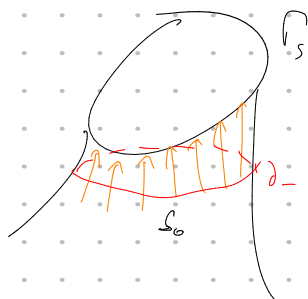
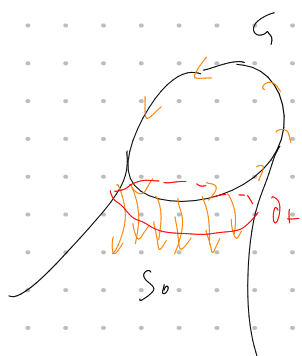
proof: For $G = 1$ -mfld (gen case is HW)

By Thm 14 it is enough to construct a singular foliation that contains G and is divided by Γ_S .

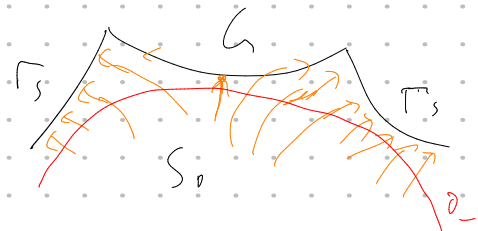
Let S_0 be a component of $S \setminus (G \cup \Gamma_S)$.

Wlog $S_0 \subset S_+$

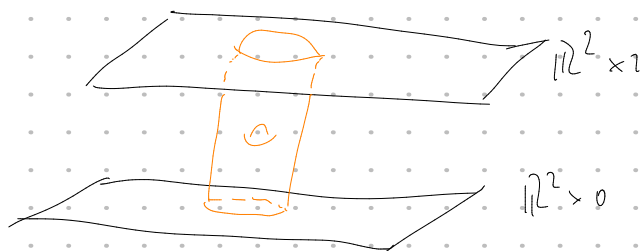
Near ∂S_0 :



non-rotating $\Rightarrow \partial_- \neq \emptyset$



Embed $S'_0 = S_0 \setminus \nu(\partial S_0)$ in $\mathbb{R}^2 \times [0, 1]$ s.t. $\partial_- \subset \mathbb{R}^2 \times 0$ & $\partial_+ \subset \mathbb{R}^2 \times 1$



s.t. height function h is Morse and has at most one maximum & no minima

$\Rightarrow \partial h$ extends the framing with at most one on elliptic point. \square

Thm 76: Let L be Legendrian in a convex surface S . Then

(1) $\lambda_c \cdot S = -\frac{1}{2} \#(L \pitchfork \Gamma_S)$

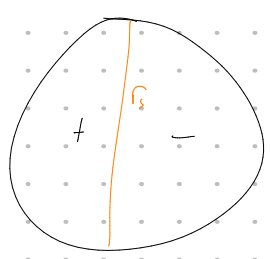
if S is a sectant surface of L

(2) $TB(L) = -\frac{1}{2} \#(L \pitchfork \Gamma_S)$

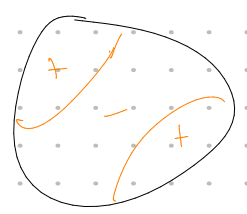
(3) $\text{Rot}(L, \Gamma_S) = X(S_+) - X(S_-)$

Remark: Thm 74 is also true for surfaces S w/ Legendrian boundary ∂S
 $\Leftrightarrow TB(\partial S) \leq 0$

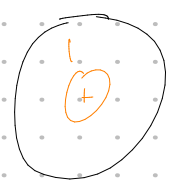
Examples



$TB(\partial S) = -1$
 $\text{rot}(\partial S) = 0$



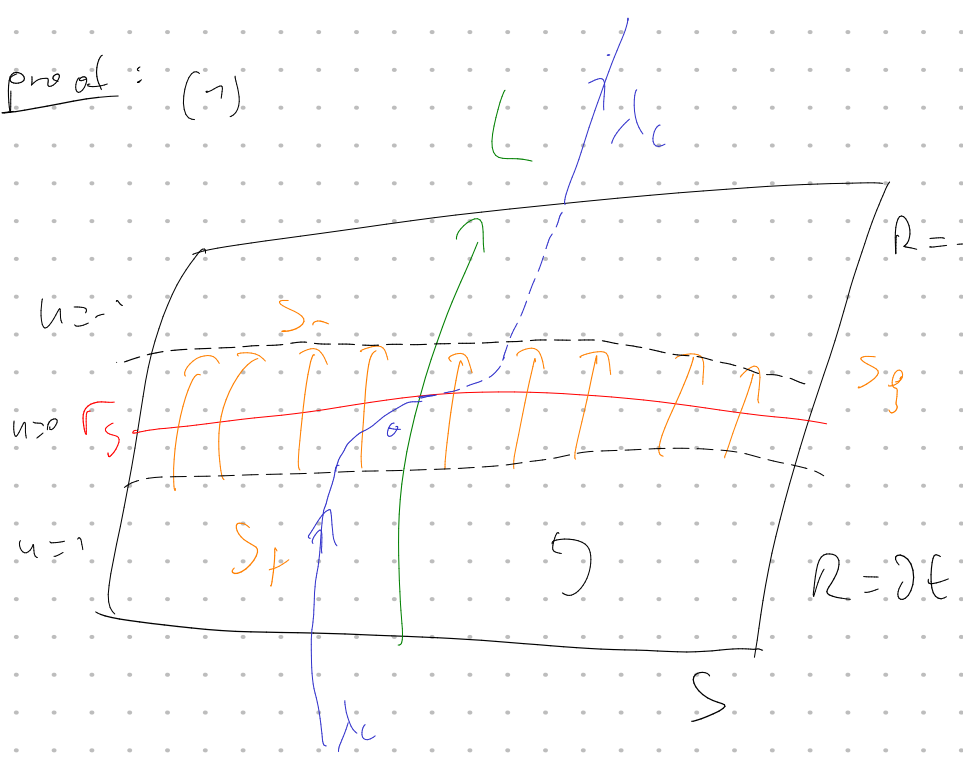
$TB(\partial S) = 2$
 $\text{rot}(\partial S) = 1$



$TB = 0$
 $\text{rot} = 1$

\Rightarrow overruled, because it contradicts the Bennequin inequality

proof: (1)



$\oint = \int_0^1 (\beta + u dt)$

$R \in TS \Leftrightarrow u = 0$

$\Rightarrow \lambda_c \cdot S = -\frac{1}{2} \#(L \pitchfork \Gamma_S)$

(2) if $\partial S = L \Rightarrow \text{fb}(L) = \text{Ch}(L, \lambda_C) = S \cdot \lambda_C \stackrel{(\sim)}{=} -\frac{1}{2} \#(L \cap \Gamma_S)$

(3) $\text{rat}(L, [S]) =$ signed count of singularities of an extension of TL_C over S .
 $= r_+ - r_-$

with $r_{\pm} := e_{\pm}^{\text{int}} - h_{\pm}^{\text{int}} + \frac{1}{2} (e_{\pm}^{\partial} - h_{\pm}^{\partial})$

$e_{\pm}^{\text{int}} := \#$ interior \pm elliptic points

$h_{\pm}^{\text{int}} := \#$ interior \pm hyperbolic points

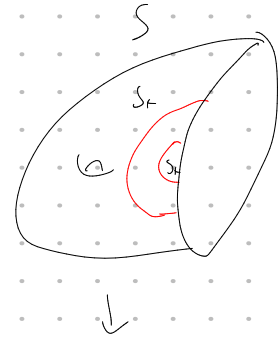
$e_{\pm}^{\partial} := \#$ boundary \pm elliptic points

$h_{\pm}^{\partial} := \#$ boundary \pm hyperbolic points

Claim: $\chi(S_{\pm}) = r_{\pm} + \frac{1}{2} \text{TB}(L)$

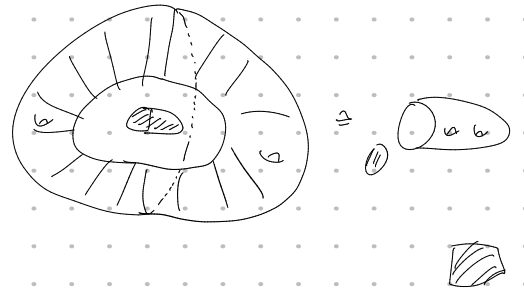
$S' = S_+ \cup_{(\partial S \cap S_+)} S_+ \xrightarrow{\text{Poincaré-Lefschetz}} \chi(S') = 2r_+$

$\Rightarrow \chi(S') = \chi(S_+) + \chi(S_-) - \chi(\partial S \cap S_+)$
 $\stackrel{(\sim)}{=} 2\chi(S_+) - \text{TB}(L)$
 $\stackrel{\text{Poincaré-Lefschetz}}{=} 2r_+ - \text{TB}(L)$



Analogous for $S_- \Rightarrow \chi(S_{\pm}) = r_{\pm} + \frac{1}{2} \text{TB}(L)$

$\Rightarrow \text{rat}(L, [S]) = r_+ - r_- = \chi(S_+) - \chi(S_-)$

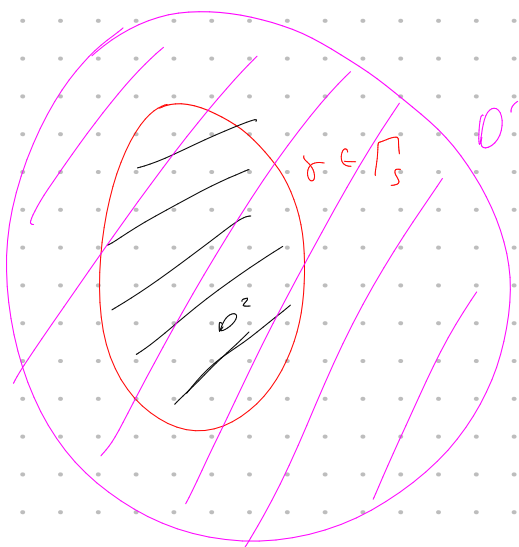


Thm 17: [Gromov criterion]

$S \subset (M, \xi)$ convex. Then a vertical point MBLH of S is tight

$\Leftrightarrow S \stackrel{\text{diffeomorphic}}{\neq} S^2$ & Γ_S contains no contractible curve
 or: $S = S^2$ & Γ_S is connected

proof: " \Rightarrow " 1) Let $S \neq S^2$ & assume $\exists \gamma \in \Gamma_S$ simple closed curve
 s.t. $\exists D^2 \subset S$ with $\partial D^2 = \gamma$

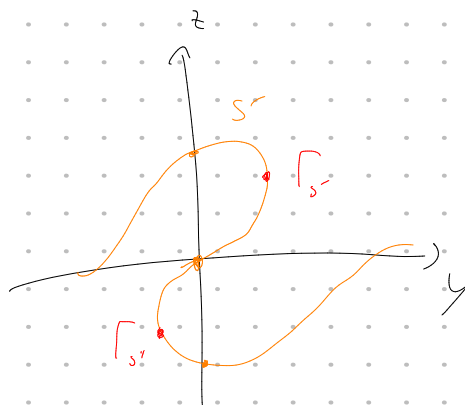
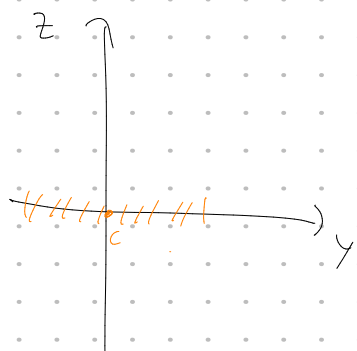
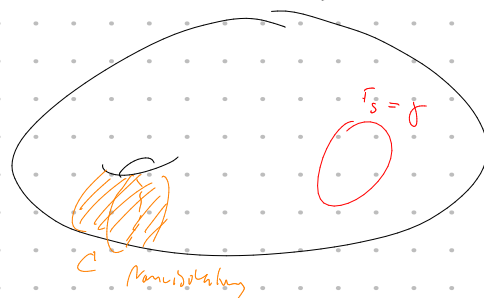


\Rightarrow If $\Gamma_S \setminus \gamma \neq \emptyset$
 $\xrightarrow{\text{Thm 15}} \partial O' \text{ is l.g.}$
 $\xrightarrow{\text{Thm 16}} \Gamma_B(\partial O') = \emptyset$
 $\Rightarrow \text{OT}$

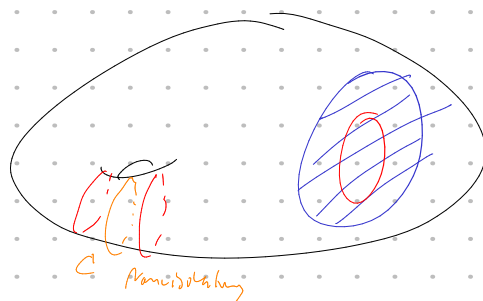
\Rightarrow If $\Gamma_S \setminus \gamma = \emptyset$ $\xrightarrow{S \neq S^2} \exists$ curve C disjoint from Γ_S & noncontractible

$\xrightarrow{\text{Thm 15}} C$ is legend

$\Rightarrow \nu C \cong$ annulus A in the
 (x,y) -plane of $(S^1 \times \mathbb{R}_{x,z}^2, \nu(xdy + dz))$



$\Rightarrow \Gamma_{S'} = C \cup \Gamma_S$

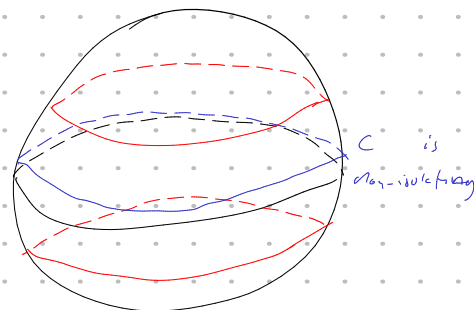


$\xrightarrow{\text{Thm 15 & 16}} \nu S' \text{ is OT}$

$\Rightarrow \nu S \text{ is OT}$

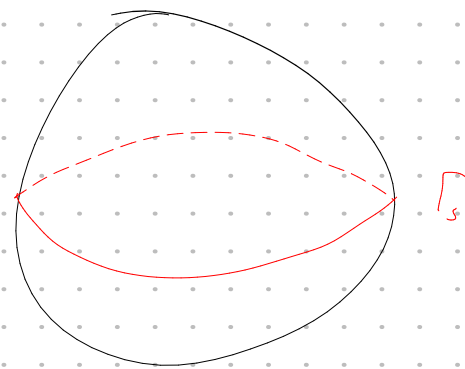
(2) If $S = S^2$ & Γ_S disconnected

$\Rightarrow C$ gives another disk



" \Leftarrow " (2) if $S=S^2$ & Γ_S is connected

$\xrightarrow{\text{Gromov flexibility}}$ $V_S \cong V_{S^2} \subset (\mathbb{R}^3, \mathcal{F}_{st})$ is tight



(1) HW: Idea Look at univ. cover \tilde{S}

4.7 The Beeneken inequality

Thm 18 [Thurston]

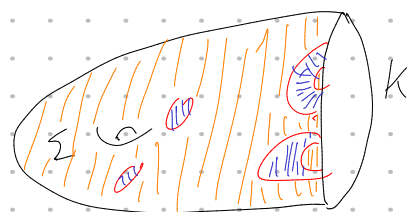
$(\mathcal{M}, \mathcal{F})$ is tight $\iff \forall K \subset (\mathcal{M}, \mathcal{F})$ nullhomologous & Legendrian
 $\forall \Sigma$ Seifert surface of K : $t_b(K) \pm \text{rot}(h, [\Sigma]) \leq 2g(h)$

proof: " \Leftarrow " if $(\mathcal{M}, \mathcal{F})$ is OT $\Rightarrow \exists$ Legendrian unknot with $t_b(h)=0$

" \Rightarrow " Let Σ be a Seifert surface of minimal genus of $\text{Lg. } K$.
 After stabilization (not changing $t_b(K) \pm \text{rot}(K)$) we can assume that
 $t_b(K) \leq 0$. \Rightarrow we can perturb Σ to be convex

Thm 16
 $\Rightarrow t_b(K) = -\frac{1}{2} \#(K \cap \Gamma_S)$
 $\& \text{rot}(K) = \chi(\Sigma_+) - \chi(\Sigma_-)$

$$\begin{aligned} * \chi(\Sigma = \Sigma_+ \cup_{\Gamma_S} \Sigma_-) &= \chi(\Sigma_+) + \chi(\Sigma_-) - \chi(\Gamma_S) \\ \chi(S^1) &= 0 \\ &= \chi(\Sigma_+) + \chi(\Sigma_-) - \underbrace{\# \text{ Arcs on } \Gamma_S}_{+T_b(K)} \end{aligned}$$



$$\Rightarrow 2g(K) - 1 = -\chi(\Sigma) = -\chi(\Sigma_+) - \chi(\Sigma_-) - t_b(K)$$

$$\geq t_b(K) - \chi(\Sigma_+) - \chi(\Sigma_-)$$

(Thm 17 + $(\mathcal{M}, \mathcal{F})$ is tight) $\Rightarrow \Sigma_{\pm} \in \mathcal{C}$ have no disk comp.

$$\geq t_b(K) + \chi(\Sigma_+) - \chi(\Sigma_-)$$

$$= t_b(K) + \text{rot}(K)$$

$\chi \leq 0$ if $\Sigma \neq \emptyset, \mathbb{R}^2$

