

## Abstract

Contact geometry is the study of odd-dimensional smooth manifolds equipped with contact structures, i.e. hyperplane distributions  $\xi = \ker \alpha$  satisfying the contact condition

$$\alpha \wedge (d\alpha)^n \neq 0.$$

While they originally arise in the study of ODEs and in classical mechanics, the topological study of contact manifolds is a more recent and very active field of research.

A manifold can have multiple different contact structures, which can be either rigid (in which case one speaks of a "tight" manifold) or flexible (in the sense that they satisfy an h-principle). The latter contact manifolds are then called overtwisted. A foundational result of Eliashberg and Borman–Eliashberg–Murphy, roughly speaking, states that overtwisted contact manifolds exist in abundance, namely whenever the manifold admits the topological version of a contact structure (an *almost* contact structure), which is a first obvious obstruction. In dimension three, an almost contact structure is simply an oriented 2-plane field.

To illustrate this dichotomy, consider the sphere  $S^3$ . By a result of Eliashberg, it has precisely one tight contact structure. On the other hand, it has infinitely many overtwisted contact structures, corresponding to the infinitely many homotopy classes of 2-plane fields on the 3-sphere. There are other examples where there are infinitely many or no tight contact structures on a contact manifold.

A further interesting property of contact manifolds comes from the fact that contact geometry is the odd-dimensional counterpart to symplectic geometry. Often, it is possible to view a contact manifold as the boundary of a symplectic manifold. Manifolds that are in this sense "fillable" are always tight. The contrary, however, doesn't need to hold and one can ask the question under which conditions such tight, but non-fillable manifolds exist. The first examples of tight and non-fillable contact manifolds were constructed by Etnyre–Honda in dimension three, and by Massot–Niederkrueger–Wendl in higher dimensions.

More recently, Bowden–Gironella–Moreno–Zhou have shown that there exist homotopically standard, non-fillable but tight contact structures on all spheres  $S^{2n+1}$  with  $n \geq 2$ . Starting with a specific open book decomposition of  $S^{2n-1}$ , one can construct a contact form on this manifold using a well-known construction by Thurston–Winkelnkemper. Then, according to Bourgeois, this contact structure can be extended to a tight contact structure on  $S^{2n-1} \times T^2$ . Applying subcritical surgery (preserving the tightness), one can kill the topology of the  $T^2$ -factor and obtain a tight contact structure on  $S^{2n+1}$ . Because of the special way of constructing it, one can show that it is non-fillable, but still homotopically standard.

The goal of my master thesis is to give a streamlined explanation of the results of Bowden–Gironella–Moreno–Zhou, including the necessary background needed to understand the main ideas.

## Preliminaries

### The Bourgeois construction

**Definition 1.** *An open book decomposition of a manifold  $M$  is a pair  $(B, p)$  where the binding  $B$  is a closed codim-2-submanifold of  $M$  and the map  $p$  is a smooth, locally trivial fibration i.e. a fiber bundle?*

$$p : M \setminus B \rightarrow S^1.$$

The fibres  $p^{-1}(\varphi), \varphi \in S^1$  are called the pages. Moreover, it is required that the binding  $B$  have a trivial tubular neighborhood  $B \times D^2$  in which  $p$  is given by the angular coordinate in the  $D^2$ -factor.

**Definition 2** (another definition of open book). *An open book decomposition of a manifold  $M$  is a pair  $(B, p)$ , together with a defining map  $\Phi : M \rightarrow \mathbb{R}^2$  so that each  $z \in \text{int}(D^2)$  is a regular value. Here,  $B \subset M$  is a closed codimension-2 submanifold,  $p : M \setminus B \rightarrow S^1$  is a fiber bundle, and  $\Phi$  is such that  $\Phi^{-1}(0) = B$  and  $p = \Phi/|\Phi|$ .*

**Lemma 1.** *Definition 1 and definition 2 are equivalent.*

*Proof.*

- "  $\implies$  " First, we need to construct a defining map  $\Phi$ . On the trivial tubular neighborhood take  $\Phi$  to be the  $D^2$ -component. Outside this neighborhood, just set  $\Phi = p$ . Then,  $\Phi^{-1}(0) = B, p = \Phi/|\Phi|$  and for all  $z \in \text{int}(D^2)$ , there is always a small neighborhood s.t.  $\Phi$  is just a projection map and hence,  $z$  is a regular value.
- "  $\impliedby$  " We need to show the existence of the trivial tubular neighborhood of  $B$ .  $\Phi^{-1}(D^2) \rightarrow D^2$  is a smooth submersion, as all points are regular points. According to the rank theorem, we can choose local coordinates for a neighborhood of any  $x \in B$  s.t.  $\Phi$  is just a projection to the  $D^2$ -component. Due to the compactness of  $B$ , finitely many such neighborhoods suffice to cover  $B$ . Moreover, the coordinate maps can be chosen in a compatible way and glued together. In total, we find a chart  $\psi$  of a neighborhood  $U$  of  $B$  s.t.

$$\psi(U) \subset \psi(B) \times D^2.$$

As there are only finitely many neighborhoods involved, we find an  $\epsilon > 0$  s.t.  $\Phi^{-1}(D_\epsilon^2) \subset \psi(U)$ . Therefore,  $U$  contains a neighborhood of the form  $B \times D^2$ . This is the desired trivial tubular neighborhood.

□

**Remark 1.** *Let  $M$  be an oriented manifold with an open book decomposition  $(B, p)$ . Choose a parametrization of  $S^1$  (i.e. the direction of  $\partial_\varphi$ ). Then, there are two equivalent ways of defining the orientation of  $B$  and the pages.*

- Via the orientation of the pages: The orientation of  $M$  is the same as the orientation of the page together with  $\partial_\varphi$ .
- Via the orientation of the binding:  $B \times D^2$  is an embedded submanifold of the same dimension as  $M$ . Therefore, it inherits the orientation. As  $B \times D^2$  carries the product orientation and the orientation of  $D^2$  is given by  $\text{rdr} \wedge \text{d}\varphi$ , we can deduce the orientation of  $B$  from the orientation of  $M$ .

In both cases, the orientation of the pages and the binding have to agree in the sense that the induced orientation on the boundary of (the closure) of each page coincides with the orientation of  $B$ . To see that both options are in fact equivalent, choose a positive basis  $\mathfrak{b}$  of the binding  $B$ . Then,  $\mathfrak{b}, \partial_r, \partial_\varphi$  is a positive basis for  $M$  at that point according to the first option. As  $-\partial_r$  is the outer normal vector of the page at this point,  $-\partial_r, \mathfrak{b}$  is a positive basis of the page and so we get  $-\partial_r, \mathfrak{b}, \partial_\varphi$  as a positive basis for  $M$ , which agrees with option 1 because  $B$  is odd-dimensional.

**Definition 3.** Let  $(B, p)$  be an oriented open book decomposition of the oriented manifold  $M$ . A contact structure  $\xi = \ker \alpha$  on  $M$  is said to be **supported** by the open book decomposition  $(B, p)$  of  $M$  if

- (i) the contact form  $\alpha$  induces the positive orientation of  $M$  ( $\alpha \wedge (\text{d}\alpha)^n > 0$ ).
- (ii) the 2-form  $\text{d}\alpha$  induces a symplectic form on each page, defining its positive orientation
- (iii) the 1-form  $\alpha$  induces a positive contact form on  $B$ , i.e.

$$\alpha|_{TB} \wedge (\text{d}\alpha|_{TB})^{(n-2)} > 0.$$

**Theorem 1.** Let  $(M, \xi = \ker \alpha)$  be a closed contact manifold of dimension  $2n-1, n \geq 2$ . One can find an open book decomposition  $(B, p)$  of  $M$  supporting  $\xi$ . According to Bourgeois, ([Bou02]) there is a contact structure  $\tilde{\xi}$  on  $M \times T^2$  (where  $\tilde{\xi}$  massively depends on the choice of open book).

*Proof.* We follow the proof of [Gei08, Thm 7.3.6]. Wlog let  $M$  be connected. The existence of an open book decomposition for  $M$  is the theorem of Giroux-Mohsen as in [Gei08, Thm 7.3.5]. By definition of an open book, there exists a tubular neighborhood  $B \times D^2$  with polar coordinates  $(r, p)$  on the  $D^2$ -part of the binding  $B$ , i.e. the angular coordinate is simply given by  $p : M \setminus B \rightarrow S^1$  in that neighborhood. Now, we want to define smooth functions  $x_1, x_2$  on  $M$  that coincide with the cartesian coordinate functions on  $D^2$  close to the binding  $B$ . In order to do that, choose a smooth function  $\rho(r)$  on  $B \times D^2$ , s.t.

- $\rho = r$  near the binding  $B$ ,
- $\rho'(r) \geq 0$
- $\rho \equiv 1$  near  $B \times \partial D^2$ .

We extend this function to a smooth function  $\rho : M \rightarrow [0, 1]$  by setting  $\rho \equiv 1$  outside  $B \times D^2$ . Now,  $x_1 := \rho \cos p$  and  $x_2 := \rho \sin p$  are the desired smooth functions on  $M$  that coincide with the Cartesian coordinate functions on the  $D^2$ -factor near  $B$ . We compute

$$\begin{aligned} x_1 dx_2 - x_2 dx_1 &= \rho^2 \cos^2 p dp + \rho \cos p \sin p d\rho + \rho^2 \sin^2 p dp - \rho \cos p \sin p d\rho \\ &= \rho^2 (\cos^2 p + \sin^2 p) dp \\ &= \rho^2 dp \end{aligned}$$

and, analogously,

$$dx_1 \wedge dx_2 = \rho d\rho \wedge dp.$$

On  $M \times T^2$ , choose coordinates  $(\theta_1, \theta_2)$  on the torus part of the manifold. Now we have all ingredients together to construct our contact form. Let

$$\tilde{\alpha} := x_1 d\theta_1 - x_2 d\theta_2 + \alpha.$$

This is a well-defined 1-form on  $M \times T^2$  ( $\alpha$  is extended to  $M \times T^2$  as the pullback  $\pi_1^* \alpha$ ) and we can compute the derivative

$$d\tilde{\alpha} = dx_1 \wedge d\theta_1 - dx_2 \wedge d\theta_2 + d\alpha,$$

hence

$$\begin{aligned} (d\tilde{\alpha})^n &= (n-1)(d\alpha)^{n-1} \wedge (dx_1 \wedge d\theta_1 - dx_2 \wedge d\theta_2) \\ &\quad - n(n-1)(d\alpha)^{n-2} \wedge dx_1 \wedge d\theta_1 \wedge dx_2 \wedge d\theta_2. \end{aligned}$$

In order to verify the contact condition, we compute

$$\begin{aligned} \tilde{\alpha} \wedge (d\tilde{\alpha})^n &= (x_1 d\theta_1 - x_2 d\theta_2 + \alpha) \wedge (n-1)(d\alpha)^{n-1} \wedge (dx_1 \wedge d\theta_1 - dx_2 \wedge d\theta_2) \\ &\quad - (x_1 d\theta_1 - x_2 d\theta_2 + \alpha) \wedge n(n-1)(d\alpha)^{n-2} \wedge dx_1 \wedge d\theta_1 \wedge dx_2 \wedge d\theta_2 \\ &= (n-1)(d\alpha)^{n-1} \wedge (x_1 dx_2 - x_2 dx_1) \wedge d\theta_1 \wedge d\theta_2 \\ &\quad + \underbrace{\alpha \wedge (n-1)(d\alpha)^{n-1} \wedge dx_1 \wedge d\theta_1}_{2n\text{-form on } M} - \underbrace{\alpha \wedge (n-1)(d\alpha)^{n-1} \wedge dx_2 \wedge d\theta_2}_{2n\text{-form on } M} \\ &\quad + n(n-1)\alpha \wedge (d\alpha)^{n-2} \wedge dx_1 \wedge d\theta_1 \wedge dx_2 \wedge d\theta_2 \end{aligned}$$

$M$  has dimension  $2n-1$ , i.e. the middle term is 0

$$\begin{aligned} &= (n-1)(d\alpha)^{n-1} \wedge \rho^2 d\phi \wedge d\theta_1 \wedge d\theta_2 \\ &\quad + n(n-1)\alpha \wedge (d\alpha)^{n-2} \wedge \rho d\rho \wedge d\phi \wedge d\theta_1 \wedge d\theta_2 \end{aligned}$$

By condition (ii) of definition 3,  $(d\alpha)^{n-1}$  must be a positive volume form on the pages. As explained in that definition, the orientation on  $M$  is given by  $\partial_\phi$  and the orientation of the page. In particular,  $(d\alpha)^{n-1} \wedge \rho d\phi$  is a positive volume

form on  $M$ . Multiplied with a second  $\rho$ -factor, it vanishes along  $B$ . As  $\theta_1 \wedge \theta_2$  is a positive volume form on  $T^2$ , the first term is non-negative everywhere and positive away from

$$\underbrace{B \times 0}_{\subset B \times D^2 \subset M} \times T^2.$$

Let  $\mathfrak{b}$  be a basis of the binding  $B$  that is positively ordered. Then,  $-\partial_r, \mathfrak{b}$  and (because the binding is odd-dimensional)  $\mathfrak{b}, \partial_r$  are positive bases of the page. Clearly, then,

$$\mathfrak{a} := \mathfrak{b}, \partial_r, \partial_\phi, \partial_{\theta_1}, \partial_{\theta_2}$$

is an ordered basis of  $M \times T^2$ . Using  $\rho'(r) \geq 0$  everywhere, we deduce that  $d\rho(\partial_r)$  is non-negative. Hence, plugging  $\mathfrak{a}$  into the second term, we conclude

$$\begin{aligned} & (n(n-1)\alpha \wedge (d\alpha)^{n-2} \wedge \rho d\rho \wedge d\phi \wedge d\theta_1 \wedge d\theta_2) (\mathfrak{a}) \\ &= n(n-1)\rho \cdot (\alpha \wedge (d\alpha)^{n-2})(\mathfrak{b}) \cdot d\rho(\partial_r) \cdot d\phi(\partial_\phi) \cdot d\theta_1(\partial_{\theta_1}) \cdot d\theta_2(\partial_{\theta_2}) \\ &\geq 0. \end{aligned}$$

By condition (iii) of definition 3,  $\alpha \wedge (d\alpha)^{n-2}$  is positive on  $B$ . Therefore, the second term is positive on  $B \times 0 \times T^2$  (hence also on a neighborhood) and non-negative everywhere else. In total, we have checked the contact condition and  $\tilde{\alpha}$  is indeed a contact form on  $M \times T^2$ .  $\square$

**Definition 4.** *An almost contact structure is a cooriented hyperplane field  $\eta$  (with an oriented trivial line bundle complementary to  $\eta$  defining the coorientation) and a complex bundle structure  $J$  on  $\eta$ . According to [Gei08, Prop 2.4.5], the space of complex bundle structures compatible to a symplectic form on  $\eta$  is non-empty and contractible. Thus, the almost contact structure can (up to homotopy) be defined by equipping  $\eta$  with a symplectic form.*

*If we don't care about the coorientation, we can therefore talk of the almost contact structure  $(\eta, \omega)$  where  $\omega$  is a symplectic form on  $\eta$ .*

**Remark 2.** *Let  $\epsilon > 0$ . The,  $\tilde{\alpha}_\epsilon := \epsilon x_1 d\theta_1 - \epsilon x_2 d\theta_2 + \alpha$  is a contact form. By Gray stability, all these contact structures are isotopic, so the underlying almost contact structures are homotopic, too. For  $\epsilon = 0$ , we have the almost contact structure  $\xi \oplus TT^2, d\alpha \wedge \Omega$  where  $\Omega$  is a volume form on  $T^2$ . This is homotopic to the almost contact structures for  $\epsilon > 0$ . Thus, up to homotopy, the almost contact structure of the Bourgeois form is given by*

$$\xi \oplus TT^2, d\alpha \wedge \Omega.$$

## The Thurston-Winkelnkemper construction

**Definition 5** (mapping torus). *Let  $\Sigma$  be a smooth manifold with boundary  $\partial\Sigma$  and  $\phi : \Sigma \rightarrow \Sigma$  a diffeomorphism that is equal to the identity close to  $\partial\Sigma$ . The mapping torus  $\Sigma(\phi)$  is given by  $\Sigma \times [0, 2\pi] / \sim$  where*

$$(x, 2\pi) \sim (\phi(x), 0).$$

The generalized mapping torus requires as additional data a smooth function  $\bar{\varphi} : \Sigma \rightarrow \mathbb{R}^+$  that is constant near  $\partial\Sigma$ . Then,

$$\Sigma_{\bar{\varphi}}(\phi) := \Sigma \times \mathbb{R} / \sim \quad \text{where} \quad (x, \theta) \sim (\phi(x), \theta - \bar{\varphi}(x)).$$

**Abstract open books** Starting with a mapping torus  $\Sigma(\phi)$ , we can construct an abstract open book  $M(\phi)$  with binding  $\partial\Sigma$  (see fig. 1)

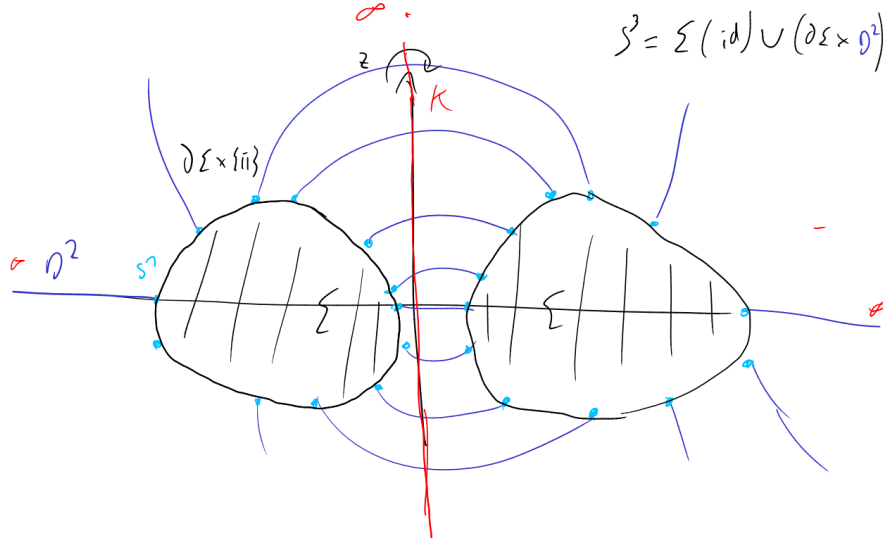


Figure 1: highly professional drawing of an abstract open book

We define

$$M(\phi) := (\Sigma(\phi) \cup \partial\Sigma \times D^2) / \sim$$

where we identify

$$[x \in \partial\Sigma, \theta] \sim (x, r = 1, \varphi = \theta)$$

**The construction** Let  $\Sigma^{2n}$  be a compact manifold admitting an exact symplectic form  $\omega = d\beta$  s.t. on the boundary  $\partial\Sigma$ , a contact form  $\beta_\partial$  is induced (this follows from the conditions requested in Geiges). Let the boundary be connected (i.e. the binding is also connected). Let the monodromy map  $\phi$  be an exact symplectomorphism of  $(\Sigma, \omega)$ , equal to the identity near the boundary  $\partial\Sigma$  (exactness is not necessary according to Geiges, as it can be obtained via a suitable isotopy of the symplectomorphism). An exact symplectomorphism  $\phi$  of  $(\Sigma, \omega)$  is such that

$$\phi^*(\beta) - \beta =: d\bar{\varphi}$$

is exact, i.e. there exists such a function  $\bar{\varphi}$  on  $\Sigma$  (of course only defined up to adding a locally constant function. Choose it in such a way that it only takes

positive values). The 1-form

$$\alpha := \beta + d\varphi$$

is a contact form on  $\Sigma \times \mathbb{R}$ :

$$\alpha \wedge (d\alpha)^n = (\beta + d\varphi) \wedge \underbrace{(d\beta)^n}_{=\Omega} = \beta \wedge \Omega + d\varphi \wedge \Omega = d\varphi \wedge \Omega,$$

where  $\Omega$  is a volume form on  $\Sigma$  (as  $\beta$  is a symplectic form). The  $\beta \wedge \Omega$  term vanishes because both are forms on  $\Sigma$ , but  $\Omega$  is already a top-level form. The resulting form is a wedge product of two volume forms on the product manifolds and therefore a volume form on  $\Sigma \times \mathbb{R}$ .

Now consider the transformation that induces the generalized mapping torus

$$F := (x, \varphi) \mapsto (\phi(x), \varphi - \bar{\varphi}(x)).$$

Remember that  $\varphi$  only takes positive values, i.e. the mapping torus is well-defined. The 1-form  $\alpha$  is invariant under this transformation:

$$\begin{aligned} F^*(\alpha) &= F^*(\beta) + F^*(d\varphi) && | \beta \text{ is independent of } \varphi \\ &= \phi^*(\beta) + dF(\varphi) && | \text{ definition of } \bar{\varphi}, F \\ &= \beta + d\bar{\varphi} + d\varphi - d\bar{\varphi} \\ &= \alpha. \end{aligned}$$

It follows that  $\alpha$  descends to a contact form on  $\Sigma_{\bar{\varphi}}(\phi)$ .

Now, we describe an adapted gluing construction for the abstract open book coming from a generalized mapping torus. Therefore, we construct a collar neighborhood on the generalized mapping torus s.t. on  $[-1, 0] \times \partial\Sigma$ , the symplectic form is given by  $d(e^s \beta_{\partial})$  where  $s$  is the collar parameter, d.h.  $\beta = e^s \beta_{\partial}$ . Why does such a neighborhood exist?

Close to  $\partial\Sigma$ ,  $\phi$  is equal to the identity and therefore  $d\bar{\varphi}$  is locally constant (hence constant, as  $\partial\Sigma$  is connected). Parametrize the neighborhood so that  $\bar{\varphi}$  is constant on  $[-1, 0] \times \partial\Sigma$ .

Now, take a look at

$$M := (\Sigma_{\bar{\varphi}}(\phi) \dot{\cup} (\partial\Sigma \times D_2^2)) / \sim.$$

A simple linear reparametrization will make the notation a lot easier: As  $\bar{\varphi}$  is constant on the neighborhood under consideration, we just pretend  $\bar{\varphi} = 2\pi$ . Furthermore, we parametrize the boundary  $\partial\Sigma$  with  $\theta \in S^1$ . Then we identify

$$(s, \theta, \varphi) \in [-1, 0] \times \partial\Sigma \times S^1 \subset \Sigma_{\bar{\varphi}}(\phi)$$

with

$$(\theta, s = 1 - r, \varphi) \in \partial\Sigma \times D_2^2 =: \mathcal{N}$$

where  $(r, \varphi)$  are polar coordinates on  $D_2^2$ , i.e. we identify a collar neighborhood of  $\Sigma$  with an annulus in  $D_2^2$ . (See fig. 2)

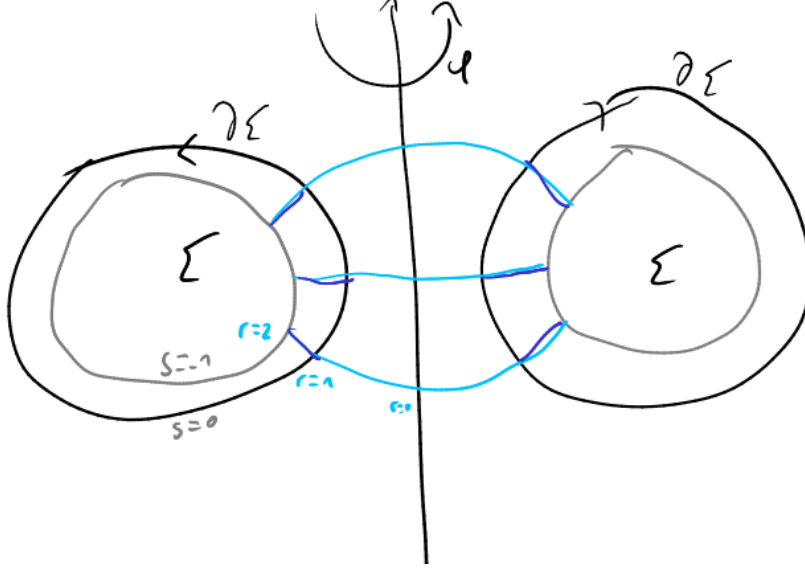


Figure 2: Detailed gluing process of the generalized abstract open book

Now we choose the ansatz

$$\alpha_{\text{ext}} := h_1(r)\beta_{\partial} + h_2(r)d\varphi.$$

for the extension of the contact form over  $\mathcal{N}$ . On the gluing area (i.e.  $1 \leq r \leq 2$ ),  $\alpha_{\text{ext}}$  has to agree with  $\alpha = \beta + d\varphi = e^s\beta_{\partial} + d\varphi$ , i.e.

$$h_1(r) = e^s = e^{1-r} \quad h_2(r) = 1.$$

In order to ensure smoothness at  $r = 0$ , in a small neighborhood of  $r = 0$  we set  $h_1(r) = 2 - r^2$  and  $h_2(r) = r^2$ , obtaining

$$\alpha_{\text{ext}}|_0 = 2\beta_{\partial}$$

We compute

$$d\alpha_{\text{ext}} = h'_1(r)dr \wedge \beta_{\partial} + h_1(r)d\beta_{\partial} + h'_2(r)dr \wedge d\varphi.$$

and

$$(d\alpha_{\text{ext}})^n = n \cdot dr \wedge (h'_1(r)\beta_{\partial} + h'_2(r)d\varphi) \cdot h_1(r)^{n-1}(d\beta_{\partial})^{n-1} + \underbrace{h_1(r)^n(d\beta_{\partial})^n}_{=0},$$



where the second term vanishes because  $(d\beta_\partial)^n$  is a  $2n$ -form on  $\partial\Sigma^{2n-1}$ . Finally,

$$\begin{aligned}\alpha_{\text{ext}} \wedge (d\alpha_{\text{ext}})^n &= h_1(r)nh_1(r)^{n-1}h_2'(r) \cdot & \beta_\partial \wedge dr \wedge d\varphi \wedge (d\beta_\partial)^{n-1} \\ &+ h_2(r)nh_1(r)^{n-1}h_1'(r) \cdot & d\varphi \wedge dr \wedge \beta_\partial \wedge (d\beta_\partial)^{n-1} \\ &= nh_1(r)^{n-1}(h_1h_2'(r) - h_2h_1'(r)) \cdot & \beta_\partial \wedge (d\beta_\partial)^{n-1} \wedge dr \wedge d\varphi \\ &= nh_1(r)^{n-1} \det \begin{pmatrix} h_1(r) & h_2(r)/r \\ h_1'(r) & h_2'(r)/r \end{pmatrix} \cdot & \beta_\partial \wedge (d\beta_\partial)^{n-1} \wedge r dr \wedge d\varphi\end{aligned}$$

As  $\beta_\partial$  is a contact form on  $\partial\Sigma$ ,  $\beta_\partial \wedge (d\beta_\partial)^{n-1}$  is a positive volume form on  $\partial\Sigma$ . Furthermore,  $rdr \wedge d\varphi$  is a positive volume form on the disk  $D_2^2$ . As a result, the right term of our result is a volume form on  $\mathcal{N} = \partial\Sigma \times D_2^2$ . The left term tells us that  $h_1(r)$  musn't have any zeros for  $r \in [0, 2]$  and that  $(h_1(r), h_2(r))$  must never be parallel to  $(h_1'(r), h_2'(r))$ , i.e.

$$h_1(r)^{n-1} \det \begin{pmatrix} h_1(r) & h_2(r)/r \\ h_1'(r) & h_2'(r)/r \end{pmatrix} > 0 \quad \forall r \in [0, 2].$$

(Close to zero, the determinant is given by  $2 \cdot 2 - 0 \cdot 0 = 4 > 0$ ). Figure 4.7 in [Gei08] proves the existence of such a pair of functions  $h_1$  and  $h_2$ .

In total, we obtain that  $\alpha_{\text{ext}}$  induces the correct orientation on the extension and, as  $M$  is connected and orientable, on all of  $M$ . In particular, condition (i) of definition 3 holds and  $\alpha \wedge (d\alpha)^n = d\varphi \wedge \Omega$  is a positive volume form on the mapping torus. As  $\Omega = (d\beta)^n = \omega^n$  for  $\omega$  the symplectic form on  $\Sigma$ , it is a  $2n$ -form and we see that  $\Omega$  is a positive volume form on  $\Sigma$ . Thus, on  $\Sigma$ ,  $d\alpha = d\beta = \omega$  is a symplectic form that induces the positive orientation of  $\Sigma$ . On  $\mathcal{N}$ , we need to check that the form induced by  $d\alpha_{\text{ext}}$  on the pages is symplectic with the right orientation. Inside  $\mathcal{N}$ , a page is given by the condition  $\varphi = \text{const}$ , i.e.  $d\varphi = 0$ . We have

$$\begin{aligned}(d\alpha_{\text{ext}})^n &= n \cdot dr \wedge (h_1'(r)\beta_\partial + h_2'(r)d\varphi) \cdot h_1(r)^{n-1}(d\beta_\partial)^{n-1} \\ &= nh_1'(r)h_1(r)^{n-1}dr \wedge \beta_\partial \wedge (d\beta_\partial)^{n-1}\end{aligned}$$

A positive volume form on  $\Sigma$  must be positive on  $-\partial_r, \mathfrak{b}$  where  $\mathfrak{b}$  is a positive basis of a point in  $\partial\Sigma$ . As  $\beta_\partial \wedge (d\beta_\partial)^{n-1}$  is a positive volume form on  $\partial\Sigma$ , we obtain

$$\begin{aligned}(d\alpha_{\text{ext}})^n(-\partial_r, \mathfrak{b}) &= \underbrace{nh_1(r)^{n-1}}_{=:A>0} \cdot h_1'(r)dr(-\partial_r) \wedge \underbrace{[\beta_\partial \wedge (d\beta_\partial)^{n-1}]}_{=:B>0}(\mathfrak{b}) \\ &= AB \cdot h_1'(r) \cdot -1 \\ &> 0,\end{aligned}$$

where in the last line we've used  $h_1'(r) = \frac{d}{dr}(2 - r^2) = -2r < 0$ . We have thus verified condition (ii) inside  $\mathcal{N}$  and outside  $\mathcal{N}$ , on  $\Sigma$ . As a result, it must hold on the whole page. Condition (iii) follows from the fact that on  $B$ ,  $\alpha_{\text{ext}} = 2\beta_\partial$  which is a positive contact form on  $\partial\Sigma$  and therefore also on  $B$ .

## Bourgeois & Thurston–Winkelnkemper

Applying the Thurston–Winkelnkemper construction yields a contact form  $\alpha$  on

$$M = (\Sigma_{\overline{\varphi}}(\phi) \dot{\cup} \mathcal{N}) / \sim.$$

Now, we apply the Bourgeois construction to it and obtain a contact form

$$\tilde{\alpha} = \alpha + x_1 d\theta_1 - x_2 d\theta_2$$

on  $M \times T^2$  where  $\theta_1, \theta_2$  are coordinates on  $T^2$  and  $x_1, x_2$  are coordinates as described in the section on the Bourgeois construction, i.e. there is a function  $\rho : M \rightarrow [0, 1]$  that agrees with  $r$  near the binding  $B$  and we define

$$x_1 := \rho \cos(p); \quad x_2 := \rho \sin(p).$$

Inside  $\mathcal{N}$ , our projection map  $p$  is given by the angular coordinate  $\varphi$ . Therefore,

$$\alpha|_{\mathcal{N}} = \alpha_{\text{ext}} = h_1(r)\beta_{\partial} + h_2(r)d\varphi = h_1(r)\beta_{\partial} + h_2(r)dp.$$

In total, we obtain for the contact form on  $\mathcal{U} := \mathcal{N} \times T^2$

$$\tilde{\alpha} = h_1(r)\beta_{\partial} + h_2(r)dp + \rho(r)(\cos(p)d\theta_1 - \sin(p)d\theta_2)$$

Outside  $\mathcal{U}$ , the form is given by

$$\tilde{\alpha} = \beta + dp + \cos(p)d\theta_1 - \sin(p)d\theta_2,$$

as  $\rho(r) = 1 \ \forall r \geq 1$ . Collecting all the conditions on  $h_1$  and  $h_2$  from the last section, we require that

- $h_1(r) = e^s = e^{1-r}, h_2(r) = 1$  in the gluing area ( $1 \leq r \leq 2$ ).
- Smoothness around the binding and contact condition on the binding:  $h_1(r) = 2 - r^2$  and  $h_2(r) = r^2$  around  $r = 0$ . The  $-r^2$ -part is only important to have  $h'_1(r) < 0$  around 0 to satisfy the symplectic condition on the pages.
- Contact condition on the tubular neighborhood:

$$h_1(r)^{n-1} \det \begin{pmatrix} h_1(r) & h_2(r)/r \\ h'_1(r) & h'_2(r)/r \end{pmatrix} > 0 \quad \forall r \in [0, 2].$$

We now define two functions  $\mu, \nu : M \rightarrow \mathbb{R}$  as follows:

$$\mu = \begin{cases} \frac{\rho'}{\rho' h_1 - \rho h'_1} & \text{inside } \mathcal{U} \\ 0 & \text{outside } \mathcal{U} \end{cases} \quad \text{and} \quad \nu = \begin{cases} \frac{-h'_1}{\rho' h_1 - \rho h'_1} & \text{inside } \mathcal{U} \\ 1 & \text{outside } \mathcal{U} \end{cases}.$$

Both  $\mu$  and  $\nu$  are smooth. First of all, we note that both are quotients of smooth functions. We have to make sure that  $\rho' h_1 \neq \rho h'_1$ . All of the involved functions

are non-negative except for  $h'_1$ , which is constantly negative ( $h_1$  needs to be chosen in such a way that  $h'_1$  is nonzero). As a result,

$$\rho' h_1 > 0 > \rho h'_1,$$

so the denominator is always nonzero. As  $\rho$  is constantly 1 for  $r \geq 1$ , we see that  $\mu$  is 0 for  $r \geq 1$ . In particular, it is smooth everywhere. For  $\nu$ , setting  $\rho \equiv 1$  implies  $\rho' = 0$  on  $r \geq 1$ . Therefore, the denominator simplifies to 1 and  $\nu$  is smooth, too.

**Lemma 2.** *The Reeb vector field of the contact form  $\tilde{\alpha}$  is given by*

$$R = \mu(r)R_B + \nu(r)[\cos(p)\partial_{\theta_1} - \sin(p)\partial_{\theta_2}],$$

where  $R_B$  is the Reeb vector field of  $\beta_{\partial}$ , the contact form on  $\partial\Sigma$ .

*Proof.* Inside  $\mathcal{U}$ , we compute

$$\tilde{\alpha}(R) = h_1(r)\beta_{\partial}(R) + h_2(r)dp(R) + \rho(r)(\cos(p)d\theta_1 - \sin(p)d\theta_2)(R)$$

$\beta_{\partial}$  and  $dp$  are 0 on  $\partial_{\theta_i}$ ,  $p$  and  $\theta_i$  are constant on  $R_B$

$$\begin{aligned} &= h_1\mu\beta_{\partial}(R_B) + \rho\nu(\cos(p)d\theta_1 - \sin(p)d\theta_2)(\cos(p)\partial_{\theta_1} - \sin(p)\partial_{\theta_2}) \\ &= h_1\mu + \rho\nu[\cos(p)\cos(p) + \sin(p)\sin(p)] \\ &= \frac{h_1\rho'}{\rho'h_1 - \rho h'_1} + \frac{-\rho h'_1}{\rho'h_1 - \rho h'_1} \\ &= 1. \end{aligned}$$

Computing  $d\tilde{\alpha}$ , we immediately drop the  $dr$ -terms, as they evaluate to 0 on  $R$ .

$$d\tilde{\alpha}(R, \cdot) = h_1(r)d\beta_{\partial}(R, \cdot) - \rho(r)[\sin(p)dp \wedge d\theta_1 + \cos(p)dp \wedge d\theta_2](R, \cdot)$$

$\beta_{\partial}$  and  $dp$  are 0 on  $\partial_{\theta_i}$ ,  $d\beta_{\partial}(R_B) = 0$ ,  $p$  is 0 on  $R_B$

$$= \rho(r)[\sin(p)d\theta_1(R)dp + \cos(p)d\theta_2(R)dp]$$

$d\theta_i$  is 0 on  $R_B$

$$\begin{aligned} &= \rho(r)\nu(r)[\sin(p)\cos(p) - \cos(p)\sin(p)]dp \\ &= 0 \end{aligned}$$

Outside  $\mathcal{U}$ , we have

$$R = \cos(p)\partial_{\theta_1} - \sin(p)\partial_{\theta_2}.$$

Therefore,

$$\tilde{\alpha}(R) = \beta(R) + dp(R) + \cos(p)d\theta_1(R) - \sin(p)d\theta_2(R)$$

$\beta$  and  $dp$  are 0 on  $\partial_{\theta_i}$

$$\begin{aligned} &= [\cos(p)d\theta_1 - \sin(p)d\theta_2](\cos(p)\partial_{\theta_1} - \sin(p)\partial_{\theta_2}) \\ &= \cos(p)\cos(p) + \sin(p)\sin(p) \\ &= 1 \end{aligned}$$

Also,

$$d\tilde{\alpha}(R) = d\beta(R, \cdot) - \sin(p)dp \wedge d\theta_1(R, \cdot) - \cos(p)dp \wedge d\theta_2(R, \cdot)$$

$\beta$  and  $dp$  are 0 on  $\partial_{\theta_i}$

$$\begin{aligned} &= \sin(p)d\theta_1(R)dp + \cos(p)d\theta_2(R)dp \\ &= \sin(p)\cos(p)dp - \cos(p)\sin(p)dp \\ &= 0 \end{aligned}$$

□

**Claim:** The computation in [Bou02] shows that, in order for  $\tilde{\alpha}$  to be a contact form in the neighborhood  $\mathcal{U}$  of  $B \times T^2$  where  $\alpha = h_1(r)\beta_{\partial} + h_2(r)dp$ , it is in fact enough that  $h_1(h_1 - h'_1) > 0$ , a condition which only depends on  $h_1$ .

*Proof.* From our computations for the Bourgeois construction, we see that  $(d\alpha)^n$  must be a positive volume form on the pages and  $\alpha \wedge (d\alpha)^{n-1}$  must be a positive volume form on  $B$  for  $\alpha$  to be a contact form. (The first condition is needed for the first term and the second condition for the second term in [Bou02, Equation (6)]). We have seen that

$$d\alpha = h'_1(r)dr \wedge \beta_{\partial} + h_1(r)d\beta_{\partial} + h'_2(r)dr \wedge dp.$$

and

$$(d\alpha)^n = n \cdot dr \wedge (h'_1(r)\beta_{\partial} + h'_2(r)d\varphi) \cdot h_1(r)^{n-1}(d\beta_{\partial})^{n-1}$$

In order for the latter to be a positive volume form on the pages, the  $\phi$  coordinate doesn't matter. As a result, it is enough that  $h'_1(r) \cdot h_1(r)^{n-1}$  is positive. Under the pullback to  $B$ ,  $dr$  and  $dp$ -components vanish. Therefore,

$$i^*(\alpha \wedge (d\alpha)^{n-1}) = h_1(r)^n \beta_{\partial} \wedge (d\beta_{\partial})^{n-1}.$$

In order for this to be a positive volume form, we need  $h_1(r)^n > 0$ . Clearly,  $h_1(r)^{2k}$  is always positive. Therefore, if  $n$  is odd, we need  $h_1(r) > 0$  and  $h'_1(r) > 0$ . If  $n$  is even, we only need  $h'_1(r) \cdot h_1(r) > 0$ . In fact, these conditions only depend on  $h_1$ . However, it is not the same condition as claimed. □

In particular, one can homotope the pair  $(h_1, h_2)$  in a compactly supported way Why compactly supported? among pairs of functions satisfying this condition, and this will result in a homotopy of contact forms (hence isotopy by Gray's stability) on  $M \times T^2$ , independently of the fact that the resulting  $\alpha$  on  $M$

might not be adapted to the open book or even a contact form. Moreover, the explicit formula in the last lemma still holds for the homotoped contact form, as the explicit computation does not use any specific property of the pair of functions  $(h_1, h_2)$  listed above. In particular, up to homotopy one can achieve the following form, which will be useful below: Where?

**Lemma 3.** *For  $\delta > 0$  sufficiently small, up to a deformation among contact structures on  $M \times T^2$  supported in the neighbourhood of radius  $2\delta$  of  $B \times T^2$ , one can assume that*

- $h_1(r) = 1$  for  $r \leq \delta$
- $h_1(h_1 - h'_1) > 0$  everywhere
- $h_2(r) = 0$  for  $r \leq 3\delta/2$

*Our description of the Reeb vector field also holds for the deformed contact form; in particular, it coincides with  $R_B$  for  $r \leq \delta$ .*

## The Milnor open book

**The open book coming from an  $A_{k-1}$ -singularity** Define  $f: \mathbb{C}^n \rightarrow \mathbb{C}$  by sending

$$(z_0, \dots, z_{n-1}) \mapsto z_0^k + z_1^2 + \dots + z_{n-1}^2.$$

Consider the sphere  $S^{2n-1} \subset \mathbb{C}^n$ . The intersection  $f^{-1}(0) \cap S^{2n-1}$  is the so called Brieskorn sphere  $B = \Sigma_{n-1}(k, 2, \dots, 2)$ . On the complement  $S^{2n-1} \setminus B$ , the map

$$\pi_f: S^{2n-1} \setminus B \rightarrow S^1: (z_0, \dots, z_{n-1}) \mapsto \frac{f(z_0, \dots, z_{n-1})}{|f(z_0, \dots, z_{n-1})|}$$

is a fibration over  $S^1$ , the Milnor fibration. It's fibers (the Milnor fibers) form the pages of the open book.

Alternative description: Positive stabilization of the standard open book  $OB(D^{2n-2}, \text{Id})$ , i.e.  $S^{2n-1} = OB(D^*S^{n-1}, \tau)$  where  $\tau$  is the Dehn-Seidel-twist. Is this really the stabilization? Attach a  $D^{n-1} \times D^{n-2}$  ( $n-1$ -handle) to the page. The 2-dimensional drawing suggests that at least the topology works out, i.e.

$$D^{2n-2} \cup h_{n-1} = D^*S^{n-1}.$$

Why (and in what sense) are the two descriptions equal? This is proven somewhere, see questions.md.

Summarizing the previous paragraph, we now have an open book decomposition of  $S^{2n-1}$ . We can apply the Thurston-Winkelnkemper construction to construct a supported contact structure and then, via the Bourgeois construction, obtain a contact structure on  $S^{2n-1} \times T^2$ .

## Surgery

Our goal is to construct a contact structure on  $S^{2n+1}$ . Currently, we have a contact structure on  $S^{2n-1} \times T^2$ . Therefore, we perform multiple contact surgeries.

## Smooth

## References

- [Bou02] Frédéric Bourgeois. Odd dimensional tori are contact manifolds. *International Mathematics Research Notices*, 2002(30):1571–1574, January 2002.
- [Gei08] Hansjörg Geiges. *An Introduction to Contact Topology*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, March 2008.