All rings are assumed to be commutative. Rings and ring homomorphisms are supposed to be unitary. For a local ring A, its unique maximal ideal will be denoted $\mathfrak{m} = \mathfrak{m}_A$ and its residue field $k = A/\mathfrak{m}$. We mostly follow chapter 4 in [Wei94].

1. Regular rings

1.1. Motivation. In algebraic geometry one considers solutions of polynomial equations over fields. For example

$$S = \{(x, y) \in \mathbb{C}^2 \colon y^2 - x^3 = 0\}.$$

For a drawing over $\mathbb{R}$, see 1. Intuitively speaking, this curve has a singularity at the origin, but how can we detect this algebraically? One option is to consider the tangent space. Following our geometric intuition, S should have dimension 1. If S was regular at the origin, the tangent space there should have equal dimension. We thus need to establish a notion of tangent space at a point. Again following our intuition, the tangent space at a point $p \in S$ should be some linear approximation of S at p. Elements of its vector space dual, the cotangent space, should then be linear functionals on the tangent space, in other words linear approximations of functions of S locally defined at p.

What are functions on S? Let us first consider the plane $\mathbb{C}^2$. Since we are doing algebraic geometry, functions should be polynomials, so global functions on $\mathbb{C}^2$ are given by $\mathbb{C}[X, Y]$. Since linear functionals vanish at 0, to obtain elements of the cotangent space, we restrict to functions vanishing at $(0, 0)$, i.e.

$$\mathfrak{m} = \{f \in \mathbb{C}[X, Y] \colon f(0, 0) = 0\}.$$

Now to obtain a linear approximation of a polynomial $f \in \mathfrak{m}$, we „pick out the linear terms", e.g. for $f = X + 2Y + 3X^2 + 2Y^2$ we obtain $X + 2Y$, in other words we consider $f \bmod \mathfrak{m}^2$. Thus a natural candidate for the cotangent space of $\mathbb{C}^2$ at $(0, 0)$ is

$$\mathfrak{m}/\mathfrak{m}^2$$

which is naturally a $\mathbb{C}$ vector space[1]. Since $\mathfrak{m}^2 = (X^2, Y^2, XY)$, we obtain

$$\begin{aligned} \mathfrak{m}/\mathfrak{m}^2 = \{aX + bY + cX^2 + dY^2 + eXY + \text{ higher order terms}\}/\mathfrak{m}^2 &\xrightarrow{\simeq} \mathbb{C}^2 \\ aX + bY + cX^2 + dY^2 + eXY + \ldots &\longmapsto (a, b), \end{aligned}$$

which matches with our intuition: The dimension of the plane $\mathbb{C}^2$ is two and so is the dimension of its cotangent space at the non-singular point $(0, 0)$.

Now how does the picture change for S? Functions on S should still be polynomials, but two functions $f, g \in \mathbb{C}[X, Y]$ should be considered equal if their difference vanishes on S, in other words if

$$\forall (x, y) \in \mathbb{C}^2 \text{ s.t. } y^2 - x^3 = 0 \colon (f - g)(x, y) = 0.$$

This is equivalent to saying that $Y^2 - X^3 \mid (f - g)$, or in other words that $f - g \in (Y^2 - X^3)$. So the ring of functions on S should be

$$A = \mathbb{C}[X, Y]/(Y^2 - X^3).$$

Functions vanishing at $(0, 0)$ are then given by

$$\mathfrak{m}_A = \mathfrak{m}/(Y^2 - X^3),$$

so the cotangent space of S at $(0, 0)$ is finally given by

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \simeq \left(\mathfrak{m}/\mathfrak{m}^2\right) / \left((Y^2 - X^3)/\mathfrak{m}^2\right).$$

[1] We are simplifying here: To be more precise, since we are only interested in the local behaviour of functions at the origin, we should also consider functions that are only locally defined in a neighbourhood of $(0, 0)$ and regard two functions as equal, if they agree on a neighbourhood of $(0, 0)$. Assuming we have defined a suitable structure of topological space on $\mathbb{C}^2$, this can be expressed as $\varinjlim_{(0,0) \in U \subseteq \mathbb{C}^2} \mathcal{O}(U)$ where U runs through all open subsets of $\mathbb{C}^2$ containing the origin and $\mathcal{O}(U)$ denotes the functions defined on the open set U. If all these objects are set up in the correct fashion, one obtains the local ring $\mathbb{C}[X, Y]_{\mathfrak{m}}$. The interested reader can replace every occurence of $\mathbb{C}[X, Y]$ by $\mathbb{C}[X, Y]_{\mathfrak{m}}$ and will observe, that the numerical result stays the same.

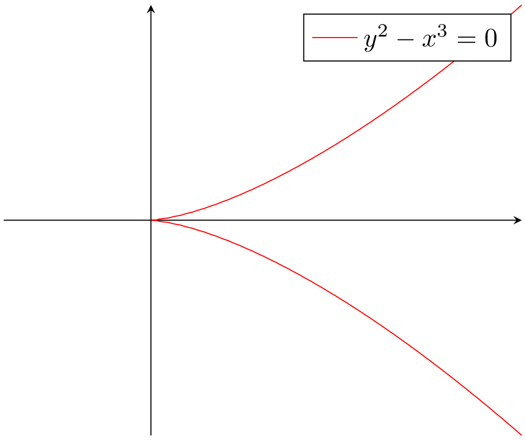

FIGURE 1. An algebraic curve over $\mathbb{C}$ drawn over $\mathbb{R}$

So the cotangent space of S at the origin is defined by cutting out the reduction of $Y^2 - X^3 \mod \mathfrak{m}^2$ out of the cotangent space of $\mathbb{C}^2$. But $\mathfrak{m}^2 = (X^2, Y^2, XY)$, so $Y^2 - X^3 \equiv 0 \mod \mathfrak{m}^2$. Finally we obtain

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \simeq \mathfrak{m}/\mathfrak{m}^2 \simeq \mathbb{C}^2.$$

Thus we obtain

$$\dim(S) = 1 < 2 = \dim_{\mathbb{C}} \left(\mathfrak{m}_A/\mathfrak{m}_A^2\right)$$

and S has a singularity at $(0,0)$.

To formalise this, we need to establish the notion of dimension. Instead of working with S, we define this for its ring of functions.

1.2. **Krull dimension.**

Definition 1.1 (Krull dimension)**.** Let A be a ring. The *Krull dimension* of A, denoted $\dim(A)$, is the length d of the longest chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_d$ of prime ideals in A. If no such maximal chain exists, set $\dim(A) = \infty$.

Example 1.2. Any field has Krull dimension 0. If A is a PID that is not a field, then $\dim(A) = 1$, since every prime ideal $\neq (0)$ is maximal and (0) is not maximal if A is not a field. One can show that for a field k

$$\dim k[T_1, \ldots, T_n] = n.$$

In the ring $k[T_1, T_2, \ldots]$ we have the strictly ascending chain of prime ideals

$$(T_1) \subsetneq (T_1, T_2) \subsetneq (T_1, T_2, T_3) \ldots,$$

so $\dim k[T_1, T_2, \ldots] = \infty$.

Example 1.3. In the introductory example we had the rings of functions $\mathbb{C}[X, Y]$ for $\mathbb{C}^2$ which has by 1.2 dimension two and $\mathbb{C}[X, Y]/(Y^2 - X^3)$ for S which has dimension 1. Indeed, this can be derived[2] from the general remark: If A is a local noetherian ring and $x \in \mathfrak{m}$ is a nonzerodivisor, then $\dim A/(x) = \dim A - 1$ (Cor. 11.18 in [AM69]).

We have seen that in general, the Krull dimension of a ring can be infinite. This can even happen for noetherian rings. An important result in dimension theory is that the situation is better for noetherian *local* rings.

Theorem 1.4 (Thm. 5.12, Cor. 5.14 [Liu02])**.** *If A is a local noetherian ring and $\mathfrak{m}$ can be generated by r elements, then $\dim(A) \leq r$. In particular, $\dim(A)$ is finite.*

[2]Since $\mathbb{C}[X, Y]$ is not local, one needs to consider all localisations $\mathbb{C}[X, Y]_{\mathfrak{p}}$ for $\mathfrak{p} \subseteq \mathbb{C}[X, Y]$ a prime ideal and use the observation that for any ring A, $\dim(A) = \sup_{\mathfrak{p} \subseteq A} \dim(A_{\mathfrak{p}})$.

Corollary 1.5. *If A is a local noetherian ring. Then*
$$\dim(A) \leq \dim_k \left(\mathfrak{m}/\mathfrak{m}^2\right).$$

Proof. Let $d = \dim_k \left(\mathfrak{m}/\mathfrak{m}^2\right)$. By Nakayama, $\mathfrak{m}$ can be generated by d elements. The result now follows from 1.4. □

Definition 1.6 (Regular ring)**.** A noetherian local ring is *regular* if $\dim(A) = \dim_k \left(\mathfrak{m}/\mathfrak{m}^2\right)$[3].

Remark 1.7. One can show that regular local noetherian rings are integral domains. A possible reference is Prop. 2.11 in [Liu02].

Whenever one introduces a notion for certain objects, one should explore permanence properties of said notion under standard constructions. For example, given a regular local noetherian ring A and a prime ideal $\mathfrak{p} \subseteq A$, is its localisation $A_\mathfrak{p}$ again regular? While our definition of regularity is useful for hands on calculations as we did in the motivation of this section, answering said questions turns out to be difficult. Fortunately, Serre found a strong characterisation of regular local noetherian rings by homological algebra. The required concepts are introduced in the next section.

2. Homological characterisation

2.1. **Homological dimension.**

Definition 2.1 (Projective Dimension)**.** Let A be a ring and M an A-module. The *projective dimension* $\mathrm{pd}_A(M)$ is the minimum integer n (if it exists) such that there is a resolution of M by projective modules
$$0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0.$$
If no such n exists, we set $\mathrm{pd}_A(M) = \infty$. If there is no risk of confusion we write $\mathrm{pd}(M) = \mathrm{pd}_A(M)$.

Definition 2.2. Let A be a ring. The (possibly infinite) number
$$\mathrm{hd}(A) = \sup\{\mathrm{pd}(M) \colon M \ A\text{-module}\}$$
is called the *homological dimension* of A.

Example 2.3. For a field k, every module is free, so $\mathrm{hd}(k) = 0$. If A is a PID, every submodule of a free module is free, in particular every module has a free resolution of length 1, so $\mathrm{hd}(A) \leq 1$.

2.2. **A criterion for regularity.** We can now state Serre's criterion:

Theorem 2.4 (Serre)**.** *A noetherian local ring A is regular if and only if* $\mathrm{hd}(A)$ *is finite. In this case* $\dim(A) = \mathrm{hd}(A)$.

This enables us to answer the introductory question:

Corollary 2.5. *If A is a regular local ring and $\mathfrak{p}$ is a prime ideal in A, then the localization $A_\mathfrak{p}$ is regular and local.*

Proof. Let S be a multiplicative set in A. It suffices to show, that $S^{-1}A$ has finite homological dimension (we may set $A_\mathfrak{p} = S^{-1}A$ for $S = A \setminus \mathfrak{p}$). Let N be a $S^{-1}A$-module. By interpreting N as an A-module, we get the existence of a projective resolution $P \to N$ of length $\leq \mathrm{hd}(A)$. Because $S^{-1}A$ is a flat A-module and $S^{-1}N = N$, $S^{-1}P \to N$ is a projective $S^{-1}A$-module resolution of length $\leq \mathrm{hd}(A)$. □

The rest of this talk is devoted to proving 2.4.

[3]The attentive reader will ask themself, why we now restrict to *local* rings. The reason is contained in 1.

2.3. **Properties of homological dimension.**

Lemma 2.6. *Let A be a ring. The following are equivalent for an A-module M:*

(i) $\mathrm{pd}(M) \leq d$.
(ii) $\mathrm{Ext}_A^n(M, N) = 0$ *for all* $n > d$ *and all A-modules N.*
(iii) $\mathrm{Ext}_A^{d+1}(M, N) = 0$ *for all A-modules N.*
(iv) if $0 \to M_d \to P_{d-1} \to \ldots \to P_1 \to P_0 \to M \to 0$ *is a resolution of M with P_i projective for all i, then M_d is also projective.*

Proof. *(iv)⇒(i)*: Let $P^\bullet \to M$ be any projective resolution of M. Then we have an exact sequence

$$0 \to K \to P_{d-1} \to \ldots \to P_1 \to P_0 \to M \to 0$$

where K is the kernel of $P_{d-1} \to P_{d-2}$. Thus K is projective and $\mathrm{pd}(M) \leq d$.

(i)⇒(ii): for an A-module N, $\mathrm{Ext}_A^\bullet(M, N)$ can be calculated by a projective resolution of M.

(ii)⇒(iii): immediate.

(iii)⇒(iv): Let

$$0 \to M_d \to P_{d-1} \to \ldots \to P_1 \to P_0 \to M \to 0$$

be a resolution of M as in (iv) and let N be an A-module. By dimension shifting, we obtain

$$0 = \mathrm{Ext}_A^{d+1}(M, N) \simeq \mathrm{Ext}_A^1(M_d, N).$$

Since N was arbitrary, M_d is projective. □

Proposition 2.7. *Let A be a ring. The following numbers are the same:*

(1) $\mathrm{hd}(A)$
(2) $\sup\{\mathrm{pd}(M)\colon M$ *finitely generated A-module*$\}$
(3) $\sup\{\mathrm{pd}(A/\mathfrak{a})\colon \mathfrak{a} \subseteq A$ *ideal*$\}$

Proof. Since for any ideal $\mathfrak{a} \subseteq A$, $A/\mathfrak{a}$ is finitely generated we have *(1)≥(2)≥(3)*. It thus suffices to show *(3)≥(1)*. We may assume $d = \sup\{\mathrm{pd}(A/\mathfrak{a})\colon \mathfrak{a} \subseteq A \text{ ideal}\} < \infty$. Let M, N A-modules and choose a resolution

$$0 \to N \to I_0 \to I_1 \to \ldots \to I_{d-1} \to N_d \to 0$$

with the I_j injective. Then by dimension shifting for any ideal $\mathfrak{a} \subseteq A$:

$$0 = \mathrm{Ext}_A^{d+1}(A/\mathfrak{a}, N) \simeq \mathrm{Ext}_A^1(A/\mathfrak{a}, N_d).$$

By Baer's criterion, N_d is injective. Thus N has an injective resolution of length d, in particular $\mathrm{Ext}_A^{d+1}(M, N) = 0$. Since N was arbitrary, $\mathrm{pd}(M) \leq d$ by 2.6. □

Remark 2.8. Let A be a ring and $(M_i)_{i \in I}$ a family of A-modules. Then

$$\mathrm{pd}\left(\bigoplus_{i \in I} M_i\right) = \sup_{i \in I}\{\mathrm{pd}(M_i)\}.$$

Proof. Let N be an A-module and $n \geq 0$. Then we have a natural isomorphism

$$\mathrm{Ext}_A^n\left(\bigoplus_{i \in I} M_i, N\right) = \prod_{i \in I}^{n} \mathrm{Ext}_A^n(M_i, N).$$

The right hand side is zero if and only if $\mathrm{Ext}_A^n(M_i, N) = 0$ for some $i \in I$. □

Lemma 2.9. *If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence of A-modules. Then*

$$\mathrm{pd}(M) \leq \max\{\mathrm{pd}(M'), \mathrm{pd}(M'')\}$$

with equality except when $\mathrm{pd}(M'') = \mathrm{pd}(M') + 1$.

Proof. Let $d = \max\{\mathrm{pd}(M'), \mathrm{pd}(M'')\}$ For any A-module N we have a long exact sequence

$$\mathrm{Ext}_A^{d+1}(M'', N) \to \mathrm{Ext}_A^{d+1}(M, N) \to \mathrm{Ext}_A^{d+1}(M', N).$$

Since the outer terms are zero, the middle term also vanishes, so $\mathrm{pd}(M) \leq d$. If $\mathrm{pd}(M'') \leq \mathrm{pd}(M')$, then $\mathrm{pd}(M') = d$ and there exists N such that $\mathrm{Ext}_A^d(M', N) \neq 0$. By exactness of

$$\mathrm{Ext}_A^d(M, N) \to \underbrace{\mathrm{Ext}_A^d(M', N)}_{\neq 0} \to \underbrace{\mathrm{Ext}_A^{d+1}(M'', N)}_{=0},$$

we obtain $\mathrm{Ext}_A^d(M, N) \neq 0$ and therefore $\mathrm{pd}(M) \geq d$. If $\mathrm{pd}(M'') > \mathrm{pd}(M')+1$, then $\mathrm{pd}(M'') = d$ and $\mathrm{pd}(M') \leq d-2$. Thus there exists N such that $\mathrm{Ext}_A^d(M'', N) \neq 0$. By exactness of

$$\underbrace{\mathrm{Ext}_A^{d-1}(M', N)}_{=0} \to \underbrace{\mathrm{Ext}_A^d(M'', N)}_{\neq 0} \to \mathrm{Ext}_A^d(M, N),$$

we obtain $\mathrm{Ext}_A^d(M, N) \neq 0$ and therefore $\mathrm{pd}(M) \geq d$. □

Remark 2.10. In the next section, we will see many examples where the inequality in 2.9 is strict.

Proposition 2.11. *Let A be a local noetherian ring and M a finitely generated A-module. Then the following holds:*

(i) If $\mathrm{Tor}_1^A(M, k) = 0$, then M is free.

(ii) For any $d \geq 0$

$$\mathrm{pd}(M) \leq d \iff \mathrm{Tor}_{d+1}^A(M, k) = 0.$$

In particular, pd(M) is the largest d such that $\mathrm{Tor}_d^A(M, k) \neq 0$.

Proof. We prove (ii) and (i) will follow on the way. ($\Rightarrow$): Immediate, since Tor is a left derivative. ($\Leftarrow$): Since M is finitely generated, there exist $u_1, \ldots, u_m \in M$ such that $\overline{u_1}, \ldots, \overline{u_m}$ form a k basis of $M/\mathfrak{m}M$. Then by Nakayama, M is generated by $u_1, \ldots, u_n$. Thus we have a surjection $A^m \to M$ given by $e_i \mapsto u_i$. Let K be the kernel, so we obtain an exact sequence

$$0 \longrightarrow K \longrightarrow A^m \longrightarrow M \longrightarrow 0\ .$$

Proceed by induction on d. If $d = 0$, then tensoring with k yields exact rows

$$\begin{tikzcd}
\underbrace{\mathrm{Tor}_1^A(M,k)}_{=0} \arrow[r] & K \otimes_A k \arrow[r] \arrow[d, "\simeq"] & A^m \otimes_A k \arrow[r] \arrow[d, "\simeq"] & A \otimes_A k \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & K/\mathfrak{m}K \arrow[r] & k^m \arrow[r] & M/\mathfrak{m}M &
\end{tikzcd},$$

where the bottom map $k^m \to M/\mathfrak{m}M$ is given by $e_i \mapsto \overline{u_i}$. Since the $\overline{u}_i$ form a basis, this map is an isomorphism, so $K/\mathfrak{m}K = 0$. As a submodule of the finitely generated A^m over the noetherian ring A, K is finitely generated and therefore by Nakayama $K = 0$. Thus $A^m \simeq M$. This shows (i) and $\mathrm{pd}(M) = 0$. Now suppose $d \geq 1$. Consider the exact sequence

$$\underbrace{\mathrm{Tor}_{d+1}^A(M, k)}_{=0} \to \mathrm{Tor}_d^A(K, k) \to \underbrace{\mathrm{Tor}_d^A(A^m, k)}_{=0},$$

where the outer terms vanish by assumption and since A^m is free. Induction thus yields $\mathrm{pd}(K) \leq d-1$. By 2.9 we have

$$\mathrm{pd}(M) \leq \max\{\mathrm{pd}(K), \mathrm{pd}(M)\} = \mathrm{pd}(A^m) = 0 \leq d$$

or $\mathrm{pd}(M) = \mathrm{pd}(K) + 1 \leq d - 1 + 1 = d$. □

Corollary 2.12. *If A is local noetherian and P is a finitely generated, projective A-module. Then P is free*[4].

[4]This also holds if we drop the noetherianity condition. A proof of this can be obtained by a slight modification of the proof of 2.11

Corollary 2.13. *If A is local noetherian, then* $\mathrm{hd}(A) = \mathrm{pd}_A(A/\mathfrak{m})$.

Proof.

$$\begin{aligned}
\mathrm{pd}(k) &\leq \mathrm{hd}(A) \\
&\overset{2.7}{=} \sup\{\mathrm{pd}(M)\colon M \text{ finitely generated } A\text{-module}\} \\
&\overset{2.11}{=} \sup\{d\colon \mathrm{Tor}_d^A(M,k) \neq 0, M \text{ finitely generated } A\text{-module}\} \\
&= \mathrm{fd}(k) \\
&= \inf\{d\colon \text{there exists a flat resolution of length } d \text{ of } k\} \\
&\leq \inf\{d\colon \text{there exists a projective resolution of length } d \text{ of } k\} \\
&= \mathrm{pd}(k).
\end{aligned}$$

□

3. Change of Rings

Lemma 3.1. *Let A be a ring, $x \in A$ a nonzerodivisor and M an A-module. Then there is a canonical isomorphism*

$$\mathrm{Tor}_1^A(A/(x), M) \simeq \{m \in M \mid xm = 0\}.$$

Proof. Since x is a nonzerodivisor, the following sequence is exact:

$$0 \longrightarrow A \xrightarrow{x\cdot} A \longrightarrow A/(x) \longrightarrow 0\ .$$

Tensoring with M yields

$$\begin{tikzcd}
\underbrace{\mathrm{Tor}_1^A(A,M)}_{=0} \arrow[r] & \mathrm{Tor}_1^A(A/(x),M) \arrow[r] & A \otimes_A M \arrow[r] \arrow[d, "\simeq"] & A \otimes_A M \arrow[d, "\simeq"] \\
& & M \arrow[r] & M
\end{tikzcd},$$

where the map $M \to M$ is given by $m \mapsto xm$. Thus

$$\mathrm{Tor}_1^A(M, A/(x)) = \ker(M \to M, m \mapsto xm) = \{m \in M \mid xm = 0\}.$$

□

Lemma 3.2. *Let A be a ring and $x \in A$ a nonzerodivisor. If M is a projective $A/(x)$-module, then* $\mathrm{pd}_A(M) = 1$.

Proof. Since $xM = 0$, M is not torsion-free as A-module, in particular not a submodule of a free A-module. Thus M is not a projective A-module, i.e. $\mathrm{pd}_A(M) \geq 1$. Since M is projective as $A/(x)$-module, there exists an $A/(x)$-module N and an index set I such that

$$(A/(x))^{(I)} \simeq M \oplus N$$

as $A/(x)$-modules and therefore in particular as A-modules. Thus

$$\mathrm{pd}_A(M) \leq \max\{\mathrm{pd}_A(M), \mathrm{pd}_A(N)\} \overset{2.8}{=} \mathrm{pd}_A(M \oplus N) = \mathrm{pd}_A\left((A/(x))^{(I)}\right) \overset{2.8}{=} \mathrm{pd}_A(A/(x)).$$

We thus have reduced to the case $M = A/(x)$. Since x is a nonzerodivisor, the sequence

$$0 \longrightarrow A \xrightarrow{\cdot x} A \longrightarrow A/(x) \longrightarrow 0$$

is exact. By 2.9, we obtain

$$0 = \mathrm{pd}_A(A) = \max\{\mathrm{pd}_A(A), \mathrm{pd}_A(A/(x))\}$$

or

$$\mathrm{pd}_A(A/(x)) = \underbrace{\mathrm{pd}_A(A)}_{=0} + 1.$$

Since $\mathrm{pd}_A(A/(x)) \geq 1$, the second equation must hold. □

Theorem 3.3 (First Change of Rings Theorem). *Let A be a ring and $x \in A$ a nonzerodivisor. If $0 \neq M$ is a A/x-module with $\mathrm{pd}_{A/x}(M)$ finite, then*

$$\mathrm{pd}_A(M) = 1 + \mathrm{pd}_{A/x}(M).$$

Proof. We proceed by induction on $d = \mathrm{pd}_{A/(x)}(M) < \infty$. If $d = 0$, then M is projective $A/(x)$-module, so by 3.2 $\mathrm{pd}_A(M) = 1 = 1 + d$. Now suppose $d \geq 1$ and choose an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

of $R/(x)$-modules where F is free. Since $\mathrm{pd}_{A/(x)}(F) = 0 < d = \mathrm{pd}_{A/(x)}(M)$, by 2.9 we obtain

$$d = \mathrm{pd}_{A/(x)}(M) = \mathrm{pd}_{A/(x)}(K) + 1.$$

By induction, $\mathrm{pd}_A(K) = 1 + \mathrm{pd}_{A/(x)}(K) = 1 + (d-1) = d$. Again by 2.9 we have

$$\mathrm{pd}_A(M) = \mathrm{pd}_A(K) + 1 = d + 1$$

or

$$1 \overset{3.2}{=} \mathrm{pd}_A(F) = \max\{\mathrm{pd}_A(K), \mathrm{pd}_A(M)\} = \max\{\underbrace{d}_{\geq 1}, \underbrace{\mathrm{pd}_A(M)}_{\geq 1}\}.$$

It remains to show that the second case does not occur. Suppose it does, then $\mathrm{pd}_A(M) = 1 = d$. Choose an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

of R-modules where F is free. Since $\mathrm{pd}_A(M) = 1$, K is projective A-module. Tensoring with $A/(x)$ yields the exact sequence:

$$\begin{tikzcd}
\underbrace{\mathrm{Tor}_1^A(F, A/(x))}_{=0} \arrow[r] & \mathrm{Tor}_1^A(M, A/(x)) \arrow[r] \arrow[d, "\simeq"] & K \otimes_A A/(x) \arrow[r] \arrow[d, "\simeq"] & F \otimes_A A/(x) \arrow[r] \arrow[d, "\simeq"] & M \otimes_A A/(x) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & M = \{m \in M \mid xm = 0\} \arrow[r] & K/xK \arrow[r] & F/xF \arrow[r] & M/xM = M \arrow[r] & 0.
\end{tikzcd}$$

In the bottom row we have used $xM = 0$ and 3.1. K/xK and F/xF are projective $A/(x)$ modules. Since $d = \mathrm{pd}_{A/x}(M) \leq 2$, the syzygy M is projective $A/(x)$-module which is a contradiction to $d \geq 1$. ☐

Theorem 3.4 (Second Change of Rings Theorem). *Let x be a nonzerodivisor in a ring A and let M be an A-module such that x is a nonzerodivisor on A. Then*

$$\mathrm{pd}_A(M) \geq \mathrm{pd}_{A/(x)}(M/xM).$$

Proof. If $\mathrm{pd}_A(M) = \infty$, there is nothing to show. Thus suppose $d = \mathrm{pd}_A(M) < \infty$ and proceed by induction on d. If $d = 0$, then M is projective A-module, so M/xM is projective $A/(x)$ module, i.e. $\mathrm{pd}_{A/(x)}(M/xM) = 0 = d$. If $d > 0$, then choose an exact sequence of A-modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

where F is free. By 2.9, we obtain $\mathrm{pd}_A(K) = \mathrm{pd}_A(M) - 1 = d - 1$. Since K is a submodule of a free module and x is a nonzerodivisor in A, x is also nonzerodivisor on K. By induction we thus have $\mathrm{pd}_{A/(x)}(K/xK) \leq \mathrm{pd}_A(K) = d - 1$. Tensoring the above exact sequence yields the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^A(M, A/(x)) \longrightarrow K/xK \longrightarrow F/xF \longrightarrow M/xM \longrightarrow 0\ .$$

Since x is a nonzerodivisor by 3.1

$$\mathrm{Tor}_1^A(M, A/(x)) \simeq \{m \in M \colon xm = 0\} = 0,$$

since x is a nonzerodivisor on M by assumption. Again by 2.9, M/xM is projective $A/(x)$ module or

$$\mathrm{pd}_{A/(x)}(M/xM) = \mathrm{pd}_{A/(x)}(K/xK) + 1 = d - 1 + 1 = d.$$

□

Lemma 3.5. *Let A be a noetherian local ring and M a finitely generated A-module. Suppose $x \in \mathfrak{m}$ is a nonzerodivisor on M. Then M/xM is free $A/(x)$-module if and only if M is free A-module.*

Proof. Tensorproduct preserves direct sums[5], so if M is free, so is M/xM. Suppose M/xM is free $A/(x)$-module. Then there exist $u_1, \dots, u_n \in M$ (possibly $n = 0$) such that $\overline{u_1}, \dots, \overline{u_n}$ are a basis of M/xM. Thus by Nakayama, $u_1, \dots, u_n$ generate M. It remains to show that the u_i are linearly independent. Denote by K the kernel of $R^n \to A$, given by $e_i \mapsto u_i$. For $(r_1, \dots, r_n) \in K$, we have $\sum_{i=1}^n r_i u_i = 0$. In particular $\sum_{i=1}^n \overline{r_i u_i} = 0$. Since the $\overline{u}_i$ are linearly independent over $A/(x)$, $r_i \in (x)$ for all $i \in \{1, \dots, n\}$. So there exist $s_i \in R$ such that $r_i = s_i x$ for all i. Thus

$$0 = \sum_{i=1}^n r_i u_i = \sum_{i=1}^n s_i x u_i = x \sum_{i=1}^n s_i u_i.$$

Since x is a nonzerodivisor on M, we obtain $\sum_{i=1}^n s_i u_i = 0$, so $(s_1, \dots, s_n) \in K$. Therefore $(r_1, \dots, r_n) = x(s_1, \dots, x_n) \in xK \subseteq \mathfrak{m}K$. Thus $K = \mathfrak{m}K$ and since A is noetherian, K is finitely generated, so by Nakayama $K = 0$. □

Theorem 3.6 (Third Change of Rings Theorem)**.** *Let A be a noetherian local ring and M a finitely generated A-module. If $x \in \mathfrak{m}$ is a nonzerodivisor on both A and M, then*

$$\mathrm{pd}_A(M) = \mathrm{pd}_{A/x}(M/xM).$$

Proof. $\geq$ holds by 3.4, in particular if $\mathrm{pd}_{A/(x)}(M/xM) = \infty$ there is nothing to show. Suppose $d = \mathrm{pd}_{A/(x)}(M/xM) < \infty$ and proceed by induction. If $d = 0$, then M/xM is projective, hence free over the local ring $A/(x)$. Thus M is free A-module by 3.5, so $\mathrm{pd}_A(M) = 0 = d$. Now let $d > 0$ and choose an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

where F is a free and finitely generated A-module. Since $\mathrm{Tor}_1^A(M, A/(x)) = \{m \in M \mid xm = 0\} = 0$, tensoring the above sequence with $A/(x)$ yields the exact sequence

$$0 \longrightarrow K/xK \longrightarrow F/xF \longrightarrow M/xM \longrightarrow 0\ .$$

Since F/xF is free, by 2.9

$$\mathrm{pd}_{A/(x)}(K/xK) = \mathrm{pd}_{A/(x)}(M/xM) - 1 = d - 1.$$

Since A is noetherian, the submodule K of F is finitely generated and x is a nonzerodivisor on K. Therefore

$$d - 1 = \mathrm{pd}_{A/(x)}(K/xK) = \mathrm{pd}_A(K)$$

by induction. By 2.9, we thus have

$$\mathrm{pd}_A(M) = \mathrm{pd}_A(K) + 1 = d - 1 + 1 = d.$$

□

Corollary 3.7. *If A is local noetherian and $x \in \mathfrak{m}$ a nonzerodivisor, then either* $\mathrm{hd}(A/xA) = \infty$ *or* $\mathrm{hd}(A) = 1 + \mathrm{hd}(A/xA)$.

[5]Since it is a left adjoint, it commutes with arbitrary colimits.

Proof. Let $B = A/xA$ have finite homological dimension, then in particular $\mathrm{pd}_B(k) < \infty$, so by the first Change of Rings Theorem 3.3 we get

$$\mathrm{pd}_A(k) = 1 + \mathrm{pd}_B(k)$$

for the residue field $k = A/\mathfrak{m} = B/\mathfrak{m}B$. The result follows, since $\mathrm{pd}_A(k) = \mathrm{hd}(A)$ and $\mathrm{pd}_B(k) = \mathrm{hd}(A/xA)$ by 2.13. □

4. Proof of Serre's criterion

In this section we prove 2.4.

Remark 4.1 (Lem. 17.3.1.2 in [Gro64]). If A is a local noetherian ring and every element of $\mathfrak{m} \setminus \mathfrak{m}^2$ is a zerodivisor, then there exists some $s \in A \setminus \{0\}$ such that $s\mathfrak{m} = 0$.

Lemma 4.2 (Grade 0 Lemma). *Let A be a local noetherian ring such that every element of $\mathfrak{m} \setminus \mathfrak{m}^2$ is a zerodivisor. Then* $\mathrm{hd}(A) = 0$ *or* $\mathrm{hd}(A) = \infty$.

Proof. Suppose $0 < \mathrm{hd}(A) < \infty$. Then by 2.7, there exists a finitely generated A-module M such that $1 < \mathrm{pd}(M) = d < \infty$. Choose any projective resolution

$$0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0.$$

Since A is noetherian and M is finitely generated, we may assume all P_j are finitely generated[6]. Now let $K = \ker(P_{d-2} \to P_{d-3})$. Then $\mathrm{pd}(K) = 1$. Indeed: we have an exact sequence

$$0 \to P_d \to P_{d-1} \to K \to 0,$$

so $\mathrm{pd}(K) \leq 1$. On the other hand,

$$0 \to K \to P_{d-2} \to P_{d-3} \to \ldots \to P_0 \to M \to 0$$

is exact, thus K is not projective, since otherwise $\mathrm{pd}(M) < d$. Thus $\mathrm{pd}(K) \geq 1$. Since A is noetherian, $K \subseteq P_{d-2}$ is finitely generated, so we may replace M by K and assume $\mathrm{pd}(M) = 1$.

Choose $u_1, \ldots, u_m \in M$ such that $\overline{u_1}, \ldots, \overline{u_m}$ form a k-basis of $M/\mathfrak{m}M$. By Nakayama, u_1, $\ldots$, u_m then generate M as A-module. We obtain an exact sequence $0 \to P \to A^m \to M \to 0$ where $A^m \to M$ is given by $e_i \mapsto u_i$. Tensoring with $k = A/\mathfrak{m}$ yields exact sequences

$$\begin{array}{ccccc} P \otimes_A A/\mathfrak{m} & \longrightarrow & A^m \otimes_A A/\mathfrak{m} & \longrightarrow & M \otimes_A A/\mathfrak{m} \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ P/\mathfrak{m}P & \longrightarrow & A^m/\mathfrak{m}A^m = k^m & \xrightarrow{\simeq} & M/\mathfrak{m}M \end{array}.$$

The map $k^m \to M/\mathfrak{m}M$ is given by $e_i \mapsto \overline{u_i}$ and is therefore an isomorphism. Thus $P/\mathfrak{m}P \to A^m/\mathfrak{m}A^m$ is the zero map, which means $P \subseteq \mathfrak{m}A^m$. Since $\mathfrak{m} \setminus \mathfrak{m}^2$ only contains zerodivisors, by 4.1 $s\mathfrak{m} = 0$ for some $s \in A \setminus \{0\}$. So $sP = 0$. Since $\mathrm{pd}(M) = 1$, P is projective and thus free over the local ring A, so $sP = 0$ implies $P = 0$. Contradiction to $\mathrm{pd}(M) > 0$. □

We now have all the ingredients to show Serre's criterion for regularity.

Proof of 2.4. ($\Rightarrow$) Proceed by induction on $d = \dim(A)$. If $d = 0$, then A is a field since A is integral domain. Then every A-module is free and $\mathrm{hd}(A) = 0$. Now suppose $d > 0$. Since A is regular, $\dim_k \left(\mathfrak{m}/\mathfrak{m}^2\right) = d$. Moreover since A is noetherian, $\mathfrak{m}$ is finitely generated, so by Nakayama $\mathfrak{m} = (x_1, \ldots, x_d)$ for elements $x_i \in A$. Denote by $B = A/(x_1)$. Since $(x_1) \subseteq \mathfrak{m}$, B is again local and noetherian. Since A is an integral domain, x_1 is non-zero divisor, so $\dim(B) \overset{1.3}{=} \dim(A) - 1 = d - 1$and $x_2, \ldots, x_d$ generate $\mathfrak{m}_B = \mathfrak{m}/x_1 A$. Again by Nakayama, their images generate $\mathfrak{m}_B/\mathfrak{m}_B^2$ as $B/\mathfrak{m}_B$-vector space. Thus

$$\dim(B) \overset{1.5}{\leq} \dim_k \left(\mathfrak{m}_B/\mathfrak{m}_B^2\right) \leq d - 1 = \dim(B).$$

[6]One constructs such a resolution inductively: Since M is finitely generated, there exists a surjection $R^{m_0} \to M \to 0$. The kernel K_0 is finitely generated since A is noetherian. If K_0 is not projective, choose new surjection $R^{m_1} \to K_0$. This gives an exact sequence $R^{m_1} \to R^{m_0} \to M \to 0$. Proceed inductively.

So B is regular of dimension $d-1$. The result applied to B thus yields

$$\mathrm{hd}(A) \overset{3.7}{=} 1 + \mathrm{hd}(B) = 1 + \dim(B) = 1 + (d-1) = d.$$

($\Leftarrow$) Proceed by induction on $\mathrm{hd}(A)$. If $\mathrm{hd}(A) = 0$, then every A-module is projective. Over local rings, finitely generated projective modules are free, thus every finitely generated A-module is free, i.e. A is a field[7] and in particular regular. Now suppose $\mathrm{hd}(A) > 0$. Since $\mathrm{hd}(A) < \infty$ by assumption

$$0, \infty \neq \mathrm{hd}(A).$$

So by 4.2 there exists a nonzerodivisor $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$. Let $d = \dim(A)$ and $B = A/(x_1)$. We claim that $k \oplus \mathfrak{m}_B \simeq \mathfrak{m}/x_1\mathfrak{m}$ as B modules. Supposing the claim is true, we obtain

$$\begin{aligned} \mathrm{hd}(B) &\overset{2.13}{=} \mathrm{pd}_B(k) \\ &\leq \sup\{\mathrm{pd}_B(k), \mathrm{pd}_B(\mathfrak{m}_B)\} \\ &\overset{2.8}{=} \mathrm{pd}_B(k \oplus \mathfrak{m}_B) \\ &= \mathrm{pd}_B(\mathfrak{m}/x_1\mathfrak{m}) \\ &\overset{3.6}{=} \mathrm{pd}_A(\mathfrak{m}) \\ &\overset{2.9}{=} \mathrm{pd}_A(k) - 1 \\ &\overset{2.13}{=} \mathrm{hd}(A) - 1. \end{aligned}$$

The result applied to B therefore yields B regular, thus

$$\dim_k(\mathfrak{m}_B/\mathfrak{m}_B^2) = \dim(B) = \dim(A) - 1 = d - 1.$$

Hence there exist $y_2, \ldots, y_d \in \mathfrak{m}_B$ mapping to a k-basis in $\mathfrak{m}_B/\mathfrak{m}_B^2$. By Nakayama, $y_2, \ldots, y_d$ then generate $\mathfrak{m}_B = \mathfrak{m}/(x_1)$. Taking pre-images $x_i \in A$ of y_i yields

$$\mathfrak{m} = (x_2, \ldots, x_d) + (x_1) = (x_1, \ldots, x_d).$$

Again by Nakayama

$$\dim(A) \overset{1.5}{\leq} \dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq d = \dim(A).$$

Thus A is regular. It remains to prove the claim. Denote by $r = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. Since $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$, we find $x_2, \ldots, x_r \in \mathfrak{m}$ such that the images of $x_1, \ldots, x_r$ form a k-basis of $\mathfrak{m}/\mathfrak{m}^2$. Let $I = (x_2, \ldots, x_r)A + x_1\mathfrak{m} \subseteq \mathfrak{m}$. Then

$$I/x_1\mathfrak{m} \subseteq \mathfrak{m}/x_1\mathfrak{m}.$$

We obtain a map

$$I/x_1\mathfrak{m} \hookrightarrow \mathfrak{m}/x_1\mathfrak{m} \twoheadrightarrow (\mathfrak{m}/x_1\mathfrak{m}) \,/\, (x_1A/x_1\mathfrak{m}) \simeq \mathfrak{m}/x_1A = \mathfrak{m}_B.$$

The composition is surjective, since any element in $\mathfrak{m}_B$ is image of a $y = \sum_{i=1}^d a_i x_i \in \mathfrak{m}$ and $y' = \sum_{i=2}^d a_i x_i \in I$ has the same image in $\mathfrak{m}_B$ since $y - y' = a_1 x_1 \in x_1 A$. Moreover we have an exact sequence

$$0 \to x_1A/x_1\mathfrak{m} \to \mathfrak{m}/x_1\mathfrak{m} \to \mathfrak{m}_B \to 0.$$

Any $x \in x_1A/x_1\mathfrak{m} \cap I/x_1\mathfrak{m}$ is the image of a_1x_1 for some $a_1 \in A$ and of $\sum_{i=2}^r a_i x_i$ for some $a_2, \ldots, a_r \in A$ in $\mathfrak{m}/x_1\mathfrak{m}$. Thus

$$a_1 x_1 = \sum_{i=2}^r a_i x_i + b_1 x_1$$

for some $b_1 \in \mathfrak{m}$. Thus

$$0 = -a_1 x_1 + \sum_{i=2}^r a_i x_i$$

[7] If $x \neq 0$ in A, then $A/(x)$ is finitely generated, hence free A-module. But $x \cdot A/(x) = 0$, so $A/(x) = 0$, since otherwise it would be a free module with torsion.

in $\mathfrak{m}/\mathfrak{m}^2$. Since $(x_1,\dots,x_r)$ form a $A/\mathfrak{m}$-basis of $\mathfrak{m}/\mathfrak{m}^2$, we have $a_j = 0$ in $A/\mathfrak{m}$, so $a_j \in \mathfrak{m}$ for all j. In particular $x \in x_1\mathfrak{m}$ so $x = 0$ in $\mathfrak{m}/x_1\mathfrak{m}$. Thus $x_1A/x_1\mathfrak{m} \cap I/x_1\mathfrak{m} = 0$. Moreover

$$x_1A + I = x_1A + (x_2,\dots,x_r)A + x_1\mathfrak{m} = \mathfrak{m},$$

so

$$x_1A/x_1\mathfrak{m} \oplus I/x_1\mathfrak{m} = \mathfrak{m}/x_1\mathfrak{m}.$$

Since $\ker(\mathfrak{m}/x_1\mathfrak{m} \to \mathfrak{m}_B) = x_1A/x_1\mathfrak{m}$, we obtain $I/x_1\mathfrak{m} \simeq \mathfrak{m}_B$.

Let $x = x_1$. Consider the exact sequence

$$0 \to \mathfrak{m} \to A \to A/\mathfrak{m} = k \to 0$$

and apply $A/(x) \otimes_A -$ to obtain an exact sequence

$$\begin{tikzcd} \underbrace{\mathrm{Tor}_1^A(A/(x),A)}_{=0} \arrow[r] & \mathrm{Tor}_1^A(A/(x),k) \arrow[r] & A/(x)\otimes_A \mathfrak{m} \arrow[r] \arrow[d, "\simeq"] & A/(x)\otimes_A A \arrow[d, "\simeq"] \\ & & \mathfrak{m}/x\mathfrak{m} \arrow[r] & A/(x) \end{tikzcd}.$$

The kernel of $\mathfrak{m}/x\mathfrak{m} \to A/(x)$ is given by $xA/x\mathfrak{m}$. The exactness of the sequence then yields

$$xA/x\mathfrak{m} \simeq \mathrm{Tor}_1^A(A/(x),k) \overset{3.1}{\simeq} \{a \in k \mid xa = 0\} = k.$$

In total we get

$$\mathfrak{m}/x_1\mathfrak{m} = x_1A/x_1\mathfrak{m} \oplus I/x_1\mathfrak{m} \simeq k \oplus \mathfrak{m}_B$$

as claimed. $\square$

References

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[Gro64] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I. *Inst. Hautes Études Sci. Publ. Math.*, (20):259, 1964.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.