Tight and non-Fillable Contact Structures on the Sphere

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Background

Contact topology

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Dichotomy: Rigidity vs. Flexibility.

- **tight** *(rigid/geometric)*;
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Theorem (Eliashberg–Gromov)

Fillable contact manifolds are tight.

Converse is false (Etnyre–Honda, Massot–Niederkrueger–Wendl).

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Theorem (Eliashberg, '91)

On S^3, it is the unique tight contact structure.

In particular, no tight and non-fillable contact structures on S^3.

Tight and non-fillable structures in $\dim \geqslant 5$

Theorem (Bowden–Gironella–Moreno–Zhou '22-'24)

For every $n \geqslant 2$, the sphere $\mathbb{S}^{2n+1}$ admits a tight, non-fillable contact structure that is homotopically standard.

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In $\dim = 5$, the same holds, if the first Chern class vanishes.

Tight and non-fillable spheres

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- Milnor open book on $\mathbb{S}^{2n-1} \overset{\text{Bourgeois}}{\rightsquigarrow}$ contact structure on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ $\rightsquigarrow$ two 1-surgeries = $\mathbb{S}^{2n-1} \times \mathbb{S}^2 \rightsquigarrow$ one 2-surgery = $\mathbb{S}^{2n+1}$.

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Claim: $(\mathbb{S}^{2n+1}, \xi_{ex})$ is tight and non-fillable.

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3. Algebraically tight $\implies$ tight.

> Milnor open book $\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex})$ is *tight*.

Fillability

Observation: Bourgeois manifolds have convex decomposition

$$\mathrm{OB}(\Sigma,\phi)\times\mathbb{T}^2=(\mathrm{OB}(\Sigma,\phi)\times\mathbb{S}^1)\times\mathbb{S}^1=V_+\times\mathbb{S}^1\cup_\phi\overline{V}_-\times\mathbb{S}^1,$$

with $V_\pm=\Sigma\times D^*\mathbb{S}^1$, $\Sigma=$ page of the open book, $\phi=$ monodromy.

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with $V_\pm = \Sigma \times D^*\mathbb{S}^1$, $\Sigma =$ page of the open book, $\phi =$ monodromy.

Theorem (Bowden–Gironella–Moreno)

$M = V \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V_-} \times \mathbb{S}^1$ with convex decomposition, $N = \partial V_\pm$ dividing set. If W is a symplectic filling of M, then

$$H_*(N) \to H_*(V_\pm) \to H_*(W),$$

induced by inclusion. Then second map is injective on image of the first.

Namely, if a homology class in N survives in $V_\pm$, then it survives in the filling.

Proof: W filling of $(\mathbb{S}^{2n+1}, \xi_{ex})$:

$$0 \neq H_n(N) \xrightarrow{\text{nontrivial}} H_n(W).$$

However, this factors as

$$0 \neq H_n(N) \to H_n(\partial W = \mathbb{S}^{2n+1}) = 0 \to H_n(W),$$

contradiction.

Thank you!