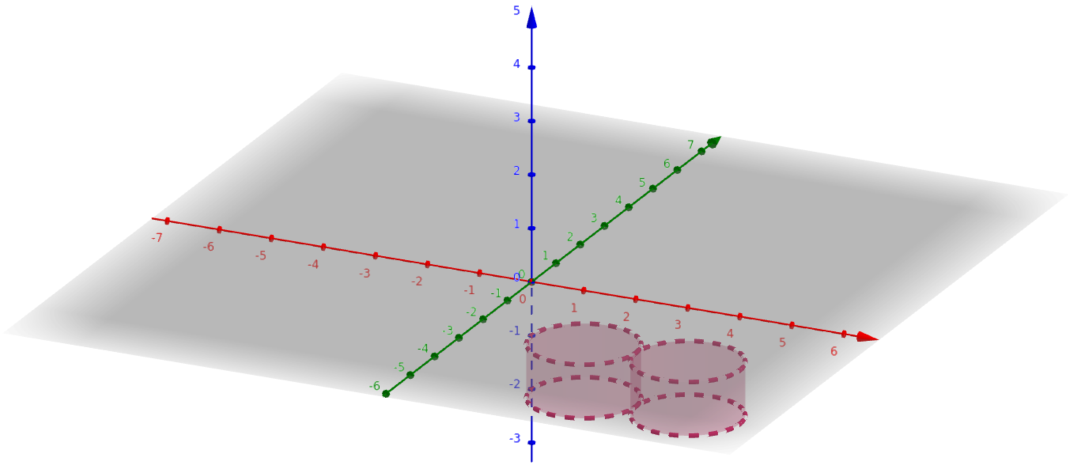

Abbildung 1: Sketch of F

Nonlinear Optimization – Sheet 08

Exercise 1

We compute

$$g(x^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Therefore, the active indices are $\mathcal{A}(x^*) = \{1, 2, 3\}$. By taking a close look at the inequality constraints, we see

$$F = \overline{B_1(1) \cup B_1(3)} \times [-2, -1].$$

We first consider the tangent cone in the $x_1 x_2$-space. Consider the sequences

$$x_k = (1 + \cos(\alpha_k), \sin(\alpha_k); \qquad t_k = \sin(\alpha_k)$$

for a zero sequence α_k. Then we compute

$$d := \lim_{k\to\infty} \frac{x_k - x}{t_k} = \lim_{k\to\infty} \left(\frac{\cos(\alpha_k) - 1}{\sin(\alpha_k)}, 1 \right)$$

As

$$\lim_{k\to\infty} \frac{-\sin(\alpha)}{\cos(\alpha)} = 0,$$

we conclude with L'Hospital that $d = (0, 1)$. Considering the geometry of the problem (see 1), we see that the tangent cone is already all of $\mathbb{R}^2$.

In the third dimension, the only active constraint is $x_3 \leq -1$. Therefore, the cone in x_3-direction is given by $\mathbb{R}_{\leq 0}$.

$$\mathcal{T}_F(x^*) = \mathbb{R}^2 \times \mathbb{R}_{\leq 0}.$$

The normal cone is therefore given by

$$\mathcal{T}_F(x^*)^\circ = \{0\} \times \{0\} \times \mathbb{R}_{\geq 0}.$$

We finally compute the linearizing cone,

$$\begin{aligned}
\mathcal{T}_F^{\text{lin}}(x^*) &= \{d \in \mathbb{R}^3 \colon 2(x_1 - 1, x_2, 0) \cdot d \leq 0, 2(x_1 - 3, x_2, 0) \cdot d \leq 0, (0,0,1) \cdot d \leq 0\} \\
&= \{d \in \mathbb{R}^3 \colon (1,0,0) \cdot d \leq 0, (-1,0,0) \cdot d \leq 0, (0,0,1) \cdot d \leq 0\} \\
&= \{(d_1, d_2, d_3) \in \mathbb{R}^3 \colon d_1 \leq 0, -d_1 \leq 0, d_3 \leq 0\} \\
&= \{0\} \times \mathbb{R} \times \mathbb{R}_{\leq} 0
\end{aligned}$$

The three cones are easy to imagine and don't need a sketch.

Exercise 2

(i) Prove Lemma 7.9 i.e. for arbitrary sets $M_1, M_2, M \subset \mathbb{R}^n$ the statements

(a) M° ist a closed convex cone.

Proof. The cone property is clear and closedness follows from continuity of the inner product on $\mathbb{R}^n$. □

(b) $M_1 \subset M_2$ implies $M_2^\circ \subset M_1^\circ$.

Proof. Clear on inspection. □

(ii) Verify the claimed forms of the polar cones in Example 7.10, i.e.

(a) Suppose $A = U + \{\overline{x}\}$ with $U \subset \mathbb{R}^n$ linear subspace, then $A^\circ = \{\overline{x}\}^\circ \cap U^\perp$.

Proof. The inclusion "$\supset$" is immediate. Let $x \in A^\circ$. Thus,

$$\forall u \in U : x^t(\overline{x} + u) = x^t\overline{x} + x^t u \leq 0$$

setting $u = 0$ we see $x \in \{\overline{x}\}^\circ$. We note $x^t u \leq -x^t\overline{x}$ and $x^t(-u) \leq -x^t\overline{x}$ i.e. $x^t u \geq x^t\overline{x}$. As $\forall n \in \mathbb{N}$ we have $nu \in U$ we get

$$\frac{1}{n} x^t\overline{x} \leq x^t u \leq -\frac{1}{n} x^t\overline{x}$$

taking the limit as $n \to \infty$ we see $x^t u = 0$ i.e. $x \in U^\perp$. □

(b) In the absence of inequality constraints the polar of the linearizing cone $\mathcal{T}_F^{\text{lin}}(x)$, $x \in F$ has the representation

$$\mathcal{T}_F^{\text{lin}}(x)^\circ = \operatorname{range} h'(x)^t = \{s \in \mathbb{R}^n : s \text{ a linear combination of } h_j(x)^t, j = 1, \ldots, n_{eq}\}$$

Proof. Directly from 7.13. □

(c) Let $N = (\mathbb{R}_{\geq 0})^n$ denote the non-negative orthant. Then $N^\circ = (\mathbb{R}_{\leq 0})^n$.

Proof. Let $x = (x_1, \ldots, x_n) \in N^\circ$ then $x^t e_i = x_i \leq 0$ as $e_i \in N$. So $x \in (\mathbb{R}_{\leq 0})^n$. The other inclusion is clear. □

Exercise 3

The optimization problems

$$\text{Minimize} \quad f(x), x \in \mathbb{R} \quad \text{s.t.} \quad x = 0 \tag{P1}$$

and

$$\text{Minimize} \quad f(x), x \in \mathbb{R} \quad \text{s.t.} \quad x^2 = 0 \tag{P2}$$

for any $f \in C^1(\mathbb{R})$ have their obvious solution $x^* = 0$. Show that ACQ and GCQ are fulfilled for (P1) at x^* but neither of them is fulfilled for (P2) at x^*.

Proof. (P1): It holds that

$$\mathcal{T}_F^{\text{lin}}(x^*) = \left\{ d \in \mathbb{R} \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}x}(x)\big|_{x=0} \cdot d = 0 \right\} = \{0\}$$

as $0 \in \mathcal{T}_F(x^*) \subset \mathcal{T}_F^{\text{lin}}(x^*) = \{0\}$ we have equality and thus ACQ and GCQ hold.

(P2): Similarly

$$\mathcal{T}_F^{\text{lin}}(x^*) = \left\{ d \in \mathbb{R} \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}x}(x^2)\big|_{x=0} \cdot d = 0 \right\} = \mathbb{R}$$

and $\mathcal{T}_F(x^*) = \{0\}$ because $F = \{0\}$. Thus, neither ACQ nor GCQ is fulfilled. □

Exercise 4

Consider

$$F = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, h_j(x) = 0, \forall 1 \leq i \leq n_{\text{ineq}}, 1 \leq j \leq n_{\text{eq}}\}$$

and

$$F^{\text{lin}}(x) = \left\{ y \in \mathbb{R}^n \;\middle|\; g_i(x) + g_i'(y - x) \leq 0, h_j(x) + h_j'(x)(y - x) = 0 \quad \forall i, j \right\}$$

for $x \in F$.

(i) Show that $\mathcal{T}_F^{\text{lin}}(x) = \mathcal{T}_{F^{\text{lin}}(x)}(x)$ for $x \in F$.

Proof. "$\subset$": $d \in \mathbb{R}^n$, $d \in \mathcal{T}_F^{\text{lin}}(x)$ i.e. $g_i'(x)d \leq 0$, $\forall i \in \mathscr{A}(x)$ and $h_j'(x)d = 0$, $\forall j$. Clearly $\frac{1}{n}d + x \in F^{\text{lin}}(x)$ and thus

$$\frac{\frac{1}{n}d + x - x}{\frac{1}{n}} = d \implies d \in \mathcal{T}_{F^{\text{lin}}(x)}(x).$$

"$\supset$": $d \in \mathcal{T}_{F^{\text{lin}}(x)}(x)$ and $x^k \in F^{\text{lin}}(x)$, $t^k \searrow 0$ sequences s.t.

$$d = \lim_{k \to \infty} \frac{x^k - x}{t^k}.$$

For all $i \in \mathscr{A}(x)$ i.e. $g_i(x) = 0$ we get

$$g_i'(x)\frac{x^k - x}{t^k} = \frac{1}{t^k}(g_i'(x)(x^k - x) + g_i(x)) \leq 0 \implies g_i'(x)d \leq 0$$

similarly we see $h_j'(x)d = 0$, $\forall j$. □

(ii) Show that $\mathcal{T}_F^{\text{lin}}(x)$ is a closed convex cone.

Proof. Nothing to prove. □

(iii) Prove Thm. 8.9 by showing that ACQ holds at any feasible point of problems of the form

$$\begin{array}{ll} \text{Minimize} & f(x) \quad \text{where } x \in \mathbb{R}^n \\ \text{subject to} & A_{\text{ineq}} x \le b_{\text{ineq}} \\ \text{and} & A_{\text{eq}} x = b_{\text{eq}}. \end{array}$$

Proof. As $(A_{\text{ineq}} x - b_{\text{ineq}} + A_{\text{ineq}}(y - x))_i = (A_{\text{ineq}} y - b_{\text{ineq}})_i$ for all y and all i and similarly for the equality constraint we see

$$F^{\text{lin}}(x) = F.$$

Thus $\mathcal{T}_F^{\text{lin}}(x) = \mathcal{T}_{F^{\text{lin}}(x)}(x) = \mathcal{T}_F(x)$ i.e. ACQ holds. □