Preliminaries

The Bourgeois construction

Definition 1. *Let M be an oriented manifold with an open book decomposition (B, p) with oriented binding B. The pages are oriented by the requirement that the induced orientation on the boundary of (the closure) of each page coincides with the orientation of B.*

Question: I don't fully understand Geiges remark there.

A contact structure $\xi = \ker \alpha$ on M is said to be ***supported*** *by the open book decomposition (B, p) of M if*

(i) the contact form α induces the positive orientation of M $(\alpha \wedge (\mathrm{d}\alpha)^n > 0)$.

(ii) the 2-form $\mathrm{d}\alpha$ induces a symplectic form on each page, defining its positive orientation

(iii) the 1-form α induces a positive contact form on B, i.e. $\alpha \wedge (\mathrm{d}\alpha)^{(n-2)} > 0$ on B. Question: Is that i.e.-part correct?

Theorem 1. *Let $(M, \xi = \ker \alpha)$ be a closed contact manifold of dimension $2n-1, n \geq 2$. One con find an open book decomposition (B, p) of M supporting ξ. According to Bourgeois, ([Bou02]) there is a contact structure $\tilde{\xi}$ on $M \times T^2$ (where $\tilde{\xi}$ massively depends on the choice of open book).*

Proof. We follow the proof of [Gei08, Thm 7.3.6]. Wlog let M be connected. The existence of an open book decomposition for M is the theorem of Giroux-Mohsen as in [Gei08, Thm 7.3.5]. By definition of an open book, there exists a tubular neighborhood $B \times D^2$ with polar coordinates (r, ϕ) on the D^2-part of the binding B s.t. $p : M \setminus B \to S^1$ is givenby ϕ in that neighborhood. Now, we want to define smooth functions x_1, x_2 on M that coincide with the cartesian coordinate functions on D^2 close to the binding B. In order to do that, choose a smooth function $\rho(r)$ on $B \times D^2$, s.t.

- $\rho = r$ near the binding B,
- $\rho'(r) \geq 0$
- $\rho \equiv 1$ near $B \times \partial D^2$.

We extend this function to a smooth function $\rho : M \to [0, 1]$ by setting $\rho \equiv 1$ outside $B \times D^2$. Now, $x_1 := \rho \cos \phi$ and $x_2 := \rho \sin \phi$ are the desired smooth functions on M that coincide with the Cartesian coordinate functions on the D^2-factor near B. We compute

$$\begin{aligned} x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1 &= \rho^2 \cos^2 \phi \mathrm{d}\phi + \rho \cos \phi \sin \phi \mathrm{d}\rho + \rho^2 \sin^2 \phi \mathrm{d}\phi - \rho \cos \phi \sin \phi \mathrm{d}\rho \\ &= \rho^2 (\cos^2 \phi + \sin^2 \phi) \mathrm{d}\phi \\ &= \rho^2 \mathrm{d}\phi \end{aligned}$$

and, analogously,

$$\mathrm{d}x_1 \wedge \mathrm{d}x_2 = \rho \mathrm{d}\rho \wedge \mathrm{d}\phi.$$

On $M \times T^2$, choose coordinates (θ_1, θ_2) on the torus part of the manifold. Now we have all ingredients together to construct our contact form. Let

$$\tilde{\alpha} \coloneqq x_1 \mathrm{d}\theta_1 - x_2 \mathrm{d}\theta_2 + \alpha.$$

This is a well-defined 1-form on $M \times T^2$ (α is extended to $M \times T^2$ in the obvious way) and we can compute the derivative

$$\mathrm{d}\tilde{\alpha} = \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 - \mathrm{d}x_2 \wedge \mathrm{d}\theta_2 + \mathrm{d}\alpha,$$

hence

$$\begin{aligned}(\mathrm{d}\tilde{\alpha})^n =&(n-1)(\mathrm{d}\alpha)^{n-1} \wedge (\mathrm{d}x_1 \wedge \mathrm{d}\theta_1 - \mathrm{d}x_2 \wedge \mathrm{d}\theta_2)\\ &- n(n-1)(\mathrm{d}\alpha)^{n-2} \wedge \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\theta_2.\end{aligned}$$

In order to verify the contact condition, we compute

$$\begin{aligned}\tilde{\alpha} \wedge (\mathrm{d}\tilde{\alpha})^n =&(x_1 \mathrm{d}\theta_1 - x_2 \mathrm{d}\theta_2 + \alpha) \wedge (n-1)(\mathrm{d}\alpha)^{n-1} \wedge (\mathrm{d}x_1 \wedge \mathrm{d}\theta_1 - \mathrm{d}x_2 \wedge \mathrm{d}\theta_2)\\ &- (x_1 \mathrm{d}\theta_1 - x_2 \mathrm{d}\theta_2 + \alpha) \wedge n(n-1)(\mathrm{d}\alpha)^{n-2} \wedge \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\theta_2\\ =&(n-1)(\mathrm{d}\alpha)^{n-1} \wedge (x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1) \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2\\ &+ \underbrace{\alpha \wedge (n-1)(\mathrm{d}\alpha)^{n-1} \wedge \mathrm{d}x_1}_{2n-\text{form on } M} \wedge \mathrm{d}\theta_1 - \underbrace{\alpha \wedge (n-1)(\mathrm{d}\alpha)^{n-1} \wedge \mathrm{d}x_2}_{2n-\text{form on } M} \wedge \mathrm{d}\theta_2\\ &+ n(n-1)\alpha \wedge (\mathrm{d}\alpha)^{n-2} \wedge \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\theta_2\end{aligned}$$

M has dimension $2n-1$, i.e. the middle term is 0

$$\begin{aligned}=&(n-1)(\mathrm{d}\alpha)^{n-1} \wedge \rho^2 \mathrm{d}\phi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2\\ &+ n(n-1)\alpha \wedge (\mathrm{d}\alpha)^{n-2} \wedge \rho \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2\end{aligned}$$

By condition (ii) of definition 1, $(\mathrm{d}\alpha)^{n-1}$ must be a positive volume form on the pages. As explained in that definition, the orientation on M is given by ∂_ϕ and the orientation of the page. In particular, $(\mathrm{d}\alpha)^{n-1} \wedge \rho \mathrm{d}\phi$ is a positive volume form on M. Multiplied with a second ρ-factor, it vanishes along B. As $\theta_1 \wedge \theta_2$ is a positive volume form on T^2, the first term is non-negative everywhere and positive away from

$$\underbrace{B \times 0}_{\subset B \times D^2 \subset M} \times T^2.$$

Let $\mathfrak{b}$ be a basis of the binding B that is positively ordered. Then, $-\partial_r, \mathfrak{b}$ and (because the binding is odd-dimensional) $\mathfrak{b}, \partial_r$ are positive bases of the page. Clearly, then,

$$\mathfrak{a} \coloneqq \mathfrak{b}, \partial_r, \partial_\phi, \partial_{\theta_1}, \partial_{\theta_2}$$

is an ordered basis of $M \times T^2$. Using $\rho'(r) \geq 0$ everywhere, we deduce that $\mathrm{d}\rho(\partial_r)$ is non-negative. Hence, plugging $\mathfrak{a}$ into the second term, we conclude

$$\begin{aligned}
&\left(n(n-1)\alpha \wedge (\mathrm{d}\alpha)^{n-2} \wedge \rho \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2\right)(\mathfrak{a}) \\
=&n(n-1)\rho \cdot (\alpha \wedge (\mathrm{d}\alpha)^{n-2})(\mathfrak{b}) \cdot d\rho(\partial_r) \cdot \mathrm{d}\phi(\partial_\phi) \cdot \mathrm{d}\theta_1(\partial_{\theta_1}) \cdot \mathrm{d}\theta_2(\partial_{\theta_2}) \\
\geq&0.
\end{aligned}$$

By condition (iii) of definition 1, $\alpha \wedge (\mathrm{d}\alpha)^{n-2}$ is positive on B. Therefore, the second term is positive on $B \times 0 \times T^2$ (hence also on a neighborhood) and non-negative everywhere else. In total, we have checked the contact condition and $\tilde{\alpha}$ is indeed a contact form on $M \times T^2$. □

In the following, we want to apply this construction to a sphere S^{2n+1} with respect to a specific Milnor open book decomposition, defining a Briescorn manifold.

References

[Bou02] Frédéric Bourgeois. Odd dimensional tori are contact manifolds. *International Mathematics Research Notices*, 2002(30):1571–1574, January 2002.

[Gei08] Hansjörg Geiges. *An Introduction to Contact Topology.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, March 2008.