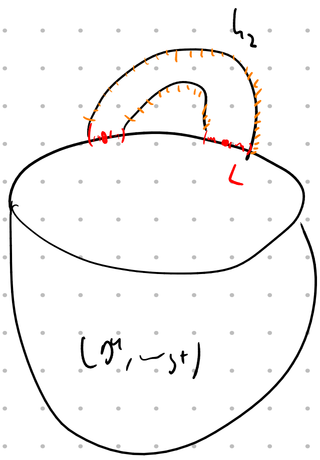
7. Dehn surgery on Legendrian knots

7.1 Contact Dehn surgery & fillings

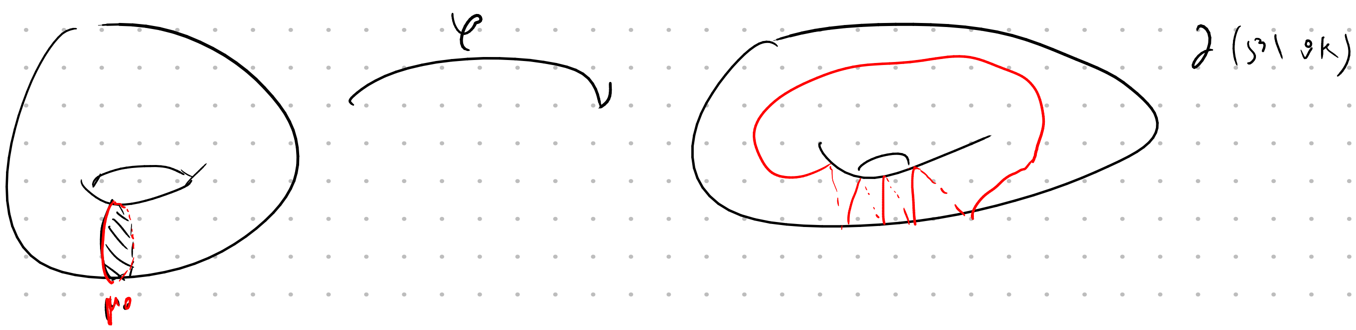


$(S^3, \xi_{st}) \longrightarrow (D^2 \times \partial D^2, \xi') \cup_\varphi (S^3 \setminus \nu L, \xi_{st})$

Theorem 1: Let $K \subset (S^3, \xi_{st})$ be a Legendrian knot & νK a std NBHD. & p, q coprime.

$$K(p/q) := S^1 \times D^2 \quad \cup_\varphi \quad S^3 \setminus \mathring{\nu} K$$

$$\mu_0 \xmapsto{\ \varphi\ } p\mu + q\lambda_c$$



Topology:

(1) $K(p/q)$ is again a 3-mfd that only depends on $p/q \in \mathbb{Q}$ & K.

Contact geometry:

(2) $\forall\, p/q \in \mathbb{Q} \setminus \{0\}$ $\exists$ finitely many tight contact structures ξ' on $S^1 \times D^2$ that induce contact structures on $K(p/q)$

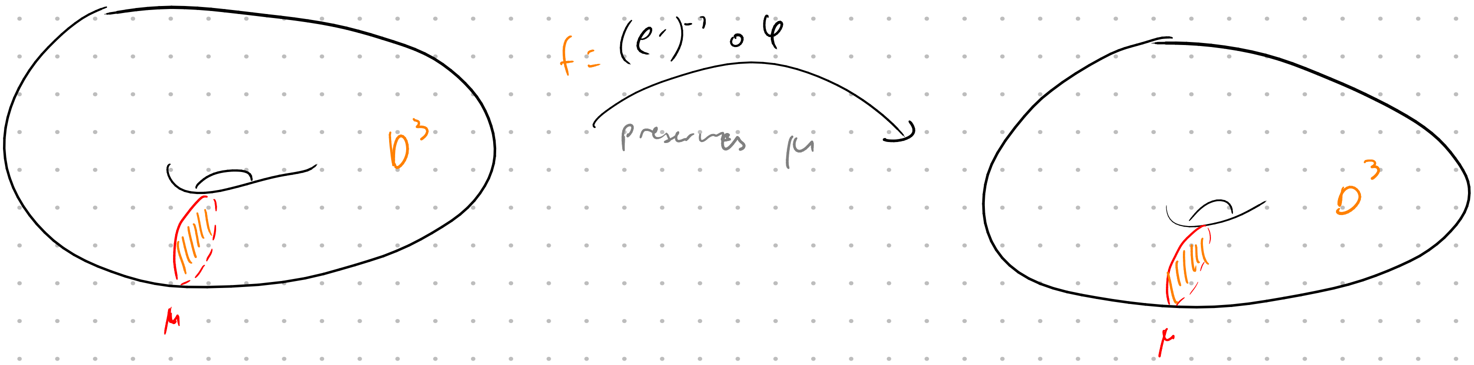
(3) $\forall\, n \in \mathbb{Z}$ $\exists!$ tight c.s. that induces c.s. on $K(1/n)$

proof: (1) Γ Alexander trick: $n=1,2$

$\forall f: S^n \xrightarrow{\cong C^\infty} S^n$

$\exists F: D^{n+1} \xrightarrow{\cong C^\infty} D^{n+1}$ s.t. $F|_\partial = f$ ⌟

$$\begin{array}{ccccc} K_\varphi(p/q) = & S^1 \times D^2 & \cup_\varphi & S^3 \setminus \nu K \\ & \Big\downarrow {\scriptstyle \text{Extension of } (\varphi')^{-1}\circ\varphi} & \circlearrowright & \Big\downarrow {\scriptstyle \cong\ id} \\ K_{\varphi'}(p/q) = & S^1 \times D^2 & \cup_{\varphi'} & S^3 \setminus \nu K \end{array}$$



(i) Extend f over D^2

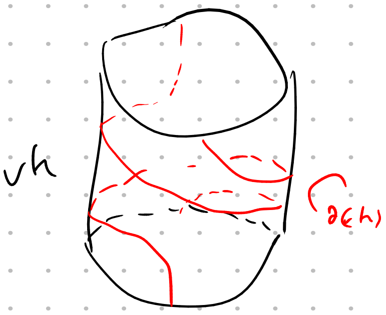
→ cut open along D^2

(ii) Extend f over D^3

(2) & (3) $(\nu K, \xi_{st}) \overset{\text{cont}}{\cong} (S^1 \times D^2, \cos\theta\, dx - \sin\theta\, dy)$

$\Rightarrow \partial(\nu K)$ is convex with dividing set $\Gamma_{\partial(\nu K)}$ of slope $= -1$,

i.e. $\Gamma_{\partial(K)}$ is given by 2 parallel copies of λ_c



$S^1 \times D^2 \cup_\varphi S^3 \setminus \mathring{\nu} K$

$$\mu_0 \xmapsto{\varphi} p\mu + q\lambda_c$$
$$\lambda_0 \xmapsto{\varphi} r\mu + s\lambda_c \qquad \text{with } \det\begin{pmatrix} p & q \\ r & s \end{pmatrix} = 1$$

$$s\mu_0 - q\lambda_0 \xleftarrow{\varphi^{-1}} \mu$$
$$-r\mu_0 + p\lambda_0 \xleftarrow{\varphi^{-1}} \lambda_c$$

$\Rightarrow$ $\Gamma_{\partial(\nu K)} = 2$ copies of $\lambda_c \xmapsto{\varphi^{-1}}$ two copies of $-r\mu + p\lambda_c$
$$\rotatebox{90}{=}$$
$$\Gamma_{\partial(S^1 \times D^2)}$$

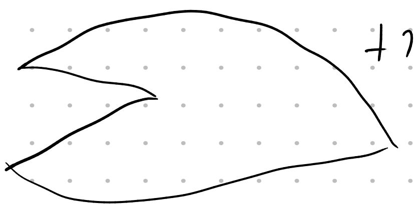
$\Rightarrow$ slope of $\Gamma_{\partial(S^1\times D^2)} = -\frac{p}{r}$

thm 5.5
$\Rightarrow$ if $p \neq 0$ $\Rightarrow$ $\exists$ finitely many tight c.s. on $S^1 \times D^2$ s.t.

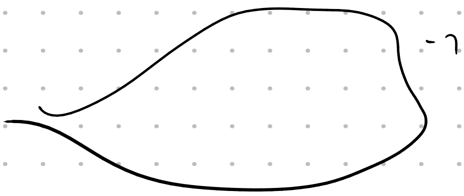
$$\Gamma_{\partial(S^1\times D^2)} \xrightarrow{\varphi} \Gamma_{\partial \nu K}$$

if $p = \pm 1$ $\Rightarrow$ $\exists !$ tight c.s. on $S^1 \times D^2$ s.t. $\square$

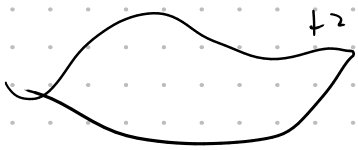
Ex:



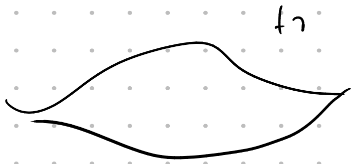
$= (S^3, \xi_{ot})$



$= (\mathbb{R}P^3, \xi_{st})$



$= (S^3, \xi_{ot})$



$= (S^1 \times S^2, \xi_{st})$