

## Abstract

Contact geometry is the study of odd-dimensional smooth manifolds equipped with contact structures, i.e. hyperplane distributions  $\xi = \ker \alpha$  satisfying the contact condition

$$\alpha \wedge (d\alpha)^n \neq 0.$$

While they originally arise in the study of ODEs and in classical mechanics, the topological study of contact manifolds is a more recent and very active field of research.

A manifold can have multiple different contact structures, which can be either rigid (in which case one speaks of a "tight" manifold) or flexible (in the sense that they satisfy an h-principle). The latter contact manifolds are then called overtwisted. A foundational result of Eliashberg and Borman–Eliashberg–Murphy, roughly speaking, states that overtwisted contact manifolds exist in abundance, namely whenever the manifold admits the topological version of a contact structure (an *almost* contact structure), which is a first obvious obstruction. In dimension three, an almost contact structure is simply an oriented 2-plane field.

To illustrate this dichotomy, consider the sphere  $S^3$ . By a result of Eliashberg, it has precisely one tight contact structure. On the other hand, it has infinitely many overtwisted contact structures, corresponding to the infinitely many homotopy classes of 2-plane fields on the 3-sphere. There are other examples where there are infinitely many or no tight contact structures on a contact manifold.

A further interesting property of contact manifolds comes from the fact that contact geometry is the odd-dimensional counterpart to symplectic geometry. Often, it is possible to view a contact manifold as the boundary of a symplectic manifold. Manifolds that are in this sense "fillable" are always tight. The contrary, however, doesn't need to hold and one can ask the question under which conditions such tight, but non-fillable manifolds exist. The first examples of tight and non-fillable contact manifolds were constructed by Etnyre–Honda in dimension three, and by Massot–Niederkrueger–Wendl in higher dimensions.

More recently, Bowden–Gironella–Moreno–Zhou have shown that there exist homotopically standard, non-fillable but tight contact structures on all spheres  $S^{2n+1}$  with  $n \geq 2$ . Starting with a specific open book decomposition of  $S^{2n-1}$ , one can construct a contact form on this manifold using a well-known construction by Thurston–Winkelnkemper. Then, according to Bourgeois, this contact structure can be extended to a tight contact structure on  $S^{2n-1} \times T^2$ . Applying subcritical surgery (preserving the tightness), one can kill the topology of the  $T^2$ -factor and obtain a tight contact structure on  $S^{2n+1}$ . Because of the special way of constructing it, one can show that it is non-fillable, but still homotopically standard.

The goal of my master thesis is to give a streamlined explanation of the results of Bowden–Gironella–Moreno–Zhou, including the necessary background needed to understand the main ideas.

## Preliminaries

### The Bourgeois construction

**Definition 1.** Let  $M$  be an oriented manifold with an open book decomposition  $(B, p)$  with oriented binding  $B$ . The pages are oriented by the requirement that the induced orientation on the boundary of (the closure) of each page coincides with the orientation of  $B$ .

*Question: I don't fully understand Geiges remark there (in Def 4.4.7). See Agustins remarks in workflowy notes.*

A contact structure  $\xi = \ker \alpha$  on  $M$  is said to be **supported** by the open book decomposition  $(B, p)$  of  $M$  if

- (i) the contact form  $\alpha$  induces the positive orientation of  $M$  ( $\alpha \wedge (d\alpha)^n > 0$ ).
- (ii) the 2-form  $d\alpha$  induces a symplectic form on each page, defining its positive orientation
- (iii) the 1-form  $\alpha$  induces a positive contact form on  $B$ , i.e.

$$\alpha|_{TB} \wedge (d\alpha|_{TB})^{(n-2)} > 0.$$

**Theorem 1.** Let  $(M, \xi = \ker \alpha)$  be a closed contact manifold of dimension  $2n - 1, n \geq 2$ . One can find an open book decomposition  $(B, p)$  of  $M$  supporting  $\xi$ . According to Bourgeois, ([Bou02]) there is a contact structure  $\tilde{\xi}$  on  $M \times T^2$  (where  $\tilde{\xi}$  massively depends on the choice of open book).

*Proof.* We follow the proof of [Gei08, Thm 7.3.6]. Wlog let  $M$  be connected. The existence of an open book decomposition for  $M$  is the theorem of Giroux-Mohsen as in [Gei08, Thm 7.3.5]. By definition of an open book, there exists a tubular neighborhood  $B \times D^2$  with polar coordinates  $(r, \phi)$  on the  $D^2$ -part of the binding  $B$  s.t.  $p : M \setminus B \rightarrow S^1$  is given by  $\phi$  in that neighborhood. Now, we want to define smooth functions  $x_1, x_2$  on  $M$  that coincide with the cartesian coordinate functions on  $D^2$  close to the binding  $B$ . In order to do that, choose a smooth function  $\rho(r)$  on  $B \times D^2$ , s.t.

- $\rho = r$  near the binding  $B$ ,
- $\rho'(r) \geq 0$
- $\rho \equiv 1$  near  $B \times \partial D^2$ .

We extend this function to a smooth function  $\rho : M \rightarrow [0, 1]$  by setting  $\rho \equiv 1$  outside  $B \times D^2$ . Now,  $x_1 := \rho \cos \phi$  and  $x_2 := \rho \sin \phi$  are the desired smooth functions on  $M$  that coincide with the Cartesian coordinate functions on the  $D^2$ -factor near  $B$ . We compute

$$\begin{aligned} x_1 dx_2 - x_2 dx_1 &= \rho^2 \cos^2 \phi d\phi + \rho \cos \phi \sin \phi d\rho + \rho^2 \sin^2 \phi d\phi - \rho \cos \phi \sin \phi d\rho \\ &= \rho^2 (\cos^2 \phi + \sin^2 \phi) d\phi \\ &= \rho^2 d\phi \end{aligned}$$

and, analogously,

$$dx_1 \wedge dx_2 = \rho d\rho \wedge d\phi.$$

On  $M \times T^2$ , choose coordinates  $(\theta_1, \theta_2)$  on the torus part of the manifold. Now we have all ingredients together to construct our contact form. Let

$$\tilde{\alpha} := x_1 d\theta_1 - x_2 d\theta_2 + \alpha.$$

This is a well-defined 1-form on  $M \times T^2$  ( $\alpha$  is extended to  $M \times T^2$  in the obvious way) and we can compute the derivative

$$d\tilde{\alpha} = dx_1 \wedge d\theta_1 - dx_2 \wedge d\theta_2 + d\alpha,$$

hence

$$\begin{aligned} (d\tilde{\alpha})^n &= (n-1)(d\alpha)^{n-1} \wedge (dx_1 \wedge d\theta_1 - dx_2 \wedge d\theta_2) \\ &\quad - n(n-1)(d\alpha)^{n-2} \wedge dx_1 \wedge d\theta_1 \wedge dx_2 \wedge d\theta_2. \end{aligned}$$

In order to verify the contact condition, we compute

$$\begin{aligned} \tilde{\alpha} \wedge (d\tilde{\alpha})^n &= (x_1 d\theta_1 - x_2 d\theta_2 + \alpha) \wedge (n-1)(d\alpha)^{n-1} \wedge (dx_1 \wedge d\theta_1 - dx_2 \wedge d\theta_2) \\ &\quad - (x_1 d\theta_1 - x_2 d\theta_2 + \alpha) \wedge n(n-1)(d\alpha)^{n-2} \wedge dx_1 \wedge d\theta_1 \wedge dx_2 \wedge d\theta_2 \\ &= (n-1)(d\alpha)^{n-1} \wedge (x_1 dx_2 - x_2 dx_1) \wedge d\theta_1 \wedge d\theta_2 \\ &\quad + \underbrace{\alpha \wedge (n-1)(d\alpha)^{n-1} \wedge dx_1 \wedge d\theta_1}_{2n\text{-form on } M} - \underbrace{\alpha \wedge (n-1)(d\alpha)^{n-1} \wedge dx_2 \wedge d\theta_2}_{2n\text{-form on } M} \\ &\quad + n(n-1)\alpha \wedge (d\alpha)^{n-2} \wedge dx_1 \wedge d\theta_1 \wedge dx_2 \wedge d\theta_2 \end{aligned}$$

$M$  has dimension  $2n-1$ , i.e. the middle term is 0

$$\begin{aligned} &= (n-1)(d\alpha)^{n-1} \wedge \rho^2 d\phi \wedge d\theta_1 \wedge d\theta_2 \\ &\quad + n(n-1)\alpha \wedge (d\alpha)^{n-2} \wedge \rho d\rho \wedge d\phi \wedge d\theta_1 \wedge d\theta_2 \end{aligned}$$

By condition (ii) of definition 1,  $(d\alpha)^{n-1}$  must be a positive volume form on the pages. As explained in that definition, the orientation on  $M$  is given by  $\partial_\phi$  and the orientation of the page. In particular,  $(d\alpha)^{n-1} \wedge \rho d\phi$  is a positive volume form on  $M$ . Multiplied with a second  $\rho$ -factor, it vanishes along  $B$ . As  $\theta_1 \wedge \theta_2$  is a positive volume form on  $T^2$ , the first term is non-negative everywhere and positive away from

$$\underbrace{B \times 0}_{\subset B \times D^2 \subset M} \times T^2.$$

Let  $\mathfrak{b}$  be a basis of the binding  $B$  that is positively ordered. Then,  $-\partial_r, \mathfrak{b}$  and (because the binding is odd-dimensional)  $\mathfrak{b}, \partial_r$  are positive bases of the page. Clearly, then,

$$\mathfrak{a} := \mathfrak{b}, \partial_r, \partial_\phi, \partial_{\theta_1}, \partial_{\theta_2}$$

is an ordered basis of  $M \times T^2$ . Using  $\rho'(r) \geq 0$  everywhere, we deduce that  $d\rho(\partial_r)$  is non-negative. Hence, plugging  $\mathbf{a}$  into the second term, we conclude

$$\begin{aligned} & (n(n-1)\alpha \wedge (d\alpha)^{n-2} \wedge \rho d\rho \wedge d\phi \wedge d\theta_1 \wedge d\theta_2)(\mathbf{a}) \\ &= n(n-1)\rho \cdot (\alpha \wedge (d\alpha)^{n-2})(\mathbf{b}) \cdot d\rho(\partial_r) \cdot d\phi(\partial_\phi) \cdot d\theta_1(\partial_{\theta_1}) \cdot d\theta_2(\partial_{\theta_2}) \\ &\geq 0. \end{aligned}$$

By condition (iii) of definition 1,  $\alpha \wedge (d\alpha)^{n-2}$  is positive on  $B$ . Therefore, the second term is positive on  $B \times 0 \times T^2$  (hence also on a neighborhood) and non-negative everywhere else. In total, we have checked the contact condition and  $\tilde{\alpha}$  is indeed a contact form on  $M \times T^2$ .  $\square$

## The Thurston-Winkelnkemper construction

**Definition 2** (mapping torus). *Let  $\Sigma$  be a smooth manifold with boundary  $\partial\Sigma$  and  $\phi : \Sigma \rightarrow \Sigma$  a diffeomorphism that is equal to the identity close to  $\partial\Sigma$ . The mapping torus  $\Sigma(\phi)$  is given by  $\Sigma \times [0, 2\pi] / \sim$  where*

$$(x, 2\pi) \sim (\phi(x), 0).$$

*The generalized mapping torus requires as additional data a smooth function  $\bar{\varphi} : \Sigma \rightarrow \mathbb{R}^+$  that is constant near  $\partial\Sigma$ . Then,*

$$\Sigma_{\bar{\varphi}}(\phi) := \Sigma \times \mathbb{R} / \sim \quad \text{where} \quad (x, \theta) \sim (\phi(x), \theta - \bar{\varphi}(x)).$$

**Abstract open books** Starting with a mapping torus  $\Sigma(\phi)$ , we can construct an abstract open book  $M(\phi)$  with binding  $\partial\Sigma$  (see fig. 1)

We define

$$M(\phi) := (\Sigma(\phi) \cup \partial\Sigma \times D^2) / \sim$$

where we identify

$$[x \in \partial\Sigma, \theta] \sim (x, r = 1, \varphi = \theta)$$

**The construction** Roadmap:

- Let 1-form on  $\Sigma$  descend to mapping torus and add  $+d\varphi$
- Extend this 1-form over the binding neighborhood of the open book
- Check that this actually gives a contact form with the desired properties

**Create 1-form on mapping torus** Let  $\Sigma^{2n}$  be a compact manifold admitting an exact symplectic form  $\omega = d\beta$  s.t. on the boundary  $\partial\Sigma$ , a contact form  $\beta_\partial$  is induced (this follows from the conditions requested in Geiges). Let the boundary be connected (i.e. the binding is also connected). Let the monodromy map  $\phi$  be an exact symplectomorphism of  $(\Sigma, \omega)$ , equal to the identity near the boundary  $\partial\Sigma$  (exactness is not necessary according to Geiges, as it

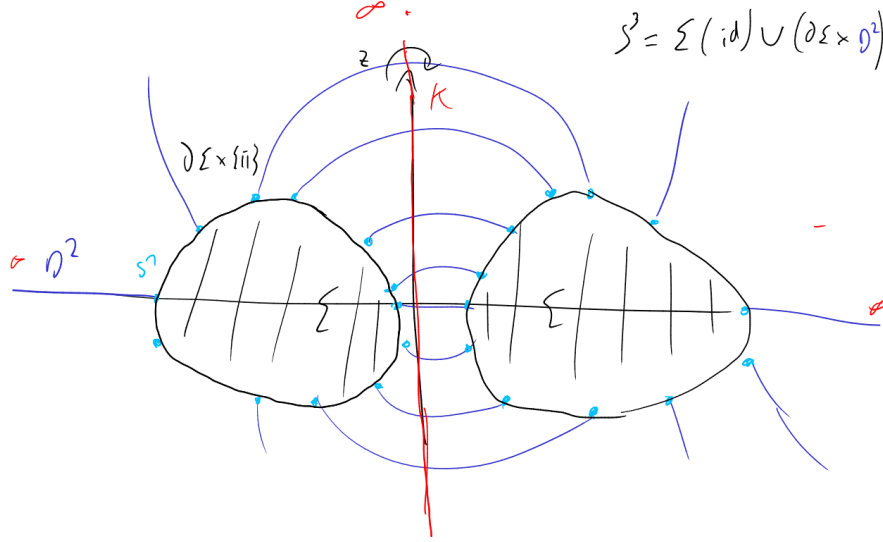


Figure 1: highly professional drawing of an abstract open book

can be obtained via a suitable isotopy of the symplectomorphism). An exact symplectomorphism  $\phi$  of  $(\Sigma, \omega)$  is such that

$$\phi^*(\beta) - \beta =: d\bar{\varphi}$$

is exact, i.e. there exists such a function  $\bar{\varphi}$  on  $\Sigma$  (of course only defined up to adding a locally constant function. Choose it in such a way that it only takes positive values). The 1-form

$$\alpha := \beta + d\varphi$$

is a contact form on  $\Sigma \times \mathbb{R}$ :

$$\alpha \wedge (d\alpha)^n = (\beta + d\varphi) \wedge \underbrace{(d\beta)^n}_{=: \Omega} = \beta \wedge \Omega + d\varphi \wedge \Omega,$$

where  $\Omega$  is a volume form on  $\Sigma$  (as  $\beta$  is a symplectic form). The second term is a wedge product of two volume forms on the product manifolds and therefore a volume form on  $\Sigma \times \mathbb{R}$ . Why is the sum of these forms nonzero? This question could be easily fixed by just adding a large enough constant to the  $d\phi$ -term.

Now consider the transformation that induces the generalized mapping torus

$$F := (x, \varphi) \mapsto (\phi(x), \varphi - \bar{\varphi}(x)).$$

Remember that  $\varphi$  only takes positive values, i.e. the mapping torus is wellde-

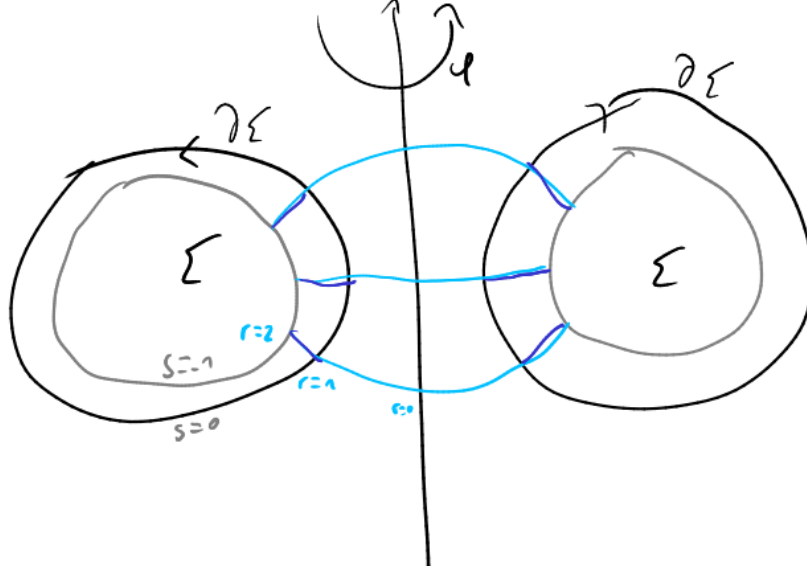


Figure 2: Detailed glueing process of the generalized abstract open book

finer. The 1-form  $\alpha$  is invariant under this transformation:

$$\begin{aligned}
 F^*(\alpha) &= F^*(\beta) + F^*(d\varphi) && | \beta \text{ is independent of } \varphi \\
 &= \phi^*(\beta) + dF(\varphi) && | \text{ definition of } \bar{\varphi}, F \\
 &= \beta + d\bar{\varphi} + d\varphi - d\bar{\varphi} \\
 &= \alpha.
 \end{aligned}$$

It follows that  $\alpha$  descends to a contact form on  $\Sigma_{\bar{\varphi}}(\phi)$ .

### Extend $\beta$ to the whole abstract open book

**Gluing together the abstract open book** First, we construct a collar neighborhood on the generalized mapping torus s.t. on  $[-1, 0] \times \partial\Sigma$ , the symplectic form is given by  $d(e^s \beta_{\partial})$  where  $s$  is the collar parameter. Why does such a neighborhood exist?

Close to  $\partial\Sigma$ ,  $\phi$  is equal to the identity and therefore  $d\bar{\varphi}$  is locally constant (hence constant, as  $\partial\Sigma$  is connected). Parametrize the neighborhood so that  $\bar{\varphi}$  is constant on  $[-1, 0] \times \partial\Sigma$ .

Now, take a look at

$$(\Sigma_{\bar{\varphi}}(\phi) \dot{\cup} (\partial\Sigma \times D_2^2)) / \sim.$$

A simple linear reparametrization will make notation a lot easier: As  $\bar{\varphi}$  is constant on the neighborhood under consideration, we just pretend  $\bar{\varphi} = 2\pi$ .

Furthermore, we parametrize the boundary  $\partial\Sigma$  with  $\theta \in S^1$ . Then we identify

$$(s, \theta, \varphi) \in [-1, 0] \times \partial\Sigma \times S^1 \subset \Sigma_{\overline{\varphi}}(\phi)$$

with

$$(\theta, s = 1 - r, \varphi) \in \partial\Sigma \times D_2^2$$

where  $(r, \varphi)$  are polar coordinates on  $D_2^2$  (See fig. 2). Now we choose the ansatz

$$h_1(r)d\theta + h_2(r)d\varphi.$$

for the contact form.

## References

- [Bou02] Frédéric Bourgeois. Odd dimensional tori are contact manifolds. *International Mathematics Research Notices*, 2002(30):1571–1574, January 2002.
- [Gei08] Hansjörg Geiges. *An Introduction to Contact Topology*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, March 2008.