Nonlinear Optimization – Sheet 03

Exercise 1

We set $f = \frac{1}{2}x^T Ax - b^T x + c$ and compute:

$$f(x^{(k)}) - f(x^{(k)} + \alpha^{(k)} d^{(k)}) = -\frac{1}{2}||\alpha^{(k)} d^{(k)}||_A^2 + b^T(\alpha^{(k)} d^{(k)}) - (\alpha^{(k)} d^{(k)})^T Ax^{(k)}$$

First case: Constant step sizes with $\alpha = \alpha^{(k)} = \frac{2}{\beta} - c$ for some $0 < c < \frac{2}{\beta}$, where β is the largest generalized eigenvalue of A w.r.t. M, so in particular $||d^{(k)}||_A^2 \leq \beta||d^{(k)}||_M^2$. We obtain:

$$\begin{aligned}
&f(x^{(k)}) - f(x^{(k)} + \alpha^{(k)} d^{(k)}) = -\frac{1}{2}||\alpha^{(k)} d^{(k)}||_A^2 + b^T(\alpha^{(k)} d^{(k)}) - (\alpha^{(k)} d^{(k)})^T Ax^{(k)} \\
&= \alpha(-\frac{1}{2}\alpha||d^{(k)}||_A^2 - f'(x^{(k)})d^{(k)}) \\
&\geq \alpha(-\frac{1}{2}(\frac{2}{\beta} - c)\beta||d^{(k)}||_M^2 - f'(x^{(k)})d^{(k)}) \\
&= \alpha((1 - \frac{c\beta}{2})(M^{-1}(Ax^{(k)} - b))^T M d^{(k)} - f'(x^{(k)})d^{(k)}) \\
&= -\frac{\alpha c\beta}{2} f'(x^{(k)})d^{(k)} = -\frac{\alpha\beta c}{2} \frac{f'(x^{(k)})d^{(k)}||d^{(k)}||_M^2}{||d^{(k)}||_M^2} \\
&= \frac{\alpha\beta c}{2} \frac{(f'(x^{(k)})d^{(k)})^2}{||d^{(k)}||_M^2}
\end{aligned}$$

Second case: Cauchy step sizes with $\alpha^{(k)} = \frac{(b - Ax^{(k)})^T d^{(k)}}{||d^{(k)}||_A^2}$. We compute:

$$\begin{aligned}
&f(x^{(k)}) - f(x^{(k)} + \alpha^{(k)} d^{(k)}) = -\frac{1}{2}||\alpha^{(k)} d^{(k)}||_A^2 + b^T(\alpha^{(k)} d^{(k)}) - (\alpha^{(k)} d^{(k)})^T Ax^{(k)} \\
&= \alpha^{(k)}(\frac{1}{2}(Ax^{(k)} - b)^T d^{(k)} + b^T d^{(k)} - d^{(k)T} Ax^{(k)}) \\
&= \frac{\alpha^{(k)}}{2}(b^T d^{(k)} - x^{(k)T} Ad^{(k)}) = -\frac{1}{2}\frac{(b - Ax^{(k)})^T d^{(k)}}{||d^{(k)}||_A^2} f'(x^{(k)})d^{(k)} \\
&= \frac{1}{2}\frac{(f'(x^{(k)})d^{(k)})^2}{||d^{(k)}||_A^2} \geq \frac{1}{2\beta}\frac{(f'(x^{(k)})d^{(k)})^2}{||d^{(k)}||_M^2}
\end{aligned}$$

Both $\frac{1}{2\beta}$ and $\frac{\alpha\beta c}{2}$ are stricly positive.

Exercise 2

Let $f \in C^1$ and $x^{(0)}$ be an initial iterate of Algorithm 5.2 with step sizes $\alpha^{(k)}$ satisfying the Wolfe-Powell conditions and descent directions $d^{(k)}$. Let f' be Lipschitz-continuous on the sublevel set $\mathcal{M}_f(x^{(0)})$, ie.

$$\|f'(x) - f'(y)\|_{M^{-1}} \leq L\|x - y\|_{M^{-1}} \tag{1}$$

for all $x, y \in \mathcal{M}_f(x^{(0)})$ and some constant $L > 0$. In particular, this is true for all pairs of iterates $x^{(k)}, x^{(l)}$, since they are elements of the sublevel set.

Let $\|.\|$ denote the M^{-1}-norm. Then we have

$$\begin{aligned}
(\tau - 1)f'(x^{(k)})d^{(k)} &\overset{(5.17)}{\leq} (f'(x^{(k)} + \alpha^{(k)} d^{(k)}) - f'(x^{(k)}))d^{(k)} \\
&\overset{\text{(CSU)}}{\leq} \|f'(x^{(k)} + \alpha^{(k)} d^{(k)}) - f'(x^{(k)})\| \cdot \|d^{(k)}\| \\
&\overset{\text{eq. (1)}}{\leq} L\alpha^{(k)}\|d^{(k)}\|^2 \,,
\end{aligned}$$

which implies

$$\alpha^{(k)} \geq \frac{(\tau - 1)}{L} \cdot \frac{f'(x^{(k)})d^k}{\|d^{(k)}\|^2}.$$

Using $f'(x^{(k)})d^{(k)} < 0$, we obtain

$$\begin{aligned} f(x^{(k)} + \alpha^{(k)}d^{(k)}) - f(x^{(k)}) &\overset{(5.12)}{\leq} \sigma\alpha^{(k)}f'(x^{(k)})d^{(k)} \\ &\leq -\frac{(\tau - 1)}{L} \cdot \frac{(f'(x^{(k)})d^k)^2}{\|d^{(k)}\|^2}, \end{aligned}$$

which is the definition of efficiency.

From (5.8) and $\|\nabla_M f(x^k)\| = f'(x^{(k)})M^{-1}MM^{-1}f(x^k)^t = \|\nabla f(x^{(k)})\|$ we get

$$\cos\angle(-\nabla_M f(x^{(k)}), d^{(k)})^2\|f'(x^{(k)})\|^2 = \frac{(f'(x^{(k)})d^k)^2}{\|\nabla_M f(x^{(k)})\|_M^2\|d^{(k)}\|^2}\|f'(x^{(k)})\|^2 = \frac{(f'(x^{(k)})d^k)^2}{\|d^{(k)}\|^2},$$

which implies the second assertion.

Exercise 3

Let $\alpha > 0$ statisfy (5.12) and (5.17) for $g(x) = \gamma f(Ax + b) + \delta$ at x with direction d. Then f statisfies (5.12) and (5.17) at $Ax + b$ with direction d.

Proof. We compute by chain rule

$$\gamma(f(Ax + b + \alpha Ad) - f(Ax + b)) = g(x + \alpha d) - g(x) \leq \sigma\alpha g'(x) = \sigma\alpha\gamma f'(Ax + b)Ad$$

which shows $f(Ax + b + \alpha Ad) - f(Ax + b) \leq \sigma\alpha f'(Ax + b)Ad$. Similarly we get

$$\tau\gamma f'(Ax + b)d = \tau g'(x)d \leq g'(x + \alpha d) = \gamma f'(Ax + b + \alpha d)Ad$$

implying $\tau f'(Ax + b)d \leq f'(Ax + b + \alpha d)Ad$, proving the claim. □

Exercise 4

In order to use the line search method, we implement the Armijo procedures.

```python
"""
define armijo algorithms to find a suitable step size
"""

def armijo_condition(alpha, phi_alpha, phi_0, phi_prime_0, sigma):
    """evaluate armijo condition"""
    return phi_alpha <= phi_0 + sigma * alpha * phi_prime_0


def armijo_backtracking(alpha_0, phi, phi_0, phi_prime_0, sigma, beta):
    """
    return a step size alpha that satisfies the armijo condition
    using the simple backtracking approach
    """
    alpha = alpha_0
    while not armijo_condition(alpha, phi(alpha), phi_0, phi_prime_0,
        sigma):
        alpha = beta * alpha
    return alpha

def armijo_interpolation(alpha_0, phi, phi_0, phi_prime_0, sigma,
    beta_lower, beta_upper):
```

```python
    """
    return a step size alpha that satisfies the armijo condition
    use the interpolation approach
    """
    alpha = alpha_0
    phi_alpha = phi(alpha)
    while not armijo_condition(alpha, phi_alpha, phi_0, phi_prime_0,
        sigma):
        alpha_star = (-phi_prime_0 * alpha**2)/2*(phi_alpha - phi_0 -
            phi_prime_0 * alpha)
        alpha = min(max(alpha_star, beta_lower * alpha), beta_upper *
            alpha) #clip alpha to interval
        phi_alpha = phi(alpha)
    return alpha
```

Now we can implement the whole algorithm

```python
import numpy as np
from armijo_procedures import armijo_backtracking, armijo_interpolation

def gradient_descent_UP(
    x_0, f, f_prime, M_inv, sigma, alpha_lower_bound, beta, beta_upper=
        None, eps=1e-5, max_iter=100
):
    """
    if beta_upper is given, we use the interpolating armijo algorithm
        with beta as beta_lower,
    otherwise the original armijo algorithm
    """

    interpolate = False
    if beta_upper:
        interpolate = True
        beta_lower = beta

    k = 0
    f_new = f(x_0)
    r = f_prime(x_0)
    d = -M_inv @ r
    delta = -r.transpose() @ d
    history = {
        "iterates": [x_0],
        "objective_values": [f_new],
        "gradient_norms": [np.sqrt(delta)],
        "step_lengths": [],
    }
    x = x_0
    while delta > eps**2 and k < max_iter:
        if k == 0:
            alpha_0 = alpha_lower_bound
        else:
            alpha_0 = max(alpha_lower_bound, (f_new - f_old) / delta)
        phi = lambda alpha: f(x + alpha * d)
        phi_0 = f_new  # f(x + 0*d) = f(x)
        phi_prime_0 = -delta
        if interpolate:
            alpha = armijo_interpolation(
```

```
                alpha_0, phi, phi_0, phi_prime_0, sigma, beta_lower,
                    beta_upper
            )
        else:
            alpha = armijo_backtracking(alpha_0, phi, phi_0,
                phi_prime_0, sigma, beta)

        x = x + alpha * d
        f_old = f_new
        f_new = f(x)
        r = f_prime(x)
        d = -M_inv @ r
        delta = -r.transpose() @ d
        k = k + 1

        history["step_lengths"].append(alpha)
        history["iterates"].append(x)
        history["objective_values"].append(f_new)
        history["gradient_norms"].append(np.sqrt(delta))

    return history
```

As examples we use the rosenbrock function from example_functions.py and some random quadratic problems from rand_problem.py.

```
import numpy as np

def rosenbrock(a,b,x):
        """
        Implements the rosenbrock function
        Accepts:
                        a,b: scalar parameters
                        x: array of length 2
        Returns:
                        f:  function value
                        df: derivative value
        """
        f  = (a - x[0])**2 + b * (x[1] - x[0]**2)**2
        df = np.array(
                [2*(a - x[0]) + 2*b*(x[1] - x[0]**2)*(-2*x[0]),
                 2*b*(x[1] - x[0]**2)
                ]
                )
        return f, df

def himmelblau(x):
        f = (x[0]**2 + x[1]-11)**2 + (x[0]+x[1]**2-7)**2
        df = 0 #if I have some spare time maybe I'll compute it

import numpy as np

class rand_problem():
    def __init__(self, n) -> None:
        self.n = n
        self.A = self.create_random_A()
        self.b = np.random.rand(n)
        self.c = np.random.rand()
```

```
        self.f = self.quadratic_function()
        self.f_prime = lambda x : self.A @ x - self.b
        self.Pinv = np.identity(n)

        self.x0 = np.random.rand(n)

    def create_random_A(self):
        """create random spd matrix in dimension n x n"""
        M = np.random.rand(self.n, self.n)
        return np.dot(M, M.T)

    def quadratic_function(self):
        f = lambda x : 0.5 * x.T @ self.A @ x - self.b.T @ x + self.c
        return f

if __name__ == "main":
    x = rand_problem(4)
    print(x.A, x.b, x.c)
```

Finally, we can put all of it together to create some plots

```
import numpy as np
import matplotlib.pyplot as plt

from visualization_functions import (
    plot_2d_iterates_contours,
    plot_f_val_diffs,
    plot_step_sizes,
    plot_grad_norms,
)
from gradient_descent_UP import gradient_descent_UP
from rand_problem import rand_problem
from example_functions import rosenbrock


# compare gradient norms
N = 10
problems = []

for i in range(N):
    problems.append(rand_problem(2))

# normalize norm of random start point
for problem in problems:
    problem.x0 = problem.x0 * (np.linalg.norm(problem.x0)) ** (-1)

histories = []
# labels = []
for problem in problems:
    histories.append(
        gradient_descent_UP(
            problem.x0,
            problem.f,
            problem.f_prime,
            problem.Pinv,
            sigma=1e-3,
            alpha_lower_bound=10,
```

```
                beta=0.5,
                #beta_upper=0.9,
            )
        )

plot_grad_norms(
    histories=histories, labels=range(len(histories))
)  # Gradient norms - gradient descent algorithm

# for i, problem in enumerate(problems):
#     plot_2d_iterates_contours(problem.f, histories=[histories[i]],
    labels=[str(i)], xlims=[-10,10], ylims=[-10,10])

a = 1
b = 100
rosenbrock_f = lambda x: rosenbrock(a, b, x)[0]
rosenbrock_prime = lambda x: rosenbrock(a, b, x)[1]

rosenbrock_histories = []
rosenbrock_labels = []
configurations = [
    ([2, 2], 1e-2, 0.01, 0.5),
    ([2, 2.5], 1e-2, 0.1, 0.5),
    ([2.5, 2], 1e-4, 0.01, 0.5),
    ([2.5, 2.5], 1e-4, 0.1, 0.5),
]
for configuration in configurations:
    rosenbrock_histories.append(
        gradient_descent_UP(
            configuration[0],
            rosenbrock_f,
            rosenbrock_prime,
            np.identity(2),
            sigma=configuration[1],
            alpha_lower_bound=configuration[2],
            beta=configuration[3],
            max_iter=100,
        )
    )
    rosenbrock_labels.append(
        f"x0: {configuration[0]}, sigma: {configuration[1]}, alpha: {
            configuration[2]}, beta: {configuration[3]}"
    )


plot_grad_norms(
    histories=rosenbrock_histories,
    labels=rosenbrock_labels,
)

plot_2d_iterates_contours(
    rosenbrock_f,
    histories=rosenbrock_histories,
    labels=rosenbrock_labels,
    xlims=[-3, 3],
    ylims=[-2,4],
```

```
    title="Iterates and iso-lines of Rosenbrock function"
)

plt.show()
```

First, we look at the rosenbrock function. We use $\beta = .5$ and vary some other parameters, see 1. Then, we consider some random quadratic problems with varying parameters.

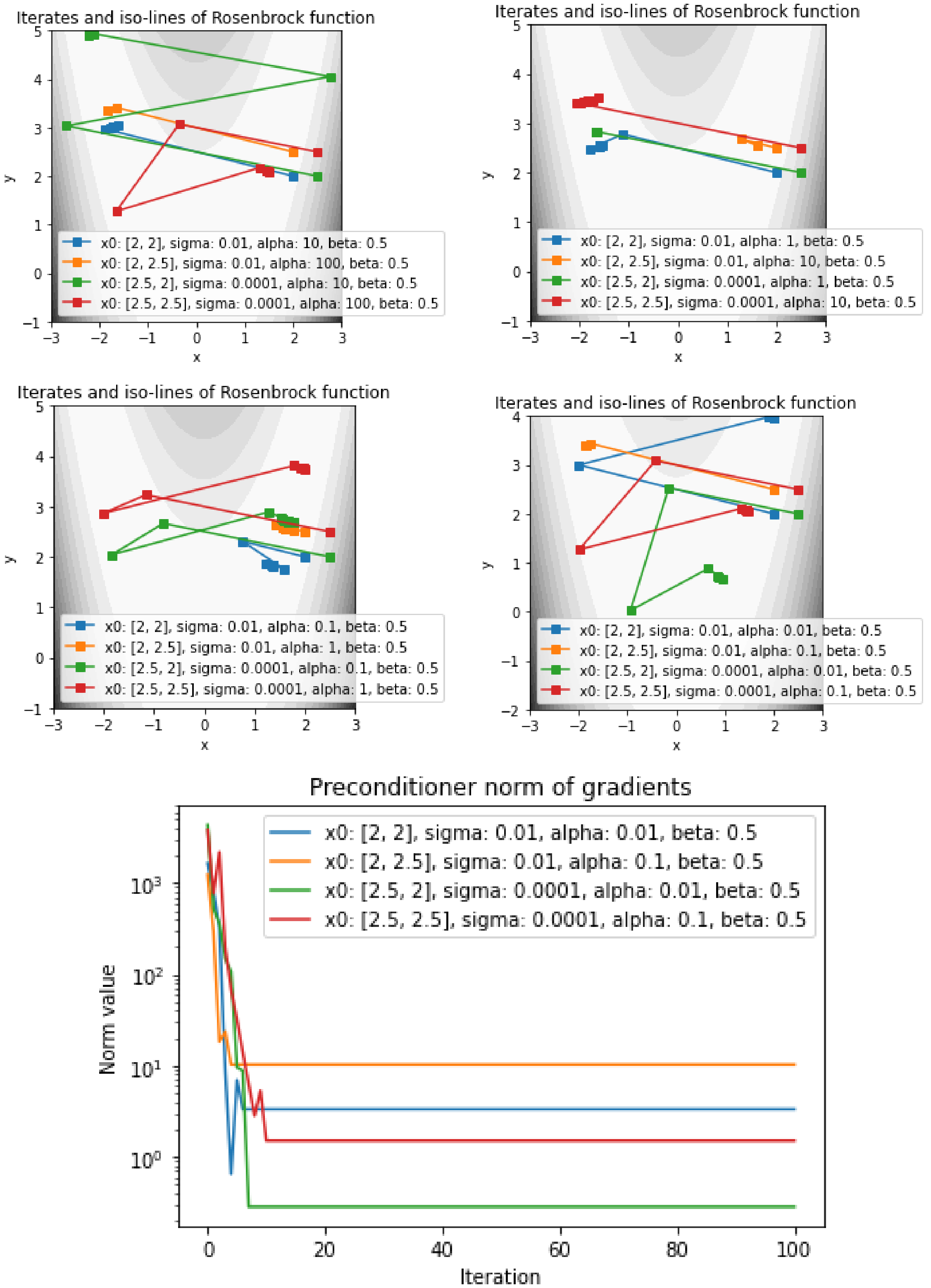
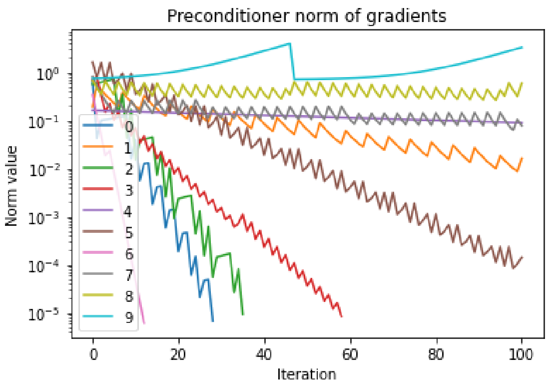
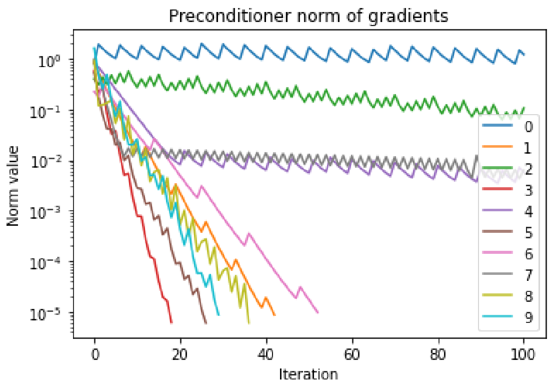


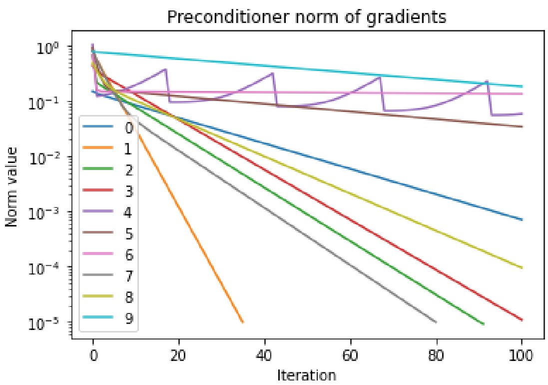
Abbildung 1: (a) - (d): Varying α and σ for the rosenbrock function, (e): gradient norms displayed for the configurations from plot (d)



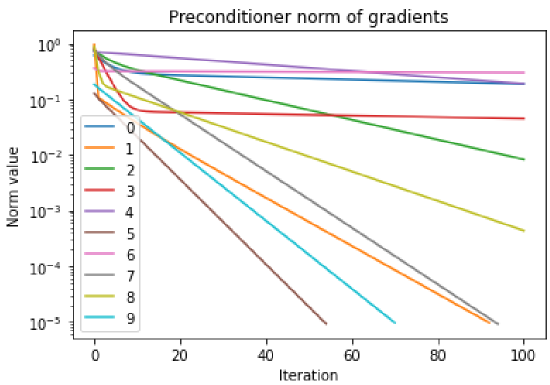
(a) $\alpha = 10, \sigma = 10^{-3}$



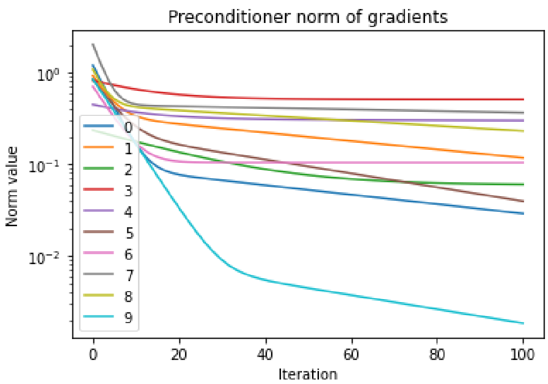
(b) $\alpha = 10, \sigma = 10^{-1}$



(c) $\alpha = 1, \sigma = 10^{-2}$



(d) $\alpha = 1, \sigma = 10^{-1}$



(e) $\alpha = 0.1, \sigma = 10^{-2}$

Abbildung 2: Varying α and σ for 10 random quadratic problems (the random problems differ for each plot)