UNIVERSITY OF WOLLONGONG IN DUBAI

ENGG102

Centroids & Moment of Inertia

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Week 8 and 9

Centroids – Procedure for Analysis

The Centroid or the Centre of Gravity of any shape or object is determined using the following steps:

1. Select an appropriate coordinate system (axes)
2. Choose a differential element for integration
3. Locate the element so that it touches any arbitrary point on the curve, such that it defines the boundary of the shape.
4. For areas the element is generally a rectangle of area dA, having a finite length and differential width.
5. Express the area dA, of the element in terms of the coordinates describing the curve.
6. Express the moment arms $\tilde{x}, \tilde{y}, \tilde{z}$ for the centroid of the element in terms of the coordinates describing the curve.
7. Integrate in terms of the **same variable as the differential thickness of the element.**
8. The integral limits will be the 2 extreme locations.

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA}$$

Example 1

Locate the **centroid** of the rod bent into the shape of a parabolic arc

$$y = x^3$$

$$x = y^{1/3}$$

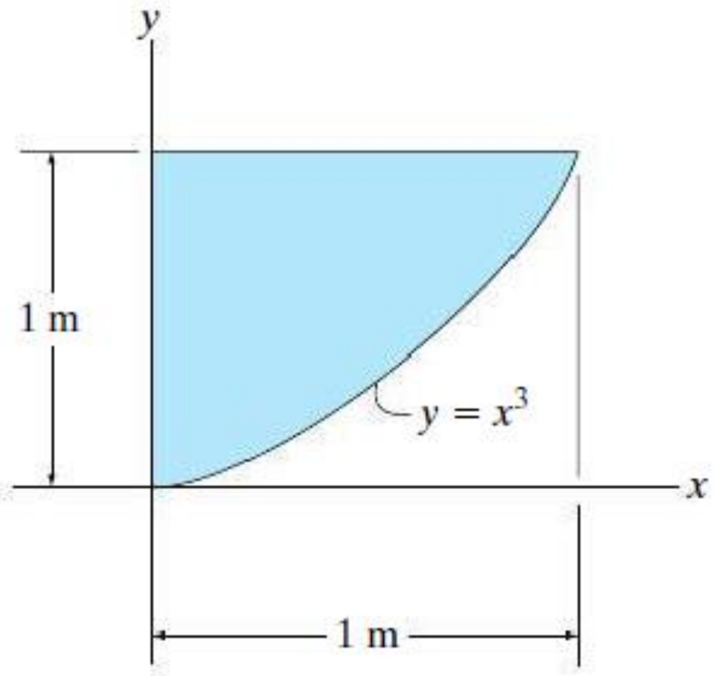
$$\tilde{x} = \frac{1}{2}x$$

$$dA = xdy = y^{1/3}dy$$

$$\bar{x} = \frac{\int \tilde{x}\, dA}{\int dA} = \frac{\frac{1}{2}\int y^{1/3}\, y^{1/3}dy}{\int y^{1/3}dy}$$

$$\bar{x} = \frac{\frac{1}{2}\int_0^1 y^{2/3}dy}{\int_0^1 y^{1/3}dy} = \frac{\left[\frac{3}{10}y^{5/3}\right]_0^1}{\left[\frac{3}{4}y^{4/3}\right]_0^1}$$

$$\boxed{\bar{x} = \frac{4}{10} = 0.4\text{m}}$$



Example 1

Locate the **centroid** of the rod bent into the shape of a parabolic arc

$$y = x^3$$

$$x = y^{1/3}$$

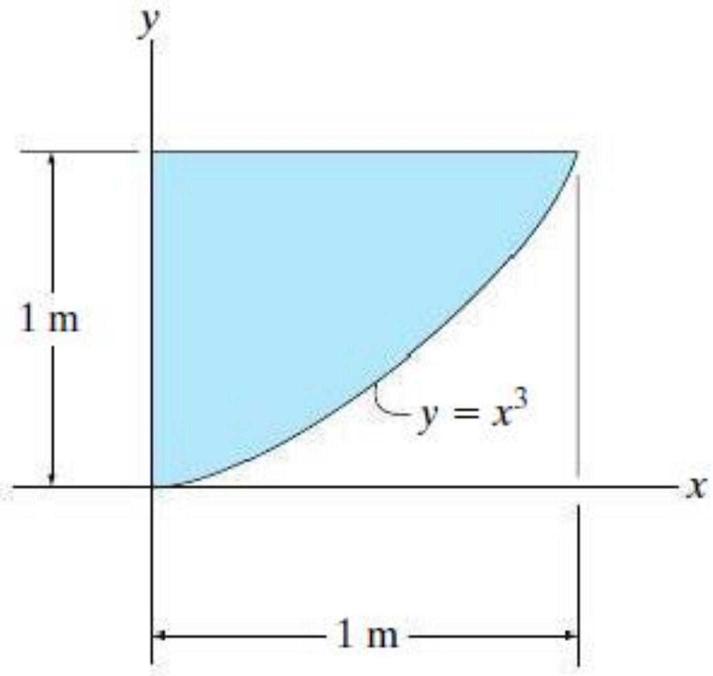
$$\tilde{y} = y$$

$$dA = xdy = y^{1/3}dy$$

$$\bar{y} = \frac{\int \tilde{y}\, dA}{\int dA} = \frac{\int y\, y^{1/3} dy}{\int y^{1/3} dy}$$

$$\bar{y} = \frac{\int_0^1 y^{4/3} dy}{\int_0^1 y^{1/3} dy} = \frac{\frac{3}{7} y^{7/3}}{\frac{3}{4} y^{4/3}}$$

$$\bar{y} = \frac{4}{7} = 0.571\text{m}$$



Example 2

Locate the centroid of the area

$$y = 1 - \frac{1}{4}x^2$$

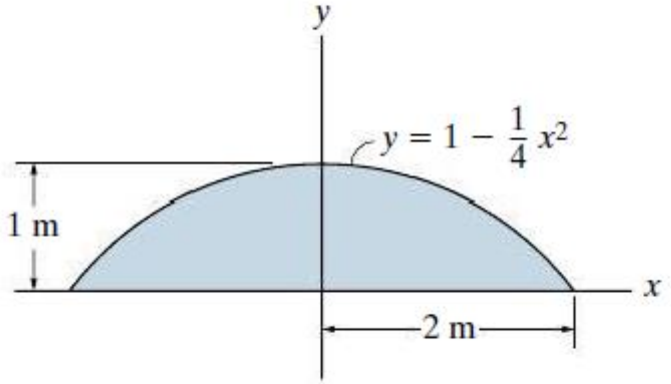
$$\tilde{y} = \frac{1}{2}y$$

$$dA = y dx = 1 - \frac{1}{4}x^2 dx$$

$$\bar{y} = \frac{\int \tilde{y}\, dA}{\int dA} = \frac{\frac{1}{2}\int \left(1 - \frac{1}{4}x^2\right)\left(1 - \frac{1}{4}x^2\right)dx}{\int 1 - \frac{1}{4}x^2 dx}$$

$$\bar{y} = \frac{\frac{1}{2}\int_{-2}^{2}\left(1 - \frac{2}{4}x^2 + \frac{1}{16}x^4\right)dx}{\int_{-2}^{2} 1 - \frac{1}{4}x^2 dx} = \frac{\frac{1}{2}\left[x - \frac{2}{12}x^2 + \frac{1}{80}x^5\right]_{-2}^{2}}{\left[x - \frac{1}{12}x^3\right]_{-2}^{2}}$$

$$\boxed{\bar{y} = 0.4\text{m}}$$



$$\boxed{\bar{x} = 0\text{m}}$$

Centroid – Composite Bodies

The location of the Centroid of Centre of Gravity of a composite geometrical object represented by an area, or volume can be determined using the following procedure:

1. Divide the body or object into a finite number of composite parts that have simpler shapes.
2. If a composite body has a hole, or a geometric region having no material, then consider the composite body without the hole and consider the hole as an additional composite part having negative weight or size.
3. Determine the coordinate axes $\tilde{x}, \tilde{y}, \tilde{z}$ of the center of gravity or centroid of each part.
4. Determine $\bar{x}, \bar{y}, \bar{z}$ by applying the center of gravity equations. $\bar{x} = \frac{\Sigma \tilde{x} A}{\Sigma A}$ $\bar{y} = \frac{\Sigma \tilde{y} A}{\Sigma A}$

TIP 1 – if the part is symmetrical about the axis, the centroid of the part lies on this axis

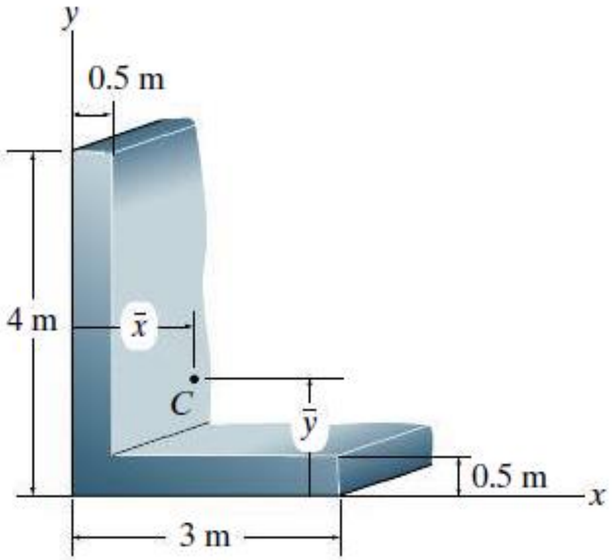
TIP 2 – arrange the calculations is a table form

Example 3

Locate the centroid of the cross-sectional area

$$\bar{x} = \frac{\sum \tilde{x} A}{\sum A} \qquad \bar{y} = \frac{\sum \tilde{y} A}{\sum A}$$

	$\tilde{x}$	$\tilde{y}$	A	$\tilde{x}A$	$\tilde{y}A$
1	0.25	2	$4 \times 0.5 = 2$	$0.25 \times 2 = 0.5$	$2 \times 2 = 4$
2	1.75	0.25	$2.5 \times 0.5 = 1.25$	$1.75 \times 1.25 = 2.1785$	$0.25 \times 1.25 = 0.3125$
		Σ	$\mathbf{3.25}$	$\mathbf{2.6785}$	$\mathbf{4.3125}$



$$\bar{x} = \frac{2.6785}{3.25} = 0.824m \qquad \bar{y} = \frac{4.3125}{3.25} = 1.33m$$

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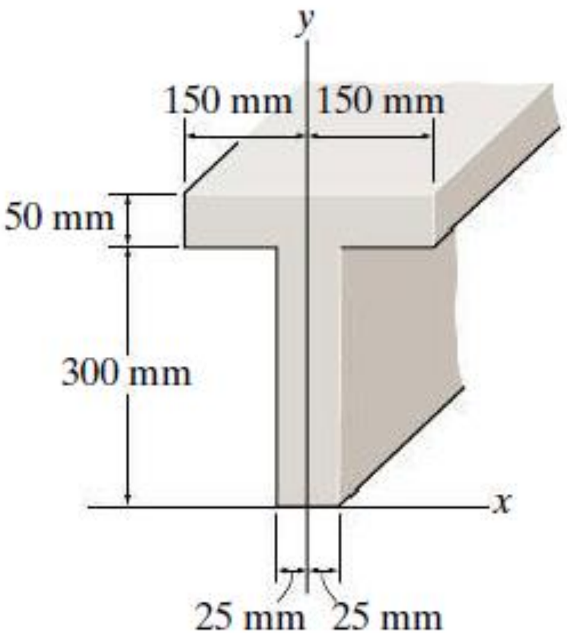
Example 4

Locate the **centroid** of the cross-sectional area

$$\bar{x} = \frac{\sum \tilde{x}A}{\sum A} \qquad \bar{y} = \frac{\sum \tilde{y}A}{\sum A}$$



	$\tilde{x}$	$\tilde{y}$	A	$\tilde{x}A$	$\tilde{y}A$
1	0	150	$300 \times 50 = 15000$	0	$150 \times 15{,}000 = 2250000$
2	0	325	$300 \times 50 = 15000$	0	$325 \times 15000 = 4875000$
		Σ	**30000**	**0**	**7125000**

$$\bar{y} = \frac{7125000}{30000} = 237.5mm$$

Moment of Inertia

- Geometric property of an area, used to determine the strength of a structural member or the location of force acting on a plate submerged in a fluid.
- If the moment of inertia of an area is known about its centroidal axis, then the moment of inertia about a corresponding parallel axis can be determined using the parallel-axis theorem.

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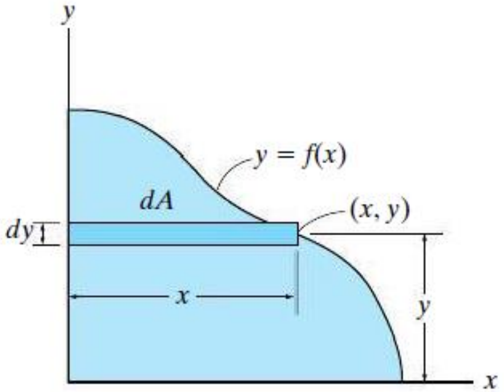
Moment of Inertia – Procedure

- If the curve defining the boundary of the area is expressed as $y = f(x)$, then select a rectangular differential element such that it has a finite length and differential width.
- The element should be located so that it intersects the curve at the arbitrary point (x, y).
- The element should be such that its length is parallel to the axis about which the moment of inertia is computed.
- This situation occurs when the rectangular element is used to determine I_x for the area. Here the entire element is at a distance y from the x axis since it has a thickness dy.

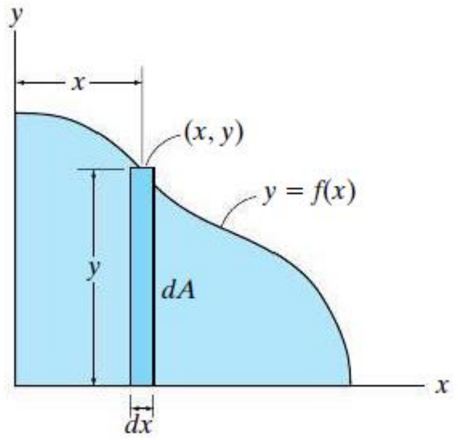


$$I_x = \int_A y^2 \, dA$$

$$I_y = \int_A x^2 \, dA$$

Moment of Inertia – Procedure

- Similary for I_y the element is selected such that it lies at the same distance x from the y axis since it has a thickness dx.



Moment of Inertia – Composite Bodies

The moment of inertia of a composite area about a reference axis:

1. Divide the area into composite parts and determine the perpendicular distance from the centroid of each part to the reference axis.

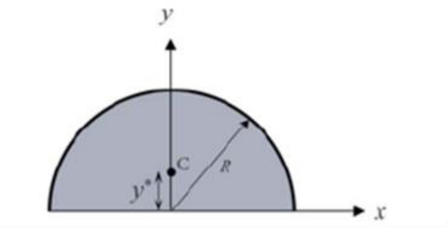
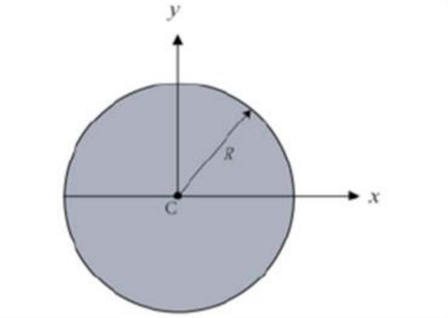
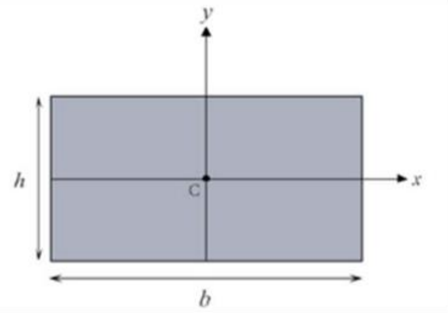
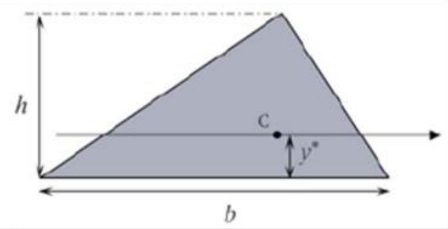
2. If the centroidal axis for each part does not coincide with the reference axis, use the parallel-axis theorem, $I = \bar{I} + Ad^2$, to determine the moment of inertia of the part about the reference axis.

3. Sum all the results to give the moment of inertia for the entire area.

TIP 1 – $\bar{I}$ is determined using the table on the next slide.

$$\boxed{I_x = \bar{I}_{x'} + Ad_y^2} \qquad \boxed{I_y = \bar{I}_{y'} + Ad_x^2}$$

Moment of Inertia

Semicircular area		$x^* = y^* = \frac{4R}{3\pi}$ $A = \frac{1}{2}\pi R^2$	$I_x = \frac{1}{8}\pi R^4$ $I_y = \frac{1}{8}\pi R^4$
Circular area		$A = \pi R^2$	$I_x = \frac{1}{4}\pi R^4$ $I_y = \frac{1}{4}\pi R^4$
Rectangular area		$A = bh$	$I_x = \frac{1}{12}bh^3$ $I_y = \frac{1}{12}hb^3$
Triangular area		$A = \frac{1}{2}bh$	$I_x = \frac{1}{36}bh^3$

Example 5

Determine the moment of inertia of the cross sectional area of the channel with respect to the y-axis.

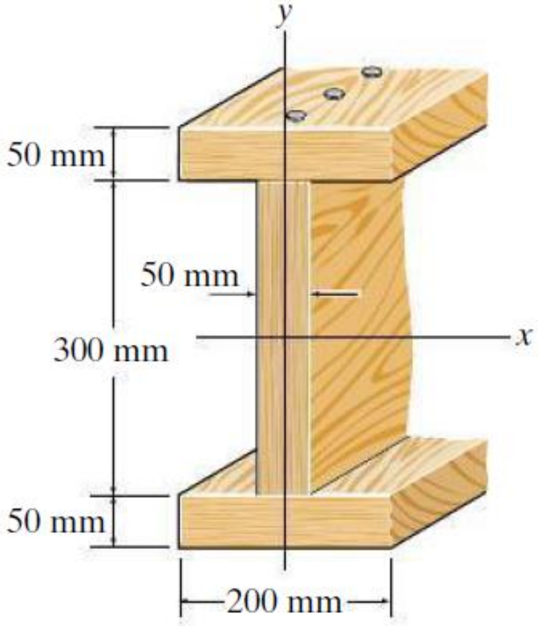
$$I_y = \bar{I}_y + Ad_x^2$$

$$d_x = 0$$

$$\bar{I}_y = \frac{1}{12}hb^3$$

$$I_y = \frac{1}{12}50(200)^3 = 33.33 \times 10^6 mm^4$$

$$I_y = \frac{1}{12}300(50)^3 = 3.125 \times 10^6 mm^4$$

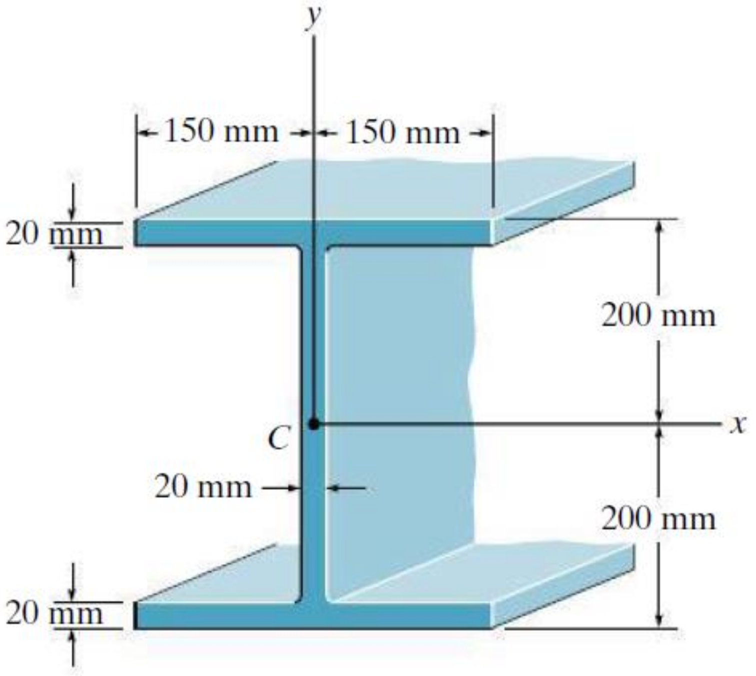


$$\sum I_y = 2 \times 33.33 \times 10^6 + 3.125 \times 10^6$$

$$\sum I_y = 69.79 \times 10^6 mm^4$$

Example 6

Determine the moment of inertia of the composite area about the x-axis
Determine the moment of inertia of the composite area about the y-axis



$$I_x = \bar{I}_x + Ad_y^2$$

$$\bar{I}_x = \frac{1}{12}bh^3$$

$d_y = 190$ $\quad I_x = \frac{1}{12}300(20)^3 + (300 \times 20)(190)^2 = 216.8 \times 10^6 mm^4$

$d_y = 0$ $\quad I_x = \frac{1}{12}20(360)^3 = 77.76 \times 10^6 mm^4$

$$\sum I_x = 2 \times 216.8 \times 10^6 + 77.76 \times 10^6$$

$$\sum I_x = 511.36 \times 10^6 mm^4$$

Example 6

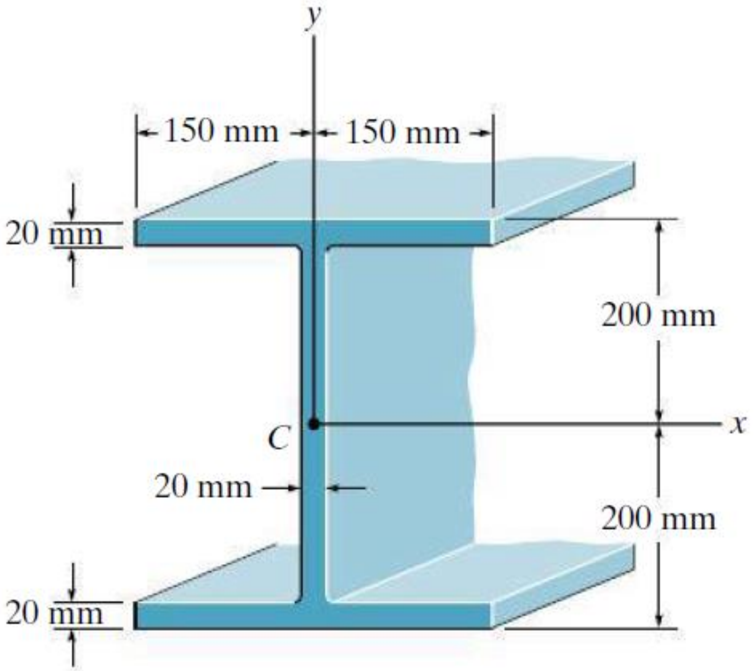
Determine the moment of inertia of the composite area about the x-axis
Determine the moment of inertia of the composite area about the y-axis

$$I_y = \bar{I}_y + Ad_x^2$$

$$\bar{I}_y = \frac{1}{12}hb^3$$

$$d_x = 0 \qquad I_y = \frac{1}{12}20(300)^3 + 0 = 45 \times 10^6 mm^4$$

$$d_x = 0 \qquad I_y = \frac{1}{12}360(20)^3 + 0 = 0.24 \times 10^6 mm^4$$

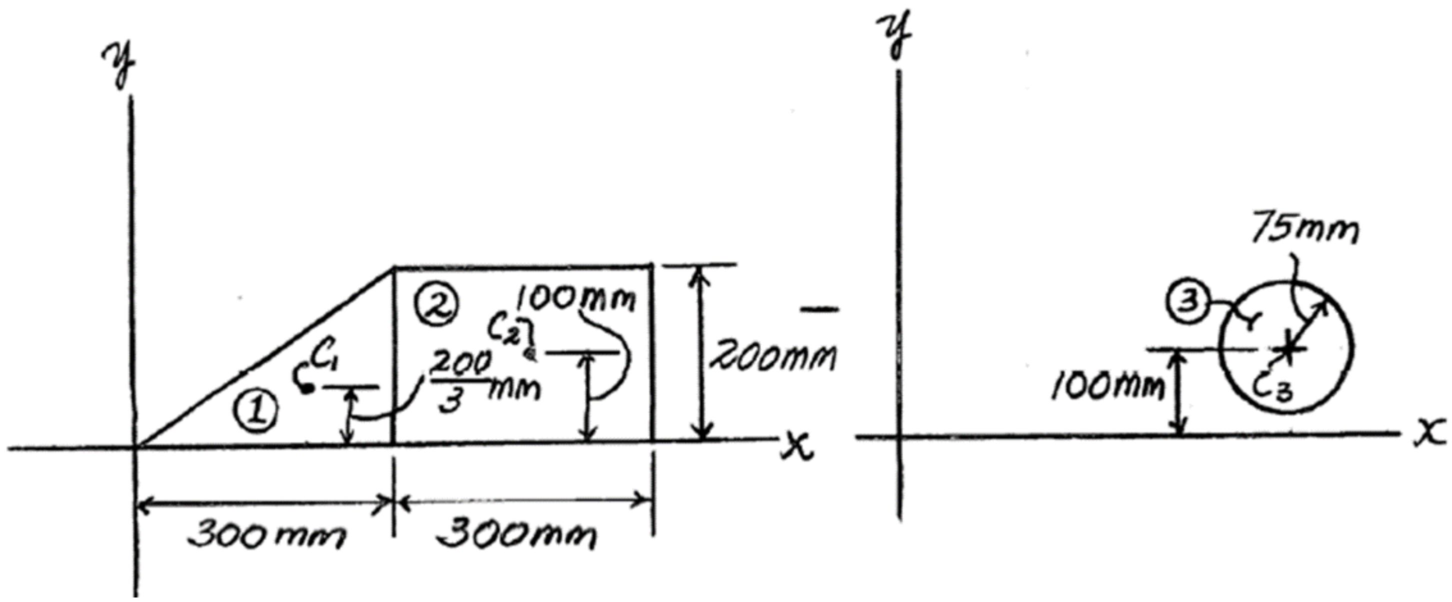


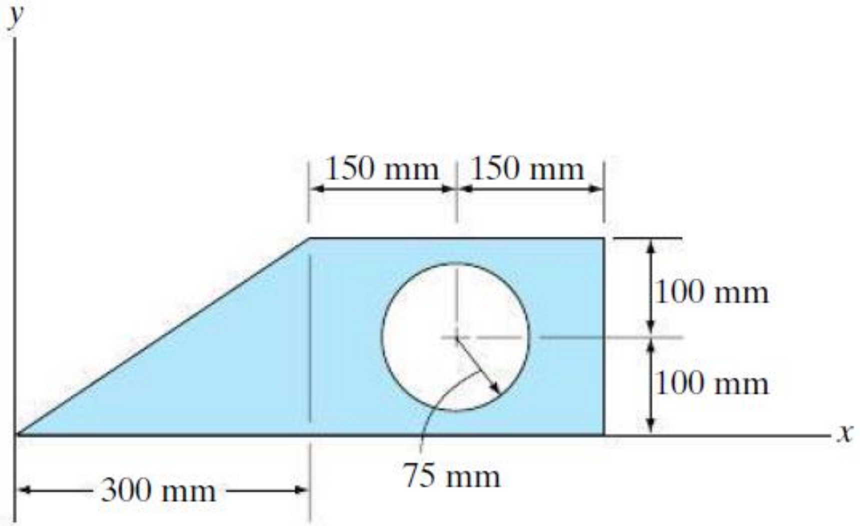
$$\sum I_y = 2 \times 45 \times 10^6 + 0.24 \times 10^6$$

$$\sum I_y = 90.24 \times 10^6 mm^4$$

Example 7

Determine the moment of inertia of the composite area about the x-axis
Determine the moment of inertia of the composite area about the y-axis





Example 7

Determine the moment of inertia of the composite area about the x-axis
Determine the moment of inertia of the composite area about the y-axis

$$I_x = \bar{I}_x + Ad_y^2$$

$$d_y = \frac{200}{3} \qquad I_x = \frac{1}{36}300(200)^3 + \frac{1}{2}(300 \times 200)\left(\frac{200}{3}\right)^2 = 200 \times 10^6 mm^4$$

$$d_y = 100 \qquad I_x = \frac{1}{12}(300)(200)^3 + (300 \times 200)(100)^2 = 800 \times 10^6 mm^4$$

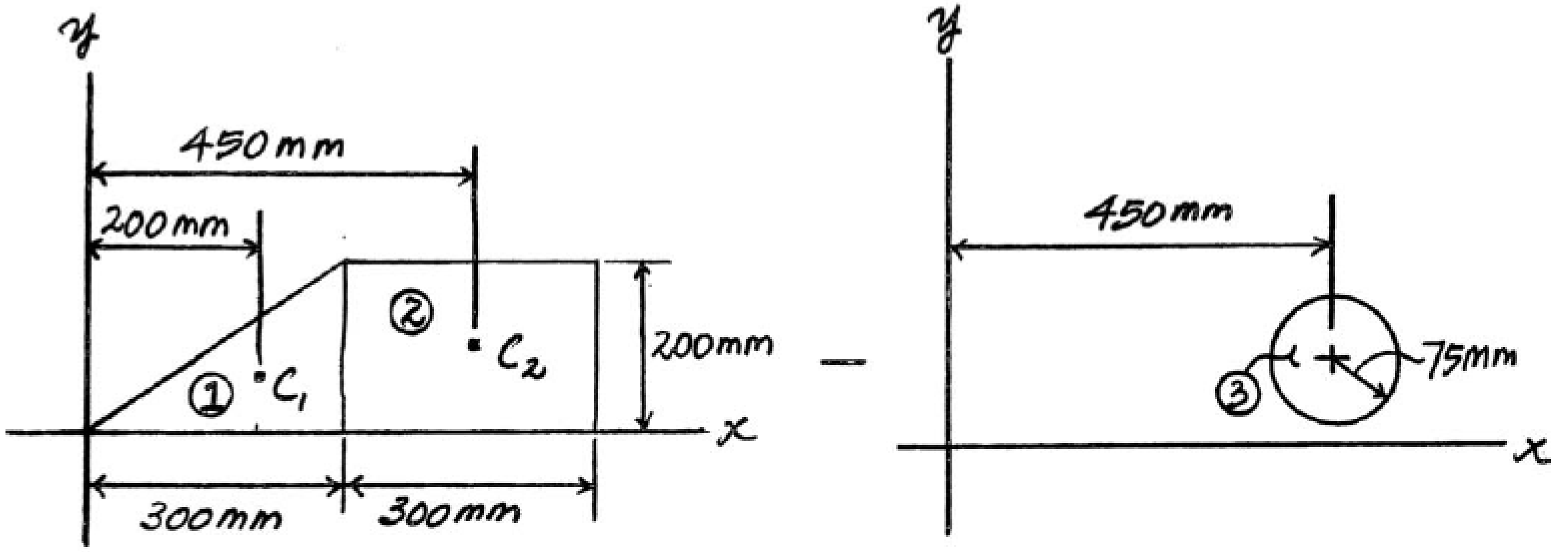
$$d_y = 100 \qquad I_x = \frac{1}{4}(\pi)(75)^4 + (\pi)(75)^2(100)^2 = 201.57 \times 10^6 mm^4$$

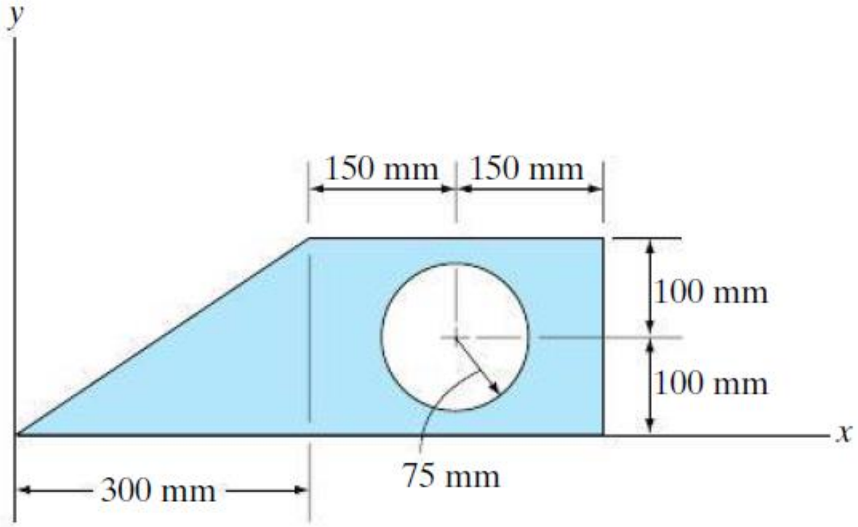
$$\sum I_x = 200 \times 10^6 + 800 \times 10^6 - 201.57 \times 10^6$$

$$\boxed{\sum I_x = 798 \times 10^6 mm^4}$$

Example 7

Determine the moment of inertia of the composite area about the x-axis
Determine the moment of inertia of the composite area about the y-axis





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Example 7

Determine the moment of inertia of the composite area about the x-axis
Determine the moment of inertia of the composite area about the y-axis

$$I_y = \bar{I}_y + Ad_x^2$$

$$d_x = 200 \qquad I_y = \frac{1}{36}200(300)^3 + \frac{1}{2}(200 \times 300)(200)^2 = 1.35 \times 10^9 mm^4$$

$$d_x = 450 \qquad I_y = \frac{1}{12}(200)(300)^3 + (200 \times 300)(450)^2 = 1.26 \times 10^{10} mm^4$$

$$d_x = 450 \qquad I_y = \frac{1}{4}(\pi)(75)^4 + (\pi)(75)^2(450)^2 = 3.58 \times 10^9 mm^4$$

$$\sum I_y = 1.35 \times 10^9 + 1.26 \times 10^{10} - 3.58 \times 10^9$$

$$\sum I_y = 10.37 \times 10^9 mm^4$$