Homotopically Standard Tight Non-fillable Contact Structures on the Sphere

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results by Bowden, Gironella, Moreno and Zhou

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Background

Contact topology

Contact manifold:

- odd-dimensional smooth manifold
- codim-1 hyperplane distribution $\ker \alpha$
- contact condition:

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0$$

Contact topology

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Eliashberg, Borman–Eliashberg–Murphy:

Dichotomy: Rigidity vs. Flexibility.

- **tight** *(rigid/geometric)*;
- **overtwisted** *(flexible/topological)*.

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- **tight** *(rigid/geometric)*;
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Theorem (Eliashberg–Gromov)

Fillable contact manifolds are tight.

Converse is false (Etnyre–Honda, Massot–Niederkrueger–Wendl).

Existence and classification

Topological obstruction: *almost* contact structure, i.e. reduction of structure group to $U(n) \times \mathbb{1}$.

Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))

Almost contact manifolds are contact, where the contact structure is overtwisted.

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Tight manifolds

How can we understand **tight** contact manifolds?

Contact topology: fillability

Hierarchy of fillability:

$$\{Stein\} \overset{①}{=} \{Weinstein\} \overset{②}{\subsetneq} \{Liouville\} \overset{③}{\subsetneq} \{strong\}$$

$$\overset{④}{\subsetneq} \{weak\} \overset{⑤}{\subsetneq} \{tight\}$$

- $dim = 3$: ① Cieliebak–Eliashberg, ② Bowden, ③ Ghiggini, ④ Eliashberg, ⑤ Etnyre–Honda.
- $dim \geqslant 5$: ① Cieliebak–Eliashberg, ② Bowden–Crowley–Stipsicz, ③ Zhou, ④ Bowden–Gironella–Moreno, ⑤ Massot–Niederkrüger–Wendl.

Contact structures on spheres

First step in classification: contact structures on spheres.

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Theorem (Eliashberg, '91)

On S^3, it is the unique tight contact structure.

In particular, no tight and non-fillable contact structures on S^3.

Exotic spheres

Theorem (Bowden–Gironella–Moreno–Zhou '22-'24)

For every $n \geqslant 2$, the sphere $\mathbb{S}^{2n+1}$ admits a tight, non-fillable contact structure that is homotopically standard.

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In $\dim = 5$, the same holds, if the first Chern class vanishes.

General remarks

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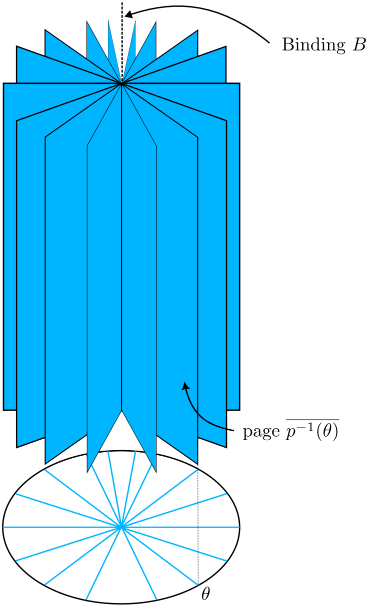
- Suggests that higher-dimensional contact phenomena should occur independently of underlying smooth topology.

Tight and non-fillable spheres

Open books



Giroux correspondence

Giroux: Contact structures are *supported* by open books.

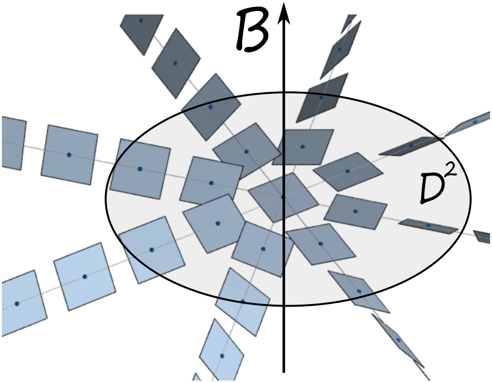


Figure: Supported contact structure.

Bourgeois contact structures

Theorem (Bourgeois '02)

Open book supporting $(M, \xi) \rightsquigarrow$ *contact structure on* $M \times \mathbb{T}^2$*.*

The resulting contact structure is $\mathbb{T}^2$-equivariant.

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Claim: $(\mathbb{S}^{2n+1}, \xi_{ex})$ is homotopically standard, tight and non-fillable.

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Milnor open book $\Rightarrow$ $(\mathbb{S}^{2n+1}, \xi_{ex})$ is *tight*.

Non-fillability

Non-fillability of $(\mathbb{S}^{2n+1}, \xi_{ex})$ can be proven via:

1. Homological obstruction and cobordisms as in [Bowden–Gironella–Moreno], building on [Massot–Niederkrüger–Wendl].
2. Symplectic cohomology computations as in [Zhou].

Convex Decomposition and Capping Cobordism

Observations:

- Bourgeois manifolds have convex decomposition

$$M \times \mathbb{T}^2 = (M \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V}_- \times \mathbb{S}^1,$$

with $V_\pm = \Sigma \times D^*\mathbb{S}^1$, $\Sigma =$ page of the open book, $\phi =$ monodromy.

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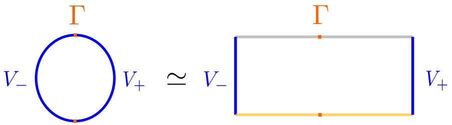
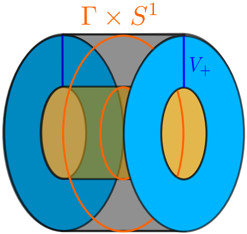
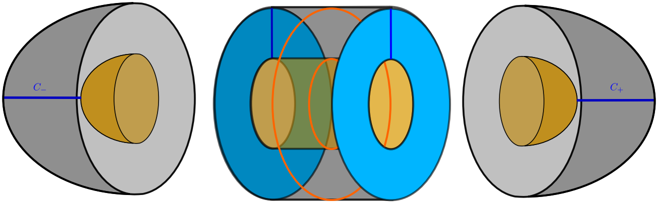
  with $V_\pm = \Sigma \times D^*\mathbb{S}^1$, $\Sigma =$ page of the open book, $\phi =$ monodromy.
- Capping cobordism from $M \times T^2$ to $\Gamma \times \mathbb{S}^2$, via handles $V_\pm \times D^2$ with co-core $C_\pm \simeq V_\pm$.

Capping Cobordism: Example







Homological obstructions

Theorem (Bowden–Gironella–Moreno)

$M \times T^2 = V \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V_-} \times \mathbb{S}^1$ with convex decomposition, $\Gamma = \partial V_\pm$ dividing set. If W is a symplectic filling of $M \times T^2$, then

$$H_*(\Gamma) \to H_*(V_\pm) \to H_*(W),$$

induced by inclusion. Then second map is injective on image of the first.

Namely, if a homology class in Γ survives in $V_\pm$, then it survives in the filling.

Idea of proof

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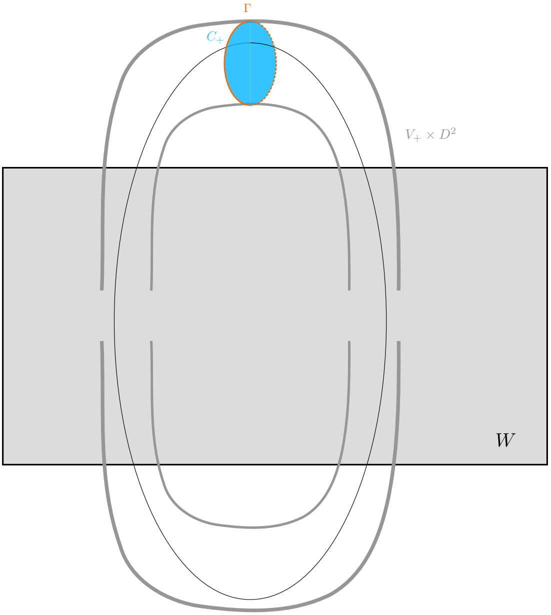

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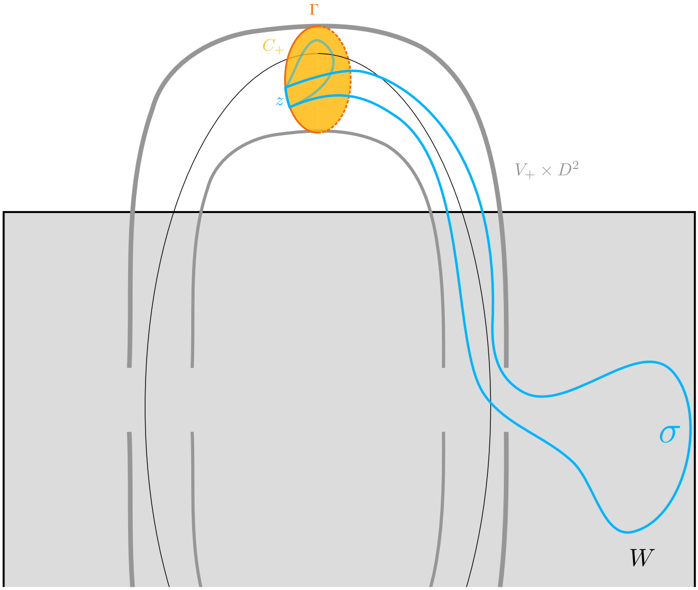

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- Spheres intersect $C_\pm$ precisely once $\rightsquigarrow$ intersection map $\mathcal{I}_\pm : \mathcal{M}_* \to C_\pm$.
- If $\sigma \subset W$ satisfies $\partial\sigma = z$ with z cycle in Γ, then z also bounds $b = \mathcal{I}_\pm ev^{-1}(\sigma) \subset V_\pm$. $\square$

Homological obstructions

Fact:

1. If $\dim \geqslant 7$, subcritical surgeries on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ can be pushed away from dividing set to V_+.

$$\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex}) \text{ still has a dividing set } \Gamma,$$

with $H_n(\Gamma) \neq 0$.

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   $$\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex}) \text{ still has a dividing set } \Gamma,$$

   with $H_n(\Gamma) \neq 0$.
2. Homological obstruction theorem persists under surgery away from dividing set (capping cobordisms).

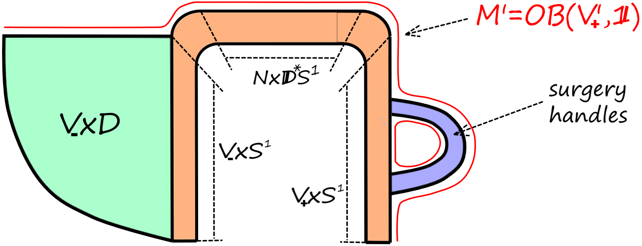

Figure: Capping cobordism.

End of the proof: Assume W filling of $(\mathbb{S}^{2n+1}, \xi_{ex})$.

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- $\exists$ homology class $z \subset \Gamma$ s.t. z survives in $V_\pm$.
- Homological obstruction theorem: $H_n(\Gamma) \to H_n(W) \neq 0$.
- However, this factors as

$$0 \neq H_n(\Gamma) \to H_n(\mathbb{S}^{2n+1}) = 0 \to H_n(W),$$

contradiction.

Thank you!