Ecuaciones del péndulo invertido

(1) $F=(M+m)\ddot{x}+b\dot{x}-m\ell\ddot{\theta}\cos(\theta)+m\ell\dot{\theta}^2\operatorname{sen}(\theta)$

(2) $(I+m\ell^2)\ddot{\theta}-mg\ell\operatorname{sen}(\theta)=m\ell\ddot{x}\cos(\theta)$

definimos nuestro vector de estados y nuestra entrada como

$$\vec{x}=[x,\dot{x},\theta,\dot{\theta}]^T \quad \wedge \quad u=F$$

buscamos una ecuación de estados de la forma

$$\dot{\vec{x}}(t,\vec{x},u)=[\dot{x},\ddot{x},\dot{\theta},\ddot{\theta}]^T=\frac{d}{dt}[x,\dot{x},\theta,\dot{\theta}]^T$$

note que debemos obtener un sistema de cuatro ecuaciones dependientes de $\vec{x}$ y de u. Para ello, despejamos $\ddot{x}$ de (2)

$$\ddot{x}=\frac{(I+m\ell^2)\ddot{\theta}-mg\ell\operatorname{sen}(\theta)}{m\ell\cos(\theta)}=\frac{(I+m\ell^2)\ddot{\theta}}{m\ell\cos(\theta)}-g\tan(\theta)$$

El resultado anterior lo sustituimos en (1)

$$F=(M+m)\left[\frac{(I+m\ell^2)\ddot{\theta}}{m\ell\cos(\theta)}-g\tan(\theta)\right]+b\dot{x}-m\ell\ddot{\theta}\cos(\theta)+m\ell\dot{\theta}^2\operatorname{sen}(\theta)$$

$$\Longleftrightarrow F=\frac{(M+m)(I+m\ell^2)\ddot{\theta}}{m\ell\cos(\theta)}-(M+m)g\tan(\theta)+b\dot{x}-m\ell\ddot{\theta}\cos(\theta)+m\ell\dot{\theta}^2\operatorname{sen}(\theta)$$

$$\Longleftrightarrow \ddot{\theta}\left[\frac{(M+m)(I+m\ell^2)}{m\ell\cos(\theta)}-m\ell\cos(\theta)\right]=F+(M+m)g\tan(\theta)-b\dot{x}-m\ell\dot{\theta}^2\operatorname{sen}(\theta)$$

$x_1 = x \quad x_3 = \theta$
$x_2 = \dot{x} \quad x_4 = \dot{\theta}$

Note que

$$\frac{(M+m)(I+ml^2)}{ml\cos(\theta)} - ml\cos(\theta) = \frac{(M+m)(I+ml^2) - (ml)^2\cos^2(\theta)}{ml\cos(\theta)}$$

Por lo tanto,

$$\ddot{\theta}\left[\frac{(M+m)(I+ml^2) - (ml)^2\cos(\theta)}{ml\cos(\theta)}\right] = F + (M+m)g\tan(\theta) - b\dot{x} - ml\dot{\theta}^2\operatorname{sen}(\theta)$$

$$\iff \ddot{\theta} = \frac{[ml\cos(\theta)][F + (M+m)g\tan(\theta) - b\dot{x} - ml\dot{\theta}^2\operatorname{sen}(\theta)]}{(M+m)(I+ml^2) - (ml)^2\cos^2(\theta)}$$

$$\boxed{\therefore\ \ddot{\theta} = \frac{ml[F\cos(\theta) + (M+m)g\operatorname{sen}(\theta) - b\dot{x}\cos(\theta) - ml\dot{\theta}^2\operatorname{sen}(\theta)\cos(\theta)]}{(M+m)(I+ml^2) - (ml)^2\cos^2(\theta)}}$$

sustituimos este resutado en la siguiente expresión para obtener

$$\ddot{x} = \frac{(I+ml^2)\ddot{\theta}}{ml\cos(\theta)} - g\tan(\theta)$$

$$\iff \ddot{x} = \frac{(I+ml^2)\,\cancel{[ml\cos(\theta)]}[F + (M+m)g\tan(\theta) - b\dot{x} - ml\dot{\theta}^2\operatorname{sen}(\theta)]}{\cancel{[ml\cos(\theta)]}\,[(M+m)(I+ml^2) - (ml)^2\cos^2(\theta)]} - g\tan(\theta)$$

Por lo tanto,

$$\boxed{\ddot{x} = \frac{(I+ml^2)(F + (M+m)g\tan(\theta) - b\dot{x} - ml\dot{\theta}^2\operatorname{sen}(\theta))}{(M+m)(I+ml^2) - (ml)^2\cos^2(\theta)} - g\tan(\theta)}$$

Punto de Operación

El punto de operación está dado por

$$\dot{\vec{x}} = [\dot{x}, \ddot{x}, \dot{\theta}, \ddot{\theta}] = \vec{0}$$

Inicialmente, tomamos el punto de operación

$$[\theta_0, \dot{\theta}_0, x_0, \dot{x}_0]^T = \vec{0}$$

y la entrada

$$u_0(t) = F_0(t)$$

Linealización del sistema

Buscamos expresar el sistema de la forma

$$\delta\vec{\dot{x}} = A'\delta\vec{x} + B'\delta u$$

Para ello, tomamos que

$$f_1 = \dot{x}, \; f_2 = \ddot{x}, \; f_3 = \dot{\theta}, \; f_4 = \ddot{\theta}$$

Con esto, calculamos las respectivas derivadas

- Para f_1

$$\frac{\partial f_1}{\partial x} = \frac{\partial \dot{x}}{\partial x} = 0$$

$$\frac{\partial f_1}{\partial \dot{x}} = \frac{\partial \dot{x}}{\partial \dot{x}} = 1$$

$$\frac{\partial f_1}{\partial \theta} = \frac{\partial \dot{x}}{\partial \theta} = 0$$

$$\frac{\partial f_1}{\partial \dot{\theta}} = \frac{\partial \dot{x}}{\partial \dot{\theta}} = 0$$

Para f_2:

$$\frac{\partial f_2}{\partial x} = \frac{\partial \ddot{x}}{\partial x} = 0$$

$$\frac{\partial f_2}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{(I+ml^2)(F+(M+m)g\tan(\theta)-b\dot{x}-ml\dot{\theta}^2\operatorname{sen}(\theta)}{(M+m)(I+ml^2)-(ml)^2\cos^2(\theta)} - \cancelto{0}{g\tan(\theta)}\right)$$

$$= \frac{(I+ml^2)}{(M+m)(I+ml^2)-(ml)^2\cos^2(\theta)}\frac{\partial}{\partial \dot{x}}\left(\cancelto{0}{F}+\cancelto{0}{(M+m)g\tan(\theta)}-b\dot{x}-\cancelto{0}{ml\operatorname{sen}(\theta)}\right)$$

$$= \frac{-(I+ml^2)b}{(M+m)(I+ml^2)-(ml)^2\cos^2(\theta)}$$

$$\frac{\partial f_2}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{(I+ml^2)(F+(M+m)g\tan(\theta)-b\dot{x}-ml\dot{\theta}^2\operatorname{sen}(\theta)}{(M+m)(I+ml^2)-(ml)^2\cos^2(\theta)} - g\tan(\theta)\right)$$

$$= (I+ml^2)\frac{\partial}{\partial \theta}\left(\frac{(F+(M+m)g\tan(\theta)-b\dot{x}-ml\dot{\theta}^2\operatorname{sen}(\theta)}{(M+m)(I+ml^2)-(ml)^2\cos^2(\theta)} - g\tan(\theta)\right)$$

Recordando la regla del cociente

$$\left(\frac{u}{v}\right)' = \frac{u'v-v'u}{v^2}$$

consideramos

$$u = F+(M+m)g\tan(\theta)-b\dot{x}-ml\dot{\theta}^2\operatorname{sen}(\theta) \wedge v=(M+m)(I+ml^2)-(ml)^2\cos^2(\theta)$$

entonces

$$\frac{\partial u}{\partial \theta} = 0+(M+m)g\sec^2(\theta)-0-ml\dot{\theta}^2\cos(\theta)$$

$$\frac{\partial v}{\partial \theta} = (m\ell)^2 \,\text{sen}(2\theta)$$

$$\frac{\partial(-g\tan(\theta))}{\partial \theta} = -g\sec^2(\theta)$$

Por lo tanto,

$$\frac{\partial f_2}{\partial \theta} = (I+m\ell^2)\frac{(u'v - v'u)}{v^2} - g\sec^2(\theta)$$

de forma extendida

$$\frac{\partial f_2}{\partial \theta} = (I+m\ell^2)\left[\frac{\overset{u'}{\left[(M+m)g\sec^2(\theta) - m\ell\dot{\theta}^2\cos(\theta)\right]}\overset{v}{\left[(M+m)(I+m\ell^2)-(m\ell)^2\cos(\theta)\right]}}{\left[(M+m)(I+m\ell^2)-(m\ell)^2\cos(\theta)\right]^2}\right.$$

$$\left. - \frac{\overset{v'}{\left[(m\ell)^2\,\text{sen}(2\theta)\right]}\overset{u}{\left[F+(M+m)g\tan(\theta)-b\dot{x}-m\ell\dot{\theta}^2\,\text{sen}(\theta)\right]}}{\left[(M+m)(I+m\ell^2)-(m\ell)^2\cos(\theta)\right]^2}\right] - g\sec^2(\theta)$$

$$\frac{\partial f_2}{\partial \dot{\theta}} = \frac{(I+m\ell^2)}{(M+m)(I+m\ell^2)-(m\ell)^2\cos^2(\theta)}\frac{\partial}{\partial \dot{\theta}}\left(\cancelto{0}{F} + (M+m)\cancelto{0}{g}\tan(\theta) - \cancelto{0}{b}\dot{x} - m\ell\dot{\theta}^2\,\text{sen}(\theta)\right)$$

$$= \frac{-2(I+m\ell^2)(m\ell\dot{\theta}\,\text{sen}(\theta))}{(M+m)(I+m\ell^2)-(m\ell)^2\cos^2(\theta)}$$

Para f_3:

$$\frac{\partial f_3}{\partial x} = \frac{\partial \dot{\theta}}{\partial x} = 0 \qquad \frac{\partial f_3}{\partial \theta} = \frac{\partial \dot{\theta}}{\partial \theta} = 0$$

$$\frac{\partial f_3}{\partial \dot{x}} = \frac{\partial \dot{\theta}}{\partial \dot{x}} = 0 \qquad \frac{\partial f_3}{\partial \dot{\theta}} = \frac{\partial \dot{\theta}}{\partial \dot{\theta}} = 1$$

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Para f_4:

$$\frac{\partial f_4}{\partial x} = \frac{\partial \ddot{\theta}}{\partial x} = 0$$

$$\frac{\partial f_4}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{m\ell\,[F\cos(\theta) + (M+m)g\,\mathrm{sen}(\theta) - b\dot{x}\cos(\theta) - m\ell\dot{\theta}^2\mathrm{sen}(\theta)\cos(\theta)]}{(M+m)(I+m\ell^2) - (m\ell)^2\cos^2(\theta)} \right)$$

$$= \frac{-bm\ell\cos(\theta)}{(M+m)(I+m\ell^2) - (m\ell)^2\cos^2(\theta)}$$

$$\frac{\partial f_4}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{m\ell\,[F\cos(\theta) + (M+m)g\,\mathrm{sen}(\theta) - b\dot{x}\cos(\theta) - m\ell\dot{\theta}^2\mathrm{sen}(\theta)\cos(\theta)]}{(M+m)(I+m\ell^2) - (m\ell)^2\cos^2(\theta)} \right)$$

considere

$$u = F\cos(\theta) + (M+m)g\,\mathrm{sen}(\theta) - b\dot{x}\cos(\theta) - m\ell\dot{\theta}^2\mathrm{sen}(\theta)\cos(\theta)$$

$$\Rightarrow \frac{du}{d\theta} = -F\mathrm{sen}(\theta) + (M+m)g\cos(\theta) + b\dot{x}\,\mathrm{sen}(\theta) - m\ell\dot{\theta}^2\cos(2\theta)$$

$$v = (M+m)(I+m\ell^2) - (m\ell)^2\cos^2(\theta)$$

$$\Rightarrow \frac{dv}{d\theta} = (m\ell)^2\mathrm{sen}(2\theta)$$

Por lo tanto

$$\frac{\partial f_4}{\partial \theta} = m\ell\,\frac{[u'v - v'u]}{v^2}$$

o de forma extendida

$$\frac{\partial f_4}{\partial \theta} = mL\left[\frac{[-F\,sen(\theta)+(M+m)g\cos(\theta)+b\dot{x}\,sen(\theta)-mL\dot{\theta}^2\cos(2\theta)]\,[(M+m)(I+mL^2)-(mL)^2\cos^2(\theta)]}{[(M+m)(I+mL^2)-(mL)^2\cos^2(\theta)]^2}\right.$$

$$\left.-\frac{[(mL)^2 sen(2\theta)]\,[F\cos(\theta)+(M+m)g\,sen(\theta)-b\dot{x}\cos(\theta)-mL\dot{\theta}^2 sen(\theta)\cos(\theta)]}{[(M+m)(I+mL^2)-(mL)^2\cos^2(\theta)]^2}\right]$$

$$\frac{\partial f_4}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{mL[\overset{\to 0}{F\cos(\theta)}+\overset{\to 0}{(M+m)g\,sen(\theta)}-\overset{\to 0}{b\dot{x}\cos(\theta)}-mL\dot{\theta}^2 sen(\theta)\cos(\theta)]}{(M+m)(I+mL^2)-(mL)^2\cos^2(\theta)}\right)$$

$$= \frac{-2(mL)(mL\dot{\theta}\,sen(\theta)\cos(\theta))}{(M+m)(I+mL^2)-(mL)^2\cos^2(\theta)}$$

$$= \frac{-2(mL)^2\dot{\theta}\,sen(\theta)\cos(\theta)}{(M+m)(I+mL^2)-(mL)^2\cos^2(\theta)}$$