Exercise 11 - Solution

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Homework Problem 11.1 (Examples for generalized Newton) 5 Points

For the nonlinear functions $F\colon \mathbb{R} \to \mathbb{R}$ and the set valued functions $N\colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$ below, find all solutions z^* of the generalized equation

$$0 \in F(z) + N(z)$$

and determine, at which solutions the problem is strongly regular.

(i) $F(z) \coloneqq z^2 - 1$ and $N(z) \coloneqq \{0\}$

(ii) $F(z) \coloneqq z^2 - 1$ and $N(z) \coloneqq \mathbb{R}_\geq$

(iii) $F(z) \coloneqq (z-1)^2$ and $N(z) \coloneqq \mathcal{N}_{\mathbb{R}_\geq}(z)$

Solution.

(i) When $N(z) = \{0\}$ for all $z \in \mathbb{R}$, then the inclusion reduces to the standard root-finding problem

$$F(z) = 0$$

which, in the case of the shifted quadratic function has the obvious solution set $\{-1, 1\}$. Note that the linearized inclusion

$$0 \in F(z^{(k)}) + F'(z^{(k)})(z^{(k+1)} - z^{(k)}) + N(z^{(k+1)}) \tag{11.17}$$

that defines the iterates $z^{(k+1)}$ of the generalized Newton method also reduces to

$$0 = F(z^{(k)}) + F'(z^{(k)})(z^{(k+1)} - z^{(k)}) \tag{0.1}$$

and hence to the standard root finding version of Newton's method.

The perturbed system, that we need to analyze for strong regularity is the linear system

$$\Delta = F(z^*) + F'(z^*)(w - z^*)$$

which, due to regularity of $F'(z*) \in \{-2, 2\}$ for all solutions $z^* \in \{-1, 1\}$ has the unique solution

$$w(\Delta) = z^* + \frac{\Delta - F(z^*)}{F'(z^*)}$$

which depend on Δ Lipschitz continuously with modulus $\frac{1}{|F'(z^*)|} = \frac{1}{2}$, so both solutions are strongly regular.

(ii) When $N(z) = \mathbb{R}_{\geq}$, then the inclusion

$$0 \in F(z) + N(z)$$

is equivalent to the problem

$$F(z) = z^2 - 1 \leq 0,$$

i. e., finding all arguments that produce a nonpositive function value for the shifted quadratic function, which has the obvious solution set $[-1, 1]$. However, the linearized and perturbed system we want to analyze for strong regularity reads

$$F(z^*) - \Delta + F'(z^*)(w - z^*) = (z^*)^2 - 1 - \Delta + 2z^*(w - z^*) \leq 0,$$

i. e.

$$F(z^*) + F'(z^*)(w - z^*) = (z^*)^2 - 1 + 2z^*(w - z^*) \leq \Delta,$$

which is not uniquely solvable for any of the z^* in the solution set.

For $z^* = 0$, this is clear, because $F'(z^*) = 0$, so for $\Delta \geq F(z^*)$, the perturbed system is solved by and $w \in \mathbb{R}$, while it is not solvable at all for $\Delta < F(z^*)$. For $z^* \in (0, 1]$ we can take, e. g., $\Delta = 0$ and observe that

$$F(z^*) + F'(z^*)(w - z^*) = F(z^*) + 2z^*(w - z^*) \leq F(z^*) \leq 0$$

for any $w \leq z^*$. For $z^* \in [-1, 0)$ we can proceed analogously. Accordingly, we can show that moving to the correct side (depending on whether the linearization of F at z^* is monotonically increasing or decreasing) we decrease the functional value so the inclusion remains satisfied.

Note: The simple idea that standard Newton's method is equivalent to generalized Newton with the trivial cone valued map and finds zeros of functions so generalized Newton with a half space as the relevant cone should be able to find solution of sign problems is not necessarily correct. In this example, none of the solutions are isolated so strong regularity will not be satisfied. This is not an application that generalized Newton methods are "designed for".

(iii) Note that the standard quadratic function is no longer shifted down, but to the right with apex at $(1, 0)$. The inclusion problem

$$0 \in F(z) + N(z)$$

means that $z \in \mathbb{R}_{\geq}$ (because otherwise the normal cone is empty), that $F(z) \geq 0$ if $z = 0$ and $F(z) = 0$ if $z > 0$, so the inclusion is equivalent to the nonlinear complementarity problem

$$F(z) \geq 0 \perp z \geq 0.$$

The single zero of $F(z)$ is at $z = 1 \geq 0$ while the single zero of the identity is at $z = 0$ with $F(0) = 1$ so the solution set is $\{0, 1\}$.

The point $z^* = 1$ is not strongly regular, because $F'(1) = 0$, so the perturbed, linear inclusion

$$\Delta \in F(z^*) + F'(z^*)\,(w - z^*) + N(w)$$

reduces to

$$\Delta \in \mathcal{N}_{\mathbb{R}_{\geq}}(w)$$

which, for $\Delta = 0$, is solved by any $w \in \mathbb{R}_{\geq}$.

At the point $z^* = 0$, we have $F(z^*) = 1$ and $F'(z^*) = -2$, so the linear perturbed system becomes

$$\Delta \in 1 - 2w + \mathcal{N}_{\mathbb{R}_{\geq}}(w)$$

For $\Delta \neq 1$, this is uniquely solvable by $w(\Delta) := \frac{1-\Delta}{2} \neq 0$ (the cone collapses to the trivial $\{0\}$ in this case), which Lipschitz continuously depends on Δ with modulous $\frac{1}{2}$. I. e., we can take $\delta < 1$ to show strong regularity.

Note: The two possible solutions are isolated so strong regularity is possible. Only where the tangent degenerates, we loose information, similarly to standard Newton.

(5 Points)

Homework Problem 11.2 (Generalized Newton for nonlinear complementarity problems) 5 Points

Implement the generalized Newton's method for solving inclusion problems of the form

$$0 \in F(z) + N(z),$$

where $F \colon \mathbb{R}^n \to \mathbb{R}^n$ is a function of class C^1 and $N \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map.

Make the termination criterion and the subproblem solver user-supplied parameters and apply your implementation to the problem in homework problem 11.1 item *(iii)*. For the subsystem solves, you can employ a linear complementarity problem solver such as Lemke's method, see, e. g., `https://github.com/AndyLamperski/lemkelcp` and you can choose the termination criterion based on the violation of complementarity in terms of $\min(F(x), 0), \min(x, 0)$ and $|F(x)x|$.

Solution.

The solution to this problem will be delayed.[GM] (5 Points)

Homework Problem 11.3 (Projected CG actually projects) 4 Points

Prove that Line 13 of the projected M-preconditioned CG method Algorithm 13.2 applied to the problem

$$\begin{aligned} &\text{Minimize} && \frac{1}{2} d^\intercal A\, d - b^\intercal d \\ &\text{subject to} && B\, d = c \end{aligned} \tag{13.1}$$

in fact computes the unique projection of the cost functionals M-steepest descent direction onto $\ker(B)$.

Solution.

The algorithm sets $\zeta = Ad - b \in \mathbb{R}^n$ (the residual vector), so the M-steepest descent direction is $-M^{-1}\zeta$. It then solves the linear system

$$\begin{bmatrix} M & B^\intercal \\ B & 0 \end{bmatrix} \begin{pmatrix} p \\ \lambda \end{pmatrix} = - \begin{pmatrix} \zeta \\ 0 \end{pmatrix}. \tag{0.2}$$

The projection problem for the M-steppest descent direction is the quadratic problem

$$\begin{aligned} &\text{Minimize} && \frac{1}{2} \|p - (-M^{-1}\zeta)\|_M^2 \\ &\text{subject to} && B\, p = 0 \end{aligned}$$

which, expanding the squared norm, can be found to be equivalent to

$$\begin{aligned}
\text{Minimize} \quad & \frac{1}{2}p^\intercal M p + (M^{-1}\zeta)^\intercal p + \text{const} \\
\text{subject to} \quad & B\,p = 0.
\end{aligned}$$

Since M is s. p. d., this is a strongly convex problem with a linear constraint satisfying where LICQ is satisfied at all feasible points by assumption (full row rank), i. e., the problem is equivalent to its KKT system (see Lemma 11.2, which is exactly (0.2).

(4 Points)

Homework Problem 11.4 (On the stopping criterion for the projected preconditioned CG) 6 Points

Consider the equality constrained linear-quadratic problem

$$\begin{aligned}
\text{Minimize} \quad & f(d) := \frac{1}{2}d^\intercal A\, d - b^\intercal d \\
\text{subject to} \quad & B\, d = c
\end{aligned} \tag{13.1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric and $B \in \mathbb{R}^{m\times n}$ has full rank. Additionally, let $M \in \mathbb{R}^{n\times n}$ be s. p. d..

(i) Show that if d^* is a solution to (13.1), then $\operatorname{proj}^M_{\ker B}(M^{-1}\nabla f(d^*)) = 0$.

Hint: You can for example argue using convexity.

(ii) Let $Z \in \mathbb{R}^{n\times n-m}$ be an M-orthonormal matrix whose columns span the space $\ker(B)$, assume that $Z^\intercal A Z \in \mathbb{R}^{n-m\times n-m}$ is positive definite and let $d^* \in \mathbb{R}^n$ denote the unique solution to (13.1). Show that

$$\|d - d^*\|_A \le \sqrt{\|(Z^\intercal A Z)^{-1}\|}\; \left\|\operatorname{proj}^M_{\ker B}(\nabla_M f(d))\right\|_M$$

for all $d \in \{d \in \mathbb{R}^n \mid Bd = c\}$.

Solution.

(6 Points)

For the remainder of this exercise we simplify the notation and let $P_M(\cdot)$ denote the projection $\operatorname{proj}^M_{\ker B}(\cdot)$.

First off, note that both (13.1) and the projection problem over $\ker(B)$ are optimization problems for a differentibale objective function $g\colon \mathbb{R}^n \to \mathbb{R}$ over a convex set Ω. We can extend the first order characterization of convexity in Statement (a) in Theorem 2.9 for the constrained case where $\Omega \subsetneq \mathbb{R}^n$ to show that convexity of a differentiable g on Ω is equivalent to

$$g(x) - g(y) \ge g'(y)(x - y)\ \forall x, y \in \Omega.$$

In fact, when g is convex on Ω, let x and y be in Ω, then

$$\underbrace{\frac{1}{t}\left(g(\underbrace{y + t(x - y)}_{\in\Omega}) - g(y)\right)}_{\le g(x)-g(y)} \xrightarrow{t\searrow 0} g'(y)(x - y).$$

Conversely, if $g'(y)(x - y) \le g(x) - g(y) \forall x, y \in \Omega$, then let $x, y \in \Omega$, $z := tx + (1 - t)y$ so adding

$$\begin{aligned} tg(x) - tg(z) &\le tg'(z)(x - z) \\ (1 - t)g(y) - (1 - t)g(z) &\le (1 - t)g'(z)(y - z) \end{aligned}$$

yields

$$tg(x) + (1 - t)g(y) \le g(z) + g'(z)(tx - tz + (1 - t)y - (1 - t)z) = g(z)$$

showing convexity.

Accordingly, in this situation, y^* is a minimizer of g over Ω if and only if $g'(y^*)(x - y^*) \ge 0$ for all $x \in \Omega$ (one implication is standard because of the sign of the finite difference quotients, the other is due to the first order characterization of convexity).

(i) Let d^* denote an optimizer of (13.1). This means that

$$\begin{aligned} (\nabla_M f(d^*), v - d^*)_M &\ge 0 \quad \forall v \in \{v \in \mathbb{R}^n \mid Bv = c\}, \\ (P_M(x) - x, v - P_M(x))_M &\ge 0 \quad \forall x \in \mathbb{R}^n,\ v \in \ker B. \end{aligned}$$

Since d^* is feasible and $\ker B$ is a linear space, this is equivalent to

$$\begin{aligned} (\nabla_M f(d^*), v)_M &= 0 \quad \forall v \in \ker B, \\ (P_M(d) - d, v)_M &= 0 \quad \forall d \in \mathbb{R}^n, v \in \ker B. \end{aligned} \tag{0.3}$$

evaluating the second line in $d = \nabla_M f(d^*)$, we obtain

$$\begin{aligned} (\nabla_M f(d^*), v)_M &= 0 \quad \forall v \in \ker B, \\ (P_M(\nabla_M f(d^*)) - \nabla f(d^*), v)_M &= 0 \quad \forall v \in \ker B, \end{aligned}$$

which means that $(P_M(\nabla_M f(d^*)), v)_M = 0\ \forall v \in \ker B$ and choosing $v = P_M(\nabla f(d^*)) \in \ker B$, we obtain that $\|P_M(\nabla_M f(d^*))\|_M = 0$.

(ii) First off, A may not be positive definite, but A is positive definite on $\ker B$ according to the assumptions, i.e., $\|x\|_A := \sqrt{d^\intercal A d}$ defines a norm on $\ker B$. Since d^* and d are assumed to be feasible, we know that the error $e := d - d^* \in \ker B$, so the norm is well defined.

Note that sufficiency of (0.3) shows that the projection onto the linear subspace must be linear.

Now, let $y \in \mathbb{R}^{n-m}$ such that $e = Zy$ and set the spd matrix $H = Z^\intercal A Z$ (which has an s. p. d. square root) for brevity. Then we have that

$$\begin{aligned}\|e\|_M^2 &= (e,e)_M = (Zy, Zy)_M = (y, Z^\intercal M Z y) = (y,y) = (H^{-\frac{1}{2}} H^{\frac{1}{2}} y, H^{-\frac{1}{2}} H^{\frac{1}{2}} y) \\ &= (H^{-1} H^{\frac{1}{2}} y, H^{\frac{1}{2}} y) \le \|H^{-1}\| \|H^{\frac{1}{2}} y\|^2 = \|H^{-1}\| y^\intercal H y = \|H^{-1}\| \|e\|_A^2.\end{aligned}$$

Additionally, we see that

$$M^{-1} A e = M^{-1}(Ae - b + b) = M^{-1}(Ad - b) - M^{-1}(Ad^* - b) = \nabla_M f(d) - \nabla_M f(d^*)$$

so that $P_M M^{-1} A e = P_M \nabla_M f(d)$.

Accordingly, using the projection orthogonality once again, we obtain

$$\begin{aligned}\|e\|_A^2 = (e, Ae) = (Me, M^{-1}Ae) = (e, M^{-1}Ae)_M &= (e, P_M M^{-1} A e)_M \\ &\le \|e\|_M \|P_M \nabla_M f(d)\|_M \\ &\le \sqrt{\|(Z^\intercal A Z)^{-1}\|} \|e\|_A \|P_M \nabla_M f(d)\|_M.\end{aligned}$$

Please submit your solutions as a single pdf and an archive of programs via moodle.