Übungsblatt 8

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8.1

(i): Lösungskurven $x(t)$, $y(t)$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial H}{\partial y}\frac{\partial y}{\partial t} \overset{!}{=} 0$$

$$= \frac{\partial H}{\partial t} + \cancel{\frac{\partial H}{\partial x}\frac{\partial H}{\partial y}} - \cancel{\frac{\partial H}{\partial y}\frac{\partial H}{\partial x}} = 0 \checkmark$$

(ii) $P_0 = (x_0, y_0)$ Fixpunkt, d.h. $\frac{\partial H}{\partial y}(x_0,y_0) = 0 = \frac{\partial H}{\partial x}(x_0,y_0)$

Linearisiertes System:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = f(x_0,y_0) + \nabla f \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 H}{\partial x \partial y}\Big|_{(x_0,y_0)} & \frac{\partial^2 H}{\partial y^2}\Big|_{(x_0,y_0)} \\ -\frac{\partial^2 H}{\partial x^2}\Big|_{(x_0,y_0)} & -\frac{\partial^2 H}{\partial y \partial x}\Big|_{(x_0,y_0)} \end{pmatrix} \cdot \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial^2 H}{\partial x \partial y}\Big|_{(x_0,y_0)} & \frac{\partial^2 H}{\partial y^2}\Big|_{(x_0,y_0)} \\ -\frac{\partial^2 H}{\partial x^2}\Big|_{(x_0,y_0)} & -\frac{\partial^2 H}{\partial y \partial x}\Big|_{(x_0,y_0)} \end{pmatrix}$$ Eigenwerte:

gleich nach Satz von Schwarz

$$\Rightarrow \left| \begin{pmatrix} \lambda - \frac{\partial^2 H}{\partial x \partial y}\Big|_{(x_0,y_0)} & -\frac{\partial^2 H}{\partial y^2}\Big|_{(x_0,y_0)} \\ +\frac{\partial^2 H}{\partial x^2}\Big|_{(x_0,y_0)} & \lambda + \frac{\partial^2 H}{\partial y \partial x}\Big|_{(x_0,y_0)} \end{pmatrix} \right| = \Big(\lambda - \overbrace{\frac{\partial^2 H}{\partial x \partial y}}^{a}\Big)\Big(\lambda + \overbrace{\frac{\partial^2 H}{\partial y \partial x}}^{a}\Big) + \overbrace{\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial x^2}}^{b} = 0$$

$\Rightarrow$ passt $\quad \lambda^2 - a^2 + b = 0$

$$\lambda^2 = -b + a^2$$

$$\Rightarrow \lambda = \pm\sqrt{|-b+a^2|} \text{ oder } \pm i\sqrt{|-b+a^2|}$$

(iii): Wie man durch partielles Ableiten von H sehen kann, sind die Differentialgleichungen für die folgenden H gegeben.

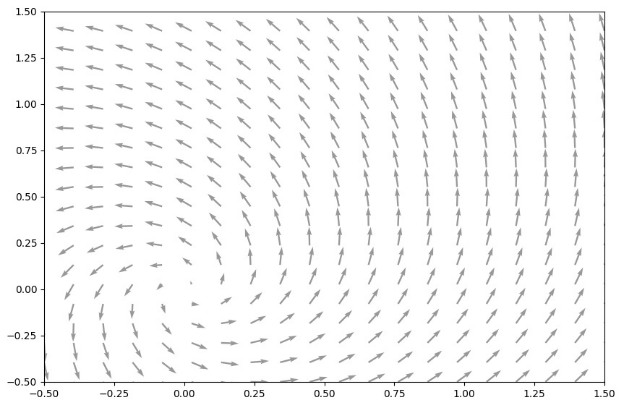
(a) $H = xy + \frac{1}{3}y^3$

(b) $H = xy - x^4$

8.2

(i)

$$\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}\frac12 x - y - \frac12(x^3+y^2x)\\ x+\frac12 y - \frac12 y^3 - \frac12 x^2 y\end{pmatrix} = \begin{pmatrix}\frac12-\frac32x^2 & -1+yx\\ 1-xy & \frac12-\frac32y^2-\frac12x^2\end{pmatrix}\Bigg|_{(0,0)} \begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\frac12 & -1\\ 1 & \frac12\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$



(ii)

$$\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}\frac12 x - y\\ x+\frac12 y\end{pmatrix}$$

$r^2 = x^2+y^2$

- $$\frac{dr}{dt} = \frac{dx}{dt}\cdot\frac{dr}{dx} + \frac{dy}{dt}\frac{dr}{dy} = \frac{dx}{dt}\frac{x}{r} + \frac{dy}{dt}\frac{y}{r} = \left(\frac12 x - y - \frac12(x^3+y^2x)\right)\frac{x}{r} + \left(x+\frac12 y-\frac12 y^3-\frac12 x^2 y\right)\frac{y}{r}$$

$$= \frac12\frac{x^2}{r} - \cancel{\frac{xy}{r}} - \frac12\frac{x^4}{r} - \frac12\frac{y^2x^2}{r} + \cancel{\frac{yx}{r}} + \frac12\frac{y^2}{r} - \frac12\frac{y^4}{r} - \frac12\frac{x^2y^2}{r}$$

$$= \frac12\frac1r(x^2+y^2) - \frac12\frac1r(x^4+y^4) - \frac{x^2y^2}{r}$$

$$= \frac12 r - \frac12\frac1r(x^2+y^2)^2 = \frac12 r - \frac12 r^3 \quad \checkmark$$

$\theta = \arctan\left(\frac{y}{x}\right)$

- $$\frac{d\theta}{dt} = \frac{d\theta}{dx}\dot x + \frac{d\theta}{dy}\dot y = -\frac{y}{r^2}\dot x + \frac{x}{r^2}\dot y = \frac{1}{r^2}\left(-y\left(\frac12 x - y - \frac12(x^3+y^2x)\right) + x\cdot\left(x+\frac12 y-\frac12 y^3-\frac12 x^2y\right)\right)$$

$$= \frac{1}{r^2}\left(-\cancel{\frac12 xy} + y^2 + \cancel{\frac12 x^3 y} + \cancel{\frac12 y^3 x} + x^2 + \cancel{\frac12 xy} - \frac12 y^3 x - \cancel{\frac12 x^3 y}\right)$$

$$= 1$$

(iii) $\theta(t) = t + C$

$$\frac{dr}{dt} = \frac{1}{2} r - \frac{1}{2} r^3$$

$$\frac{dr}{r - r^3} = \frac{1}{2} dt$$

Partialbruchzerlegung:

$$\frac{1}{r(1-r^2)} = \frac{A+}{r} + \frac{Br}{1-r^2} = \frac{1}{r} + \frac{1}{1-r^2} \qquad B=A \quad , A=1$$

$$\int \frac{1}{r-r^3} dr = \int \frac{1}{r} dr + \int \frac{r}{1-r^2} dr$$

$$\overset{u=r^2}{=} \ln r + \frac{1}{2} \int \frac{1}{1-u} du$$

$$= \ln r - \frac{1}{2} \ln(1-u)$$

$$= \ln r - \frac{1}{2} \ln(1-r^2) = \ln\left(\frac{r}{\sqrt{1-r^2}}\right)$$

$$\Rightarrow \quad \int \frac{dr}{r-r^3} = \int \frac{1}{2} dt \quad \Rightarrow \quad \ln\left(\frac{r}{\sqrt{1-r^2}}\right) = \frac{1}{2} t + D$$

$$\frac{r}{\sqrt{1-r^2}} = De^{\frac{t}{2}}$$

$$r^2 = De^t (1-r^2)$$

$$r^2 (1 + De^t) = De^t$$

$$r = \sqrt{\frac{De^t}{1+De^t}}$$

Anfangswerte: $\theta(0) = \theta_0 \Rightarrow C = \theta_0$

$$r(0) = r_0 = \sqrt{\frac{D}{1+D}}$$

$$r_0^2 = \frac{D}{1+D} \qquad r_0^2 + Dr_0^2 = D$$

$$D = \frac{r_0^2}{1-r_0^2}$$

$$\Rightarrow \boxed{\Phi_t \begin{pmatrix} r_0 \\ \theta_0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{r_0^2 e^t}{1-r_0^2+r_0^2e^t}} \\ t + \theta_0 \end{pmatrix}}$$

(iv)

$$\sqrt{\frac{r_0^2 e^t}{1-r_0^2+r_0^2e^t}} \overset{!}{=} \frac{1}{2} \Rightarrow \frac{r_0^2 e^t}{1-r_0^2+r_0^2e^t} = \frac{1}{4}$$

$$r_0^2 e^t = \frac{1}{4} - \frac{r_0^2}{4} + \frac{r_0^2}{4} e^t$$

$$\frac{3}{4} r_0^2 e^t = \frac{1}{4} - \frac{r_0^2}{4}$$

$$e^t = \frac{\frac{1}{3} - \frac{r_0^2}{3}}{r_0^2} > 0 \quad \text{für } r_0 \in (0,1)$$

$\Rightarrow$ eindeutig nach t auflösbar

Stetigkeit: $r_0 \mapsto \ln\left(\frac{\frac{1}{3} - \frac{r_0^2}{3}}{r_0^2}\right)$ stetig