Contents

Abstract

Contact geometry is the study of odd-dimensional smooth manifolds equipped with contact structures, i.e. hyperplane distributions $\xi = \ker\alpha$ satisfying the contact condition

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0.$$

While they originally arise in the study of ODEs and in classical mechanics, the topological study of contact manifolds is a more recent and very active field of research.

A manifold can have multiple different contact structures, which can be either rigid (in which case one speaks of a "tight" manifold) or flexible (in the sense that they satisfy an h-principle). The latter contact manifolds are then called overtwisted. A foundational result of Eliashberg and Borman–Eliashberg–Murphy, roughly speaking, states that overtwisted contact manifolds exist in abundance, namely whenever the manifold admits the topological version of a contact structure (an *almost* contact structure), which is a first obvious obstruction. In dimension three, an almost contact structure is simply an oriented 2-plane field.

To illustrate this dichotomy, consider the sphere S^3. By a result of Eliashberg, it has precisely one tight contact structure. On the other hand, it has infinitely many overtwisted contact structures, corresponding to the infinitely many homotopy classes of 2-plane fields on the 3-sphere. There are other examples where there are infinitely many or no tight contact structures on a contact manifold.

A further interesting property of contact manifolds comes from the fact that contact geometry is the odd-dimensional counterpart to symplectic geometry. Often, it is possible to view a contact manifold as the boundary of a symplectic manifold. Manifolds that are in this sense "fillable" are always tight. The contrary, however, doesn't need to hold and one can ask the question under which conditions such tight, but non-fillable manifolds exist. The first examples of tight and non-fillable contact manifolds were constructed by Etnyre–Honda in dimension three, and by Massot–Niederkrueger–Wendl in higher dimensions.

More recently, Bowden–Gironella–Moreno–Zhou have shown that there exist homotopically standard, non-fillable but tight contact structures on all spheres S^{2n+1} with $n >= 2$. Starting with a specific open book decomposition of S^{2n-1}, one can construct a contact form on this manifold using a well-known construction by Thurston–Winkelnkemper. Then, according to Bourgeois, this contact structure can be extended to a tight contact structure on $S^{2n-1} \times T^2$. Applying subcritical surgery (preserving the tightness), one can kill the topology of the T^2-factor and obtain a tight contact structure on S^{2n+1}. Because of the special way of constructing it, one can show that it is non-fillable, but still homotopically standard.

The goal of my master thesis is to give a streamlined explanation of the results of Bowden–Gironella–Moreno–Zhou, including the necessary background needed to understand the main ideas.

CHAPTER 1

Introduction

- Where does this fit into the big picture research?
- What questions are answered?
- which questions remain open?
- interesting future questions?

Maybe copy things from the Expose for the scholarship application.

CHAPTER 2

The Construction: A homotopically standard contact structure on the Sphere

1. Outline

The goal is to find homotopically standard, tight, non-fillable contact structures on the sphere S^{2n+1} for $n \geq 2$. In this work, only the easier case $n \geq 3$ will be covered. As $(S^{2n+1}, \xi_{\text{std}})$ is defined as the contact boundary of the standard symplectic ball $(D^{2n+2}, \omega_{\text{std}})$, it is fillable by definition. Hence, one needs a different contact structure. This chapter explains how to construct a homotopically standard contact structure on the sphere in a way that at first doesn't seem very intuitive, hoping that in the end it will be different from ξ_{std}. Later, in chapters 3 and 4, it will be shown that, as desired, this contact structure really is tight and non-fillable (and hence different from ξ_{std}). The starting point for the whole construction is a Milnor open book, i.e. a certain kind of decomposition of S^{2n-1} that comes from Milnors work on hypersurface singularities [**Mil69**]. By the Giroux correspondence, there is a contact structure on this manifold. In this case, the Giroux correspondence can be realized by an explicit construction due to Thurston-Winkelnkemper. Now, a construction due to Bourgeois ([**Bou02**]) extends the resulting contact structure on S^{2n-1} to $S^{2n-1} \times T^2$. By applying smooth surgery, one can kill the homology in the T^2-factor and so obtain S^{2n+1}. By an h-principle due to Eliashberg–Murphy ([**EM02**, section 12.4]), the surgeries can be realized as contact surgeries. This completes the construction as now one has a contact structure on S^{2n+1}. Finally in section 6, it will be explained why the contact structure is homotopically standard.

2. Raw material for the construction: The Milnor open book

Define $f\colon \mathbb{C}^n \to \mathbb{C}$ by

$$(z_0, \dots, z_{n-1}) \mapsto z_0^k + z_1^2 + \dots z_{n-1}^2.$$

Consider the sphere $S^{2n-1} \subset \mathbb{C}^n$. The intersection $f^{-1}(0) \cap S^{2n-1}$ is the so called Brieskorn sphere $B = \Sigma_{n-1}(k, 2, \dots, 2)$. On the complement $S^{2n-1} \setminus B$, the map

$$\pi_f\colon S^{2n-1} \setminus B \to S^1\colon (z_0, \dots, z_{n-1}) \mapsto \frac{f(z_0, \dots, z_{n-1})}{|f(z_0, \dots, z_{n-1})|}$$

is a fibration over S^1, the Milnor fibration. According to [**Mil69**, Lemma 6.1], the fibers are smooth $2n-2$-dimensional manifolds with boundary B. It is well-known that such a fibration is precisely an open book decomposition (of S^{2n-1}).

If the dimension is high enough, the Brieskorn sphere is not just connected, but actually a topological sphere.

According to [**Mil69**, Thm 6.5], the pages of this open book (i.e. the Milnor fibers) have the homotopy type of a bouquet of spheres $S^{n-1} \vee \cdots \vee S^{n-1}$. It follows from the Hurewicz theorem that

$$H_0 = \mathbb{Z}, \qquad H_i = 0,\ 0 < i < n \qquad \text{and } H_n = \pi_n.$$

[**Mil69**, Theorem 9.1] implies that $H_n = \pi_n = \mathbb{Z}^{k-1}$.

3. From open books to contact structures: The Thurston-Winkelnkemper construction

DEFINITION 1. *Let (B, p) be an oriented open book decomposition of the oriented manifold M. A contact structure $\xi = \ker \alpha$ on M is said to be* ***supported*** *by the open book decomposition (B, p) of M if*

(i) the contact form α induces the positive orientation of M $(\alpha \wedge (\mathrm{d}\alpha)^n > 0)$.

(ii) the 2-form $\mathrm{d}\alpha$ induces a symplectic form on each page, defining its positive orientation

(iii) the 1-form α induces a positive contact form on B, i.e.

$$\alpha|_{TB} \wedge (\mathrm{d}\alpha|_{TB})^{(n-2)} > 0.$$

DEFINITION 2 (mapping torus). *Let Σ be a smooth manifold with boundary $\partial\Sigma$ and $\phi : \Sigma \to \Sigma$ a diffeomorphism that is equal to the identity close to $\partial\Sigma$. The mapping torus $\Sigma(\phi)$ is given by $\Sigma \times [0, 2\pi]/\sim$ where*

$$(x, 2\pi) \sim (\phi(x), 0).$$

The generalized mapping torus requires as additional data a smooth function $\overline{\varphi} : \Sigma \to \mathbb{R}^+$ that is constant near $\partial\Sigma$. Then,

$$\Sigma_{\overline{\varphi}}(\phi) := \Sigma \times \mathbb{R}/\sim \quad \textit{where} \quad (x, \theta) \sim (\phi(x), \theta - \overline{\varphi}(x)).$$

Abstract open books. Starting with a mapping torus $\Sigma(\phi)$, one can construct an abstract open book $M(\phi)$ with binding $\partial\Sigma$ (see fig. 1)

Define

$$M(\phi) := \left(\Sigma(\phi) \cup \partial\Sigma \times D^2\right) / \sim$$

and identify

$$[x \in \partial\Sigma, \theta] \sim (x, r = 1, \varphi = \theta).$$

The construction. Let Σ^{2n} be a compact manifold admitting an exact symplectic form $\omega = \mathrm{d}\beta$ s.t. on the boundary $\partial\Sigma$, a contact form β_∂ is induced (this follows from the conditions requested in Geiges). Let the boundary be connected (i.e. the binding is also connected). Let the monodromy map ϕ be an exact symplectomorphism of (Σ, ω), equal to the identity near the boundary $\partial\Sigma$ (exactness is not necessary according to Geiges, as it can be obtained via a suitable isotopy of the symplectomorphism). An exact symplectomorphism ϕ of (Σ, ω) is such that

$$\phi^*(\beta) - \beta =: \mathrm{d}\overline{\varphi}$$

is exact, i.e. there exists such a function $\overline{\varphi}$ on Σ (of course only defined up to adding a locally constant function. Choose it in such a way that it only takes positive values). The 1-form

$$\alpha := \beta + \mathrm{d}\varphi$$

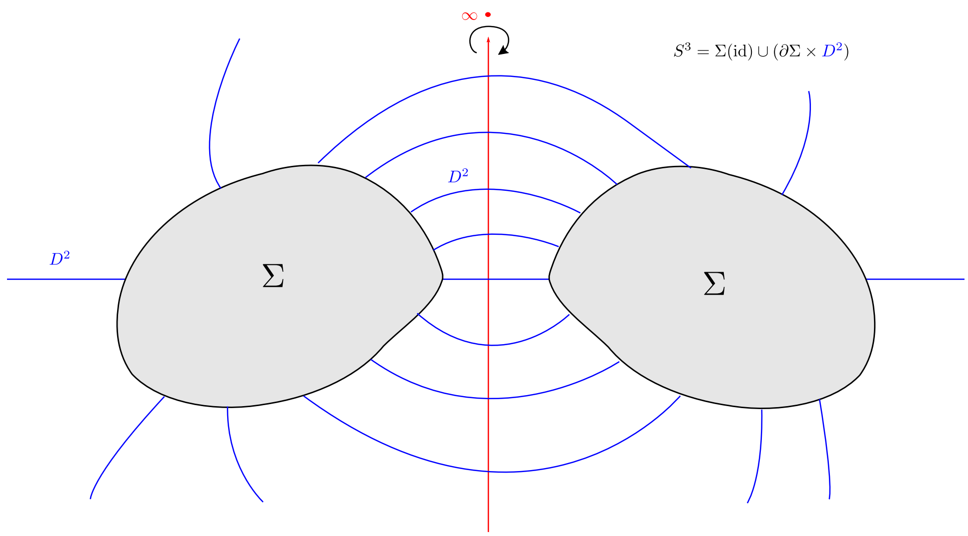


FIGURE 1. abstract open book

is a contact form on $\Sigma \times \mathbb{R}$:

$$\alpha \wedge (\mathrm{d}\alpha)^n = (\beta + \mathrm{d}\varphi) \wedge \underbrace{(\mathrm{d}\beta)^n}_{=:\Omega} = \beta \wedge \Omega + \mathrm{d}\varphi \wedge \Omega = \mathrm{d}\varphi \wedge \Omega,$$

where Ω is a volume form on Σ (as β is a symplectic form). The $\beta \wedge \Omega$ term vanishes because both are forms on Σ, but Ω is already a top-level form. The resulting form is a wedge product of two volume forms on the product manifolds and therefore a volume form on $\Sigma \times \mathbb{R}$.

Now consider the transformation that induces the generalized mapping torus

$$F := (x, \varphi) \mapsto (\phi(x), \varphi - \overline{\varphi}(x)).$$

Remember that φ only takes positive values, i.e. the mapping torus is welldefined. The 1-form α is invariant under this transformation:

$$\begin{aligned} F^*(\alpha) &= F^*(\beta) + F^*(\mathrm{d}\varphi) && \mid \beta \text{ is independent of } \varphi \\ &= \phi^*(\beta) + \mathrm{d}F(\varphi) && \mid \text{definition of } \overline{\varphi}, F \\ &= \beta + \mathrm{d}\overline{\varphi} + \mathrm{d}\varphi - \mathrm{d}\overline{\varphi} \\ &= \alpha. \end{aligned}$$

It follows that α descends to a contact form on $\Sigma_{\overline{\varphi}}(\phi)$.

In the following, it will be necessary to have an adapted gluing construction for the abstract open book coming from a generalized mapping torus. Therefore, choose a collar neighborhood on the generalized mapping torus s.t. on $[-1, 0] \times \partial\Sigma$, the symplectic form is given by $\mathrm{d}\left(e^s \beta_\partial\right)$ where s is the collar parameter, i.e. $\beta = e^s \beta_\partial$. Why does such a neighborhood exist?

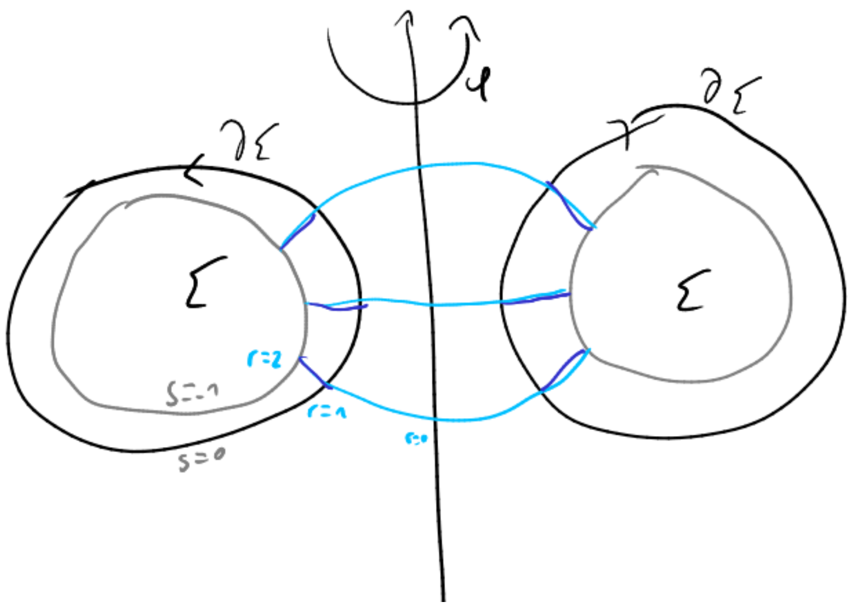


FIGURE 2. Detailled gluing process of the generalized abstract open book

Close to $\partial\Sigma$, ϕ is equal to the identity and therefore $\mathrm{d}\overline{\varphi}$ is locally constant (hence constant, as $\partial\Sigma$ is connected). Parametrize the neighborhood so that $\overline{\varphi}$ is constant on $[-1,0]\times\partial\Sigma$.

Now, take a look at

$$M := \left(\Sigma_{\overline{\varphi}}(\phi)\ \dot\cup\ \left(\partial\Sigma\times D_2^2\right)\right)/\sim .$$

A simple linear reparametrization will make the notation a lot easier: As $\overline{\varphi}$ is constant on the neighborhood under consideration, just pretend that $\overline{\varphi}=2\pi$. Furthermore, parametrize the boundary $\partial\Sigma$ with $\theta\in S^1$. Then identify

$$(s,\theta,\varphi)\in[-1,0]\times\partial\Sigma\times S^1\subset\Sigma_{\overline{\varphi}}(\phi)$$

with

$$(\theta, s=1-r,\varphi)\in\partial\Sigma\times D_2^2 =: \mathcal{N}$$

where (r,φ) are polar coordinates on D_2^2, i.e. a collar neighborhood of Σ is identified with an annulus in D_2^2. (See fig. 2)

Now choose the ansatz

$$\alpha_{\mathrm{ext}} := h_1(r)\beta_\partial + h_2(r)\mathrm{d}\varphi.$$

for the extension of the contact form over $\mathcal{N}$. On the gluing area (i.e. $1\le r\le 2$), α_{ext} has to agree with $\alpha=\beta+\mathrm{d}\varphi=e^s\beta_\partial+\mathrm{d}\varphi$, i.e.

$$h_1(r)=e^s=e^{1-r}\qquad h_2(r)=1.$$

In order to ensure smoothness at $r=0$, set $h_1(r)=2-r^2$ and $h_2(r)=r^2$ in a small neighborhood of $r=0$. Then,

$$\alpha_{\mathrm{ext}}|_0 = 2\beta_\partial$$

Further,

$$\mathrm{d}\alpha_{\text{ext}} = h_1'(r)\mathrm{d}r \wedge \beta_\partial + h_1(r)\mathrm{d}\beta_\partial + h_2'(r)\mathrm{d}r \wedge \mathrm{d}\varphi.$$

and

$$(\mathrm{d}\alpha_{\text{ext}})^n = n \cdot \mathrm{d}r \wedge (h_1'(r)\beta_\partial + h_2'(r)\mathrm{d}\varphi) \cdot h_1(r)^{n-1}(\mathrm{d}\beta_\partial)^{n-1} + \underbrace{h_1(r)^n(\mathrm{d}\beta_\partial)^n}_{=0},$$

where the second term vanishes because $(\mathrm{d}\beta_\partial)^n$ is a $2n$-form on $\partial\Sigma^{2n-1}$. Finally,

$$\begin{aligned}
\alpha_{\text{ext}} \wedge (\mathrm{d}\alpha_{\text{ext}})^n = {} & h_1(r)nh_1(r)^{n-1}h_2'(r)\cdot && \beta_\partial \wedge \mathrm{d}r \wedge \mathrm{d}\varphi \wedge (\mathrm{d}\beta_\partial)^{n-1} \\
& + h_2(r)nh_1(r)^{n-1}h_1'(r)\cdot && \mathrm{d}\varphi \wedge \mathrm{d}r \wedge \beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1} \\
= {} & nh_1(r)^{n-1}(h_1h_2'(r) - h_2h_1'(r))\cdot && \beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1} \wedge \mathrm{d}r \wedge \mathrm{d}\varphi \\
= {} & nh_1(r)^{n-1}\det\begin{pmatrix} h_1(r) & h_2(r)/r \\ h_1'(r) & h_2'(r)/r \end{pmatrix}\cdot && \beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1} \wedge r\mathrm{d}r \wedge \mathrm{d}\varphi
\end{aligned}$$

As β_∂ is a contact form on $\partial\Sigma$, $\beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1}$ is a positive volume form on $\partial\Sigma$. Furthermore, $r\mathrm{d}r \wedge \mathrm{d}\varphi$ is a positive volume form on the disk D_2^2. As a result, the right term of the result is a volume form on $\mathcal{N} = \partial\Sigma \times D_2^2$. The left term shows that $h_1(r)$ musn't have any zeros for $r \in [0, 2]$ and that $(h_1(r), h_2(r))$ must never be parallel to $(h_1'(r), h_2'(r))$, i.e.

$$h_1(r)^{n-1}\det\begin{pmatrix} h_1(r) & h_2(r)/r \\ h_1'(r) & h_2'(r)/r \end{pmatrix} > 0 \qquad \forall r \in [0, 2].$$

(Close to zero, the determinant is given by $2 \cdot 2 - 0 \cdot 0 = 4 > 0$). Figure 4.7 in [**Gei08**] proves the existence of such a pair of functions h_1 and h_2.

In total, α_{ext} induces the correct orientation on the extension and, as M is connected and orientable, on all of M. In particular, condition (i) of definition 1 holds and $\alpha \wedge (\mathrm{d}\alpha)^n = \mathrm{d}\varphi \wedge \Omega$ is a positive volume form on the mapping torus.

Recall that ω is the symplectic form on Σ. As $\Omega = (\mathrm{d}\beta)^n = \omega^n$, it is a $2n$-form and hence Ω is a positive volume form on Σ. Thus, on Σ, $\mathrm{d}\alpha = \mathrm{d}\beta = \omega$ is a symplectic form that induces the positive orientation of Σ. On $\mathcal{N}$, it is necessary to check that the form induced by $\mathrm{d}\alpha_{\text{ext}}$ on the pages is symplectic with the right orientation. Inside $\mathcal{N}$, a page is given by the condition $\varphi = \text{const}$, i.e. $\mathrm{d}\varphi = 0$. Also,

$$\begin{aligned}
(\mathrm{d}\alpha_{\text{ext}})^n &= n \cdot \mathrm{d}r \wedge (h_1'(r)\beta_\partial + h_2'(r)\mathrm{d}\varphi) \cdot h_1(r)^{n-1}(\mathrm{d}\beta_\partial)^{n-1} \\
&= nh_1'(r)h_1(r)^{n-1}\mathrm{d}r \wedge \beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1}
\end{aligned}$$

A positive volume form on Σ must be positive on $-\partial_r, \mathfrak{b}$ where $\mathfrak{b}$ is a positive basis of a point in $\partial\Sigma$. As $\beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1}$ is a positive volume form on $\partial\Sigma$, it follows

$$\begin{aligned}
(\mathrm{d}\alpha_{\text{ext}})^n(-\partial_r, \mathfrak{b}) &= \underbrace{nh_1(r)^{n-1}}_{=:A>0} \cdot h_1'(r)\mathrm{d}r(-\partial_r) \wedge \underbrace{\left[\beta_\partial \wedge (\mathrm{d}\beta_\partial)^{n-1}\right](\mathfrak{b})}_{=:B>0} \\
&= AB \cdot h_1'(r) \cdot -1 \\
&> 0,
\end{aligned}$$

where in the last line $h_1'(r) = \frac{\mathrm{d}}{\mathrm{d}r}(2 - r^2) = -2r < 0$. This verifies condition (ii) inside $\mathcal{N}$ and outside $\mathcal{N}$, on Σ. As a result, is must hold on the whole page.

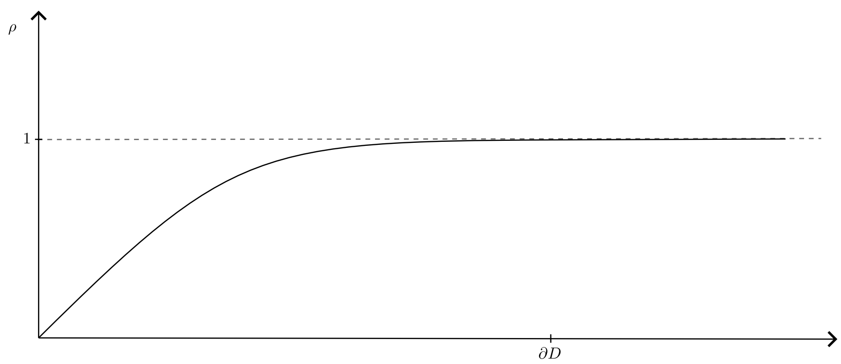


FIGURE 3. Onedimensional sketch of ρ

Condition (iii) follows from the fact that on B, $\alpha_{\text{ext}} = 2\beta_\partial$ which is a positive contact form on $\partial\Sigma$ and therefore also on B.

4. From M to $M \times T^2$: The Bourgeois construction

It is not very hard to construct a contact structure on the three-torus. When Lutz [**Lut79**] discovered a contact structure on T^5, however, it was natural to wonder whether there exists a contact struture on T^{2n+1} for all $n \in \mathbb{N}$. In order to answer this long-standing question, Bourgeois [**Bou02**] came up with a construction that takes as input a contact structure on M and as output returns a contact structure on $M \times T^2$. To be more precise, it requires a contact structure that is supported by an open book decomposition as an input. However, a result by Giroux and Mohsen [**Gei08**, Theorem 7.3.5] shows that to every contact structure one can find such an open book decomposition, so that is not an obstruction, but rather part of the construction data.

4.1. General construction. Let $\dim M \geq 3$ and (B, π) an open book decomposition of M supporting $(M, \xi = \ker\alpha)$. By definition of an open book, there is a trivial tubular neighborhood $B \times D^2$ around B and there exist a radial coordinate r with $r = 0$ precisely on B s.t. (r, π) form polar coordinates on this neighborhood. Choose a smooth function ρ of r s.t. $\rho = r$ close to $r = 0$, $\rho'(r) > 0$ and $\rho = 1$ at ∂D. Extend this function to M by setting $\rho = 1$ on $M \setminus B \times D$ (see fig. 3). As π and ρ are smooth functions on M, one can define the smooth functions $x_1 \coloneqq \rho\cos\pi$ and $x_2 \coloneqq \rho\sin\pi$ on all of M. On $B \times D^2$, they coincide with the Cartesian coordinate functions near B. As always with corresponding polar- and cartesian coordinates, they satisfy

$$\begin{aligned} x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1 &= \rho^2\cos^2\pi\mathrm{d}\pi + \rho\cos\pi\sin\pi\mathrm{d}\rho + \rho^2\sin^2\pi\mathrm{d}\pi - \rho\cos\pi\sin\pi\mathrm{d}\rho \\ &= \rho^2(\cos^2\pi + \sin^2\pi)\mathrm{d}\pi \\ &= \rho^2\mathrm{d}\pi \end{aligned}$$

and, analogously,

$$\mathrm{d}x_1 \wedge \mathrm{d}x_2 = \rho \mathrm{d}\rho \wedge \mathrm{d}\pi.$$

On $M \times T^2$, choose coordinates (θ_1, θ_2) on the torus part of the manifold. Define

$$\tilde{\alpha} \coloneqq x_1 \mathrm{d}\theta_1 - x_2 \mathrm{d}\theta_2 + \alpha.$$

to be a 1-form on M (where α is extended in the obvious way to $M \times T^2$ as the pullback $\pi_1^* \alpha$). This is the candidate for the contact form on $M \times T^2$. Having computed the exterior derivative and its n-th power

$$\begin{aligned} \mathrm{d}\tilde{\alpha} &= \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 - \mathrm{d}x_2 \wedge \mathrm{d}\theta_2 + \mathrm{d}\alpha, \\ (\mathrm{d}\tilde{\alpha})^n =&(n-1)(\mathrm{d}\alpha)^{n-1} \wedge (\mathrm{d}x_1 \wedge \mathrm{d}\theta_1 - \mathrm{d}x_2 \wedge \mathrm{d}\theta_2) \\ &- n(n-1)(\mathrm{d}\alpha)^{n-2} \wedge \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\theta_2, \end{aligned}$$

one can check the contact condition:

$$\begin{aligned} \tilde{\alpha} \wedge (\mathrm{d}\tilde{\alpha})^n =&(x_1 \mathrm{d}\theta_1 - x_2 \mathrm{d}\theta_2 + \alpha) \wedge (n-1)(\mathrm{d}\alpha)^{n-1} \wedge (\mathrm{d}x_1 \wedge \mathrm{d}\theta_1 - \mathrm{d}x_2 \wedge \mathrm{d}\theta_2) \\ &- (x_1 \mathrm{d}\theta_1 - x_2 \mathrm{d}\theta_2 + \alpha) \wedge n(n-1)(\mathrm{d}\alpha)^{n-2} \wedge \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\theta_2 \\ =&(n-1)(\mathrm{d}\alpha)^{n-1} \wedge (x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1) \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \\ &+ \underbrace{\alpha \wedge (n-1)(\mathrm{d}\alpha)^{n-1} \wedge \mathrm{d}x_1}_{2n-\text{form on } M} \wedge \mathrm{d}\theta_1 - \underbrace{\alpha \wedge (n-1)(\mathrm{d}\alpha)^{n-1} \wedge \mathrm{d}x_2}_{2n-\text{form on } M} \wedge \mathrm{d}\theta_2 \\ &+ n(n-1)\alpha \wedge (\mathrm{d}\alpha)^{n-2} \wedge \mathrm{d}x_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\theta_2 \end{aligned}$$

M has dimension $2n-1$, i.e. the middle term is 0

$$\begin{aligned} =&(n-1)(\mathrm{d}\alpha)^{n-1} \wedge \rho^2 \mathrm{d}\phi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \\ &+ n(n-1)\alpha \wedge (\mathrm{d}\alpha)^{n-2} \wedge \rho \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2. \end{aligned}$$

As this expression is a top-dimensional form, it suffices to show that its nowhere zero. For that, one needs to employ the fact that α is supported by (B, π). By condition (ii) of definition 1, $(\mathrm{d}\alpha)^{n-1}$ must be a positive volume form on the pages. As explained in that definition, the orientation on M is given by ∂_ϕ and the orientation of the page. In particular, $(\mathrm{d}\alpha)^{n-1} \wedge \rho \mathrm{d}\phi$ is a positive volume form on M. Multiplied with a second ρ-factor, it vanishes along B. As $\theta_1 \wedge \theta_2$ is a positive volume form on T^2, the first term is non-negative everywhere and positive away from

$$\underbrace{B \times 0}_{\subset B \times D^2 \subset M} \times T^2.$$

Let $\mathfrak{b}$ be a basis of the binding B that is positively ordered. Then, $-\partial_r, \mathfrak{b}$ and (because the binding is odd-dimensional) $\mathfrak{b}, \partial_r$ are positive bases of the page. Clearly, then,

$$\mathfrak{a} \coloneqq \mathfrak{b}, \partial_r, \partial_\phi, \partial_{\theta_1}, \partial_{\theta_2}$$

is an ordered basis of $M \times T^2$. Using $\rho'(r) \geq 0$ everywhere, it follows that $\mathrm{d}\rho(\partial_r)$ is non-negative. Hence, plugging $\mathfrak{a}$ into the second term,

$$\begin{aligned} &\left(n(n-1)\alpha \wedge (\mathrm{d}\alpha)^{n-2} \wedge \rho \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2\right)(\mathfrak{a}) \\ =&n(n-1)\rho \cdot (\alpha \wedge (\mathrm{d}\alpha)^{n-2})(\mathfrak{b}) \cdot d\rho(\partial_r) \cdot \mathrm{d}\phi(\partial_\phi) \cdot \mathrm{d}\theta_1(\partial_{\theta_1}) \cdot \mathrm{d}\theta_2(\partial_{\theta_2}) \\ \geq&0. \end{aligned}$$

By condition (iii) of definition 1, $\alpha\wedge(\mathrm{d}\alpha)^{n-2}$ is positive on B. Therefore, the second term is positive on $B\times 0\times T^2$ (hence also on a neighborhood) and non-negative everywhere else. This proves the contact condition and $\tilde{\alpha}$ is indeed a contact form on $M\times T^2$.

4.2. Example: Bourgeois and Thurston-Winkelnkemper. Applying the Thurston–Winkelnkemper construction yields a contact form α_{ext} on

$$M = (\Sigma_{\overline{\varphi}}(\phi)\ \dot{\cup}\ \mathcal{N})\ /\sim .$$

Now, carry out the Bourgeois construction and obtain a contact form

$$\tilde{\alpha}_{\text{ext}} = \alpha_{\text{ext}} + x_1\mathrm{d}\theta_1 - x_2\mathrm{d}\theta_2$$

on $M\times T^2$ where θ_1,θ_2 are coordinates on T^2 and x_1,x_2 are coordinates as described in the section on the Bourgeois construction, i.e. there is a function $\rho : M\to[0,1]$ that agrees with r near the binding B and define

$$x_1 \coloneqq \rho\cos(p); \qquad x_2 \coloneqq \rho\sin(p).$$

Inside $\mathcal{N}$, the projection map p is given by the angular coordinate φ. Therefore,

$$\alpha|_{\mathcal{N}} = \alpha_{\text{ext}} = h_1(r)\beta_\partial + h_2(r)\mathrm{d}\varphi = h_1(r)\beta_\partial + h_2(r)\mathrm{d}p.$$

In total, the contact form on $\mathcal{U}\coloneqq\mathcal{N}\times T^2$ is

$$\tilde{\alpha} = h_1(r)\beta_\partial + h_2(r)\mathrm{d}p + \rho(r)(\cos(p)\mathrm{d}\theta_1 - \sin(p)\mathrm{d}\theta_2)$$

Outside $\mathcal{U}$, the form is given by

$$\tilde{\alpha} = \beta + \mathrm{d}p + \cos(p)\mathrm{d}\theta_1 - \sin(p)\mathrm{d}\theta_2,$$

as $\rho(r)=1\ \forall r\geq 1$. Collecting all the conditions on h_1 and h_2 from the last section, the requirements are

- $h_1(r)=e^s=e^{1-r}, h_2(r)=1$ in the gluing area $(1\leq r\leq 2)$.
- Smoothness around the binding and contact condition on the binding: $h_1(r) = 2-r^2$ and $h_2(r)=r^2$ around $r=0$. The $-r^2$-part is only important to have $h_1'(r)<0$ around 0 to satisfy the symplectic condition on the pages.
- Contact condition on the tubular neighborhood:
$$h_1(r)^{n-1}\det\begin{pmatrix} h_1(r) & h_2(r)/r \\ h_1'(r) & h_2'(r)/r\end{pmatrix} > 0 \qquad \forall r\in[0,2].$$

4.3. Reeb dynamics. We now define two functions $\mu,\nu : M\to\mathbb{R}$ as follows:

$$\mu = \begin{cases} \frac{\rho'}{\rho' h_1 - \rho h_1'} & \text{inside } \mathcal{U} \\ 0 & \text{outside } \mathcal{U}\end{cases} \quad \text{and} \quad \nu = \begin{cases} \frac{-h_1'}{\rho' h_1 - \rho h_1'} & \text{inside } \mathcal{U} \\ 1 & \text{outside } \mathcal{U}\end{cases}.$$

Both μ and ν are smooth. First of all, note that both are quotients of smooth functions. We have to make sure that $\rho' h_1 \neq \rho h_1'$. All of the involved functions are non-negative except for h_1', which is constantly negative (h_1 needs to be chosen in such a way that h_1' is nonzero). As a result,

$$\rho' h_1 > 0 > \rho h_1',$$

so the denominator is always nonzero. As ρ is constantly 1 for $r \geq 1$, μ must be 0 for $r \geq 1$. In particular, it is smooth everywhere. For ν, setting $\rho \equiv 1$ implies $\rho' = 0$ on $r \geq 1$. Therefore, the denominator simplifies to 1 and ν is smooth, too.

LEMMA 1. *The Reeb vector field of the contact form $\tilde{\alpha}$ is given by*

$$R = \mu(r) R_B + \nu(r)[\cos(p)\partial_{\theta_1} - \sin(p)\partial_{\theta_2}],$$

where R_B is the Reeb vector field of β_∂, the contact form on $\partial\Sigma$.

PROOF. Inside $\mathcal{U}$,

$$\tilde{\alpha}(R) = h_1(r)\beta_\partial(R) + h_2(r)\mathrm{d}p(R) + \rho(r)(\cos(p)\mathrm{d}\theta_1 - \sin(p)\mathrm{d}\theta_2)(R)$$

β_∂ and $\mathrm{d}p$ are 0 on ∂_{θ_i}, p and θ_i are constant on R_B

$$\begin{aligned} &= h_1\mu\beta_\partial(R_B) + \rho\nu(\cos(p)\mathrm{d}\theta_1 - \sin(p)\mathrm{d}\theta_2)(\cos(p)\partial_{\theta_1} - \sin(p)\partial_{\theta_2}) \\ &= h_1\mu + \rho\nu[\cos(p)\cos(p) + \sin(p)\sin(p)] \\ &= \frac{h_1\rho'}{\rho' h_1 - \rho h_1'} + \frac{-\rho h_1'}{\rho' h_1 - \rho h_1'} \\ &= 1. \end{aligned}$$

Computing $\mathrm{d}\tilde{\alpha}$, one can immediately drop the $\mathrm{d}r$-terms, as they evaluate to 0 on R.

$$\mathrm{d}\tilde{\alpha}(R, \cdot) = h_1(r)\mathrm{d}\beta_\partial(R, \cdot) - \rho(r)[\sin(p)\mathrm{d}p \wedge \mathrm{d}\theta_1 + \cos(p)\mathrm{d}p \wedge \mathrm{d}\theta_2](R, \cdot)$$

β_∂ and $\mathrm{d}p$ are 0 on ∂_{θ_i}, $\mathrm{d}\beta_\partial(R_B) = 0$, p is 0 on R_B

$$= \rho(r)\left[\sin(p)\mathrm{d}\theta_1(R)\mathrm{d}p + \cos(p)\theta_2(R)\mathrm{d}p\right]$$

$\mathrm{d}\theta_i$ is 0 on R_B

$$\begin{aligned} &= \rho(r)\nu(r)\left[\sin(p)\cos(p) - \cos(p)\sin(p)\right]\mathrm{d}p \\ &= 0 \end{aligned}$$

Outside $\mathcal{U}$,

$$R = \cos(p)\partial_{\theta_1} - \sin(p)\partial_{\theta_2}.$$

Therefore,

$$\tilde{\alpha}(R) = \beta(R) + \mathrm{d}p(R) + \cos(p)\mathrm{d}\theta_1(R) - \sin(p)\mathrm{d}\theta_2(R)$$

β and $\mathrm{d}p$ are 0 on ∂_{θ_i}

$$\begin{aligned} &= [\cos(p)\mathrm{d}\theta_1 - \sin(p)\mathrm{d}\theta_2](\cos(p)\partial_{\theta_1} - \sin(p)\partial_{\theta_2}) \\ &= \cos(p)\cos(p) + \sin(p)\sin(p) \\ &= 1 \end{aligned}$$

Also,

$$\mathrm{d}\tilde{\alpha}(R) = \mathrm{d}\beta(R, \cdot) - \sin(p)\mathrm{d}p \wedge \mathrm{d}\theta_1(R, \cdot) - \cos(p)\mathrm{d}p \wedge \mathrm{d}\theta_2(R, \cdot)$$

β and $\mathrm{d}p$ are 0 on ∂_{θ_i}

$$\begin{aligned} &= \sin(p)\mathrm{d}\theta_1(R)\mathrm{d}p + \cos(p)\mathrm{d}\theta_2(R)\mathrm{d}p \\ &= \sin(p)\cos(p)\mathrm{d}p - \cos(p)\sin(p)\mathrm{d}p \\ &= 0 \end{aligned}$$

$\square$

This computation is needed in the proof of lemma 4, cf. [**BGMZ22**, Lemma 2.8]

5. Surgery

Carrying out the constructions of the previous sections, one starts with the Milnor open book decomposition of S^{2n-1}, turns it into a contact manifold via the Thurston-Winkelnkemper construction and then constructs a contact structure on $S^{2n-1} \times T^2$. However, a contact structure on S^{2n+1} is needed!

This is where surgery is useful: In general, surgery is a procedure to change manifolds in a controlled way (see [**Mil61**]). The rough idea is that the product manifold $S^p \times S^q$ can be considered either

$$\text{as the boundary of } S^p \times D^{q+1}, \tag{1}$$

or

$$\text{as the boundary of } D^{p+1} \times D^q. \tag{2}$$

Hence, given any imbedding of $S^p \times D^{q+1}$ in a manifold M, the interior can be removed and replaced with $D^{p+1} \times S^q$. This can be done in a smooth way (i.e. preserving the smooth structure of the manifold).

In the situation of this paper, $S^{2n-1} \times T^2$ has to be changed into S^{2n+1}. This can be done via surgery and the details will be explained further below. However, preserving the smooth structure isn't enough: The contact structure has to be preserved, too. Fortunately, there is suitable h-principle that allows to realize all the required surgeries as contact surgeries.

5.1. Smooth surgery.

For the details of how to realize surgery in a smooth way, consult [**Mil61**, paragraph 1]. Surgery is a quite general operation, in fact [**Mil61**, Theorem 1] states that two manifolds can be transferred into one another by a sequence of surgeries if and only if they belong to the same cobordism class. In particular, the Stiefel-Whitney numbers (and hence orientability) are preserved under surgery.

Let W^n be a manifold, $\lambda \in \pi_p(W)$ a homotopy class and $f_0 : S^p \to W \in \lambda$. When can λ be killed? According to [**Mil61**, Lemma 3], a homotopy group can be killed if $n \geq 2p+1$ and the induced S^p-bundle $f_0^*(TW)$ is trivial. In this case, all homotopy groups $\pi_i(W)$ for $i < p$ stay unchanged, but $\pi_p(W)$ changes to $\pi_p(W)/G$ where G is a subgroup containing λ.

Pick any base point $x \in M \times S^1$ and then embed $x \times S^1 \hookrightarrow M \times T^2$. As the normal bundle to $S^1 \hookrightarrow T^2$ is trivial, it follows that also the normal bundle to $S^1 \hookrightarrow M \times T^2$ is trivial. This yields the desired embedding and so one can kill the respective homotopy class (which in this case is the same as a homology class because $\pi_1 = \mathbb{Z}^2$ is already an abelian group). It turns out that the second generator in H_1 can be killed like that, too. H_2 is then isomorphic to π_2 by the Hurewicz theorem. After proving that the respective 2-surgery is possible, one has created a manifold whose H_1 and H_2 homology groups are zero. In fact, one can prove that all homology groups are zero after the described surgery operations. FInally, the following two lemmata conclude the proof that the resulting manifold is diffeomorphic to a sphere.

Lemma 2. *A simply connected homology sphere is homeomorphic to the sphere.*

PROOF. Let M^n be a simply connected CW-complex that is a homology sphere, i.e. $H^0(M) = \mathbb{Z}$ and $H^n(M) = \mathbb{Z}$. As M is simply connected, i.e. $\pi_1(M) = 0$, the Hurewicz theorem states that

$$\pi_k(M) = 0 \; \forall 1 < k < n \text{ and } \pi_n(M) = H^n(M) = \mathbb{Z}.$$

As a result, M is a homotopy sphere. Consider a generator $f : S^n \to M$ of $\pi_n(M)$. On π_0 level, it maps one connected component to one connected component, so here the induced map is obviously bijective. On π_k level with $0 < k < n$, it just maps 0 to 0 which is an isomorphism. On π_n level, one needs to show that

$$f_* : \mathbb{Z} = \pi_n(S^n) \to \pi_n(M) = \mathbb{Z}$$

is an isomorphism. The identity map is a generator for $\pi_n(S^n)$. Now $f_*(\mathrm{id}) = f$, so one obtains a generator of $\pi_n(M)$. A map from $\mathbb{Z} \to \mathbb{Z}$ that sends 1 to 1 is a group isomorphism. These considerations show that f is a weak homotopy equivalence. As a smooth manifold, M is a CW-complex. By Whiteheads theorem it follows that f is a homotopy equivalence, so the generalized Poincaré conjecture shows that M is homeomorphic to the sphere. □

LEMMA 3. *A simply connected homology sphere that bounds a homology ball is diffeomorphic to the sphere.*

PROOF. According to the last lemma, the homology sphere M is homotopy equivalent to a sphere.

As W is a simply connected homology ball, all homotopy groups $\pi_{\geq 1}$ are 0 by the Hurewicz theorem. Taking any constant map on W therefore is a homotopy equivalence by Whiteheads theorem, i.e. W is contractible. Then, cut out a ball inside W and obtain a cobordism W' from a sphere to M. One can prove, then, that W' is an h-cobordism. Thus, by the h-cobordism theorem (Ranicki, Theorem 1.9), M is diffeomorphic to a sphere. □

5.2. What is an *h*-principle? h-principle stands for homotopy-principle. The term often appears in the following setting: There is an underlying set of topological objects (whatever that may be) T and among them a subset G of geometric (more special) objects. An h-principle would state that every object $x \in T$ is homotopic to an object $x \in G$. For example, there is an h-principle for overtwisted contact structures: Any manifold that admits an almost contact structure will also admit an overtwisted contact structure in the same homotopy class, i.e. every homotopy class of almost contact structures contains an overtwisted contact structure.

5.3. The *h*-principle for this specific case. In this case, one needs an h-principle guaranteeing that the necessary surgeries can be realized as contact surgeries. As prerequisite for any surgery, an embedding of the neighborhood of a sphere is required. A subcritical contact surgery is posssible if and only if there exists an isotropic embedding of such a sphere. Now by the h-principle for isotropic embeddings [**EM02**, section 12.4], any embedding of a sphere can be realized in an isotropic way, i.e. all of the surgeries can be realized as contact surgeries.

6. Homotopy Class of the Contact Structure

DEFINITION 3. *An almost contact structure is a cooriented hyperplane field η (with an oriented trivial line bundle complementary to η defining the coorientation) and a complex bundle structure J on η. According to* [**Gei08**, Prop 2.4.5]*, the space of complex bundle structures compatible with a symplectic form on η is non-empty and contractible. Hence, it suffices to choose a symplectic form ω on η to determine the almost contact structure up to homotopy (as the space of trivial line bundles complementary to η is non-empty and contractible, too).*

CHAPTER 3

Tightness

1. Proving tightness in general

In general, for a contact structure to be tight means that it's *not overtwisted*, i.e. in dimension 3 it doesn't contain an overtwisted disk. In higher dimensions, the notion of overtwistedness is a little more complicated. In the important paper [**CMP19**], the authors prove the equivalence of many criteria for overtwistedness and thereby provide a good understanding. In any case, it seems easier to prove that a manifold is overtwisted than to prove that it is not. Usually, tightness is proved via holomorphic curves. Gromov and Eliashberg were the first to use that technique [**Gro85, Eli91**] when they proved that fillable manifolds are tight. This has since then often been used as a criterion to show tightness. Recent approaches like contact homology are also based on holomorphic curves techniques.

Maybe I can write something on holomorphic curves here?

2. The proof in this situation

For a given contact manifold, one can define a partial order by the following relation

$$\alpha \geq \beta \Leftrightarrow \exists \text{ smooth function } f : M \to \mathbb{R}_{\geq 1} \text{ s.t. } \alpha = f \cdot \beta.$$

Definition 4 (k-ADC). *cf.* [**BGMZ22**, Definition 2.5] *A contact structure (M, ξ) is called asymptotically dynamically convex, if there is an ordered sequence of contact forms $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_i \geq \ldots$ and a sequence of real numbers $D_i \to \infty$ with the following property: All contractible periodic orbits of the Reeb vector field R_{α_i} with α_i-action $\leq D_i$ are non-degenerate and have degree $> k$.*

Let $n \geq 4$ (for $n = 3$, a similar but more complicated argument gives the same result). According to [**Koe08**], the Brieskorn sphere Σ_{n-1} is index-positive, i.e. for a given contact form α, all contractible periodic orbits are non-degenerate and have positive degree. In particular, by choosing $\alpha_i = \alpha$ and D_i to be any ascending sequence, Σ_{n-1} is 0-ADC. Then, by the following lemma, the Bourgeois manifold $S^{2n-1} \times T^2$ is 1-ADC (in this case it actually shows that it's 2-ADC but one only needs 1-ADC).

Lemma 4. *cf.* [**BGMZ22**, Lemma 2.8] *If the binding of an open book decomposition is k-ADC, then the corresponding Bourgeois manifold is $(k+2)$-ADC.*

The proof uses the specific Reeb dynamics as computed in section section 4.3, but it is omitted here.

According to [**BGMZ22**, Proposition 3.2 5 (c)], 1-ADC contact manifolds are algebraically tight. A manifold is algebraically tight if the contact homology doesn't vanish, i.e. this is the step where the holomorphic curves are hidden.

As algebraic tightness is preserved under contact surgery, the contact structure on S^{2n+1} is algebraically tight, too. Finally, algebraic tightness implies tightness.

DEFINITION 5. *Let* (Σ, j) *be a Riemann surface and* (M, J) *an almost-complex manifold. Then a smooth map*

$$u\colon \Sigma \to M$$

is called J*-holomorphic (or pseudoholomorphic) if its differential at every point is complex-linear, i.e.*

$$Tu \circ j = J \circ Tu.$$

Pseudoholomorphic curves are used because holomorphic curves (where M has to be a complex manifold) are to specific: A lot of manifolds don't even have a complex structure, whereas all symplectic manifolds admit an almost-complex structure.

CHAPTER 4

Non-Fillability

1. Convex hypersurfaces and convex decompositions

DEFINITION 6 (Convex hypersurface). *cf.* [**HH19**, Definition 1.1.4] *A hypersurface $\Sigma \subset (M, \xi)$ is convex if there exists a contact vector field v, i.e., a vector field whose flow preserves ξ, which is transverse to Σ everywhere.*

DEFINITION 7 (Dividing set). *cf.* [**HH19**, Definition 1.2.1] *With notation as above, define the dividing set $\Gamma(\Sigma) := \{\alpha(v) = 0\}$ and $R_\pm(\Sigma) := \{\pm\alpha(v) > 0\}$ as subsets of Σ.*

It turns out that this is a codimension 2 submanifold and $R_\pm(\Sigma)$ are Liouville manifolds.

DEFINITION 8 (ideal Liouville domain). [**Gir20**, Definition 1] *An ideal Liouville domain (F, ω) is a domain F endowed with an ideal Liouville structure ω. This ideal Liouville structure is an exact symplectic form on* $\operatorname{int} F$ *admitting a primitive λ such that: for some (and then any) function $u\colon F \to \mathbb{R}_{\geq 0}$ with regular level set $\partial F = \{u = 0\}$, the product $u\lambda$ extends to a smooth 1-form on F which induces a contact form on ∂F.*

DEFINITION 9 (corresponding Giroux domain). [**MNW13**, Section 5.3] *Given an ideal Liouville domain (F, ω) with primitive λ and function $u\colon F \to \mathbb{R}_{\geq 0}$ as above, the corresponding Giroux domain is given by*

$$F \times S^1_\theta$$

endowed with contact structure

$$\ker(u\mathrm{d}\theta + u\lambda)$$

∂_θ is a contact vector field, as it doesn't change the contact structure. Also, it is transversal to F. Hence, F is a convex surface and there always is a convex decomposition.

Start with a Bourgeois manifold $\mathrm{BO}(\Sigma, \dots)$. Smoothly,

$$\mathrm{BO}(\Sigma, \dots) = \mathrm{OB}(\Sigma, \dots) \times T^2 = \left[\mathrm{OB}(\Sigma, \dots) \times S^1\right] \times S^1 =: V \times S^1.$$

In the convex decomposition of the first factor

$$V = V_+ \bigcup_\Gamma \overline{V}_-,$$

where Γ is the dividing set, $V_\pm$ turn out to be ideal Liouville domains according to [**DG12**, Section 6]. In [**DG12**, Section 5.3], the authors explicitly compute Γ and $V_\pm$ for the Bourgeois construction and obtain

$$\Gamma = \{y = 0\} = p^{-1}(\{0\}) \cup_B p^{-1}(\{\pi\})$$

and

$$V_+ = p^{-1}([0,\pi]) \times S^1, \qquad V_- = p^{-1}([\pi, 2\pi]) \times S^1,$$

i.e. topologically $V_\pm = \Sigma \times D^* S_1$. If $\alpha + x\mathrm{d}\phi + y\mathrm{d}\theta$ is the contact structure on $\mathrm{OB}(\Sigma, \dots)$, then as explained in [**DG12**, Section 5.3], $\alpha + x\mathrm{d}\phi$ is a S^1_ϕ-invariant contact form on Γ,

$$\omega_\pm = \pm \mathrm{d}\left(\frac{\alpha}{y} + \frac{x}{y}\mathrm{d}\phi\right)$$

is an S^1_ϕ-invariant symplectic form on $V_\pm$ and y is a function with zero level set $\pm\Gamma = \partial V_\pm$. Hence, $(V_\pm, \omega_\pm)$ is an ideal Liouville domain with Liouville form

$$\beta_\pm = \pm\left(\frac{\alpha}{y} + \frac{x}{y}\mathrm{d}\phi\right).$$

According to definition 9, $V_\pm \times S^1_\theta$ endowed with the contact structure

$$\ker(y\mathrm{d}\theta + y\beta_\pm) = \alpha + x\mathrm{d}\phi + y\mathrm{d}\theta$$

is the corresponding Giroux domain. Clearly, this is just the restriction of the open book contact structure. Hence, the whole procedure actually yields a splitting into two Giroux domains

$$\mathrm{OB}(\Sigma, \dots) = V_+ \times S^1_\theta \bigcup_{\Gamma \times S^1_\theta} V_- \times S^1_\theta$$

2. Surgery along embedded Giroux domains

Given a embedded Giroux domain, this section describes a procedure to remove its interior and "blow down" the resulting boundary. We will refer to the procedure as "clean cut-out" of the Giroux domain.

These boundary components are always of the form $B = S^1 \times M$. Topologically, blowing down is equivalent to simply gluing in $D^2 \times M$.

This operation can be performed in a way that respects the contact structure, provided that $S^1 \times M$ has a neighborhood of the form $[0, \epsilon)_s \times S^1_t \times M$ where $\alpha_M + s\mathrm{d}t$ defines a contact form. In general this holds by [**MNW13**, Lemma 5.1] if the boundary components are ξ-round hypersurfaces, but it is also possible to show the existence of that neighborhood directly.

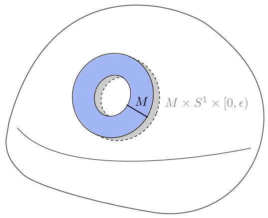
Let D be the disk of radius ϵ in $\mathbb{R}^2$. The map

$$\Psi\colon (re^{i\theta}, m) \mapsto (s = r^2, t = \theta, m)$$

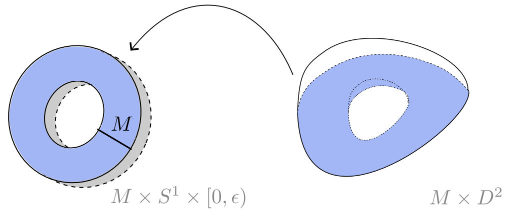
is a diffeomorphism from $(D \setminus \{0\}) \times M$ to $(0, \epsilon)_s \times S^1_t \times M$ s.t.

$$\Psi^*(\alpha_M + s\mathrm{d}t) = \Psi^*(\alpha_M) + \Psi^*(s\mathrm{d}t) = \alpha_M + r^2\mathrm{d}\theta,$$

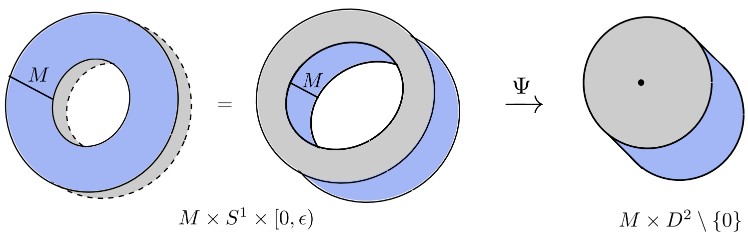
where the latter contact form can be extended to all of $D^2 \times M$. In summary: If there is such a neighborhood of $M \times S^1$ as described above, we can glue $D \times M$ to $V \setminus B$ to get a new contact manifold in which B has been replaced by M.



(A) The blue surface is a ξ-round boundary surface with (gray) neighborhood



(B) Topologically, the blowdown corresponds to gluing $M \times D^2$ on top of the blue surface.



(C) Visualization of Ψ: The neighborhood of the blue area $M \times S^1 \times \{0\}$ on the left is given by $M\times$ the gray annulus. In the middle, twist it so that the blue area sits on the inside. Applying Ψ is simply retracting the inner circle of the gray annulus to a point. The blowdown effectively reduces the inner blue surface $M \times S^1$ to $M = M \times \{0\} \subset M \times D^2$.

Boundary components of Giroux domains are ξ-round hypersurfaces ([**MNW13**, Section 5.3]), Therefore, after removal of a Giroux domain, its boundary components can be blown down. These two steps together form the clean cut-out.

3. A capping cobordism

The following sections closely follow [**BGM22**, Section 6].

For now, consider the easier case where no additional surgery is carried out on the Bourgeois manifold. That makes the proof for the homological obstruction theorem easier to understand. It can be generalized to the case with surgery later. The idea is to cap off both sides (i.e. V_+ and V_-) of the convex decomposition using the surgery procedure introduced in the last section. Topologically, this will result in $\Gamma \times S^2$ and it will turn out to carry a stable Hamiltonian structure.

In [**MNW13**, Section 6], where the clean cut-out is first introduced, it is shown that it corresponds to a symplectic cobordism. The setting there is actually more general: The authors consider a Giroux domain where already some of the boundary components have been blown down. In this case, the situation is simpler: The Giroux domain $V_\pm \times S^1 \subset \mathrm{BO}(\Sigma, \dots)$ is directly obtained from the corresponding ideal Liouville domain $V_\pm$ by round contactization. Its boundary is given by

$$\partial V_\pm \times S^1 = \Gamma \times S^1.$$

Now, considering the manifold $(V_+ \cup_\Gamma V_-) \times S^1$, perform a clean cut-out on both ends, i.e. remove the interior of $V_\pm \times S^1$ and blow down the boundary $\Gamma \times S^1$. In order to do this properly, it is necessary to consider a neighborhood $\Gamma \times (-\delta, \delta)$ around the dividing set and instead of $V_\pm$ take $V'_\pm \coloneqq V_\pm \setminus \Gamma \times [0, \pm\delta]$. Topologically, blowing down $\Gamma \times S^1$ yields $\Gamma \times D^2$. As this is done on both sides, the result is

$$\Gamma \times D^2 \cup_{\Gamma \times S^1 \times -\delta} \Gamma \times (-\delta, \delta) \cup_{\Gamma \times S^1 \times +\delta} \Gamma \times D^2 \cong \Gamma \times S^2.$$

This whole surgery procedure can be realized as applying a certain symplectic cobordism. This is made precise in the following

Lemma 5. *(follows from* [**MNW13**, Theorem 6.1], *cf.* [**BGM22**, Lemma 6.1]*)* *There is a symplectic cobordism (W, ω) with negative (i.e. concave) contact boundary $\partial_- W = V \times S^1$ and weakly convex positive boundary $\partial_+ W = \Gamma \times S^2$ where $\Gamma = \partial V$. Moreover, there is a tubular neighborhood $(-\delta, 0] \times \partial_+ W$ such that ω is of the form $d(e^t \alpha_\Gamma) + \omega_S$, where $t \in (-\delta, 0]$ and*

- *ω_S is an area form on S^2,*
- *α_Γ is a contact form on Γ.*

Lastly, there are symplectic submanifolds $W_\pm$, diffeomorphic to $V_\pm$, such that $W \setminus W_\pm$ deformation retracts onto its negative boundary, and such that $W_\pm$ intersect transversely, positively and in exactly one point, each symplectic sphere in the previously described neighborhood of the positive boundary $\partial_+ W$.

The submanifolds $W_\pm$ are basically $V_\pm \times 0 \subset V_\pm \times D^2$, i.e. the co-cores of the attached handles. If these submanifolds are taken out, the effect of the capping is topologically trivial. Therefore, the remaning cobordism is topologically just $\partial_- W \times [0, 1]$. This clearly deformation retracts to the negative boundary. One can also construct a slightly modified deformation retract where instead of removing the co-cores, we remove slight push-offs (i.e. submanifolds that are obtained by shifting $W_\pm$ aside by a section of its normal bundle). Then, $W_\pm$ retracts to $V_\pm$.

4. Holomorphic spheres

W, like any symplectic manifold, admits a compatible almost complex structure J. Now, the maps

$$\begin{aligned} u_{(t,q)} \colon S^2 &\to W, \\ y &\mapsto (t, q, y) \in (-\delta, 0] \times \underbrace{\Gamma \times S^2}_{= \partial_+ W}. \end{aligned}$$

are J-holomorphic curves.

Consider the connected component $\mathcal{M}$ of the moduli space of J-holomorphic spheres containing these $u_{t,q}$ and denote with $\mathcal{M}_*$ the corresponding marked moduli space with evaluation map

$$\mathrm{ev} \colon \mathcal{M}_* \to W.$$

Define the respective Gromov compactifications $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}_*}$. Gromov proved that any sequence of holomorphic curves with finite energy bound converges to a holomorphic curve that possibly has nodes or bubbles. For now, just think of the Gromov compactification as adding all such limits of sequences in our moduli space to the space. The finite energy bound follows from the fact that the homology class of the curve doesn't change.

$$E(u_k) = \int_{S^2} u_k^*(\omega) = \langle [u_k], [\omega] \rangle = \langle [S^2], [\omega] \rangle = \text{const}\,.$$

The following lemma describes the compactified moduli space close to the boundary.

Lemma 6. *(*[**BGM22**, Lemma 6.3]*, cf.* [**MNW13**, page 334]*) Any curve in $\overline{\mathcal{M}_*}$ that intersects a small enough collar neighborhood*

$$(-\epsilon, 0] \times \Gamma \times S^2$$

is already a reparametrization of a sphere $u_{t,q}$ (i.e. equivalent in the moduli space).

Now consider the evaluation map $\mathrm{ev}\colon \overline{\mathcal{M}_*} \to W$ close to the boundary (in the neighborhood of lemma 6). There, the map is a diffeomorphism: Take any point $p = (t, q, y) \in (-\epsilon, 0] \times \Gamma \times S^2$. Then, by construction there exists at least one holomorphic sphere $(u_{t,q})$ that intersects this point and even has this point as marked point. Now take any such curve. By the lemma, it is equivalent in the moduli space $\overline{\mathcal{M}_*}$ to $u_{t,q}$, hence the map is bijective. The proof of smoothness is skipped here.

According to lemma 5, any such curve intersects $W_\pm$ transversely, positively and in exactly one point. This gives rise to an intersection map

$$\mathcal{I}^\pm \colon \overline{\mathcal{M}} \to W_\pm.$$

As $\partial W_\pm \subset \partial W$, a collar neighborhood of $\partial W_\pm$ is contained in a collar neighborhood of ∂W. Therefore, the uniqueness lemma applies and it turns out that this map is a diffeomorphism close to the boundary $\partial\overline{\mathcal{M}}$. We skip the smoothness and show that it's bijective.

- Surjectivity: For $x \in W_\pm \cap (-\epsilon, 0] \times \Gamma \times S^2$, consider the coordinate representation $x = (t, q, y)$. Clearly, a preimage is given by $u_{t,q} \in \overline{\mathcal{M}_*}$ with marked point x.
- Injectivity: The preimage is unique due to the uniqueness lemma.

5. Proof sketch of the homological obstruction theorem

The homology obstruction theorem states that any homology class in the dividing set Γ that dies in the filling W, will also die in $V_\pm$. In order to make this more precise, define the inclusion maps $i'\colon \Gamma \to \Gamma \times \{\mathrm{pt}\} \subset \Gamma \times S^2$ where pt is a point on the equator of S^2 and i as the composition of i' with the inclusion $\Gamma \times S^2 \hookrightarrow \partial W \hookrightarrow W$.

Now, consider a homology class $z \in H_*(\Gamma, \mathbb{Q})$ such that $i_*(z) = 0 \in H_*(W, \mathbb{Q})$. Then the obstruction theorem states that already $f_{\pm,*}(z) = 0$.

Start with a cycle $z \in H_*(\Gamma, \mathbb{Q})$ with $i_*(z) = 0 \in H_*(W, \mathbb{Q})$. Consequently, there is a homology class $\sigma \in H_*(W, \mathbb{Q})$ with boundary $i_*(z) = \partial\sigma$. Any homology class in Γ that dies in W can be described in this way. Intuitively, σ can be pushed into $W_\pm$ in order to kill z in $H_*(W_\pm, \mathbb{Q}) = H_*(V_\pm, \mathbb{Q})$.

More precisely, consider the commutative diagram

$$\begin{tikzcd}
\Gamma \arrow[r, hook, "i'"] \arrow[rrrr, bend left, "i"] \arrow[ddrrr, "f_\pm"'] & \Gamma \times S^2 = \partial W \arrow[r, "\mathrm{ev}_\partial^{-1}", "\sim"'] & \partial\overline{\mathcal{M}}_* \arrow[r, hook, "j"] & \overline{\mathcal{M}}_* \arrow[r, "\mathrm{ev}"] \arrow[d, two heads, "\pi"] \arrow[dd, bend right, "\mathcal{I}_\pi^\pm"'] & W \\
& & & \overline{\mathcal{M}} \arrow[d, "\mathcal{I}^\pm"] & \\
& & & W_\pm \simeq V_\pm &
\end{tikzcd}$$

where j and π are the obvious inclusion respectively forgetful maps. Recall that ev_∂ is the evaluation map restricted to the boundary of the moduli space. As proved in the last section, this map is a diffeomorphism close to the boundary and in particular on the boundary. $\mathcal{I}^\pm \colon \overline{\mathcal{M}} \to W_\pm$ is the intersection map defined in the last section. Finally, choose $\mathcal{I}_\pi^\pm$ and $f_\pm$ such that the diagram commutes.

Consider the set preimage $\mathrm{ev}^{-1}(\sigma) \subset \overline{\mathcal{M}}_*$. As the boundary $\partial\sigma$ is contained in ∂W, the evaluation map is a diffeomorphism and a neighborhood of $\partial\sigma$ is preserved under taking the preimage. Away from this neighborhood, a slight perturbation might be necessary to make the resulting set a cycle in $\overline{\mathcal{M}}_*$. Clearly, the resulting cycle τ has boundary $\partial\tau \cong b$ and $\partial\tau$ is contained in the boundary $\partial\overline{\mathcal{M}}_*$.

The following equation holds:

$$\begin{aligned}
\partial I^\pm_{\pi,*}(\tau) &= I^\pm_{\pi,*}(\partial\tau) \\
&= I^\pm_{\pi,*}(\mathrm{ev}^{-1}_{\partial,*}(\partial\sigma)) \\
&= I^\pm_{\pi,*}(\mathrm{ev}^{-1}_{\partial}(i_*(z))) \\
&= I^\pm_{\pi,*}\left((j \circ \mathrm{ev}^{-1}_{\partial} \circ i')_*(z)\right) \\
&= f^\pm(z)
\end{aligned}$$

As a result, $[f^\pm(z)] = \left[\partial I^\pm_{\pi,*}(\tau)\right] = 0 \in H_*(W_\pm, \mathbb{Q})$.

6. How to generalize the obstruction theorem to the case with surgery

7. Existence of the required homology class

The obstruction theorem says that any homology class in the dividing set that dies in the filling W, will also die in $V_\pm$. The goal is to prove that there is a nontrivial homology class $\beta \subset V_\pm$ that comes from Γ, but dies in W. This contradicts the obstruction theorem and shows that there can be no filling.

Lemma 7. *There exists a homology class in Γ that survives in $V_\pm$.*

Proof. By construction,

$$V_{\pm} = \Sigma \times D^* S^1$$

and

$$\Gamma = \partial V_{\pm} = \partial\Sigma \times D^* S^1 \cup \Sigma \times \left(S^1 \sqcup S^1\right).$$

First construct a homology class in $\Sigma \times S^1$. The Künneth-formula for coefficient ring $\mathbb{Z}$ shows that

$$0 \to \bigoplus_{i+j=n} H_i(\Sigma) \otimes_{\mathbb{Z}} H_j(S^1) \to H_n(\Sigma \times S^1) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^{\mathbb{Z}}\left(H_i(\Sigma), H_j(S^1)\right) \to 0 \tag{3}$$

is an exact split sequence. Consequently,

$$H_n(V_{\pm}) \cong \bigoplus_{i+j=n} H_i(\Sigma) \otimes_{\mathbb{Z}} H_j(S^1) \oplus \bigoplus_{i+j=n-1} \mathrm{Tor}_1^{\mathbb{Z}}\left(H_i(\Sigma), H_j(S^1)\right).$$

In section 2, homotopy and homology properties of Σ (the Milnor fiber) were described. In particular, $H_{n-1}(\Sigma^{2n-2}) \neq 0$, as this is precisely half the dimension of the page. Together with $H_1(S^1) \neq 0$, this proves that the term $H_{n-1}(\Sigma) \otimes_{\mathbb{Z}} H_1(S^1)$ is nonzero. As a result, there is a nonzero homology class in $H_n(\Sigma \times S^1) \hookrightarrow H_n(\Gamma)$. Inlusion into $V_{\pm}$ doesn't kill this homology group, as $S^1 \hookrightarrow D^* S^1$ is a homotopy equivalence, i.e. induces an isomorphism on homology level. This proves the existence of a nontrivial homology class in Γ that survives in $V_{\pm}$. □

Now, in the surgery case, with several modifications the inclusion $V_+ \to W$ factors via $V_+ \hookrightarrow \partial W = S^{2n+1} \hookrightarrow W$. On homology level, one gets $H_n(V_+) \hookrightarrow H_n(S^{2n+1}) \hookrightarrow H_n(W)$. As the middle term is 0, any homology class coming from V_+ dies in the filling W. This is the desired contradiction.

Bibliography

[BGM22] Jonathan Bowden, Fabio Gironella, and Agustin Moreno. Bourgeois contact structures: Tightness, fillability and applications. *Inventiones mathematicae*, 230(2):713–765, November 2022.

[BGMZ22] Jonathan Bowden, Fabio Gironella, Agustin Moreno, and Zhengyi Zhou. Tight contact structures without symplectic fillings are everywhere. 2022.

[Bou02] Frédéric Bourgeois. Odd dimensional tori are contact manifolds. *International Mathematics Research Notices*, 2002(30):1571–1574, January 2002.

[CMP19] Roger Casals, Emmy Murphy, and Francisco Presas. Geometric criteria for overtwistedness. *Journal of the American Mathematical Society*, 32(2):563–604, April 2019.

[DG12] Fan Ding and Hansjörg Geiges. Contact structures on principal circle bundles. *Bulletin of the London Mathematical Society*, 44(6):1189–1202, December 2012.

[Eli91] Yakov Eliashberg. Filling by holomorphic discs and its applications. volume 2 of *London Mathematical Society Lecture Note Series*, pages 45–68. Cambridge University Press, Cambridge, 1991.

[EM02] Y. Eliashberg and Nikolai M. Mishachev. *Introduction to the \$h\$-Principle*. American Mathematical Soc., 2002. Google-Books-ID: ZNyFAwAAQBAJ.

[Gei08] Hansjörg Geiges. *An Introduction to Contact Topology*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, March 2008.

[Gir20] Emmanuel Giroux. Ideal Liouville Domains, a cool gadget. *Journal of Symplectic Geometry*, 18(3):769–790, May 2020.

[Gro85] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Inventiones mathematicae*, 82(2):307–347, June 1985.

[HH19] Ko Honda and Yang Huang. Convex hypersurface theory in contact topology. Technical report, July 2019. ADS Bibcode: 2019arXiv190706025H Type: article.

[Koe08] Otto van Koert. Contact homology of Brieskorn manifolds. 20(2):317–339, March 2008.

[Lut79] Robert Lutz. Sur la géométrie des structures de contact invariantes. *Annales de l'Institut Fourier*, 29(1):283–306, 1979.

[Mil61] John Milnor. A procedure for killing homotopy groups of differentiable manifolds. *Proceedings of Symposia in Pure Mathematics*, pages 39–55, 1961.

[Mil69] John Milnor. *Singular Points of Complex Hypersurfaces (AM-61), Volume 61*. Princeton University Press, March 1969.

[MNW13] Patrick Massot, Klaus Niederkrüger, and Chris Wendl. Weak and strong fillability of higher dimensional contact manifolds. *Inventiones mathematicae*, 192(2):287–373, May 2013.