The standard multiplicative coalescent revisited

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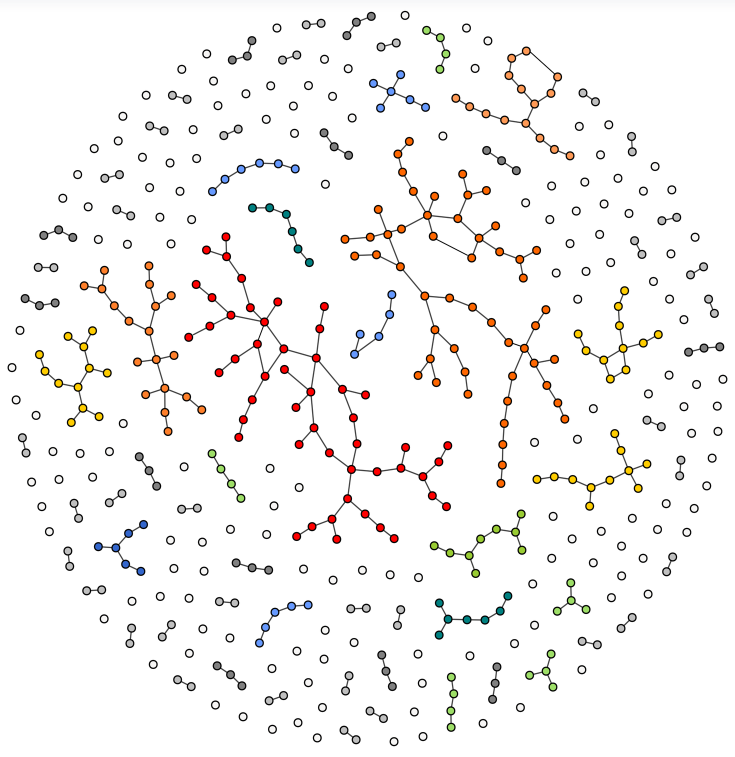
Main result

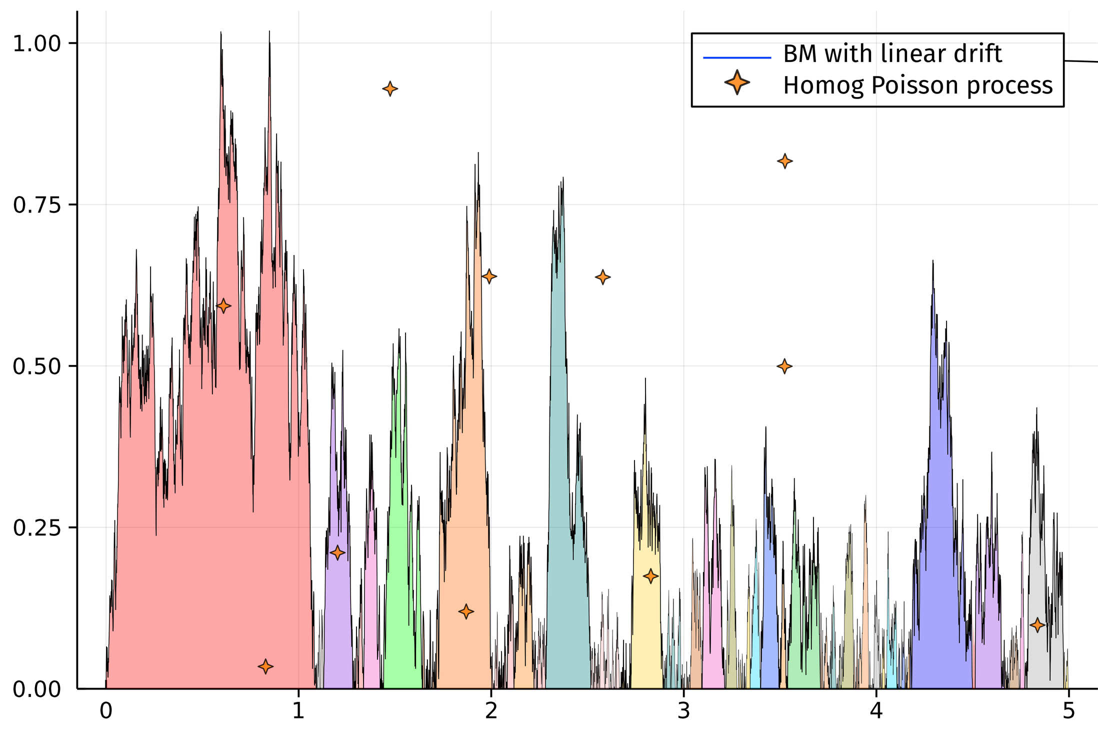
Erdös-Rényi random graph :

- n vertices
- each edge exists with probability

$$\frac{1}{n} + \frac{t}{n^{4/3}}$$



convergence in law $n \to \infty$



Brownian motion $(W(s), s \geq 0)$

BM with linear drift $W^t(s) := W(s) - \frac{1}{2}s^2 + t \cdot s$

Reflected BM with linear drift $B^t(s) := W^t(s) - \inf_{0 \leq u \leq s} W^t(u)$

($n^{-2/3}$ size of the connected components, number of surplus edges)

(length of the excursions, number of marks below the curve)

Erdös-Rényi (1960), Bollobas (1985), Aldous (1997)

size of the largest component

$\Theta(\ln n)$ $\Theta(n)$

0 $\frac{1}{n}$

within this time window the k-th ($k \in \mathbb{N}$) connected component is of order $\Theta(n^{2/3})$ w.h.p.

$t_n = \frac{1}{n} + \frac{t}{n^{4/3}}$ with $t \in \mathbb{R}$

a simulation

$\left(\mathbb{MG}^{(n)}(t), t \geq 0\right)$ multi-graph-valued Markov chain

- each vertex has size $n^{-2/3}$
- $\#\,(\,i \longrightarrow j\,) = N_{\{i,j\}}(t)$ — Poisson process with rate $n^{-4/3}/2$

directed edge

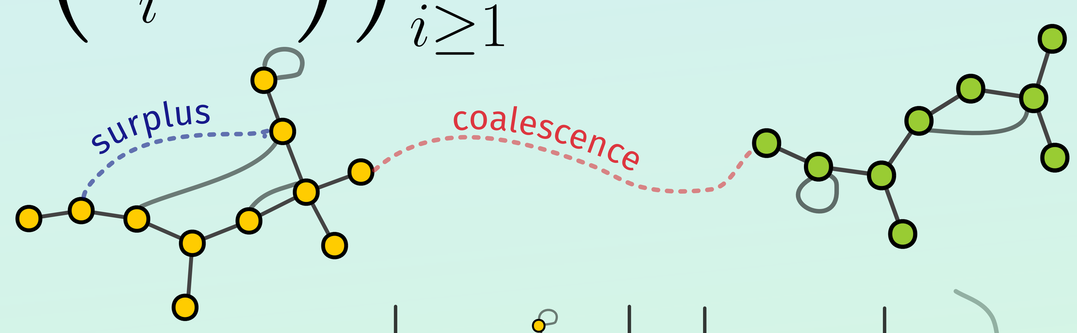
$$\mathbb{MG}^{(n)}(t) \xrightarrow[\text{removing self-loops}]{\text{unifying multi-edges}} \text{Erdös-Rényi}\left(n, 1 - \mathrm{e}^{-n^{-4/3}\,t}\right)$$

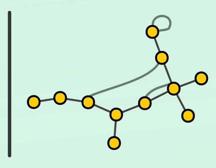
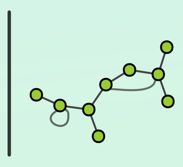
critical time : $n^{1/3} + t$

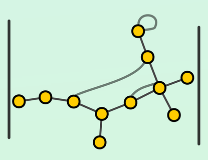
$\mathcal{C}_i^{(n)}(t)$ i-th largest connected component of $\mathbb{MG}^{(n)}(t)$

$|\mathcal{C}_i^{(n)}(t)|$ size and $\mathrm{SP}\left(\mathcal{C}_i^{(n)}(t)\right)$ number of surplus edges

$\left(|\mathcal{C}_i^{(n)}(t)|, \mathrm{SP}\left(\mathcal{C}_i^{(n)}(t)\right)\right)_{i \geq 1}$ Markov process with dynamic



coalescence: with rate $\left|\right.$  $\left.\right| \cdot \left|\right.$  $\left.\right|$

surplus creation: with rate $\left|\right.$  $\left.\right|^2 / 2$

augmented multiplicative coalescent (AMC)

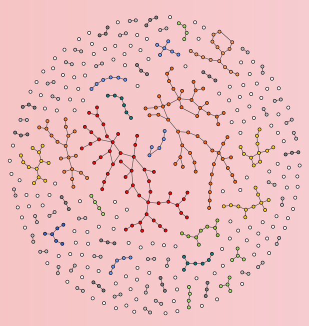
Space $\left\{ (x_i, n_i)_{i \geq 1} \in l^2_{\searrow} \times \mathbb{N}^{\infty} : \sum_{i=1}^{\infty} x_i\, n_i < \infty \text{ and } n_i = 0 \text{ whenever } x_i = 0,\ i \geq 1 \right\}$

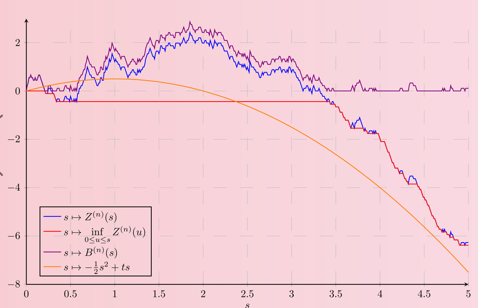
size of the i-th connected component; number of surplus edges in the i-th connected component

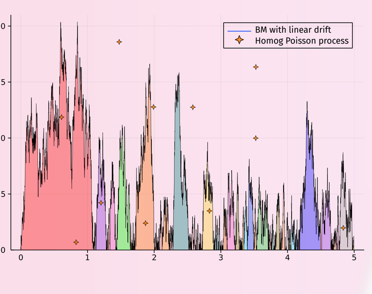
Metric $\mathrm{d}_{\mathbb{U}}((\boldsymbol{x}, \boldsymbol{n}), (\boldsymbol{x}', \boldsymbol{n}')) = \left(\sum_{i=1}^{\infty} (x_i - x_i')^2\right)^{1/2} + \sum_{i=1}^{\infty} |x_i \cdot n_i - x_i' \cdot n_i'|$

(making a process with the AMC dynamic a Feller process)

Proof sketch







Main difficulty: control of the tail $\lim_{\delta \to 0} \limsup_n \mathbb{E}\left[\sum_{i \geq 1} X_i^{(n)} \cdot N_i^{(n)} \cdot \mathbf{1}_{\{X_i^{(n)} < \delta\}}\right] = 0$

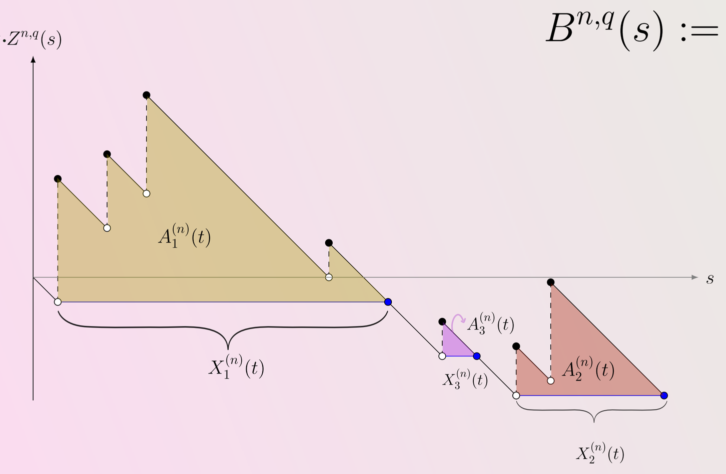
Methods

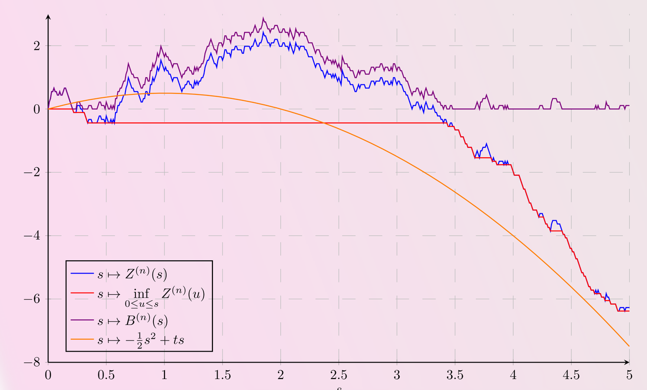
- bringing the tail to the beginning: $\mathbb{E}\left[\sum_{i \geq 1} X_i^{(n)} \cdot N_i^{(n)} \cdot \mathbf{1}_{\{X_i^{(n)} < \delta\}}\right] = n^{1/3} \cdot \mathbb{E}\left[\mathrm{SP}(\mathcal{C}(V_n))\mathbf{1}_{\{|\mathcal{C}(V_n)| < \delta\}}\right]$ — randomly chosen vertex
- encoding: $\mathbb{E}\left[\mathrm{SP}(\mathcal{C}(V_n))\mathbf{1}_{\{|\mathcal{C}(V_n)| < \delta\}}\right] = \mathbb{E}\left[q_n(t) \int_{\text{First excursion}} B^{n, q_n(t)}(s)\,\mathrm{d}s \cdot \mathbf{1}_{\{|\text{ First excursion }| < \delta\}}\right]$
- controling the expected area below the curve: $\leq \delta\, n^{-1/3}$
- counting the multi-edges and self-loop to get the result for the Erdös-Rényi model

Simultaneous breadth-first walk

$$Z^{n,q}(s) = \sum_{i=1}^{n} \frac{1}{n^{2/3}} \mathbf{1}_{(\xi_i / q \leq\ s)} - s, \text{ where } \xi_i \sim \exp\left(\text{rate } = \frac{1}{n^{2/3}}\right)$$

$$B^{n,q}(s) := Z^{n,q}(s) - \inf_{u \leq s} Z^{n,q}(u)$$





Encoding the AMC

$X_i^{(n)}(t)$ size of the i-th excursion

$N_i^{(n)}(t) = \underbrace{\mathrm{Poisson}(A_i^{(n)}(t))}_{\text{area below the curve under the i-th excursion of } q \cdot B^{n,q}(s)}$

Theorem (C. and Limic 2023+)

$\left(X_i^{(n)}(t), N_i^{(n)}(t)\right)_{i \geq 1}$ is equal in law to $\left(|\mathcal{C}_i^{(n)}(t)|, \mathrm{SP}\left(\mathcal{C}_i^{(n)}(t)\right)\right)_{i \geq 1}$

$$q_n(t) = n^{1/3} + t$$

Theorem (Limic 2019)

$q_n(t) Z^{n, q_n(t)}(s)$ converges in distribution towards $W^t(s)$

$\mathcal{X}_i(t)$ size of the i-th excursion of $W^t(s)$ $\quad \mathcal{N}_i(t) = \underbrace{\mathrm{Poisson}\left(\mathcal{A}_i(t)\right)}_{\text{area below the curve under the i-th excursion of } B^t(s)}$

We need to prove: $\mathrm{d}_{\mathbb{U}}\left((X_i^{(n)}, N_i^{(n)})_{i \geq 1}, (\mathcal{X}_i(t), \mathcal{N}_i(t))_{i \geq 1}\right) \xrightarrow[n \to \infty]{\mathbb{P}} 0$

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Read this poster online

This poster is based on the preprints:
Corujo and Limic *The standard augmented multiplicative coalescent revisited* (2023)

arXiv: 2304.07545

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