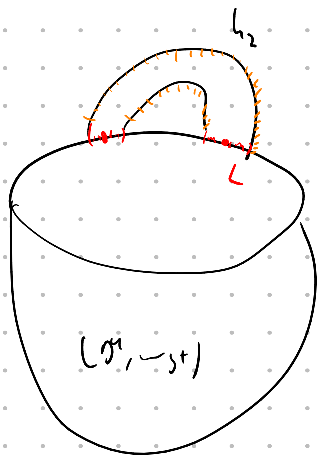
# 7. Dehn surgery on Legendrian knots

## 7.1 Contact Dehn surgery & fillings

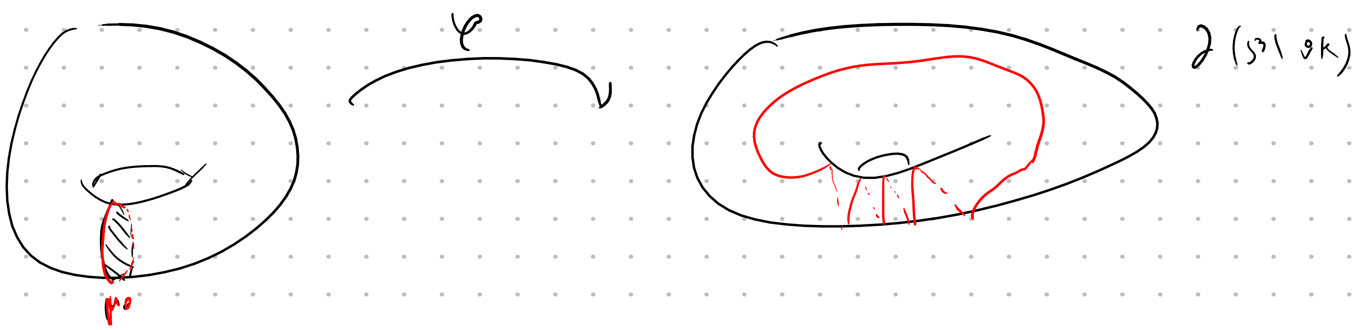



$(S^3, \xi_{st}) \longrightarrow (D^2 \times \partial D^2, \xi') \cup_\varphi (S^3 \setminus \nu L, \xi_{st})$

**Theorem 1:** Let $K \subset (S^3, \xi_{st})$ be a Legendrian knot & $\nu K$ a std NBHD. & $p, q$ coprime.

$$K(p/q) := S^1 \times D^2 \quad \cup_\varphi \quad S^3 \setminus \mathring{\nu} K$$

$$\mu_0 \xmapsto{\varphi} p\mu + q\lambda_c$$




Topology:

(1) $K(p/q)$ is again a 3-mfd that only depends on $p/q \in \mathbb{Q}$ & $K$.

Contact geometry:

(2) $\forall\, p/q \in \mathbb{Q} \setminus \{0\}$ $\exists$ finitely many tight contact structures $\xi'$ on $S^1 \times D^2$ that induce contact structures on $K(p/q)$

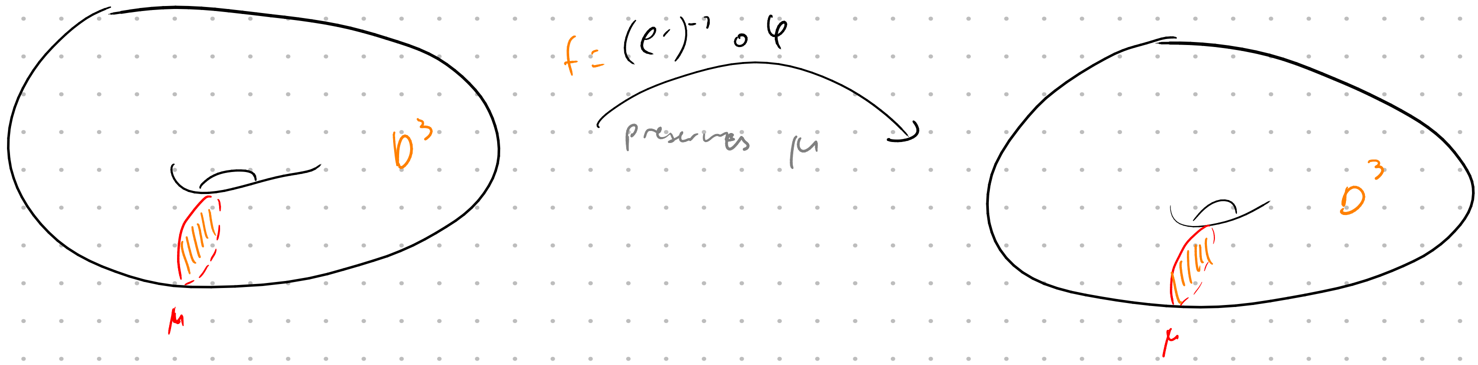
(3) $\forall\, n \in \mathbb{Z}$ $\exists!$ tight c.s. that induces c.s. on $K(1/n)$

proof: (1) Γ Alexander trick:  $n=1,2$

$\forall f: S^n \xrightarrow{\cong C^\infty} S^n$

$\exists F: D^{n+1} \xrightarrow{\cong C^\infty} D^{n+1}$ s.t. $F|_\partial = f$ ⌟

$$\begin{tikzcd}
K_\varphi(p/q) = S^1 \times D^2 \cup_\varphi S^3 \setminus \mathring{\nu} K \arrow[d, "\text{Extension of } (\varphi')^{-1} \circ \varphi"'] \arrow[r, phantom, "\circlearrowright"] & \arrow[d, "\cong", "id"'] \\
K_{\varphi'}(p/q) = S^1 \times D^2 \cup_{\varphi'} S^3 \setminus \mathring{\nu} K &
\end{tikzcd}$$




(i) Extend $f$ over $D^2$

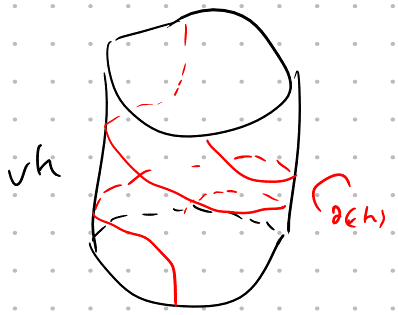
→ cut open along $D^2$

(ii) Extend $f$ over $D^3$

(2) & (3) $(\nu K, \xi_{st}) \overset{\text{Cont}}{\cong} (S^1 \times D^2, \cos\theta\, dx - \sin\theta\, dy)$

⟹ $\partial(\nu K)$ is convex with dividing set $\Gamma_{\partial \nu K}$ of slope $= -1$,

i.e. $\Gamma_{\partial \nu K}$ is given by 2 parallel copies of $\lambda_c$



$S^1 \times D^2 \quad \cup_\varphi \quad S^3 \setminus \mathring{N}$

$\mu_0 \xmapsto{\ \varphi\ } p\mu + q\lambda_c$

$\lambda_0 \xmapsto{\ \varphi\ } r\mu + s\lambda_c$ $\quad$ with $\det\begin{pmatrix} p & q \\ r & s \end{pmatrix} = 1$

$s\mu_0 - q\lambda_0 \xleftarrow{\ \varphi^{-1}\ } \mu$

$-r\mu_0 + p\lambda_0 \xleftarrow{\ \varphi^{-1}\ } \lambda_c$

$\Rightarrow \Gamma_{\partial(\nu h)} = 2$ copies of $\lambda_c \xmapsto{\ \varphi^{-1}\ }$ two copies of $-r\mu + p\lambda_c$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Gamma_{\partial(S^1\times D^2)}$

$\Rightarrow$ slope of $\Gamma_{\partial(S^1\times D^2)} = -\frac{p}{r}$

thm 5.5

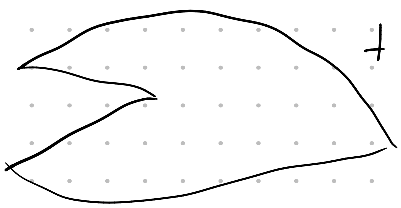
$\Rightarrow$ if $p \neq 0 \Rightarrow \exists$ finitely many tight c.s. on $S^1\times D^2$ s.t.

$$\Gamma_{\partial(S^1\times D^2)} \xrightarrow{\ \varphi\ } \Gamma_{\partial\nu K}$$

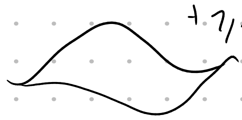
if $p = \pm 1 \Rightarrow \exists!$ tight c.s. on $S^1 \times D^2$ s.t.

$\square$

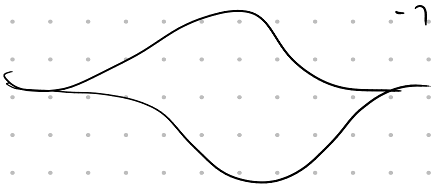
**Ex:**



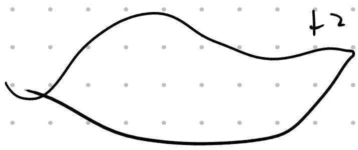
$+1$ $\quad = (S^3, \xi_{ot}) =$



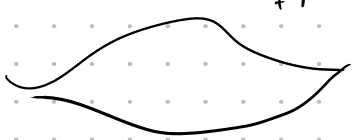
$+1/2$



$-1$ $\quad = (\mathbb{RP}^3, \xi_{st})$



$+2$ $\quad = (S^3, \xi_{ot})$



$+1$ $\quad = (S^1 \times S^2, \xi_{st})$

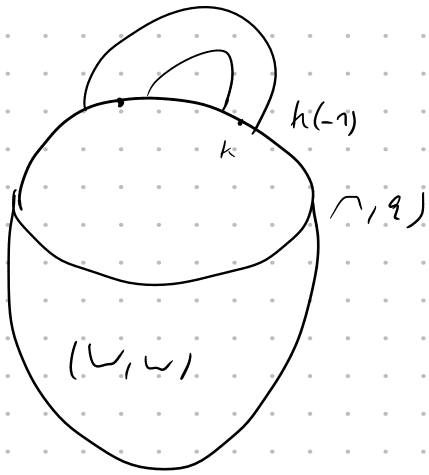
**Thm 2:** Let $K \subset (S^3, \xi_{st}) = \partial(D^4, \omega_{st})$ Leg.

$$\partial(W_K, \omega) = K(-1)$$

$W_K = D^4 \cup$ Weinstein 2-handle attached along $K$

**proof:** HW: compute the framing in the local model w.r.t. the contact framing, see that it matches $-1$ twist

**Cor 3:** $(-1)$-surgery preserves fillability




∎

**Thm 4** [Wand] $(-1)$-surgery preserves tightness. ∎

**Thm 5** [Lickorish–Wallace] Any connected, orientable, closed (smooth) 3-mfd can be obtained by smooth (integer) surgery along a link in $S^3$ ∎

**Corollary 6:** [Martinet] Every 3-mfd carries a c.s.

**proof:** Let $L$ be a smooth surgery description of $M$. (Thm 5)
Choose a Legendrian approximation of $L$ s.t. contact surgery coefficient $\neq 0$
( Stabilizing a knot: $\lambda_c \longrightarrow \lambda_c - \mu$

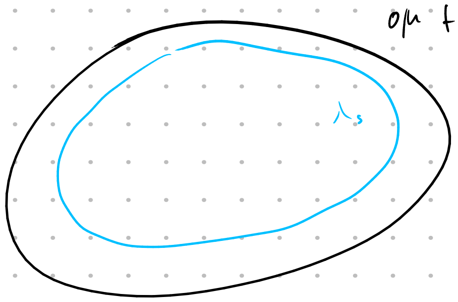
) □

**Thm 7** [Ding–Geiges] Any contact 3-mfd can be obtained by contact $(\pm 1)$-surgery along a Legendrian link from $(S^3, \xi_{st})$

**proof:** In section 7.4 ∎

# 7.2 Surgeries on the unknot

Ex: (1)




$0\mu + 1\cdot\lambda_s$

$$= S^1\times D^2 \cup_\varphi S^3 \setminus \mathring{\nu} U$$

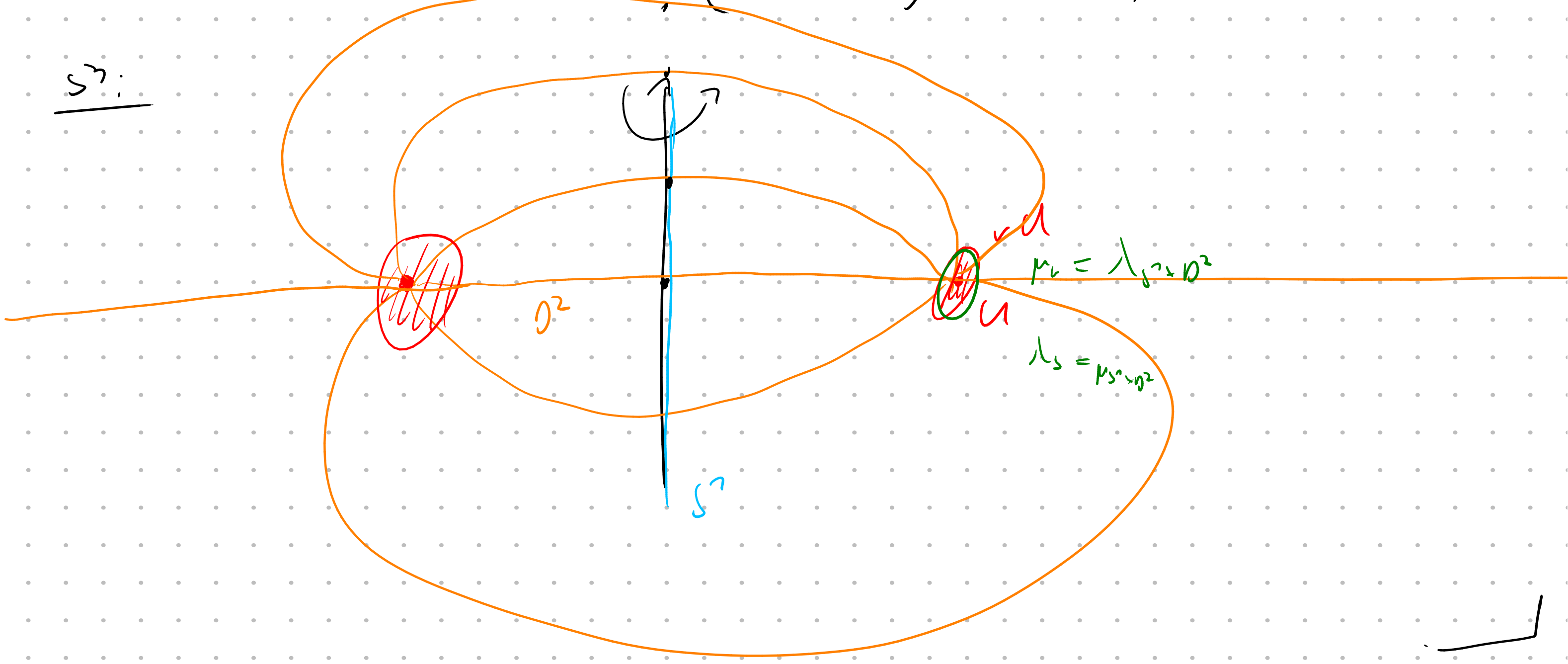
$$\mu_0 \longmapsto \lambda_s$$

$$\lambda_0 \longmapsto \mu_U$$

Claim: $S^3 \setminus \mathring{\nu} U = S^1 \times D^2$

$$S^3 = \partial D^4 = \partial(D^2\times D^2) = (\partial D^2 \times D^2) \cup (D^2 \times \partial D^2)$$

$S^3$:



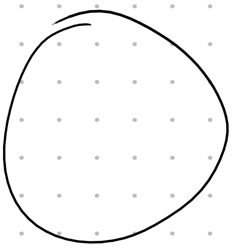

$$= S^1\times D^2 \cup S^1\times D^2$$

$$\mu_0 \longmapsto \mu_1$$

$$\lambda_0 \longmapsto \lambda_1$$

$$= S^1 \times (D^2 \cup D^2)$$

$$= S^1 \times S^2$$

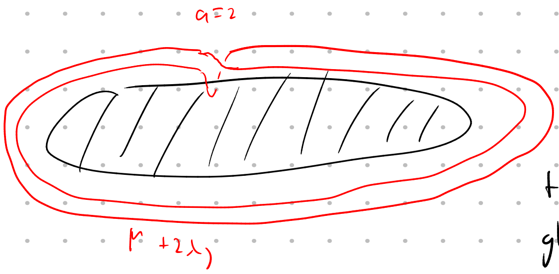
(2)  $\mu + q\lambda_s$ $= S^3$

$$U(\mu+q\lambda_s) = S^1\times D^2 \cup_f S^3\setminus i\,U$$

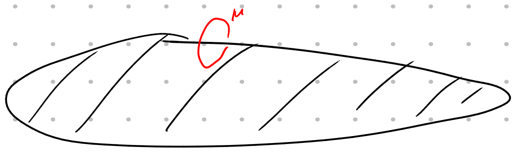
$$\begin{tikzcd} \mu_0 \arrow[r, mapsto] \arrow[d, "\cong"', "id"] \arrow[dr, phantom, "\circlearrowright"] & \mu + q\lambda_s \arrow[d, bend left] \\ \mu_0 \arrow[r, mapsto] & \mu \end{tikzcd}$$

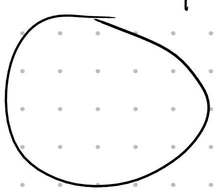
$$S^3 = U(\mu) = S^1\times D^2 \cup_{id} S^3\setminus i\,U$$

$$f: S^3\setminus i\,U \xrightarrow{\cong} S^3\setminus i\,U$$

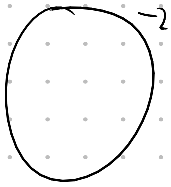



cut along $D$
twist $q$ times
glue along $D$



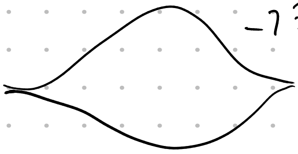
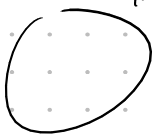

(3) HW:  $-p\mu + q\lambda_s$ $= L(p,q) = S^3/\mathbb{Z}_p$

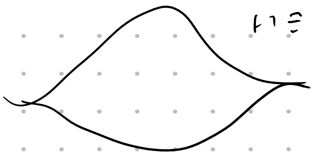
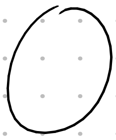
check that group action preserves splitting of $S^3$ into two solid tori

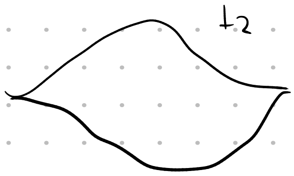
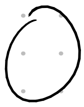
(4) HW:  $-2\mu+\lambda_s$ $= L(2,1) = \mathbb{RP}^3$

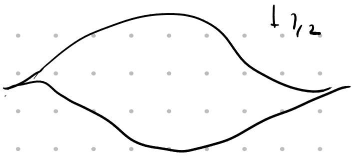
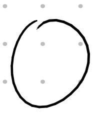
Let $K$ be a Legendrian knot.

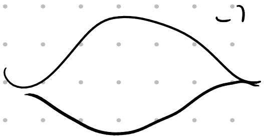
$\Rightarrow \boxed{\lambda_c = tb(K)\,\mu + \lambda_s}$

 $-1 \mathrel{\hat{=}} -\mu + \lambda_c$ $\overset{C^\infty}{\cong}$  $-2\mu + \lambda_s$ $= \mathbb{RP}^3$

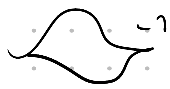
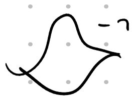
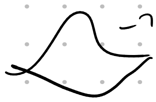
 $+1 \mathrel{\hat{=}} +\mu + \lambda_c$ $\overset{C^\infty}{\cong}$  $\lambda_s$ $= S^1 \times S^2$

 $+2$ $\overset{C^\infty}{\cong}$  $\mu + \lambda_s$ $= S^3$

 $+1/2$ $\overset{C^\infty}{\cong}$  $\mu + 2(-\mu + \lambda_s)$ $=$  $-\mu + 2\lambda_s$ $= S^3$

**Lemma 8:**  $-1$ $\overset{cont}{\cong} (\mathbb{RP}^3, \xi_{st})$

proof:

-  $-1$ $\overset{C^\infty}{\cong} \mathbb{RP}^3$
- Corollary 3 $\Rightarrow$  $-1$ is fillable $\overset{\text{Thm 6.3}}{\Rightarrow}$  $-1$ is tight.
- $(\mathbb{RP}^3, \xi_{st}) := (S^3, \xi_{st})/\mathbb{Z}_2 \Rightarrow$ univ. cover is tight $\Rightarrow (\mathbb{RP}^3, \xi_{st})$ is tight
- $\exists!$ tight contact structure on $\mathbb{RP}^3$

$\ulcorner\ \mathbb{RP}^3 = \overset{V_1}{S^1 \times D^2} \cup_\varphi S^3 \setminus \mathring{U} = \overset{V_2}{S^1 \times D^2}\ \urcorner$

$\mu_1 \longmapsto -2\mu_u + \lambda_s = +\mu_2 - 2\lambda_2$

$\lambda_1 \longmapsto \lambda_2$

Let $\xi$ be a tight c.s. on $\mathbb{RP}^3$.

$\Rightarrow B_1$ by app thm $\Rightarrow$ $\cup$ by $(V_1, \xi) = (V_{\varepsilon_1}, \xi_{st})$
with slope $1/n$

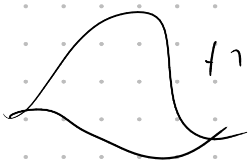
$\Rightarrow (V_2, \xi)$ is a tight solid torus with convex $\partial$

$\Rightarrow \Gamma_1 = \mu_1 + n\lambda_1 \xrightarrow{\varphi} \mu_2 - 2\lambda_2 + n\lambda_2 = \mu_2 + (n-2)\lambda_2$

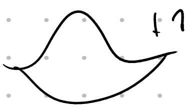
$\Rightarrow$ slope of $(V_2, \xi)$ is $\frac{1}{n-2}$

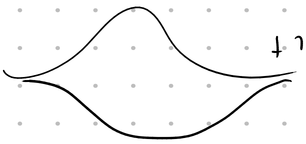
$\overset{\text{Thm 5.5}}{\Rightarrow} \exists! \, (V_2, \xi)$

**Lemma 9:**



$= (S^1 \times S^2, \xi_{st})$

**Proof:** •  $\overset{C^\infty}{\cong} S^1 \times S^2$

•  $= \underset{V_1}{\overset{\shortparallel}{S^1 \times D^2}} \quad \cup_\varphi \quad \underset{V^2}{\overset{\shortparallel}{S^3 \setminus \mathring{U}}}$

$\mu \longmapsto \mu_n + \lambda_L = \lambda_S = \mu_2$

$\lambda_1 \longmapsto \lambda_2$

$\Gamma_1 = \lambda_1 - \mu_1 \xrightarrow{\varphi} \lambda_2 - \mu_2 = \Gamma_2$

$\Rightarrow V_1$ & $V_2$ are tight solid tori of slope $= -1$

• claim: $(S^1 \times S^2, \ker(x\,d\theta + y\,dz - z\,dy)) = V_1 \cup_\varphi V_2$

$T : T^2 \hookrightarrow S^1 \times S^2$

$(\theta, \varphi) \longmapsto (\theta, f(\varphi), \sqrt{1 - f^2(\varphi)} \cos\varphi, \sqrt{1 - f^2(\varphi)} \sin\varphi)$

with $f(\varphi) = \frac{1}{2}\sin(\varphi)$

$T T = \langle \partial_\theta, \; V = \left(0, f', -\frac{ff'}{\sqrt{1-f^2}}\cos\varphi - \sqrt{1-f^2}\sin\varphi, \;\; \text{''} \sin\varphi + \text{''}\cos\varphi\right) \rangle$

$\alpha(\partial_\theta) = f \qquad\qquad \alpha(V) = 1 - f^2$

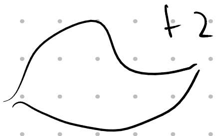
$\Rightarrow$ char. foliation $T_\xi$ is given by $\partial_\theta - \frac{f}{1-f^2}\partial_\varphi$.

$\Rightarrow T$ is convex with dividing set

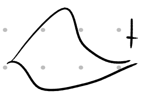
$\{\varphi = \frac{\pi}{2}\}$ & $\{\varphi = \frac{3}{2}\pi\}$

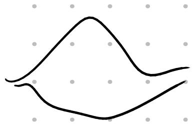
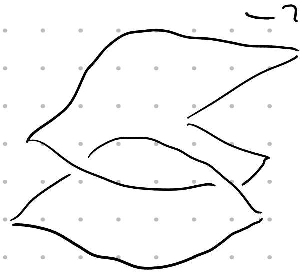
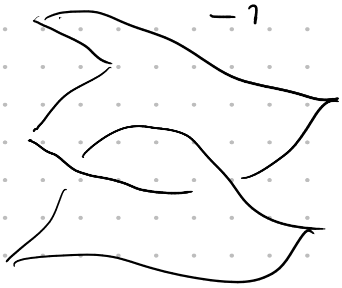
$\Rightarrow$ T is a convex torus in $(S^1 \times S^2, \xi_{st})$, dividing set [illegible] two solid tori of slope $1/n$

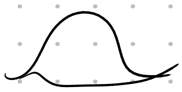
$\Rightarrow$ there is a contactomorphism $\square$

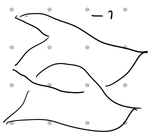
Lemma 10:  $+2$ $= (S^3, \xi_{st})$

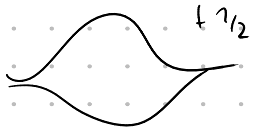
proof:

-  $+2 \overset{C^\infty}{=} S^3$

-  $+2$ $\overset{\text{cont}}{\cong}$  $-1$, $+1$ or  $-1$, $+1$

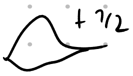
-  $+1$ $= (S^1 \times S^2, \xi_{st})$

cor 3 $\Rightarrow$  $-1$, $+1$ is fillable $\Rightarrow$ tight $\Rightarrow \xi_{st}$ $\square$

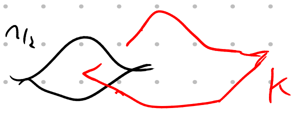
**Lemma 11:** 

= is an OT c.s. on $S^3$

**Proof:**

- 

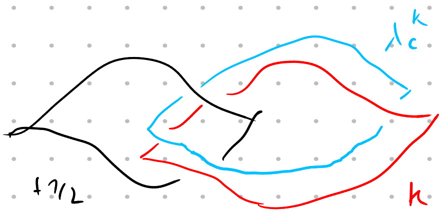
$\cong^{C^\infty} S^3$

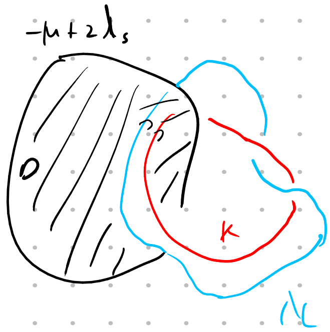
- 

K yields a Legendrian knot in the surgered mfd

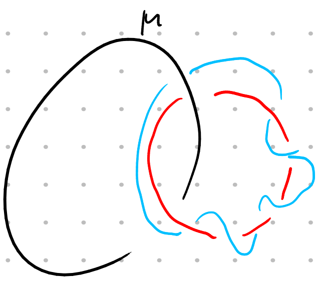
claim: K violates the Bennequin ineq.

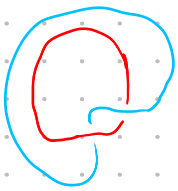
$tb(K) = lk(K, \lambda_c^K)$



$\cong^{C^\infty}$



$\cong$ 2 twists along 0



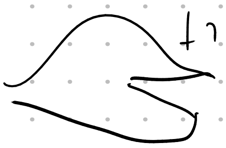
=


$lk = +1$

$\Rightarrow tb(K) = +1$ & K is an unknot

$\Rightarrow$ 

is OT ∎

**Thm:** 

= OT