Homotopically Standard Tight Non-fillable Contact Structures on the Sphere

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results by Bowden, Gironella, Moreno and Zhou

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Background

Contact topology

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Fillability: *fillable* contact mflds are boundaries of symplectic mflds.

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Eliashberg, Borman–Eliashberg–Murphy:

Dichotomy: Rigidity vs. Flexibility.

- **tight** *(rigid/geometric)*;
- **overtwisted** *(flexible/topological)*.

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Theorem (Eliashberg–Gromov)

Fillable contact manifolds are tight.

Converse is false (Etnyre–Honda, Massot–Niederkrueger–Wendl).

Existence and classification

Topological obstruction: *almost* contact structure, i.e. reduction of structure group to $U(n) \times \mathbb{1}$.

Theorem (Lutz–Martinet (dim 3), Casals–Pancholi–Presas (dim 5), Borman–Eliashberg–Murphy (any dim))

Almost contact manifolds are contact, where the contact structure is overtwisted.

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Tight manifolds

How can we understand **tight** contact manifolds?

Contact topology: fillability

Hierarchy of fillability:

$$\{\mathit{Stein}\} \overset{①}{=} \{\mathit{Weinstein}\} \overset{②}{\subsetneq} \{\mathit{Liouville}\} \overset{③}{\subsetneq} \{\mathit{strong}\}$$

$$\overset{④}{\subsetneq} \{\mathit{weak}\} \overset{⑤}{\subsetneq} \{\mathit{tight}\}$$

- $\mathit{dim} = 3$: ① Cieliebak–Eliashberg, ② Bowden, ③ Ghiggini, ④ Eliashberg, ⑤ Etnyre–Honda.

- $\mathit{dim} \geqslant 5$: ① Cieliebak–Eliashberg,
② Bowden–Crowley–Stipsicz, ③ Zhou,
④ Bowden–Gironella–M., ⑤ Massot–Niederkrüger–Wendl.

Contact structures on spheres

First step in classification: contact structures on spheres.

Standard contact structure

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Theorem (Eliashberg, '91)

On S^3, it is the unique tight contact structure.

In particular, no tight and non-fillable contact structures on S^3.

Tight and non-fillable structures in $\dim \geqslant 5$

Theorem (Bowden–Gironella–M.–Zhou '22-'24)

In $\dim \geqslant 7$*, if* M *admits a tight structure, it also admits a tight and non strongly-fillable structure, in the same almost contact class.*

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In $\dim = 5$*, the same holds, if the first Chern class vanishes.*

Case of spheres

The general theorem follows by connected sum with an “exotic” sphere:

Theorem (Bowden–Gironella–M.–Zhou ’22-’24)

For every $n \geqslant 2$, the sphere $\mathbb{S}^{2n+1}$ admits a tight, non-fillable contact structure that is homotopically standard.

General remarks

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- Suggests that higher-dimensional contact phenomena should occur independently of underlying smooth topology.

Tight and non-fillable spheres

Giroux correspondence

Giroux: Contact structures are *supported* by open books.

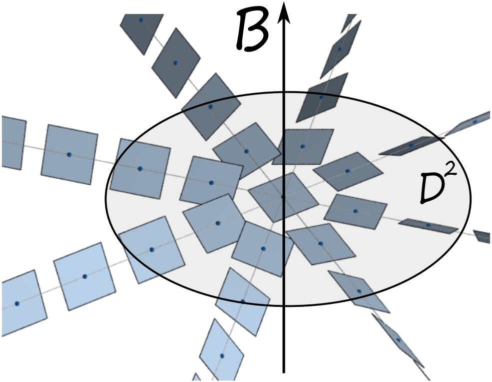

Figure: Supported contact structure.

Bourgeois contact structures

Theorem (Bourgeois '02)

Open book supporting $(M, \xi) \rightsquigarrow$ contact structure on $M \times \mathbb{T}^2$.

These are $\mathbb{T}^2$-equivariant.

Geometric construction

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- Milnor open book on $\mathbb{S}^{2n-1}$ $\rightsquigarrow$ Bourgeois manifold on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ $\rightsquigarrow$ two 1-surgeries = $\mathbb{S}^{2n-1} \times \mathbb{S}^2$ $\rightsquigarrow$ one 2-surgery = $\mathbb{S}^{2n+1}$.

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Claim: $(\mathbb{S}^{2n+1}, \xi_{ex})$ is tight and non-fillable.

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Milnor open book $\Rightarrow$ $(\mathbb{S}^{2n+1}, \xi_{ex})$ is *tight*.

Non-fillability

Non-fillability of $(\mathbb{S}^{2n+1}, \xi_{ex})$ can be proven via:

1. Homological obstruction and cobordisms as in [Bowden–Gironella–M.], building on [Massot–Niederkrüger–Wendl].
2. Symplectic cohomology computations as in [Zhou].

Homological obstructions

Observation: Bourgeois manifolds have convex decomposition

$$M \times \mathbb{T}^2 = (M \times \mathbb{S}^1) \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V}_- \times \mathbb{S}^1,$$

with $V_\pm = \Sigma \times D^*\mathbb{S}^1$, $\Sigma =$ page of the open book, $\phi =$ monodromy.

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Theorem (Bowden–Gironella–M.)

$M = V \times \mathbb{S}^1 = V_+ \times \mathbb{S}^1 \cup_\phi \overline{V_-} \times \mathbb{S}^1$ with convex decomposition, $N = \partial V_\pm$ dividing set. If W is a symplectic filling of M, then

$$H_*(N) \to H_*(V_\pm) \to H_*(W),$$

induced by inclusion. Then second map is injective on image of the first.

Namely, if a homology class in N survives in $V_\pm$, then it survives in the filling.

Idea of proof

- Capping cobordism from M to $N \times \mathbb{S}^2$ with a SHS, via handles $H_\pm$ with co-core $V_\pm$.

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- Spheres intersect $H_\pm$ precisely once $\rightsquigarrow$ intersection map $\mathcal{I}_\pm : \mathcal{M}_* \to V_\pm$.
- If $\sigma \subset W$ satisfies $\partial\sigma = c$ with c cycle in N, then $b = \mathcal{I}_\pm ev^{-1}(\sigma)$ bounds σ in $V_\pm$. $\square$

Homological obstructions

Fact:

1. If $\dim \geqslant 7$, subcritical surgeries on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ can be pushed away from dividing set to V_+.

$$\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex}) \text{ still has a dividing set } N,$$

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$$\Rightarrow (\mathbb{S}^{2n+1}, \xi_{ex}) \text{ still has a dividing set } N,$$

with $H_n(N) \neq 0$.

2. Homological obstruction theorem persists under surgery away from dividing set (capping cobordisms).

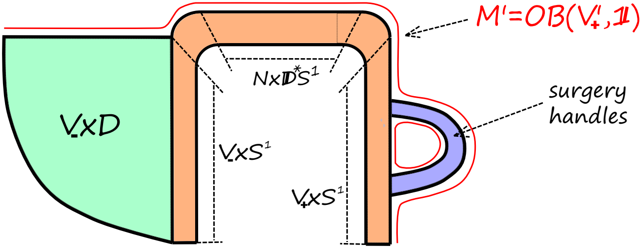

Figure: Capping cobordism.

End of the proof: W filling of $(\mathbb{S}^{2n+1}, \xi_{ex}) \Rightarrow$ Homological obstruction:

$$0 \neq H_n(N) \hookrightarrow H_n(W).$$

However, this factors as

$$0 \neq H_n(N) \to H_n(\mathbb{S}^{2n+1}) = 0 \to H_n(W),$$

contradiction.

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1. **Homotopically standard:** Fixed ξ, Bourgeois manifolds have same almost contact class $\xi \oplus T\mathbb{T}^2$, so suffices with trivial open book. *h*-cobordism theorem gives standard smooth topology on sphere.
2. **Dimension** 5**:** Needs careful *flexible* version of the homological obstruction theorem.
3. **Symplectic cohomology:** Capping cobordisms reach $\partial(V \times \mathbb{D}^2)$. Zhou's computations of $SH_+(\partial(V \times \mathbb{D}^2))$ and SH_+ computations of Brieskorn spheres as by [Kwon–van-Koert] can be used.

Thank you!