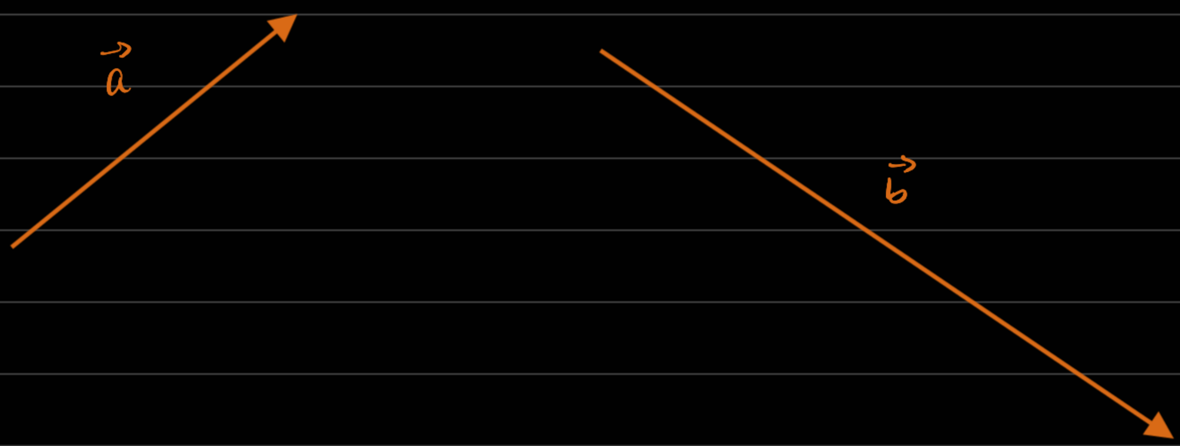
Vectors

A vector is a directed line segment characterized by
- Its direction
- Its orientation
- Its magnitude

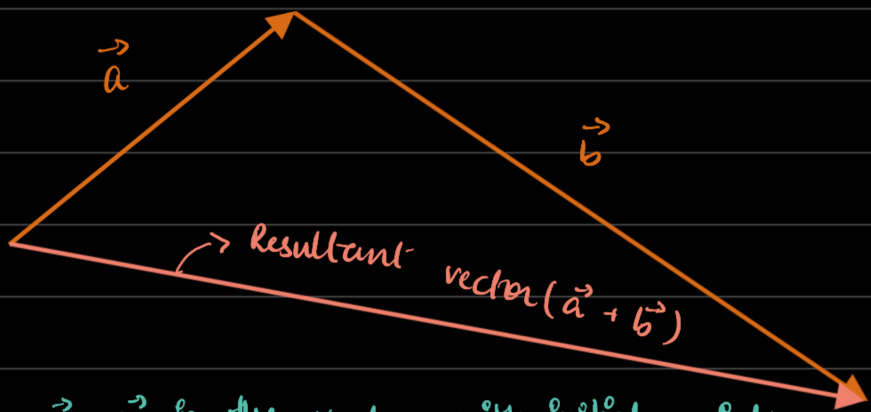
Addition of Vectors

Let $\vec{a}$ and $\vec{b}$ be two vectors.



To find $\vec{a}+\vec{b}$ do the following

1. Move vector $\vec{b}$ so that its initial point is at the terminal point of $\vec{a}$



2. $\vec{a}+\vec{b}$ is the vector with initial point the initial point of $\vec{a}$ and terminal point the terminal point of $\vec{b}$.

Scalar Multiplication

Let $\vec{a}$ be a vector and λ be a number.

$\lambda\vec{a}$ is the vector with following characteristics

- $\lambda\vec{a}$ has the direction of $\vec{a}$
- If λ is positive, then $\lambda\vec{a}$ and $\vec{a}$ have the same orientation.
- If λ is negative, then $\lambda\vec{a}$ has the opposite orientation of $\vec{a}$.
- $\|\lambda\vec{a}\| = |\lambda|\,\|\vec{a}\|$

Vectors in Component Form

Position Vector

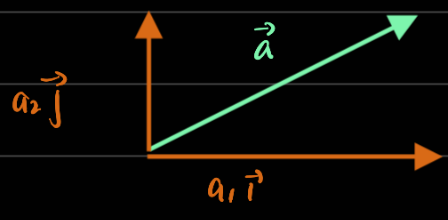
Any vector, whose initial point is at the origin is called a position vector.

Unit Vector

Any vector, whose magnitude is 1 is called a unit vector.

On the xy-plane, the unit vectors that give the direction of the x and y axis are respectively denoted by $\vec{i}$ and $\vec{j}$

$$\vec{a} = a_1\vec{i} + a_2\vec{j}$$
$$= \langle a_1, a_2 \rangle$$



Any vector $\vec{a}$ on the two dimensional plan can be represented in component form as

$$\vec{a} = a_1\vec{i} + a_2\vec{j} = \langle a_1, a_2 \rangle$$

Any vector $\vec{a}$ in the three dimensional plan can be represented in component form as

$$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k} = \langle b_1, b_2, b_3 \rangle$$

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ then

$$\vec{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Operations of vectors given in component form

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ and λ a number

$$\vec{a} + \vec{b} = \langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle$$
$$= \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\lambda\vec{a} = \lambda \langle a_1, a_2, a_3 \rangle$$
$$= \langle \lambda a_1, \lambda a_2, \lambda a_3 \rangle$$

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Example

Let $\vec{a} = \langle 2, -1, 5 \rangle$

- Find a unit vector $\vec{b}$ having the same direction as $\vec{a}$

$$\|\vec{a}\| = \sqrt{4+1+25}$$
$$= \sqrt{30}$$

$$\vec{b} = \frac{\vec{a}}{\|\vec{a}\|} = \overset{\lambda}{\boxed{\frac{1}{\|\vec{a}\|}}}\, a$$

$$\vec{b} = \frac{1}{\sqrt{30}} \langle 2, -1, 5 \rangle$$

$$= \left\langle \frac{2}{\sqrt{30}}, \frac{-1}{\sqrt{30}}, \frac{5}{\sqrt{30}} \right\rangle$$

$$\|\vec{b}\| = \left\| \frac{1}{\|\vec{a}\|} \vec{a} \right\| = \frac{1}{\cancel{\|\vec{a}\|}} \cdot \cancel{\|\vec{a}\|}$$
$$= 1$$

- Find a vector $\vec{b}$ with magnitude 15 having the same direction as $\vec{a}$.

$$\vec{b} = \frac{15}{\sqrt{30}} \langle 2, -1, 5 \rangle$$

$$= \left\langle \sqrt{30}, \frac{-15}{\sqrt{30}}, \frac{75}{\sqrt{30}} \right\rangle$$

Example

Find the terminal point of the vector $\vec{a}$ given that the initial point is $P(-2, 3, 5)$

$\vec{a} = \langle 1, 2, -4 \rangle$

Let $Q \langle x, y, z \rangle$ be the terminal point

$$\vec{PQ} = \vec{a}$$

$$\langle x+2, y-3, z-5 \rangle = \langle 1, 2, -4 \rangle$$

$x + 2 = 1$ $\quad$ $y - 3 = 2$ $\quad$ $z - 5 = -4$

$x = -1$ $\quad$ $y = 5$ $\quad$ $z = 1$

$\therefore$ Terminal point is $(-1, 5, 1)$

Dot Product

Valid in any dimension

Cross

Valid only in 3D

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ be two vectors.

The dot product of $\vec{a} \cdot \vec{b}$ denoted by

$\vec{a} \cdot \vec{b} \longrightarrow$ is a number

is the number given by

$$\vec{a} \cdot \vec{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example

Evaluate $\langle 2, -1, 6 \rangle \cdot \langle 3, 5, -1 \rangle$

$6 - 5 - 6$

$= -5$

Properties of Dot Product

$\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

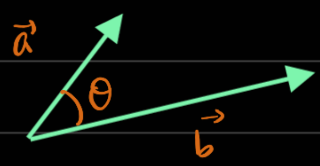
$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

$(\lambda \vec{a}) \cdot \vec{b} = \vec{a} \cdot (\lambda \vec{b}) = \lambda (\vec{a} \cdot \vec{b})$

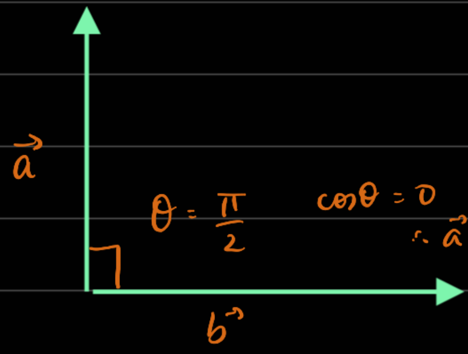
$\vec{0} \cdot \vec{a} = 0$ $\qquad \vec{0} = \langle 0, 0, 0 \rangle$

The angle between two non-zero vectors is the angle θ $(0 \leq \theta \leq \pi)$ between their respective position



$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|}$$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$



$\cos \theta = 0$

$\therefore \vec{a} \cdot \vec{b} = 0$

If two vectors are perpendicular, $\vec{a} \cdot \vec{b} = 0$

Two vectors $\vec{a}$ and $\vec{b}$ are perpendicular if and only if $\vec{a} \cdot \vec{b} = 0$

Example

Let $\vec{a} = \langle 1, 0, -1 \rangle$ and $b = \langle 5, 3, 2 \rangle$. Find the angle between $\vec{a}$ and $\vec{b}$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$$

$$5 + 0 - 2 = (\sqrt{2})(\sqrt{38}) \cos\theta$$

$$\cos\theta = \frac{3}{\sqrt{76}}$$

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{76}}\right)$$

Example

Let $\vec{a}$ be a unit vector. $\|\vec{b}\| = 3$

If the angle θ between $\vec{a}$ and $\vec{b}$ is $\theta = \frac{2\pi}{3}$

Find $\|2\vec{a} - \vec{b}\|$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|}$$

$$\cos\frac{2\pi}{3} = \frac{\vec{a} \cdot \vec{b}}{1 \times 3}$$

$$\frac{-1}{2} = \vec{a} \cdot \vec{b}$$

$$\vec{a} \cdot \vec{b} = -\frac{3}{2}$$

$$\|2\vec{a} - \vec{b}\|^2 = (2\vec{a} - \vec{b}) \cdot (2\vec{a} - \vec{b})$$

$$= 4\|\vec{a}\|^2 - 4\,\vec{a} \cdot \vec{b} + \|\vec{b}\|^2$$

$$= 4 - \cancel{4}^{2}\left(\frac{-3}{\cancel{2}}\right) + 9$$

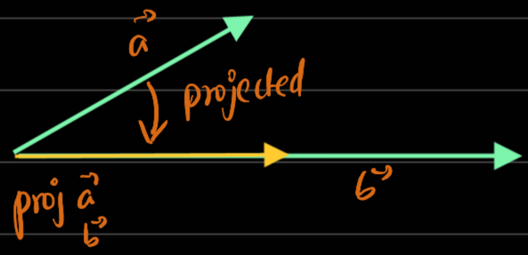
$$= 4 + 6 + 9$$

$$= 19$$

$\|2a - b\| = \sqrt{19}$

Projection Vector

Using dot product, we can compute $\text{proj}_{\vec{b}}\,\vec{a}$ as following



$$\text{proj}_{\vec{b}}\,\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$$

Example

Let $\vec{a} = \langle 1, -1, 2 \rangle$ $\vec{b} = \langle -1, 2, 1 \rangle$

Find $\text{proj}_{\vec{b}}\,\vec{a}$, $\vec{a} - \text{proj}_{\vec{b}}\,\vec{a}$ and verify that $\vec{a} - \text{proj}_{\vec{b}}\,\vec{a}$ is perpendicular to $\vec{b}$

$$\text{proj}_{\vec{b}}\,\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$$

$$= \left(\frac{-1-2+2}{6}\right)\vec{b}$$

$$= \left(\frac{-1}{6}\right)\vec{b}$$

$$= \left\langle \frac{1}{6}, \frac{-1}{3}, \frac{-1}{6} \right\rangle$$

$$\vec{a} - \text{proj}_{\vec{b}}\,\vec{a} = \langle 1, -1, 2 \rangle - \left\langle \frac{1}{6}, \frac{-1}{3}, \frac{-1}{6} \right\rangle$$

$$= \left\langle \frac{5}{6}, \frac{-2}{3}, \frac{13}{6} \right\rangle$$

Verifying perpendicularity

$$\left\langle \frac{5}{6}, \frac{-2}{3}, \frac{13}{6} \right\rangle \cdot \langle -1, 2, 1 \rangle$$

$$= \frac{-5}{6} - \frac{4}{3} + \frac{13}{6}$$

$$= \frac{-5 - 8 + 13}{6}$$

$$= \frac{0}{6}$$

$$= \underline{\underline{0}}$$

$\therefore$ $\vec{a} - \text{proj}_{\vec{b}}\,\vec{a}$ is perpendicular to $\vec{b}$