8 Open books & the Giroux correspondence

8.1 Open books

An open book decomposition of M^n is

- $B^{n-2} \subset M$ with $\nu B \cong B \times D^2$
- $p : M \setminus B \longrightarrow S^1$ a (locally trivial) fibration with

$$p = \theta \quad \text{on} \quad \begin{matrix} B \times D^2 \\ (b, (r,\theta)) \end{matrix}$$

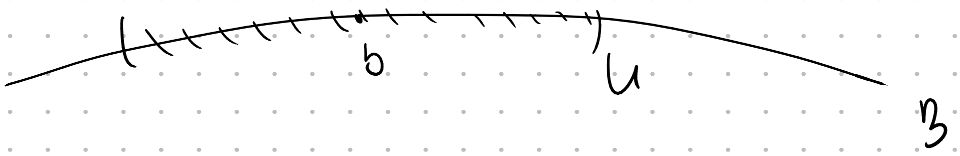
B = binding

p = page

$f : M \longrightarrow B$ = base is called fibration $:\Leftrightarrow$ $\forall\, b \in B \ \exists\, U \subset B$ open nbhd s.t.

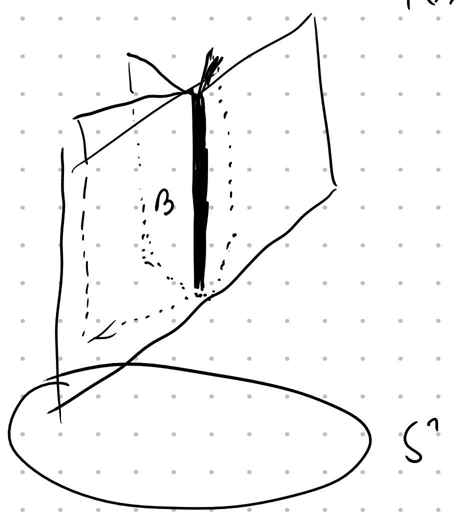
$$\begin{tikzcd} f^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "f"'] & & U \times F \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

$F := f^{-1}(b)$ = fiber

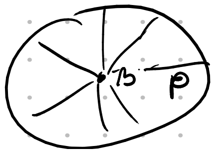


Ex: • $M = \mathbb{R}^3$, $B = \{x = y = 0\}$

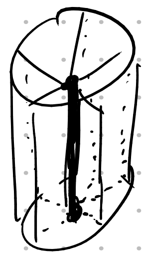
$$p : \mathbb{R}^2 \setminus B \longrightarrow S^1$$
$$(x,y,z) \longmapsto \frac{(x,y)}{|(x,y)|}$$



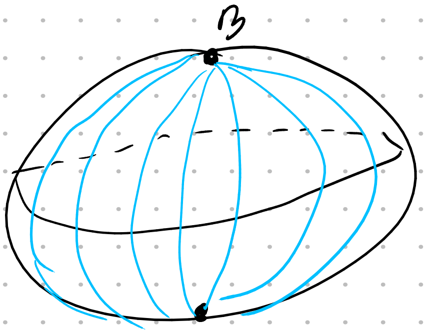
• D^2



- $D^n = D^2 \times D^{n-2}$



- S^2

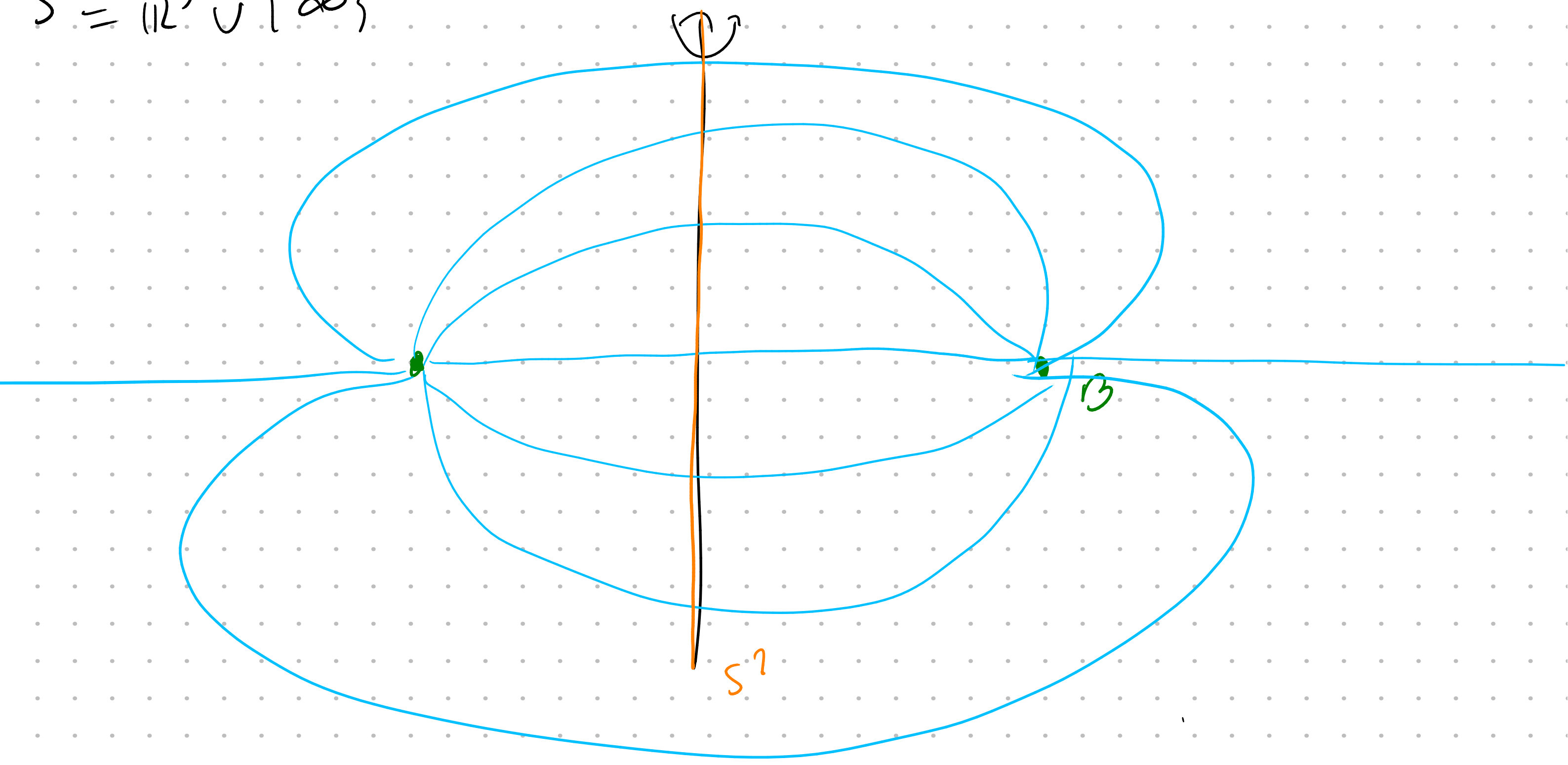


- $S^{n-1} = \partial D^n$

- $S^3 \subset \mathbb{C}^2$ $\quad B = \{z_1 = 0\}$ Unknot

$$p: S^3 \setminus B \longrightarrow S^1$$
$$(z_1, z_2) \longmapsto \frac{z_1}{|z_1|}$$

$S^3 = \mathbb{R}^3 \cup \{\infty\}$

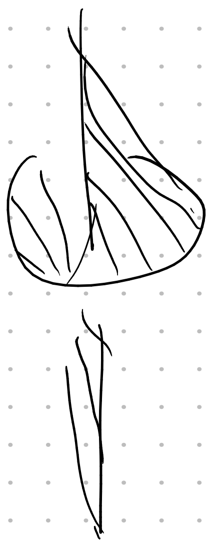


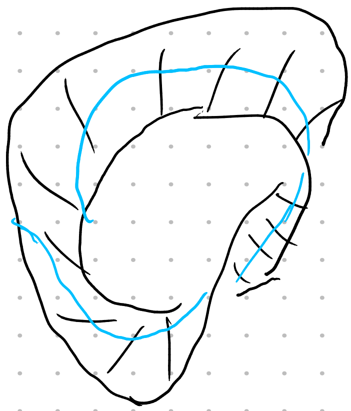
$B_+ = \{ z_1 \cdot z_2 = 0 \}$ $B_- = \{ z_1 \cdot \bar{z}_2 = 0 \} \subset S^3$

$$p_+ : S^3 \setminus B_+ \longrightarrow S^1 \qquad p_- : S^3 \setminus B_- \longrightarrow S^1$$

$$(z_1, z_2) \longmapsto \frac{z_1 \cdot z_2}{|z_1 \cdot z_2|} \qquad (z_1, z_2) \longmapsto \frac{z_1 \bar{z}_2}{|z_1 \bar{z}_2|}$$

B_+, B_- are Hopf links





Non-trivial fibration

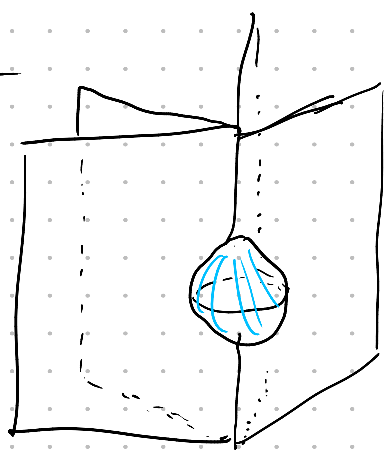
Exercise: Describe an open book on $S^1 \times S^2$

- take open book on S^2
- take everything $\times S^1$
- try to draw a nice picture

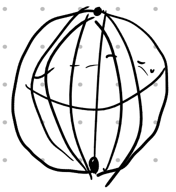
Stabilization / book connected sum

$\Lambda_i := (B_i, p_i)$ open books

(B_i, p_i) locally looks like:



$M_1 \# M_2 = M_1 \setminus \mathring{D}_1 \cup M_2 \setminus \mathring{D}_2$

$D_1 =$ 

$(B, p) \# (B_\pm, p_\pm)$ stabilization

Thm: [Alexander] $\forall M^3$ closed, connected, oriented

$\exists (B, p) = M$

8.2 Giroux correspondence

Thm [Giroux] M^3 closed, conn., oriented

$\Rightarrow$ $\{$open books on $M^3\}$ / isotopy, positive stabilization $\xleftrightarrow{1:1}$ $\{$contact structures on $M\}$ / isotopy

Def: $\xi = \ker(\alpha)$ on M is supported by an open book (B, p)

$:\Leftrightarrow$
- $d\alpha$ is a symplectic form on the pages (in dim $= 3$ $\Leftarrow$ $d\alpha$ is volume form on the pages)
- α is a contact form on B
  (in dim $= 3$ $\Leftarrow$ B is a transverse link)

$\Leftarrow$) ξ is transverse to B
ξ is ε-close to TP

Lemma: $\xi_i = \ker(\alpha_i)$ suppose by (B, p)
$\Rightarrow$ ξ_1 is isotopic to ξ_2

proof: $\alpha_t = t\alpha_1 + (1-t)\alpha_2 + f(r)\, d\varphi$ for $f(r)$ correctly chosen

$\Rightarrow \alpha_t$ is a contact form supported by (B, p) $\xRightarrow{\text{Gray th.}}$ claim

Ex: $(B_+, p_+) = (S^3, \xi_{st}) = (B, p)$

$(B_-, p_-) = (S^3, \xi_{OT})$

8.3 The Thurston – Winkelkemper construction

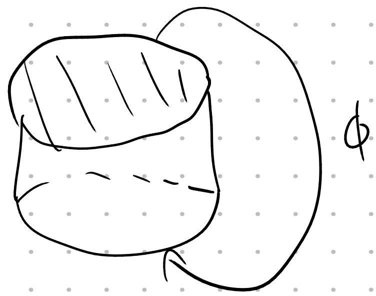
Thm: Any open book carries a contact str.

proof: Abstract open book:

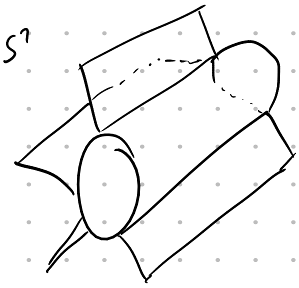
F surface with $\partial F \neq 0$ $\quad \phi : F \longrightarrow F$ with $\phi|_{\partial F} = id$

mapping torus:

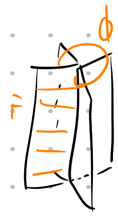
$$F_\phi = F \times [0,1] \big/ (p,1) \sim (\phi(p), 0)$$



$\Rightarrow \partial F_\phi = \partial F \times S^1$

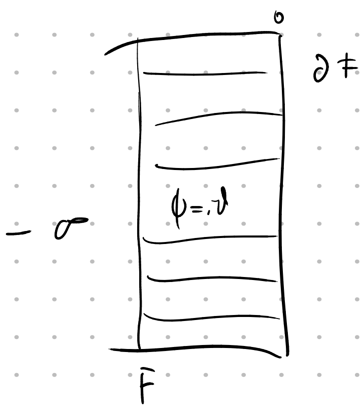


$$M(F, \phi) := F_\phi \big/ (p,t) \sim (p,t') \quad \text{for } p \in \partial F$$



Case: $|\partial F| = 1$ (binding is connected)

Let $(-\infty, 0] \times \partial F \subset F$ "collar nbhd"
(s, φ)



s.t. $\phi|_{(-2,0] \times \partial F} = id$

Lemma:

$\left\{ \beta \text{ 1-form on } F \;\middle|\; \begin{array}{l} d\beta \text{ is a volume form of vol } 2\pi \\ \beta = e^s d\varphi \text{ on } [-3/2, 0] \times \partial F \end{array} \right\} \neq \emptyset$ & convex

Hint: Stokes ☒

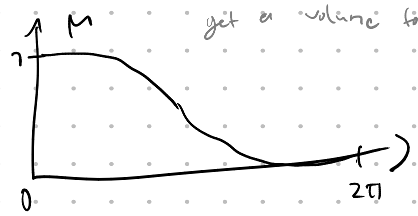
- For $\phi = id$: $\alpha := \beta + d\theta$ is a contact form on F_ϕ

$\left[\alpha \wedge d\alpha = (\beta + d\theta) \wedge d\beta = d\theta \wedge d\beta > 0 \text{ on } F \times S^1 \right]$

"Thurston trick"

get a volume form

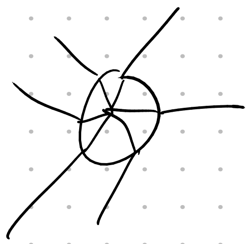
- For $\phi \neq id$: $\tilde{\beta} := \mu(\theta)\, \beta + (1 - \mu(\theta))\, \phi^* \beta$



$\alpha := \tilde{\beta} + c\, d\theta$ for $c \gg 0$, this is a cont. form on F_ϕ

$\left[\alpha \wedge d\alpha = (\tilde{\beta} + c\, d\theta) \wedge d\tilde{\beta} > 0 \text{ for } c \gg 0 \right]$

$$M = \left(F_\phi \cup \partial F \times D_2^2 \right) / \sim$$



$(s, \varphi, \theta) \sim (\varphi, r = 1 - s, \theta)$

$\alpha = e^s d\varphi + c\, d\theta$ near ∂F_ϕ

<u>Ansatz:</u> $\alpha = h_1(r)\, d\varphi + h_2(r)\, d\theta$

<u>conditions:</u> $h_1(r) = 2$, $h_2(r) = r^2$ near $r = 0$ (transversality)

$h_1(r) = e^{1-r}$, $h_2(r) = c$ for $r \in [1, 2]$ (fit to $(\cdot, s_1)$)

$$0 \neq \alpha \wedge d\alpha = (h_1\, d\varphi + h_2\, d\theta) \wedge (h_1'\, dr \wedge d\varphi + h_2'\, dr \wedge d\theta)$$
$$= h_1 h_2'\, d\varphi \wedge dr \wedge d\theta + h_2 h_1'\, d\theta \wedge dr \wedge d\varphi$$

$$\iff \det \begin{pmatrix} h_1 & h_2 \\ h_1' & h_2' \end{pmatrix} \neq 0$$ (contact condition)

$r \longmapsto (h_1(r), h_2(r))$

$\det \begin{pmatrix} h_1 & h_2 \\ h_1' & h_2' \end{pmatrix} \neq 0 \iff \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ is never parallel to $\begin{pmatrix} h_1' \\ h_2' \end{pmatrix}$

$\exists$ well defined map:

$$\{\text{open books}\}/\text{isotopy} \longrightarrow \{\text{cont. str.}\}/\text{isotopy}$$

Hard: $\longrightarrow\!\!\!\!\rightarrow$

ker = {pos. stab.}

Etnyre: Lecture notes on open books