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Theoretische Physik III: Elektrodynamik | Wintersemester 2020/2021

2. Übungsblatt

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1. Lösung: Delta Distribution

(a)

$$g(x) = x\theta(x) \tag{1}$$
$$g'(x) = x\theta'(x) + \theta(x) \tag{2}$$
$$g''(x) = 2\theta'(x) + x\theta''(x) \tag{3}$$

Da $\theta'(x) = \delta(x)$ kann man zeigen, dass

$$x\theta''(x) = x\delta'(x) = -\delta(x) \tag{4}$$

Man verwendet dafür, dass für alle Testfunktionen $h(x)$ gilt:

$$\int \mathrm{d}x\, h(x)x\delta'(x) = -\int \mathrm{d}x\, \delta(x)\,[h(x)x]' \tag{5}$$
$$= -\int \mathrm{d}x\, \delta(x)\,\big[(h(x) + xh'(x)\big) \tag{6}$$
$$= -\int \mathrm{d}x\, \delta(x)h(x) \tag{7}$$

(b) Sei $f(x)$ eine Testfunktion.
Angenommen $a > 0$:

$$\int_c^d \mathrm{d}x\, f(x)\delta(ax) = \int_{c\cdot a}^{d\cdot a} \frac{\mathrm{d}x}{\mathrm{d}y}\mathrm{d}y\, f\left(\frac{y}{a}\right)\delta(y) \tag{8}$$
$$= \frac{1}{a}\int_{c\cdot a}^{d\cdot a} \mathrm{d}y\, f\left(\frac{y}{a}\right)\delta(y) \tag{9}$$
$$= \begin{cases} \frac{1}{a}f(0) & \text{falls } 0 \in [ac, ad] \\ 0 & \text{falls } 0 \notin [ac, ad] \end{cases} \tag{10}$$

Angenommen $a < 0$:

$$\int_c^d \mathrm{d}x\, f(x)\delta(ax) = \int_{c\cdot a}^{d\cdot a} \frac{\mathrm{d}x}{\mathrm{d}y}\mathrm{d}y\, f\left(\frac{y}{a}\right)\delta(y) \tag{11}$$

$$= \frac{1}{a}\int_{c\cdot a}^{d\cdot a} \mathrm{d}y\, f\left(\frac{y}{a}\right)\delta(y) \tag{12}$$

$$= -\frac{1}{a}\int_{d\cdot a}^{c\cdot a} \mathrm{d}y\, f\left(\frac{y}{a}\right)\delta(y) \tag{13}$$

$$= \begin{cases} -\frac{1}{a}f(0) & \text{falls } 0 \in [ad, ac] \\ 0 & \text{falls } 0 \notin [ad, ac] \end{cases} \tag{14}$$

(c) Sei $\epsilon > 0$ so klein, dass in $[x_i - \epsilon, x_i + \epsilon]$ nur eine Nullstelle x_i liegt. Dann ist

$$f(x) = f(x_i) + f'(x_i)(x - x_i) + \dots \tag{15}$$

$$= f'(x_i)(x - x_i) + \dots \tag{16}$$

und

$$\int_{x_i-\epsilon}^{x_i+\epsilon} \mathrm{d}x\delta(f(x))g(x) = \int_{x_i-\epsilon}^{x_i+\epsilon} \mathrm{d}x\delta(f'(x_i)(x - x_i))g(x) \tag{17}$$

$$= \int_{\epsilon}^{\epsilon} \mathrm{d}y\delta(f'(x_i)y)g(y + x_i) \tag{18}$$

$$= \int_{\epsilon}^{\epsilon} \mathrm{d}y\frac{1}{|f'(x_i)|}\delta(y)g(y + x_i) \tag{19}$$

$$= \frac{g(x_i)}{|f'(x_i)|} \tag{20}$$

Daraus folgt, dass

$$\int \mathrm{d}x\delta(f(x))g(x) = \sum_i \frac{g(x_i)}{|f'(x_i)|} \tag{21}$$

(d) (i) Es gilt

$$\lim_{\epsilon\to 0} f_\epsilon(x) = 0 \qquad \text{für } x \neq 0 \tag{22}$$

und

$$\int_{-\infty}^{\infty} \mathrm{d}x\, f_\epsilon(x)g(x) = \int_{-\epsilon}^{\epsilon} \mathrm{d}x\, \frac{1}{2\epsilon}g(x) \tag{23}$$

$$= \int_{-\epsilon}^{\epsilon} \mathrm{d}x\, \frac{1}{2\epsilon}[g(0) + g'(0)x + \dots] \tag{24}$$

$$= g(0) + \mathcal{O}(\epsilon^2) \xrightarrow{\epsilon\to 0} 0 \tag{25}$$

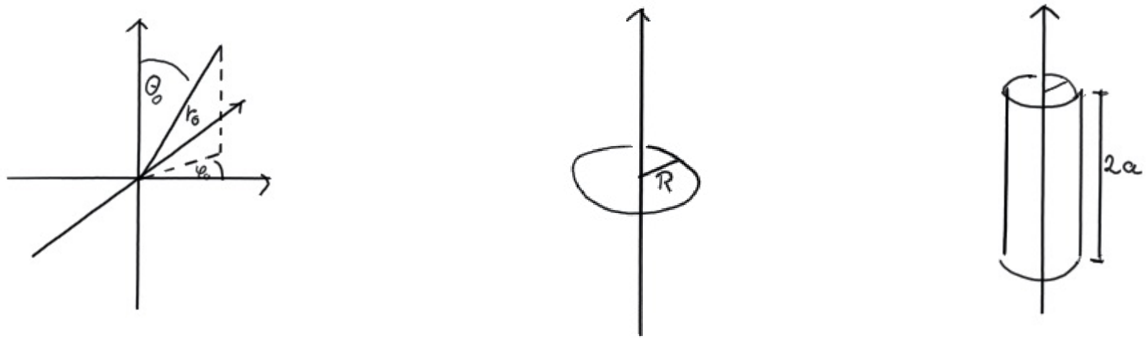

Abbildung 1: Ladungsverteilungen

(ii) Es gilt

$$\lim_{\epsilon\to 0} f_\epsilon(x) = \begin{cases} 0 & \text{für } x \neq 0 \\ \infty & \text{für } x = 0 \end{cases} \tag{26}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, f_\epsilon(x) g(x) = \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{1}{\sqrt{2\pi\epsilon}} \mathrm{e}^{-\frac{x^2}{2\epsilon}} g(x) \tag{27}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}y\, \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} g(y\sqrt{\epsilon}) \tag{28}$$

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}x\, f_\epsilon(x) g(x) = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}y\, \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} g(y\sqrt{\epsilon}) \tag{29}$$

$$= g(0) \int_{-\infty}^{\infty} \mathrm{d}y\, \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} \tag{30}$$

$$= g(0) \tag{31}$$

2. Lösung: Ladungsverteilungen

Kugelkoordinaten

$$\boldsymbol{x} = r \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \qquad \mathrm{d}^3x = r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \tag{32}$$

Polarkoordinaten

$$\boldsymbol{x} = r \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix} \qquad \mathrm{d}^3x = \rho \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}z \tag{33}$$

(a)

$$\rho(\boldsymbol{x}) = N \cdot Q \cdot \delta(r - r_0)\delta(\phi - \phi_0)\delta(\theta - \theta_0) \tag{34}$$

Die Normierung N folgt aus:

$$
\begin{aligned}
Q &= \int \mathrm{d}^3x\, \rho(\boldsymbol{x}) && (35)\\
&= \int \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi\, r^2 \sin\theta \rho(r,\theta,\phi) && (36)\\
&= \int \mathrm{d}^3x\, \rho(\boldsymbol{x}) && (37)\\
&= r_0^2 \sin\theta_0 \cdot N \cdot Q && (38)\\
&\to N = \frac{1}{r_0^2 \sin\theta_0} && (39)
\end{aligned}
$$

(b)

$$\rho(\boldsymbol{x}) = \frac{Q}{4\pi R^2}\delta(r-R) \tag{40}$$

Man erhält die Normierung wie in (a) mit

$$\int_0^\pi \int_0^{2\pi} \sin\theta \mathrm{d}\theta\mathrm{d}\phi = 4\pi \tag{41}$$

(c)

$$\rho(\boldsymbol{x}) = Q \cdot N \cdot \delta(z)\theta(R-r) \tag{42}$$

Die Normierung folgt aus:

$$
\begin{aligned}
\int \mathrm{d}r\mathrm{d}\phi\mathrm{d}z\, \rho(r,\phi,z) r &= \int_0^{2\pi} \int \mathrm{d}\phi\mathrm{d}r\, Q \cdot N \cdot \theta(R-r) r && (43)\\
&= Q \cdot N \cdot \frac{R^2}{2} \cdot 2\pi && (44)\\
&\to N = \frac{1}{R^2\pi} && (45)
\end{aligned}
$$

Hinweis:

$$
\begin{aligned}
\int \mathrm{d}x\, \theta(x) x &= \left[\theta(x)\frac{x^2}{2}\right] - \int \mathrm{d}x\, \delta(x)\frac{x^2}{2} && (46)\\
&= \frac{x^2}{2} && (47)
\end{aligned}
$$

(d)

$$\rho(\boldsymbol{x}) = Q \cdot N \cdot \delta(r-b)\theta(a-|z|) \tag{48}$$

Normierung folgt aus:

$$
\begin{aligned}
Q &= \int \mathrm{d}r\mathrm{d}\phi\mathrm{d}z\, r\rho(r,\phi,z) && (49)\\
&= b\int_0^{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\phi\mathrm{d}z\, Q \cdot N \cdot \theta(a-|z|) && (50)\\
&= b \cdot 2\pi \cdot Q \cdot N \cdot 2a && (51)
\end{aligned}
$$

3. Lösung: Radialfeld

(a)

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z \tag{52}$$
$$= \sin\theta\cos\phi\partial_x + \sin\theta\sin\phi\partial_y + \cos\theta\partial_z \tag{53}$$
$$\frac{1}{r}\partial_\theta = \frac{1}{r}\left(\frac{\partial x}{\partial\theta}\partial_x + \frac{\partial y}{\partial\theta}\partial_y + \frac{\partial z}{\partial\theta}\partial_z\right) \tag{54}$$
$$= \cos\theta\cos\phi\partial_x + \cos\theta\sin\phi\partial_y - \sin\theta\partial_z \tag{55}$$
$$\frac{1}{r\sin\theta}\partial_\phi = \frac{1}{r\sin\theta}\left(\frac{\partial x}{\partial\phi}\partial_x + \frac{\partial y}{\partial\phi}\partial_y + \frac{\partial z}{\partial\phi}\partial_z\right) \tag{56}$$
$$= -\sin\phi\partial_x + \cos\phi\partial_y \tag{57}$$

Wenn wir nun die Einheitsvektoren der Kugelkoordinaten in kartesischen Einheitsvektoren ausdrücken folgt:

$$\hat{\boldsymbol{e}}_r\partial_r + \frac{1}{r}\hat{\boldsymbol{e}}_\theta\partial_\theta + \frac{1}{r\sin\theta}\hat{\boldsymbol{e}}_\phi\partial_\phi \tag{58}$$

$$\begin{aligned}
=&[\sin\theta\cos\phi\hat{\boldsymbol{e}}_x + \sin\theta\sin\phi\hat{\boldsymbol{e}}_y + \cos\theta\hat{\boldsymbol{e}}_z]\,(\sin\theta\cos\phi\partial_x + \sin\theta\sin\phi\partial_y + \cos\theta\partial_z)\\
&+[\cos\theta\cos\phi\hat{\boldsymbol{e}}_x + \cos\theta\sin\phi\hat{\boldsymbol{e}}_y - \sin\theta\hat{\boldsymbol{e}}_z]\,(\cos\theta\cos\phi\partial_x + \cos\theta\sin\phi\partial_y - \sin\theta\partial_z)\\
&+[-\sin\phi\hat{\boldsymbol{e}}_x + \cos\phi\hat{\boldsymbol{e}}_y]\,(-\sin\phi\partial_x + \cos\phi\partial_y)
\end{aligned} \tag{59}$$

$$\begin{aligned}
=&\left[\sin\theta^2\cos\phi^2 + \cos\theta^2\cos\phi^2 + \sin\phi^2\right]\hat{\boldsymbol{e}}_x\partial_x\\
&+\left[\sin\theta^2\sin\phi^2 + \cos\theta^2\sin\phi^2 + \cos\phi^2\right]\hat{\boldsymbol{e}}_y\partial_y\\
&+\left[\cos\theta^2 + \sin\theta^2\right]\hat{\boldsymbol{e}}_z\partial_z\\
&+[\sin\theta\cos\theta\cos\phi - \sin\theta\cos\theta\cos\phi]\,(\hat{\boldsymbol{e}}_x\partial_z + \hat{\boldsymbol{e}}_z\partial_x)\\
&+[\sin\theta\cos\theta\sin\phi - \sin\theta\cos\theta\sin\phi]\,(\hat{\boldsymbol{e}}_y\partial_z + \hat{\boldsymbol{e}}_z\partial_y)\\
&+\left[\sin\theta^2\cos\phi\sin\phi + \cos\theta^2\cos\phi\sin\phi + \sin\phi\cos\phi\right](\hat{\boldsymbol{e}}_x\partial_y + \hat{\boldsymbol{e}}_y\partial_x)
\end{aligned} \tag{60}$$

$$=\hat{\boldsymbol{e}}_x\partial_x + \hat{\boldsymbol{e}}_y\partial_y + \hat{\boldsymbol{e}}_z\partial_z \tag{61}$$

(b) Um die Rotation des Vektorfeldes auszurechnen benötigen wir die Ableitung von $\hat{\boldsymbol{e}}_r$.

$$\partial_r\hat{\boldsymbol{e}}_r = 0 \tag{62}$$
$$\partial_\theta\hat{\boldsymbol{e}}_r = \hat{\boldsymbol{e}}_\theta \tag{63}$$
$$\partial_\phi\hat{\boldsymbol{e}}_r = \sin\theta\hat{\boldsymbol{e}}_\phi \tag{64}$$

$$
\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{V} =& (\hat{\boldsymbol{e}}_r \partial_r + \frac{1}{r}\hat{\boldsymbol{e}}_\theta \partial_\theta + \frac{1}{r\sin\theta}\hat{\boldsymbol{e}}_\phi \partial_\phi) \times V_r \hat{\boldsymbol{e}}_r && (65)\\
=& \hat{\boldsymbol{e}}_r \times (\hat{\boldsymbol{e}}_r \partial_r V_r + V_r \partial_r \hat{\boldsymbol{e}}_r) + \frac{1}{r}\hat{\boldsymbol{e}}_\theta \times (\hat{\boldsymbol{e}}_r \partial_\theta V_r + V_r \partial_\theta \hat{\boldsymbol{e}}_r)\\
&+ \frac{1}{r\sin\theta}\hat{\boldsymbol{e}}_\phi \times (\hat{\boldsymbol{e}}_r \partial_\phi V_r + V_r \partial_\phi \hat{\boldsymbol{e}}_r) \qquad \text{(Produktregel)} && (66)\\
=& \hat{\boldsymbol{e}}_r \times (\hat{\boldsymbol{e}}_r \partial_r V_r + 0) + \frac{1}{r}\hat{\boldsymbol{e}}_\theta \times (\hat{\boldsymbol{e}}_r \partial_\theta V_r + V_r \hat{\boldsymbol{e}}_\theta)\\
&+ \frac{1}{r\sin\theta}\hat{\boldsymbol{e}}_\phi \times (\hat{\boldsymbol{e}}_r \partial_\phi V_r + V_r \sin\theta \hat{\boldsymbol{e}}_\phi) \qquad \text{(Ableitung } \hat{\boldsymbol{e}}_r \text{)} && (67)\\
=& \frac{1}{r}\hat{\boldsymbol{e}}_\theta \times \hat{\boldsymbol{e}}_r \partial_\theta V_r + \frac{1}{r\sin\theta}\hat{\boldsymbol{e}}_\phi \times \hat{\boldsymbol{e}}_r \partial_\phi V_r \qquad \boldsymbol{a} \times \boldsymbol{a} = 0 && (68)\\
=& -\frac{\partial_\theta V_r}{r}\hat{\boldsymbol{e}}_\phi + \frac{\partial_\phi V_r}{r\sin\theta}\hat{\boldsymbol{e}}_\theta && (69)
\end{aligned}
$$

Während in unserem Spezialfall die meisten Terme verschwinden, lautet die allgemeine Form der Rotation in Kugelkoordinaten:

$$
\boldsymbol{\nabla} \times \boldsymbol{V} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(V_\phi \sin\theta) - \frac{\partial V_\theta}{\partial\phi}\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial}{\partial r}(rV_\phi)\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial\theta}\right)\mathbf{e}_\phi
$$