Centroid and Moment of Inertia

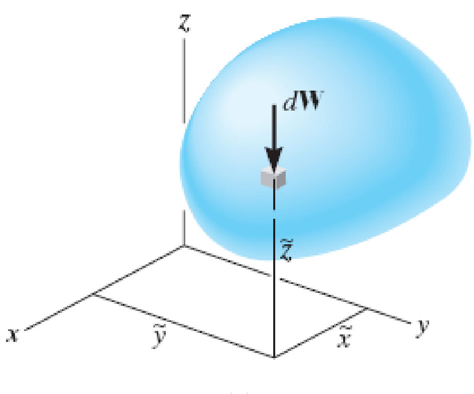
Chapter Outline

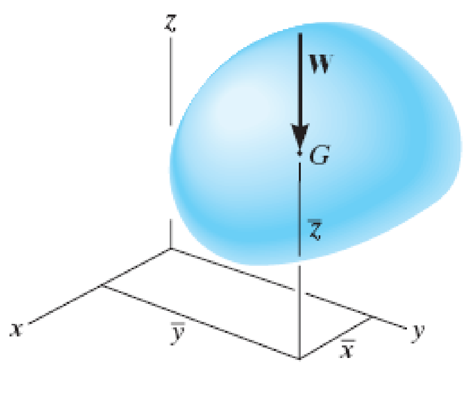
1. Center of gravity
2. Center of area
3. Centroid of composite bodies
4. Moment of Inertia
5. Parallel axis theorem

9.1 Center of Gravity and Center of Mass for a System of Particles

Center of Gravity

- Locates the resultant weight of a system of particles
- Consider system of n particles fixed within a region of space
- The weights of the particles can be replaced by a single (equivalent) resultant weight having defined point G of application





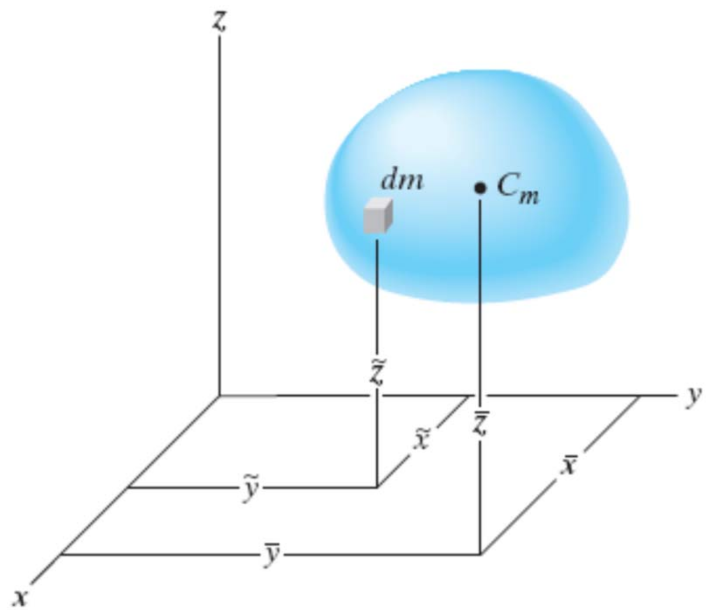
9.1 Center of Gravity and Center of Mass for a System of Particles

Center of Gravity and center of Mass

- A rigid body is composed of an infinite number of particles
- Consider arbitrary particle having a weight of dW = g dm

$$\bar{x} = \frac{\int \tilde{x} dW}{\int dW}; \quad \bar{y} = \frac{\int \tilde{y} dW}{\int dW}; \quad \bar{z} = \frac{\int \tilde{z} dW}{\int dW}$$

$$\bar{x} = \frac{\int \tilde{x} dm}{\int dm}; \quad \bar{y} = \frac{\int \tilde{y} dm}{\int dm}; \quad \bar{z} = \frac{\int \tilde{z} dm}{\int dm}$$

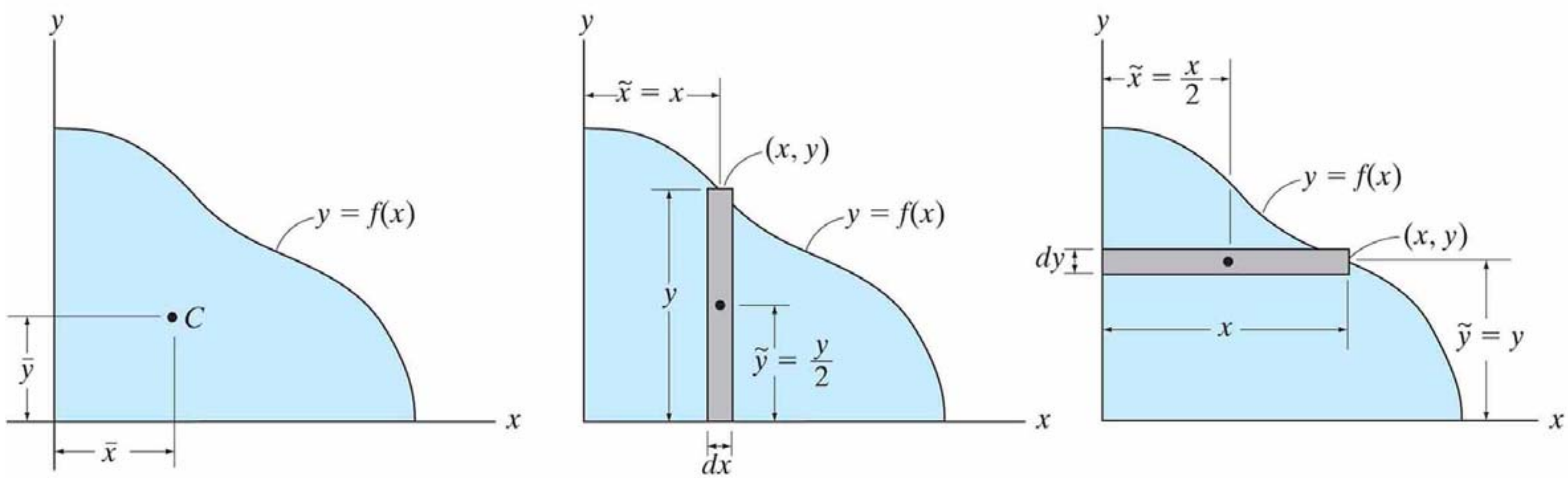


9.1 Center of Gravity and Center of Mass for a System of Particles

Centroid of an Area

- For centroid for surface area of an object, such as plate and shell, subdivide the area into differential elements dA = dx dy

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA}; \quad \bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA}; \quad \bar{z} = \frac{\int_A \tilde{z}\,dA}{\int_A dA}$$



Determine the distance $\bar{y}$ from the x axis to the centroid of the area of the triangle shown in Fig. 9–11.

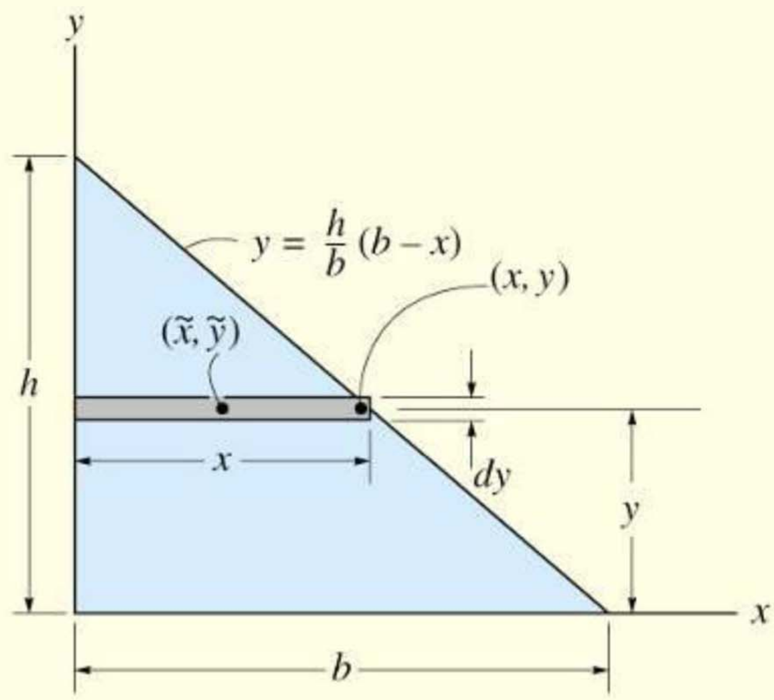
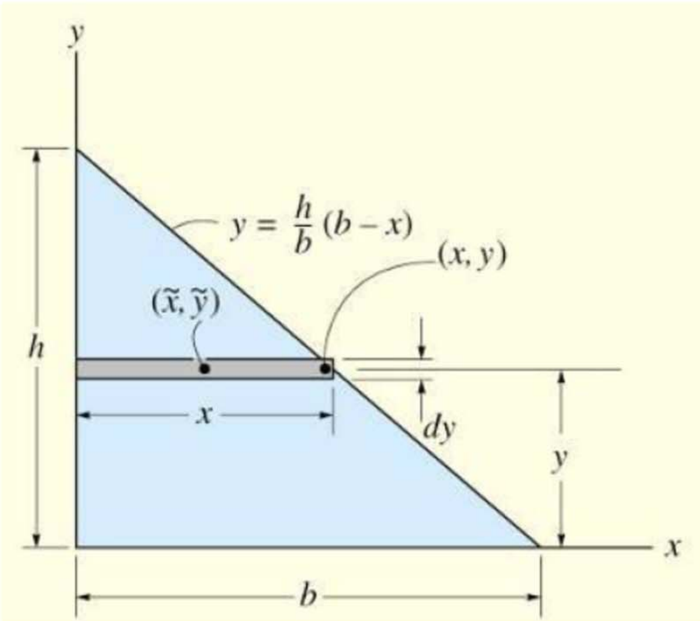


Fig. 9–11



$$x = \frac{h-y}{h} b$$

$$dA = x dy = \frac{h-y}{h} b dy$$

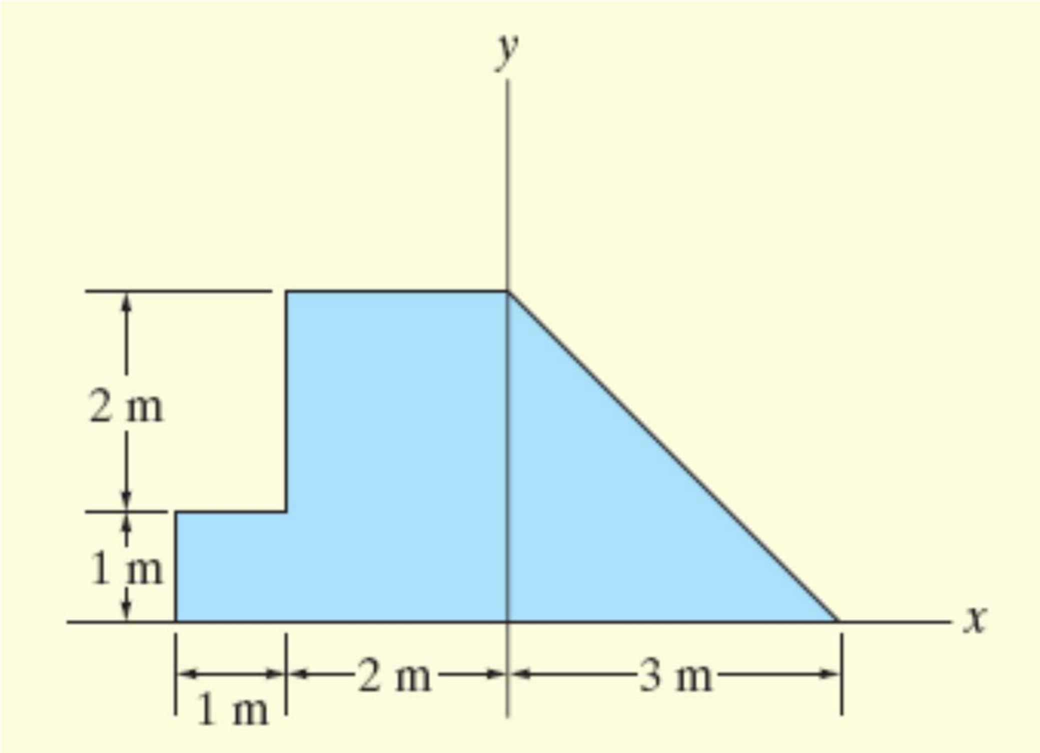
$$\bar{y} = \frac{\int_A \tilde{y} dA}{\int_A dA} = \frac{\int_0^h y \frac{(h-y)}{h} b dy}{\int_0^h \frac{(h-y)}{h} b dy} = \frac{\frac{1}{6} b h^2}{\frac{1}{2} b h} = \frac{h}{3}$$

9.2 Composite Bodies

- Consists of a series of connected "simpler" shaped bodies, which may be rectangular, triangular or semicircular
- A body can be sectioned or divided into its composite parts
- Accounting for finite number of weights

Example 9.10

Locate the centroid of the plate area.

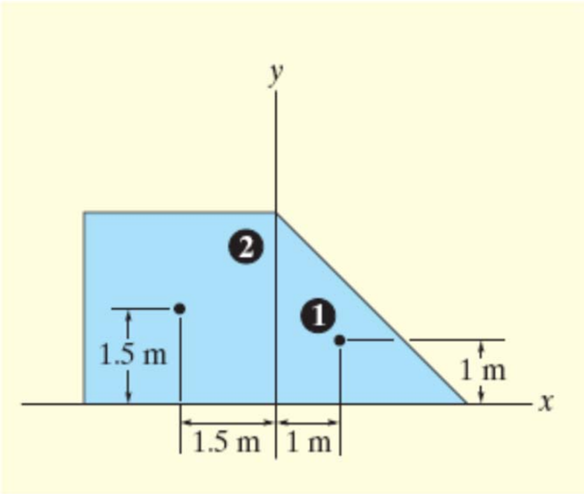


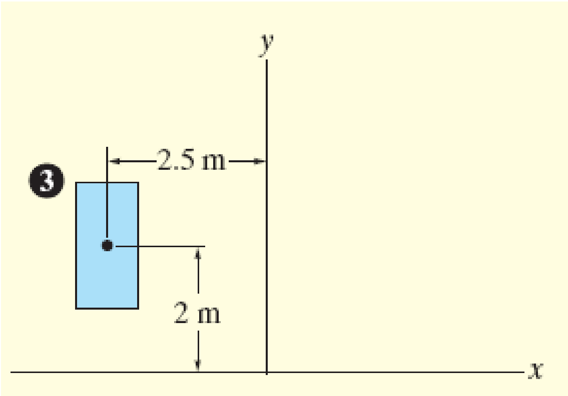
Solution

<u>Composite Parts</u>

Plate divided into 3 segments.

Area of small rectangle considered "negative".





Solution

Moment Arm

Location of the centroid for each piece is determined and indicated in the diagram.

Segment	A (m^2)	$\tilde{x}$ (m)	$\tilde{y}$ (m)	$\tilde{x}A$ (m^3)	$\tilde{y}A$ (m^3)
1	$\frac{1}{2}(3)(3) = 4.5$	1	1	4.5	4.5
2	$(3)(3) = 9$	−1.5	1.5	−13.5	13.5
3	$-(2)(1) = -2$	−2.5	2	5	−4
	$\Sigma A = 11.5$			$\Sigma \tilde{x}A = -4$	$\Sigma \tilde{y}A = 14$

Summations

$$\bar{x} = \frac{\sum \tilde{x}A}{\sum A} = \frac{-4}{11.5} = -0.348mm$$

$$\bar{y} = \frac{\sum \tilde{y}A}{\sum A} = \frac{14}{11.5} = 1.22mm$$

10.1 Definition of Moments of Inertia for Areas

Moment of Inertia

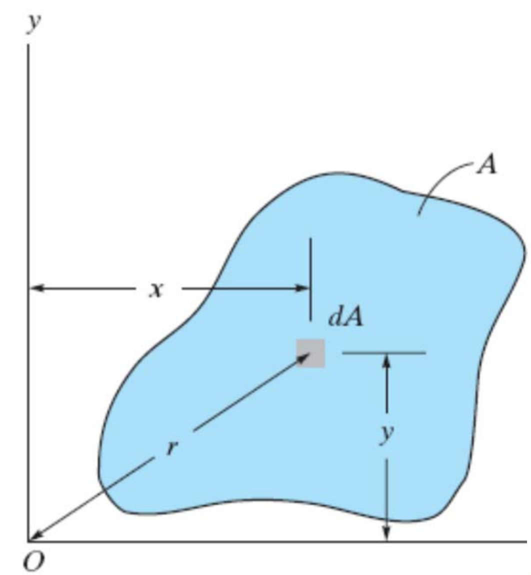
- Consider area A lying in the x-y plane
- Be definition, moments of inertia of the differential plane area dA about the x and y axes

$$dI_x = y^2 dA \quad dI_y = x^2 dA$$

- For entire area, moments of inertia are given by

$$I_x = \int_A y^2 dA$$

$$I_y = \int_A x^2 dA$$



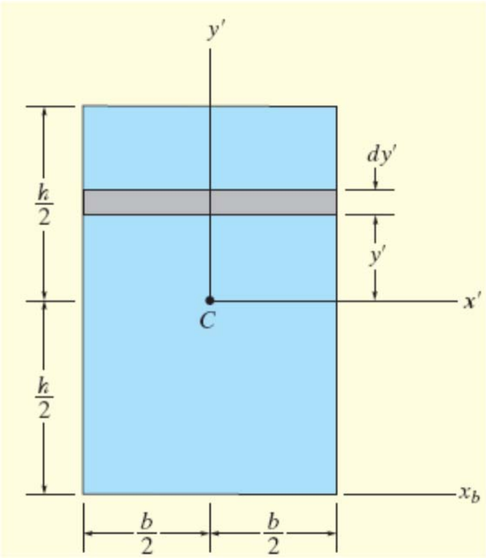
Polar MoI

Second moment of area about z axis

- Relationship between J_O, I_x and I_y is possible since $r^2 = x^2 + y^2$
- J_O, I_x and I_y will always be positive since they involve the product of the distance squared and area
- Units of inertia involve length raised to the fourth power eg m^4, mm^4

Example 10.1

Determine the moment of inertia for the rectangular area with respect to (a) the centroidal x' axis, (b) the axis x_b passing through the base of the rectangular, and (c) the pole or z' axis perpendicular to the x'-y' plane and passing through the centroid C.



Solution

Part (a)

Differential element chosen, distance y′ from x′ axis.

Since dA = b dy′,

$$\bar{I}_x = \int_A y'^2\, dA = \int_{-h/2}^{h/2} y'^2 (b\,dy') = b\int_{-h/2}^{h/2} y'^2\, dy' = \frac{1}{12} bh^3$$

Part (b)

By applying parallel axis theorem,

$$I_{x_b} = \bar{I}_x + Ad^2 = \frac{1}{12} bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3} bh^3$$

Solution

Part (c)

For polar moment of inertia about point C,

$$\bar{I}_{y'} = \frac{1}{12} hb^3$$

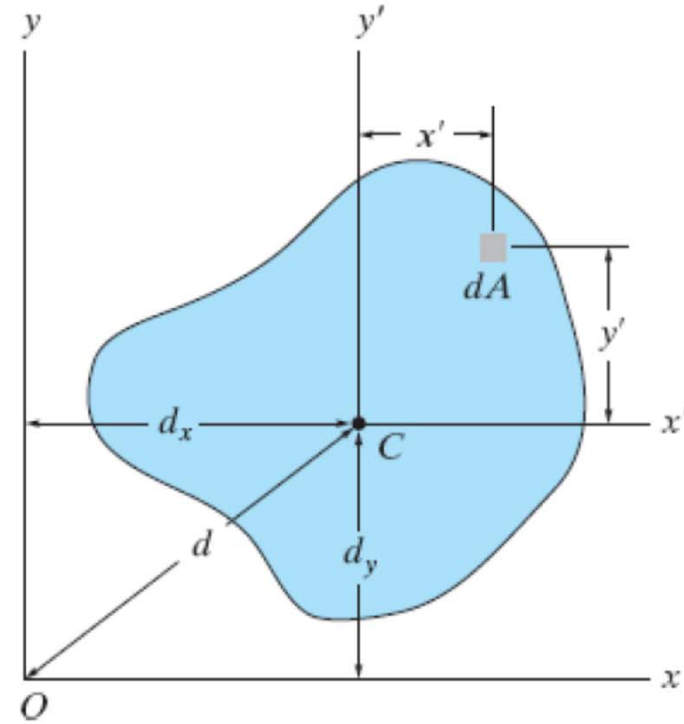
$$J_C = \bar{I}_x + \bar{I}_{y'} = \frac{1}{12} bh(h^2 + b^2)$$

Parallel Axis Theorem

$$I_x = I_{x'} + Ad_y^2$$

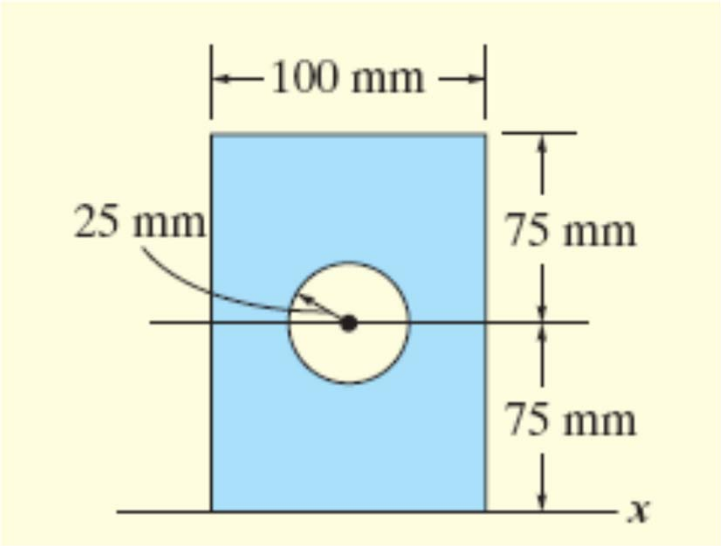
Similarly

$$I_y = I_{y'} + Ad_x^2$$



Example 10.4

Compute the moment of inertia of the composite area about the x axis.

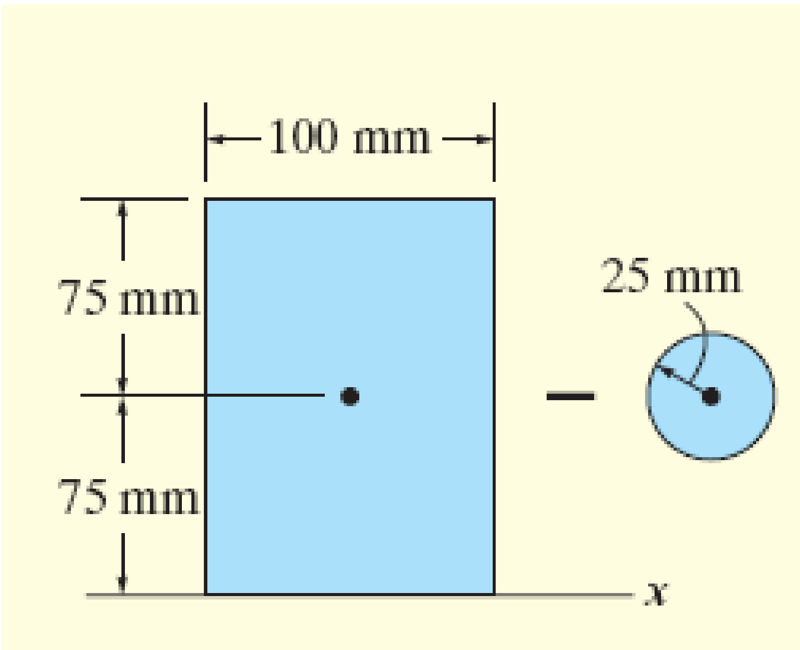


Solution

<u>Composite Parts</u>

Composite area obtained by subtracting the circle form the rectangle.

Centroid of each area is located in the figure below.



Solution

<u>Parallel Axis Theorem</u>

Circle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4\left(10^6\right)mm^4$$

Rectangle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5\left(10^6\right)mm^4$$

Solution

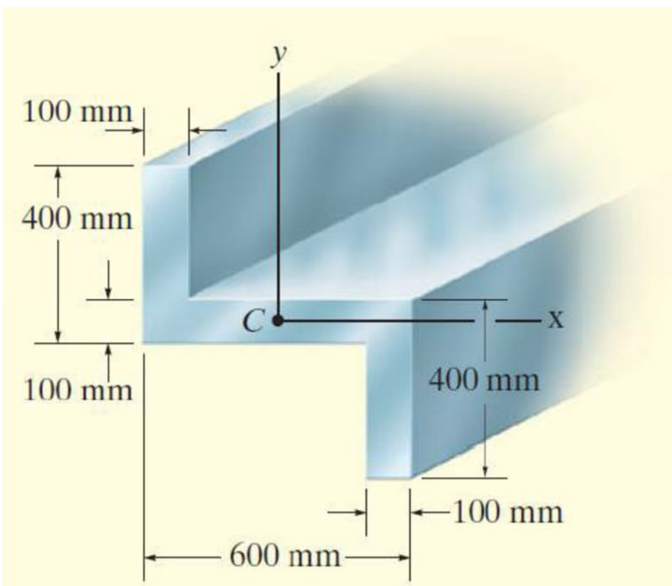
Summation

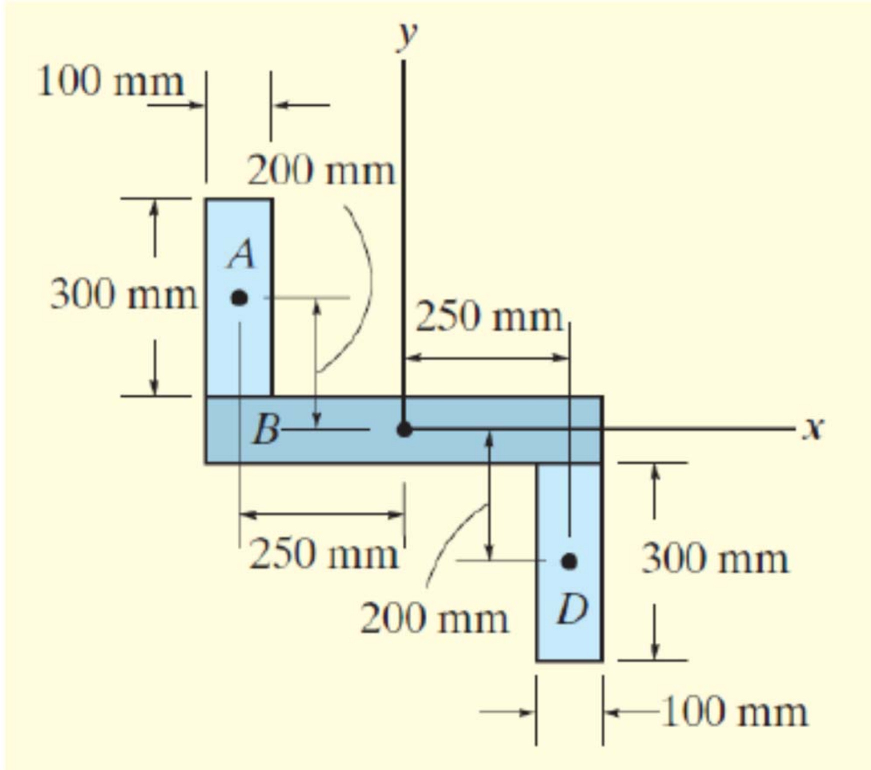
For moment of inertia for the composite area,

$$I_x = -11.4\left(10^6\right) + 112.5\left(10^6\right)$$
$$= 101\left(10^6\right) mm^4$$

Q-Find out the MoI about x and y axis.

The second moments of area of the compound area about the *x* and *y* axes are





y
100 mm
200 mm
A
300 mm
250 mm
B
x
250 mm
200 mm
300 mm
D
100 mm

Rectangles A and D

$$I_x = \bar{I}_{x'} + Ad_y^2 = \frac{1}{12}(100)(300)^3 + (100)(300)(200)^2$$

$$= 1.425(10^9)\ \text{mm}^4$$

$$I_y = \bar{I}_{y'} + Ad_x^2 = \frac{1}{12}(300)(100)^3 + (100)(300)(250)^2$$

$$= 1.90(10^9)\ \text{mm}^4$$

Rectangle B

$$I_x = \frac{1}{12}(600)(100)^3 = 0.05(10^9)\ \text{mm}^4$$

$$I_y = \frac{1}{12}(100)(600)^3 = 1.80(10^9)\ \text{mm}^4$$

Summation. The moments of inertia for the entire cross section are thus

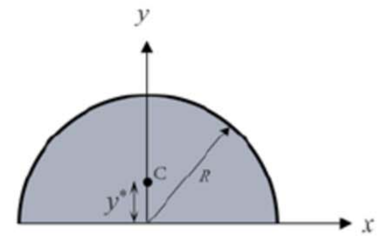
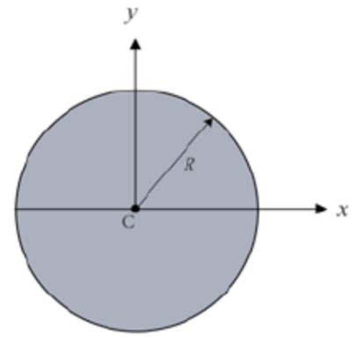
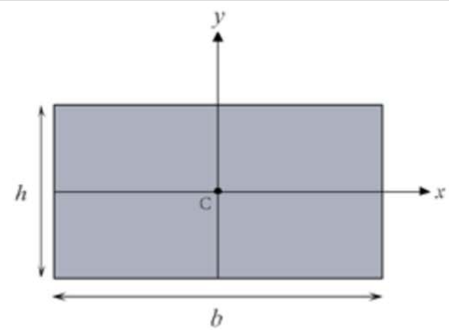
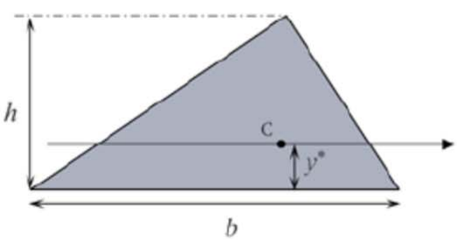
$$I_x = 2[1.425(10^9)] + 0.05(10^9)$$

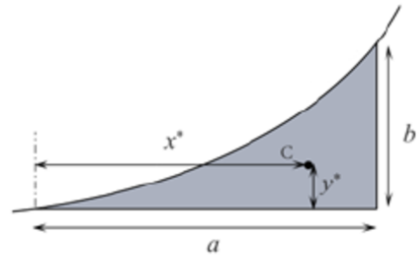
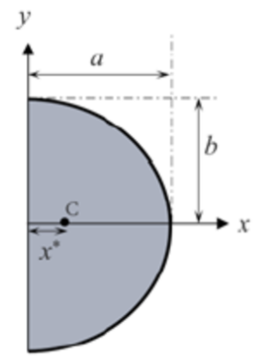
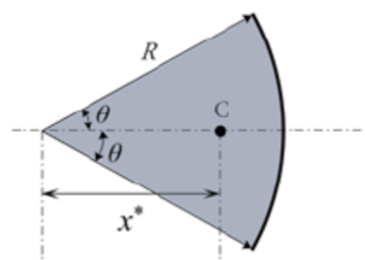
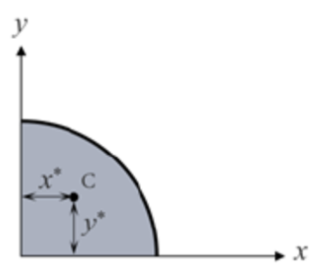
$$= 2.90(10^9)\ \text{mm}^4 \qquad \textit{Ans.}$$

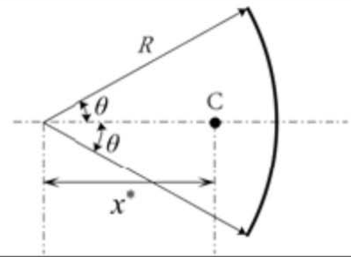
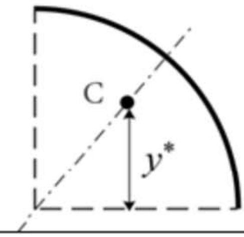
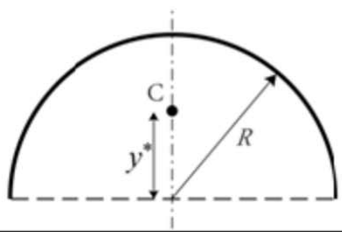
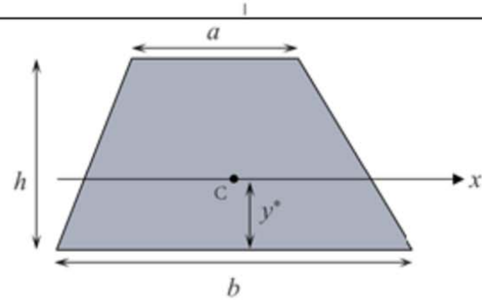
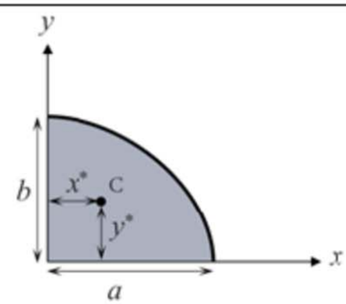
$$I_y = 2[1.90(10^9)] + 1.80(10^9)$$

$$= 5.60(10^9)\ \text{mm}^4 \qquad \textit{Ans.}$$

Table for centroids and second moment of areas

Semicircular area		$x^* = y^* = \frac{4R}{3\pi}$ $A = \frac{1}{2}\pi R^2$	$I_x = \frac{1}{8}\pi R^4$ $I_y = \frac{1}{8}\pi R^4$
Circular area		$A = \pi R^2$	$I_x = \frac{1}{4}\pi R^4$ $I_y = \frac{1}{4}\pi R^4$
Rectangular area		$A = bh$	$I_x = \frac{1}{12}bh^3$ $I_y = \frac{1}{12}hb^3$
Triangular area		$A = \frac{1}{2}bh$	$I_x = \frac{1}{36}bh^3$

Exparabolic area		$x^* = \frac{3}{4}a$ $y^* = \frac{3}{10}b$	$A = \frac{1}{3}ab$
Parabolic area		$x^* = \frac{2}{5}a$	$A = \frac{4}{3}ab$
Circular sector area		$x^* = \frac{2}{3}\frac{R\sin(\theta)}{\theta}$ $A = \theta R^2$	$I_x = \frac{1}{4}R^4\left(\theta - \frac{1}{2}\sin(2\theta)\right)$ $I_y = \frac{1}{4}R^4\left(\theta + \frac{1}{2}\sin(2\theta)\right)$
Quarter circle area		$x^* = y^* = \frac{4R}{3\pi}$ $A = \frac{1}{4}\pi R^2$	$I_x = \frac{1}{16}\pi R^4$ $I_y = \frac{1}{16}\pi R^4$

Circular arc segment		$x^* = \dfrac{R\sin(\theta)}{\theta}$	$L = 2R\theta$
Quarter-circle arc		$y^* = \dfrac{2R}{\pi}$	$L = \dfrac{\pi R}{2}$
Semi-circle arc		$y^* = \dfrac{2R}{\pi}$	$L = \pi R$
Trapezoidal area		$y^* = \dfrac{1}{3}\left(\dfrac{2a+b}{a+b}\right)h$	$A = \dfrac{1}{2}h(a+b)$
Semi parabolic area		$x^* = \dfrac{2}{5}a$ $y^* = \dfrac{3}{8}b$	$A = \dfrac{2}{3}ab$