# The standard multiplicative coalescent revisited


**Josué Corujo and Vlada Limic**

josue.corujorodriguez@math.unistra.fr

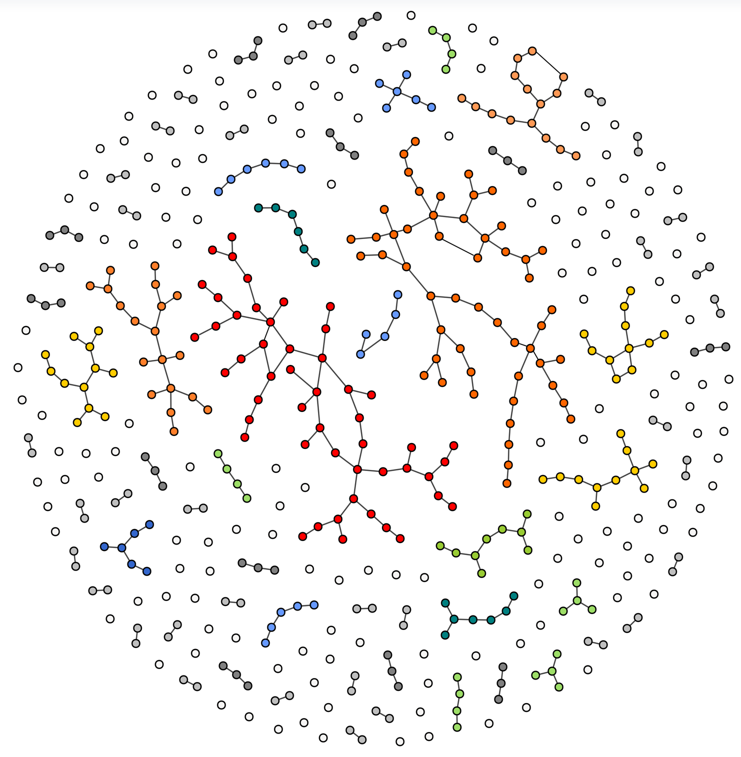
Institut de Recherche Mathématique Avancée, Université de Strasbourg and CNRS


## Main result

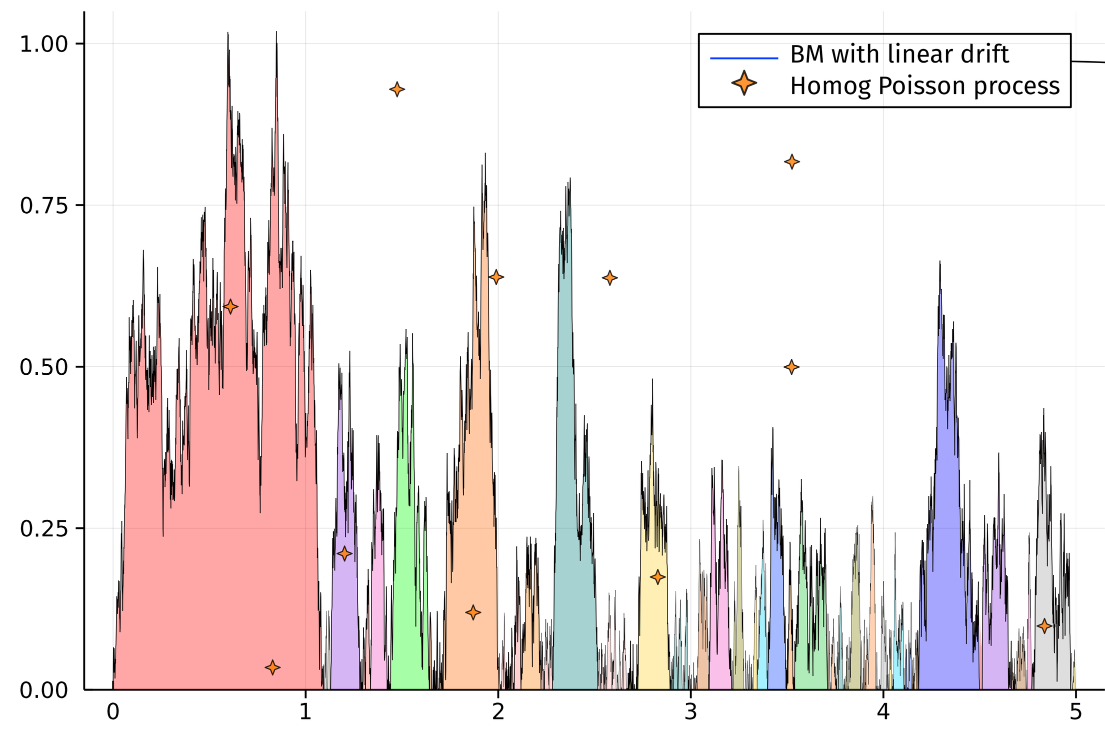
**Erdös-Rényi random graph :**

- $n$ vertices
- each edge exists with probability

$$\frac{1}{n} + \frac{t}{n^{4/3}}$$



convergence in law $n \to \infty$




Brownian motion $(W(s), s \geq 0)$

BM with linear drift $W^t(s) := W(s) - \frac{1}{2}s^2 + t \cdot s$

Reflected BM with linear drift $B^t(s) := W^t(s) - \inf_{0 \leq u \leq s} W^t(u)$

( $n^{-2/3}$ size of the connected components, number of surplus edges )

( length of the excursions, number of marks below the curve )

### Erdös-Rényi (1960), Bollobas (1985), Aldous (1997)

size of the largest component

$\Theta(\ln n)$ $\Theta(n)$

$0$ $\frac{1}{n}$

within this time window the k-th ($k \in \mathbb{N}$) connected component is of order $\Theta(n^{2/3})$ w.h.p.

$t_n = \frac{1}{n} + \frac{t}{n^{4/3}}$ with $t \in \mathbb{R}$

a simulation

$\left(\mathbb{MG}^{(n)}(t), t \geq 0\right)$ multi-graph-valued Markov chain

- each vertex has size $n^{-2/3}$
- $\#\left(i \longrightarrow j\right) = N_{\{i,j\}}(t)$ Poisson process with rate $n^{-4/3}/2$

directed edge

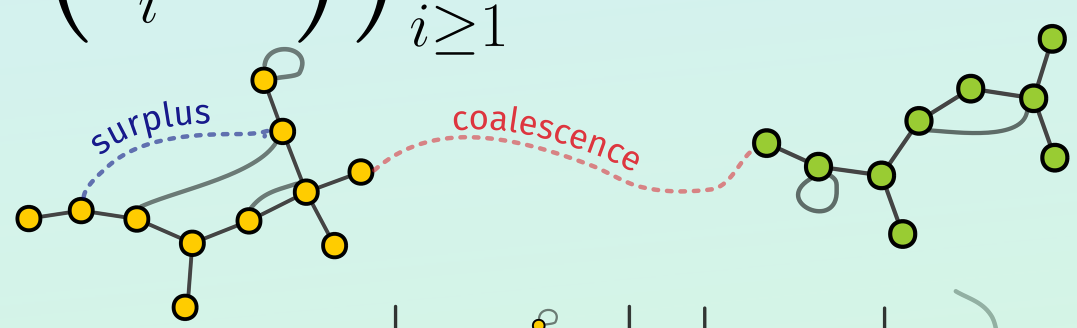
$\mathbb{MG}^{(n)}(t) \xrightarrow[\text{removing self-loops}]{\text{unifying multi-edges}}$ Erdös-Rényi$\left(n, 1 - e^{-n^{4/3}\, t}\right)$

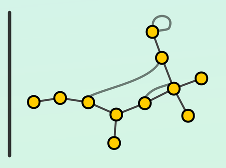
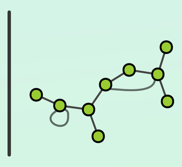
critical time : $n^{1/3} + t$

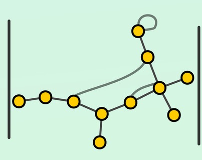
$C_i^{(n)}(t)$ i-th largest connected component of $\mathbb{MG}^{(n)}(t)$

$|C_i^{(n)}(t)|$ size and $\mathrm{SP}\left(C_i^{(n)}(t)\right)$ number of surplus edges

$\left(|C_i^{(n)}(t)|, \mathrm{SP}\left(C_i^{(n)}(t)\right)\right)_{i \geq 1}$ Markov process with dynamic




coalescence: with rate $\left|\right.$  $\left|\right. \cdot \left|\right.$  $\left|\right.$

surplus creation: with rate $\left|\right.$  $\left|\right.^2 / 2$

augmented multiplicative coalescent (AMC)

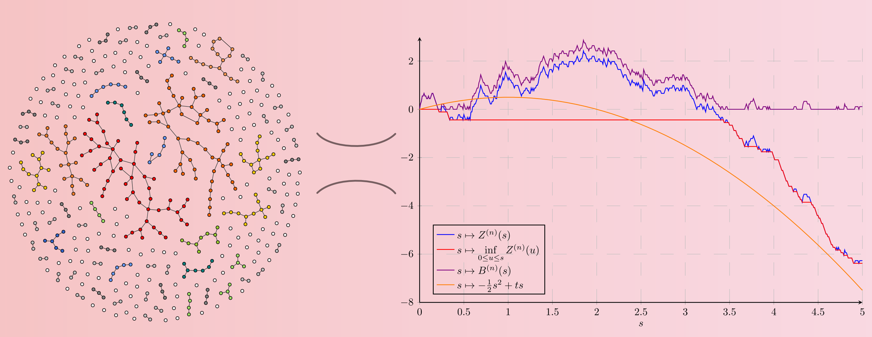
Space $\left\{(x_i, n_i)_{i \geq 1} \in l^2_{\searrow} \times \mathbb{N}^\infty : \sum_{i=1}^\infty x_i n_i < \infty \text{ and } n_i = 0 \text{ whenever } x_i = 0,\ i \geq 1\right\}$

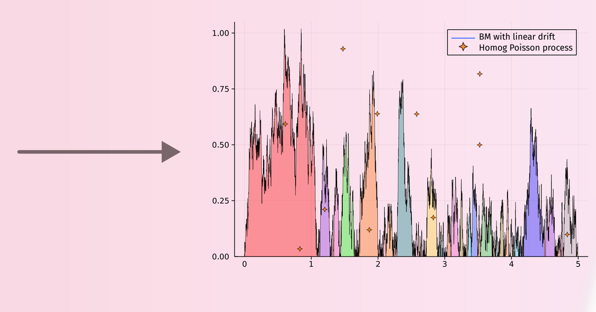
size of the i-th connected component; number of surplus edges in the i-th connected component

Metric $\mathbf{d}_{\mathbb{U}}((\boldsymbol{x}, \boldsymbol{n}), (\boldsymbol{x}', \boldsymbol{n}')) = \left(\sum_{i=1}^\infty (x_i - x_i')^2\right)^{1/2} + \sum_{i=1}^\infty |x_i \cdot n_i - x_i' \cdot n_i'|$

(making a process with the AMC dynamic a Feller process)

### Proof sketch





Main difficulty: control of the tail $\lim_{\delta \to 0} \limsup_n \mathbb{E}\left[\sum_{i \geq 1} X_i^{(n)} \cdot N_i^{(n)} \cdot \mathbf{1}_{\{X_i^{(n)} < \delta\}}\right] = 0$

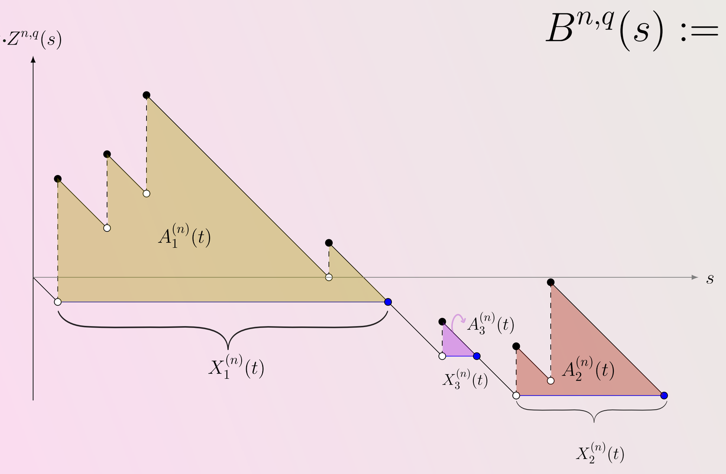
Methods

- bringing the tail to the beginning: $\mathbb{E}\left[\sum_{i \geq 1} X_i^{(n)} \cdot N_i^{(n)} \cdot \mathbf{1}_{\{X_i^{(n)} < \delta\}}\right] = n^{1/3} \cdot \mathbb{E}\left[\mathrm{SP}(\mathcal{C}(V_n)) \mathbf{1}_{\{|\mathcal{C}(V_n)| < \delta\}}\right]$ (randomly chosen vertex)
- encoding: $\mathbb{E}\left[\mathrm{SP}(\mathcal{C}(V_n)) \mathbf{1}_{\{|\mathcal{C}(V_n)| < \delta\}}\right] = \mathbb{E}\left[q_n(t) \int_{\text{First excursion}} B^{n, q_n(t)}(s)\, \mathrm{d}s \cdot \mathbf{1}_{\{|\text{First excursion}| < \delta\}}\right]$
- controling the expected area below the curve: $\leq \delta\, n^{-1/3}$
- counting the multi-edges and self-loop to get the result for the Erdös-Rényi model

### Simultaneous breadth-first walk

$$Z^{n,q}(s) = \sum_{i=1}^n \frac{1}{n^{2/3}} \mathbf{1}_{(\xi_i / q \leq s)} - s, \text{ where } \xi_i \sim \exp\left(\text{rate } = \frac{1}{n^{2/3}}\right)$$

$$B^{n,q}(s) := Z^{n,q}(s) - \inf_{u \leq s} Z^{n,q}(u)$$




### Encoding the AMC

$X_i^{(n)}(t)$ size of the i-th excursion

$N_i^{(n)}(t) = \mathrm{Poisson}(A_i^{(n)}(t))$ — area below the curve under the i-th excursion of $q \cdot B^{n,q}(s)$

**Theorem (C. and Limic 2023+)**
$\left(X_i^{(n)}(t), N_i^{(n)}(t)\right)_{i \geq 1}$ is equal in law to $\left(|C_i^{(n)}(t)|, \mathrm{SP}\left(C_i^{(n)}(t)\right)\right)_{i \geq 1}$

$$q_n(t) = n^{1/3} + t$$

**Theorem (Limic 2019)**
$q_n(t) Z^{n, q_n(t)}(s)$ converges in distribution towards $W^t(s)$

$\mathcal{X}_i(t)$ size of the i-th excursion of $W^t(s)$

$\mathcal{N}_i(t) = \mathrm{Poisson}\left(\mathcal{A}_i(t)\right)$ — area below the curve under the i-th excursion of $B^t(s)$

We need to prove: $\mathrm{d}_{\mathbb{U}}\left((X_i^{(n)}, N_i^{(n)})_{i \geq 1}, (\mathcal{X}_i(t), \mathcal{N}_i(t))_{i \geq 1}\right) \xrightarrow[n \to \infty]{\mathbb{P}} 0$


**Bibliography:**

- Aldous, *Brownian excursions, critical random graphs and the multiplicative coalescent* (1997)
- Bhamidi, Budhiraja and Wang, *The augmented multiplicative coalescent, bounded size rules and critical dynamics of random graphs* (2014)
- Broutin and Marckert, *A new encoding of coalescent processes: applications to the additive and multiplicative cases* (2016)
- Corujo and Limic, *A dynamical approach to spanning and surplus edges of random graphs*, arXiv: 2305.04716
- Limic, *The eternal multiplicative coalescent encoding via excursions of Lévy-type processes* (2019)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This poster is based on the preprints:
Corujo and Limic *The standard augmented multiplicative coalescent revisited* (2023)

arXiv: 2304.07545


Acknowledgments:
this research was funded by the ITI IRMIA++ and the participation in the INFORMS APS Conference 2023 was funded by the ITI IRMIA++ and the APS