Nonlinear Optimization – Sheet 09

Exercise 1

We have

$$g_1(x) = x_1 + 4x_2 - 3; \qquad \nabla g_1(x) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

and

$$g_2(x) = x_2 - x_1; \qquad \nabla g_2(x) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

As $\nabla g_1(x)$ and $\nabla g_2(x)$ are linearly independent for every x, the LICQ is satisfied everywhere, regardless of how many of the constraints are active. Therefore, the KKT-conditions for feasible x,

$$\nabla f(x) + g'(x)^\top \mu = 0 \qquad (I)$$

$$\begin{pmatrix} -2(x_1 - 2) \\ -4(x_2 - 1) \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = 0 \qquad (I)$$

$$\mu^\top g(x) = 0 \qquad (II),$$

are necessary optimality conditions. The second condition can be extended to yield two conditions (due to the complementarity condition),

$$\mu_1(x_1 + 4x_2 - 3) = 0$$
$$\mu_2(x_2 - x_1) = 0.$$

We make a case distinction

- $\mu_1 = 0 \wedge \mu_2 = 0$ In this case we find that $x = (2, -1)$ with $\mu = (0, 0)$ is the unique point satisfying the KKT-conditions.
- $\mu_1 = 0 \wedge \mu_2 > 0$ In this case we find that $x = (4/3, 4/3)$ with $\mu = (0, 4/3)$ is the unique point satisfying the KKT-conditions.
- $\mu_1 > 0 \wedge \mu_2 = 0$ In this case we find that that the only solution of the system of equations is $x = (5/3, 1/3)$ with $\mu = (-2/3, 0)$. However, μ_1 is not positive. Therefore, there are no KKT-points in this case.
- $\mu_1 > 0 \wedge \mu_2 > 0$ In this case we find $x = (3/5, 3/5)$ with $\mu = (-22/25, 48/25)$. However, μ_1 is not positive. Therefore, there are no KKT-points in this case.

Under the LICQ, KKT-conditions are necessary. Therefore, the only two candidates for local minima are $x = (2, -1)$ with $\mu = (0, 0)$ and $x = (4/3, 4/3)$ with $\mu = (0, 4/3)$.

For $x^* = (2, -1)$ with $\mu^* = (0, 0)$, we compute

$$\mathcal{L}_{xx}(x^*, \mu^*) = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}.$$

At x^*, there are no active inequality constraints. Therefore, the critical cone is $\mathbb{R}^2$, in particular $d = (1, 0)$ is contained in the critical cone and we obtain

$$d^\top \cdot \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix} \cdot d = -2 < 0.$$

As a result, the necessary second order optimality conditions are not satisfied.

For $x^* = (4/3, 4/3)$ with $\mu^* = (0, 4/3)$, we compute

$$\mathcal{T}_{\text{NLP}}^{\text{critical}}(x) = \{d \in \mathbb{R}^2 |\ d_1 + 4d_2 \le 0,\ d_1 = d_2\} = \mathbb{R}_{>0} \cdot (-1, -1),$$

in particular $d = (-1, -1) \in \mathcal{T}_{\text{NLP}}^{\text{critical}}(x)$. Therefore,

$$d^\top \cdot \mathcal{L}(x^*, \mu^*) \cdot d = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix} = -6 < 0.$$

Again, the necessary second-order optimality conditions are not satisfied. Therefore, we don't have any local minimizers in this problem.

Exercise 2

Exercise 3

Exercise 4