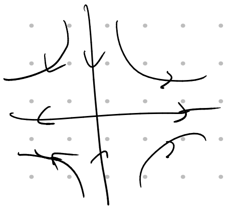
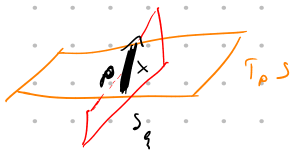
4. Surfaces in contact 3-mfds

Def A singular foliation on M is the equivalence class of a vector field X s.t. $X \sim X' \iff \exists f : M \to \mathbb{R}_+$ s.t. $X = f X'$



Let $S \subset (M, \xi = \ker(\alpha))$ be an oriented surface. The characteristic foliation S_ξ of S is given by $TS \cap \xi$



If $T_pS = \xi_p \implies X(p) = 0$

Ex: $S = S^2 \subset (\mathbb{R}^3, \ker(x\,dy - y\,dx + dz))$

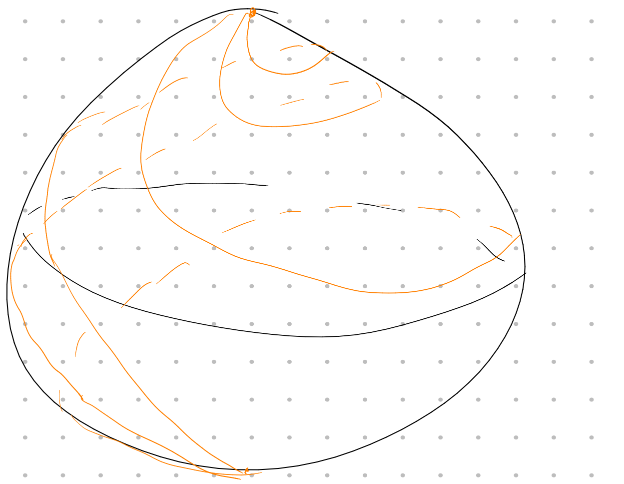
S_ξ is spanned by $X := (xz - y)\partial_x + (yz + x)\partial_y - (x^2 + y^2)\partial_z$

$X \in TS^2$ & $X \in \xi$

$X \in TS^2$: $x^2 z - yx + y^2 z + xy - (x^2 z + y^2 z) = 0$ ✓

$X \in \xi$: $(xyz + x^2) - (yxz - y^2) - (x^2 + y^2) = 0$ ✓

$X(x,y,z) = 0 \iff (x,y,z) = (0,0,\pm 1)$



Identify $\nu S \cong S \times \mathbb{R}$

$S \hookrightarrow S \times 0$

write $\alpha = \beta_z + u_z\,dz$

with $z \in \mathbb{R}$, β_z 1-forms on S

& $u_z : S \to \mathbb{R}$

$\implies d\alpha = d\beta_z - \dot\beta_z \wedge dz + du_z \wedge dz$

contact condition: $u_z\,d\beta_z + \beta_z \wedge (du_z - \dot\beta_z) > 0$

Let Ω be an area form on S.

Lemma 1: S_ξ is given by a vf defined by

$$i_X \Omega = \beta_0 = \alpha|_{TS}$$

proof: $*$ $X(p)=0 \iff \beta_{0,p}=0 \iff T_pS = \xi_p$

$*$ $0 = L_X(i_X\Omega) = L_X\beta_0 \Rightarrow X \in \xi$ $\square$

Lemma 2: A vector field X on S defines a characteristic foliation of a contact structure ξ

$\iff$ $\forall p \in S$ with $X(p)=0 \Rightarrow \operatorname{div}_\Omega(X)(p) \neq 0$ $(*)$

Divergence: $\operatorname{Div}_\Omega(X)$ of a vector field X on S wrt. an area form Ω is defined by

$$\operatorname{Div}_\Omega(X)\cdot\Omega := L_X\Omega = d(i_X\Omega)$$

Exercise: Check that this gives the old defn in coordinates

proof: "$\Rightarrow$" If $X(p)=0 \Rightarrow \beta_{0,p}=0 \Rightarrow \alpha_p = u_0(p)\,dz \Rightarrow \xi_p = T_pS$

$$\Rightarrow d(i_X\Omega) = (d\beta_0)_p = d\alpha|_{\xi_p} \neq 0$$

"$\Leftarrow$" Let X with $(*)$. Let $\beta := i_X\Omega$ & $u: S \to \mathbb{R}$ def by $d\beta = u\Omega$ (One-dimensional vector space $\Rightarrow \exists$ such a function)

$(*) \Rightarrow$ if $\beta_p = 0 \Rightarrow u(p) \neq 0$

choose 1-form γ on S

$$\beta\wedge\gamma \geq 0 \geq (\beta\wedge\gamma)(p) > 0 \quad \text{if } \beta(p)\neq 0$$

$\beta_z := \beta + z(du - \gamma)$, $d\beta_0 = d\beta = u\Omega$ & $\dot\beta_0 = du - \gamma$;

$\alpha = \beta_z + u\,dz$ is C.F. near S. $[u\,d\beta_0 + \beta_0\wedge(du - \dot\beta_0) = u^2\Omega + \beta\wedge\gamma > 0]$ $\square$

Ex: $S_1 = S^1 \times S^1 \subset (S^1 \times \mathbb{R}^2, \xi_n = \ker(\underbrace{\cos(n\theta)dx - \sin(n\theta)dy}_{\alpha_n}))$

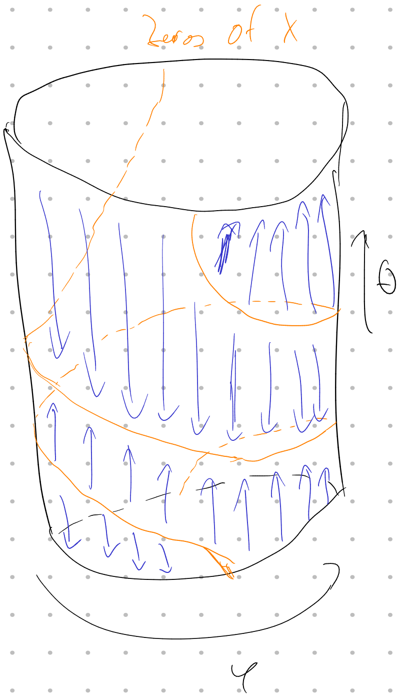
(r,φ) polar coordinates on $\mathbb{R}^2$

$\Omega := d\theta \wedge d\varphi$ defines the standard orientation on S

find X s.t. $\iota_X \Omega = \alpha_n|_{TS} = -(\cos(n\theta)\sin\varphi - \sin(n\theta)\cos\varphi)\,d\varphi$

Ansatz: $X(\theta,\varphi) = a(\theta,\varphi)\partial_\theta + b(\theta,\varphi)\partial_\varphi$

$$\Rightarrow \boxed{X = -(x\sin(n\theta) + y\cos(n\theta))\partial_\theta}$$



Thm 3: Let $S_i \subset (M_i, \xi_i)$, $i = 0,1$

& $\phi : S_0 \xrightarrow[\cong]{C^\infty} S_1$ s.t.

$\phi(S_{0,\xi_0}) = S_{1,\xi_1}$ as oriented foliations

$\Rightarrow \exists$ tubular neighborhoods νS_0 & νS_1 in M_0 & M_1

& $\Phi : (\nu S_0, \xi_0) \xrightarrow{\cong} (\nu S_1, \xi_1)$ s.t. $\Phi|_{S_0} = \phi$

proof: the (Moser trick). ∎

(4.2) Singularities of S_ξ

$X(x,y) = a(x,y)\partial_x + b(x,y)\partial_y$ on a NBHD of $0 \in \mathbb{R}^2$

with isolated zero at 0 $\quad \Omega := dx \wedge dy$

$d(\iota_X \Omega) = d(a\,dy - b\,dx) = (a_x + b_y)\,dx \wedge dy$

$\Rightarrow \operatorname{div}(X) = a_x + b_y$

characteristic foliation is given by flow lines φ_t of

$$(\dot x, \dot y) = X(x,y)$$

Linearized equation:

$$\begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = \underbrace{\begin{pmatrix} a_x(0) & a_y(0) \\ b_x(0) & b_y(0) \end{pmatrix}}_{=:A} \begin{pmatrix} x \\ y \end{pmatrix}$$

A singularity is called <u>non-degenerate</u> $\Leftrightarrow$ every eigenvalue of A has real part $\neq 0$.

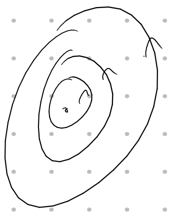
Remark [Hartman–Grobman thm]

If a singularity is non-deg. $\Rightarrow \exists\, C^1$-diffeo h on a NBHD of $0 \in \mathbb{R}^2$ s.t.

$$\varphi_t(h(x,y)) = h\left(e^{At}\cdot \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

Ex: $X = x\partial_y - y\partial_x$

$$\Rightarrow A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad EV = \pm i$$



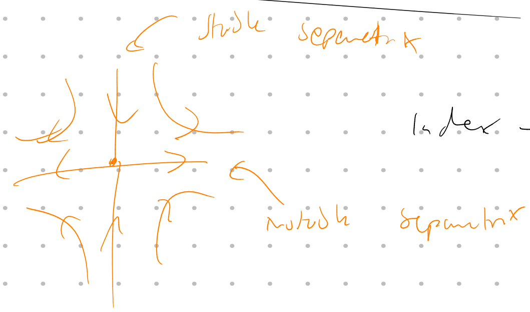
degenerate singularity

$\operatorname{Div} = 0$

$\Rightarrow$ does not appear in characteristic foliation

If a singularity is non-degenerate we call it

* elliptic $:\Leftrightarrow$ $\exists$ only one eigenvalue or $\exists$ two eigenvalues with real parts of the same sign
* hyperbolic $:\Leftrightarrow$ $\exists$ two real eigenvalues of opposite signs.

Elliptic	Hyperbolic
	
Index $+1$	Index -1
	Lin model: $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\quad \lambda_1 > 0 > \lambda_2$

$$\operatorname{Index}_X(p) := \deg\left(\begin{aligned} S^1_q &\longrightarrow S^1 \\ q &\longmapsto \frac{X(q)}{|X(q)|} \end{aligned} \right)$$

Sign: $+1$ if source, -1 if sink	sign of $(\lambda_1 + \lambda_2)$

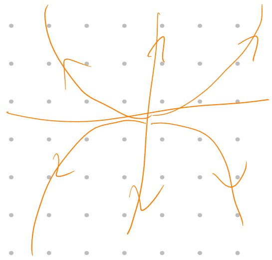
Example: Hyperbolic point

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \lambda_1 > 0 > \lambda_2$$

$$\Rightarrow X(x,y) = \lambda_1 x \partial_x + \lambda_2 y \partial_y$$

$\Rightarrow \beta = \iota_X(dx \wedge dy) = \lambda_1 x\, dy - \lambda_2 y\, dx$

$\Rightarrow \alpha = dz + \lambda_1 x\, dy - \lambda_2 y\, dx$ is a contact form $\Leftrightarrow \lambda_1 + \lambda_2 \neq 0$



Example: $X(x,y) = x\partial_x + y^3\partial_y \Rightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & 3y^2 \end{pmatrix}$

$\Rightarrow$ degenerated

But $\operatorname{div}(X)(0) = 1 \neq 0$

$\Rightarrow X$ defines the char. foliation of a contact str.

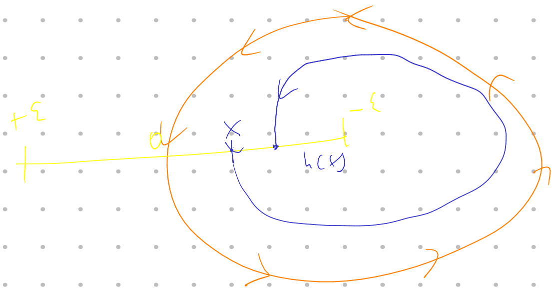
$(\alpha = -y^3\, dx + x\, dy + dz)$

By a C^∞-perturbation of X we get a non-deg. elliptic point.

A vector field X on a closed surface S is called Morse-Smale $:\Leftrightarrow$

(i) $\exists$ only finitely many singularities & finitely many closed orbits, all non-deg.

$\ulcorner$ a closed orbit is non-degenerate $:\Leftrightarrow$ Poincaré return map h satisfies $h'(0) \neq 1$



$\lrcorner$

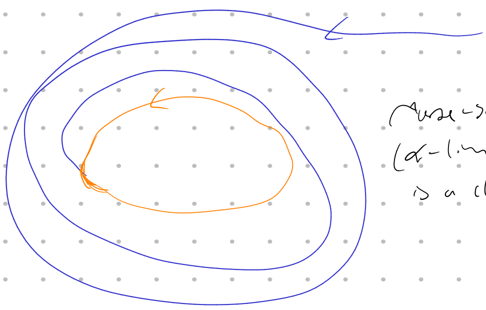
(ii) The α- & ω-limit sets of every flow-line are a single point or a closed orbit.

$\ulcorner$ φ_t flow of X

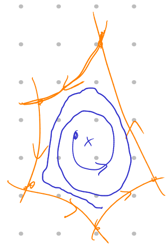
α-limit set of the orbit through $x_0 := \{ \lim_{n\to\infty} \varphi_{t_n}(x_0) \mid t_n \nearrow \infty \}$

ω- " " $\{ \quad " \quad \mid t_n \searrow -\infty \}$

$\urcorner$



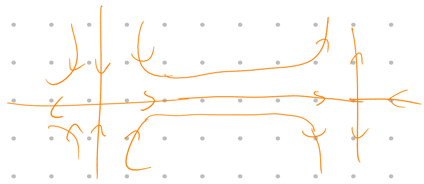
Morse-Smale (α-limit-set is a closed orbit)



not Morse-Smale

$\lrcorner$

(iii) $\nexists$ flow line connecting hyperbolic points



not morse-smale

Thm 4: After a C^∞-perturbation of the surface S we can assume that S_ξ is morse-smale

Proof: not easy $\longrightarrow$ see dynamical systems ∎

4.3 Convex surfaces (Giroux)

Def: $S \subset (M, \xi)$ is called convex $:\Longleftrightarrow$ $\exists$ contact vector field Y near S s.t. $Y \pitchfork S$

Ex: $S^1 \times S^1 \subset (S^1 \times \mathbb{R}^2, \ker(\underbrace{\cos(n\theta)dx - \sin(n\theta)dy}_{\alpha_n}))$

$Y = x\partial x + y\partial y$ $\quad$ $Y \pitchfork S$

$L_Y \alpha_n = i_Y(d\alpha_n) + d(i_Y \alpha_n) = \alpha_n \implies Y$ is a contact vector field

Ex: unit sphere in $(\mathbb{R}^3, \xi_{ot})$.

Lemma 5: $S \subset (M, \xi)$ closed is convex

$\Longleftrightarrow \exists\ \psi : S \times \mathbb{R} \longrightarrow M$ s.t. $p \longmapsto \psi(p, 0)$ is the inclusion $S \hookrightarrow M$
& $\ker(\psi^* \alpha)$ is an $\mathbb{R}$-inv. contact structure on $S \times \mathbb{R}$

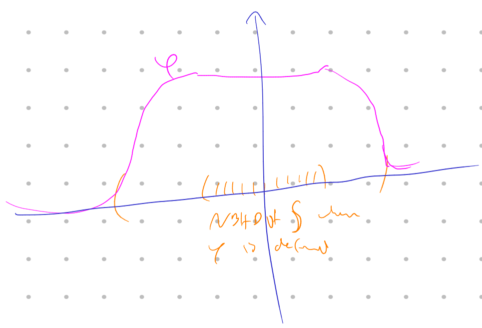
Proof:

"$\Longleftarrow$" $T\psi(\partial_t)$ is a contact vector field $\pitchfork S$

"$\Longrightarrow$" Let Y be a contact vector field s.t. $Y \pitchfork S$

$H := \alpha(Y)$ defined near S

Let $\varphi : M \longrightarrow \mathbb{R}$ s.t. $\varphi \equiv 1$ near S
$\varphi \equiv 0$ on $M \setminus \sim S$



$\overline{Y}$ The contact vector field corresponding to $\varphi \cdot H$

$\psi_t :=$ flow of $\overline{Y}$

$\psi : S \times \mathbb{R} \longrightarrow M$
$(p, t) \longmapsto \psi_t(p)$

$\rightarrow T\psi_{(p,t)}(\partial_t) = \dot{\psi}_t(p) = Y(\psi_t(p))$ near $S \implies \ker \psi^* \alpha$ is $\mathbb{R}$-inv. ∎

Write $\alpha = \beta + u\,dt$

$\Rightarrow \alpha \wedge d\alpha = (\beta + u\,dt) \wedge (d\beta + du \wedge dt)$
$= (u\,d\beta + \beta \wedge du) \wedge dt$

contact condition: $\boxed{u\,d\beta + \beta \wedge du > 0}$

* $d\beta = d(\iota_X \Omega) = \mathrm{Div}_\Omega(X)\,\Omega$
* $du \wedge \Omega = 0$ (3-form on a surface)

$\Rightarrow 0 = \iota_X(du \wedge \Omega) = X(u)\,\Omega - du \wedge \iota_X \Omega = X(u)\,\Omega + \beta \wedge du$

$\Rightarrow$ <u>contact condition</u> $u\,\mathrm{Div}_\Omega(X) - X(u) > 0$

Ex: $S^3 \subset \mathbb{C}^2 \qquad \alpha = r_1^2\,d\varphi_1 + r_2^2\,d\varphi_2$

$S := \{ r_1^2 = c,\ r_2^2 = 1 - c \} \cong T^2$ for $c \in (0,1)$

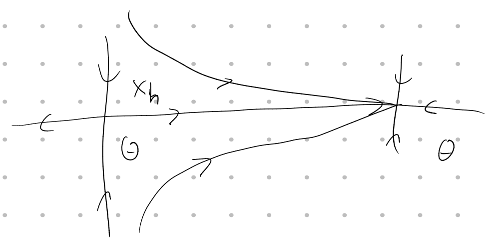
$\beta = \alpha|_{T_S} = c\,d\varphi_1 + (1-c)\,d\varphi_2 \qquad \Omega = d\varphi_1 \wedge d\varphi_2$

$\Rightarrow X = (1-c)\,\partial\varphi_1 - c\,\partial\varphi_2$
$d\beta = 0 \Rightarrow \mathrm{Div}_\Omega(X) = 0$

$\Rightarrow S$ is not convex since $-X(u) > 0$ admits no solution u on S.

6.4 The elimination lemma

An elliptic point x_e & a hyperbolic point x_h are in <u>elimination position</u> :$\Leftrightarrow$ $\mathrm{sign}(x_e) = \mathrm{sign}(x_h)$ & $\exists$ a separatrix of x_h connecting x_h & x_e

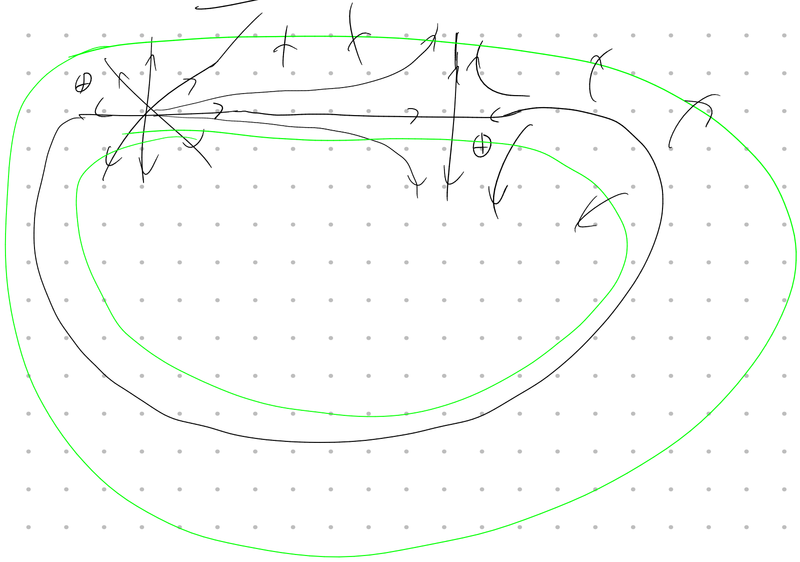


<u>Idea</u>: replace by

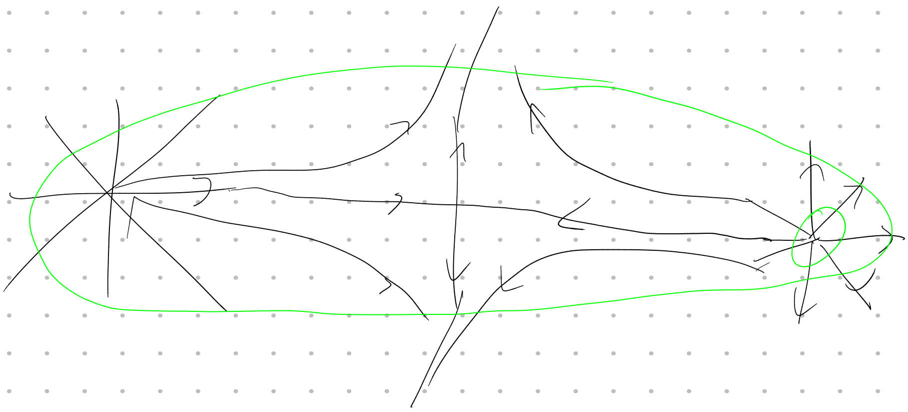
<u>Lemma 6</u>: Let x_e & x_h be in elim pos $\Rightarrow \exists$ an annulus $A \subset S$ s.t.

* x_e & x_h are the only singularities in A_ξ
* A_ξ has no closed orbit
* A_ξ is transversal to ∂A

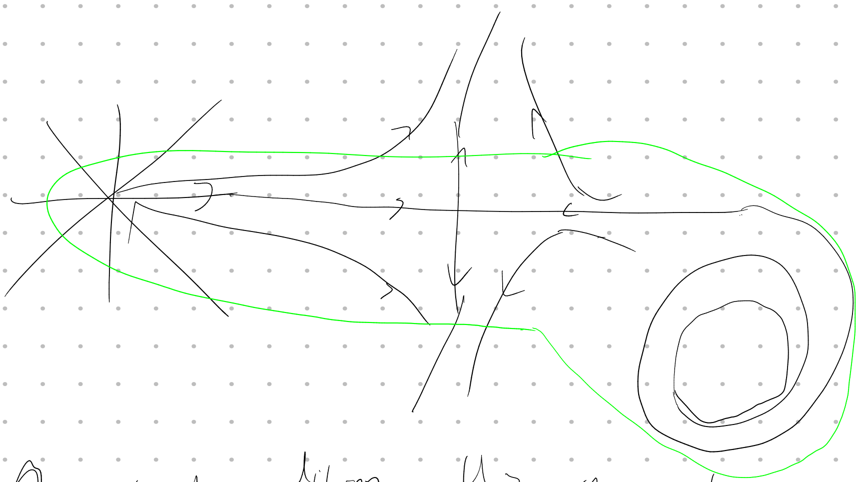
proof: Case 1:



Case 2:



Case 3



By the Morse-Smale-condition this are all cases (it can't go into a hyperbolic point)

Lemma 7: $\exists f: S \to \mathbb{R}^+$ s.t. $\operatorname{div}(f x) > 0$ on A

proof: $\operatorname{div}(f \cdot x)\, \Omega = d(i_{fx} \Omega) = d(f i_x \Omega)$

$= df \wedge i_x \Omega + f\, d(i_x \Omega) = df \wedge i_x \Omega + f \operatorname{div}(x) \Omega$

$df \wedge \Omega = 0 \Rightarrow 0 = i_x(df \wedge \Omega) = X(f)\Omega - df \wedge i_x \Omega$

$\Rightarrow \operatorname{Div}(fX) = X(f) + f \operatorname{div}(X)$

choose f constant on a nbhd of x_e^+ & x_h^+ s.t.

$X(f) + f\,\mathrm{div}(X) > 0$ on A

[$\mathrm{div}(X) \geq -C$ on A (as a continuous function on the compact set A)

$X(f) > Cf$ admits a solution f]

↑ simple differential inequality, can always be solved. □

Thm 8: [Elimination Lemma, Giroux]

Let $S \subset (M, \xi = \ker(\alpha))$ & $x_e, x_h \in S$ in elim. position

$\Rightarrow$ $\exists$ a small C^0 isotopy $\varphi_t : S \to M$, $t \in [0,1]$ s.t.

* φ_0 = inclusion $S \hookrightarrow M$
* $\varphi_t = id$ on $S \setminus A$
* $(\varphi_1(A))_\xi$ admits no singularities

Proof: replace α by $f \cdot \alpha$ $\Rightarrow$ $\beta = \alpha|_{TS}$ & X def. by $\iota_X \Omega = \beta$ satisfies $\mathrm{div}(X) > 0$ on A

$\Rightarrow$ $d\beta > 0$ on A

$\Rightarrow$ $\beta + dz$ is a contact form on $A \times \mathbb{R}$ (i.e. A is convex)

$\Rightarrow$ $\exists$ C^0-small nbhd of A in (M, ξ) is contactomorphic to $(A \times \mathbb{R}, \ker(\beta + dz))$

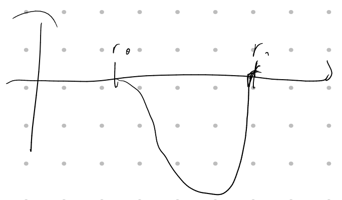
replace $A = A \times 0 \hookrightarrow A \times \mathbb{R}$ by the graph of a function

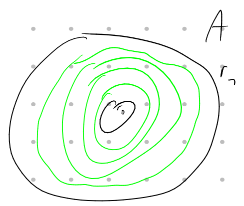
$g : A \to \mathbb{R}$ s.t. $g \equiv 0$ near ∂A

$\exists$ isotopy $\varphi_t(p) := (p, t g(p))$ $\quad (\varphi_1^* \alpha = \beta + dg)$ ↓

$\Rightarrow$ $(\varphi_1(A))_\xi$ is given by $X + X_g$ with $\iota_{(X + X_g)} \Omega = \beta + dg$

choose $g : A \to (-\infty, 0]$ as follows





$\iota_{X_g} \Omega = dg$ $\Rightarrow$ $dg(X_g) = \Omega(X_g, X_g) = 0$ $\Rightarrow$ X_g is tangent to $S^1 \times pt \subset S^1 \times I = A$

$X \pitchfork \partial A$

$\Longrightarrow$ $X + X_g \neq 0$ near ∂A on $A \setminus V(\partial A)$ choose d_g so large that $\beta + d_g \neq 0$

$\Longrightarrow$ $X + X_g \neq 0$ on $A \setminus V(\partial A)$ $\square$

$\Longrightarrow$ no singularities

4.5: The dividing set

Def: The *dividing set* Γ_S of a convex surface $S \subset (M, \xi)$ is

$$\Gamma_S := \{ p \in S \mid Y \in \xi \} \quad \text{(w.r.t. to a contact v.f. } Y)$$

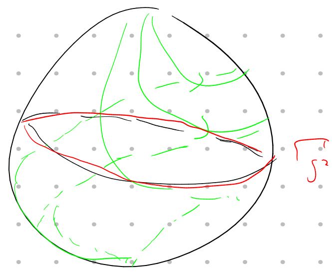
Ex 1 $S^2 \subset (\mathbb{R}^3, \ker(dz - y\,dx + dz))$

$Y := x\partial_x + y\partial_y + 2z\partial_z$ $\qquad Y \pitchfork S^2$

$L_Y \alpha = \iota_Y d\alpha + d(\iota_Y \alpha) = 2\alpha$

$\Rightarrow S^2$ is convex

$\alpha(Y) = 2z \Rightarrow \Gamma_{S^2} = \{z = 0\} \subset S^2$

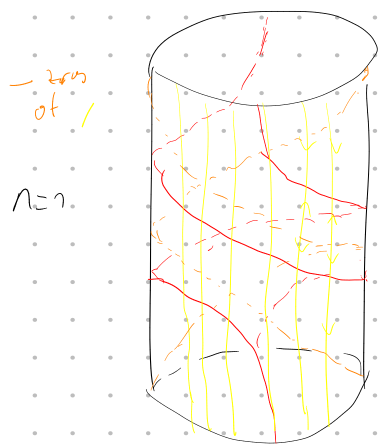


Ex 2 $S^1 \times S^1 \subset \left(S^1 \times \mathbb{R}^2, \ker(\underbrace{\cos(n\theta)\,dx - \sin(n\theta)\,dy}_{\alpha_n})\right)$

$Y = x\partial_x + y\partial_y$ is a contact v.f. $\pitchfork S^1 \times S^1$

$\alpha_n(Y) = x\cos(n\theta) - y\sin(n\theta)$

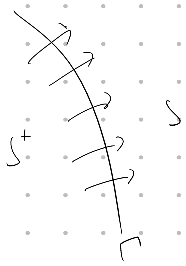
$\Rightarrow \Gamma_{S^1 \times S^1} = \{(\theta, \pm\sin(n\theta), \pm\cos(n\theta))\}$



Def: Let X be a v.f. repr. a singular foliation on a surface S with volume form Ω

$\Gamma \subset S$ *dividing* X $:\Longleftrightarrow$

* $\Gamma \subset S$
* $X \pitchfork \Gamma$
* $\Gamma \neq \emptyset$
* $L_X \Omega \neq 0$ on $S \setminus \Gamma$
* For $S_\pm := \{p \in S \mid \pm \operatorname{Div}_\Omega(X) > 0\}$ then X points out of S_+ along Γ



Corollary 9: Γ_S of a convex surface S divides S_ξ

proof: * $\Gamma_S = \{u(p) = 0\}$ contact condition $du \neq 0$ on Γ_S

$\Rightarrow \Gamma_S$ is a 1-mfd

* If $p \in \Gamma_S = u^{-1}(0)$ s.t. $X \in T_p\Gamma_S$

$\Rightarrow T_p\Gamma_S \subset \ker(\beta)$ (because $X \subset \ker(\beta)$)

$\Rightarrow \forall v \in T_p\Gamma_S = i_v(u\,d\beta + \beta \wedge du) = 0$ ↯ contact condition

* If $\Gamma = \emptyset \Rightarrow u \neq 0 \Rightarrow \beta/u + dz$ is a contact form

$\Rightarrow d(\beta/u)$ is an exact area form on S ↯

Γ $0 \neq \text{area}(S) = \int_S d(\beta/u) \overset{\text{Stokes}}{=} \int_{\partial S} (\beta/u) = 0$?

* on $S_\pm \times \mathbb{R}$ we can write $\xi = \ker\big(\underbrace{\beta/|u| \pm dz}_{=:\tilde\beta}\big)$

$\Rightarrow$ contact condition $= \pm\, d\tilde\beta > 0$ on $S_\pm$

$\Rightarrow \pm d\tilde\beta$ is an area form Ω_0 on $S_\pm$

$\Rightarrow d(i_X \Omega_0) = d\tilde\beta = \pm \Omega_0$

$\Rightarrow \operatorname{div}_{\Omega_0}(X) = \pm 1$ on S

$\Omega = f\,\Omega_0$ on $S_\pm$ for $f: S_\pm \to \mathbb{R}$

$\overset{\text{continuity}}{\Rightarrow} L_X \Omega \neq 0$ on $S \setminus \Gamma$ & X points out of S_+ along Γ

Thm 20: (a) $S \subset (M, \xi)$ is convex $\Leftrightarrow S_\xi$ is divided by a 1-mfd $\Gamma \subset S$

(b) Γ is (up to isotopy) determined by S_ξ

proof: (a) "$\Rightarrow$" $\Gamma = \Gamma_S$ by Corollary 9

"$\Leftarrow$" Let Ω be a volume form & X representative of S_ξ

$\beta := i_X \Omega$ & $\alpha := \beta + u\,dz$ on $S \times \mathbb{R}$.

α is contact $\Leftrightarrow u \operatorname{div}_\Omega(X) - X(u) > 0$

Γ divides $S_\xi \Rightarrow \exists$ suitable choice of u s.t. $u \operatorname{div}_\Omega(X) - X(u) > 0$.

$\Rightarrow \ker(\alpha)$ is an $\mathbb{R}$-invariant contact structure on $S \times \mathbb{R}$ & $S_\xi = S_{\ker\alpha}$

$\overset{\text{Thm 3}}{\Rightarrow} \xi$ & $\ker\alpha$ are contactomorphic near S

$\overset{S \text{ convex w.r.t. } \ker\alpha}{\Rightarrow}$ S convex w.r.t. ξ

(5) Let Y_0, Y_1 be contact vector fields transverse to S

$$\Gamma_i := \{ Y_i(p) \in \xi_p \}$$

Write $\xi_i = \ker(\beta + u_i \, dz)$ (u_i corresponding to Γ_i)

$\Rightarrow \Gamma_i = \{ u_i(p) = 0 \}$

Consider $\alpha_t = \beta + ((1-t)u_0 + t u_1)\, dz$

Apply the Moser trick to α_t to get a contact isotopy ψ_t

s.t. $\psi_1(\Gamma_0) = \Gamma_1$ $\square$