

Homotopically Standard Tight Non-fillable Contact Structures on the Sphere

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results by Bowden, Gironella, Moreno and Zhou

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Background

Contact topology

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Fillability: *fillable* contact mflds are boundaries of symplectic mflds.

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Eliashberg, Borman–Eliashberg–Murphy:

Dichotomy: Rigidity vs. Flexibility.

- **tight** (*rigid/geometric*);
- **overtwisted** (*flexible/topological*).

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Theorem (Eliashberg–Gromov)

Fillable contact manifolds are tight.

Converse is false (Etnyre–Honda, Massot–Niederkrueger–Wendl).

Existence and classification

Topological obstruction: *almost* contact structure, i.e. reduction of structure group to $U(n) \times \mathbb{1}$.

Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))

Almost contact manifolds are contact, where the contact structure is overtwisted.

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Tight manifolds

How can we understand **tight** contact manifolds?

Contact topology: fillability

Hierarchy of fillability:

$$\begin{array}{ccccccc} \{Stein\} & \overset{\textcircled{1}}{=} & \{Weinstein\} & \overset{\textcircled{2}}{\subsetneq} & \{Liouville\} & \overset{\textcircled{3}}{\subsetneq} & \{strong\} \\ & & & & \overset{\textcircled{4}}{\subsetneq} & \{weak\} & \overset{\textcircled{5}}{\subsetneq} \{tight\} \end{array}$$

- $dim = 3$: $\textcircled{1}$ Cieliebak–Eliashberg, $\textcircled{2}$ Bowden, $\textcircled{3}$ Ghiggini, $\textcircled{4}$ Eliashberg, $\textcircled{5}$ Etnyre–Honda.
- $dim \geq 5$: $\textcircled{1}$ Cieliebak–Eliashberg,
 $\textcircled{2}$ Bowden–Crowley–Stipsicz, $\textcircled{3}$ Zhou,
 $\textcircled{4}$ Bowden–Gironella–M., $\textcircled{5}$ Massot–Niederkrüger–Wendl.

Contact structures on spheres

First step in classification: contact structures on spheres.

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Theorem (Eliashberg, '91)

On S^3 , it is the unique tight contact structure.

In particular, no tight and non-fillable contact structures on S^3 .

Tight and non-fillable structures in $\dim \geq 5$

Theorem (Bowden–Gironella–M.–Zhou '22-'24)

In $\dim \geq 7$, if M admits a tight structure, it also admits a tight and non strongly-fillable structure, in the same almost contact class.

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In $\dim = 5$, the same holds, if the first Chern class vanishes.

We get infinitely many if $\dim \geq 11$, and M is Weinstein fillable with torsion first Chern class.

Case of spheres

The general theorem follows by connected sum with an “exotic” sphere:

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General remarks

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- Suggests that higher-dimensional contact phenomena should occur independently of underlying smooth topology.

Liouville but not Weinstein

Theorem (Bowden–Gironella–M.–Zhou '22-'24)

In $\dim \geq 7$, if M admits a Weinstein fillable structure with torsion first Chern class, then it also admits infinitely many Liouville but non-Weinstein fillable structures in the same formal class.

Case of spheres

This again follows by connected sum with an "exotic" sphere:

Theorem (Bowden–Gironella–M.–Zhou '22)

For any $n \geq 3$, there exist infinitely many Liouville fillable contact structures on \mathbb{S}^{2n+1} that are not Weinstein fillable, and are homotopically standard.

Open questions

- Is there a Liouville but not Weinstein fillable structure on \mathbb{S}^5 ?
- Is there a strong but not Liouville fillable structure on \mathbb{S}^{2n+1} , $n \geq 2$?

Tight and non-fillable spheres

Giroux correspondence

Giroux: Contact structures are *supported* by open books.

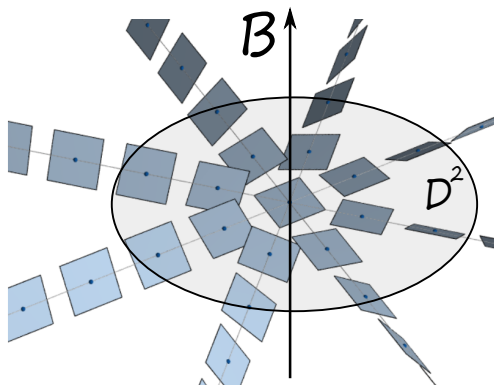


Figure: Supported contact structure.

Bourgeois contact structures

Theorem (Bourgeois '02)

Open book supporting $(M, \xi) \rightsquigarrow$ contact structure on $M \times \mathbb{T}^2$.

These are \mathbb{T}^2 -equivariant.

Geometric construction

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- Milnor open book on $\mathbb{S}^{2n-1} \rightsquigarrow$ Bourgeois manifold on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$
 \rightsquigarrow two 1-surgeries = $\mathbb{S}^{2n-1} \times \mathbb{S}^2 \rightsquigarrow$ one 2-surgery = \mathbb{S}^{2n+1} .

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Claim: $(\mathbb{S}^{2n+1}, \xi_{ex})$ is tight and non-fillable.

Tightness and fillability from algebraic perspective

Contact homology algebra $CHA(Y)$ (homology well-defined by Pardon).

Definition

- 1 Y is *algebraically* tight if $CHA(Y) \neq 0$.
- 2 Y is *algebraically* fillable if there is a DGA augmentation of $CHA(Y)$ at the chain level.

Similarly for *algebraically* overtwisted/non-fillable.

Note: This definition is well-defined, due to functoriality of the DGA, even though homotopy type of chain level is not.

Formal algebraic properties

Lemma

- ① *Algebraically tight* \Rightarrow *tight*.
- ② *Algebraically fillable* \Rightarrow *algebraically tight*.
- ③ *Algebraically non-fillable* \Rightarrow *non-fillable*.
- ④ *1-ADC* \Rightarrow *algebraically tight*.

1-ADC is an *index-positivity* condition (Lazarev, Zhou).

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Facts:

- 1 (Advek '22) tight contact manifolds can be algebraically overtwisted, in dim 3.
- 2 Algebraic tightness is preserved under surgeries. Tightness is also, in dim 3 (Wand '14).
- 3 1-ADC binding of fillable open book \Rightarrow 1-ADC algebraically fillable
Bourgeois manifold \Rightarrow algebraically tight.
- 4 E.g. Milnor A_k -singularity open book has 1-ADC binding.

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Note: Heuristically, there is a priori *many* choices of open book. This suggests *many* non-standard structures. However, distinguishing is subtle.

Non-fillability

Non-fillability of $(\mathbb{S}^{2n+1}, \xi_{ex})$ can be proven via:

- 1 Homological obstruction and cobordisms as in [Bowden–Gironella–M.], building on [Massot–Niederkrüger–Wendl].
- 2 Symplectic cohomology computations as in [Zhou].

Homological obstructions

Observation: Bourgeois manifolds have convex decomposition

$$M \times \mathbb{T}^2 = (M \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_{\phi} \overline{V}_- \times \mathbb{S}^1,$$

with $V_{\pm} = \Sigma \times D^*\mathbb{S}^1$, Σ = page of the open book, ϕ = monodromy.

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Theorem (Bowden–Gironella–M.)

$M = V \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_{\phi} \overline{V_-} \times \mathbb{S}^1$ with convex decomposition, $N = \partial V_{\pm}$ dividing set. If W is a symplectic filling of M , then

$$H_*(N) \rightarrow H_*(V_{\pm}) \rightarrow H_*(W),$$

induced by inclusion. Then second map is injective on image of the first.

Namely, if a homology class in N survives in V_{\pm} , then it survives in the filling.

Idea of proof

- Capping cobordism from M to $N \times \mathbb{S}^2$ with a SHS, via handles H_{\pm} with co-core V_{\pm} .

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- Second factor gives moduli space of spheres \mathcal{M}_* with evaluation map $ev : \mathcal{M}_* \rightarrow W$.
- Spheres intersect H_{\pm} precisely once \rightsquigarrow intersection map $\mathcal{I}_{\pm} : \mathcal{M}_* \rightarrow V_{\pm}$.
- If $\sigma \subset W$ satisfies $\partial\sigma = c$ with c cycle in N , then $b = \mathcal{I}_{\pm} ev^{-1}(\sigma)$ bounds σ in V_{\pm} . □

Homological obstructions

Fact:

- ① If $\dim \geq 7$, subcritical surgeries on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ can be pushed away from dividing set to V_+ .

$\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex})$ still has a dividing set N ,

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 $\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex})$ still has a dividing set N ,
 with $H_n(N) \neq 0$.
- ② Homological obstruction theorem persists under surgery away from dividing set (capping cobordisms).

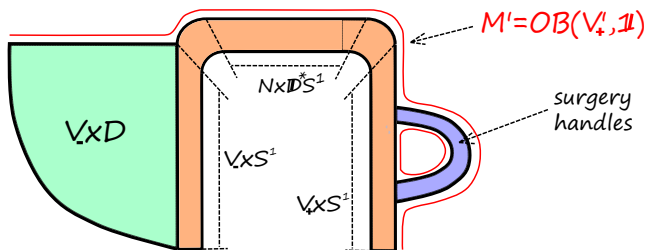


Figure: Capping cobordism.

End of the proof: W filling of $(\mathbb{S}^{2n+1}, \xi_{ex}) \Rightarrow$ Homological obstruction:

$$0 \neq H_n(N) \hookrightarrow H_n(W).$$

However, this factors as

$$0 \neq H_n(N) \rightarrow H_n(\mathbb{S}^{2n+1}) = 0 \rightarrow H_n(W),$$

contradiction.

Remarks

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- ③ **Dimension 5:** Needs careful *flexible* version of the homological obstruction theorem.
- ④ **Symplectic cohomology:** Capping cobordisms reach $\partial(V \times \mathbb{D}^2)$. Zhou's computations of $SH_+(\partial(V \times \mathbb{D}^2))$ and SH_+ computations of Brieskorn spheres as by [Kwon–van-Koert] can be used.

Liouville but not Weinstein fillable spheres

Geometric construction

One example:

- $V = N^{2n-1} \times [-1, 1]$ Liouville domain (MNW) $\rightsquigarrow M = \partial(V \times \mathbb{D}^2)$, which is ADC (Lazarev, Zhou).

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- Stacking W on top of $V \times \mathbb{D}^2 \rightsquigarrow (\mathbb{S}^{2n+1}, \xi)$ has Liouville filling $X^{2n+2} = V \times \mathbb{D}^2 \cup W$.

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X' another filling, ADC $\rightsquigarrow H_*(W) \cong H_*(W')$ (Zhou) \Rightarrow **not** Weinstein fillable.

Thank you!