

The standard multiplicative coalescent revisited

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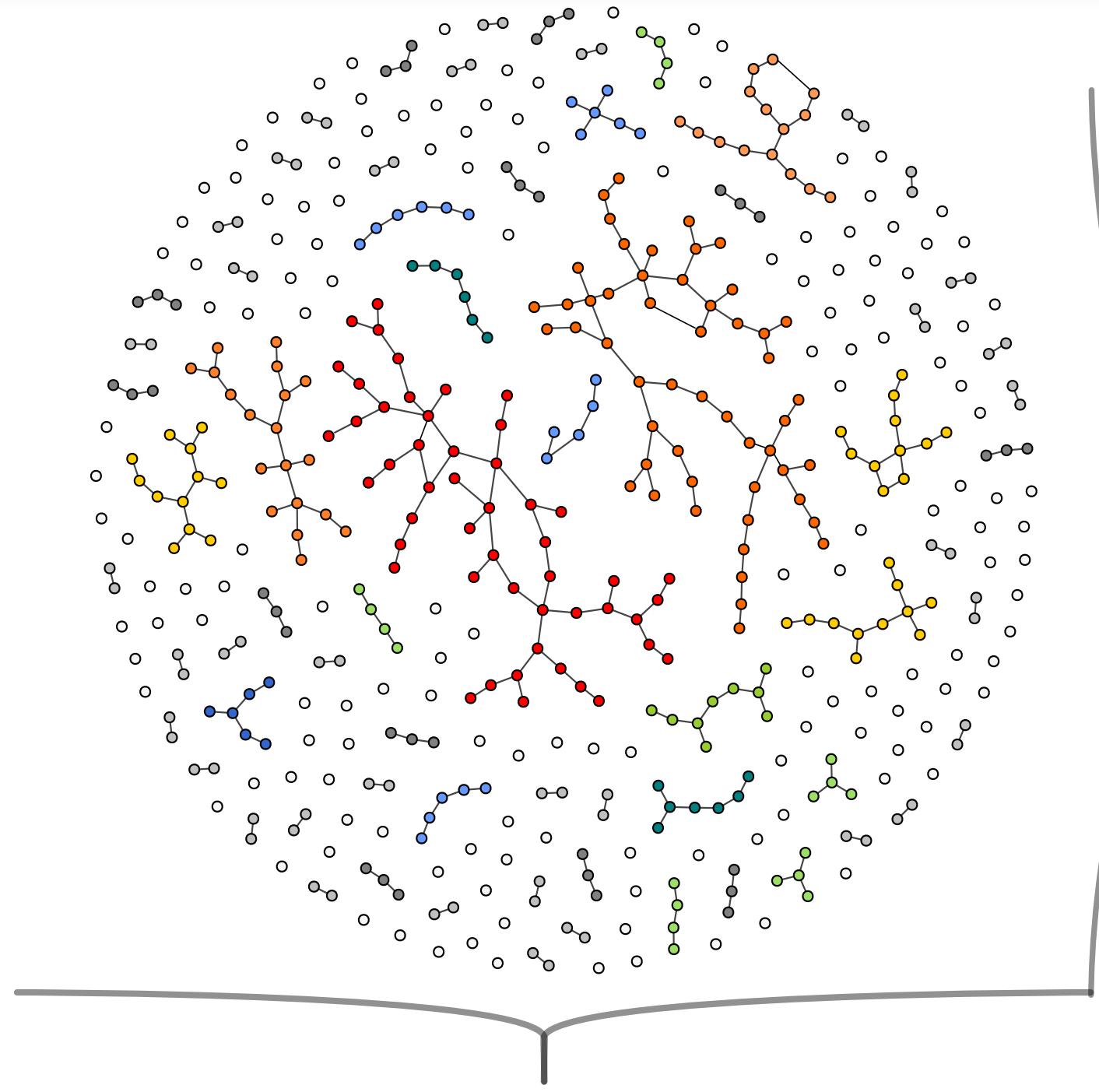


Main result

Erdős-Rényi
random graph :

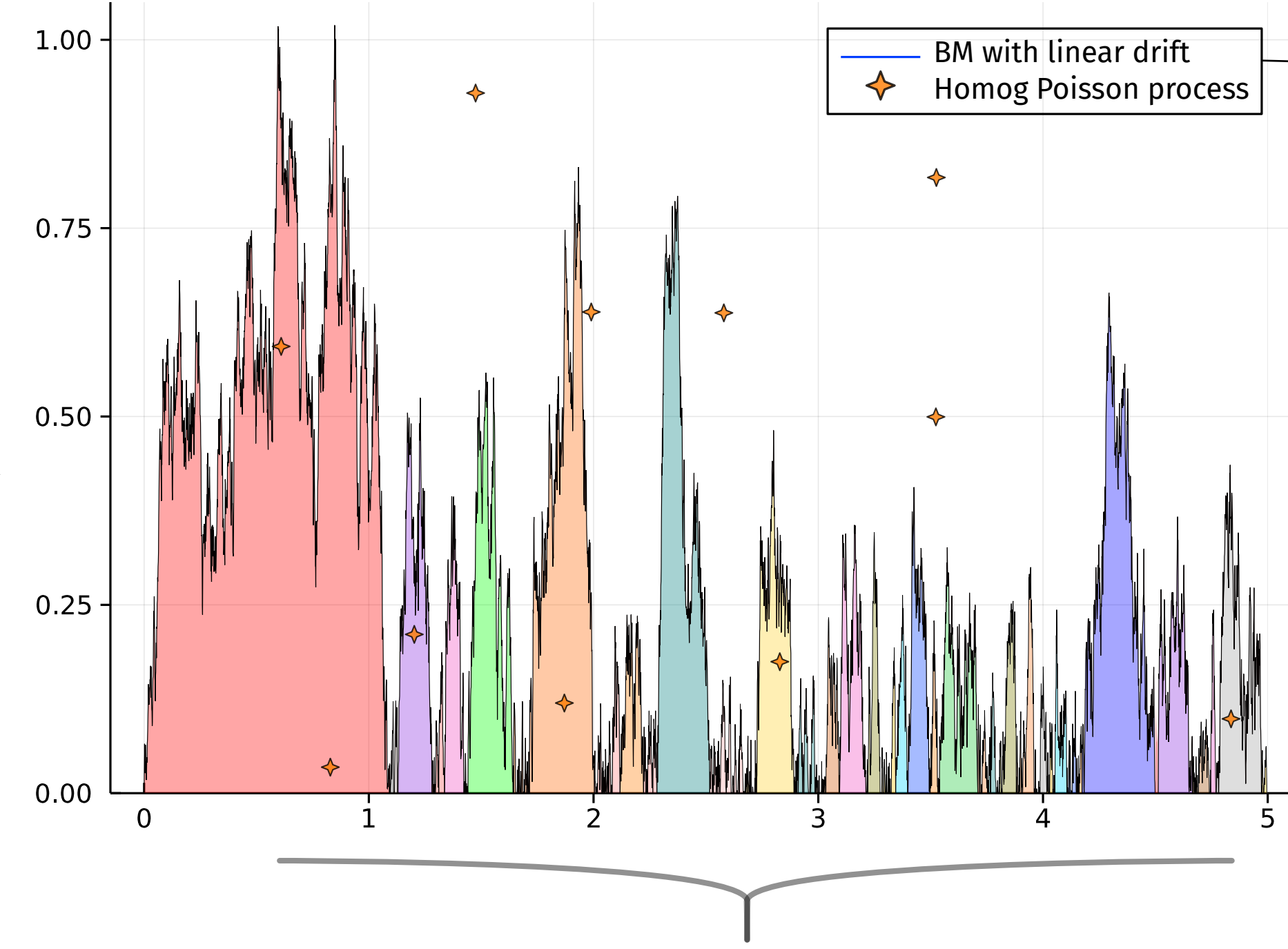
- n vertices
- each edge exists with probability

$$\frac{1}{n} + \frac{t}{n^{4/3}}$$



($n^{-2/3}$ size of the connected components, number of surplus edges)

convergence
in law



(length of the excursions, number of marks below the curve)

Brownian motion

$$(W(s), s \geq 0)$$

BM with linear drift

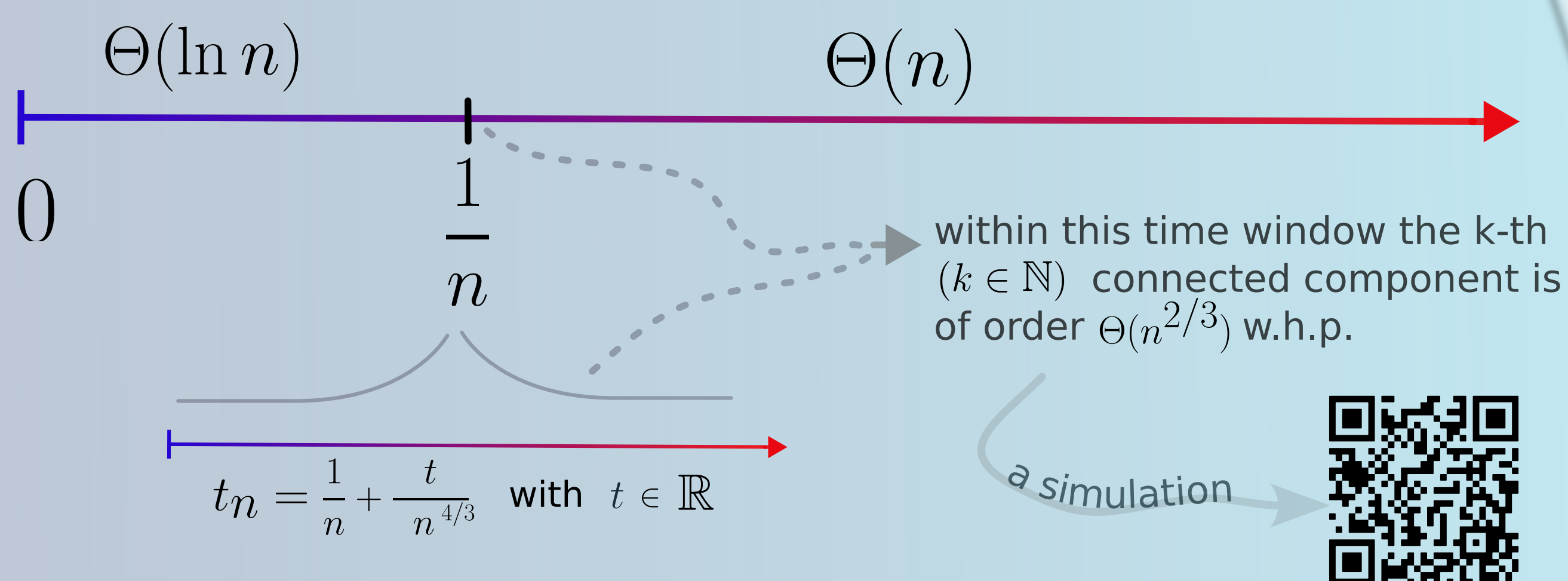
$$W^t(s) := W(s) - \frac{1}{2}s^2 + t \cdot s$$

Reflected BM with linear drift

$$B^t(s) := W^t(s) - \inf_{0 \leq u \leq s} W^t(u)$$

Erdős-Rényi (1960), Bollobas (1985), Aldous (1997)

size of the largest component



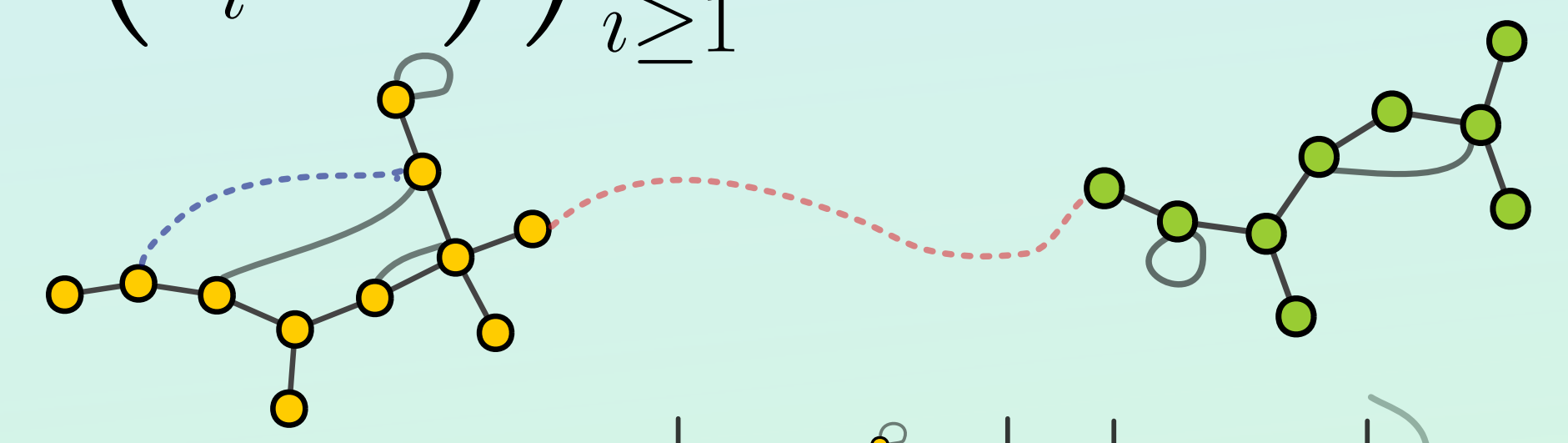
($\text{MG}^{(n)}(t), t \geq 0$) multi-graph-valued Markov chain

- each vertex has size $n^{-2/3}$
 - $\#(i \longrightarrow j) = N_{\{i,j\}}(t)$ Poisson process with rate $n^{4/3}/2$
- directed edge
- $\text{MG}^{(n)}(t) \xrightarrow[\text{removing self-loops}]{\text{unifying multi-edges}} \text{Erdős-Rényi}(n, 1 - e^{-n^{4/3}t})$
- critical time : $n^{1/3} + t$

$C_i^{(n)}(t)$ i-th largest connected component of $\text{MG}^{(n)}(t)$

$|C_i^{(n)}(t)|$ size and $\text{SP}(C_i^{(n)}(t))$ number of surplus edges

$(|C_i^{(n)}(t)|, \text{SP}(C_i^{(n)}(t)))_{i \geq 1}$ Markov process with dynamic



coalescence jump: with rate $|C_i^{(n)}(t)| \cdot |C_j^{(n)}(t)|$

surplus jump: with rate $|C_i^{(n)}(t)|^2/2$

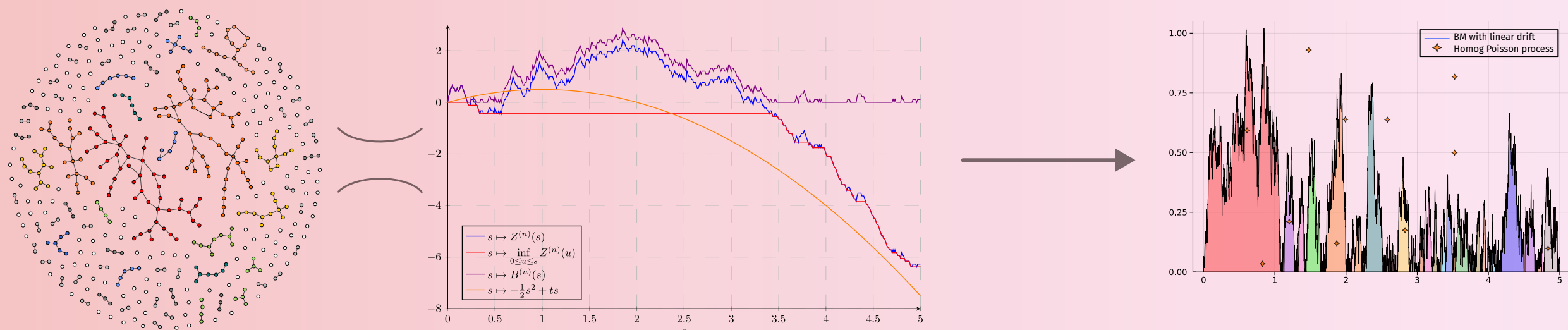
augmented multiplicative coalescent (AMC)

Space $\left\{ (x_i, n_i)_{i \geq 1} \in \mathbb{R}_+^2 \times \mathbb{N}^\infty : \sum_{i=1}^\infty x_i n_i < \infty \text{ and } n_i = 0 \text{ whenever } x_i = 0, i \geq 1 \right\}$

Metric $d_{\text{W}}((x, n), (x', n')) = \left(\sum_{i=1}^\infty (x_i - x'_i)^2 \right)^{1/2} + \sum_{i=1}^\infty |x_i \cdot n_i - x'_i \cdot n'_i|$

(making a process with the AMC dynamic a Feller process)

Proof sketch



Main difficulty: control of the tail $\lim_{\delta \rightarrow 0} \limsup_n \mathbb{E} \left[\sum_{i \geq 1} X_i^{(n)} \cdot N_i^{(n)} \cdot \mathbf{1}_{\{X_i^{(n)} < \delta\}} \right] = 0$

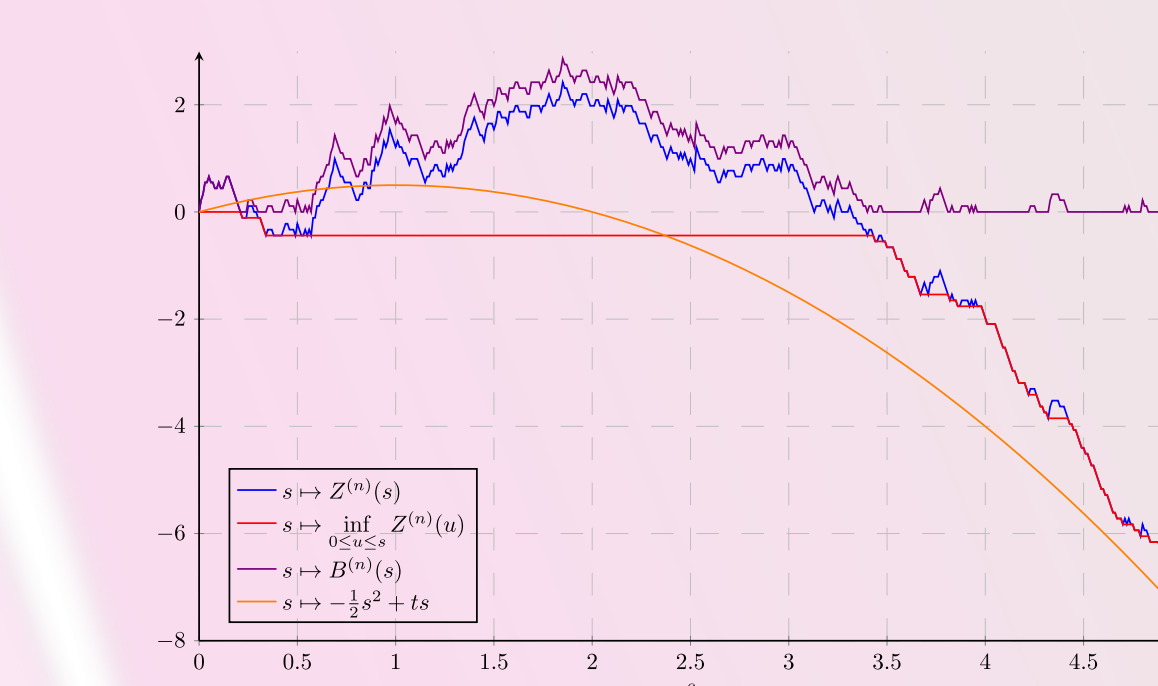
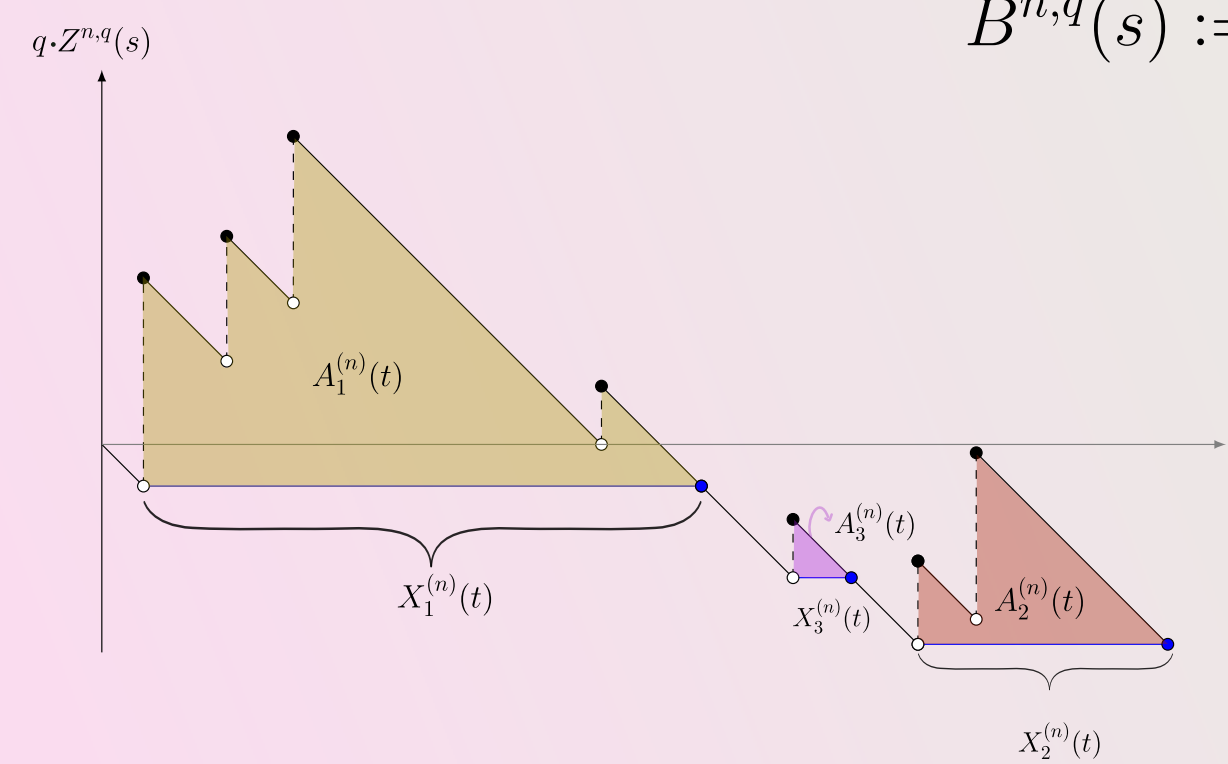
Methods

- bringing the tail to the beginning: $\mathbb{E} \left[\sum_{i \geq 1} X_i^{(n)} \cdot N_i^{(n)} \cdot \mathbf{1}_{\{X_i^{(n)} < \delta\}} \right] = n^{1/3} \cdot \mathbb{E} [\text{SP}(\mathcal{C}(V_n)) \mathbf{1}_{\{|\mathcal{C}(V_n)| < \delta\}}]$ randomly chosen vertex
- encoding: $\mathbb{E} [\text{SP}(\mathcal{C}(V_n)) \mathbf{1}_{\{|\mathcal{C}(V_n)| < \delta\}}] = \mathbb{E} \left[\int_{\text{First excursion}} B^{(n)}(s) ds \cdot \mathbf{1}_{\{|\text{First excursion}| < \delta\}} \right]$
- controlling the expected area below the curve: $\leq \delta n^{-1/3}$
- counting the multi-edges and self-loop

Simultaneous breadth-first walk

$$Z^{n,q}(s) = \sum_{i=1}^n \frac{1}{n^{2/3}} \mathbf{1}_{\{\xi_i/q \leq s\}} - s, \text{ where } \xi_i \sim \exp(\text{rate} = \frac{1}{n^{2/3}})$$

$$B^{n,q}(s) := Z^{n,q}(s) - \inf_{u \leq s} Z^{n,q}(u)$$



Encoding the AMC

$X_i^{(n)}(t)$ size of the i-th excursion

$N_i^{(n)}(t) = \text{Poisson}(A_i^{(n)}(t))$

area below the curve under the i-th excursion of $q \cdot Z^{n,q}(s)$

Theorem (C. and Limic 2023+)

$(X_i^{(n)}(t), N_i^{(n)}(t))_{i \geq 1}$ is equal in law to $(|C_i^{(n)}(t)|, \text{SP}(C_i^{(n)}(t)))_{i \geq 1}$

$$q_n(t) = n^{1/3} + t$$

Theorem (Limic 2019)

$q_n(t) Z^{n,q_n(t)}(s)$ converges in distribution towards $W^t(s)$

$\mathcal{X}_i(t)$ size of the i-th excursion of $W^t(s)$ $\mathcal{N}_i(t) = \text{Poisson}(A_i(t))$

area below the curve under the i-th excursion of $B^t(s)$

We need to prove: $d_{\text{W}}((X_i^{(n)}, N_i^{(n)})_{i \geq 1}, (\mathcal{X}_i(t), \mathcal{N}_i(t))_{i \geq 1}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$

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Read this poster online

This poster is based on the preprints:
Corujo and Limic *The standard augmented multiplicative coalescent revisited* (2023)

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arXiv: 2304.07545