HOMOTOPICALLY STANDARD TIGHT NON-FILLABLE CONTACT STRUCTURES ON THE SPHERE

Introduction and Background

Contact topology is the study of contact manifolds up to isotopy. It can be viewed as the odd-dimensional counterpart to symplectic topology. In fact, many contact manifolds are fillable, i.e. they are the (contact) boundary of a corresponding symplectic manifold. This is an important tool in constructing and classifying contact structures. For the general question of existence and classification, the dichotomy between tight and overtwisted contact manifolds is essential [Eli89, BEM15]. It turns out that overtwisted contact structures are very much topological objects: The topological obstruction for the existence of a contact structure is the existence of an almost contact structure (which can be described as reduction of the structure group to $\mathrm{U}(n) \times \mathrm{id}$). In any such almost contact class (almost contact structures up to homotopy) there is precisely one overtwisted contact structure up to isotopy.

However, it is much harder to understand tight manifolds. Thanks to groundbreaking work by Gromov and Eliashberg [Gro85, Eli91] We know that fillable contact manifolds are tight. The standard contact structure ξ_{std} on S^{2n+1} is just the contact boundary of $(B^{2n+2}, \omega_{\mathrm{std}})$. On S^3, this is the unique tight contact structure, so there are no tight but non-fillable contact structures on S^3 [Eli92]. However, this is not true in general: The first examples of non-fillable tight contact structures were provided by Etnyre-Honda ($\dim = 3$, [EH02]) and Massot-Niederkrueger-Wendl ($\dim \geq 5$, [MNW13]). More recently, Bowden–Gironella–Moreno–Zhou [BGMZ22] have shown that for any tight manifold in $\dim \geq 7$, there exists a tight non-fillable contact structure in the same almost contact class (for $\dim = 5$ if the first Chern class vanishes). The first step towards this result is to construct a tight non-fillable contact structure on S^{2n+1} that is homotopically standard, i.e. in the same almost contact class as the standard contact structure ξ_{std}. The goal of my Master's thesis is to give a streamlined explanation of the proof of this result.

Sketch of proof

The proof consists of mainly 3 steps:

- Construction of the contact structure ξ on S^{2n+1},
- Establishing tightness of ξ,
- and showing that ξ is not symplectically fillable.

Construction. According to [Gir02], there is a correspondence between open books up to positive stabilization and contact structures that are supported by the open book, up to isotopy. For concreteness, we start with a very specific open book, the Milnor open book coming from the A_{k-1}-singularity, i.e. the binding is given by the Brieskorn sphere

$$B = \Sigma_{n-1}(k, 2, \dots, 2),$$

i.e. $f^{-1}(0) \cap S^{2n-1} \subset \mathbb{C}^n$ for $f(z_1, \dots, z_n) = z_1^k + z_2^2 + \cdots +$.

By the Thurston-Winkelnkemper construction, we explicitly construct a contact form supported this open book decomposition of S^{2n-1}. By a construction due to Bourgeois [Bou02], we use this contact form to build a contact form on $S^{2n-1} \times T^2$. What we really want, however, is a contact structure on S^{2n+1}. Therefore, we perform several surgeries on $S^{2n-1} \times T^2$ to correct that. By two 1-surgeries we kill the two homotopy classes in π_1. As 1-surgery doesn't change the higher homotopy groups by a general position argument, we can then proceed with a 2-surgery to kill the one homotopy class in π_2 and obtain a homology sphere M that bounds a homology ball $\partial W = M$. Using the Whitehead theorem we obtain that M is homotopy equivalent to a sphere and with the h-cobordism theorem we conclude that M is diffeomorphic to a sphere. By so called h-principles, the three surgeries can be realized as contact surgeries. Therefore, we obtain a contact manifold that is smoothly diffeomorphic to a sphere (by the h-cobordism theorem). The underlying almost contact structure can also be shown to coincide with the standard almost contact structure on S^{2n+1}, so that in total we have constructed a homotopically standard contact structure on the sphere. This is what I am working on currently. For the next months, I plan to cover the remaining two parts of the proof:

Tightness. In order to prove tightness, we use the concept of algebraic tightness which implies tightness, but is preserved under surgery. (Algebraic tightness is a statement about a certain contact homology algebra being nonzero). We need the following lemma.

Lemma 1. *The Bourgeois construction starting from our Milnor open book is algebraically tight.*

The proof is quite technical, so we don't go into details here. The surgeries from part one of the proof don't affect algebraic tightness, so the resulting contact manifold $(S^{2n+1}, \xi_{\mathrm{ex}})$ is algebraically tight, hence tight.

Obstructions to symplectic fillability. We use the fact that Bourgeois manifolds have a convex decomposition

$$M \times T^2 = (M \times S^1) \times S^1 = V_+ \times S^1 \cup_\phi \overline{V_-} \times S^1,$$

with $V_\pm = \Sigma \times D^* S^1$ where Σ is the page of our Milnor open book. The dividing set is $N := \partial V_\pm$. For the sake of contradiction, assume that there exists a symplectic filling W of (S^{2n+1}, ξ). It can then be shown via symplectic cohomology that $H_n(N) \hookrightarrow H_n(W)$. By clever construction of a certain symplectic cobordism, this turns out to factor as $H_n(N) \to H_n(S^{2n+1}) \to H_n(W)$. As $H_n(S^{2n+1}) = 0$, together with the injectivity of the first map, this implies that $H_n(N) = 0$. However, in the case under consideration one can construct a nontrivial homology class in $H_n(N)$, contradiction.

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