5. Existence & classification of contact 3-mfds & Leg. knots

5.1 big picture

OT = flexible:

$$\{\xi \text{ OT}\}/\text{Isotopy} \xleftrightarrow{1:1} \{\text{Tang } 2\text{-plane fields}\}/\text{Hom.} \xleftrightarrow{1:1} \text{Alg top}$$

i.e. 'solved'

Tight = rigid

$\exists$ mfds with a unique tight c.s.
$\exists$ mfds with no " "
$\exists$ " " ∞ " "

5.2 classification of tight contact structures

Thm 1: $(\mathbb{R}^3, \xi_{st})$, (S^3, ξ_{st}), $(S^1\times \mathbb{R}^2, \xi_{st})$, $(S^1\times S^2, \xi_{st})$ are tight

proof: in sect 6

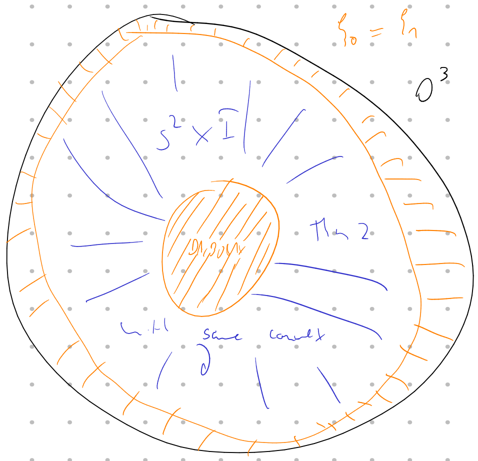
Thm 2: A tight contact structure ξ on $S^2\times[0,1]$ with convex boundary is determined (up to isotopy fixing ∂) by the dividing set of ∂

Thm 3 [Eliashberg]

Let ξ_0, ξ_1 be tight c.s. on D^3 s.t. ∂D^3 is convex and $\Gamma^0_{\partial D^3} = \Gamma^1_{\partial D^3}$

$\Rightarrow$ ξ_0 is isotopic to ξ_1 rel ∂D^3

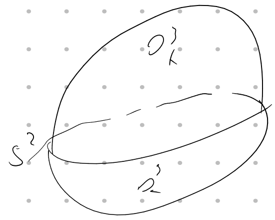
proof:



Thm 4 [Eliashberg]

S^3, $\mathbb{R}^3$, $S^1 \times S^2$ admit unique tight c.s.

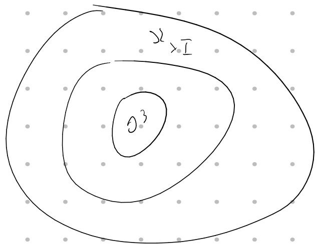
Proof: (1) $S^3 = D^3_+ \cup_{S^2} D^3_-$ wlog: S^2 convex



ξ tight $\Longrightarrow$ Γ_{S^2} is connected

Thm 3 $\Longrightarrow$ ξ_0 is isotopic to ξ_1

(2) $\mathbb{R}^3 = D^3(0) \cup S^2 \times [1,2] \cup S^2 \times [2,3] \cup \cdots$



wlog $S^2 \times n$ convex $\forall n \in \mathbb{Z}$

ξ tight $\Longrightarrow$ $\Gamma_{S^2 \times n}$ is connected

Thm 2&3 $\Longrightarrow$ ξ_0 is isotopic to ξ_1

(3) $S^1 \times S^2 = [-1,1] \times S^2 \big/ -1 \times S^2 \sim 1 \times S^2$

$\Longrightarrow$ wlog $\pm 1 \times S^2$ is convex with connected Γ

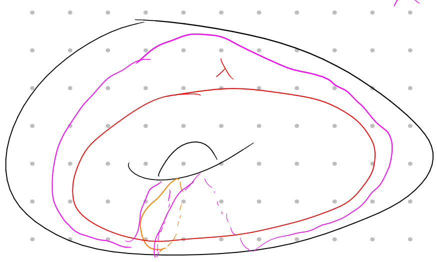
Let V be a solid torus with tight c.s. s.t. ∂V is convex

Fix: meridian μ: s.c.c. on ∂V non-trivial in ∂V but trivial in V.

Longitude λ: s.c.c. on ∂V nontrivial in ∂V s.t. $\mu \cap \lambda = pt$

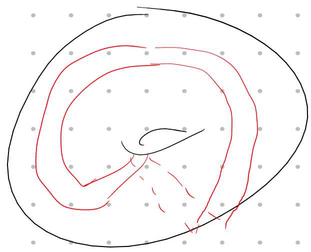
$\lambda' = \lambda + n\mu$

ξ tight $\Longrightarrow$ $\Gamma_{\partial V}$ consist of linear curves i.e. parallel copies of $p\mu + q\lambda$

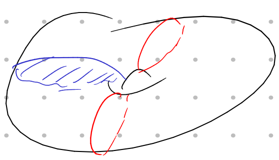


$$\text{Slope}(\partial V) = \text{slope}(\Gamma_{\partial V})$$

Ex: $\text{slope}(S^1 \times D^2, \xi_n) = -\frac{1}{n}$



no slope 0:



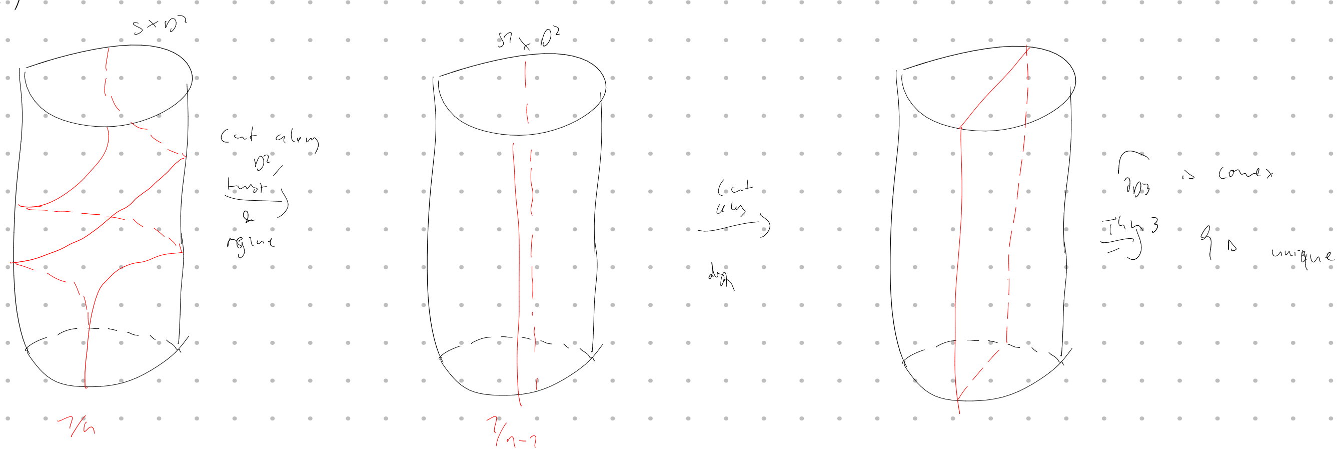
Thm 5: [Giroux, Honda]

(1) $\forall n \in \mathbb{Z}$ $\exists!$ tight c.s. on $S^1 \times D^2$ with convex ∂ with 2 dividing curves of slope $= \frac{1}{n}$

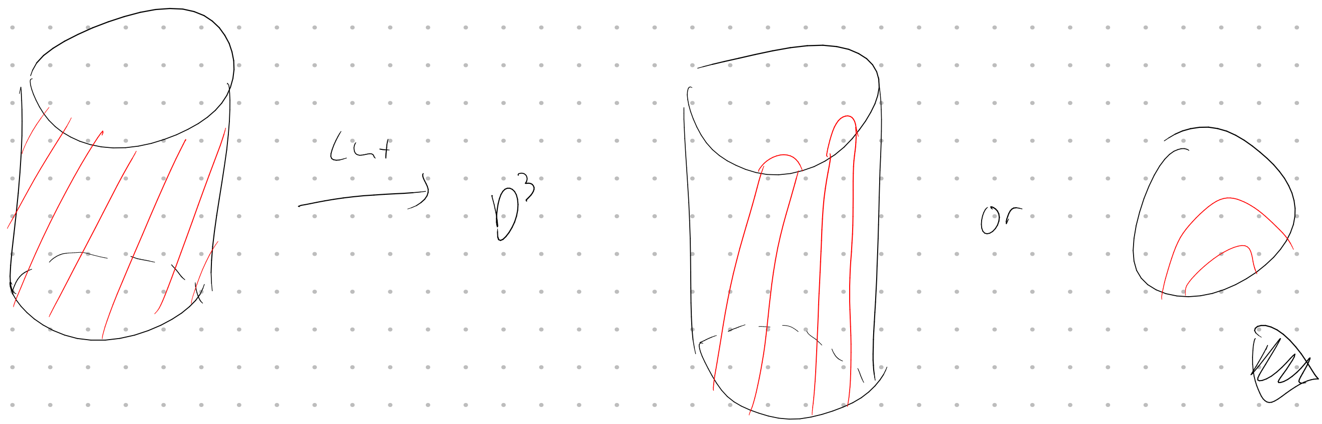
(2) $\forall r \in \mathbb{Q} \setminus \{0\}$: $\exists$ finitely many tight c.s. " " " of slope $= r$

proof sketch:

(1)



(2)



Thm 2: A tight c.s. on $S^2 \times [-1, 1]$ with convex ∂ is determined up to isotopy rel ∂ by the dividing set of ∂

proof sketch (Thm 2): Write $S^z = S^2 \times \{z\}$.

<u>Idea</u>: make S^z convex $\forall z$.

Problem: Morse-Smale is not generic for one-parameter families

ξ tight $\Rightarrow$ $\nexists$ closed orbit in S^z_ξ

$\Rightarrow$ if S^z is convex $\Rightarrow$ Γ_s is connected

By general position (isotopy of contact structures by Gray stability):
(use "transversality theory")

(1) $\forall z$: S^z_ξ contains only finitely many singularities & on finitely many z-levels there are at most one degenerated one

(2) $\nexists$ orbits from H_- to H_+ except for finitely many z-levels where there exist a unique such orbit.

Write $z_1, \dots, z_n$ for these exceptions.

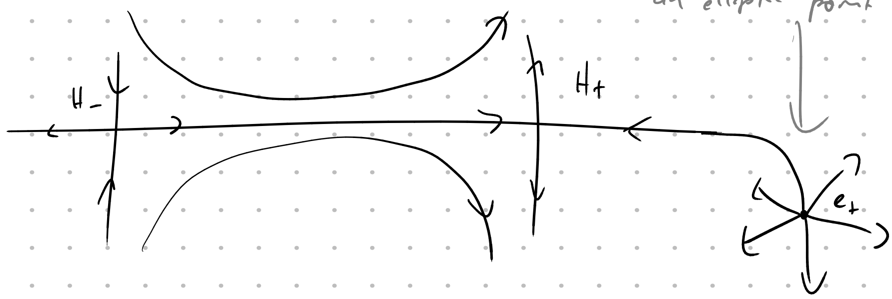
If $z \notin \{z_1, \dots, z_n\}$ $\Rightarrow$ S^z is convex & Γ_{S^z} is on equator

<u>Goal</u>: Remove (1) & (2)

(2)

This flow line has to come from an elliptic point

(tightness, uniqueness of flow line from $H^- \to H^+$)

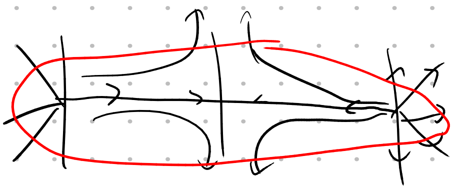


$\Rightarrow$ H_+ & e_+ are in elim. pos.

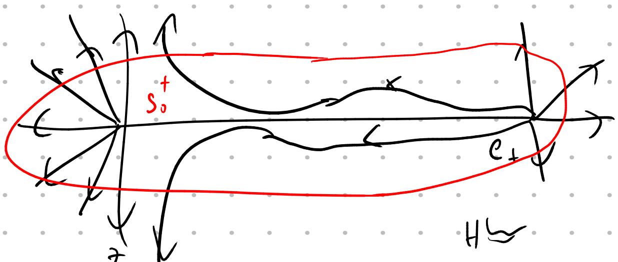
$\Rightarrow$ The proof of the elimination lemma can be done s.t. the nearby surfaces stay convex

$\Rightarrow$ remove all z_i with (2)

(1) S^{z_i} is actually convex



$\longrightarrow$

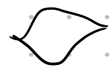


1. make the char. fol the same
2. they are the same in nbhds

$\Rightarrow$ $\forall z$: S^z is convex $\Rightarrow$ ξ_0 is isotopic to ξ_1. $\blacksquare$

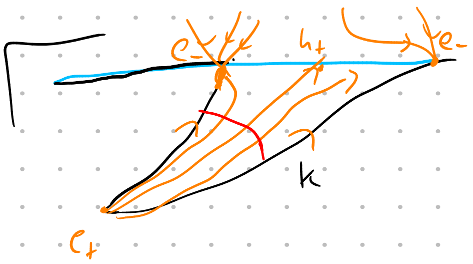
5.3 classification of Legendrian unknots

Thm 6 [Eliashberg–Fraser]:
If $K \subset (S^3, \xi_{st})$ is a Legendrian unknot $\Rightarrow K$ is isotopic to a stabilization of



Proof: Bennequin inequality $\Rightarrow$ $tb(K) \leq -1$

Step 1: If $tb(K) < -1 \Rightarrow K$ destabilizes.



Can't appear in a tight nbd

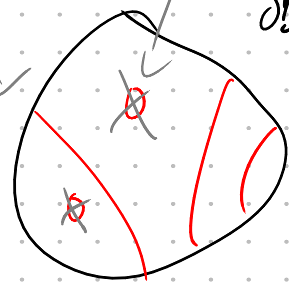
Bypass := a boundary parallel dividing curve $\hat{=}$ stabilization

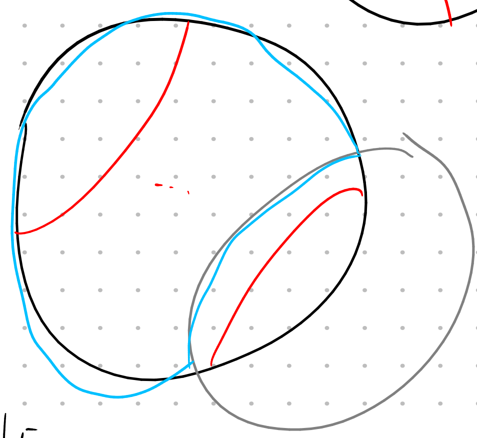
$\partial D = K$

Let D be a Seifert disk of K

$\xrightarrow{tb \leq 0}$ wlog D is convex

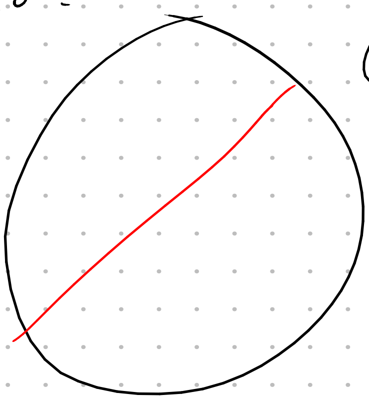
$\Rightarrow -1 > tb(K) = -\#(\Gamma_s)$





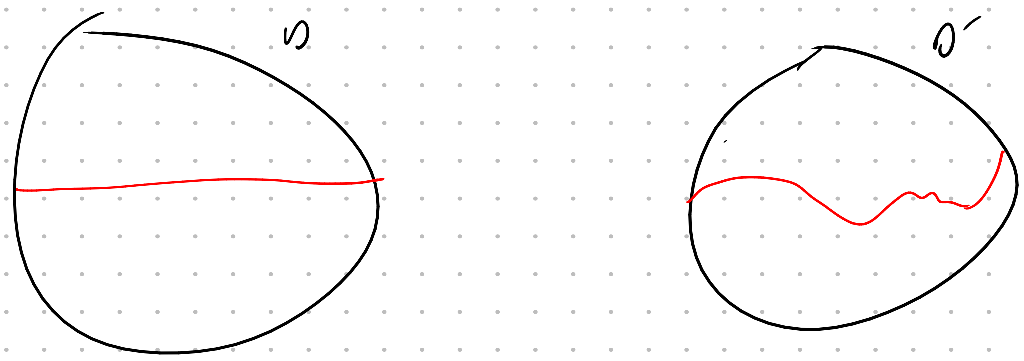
Destabilizes

$tb = -1$



Step 2 $\exists!$ Legendrian unknot with $tb = -1$

$\ulcorner$ Let K & K' be Leg. unknots with $tb = -1$ & D & D' be convex Seifert disks



$tb = -1$ & ξ is tight, so it has to look like the above

$\Rightarrow \exists$ contact isotopy $(\nu D, \xi_{st}) \longrightarrow (\nu D', \xi_{st})$

$\Rightarrow S^3 \setminus \nu D$ & $S^3 \setminus \nu D'$ are 3-balls with the same c. s. on ∂

i.e. $\exists$ contact isotopy $S^3 \setminus \nu D \longrightarrow S^3 \setminus \nu D'$

$\Rightarrow \exists$ contact isotopy $(S^3, \xi_{st}) \longrightarrow (S^3, \xi_{st})$

$K \longmapsto K'$ $\lrcorner$ $\boxtimes$

5.4 Classification of Tangential 2-Plane-Fields

Let M^3 be closed, oriented, connected.

Thm 7: $\left\{ \begin{array}{l} \text{Tangential} \\ \text{2-Plane} \\ \text{Fields } \xi \\ \text{on } M^3 \end{array} \right\} \Big/ \text{Homotopy} \longleftrightarrow \left\{ M \xrightarrow{C^0} S^2 \right\} \Big/ \text{Homotopy}$

proof: Stiefel $\Rightarrow TM \overset{\Phi}{\cong} M \times \mathbb{R}^3$

$\xi : M \longrightarrow S^2$

$p \longmapsto$ normal vector of ξ_p

Warning: This depends on Φ

$\{M \longrightarrow S^2\}/_{\text{Homotopy}}$ is classified by obstruction theory via two homology classes d_2, d_3

$d_2 \to$ Euler class, $d_3 \to$ Hopf-invariant

(see chapter 7) $\blacksquare$

5.5 Classification of overtwisted contact manifolds

$\Delta \subset M$ a 2-disk & $O_\Delta \in \Delta$ a point

$$OT(M, \Delta) := \left\{ \begin{array}{l} \text{contact structures} \\ \text{with } \Delta \text{ as std OT} \\ \text{disk with unique} \\ \text{singularity at } O_\Delta \end{array} \right\}$$

$$DIST(M, \Delta) := \left\{ \begin{array}{l} \text{Tangential 2-plane} \\ \text{fields pos. tangent} \\ \text{to } \Delta \text{ in } O_\Delta \end{array} \right\}$$

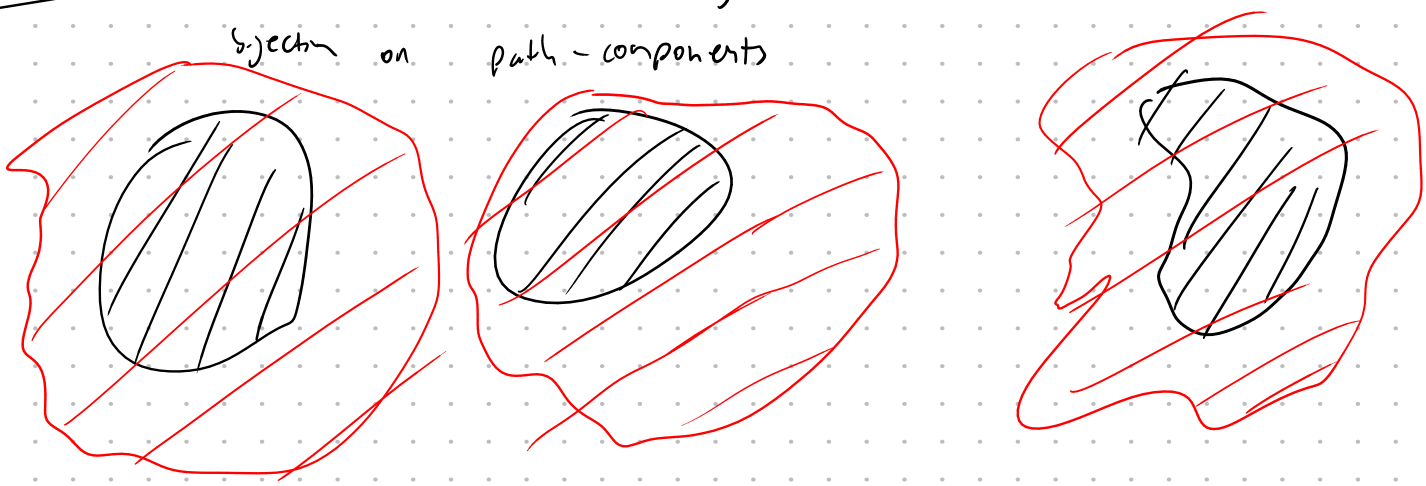
Thm 8: [Eliashberg]

The inclusion

$$i_\Delta : OT(M, \Delta) \longrightarrow DIST(M, \Delta)$$

is a homotopy equivalence.

Corollary 9: The inclusion $i: OT(M) \hookrightarrow DIST(M)$ induces a bijection on path-components



proof idea of Thm 8

Gromov (h-principle): If N^3 is open:

$$\{\text{contact structures on } M\} \hookrightarrow \{\text{tangent 2-plane fields on } M\}$$

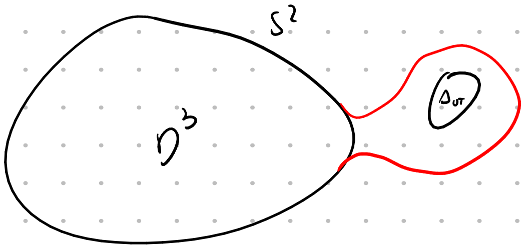
is a homotopy eq.

If M^3 is closed consider $M \setminus pt$ (open)

1. Apply Gromov
2. study extension problem:

Given S^2_q, extend to c.s. on D^3

In general, this is not possible, but if $\exists$ OT disk near S^2, then it is possible



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