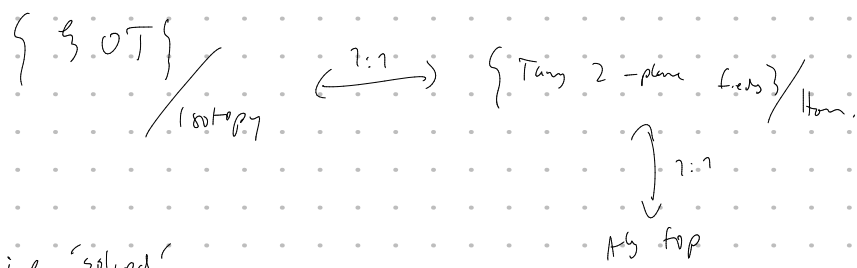


5. Existence & classification of contact 3-manifolds & leg knots

5.1 by picture

OT = flexible:



Tight = rigid

\exists tight with a unique tight c.s.

\exists tight with no c.s.

\exists tight with ∞ c.s.

5.2 classification of tight contact structures

Thm 1: $(\mathbb{R}^3, \xi_{st}), (S^2, \xi_{st}), (S^2 \times \mathbb{R}, \xi_{st}), (S^2 \times S^2, \xi_{st})$ are tight

proof: in sect 6

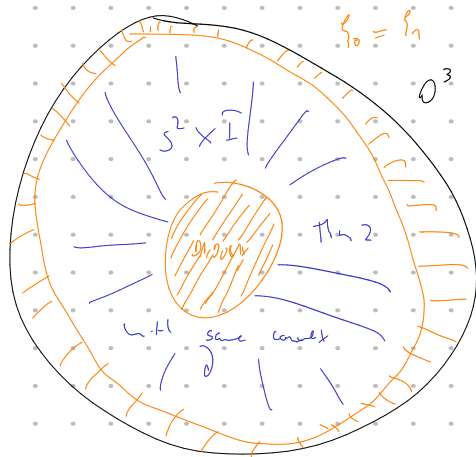
Thm 2: A tight contact structure ξ on $S^2 \times E(2)$ with convex boundary is determined (up to isotopy fixing ∂) by the dividing set of ∂

Thm 3 [Bourbaki]

Let ξ_0, ξ_1 be tight c.s. on D^3 s.t. ∂D^3 is convex and $\Gamma_{\partial D^3}^0 = \Gamma_{\partial D^3}^1$

$\Rightarrow \xi_0$ is isotopic to ξ_1 rel ∂D^3

proof:



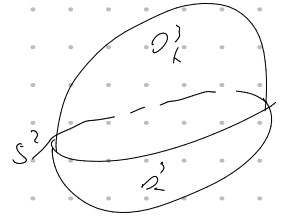
Thm 4 [Eckberg]

$S^3, \mathbb{R}^3, S^1 \times S^2$ admit unique total C.S.

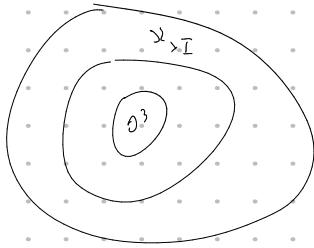
Proof: (1) $S^3 = D^3_+ \cup_{S^2} D^3_-$ why: S^2 convex

$\xrightarrow{\text{q.t.}} \Gamma_{S^2}$ is connected

Thm 3
 $\xrightarrow{\text{q.t.}} q_0$ is isotopic to q_1



(2) $\mathbb{R}^3 = D^3(0) \cup S^2 \times [0,1] \cup S^2 \times [1,2] \cup \dots$



why $S^2 \times n$ convex $\forall n \in \mathbb{Z}$
 $\xrightarrow{\text{q.t.}} \Gamma_{S^2 \times n}$ is connected
 $\xrightarrow{\text{Thm 28}} q_0$ is isotopic to q_1

(3) $S^1 \times S^2 = [-1,1] \times S^2$
 $-1 \times S^2 \sim 1 \times S^2$

\Rightarrow why $\pm 1 \times S^2$ is convex with connected Γ

Let V be a solid torus with tight C.S. st. ∂V is convex

Fix: meridian μ : S.C.C. on ∂V non-trivial in ∂V but trivial in V .

Longitude λ : S.C.C. on ∂V non-trivial in ∂V st. $\mu \neq \lambda = p\mu$

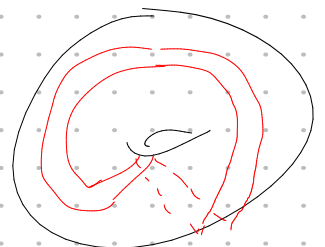
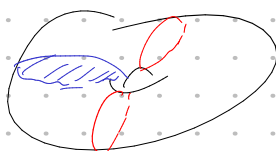
$$\lambda' = \lambda + n\mu$$

$\xrightarrow{\text{q.t.}} \Gamma_{\partial V}$ consists of (non-convex) parallel copies of $p\mu + q\lambda$

$$\text{slope}(\partial V) = \text{slope}(\Gamma_{\partial V})$$

$$\text{Ex: } \text{slope}(S^1 \times D^2, q_1) = -\frac{1}{n}$$

no slope 0:



Thm 5: [Carvery Lemma]

(1) $\forall n \in \mathbb{Z} \exists!$ Tight C.S. on $S^n \times D^2$ with convex ∂ with 2 boundary curves of slope $= \frac{2}{n}$

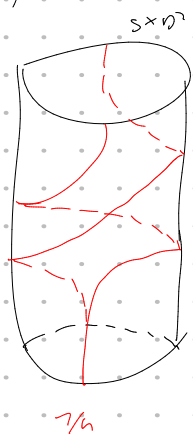
(2) $\forall r \in \mathbb{R} \setminus \{0\} \exists$ further many tight C.S. !!

!!

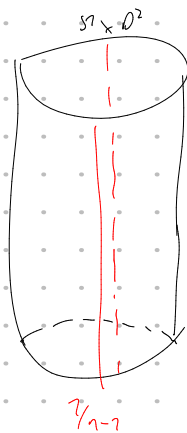
of slope $= r$

proof sketch:

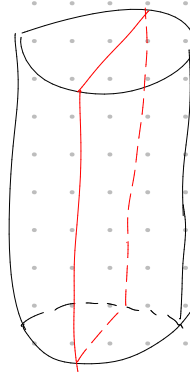
(1)



cut along ∂^2
twist
&
reglue

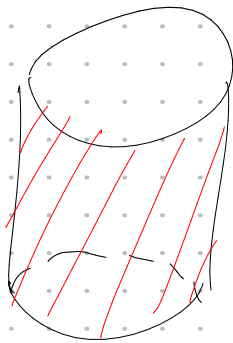


cut
along
down



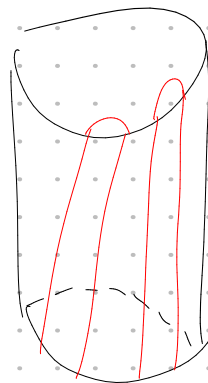
\cap is convex
 \Rightarrow q is unique

(2)

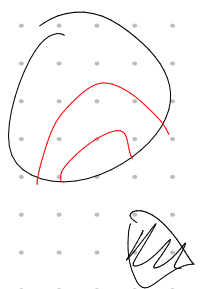


cut

D^3



or



Thm 2: A tight C.S. on $S^2 \times [-1, 1]$ with convex ∂ is defined by isotopy rel ∂ by the family set of ∂

Proof sketch (Thm 2): Write $S^Z = S^2 \times \{Z\}$.

Idea: Make S^Z convex w.r.t. Z .

Problem: Morse-Smale is not generic for one-parameter families.

\S tight \Rightarrow \exists closed orbit in S^2_ξ

\Rightarrow if S^2 is convex $\Rightarrow \Gamma_S$ is connected

By general position (noting of contact structures by any subalgebra):
(use "transversality theory")

(1) $\forall Z$: S^2_ξ contains only finitely many singularities & on finitely many Z -levels there are at most one degenerated one

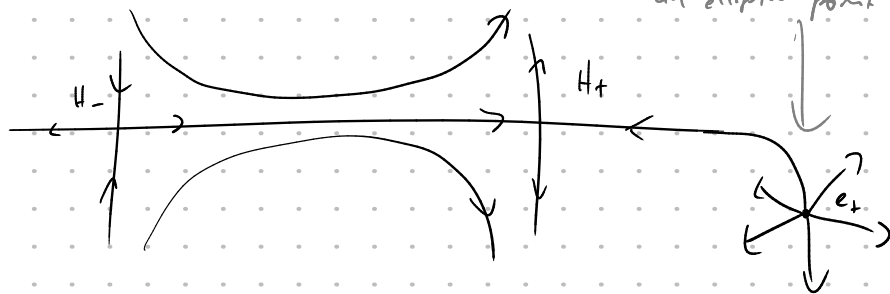
(2) \exists orbits from H_- to H_+ except for finitely many Z -levels where there exist a unique such orbit.

Write z_1, \dots, z_n for these exceptions.

If $Z \notin \{z_1, \dots, z_n\} \Rightarrow S^Z$ is convex & Γ_Z is an equator

Goal: Remove (1) & (2)

(2)



This plan line is to come from an elliptic point

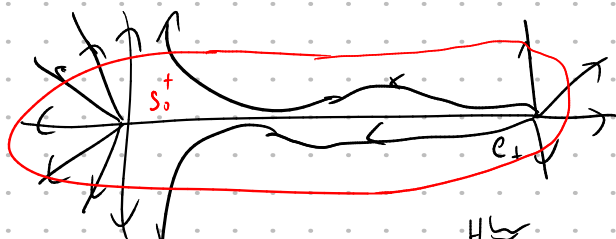
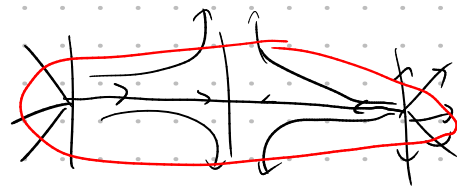
(tightness, uniqueness of flow from $H_- \rightarrow H_+$)

$\Rightarrow H_+$ & e_+ are in elim. pos.

\Rightarrow The proof of the elimination lemma can be done s.t. the nearby surfaces stay convex

\Rightarrow remove all z_i with (2)

(1) S^Z is actually convex



$\Rightarrow \forall Z$: S^Z is convex. \Rightarrow

1. make the chain fol the same
2. they are the same in orbits

\S_0 is isotopic to \S_1 .



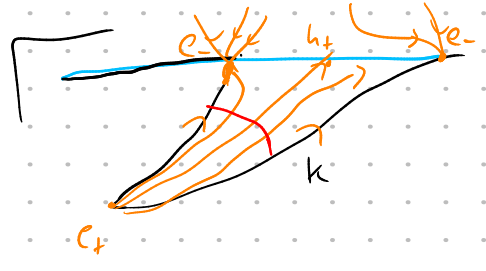
5.3 classification of Legendrian unknots

Thm 6 [Sikirić - Fraser]:

If $K \subset (S^3, \xi_{\text{st}})$ is a Legendrian unknot $\Rightarrow K$ is isotopic to a stabilizer of \bigcirc

Proof: Bennequin inequality $\Rightarrow tb(K) \leq -1$

Step 1: If $tb(K) < -1 \Rightarrow K$ destabilizes.



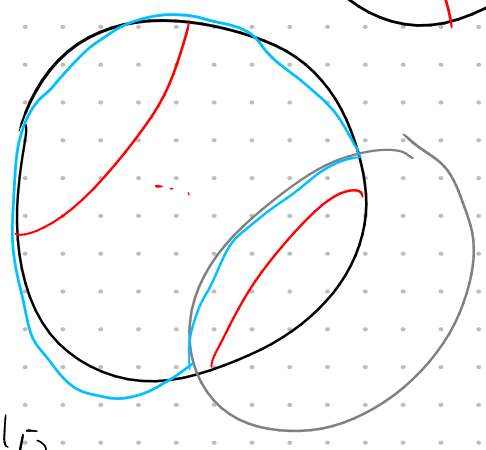
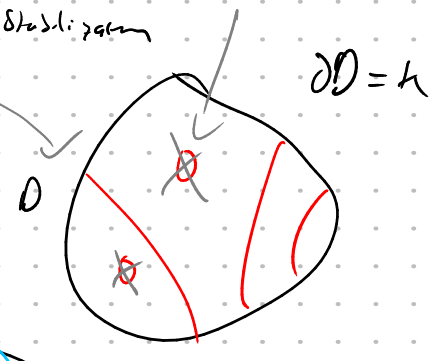
Can't appear in \hookrightarrow $tb(K) \leq -1$

By-pass: a boundary parallel disk \Rightarrow stabilizer

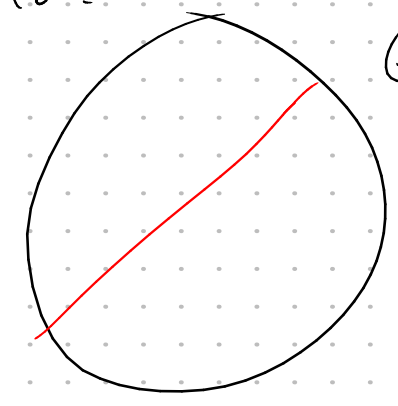
Let D be a disk of K

$tb \leq 0 \Rightarrow \hookrightarrow D$ is convex

$\Rightarrow -1 > tb(K) = -\#(\Gamma_s)$



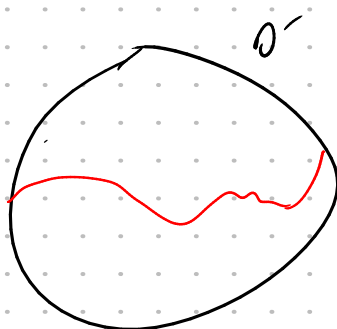
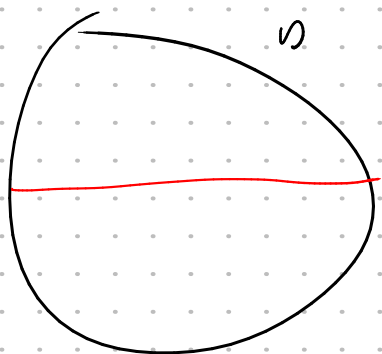
$tb = -1$



Destabilizes

Step 2 $\exists!$ Legendrian unknot with $tb = -1$

Let K & K' be leg. unknots with $tb = -1$ & ∂ & ∂' be com-
 pacted links



$tb = -1$ & ∂ is tight, so K has to look like the above

$\Rightarrow \exists$ contact isotopy $(\nu D, \eta_4) \longrightarrow (\nu D', \eta_{st})$

$\Rightarrow S^3 \setminus \nu D$ & $S^3 \setminus \nu D'$ are 3-balls with the same C.S. on ∂

i.e. \exists contact isotopy $S^3 \setminus \nu D \longrightarrow S^3 \setminus \nu D'$

$\Rightarrow \exists$ contact isotopy $(S^3, \eta_x) \longrightarrow (S^3, \eta_a)$

$K \longrightarrow K'$



5.4 Classification of Tangential 2-Plane-Fields

Let M^3 be closed, oriented, connected.

Thm 7:

$$\left\{ \begin{array}{l} \text{Tangential} \\ \text{2-plane} \\ \text{fields } \xi \\ \text{on } M^3 \end{array} \right\} / \text{Homotopy} \longleftrightarrow \left\{ M \xrightarrow{c} S^2 \right\} / \text{Homotopy}$$

proof: Startel $\Rightarrow T M \cong \oplus M \times \mathbb{R}^3$

$$\begin{array}{ccc} \xi : M & \longrightarrow & S^2 \\ p & \longmapsto & \text{normal vector of } \xi_p \end{array}$$

Warning: This depends on Φ

$\{\curvearrowright \rightarrow S^2\} / \text{homotopy}$ is classified by obstruction theory via two homology classes d_2, d_3 ,
 \downarrow \downarrow
 Euler class Hopf-invariant

(see chapter 7)



9.5 Classification of overtwisted contact manifolds

$\Delta \subset \curvearrowright$ A 2-disk & $O_\Delta \subset \Delta$ a point

$$OT(\curvearrowright, \Delta) := \left\{ \begin{array}{l} \text{contact structures} \\ \text{with } \Delta \text{ as std. OT} \\ \text{disk with unique} \\ \text{singularity at } O_\Delta \end{array} \right\}$$

$$DIST(\curvearrowright, \Delta) := \left\{ \begin{array}{l} \text{Tangential 2-plane} \\ \text{fields pos. tangent} \\ \text{to } \Delta \text{ in } O_\Delta \end{array} \right\}$$

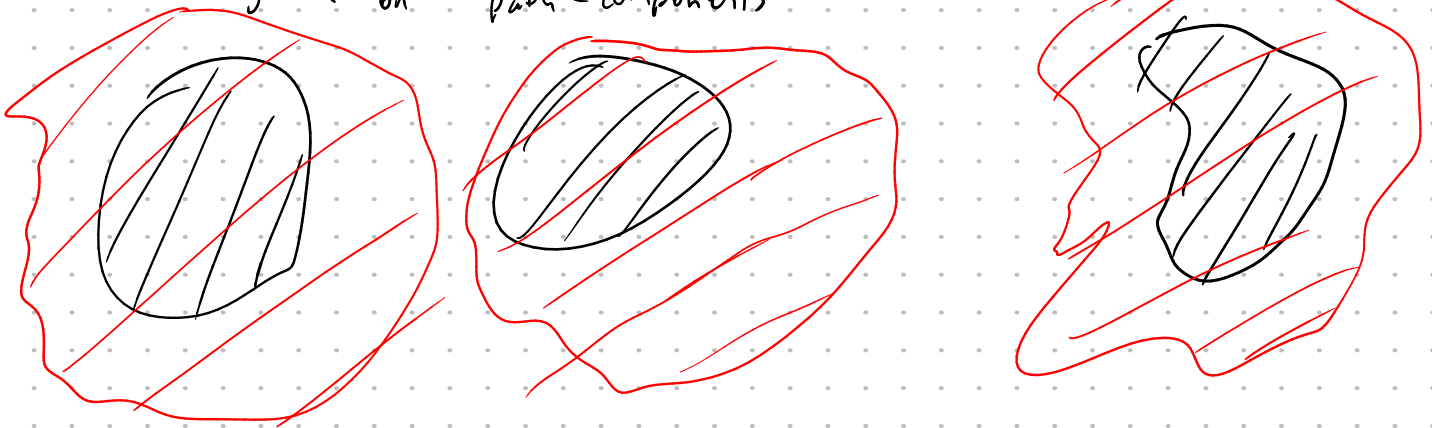
Thm 8: [Blaugberg]

The inclusion

$$i_\Delta : OT(\curvearrowright, \Delta) \longrightarrow DIST(\curvearrowright, \Delta)$$

is a homotopy equivalence.

Corollary 2: The inclusion $i : OT(\curvearrowright) \hookrightarrow DIST(\curvearrowright)$ induces a bijection on path-components



proof idea of Thm 8

Gromov (h-principle): If N^3 is open:

$\{ \text{contact structures on } M \} \hookrightarrow \{ \text{tangent 2-plane fields on } M \}$
is a homotopy eq.

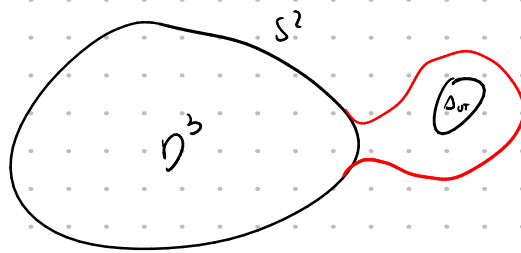
If M^3 is closed consider $M \cup \{pt\}$ (open)

① Apply Gromov

② study extension problem:

Given S^2 , extend to C.S. on D^3

In general, this is not possible, but if \exists OT def near S^2 , then it is possible



□