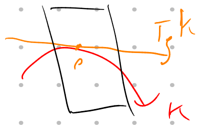
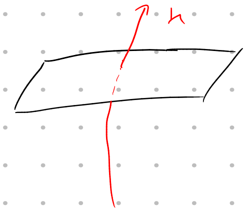
3. Knots in contact 3-manifolds

An embedding $K: S^1 \hookrightarrow (M^3, \xi)$ is called



* Legendrian knot $:\Leftrightarrow$ $TK \subset \xi$
* Transverse knot $:\Leftrightarrow$ $TK \oplus \xi = TM$



K_0 is isotopic to K_1 $:\Leftrightarrow$
$\exists K_t$, $t \in I$, K_t is Legendrian (transverse) $\forall t$.

Notation: $K \subset (M, \xi)$ for the isotopy class

Examples: (0) $S^1 = \mathbb{R}/2\pi\mathbb{Z} \ni t \xmapsto{K} (3\sin(t)\cos(t), \cos(t), \sin^3(t)) \in (\mathbb{R}^3, \xi_{st} = \ker(x\,dy + dz))$
is a Legendrian unknot.

it's Legendrian: take the derivative, plug it into α
if it's $0 \Rightarrow$ it's Legendrian

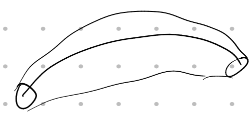
(1) $S^1 \ni \theta \mapsto (\theta, 0, 0) \in (S^1 \times \mathbb{R}^2, \xi_n = \ker(\cos(n\theta)\,dx - \sin(n\theta)\,dy))$ is Legendrian.

(2) $S^1 \ni \theta \mapsto (\theta, 0, 0) \in (S^1 \times \mathbb{R}^2, \xi = \ker(d\theta + r^2\,d\varphi))$ is transverse.

3.1) Neighborhood & isotopy extension theorems

Thm 1: (1) Let $K \subset (M, \xi)$ be Legendrian. $\Rightarrow \exists$ tubular NBHD V_K of K in M s.t.

$\forall r > 0$ $\forall n \in \mathbb{Z} \setminus \{0\}$: $(V_K, \xi) \overset{\text{cont}}{\cong} (S^1 \times D^2_r, \xi_n)$



$K \mapsto S^1 \times 0$

(2) Let $K \subset (M, \xi)$ be transverse. $\Rightarrow \exists$ tubular NBHD V_K of K in M
& $\exists\, \varepsilon > 0$ s.t. $(V_K, \xi) \overset{\text{cont}}{\cong} (S^1 \times D^2_\varepsilon, \ker(d\theta + r^2\,d\varphi))$

$K \mapsto S^1 \times 0$

Proof: HW Hint: Use Moser trick as in Darboux's theorem ∎

Thm 2: Let $K_t: S^1 \hookrightarrow (M^3, \xi)$ be an isotopy of Legendrian (transverse) knots. $\Rightarrow \exists$ isotopy of contactomorphisms

$\psi_t: (M, \xi) \xrightarrow{\cong \text{ cont}} (M, \xi)$ s.t.

$\psi_0 = id$ & $\psi_t \circ K_0 = K_t$

Proof: Construct φ_t as flow of a contact vectorfield from a hamiltonian function.
(HW) ▢

9.2 The front projection

Let $K \subset (S^3, \xi_{ot})$ be a knot

Thm 2.4 $\Rightarrow$ We can see $K \subset (\mathbb{R}^3, \xi_{ot})$

$(x,y,z) \longmapsto (y,z)$ <u>front projection</u>

$S^1 \ni t \longmapsto (x(t), y(t), z(t)) \in (\mathbb{R}^3, \xi_{ot} = \ker(x\,dy + dz))$

is Legendrian $\Leftrightarrow$ $0 = \alpha(TK) = x(t) \cdot y'(t) + z'(t) \quad \forall t$

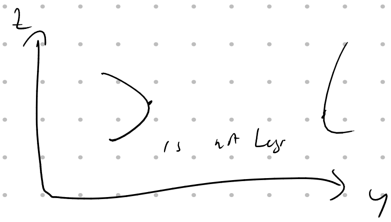
If K is generic (i.e. $y'(t) = 0$ only for finitely many t)

$\Rightarrow x(t) = -\frac{z'(t)}{y'(t)} = -\frac{dz}{dy}$ (if $y'(t) = 0 \Rightarrow z'(t) = 0$

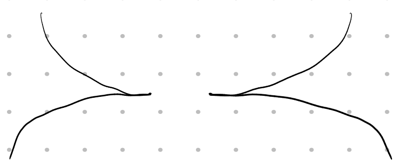
i.e. we can recover K from it's front projection via ⊛

Certain configurations do <u>not</u> appear as front projections of Legendrian knots.

* if $y' = 0 \Rightarrow z' = 0 \Rightarrow$ there can't be vertical tangencies



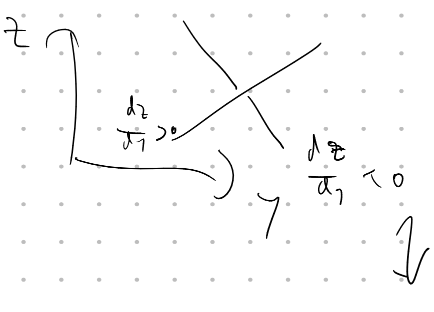
Instead: <u>semicubical cusps!</u>



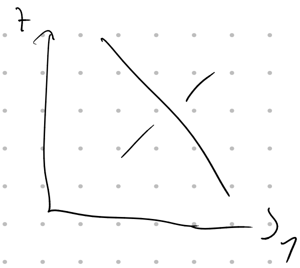
$(x(t), y(t), z(t)) = (t, t^2, -\frac{2}{3}t^3)$

(every Legendrian knot has at least 2 of them)

* No crossing as follows:



Instead:

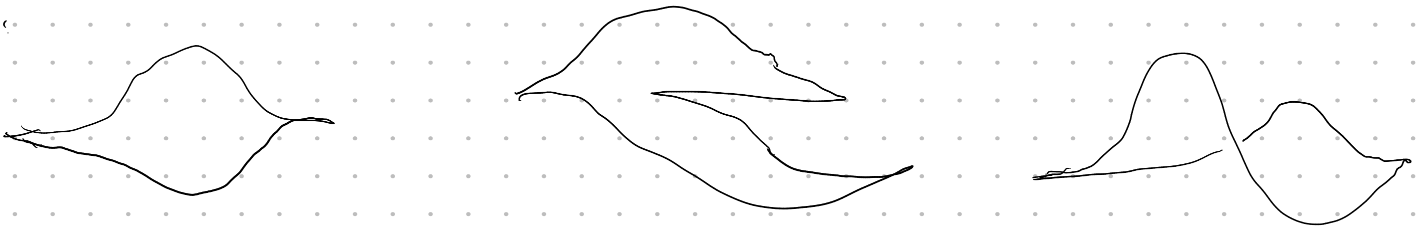


✓

$\Rightarrow$ Any such front projection determines a unique Legendrian knot.

Conventions: We draw Legendrian knots in the front projection

Examples:



is isotopic to one of those on the left

trefoil trefoil

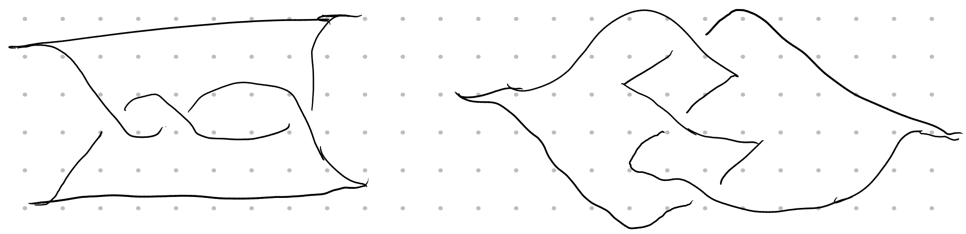
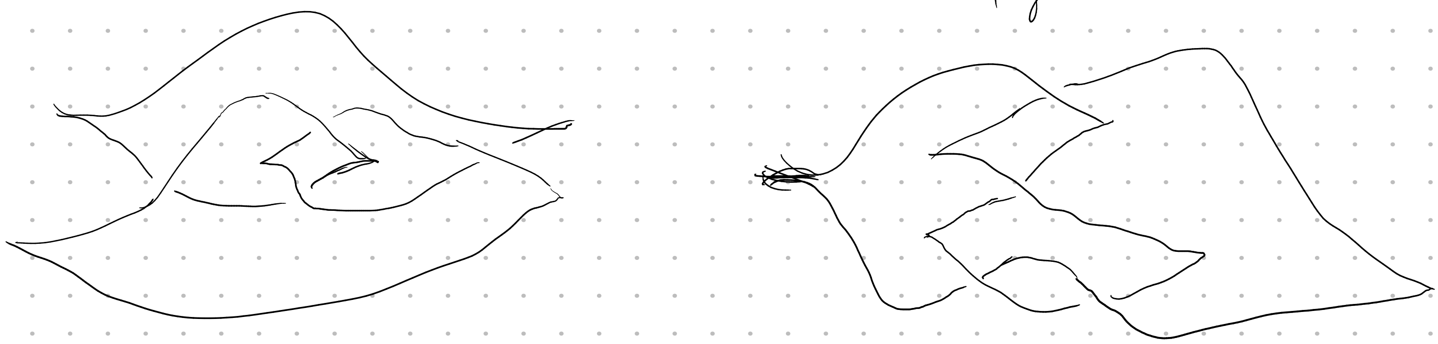


figure 8 figure 8



Corollary 3: For every smooth knot $K \subset (\mathbb{R}^3, \xi_{st})$ $\exists$ isotopic knot $K' \subset (\mathbb{R}^3, \xi_{st})$ Leg.

Proof: front of K