Revista digital
Matemática, Educación e Internet
https://revistas.tec.ac.cr/index.php/matematica
Vol 25, No 2. Marzo, 2025 – Agosto, 2025
ISSN 1659 -0643
DOI: 10.18845/rdmei.v25i2.7821

GeoGebra as a Potential Tool for Exploring the Concepts of Continuity and Convergence

GeoGebra como una Herramienta Potencial para Explorar los Conceptos de Continuidad y Convergencia

André Luiz Araújo da Costa[1]
and.costa@uece.br
Universidade Estadual do Ceará - UECE
Fortaleza, Brasil

Francisco Régis Vieira Alves[2]
fregis@ifce.edu.br
Instituto Federal de Educação, Ciência e Tecnologia do Ceará - IFCE
Fortaleza, Brasil

Recibido: 20 de setiembre de 2024 Aceptado: 10 de diciembre de 2024

Abstract: This paper exemplifies the potential of GeoGebra as a didactic resource for teaching Real and Complex Analysis. To be more precise, our main goal is to demonstrate how useful GeoGebra is in providing a visual approach for understanding the concepts of continuity, equicontinuity, and the convergence of sequences of real functions of two variables and complex functions of a single variable. The complexity of the definition of these concepts, which rely on various parameters such as classical delta and epsilon, motivated the choice of the subject of this article. Additionally, their connection to the Ascoli-Arzelà Theorem, which is significant in many areas of mathematics, also influenced this decision. Throughout the paper we present some applets developed using GeoGebra which allow a satisfactory exploration of those concepts according to our analysis. This exploration is made along a sequence of examples and counterexamples for the concepts of continuity, equicontinuity and convergence addressed. In the Appendix, we provide some instructions for designing the applets used along the text.

Keywords: GeoGebra, visualization, Real and Complex Analysis, equicontinuity.

Resumen: Este artículo ejemplifica el potencial de GeoGebra como recurso didáctico para la enseñanza del Análisis Real y Complejo. Para ser más precisos, nuestro objetivo principal es demostrar lo útil que es GeoGebra al proporcionar un enfoque visual para entender los conceptos de continuidad, equicontinuidad y la convergencia de secuencias de funciones reales de dos variables y funciones complejas de una variable. La complejidad de la definición de estos conceptos, que dependen de varios parámetros como los clásicos delta y épsilon, motivó la elección del tema de este artículo. Además, su conexión con el Teorema de Ascoli-Arzelà, que es significativo en muchas áreas de las matemáticas, también influyó en esta decisión. A lo largo del artículo, presentamos algunos applets desarrollados con GeoGebra que permiten una exploración satisfactoria de esos conceptos según nuestro análisis. Esta

[1]André Luiz Araújo da Costa. Associate Professor at UECE - Center of Science and Technology. Postdoctoral researcher (RENOEN - IFCE). Postal Address: Dr. Silas Munguba Avenue, 1700 - Itaperi, Fortaleza - CE - Brazil. Postal Code: 60714-903. E-mail: and.costa@uece.br.

[2]Francisco Régis Vieira Alves. Full Professor at IFCE – Department of Mathematics and Physics, CNPQ productivity scholarship holder. Academic Coordinator of the Doctorate in the RENOEN network, IFCE hub. Postal Address: Treze de Maio Avenue, 2081 - Benfica, Fortaleza - CE - Brazil. Postal Code: 60040-531. E-mail: fregis@ifce.edu.br.

exploración se realiza a lo largo de una secuencia de ejemplos y contraejemplos para los conceptos de continuidad, equicontinuidad y convergencia tratados. En el Apéndice, proporcionamos algunas instrucciones para diseñar los applets utilizados a lo largo del texto.

Palabras Clave: GeoGebra, visualización, Análisis Real y Complejo, equicontinuidad.

1. Introduction

The presence of quantifiers and multiple parameters found in the definitions of concepts related to the continuity and convergence of sequences of functions are elements that complicate the assimilation of these concepts by students (Boero 2015; Pinto 1998). In the case of sequences of functions, the definitions of equicontinuity and uniform convergence, present in important theorems like the Ascoli-Arzelà Theorem, carry technical details like a fixed value ε, the index n of a function f_n, the point where the property is being analyzed, and a value δ (which may depend on the previous parameters). The study of equicontinuity and uniform convergence of sequences of complex functions presents another obstacle to learning: the high dimension of the graphs of the functions (Needham 2000).

The use of visual resources as an alternative to enhance learning in Mathematics, establishing a connection between rigorous mathematical statements and intuitive notions, has been widely studied (Martín-Caraballo and Tenorio-Villalón 2015; Martins et al. 2023; Yilmaz and Argun 2018), especially on Real (Adhikari 2021; Costa and Alves 2024; Hanh et al. 2021; Hanifah and Istikomar 2023; Igliori and Almeida 2018) and Complex Analysis (Alves 2014; Breda and Santos 2016; Valíková and Chalmovianský 2015; Wegert 2016). The aim of this paper is to analyze the potential use of GeoGebra software as a tool for the geometric visualization of the concepts addressed. For this purpose, we rely on GeoGebra's Slider tool, which is essential for visualizing the variation of parameters present in the definitions of continuity and convergence concepts. We based our approach to GeoGebra on the notion of generic organizer developed by Tall (1989): an environment (or microworld) which enables the learner to manipulate examples and (if possible) non-examples of a specific mathematical concept or a related system of concepts.

The article is organized as follows: in Section 2, we discuss the importance of the Ascoli-Arzelà Theorem and present two versions of this theorem, one for functions of two real variables and another for complex functions; in Section 3, we present some definitions of continuity and convergence concepts and a didactic sequence of examples and their respective visualizations using GeoGebra; in Section 4, we demonstrate that the equicontinuity of families of complex functions can be visualized through the equicontinuity of the real and imaginary parts of these functions, and finally, we present an example and its visualization in GeoGebra; Section 5 presents the conclusions on the developments made in this article. In the Appendix, we provide detailed descriptions of the specific GeoGebra commands and techniques used in the design of the applets presented throughout this text.

2. The Ascoli-Arzelà Theorem

The Ascoli-Arzelà Theorem is a fundamental tool in various areas of mathematics, including Real Analysis, Ordinary Differential Equations (ODEs), and Complex Analysis (Kolmogorov and Fomin 1975). This theorem provides conditions under which a sequence of continuous functions has a uniformly convergent subsequence. The importance of this theorem lies in its ability to ensure the compactness of sets of functions, which is essential for obtaining significant and practical results in these areas.

In the theory of Ordinary Differential Equations, the Ascoli-Arzelà Theorem plays a central role in the study of the existence and uniqueness of solutions. For example, in the proof of the Cauchy-Peano Existence Theorem (Coddington and Levinson 1955; Kolmogorov and Fomin 1975), which establishes conditions for the existence and uniqueness of solutions to ODEs, it is used to show that the sequence of Picard's successive approximations converges uniformly to a function that is a solution to the ODE.

In Complex Analysis, the Ascoli-Arzelà Theorem is equally significant, especially in the context of holomorphic functions. For instance, in the proof of Montel's Theorem (Conway 1978), which states that a family of holomorphic and uniformly bounded functions in a domain is normal (i.e., every sequence has a subsequence that converges uniformly on compact sets), the Ascoli-Arzelà Theorem is a fundamental component.

Observe that, unlike convergence phenomena, the conditions of uniform boundedness and equicontinuity do not depend on the ordering of the functions f_n. In this context, we distinguish the notations $\{f_n\}_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$, where the latter considers the ordering of the functions f_n while the first does not. In this article, we consider the following versions of the Ascoli-Arzelà Theorem:

Theorem 1 (Two real variables)

Let $K \subset \mathbb{R}^2$ be a compact set and $\{f_n\}_{n\in\mathbb{N}}$ a family of functions $f_n : K \to \mathbb{R}$. If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded and (uniformly) equicontinuous, then $(f_n)_{n\in\mathbb{N}}$ admits a uniformly convergent subsequence.

Theorem 2 (Complex functions)

Let $K \subset \mathbb{C}$ be a compact set and $\{f_n\}_{n\in\mathbb{N}}$ a family of functions $f_n : K \to \mathbb{C}$. If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded and (uniformly) equicontinuous, then $(f_n)_{n\in\mathbb{N}}$ admits a uniformly convergent subsequence.

3. Visualization with GeoGebra

The importance of visualization in the study of real analysis has historically been underestimated due to an excessive emphasis on rigorous methods. However, visual thinking is valuable in Analysis when it is paired with the appropriate 'epsilon-delta' approach (Giaquinto 2007). In this section, we explore a sequence of examples that provide visualizations of important concepts related to the Ascoli-Arzelà Theorem, using GeoGebra applets, after recalling their definitions.

The concepts of continuity relate a neighborhood of a point in the domain of a function to a neighborhood of the image of this point. Thus, to elaborate the visualization of continuity concepts, we will observe the Cartesian product of these neighborhoods, obtaining a cylinder. Therefore, the conditions involving the choice of an $\varepsilon > 0$ will be translated into the choice of the height of the cylinder, according to which we analyze the existence of an appropriate $\delta > 0$, which is translated into the existence of an appropriate radius for the circular base of this cylinder.

The visual representation of the parameters' relationships (between ε and δ, for instance) by the cylinder cited above and the physical manipulation of them by using the Slider tool of GeoGebra are suitable examples of what led Tall to design his notion of generic organizer (Tall 2000).

3.1. Continuity of two-variable functions

From now on, we denote the 2-dimensional open ball centered at a point $(x_0, y_0) \in \mathbb{R}^2$ and with radius r as below

$$B^2((x_0, y_0), r) = \{(x, y) \in \mathbb{R}^2 : |(x, y) - (x_0, y_0)| < r\}.$$

Definition 1

Let $A \subset \mathbb{R}^2$, $f : A \to \mathbb{R}$ a function, and $(x_0, y_0) \in A$. We say that f is continuous at (x_0, y_0) if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(x, y) \in A,\ |(x, y) - (x_0, y_0)| < \delta \implies |f(x, y) - f(x_0, y_0)| < \varepsilon. \tag{1}$$

To develop a visualization for Definition 1, let us consider the graph of the function f in $\mathbb{R}^3$ and observe the following: for f to be continuous at the point (x_0, y_0), given $\varepsilon > 0$, it must be possible to obtain a circle centered at (x_0, y_0) with radius $\delta > 0$ such that $f(x, y)$ lies between $f(x_0, y_0) - \varepsilon$ and $f(x_0, y_0) + \varepsilon$ for all (x, y) within that circle. Thus, the continuity of f is confirmed by observing the possibility of obtaining a cylinder

$$B^2((x_0, y_0), \delta) \times [f(x_0, y_0) - \varepsilon, f(x_0, y_0) + \varepsilon]$$

such that the points on the graph of f over $B^2((x_0, y_0), \delta)$ lie inside this cylinder. Note that if the graph of the function is connected, the previous condition translates to checking whether the graph of the function intersects the cylinder at points that are not on its bases (bottom and top). For visualizing these conditions, we propose the following example:

Example 1

The function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \begin{cases} x^2, & x < 1 \\ x + 1, & x \geq 1 \end{cases}$$

is not continuous at points of the form $(1, y)$. As we can see in Fig. 1, for ε values less than 1, such as $\varepsilon = \frac{1}{2}$ shown in Fig. 1, the condition (1) of continuity is not satisfied, since there are points on the graph of f over the ball $B^2((1, 0), \delta)$ but not inside the cylinder, regardless of the value of δ.

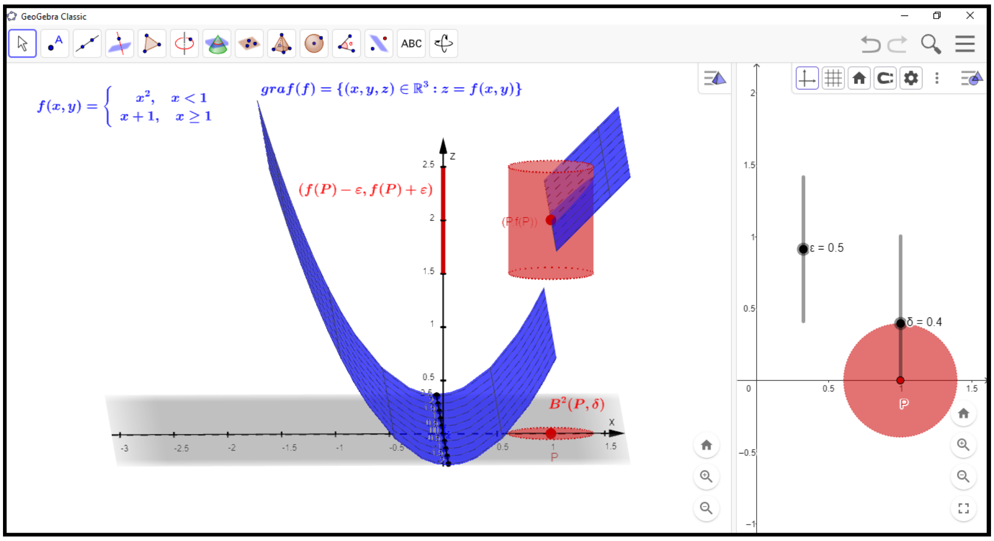

Figure 1: Discontinuity at points $(1, y)$. Created by the authors.

We constructed a dynamic visualization of the situation in Example 1 and made it available at the link: https://www.geogebra.org/m/sywppdzb.

Using the Sliders to vary the values of δ and ε, we easily conclude that there are points of the graph of f over $B^2((x1, 0), \delta) \times [2 - \varepsilon, 2 + \varepsilon]$ but outside the cylinder $B^2((x1, 0), \delta) \times [2 - \varepsilon, 2 + \varepsilon]$.

3.2. Pointwise and uniform equicontinuity

Definition 2

Let $A \subset \mathbb{R}^2$, $\{f_n\}_{n\in\mathbb{N}}$ a family of functions $f_n : A \to \mathbb{R}$, and $(x_0, y_0) \in A$. The family $\{f_n\}$ is equicontinuous at (x_0, y_0) if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(x, y) \in A,\ |(x, y) - (x_0, y_0)| < \delta \implies |f_n(x, y) - f_n(x_0, y_0)| < \varepsilon, \quad \forall n \in \mathbb{N}. \tag{2}$$

If $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous at every point (x, y) of the domain A, we simply say that $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous.

The equicontinuity of $\{f_n\}_{n\in\mathbb{N}}$ at (x_0, y_0) implies that each function f_n is continuous at (x_0, y_0). The distinctive aspect that the condition of equicontinuity at (x_0, y_0) brings is that, for a fixed value of ε, the same δ satisfies the condition (1) of continuity for all functions in the family $\{f_n\}_{n\in\mathbb{N}}$.

To visualize the pointwise equicontinuity of $\{f_n\}_{n\in\mathbb{N}}$, we observe the following: for $\{f_n\}_{n\in\mathbb{N}}$ to be equicontinuous at (x_0, y_0), given $\varepsilon > 0$, it must be possible to obtain a circle centered at (x_0, y_0) with radius $\delta > 0$ such that $f_n(x, y)$ lies between $f(x_0, y_0) - \varepsilon$ and $f(x_0, y_0) + \varepsilon$, for all $(x, y) \in A$ within that circle and for all $n \in \mathbb{N}$. Thus, the equicontinuity of $\{f_n\}_{n\in\mathbb{N}}$ is confirmed by observing the possibility of obtaining cylinders

$$B^2((x_0, y_0), \delta) \times [f_n(x_0, y_0) - \varepsilon, f_n(x_0, y_0) + \varepsilon]$$

such that the points on the graph of f_n over $B^2((x_0, y_0), \delta)$ lie inside this cylinder, for all $n \in \mathbb{N}$. To visualize the above conditions, we initially propose an example where condition (2) of pointwise equicontinuity is not satisfied.

Example 2

The family $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n : \mathbb{R}^2 \to \mathbb{R}$ given by $f_n(x, y) = x^n$ is not equicontinuous at $(1, 0)$. Given $\varepsilon > 0$, for each $n \in \mathbb{N}$, the largest value of δ satisfying the continuity condition is $\delta = \sqrt[n]{\varepsilon + 1} - 1$. However, since the sequence $(\sqrt[n]{\varepsilon + 1} - 1)_{n\in\mathbb{N}}$ has limit zero, we conclude that it is not possible to obtain a positive value of δ satisfying the condition (2) of pointwise equicontinuity (see Fig. 2).

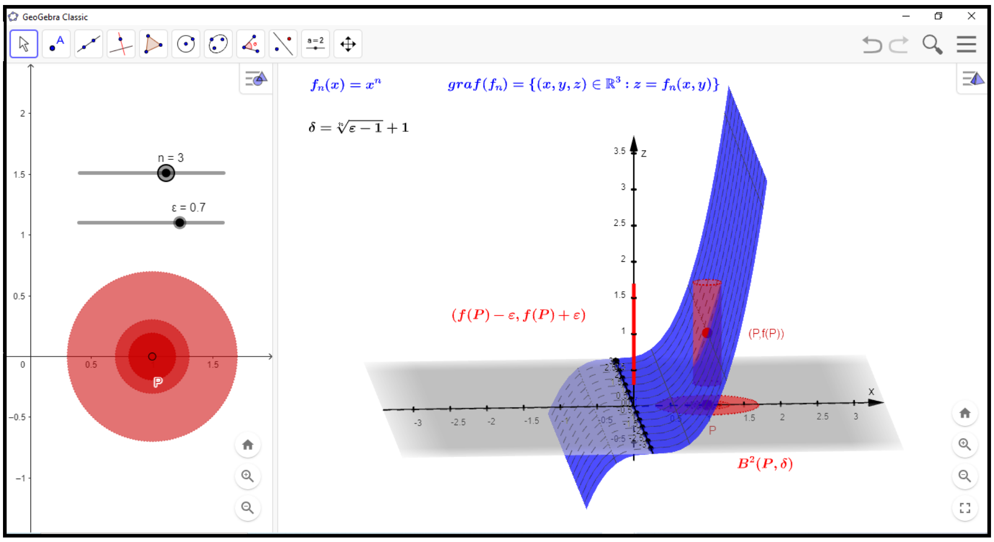

Figure 2: Decrease in the value of δ with the variation of ε. Created by the authors.

We constructed a dynamic visualization of the situation in Example 2 and made it available at the link: https://www.geogebra.org/m/ssjwby4s.

The fact that $\sqrt[n]{\varepsilon+1}-1$ is the highest value of δ satisfying the continuity condition for f_n is visualized as we see the graph of f_n on the verge of touching the top of the cylinder. Moving the Slider to higher values of n, we observe that the radius of the cylinder tends to zero meaning that we cannot obtain a cylinder satisfying the equicontinuity condition (2) as in Definition 2. The visualization of these facts can be improved by observing the trace of the circle $B^2((1,0),\delta)$ which is accessed by clicking with the right mouse button on the circle.

Example 3

The family $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n:\mathbb{R}^2\to\mathbb{R}$ given by $f_n(x,y)=x^2+\frac{1}{n}$ is equicontinuous (at all points in $\mathbb{R}^2$). Given $\varepsilon>0$, for each $n\in\mathbb{N}$, the largest value of δ satisfying the continuity condition for f_n at (x_0,y_0) is $\delta=\sqrt{\varepsilon+x_0^2}-|x_0|$. Since this value for δ does not depend on n, it satisfies the condition (2) of equicontinuity presented in Definition 2. This means that the dimensions of the cylinder associated with the continuity of the functions f_n does not change when the value of n varies (see Fig. 3).

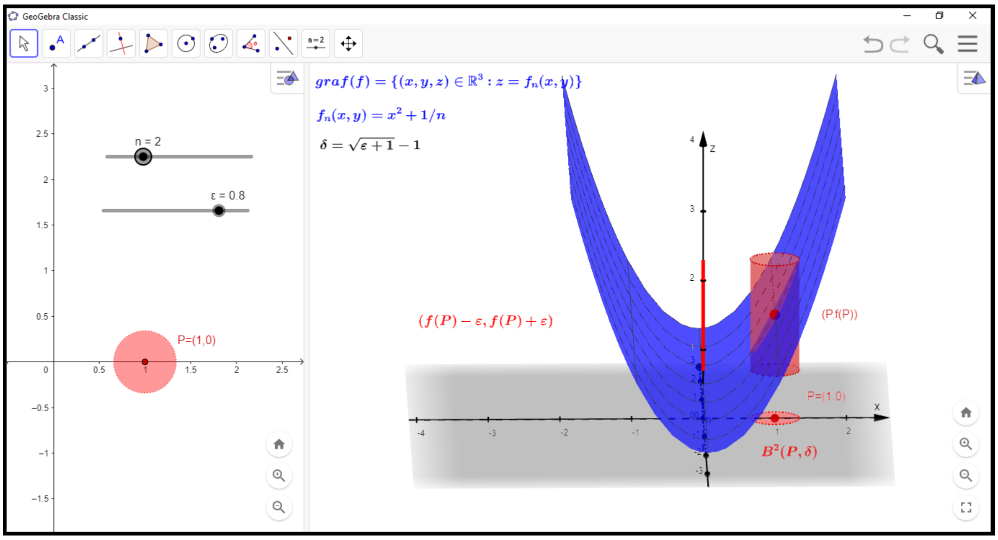

Figure 3: Equicontinuity of $\{f_n\}_{n\in\mathbb{N}}$ at $(1,0)$. Created by the authors.

We constructed a dynamic visualization of the situation in Example 3 and made it available at the link: https://www.geogebra.org/m/xbnrzbmp.

The fact that $\delta=\sqrt{\varepsilon+r^2}-|r|$ is the highest value of δ satisfying the continuity condition for f_n is visualized as we see the graph of f_n on the verge of touching the top of the cylinder.

Moving the Slider to higher values of n, we observe that the dimensions of the cylinder do not change and hence we conclude that the family $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous at each point (r,r). However, moving the Slider for the value of r to higher values we observe that the dimensions of the cylinder change. This motivates the next definition.

Definition 3

Let $A\subset\mathbb{R}^2$, and $\{f_n\}_{n\in\mathbb{N}}$ a family of functions $f_n:A\to\mathbb{R}$. The family $\{f_n\}_{n\in\mathbb{N}}$ is uniformly equicontinuous if, for every $\varepsilon>0$, there exists $\delta>0$ such that whenever $(x,y)\in A$ we have

$$|(x,y)-(x_0,y_0)|<\delta \implies |f_n(x,y)-f_n(x_0,y_0)|<\varepsilon,\ \forall n\in\mathbb{N},\forall(x_0,y_0)\in A. \tag{3}$$

To visualize the uniform equicontinuity of $\{f_n\}_{n\in\mathbb{N}}$, we observe the following: for $\{f_n\}_{n\in\mathbb{N}}$ to be uniformly equicontinuous, given $\varepsilon > 0$, it must be possible to obtain a radius $\delta > 0$ for the circles centered at the points $(x_0, y_0) \in A$ such that $f_n(x, y)$ lies between $f_n(x_0, y_0) - \varepsilon$ and $f_n(x_0, y_0) + \varepsilon$, for all (x, y) within that circle and all $n \in \mathbb{N}$. Thus, the uniform equicontinuity of $\{f_n\}_{n\in\mathbb{N}}$ is confirmed by observing the possibility of obtaining cylinders $B^2((x_0, y_0), \delta) \times [f_n(x_0, y_0) - \varepsilon, f_n(x_0, y_0) + \varepsilon]$ such that the points on the graph of f_n over $B^2((x_0, y_0), \delta)$ lie inside this cylinder, for all $(x_0, y_0) \in A$ and all $n \in \mathbb{N}$. This means that the dimensions of the cylinder should not change as (x_0, y_0) and n vary.

The family of functions studied in Example 3 is not uniformly equicontinuous as we can observe that the radius of the base of the cylinder tends to zero as the Slider for the value of is moved towards higher values (see Fig. 4).

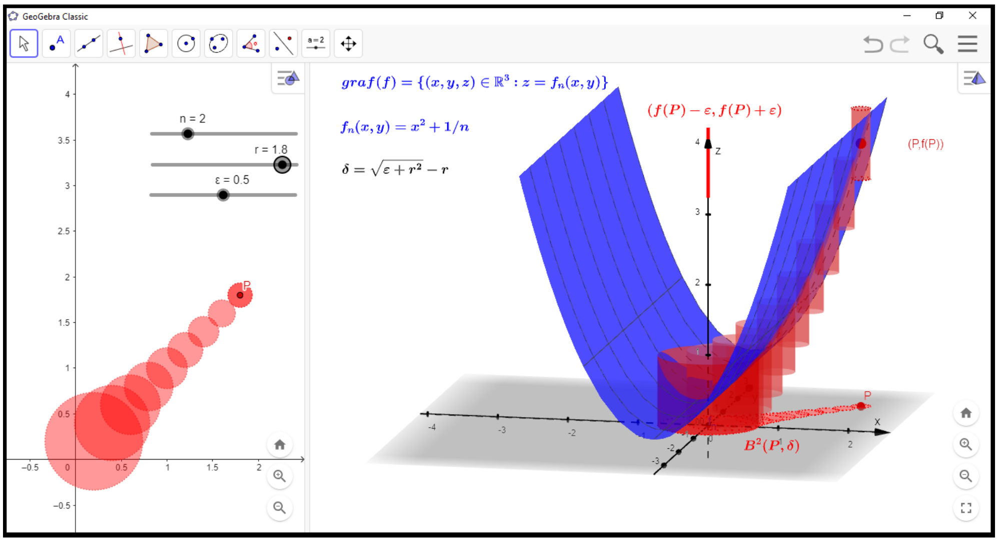

Figure 4: Decrease of δ as x_0 increases.

The restriction of the domain of an equicontinuous family of functions to a compact subset defines a uniformly equicontinuous family of functions. Indeed, Kumaresan (2005) shows that:

Theorem 3

Let X be a compact metric space and $\mathcal{F}$ be a family of equicontinuous functions from X to another metric space Y. Then $\mathcal{F}$ is a uniformly equicontinuous family.

In the context of Theorem 3, we present the following example.

Example 4

The family $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n : [-1, 1] \times [-1, 1] \to \mathbb{R}$ given by

$$f_n(x, y) = x^3 + \frac{1}{n}$$

is uniformly equicontinuous. Given $\varepsilon > 0$, the choice $\delta = \frac{\varepsilon}{3}$ satisfies the continuity condition for all $(x_0, y_0) \in [-1, 1] \times [-1, 1]$ and all $n \in \mathbb{N}$. The points of the graph of f_n which are over $[-1, 1] \times [-1, 1] \cap B^2((x_0, y_0), \delta)$ lie inside the cylinder $B^2((x_0, y_0), \delta) \times [f_n(x_0, y_0) - \varepsilon, f_n(x_0, y_0) + \varepsilon]$ for (see Fig. 5).

We constructed a dynamic visualization of the situation in Example 4 and made it available at the link: https://www.geogebra.org/m/kxht6mnz.

Moving the Sliders for the values of n and r we may visualize the fact that the cylinder $B^2((x_0, y_0), \delta) \times$

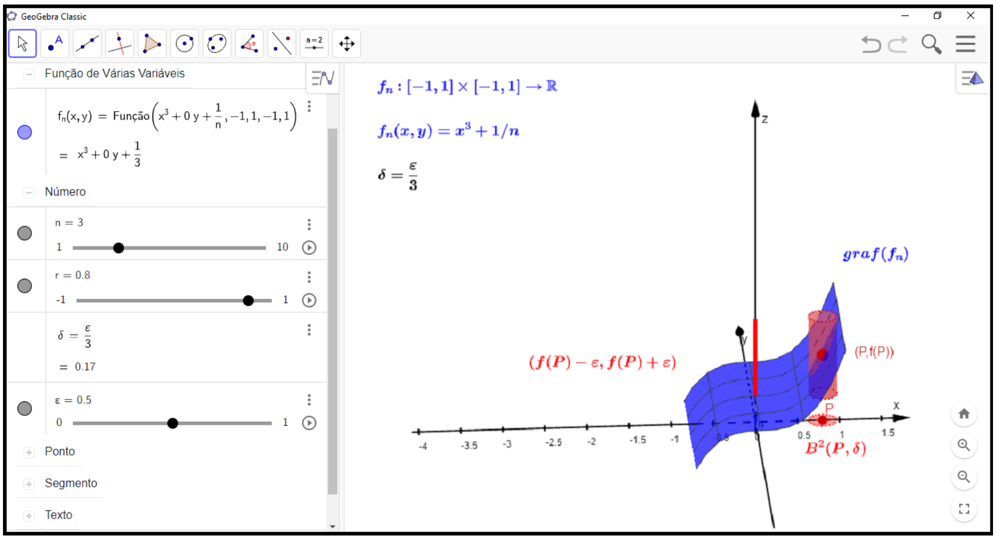

Figure 5: Uniform equicontinuity of $\{f_n\}_{n\in\mathbb{N}}$. Created by the authors.

$[f_n(x_0,y_0)-\varepsilon, f_n(x_0,y_0)+\varepsilon]$ with radius $\delta = \frac{\varepsilon}{3}$ contains all the points of the graph of f_n which are over $B^2((x_0,y_0),\delta)$ and hence, the family $\{f_n\}_{n\in\mathbb{N}}$ is uniformly equicontinuous.

3.3. Pointwise and uniform convergence

Definition 4

Let $A \subset \mathbb{R}^2$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of functions $f_n : A \to \mathbb{R}$. We say that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to a function $f : A \to \mathbb{R}$ if, for each $(x_0,y_0) \in A$, the numerical sequence $(f_n(x_0,y_0))_{n\in\mathbb{N}}$ converges to $f(x_0,y_0)$. In other words, given $\varepsilon > 0$, for each (x_0,y_0) there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \implies |f_n(x_0,y_0) - f(x_0,y_0)| < \varepsilon. \tag{4}$$

Observe that, in the definition of pointwise convergence, there is no requirement that the same n_0 satisfies the condition (4) of pointwise convergence for all $(x_0,y_0) \in A$, unlike the definition of uniform convergence presented below.

Definition 5

Let $A \subset \mathbb{R}^2$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n : A \to \mathbb{R}$. We say that $(f_n)n \in \mathbb{N}$ converges uniformly to a function $f : A \to \mathbb{R}$ if, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \implies |f_n(x_0,y_0) - f(x_0,y_0)| < \varepsilon, \quad \forall (x_0,y_0) \in A. \tag{5}$$

To create a visual approach for the definition of uniform convergence, we consider the graph of the function f in $\mathbb{R}^3$ and, for each given $\varepsilon > 0$, consider the following set:

$$R(f,\varepsilon) = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in A, \;\; f(x,y) - \varepsilon < z < f(x,y) + \varepsilon\}.$$

The set $R(f,\varepsilon)$ is the union of vertical segments centered at points of the graph of f with radius ε. For $(f_n)_{n\in\mathbb{N}}$ to converge uniformly to f, there must exist $n_0 \in \mathbb{N}$ such that the graph of f_n lies within the set $R(f,\varepsilon)$ for all $n \leq n_0$.

Example 5

The sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n : [0,1] \times [0,1] \to \mathbb{R}$ given by $f_n(x,y) = x^n$ converges pointwise to the function $f : [0,1] \times [0,1] \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} 0, & x < 1 \\ 1, & x = 1, \end{cases}$$

but does not converge uniformly. In fact, whenever $0 < \varepsilon < 1$, there will exist $n \in \mathbb{N}$ for which the graph of f_n is not completely inside of $R(f, \varepsilon)$ (see Fig. 6).

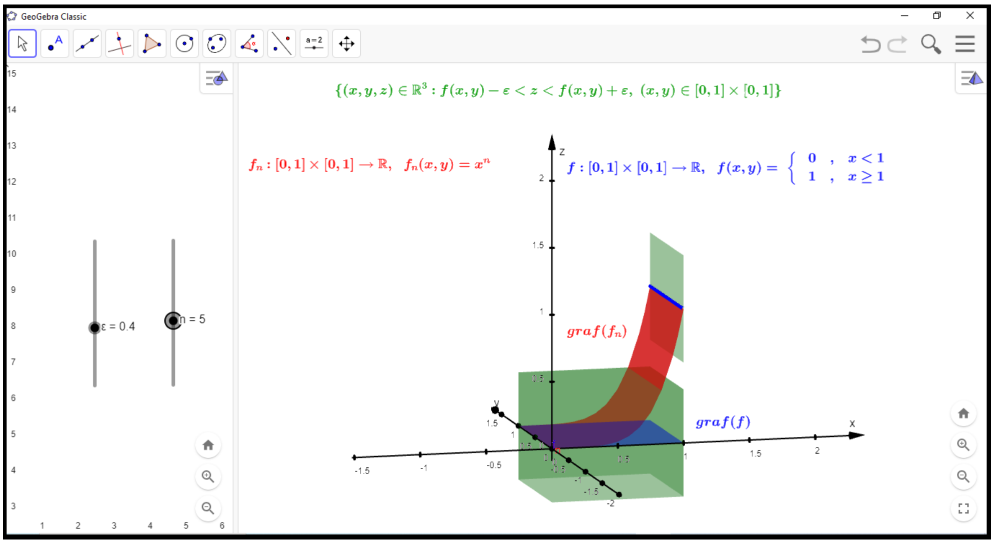

Figure 6: Non-uniform convergence of $(f_n)_{n\in\mathbb{N}}$ to f. Created by the authors.

We constructed a dynamic visualization of the situation in Example 5 and made it available at the link: https://www.geogebra.org/m/e48kdzse.

It is easily observed, with the support of the Slider of ε, that whenever $0 < \varepsilon < 1$, there will exist $n \in \mathbb{N}$ such that the graph of f_n has points outside $R(f, \varepsilon)$ and hence we conclude that $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly to f. However, with the support of the Slider of n, we observe that, for a fixed point (x_0, y_0), the point $(x_0, y_0, f_n(x_0, y_0))$ of the graph of f_n lies inside $R(f, \varepsilon)$ whenever n is large enough meaning that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f.

4. Equicontinuity in the case of Complex Functions

Since the set $\mathbb{C}$ can be geometrically seen as $\mathbb{R}^2$, visualizing some properties of complex functions can become challenging because the graph of these functions lies in $\mathbb{C}^2$, which has real dimension 4. A valid alternative to remedy this difficulty is to consider separately the real and imaginary parts of the complex function, which can be viewed as real functions of two variables, as those considered in the previous section.

To take advantage of the visualization techniques developed in the previous section, it is important to clarify the relationship between the equicontinuity of a family $\{f_n\}_{n\in\mathbb{N}}$ of complex functions $f_n = a_n + i \cdot b_n$ and the equicontinuity of the families $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ of real functions of two variables and defined by the real and imaginary parts of the complex functions f_n. We present a proof of the following theorem since we have not found any reference for this.

Theorem 4

Let $A \subset \mathbb{C}$ and $\{f_n\}_{n\in\mathbb{N}}$ be a family of functions $f_n : A \to \mathbb{C}$ given by $f_n = a_n + i \cdot b_n$ where a_n and b_n are real functions defined on A. Then $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous at a point $(x_0, y_0) \in A$ if and only if $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are equicontinuous at (x_0, y_0).

Proof. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous at (x_0, y_0). Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that, whenever $(x, y) \in A$, we have

$$|(x, y) - (x_0, y_0)| < \delta \implies |f_n(x, y) - f_n(x_0, y_0)| < \varepsilon, \ \ \forall n \in \mathbb{N}.$$

Since $|a_n(x, y) - a_n(x_0, y_0)|, |b_n(x, y) - b_n(x_0, y_0)| \leq |f_n(x, y) - f_n(x_0, y_0)|$, it follows that

$$|(x, y) - (x_0, y_0)| < \delta \implies \begin{cases} |a_n(x, y) - b_n(x_0, y_0)| < \varepsilon, & \forall n \in \mathbb{N} \\ |b_n(x, y) - b_n(x_0, y_0)| < \varepsilon, & \forall n \in \mathbb{N} \end{cases}$$

and therefore, $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are equicontinuous at (x_0, y_0).

Conversely, suppose that $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are equicontinuous at (x_0, y_0). Then, given $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that, whenever $(x, y) \in A$, we have

$$|(x, y) - (x_0, y_0)| < \delta_1 \implies |a_n(x, y) - b_n(x_0, y_0)| < \frac{\varepsilon}{2}, \ \ \forall n \in \mathbb{N},$$

$$|(x, y) - (x_0, y_0)| < \delta_2 \implies |b_n(x, y) - b_n(x_0, y_0)| < \frac{\varepsilon}{2}, \ \ \forall n \in \mathbb{N}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Since

$$|f_n(x, y) - f_n(x_0, y_0)| \leq |a_n(x, y) - a_n(x_0, y_0)| + |b_n(x, y) - b_n(x_0, y_0)|,$$

it follows that

$$|(x, y) - (x_0, y_0)| < \delta \implies |f_n(x, y) - f_n(x_0, y_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous. □

Note that the same arguments used in the proof of Theorem 4 prove the same property for uniform equicontinuity.

Although it is not possible to visualize the equicontinuity of the family $\{f_n\}_{n\in\mathbb{N}}$ since the graph of these functions lies in $\mathbb{R}^4$, we can visualize the equicontinuity of the families $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ of the real and imaginary parts of the functions f_n, which characterizes the equicontinuity of $\{f_n\}_{n\in\mathbb{N}}$ according to Theorem 4.

Example 6

The family $\{f_n\}_{n\in\mathbb{N}}$ of complex functions $f_n : A \to \mathbb{C}$ defined by

$$f_n(z) = z^2 + \frac{1}{n}$$

on the compact set

$$A = \{z = x + i \cdot y \in \mathbb{C} : x, y \in [0, 1]\}$$

is uniformly equicontinuous. Note that $f_n = a_n + i \cdot b_n$ where

$$a_n(x, y) = x^2 + y^2 + \frac{1}{n} \ \text{ and } \ b_n(x, y) = 2xy.$$

Given $\varepsilon > 0$, $\delta = \frac{\varepsilon}{4}$ satisfies condition (3) of uniform equicontinuity for both $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$. The points of the graph of a_n which are over $[0,1] \times [0,1] \cap B^2((x_0,y_0),\delta)$ lie inside the cylinder $B^2((x_0,y_0),\delta) \times [a_n(x_0,y_0) - \varepsilon, a_n(x_0,y_0) + \varepsilon]$ (see Fig. 7), and the points of the graph of b_n which are over $[0,1] \times [0,1] \cap B^2((x_0,y_0),\delta)$ lie inside the cylinder $B^2((x_0,y_0),\delta) \times [b_n(x_0,y_0) - \varepsilon, b_n(x_0,y_0) + \varepsilon]$ (see Fig. 8).

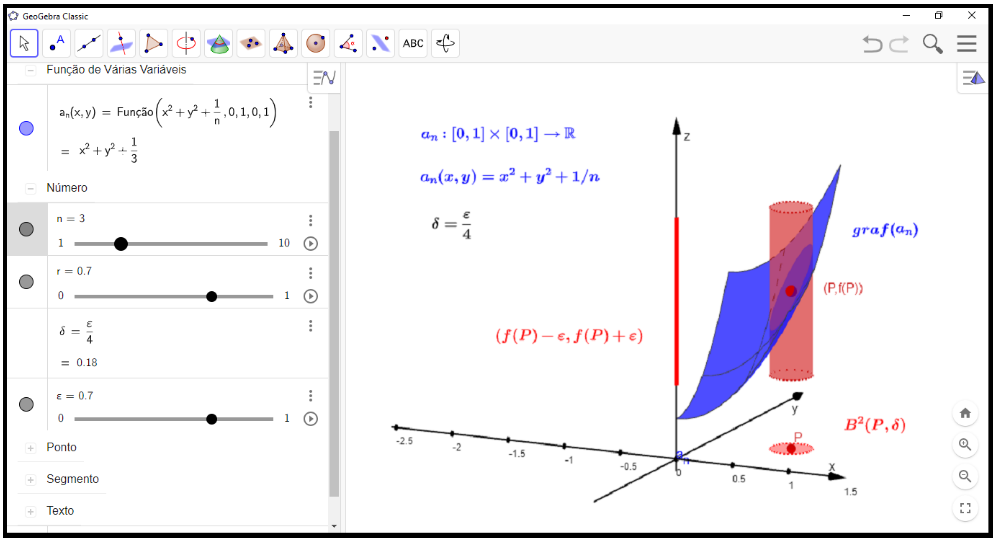

Figure 7: Uniform equicontinuity of the real part $\{a_n\}_{n\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$. Created by the authors.

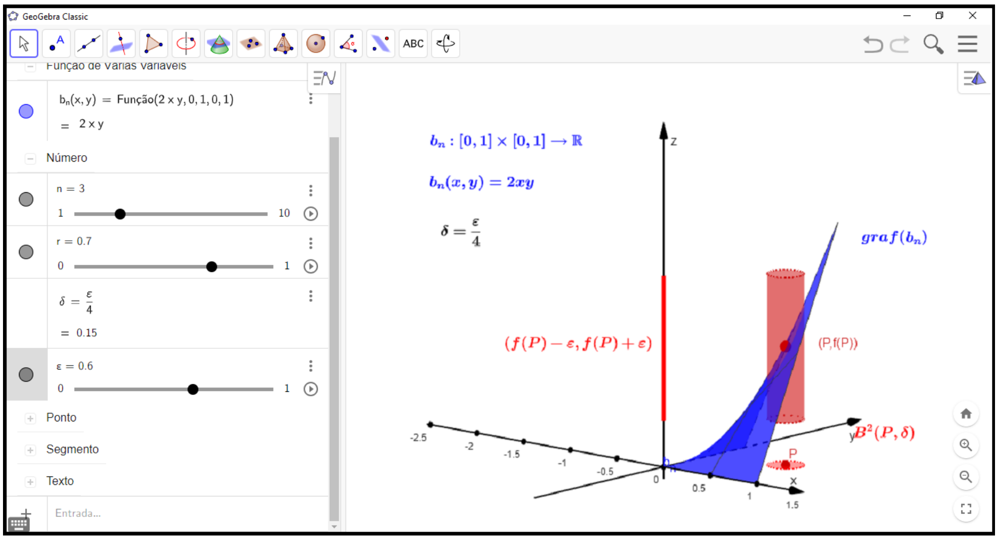

Figure 8: Uniform equicontinuity of the imaginary part $\{b_n\}_{n\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$. Created by the authors.

We constructed a dynamic visualization of the situation of the families $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ in Example 6 and made it available respectively at the links: https://www.geogebra.org/m/axvzddsd and https://www.geogebra.org/m/ezve8mcj.

The analysis with the applets for real and imaginary parts for Example 6 are analogous to the described for Example 4, so we will not repeat them here.

5. Conclusions

We analyzed the potential use of GeoGebra software as a tool to provide visualization of the concepts addressed. Our methodology was based on the geometric significance of the choice and existence of the values of ε and δ present in the definitions of equicontinuity and convergence. At this point, GeoGebra's Slider tool was essential for providing a visualization of the variation of the parameters.

Initially, we worked within the context of real functions of two variables. Supported by Theorem 4, we presented the possibility of applying the developed technique of visualization to the case of complex functions. This was achieved by separately observing the real and imaginary parts of these functions. This shows an example of how theoretical results in mathematics can be applied to the teaching context.

The examples and counterexamples presented were chosen to validate the proposal for visualizing the addressed concepts. This approach allows a visualization of what occurs when the conditions of continuity, equicontinuity, uniform equicontinuity, and uniform convergence are not satisfied.

We conclude that GeoGebra has the potential to be adopted as a didactic resource for teaching content, even with considerable levels of abstraction. When well used, it presents satisfactory results in shortening the path between visual perception and theoretical understanding.

A. Appendix: Notes on the design of GeoGebra applets

In this appendix, we provide detailed descriptions of the specific GeoGebra commands and techniques used in the design of the applets presented throughout this text. Below, we outline the steps and commands used to create each applet, ensuring that the readers can replicate and explore these visualizations on their own using GeoGebra.

Applet used in Example 1

The first standard proceeding to build the GeoGebra applet for this visualization consists of plotting the graph of the function and the point $P = (1,0,0)$ to represent the point $(1,0)$ in the xy-plane. The software was not able to plot the graph of f satisfactorily via the command **f(x,y)=If(x<1,x^2,x+1)** since it somehow connected the two connected components of the graph of f (disconnected blue pieces of graph in Fig. 1). Then we build the components of the graph of f separately with the commands **f_1(x,y)=If(x>=1,x+1)** and **f_2(x,y)=If(x<1,x^2)**. Since $f(1,0) = 2$, we define the points $A = (1,0,2+\varepsilon)$ and $B = (1,0,2-\varepsilon)$ automatically creating the Slider for the value of ε. After that, we build the circle $B^2((1,0),\delta)$ inserting the command **Circle(P,δ,xOyPlane)** which will also create the Slider for the value of δ , and the cylinder $B^2((x1,0),\delta) \times [2-\varepsilon, 2+\varepsilon]$ by inserting **Cylinder(A,B,δ)**.

Applet used in Example 2

To build this applet we first insert **f_n(x,y)=x^n** that creates a Slider for the value of n together with the graph of f_n for the respective value of n controlled by the Slider tool. The variation of the graph while the Slider is moved gives a satisfactory perception of the sequence $(f_n)_{n\in\mathbb{N}}$. Then we plot four points: the point $P = (1,0,0)$ to represent the point $(1,0)$ in the xy-plane, the point $Q = (1,0,1)$ since $f_n(1,0) = 1$ for all $n \in \mathbb{N}$ and the points $A = (1,0,1+\varepsilon)$ and $B = (1,0,1-\varepsilon)$ which will be used to determine the height of the cylinder. Before we start to build the cylinder to analyze the condition of equicontinuity, we need to define the value of δ by inserting **δ =(ε+1)^(1/n)-1** in the input box. Note that the value of δ also varies when the Slider for the value of n is moved. Finally, we insert **Circle(P,δ,xOyPlane)** and **Cylinder(A,B,δ)** plotting the circle $B^2((1,0),\delta)$ in the xy-plane and the cylinder $B^2((1,0),\delta) \times [1-\varepsilon, 1+\varepsilon]$.

Applet used in Example 3

To build this applet we insert **f_n(x,y)=x^2+1/n** that creates a Slider for the value of n together with the graph of f_n for the respective value of n controlled by the Slider tool. In order to analyze the points in the diagonal of the odd quadrants, we first define the point $P = (r, r, 0)$ to represent the point (r) in the xy-plane. This automatically creates a Slider for the value of r which provides a dynamic way to analyze the points in the chosen diagonal by moving the Slider. Then we plot the point $Q = (r, r, f_n(r, r))$ to represent the visualization of the image of (r, r), and the points $A = (r, r, f_n(r, r) + \varepsilon)$ and $B = (r, r, f_n(r, r) - \varepsilon)$ which will be used to determine the height of the cylinder. Note that all these points move as the Slider for the value of r. After that, we define the value of δ by inserting **δ=sqrt(ε+r^2)-|r|** in the input box. Note that the value of δ does not change when the Slider for the value of n is moved. Finally, we use **Circle(P,δ,xOyPlane)** and **Cylinder(A,B,δ)** to plot the circle $B^2((r, r), \delta)$ in the xy-plane and the cylinder $B^2((r, r), \delta) \times [f_n(r, r) - \varepsilon, f_n(r, r) + \varepsilon]$.

Applet used in Example 4

The preparation of this applet essentially follows the same procedure as the previous one except for a couple details that we highlight here. To plot the family of functions f_n, we type the command **f_n(x,y)=Function** in the input box and then we choose the option **Function(Expression, Parameter Variable 1, Start Value, End Value, Parameter Variable 2, Start Value, End Value)** and fill the fields with the data **(x^3+0y+1/n,x,-1,1,y,-1,1)** in order to establish the domain $[-1, 1] \times [-1, 1]$. We warn that the variable y should appear in the field **Expression**, otherwise the software will not work, this is the reason why we typed **x^3+0y+1/n** instead of just **x^3+1/n** in the field **Expression**. Moreover, we defined the point $P = (r, 0)$ instead of (r, r). This choice does not affect the analyzes since the functions f_n essentially depend only on the variable x.

Applet used in Examples 5 and 6

To build the applet used in Example 5 we plot the family of functions f_n similarly as previous examples using the command **f_n(x,y)=Function(x^n+0y,x,0,1,y,0,1)** which also creates the Slider for the value of n. The graph of the function f consists of the union of two disconnected pieces (components): the rectangle $C_1 = [0, 1) \times [0, 1] \times \{0\}$ and the segment $C_2 = \{1\} \times [0, 1] \times \{1\}$. We warn again that the software used to have problems plotting graphs with more than one connected component. To plot C_1 we used **Function(0x+0y,0,1,0,1)** which plots $[0, 1] \times [0, 1] \times \{0\}$ that differs from C_1 only by the null measure set $\{1\} \times [0, 1] \times \{0\}$ (which does not affect the visualization). To plot the segment C_2, we use the command **g=Segment((1,0,1),(1,1,1))**. The desired set $R(f, \varepsilon)$ consists of the union of the block $B_1 = [0, 1) \times [0, 1] \times (0 - \varepsilon,) + \varepsilon)$, since $f(x, y) = 0$ for any $(x, y) \in [0, 1) \times [0, 1]$, and the rectangle $B_2 = \{1\} \times [0, 1] \times (1 - \varepsilon, 1 + \varepsilon)$, since $f(x, y) = 1$ for any $(x, y) \in \{1\} \times [0, 1]$. To plot B_1, we first define the points $A = (0, 0, -\varepsilon)$, $B = (1, 0, -\varepsilon)$, $C = (1, 1, -\varepsilon)$ and $D = (0, 0, -\varepsilon)$ together with the Slider for the value of ε. Then we use **q1=Polygon(A,B,C,D)** to define what will be the basis of the block B_1 and hence we use **e=Prism(q1,2ε)** to plot a prism with basis **q1** and height 2ε obtaining the visualization of B_1. Finally, to plot B_2 we first define the points $A_1 = (1, 0, 1 - \varepsilon)$, $A_2 = (0, 0, 1 + \varepsilon)$, $A_3 = (1, 1, 1 + \varepsilon)$ and $A_4 = (1, 1, 1 - \varepsilon)$ and then we use the command **q2=Polygon(A_1,A_2,A_3,A_4)**.

The construction procedure for the applets presented in Example 6 is analogous to the described for Example 4, so we will not repeat them here.

Acknowledgments: This work was funded by the National Council for Scientific and Technological Development (CNPQ), an entity affiliated with the Ministry of Science, Technology, and Innovations, to support research in Brazil.

Authors' contribution: Conceptualization: A.L.A.C., F.R.V.A. Formal analysis: A.L.A.C., F.R.V.A. Methodology: A.L.A.C., F.R.V.A. Writing – original draft: A.L.A.C., F.R.V.A. Writing – review and editing: A.L.A.C., F.R.V.A.

Data availability: The data used in this study consist of applets constructed using the GeoGebra software platform. Links to these applets are provided throughout the text, allowing readers to access and interact with the materials directly.

6. References

Adhikari, G. P. (2021). Visualization of the riemann darboux sum and its properties with geogebra. *Journal of Ramanujan Society of Mathematics & Mathematical Sciences*, *9*(1), 145–156.

Alves, F. R. V. (2014). Visualizing the behavior of infinite series and complex power series with the geogebra. *GeoGebra International Journal of Romania*, *4*(1), 11–20.

Boero, P. (2015). Analyzing the transition to epsilon-delta Calculus: A case study. In K. Krainer & N. Vondrová (Eds.), *CERME 9 - Ninth Congress of the European Society for Research in Mathematics Education* (pp. 93–99). https://hal.science/hal-01280542

Breda, A. M. D., & Santos, J. M. D. S. D. (2016). Complex functions with GeoGebra. *Teaching Mathematics and its Applications: An International Journal of the IMA*, *35*(2), 102–110. https://doi.org/10.1093/teamat/hrw010

Coddington, E. A., & Levinson, N. (1955). *Theory of ordinary differential equations*. McGraw-Hill.

Conway, J. (1978). *Functions of one complex variable*. Springer-Verlag.

Costa, A. L. A., & Alves, F. R. V. (2024). Visualização de elementos do teorema de ascoli-arzelà com ferramentas do software geogebra. *Revista do Instituto GeoGebra Internacional de São Paulo*, *13*(3), 141–154. https://doi.org/10.23925/2237-9657.2024.v13i3p141-154

Giaquinto, M. (2007). *Visual thinking in mathematics: An epistemological study*. Oxford University Press.

Hanh, N. T. H., et al. (2021). Using geogebra software application in teaching definite integral. In M. N. Favorskaya, S.-L. Peng, M. Simic, B. Alhadidi, & S. Pal (Eds.), *Intelligent computing paradigm and cutting-edge technologies* (pp. 327–335). Springer International Publishing.

Hanifah, H., & Istikomar, I. (2023). Validity and practicality of student activity sheets (lam) of real number sequences with the aid of geogebra based on the apos model. *Proceedings of the Mathematics and Science Education International Seminar 2021 (MASEIS 2021)*, 97–106. https://doi.org/10.2991/978-2-38476-012-1_14

Igliori, S. B. C., & Almeida, M. V. d. (2018). Desenvolvendo abordagens de ensino para conceitos de cálculo diferencial e integral com geogebra. *Educação e Fronteiras*, *8*(23), 164–175. https://doi.org/10.30612/eduf.v8i23.9450

Kolmogorov, A. N., & Fomin, S. V. (1975). *Introductory real analysis*. Dover Publications.

Kumaresan, S. (2005). *Topology of metric spaces*. Alpha Science International.

Martín-Caraballo, A. M., & Tenorio-Villalón, A. F. (2015). Teaching Numerical Methods for Non-linear Equations with GeoGebra-Based Activities. *International Electronic Journal of Mathematics Education*, *10*(2), 53–65. https://doi.org/https://doi.org/10.29333/iejme/291

Martins, E. M., et al. (2023). The geogebra software in the introductory teaching of dynamic systems: Research with students of bachelor's degree in mathematics. *Revista do Instituto GeoGebra Internacional de São Paulo*, *12*(1), 004–028. https://doi.org/10.23925/2237-9657.2023.v12i1p004-028

Needham, T. (2000). *Visual complex analysis*. Oxford University Press.

Pinto, M. M. F. (1998). *Students' understanding of real analysis* [PhD thesis]. Warwick University [Available at https://www.proquest.com/openview/bf3b8c4f1c69239a4d2a893b0c0c2151/1?pq-origsite=gscholar&cbl=18750&diss=y].

Tall, D. (1989). Concept images, generic organizers, computers, and curriculum change. *For the Learning of Mathematics*, *9*(3), 37–42. Retrieved July 17, 2024, from http://www.jstor.org/stable/40248161

Tall, D. (2000). Biological brain, mathematical mind & computational computers (how the computer can support mathematical thinking and learning). *Proceedings of the 5th Asian Technology Conference in Mathematics (ATCM 2000, Chiang Mai, Thailand)*. https://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot2000h-plenary-atcm2000.pdf

Valíková, M., & Chalmovianský, P. (2015). Visualisation of complex functions on Riemann sphere. *The Visual Computer*, *31*(2), 141–154. https://doi.org/10.1007/s00371-014-0928-3

Wegert, E. (2016). Visual exploration of complex functions. In T. Qian & L. G. Rodino (Eds.), *Mathematical analysis, probability and applications – plenary lectures* (pp. 253–279). Springer International Publishing.

Yilmaz, R., & Argun, Z. (2018). Role of visualization in mathematical abstraction: The case of congruence concept. *International Journal of Education in Mathematics, Science and Technology (IJEMST)*, *6*(1), 41–57. https://www.ijemst.com/index.php/ijemst/article/view/427