# Tight and non-fillable contact manifolds are everywhere

Josua Kugler
results by Bowden[1], Gironella[2], Moreno[3], Zhou[4]

Heidelberg University

[1] University of Regensburg
[2] University of Nantes
[3] Heidelberg University
[4] Morningside Center of Mathematics, CAS

# Background

# Contact topology

**Contact topology:** The study of contact manifolds, up to isotopy.

# Contact topology

**Contact topology:** The study of contact manifolds, up to isotopy.

**Fillability:** *fillable* contact mflds are boundaries of symplectic mflds.

# Contact topology

**Contact topology:** The study of contact manifolds, up to isotopy.

**Fillability:** *fillable* contact mflds are boundaries of symplectic mflds.

### Fillability question

Which contact manifolds are **fillable**?

# Contact topology

**Contact topology:** The study of contact manifolds, up to isotopy.

**Fillability:** *fillable* contact mflds are boundaries of symplectic mflds.

**Fillability question**

Which contact manifolds are **fillable**?

Eliashberg, Borman–Eliashberg–Murphy:

**Dichotomy:** Rigidity vs. Flexibility.

- **tight** *(rigid/geometric)*;
- **overtwisted** *(flexible/topological)*.

# Contact topology

**Contact topology:** The study of contact manifolds, up to isotopy.

**Fillability:** *fillable* contact mflds are boundaries of symplectic mflds.

## Fillability question

Which contact manifolds are **fillable**?

Eliashberg, Borman–Eliashberg–Murphy:

**Dichotomy:** Rigidity vs. Flexibility.

- **tight** *(rigid/geometric)*;
- **overtwisted** *(flexible/topological)*.

## Theorem (Eliashberg–Gromov)

*Fillable contact manifolds are tight.*

Converse is false (Etnyre–Honda, Massot–Niederkrueger–Wendl).

# Existence and classification

*Topological* obstruction: *almost* contact structure, i.e. reduction of structure group to $U(n) \times \mathbb{1}$.

**Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))**

*Almost contact manifolds are contact, where the contact structure is overtwisted.*

# Existence and classification

*Topological* obstruction: *almost* contact structure, i.e. reduction of structure group to $U(n) \times \mathbb{1}$.

**Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))**

*Almost contact manifolds are contact, where the contact structure is overtwisted.*

**Tight manifolds**

How can we understand **tight** contact manifolds?

# Contact topology: fillability

**Hierarchy of fillability:**

$$\{Stein\} \overset{①}{=} \{Weinstein\} \overset{②}{\subsetneq} \{Liouville\} \overset{③}{\subsetneq} \{strong\}$$

$$\overset{④}{\subsetneq} \{weak\} \overset{⑤}{\subsetneq} \{tight\}$$

- *dim* $=$ 3: ① Cieliebak–Eliashberg, ② Bowden, ③ Ghiggini, ④ Eliashberg, ⑤ Etnyre–Honda.
- *dim* $\geqslant$ 5: ① Cieliebak–Eliashberg, ② Bowden–Crowley–Stipsicz, ③ Zhou, ④ Bowden–Gironella–M., ⑤ Massot–Niederkrüger–Wendl.

# Contact structures on spheres

**First step in classification:** contact structures on spheres.

## Standard contact structure

The standard contact structure is $(S^{2n-1}, \xi) = \partial(B^{2n}, \omega_{std})$.

# Contact structures on spheres

**First step in classification:** contact structures on spheres.

### Standard contact structure

The standard contact structure is $(S^{2n-1}, \xi) = \partial(B^{2n}, \omega_{std})$.

### Theorem (Eliashberg, '91)

*On $S^3$, it is the unique tight contact structure.*

In particular, no tight and non-fillable contact structures on $S^3$.

# Tight and non-fillable structures in $\dim \geqslant 5$

**Theorem (Bowden–Gironella–M.–Zhou '22-'24)**

*In* $\dim \geqslant 7$*, if* $M$ *admits a tight structure, it also admits a tight and non strongly-fillable structure, in the same almost contact class.*

# Tight and non-fillable structures in $\dim \geqslant 5$

**Theorem (Bowden–Gironella–M.–Zhou '22-'24)**

*In* $\dim \geqslant 7$*, if* $M$ *admits a tight structure, it also admits a tight and non strongly-fillable structure, in the same almost contact class.*

*In* $\dim = 5$*, the same holds, if the first Chern class vanishes.*

We get infinitely many if $\dim \geqslant 11$, and $M$ is Weinstein fillable with torsion first Chern class.

# Case of spheres

The general theorem follows by connected sum with an "exotic" sphere:

**Theorem (Bowden–Gironella–M.–Zhou '22-'24)**

*For every $n \geqslant 2$, the sphere $\mathbb{S}^{2n+1}$ admits a tight, non-fillable contact structure that is homotopically standard.*

# Case of spheres

The general theorem follows by connected sum with an “exotic” sphere:

**Theorem (Bowden–Gironella–M.–Zhou ’22-’24)**

*For every $n \geqslant 2$, the sphere $\mathbb{S}^{2n+1}$ admits a tight, non-fillable contact structure that is homotopically standard.*

Infinitely many if $n \geqslant 5$.

# General remarks

- This is a novel and strictly higher-dimensional phenomenon (false in dim 3).

# General remarks

- This is a novel and strictly higher-dimensional phenomenon (false in dim 3).
- Suggests that higher-dimensional contact phenomena should occur independently of underlying smooth topology.

# Liouville but not Weinstein

**Theorem (Bowden–Gironella–M.–Zhou '22-'24)**

*In* $\dim \geqslant 7$*, if* $M$ *admits a Weinstein fillable structure with torsion first Chern class, then it also admits infinitely many Liouville but non-Weinstein fillable structures in the same formal class.*

# Case of spheres

This again follows by connected sum with an "exotic" sphere:

**Theorem (Bowden–Gironella–M.–Zhou '22)**

*For any $n \geqslant 3$, there exist infinitely many Liouville fillable contact structures on $\mathbb{S}^{2n+1}$ that are not Weinstein fillable, and are homotopically standard.*

## Open questions

- Is there a Liouville but not Weinstein fillable structure on $\mathbb{S}^5$?
- Is there a strong but not Liouville fillable structure on $\mathbb{S}^{2n+1}$, $n \geqslant 2$?

# Tight and non-fillable spheres

# Giroux correspondence

**Giroux:** Contact structures are *supported* by open books.

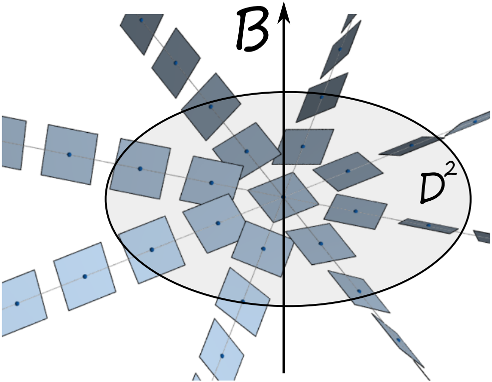


Figure: Supported contact structure.

# Bourgeois contact structures

**Theorem (Bourgeois '02)**

*Open book supporting* $(M, \xi) \rightsquigarrow$ *contact structure on* $M \times \mathbb{T}^2$*.*

These are $\mathbb{T}^2$-equivariant.

# Geometric construction

**Geometric construction:** We now construct **one** tight and non-fillable contact structure on $\mathbb{S}^{2n+1}$.

# Geometric construction

**Geometric construction:** We now construct **one** tight and non-fillable contact structure on $\mathbb{S}^{2n+1}$.

- Milnor open book on $\mathbb{S}^{2n-1}$ $\rightsquigarrow$ Bourgeois manifold on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ $\rightsquigarrow$ two 1-surgeries = $\mathbb{S}^{2n-1} \times \mathbb{S}^2$ $\rightsquigarrow$ one 2-surgery = $\mathbb{S}^{2n+1}$.

# Geometric construction

**Geometric construction:** We now construct **one** tight and non-fillable contact structure on $\mathbb{S}^{2n+1}$.

- Milnor open book on $\mathbb{S}^{2n-1}$ $\rightsquigarrow$ Bourgeois manifold on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ $\rightsquigarrow$ two 1-surgeries = $\mathbb{S}^{2n-1} \times \mathbb{S}^2$ $\rightsquigarrow$ one 2-surgery = $\mathbb{S}^{2n+1}$.
- If $n \geqslant 3$, surgeries are *subcritical* $\rightsquigarrow$ by 'Eliashberg's' h-pple, Weinstein cobordism $\rightsquigarrow$ contact manifold $(\mathbb{S}^{2n+1}, \xi_{ex})$.

# Geometric construction

**Geometric construction:** We now construct **one** tight and non-fillable contact structure on $\mathbb{S}^{2n+1}$.

- Milnor open book on $\mathbb{S}^{2n-1} \rightsquigarrow$ Bourgeois manifold on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ $\rightsquigarrow$ two 1-surgeries = $\mathbb{S}^{2n-1} \times \mathbb{S}^2 \rightsquigarrow$ one 2-surgery = $\mathbb{S}^{2n+1}$.
- If $n \geqslant 3$, surgeries are *subcritical* $\rightsquigarrow$ by 'Eliashberg's' h-pple, Weinstein cobordism $\rightsquigarrow$ contact manifold $(\mathbb{S}^{2n+1}, \xi_{ex})$.

**Claim:** $(\mathbb{S}^{2n+1}, \xi_{ex})$ is tight and non-fillable.

# Tightness and fillability from algebraic perspective

Contact homology algebra $CHA(Y)$ (homology well-defined by Pardon).

## Definition

1. $Y$ is *algebraically* tight if $CHA(Y) \neq 0$.
2. $Y$ is *algebraically* fillable if there is a DGA augmentation of $CHA(Y)$ at the chain level.

Similarly for *algebraically* overtwisted/non-fillable.

**Note:** This definition is well-defined, due to functoriality of the DGA, even though homotopy type of chain level is not.

# Formal algebraic properties

**Lemma**

1. *Algebraically tight* $\Rightarrow$ *tight.*
2. *Algebraically fillable* $\Rightarrow$ *algebraically tight.*
3. *Algebraically non-fillable* $\Rightarrow$ *non-fillable.*
4. 1*-ADC* $\Rightarrow$ *algebraically tight.*

1-ADC is an *index-positivity* condition (Lazarev, Zhou).

# Formal algebraic properties

**Lemma**

1. *Algebraically tight* $\Rightarrow$ *tight.*
2. *Algebraically fillable* $\Rightarrow$ *algebraically tight.*
3. *Algebraically non-fillable* $\Rightarrow$ *non-fillable.*
4. 1*-ADC* $\Rightarrow$ *algebraically tight.*

1-ADC is an *index-positivity* condition (Lazarev, Zhou).

**Facts:**

1. (Advek '22) tight contact manifolds can be algebraically overtwisted, in dim 3.
2. Algebraic tightness is preserved under surgeries. Tightness is also, in dim 3 (Wand '14).
3. 1-ADC binding of fillable open book $\Rightarrow$ 1-ADC algebraically fillable Bourgeois manifold $\Rightarrow$ algebraically tight.
4. E.g. Milnor $A_k$-singularity open book has 1-ADC binding.

# Tightness

Milnor $A_k$ open book $\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex})$ is *tight*.

# Tightness

Milnor $A_k$ open book $\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex})$ is *tight*.

**Note:** Heuristically, there is a priori *many* choices of open book. This suggests *many* non-standard structures. However, distinguishing is subtle.

# Non-fillability

**Non-fillability** of $(\mathbb{S}^{2n+1}, \xi_{ex})$ can be proven via:

1. Homological obstruction and cobordisms as in [Bowden–Gironella–M.], building on [Massot–Niederkrüger–Wendl].
2. Symplectic cohomology computations as in [Zhou].

# Homological obstructions

**Observation:** Bourgeois manifolds have convex decomposition

$$M \times \mathbb{T}^2 = (M \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V}_- \times \mathbb{S}^1,$$

with $V_\pm = \Sigma \times D^*\mathbb{S}^1$, $\Sigma =$ page of the open book, $\phi =$ monodromy.

# Homological obstructions

**Observation:** Bourgeois manifolds have convex decomposition

$$M \times \mathbb{T}^2 = (M \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V}_- \times \mathbb{S}^1,$$

with $V_\pm = \Sigma \times D^*\mathbb{S}^1$, $\Sigma =$ page of the open book, $\phi =$ monodromy.

**Theorem (Bowden–Gironella–M.)**

*$M = V \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V_-} \times \mathbb{S}^1$ with convex decomposition, $N = \partial V_\pm$ dividing set. If $W$ is a symplectic filling of $M$, then*

$$H_*(N) \to H_*(V_\pm) \to H_*(W),$$

*induced by inclusion. Then second map is injective on image of the first.*

Namely, if a homology class in $N$ survives in $V_\pm$, then it survives in the filling.

# Idea of proof

- Capping cobordism from $M$ to $N \times \mathbb{S}^2$ with a SHS, via handles $H_\pm$ with co-core $V_\pm$.

# Idea of proof

- Capping cobordism from $M$ to $N \times \mathbb{S}^2$ with a SHS, via handles $H_\pm$ with co-core $V_\pm$.
- Second factor gives moduli space of spheres $\mathcal{M}_*$ with evaluation map $ev : \mathcal{M}_* \to W$.

# Idea of proof

- Capping cobordism from $M$ to $N \times \mathbb{S}^2$ with a SHS, via handles $H_\pm$ with co-core $V_\pm$.
- Second factor gives moduli space of spheres $\mathcal{M}_*$ with evaluation map $ev : \mathcal{M}_* \to W$.
- Spheres intersect $H_\pm$ precisely once $\rightsquigarrow$ intersection map $\mathcal{I}_\pm : \mathcal{M}_* \to V_\pm$.

# Idea of proof

- Capping cobordism from $M$ to $N \times \mathbb{S}^2$ with a SHS, via handles $H_\pm$ with co-core $V_\pm$.
- Second factor gives moduli space of spheres $\mathcal{M}_*$ with evaluation map $ev : \mathcal{M}_* \to W$.
- Spheres intersect $H_\pm$ precisely once $\rightsquigarrow$ intersection map $\mathcal{I}_\pm : \mathcal{M}_* \to V_\pm$.
- If $\sigma \subset W$ satisfies $\partial\sigma = c$ with $c$ cycle in $N$, then $b = \mathcal{I}_\pm ev^{-1}(\sigma)$ bounds $\sigma$ in $V_\pm$. $\square$

# Homological obstructions

**Fact:**

1. If $\dim \geqslant 7$, subcritical surgeries on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ can be pushed away from dividing set to $V_+$.

   $$\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex}) \text{ still has a dividing set } N,$$

   with $H_n(N) \neq 0$.

# Homological obstructions

**Fact:**

1. If $\dim \geqslant 7$, subcritical surgeries on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ can be pushed away from dividing set to $V_+$.

   $$\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex}) \text{ still has a dividing set } N,$$

   with $H_n(N) \neq 0$.
2. Homological obstruction theorem persists under surgery away from dividing set (capping cobordisms).

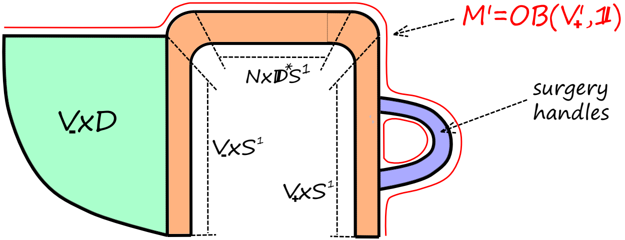


Figure: Capping cobordism.

**End of the proof:** $W$ filling of $(\mathbb{S}^{2n+1}, \xi_{ex}) \Rightarrow$ Homological obstruction:

$$0 \neq H_n(N) \hookrightarrow H_n(W).$$

However, this factors as

$$0 \neq H_n(N) \to H_n(\mathbb{S}^{2n+1}) = 0 \to H_n(W),$$

contradiction.

# Remarks

1. **Homotopically standard:** Fixed $\xi$, Bourgeois manifolds have same almost contact class $\xi \oplus T\mathbb{T}^2$, so suffices with trivial open book. *h*-cobordism theorem gives standard smooth topology on sphere.

# Remarks

1. **Homotopically standard:** Fixed $\xi$, Bourgeois manifolds have same almost contact class $\xi \oplus T\mathbb{T}^2$, so suffices with trivial open book. *h*-cobordism theorem gives standard smooth topology on sphere.
2. **Infinitely many:** connected sums with Lazarev's non-standard flexibly fillable spheres. Distinguished by positive symplectic cohomology (Cieliebak–Oancea).

# Remarks

1. **Homotopically standard:** Fixed $\xi$, Bourgeois manifolds have same almost contact class $\xi \oplus T\mathbb{T}^2$, so suffices with trivial open book. *h*-cobordism theorem gives standard smooth topology on sphere.
2. **Infinitely many:** connected sums with Lazarev's non-standard flexibly fillable spheres. Distinguished by positive symplectic cohomology (Cieliebak–Oancea).
3. **Dimension 5:** Needs careful *flexible* version of the homological obstruction theorem.

# Remarks

1. **Homotopically standard:** Fixed $\xi$, Bourgeois manifolds have same almost contact class $\xi \oplus T\mathbb{T}^2$, so suffices with trivial open book. *h*-cobordism theorem gives standard smooth topology on sphere.
2. **Infinitely many:** connected sums with Lazarev's non-standard flexibly fillable spheres. Distinguished by positive symplectic cohomology (Cieliebak–Oancea).
3. **Dimension 5:** Needs careful *flexible* version of the homological obstruction theorem.
4. **Symplectic cohomology:** Capping cobordisms reach $\partial(V \times \mathbb{D}^2)$. Zhou's computations of $SH_+(\partial(V \times \mathbb{D}^2))$ and $SH_+$ computations of Brieskorn spheres as by [Kwon–van-Koert] can be used.

# Liouville but not Weinstein fillable spheres

# Geometric construction

**One** example:

- $V = N^{2n-1} \times [-1, 1]$ Liouville domain (MNW) $\rightsquigarrow M = \partial(V \times \mathbb{D}^2)$, which is ADC (Lazarev, Zhou).

# Geometric construction

**One** example:

- $V = N^{2n-1} \times [-1,1]$ Liouville domain (MNW) $\rightsquigarrow$ $M = \partial(V \times \mathbb{D}^2)$, which is ADC (Lazarev, Zhou).
- Bowden–Crowley–Stipsicz $\rightsquigarrow$ cobordism $W$ to sphere $\mathbb{S}^{2n+1}$.

# Geometric construction

**One** example:

- $V = N^{2n-1} \times [-1, 1]$ Liouville domain (MNW) $\rightsquigarrow$ $M = \partial(V \times \mathbb{D}^2)$, which is ADC (Lazarev, Zhou).
- Bowden–Crowley–Stipsicz $\rightsquigarrow$ cobordism $W$ to sphere $\mathbb{S}^{2n+1}$.
- Cieliebak–Eliashberg $\rightsquigarrow$ $W$ can be taken flexible Weinstein $\rightsquigarrow$ contact sphere $(\mathbb{S}^{2n+1}, \xi)$, which is ADC.

# Geometric construction

**One** example:

- $V = N^{2n-1} \times [-1, 1]$ Liouville domain (MNW) $\rightsquigarrow M = \partial(V \times \mathbb{D}^2)$, which is ADC (Lazarev, Zhou).
- Bowden–Crowley–Stipsicz $\rightsquigarrow$ cobordism $W$ to sphere $\mathbb{S}^{2n+1}$.
- Cieliebak–Eliashberg $\rightsquigarrow$ $W$ can be taken flexible Weinstein $\rightsquigarrow$ contact sphere $(\mathbb{S}^{2n+1}, \xi)$, which is ADC.
- Stacking $W$ on top of $V \times \mathbb{D}^2 \rightsquigarrow (\mathbb{S}^{2n+1}, \xi)$ has Liouville filling $X^{2n+2} = V \times \mathbb{D}^2 \cup W$.

# Geometric construction

**One** example:

- $V = N^{2n-1} \times [-1, 1]$ Liouville domain (MNW) $\rightsquigarrow M = \partial(V \times \mathbb{D}^2)$, which is ADC (Lazarev, Zhou).
- Bowden–Crowley–Stipsicz $\rightsquigarrow$ cobordism $W$ to sphere $\mathbb{S}^{2n+1}$.
- Cieliebak–Eliashberg $\rightsquigarrow$ $W$ can be taken flexible Weinstein $\rightsquigarrow$ contact sphere $(\mathbb{S}^{2n+1}, \xi)$, which is ADC.
- Stacking $W$ on top of $V \times \mathbb{D}^2 \rightsquigarrow (\mathbb{S}^{2n+1}, \xi)$ has Liouville filling $X^{2n+2} = V \times \mathbb{D}^2 \cup W$.

**Note:** $H_{2n-1}(X) \neq 0$, coming from $[N]$, and $2n - 1 > n + 1$ if $n \geqslant 3$ $\Rightarrow X$ **not** Weinstein (if $n = 2$, it is by Breen–Christian).

# Geometric construction

**One** example:

- $V = N^{2n-1} \times [-1, 1]$ Liouville domain (MNW) $\rightsquigarrow M = \partial(V \times \mathbb{D}^2)$, which is ADC (Lazarev, Zhou).
- Bowden–Crowley–Stipsicz $\rightsquigarrow$ cobordism $W$ to sphere $\mathbb{S}^{2n+1}$.
- Cieliebak–Eliashberg $\rightsquigarrow$ $W$ can be taken flexible Weinstein $\rightsquigarrow$ contact sphere $(\mathbb{S}^{2n+1}, \xi)$, which is ADC.
- Stacking $W$ on top of $V \times \mathbb{D}^2 \rightsquigarrow (\mathbb{S}^{2n+1}, \xi)$ has Liouville filling $X^{2n+2} = V \times \mathbb{D}^2 \cup W$.

**Note:** $H_{2n-1}(X) \neq 0$, coming from $[N]$, and $2n - 1 > n + 1$ if $n \geqslant 3$ $\Rightarrow X$ **not** Weinstein (if $n = 2$, it is by Breen–Christian).

$X'$ another filling, ADC $\rightsquigarrow H_*(W) \cong H_*(W')$ (Zhou) $\Rightarrow$ **not** Weinstein fillable.

Thank you!