Contact Geometry — Exercises

Contact Geometry

Exercise sheet 1

Exercise 1.
Prove (as stated in the lecture) that the standard contact structure on $S^{2n-1} \subset \mathbb{R}^{2n}$ is given by the kernel of

$$\alpha_{st} = \sum_{j=1}^{n} x_j \, dy_j - y_j \, dx_j,$$

and compute the Reeb flow of α_{st}. Which orbits of the Reeb flow are closed?

Exercise 2.
Describe an explicit contact form on $S^1 \times S^2$, draw its contact planes and compute the Reeb flow.

Exercise 3.
Consider $S^1 \times \mathbb{R}^2$ with an angular coordinate θ on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and cartesian coordinates (x, y) on $\mathbb{R}^2$. Let $n \in \mathbb{Z} \setminus \{0\}$. Show that

$$\alpha_n = \cos(n\theta) \, dx - \sin(n\theta) \, dy$$

is a contact form and sketch the contact structures. Show that $(S^1 \times \mathbb{R}^2, \xi_n = \ker \alpha_n)$ is contactomorphic to $(S^1 \times \mathbb{R}^2, \xi_m = \ker \alpha_m)$ for all n and m.
Challenge: Are these contact structures also isotopic?

Bonus exercise 1.

(a) Draw sketches of the three contact structures on $\mathbb{R}^3$ introduced in the lecture.

(b) Describe a 2-plane field on a 3-manifold that is not contact but to which no surface is tangent.

(c) Describe a hyperplane field $\xi = \ker \alpha$ that is not contact but for which $\alpha \wedge d\alpha$ is non-vanishing.

(d) Describe a hyperplane field that is nowhere contact and nowhere a foliation.

(e) Describe a non-coorientable hyperplane field on an orientable manifold.

(f) Describe a non-coorientable hyperplane field on a non-orientable manifold.

(g) Describe a coorientable hyperplane field on a non-orientable manifold.

Bonus exercise 2.

(a) Describe a 1-dimensional foliation on T^2 that admits only closed leaves.

(b) Describe a 1-dimensional foliation on T^2 that admits only non-closed leaves.

(c) Describe a 1-dimensional foliation on T^2 that admits closed and non-closed leaves.

(d) Describe a 2-dimensional foliation of $S^1 \times D^2$ that admits exactly one closed leave.

(e) Describe a 2-dimensional foliation of S^3 that admits exactly one closed leave.
Hint: Consider the boundary of $D^2 \times D^2$ and use (d).

(f) Show that the 2-plane field on $\mathbb{R}^3$ defined as the kernel of the 1-form

$$dz - z\,dy$$

is induced by a foliation and describe its leaves.

This sheet will be discussed on Thursday 26.10. and should be solved by then.

Ex. 1: $S^{2n-1} \subset \mathbb{R}^{2n}$ hypersurface. Consider the symplectic form $\omega_{st} := \sum_{i=1}^{n} dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$.

$\iota_Y \omega_{st} = \frac{1}{2} \sum_{i=1}^{n} x_i dy_i - y_i dx_i$

$d(\iota_Y \omega_{st}) = \frac{1}{2}\left(\sum_{i=1}^{n} dx_i \wedge dy_i - dy_i \wedge dx_i\right) = \omega_{st}$

Sheet 1

Lemma 5 $\Longrightarrow$ $\alpha_{st} = 2\iota_Y \omega_{st}$ is a contact form on S^{2n-1}

$$= \sum_{j=1}^{n} x_j dy_j - y_j dx_j,$$

where $\boxed{Y := \frac{1}{2} r \partial r} = \frac{1}{2}\left(\sum_{i=1}^{n} x_i \partial x_i + y_i \partial y_i\right)$ is the well-known Liouville vector field on $(\mathbb{R}^{2n}, \omega_{st})$.

The Reeb vectorfield is given by $R = \sum_j x_j \partial y_j - y_j \partial x_j$,

Since $\cdot)$ $\alpha_{st}(R) = \sum_{i=1}^{n} x_i^2 + y_i^2 = 1$ on S^{2n-1} and

$\cdot)$ $d\alpha_{st} = 2\sum_{j=1}^{n} dx_j \wedge dy_j$

$$\Rightarrow d\alpha_{st}(R, \cdot) = 2\left(\sum_j dx_j \wedge dy_j\right)\left(\sum_i x_i \partial y_i - y_i \partial x_i, \cdot\right)$$

$$= 2\left(\sum_j \sum_i dx_j (x_i \partial y_i - y_i \partial x_i) dy_j - dx_j \cdot \left(dy_j (x_i \partial y_i - y_i \partial x_i)\right)\right)$$

$$= 2\left(\sum_i - y_i dy_i - x_i dx_i\right) = -2\left(\sum x_i dx_i + y_i dy_i\right)$$

$$= -2 r dr = 0 \text{ on } S^{2n-1}$$

$[r \equiv 1$ on S^{2n-1} $\Rightarrow dr = 0]$

Find the Reeb flow:

Solve $\begin{cases} \dot\gamma_p(t) = R_{\gamma_p(t)} \\ \gamma_p(0) = p = (x_1, y_1, \ldots, x_n, y_n) \end{cases}$

Let $p \in S^{2n-1}$. Then, we have to solve,

$$\begin{pmatrix} \dot\gamma_1 \\ \dot\gamma_2 \\ \vdots \\ \dot\gamma_{2n-1} \\ \dot\gamma_{2n} \end{pmatrix} = \begin{pmatrix} -\gamma_2(t) \\ \gamma_1(t) \\ \vdots \\ -\gamma_{2n}(t) \\ \gamma_{2n-1}(t) \end{pmatrix}$$

$$\sum_{i=1}^n x_i^2 + y_i^2 = 1 = \alpha_{st}(R_{\alpha_{st}})$$
$$= \sum x_i dy_i - y_i dx_i \left(\sum_j a_j \partial x_j + b_j \partial y_j \right)$$
$$= \sum_i x_i b_i - y_i a_i$$
$$\Rightarrow b_i = x_i,\ a_i = -y_i \Rightarrow \checkmark$$

Consider γ_1, γ_2. We have: $\begin{cases} \ddot\gamma_1 = \dot\gamma_2 = -\gamma_1 \\ \ddot\gamma_2 = -\dot\gamma_1 = -\gamma_2 \end{cases}$

$\Rightarrow$ The solutions are given by $\gamma_1(t) = a_1 \cos t + b_1 \sin t$
$\gamma_2(t) = a_1 \sin t - b_1 \cos t$

Plug in the initial values: $x_1 = \gamma_1(0) = a_1$
$y_1 = \gamma_2(0) = -b_1$

Doing the same with $\gamma_3, \ldots, \gamma_{2n}$ yields

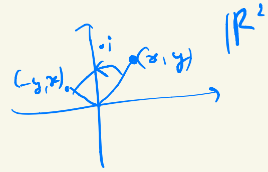
$\Rightarrow$ all orbits are closed (correct?)

$$\gamma_p(t) = \sum_{j=1}^n (x_j \cos t - y_j \sin t)\partial_{x_j} + (x_j \sin t + y_j \cos t)\partial_{y_j}$$

Why $\ker \alpha_{st} = \xi_{st} = TS^{2n-1} \cap i(TS^{2n-1})$?

$S^{2n-1} = H^{-1}(\{0\})$, $\quad H: \mathbb{R}^{2n} \to \mathbb{R}$, $(\underline{x},\underline{y}) \mapsto \langle \underline{x},\underline{x}\rangle + \langle \underline{y},\underline{y}\rangle - 1$

regular value $\qquad$ smooth



$\Rightarrow TS^{2n-1} = \ker dH = \ker\left(\sum_i x_i\, dx_i + y_i\, dy_i\right)$

$\Rightarrow i\,TS^{2n-1} = i \ker\left(\sum_i x_i\, dx_i + y_i\, dy_i\right) \overset{(*)}{=} \ker\left(\sum_i x_i\, dy_i - y_i\, dx_i\right)$

(*) Indeed, $X \in \ker\left(\sum_i x_i\, dx_i + y_i\, dy_i\right) = \sum_i a_i x_i + b_i y_i = 0$

$X = \sum_{i=1}^n a_i \partial_{x_i} + b_i \partial_{y_i}$ $\quad (\to iX = \sum_{i=1}^n -b_i \partial_{x_i} + a_i \partial_{y_i})$

$\Updownarrow$

$iX \in \ker\left(\sum_i x_i\, dy_i - y_i\, dx_i\right)$

$Y \in \ker\left(\sum x_i\, dy_i - y_i\, dx_i\right) \Longleftrightarrow -iY \in \ker\left(\sum_i x_i\, dx_i + y_i\, dy_i\right)$ and

$Y = i(-iY)$

$Y = i(-iY) \in i\,(TS^{2n-1})$

$\Longrightarrow X \in \xi_{st} = TS^{2n-1} \cap i\left(TS^{2n-1}\right) = \ker\left(\sum x_i\, dx_i + y_i\, dy_i\right)$

$\cap \ker\left(\sum x_i\, dy_i - y_i\, dx_i\right)$

$X = \sum_{i=1}^n a_i \partial_{x_i} + b_i \partial_{y_i}$

$\Longleftrightarrow X \in TS^{2n-1}$ and $X \in \ker \alpha_{st}$ #

Ex 2) Consider $S^1 \times S^2 \subset S^1 \times \mathbb{R}^3$

What are the contact planes ? and the Reeb flow?

Consider coordinates (θ, x, y, z) on $S^1 \times \mathbb{R}^3$ and let $\omega = dz \wedge d\theta + dx \wedge dy$ be the symplectic form on $S^1 \times \mathbb{R}^3$. *

Define $Y := z\partial_z + \frac{1}{2}(x\partial_x + y\partial_y)$. Then,

$$\iota_Y \omega = z\, d\theta + \frac{1}{2}(x\, dy - y\, dx)$$

$$\Rightarrow d(\iota_Y \omega) = dz \wedge d\theta + \frac{1}{2}(dx \wedge dy - dy \wedge dx)$$

$$= dz \wedge d\theta + dx \wedge dy = \omega$$

$\Rightarrow Y$ is Liouville on $(S^1 \times \mathbb{R}^3, \omega)$.

* $d\omega = 0$ ✓ ; $\omega^2 = 2 dz \wedge d\theta \wedge dx \wedge dy$

$$= -2\, d\theta \wedge dx \wedge dy \wedge dz$$

?

$= -2$ vol ✓

$S^1 \times S^2 \subset S^1 \times \mathbb{R}^3$ hypersurface which is transverse to Y. Indeed,

$$S^1 \times S^2 = H^{-1}(1), \text{ where } H : S^1 \times \mathbb{R}^3 \to \mathbb{R}$$

regular (*)

$$(\theta, x, y, z) \mapsto x^2 + y^2 + z^2$$

$$* \; dH = 2(x\,dx + y\,dy + z\,dz)$$

$$\Rightarrow dH(Y) = dH\left(z\partial_z + \tfrac{1}{2}(x\partial_x + y\partial_y)\right)$$

$$= 2z^2 + x^2 + y^2 = 1 + z^2 \neq 0 \quad ✓$$

(*) $D_{(\theta,x,y,z)}H = (0, x, y, z) \neq 0 \quad \forall (\theta, x, y, z) \in S^1 \times S^2$

$\Rightarrow D(\theta,x,y,z)H$ surjective — " —

Lemma 5 $\Longrightarrow$ $(S^1 \times S^2, \ker\alpha)$, where $\alpha := \iota_Y \omega$, is a contact manifold. #

Define $R := \frac{2}{1+z^2}\left(z\partial_\theta + x\partial_y - y\partial_x\right)$

Then: $\cdot$) $\alpha(R) = \iota_Y \omega(R)$

$$= \left(z\,d\theta + \tfrac{1}{2}(x\,dy - y\,dx)\right)(R)$$

$$= \frac{2}{1+z^2}\left(z^2 + \frac{1}{2}\left(x^2+y^2\right)\right)$$

$$= \frac{2}{1+z^2}\left(\frac{2z^2+x^2+y^2}{2}\right) \quad \left(x^2+y^2+z^2=1\right)$$

$$= \frac{2}{1+z^2}\left(\frac{1+z^2}{2}\right) = 1. \checkmark$$

$\cdot$) $d\alpha(R, \cdot) = \omega(R, \cdot) = dz\wedge d\theta + dx\wedge dy(R, \cdot)$

$$= \frac{2}{1+z^2}\left(-z\,dz - x\,dx - y\,dy\right)$$

$$= \frac{-2}{1+z^2}\left(x\,dx + y\,dy + z\,dz\right) = 0, \text{ since}$$

$T(S^1\times S^2) = \ker dH = \ker(x\,dx + y\,dy + z\,dz)$.

Solve $\dot{\theta}(t)\,\partial_\theta + \dot{x}(t)\,\partial_x + \dot{y}(t)\,\partial_y + \dot{z}(t)\,\partial_z$

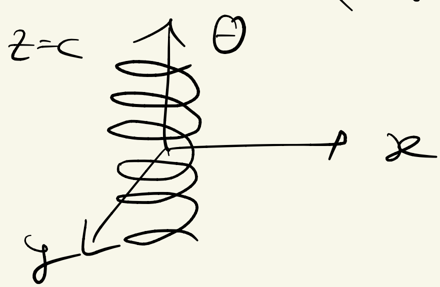
$$= \frac{2}{1+z(t)}\Big(z(t)\,\partial_\theta + x(t)\,\partial_y - y(t)\,\partial_x\Big)$$

$$\Longleftrightarrow \begin{cases} \dot{\theta} = \dfrac{2z}{1+z} \Longrightarrow \theta = \left(\dfrac{2c}{1+c}\right)t \\[2ex] \dot{x} = \dfrac{-2y}{1+z} = \dfrac{-2y}{1+c} \Rightarrow \ddot{x} = \dfrac{-2}{1+c}\,\dot{y} = -\dfrac{4}{(1+c)^2}\,x \\[2ex] \dot{y} = \dfrac{2x}{1+z} = \dfrac{2x}{1+c} \Rightarrow \ddot{y} = \dfrac{2}{1+c}\,\dot{x} = -\dfrac{4}{(1+c)^2}\,y \\[2ex] \dot{z} = 0 \Longrightarrow z = c = \text{const} \end{cases}$$

$$\Rightarrow x = \cos\left(\frac{2}{1+c}t\right),\ y = \sin\left(\frac{2}{1+c}t\right),\ z = c$$

$$\theta = \left(\frac{2c}{1+c}\right)t \quad \text{and} \quad \varphi^t_{(\theta,x,y,z)} = \begin{pmatrix} \theta(t) \\ x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Sketch:



Ex. 3: $S^1 \times \mathbb{R}^2$, angular coordinate θ on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, cartesian coordinates (x,y) on $\mathbb{R}^2$, $n \in \mathbb{Z}\setminus\{0\}$.

Clm. $\alpha_n := \cos(n\theta)\,dx - \sin(n\theta)\,dy$ contact form.

Pf. $d\alpha_n = -n\sin(n\theta)\,d\theta\wedge dx - n\cos(n\theta)\,d\theta\wedge dy$

$$\Rightarrow \alpha_n \wedge d\alpha_n = -n\cos^2(n\theta)\,dx\wedge d\theta\wedge dy + n\sin^2(n\theta)\,dy\wedge d\theta\wedge dx$$

$$= n\left(\cos^2(n\theta)+\sin^2(n\theta)\right) dx\wedge dy\wedge d\theta$$

$$= n\,dx\wedge dy\wedge d\theta \neq 0 \text{ is a volume form on } S^1\times\mathbb{R}^2$$

Consider $\varphi: S^1\times\mathbb{R}^2 \to S^1\times\mathbb{R}^2$ in coordinates given by

$$(\theta,x,y) \mapsto \left(\tfrac{m}{n}\theta, x, y\right) =: (\theta', x, y)$$ no diffeo

$f: (\theta,x,y)$

$\mapsto \left(\theta,\ x\cos(n\theta) + y\sin(n\theta),\ -x\sin(n\theta) + y\cos(n\theta)\right)$

Then, $\varphi^*\alpha_n = \cos\left(n\cdot\tfrac{m}{n}\theta\right)dx - \sin\left(n\tfrac{m}{n}\theta\right)dy = \alpha_m$

$\Rightarrow \varphi_* \ker\alpha_m = \ker\alpha_n \Rightarrow$ all $(S^1\times\mathbb{R}^2, \xi_n)$ are contactomorphic

$f^*\alpha_n = \ldots = dx + ny\,d\theta$

$g: (\theta,x,y) \mapsto \left(\theta, x, \tfrac{y}{n}\right) \to g^*(f^*\alpha_n) = dx + y\,d\theta$

$S^3 \subset \mathbb{R}^4(x_1, y_1, x_2, y_2)$ (regarding Ex3)

$$\alpha_{st} = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2$$

$$d\alpha_{st} = 2(dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$$

$$\Rightarrow \alpha_{st} \wedge d\alpha_{st} = 2x_1 dy_1 \wedge dx_2 \wedge dy_2$$
$$- 2y_1 dx_1 \wedge dx_2 \wedge dy_2$$
$$+ 2x_2 dy_2 \wedge dx_1 \wedge dy_1$$
$$- 2y_2 dx_2 \wedge dx_1 \wedge dy_1$$
$$\neq 0 \text{ on } S^3$$

$\forall p \in S^3$: $\alpha_p: T_pS^3 \to \mathbb{R}$ with smooth coordinate functions

$\xi_{st} := TS^3 \cap i(TS^3)$ Claim: $\ker \alpha_{st} = \xi_{st}$

Now: Reeb vector field: $R_\alpha \begin{cases} d\alpha(R, \cdot) = 0 \\ \alpha(R) = 0 \end{cases}$

$$R_\alpha = \sum_{i=1}^{2} x_i \partial y_i - y_i \partial x_i$$ (by Linear Algebra)

vf on S^3 ($\Leftarrow$ short calculation)

*) $$\alpha_{st}(R_{\alpha_{st}}) = \sum_{i=1}^{2} x_i^2 + y_i^2$$

*) $$d\alpha_{st}(R_{\alpha_{st}}, \cdot) = 2(-y_1)\,dy_1 + 2(-y_2)\,dy_2 - 2x_1\,dx_1 - 2x_2\,dy_2$$

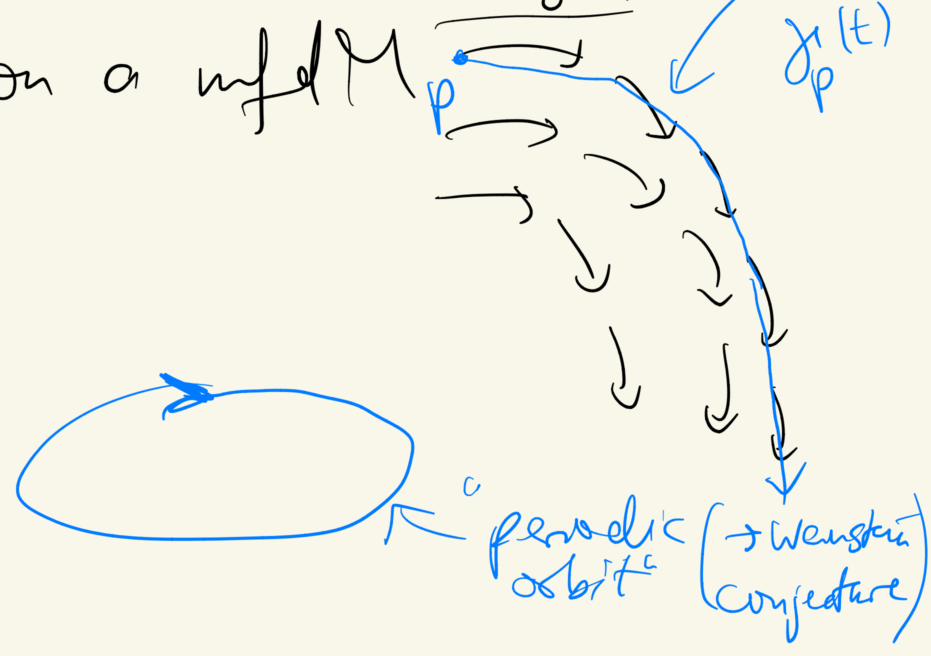
$$= -2r\,dr = 0$$

$$r = \sqrt{\sum_{i=1}^{2} x_i^2 + y_i^2}$$

$$r^2 = 1 \xRightarrow{d} 2r\,dr = 0$$
on S^3

X a vector field on a mfd M

e.g.:



$$X(\gamma_p(t)) \overset{!}{=} \dot{\gamma}_p(t)$$

$$R = \partial_z \quad \text{on } \mathbb{R}^3$$

$$\gamma_{p_0}(t) = (x_0, y_0, z_0 + t), \quad p_0 = (x_0, y_0, z_0)$$

$$\partial_z(\gamma_p(t)) = \partial_z = \dot\gamma_p(t) \checkmark$$

$$\sum_{i=1}^{2} x_i \partial_{y_i} - y_i \partial_{x_i} \qquad \gamma_1(t) = e^{it}$$

$$(x_j, y_j) = x_j + i y_j = z_j \in \mathbb{C}$$

$$\gamma_{\underset{(z_1, z_2)}{p}}(t) = e^{it}(z_1, z_2) = (\cos t + i \sin t)\begin{pmatrix} x_1 + i y_1 \\ x_2 + i y_2 \end{pmatrix}$$

$$= (\cos(t) x_1 - \sin(t) y_1, \cos(t) y_1 + \sin(t) x_1, \dots, \dots)$$

$$\dot\gamma_p(t) = i e^{it} (\partial_{z_1} + \partial_{z_2})$$

$$R_{\alpha_{st}}(\gamma_p(t)) = (\cos t\, x_1 - \sin t\, y_1)\partial_{y_1} - (\cos(t) y_1 + \sin(t) x_1)\partial_{x_1}$$

$$+ \dots = \dot\gamma_p(t)$$

$$\Rightarrow R_{\measuredangle St}\left(\gamma_{(z_1,z_2)}(t)\right)$$

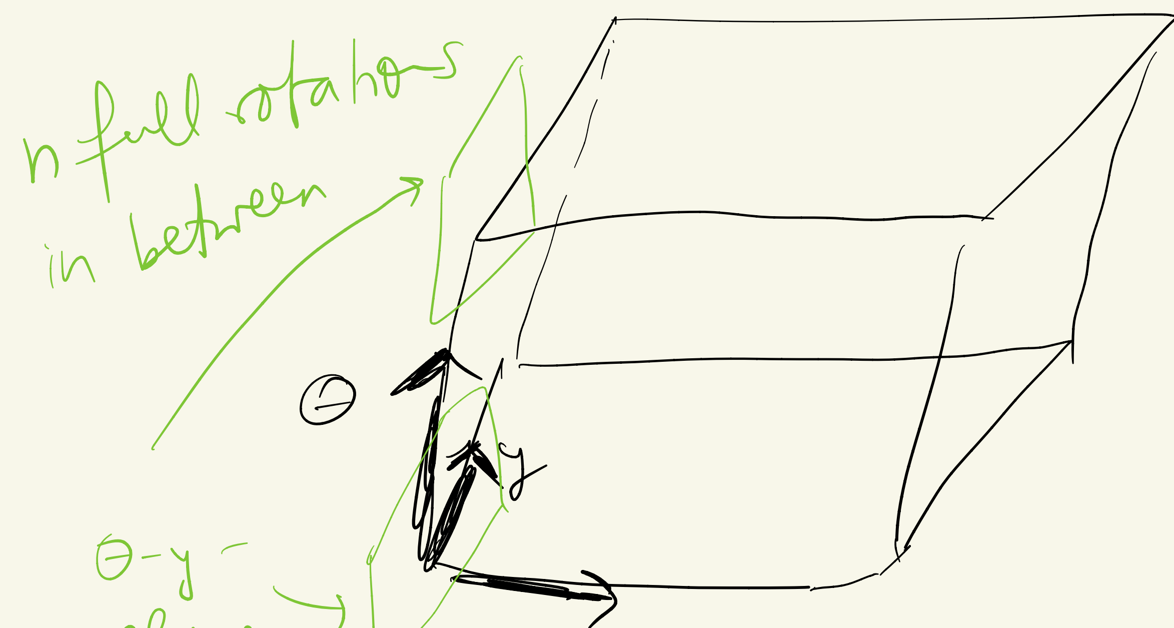
$$= \left(-\cos(t)y_1 - \sin(t)x_1,\ \cos(t)x_1 - \sin(t)y_1,\ \dots\right)$$

$$= i\left(\cos(t)x_1 - \sin(t)y_1,\ \cos(t)y_1 + \sin(t)x_1,\ \dots\right)$$

$$= i e^{it}(z_1,z_2)$$ ($\rightarrow$ Hopf fibration, Hopf flow)

$S^1 \times \mathbb{R}^2$

identify $(0)\times\mathbb{R}^2$ with $(2\pi)\times\mathbb{R}^2$



$$\cos(n\Theta)\,dx - \sin(n\Theta)\,dy$$

$f: (\theta, x, y)$

$\mapsto (\theta,\ x\cos(n\theta) + y\sin(n\theta),$
$-x\sin(n\theta) + y\cos(n\theta))$

$\alpha_n = \cos(n\theta)\,dx - \sin(n\theta)\,dy$

$$Dpf = \begin{pmatrix} 1 & 0 & 0 \\ -nx\sin() + ny\cos() & \cos() & \sin() \\ -nx\cos() - ny\sin() & -\sin() & \cos() \end{pmatrix} \in GL_3(\mathbb{R}) \ \forall p. \checkmark$$

$$f^*\alpha_n = \cos(n\theta)\, d\big(x\cos(n\theta) + y\sin(n\theta)\big)$$
$$- \sin(n\theta)\, d\big(-x\sin(n\theta) + y\cos(n\theta)\big)$$

$$= \cos(n\theta)\big(\cos(n\theta)\,dx - nx\sin(n\theta)\,d\theta$$
$$+ \sin(n\theta)\,dy + ny\cos(n\theta)\,d\theta\big)$$

$$- \sin(n\theta)\big(-\sin(n\theta)\,dx - nx\cos(n\theta)\,d\theta$$
$$+ \cos(n\theta)\,dy - ny\sin(n\theta)\,d\theta\big)$$

$$= \cos^2(n\theta)\,dx + \sin^2(n\theta)\,dx$$
$$+ ny\cos^2(n\theta)\,d\theta + ny\sin^2(n\theta)\,d\theta$$

$$= dx + ny\,d\theta \quad \checkmark$$

Contact Geometry

Exercise sheet 2

Exercise 1. ← most important

On $S^3 \subset \mathbb{R}^4$ consider the 1-form

"scaling" by t

$$\alpha = x_1\,dy_1 - y_1\,dx_1 + \sqrt{2}(x_2\,dy_2 - y_2\,dx_2). \tag{1}$$

(a) Show that α is a contact form. Exam!

(b) Compute the Reeb flow of α. How many closed orbits has it?

(c) Show that $(S^3, \ker\alpha)$ is contactomorphic to (S^3, ξ_{st}).
Hint: Use Gray stability.

(d) A strict contactomorphism preserves the Reeb flow.

(e) Deduce that (S^3, α) and (S^3, α_{st}) are not strictly contactomorphic, that a contactomorphism does in general not preserve the Reeb flow, and that Gray stability does not hold true for contact forms.

Exercise 2.

Two contact structures ξ_0 and ξ_1 on a manifold M are called **isotopic** if there exist a family of diffeomorphisms $\psi_t\colon M \to M$ such that $\psi_0 = \mathrm{Id}$ and ψ_1 is a contactomorphism from (M, ξ_0) to (M, ξ_1).

ψ_t $\forall t$?

(a) Consider on $\mathbb{R}^{2n+1}$ the contact forms

$$\alpha_0 = dz + \sum_{j=1}^{n} x_j\,dy_j \quad \text{and} \quad \alpha_0 = 2dz + \sum_{j=1}^{n} x_j\,dy_j - y_j\,dx_j.$$

Show that the induced contact structures are isotopic.

(b) Can the isotopy be chosen such that ψ_1 is a strict contactomorphism?

Exercise 3.

Prove Theorem 2.4 from the lecture.
Hint: Use the stereographic projection.

Exercise 4.

(a) Describe a contact structure on $\mathbb{R}P^3$. $\cong S^*S^2$
Hint: Write $\mathbb{R}P^3$ as the unit cotangent bundle of a surface and use the canonical contact structure.

(b) Let M and N be diffeomorphic manifolds. Show that their unit cotangent bundles (with their canonical contact structures) are contactomorphic.

Exercise 5.

Let p and q be coprime integers and consider S^3 as the unit sphere in $\mathbb{C}^2$. Consider the $\mathbb{Z}_p$ action on S^3 generated by the diffeomorphism

$\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, $(z,w) \in S^3$

$$f: (z, w) \longmapsto (e^{2\pi i/p}z, e^{2\pi iq/p}w). \quad \to f^p = id_{\mathbb{C}^2}$$

(a) Show that this group action is free and thus the quotient is a smooth 3-manifold. We call that manifold the **lens space** $L(p, q)$.

(b) Show that $L(p, q)$ admits a contact structure.

(c) Let (M, ξ) be a contact manifold and let M' be a cover of M. Then M' admits a contact structure.

Need to show that f is isotopic to contactomorphisms of (S^3, ξ_{st})

Bonus exercise.

Prove Lemma 2.8 from the lecture.

$L(1,q) = S^3$

$L(2,1) = \mathbb{R}P^3$

3–5 more challenging

This sheet will be discussed on Thursday 9.11. and should be solved by then.

Sheet 2 $\quad 2\cdot 1+1 \to n=1$

Ex 1: $S^3 \subset \mathbb{R}^4$, $\quad \alpha = x_1 dy_1 - y_1 dx_1 + \sqrt{2}(x_2 dy_2 - y_2 dx_2)$

(a) Define $Y := x_1 \partial x_1 + y_1 \partial y_2 + x_2 \partial x_2 + y_2 \partial y_2 \sim r\partial_r \pitchfork S^3$

$TS^3 = \ker(\underbrace{\sum x_i dx_i}_{= r\,dr})$ and

$(\sum x_i dx_i)(Y) = \sum x_i^2 = r^2 = 1 \neq 0$

$\Rightarrow Y \pitchfork S^3$

alternative: $\alpha \wedge d\alpha = \alpha \wedge (dx_1 \wedge dy_1 - dy_1 \wedge dx_1 + \sqrt{2}(dx_2 \wedge dy_2 - dy_2 \wedge dx_2))$

$= 2\sqrt{2} x_1 dy_1 \wedge dx_2 \wedge dy_2 - \sqrt{2} y_1 dx_1 \wedge dx_2 \wedge dy_2$

$+ 2\sqrt{2} x_2 dy_2 \wedge dx_1 \wedge dy_1 - 2\sqrt{2} y_2 dx_2 \wedge dx_1 \wedge dy_1$

$= 0$ iff $(x_1, y_1, x_2, y_2) = 0 \notin S^3$

Write $\omega = \frac{1}{2} d\alpha = \frac{1}{2} 2 (dx_1 \wedge dy_1 + \sqrt{2}\, dx_2 \wedge dy_2)$. Then,

$\iota_Y \omega = (dx_1 \wedge dy_1 + \sqrt{2}\, dx_2 \wedge dy_2)(x_1 \partial_{x_1} + y_1 \partial_{y_1} + x_2 \partial_{x_2} + y_2 \partial_{y_2})$

$= x_1 dy_1 - y_1 dx_1 + \sqrt{2}(x_2 dy_2 - y_2 dx_2) = \alpha$

$\Rightarrow L_Y \omega = d(\iota_Y \omega) = d\alpha = \omega \Rightarrow \alpha$ contact form on S^3. #

(b) Compute the Reeb vectorfield

Define $\underbrace{\hat{R}_{(x_1, y_1, x_2, y_2)}}_{=: \frac{1}{r}} := y_1 \partial_{x_1} - x_1 \partial_{y_1} + \frac{1}{\sqrt{2}}(y_2 \partial_{x_2} - x_2 \partial_{y_2})$.

Then: $d\alpha(\hat{R}, \cdot)|_{\frac{1}{r}} = 2\underbrace{(y_1 dy_1 + x_1 dx_1 + \frac{1}{\sqrt{2}} \cdot \sqrt{2}(y_2 dy_2 + y_1 dy_1))}_{= 0 \text{ on } S^3}$

*) $\alpha(\hat{R})|_{\vec{r}} = \left(x_1\, dy_1 - y_1 dx_1 + \sqrt{2}\,(x_2 dy_2 - y_2 dx_2)\right)(\hat{R}_{\vec{r}})$

$$= -x_1^2 - y_1^2 + \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}\left(-x_2^2 - y_2^2\right)$$

$$\hat{R}_{\vec{r}} = y_1 \partial x_1 - x_1 \partial y_1 + \frac{1}{\sqrt{2}}\left(y_2 \partial x_2 - x_2 \partial y_2\right)$$

$$= -1$$

$\Rightarrow \quad R = -\hat{R} = x_1 \partial y_1 - y_1 \partial x_1 + \frac{1}{\sqrt{2}}\left(x_2 \partial y_2 - y_2 \partial x_2\right)$

is the correct Reeb vector field for α.

comparing with Sheet 1/Ex. 1 delivers the Reeb flow:

$$\gamma_{\vec{r}}(t) = (x_1 \cos t - y_1 \sin t)\,\partial x_1 + (x_1 \sin t + y_1 \cos t)\,\partial y_1$$
$$+ \left(x_2 \cos\left(\tfrac{t}{\sqrt{2}}\right) - y_2 \sin\left(\tfrac{t}{\sqrt{2}}\right)\right)\partial x_2 + \left(x_2 \sin\left(\tfrac{t}{\sqrt{2}}\right) + y_2 \cos\left(\tfrac{t}{\sqrt{2}}\right)\right)\partial y_2$$

Sanity check: $\gamma_{\vec{r}}(0) = x_1 \partial x_1 + y_1 \partial y_1 + x_2 \partial x_2 + y_2 \partial y_2$ ✓

(I) $x_2(0) = y_2(0) = 0 \rightarrow$
(II) $x_1(0) = y_1(0) = 0 \rightarrow$

} closed orbit !

pts satisfying (I) or (II) lie on the same orbit

$\sqrt{2} \notin \mathbb{Q}$!

c) Claim: $(S^3, \ker\alpha) = (S^3, \xi_{st})$

Remember (Sheet 1): $\xi_{st} = \ker\alpha_{st}$, $\alpha_{st} = \sum_{i=1}^{2} x_i dy_i - y_i dx_i$

Set $\alpha_t := (1-t)\alpha + t\alpha_{st}$

$$\Rightarrow d\alpha_t = (1-t)d\alpha + t\,d\alpha_{st}$$
$$= (1-t)\cdot 2\left(dx_1 \wedge dy_1 + \sqrt{2}\,(dx_2 \wedge dy_1)\right)$$
$$+ t\cdot 2(dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$$

$$\Rightarrow \alpha_t \wedge d\alpha_t =$$
$$= \left[(1-t)\left(x_1 dy_1 - y_1 dx_1 + \sqrt{2}\,(x_2 dy_2 - y_2 dx_2)\right) + t\left(\sum_{i=1}^{2} x_i dy_i - y_i dx_i\right)\right]$$
$$\wedge\, 2\Big[(1-t)\left(dx_1 \wedge dy_1 + \sqrt{2}\,(dx_2 \wedge dy_2)\right)$$
$$+ t\,(dx_1 \wedge dy_1 + dx_2 \wedge dy_2)\Big]$$

d) Let $\varphi: M \to N$ be a strict contacto, i.e. $\varphi^* \alpha' = \alpha$. Clm: φ preserves the Reeb flow. [more general: $\varphi: (M, \ker\alpha) \to (N, \ker\alpha')$]

pf: Let R be the Reeb vf of α. Then, $\varphi^*(d\alpha') = d(\varphi^*\alpha') = d\alpha$

$$*) \quad \alpha'_{\varphi(p)}\left((T\varphi)_p R^\alpha_p\right) \overset{\text{def}}{=} (\varphi^*\alpha')_p (R^\alpha_p) = \alpha_p(R^\alpha_p) = 1$$

$$*) \quad (d\alpha')_{\varphi(p)}\left((T\varphi)_p R^\alpha_p, \cdot\right) \underset{\uparrow}{=} (\varphi^* d\alpha')_p \left(R^\alpha_p, (T\varphi^{-1})_{\varphi(p)} \cdot\right)$$

φ diffeo

[Habe gezeigt]: $\frac{\partial}{\partial t} \varphi(\gamma(t)) = R^{\alpha'}_{\varphi(\gamma(t))}$

$$= (d\alpha)_p \left(R^\alpha_p, (T\varphi^{-1})_{\varphi(p)} \cdot\right) = 0$$

$\Rightarrow R^{\alpha'}_{\varphi(p)} = (T\varphi)_p R^\alpha_p$. Let $\gamma(t)$ be an integral curve of R at $p \in M$. Then, $\frac{\partial}{\partial t}\varphi(\gamma(t)) = (T\varphi)_{\gamma(t)}(\dot\gamma(t)) = (T\varphi)_{\gamma(t)}(R_{\gamma(t)})$

$$= R^{\alpha'}_{\varphi(\gamma(t))} \Rightarrow \varphi(\gamma(t)) \text{ integral curve of } \varphi(p) \in N. \quad \blacksquare$$

e) Claim: $(S^3, \alpha), (S^3, \alpha_{st})$ are not strictly contactomorphic, contacto does not in general preserves the Reeb flow, Gray stability does not hold true for contact forms.

pf: Assume $(S^3, \alpha), (S^3, \alpha_{st})$ are strictly contactomorphic.

$\overset{(d)}{\Longrightarrow}$ They have the same closed Reeb orbits (with same periode)

(b) $\Rightarrow$ On $S^3 \cap \{x_1 = 0 = y_1\}$ there exist Reeb orbits with period $2\sqrt{2}\pi$,

but the Reeb orbits of α_{st} have all period 2π (Sheet 1 (Ex. 1))

$\lightning$

Ex 2: Consider $\mathbb{R}^{2n+1}$, $\alpha_0 = dz + \sum_{j=1}^n x_j dy_j$, $\alpha_1 = 2dz + \sum_{j=1}^n x_j dy_j - y_j dx_j$

Clm: $\ker\alpha_0$, $\ker\alpha_1$ are isotopic.

Pf: Define $\alpha_t := (1-t)\alpha_0 + t\alpha_1 = (1-t)\left(dz + \sum_j x_j dy_j\right) + t\left(2dz + \sum_j x_j dy_j - y_j dx_j\right)$

$$= \sum_j x_j dy_j + (1+t)dz - t\sum y_j dx_j$$

$(\Rightarrow d\alpha_t = \sum_j dx_j \wedge dy_j - t \sum_j dy_j \wedge dx_j)$

Then, α_t coincides with α_0 / α_1 for $t=0 / t=1$.

Find an isotopy φ_t, family of functions $f_t : \mathbb{R}^{2n+1} \to (0,\infty)$ with $\varphi_t^* \alpha_t = f_t \alpha_0$ ($\star$)

i.e. $\varphi_t : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ $\forall t \in I := [0,1]$, $\varphi_0 = id_{\mathbb{R}^{2n+1}}$, φ_1 is a contacto from $(\mathbb{R}^{2n+1}, \alpha_0)$ to $(\mathbb{R}^{2n+1}, \alpha_1)$

(Then $T\varphi_1(\ker\alpha_0) = \ker\alpha_1$)

Assume, we have found φ_t. Then φ_t is the flow of a time dependent vector field X_t: $X_t(\varphi_t) = \dot\varphi_t \in T\varphi_t(\xi_0) = \xi_t$, $\xi_0 = \ker\alpha_0$, $\xi_1 = \ker\alpha_1$

$$\varphi_t^*\alpha_t = f_t\alpha_0 \overset{\frac{d}{dt}}{\Longleftrightarrow} \varphi_t^*(\dot\alpha_t + L_{X_t}\alpha_t) \overset{\text{Lemma 9}}{=} \frac{d}{dt}\varphi_t^*\alpha_t = \frac{d}{dt} f_t \alpha_0 = \dot f_t \alpha_0 \quad \forall t \in I$$

Cartan's magic formula $\Rightarrow$

$$\dot f_t \alpha_0 = \varphi_t^*\left(\dot\alpha_t + d(\alpha_t(X_t)) + \iota_{X_t}(d\alpha_t)\right)$$

$\overset{(\star)}{=}$

$$\frac{\dot f_t}{f_t}\varphi_t^*\alpha_t = \varphi_t^*(g_t\alpha_t) \text{ for } g_t := \frac{d}{dt}(\log f_t)\circ\varphi_t^{-1},$$

since $\varphi_t^*(g_t\alpha_t) = \varphi_t^* g_t \cdot \varphi_t^*\alpha_t = g_t \circ \varphi_t \cdot \varphi_t^*\alpha_t$

$$= \frac{d}{dt}(\log f_t)\circ\varphi_t^{-1}\circ\varphi_t \cdot \varphi_t^*\alpha_t = \frac{\dot f_t}{f_t}\cdot\varphi_t^*\alpha_t \ (\checkmark)$$

φ_t diffeo

$$\Rightarrow \dot\alpha_t + d(\underbrace{\alpha_t(X_t)}_{=0,\text{ since } X_t \in \xi_t = \ker\alpha_t}) + \iota_{X_t}(d\alpha_t) = g_t\alpha_t \Rightarrow \dot\alpha_t + \iota_{X_t}(d\alpha_t) = g_t\alpha_t \ (1)$$

Plug in $R_t := R_{\alpha_t}$ (Reeb v.f.) $\Rightarrow \dot{\alpha}_t(R_t) = g_t$

*) R_t is given by $R_t = \frac{1}{1+t}\partial_z$

$\Big[d\alpha_t = (1+t)\sum_j dx_j \wedge dy_j \Rightarrow d\alpha_t(R_t, \cdot)$

$\alpha_t(R_t) = (1+t)\cdot\frac{1}{1+t} = 1 \Big]$

$\Rightarrow g_t = \dot{\alpha}_t(R_t) = \frac{1}{1+t}$

*) $\dot{\alpha}_t = dz - \sum_j y_j dx_j$

$(1) \Rightarrow \cancel{dz} - \sum_j y_j dx_j + \iota_{X_t}\left((1+t)\sum_j dx_j \wedge dy_j\right)$

$$= \frac{1}{1+t}\left(\sum_j x_j dy_j + \cancel{(1+t)dz} - t\sum_j y_j dx_j\right)$$

$$\Rightarrow \iota_{X_t}\left((1+t)\sum_j dx_j \wedge dy_j\right) \overset{!}{=} \frac{1}{1+t}\left(\sum_j x_j dy_j - t\sum_j y_j dx_j\right) + \sum_j y_j dx_j$$

$$\Rightarrow \sum_j dx_j(X_t)dy_j - dy_j(X_t)dx_j$$

$$\overset{!}{=} \frac{1}{(1+t)^2}\left(\sum_j x_j dy_j - t\sum_j y_j dx_j\right) + \frac{1}{1+t}\sum_j y_j dx_j$$

$$\Rightarrow \begin{cases} dx_j(X_t) = \frac{1}{(1+t)^2}x_j \\ dy_j(X_t) = \frac{t}{(1+t)^2}y_j - \frac{1}{1+t}y_j \end{cases}$$

$$\Rightarrow X_t = \frac{1}{(1+t)^2}\sum_j x_j\partial_{x_j} + \frac{1}{1+t}\left(\frac{t}{1+t} - 1\right)\sum_j y_j\partial_{y_j} + c_t\partial_z$$

works. Find c_t

Sei X_t in α_t ein:

$$\Longrightarrow \quad 0 \overset{!}{=} \alpha_t(X_t) =$$

$$= \left(\sum_i x_i \, dy_i + (1+t) dz - t \sum_i y_i \, dx_i \right)$$

$$\left(\frac{1}{(1+t)^2} \sum_j x_j \partial_{x_j} + \frac{1}{1+t}\left(\frac{t}{1+t} - 1\right) \sum_j y_j \partial_{y_j} + c_t \partial_z \right)$$

$$= \sum_i x_i y_i \frac{1}{1+t}\left(\frac{t}{1+t} - 1\right) + (1+t) c_t - \frac{t}{(1+t)^2} \sum x_j y_j$$

$$= (1+t) c_t - \sum_i x_i y_i = (1+t) c_t - \langle \underline{x}, \underline{y} \rangle$$

$$\rightarrow \quad c_t = \frac{\langle \underline{x}, \underline{y} \rangle}{1+t}$$

$$\Longrightarrow \quad X_t(\underline{x}, \underline{y}, z) = \frac{1}{1+t}\left(\frac{1}{1+t} \sum_j x_j \partial_{x_j} + \left(\frac{t}{1+t} - 1\right) \sum_j y_j \partial_{y_j} + \langle \underline{x}, \underline{y} \rangle \partial_z \right)$$

Find the flow:

$$\varphi_t(\underline{x}, \underline{y}, z) = \begin{pmatrix} \left(\exp\left(\int_0^t \frac{1}{(1+s)^2} ds \right) x_j \right)_j \\ \left(\exp\left(\int_0^t \frac{1}{1+s}\left(\frac{s}{1+s} - 1\right) ds \right) y_j \right)_j \\ \langle \underline{x}, \underline{y} \rangle t + z \end{pmatrix}$$

Ex. 3: Clm: $(S^{2n-1}\setminus\{p\}, \xi_{st}) \overset{\text{cont}}{\cong} (\mathbb{R}^{2n-1}, \xi_{st})$

$\xi_{st} = \ker\alpha_0$ on $S^{2n-1}\setminus\{p\}$, $\xi_{st} = \ker\alpha_2$ on $\mathbb{R}^{2n-1}$

Pf: Wlog $p = (0,\dots,0,1)$. Let $\varphi: S^{2n-1}\setminus\{p\} \to \mathbb{R}^{2n-1}$ be the stereographic projection from p, i.e.

with coordinates $(x_1, y_1, \dots, x_n, y_n)$ on $\mathbb{R}^{2n} \supset S^{2n-1}$ and $(u_1, v_1, \dots, u_{n-1}, v_{n-1}, w)$ on $\mathbb{R}^{2n-1}$

φ is given by $u_j = \dfrac{x_j}{1-y_n}$, $v_j = \dfrac{y_j}{1-y_n}$, $j = 1,\dots,n-1$, $w = \dfrac{x_n}{1-y_n}$

The inverse is given by $x_j = \lambda u_j$, $y_j = \lambda v_j$, $j = 1,\dots,n-1$,

$x_n = \lambda w$, $y_n = 1-\lambda$, where

$$\lambda = 2\Big/\Big(1 + w^2 + \sum_{j=1}^{n-1} u_j^2 + v_j^2\Big) = \frac{2}{\mu}$$

$d\lambda = 2\cdot(-1)\frac{1}{\mu^2}d\mu = \frac{-2}{\mu^2}d\mu$

$= -\frac{\lambda^2}{\cancel{2}}\Big(\cancel{2}w\,dw + \cancel{2}\sum_{j=1}^{n-1} u_j\,du_j + v_j\,dv_j\Big)$

Let (r_j, φ_j) be the polar coordinates in the (u_j, v_j)-plane, $1 \le j \le n-1$ $\Rightarrow$ $\alpha_0 = \sum_{j=1}^{n} x_j\,dy_j - y_j\,dx_j$ and

$d\lambda = d(2/\mu)$
$= 2\cdot(-1)\frac{1}{\mu^2} = -2\left(\frac{\lambda}{2}\right)^2$
$= -\frac{1}{2}\lambda^2$

$$\alpha_2 = dw + \sum_{j=1}^{n} u_j\,dv_j - v_j\,du_j$$

$$= dw + \sum_{j=1}^{n} r_j^2\,d\varphi_j$$

We compute: $(\varphi^{-1})^*\alpha_0 = \sum_{j=1}^{n-1} \lambda u_j(v_j\,d\lambda + \lambda\,dv_j) - \lambda v_j(u_j\,d\lambda + \lambda\,du_j)$

$+ \lambda w(-d\lambda) - (1-\lambda)(w\,d\lambda + \lambda\,dw)$?

$= \lambda^2\Big(\sum_{j=1}^{n-1}(u_j\,dv_j - v_j\,du_j)\Big) + \lambda w\Big(w\,dw + \sum_j u_j\,du_j + v_j\,dv_j\Big)$

$- (1-\lambda)\Big(w\Big(-w\,dw - \sum u_j\,du_j + v_j\,dv_j\Big) + \frac{1}{\lambda}dw\Big)$

$$\dots = \lambda^2 \Big[\sum_{j=1}^{n-1} (u_j\, dv_j - v_j\, du_j) + \lambda w \big(w\, dw + \sum_j u_j\, du_j + v_j\, dv_j \big)$$

$$- (1-\lambda) \Big(w \big(- w\, dw - \sum (u_j\, du_j + v_j\, dv_j) \big) + \frac{1}{\lambda} dw \Big) \Big]$$

$$1 - \frac{1}{\lambda} = 1 - \frac{1 + w^2 + \sum u_j^2 + v_j^2}{2} = \frac{1}{2}\left(2 - 1 + w^2 + \sum u_j^2 + v_j^2\right) = \frac{1}{2}\left(1 + w^2 + \sum r_j^2\right)$$

$$= \lambda^2 \Big[\sum_{j=1}^{n-1} (u_j\, dv_j - v_j\, dv_j) + w \Big(\cancel{\lambda} w\, dw + \lambda \sum_j (u_j\, du_j + v_j\, dv_j)$$

$$+ (1 - \cancel{\lambda}) w\, \cancel{dw} + (1-\lambda) \sum_j (u_j\, du_j + v_j\, dv_j) \Big) - (1-\lambda)\, dw \Big]$$

$$= \lambda^2 \Big(\sum_{j=1}^{n-1} (u_j\, dv_j - v_j\, du_j) + w \big(w\, dw + \sum_{j=1}^{n-1} (u_j\, du_j + v_j\, dv_j) \big) + \big(1 - \tfrac{1}{\lambda}\big)\, dw \Big)$$

$$= \lambda^2 \Big(\sum_j r_j^2\, d\varphi_j + w \sum_j r_j\, dr_j + \tfrac{1}{2} \big(1 + w^2 - \sum_j r_j^2\big)\, dw \quad (*) \quad \text{(next page)}$$

$=: \lambda^2 \tilde{\alpha}_2$. Find a diffeo f of $\mathbb{R}^{2n-1}$ with $f^* \alpha_2 = \tilde{\alpha}_2$

(Then, $f^* d\alpha_2 = d\tilde{\alpha}_2 \rightsquigarrow$ Reeb v.f. $\tilde{R}_2$ of $\tilde{\alpha}_2$ is sent to Reeb v.f. R_2 of α_2

We know: $R_2 = \partial_w$

$$\tilde{R}_2 = \frac{2}{1 + w^2 + \sum_j r_j^2} \Big(\partial_w + \sum_j \partial_{\varphi_j} \Big)$$

$\Big[$ (*) $\tilde{\alpha}_2(\tilde{R}_2) = 1$ is easy.

$$(*) \quad d\tilde{\alpha}_2 = \sum_j 2 r_j\, dr_j \wedge d\varphi_j + \sum_{j=1}^{n} r_j\, dw \wedge dr_j - \sum_j r_j\, dr_j \wedge dw$$

$$\Rightarrow d\tilde{\alpha}_2(\tilde{R}_2, \cdot) = \frac{2}{1 + w^2 + \sum_j r_j^2} \Big(\sum_j r_j\, dr_j + \sum_j r_j\, dr_j - \sum 2 r_j\, dr_j \Big) = 0 \quad \checkmark \quad \Big]$$

$\Rightarrow$ flow lines of R_2 are the lines parallel to the w-axis.

(*) holds true, since:

$$*)\ \sum_j u_j\, dv_j - v_j\, du_j = \sum_j r_j\cos\varphi_j\, d(r_j\sin\varphi_j) - r_j\sin\varphi_j\, d(r_j\cos\varphi_j)$$

$$= \sum_j r_j^2\cos^2\varphi_j \underbrace{}_{d\varphi_j} + r_j^2\sin^2\varphi_j \underbrace{}_{d\varphi_j} = \sum_j r_j\, d\varphi_j \quad ✓$$

$$*)\ w\,dw + \sum_{j=1}^{n-1} u_j\, du_j + v_j\, dv_j = w\,dw + \sum_j r_j\cos\varphi_j\, d(r_j\cos\varphi_j) + r_j\sin\varphi_j\, d(r_j\sin\varphi_j)$$

$$= w\,dw + \sum_j(-r_j^2\cos\varphi_j\sin\varphi_j + r_j^2\sin\varphi_j\cos\varphi_j)$$

$$= w\,dw = \sum_{j=1}^{n-1} r_j\, dr_j \quad \text{Why?}$$

We have: $d\tilde{\alpha}_2\big|_{\{w=0\}} = 2\sum_{j=1}^{n-1} dr_j\wedge d\varphi_j = d\alpha_2|\{w=0\}$

$\Rightarrow$ f should be the identity on $\{w=0\}$ and maps the flow lines of $\tilde{R}_2$ to those of R_2

$\tilde{R}_2$: helices determined by $r_j \equiv \text{const}$, $\varphi_j = w + \text{const}$ and $\frac{dw}{dt} = \frac{2}{1+w^2+\sum_j r_j^2} =: g(w)$ (**)

$$\frac{2}{1+w^2+\sum_j r_j^2}\left(\partial_w + \sum_j \partial\varphi_j\right) \overset{!}{=} \frac{dw}{dt}\partial_w + \sum_j \frac{d\varphi_j}{dt}\partial\varphi_j$$

Solve (**) by separation of variables:

$$\left(1+w^2+\sum_j r_j^2\right)dw = 2\,dt \Rightarrow t = \frac{1}{2}\int_0^w\left(1+w'^2+\sum_j r_j^2\right)dw' + C$$

$$= \frac{1}{2}\left(\left(1+\sum_j r_j^2\right)w + \frac{1}{3}w^3\right) + C$$

$w(t=0) = 0 \Rightarrow C = 0$.

Define $f(\vec{r},\vec{\varphi},w) = \left(\vec{r}, (\varphi_j - w)_j, \int_0^w \frac{d\zeta}{g(\zeta)}\right)$

$= \left(\vec{r}, (\varphi_j - w)_j, \frac{1}{2}w\left(1+\frac{w^2}{3}+\sum_j r_j^2\right)\right)$

Do Sanity Checks !!! ($f|_{\{w=0\}} = id$)

regarding Ex. 3:

Thm: $\forall p \in S^3$, $(S^3 \setminus p, \xi_{st}) \overset{cont}{\cong} (\mathbb{R}^3, \xi_{st})$

$f: S^3 \setminus N \xleftarrow{C^\infty} \mathbb{R}^3$

$f^* \alpha_{st}$ is a contact form on $\mathbb{R}^3$

idea: find $g: \mathbb{R}^3 \to \mathbb{R}^3$ $R_{f^*\alpha_{st}}$ to R_α ($\to$ Geiges)

Ex. 4: Consider the manifold $B = \mathbb{R}^2$ with global coordinates (b_1, b_2)

Then $T^*B \cong \mathbb{R}^2 \times \mathbb{R}^2$ has coordinates $(b_1, b_2, \underbrace{r, \theta}_{\text{polar coordinates on the 2. factor}})$

$\Rightarrow S^*B \cong \mathbb{R}^2 \times \mathbb{R}P^1$ with coordinates (b_1, b_2, θ), $\theta \sim \theta + \pi \ \forall \theta \in \mathbb{R}$

Then $V_{(b_1,b_2,\theta)} = \mathbb{R} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ $\left(V_{(b_1,b_2,\theta)} = \ker\left(\lambda(\cos\theta \, db_1 + \sin\theta \, db_2)\right) \ \forall \lambda \in \mathbb{R} \right)$

We have $T_{(b_1,b_2,\theta)} S^*B \cong T_{(b_1,b_2)} \mathbb{R}^2 \oplus T_\theta \mathbb{R}P^1$

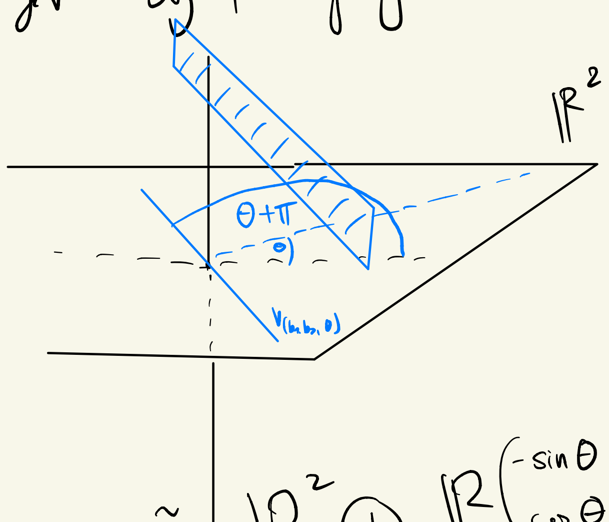
$$\cong \mathbb{R}^2 \oplus \mathbb{R}$$

Look at the differential of $\pi : S^*B \to \mathbb{R}^2$; $(b_1, b_2, \theta) \mapsto (b_1, b_2)$

$$(T\pi)_{(b_1,b_2,\theta)} : \begin{array}{ccc} T_{(b_1,b_2,\theta)}(S^*B) & \longrightarrow & T_{(b_1,b_2)}B \\ \wr\| & & \wr\| \\ \mathbb{R}^2 \oplus \mathbb{R} & \xrightarrow{pr} & \mathbb{R}^2 \end{array}$$

$$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = -\sin\theta \, \partial_{b_1} + \cos\theta \, \partial_{b_2}$$

$\Rightarrow T\pi$ is also given by the projection on the first factor.



$\Rightarrow (\xi_{can})_{(b_1,b_2,\theta)} \cong \mathbb{R}^2 \oplus \mathbb{R}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

(b) M, N manifolds, $f: M \to N$ diffeo

*) $(q_1, \dots, q_n)$ coordinates on M

$(r_1, \dots, r_n)$ coordinates on N

*) $(q_1, -, q_n, [p_1, -, p_n])$ coordinates on $S^*M \equiv \mathbb{P}T^*M$

$(r_1, -, r_n, [s_1, -, s_n])$ —— " —— $S^*N \equiv \mathbb{P}T^*N$

*) $\xi_1 = \ker\left(\sum_{j=1}^n p_j \, dq_j\right) \subset T(S^*M)$

$\xi_2 = \ker\left(\sum_{j=1}^n s_j \, dr_j\right) \subset T(S^*N)$

Define $F: S^*M \to S^*N$

$$(q_1, \dots, q_n, \underbrace{[p_1, \dots, p_n]}_{\hat{=}\left[\left(\sum_{j=1}^n p_j \, dq_j\right)_{\bar{q}}\right]}) \mapsto \left(f_1(\bar{q}), \dots, f_n(\bar{q}), \left(\sum_j p_j \, d(q_j \circ f^{-1})\right)_{f(\bar{q})}\right)$$

$$\left(d(q_j \circ f^{-1})\right)_{f(\bar{q})} = \sum_{i=1}^n \frac{\partial}{\partial r_i} q_j \circ f^{-1} \, dr_i \Rightarrow \left(\sum_j p_j \, d(q_j \circ f^{-1})\right)_{f(\bar{q})}$$

$$= \sum_j p_j \sum_i \frac{\partial}{\partial r_i} f_j^{-1}\Big|_{f(\bar{q})} dr_i$$

$$= \sum_i \underbrace{\left(\sum_j p_j \frac{\partial}{\partial r_i} f_j^{-1} \Big|_{f(\bar q)} \right)}_{\leftarrow \text{ coordinates}} dr_i$$

So, $$F(\bar q, [\bar p]) = \left(f_1(\bar q), \dots, f_n(\bar q), \left(\left[\sum_j p_j \frac{\partial}{\partial r_i} f_j^{-1} \Big|_{f(\bar q)} \right)_i \right] \right)$$

$$\Rightarrow TF\Big|_{(\bar q, [\bar p])} = \begin{pmatrix} Tf|_{\bar q} & 0 \\ & (a_{ik})_{ik} \end{pmatrix}$$

$$\frac{\partial}{\partial q_k} \left(\sum_j p_j \frac{\partial}{\partial r_i} f_j^{-1} \Big|_{f(\bar q)} \right) = \sum_j p_j \frac{\partial}{\partial q_k} \left(\frac{\partial}{\partial r_i} f_j^{-1} \Big|_{f(\bar q)} \right)$$

$$= \sum_j p_j \frac{\partial}{\partial q_k} f(\bar q) \cdot \left(\frac{\partial}{\partial r_1} \frac{\partial}{\partial r_i} f_j^{-1} \Big|_{f(\bar q)}, \dots, \frac{\partial}{\partial r_n} \frac{\partial}{\partial r_i} \dots \right)$$

$$= \sum_j p_j \sum_{l=1}^{n} \frac{\partial}{\partial q_k} f_l(\bar q) \cdot \frac{\partial^2}{\partial r_l \partial r_i} f_j^{-1} \Big|_{f(\bar q)} =: a_{ik}$$

F is differentiable, $F^{-1}(r_1, \dots, r_n, [s_1, \dots, s_n])$

$$= \left(f_1^{-1}(\bar r), \dots, f_n^{-1}(\bar r), \left[\left(\sum_{j=1}^{n} s_j d(r_j \circ f) \right)_{f^{-1}(\bar r)} \right] \right)$$

is the inverse map, which is also diff'ble $\Rightarrow F$ diffeo

Need to show that F is a ~~contacto~~: $F^* \ker\left(\sum_j p_j\, dq_j\right) = \ker\left(\sum s_j\, dr_j\right)$

Let $X \in \ker\left(\Sigma_{p_j} dq_j\right)$. Then,

$$\left(F^* X\right)_{([\bar{r}],[\bar{s}])} = X_{F^{-1}([\bar{r}],[\bar{s}])}\left(TF^{-1}\Big|_{([\bar{r}],[\bar{s}])}\right)$$

No idea

Ex. 5: (b) Define $\alpha_0 := \frac{i}{2}(z\,d\bar z - \bar z\,dz + w\,d\bar w - \bar w\,dw)$

We compute: $z\,d\bar z - \bar z\,dz = (x_1+iy_1)\,d(x_1-iy_1) - (x_1-iy_1)\,d(x_1+iy_1)$

$\uparrow$

$z = x_1+iy_1$

$= x_1dx_1 - ix_1dy_1 + iy_1dx_1 + y_1dy_1$

$- [x_1dx_1 + ix_1dy_1 - iy_1dx_1 + y_1dy_1]$

$= 2i(-x_1dy_1 + y_1dx_1)$

and analogously $w\,d\bar w - \bar w\,dw = 2i(-x_2dy_2 + y_2dx_2)$

$$\Rightarrow \alpha_0 = \frac{i}{2}\cdot 2i\left(\sum_{i=1}^{2} -x_i\,dy_i + y_i\,dx_i\right)$$

$$= \sum_{i=1}^{2} x_i\,dy_i - y_i\,dx_i = \alpha_{st} \text{ on } S^3 \subset \mathbb{C}^2$$

The form $z\,d\bar z - \bar z\,dz$ is invariant under the given action $z \mapsto (e^{2\pi i/p} z)$ and $w\,d\bar w - \bar w\,dw$ is invariant under $w \mapsto (e^{2\pi i\frac{q}{p}} w)$. [Pulling back delivers:

$$e^{2\pi i/p} z\, d(e^{-2\pi i/p}\bar z) - e^{-2\pi i/p}\bar z\, d(e^{2\pi i/p} z)$$

$$= e^{2\pi i/p} z\, e^{-2\pi i/p}\,d\bar z - e^{-2\pi i/p}\bar z\, e^{2\pi i/p}\,dz = z\,d\bar z - \bar z\,dz$$

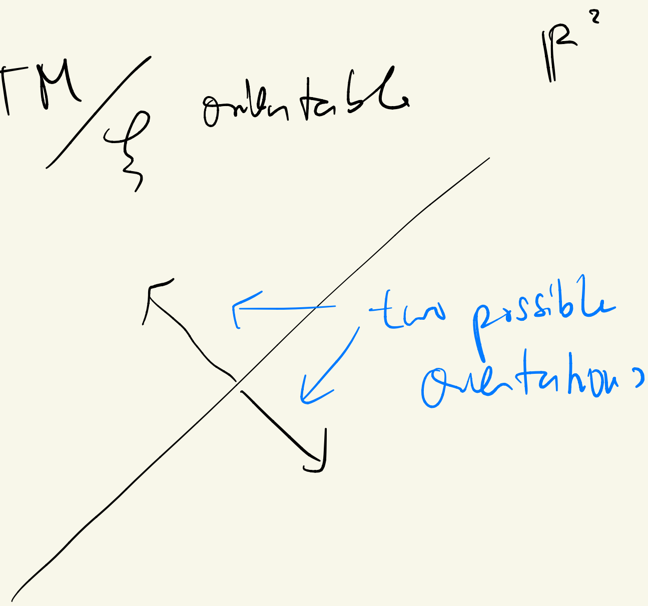
and similarly for $w\,d\bar w - \bar w\,dw$.] $\Rightarrow \exists$ contact structure on $L(q,p)$ which is induced by the cover of $S^3/\sim$ through S^3 $\blacksquare$

(c) Let $q \in \mathbb{Z}_p$, $(z,w) \in S^3$ with $(z,w) = (e^{2\pi i/p} z, e^{2\pi i q/p} w)$

$\Rightarrow e^{2\pi i/p} = 1 = e^{2\pi i q/p} \Rightarrow q = 0$ $\blacksquare$

ξ is coorientable $:\Leftrightarrow \exists$ 1-form α : $\xi = \ker \alpha$

$\Leftrightarrow TM/\xi$ orientable



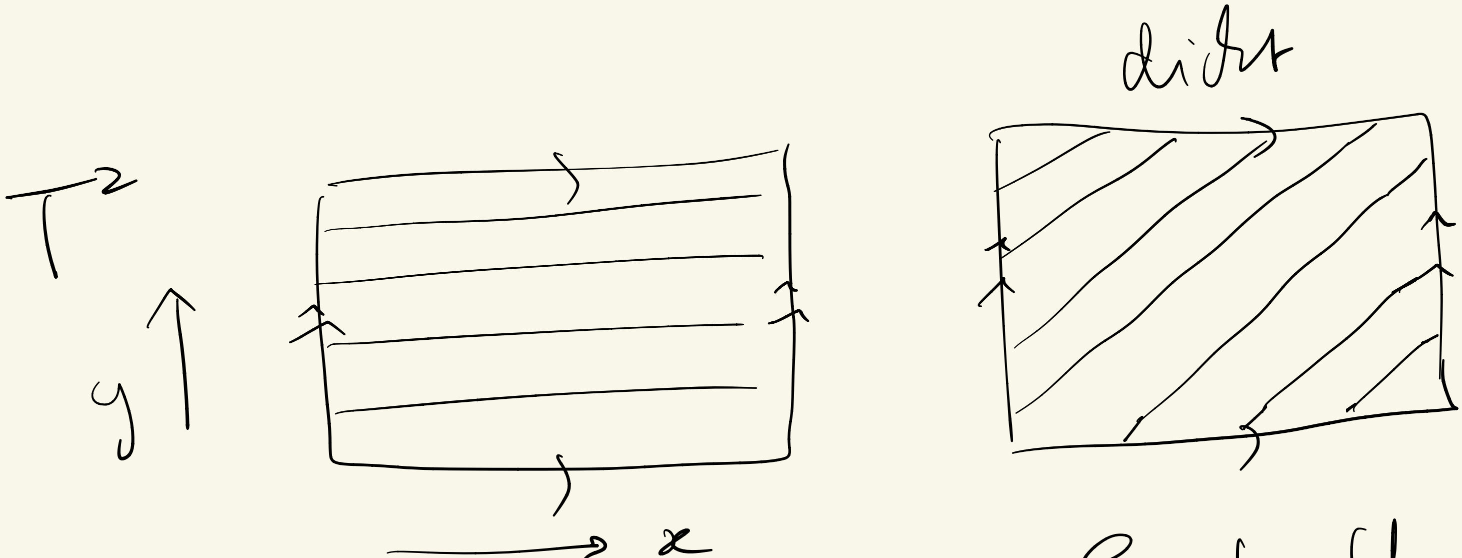
Discussion session:

$r\,dr \wedge d\varphi$ auf $\mathbb{R}^2$

$dx \wedge dy$ = vol auf $\mathbb{R}^2$

Γ $d(r\cos\varphi) \wedge d(r\sin\varphi) = (\cos\varphi\,dr - r\sin\varphi\,d\varphi) \wedge (\sin\varphi\,dr + r\cos\varphi\,d\varphi)$

$= r\,dr \wedge d\varphi \,\lrcorner$



$\xi_0 = \ker(dy)$

$\xi_1 = \ker(dy - \sqrt{2}\,dx)$

$= \langle \sqrt{2}\,\partial_y + \partial_x \rangle$

kommen von Blätterungen

$f: N^n \to f(N^n)$ inj., diff'ble,
N^n cpt, $f(N^n)$ Hausdorff'ian
$\Rightarrow f^{-1}$ diff'ble

WS 2023/24 Marc Kegel

Contact Geometry

Exercise sheet 3

$f: N^n \to f(N)$ diffeo
$\Leftrightarrow$ f inj., f diff'ble, f^{-1} diff'ble

Let ξ be a tangential 2-plane field on a 3-manifold M. An embedding $c\colon \mathbb{R} \to M$ is called **Legendrian** if it is tangent to ξ, i.e. $Tc \subset \xi$. Similarly, an embedding $c\colon \mathbb{R} \to M$ is called **transverse** if it is transverse to ξ, i.e. $Tc \oplus \xi = TM$.

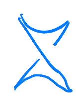
$\dot{c} = Tc \neq dc = \dot{c}\,dt$

Exercise 1.
Consider the following two curves in $(\mathbb{R}^3, \xi_{st})$ $\quad \alpha_{st} = x\,dy + dz$

$$c_1 : t \longmapsto \Big(3\sin(t)\cos(t), \cos(t), \sin^3(t)\Big),$$
$$c_2 : t \longmapsto \Big(\cos(t), \sin(2t), \frac{2}{3}\sin(t)\cos(2t) - \frac{4}{3}\cos(t)\sin(2t)\Big).$$

unknot ($\overset{\text{def.}}{:=}$ boundary of a disc)

(2) (y,z):



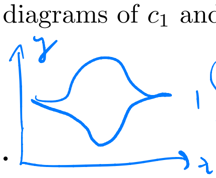
(x,y):

(a) Show that c_1 and c_2 define Legendrian knots.

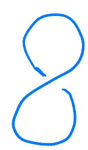
(b) Draw sketches of these curves and determine the smooth knot types.

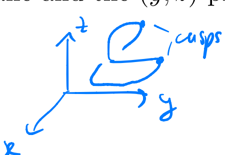
(c) Draw diagrams of c_1 and c_2 in the (x, y)-plane and the (y, z)-plane.

(c) (1) (y,z):



(x,y):





Exercise 2.

See lecture

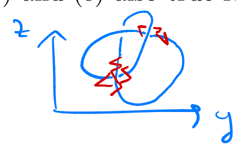
(a) Any two points in $(\mathbb{R}^3, \xi_{st})$ can be connected by a Legendrian curve.

(b) Any two points in an arbitrary connected contact 3-manifold can be connected by a Legendrian curve.

(c) Any smooth knot in a contact 3-manifold is isotopic to a Legendrian knot. (See lecture)

(d) Are (a), (b) and (c) also true for transverse curves?

zu (c)



Was ist C^0-nah? $K: S^1 \hookrightarrow \mathbb{R}^3$
$\forall \varepsilon > 0\ \exists \tilde{K}: S^1 \hookrightarrow \mathbb{R}^3\ \forall t \in S^1: |K(t) - \tilde{K}(t)| < \varepsilon$

Exercise 3.
Describe a Legendrian knot L in $(\mathbb{R}^3, \xi_{ot})$ that is the boundary of an embedded disk D such that along L the contact planes agree with the tangent spaces of D. Deduce that one can find such a Legendrian knot in any contact manifold that is contactomorphic to $(\mathbb{R}^3, \xi_{ot})$.
Remark: We will later show that in $(\mathbb{R}^3, \xi_{st})$ such a Legendrian knot cannot exist~~s~~. This will prove that these two contact structures are not contactomorphic on $\mathbb{R}^3$.

Exercise 4. (uses Ex. 2)

(a) Let ξ be a tangential 2-plane field on a 3-manifold M, that is induced by a foliation. Show that two points in M can be connected by a Legendrian curve if and only if the two points lie in the same leaf.

(b) What can be said about transverse curves and transverse knots in 2-planes fields induced by foliations?

(c) Use (a) to deduce that a contact structure cannot be induced from a foliation.

(d) Describe tangential 2-plane fields on 3-manifolds that are not contact structures but in which any two points can be connected by Legendrians.

Ex. 4) (a) ξ is induced by a foliation $\Rightarrow M = \bigsqcup_{i \in I} N_i$, $N_i \subset M$ sub-2-mfds (the leafs)

and $\forall p \in M \ \exists! i \in I$ s.t. $p \in N_i$ with $\xi_p = T_p N_i$

Let $\gamma : [0,1] \to M$ be Legendrian, $p := \gamma(0)$, $q := \gamma(1)$. Then $T\gamma \subset \xi$

$\Rightarrow \forall t \in [0,1]: \dot{\gamma}(t) \in \xi_{\gamma(t)}$

This sheet will be discussed on Wednesday 15.11. and should be solved by then.

Ex. 1) $c_2(t) := \left(\cos(t), \sin(2t), \frac{2}{3}\sin(t)\cos(2t) - \frac{4}{3}\cos(t)\sin(2t) \right) : [0,2\pi) \to \mathbb{R}^3$

$\xi_{st} = \ker \alpha_{st}$, $\alpha_{st} = dz + x\,dy$

*) $T_{c_2} = \left(-\sin(t), 2\cos(t), \frac{2}{3}\left(\cos(t)\cos(2t) - 2\sin(t)\sin(2t)\right) - \frac{4}{3}\left(-\sin(t)\sin(2t) + 2\cos(t)\cos(2t)\right) \right)$

$= \left(-\sin(t), 2\cos(2t), -\frac{6}{3}\cos(t)\cos(2t) \right)$

*) $\alpha_{st, \gamma(t)}(T_{c_2}) = -2\cos(t)\cos(2t) + \cos(t)\cdot 2\cos(2t)) = 0$

$\Rightarrow T_{c_2} \subset \ker \alpha_{st} = \xi_{st}$

*) embedding: injectivity: Let $s,t \in [0,2\pi)$ with $c_2(s) = c_2(t)$. A: $s \neq t$

Then, *) $\cos(s) = \cos(t) \Rightarrow t = 2\pi - s$

*) $\sin(2s) = \sin(2t) = \sin(4\pi - 2s) = \underbrace{\sin(4\pi)}_{=0}\cos(2s) - \underbrace{\cos(4\pi)}_{=1}\sin(2s)$

$= -\sin(2s)$

$\Rightarrow \sin(2s) = 0$

$\Rightarrow s \in \{\frac{\pi}{2}, \pi, 3\frac{\pi}{2}\}$. Due to $t \neq s$, $s \neq \pi \Rightarrow s \in \{\frac{\pi}{2}, \frac{3}{2}\pi\}$

$s = \pi/2 \Rightarrow c_{2,z}(s) = \frac{2}{3}\cdot 1 \cdot (-1) = -\frac{2}{3}$ and

$c_{2,z}(t) = c_{2,z}(\frac{3}{2}\pi) = \frac{2}{3}(-1)\cdot(-1) = \frac{2}{3}$ ↯

$s = \frac{3}{3}\pi \Rightarrow$ (similarly) ↯

$\Rightarrow c_2$ injective.

*) immersion: $0 = c_{2,x}'(t) = -\sin t \Rightarrow t \in \{0, \pi\}$

$0 = c_{2,y}'(t) = 2\cos(2t) \Rightarrow t \in \{\frac{\pi}{4}, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi\}$ } disjoint

$\Rightarrow c_2'(t) \neq 0 \;\forall t \Rightarrow c_2$ immersion.

Ex 2)(a) Let $X = p\,\partial_x + q\,\partial_y + r\,\partial_z$ be a vector field on $\mathbb{R}^3$.

Then, $X \subset \xi_{st} \iff \alpha_{st}(X) = 0 \iff r + xq = 0 \iff r = -xq$ (*)

Observe that $\xi_{st} = \ker \alpha_{st} = \ker(dz + x\,dy) = \langle \partial_x, \partial_y - x\partial_z \rangle$

Let p and q only depend on x and y.

Define $X = \sin\theta\,\partial_x + \cos\theta\,\partial_y - x\cos\theta\,\partial_z \subset \xi_{st} \rightarrow$ integral curves of X are Legendrian

We want to find an integral curve with initial conditions $(0,0,c)$ at $t=0$

We have to fulfill $x' = \sin\theta$ and $y' = \cos\theta$ $\xRightarrow[x(0)=0=y(0)]{}$ $x(t) = t\sin\theta$, $y(t) = t\cos\theta$

Plug in this in (*) yields $z' = -x y' = -t\sin\theta\cos\theta \xRightarrow[z(0)=c]{} z(t) = c - \frac{\cos\theta\sin\theta}{2}t^2$

Claim: all pts of these integral curves lie on the surface $S_c = \{z = -\frac{xy}{2} + c\}$ and, conversely, every pt on this surface lies on an integral curve of some vector field of the type above.

pf: $-\frac{xy}{2} + c = -\frac{\sin\theta\cos\theta}{2}t^2 + c = z$ ✓

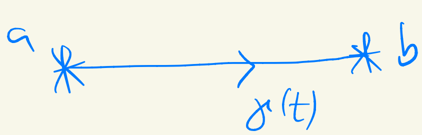
$TS = \ker\left(D\left(z + \frac{xy}{2} - c\right)\right) = \ker\left(dz + \frac{x}{2}dy + \frac{y}{2}dx\right)$ and

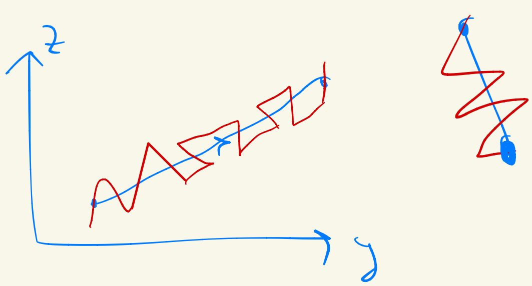
$dz + \frac{1}{2}(x\,dy + y\,dx)(X) = -x\cos\theta + \frac{1}{2}\left(x\cos\theta + y\sin\theta\right) = 0$

Let $(x,y,z) \in \mathbb{R}^3$ with $z = -\frac{xy}{2} + c$. Define $x(t) = t\sin\theta + x$, $y(t) = t\cos\theta + y$,
$z(t) = c - \frac{\cos\theta\sin\theta}{2}t^2 + z$

Then $\begin{pmatrix}\dot x \\ \dot y \\ \dot z\end{pmatrix} = \begin{pmatrix}\sin\theta \\ \cos\theta \\ -\cos\theta\sin\theta\end{pmatrix}$ ✓ #

brief proof: $(\mathbb{R}^3, \xi_{st} = \ker(x\,dy + dz))$





Define $\phi: \mathbb{R}^3 \to \mathbb{R}^3, (u,v,w) \mapsto (u, v, -\frac{uv}{2} + w)$

Then,

$$D\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -u/2 & -v/2 & 1 \end{pmatrix},$$ which has determinant $1 \Rightarrow \phi$ is a diffeo

$\uparrow$ ϕ^{-1} is given by $(x,y,z) \mapsto (x, y, \frac{xy}{2} + z)$

$$*) \ \phi(\{w = c\}) = S_c \ \left(= \{ z = -\frac{xy}{2} + c \} \right)$$

Let $\varepsilon > 0 \underset{\text{continuity}}{\Longrightarrow} \exists \delta > 0 \ \forall w \in (-\delta, \delta), (u,v) \in B^2_\delta(0): \phi(u,v,w) \in B_\varepsilon(0)$

Define $U_1 := \phi((-\delta,\delta) \times B^2_\delta(0)) \subset B_\varepsilon(0)$

$\Rightarrow \forall c \in (-\delta, \delta) \ \forall p, q \in U_1 \cap L(c) \ \exists$ Legendrian curve joining p to q.

Now, we want to find vfs being tangent to ξ_{st} and having integral curves being transverse to the surfaces S_c, $c \in \mathbb{R}$.

Consider $X = x^2 \partial_z - y \partial_x + x \partial_y$. Initial data: $(a,0,b)$ at $t=0$, assume $a > 0$.

$\Rightarrow x(t) = a\cos t, \ y(t) = a \sin t$

($\Rightarrow \dot{x} = -a\sin t = -y, \ \dot{y} = a\cos t = x$ ✓)

$\dot{z}(t) = a^2 \cos^2 t, \ z(0) = b$

$= a^2\left(\frac{\cos 2t + 1}{2}\right) \Rightarrow z(t) = \frac{a^2}{4}\sin(2t) + \frac{a^2}{2}t + b$

Define $f(x,y,z) := z + \frac{1}{2}xy$. Then, $g(t) := f \circ \gamma(t)$ is strictly increasing near $t = 0$

$\left[g'(0) = \nabla f(\gamma(0)) \cdot \gamma'(0) = \nabla f(a,0,b) \cdot (0, a, a^2)^t = \begin{pmatrix} 0 \\ a/2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ a \\ a^2 \end{pmatrix} = \frac{a^2}{2} + a^2 > 0 \right]$

We conclude: integral curve γ with initial data $(a,0,b)$ lies on

$S_c \Rightarrow b = z = c - \frac{xy}{2} \overset{y=0}{=} c \Rightarrow \exists \delta > 0 \ \forall c' \in \mathbb{R}, |c'-c| < \delta: \gamma$ contains a point of $S_{c'}$

(b) Let $p \in M^3$. Darboux $\Rightarrow \exists p \in U \subset M^3$ open, $h: U \xrightarrow{C^\infty} U_0 \subset \mathbb{R}^3$ diffeo, s.t.

$h(p) = 0, \ h^*(\xi) = \xi_{st}$

(a) $\Rightarrow$ all pts in U_0 can be connected with $0 \in U_0$ by Legendrian curves

$\Rightarrow$ — a — U ——————— u ——— $p \in U$ ——————— u ———————

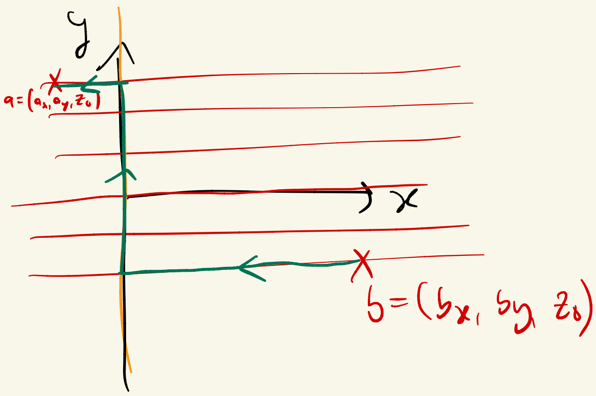
$x\,dy + dz$

$t \mapsto (t, y_0, z_0)$ Leg

$t \mapsto (0, t, z_0)$ Leg

$t \mapsto (1, -t, t)$ Leg.

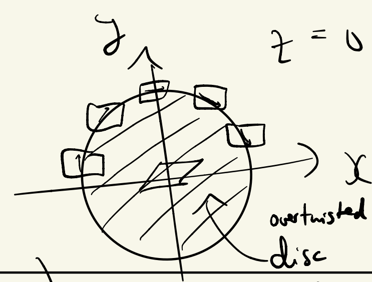
$\rightsquigarrow$ "$\underbrace{\text{pl}}_{\text{piecewise linear}}$- Kurve"



Cylindric Coordinates (r, φ, z)

Ex. 3

$$\alpha_{ot} = \cos(r)\,dz + r\sin r\,d\varphi$$



$r = \pi \longrightarrow \alpha_{ot} = dz$

$\rightsquigarrow \xi_{ot} = \langle \partial_x, \partial_y \rangle$

$T\varphi(\xi_{ot}) = \xi'_{ot}$

$\uparrow$

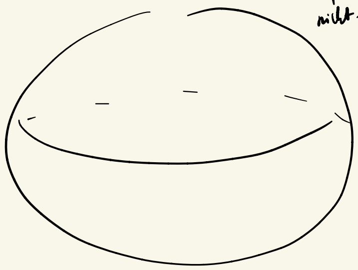
contacto

$\nexists$ in $(\mathbb{R}^3, \xi_{st}) \Longrightarrow (\mathbb{R}^3, \xi_{st})$ and $(\mathbb{R}^3, \xi_{ot})$ not contactomorphic

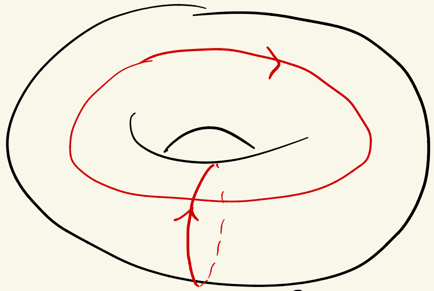
(Bennequin)

Homologie theorie

$\chi = 2$; $\pi_1 = \{1\}$ nicht trivial



$\chi = 0$, $\pi_1 = \mathbb{Z}^2$ (2 Erzeuger)



$T^2 = S^1 \times S^1$

nicht homöomorph

$0 \to \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$

$\sigma_1 \longmapsto b-a-c \neq 0$

$\sigma_2 \longmapsto a+c-b \neq 0$

$\sigma_1 + \sigma_2 \in \ker$

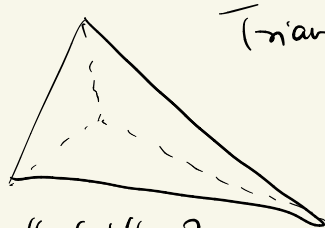
$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longmapsto 0$

2 Dreiecke, 3 Kanten, 1 Ecke

$H_0 = \mathbb{Z}$
$H_1 = \mathbb{Z}^2$
$H_2 = \mathbb{Z}$

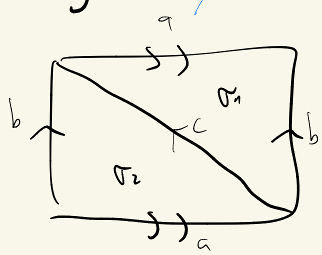
I: top. Räume / Homöo $\longrightarrow$ alg. Objekt

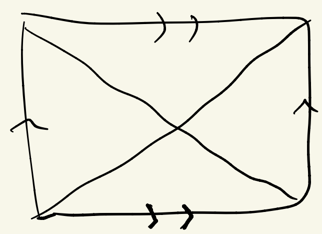
Euler-Charakteristik:



$\chi = 4 - 6 + 4 = 2$

Triangulierung



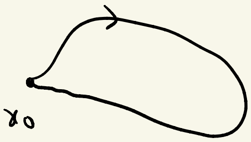


$\chi = \#\text{Ecken} - \#\text{Kanten} + \#\text{Dreiecke}$

$1 - 3 - 2 = 0$ $\quad$ $2 - 6 - 4 = 0$

$\pi_1(X, x_0)$

Gruppe



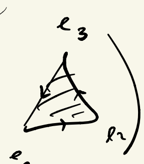
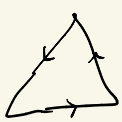
$\pi_1(X, x_0) = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$

$H_1(X) = \pi_1^{ab}(X, x_0)$ = endlich präsentierte abelsche Gruppe $\cong \mathbb{Z}^r \oplus \bigoplus_{p_i} \mathbb{Z}_{p_i}$

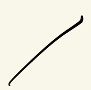
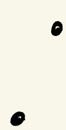
Kettengruppen:

$C_i(X) := \mathbb{Z}^{\#(i\text{-Simplizes})}$

$C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \longrightarrow \dots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C_{-1} := 0$

$\partial\Big($  $\Big) =$  $= (e_1, e_2) + (e_2, e_3) + (e_3, e_1)$

$\boxed{\partial^2 = 0}$

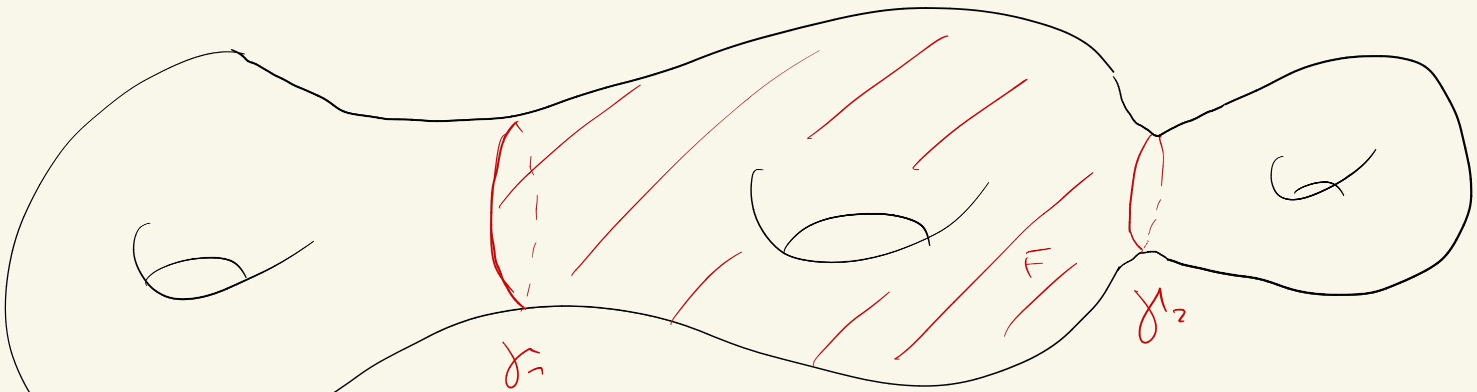
$\partial\Big($  $\Big) =$ 

$\rightsquigarrow$ **Kettenkomplex**

Homologie $H_n(X) := \dfrac{\ker(\partial_n)}{\operatorname{im}(\partial_{n-1})}$

Satz: $H_n(X)$ ist unabhängig von der Triangulierung ab

$\chi = \sum_{i=0}^{\infty} (-1)^i \operatorname{rk}(H_i(X))$



$0 \neq \gamma \in \pi_1$

$\partial F = \gamma_1 \sqcup \gamma_2 \iff [\gamma_1] = 0 \in H_1$

$\|$

$[\gamma_2]$

$[K] = 0 \in H_1(M^3)$

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Contact Geometry

Exercise sheet 4

Exercise 1.

(a) Show that the local modifications of front projections depicted in Figure 1 induce isotopies of Legendrian knots.
Remark: The Legendrian-Reidemeister theorem says that in fact two front projections describe Legendrian isotopic knots if and only if the front projections can be transformed into each other via finitely many planar isotopies and the moves from Figure 1.

(b) Show that the pairs of front projections shown in Figures 2 describe isotopic Legendrian knots.

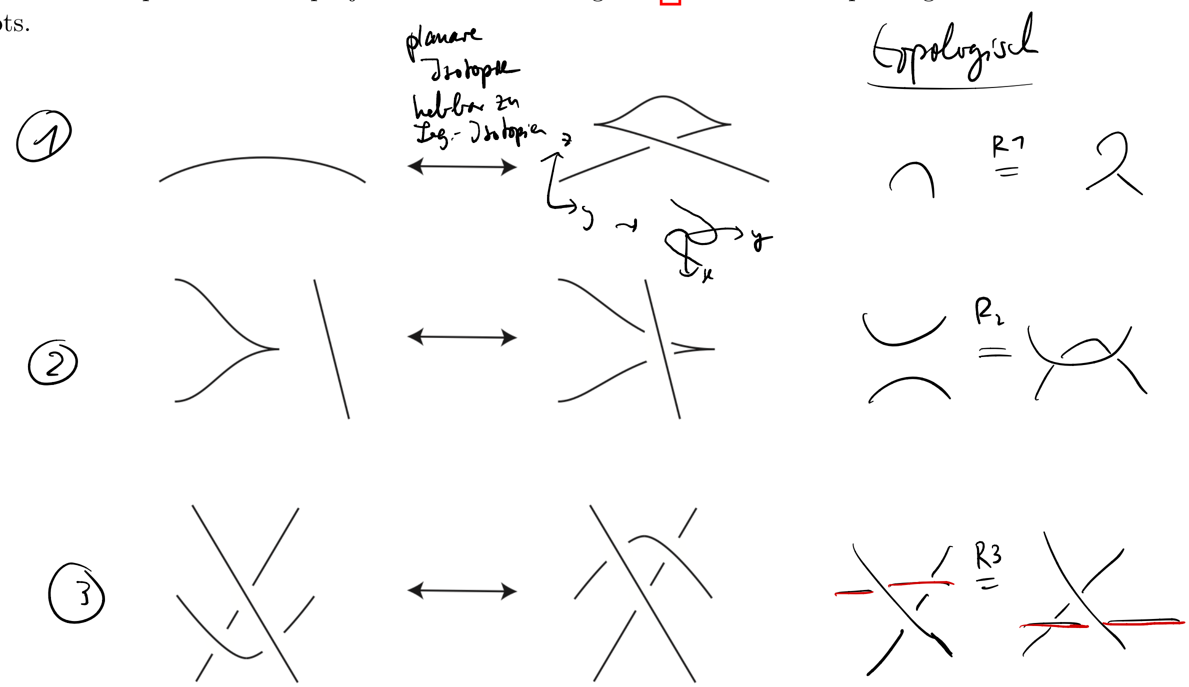


Abbildung 1: Together with the π-rotations around all coordinate axes these local modifications are the Legendrian Reidemeister moves.

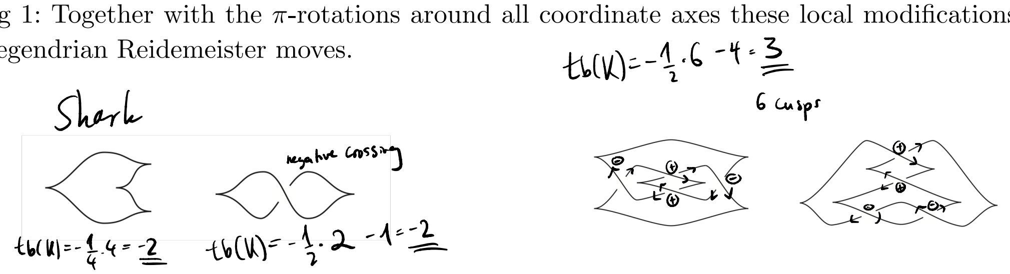
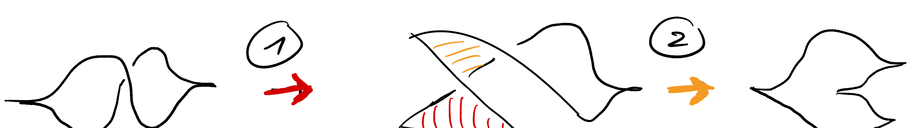


Abbildung 2: Front projections of Legendrian unknots (left) and Legendrian figure eight knots (right).



Exercise 2.

(a) Describe transverse realizations of unknots, right- and left-handed trefoils, and figure eight knots. No cusps!

(b) Describe an algorithm to construct a front projection of the transverse push-off of a Legendrian knot K from a front projection of K.

(c) Discus front projections of links consisting of Legenrian and transverse knots.

Exercise 3.

(a) Fill in the details in the argument from the lecture that the Alexander polynomial is a knot invariant.

(b) Compute the Alexander polynomial of the unknot, the trefoil, and the figure eight knot and deduce that these knots are pairwise non-isotopic.

(c) Show that the figure eight knot has genus 1.

(d) Construct for every natural number $g \in \mathbb{N}_0$ a knot K_g with genus g.

Exercise 4.
The **Lagrangian** projection is the projection to the (x, y)-plane.

(a) The Lagrangian projection of a Legendrian knot K determines K up to isotopy of Legendrian knots (in fact up to translation in z-direction).

(b) Discuss Lagrangian projections of Legendrian knots in analogy to the front projections.

(c) Use the Lagrangian projection to reprove that any smooth knot can be approximated by a Legendrian knot.

(d) Describe an algorithm to construct a Lagrangian projection of a Legendrian knot K from a front projection of K.

Bonus exercise.
Let K be a knot in a 3-manifold M.

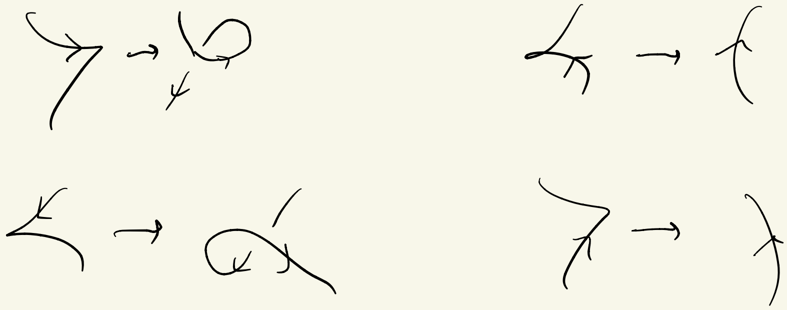
(a) Compute the homology groups of the complement of K in M.

(b) Show that K bounds a Seifert surface if and only if K is nullhomologous.
Hint: Try to generalize the proofs from the lecture for knots in S^3.

Bonus exercise.
Prove Theorems 3.1 and 3.2 from the lecture.

This sheet will be discussed on Wednesday 22.11. and should be solved by then.

Ex. 2) (b) Wähle Orientierung

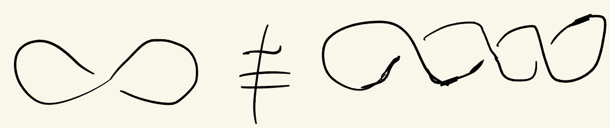


L Legendre-Knoten

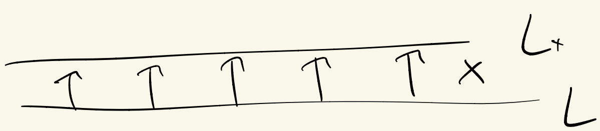
$(\nu L, \xi)$ Tubenumgebung

$$T_+ := L_+ \quad \theta \mapsto (\theta, \varepsilon \sin\theta, \varepsilon\cos\theta)$$

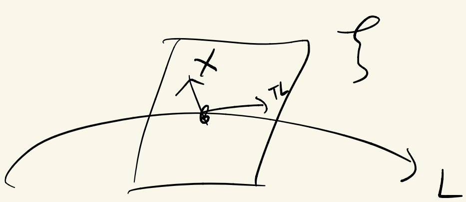
$$L \cong_{cont} \left(S^1 \times D^2, \ker(\cos(\theta)dx - \sin(\theta)dy)\right), \quad L \mapsto S^1 \times \{0\}$$

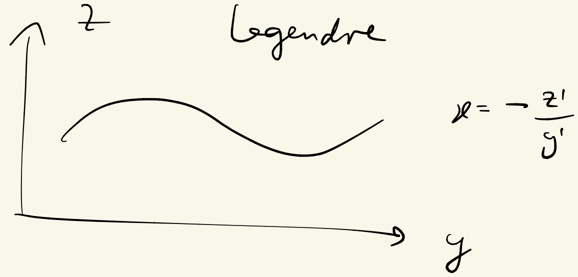


als transversale Knoten, aber „$\overset{C^0}{=}$"



$X = \varepsilon \sin(\theta)\partial_x + \varepsilon\cos(\theta)\partial_y$ is vf in ξ & $X \pitchfork L$





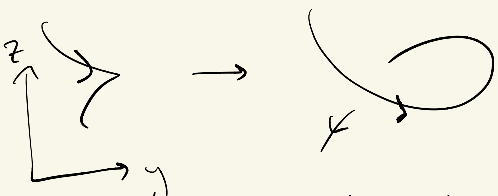
__Ansatz:__ $X := \partial_x \quad X \in \xi = \ker(x\,dy + dz)$

$X \pitchfork L \iff TL \neq \pm\partial_x \iff$ L hat keine Spitze

$\Rightarrow$ Wenn L keine Spitze hat, dann ist Front von L_+ = Front von L
Bogen von

nahe an der Spitze:
$X = \partial_y - x\partial_z \in \xi$ & $X \pitchfork L$

„Zeige":

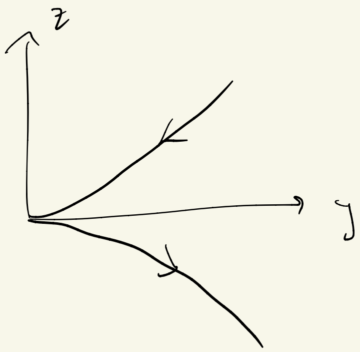


& glatt interpolieren

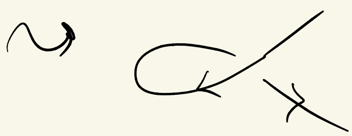
Cusp: $t \overset{c}{\longmapsto} (t, t^2, -\frac{2}{3}t^3)$

$$T\mathcal{L} = \partial_x + 2t\,\partial_y - 2t^2\partial_z$$

$$\alpha(T\mathcal{L}) = t \cdot 2t - 2t^2 = 0$$



$$L_+ : t \longmapsto (t,\ t^2+1,\ \underbrace{-\tfrac{2}{3}t^3 - t}_{-t(\frac{2}{3}t^2+1)})$$



Ex. 4) (a) $x\,dy + dz = 0 \Rightarrow x(t)\,y'(t) + z'(t) = 0$

$$\Rightarrow z' = -xy' \Rightarrow z(t) = z_0 - \int_0^t x y'\,ds$$

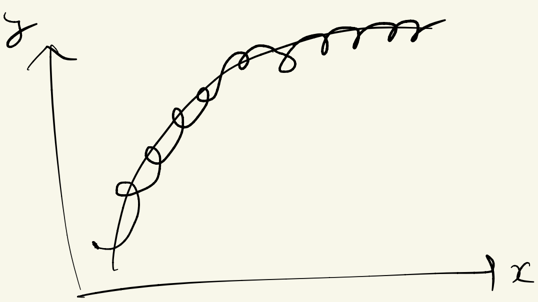
$$= z_0 - \int_{y_0}^{y(t)} x\,dy$$

(b) $0 \overset{!}{=} z(T) - z(0) = -\int_\gamma x\,dy$

$$= -\int_F dx \wedge dy$$

= Flächeninhalt von F, $\partial F = \mathrm{im}\,\gamma$

(c)



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Contact Geometry

Exercise sheet 5

Sheet 6 / Ex 1!

Exercise 1.

(a) Prove Lemma 3.12 from the lecture. $tb(K) = -\frac{1}{2} c + W$

(b) Compute the classical invariants of the Legendrian knots from Sheet 4.

(c) Verify that the classical invariants of Legendrian knots stay the same under the Legendrian Reidemeister moves (see Sheet 4).

Thm: D_1 & D_2 fronts of Leg knots L_1, L_2 $\Rightarrow \Big(L_1 \overset{\text{isotopic as Leg knots}}{\cong} L_2 \iff D_1 \xrightarrow{\Omega} \dots \rightarrow \dots \xrightarrow{\Omega} D_2 \Big)$

Exercise 2.

(a) Show that $\mathrm{tb}(K) + \mathrm{rot}(K)$ is for any Legendrian knot K in $(\mathbb{R}^3, \xi_{st})$ an odd number. (→ Geiges)

(b) Show that any odd number is realized as $\mathrm{tb}(K) + \mathrm{rot}(K)$ for some Legendrian knot K. (→ Geiges)

Exercise 3.

(a) Show that the stabilization of a Legendrian knot (as defined in the lecture) is a well-defined operation.

(b) ~~Any two Legendrian knots become Legendrian isotopic after sufficiently many stabilizations.~~
~~*Hint:* Use the Reidemeister theorem for smooth knots.~~

Exercise 4.

(a) Fill in the details in the argument from the lecture that the Alexander polynomial is a knot invariant.

(b) Compute the Alexander polynomial of the unknot, the trefoil, and the figure eight knot and deduce that these knots are pairwise non-isotopic.

(c) Show that the figure eight knot has genus 1.

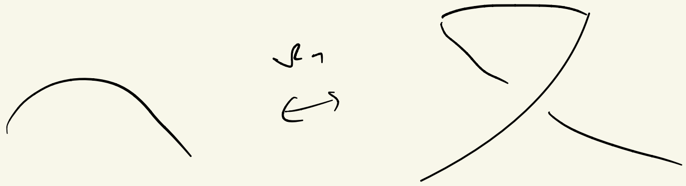
(d) Construct for every natural number $g \in \mathbb{N}_0$ a knot K_g with genus g.

(e) Verify that the combinatorial formula for the linking number is preserved under the smooth Reidemeister moves.

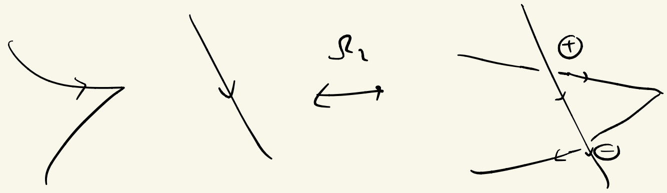
Exercise 5.
Describe formulas for computing the Thurston–Bennequin invariant and the rotation number from a Lagrangian projection (see Sheet 4).

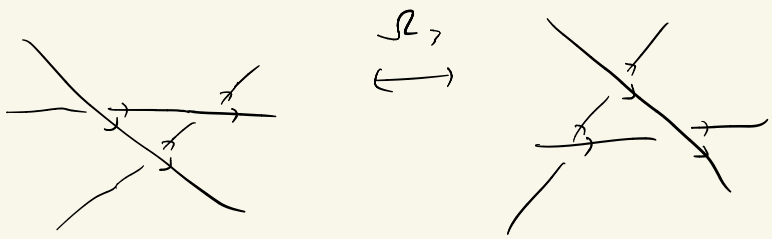
This sheet will be discussed on Wednesday 29.11. and should be solved by then.

1) c)



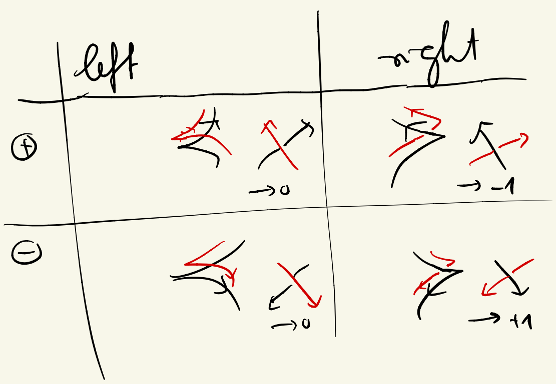
$$\boxed{\begin{aligned} tb &= -\tfrac{1}{2}C + W \quad (*) \\ rot &= \tfrac{1}{2}(C_- - C_+) \end{aligned}}$$





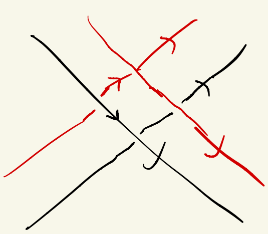
Beweis von (*): lk(K, K') = # crossings of K' under K

tb(K) = lk(K, K')



n cusps ⇒ $\frac{n}{2}$ right cusps

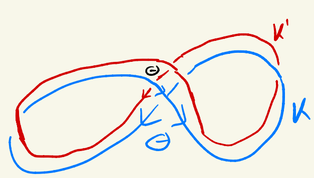
⇒ $tb(K) = -\frac{1}{2}C$



→ −1

⇝ alle Fälle betrachten!

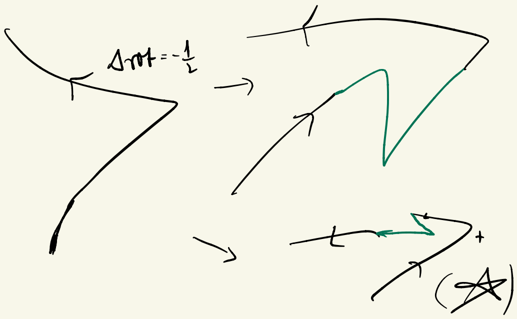
z.B.

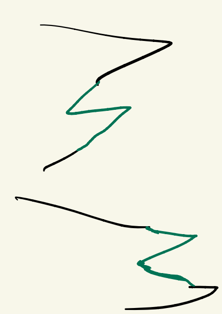


⇒ $tb = -\frac{1}{2}C + W$

---

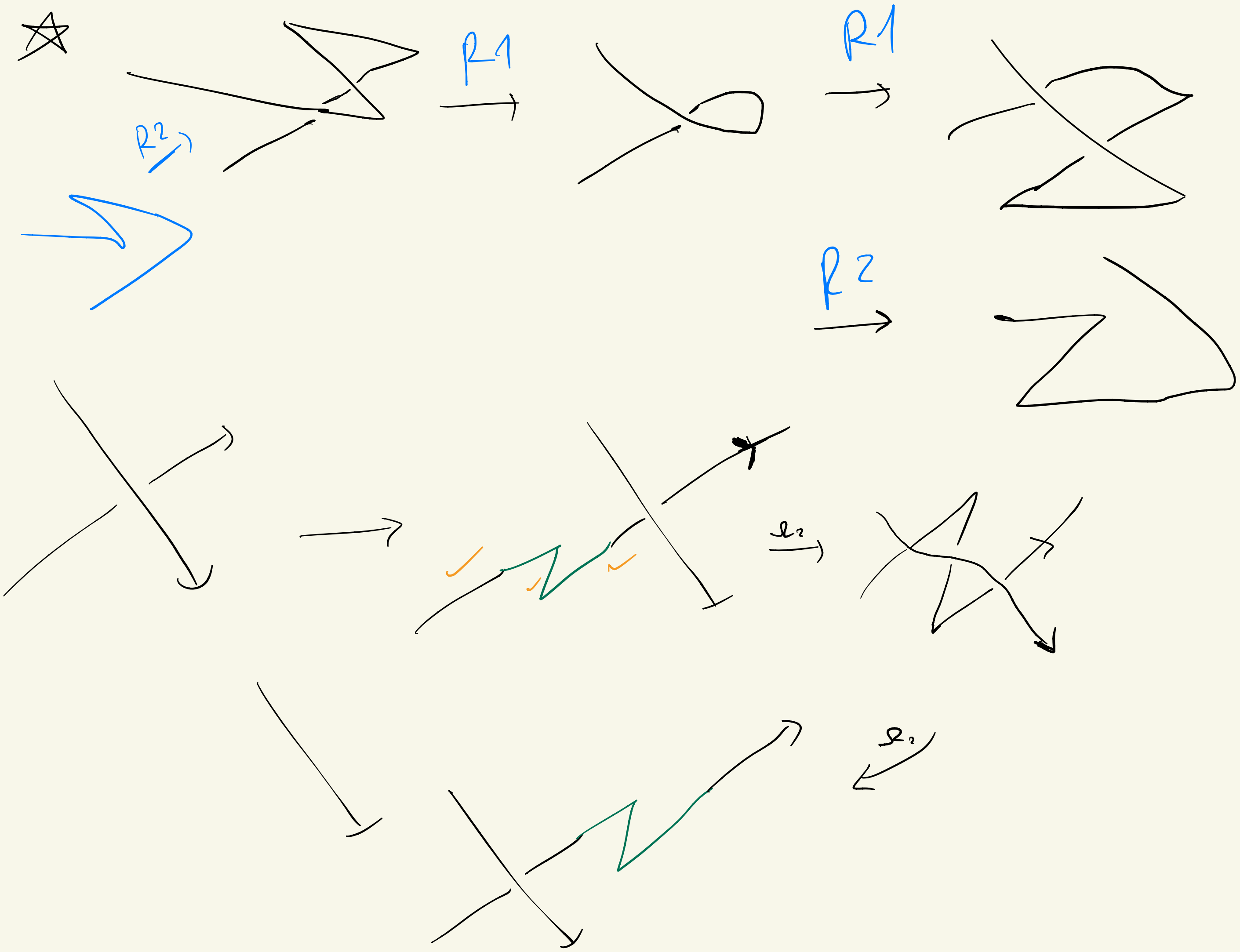
3) a)



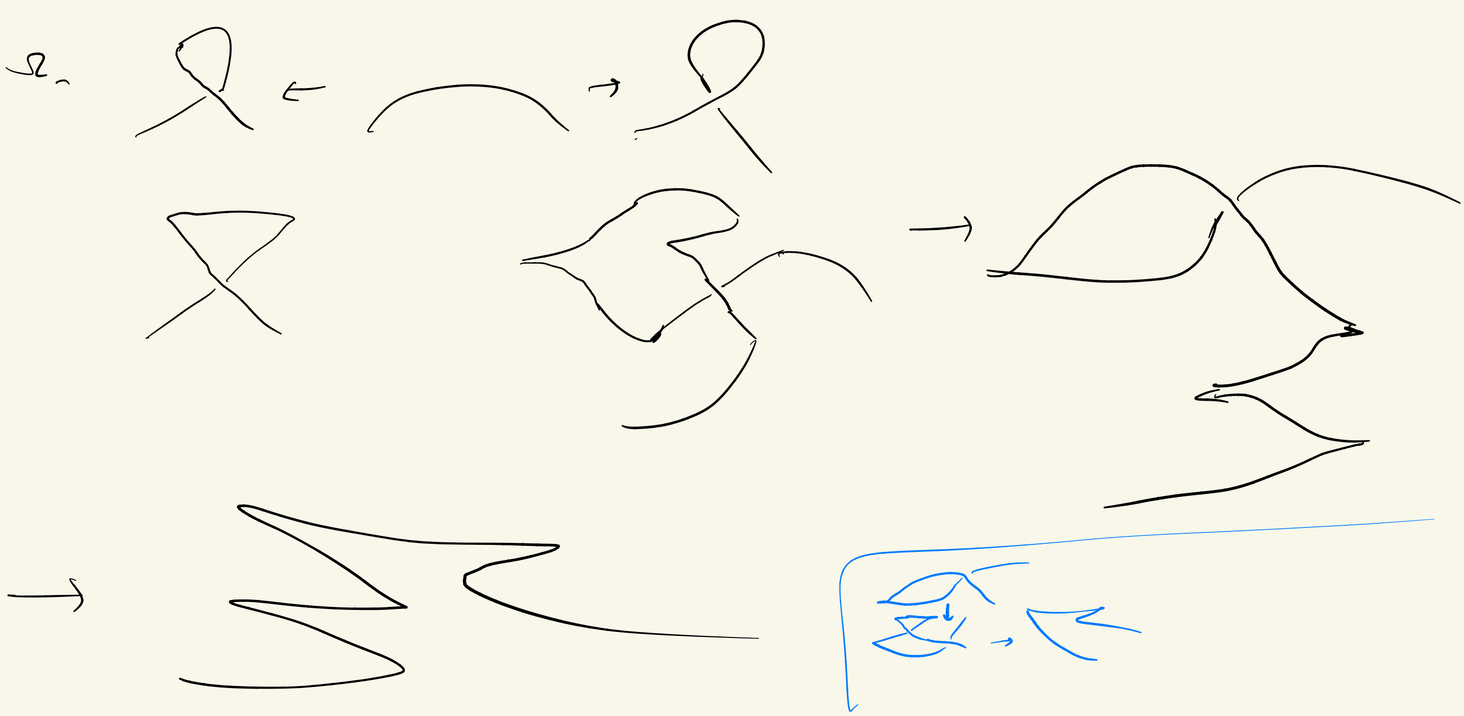


?

☆



L_1, L_2 Leg.-Knoten, glatt isotop $\Rightarrow \exists\, N_1, N_2 \in \mathbb{N}: S_{N_1}(L_1) \cong S_{N_2}(L_2)$ als Leg.-Knoten



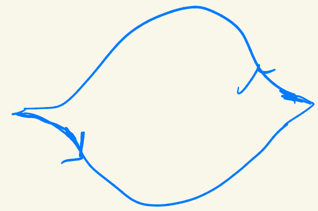
2) a)

$tb(k) + rot(k)$

$= -\frac{1}{2} c + W + \frac{1}{2}(C_- - C_+)$

$= -\frac{1}{2}(C_- + C_+) + W + \frac{1}{2}(C_- - C_+)$

$= -C_+ + W$

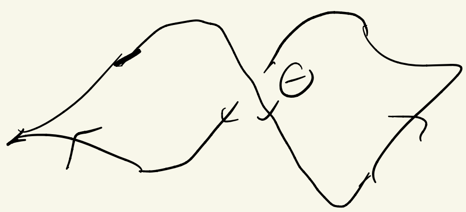


$C_+ = 1, \; W = 0$

Fall 1: C_+ gerade $\overset{??}{\Rightarrow}$ W ungerade

Fall 2: C_+ ungerade $\overset{??}{\Rightarrow}$ W gerade

Zu (1):

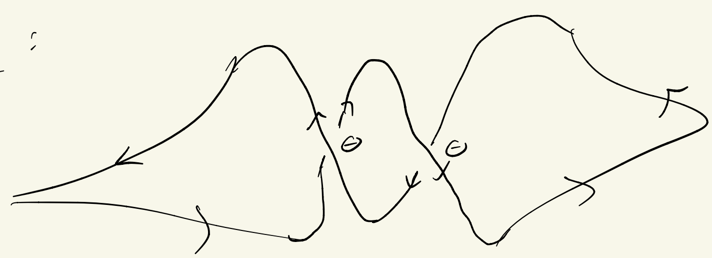


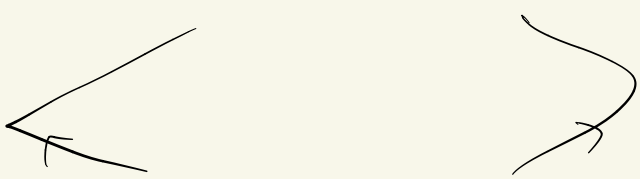
$\Rightarrow$ Clu

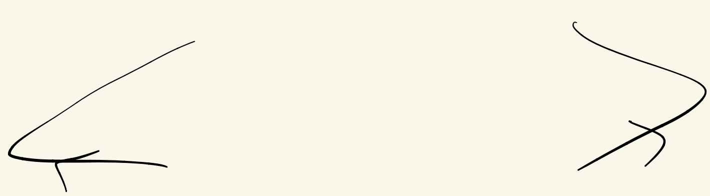
Zu (2):

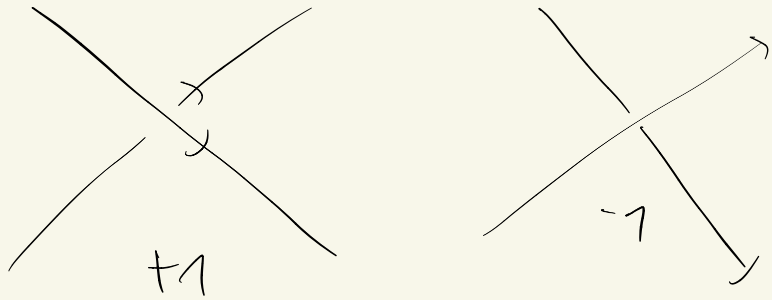
$C_+ = 1, \; W = -2$

$\to -1$

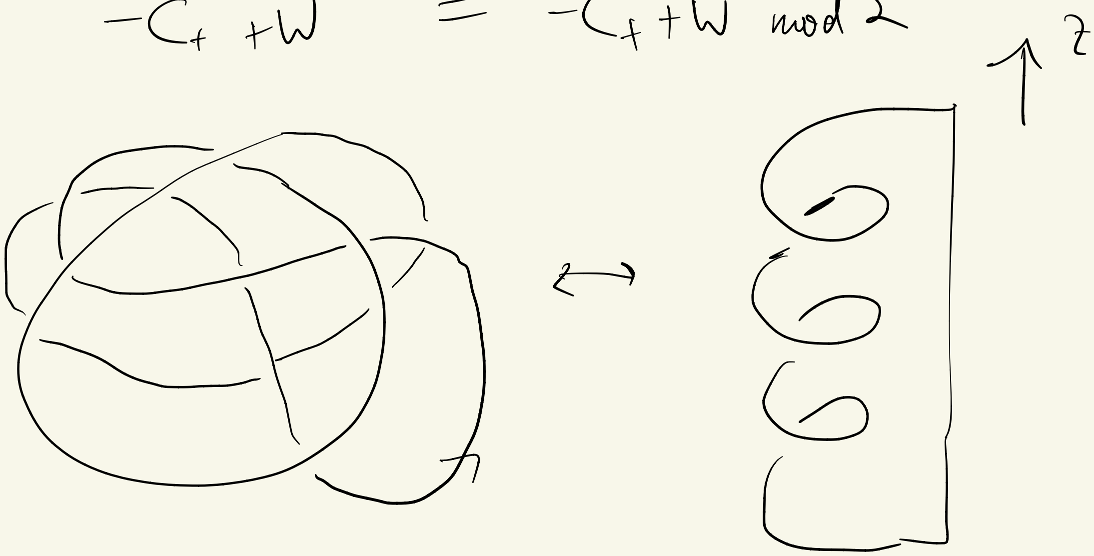




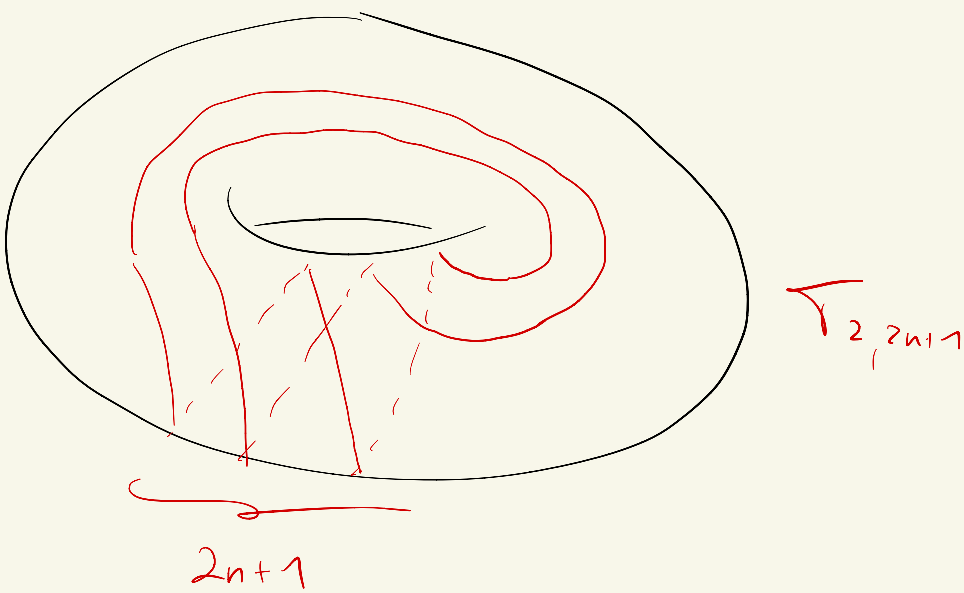




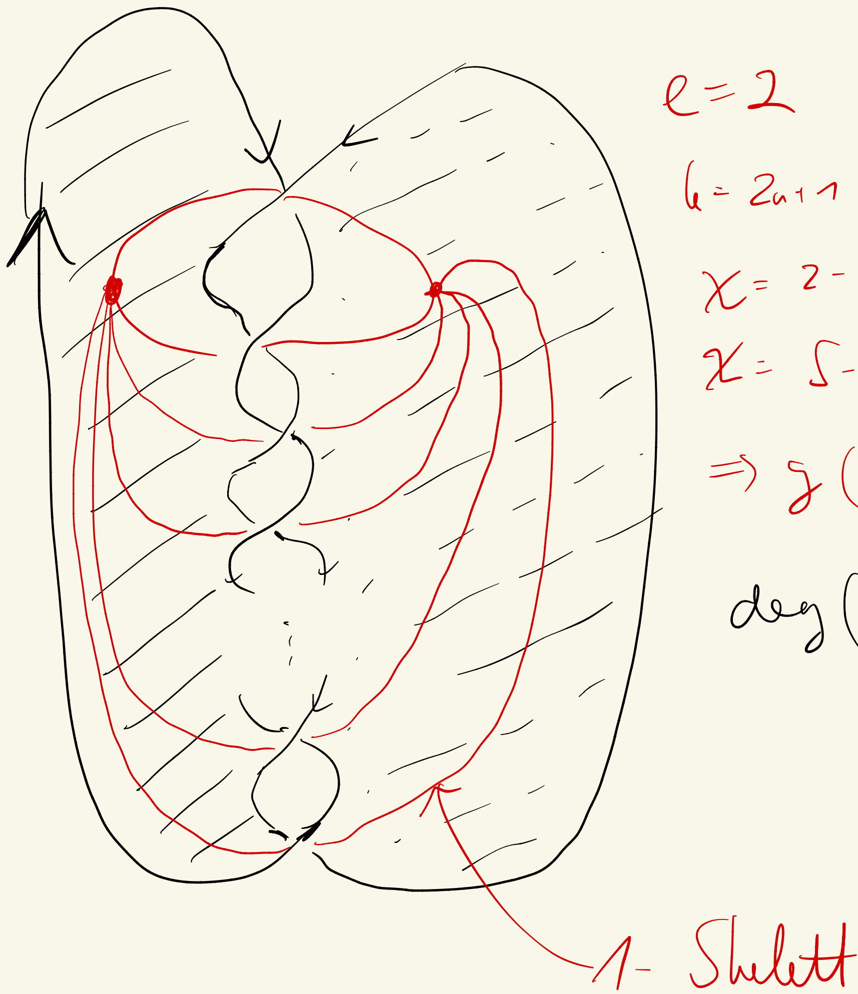
$$-C_+ + W \equiv -C_+ + W \mod 2$$



zu 4 d)



$$\deg(\Delta_{T_{2,2n+1}}) = g = n$$



$e = 2$

$k = 2n+1 = \#$Kante $\quad \#$Fläche $= 0$

$\chi = 2 - 2n - 1 = 1 - 2n$

$\chi = S - c$

$\Rightarrow g(T_{2,2n+1}) \leq n$

$\deg(\Delta_{T_{2,2n+1}}) = n \leq g(T_{2,2n+1})$

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Contact Geometry

Exercise sheet 6

Exercise 1.
Describe the characteristic foliations of

(a) the sphere of radius r in $(\mathbb{R}^3, \xi_{st})$,

(b) the sphere of radius r in $(\mathbb{R}^3, \xi_{ot})$, and

(c) the boundary of a standard tubular neighborhood of a transverse knot.

$T^2 = \mathbb{R}^2/\mathbb{Z}^2$

Exercise 2. $\Omega := d\theta \wedge d\varphi$

lecture

$(T^2 \times \{0\})_{\xi_0} \overset{!}{=} \langle -(x \sin\theta + y\cos\theta)\partial_\theta \rangle$

We consider on $T^2 \times \mathbb{R}$ the contact forms $\underset{(x,y)}{T^2}$ $\underset{z}{\mathbb{R}}$

S

$$\alpha_0 = \cos(z)dx + \sin(z)dy, \quad \to \xi_0 = \ker \alpha_0$$
$$\alpha_1 = dx + zdy. \quad \to \xi_1 = \ker \alpha_1$$

Show that $T^2 \times 0$ admits a neighborhood on which the induced contact structures are contactomorphic. $i_X \Omega \overset{!}{=} \alpha_1|_{TS} = d$ Ex: Moser trick!

Exercise 3.
The two Chekanov knots have the same classical invariants and get isotopic after a single stabilization.

Exercise 4.

(a) Express the self-linking number of the transverse push-off $L_\pm$ of a Legendrian knot K in terms of the classical invariants of L.

(b) Reformulate the Bennequin bound in terms of transverse knots.

Exercise 5.
Prove Theorem 4.3 from the lecture. (→ Geiges (in general dim.))

This sheet will be discussed on Wednesday 6.12. and should be solved by then.

Ex.4) (a) Clm: $sl(K_\pm, \Sigma) = tb(K) \mp rot(K, [\Sigma])$

Pf: X a nowhere zero section of $\xi|_\Sigma$, Σ a Seifert surface of K, $K_\pm$

① Suppose: $rot(K, [\Sigma]) = 0$. $\Rightarrow$ total nb of rotations of TK relative to X is 0, if we traverse K

Change X by a homotopy, s.t. $X \pitchfork TK$ everywhere

$\rightarrow$ let $K_\pm$ be obtained by pushing K in the dir. of $\pm X$.

$\rightarrow$ let $K'_\pm$ be obtained by pushing $K_\pm$ a little farther in the dir of $\pm X$.

$$\Rightarrow \underbrace{lk(K, K_\pm)}_{\overset{\text{def.}}{=}\, tb(K)} = lk(K_\pm, K'_\pm) = sl(K_\pm, \Sigma) \rightarrow \text{Clm}$$

Let X be a section of $\xi|_\Sigma$.

② general case: Let X_0 be a section of ξ over a nbhood of K, transverse to TK along K

$\rightarrow$ $K_\pm$ can be defined by pushing K in the direction of $\pm X_0$

$\rightarrow$ let $K'_\pm$ be defined by pushing $K_\pm$ in the direction of $\pm X$ (defined as above)

$\rightarrow$ let $K^0_\pm$ —— " —— $K_\pm$ further in the dir of. $\pm X_0$

$$\Rightarrow \underbrace{lk(K, K_\pm)}_{=\, tb(K)} = lk(K_\pm, K^0_\pm) \quad \& \quad lk(K_\pm, K'_\pm) = sl(K_\pm, \Sigma)$$

$\rightarrow$ $rot(K, [\Sigma]) = \#$ rotations of X_0 relative to X, as we traverse K once in pos dir.

	Contribute to $lk(K_\pm, K^0_\pm)$ relative to $lk(K_\pm, K'_\pm)$
K_+	$\oplus$
K_-	$\ominus$

$$\Rightarrow rot(K, [\Sigma]) = lk(K_+, K^0_+) - lk(K_+, K'_+) = lk(K_-, K'_-) - lk(K_-, K^0_-)$$

$$\Rightarrow sl(K_\pm, \Sigma) = lk(K_\pm, K'_\pm) = lk(K_\pm, K^0_\pm) \mp rot(K, [\Sigma]) = tb(K) \mp rot(K, [\Sigma]) \quad \blacksquare$$

24 (a) $\nexists \; K(t) = (\theta = t, \, x = 0, \, y = 0) \in (S^1 \times \mathbb{R}^2, \cos\theta \, dx - \sin\theta \, dy)$

$R_\alpha = -\sin\theta \, \partial_y + \cos\theta \, \partial_x$

$L_\pm = (\theta = t, \, x = \pm\varepsilon \sin t, \, y = \pm \varepsilon \cos t)$

$\lambda_c = (\theta = t, \, x = \varepsilon \cos t, \, y = -\varepsilon \sin t)$

$tb(K) = lk(K, \lambda_c) = lk(K, L_\pm)$

falsch

Solution:

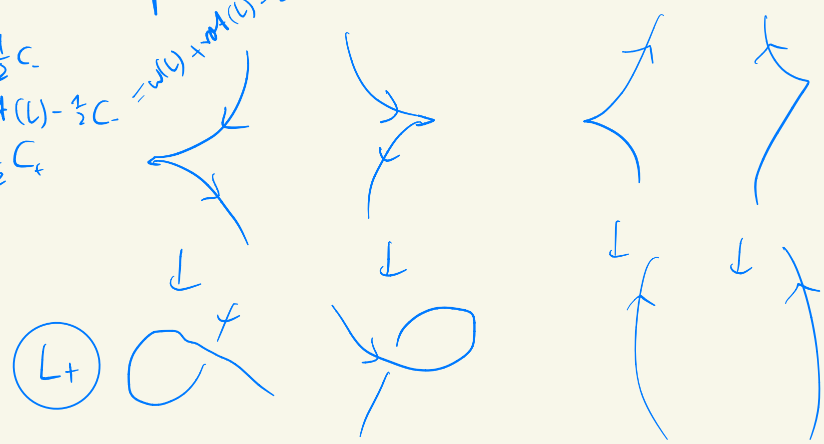
$L \subset (\mathbb{R}^3, \xi_{st})$ Legendre

$T = L_\pm \subset (\mathbb{R}^3, \xi_{st})$ transversal push-off $\Rightarrow tb(L) - rot(L) = w(L) - C_- = w(L_-) = sl(L_+)$

$tb(L) = w(L) - \frac{1}{2} C = w(L) - \frac{1}{2} C_+ - \frac{1}{2} C_- = w(L) - \frac{1}{2} C_- + rot(L) - \frac{1}{2} C_- = w(L) + rot(L) - C_-$

$rot(L) = \frac{1}{2}(C_- - C_+) = \frac{1}{2} C_- - \frac{1}{2} C_+$

$sl(L_+) = w(L_+)$

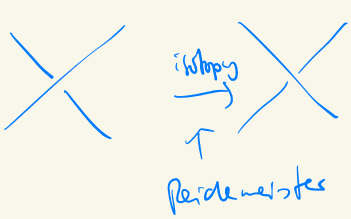


$\Rightarrow \boxed{sl(L_+) = tb(L) - rot(L)}$

analog: $sl(L_-) = tb(L) + rot(L)$

(b) K transversal in $(\mathbb{R}^3, \xi_{st}) \Rightarrow \overset{\text{Clm:}}{sl(K) \leq 2g(K) - 1}$

proof: if $\exists L \subset (\mathbb{R}^3, \xi_{st})$ Legendre s.t. $L_+ = K \Rightarrow$ Clm



Isotopie durch transversale Knoten, sodass jede Kreuzung von der Form

ist.

Legendre push-off von K, $K = S^1 \times \{0\} \subset (S^1 \times \mathbb{R}^2, \overbrace{r^2 d\varphi + dz}^{x\,dy - y\,dx + dz})$

$t \longmapsto (t, \underbrace{\varepsilon \cos(t)}_{?}, \underbrace{\varepsilon \sin(t)}_{?})$ (idea) s.t. $TL \subset \xi$ & $L_+ = K$

Ex. 1) (a) Let $\xi_{st} = \ker(dz + x\,dy - y\,dx)$. Let $p = (x, y, z) \in S_r^2$.

Then: $T_pS_r^2 = \{a\partial_x + b\partial_y + c\partial_z \mid ax + by + cz = 0\}$,
$(\Leftrightarrow (a,b,c) \perp p)$

$\xi_{st|p} = \langle \partial_y - x\partial_z, \partial_x + y\partial_z \rangle$

1.) $\xi_{st|p} = T_pS_r^2$

$$\Leftrightarrow \begin{cases} \partial_y - x\partial_z \in T_pS_r^2 & (a,b,c) = (0,1,-x) \\ \partial_x + y\partial_z \in T_pS_r^2 & (a,b,c) = (1,0,y) \end{cases}$$

$$\Leftrightarrow \begin{cases} y - xz = 0 \\ x + yz = 0 \end{cases} \quad \left\| \quad \begin{array}{l} z = 0 \Rightarrow (x,y,z) = 0 \notin S_r^2 \ \text{ᵶ} \\ z \neq 0 \Rightarrow x = \frac{y}{z} \,(1)\ \& \ y = -\frac{x}{z} \,(2) \end{array} \right.$$

$$\overset{(2)}{=} \frac{-x/z}{z} = -\frac{x}{z^2} \Rightarrow x\underbrace{\left(1 + \frac{1}{z^2}\right)}_{\neq 0} = 0 \Rightarrow x = 0 \Rightarrow y = 0 \underset{p \in S_r^2}{\Rightarrow} z = \pm r$$

$\Leftrightarrow p = \pm(0,0,r)$

$\Rightarrow S^* = S_r^2 \setminus \{\pm(0,0,r)\}$

2.) $T_pS_r \cap \xi_p = \ker(x\,dx + y\,dy + z\,dz) \cap \ker(dz + x\,dy - y\,dx)$, $p = (x,y,z) \in S_r^2$

$X_p = (a\partial_x + b\partial_y + c\partial_z)_p \overset{!}{\in} T_pS_r^2 \cap \xi_p$, $a, b, c : S_r^2 \to \mathbb{R}$

$$\Leftrightarrow \begin{cases} ax + by + cz = 0 \\ c + bx - ay = 0 \end{cases} \quad \Rightarrow c = ay - bx$$

$\Leftrightarrow ax + by + ayz - bxz = 0$

$\Leftrightarrow a(x + yz) + b(y - xz) = 0 \to \boxed{a = xz - y,\ b = x + yz}$

$\Rightarrow \boxed{c = (xz - y)y - (x + yz)x = xzy - y^2 - x^2 - yzx = -x^2 - y^2}$

Thus: $X = (xz - y)\partial_x + (x + yz)\partial_y + (-x^2 - y^2)\partial_z$ generates $(S_r^2)\xi_{st}$.

$\lceil X(x,y,z) = 0 \Leftrightarrow x = y = 0 \ \&\ z = \pm r \Leftrightarrow (x,y,z)$ singular $\rfloor$

(b) Let $\xi_{ot} = \ker\left(\cos(r)\,dz + r\sin(r)\,d\varphi\right)$, $\mathbb{R}^3 \ni (z, r, \varphi)$

In cylindric coordinates for $p \in S^2_R$, $p = (z, r, \varphi)$, we have

$$T_p S^2_R = \ker\left(D\left(z^2 + r^2 - 1\right)\right) = \ker\left(z\,dz + r\,dr\right)$$

$$= \langle \partial_\varphi, \; z\partial_r - r\partial_z \rangle$$

<u>Now</u> $X_p = (a\partial_z + b\partial_r + c\partial_\varphi)|_p \in T_p S^2 \cap \xi_{ot,p}$, $p = (z, r, \varphi) \in S^2_R$

$$\Leftrightarrow \begin{cases} az + br = 0 \;\Big\|\; r \neq 0 \Rightarrow b = -\dfrac{az}{r} \\ a\cos(r) + c\,r\sin(r) = 0 \;\Big\|\; \boxed{r = 0 \underset{\substack{\uparrow \\ (z,r,\varphi) \in S^2_R}}{\Rightarrow} z = \pm R \;;\; az = 0 \overset{z \neq 0}{\Rightarrow} a = 0} \end{cases}$$

$$\rightarrow a = r\sin(r), \quad c = -\cos(r) \Rightarrow b = \frac{-az}{r} = -\frac{r\sin(r)\cdot z}{r} = -z\sin(r)$$

$X = r\sin(r)\,\partial_z - z\sin(r)\,\partial_r - \cos(r)\,\partial_\varphi$ defines $\left(S^2_R\right)_{\xi_{ot}}$.

$$\boxed{\Gamma X|_{r=0} = -\partial_\varphi \in T S_p^2 \cap \xi_{ot}}$$

$$\Gamma X = 0 \iff \begin{cases} r\sin(r) = 0 \\ z\sin(r) = 0 \\ \cos(r) = 0 \end{cases}$$

$\Rightarrow$ ·) $r = 0 \Rightarrow p = (\pm R, 0, 0)$ (north / south pole), but $\cos(0) \neq 0$ ↯

or $\sin(r) = 0 \Rightarrow r = k\pi, k \in \mathbb{Z}$

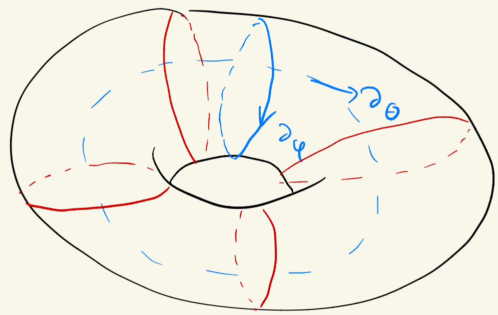
$(0, R, \varphi)$, φ arbitrary & $R = \pi/2 + k\pi, k \in \mathbb{Z}$

·) $z\sin(r) = 0 \Rightarrow z = 0 \Rightarrow$

or $\sin(r) = 0 \Rightarrow r = k\pi, k \in \mathbb{Z}$

·) $\cos(r) = 0 \Rightarrow r = \pi/2 + k\pi, k \in \mathbb{Z}$ → ∃ geschlossener Orbit, der Scheibe berandet ⇒ Kontaktstruktur überdreht

(c) $S = S^1 \times S^1_R \subset \underset{\theta}{S^1} \times \underset{(r,\varphi)}{\mathbb{R}^2}$, $\ker(r^2 d\varphi + d\theta)$



$\Omega = d\theta \wedge d\varphi$ Volumenform auf S

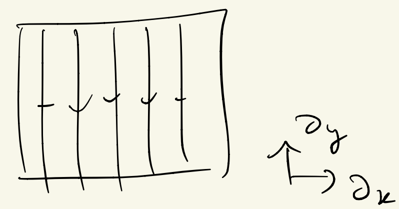
$i_X \Omega = \alpha|_S = R^2 d\varphi + d\theta$

$X = -\partial_\varphi + R^2 \partial_\theta$ (no zeros)

$i_X \Omega = R^2 d\varphi - (-d\theta) = \alpha|_S$

Ex. 2) $\alpha_0 = \cos z \, dx + \sin z \, dy$

$\Omega = dx \wedge dy \qquad i_{X_0}\Omega \overset{!}{=} \alpha_0|_{T^2} = dx$



$X_0 = -\partial_y$

$\alpha_1 = dx + z\,dy \Rightarrow \alpha_1|_{T^2} = dx = \alpha_0|_{T^2}$

$\Rightarrow X_1 = X_0 = -\partial_y \overset{\text{Thm 3}}{\Longrightarrow}$ Claim $\blacksquare$

Prfg: 30 min, mdl. Kapitel und vorgegebe — dazu selber was erzähle können, dann Nachfrage

WS 2023/24 Marc Kegel

Contact Geometry

Exercise sheet 7

Exercise 1.

(a) The boundary of a standard neighborhood of a transverse knot is not convex.

(b) Every transverse knot admits a tubular neighborhood whose boundary is convex.

(follows directly from lecture)

(c) Let S be a surface whose characteristic foliation admits a flow line from a negative singularity to a positive singularity. Can S be convex?

Exercise 2.
We consider $T^2 \times \mathbb{R}$ with 1-forms

$$\alpha_1 := \sin(\pi y)dx + 2\sin(\pi x)dy + (2\cos(\pi x) - \cos(\pi y))dz$$
$$\alpha_2 := \sin(\pi y)dx + (1 - \frac{1}{K}\cos(\pi x))\sin(\pi x)dy + (\cos(\pi x) - \frac{1}{K}\cos(2\pi x) - \cos(\pi y))dz.$$

Here x and y are coordinates on T^2 that are induced by $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and z is a coordinate on $\mathbb{R}$.

(a) Show that for a suitable choice of $K \in \mathbb{R}$ these 1-forms are contact forms inducing contact structures ξ_1 and ξ_2.

(b) Compute the characteristic foliations on $T^2 \times 0$ and draw them.

(c) Find the singularities of the foliations and compute their indices and divergences.

(d) Are these characteristic foliations homeomorphic? Are they diffeomorphic?

← Geiges, 4.6.12

Exercise 3.
Let L be a Legendrian knot in $(\mathbb{R}^3, \xi_{st})$ and D one of its front project. We denote by F the Seifert surface of L that is obtained by applying the Seifert algorithm to the diagram D.

(a) Show that $\mathrm{tb}(L) \leq 2g(F) - 1$.

(b) Show that $\mathrm{tb}(L) \pm \mathrm{rot}(L) \leq 2g(F) - 1$. $\Rightarrow g(\;) \geq 1$; $g(O) = 0$

(c) Can you deduce from (a) and (b) the Bennequin inequality for Legendrian knots in $(\mathbb{R}^3, \xi_{st})$? If yes give a prove. If no give a counterexample.

(d) Use (b) to give a contact geometric proof that the right-handed trefoil is not smoothly isotopic to the unknot.

(e) Use ~~(b) (or~~ directly the Bennequin inequality) to given an alternative proof of Exercise 4 (d) from Sheet 5, i.e. show that for every natural number $g \in \mathbb{N}_0$ there exist a knot K_g that has genus g.

This sheet will be discussed on Wednesday 13.12. and should be solved by then.

Ex. 1) (c) -) Std nbhd of transverse knot: $\exists r>0$ s.t. $(S^1 \times D^2_r, \ker(d\theta + \underbrace{xdy - ydx}_{r^2 d\varphi}))$

-) $S := S^1 \times S_R$

-) $i_X \Omega = \alpha|_{TS} = d\theta + R^2 d\varphi$ ← geschlossen!

| $\Omega = d\theta \wedge d\varphi$ area form on S

$X = R^2 \partial_\theta - \partial_\varphi$ $\left[R^2 d\varphi - (-d\theta) = \alpha|_{TS} \right]$ ⇒ X repräsentiert die charakteristische Blätterung S_ξ

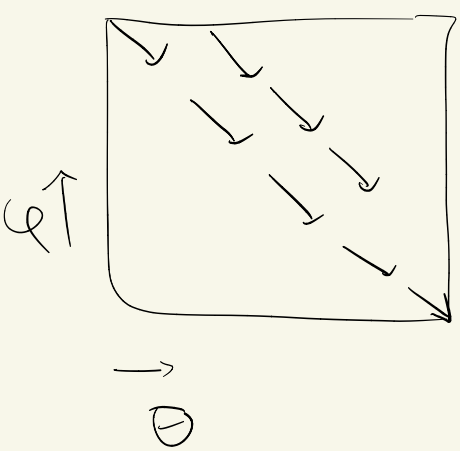
//A: S convex, then $\xi = \ker(\beta + u\,dz)$

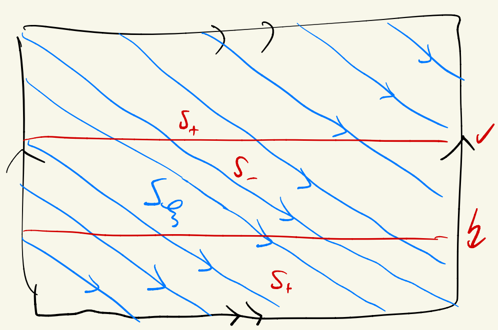
contact condition $\boxed{u \operatorname{div}_\Omega X - X(u) > 0}$

$\operatorname{div}_\Omega X \cdot \Omega = d(i_X \Omega) = 0 \Rightarrow \operatorname{div}_\Omega X = 0 \Rightarrow X(u) < 0$

$\Rightarrow u$ decreases along the flow of X

alternatively:





Γ_S teilt S_ξ $:\overset{\text{def}}{\Longleftrightarrow}$ $\Gamma \pitchfork S_\xi$, $\Gamma \neq \emptyset$, $S = S_+ \sqcup S_-$, S_ξ zeigt aus S_+ entlang Γ heraus

Bild ⇒ Γ_S teilt nicht S_ξ ↯

(b) Y contact vector field $\rightsquigarrow L_Y \alpha = \mu \alpha$, $\alpha = d\theta + r^2 d\varphi$,

$Y = a\,d\theta + b\,d\varphi + c\,dr$

contact planes: $\langle \partial_r, -r^2 \partial_\theta + \partial_\varphi \rangle$

Ex.1) (c) Assume that S is convex

$\rightsquigarrow$ $S \equiv S \times \{0\} \subset S \times \mathbb{R}$, $\alpha = u\,dz + \beta$, ($\beta = \alpha|_S$),

α contact form on $M \to \xi := \ker\alpha$

$\rightsquigarrow$ contact condition: $u\,d\beta + \beta \wedge du > 0$

$\rightsquigarrow$ S_ξ is generated by the vector field X defined by $i_X\Omega = \beta$ (Ω is an area form on S)

$\rightsquigarrow$ $\operatorname{div}_\Omega(X)\cdot\Omega = \mathcal{L}_X\Omega = d(i_X\Omega) = d\beta$

At singularities $\operatorname{div}_\Omega(X)$ doesn't vanish

$du \wedge \Omega$ is a 3-form on the 2-mfd S $\Rightarrow$ $0 = du \wedge \Omega$

$$\Rightarrow 0 = i_X(du \wedge \Omega) = du(X)\Omega - du \wedge (i_X\Omega) = du(X)\,\Omega + \beta \wedge du$$

$$\underset{\substack{\text{contact}\\\text{condition}}}{>} du(X) - \underbrace{u\operatorname{div}_\Omega(X)\,\Omega}_{u\,d\beta}.$$

$$\Rightarrow du(X) < u\operatorname{div}_\Omega(X)$$

At singularities of S_ξ we have $X = 0$ $\Rightarrow$ $0 < u\operatorname{div}_\Omega(X)$

$\Rightarrow$ signs of u and $\operatorname{div}_\Omega(X)$ coincide. (*)

Let $\gamma: \mathbb{R} \to S$ be a flow line from a negative singular pt p_- to a positive singular pt p_+, i.e. $\lim_{t\to\pm\infty}\gamma(t) = p_\pm$.

Look at the fct u along γ $\rightsquigarrow$ $f := u\circ\gamma: \mathbb{R} \to \mathbb{R}$ with

$\lim_{t\to\pm\infty} f(t) = u(p_\pm)$. Observe that $\pm u(p_\pm) > 0$ (because (*))

We compute: $\frac{d}{dt} f(t) = \frac{d}{dt} u \circ \gamma(t) = du_{\gamma(t)}(\dot{\gamma}(t)) = du_{\gamma(t)}(X_{\gamma(t)})$

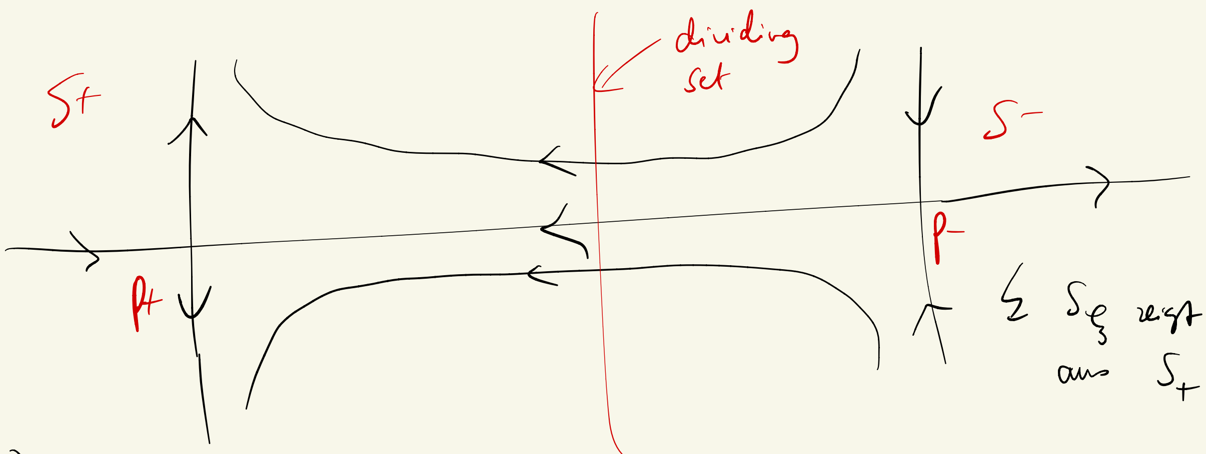
At the zeros of $\overset{=u\circ\gamma}{f}$ we have $du(X) < \overset{=0}{u}\, \mathrm{div}_\Omega(X) = 0$ $\Rightarrow \boxed{\frac{d}{dt} f(t) < 0}$

f continuous, $\lim\limits_{t \to -\infty} f(t) = u(p_-) < 0$

$\mathrm{div}_\Omega X(p_-) < 0$

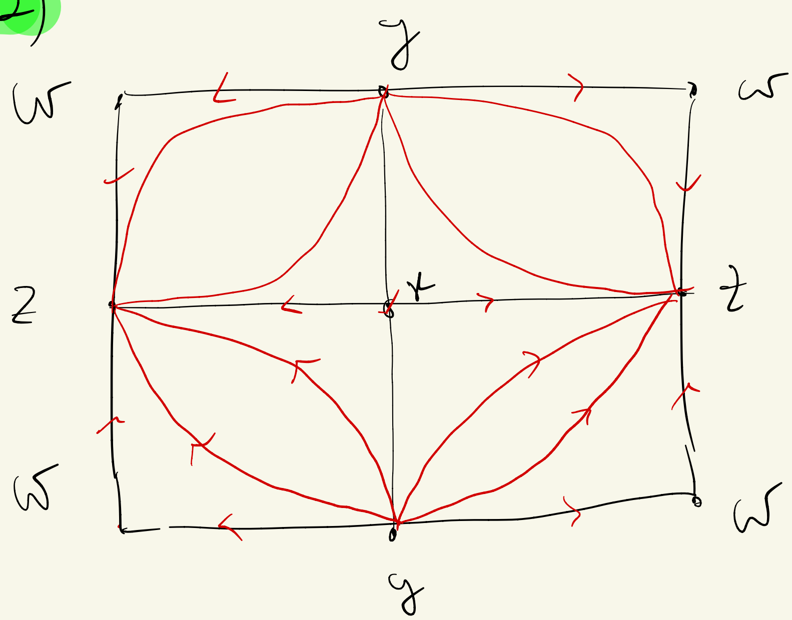
$\Rightarrow f$ can't change its sign from $\ominus$ to $\oplus$ ↯

$\Rightarrow S$ is not convex. ∎



↯ S_ξ zeigt nicht aus S_+ heraus

Zu 2)



Ex. 2) (a) Let $k \in \mathbb{R}_+$. We compute:

$$d\alpha_1 = \pi\cos(\pi y)\,dy\wedge dz + 2\pi\cos(\pi x)\,dx\wedge dy + \left(-2\pi\sin(\pi x)\,dx + \pi\sin(\pi y)\,dy\right)\wedge dz$$

$$= \pi\left(2\cos(\pi x) - \cos(\pi y)\right)dx\wedge dy + \pi\sin(\pi y)\,dy\wedge dz - 2\pi\sin(\pi x)\,dx\wedge dz$$

$$\Rightarrow \alpha_1\wedge d\alpha_1 = \left(2\cos(\pi x) - \cos(\pi y)\right)\cdot\pi\left(2\cos(\pi x) - \cos(\pi y)\right)dz\wedge dx\wedge dy$$

$$+ \sin(\pi y)\cdot\pi\sin(\pi y)\,dx\wedge dy\wedge dz - 4\pi\sin^2(\pi x)\,dy\wedge dx\wedge dz$$

$$= \pi\left[\sin^2(\pi y) + 4\sin^2(\pi x) + \left(2\cos(\pi x) - \cos(\pi y)\right)^2\right]dx\wedge dy\wedge dz$$

As $x, y \in \mathbb{Z}$

& $2\cos(\pi x) - \cos(\pi y) = 0$, $\Rightarrow \underbrace{2\cos(\pi x)}_{=\pm 2} = \cos(\pi y) = \pm 1$ $\lightning$

$\Longrightarrow \alpha_1\wedge d\alpha_1 = 0$

side computation:

$\sin(\pi x)\sin(\pi x) - \cos(\pi x)\cos(\pi x) = \cos(2\pi x)$

Moreover, we compute:

$$d\alpha_2 = \pi\cos(\pi y)\,dy\wedge dx + \left(\frac{\pi}{k}\sin(\pi x)\sin(\pi(x)) + \left(1 - \frac{1}{k}\cos(\pi x)\right)\cdot\pi\cos(\pi x)\right)dx\wedge dy$$

$$+ \left(-\pi\sin(\pi x)\,dx + \frac{2\pi}{k}\sin(2\pi x)\,dx + \pi\sin(\pi y)\,dy\right)\wedge dz$$

$$= \pi\left(\cos(\pi x) - \frac{1}{k}\cos(2\pi x) - \cos(\pi y)\right)dx\wedge dy + \pi\sin(\pi y)\,dy\wedge dz$$

$$+ \pi\left(\sin(\pi x) - \frac{2}{k}\sin(2\pi x)\right)dz\wedge dx$$

$$\Rightarrow \alpha_2 \wedge d\alpha_2 = \pi\Big[\Big(\cos(\pi x) - \frac{1}{k}\cos(2\pi x) - \cos(\pi y)\Big)\Big(\cos(\pi x) - \frac{1}{k}\cos(2\pi x) - \cos(\pi y)\Big) dz \wedge dx \wedge dy$$

$$+ \Big(1 - \frac{1}{k}\cos(\pi x)\Big)\sin(\pi x)\Big(\sin(\pi x) - \frac{2}{k}\sin(2\pi x)\Big) dy \wedge dz \wedge dx$$

$$+ \sin(\pi y)\cdot\pi\sin(\pi y)\, dy \wedge dz \wedge dx$$

$$= \pi\Big[\sin^2(\pi y) + \sin^2(\pi x)\Big(1 - \frac{1}{k}\cos(\pi x)\Big)$$

$$+ \Big(\cos(\pi x) - \cos(\pi y) - \frac{1}{k}\cos(2\pi x)\Big)^2$$

$$- \frac{2}{k}\sin(\pi x)\sin(2\pi x)\Big(1 - \frac{1}{k}\cos(\pi x)\Big)\Big] dx \wedge dy \wedge dz$$

if $k > 1 \Rightarrow$ * $1 - \frac{1}{k}\cos(\pi x) > 1 - \cos(\pi x) \geq 0$

* Consider, $\sin(\pi y) = 0$ & $\sin(\pi x)\left(1 - \frac{1}{k}\cos(\pi x)\right) = 0$

$\Rightarrow y \in \mathbb{Z}$ & $x \in \mathbb{Z} \Rightarrow |\cos(\pi x)|, |\cos(\pi y)| = 1$, $\neq |\cos(2\pi x)|$

$\Rightarrow \left|\cos(\pi x) - \frac{1}{k}\cos(2\pi x)\right| \neq 1 \Rightarrow \cos(\pi x) - \cos(\pi y) - \frac{1}{k}\cos(2\pi x) \neq 0$

* Sum of the first three summands in $[\ldots]$ is bounded away from 0
* the last summand becomes small for $k \gg 1$

$\rightsquigarrow$ Choose k large enough, the α_2 is a contact form.

$\rightsquigarrow \xi_i := \ker \alpha_i$, $i = 1, 2$

#

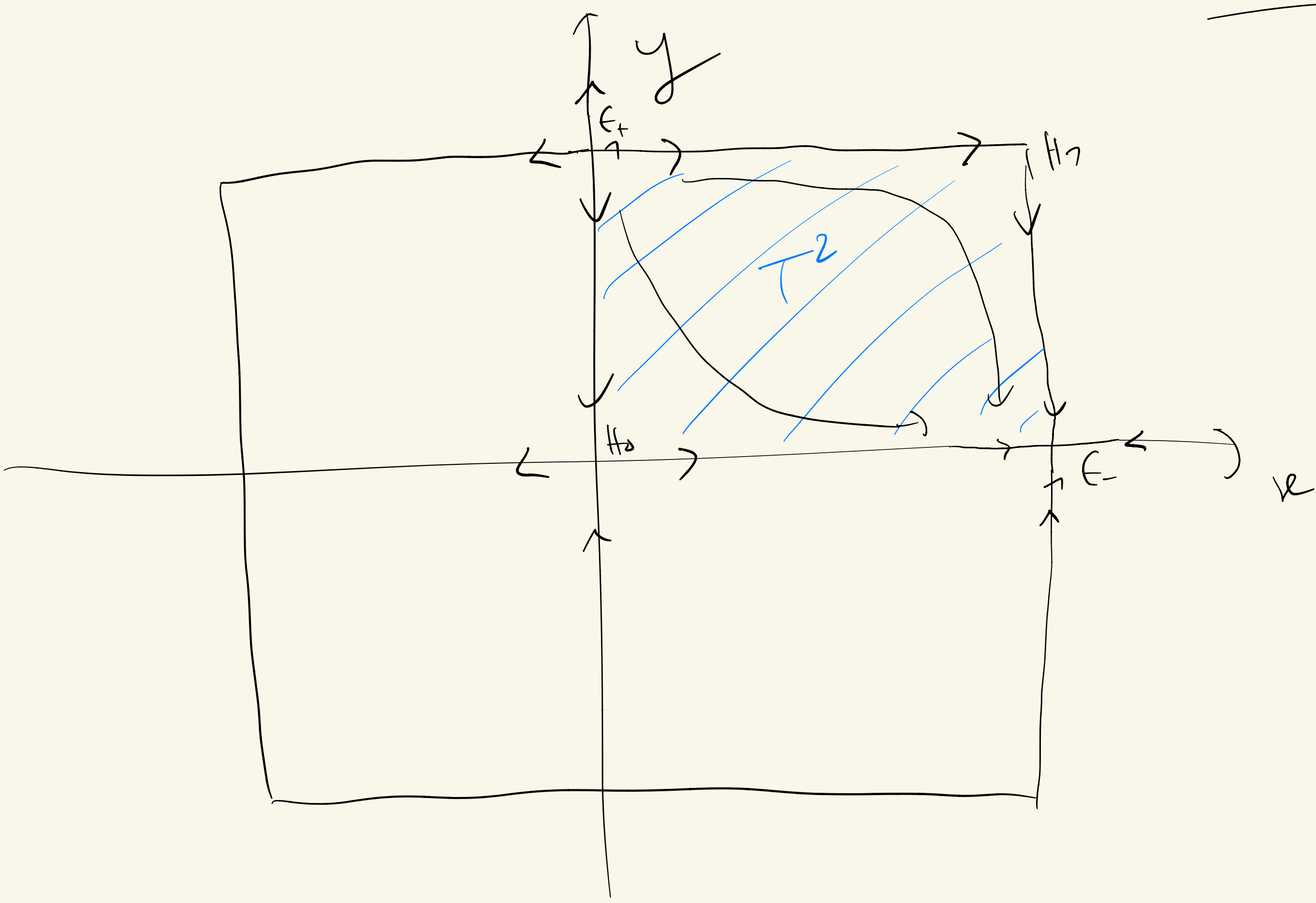
(b) Define $S := T^2 \times \{0\} \subset T^2 \times \mathbb{R}$. $\Omega := dx \wedge dy$ is an area form on S. Define:

$$X_1 := 2\sin(\pi x)\,\partial_x - \sin(\pi y)\,\partial_y$$

$$X_2 := \left(1 - \frac{1}{K}\cos(\pi x)\right)\sin(\pi x)\,\partial_x - \sin(\pi y)\,\partial_y$$

* $i_{X_1}\Omega = 2\sin(\pi x)\,dy + \sin(\pi y)\,dx = \alpha_1|_S$

* $i_{X_2}\Omega = \left(1 - \frac{1}{K}\cos(\pi x)\right)\sin(\pi x)\,dy + \sin(\pi y)\,dx = \alpha_2|_S$



(c) Look at $X_1 = 2\sin(\pi x)\,\partial_x - \sin(\pi y)\,\partial_y$

* $e_+ = (0,1)$: $A = \begin{pmatrix} 2\pi\cos(\pi x) & 0 \\ 0 & -\pi\cos(\pi y) \end{pmatrix}\Big|_{(x,y)=(0,1)} = \begin{pmatrix} 2\pi & 0 \\ 0 & \pi \end{pmatrix} \to$ elliptic (source), $\operatorname{div}_\Omega X_1 = \underline{3\pi}$

* $e_- = (1,0)$ $A = \begin{pmatrix} -2\pi & 0 \\ 0 & -\pi \end{pmatrix} \to$ elliptic (sink), $\operatorname{div}_\Omega X_1 = -\underline{3\pi}$

$* \underline{H_0 = (0,0)}$: $A = \begin{pmatrix} 2\pi & 0 \\ 0 & -\pi \end{pmatrix} \to$ hyperbolic (positive), $\mathrm{div}_\Omega X_1 = \underline{\underline{\pi}}$

$* \underline{H_1 = (1,1)}$: $A = \begin{pmatrix} -2\pi & 0 \\ 0 & \pi \end{pmatrix} \to$ hyperbolic (negative), $\mathrm{div}_\Omega X_1 = \underline{\underline{-\pi}}$

$$\mathcal{L}_{X_1}\Omega = d(i_{X_1}\Omega) = d\left(2\sin(\pi x)\,dy + \sin(\pi y)\,dx\right) = \underbrace{\pi\left(2\cos(\pi x) - \cos(\pi y)\right)}_{\mathrm{div}_\Omega X_1} dx \wedge dy$$

&

$$\mathcal{L}_{X_2}\Omega = d\left(\left(1 - \frac{1}{k}\cos(\pi x)\right)\sin(\pi x)\,dy + \sin(\pi y)\,dx\right)$$

$$= \underbrace{\pi\left(\left(1 - \frac{1}{k}\cos(\pi x)\right)\cos(\pi x) + \frac{1}{k}\sin^2(\pi x) - \cos(\pi y)\right)}_{\mathrm{div}_\Omega X_2} dx \wedge dy$$

$\Rightarrow \mathrm{div}_\Omega X_2(0,0) = \underline{\underline{\frac{-\pi}{k}}}$; $\mathrm{div}_\Omega X_2(1,1) = \underline{\underline{\frac{-\pi}{k}}}$; $\Big\}$ both negative !!!

$$\mathrm{div}_\Omega X_2(0,1) = \pi\left(\left(1 - \frac{1}{k}\right)\cdot 1 + 1\right) = \underline{\underline{\pi\left(2 - \frac{1}{k}\right)}};$$

$$\mathrm{div}_\Omega X_2(1,0) = \pi\left(\left(1 + \frac{1}{k}\right)\cdot(-1) - 1\right) = \underline{\underline{\pi\left(-2 - \frac{1}{k}\right)}}$$

Ex. 3) (a)

Clm: $tb(L) \leq 2g(F) - 1 \overset{?}{=} -\chi(F) \Leftrightarrow w - \frac{1}{2}C \leq k - s$ $*$

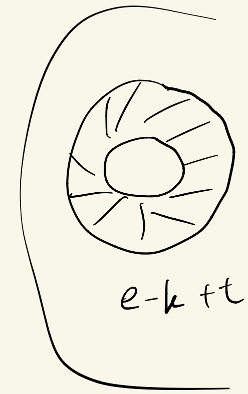
Legendre-Knoten

Seifert-Fläche aus Seifert Algorithmus

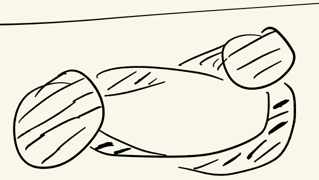
$tb(L) = w - \frac{1}{2}C$

$\boxed{\chi = 2-2g}$

Euler-charakteristik für geschlossene Fläche



e-k+t

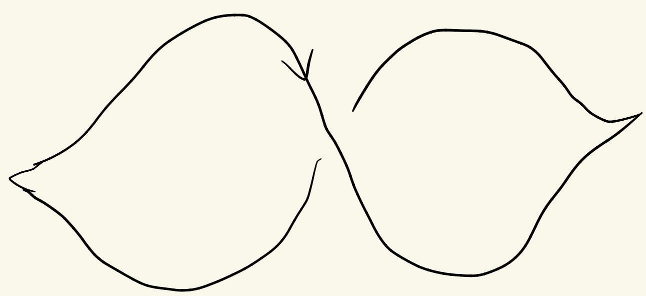
F: 

$\chi = \underbrace{\#(\text{Seifert circles})}_{=s} - \#(\text{Kreuzungen})$ (= Anzahl Bänder)

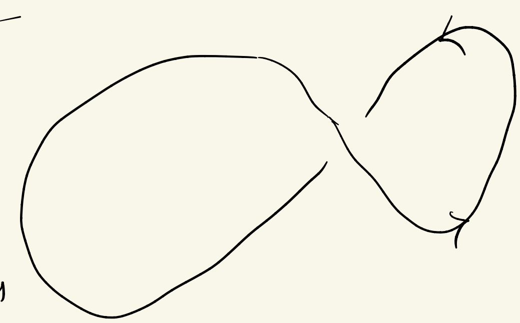
$F \cup D^2 = F_g$ ← Fläche vom Geschlecht g

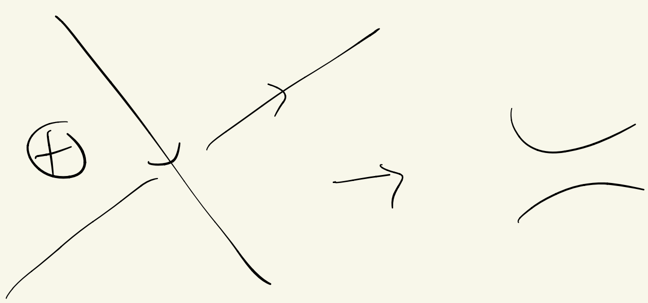
$2-2g = \chi(F_g) \overset{\text{allgemein gültig}}{=} \underbrace{\chi(F) + \overbrace{\chi(D^2)}^{=0} - \chi(\underbrace{F \cap D^2}_{S^1})}_{=0}$

allg. für geschlossene Flächen



Seifert-Algorithmus $\rightsquigarrow$

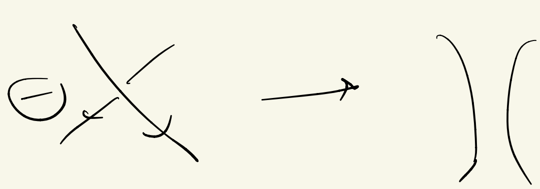




$2 \cdot s \overset{?}{\leq} \#$ vertikale Tangenten

$= 2 \cdot k^- + C \Rightarrow \boxed{s \leq k^- + \frac{C}{2}}$

neg. Kreuzungen



$*$ $\Rightarrow w - \frac{1}{2}c + s \leq k$, s.z.z. $\overbrace{w}^{k^+ - k^-} + k^- \leq k \Leftrightarrow k^+ \leq k$ ist klar

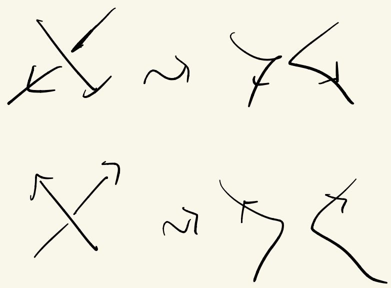
(b) mind. 1 up-cusp in jedem (Legendre-) Seifert-Kreis, d.h.

C_+ oder K_-

$\uparrow$ negative Kreuzung

$S = \#$ Seifert-Kreise $\le C_+ + K_-$

$$tb(K) + rot(K) = -\tfrac{1}{2}c + W + \tfrac{1}{2}(C_- - C_+)$$



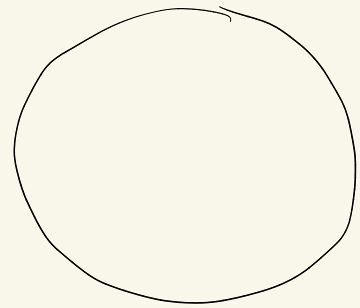
$$= W_+ + W_- - C_+$$

$$\le$$

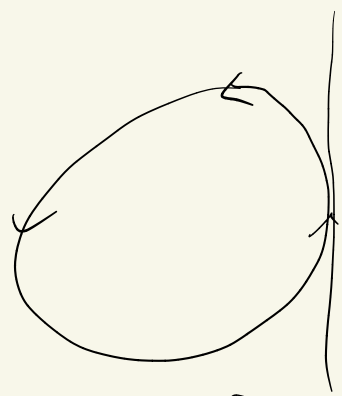
$$-\chi(F) = K - S = W_+ - W_- - S$$

$$\iff -W_- - C_+ \le W_- - S$$

$$\iff S \le 2W_- + C_+$$



$C_+ = \#$ vertical tangents in pos. direction



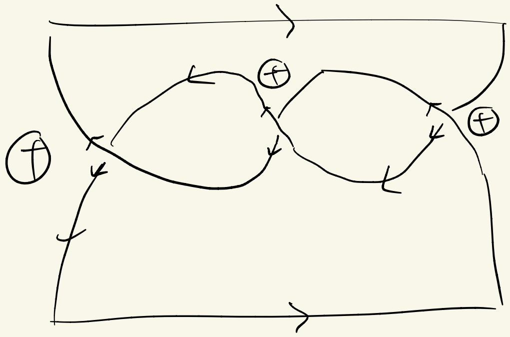
$$S \le C_+ + 2W_-$$

Thm: $\bar{g} = \min\{$Geschlechter von Seifert-Fläche, die aus front proj. von L via Seifert-Alg. erhalten werden$\}$

$$tb(L) \pm rot(L) \le 2\bar{g}(L) - 1$$

$\bar{g}(L) \ge g(L)$, aber $\overset{\text{i.A.}}{\neq}$

(d) Use $tb(L) \pm rot(L) \leq 2g(F) - 1 \Rightarrow g(F) \geq 1 \quad \forall F$ gen. by Seifert algorithm

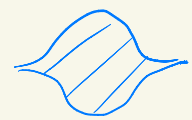


$tb = 3 - \overset{4/2}{2} = 1$

$rot = 0$

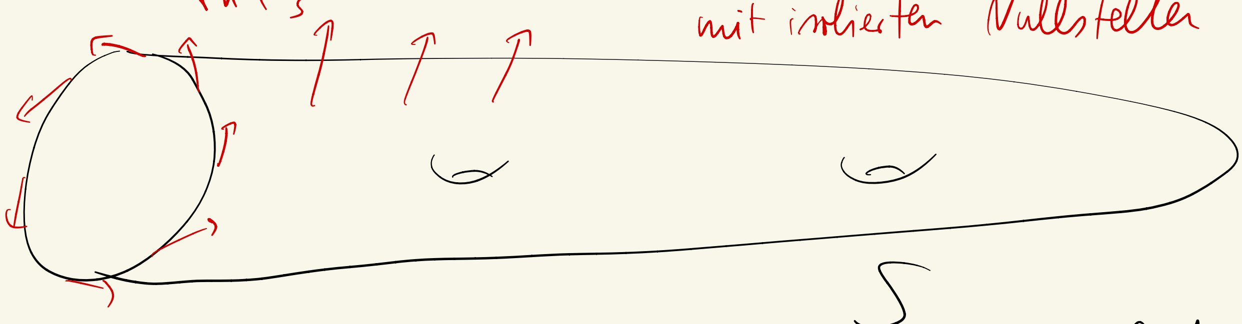
$\exists$ front projection von O, sd. $g(F) = 0$

$\bar{g}\Big($  $\Big) \geq 1$

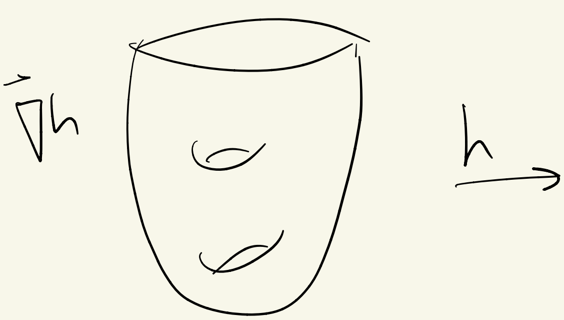


Discussion session

X vf entlang S sd $X \in \xi$
mit isolierten Nullstellen



$rot = \#$Nullstellen von X mit $\underbrace{\text{Vorzeichen}}_{\text{Indizes}}$ $= \chi(S)$, falls $X \pitchfork \partial S$ (nach Poincaré-Hopf)



$= \overset{rot}{rot}(TL, Y)$, wobei Y ist VF entlang S, sd $Y \subset \xi$, aber $Y \neq 0$

WS 2023/24 Marc Kegel

Contact Geometry

Exercise sheet 8

Exercise 1.
Consider $M = T^2 \times \mathbb{R}$ with circle-valued coordinates φ and θ on the T^2-factor, and with z denoting the $\mathbb{R}$ coordinate. Let ξ be the contact structure given as the kernel of $d\varphi + z d\theta$ on M. Let S in M be the 2-torus $T^2 \times 0$.

(a) Describe the characteristic foliation of S and deduce that S is not convex.

(b) Find an explicit perturbation S' of S, such that S' is convex. Describe its characteristic foliation and dividing set.

Exercise 2.

(a) Describe and depict the characteristic foliation of the *standard* overtwisted disk, i.e. a disk of radius π in $(\mathbb{R}^3, \xi_{ot})$.

(b) Let (M, ξ) be a contact manifold that contains a Legendrian unknot with $\mathrm{tb} = 0$. Show that (M, ξ) contains a standard overtwisted disk.
Hint: Use the elimination Lemma and the Poincaré–Hopf index theorem

Exercise 3.

(a) Let S be an embedded convex surface in a contact manifold. Let G be a properly embedded graph which is non-isolating. Then there exists an isotopy of S, rel boundary, to a surface S' such that G is contained in the characteristic foliation of S'.
Hint: Generalize the prove from the lecture of the Legendrian realization principle for knots.

(b) Use (a) to show that if $S \neq S^2$ nd Γ_S has no component that bounds a disk in S, then a vertically invariant neighborhood of S is tight.
Hint: Consider the universal cover of S and lift a Legendrian realization of the 1-skeleton and a potential overtwisted disk near S to the universal cover. Use that to realize the overtwisted disk as a convex surface from which you can construct an embedding of a neighborhood of the overtwisted disk in a tight contact manifold contradicting the assumption.

Exercise 4.
Prove Theorem 4.14 from the lecture.

This sheet will be discussed on Wednesday 20.12. and should be solved by then.

Discussion session Do., 18.01., 13 Uhr

WS 2023/24 Marc Kegel

Contact Geometry

Exercise sheet 9

Exercise 1. (Legendre-Realisierungs-Prinzip)

Let K be an isolating simple closed curve on a convex surface S then there is no isotopy of S through convex surfaces such that K lies in the characteristic foliation.

Exercise 2.

Let S be a closed oriented surface. We write S^z for $S \times z$ in $S \times [-1,1]$.

(a) Let ξ_0 and ξ_1 be two contact structures on $S \times [-1,1]$ such that their characteristic foliations $S^z_{\xi_0}$ and $S^z_{\xi_1}$, coincide for all $z \in [-1,1]$. Then ξ_0 and ξ_1 are isotopic rel boundary.

(b) Let ξ_0 and ξ_1 be two contact structures on $S \times [-1,1]$ with the following properties:

- The characteristic foliations $S^{\pm 1}_{\xi_i}$ coincide on the boundary of $S \times [-1,1]$ for $i = 0,1$.
- Each surface S^z, $z \in [-1,1]$, is convex for both contact structures, and there is a smoothly varying family of multi-curves Γ_z dividing both$S^z_{\xi_0}$ and $S^z_{\xi_1}$.

Then ξ_0 and ξ_1 are isotopic rel boundary.

Exercise 3.

For this and the next exercise we assume Bennequin's theorem, i.e. that the standard contact structure on S^3 is tight.

(a) The standard contact structure on $\mathbb{R}^3$ is tight.

(b) Let $f\colon (M', \xi') \to (M, \xi)$ be a contact covering (i.e. a covering that is a local contactomorphism). If ξ' is tight then ξ is tight.

(c) (T^3, ξ_n) and $(S^1 \times \mathbb{R}^2, \xi_n)$ are tight for all n. Überlagerung p: $\underbrace{(\mathbb{R}^3, \xi_{std})}_{\text{straff}} \to (T^3, \xi_n)$, $T^3 = \mathbb{R}^3/\mathbb{Z}^3$, $\xi_n = \ker(\cos(n\theta)dx - \sin(n\theta)dy)$

mit $p^*(\alpha_n) = v\,dw + dz$

3) (a) (S^3, ξ_{st}) straff, $\underbrace{(S^3 \setminus \{pt\}, \xi_{st})}_{}\overset{\text{cont}}{\cong} (\mathbb{R}^3, \xi_{st})$

straff als offene Untermenge einer straffe Kontaktstruktur

(b) $(M', \xi') \to (M, \xi)$ contact covering

Falls ξ' straff $\Rightarrow$ überdrehte Scheibe in (M, ξ)

wird zu überdrehter Scheibe $\overset{\text{in } (M', \xi')}{\text{geliftet}}$ $\Rightarrow$ ξ straff

$$\begin{tikzcd} & U \arrow[d, "\text{lor. cont}"] \\ M \arrow[ur, dotted, "\exists!"] \arrow[r, "\text{lor. cont}"'] \arrow[ur, phantom, "\circlearrowleft"] & M \end{tikzcd}$$

Exercise 4.
Let (M, ξ) be a contact manifold. We write $\widetilde{M}$ for the universal covering of M and $\widetilde{\xi}$ for the lift of ξ to $\widetilde{M}$. We call $(\widetilde{M}, \widetilde{\xi}) \to (M, \xi)$ the *universal contact covering* of (M, ξ). We say that a tight contact structure ξ on M is *universally tight* if its universal contact covering is tight. A tight contact structure ξ on M is *virtually overtwisted* if there exists a finite contact covering $(M', \xi') \to (M, \xi)$ such that ξ' is overtwisted.

(a) A virtually overtwisted contact structure is not universally tight. (folgt aus 3b)

(b) A tight contact structure is either universally tight or virtually overtwisted.
Hint: The geometrization theorem for 3-manifolds implies that the fundamental group of every compact 3-manifold M is *residually finite*, i.e. for every non-trivial element $g \in \pi_1(M)$ there exist a normal subgroup N in $\pi_1(M)$ of finite index that does not contain g. Use this result without proof.

universelle Überl.

(c) Show that the standard contact structures on $\mathbb{R}^3$, S^3, T^3, and $S^1 \times \mathbb{R}^2$ are universally tight.

univ. Überl. straff durch $\mathbb{R}^3$

(d) Construct a virtually overtwisted contact structure on $S^1 \times D^2$.
Hint: See [V. Colin, *Recollement de variétés de contact tendues*, Bull. Soc. Math. France **127**] or Example 2.27(2) [M. Kegel, Symplektisches Füllen von Torusbündeln, https://www.mathematik.hu-berlin.de/~kegemarc/Publications/Masterarbeit.pdf] for a exposition of that result in German.

This sheet will be discussed on Wednesday 10.1. and should be solved by then.

WS 2023/24 Marc Kegel

Contact Geometry

Exercise sheet 10

Exercise 1.

(a) Construct symplectic fillings of $(S^1 \times S^2, \xi_{st})$ and (T^3, ξ_1).

(b) Construct infinitely many pairwise non-diffeomorphic symplectic caps of (S^3, ξ_{st}).
Hint: Use Darboux's theorem.

(c) Show that S^4 carries no symplectic structure.
Hint: Use de Rham's theorem and show more generally that any 4-manifold with vanishing second homology does not carry a symplectic structure.

(d) Construct a symplectic form on $\mathbb{C}P^2$.

Bonus: Can $-\mathbb{C}P^2$ carry a symplectic structure?
Hint: Why is the symplectization $\mathbb{R} \times M$ and not $M \times \mathbb{R}$? Use the intersection form.

Exercise 2.
Prove Lemma 6.7 from the lecture.

Exercise 3.
An n-dimensional 1-handle is a copy of $D^1 \times D^{n-1}$ attached to an n-manifold M via an embedding $\varphi\colon \partial D^1 \times D^{n-1}$. $\hookrightarrow \partial M$

(a) Draw sketches of 1-handle attachments in dimension 2, 3, and 4. And analyze what is happening to the boundaries of the manifolds.

(b) Express the connected sum of two manifolds as a certain 1-handle attachement.

(c) Construct a smooth compact manifold W whose boundary is $M \# M$. Which surfaces are boundaries of compact 3-manifolds?

(d) Let W be a 4-dimensional symplectic cobordism. And let W' be the result of attaching a 1-handle to the positive boundary of W. Show that W' carries a symplectic structure making W' into a symplectic cobordism.
Hint: Use the same approach as in Theorem 6.6 from the lecture.

(e) Construct a symplectic filling of $(S^1 \times S^2, \xi_{st})$ by attaching a single 1-handle to D^4.

Exercise 4.
Prove Theorem 6.1 from the lecture.

Bonus exercise.
Prove Lemma 6.2 from the lecture.

This sheet will be discussed on Wednesday 17.1. and should be solved by then.

Ex.1) (a) $(S^1\times S^2, \xi_{st} = \ker(-x\,d\theta + y\,dz - z\,dy))$

$S^1\times S^2 = \partial(S^1\times D^2) \subset S^1\times\mathbb{R}^3 = \mathbb{R}^4/\mathbb{Z}$

$\omega = d\theta\wedge dx + dy\wedge dz$

$Y = x\partial_x + \frac{1}{2}(y\partial_y + z\partial_z)$

$\iota_Y\omega = -x\,d\theta + \frac{1}{2}(y\,dz - z\,dy)$

$d(\iota_Y\omega) = d\theta\wedge dx + dy\wedge dz = \omega$ ✓ $\Longrightarrow$ Y Liouville vectorfield for ω

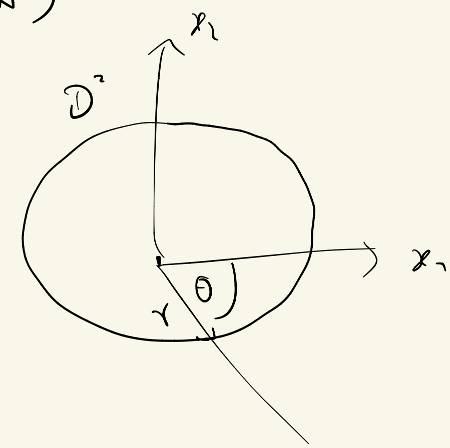
(*) $(T^3, \xi_1 = \ker(\underbrace{\cos\theta\,dx - \sin\theta\,dy}_{=\alpha_1}))$, coordinates (θ, x, y)

$T^3 = S^1\times T^2 = \partial(D^2\times T^2)$, $(x_1, x_2, \theta_1, \theta_2)$

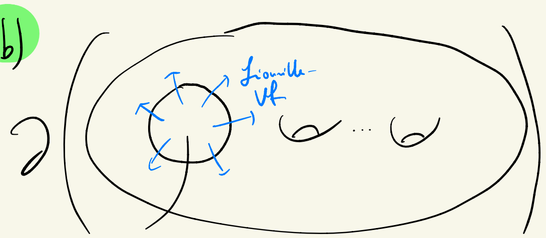
$\omega = dx_1\wedge d\theta_1 + dx_2\wedge d\theta_2$

$Y = x_1\partial_{x_1} + x_2\partial_{x_2}$ („Radialvektorfeld auf D^2-Faktor")

$\underbrace{\iota_Y\omega = x_1\,d\theta_1 + x_2\,d\theta_2}_{=\alpha_1} \Longrightarrow d(\iota_Y\omega) = \omega$ ✓



(b)



$= -S^3$

da rausgeschnitten

Darboux-Ball D^4

de-Rham & Poincaré-Dualität

(c) $0 = H_2(S^4) \overset{\perp}{=} H^2_{dR}(S^4) = \frac{\ker(d)}{\operatorname{im}(d)}$ (Poincaré-Dualität) $H_k(M) = H^{n-k}(M)$

d.h.: Jede 2-Sphäre $\subset S^4$ ist Rand eines Balls

$\Longrightarrow \ker d \subset \ker d$ (Jede geschlossene Form ist exakt) das entscheidende!

ω symplektische Form auf $S^4 \Rightarrow d\omega = 0 \Rightarrow \omega = d\alpha$ für 1-Form α

$\Longrightarrow \Omega = d\alpha\wedge d\alpha = d(\alpha\wedge d\alpha)$

$\Longrightarrow \int_{S^4}\Omega = \int_{S^4} d(\alpha\wedge d\alpha) \overset{\text{Stokes}}{=} \int_{\partial S^4 = \emptyset} \alpha\wedge d\alpha = 0$ ↯

(d) $(a,b,c) \sim (\lambda a, \lambda b, \lambda c)$, $\lambda \in \mathbb{C}^\times$

$\mathbb{C}^3\setminus\{0\} \ni (z_1, z_2, z_3)/\sim \;\cong\; S^5 / \text{Multiplikation mit } z \in \mathbb{C} \text{ mit } |z|=1 \longleftarrow S^5$

$z_j = x_j + i y_j$; $\omega = \sum_{j=1}^{3} dx_j \wedge dy_j \Big|_{TS^5}$, wollen $p^* \omega_{\mathbb{C}P^2} \overset{!}{=} \omega$

$$f\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \qquad f^*(dx_i \wedge dy_i) = dx_i \wedge dy_i$$

$\omega_{\mathbb{C}P^2}(X,Y) = \operatorname{im}(\langle X, Y \rangle)$

Trägt $-\mathbb{C}P^2$ symplektische Struktur?

Ja! Aber nicht mit gleicher Orientierung

ω auf $-\mathbb{C}P^2$ symplektisch, s.d. $\omega \wedge \omega > 0$

$H_2(\mathbb{C}P^2) = \mathbb{Z}_{\langle \mathbb{C}P^1 \subset \mathbb{C}P^2 \rangle}$

$I_{\mathbb{C}P^2}(\sigma, \eta) := \int_{\mathbb{C}P^2} \sigma \wedge \eta$, $H_2 \times H_2 \to \mathbb{R}$ positiv definit

$\sigma, \eta \in H^2(-\mathbb{C}P^2)$

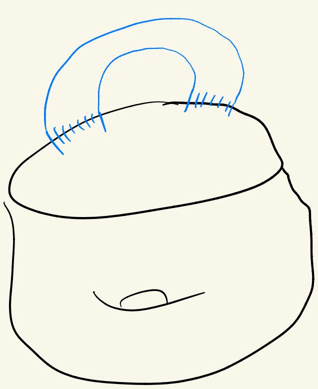
$I_{-\mathbb{C}P^2}$ ist negativ definit ↯ zu $\omega \wedge \omega > 0$

Ex. 3) k-Henkel $h_k := D^k \times D^{n-k}$, $\varphi : \partial D^k \times D^{n-k} \hookrightarrow \partial M$

$\rightarrow$ 1-Henkel: $D^1 \times D^{n-1}$ $\qquad \rightsquigarrow M \cup_\varphi h_k$

$\boxed{n=2}$

M^2

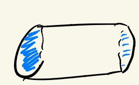


$h_1 = D^1 \times D^1$

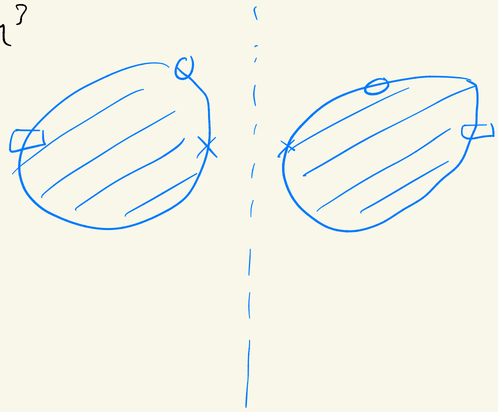
$\partial D^1 \times D^1 = \{p_\pm\} \times D^1$

$\boxed{n=3}$

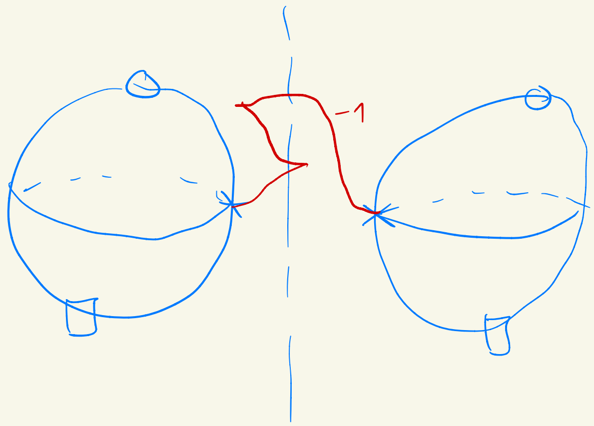
$h_1 = D^1 \times D^1$

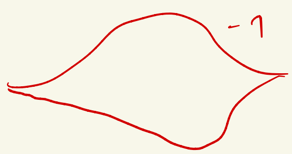


$\boxed{n=3}$ ∂M^3

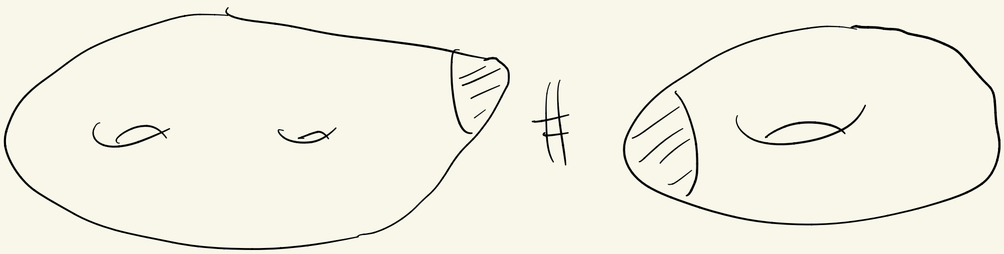


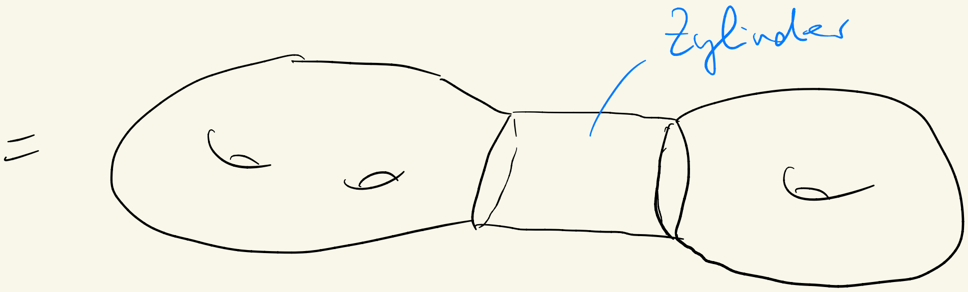
$\boxed{n=4}$ ∂M^4

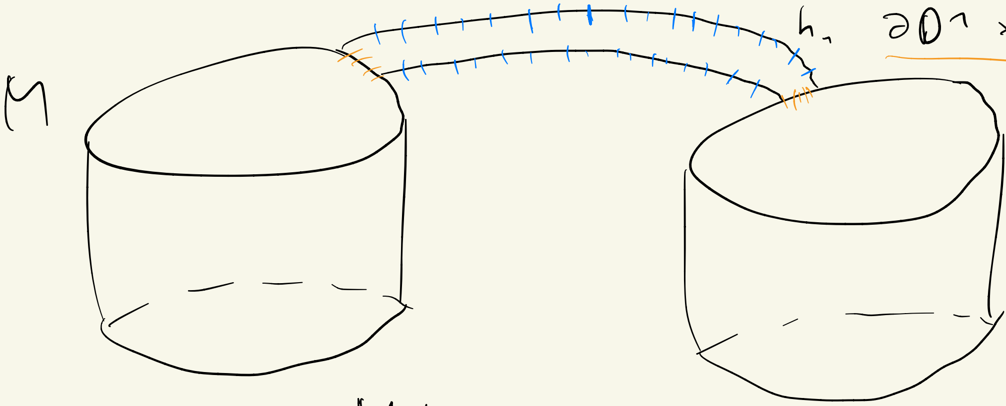




b) $M^n \# N^n$ $\quad M \setminus \mathring{D}_1^n$ $\quad N \setminus \mathring{D}_1^n$







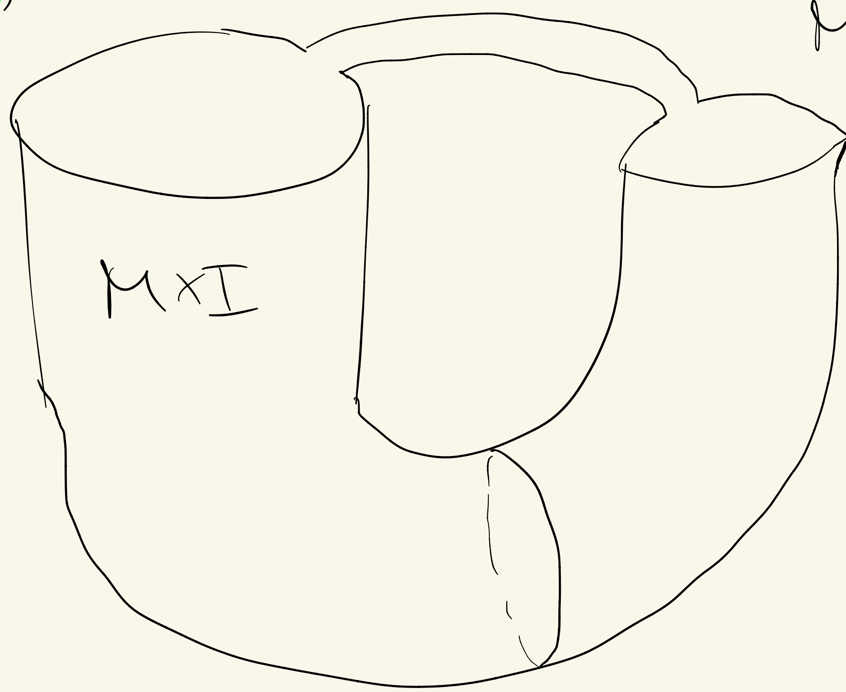
$h_1 \quad \partial D^1 \times D^n = P_N \times D^n \sqcup P_M \times D^n$

$\partial h_1 = \partial D^1 \times D^n$
$\cup \; D^1 \times \partial D^n$

$M \sqcup N$

(c)

$M \# M$



$F^2 \cong \begin{cases} S^1 \\ \#_g T^2 = \Sigma_g \\ \#_k \mathbb{RP}^2 \end{cases}$

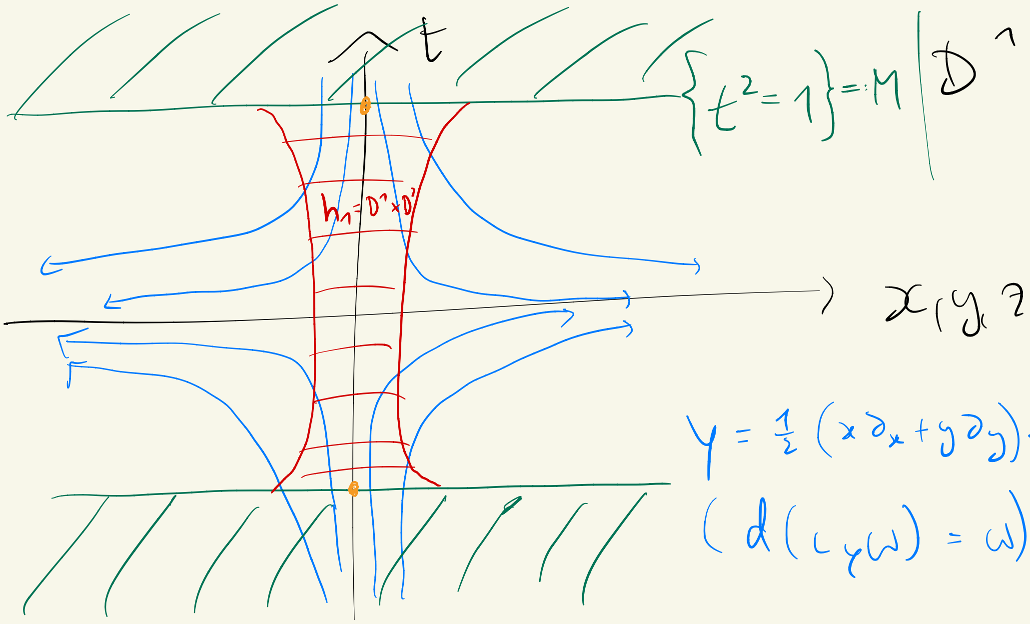
 $= \partial D^3$

 $= \partial(D^3 \cup \underbrace{h_1 \cup \dots \cup h_1}_{g \text{ Stck}})$

$\#_k \mathbb{RP}^2$ (z.B. Kleinsche Flasche)

Aus alg. Topologie $\Rightarrow \nexists M^3$ mit $\partial M = \mathbb{RP}^2$.

(d)



$\{t^2 = 1\} =: M \mid D^1 \times D^3, \ \omega = dx \wedge dy + dz \wedge dt$

$Y = \frac{1}{2}(x\partial_x + y\partial_y) + 2z\partial_z - t\partial_t \pitchfork M$

$(d(\iota_Y \omega) = \omega) \Rightarrow M$ kontakt

$W = \{t^2 > 1\}$

$p_\pm := (0, 0, 0, \pm 1)$

Contact Geometry

Exercise sheet 11

Exercise 1.
Contact $(+1)$-surgery on a stabilized Legendrian knot K yields an overtwisted contact manifold.
Hint: Describe a Legendrian knot J in the complement of K and show that J, seen as a knot in $K(+1)$, violates the Bennequin bound. Can you explicitly describe an overtwisted disk in $K(+1)$?

Exercise 2.

(a) We say that a surgery is an **integer** surgery, if the surgery coefficient is an integer. Show that this notion is independent of the choice of the longitude.

(b) Any integer surgery corresponds to a 4-dimensional handle attachment.

(c) Every closed, connecnted, oriented 3-manifold bounds a compact orientable 4-manifold.
Hint: Use (b) together with the Lickorish–Wallace theorem.

(d) Describe connected sums in surgery diagrams.

(e) The lens spaces $L(p, q)$ as defined in Exercise 5 on Sheet 2 is diffeomorphic to the result of $-p\mu + q\lambda_S$ surgery on the unknot, where λ_S denotes the Seifert longitude of the unknot.
Hint: Show that the group action yielding the lens space preserves the splitting of S^3 into two solid tori and compute the new gluing maps.

(f) $\mathbb{R}P^3$ is diffeomorphic to $L(2, 1)$.

(g) For every $n \in \mathbb{Z}$, $L(p, q)$ is diffeomorphic to $L(p, q + np)$.
Hint: Perform an n-fold twist along the Seifert disk of the unknot.

Exercise 3.
Prove Theorem 7.2 from the lecture.
Hint: Compute the surgery framing in the local model of the Weinstein 2-handle with respect to the contact framing.

Exercise 4.
Let K be a Legendrian unknot that is stabilized once positive and once negative. Show that contact (-1)-surgery on K yields a virtually overtwisted contact structure on a lens space.
Hint: Describe a Legendrian knot J in the complement of K such that J bounds an immersed overtwisted disk in $K(-1)$ that yields an embedded overtwisted disk in the universal covering.

Bonus exercise.
Let L be a Legendrian link in (S^3, ξ_{st}) along which we perform contact (± 1)-surgery to obtain a contact manifold (M, ξ). Let K be a Legendrian knot in the complement of L. Then K represents also a Legendrian knot in (M, ξ). Describe an algebraic criterion (depending only on the linking numbers of the Legendrian knots, the surgery coefficients and the classical invariants) that is equivalent to the statement that K is nullhomologous in (M, ξ). In that case, compute the Thurston–Bennequin invariant of K in (M, ξ).

This sheet will be discussed on Wednesday 24.1. and should be solved by then.

Contact Geometry

Exercise sheet 12

Exercise 1.

(a) We denote by U the Legendrian unknot with $\mathrm{tb} = -1$. In the lecture we have seen that there exist a contactomorphism $f\colon U(+2) \to (S^3, \xi_{st})$. Now let L be a Legendrian link in the complement of U. Then $f(L)$ is a Legendrian link in (S^3, ξ_{st}). Describe a front project of $f(L)$ depending on a front projection of L. How do the contact framings of L and $f(L)$ differ?
Hint: Draw a front projection of U and L and perform handle slides of L over U (after applying the transformation lemma to U).

(b) Use (a) to show that any contact manifold admits a contact (±1)-surgery diagram in which every component is a Legendrian unknot.

(c) Show that handle slides and cancellations are not enough to relate any two contact (±1)-surgery diagrams of the same contact manifold.
Hint: Describe a contact (±1)-surgery diagram of (S^3, ξ_{st}) consisting of three Legendrian knots and observe that handle slides and cancellations preserve the parity of the number of components in a surgery diagram.

Exercise 2.
Classify the number of tight contact structures on the lens spaces $L(p, 1)$.
Hint: Try to generalize the strategy on $\mathbb{R}P^3 = L(2, 1)$ from the lecture:

1. Get an upper bound by gluing together two tight contact solid tori and using the classifications of the tight contact structures on solid tori.
2. Describe as many contact surgery diagrams of tight contact structures on $L(p, 1)$ as possible.
3. Get a lower bound by distinguishing as many of the contact structures from (2) by computing their homotopical invariants.
4. Hope that the lower and upper bounds agree.

Exercise 3.
Finish the computations in Part (b) of the proof of Theorem 7.21 from the lecture, i.e. show that the contact surgery diagrams presented in that proof yields smoothly S^3 and compute its homotopical invariants.

Exercise 4.

(a) Construct for every natural number n a manifold with surgery number n.
Hint: Show that the number of generators of the first homology is a lower bound on the surgery number.

(b) Construct for every natural number n a manifold with contact surgery number n.

Bonus exercise.
Prove Lemma 7.14 and Lemma 7.15 from the lecture. Deduce that negative contact surgery preserves fillability (and tightness if you assume Wand's theorem).

This sheet will be discussed on Wednesday 31.1. and should be solved by then.