Nonlinear Optimization – Sheet 10

Exercise 1

(i) As the constraints are linear, the Abadie CQ holds everywhere and the KKT conditions are necessary optimality conditions, i.e. (a) $\implies$ (c).

The reduced problem (11.3) is of the form (4.1). By Lemma 4.1, we get (d)$\longleftrightarrow$(e)$\iff$(f).

Since we assume $h(\overline{x}) + h'(\overline{x})d = 0$ to be solvable, the problem (11.1) cannot infeasible. The objectives of (11.1) and (11.3) only differ by a constant term, therefore we have (b)$\iff$(e).

A minimizer of (11.3) minimizes (11.1) as well, and is feasible by construction, this yields (d) $\implies$ (a).

For any solution d, λ of the KKT system (11.2), $d - d_{\text{part}}$ is solution of (11.4) (Z^t annihilates $h'(\overline{x})^t$, which gives (c) $\implies$ (f), and we are done. The statement about solutions is clear from this as well.

(ii) By choosing y to be an eigenvector of an negative eigenvalue and scaling, I can get the objective of (11.3) arbitrarily low. Thus it is unbounded. The same follows for (11.1), since the objective just differs by an constant term.

(iii) yes, this is true.

Exercise 2

Consider

$$\text{Minimize} \quad \frac{1}{2}x^tAx + b^tx + c,\ x \in \mathbb{R}^n, \quad \text{s.t. } Cx = d$$

for symmetric $A \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n,\ c \in \mathbb{R}$ and $C \in \mathbb{R}^{n\times n_{eq}},\ d \in \mathbb{R}^{n_{eq}}$ and let x^* be a KKT Point of the above minimization problem.

Show: $\Lambda(x^*)$ is a singleton iff LICQ is statisfied at x^*.

Proof. An easy calculation shows

$$\text{LICQ at } x^* \iff x^* \in F \text{ and } \cdot C : \mathbb{R}^n \to \mathbb{R}^{n_{eq}} \text{ surjective.}$$

Let $f(x) = \frac{1}{2}x^tAx + b^tx + c,\ x \in \mathbb{R}^n$ and let $h(x) = Cx - d,\ x \in \mathbb{R}^n$. We surpress the parameter of the inequality constraints as there are none. So let $\lambda \in \Lambda(x^*)$ (non-empty because x^* is a KKT point) then

$$\nabla f(x) + h'(x^*)\lambda = 0 \iff Ax^* + b + C^t\lambda = 0 \iff C^t\lambda = -Ax^* - b$$

As LICQ at x^* iff C surjective iff C^t injective, the claim follows. □

Exercise 3

Prove Lemma 11.4 i.e.

$$\mu \geq 0,\ g(x) \leq 0,\ \mu^tg(x) = 0$$

is equivalent to

$$\mu \in K, \quad g(x)^t(\nu - \mu) \leq 0\ \forall \nu \in K$$

where $K = \mathbb{R}^{n_{ineq}}_{\geq 0}$.

Proof. $\mu \geq 0 \iff \mu \in K$ by definition.

„$\Rightarrow$“: $\nu \in K$:

$$g(x)^t(\nu - \mu) = g(x)^t\nu \leq 0$$

because $\nu \geq 0, g(x)^t \leq 0$ and $g(x)^t\mu = 0$.

„$\Leftarrow$“: Let $\mu \in K$ as $0, 2\mu \in K$ as well we get

$$g(x)^t(0-\mu) = -g(x)^t\mu \leq 0, \quad g(x)^t(2\mu-\mu) = g(x)^t\mu \leq 0$$

so $g(x)^t\mu = 0$. As $\mu \geq 0$ there is $\nu_i \in K$ with $\nu_i - \mu = e_i \ \forall i = 1, \ldots, n_{ineq}$. So

$$g(x)^t(\nu_i - \mu) = g(x)_i \leq 0$$

i.e. $g(x) \leq 0$. □

Exercise 4

Let $M \subset \mathbb{R}^n$ arbitrary, $x \in M$:

(i) The normal cone is a closed convex cone.

Proof. Closed: Continuity of inner product.
Convex: $\lambda \in [0,1], s, t \in N_M(x), y \in M$:

$$(\lambda s + (1-\lambda)t)^t y = \lambda s^t y + (1-\lambda)t^t y \leq 0.$$

Cone property: clear. □

(ii) $N_M(x) = (M - \{x\})^\circ$.

Counter example. Take $M = \{1, 2\} \subset \mathbb{R}$ and $x = 2$.

$$N_{\{1,2\}}(x) = \{s \in \mathbb{R} \mid s, 2s \leq 2s\} = [0, \infty)$$
$$(\{1,2\} \setminus \{2\})^\circ = \{1\}^\circ = \{s \in \mathbb{R} \mid s \cdot 1 \leq 0\} = (-\infty, 0].$$

□

(iii) $N_M(x) \subset T_M(x)^\circ$ but equality does not hold in general.

Proof. Let $s \in N_M(x)$ i.e. $s^t(y-x) \leq 0 \ \forall y \in M$. Let $d \in T_M(x)$ and $x^k \in M, t^k \searrow 0$ s.t.

$$d = \lim_{k\to\infty} \frac{x^k - x}{t^k}.$$

Then

$$s^t d = \lim_{k\to\infty} \frac{s^t(x^k - x)}{t^k} \leq 0$$

by continuity and assumption. □