2. Contact Manifolds

17.10.23

2.1 Hyperplane fields

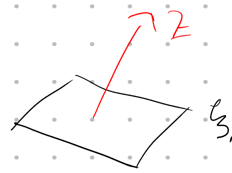
M^n = smooth n-MFD

$$\left(p \longmapsto \xi_p^{n-1} \subset T_pM^n\right)$$

- $\xi \in TM$ is called <u>hyperplane field</u> $:\Leftrightarrow$ $\forall p \in M$ $\exists$ NBHD U of p and linearly independent vector fields $x_1, \dots, x_{n-1}$ on U s.t. $\forall q \in U$:

$$\xi_q := \xi \cap T_qM = \langle x_1(q), \dots, x_{n-1}(q) \rangle$$

- ξ is called <u>co-orientable</u> $:\Leftrightarrow$ TM/ξ is orientable



$$p \mapsto T_pM/\xi_p = V_p$$

<u>Def</u> vector bundle over M^n

$$E \supset p^{-1}(U) \xrightarrow[\text{Fib. Isom}]{\cong\, C^\infty} U \times \mathbb{R}^k$$

$p^{-1}(U) \to U$, $U \times \mathbb{R}^k \xrightarrow{pr} U$

$E \xrightarrow{p} M^n \ni 0$

<u>Why is TM/ξ a bundle?</u>

$$\xi_q = \langle x_1(q), \dots, x_{n-1}(q) \rangle$$

$$N_q = \xi_q^{\perp}$$

$$p^{-1}(U) \cong U \times \langle N \rangle$$

$$p^{-1}(U) \cong U \times V_p$$

<u>Lemma 1</u>: $\xi \in TM$ is coorientable $\Leftrightarrow$ $\exists$ 1-form α on M s.t. $\xi = \ker(\alpha)$ (i.e. $\alpha_p : T_pM \to \mathbb{R}$)

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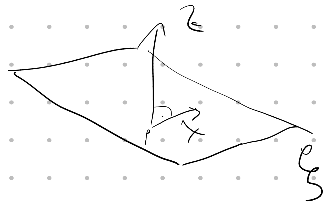
<u>proof</u>: Let g be a riemannian metric.

"$\Rightarrow$" Choose a vector field Z on M with

$$\begin{cases} g(Z, x) = 0 & \forall x \in \xi \\ g(Z, Z) = 1 \end{cases}$$



$$\alpha := g(Z, \cdot)$$

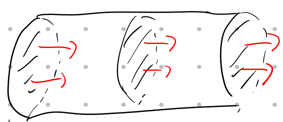
"$\Leftarrow$" A vector field Z with $\begin{cases} g(Z,Z) = 1 \\ \alpha(Z) \neq 0 \end{cases}$ defines a coorientation of ξ

$\square$

<u>Convention</u>: we restrict to coorientable hyperplane fields (often this is secretly assumed)

<u>Example</u>:

$$M^n := S^1 \times N^{n-1}$$



glue left and right side together

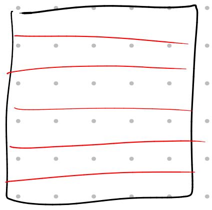
$\forall p = (\theta, x) \in M \Rightarrow \xi_p := T_xN \subset T_pM$

$\xi = \bigcup_{p \in M} \xi_p$ is a hyperplane field

$\xi = \ker(d\theta)$ (codim-1-foliation)

Foliation: $\mathcal{M}^n = \bigsqcup_{\theta \in \Lambda} F_\theta$ is called h-dim foliation
"leafs"

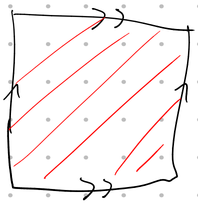
$:\Leftrightarrow$ $\exists$ Atlas of M (U_α, h_α) s.t. $\forall \theta\ \forall \alpha$: $h_\alpha(F_\theta \cap U) = \emptyset$ or $\mathbb{R}^h \times p \cap h_\alpha(U)$



$(F_\theta \overset{h}{\supset})\ \mathbb{R}^h \times (*)$ $\quad \mathbb{R}^n \ (\overset{h_\alpha}{\subset} M)$

Ex: Flow lines of a non-vanishing vector field are 1-dimensional foliation

$M = T^2$



If slope $\in \mathbb{Q} \Rightarrow$ leafs are compact
If slope $\in \mathbb{R}\setminus\mathbb{Q} \Rightarrow$ " noncompact

If $h = n-1 \Rightarrow \xi = \bigcup_{p \in M} T_p F(p)$ is a hyperplane field
"every foliation of codim 1 induces a hyperplane field"

Q: "when does a hyperplane come from a foliation?"

Theorem 2 [Frobenius]

$\ker(\alpha) = \xi \subset TM$ is induced by a foliation $\Leftrightarrow$ $\alpha \wedge d\alpha = 0$

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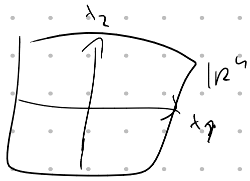
digression: $\alpha_p : T_p M \xrightarrow{1\text{-Form}} \mathbb{R}$ lin

$\alpha_p \in (T_p M)^*$

Let $x_1, \dots, x_n$ be coordinates on M

$\Rightarrow$ coordinate vector fields $\partial x_1, \dots, \partial x_n$ form a basis of $T_p M$



Dual basis $dx_1, \dots, dx_n$, i.e. $dx_i(\partial x_j) = \delta_{ij}$

$\Rightarrow \alpha_p = \sum c_i(p) x_i$

h-Form β: $\quad \beta_p : \underbrace{(T_p M) \times \dots \times (T_p M)}_{n \text{ times}} \longrightarrow \mathbb{R}$ multilinear, alternating

$\wedge$-product:

1-Forms $\quad \alpha = \sum c_i\, dx_i$

$\beta = \sum b_j\, dx_j$, $\quad \alpha \wedge \beta =$ def. to be linear

$dx_i \wedge dx_j = -dx_j \wedge dx_i$

$dx_i \wedge dx_i = 0$

Differential

d: h-Form $\longrightarrow$ $(h+1)$-Forms

$\sum c_i(p)\, dx_i \longmapsto d\alpha$

$\sum \frac{\partial c_i}{\partial x_j} dx_j \wedge dx_i$

Ex: $M = \mathbb{R}^3_{(x,y,z)}$ $\alpha = dx$ $\Rightarrow$ $\alpha \wedge d\alpha$ comes from a foliation

$d\alpha = ddx = 0$

(2) $\alpha = x\,dy + dz$

$$d\alpha = d(x\,dy + dz) = d(x\,dy) + \cancel{d(dz)} = \underbrace{\frac{\partial x}{\partial x}}_{=1} dx \wedge dy + \cancel{\frac{\partial x}{\partial y} dy \wedge dy}$$

$$\alpha \wedge d\alpha = (x\,dy + dz) \wedge (dx \wedge dy) = x\,dy \wedge (dx \wedge dy) + dz \wedge dx \wedge dy$$

$$= dx \wedge dy \wedge dz \neq 0$$

2.2 Contact structures

Let M^{2n+1}

* $\xi^{2n} = \ker(\alpha)$ is a contact structure $\Leftrightarrow$ $\alpha \wedge (d\alpha)^n$ is a volume form $\left\{ \begin{array}{l} \text{i.e. } \alpha \wedge (d\alpha)^n \neq 0 \\ \text{at every point} \end{array} \right\}$

$\alpha \wedge (d\alpha)^n = \alpha \wedge d\alpha \wedge \ldots \wedge d\alpha$

* α is called contact form

* the reeb vector field R_α of α is defined

$$\begin{cases} d\alpha(R_\alpha, \cdot) \equiv 0 \\ \alpha(R_\alpha) \equiv 1 \end{cases}$$

remark: $\alpha \wedge (d\alpha)^n$ is a vol form $\Rightarrow$ M orientable

* $\xi = \ker(\alpha) = \ker(\tilde{\alpha}) \Rightarrow \tilde{\alpha} = f\alpha$ for $f: M \to \mathbb{R} \setminus \{0\}$

$$\Rightarrow \tilde{\alpha} \wedge (d\tilde{\alpha})^n = (f\alpha) \wedge (d(f\alpha))^n = f\alpha \wedge (f\,d\alpha + df \wedge \alpha)^n$$

$\alpha \wedge \alpha = 0$

$$= \underbrace{f^{n+1}}_{\neq 0} \underbrace{\alpha \wedge (d\alpha)^n}_{\neq 0}$$

* R_α is well-defined:

$\alpha \wedge (d\alpha)^n \neq 0 \Rightarrow (d\alpha)^n\big|_{\xi} \neq 0$ $\Rightarrow$ $d\alpha$ has rank $2n$

ξ: $2n$-dimensional bundle $\Rightarrow$ $\ker(d\alpha)$ is 1-dim

& $\alpha \neq 0$ on $\ker(d\alpha)$

Weinstein conj: M closed $\overset{?}{\Rightarrow}$ $\forall$ contact form α on M: R_α has a periodic orbit

Ex) (1) Consider $\mathbb{R}^{2n+1}_{(x_1, y_1, \dots, x_n, y_n, z)}$

$\xi_{st} = \ker(\alpha_{st})$ standard contact structure

$\alpha_{st} = \left(\sum_{i=1}^{n} x_i dy_i\right) + dz$

$\alpha_{st} \wedge (d\alpha_{st})^n = (\sum x_i dy_i + dz) \wedge (d(\sum x_i dy_i + dz))^n = (\sum x_i dy_i + dz) \wedge (\sum dx_i \wedge dy_i)^n$

$= (\sum x_i dy_i + dz) \wedge \underbrace{(\sum dx_i \wedge dy_i) \wedge (\sum dx_i \wedge dy_i)} \wedge (\sum dx_i \wedge dy_i)^{n-2}$

$= (\sum x_i dy_i + dz) \wedge \sum_{i \neq j} dx_i \wedge dy_i \wedge dx_j \wedge dy_j \wedge (\sum dx_i \wedge dy_i)^{n-2}$

$= (\sum x_i dy_i + dz) \wedge n!\,(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \dots \wedge dx_n \wedge dy_n)$

$= dz \wedge n!\,(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \dots \wedge dx_n \wedge dy_n)$

$= n!\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \dots \wedge dx_n \wedge dy_n \wedge dz \neq 0$

Compute $R_{\alpha_{st}} = (\sum A_i \partial x_i + B_i \partial y_i) + C \partial_z$

$d\alpha_{st}(R_{st}, \cdot) = \sum A_i dy_i - B_i dx_i = 0$

$\alpha_{st}(R_{st}) = C + \sum x_i B_i = 1$

One solution: $R_{\alpha_{st}} = \partial_z$

(2) $\xi_{syn} = \ker\big(\underbrace{\sum (x_i dy_i - y_i dx_i) + dz}_{\alpha}\big)$

Homework:
- This is a contact structure
- $R_\alpha = \partial_z$
- for $n=1$: draw this
- read Milnor: Topology from a differential viewpoint
  Jeffrey Lee

Recall: $(M^{2n+1}, \ker(\alpha) = \xi^{2n} \subset TM)$ Contact manifold (19.10.23)

$\Leftrightarrow$ $\alpha \wedge (d\alpha)^n$ is a volume form (i.e. $\neq 0$)

$\mathbb{R}^{2n+1}$ $(x_1, y_1, \dots, x_n, y_n, z)$

(1) $\alpha_{st} = \left(\sum_{i=1}^{n} x_i dy_i\right) + dz \Rightarrow \alpha_{st} \wedge (d\alpha_{st})^n \neq 0$

(2) $\alpha_{sym} = \left(\sum_{i=1}^{n} x_i dy_i - y_i dx_i\right) + dz \Rightarrow \alpha_{sym} \wedge (d\alpha_{sym})^n \neq 0$

see homework week 1

(3) $\mathbb{R}^3$ with cylindrical coordinates (θ, r, z)

$\xi_{OT} = \ker(\alpha_{OT}) = \ker(\cos(r)dz + r\sin(r)\,d\theta)$

$d\alpha_{OT} = -\sin r\, dr \wedge dz + \underbrace{\frac{\partial \cos(r)}{\partial\theta}}_{=0} d\theta \wedge dz + (\sin(r) + r\cos(r))\, dr \wedge d\theta$

$\alpha_{OT} \wedge d\alpha_{OT} = \left[(-r\sin^2 r) - \sin r\cos r - r\cos^2 r\right] d\theta \wedge dr \wedge dz$

$= -\Big[\underbrace{1 + \frac{\sin(r)\cos(r)}{r}}_{\neq 0}\Big] \underbrace{r\, d\theta \wedge dr \wedge dz}_{\text{volume form}} \neq 0$

Exercise: $f: (r, \theta, z) \longmapsto (x, y, z)$ If you pull that back, you get the volume form

- Look at the volume form on a sphere S^2

When are two contact structures equivalent?

Definition: [Contactomorphism]

- $f: (M_1, \xi_1) \longrightarrow (M_2, \xi_2)$ is called contactomorphism

$:\Leftrightarrow$ f is a diffeomorphism s.t. $Tf(\xi_1) = \xi_2$

(i.e. $f^*(\alpha_2) = g\alpha_1$ for some $g: M_1 \to \mathbb{R}\setminus\{0\}$)

Pullback: $f: M_1 \longrightarrow M$

α_2 a 1-form on M_2

$\alpha_2: TM \longrightarrow \mathbb{R}$

$(f^*\alpha_2)_p(x_p) := (\alpha_2)_{f(p)}(T_p f(x_p))$

SHORT: $f^*\alpha_2(x) = \alpha_2(Tf(x))$

Example: $f : (\mathbb{R}^{2n+1}, \xi_{st}) \longrightarrow (\mathbb{R}^{2n+1}, \xi_{sym})$

$$(x,y,z) \longmapsto \left(\frac{x+y}{2}, \frac{y-x}{2}, z + \frac{\langle x,y\rangle}{2}\right)$$

is a contactomorphism.

proof:

$$f^* \alpha_{sym} = \left(\sum_{i=1}^n \frac{x_i+y_i}{2} d\left(\frac{y_i - x_i}{2}\right) - \left(\frac{y_i-x_i}{2}\right) d\left(\frac{x_i+y_i}{2}\right)\right) + d\left(z + \frac{\langle x,y\rangle}{2}\right)$$

$$= \sum_{i=1}^n \frac{x_i+y_i}{2} \frac{dy_i - dx_i}{2} - \left(\frac{y_i-x_i}{2}\right)\frac{dx_i + dy_i}{2} + dz$$

$$+ \sum_{i=1}^n \frac{x_i dy_i}{2} + \sum_{i=1}^n \frac{y_i dx_i}{2}$$

$$= dz + \sum_{i=1}^n \left(\frac{x_i dy_i}{2} - \frac{y_i dx_i}{2} + \frac{x_i dy_i}{2} + \frac{y_i dx_i}{2}\right)$$

$$= dz + \sum_{i=1}^n x_i dy_i$$

If $f^* \alpha_2 = \alpha_1$ then f is called strict

Thm 3 [Bennequin]: $(\mathbb{R}^3, \xi_{st}) \overset{\text{cont}}{\not\cong} (\mathbb{R}^3, \xi_{OT})$ proof maybe in section 4

$\to$ goal: distinguish contact manifolds

Example: $S^{2n-1} \subset \mathbb{C}^n$

Standard contact structure on the sphere:

$\xi_{st} := TS^{2n-1} \cap (i\, TS^{2n-1})$ HW sheet 1

Thm 4: $(S^{2n-1} \setminus \{pt\}, \xi_{st}) \overset{\text{cont}}{\cong} (\mathbb{R}^{2n-1}, \xi_{st})$

proof: HW, sheet 2

Let W^{2n} be a $(2n)$-manifold

Def: A symplectic form is a 2-form ω s.t.
$d\omega = 0$ & ω^n is a volume form

Ex: $\left(\mathbb{R}^{2n}, \omega_{st} := \sum_{i=1}^{n} dx_i \wedge dy_i\right)$

Def: a Liouville vector field Y on (W, ω) is a vector field s.t. $d(\omega(Y, \cdot)) =: d(\iota_Y \omega) = \omega$ $\qquad \iota :=$ plug in

Ex: $Y = \frac{1}{2} r \partial_r = \frac{1}{2}\left(\sum x_i \partial_{x_i} + y_i \partial_{y_i}\right)$ is Liouville on $(\mathbb{R}^{2n}, \omega_{st})$

$\iota_Y \omega_{st} = \frac{1}{2}\left(\sum x_i\, dy_i - y_i\, dx_i\right) \Rightarrow d(\iota_Y \omega_{st}) = \omega_{st}$

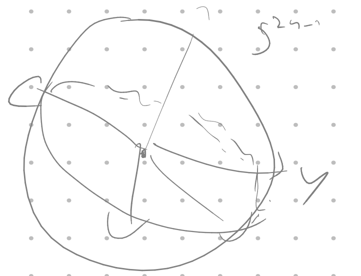
Lemma 5: Let Y be Liouville on (W^{2n}, ω)

$\Rightarrow$ $\alpha := \iota_Y \omega$ is a contact form on every hypersurface $M^{2n-1} \subset W$ transverse to Y.

Ex: $S^{2n-1} \subset (\mathbb{R}^{2n}, \omega_{st})$ is transverse to $Y = \frac{1}{2} r \partial_r$

$\Rightarrow$ $\alpha = \iota_Y \omega_{st}$ is a contact form

$= \sum x_i dy_i - y_i dx_i$



Homework:
- check directly that α is a contact form on S^{2n-1}
- sheet 1: $\ker(\alpha) = \xi_{st}$

proof (Lemma 5):

$$\alpha \wedge (d\alpha)^{n-1} = \iota_Y\omega \wedge (d(\iota_Y\omega))^{n-1}$$

$$= \iota_Y\omega \wedge \omega^{n-1}$$

$$= \frac{1}{n}\iota_Y(\omega^n)$$

write down coordinates and compute it explicitly

$\omega^n \neq 0 \Rightarrow \alpha \wedge (d\alpha)^{n-1} \neq 0$ on Σ transverse to Y $\square$

(for me to understand $(\alpha \wedge (d\alpha)^{n-1})(Y, \dots) = \frac{1}{n}\omega^n(Y, Y) = 0$)

Given a manifold B^n the space of contact elements:

$$\{ (b, V_b) \mid b \in B \text{ \& } V_b^{n-1} \subset T_bB \text{ oriented \& co-oriented hyperplanes} \}$$

Lemma 6: Space of contact elements $\cong S^*B$ (unit cotangent bundle)

proof: $(b, V_b) \overset{1:1}{\longmapsto} U_b^{V_b} : T_bB \longrightarrow \mathbb{R}$ linearly with $\ker(U_b^{V_b}) = V_b$

V_b oriented & cooriented

$U_b^{V_b}$ is unique up to scaling $\square$

Theorem 7 | $\pi : S^*B \longrightarrow B$

Define a hyperplane field ξ_{can} on S^*B as follows:

$$\begin{tikzcd}
(T_\pi)^{-1}(V_b) =: \xi(b, u_b^{V_b}) \subset T(S^*B) \arrow[r] \arrow[d, "T_\pi"] \arrow[dr, phantom, "\circlearrowright"] & S^*B \ni (b, u_b^{V_b} : T_bB \longrightarrow \mathbb{R}) \arrow[d, "\pi"] \\
V_b \subset TB \arrow[r] & B
\end{tikzcd}$$

$\Rightarrow$ ξ_{can} is a contact structure called the canonical contact structure

Proof: Let $(q_1, \dots, q_n)$ be local coordinates on B and $(p_1, \dots, p_n)$ be dual coordinates in the fibers of T^*B, i.e. $(q_1, \dots, q_n, p_1, \dots, p_n) = \left(\sum_{j=1}^{n} p_j dq_j\right) \in T^*B$

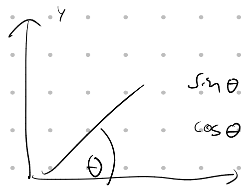
$$V_{(q_1, \dots, q_n)} = \left\{ \sum_{j=1}^{n} p_j dq_j = 0 \right\}$$ $\square$

Nearby Lagrangian conjecture $\quad M \overset{C^\infty}{\cong} N \underset{\overset{\Longleftarrow}{?}}{\Rightarrow} (S^*M, \xi_{can}) \overset{cont.}{\cong} (S^*N, \xi_{can})$

Ex: $B = T^2$ or $B = \mathbb{R}^2$ or $B = D^2$ with coordinates (x, y).

$\Rightarrow$ $S^*B \cong S^1 \times B$

$$\xi_{can} = \ker(\sin\theta \, dx - \cos\theta \, dy)$$



$\sin\theta \, dx - \cos\theta \, dy = 0$

2.3 Gray stability, the Moser trick and Darboux's theorem

Lemma 8: Let ω_t, $t \in [0,1] =: I$ a smooth family of k-forms on M and $(\psi_t)_{t \in I}$ an isotopy (i.e. $\psi_0 = id_M$, ψ_t diffeo of M)

Define a vector field X_t by $X_t \circ \psi_t = \dot{\psi}_t$

$$\Rightarrow \frac{d}{dt}(\psi_t^* \omega_t) = \psi_t^*(\dot{\omega}_t + \underbrace{L_{X_t}}_{\text{Lie derivative}} \omega_t)$$

$$L_X \omega = d(\iota_X \omega) + \iota_X d\omega$$

proof: sheet 2, bonus exercise

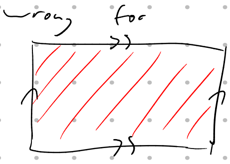
Theorem 9 (Gray stability)

Let ξ_t, $t \in I$ be a family of contact structures on M, M closed.

$\Rightarrow$ $\exists$ isotopy $(\psi_t)_{t \in I}$ on M s.t. $T\psi_t(\xi_0) = \xi_t \quad \forall t \in I$

Ex: This is wrong for foliations:

$M = T^2$



F_s = curves of slope s

$s \in \mathbb{Q}$: leaves are all closed

$s \in \mathbb{R} \setminus \mathbb{Q}$: leaves are all open

$\Rightarrow$ F_t for $s \in \mathbb{Q}$ and $s' \in \mathbb{R} \setminus \mathbb{Q}$ are not isotopic.

proof (thm 9 via moser trick)

Let α_t smooth s.t. $\ker(\alpha_t) = \xi_t$. Need to construct: ψ_t with $\psi_t^*(\alpha_t) = \lambda_t \alpha_0$,

$\lambda_t : M \to \mathbb{R}_+$

Assume ψ_t is the flow of a vector field X_t.

$\Rightarrow \frac{d}{dt}\left(\psi_t^*(\alpha_t)\right) = \dot{\lambda}_t \alpha_0 = \frac{\dot{\lambda}_t}{\lambda_t} \cdot \psi_t^* \alpha_t = \psi_t^*(\mu_t \alpha_t)$ (with $\mu_t = \frac{d}{dt}\left(\log(\lambda_t)\right) \circ \psi_t^{-1}$)

with Lemma 8, on the other hand, we have

$$\frac{d}{dt}(\psi_t^*(\alpha_t)) = \psi_t^*(\dot{\alpha}_t + L_{X_t} \alpha_t) = \psi_t^*\left(\dot{\alpha}_t + d(\alpha_t(X_t)) + \iota_{X_t}(d\alpha_t)\right).$$

As ψ_t is a diffeomorphism, this is equivalent to

$\Leftrightarrow \mu_t \cdot \alpha_t = \dot{\alpha}_t + d(\alpha_t(X_t)) + \iota_{X_t}(d\alpha_t))$

If furthermore $X_t \in \xi_t = \ker(\alpha_t)$

$\Leftrightarrow \mu_t \alpha_t = \dot{\alpha}_t + \iota_{X_t} d\alpha_t$ $(*)$

Plugging in R_{α_t}, we obtain $\mu_t = \dot{\alpha}_t(R_{\alpha_t})$.

- Define $\mu_t := \dot{\alpha}_t(R_{\alpha_t})$
- $R_{\alpha_t} \in \ker(\mu_t \alpha_t - \dot{\alpha}_t)$ & $d\alpha_t|_{\xi_t}$ non-degenerate $\Rightarrow \exists!$ solution $X_t \in \xi_t$ of $(*)$
- M closed $\Rightarrow$ Flow ψ_t of X_t is globally defined $\square$

<u>Def</u> A vector field on $(M, \xi = \ker(\alpha))$ is called <u>contact vector field</u>

$:\Leftrightarrow$ Flow ψ_t of X is a contactomorphism, i.e. $T\psi_t(\xi) = \xi$

$\overset{\text{thm}}{\Leftrightarrow}$ $L_X \alpha = \mu \cdot \alpha$ for $\mu : M \to \mathbb{R}$

<u>Ex 1</u>: $M = S^1 \times \mathbb{R}^2$ $\alpha = \cos(\theta)dx + \sin(\theta)dy$

$X = x\partial x + y \partial y$

$L_X \alpha = d\,\alpha(X) + \iota_X d\alpha$

$= d(x\cos\theta + y\sin\theta) + \iota_X(-\sin\theta\, d\theta \wedge dx + \cos\theta\, d\theta \wedge dy)$

$= \cos\theta\, dx + \sin\theta\, dy - x\sin\theta\, d\theta + y\cos\theta\, d\theta + \sin\theta\, x\, d\theta - \cos\theta\, y\, d\theta$

$= \alpha$

$\Rightarrow X$ is a contact vector field

<u>Ex 2</u>: $L_{R_\alpha} \alpha = d\,\iota_{R_\alpha}(\alpha) + \iota_{R_\alpha} d\alpha = d\,\underbrace{\alpha(R_\alpha)}_{=1} + \underbrace{d\alpha(R_\alpha, \cdot)}_{=0} = 0$

$\Rightarrow R_\alpha$ is a contact vector field

Thm 10: Let α with $\xi = \ker\alpha$. Then

$$\left\{\begin{matrix}\text{contact}\\ \text{vector}\\ \text{fields}\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} C^\infty(M, \mathbb{R})$$

$$X \longmapsto \alpha(X) =: H_X$$

$$\left.\begin{matrix}\alpha(X_H) = H\\ \iota_{X_H} d\alpha = dH(R_\alpha)\alpha - dH\end{matrix}\right\} =: X_H \longleftarrow\!\!| \; H$$

Observe: $R_\alpha \in \ker(dH(R_\alpha)\alpha - dH)$

proof: "$\longrightarrow$" Given X. Define $H_X := \alpha(X)$

$\Rightarrow dH_X + \iota_X d\alpha = L_X \alpha = \mu \cdot \alpha$

plug in R_α

$\Rightarrow dH_X(R_\alpha) + \underbrace{d\alpha(X, R_\alpha)}_{=0} = \mu \cdot \underbrace{\alpha(R_\alpha)}_{=1}$

$\Rightarrow \iota_X d\alpha = dH_X(R_\alpha)\cdot\alpha - dH_X$

$\Rightarrow X_{H_X} = X$

"$\longleftarrow$" Given H. Define X_H as above

- $L_{X_H}\alpha = d(\underbrace{\alpha(X_H)}_{H}) + \iota_{X_H} d\alpha = dH(R_\alpha)\alpha \Rightarrow X_H$ is a contact vector field
- $H_{X_H} = \alpha(X_H) = H$

Thm 11 (Darboux) Let α be a contact form on M^{2n+1} and let $p \in M$.

$\Rightarrow \exists$ NBHD $U \subset M$ of p & coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$ on U s.t.

$p := (0, \dots, 0)$ and $\alpha|_U = \left(\sum_{j=1}^n x_j dy_j\right) + dz$

proof (via Moser trick) w.l.o.g. $M = \mathbb{R}^{2n+1}$ (because we are working locally) and $p = 0$.

Choose coordinates s.t. on $T_0\mathbb{R}^{2n+1}$: $\alpha(\partial_z) = 1$; $\iota_{\partial_z} d\alpha = 0$; $\partial x_j, \partial y_j \in \ker\alpha$; $d\alpha = \sum_{j=1}^n dx_j \wedge dy_j$

Define $\alpha_0 := \left(\sum_{j=1}^n x_j dy_j\right) + dz$

$\alpha_t := (1-t)\alpha_0 + t\alpha, \quad t \in I$

$\Rightarrow \alpha_t := (1-t)\alpha_0 + t\alpha, \quad t \in I$

$\Rightarrow \alpha_t = \alpha$ & $d\alpha_t = d\alpha$ at $p = 0$ $\Rightarrow \alpha_t$ is contact form $\forall t$ on a NBHD of $p = 0$

(because it's contact at $0 \Rightarrow$ nonzero at $p=0$
$\Rightarrow$ nonzero in a NBHD of 0)

Moser trick: Assume $\Psi_t^* \alpha_t = \alpha_0$ for Ψ_t the flow of X_t

Lemma 8
$\Rightarrow \Psi_t^*(\dot\alpha_t + L_{X_t}\alpha_t) = 0 \iff \dot\alpha_t + d(\alpha_t(X_t)) + \iota_{X_t} d\alpha_t = 0$

Write $X_t = H_t R_{\alpha_t} + Y_t$ for $Y_t \in \ker(\alpha_t)$ & plug in R_{α_t}: $\dot\alpha_t(R_{\alpha_t}) + dH_t(R_{\alpha_t}) = 0$ (*)

On a NBHD of $p=0$ where R_{α_t} has no closed orbit $\exists$ solution of (*) with $H_t(0) = 0$, $dH_t|_0 = 0$ $\forall t \in I$

Define Y_t by $\dot\alpha_t + dH_t + \iota_{Y_t} d\alpha_t = 0 \Rightarrow X_t(0) = 0 \Rightarrow \Psi_t :=$ Flow of X_t is defined $\forall t \in I$ on a NBHD of $p=0$.

2.4 Order of contact

Lemma: Let $L \subset (M^{2n+1}, \xi)$ s.t. $TL \subset \xi$

$\Rightarrow \dim(L) \le n$

If $\dim(L) = n \Rightarrow L$ Legendrian

If $\dim(L) < n \Rightarrow L$ isotropic

proof: $i: L \hookrightarrow M$ $\quad \xi = \ker(\alpha)$

$TL \subset \xi \Leftrightarrow i^*\alpha = 0$

$\Rightarrow 0 = d i^*\alpha = i^* d\alpha$

$$T_pL \subset (\xi_p, d\alpha|_{\xi_p})$$

$$T_pL \subset (T_pL)^{d\alpha|_{\xi_p}} := \{ v \in \xi_p \mid d\alpha(v,u) = 0 \quad \forall u \in T_pL \}$$

Consider $\Phi: \xi_p \longrightarrow T_pL^*$

$$v \longmapsto d\alpha(v,\cdot)\big|_{T_pL}$$

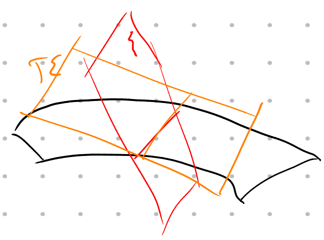
$$\Rightarrow 2n = \dim(\xi_p) = \dim(\underset{\substack{\| \\ (T_pL)^{d\alpha|_{\xi_p}} \\ \cup \\ T_pL}}{\ker \Phi}) + \dim(\underset{\substack{\| \\ T_pL^*}}{\operatorname{im} \Phi}) \ge 2 \dim(T_pL) = 2\dim(L)$$

$\square$

Let M^3 be a 3-mfld & ξ^2 a 2-plane field on M.

Let $\Sigma \subset M$ a surface & (u,v) coord. on Σ near $p = (0,0) \in \Sigma$

$$\theta(u,v) := \sphericalangle(T_{(u,v)}\Sigma, \xi_{(u,v)})$$



ξ has <u>contact of order at least k</u> with Σ at p $:\Leftrightarrow$ $\theta(u,v)$ is of type $O(|(u,v)|^k)$ for $(u,v) \longrightarrow (0,0)$, i.e. $\frac{\theta(u,v)}{|(u,v)|^k} \le C$ for $(u,v) \longrightarrow (0,0)$.

Theorem 13: ξ is contact at $p \in M \iff \forall \Sigma \subset M$ with $p \in \Sigma$: Σ has order of contact at most 1 with ξ at p.

Proof: WLOG $M = \mathbb{R}^3$, $p = 0$ & $\xi_0 = (x,y)$-Plane

$$\xi = \ker(\underbrace{dz + a(x,y,z)\,dx + b(x,y,z)\,dy}_{\alpha})$$

with $a(0) = 0 = b(0)$.

ξ is contact at $p \iff 0 \neq (\alpha \wedge d\alpha)_p = \Big(-\underbrace{\frac{\partial a}{\partial y}(0)}_{=: a_y(0)} + \underbrace{\frac{\partial b}{\partial x}(0)}_{=: b_x(0)}\Big)\, dx \wedge dy \wedge dz$

$\iff a_y(0) \neq b_x(0)$

$\underline{\text{w.t.s.}}$ $a_y(0) = b_x(0) \iff \exists \Sigma$ s.t. Σ has order of contact at least 2 with ξ at p.

If Σ has order of contact at least 1.

locally $\Rightarrow$ $\Sigma = \{(x,y,z) \mid z = f(x,y)\}$

with $f(0,0) = f_x(0,0) = f_y(0,0) = 0$

$\xi = \langle \partial_x - a\partial_z, \partial_y - b\partial_z \rangle$

$\Rightarrow n_\xi(x,y) := a(x,y,f(x,y))\partial x + b(x,y,f(x,y))\partial_y + \partial_z \quad \perp \xi_{(x,y)}$

$n_\Sigma(x,y) := -f_x(x,y)\partial x - f_y(x,y)\partial y + \partial z \quad \perp T\Sigma$

$$h(x,y) := \sin^2(\theta(x,y)) = \frac{|n_\xi(x,y) \times n_\xi(x,y)|^2}{|n_\xi|^2 \quad |n_\Sigma|^2}$$

$$= \frac{(b+f_y)^2 + (a+f_x)^2 + (bf_x - af_y)^2}{(1+a^2+b^2)\;(1+f_x^2+f_y^2)}$$

$\Rightarrow$ ξ has contact with Σ of order at least 2 $\iff$ $h(x,y)$ vanishes to order 4 at $(0,0)$.

$h(x,y)$ vanishes to order 4 at $(0,0)$,

$\Leftrightarrow$
$$\left.\begin{aligned} a_x(0) + f_{xx}(0) &= 0 \\ a_y(0) + f_{xy}(0) &= 0 \\ b_x(0) + f_{xy}(0) &= 0 \\ b_y(0) + f_{yy}(0) &= 0 \end{aligned}\right\} (\ast)$$

"$\Leftarrow$" if Σ has order of contact at least 2

$\Rightarrow a_y(0) + f_{xy}(0) = 0 = b_x(0) + f_{xy}(0)$

$\Rightarrow a_y(0) = b_x(0)$

$\Rightarrow \xi$ is not contact

"$\Rightarrow$" Assume $a_y(0) = b_x(0)$. $\quad f(x,y) := -\int_0^1 \big(a(tx, ty, 0) + b(tx, ty, 0) \big)\, dt$

we compute:
$$f_{xx}(0) = -a_x(0)$$
$$f_{xy}(0) = -\tfrac{1}{2}\, a_y(0) + b_x(0)$$
$$f_{yy}(0) = -b_y(0)$$

$\Rightarrow f_{xy}(0) = -a_y(0) = -b_x(0) \Rightarrow (\ast)$ holds $\Rightarrow \Sigma = \{ z = f(x,y) \}$ has second-order contact $\blacksquare$