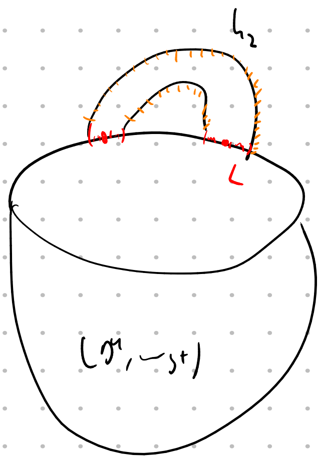
7. Dehn surgery on Legendrian knots

7.1 Contact Dehn surgery & fillings

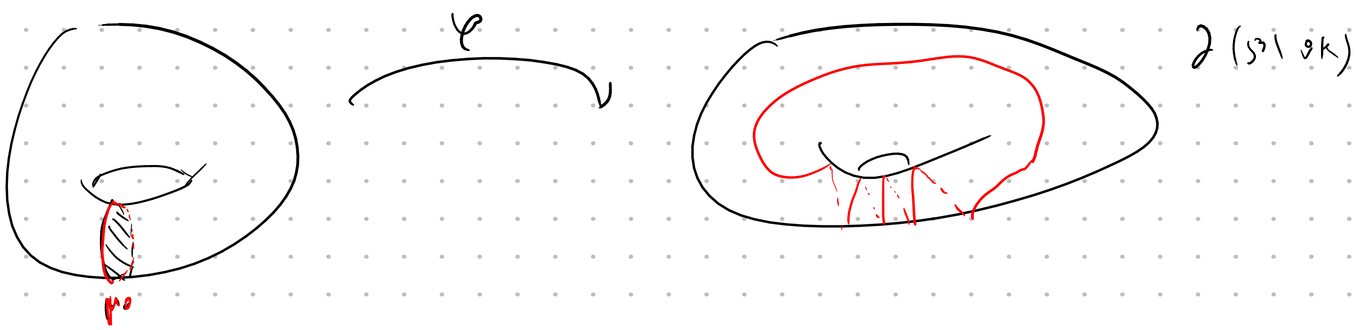


$(S^3, \xi_{st}) \longrightarrow (D^2 \times \partial D^2, \xi') \cup_\varphi (S^3 \setminus \nu L, \xi_{st})$

Theorem 1: Let $K \subset (S^3, \xi_{st})$ be a Legendrian knot & νK a std NBHD. & p, q coprime.

$$K(p/q) := S^1 \times D^2 \cup_\varphi S^3 \setminus \mathring{\nu} K$$

$$\mu_0 \xmapsto{\varphi} p\mu + q\lambda_c$$



Topology:

(1) $K(p/q)$ is again a 3-mfd that only depends on $p/q \in \mathbb{Q}$ & K.

Contact geometry:

(2) $\forall\, p/q \in \mathbb{Q} \setminus \{0\}$ $\exists$ finitely many tight contact structures ξ' on $S^1 \times D^2$ that induce contact structures on $K(p/q)$

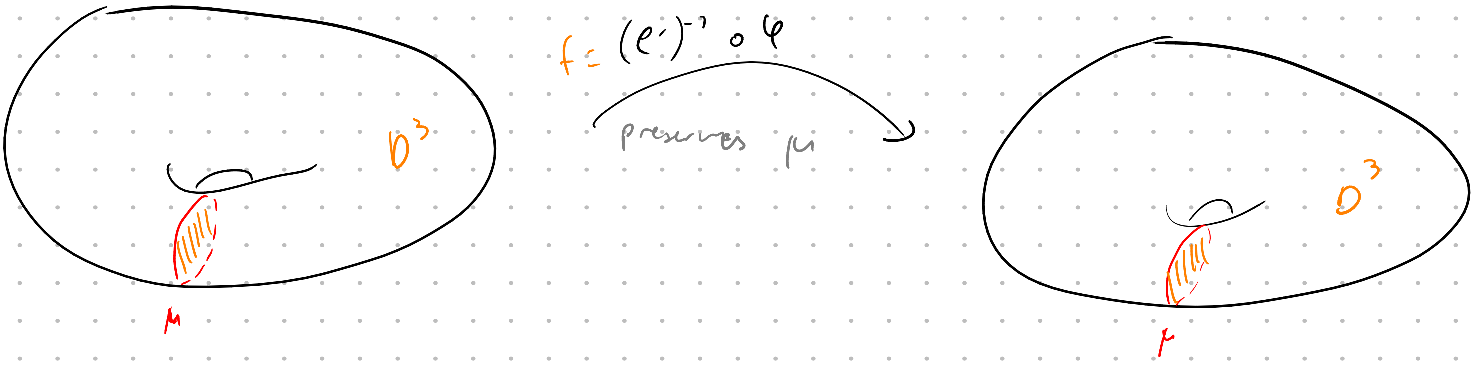
(3) $\forall\, n \in \mathbb{Z}$ $\exists!$ tight c.s. that induces c.s. on $K(1/n)$

proof: (1) Alexander trick: $n = 1, 2$

$\forall f : S^n \xrightarrow{\cong_{C^\infty}} S^n$

$\exists F : D^{n+1} \xrightarrow{\cong_{C^\infty}} D^{n+1}$ s.t. $F|_{\partial} = f$ ⌟

$$\begin{tikzcd}
K_\varphi(p/q) = S^1 \times D^2 \arrow[d, "\text{Extension of } (\varphi')^{-1} \circ \varphi"'] & \cup_\varphi & S^3 \setminus \nu K \arrow[d, "\cong \; id"] \\
K_{\varphi'}(p/q) = S^1 \times D^2 & \cup_{\varphi'} \arrow[u, phantom, "\circlearrowright"] & S^3 \setminus \nu K
\end{tikzcd}$$



(i) Extend f over D^2

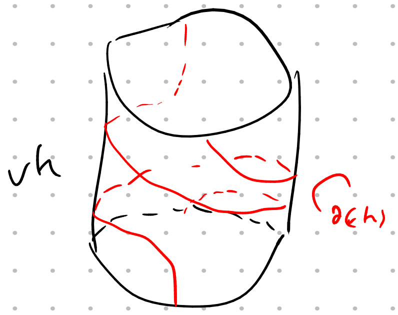
→ cut open along D^2

(ii) Extend f over D^3

(2) & (3) $(\nu K, \xi_{st}) \overset{\text{cont}}{\cong} (S^1 \times D^2, \cos\theta\, dx - \sin\theta\, dy)$

$\Rightarrow$ $\partial(\nu K)$ is convex with dividing set $\Gamma_{\partial \nu K}$ of slope $= -1$,

i.e. $\Gamma_{\partial(K)}$ is given by 2 parallel copies of λ_c



$S^1 \times D^2 \quad \cup_\varphi \quad S^3 \setminus \mathring{\nu} h$

$$\mu_0 \xmapsto{\varphi} p\mu + q\lambda_c$$
$$\lambda_0 \xmapsto{\varphi} r\mu + s\lambda_c \qquad \text{with } \det\begin{pmatrix} p & q \\ r & s \end{pmatrix} = 1$$

induces iso on homology

$$s\mu_0 - q\lambda_0 \xleftarrow{\varphi^{-1}} \mu$$
$$-r\mu_0 + p\lambda_0 \xleftarrow{\varphi^{-1}} \lambda_c$$

$\Rightarrow$ $\Gamma_{\partial(\nu h)} = 2$ copies of $\lambda_c \xmapsto{\varphi^{-1}}$ two copies of $-r\mu + p\lambda_c$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \Gamma_{\partial(S^1 \times D^2)}$

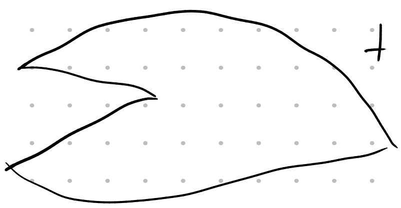
$\Rightarrow$ slope of $\Gamma_{\partial(S^1\times D^2)} = -\frac{p}{r}$

thm 5.5
$\Rightarrow$ if $p \neq 0 \Rightarrow \exists$ finitely many tight c.s. on $S^1 \times D^2$ s.t.

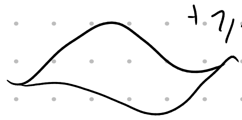
$$\Gamma_{\partial(S^1\times D^2)} \xrightarrow{\varphi} \Gamma_{\partial\nu K}$$

if $p = \pm 1 \Rightarrow \exists!$ tight c.s. on $S^1 \times D^2$ s.t. $\square$

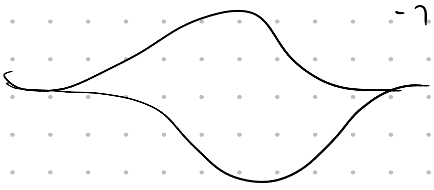
Ex:



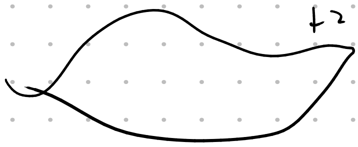
$+1$ $= (S^3, \xi_{ot}) =$



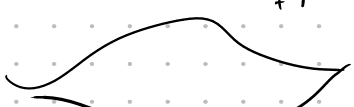
$+1/2$



-1 $= (\mathbb{R}P^3, \xi_{st})$



$+2$ $= (S^3, \xi_{ot})$



$+1$ $= (S^1 \times S^2, \xi_{st})$

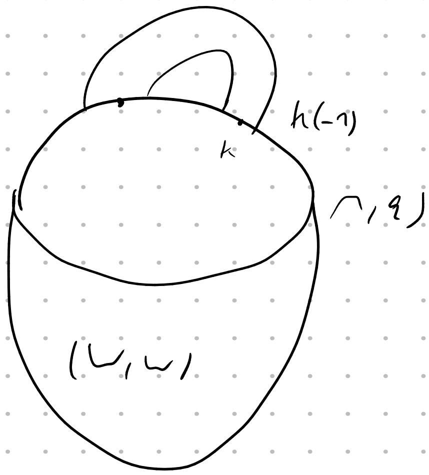
Thm 2: Let $K \subset (S^3, \xi_{st}) = \partial(D^4, \omega_{st})$ Leg.

$$\partial(W_{K,\omega}) = K(-1)$$

$W_{K,\omega} = D^4 \cup$ Weinstein 2-handle attached along K

proof: HW: compute the framing in the local model w.r.t. the contact framing, see that it makes -1 twist

Cor 3: (-1)-surgery preserves fillability



Thm 4 [Wand] (-1)-surgery preserves tightness.

Thm 5 [Lickorish–Wallace] Any connected, orientable, closed (smooth) 3-mfd can be obtained by smooth (integer) surgery along a <u>link</u> in S^3

Corollary 6: [Martinet] Every 3-mfd carries a c.s.

proof: Let L be a smooth surgery description of M. (Thm 5)
Choose a Legendrian approximation of L s.t. contact surgery coefficient $\neq 0$
(Stabilizing a knot: $\lambda_c \longrightarrow \lambda_c - \mu$

)

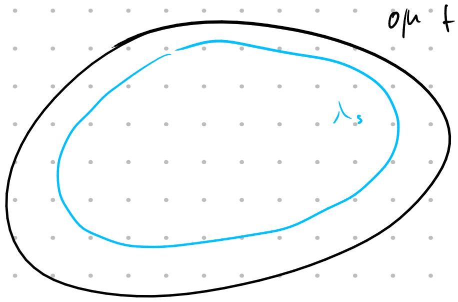
Thm 7 [Ding–Geiges] Any contact 3-mfd can be obtained by contact (± 1)-surgery along a Legendrian <u>link</u> from (S^3, ξ_{st})

proof: In section 7.4

7.2 Surgeries on the unknot

Ex: (1)

$0\mu + 1\cdot\lambda_s$



$= \quad S^1 \times D^2 \cup_\varphi S^3 \setminus \mathring{U}$

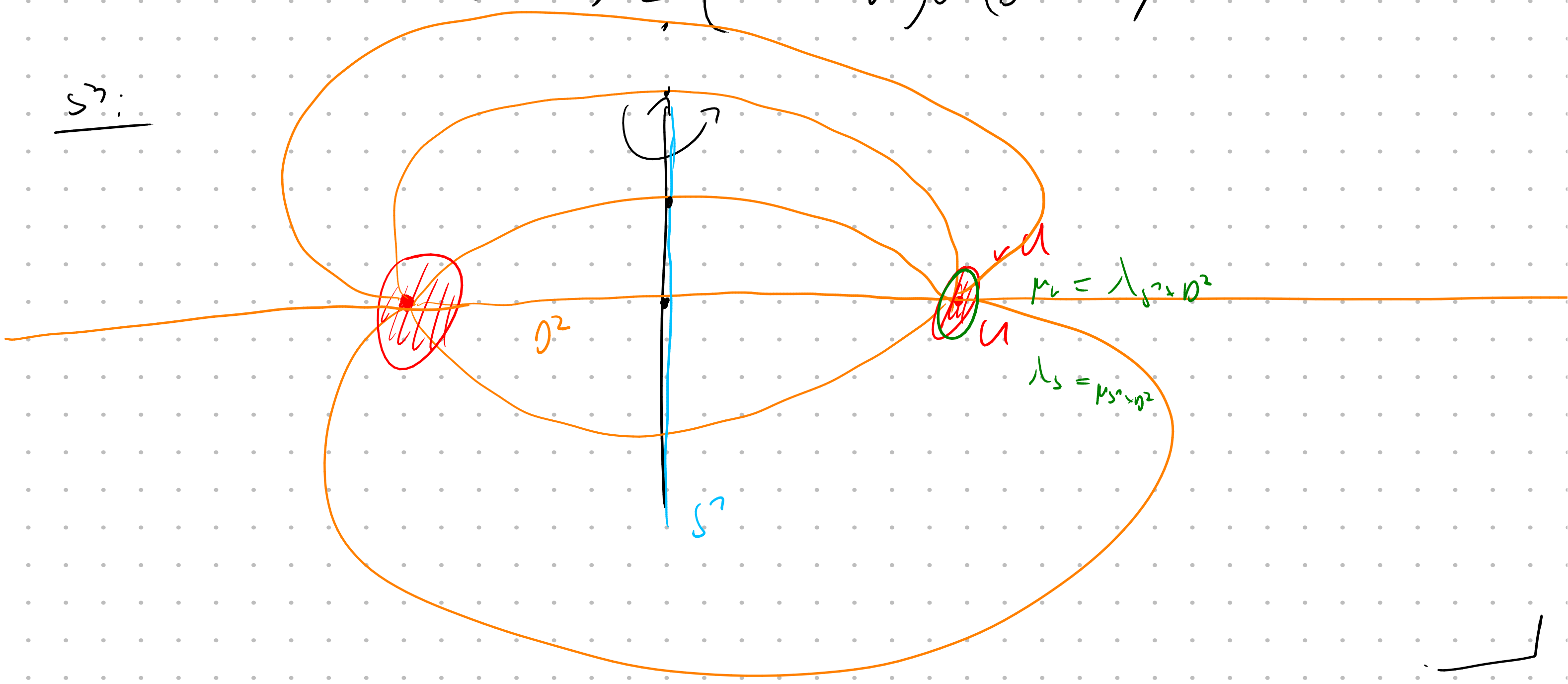
$\mu_0 \longmapsto \lambda_s$

$\lambda_0 \longmapsto \mu_U$

⌈ Claim: $S^3 \setminus \mathring{U} = S^1 \times D^2$

$S^3 = \partial D^4 = \partial(D^2 \times D^2) = (\partial D^2 \times D^2) \cup (D^2 \times \partial D^2)$

S^3:



⌋

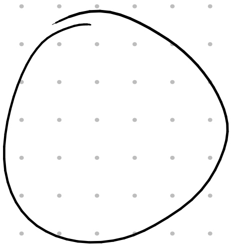
$= \quad S^1 \times D^2 \cup S^1 \times D^2$

$\mu_0 \longmapsto \mu_1$

$\lambda_0 \longmapsto \lambda_1$

$= S^1 \times (D^2 \cup D^2)$

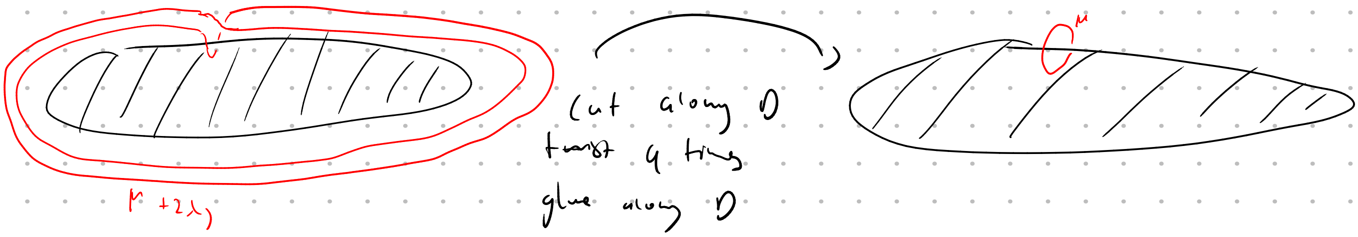
$= S^1 \times S^2$

(2)  $\mu + q\lambda_s$ $= S^3$

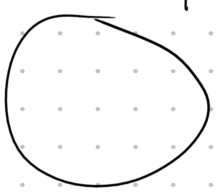
$$
\begin{tikzcd}
U(\mu+q\lambda_s) = \arrow[d, dashed] & S^1\times D^2 \arrow[d, "\cong"', "id"] & \cup_\varphi & S^3 \setminus i U \arrow[d, bend left] \\
S^3 = U(\mu) = & S^1 \times D^2 & \cup_{id} & S^3 \setminus i U
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mu_0 \arrow[r, mapsto] \arrow[d, "\cong"', "id"] \arrow[dr, phantom, "\circlearrowright"] & \mu + q\lambda_s \\
\mu_0 \arrow[r, mapsto] & \mu
\end{tikzcd}
$$

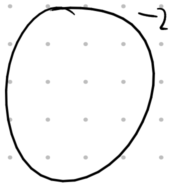
$f: S^3 \setminus i U \xrightarrow{\cong} S^3 \setminus i U$



cut along D
twist q times
glue along D

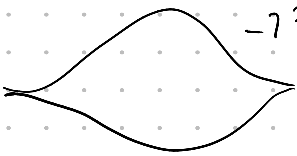
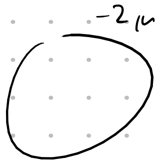
(3) HW:  $-p\mu + q\lambda_s$ $= L(p,q) = S^3/\mathbb{Z}_p$

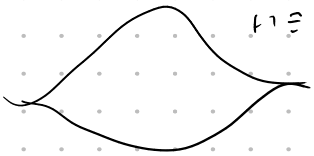
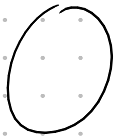
check that group action preserves splitting of S^3 into two solid tori

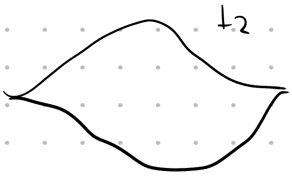
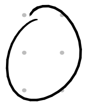
(4) HW:  $-2\mu + \lambda_s$ $= L(2,1) = \mathbb{RP}^3$

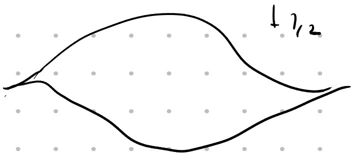
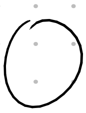
Let K be a Legendrian knot.

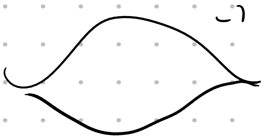
$\Rightarrow \boxed{\lambda_c = tb(K)\,\mu + \lambda_s}$

 $-1 \,\hat{=}\, -\mu+\lambda_c$ $\overset{C^\infty}{\cong}$  $-2\mu+\lambda_s$ $= \mathbb{RP}^3$

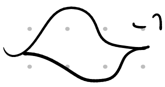
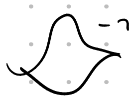
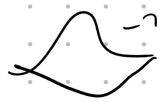
 $+1 \,\hat{=}\, +\mu+\lambda_c$ $\overset{C^\infty}{\cong}$  λ_s $= S^1 \times S^2$

 $+2$ $\overset{C^\infty}{\cong}$  $\mu+\lambda_s$ $= S^3$

 $+1/2$ $\overset{C^\infty}{\cong}$  $\mu+2(-\mu+\lambda_s)$ $=$  $-\mu+2\lambda_s$ $= S^3$

Lemma 8:  -1 $\overset{cont}{\cong} (\mathbb{RP}^3, \xi_{st})$

proof:

-  -1 $\overset{C^\infty}{\cong} \mathbb{RP}^3$
- Corollary 3 $\Rightarrow$  -1 is fillable $\overset{\text{Thm 6.3}}{\Rightarrow}$  -1 is tight.
- $(\mathbb{RP}^3, \xi_{st}) := (S^3, \xi_{st})/\mathbb{Z}_2 \Rightarrow$ univ. cover is tight $\Rightarrow (\mathbb{RP}^3, \xi_{st})$ is tight
- $\exists!$ tight contact structure on $\mathbb{RP}^3$

$\ulcorner\ \mathbb{RP}^3 = \overset{V_1}{\overbrace{S^1 \times D^2}} \cup_\varphi S^3 \setminus \mathring{U} = \overset{V_2}{\overbrace{S^1 \times D^2}}\ \urcorner$

$$\mu_1 \longmapsto -2\mu_u + \lambda_s = +\mu_2 - 2\lambda_2$$
$$\lambda_1 \longmapsto \lambda_2$$

Let ξ be a tight c.s. on $\mathbb{RP}^3$

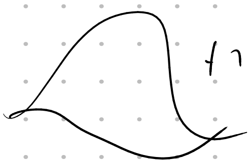
$\Rightarrow B_1$ by app thm $\Rightarrow$ $\smile$ by $(V_1, \xi) = (V_{ex}, \xi_{ot})$
with slope $1/n$

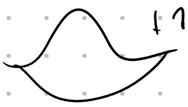
$\Rightarrow (V_2, \xi)$ is a tight solid torus with convex ∂

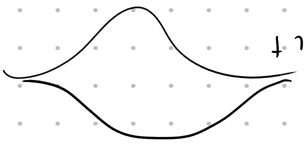
$\Rightarrow \Gamma_1 = \mu_1 + n\lambda_1 \xmapsto{\varphi} \mu_2 - 2\lambda_2 + n\lambda_2 = \mu_2 + (n-2)\lambda_2$

$\Rightarrow$ slope of (V_2, ξ) is $\frac{1}{n-2}$

Thm 5.5 $\Rightarrow \exists!\ (V_2, \xi)$

Lemma 9:  $+1 = (S^1 \times S^2, \xi_{st})$

Proof: •  $+1 \overset{C^\infty}{\cong} S^1 \times S^2$

•  $+1 = \overset{V_1 \,=}{S^1 \times D^2} \cup_\varphi \overset{V^2 \,=}{S^3 \setminus \mathring{U}}$

$$\mu \longmapsto \mu_u + \lambda_u = \lambda_s = \mu_2$$
$$\lambda_1 \longmapsto \lambda_2$$

$$\Gamma_1 = \lambda_1 - \mu_1 \xmapsto{\varphi} \lambda_2 - \mu_2 = \Gamma_2$$

$\Rightarrow V_1$ & V_2 are tight solid tori of slope $= -1$

• claim: $(S^1 \times S^2, \ker(x\,d\theta + y\,dz - z\,dy)) = V_1 \cup_\varphi V_2$

$T: T^2 \hookrightarrow S^1 \times S^2$

$(\theta, \varphi) \longmapsto (\theta, f(\varphi), \sqrt{1-f^2(\varphi)}\cos\varphi, \sqrt{1-f^2(\varphi)}\sin\varphi)$

with $f(\varphi) = \frac{1}{2}\sin(\varphi)$

$T T = \langle \partial_\theta, V = \left(0, f', -\frac{ff'}{\sqrt{1-f^2}} \cos\varphi - \sqrt{1-f^2} \sin\varphi, \right.$
$\left. \quad " \sin\varphi + " \cos\varphi \right) \rangle$

$\alpha(\partial_\theta) = f \qquad \alpha(V) = 1 - f^2$

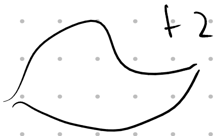
$\Rightarrow$ char. foliation T_ξ is given by $\partial_\theta - \frac{f}{1-f^2} \partial_\varphi$.

$\Rightarrow T$ is convex with dividing set

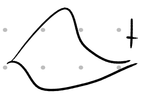
$\{\varphi = \frac{\pi}{2}\}$ & $\{\varphi = \frac{3}{2}\pi\}$

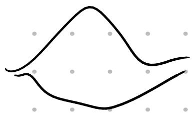
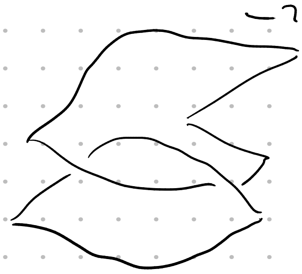
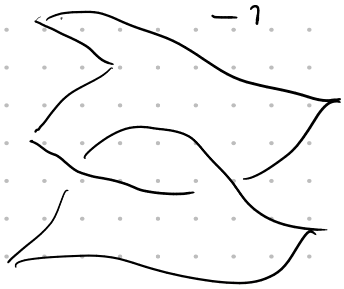
$\Rightarrow$ T is a convex torus in $(S^1 \times S^2, \xi_{st})$, dividing set two solid tori of slope $1/n$

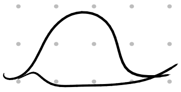
$\Rightarrow$ there is a contactomorphism ∎

Lemma 10:  $+2$ $= (S^3, \xi_{st})$

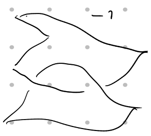
proof:

•  $+2 \overset{C^\infty}{=} S^3$

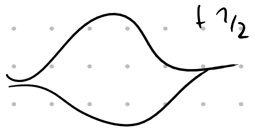
•  $+2$ $\overset{cut}{\cong}$  -1, $+1$ or  -1, $+1$?

• $+1$  $= (S^1 \times S^2, \xi_{st})$

cor 3

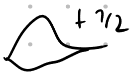
$\Rightarrow$  is fillable $\Rightarrow$ tight $\Rightarrow$ ξ_{st} ∎

Lemma 11:

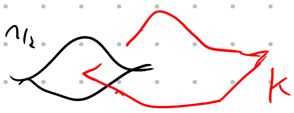


= is an OT c.s. on S^3

Proof:

- 

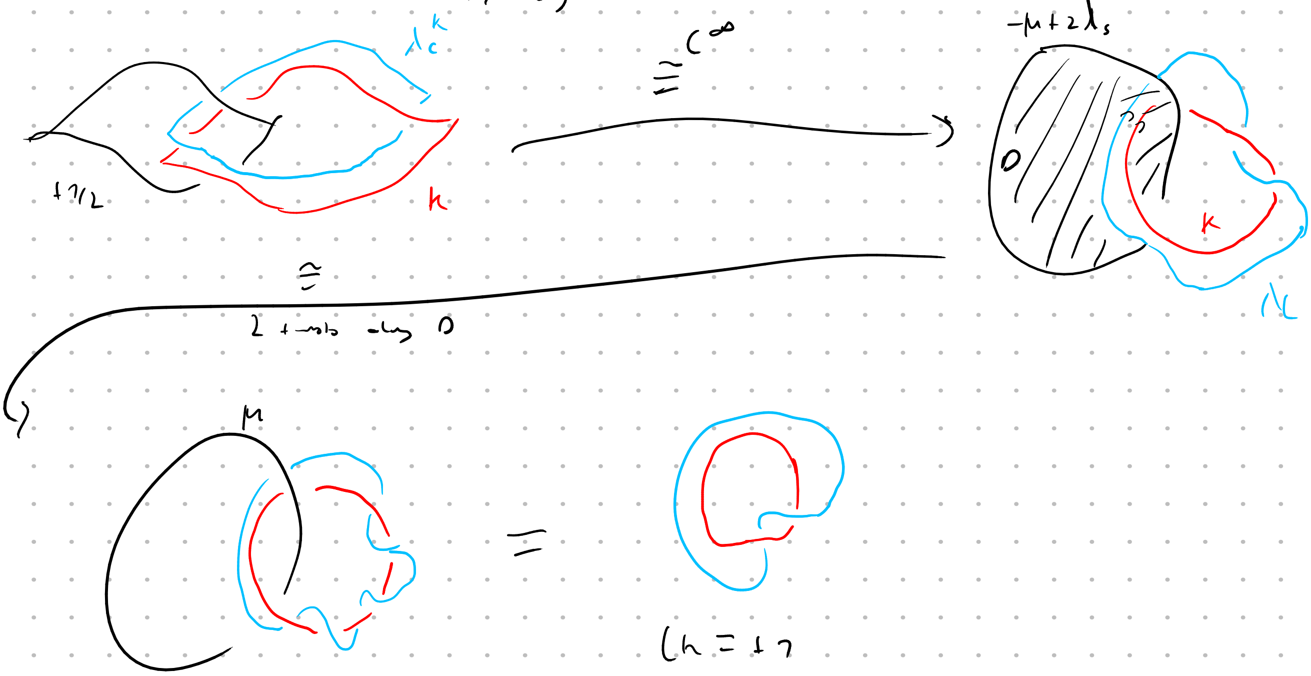
$\cong^{C^\infty}$ S^3

- 

K yields a legendrian knot in the surgered mfd

claim: K violates the Bennequin ineq.

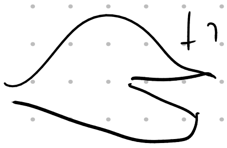
$tb(K) = lk(K, \lambda_c^K)$



$\Rightarrow$ $tb(K) = +1$ & K is an unknot

$\Rightarrow$ 

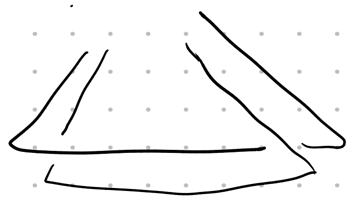
is OT ∎

Thm: 

= OT

2.3 Contact surgery moves

Notation: $K \times$ for K & a Legendrian push-off

K  $\longrightarrow K \times K$ 

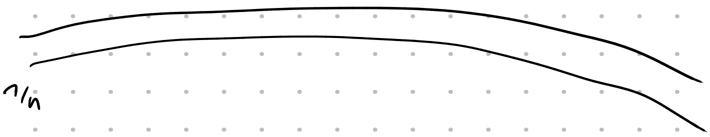
$K_n :=$ K n-times stabilized

K  K_2  or  or 

Lemma 7.3 [cancellation lemma] $\forall n \in \mathbb{Z}$:

$$K(1/n) \times K(-1/n) \cong (S^3, \xi_{st})$$

Proof:



$= (S^3, \xi_{st})$

$K' = \lambda_c$

$$\Rightarrow K(1/n) = S^1 \times D^2 \cup_\varphi S^3 \setminus \mathring{\nu} K$$

$$\mu_0 \longmapsto \mu + n\lambda_c$$

$$\lambda_0 \longmapsto \lambda_c$$

$\Rightarrow$ Surgery on K' in $K(1/n) \hat{=}$ surgery on λ_c

$$\Rightarrow K(1/n) \times K'(-1/n) = S^1 \times D^2 \cup S^3 \setminus \mathring{\nu} K$$

$$\mu_0 \xmapsto{\varphi'} \mu - n\lambda_c \qquad \mu \xmapsto{\varphi} \mu + n\lambda_c$$

$$\lambda_0 \longmapsto \lambda_c \qquad \lambda \longmapsto \lambda_c$$

$$\mu_0 \xmapsto{\varphi \circ \varphi'} \mu$$

$$\lambda_0 \xmapsto{\varphi \circ \varphi'} \lambda_c$$

$$= (S^3, \xi_{st})$$

∎

Lemma 7.4: [Replacement Lemma] $\forall n \geq 1$

$$K(\pm 1/n) \cong \underbrace{K(\pm 1) \times \cdots \times K(\pm 1)}_{n\text{-times}}$$

proof: HW

∎

Lemma 15: [Transformation Lemma]

(1) Let $r \in \mathbb{Q}\setminus\{0\}$ $\forall k \in \mathbb{Z}$

$$K(r) = K(1/k) \times K\left(\frac{1}{1/r - k}\right)$$

(2) If $r \in \mathbb{Q}_-$ write r as a continued fraction

$$r = r_1 + 1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots - \cfrac{1}{r_n}}} \quad \text{with } r_i \in -2$$

$$=: [r_1, \dots, r_n]$$

$$\Rightarrow K(r) = K_{(2+r_1)}(-1) \times K_{(2+r_1),(2+r_2)}(-1) \times \dots \times K_{(2+r_1),\dots,(2+r_n)}(-1)$$

proof: (to)

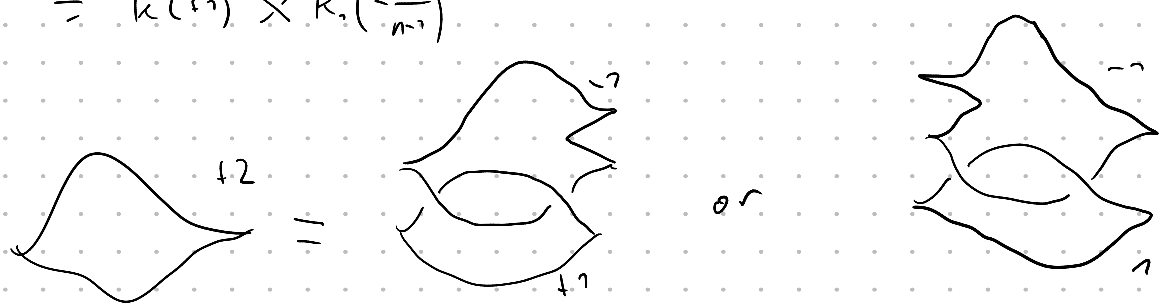
Ex: $n > 1$

$$K(n) \overset{L.15(1)}{=} K(+1) \times K\left(\frac{1}{1/n - 1}\right)$$

$$= K(+1) \times K\left(-\frac{n}{n-1}\right) \qquad \left(-\frac{n}{n-1} = [-3, \underbrace{-2, \dots, -2}_{n-2}]\right)$$

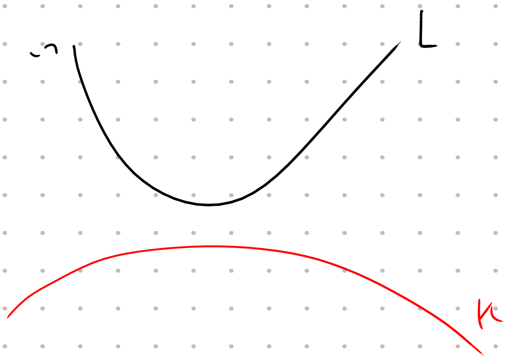
$$\overset{L.15(2)}{=} K(+1) \times \underbrace{K_1(-1) \times K_1(-1) \times \dots K_1(-1)}_{(n-1) \text{ times}}$$

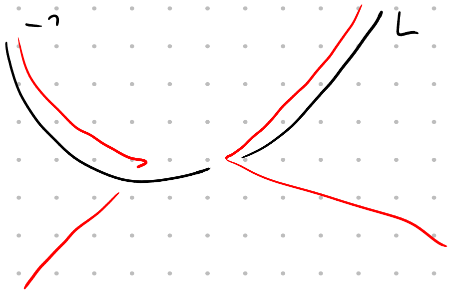
$$\overset{L.14}{=} K(+1) \times K_1\left(-\frac{1}{n-1}\right)$$



$$= (S^3, \xi_{st})$$

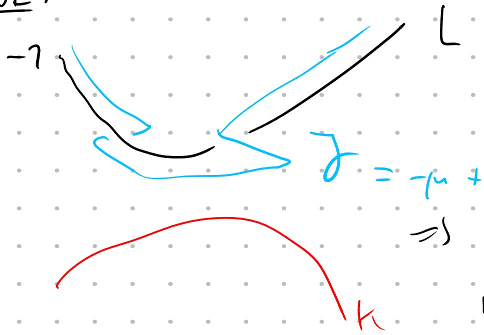
Lemma 16 [Handle slide]





$K \sim K'$ in $L(-1)$

proof:



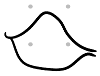
$L(-1) = S^1 \times D^2 \cup_\varphi S^3 \setminus \mathring{\nu} L$

$\mu_0 \longmapsto -\mu + \lambda_c$

$\gamma = -\mu + \lambda_c = \mu_0 = \{\overset{s^1}{*}\} \times \partial D^2$

$\Rightarrow$ γ is a Leg. unknot in $L(-1)$

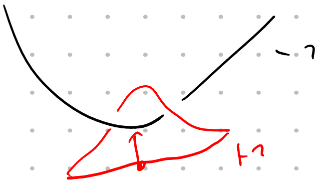
then: $tb(\gamma) = -1$

$\Rightarrow$  $\sim \gamma$ in $L(-1)$

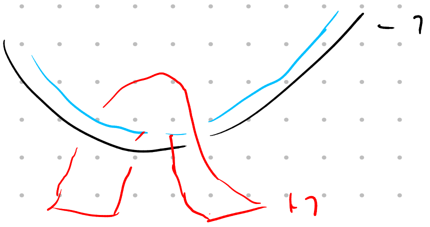
$\Rightarrow$ $K \sim K \# \gamma = K'$ in $L(-1)$

$\square$

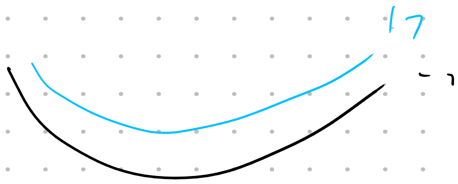
Ex:



H. S.
$\cong$



Isotopy
$\cong$



Cancel
$\cong$ (S^3, ξ_{st})

7.4 The homotopical invariants

Def: Let $L = L_1 \cup \dots \cup L_n$ be an or. Leg. link in (S^3, ξ_{st}).

(M, ξ) = contact mfd obtained by contact $(\pm 1/n_i)$-surgery on L $(n_i \in \mathbb{Z} > 0)$.

$tb_i = tb(L_i)$; $rot_i = rot(L_i)$ $\quad l_{ij} = lk(L_i, L_j)$

$\frac{p_i}{q_i} := \pm 1/n_i + tb_i$ $\quad$ Linking matrix:

$$Q := \begin{pmatrix} p_1 & q_2 l_{12} & \cdots & q_n l_{1n} \\ q_2 l_{21} & p_2 & & \vdots \\ \vdots & & \ddots & \\ q_1 l_{n1} & \cdots & & p_n \end{pmatrix}$$

Euler class:
$$e(\xi) := \sum_{i=1}^{n} n_i \, rot_i \, \mu_i \in \left\langle \mu_1, \dots, \mu_n \;\middle|\; Q \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = 0 \right\rangle_{\mathbb{Z}} = H_1(M)$$

If $e(\xi)$ is torsion (for ex. if $H_1(M)$ torsion)

$\iff \exists\, b \in \mathbb{Q}^n : \; Q \cdot b = \begin{pmatrix} rot_1 \\ \vdots \\ rot_n \end{pmatrix}$

Then:
$$d_3(\xi) = \frac{1}{4} \cdot \frac{1}{4}\left(\sum_{i=1}^{n} n_i b_i \, rot_i + (3 - n_i)\, sign_i\right) - \frac{3}{4}\underbrace{\sigma(Q)}_{\text{Signature}}$$

Thm 7.7: (1) e & d_3 are invariants of the underlying tangential 2-plane-fields.

(2) If $H_1(M)$ has no 2-torsion (& d_3 is defined) then

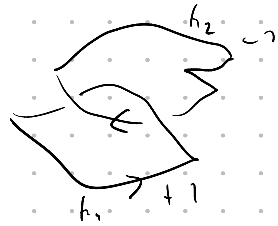
$$\xi_1 \underset{\text{hom}}{\simeq} \xi_2 \iff e(\xi_1) = e(\xi_2) \;\&\; d_3(\xi_1) = d_3(\xi_2)$$

(3) If $H_1(M) = 0 \Rightarrow d_3 \in \mathbb{Z}$ & takes all integer values

Eliashberg
$$\{\text{OT c.s.}\}/_{\text{isotopy}} \xleftrightarrow{1:1} \{\text{2-plane fields in } TM\}/_{\text{Homotopy}}$$
$$\xi \longmapsto d_3(\xi)$$

Write ξ_n for the OT c.s. on M (with $H_1(M) = 0$) with $d_3(\xi_n) = n$

Example: (1) $d_3(S^3,\xi_{st})=0$ (empty diag)



$\Rightarrow Q = \begin{pmatrix}0 & 1\\ 1 & -3\end{pmatrix} \Rightarrow \det = -1 \Rightarrow EV = +,- \Rightarrow \sigma = 0$

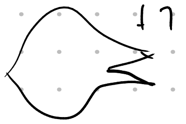
$tb_1 = -1$

$tb_2 = -2$

$Q\cdot b = \begin{pmatrix}rot_1\\ rot_2\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix} \Rightarrow b = \begin{pmatrix}1\\ 0\end{pmatrix}$

$\Rightarrow d_3 = \frac{1}{4}\left(0+0+2-2\right) - \frac{3}{4}\cdot 0 = 0$

Ex 2:



$= (S^3, \xi_1)$

$tb = -2$, $rot = 1$

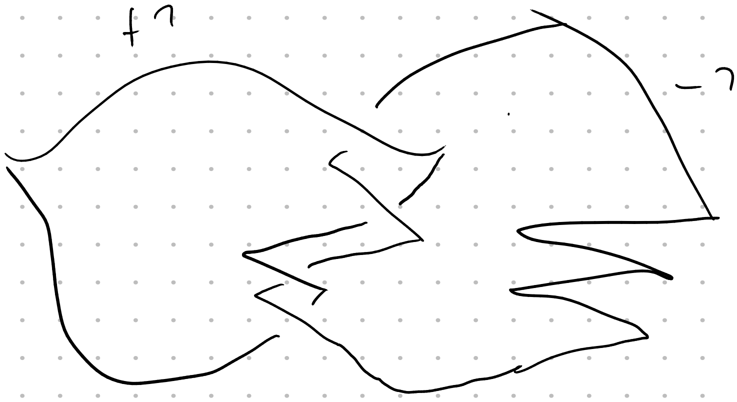
$\Rightarrow Q = (-1)$, $\sigma(Q) = -1$

$Q\cdot b = rot = 1 \Rightarrow b = -1$

$\Rightarrow d_3 = \frac{1}{4}\left(-1+2\right) + \frac{3}{4} = 1$

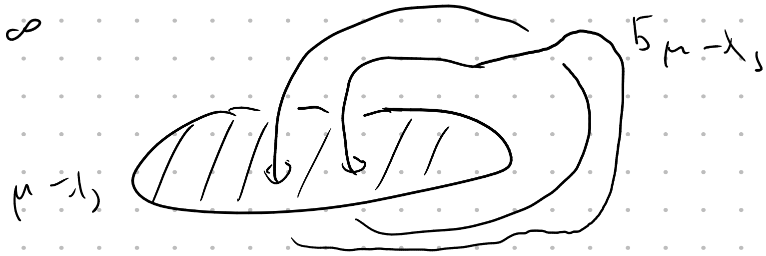
since $d_3(\xi_{st}) = 0$

$\Rightarrow$  is OT

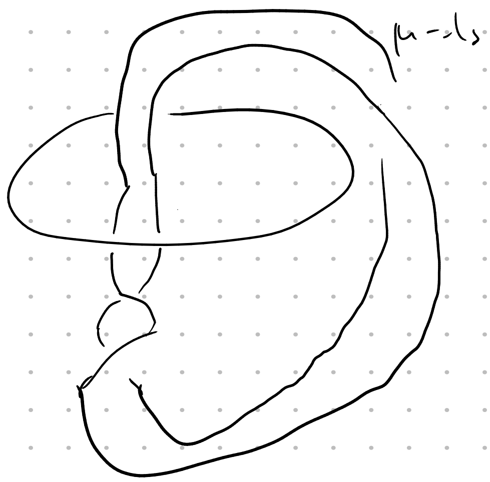


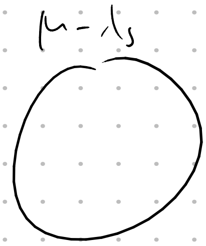
$\cong (S^3, \xi_{-1})$

$\cong_{C^\infty}$



twist

$=$ μ 

$=$ 

$= S^3$

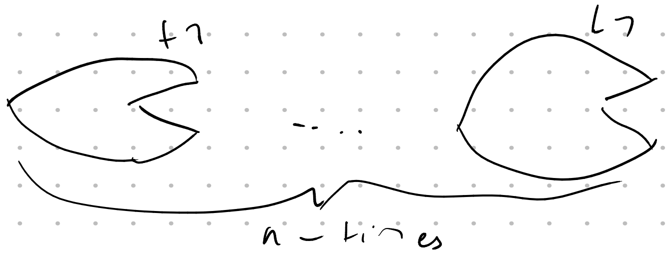
$$Q = \begin{pmatrix} -1 & 2 \\ 2 & -5 \end{pmatrix} \quad \begin{matrix} \det = 1 \\ \text{trace} = -6 \end{matrix} \quad \Rightarrow EV = -,- \Rightarrow \sigma = -2$$

$$Q \cdot b = \begin{pmatrix} rot_1 \\ rot_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \Rightarrow b = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

$$d_3 = \frac{1}{4}\left(-7 - 3 + 2 - 2\right) + \frac{6}{4} = -1$$

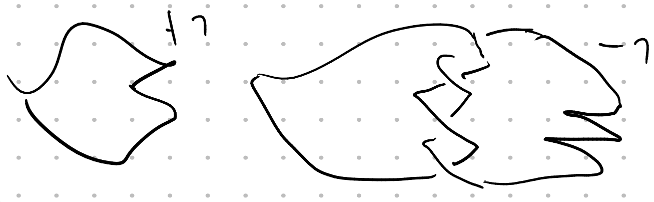
<u>Thm 18:</u>

$(S^3, \xi_n) =$



$n > 0$

$n < 0$



if $n = 0$

proof: If $(M_j, \xi_j) = L_j(\pm 1/n_j)$ $\quad j = 1, 2$

$$\Rightarrow (M_1, \xi_1) \# (M_2, \xi_2) = L_1(\pm 1/n_1) \sqcup L_2(\pm 1/n_2)$$

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \Rightarrow d_3((M_1, \xi_1) \# (M_2, \xi_2)) = d_3(M_1, \xi_1) + d_3(M_2, \xi_2)$$

7.5 The Ding–Geiges Thm

Thm 7: $\forall (M, \xi)$ $\exists$ Leg. Link $\exists$ $L \subset (S^3, \xi_{st})$ s.th. (M, ξ) is obtained by (± 1)-surgery on L

Proof: Let (M, ξ) be contact $\xRightarrow[\text{Wallace}]{\text{Lickorish}}$ $\exists$ Link $L \subset S^3$ s.t. integer surgery on L yields M.

$\Rightarrow$ $\exists$ Link $L^* \subset M$ s.t. integer surgery on L^* yields S^3.

Approx. Thm

$\Rightarrow$ $\exists$ Leg. realization of L^* s.t. contact surgery coeff. $\neq 0$. (Stabilizations)

Transformation Lemma

$\Longrightarrow$ $\exists$ Leg. Link $\tilde{L}$ in (M, ξ) s.t. contact (± 1)-Surgery on $\tilde{L}$ yields a contact str. ξ_{S^3} on S^3

$$\begin{tikzcd} (M, \xi) \arrow[r, "\pm 1"] & \cdots \arrow[l, bend left, "\mp 1"] \arrow[r, "\pm 1"] & (S^3, \xi_{S^3}) \arrow[l, bend left, "\mp 1"] \end{tikzcd} \underset{\text{Eliashberg}}{=} (S^3, \xi_n) \xleftarrow{\pm 1} \cdots \xleftarrow{\pm 1} \cdots \xleftarrow{\pm 1} (S^3, \xi_{st})$$

Canc. Lemma

$\Rightarrow$ $\exists$ Leg. Link in (S^3, ξ_{st}) s.t. contact (± 1)-surgery yields (M, ξ). $\blacksquare$

7.6 Contact surgery numbers

Surgery numbers of a 3-mfd M

$$s(M) := \min \left\{ \#(L) \;\middle|\; \begin{array}{l} L \subset S^3 \text{ a link s.t.} \\ M \text{ is obt. by surgery on } L \end{array} \right\}$$

$s_{\mathbb{Z}}(M) :=$ $\quad$ " $\quad$ integer surgery on L

Remark: Q: $\exists$ a homology sphere M s.t. $s(M) > 2$?

Q: $M \neq S^3 \Rightarrow s(\#_n M) \xrightarrow{n \to \infty} \infty$?

Q: $s_{\mathbb{Z}}(L(p, q)) = ?$

contact surgery numbers of a contact mfd (M, ξ)

$cs(M, \xi) := \min \{ \#(L) \mid L \subset (S^3, \xi_{st})$ leg. s.t. (M, ξ) is obtained by contact surgery on $L \}$

$cs_{\mathbb{Z}}(M, \xi) :=$ " integer "

$cs_{\pm 1}(M, \xi) :=$ " (± 1)- "

$$s(M) \leq s_{\mathbb{Z}}(M), \quad cs(M, \xi) \leq cs_{\mathbb{Z}}(M, \xi) \leq cs_{\pm 1}(M, \xi)$$

Thm 19

$$s_{\mathbb{Z}}(M) \leq cs_{\pm 1}(M, \xi) \leq s_{\mathbb{Z}}(M) + 3$$

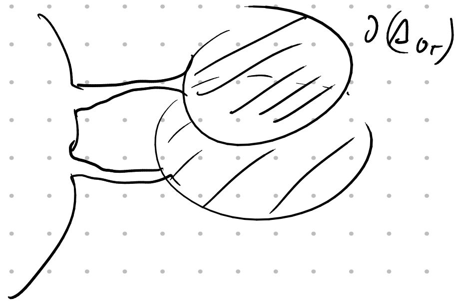
$$s(M) \leq cs(M, \xi) \leq s(M) + 3$$

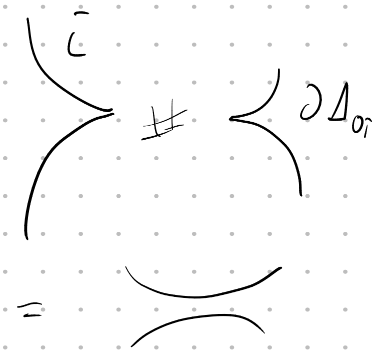
Lemma 20 Let K be a smooth knot in an OT (M, ξ).

$\Rightarrow \forall t \in \mathbb{Z} \; \exists$ leg. realization of K with $tb(L) = t$

proof: choose a leg real $\tilde{L}$ of K s.t. $M \setminus \nu \tilde{L}, \xi \supset \Delta_{OT}$

- if $tb(\tilde{L}) \geq t \Rightarrow$ stabilize $\tilde{L}$ to L
- else $tb(\tilde{L} \# \partial \Delta_{OT}) = tb(\tilde{L}) + tb(\partial \Delta_{OT}) + 1 = tb(\tilde{L}) + 1$





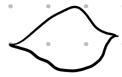
Thm 21:
- $cs_{\pm 1}(S^3, \xi_{st}) = 0$ ✓
- $cs_{\pm 1}(S^3, \xi_1) = 1$ ✓
- $cs_{\pm 1}(S^3, \xi_{n \neq 1}) = 2$

lower bound: Let $n \neq 1 \Rightarrow cs_{\pm 1}(S^3, \xi_n) > 1$

$\ulcorner$ Let $L \subset (S^3, \xi_{st})$ be a leg. knot s.t. $L(\pm 1) = (S^3, \xi_n)$.

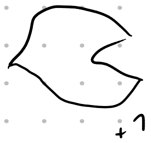
Gordon-Luecke $\Rightarrow L \overset{C^\infty}{\cong}$ unknot & slope $= \mu + t\lambda$ for some $n \in \mathbb{Z}$

[Eliashberg-Fraser] $\Rightarrow L$ is a stabilization of



$$\mu + k\lambda_s \overset{!}{=} \pm\mu + \lambda_c = \pm\mu + tb(L)\mu + \lambda_s$$

$\Rightarrow k = -1, \quad tb = -2 \qquad \boxed{\pm} = +$

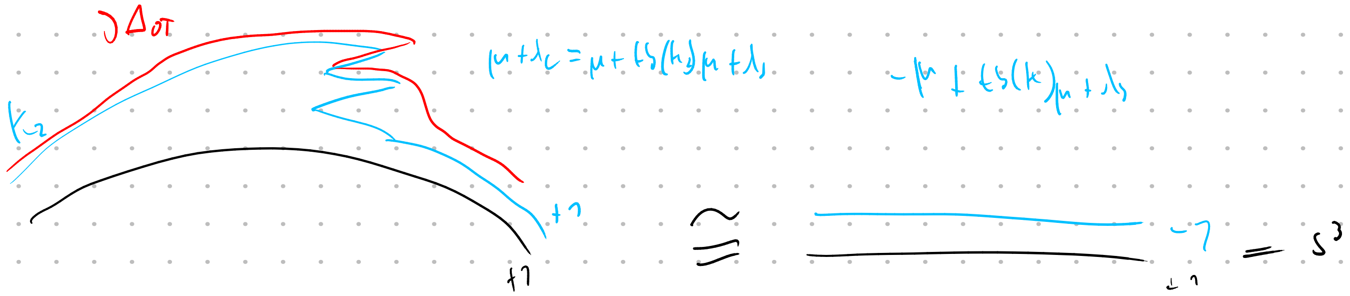
$\Rightarrow L(\pm 1) =$  $= (S^3, \xi_1)$

upper bound: $\quad cs_t(S^3, \xi_n) \le 2$

(a) Let $n \in 2\mathbb{Z}+1$

Let K be a Leg in (S^3, ξ_{st}) with $t = tb(K)$, $r = rot(K)$.

Consider: $\quad K(+1) \times K_2(+1)$



$$Q = \begin{pmatrix} t+1 & t \\ t & t-1 \end{pmatrix} \qquad Q^{-1} = \begin{pmatrix} -t+1 & t \\ t & -t-1 \end{pmatrix}$$

$\det Q = t^2 - 1 - t^2 = -1$

$\Rightarrow ev = +, - \Rightarrow \sigma(Q) = 0$

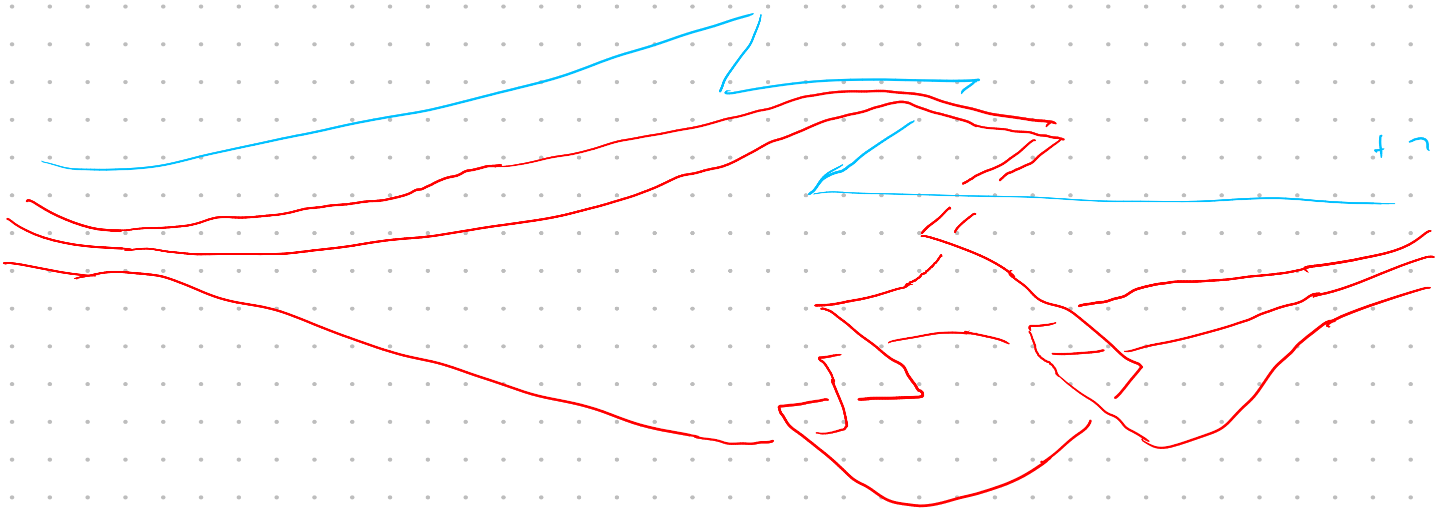
$$b = Q^{-1}\begin{pmatrix} r \\ r+2 \end{pmatrix} = \begin{pmatrix} \cancel{-tr} + r \cancel{+tr} + 2t \\ \cancel{tr - tr} - r - 2t - 2 \end{pmatrix} = \begin{pmatrix} 2t + r \\ -r - 2t - 2 \end{pmatrix}$$

$$d_3 = \frac{1}{4}\left(\cancel{r^2} \cancel{+2tr} \cancel{-2tr} - 4t \cancel{-r^2} - 2r - 2r \cancel{-4+4}\right)$$

$$= -(t+r) \in 2\mathbb{Z}+1$$

we can realize all of that (negative numbers as unknots) (positive numbers as twist-knots), see exercise sheet 4 or 5 maybe?

(5) Let $n \in 2\mathbb{Z}$. consider $L := K \# \partial\Delta_{OT} \# -\partial\Delta_{OT}$



Compute $d_3 = -(t + r + 1)$

$H \curvearrowright$

Lemma 22 Let (M, ξ) be OT

$\Rightarrow$ (i) $cs_\pm(M, \xi) \leq s_{\mathbb{Z}}(M) + 2$

(ii) $cs(M, \xi) \leq s(M) + 2$

proof Let $K \subset (L, \xi)$ be a smooth link of $s_{\mathbb{Z}}(M)$ many components s.t. surgery on K yields S^3.

Lem 20 & ξ OT

$\Rightarrow \exists$ leg. realisation L of K s.t. contact (± 1)-surgery on L yeds (S^3, ξ_n)

$$(M, \xi) \xrightarrow{\pm 1} \xrightarrow{\pm 1} \cdots \xrightarrow{\pm 1} (S^3, \xi_n)$$

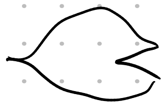
$s_{\mathbb{Z}}(M)$ - times

$\uparrow \pm 1$

$\uparrow \pm 1$

(S^3, ξ_{st})

Thm 19:

Let (M,ξ) tight. Let $K =$  $\subset$ Darboux ball $\subset (M,\xi)$

$\Rightarrow K(+1) = (M,\xi) \# (S^3,\xi_1) = (M,\xi_{OT})$

$(M,\xi_{OT}) \xleftarrow{\pm 1} \dots \xleftarrow{\pm 1} (S^3,\xi_{std})$ Lem 22

$\rightarrow (\ )+1 \qquad S_Z(M)+2$

(M,ξ)

Q: $\exists (M,\xi)$ s.t. $cs_{\pm 1}(M,\xi) = S_Z(M)+3$?