6. Symplectic fillings

6.1 Symplectic manifolds (dim = 4)

Let W be a 4-mfd. A 2-form ω on W is called symplectic

$:\Leftrightarrow$ $d\omega = 0$ and $\omega \wedge \omega \neq 0$

Ex: • $(\mathbb{R}^4, \omega_{st} := dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$

• $(\Sigma_g \times \Sigma_h, \Omega_{\Sigma_g} + \Omega_{\Sigma_h})$

• $\mathbb{C}P^2$ carries a (natural) symplectic structure (HW)

• S^4 doesn't carry a symplectic structure (HW)

• Let $(M, \xi = \ker\alpha)$ be a contact mfd.

$\Rightarrow$ $(W := \mathbb{R}_t \times M, \omega := d(e^t\alpha))$ is symplectic and is called symplectization

$[\omega \wedge \omega = (e^t dt \wedge \alpha + e^t d\alpha) \wedge (\dots) = 2e^{2t} dt \wedge \alpha \wedge d\alpha \neq 0]$

Def: A diffeo $f: (W_1, \omega_1) \longrightarrow (W_2, \omega_2)$ is called

• symplectomorphism $:\Leftrightarrow$ $f^*\omega_2 = \omega_1$

• (deformation) equivalence $:\Leftrightarrow$ $f^*\omega_2$ is isotopic to ω_1

Theorem 1 [Darboux]: $\forall p \in (W, \omega)$ $\exists$ NBHD U of p s.t. $(U, \omega) \overset{\text{symp.}}{\cong} (\mathbb{R}^4, \omega_{st})$

proof: Moser trick (HW) $\boxed{\text{HW}}$

6.2 Fillable contact manifolds

Let (M_+, ξ_+) and (M_-, ξ_-) be oriented (orientation induced by contact form) contact mfds.

Def: A (strong) symplectic cobordism from (M_-, ξ_-) to (M_+, ξ_+) is an oriented compact symplectic manifolds (W, ω) s.t.

• $\partial W = M_+ \sqcup -M_-$

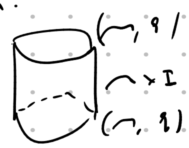
• in a NBHD of ∂W $\exists$ Liouville vector field Y $(L_Y\omega = \omega)$ transverse to ∂W, pointing out of W along M_+ and pointing into W along M_-

• $\ker(i_Y\omega|_{TM_\pm}) = \xi_\pm$

Ex: (M, ξ) is symplectically cobordant to (M, ξ).

• $(I \times M, d(e^t\alpha))$ is a symplectic cobordism.

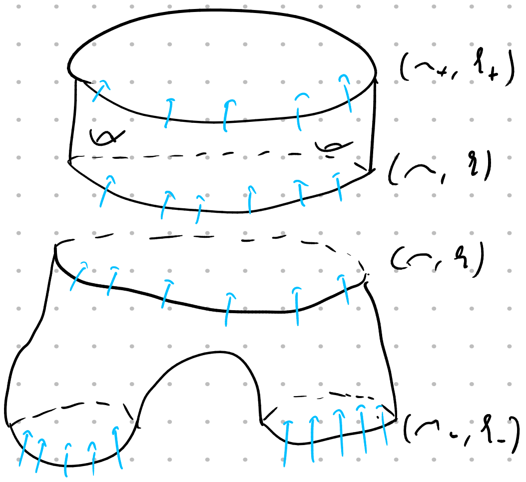
• $Y = \partial_t$ is a Liouville vector field.



Lemma 2 Let (W_+, ω_+) be a symplectic cobordism from (M,ξ) to (M_+,ξ_+)
& (W_-,ω_-) " " (M_-,ξ_-) to (M,ξ)

$\Rightarrow$ $\exists$ symplectic cobordism (W,ω) from (M_-,ξ_-) to (M_+,ξ_+)

proof:



Glue $M\times I$ in between to smoothly transition Liouville-v.f. into each other.

Def A (strong) **symplectic filling** of a contact mfd (M,ξ) is a symplectic cobordism from $\emptyset$ to (M,ξ).

Ex
- (D^4,ω_{st}) is a filling of (S^3,ξ_{st})
- $(S^1\times S^2,\xi_{st})$ & (T^3,ξ_n) are also fillable (HW)

Theorem 3: [Eliashberg, Gromov] If (M,ξ) is fillable, then it is tight.

proof: In section 6.4.

Theorem 4: [Eliashberg] $\forall (M,\xi)$ $\exists$ symplectic cap, i.e. a cobordism from (M,ξ) to $\emptyset$.

proof: maybe in section 7

Corollary 5 (of Thm 3): (S^3,ξ_{st}), $(S^1\times S^2,\xi_{st})$, (T^3,ξ_n), $(\mathbb{R}^3,\xi_{st})$ & $(S^1\times D^2,\xi_n)$ are tight.

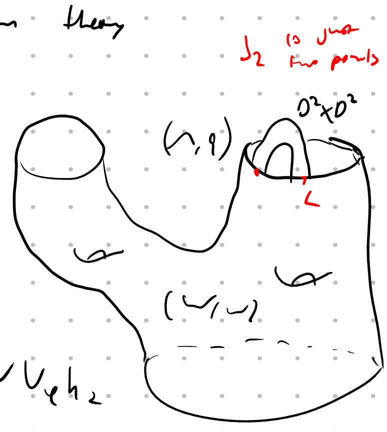
proof:
- (S^3,ξ_{st}) is fillable $\Rightarrow$ tight.
- $(S^1\times S^2,\xi_{st})$ " " "
- $(S^3\setminus\{pt\},\xi_{st}) \cong (\mathbb{R}^3,\xi_{st})$ $\rightsquigarrow$ OT disk in RHS is also in LHS.
- universal cover of (T^3,ξ_n) & $(S^1\times D^2,\xi_n)$ is $(\mathbb{R}^3,\xi_{st})$ (Sheet 9) $\square$

Remark: For every closed 3-mfd $\exists$ compact 4-mfd W s.t. $\partial W = M$ (see chapter 7)
$\rightsquigarrow$ cobordism theory

6.3 Weinstein handles and other types of fillings

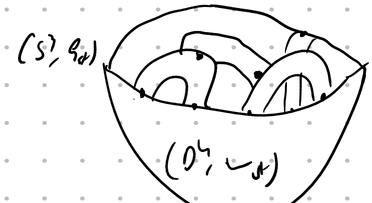
Let $L\subset(M,\xi)$ be a Legendrian knot
& (W,ω) be a symplectic cobordism with $(M_+,\xi_+)\supset(M,\xi)$
& attach a 2-handle $h_2 := D^2\times D^2$ to W along L
via an embedding

$$\varphi: \partial D^2\times D^2 \hookrightarrow \partial W$$
$$\partial D^2\times 0 \longmapsto L\,,\qquad W_L := W\cup_\varphi h_2$$



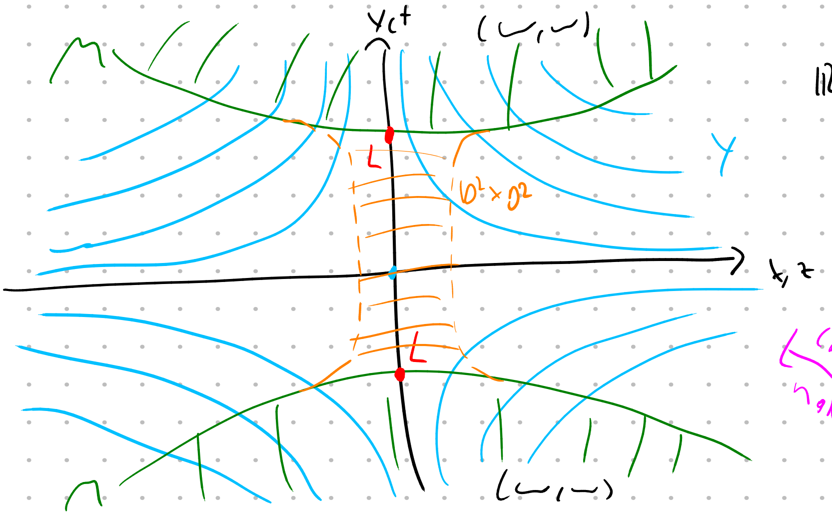
Thm 6: [Weinstein] $\forall$ L $\exists$ φ s.t. $W_L = W \cup_\varphi h_2$ carries a symplectic structure extending ω over h_2 s.t. W_L is a symplectic cobordism.

Remark: We can construct infinitely many contact mfds by starting with (D^4, ω_{st}) & attaching Weinstein two-handles along Legendrian links.



proof (Thm): $(\nu L, \xi) \overset{\text{cont.}}{\cong}$ std. model

$\Rightarrow$ wlog work in a local model & describe 2-handles there.

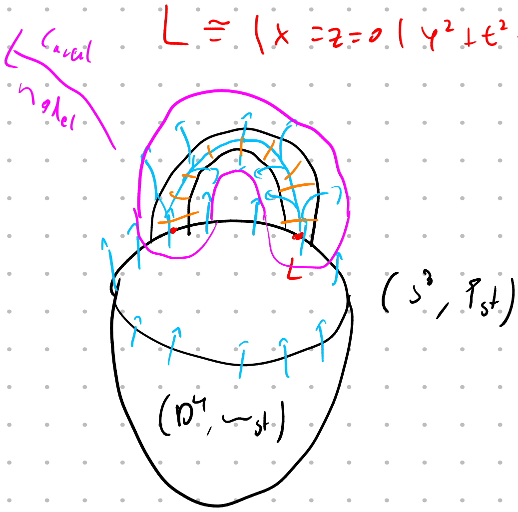


$\mathbb{R}^4$, $\omega = dx \wedge dy + dz \wedge dt$

$Y = 2x\partial_x - y\partial_y + 2z\partial_z - t\partial_t$

$M \cong \{x^2 - y^2 + z^2 - t^2 = -1\}$ contact $M \pitchfork Y$

$L \cong \{x = z = 0 \mid y^2 + t^2 = 1\}$ Legendrian

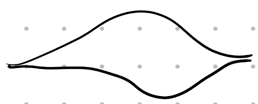


[$\alpha := \iota_Y \omega = 2x\,dy + y\,dx + 2z\,dt + t\,dz = 0$ if you plug in TL]

Remark: If $(W, \omega) = (D^4, \omega_{st})$

$\partial W_L = (S^3 \setminus \mathring{\nu} L, \xi_{st}) \cup_\varphi (D^2 \times S^1, \xi')$

Stein

Ex: $\partial(D^4 \cup h_2)$ attached along  $= (\mathbb{R}P^3, \xi_{st})$

$\Rightarrow$ $(\mathbb{R}P^3, \xi_{st})$ is fillable (see chapter 7).

Other notions of fillings

(M,ξ) is called

- **strongly fillable** $:\Longleftrightarrow$ $\exists$ compact sympl. 4-mfd (W,ω) s.t.
  - Liouville VF Y near ∂W s.t. Y points out of W along ∂W
  - $\ker\left(\iota_Y \omega|_{TM}\right) = \xi$

- **Stein fillable** $:\Longleftrightarrow$ $\exists$ strong sympl. filling that is obtained by attaching Weinstein 1- & 2-handles to (D^4,ω_{st})
- **Exactly fillable** $:\Longleftrightarrow$ $\exists$ strong symplectic filling s.t. Y is defined on all of W
- **Weakly fillable** $:\Longleftrightarrow$ $\exists$ compact sympl. 4-mfd (W,ω) s.t. $\partial W = M$ & $\omega|_\xi > 0$

Lemma 7:

$$\{\text{Stein}\} \subset \{\text{exactly}\} \subset \{\text{strongly}\} \subset \{\text{weakly}\} \subset \{\text{tight}\}$$

proof: HW (except for last inclusion) $\square$

Remark: In general, all these inclusions are $\neq$

Thm 8 [Gromov] Let (W,ω) be a Stein filling of (S^3,ξ_{st})
$\Rightarrow$ (W,ω) is def. eq. to (D^4,ω_{st})

proof: via holomorphic curves (see section 6.4) $\square$

6.4 Holomorphic curves

Let (W,ω) be a symplectic 4-mfd.

$J: TW \longrightarrow TW$ is called almost complex $:\Leftrightarrow J^2 = -id$

Ex: W is a complex mfd, $J = i$.

J is compatible with the symplectic form ω

$:\Leftrightarrow \omega(Ju, Jv) = \omega(u,v) \quad \forall u,v$

& $\omega(u, Ju) > 0 \quad \forall u \neq 0$

Lemma 9: The space of compatible almost complex structures on (W,ω) is contractible an non-empty. ☒

- A J-holomorphic disk is a map

$$u: (D^2, i) \longrightarrow (W, J) \quad \text{s.t.}$$

$$J \circ Tu = Tu \circ i \quad \& \quad \partial D^2 \subset \partial W$$

- u is called simple $:\Leftrightarrow \exists z$ s.t. $u^{-1}u(z) = \{z\}$ & $Tu_z \neq 0$
  injective

- moduli space $c > 0$

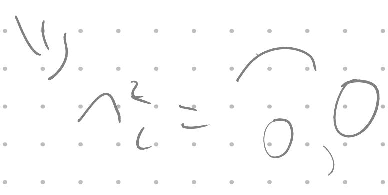
$$\mathcal{M}_c^J := \left\{ \begin{array}{l} u: (D^2, i) \longrightarrow (W, J) \text{ simple holomorphic disks} \\ \text{s.t. } \mathrm{Energy}(u) = \int_{D^2} u^*\omega < c \end{array} \right\} \Big/ \sim$$

$u \sim u \circ \varphi$ for $\varphi: (D^2, i) \longrightarrow (D^2, i)$ a biholomorphism.

no?

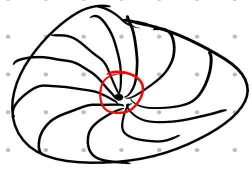
Thm 14 [Gromov]

- For a generic choice of J, $\mathcal{M}_c^J$ is a n-mfd
- by adding "stable" curves with sphere bubbles then $\mathcal{M}_c^J$ is compact ☐



$\mathcal{M}_c^2 =$

proof (of Thm 3): Let (M, ξ) be OT with std. OT disk D_{OT}



Assume: $\exists\ (W, \omega = d\lambda)$ an exact filling of (M, ξ).

Consider $S^3 \subset \mathbb{C}^2$, $\xi_{st} = TS^3 \cap i(TS^3)$

$$D' : D^2 \longrightarrow S^3$$
$$z \longmapsto (z, \sqrt{1-|z|^2})$$

near 0 the characteristic foliation of D_{OT} agrees with D'

$\overset{\text{Gromov}}{\Longrightarrow}$ D' yields a local model for D_{OT} near the origin

Bishop family: $s \in \mathbb{R}_+$ near 0

$$u_s : D^2 \longrightarrow \mathbb{C}^2 \qquad \text{sequence of holom disks.}$$
$$z \longmapsto (sz, \sqrt{1-s^2})$$

← real

Claim: these are all simple holomorphic disks with $u(\partial D^2) \subset D' \subset D_{OT}$

⌈ Let $u : D^2 \longrightarrow \mathbb{C}^2$ simple holomorphic ⌉

$$z \longmapsto (v_1(z), v_2(z))$$

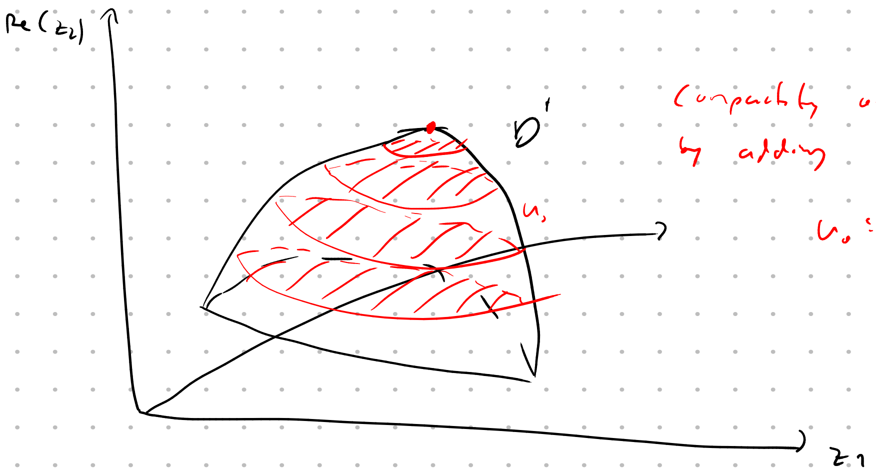
- $u(\partial D^2) \subset D' \Rightarrow v_2(\partial D^2) \subset \mathbb{R}$

harmonic functions satisfy maximum principle

$\Longrightarrow$ $v_2(D^2) \subset \mathbb{R}$

⌊ $\overset{\text{Holomorphic}}{\Longrightarrow}$ $v_2(D^2)$ is constant ⌋

$$u_s : D^2 \longrightarrow \mathbb{C} \times \mathbb{R} \subset \mathbb{C}^2$$



Compactify on one end of the component of $\mathcal{M}^{\mathcal{J}}_c$ by adding the const curve

$$u_0 : D^2 \longrightarrow \mathbb{C}^2$$
$$z \longmapsto (0, 1)$$

From the classification of 1-mfd $\overset{\text{Thm 14}}{\Longrightarrow}$ $\exists$ stable curve at the other boundary of $\mathcal{M}^{\mathcal{J}}_c$

Again by the maximum principle $\Rightarrow$ $u(\partial D^2)$ can't be tangent to q
(in particular not to ∂D_{ot})

$\Rightarrow$ Limit cannot converge to ∂D_{OT}

$\overset{\text{Gromov}}{\Longrightarrow}$ $\exists$ stable curve with sphere bubble in (W, ω)

$\Rightarrow$ $\exists\ u : (S^2, i) \longrightarrow (W, J)$ holomorphic sphere

$$\Rightarrow \quad \text{Area}(u) = \int_{S^2} u^*\omega = \int_{S^2} du^*\lambda \overset{\text{Stokes}}{=} 0$$

$\Rightarrow$ $u =$ constant

$\Rightarrow$ $\nexists$ sphere bubble $\lightning$

$\blacksquare$