§ Integrals of Rational Functions by PFD.

$$\int \frac{P(x)}{Q(x)}\,dx$$ P and Q are polynomials.

* deg $P(x) <$ deg $Q(x)$
  * Rule 1 ✓
  * Rule 2
  * Rule 3
  * Rule 4

Rule 2. If the denominator $Q(x)$ can be decomposed as a product of linear factors some of which are repeated

$$Q(x) = a(x-c_1)\cdots(x-b)^n\cdots$$

The partial fraction decomposition is given by

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-c_1} + \cdots + \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + \cdots + \frac{B_n}{(x-b)^n} + \cdots$$

$$\frac{x^2+x+1}{(x-2)(x+3)^2(x-1)^3} = \frac{A}{x-2} + \frac{B_1}{x+3} + \frac{B_2}{(x+3)^2} + \frac{C_1}{x-1} + \frac{C_2}{(x-1)^2} + \frac{C_3}{(x-1)^3}$$

Exple Evaluate

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx .$$

$$x^3+2x^2+x = x\,(x^2+2x+1)$$
$$= x\,(x+1)^2$$

$$\frac{5x^2+20x+6}{x(x+1)^2} = \frac{A}{x}+\frac{B_1}{x+1}+\frac{B_2}{(x+1)^2}$$

$$5x^2+20x+6 = A\,(x+1)^2 + B_1 x(x+1) + B_2 x$$

$$5x^2+20x+6 = A\,(x^2+2x+1) + B_1\,(x^2+x) + B_2 x$$

$$5x^2+20x+6 = (A+B_1)x^2 + (2A+B_1+B_2)x + A$$

$$\begin{cases} A+B_1 = 5 \\ 2A+B_1+B_2 = 20 \\ A = 6 \end{cases}$$

$A = 6$

$B_1 + 6 = 5$

$B_1 = -1$

$$2(6)+(-1)+B_2 = 20$$

$B_2 = 9$

$$12-1+B_2 = 20$$

$$B_2 = 20-11 = 9$$

$$\int \frac{5x^2+20x+6}{x(x+1)^2}\,dx = \int \left[\frac{6}{x} - \frac{1}{x+1} + 9\,(x+1)^{-2}\right] dx$$

$$= 6\ln|x| - \ln|x+1| - \frac{9}{x+1} + C$$

$$\int (1)(x+1)^{-2}\,dx = \frac{(x+1)^{-2+1}}{-2+1} + C = (-)\frac{1}{x+1}$$

$$\int u'u^n\,dx = \frac{u^{n+1}}{n+1} + C$$

* Irreducible quadratic function

If $b^2 - 4ac < 0$, then the quadratic function $ax^2 + bx + c$ is called irreducible.

Rule 3 : If the decomposition of $Q(x)$ contains distinct irreducible quadratic factors

$$Q(x) = m(x-c_1)\cdots(x-c_2)^n\cdots(ax^2+bx+c)\cdots$$

with $b^2 - 4ac < 0$

The partial fraction decomposition is given by

$$\frac{P(x)}{Q(x)} = \frac{A}{x-c_1} + \cdots + \frac{B_1}{x-c_2} + \frac{B_2}{(x-c_2)^2} + \cdots + \frac{B_n}{(x-c_2)^n} + \frac{Cx+D}{ax^2+bx+c} + \cdots$$

Recall $$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

$$\frac{2x^3+x-9}{x^3(x+1)(x^2+x+1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{B}{x+1} + \frac{Cx+D}{x^2+x+1}$$

<u>Ex p'le</u> Evaluate

$$\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,dx.$$

$$\frac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$$

$$2x^3-4x-8 = A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)x(x-1)$$

$$= A(x^3+4x-x^2-4) + B(x^3+4x) + (Cx+D)(x^2-x)$$

$$= A(x^3-x^2+4x-4) + B(x^3+4x) + C(x^3-x^2) + D(x^2-x)$$

$$= (A+B+C)x^3 + (-A-C+D)x^2 + (4A+4B-D)x + -4A$$

$$\begin{cases} A+B+C = 2 \\ -A-C+D = 0 \\ 4A+4B-D = -4 \\ -4A = -8 \end{cases} \longrightarrow A = \frac{8}{4} = 2 \qquad A = 2$$

$$\begin{cases} \cancel{2}+B+C = \cancel{2} \\ -C+D = 2 \\ 4B-D = -12 \end{cases} \iff \begin{cases} B = -C \\ -C+D = 2 \\ -4C-D = -12 \end{cases}$$

$$D = 2+C = 2+2 = 4 \qquad D = 4$$

$$-5C = -10$$

$$C = 2 \qquad B = -2$$

$$\int \frac{2x^3-4x-8}{x(x-1)(x^2+4)}\,dx = \int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x+4}{x^2+4} \right) dx$$

$$\int \frac{2x+4}{x^2+4}\,dx = \int \frac{2x}{x^2+4}\,dx + \int \frac{4}{x^2+4}\,dx$$

$$= \ln(x^2+4) + 4\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= 2\ln|x| - 2\ln|x-1| + \ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

Rule 4 If the decomposition of $Q(x)$ contains repeated irreducible quadratic factors

$$Q(x) = \cdots\cdots\,(ax^2+bx+c)^m \cdots\cdots$$

$$b^2-4ac<0$$

then the partial fraction decomposition is given by

$$\frac{P(x)}{Q(x)} = \cdots\cdots + \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots$$

$$\cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

Exple give the partial fraction decomposition of

$$f(x) = \frac{5x^4 - 6x + 9}{x(x+3)^3(x^2+1)(x^2+x+1)^3}.$$

(Do NOT find the value of the coefficients)

$$= \frac{A}{x} + \frac{B_1}{x+3} + \frac{B_2}{(x+3)^2} + \frac{B_3}{(x+3)^3} + \frac{Cx+D}{x^2+1} +$$

$$\frac{E_1x+F_1}{x^2+x+1} + \frac{E_2x+F_2}{(x^2+x+1)^2} + \frac{E_3x+F_3}{(x^2+x+1)^3}$$

Ex ple Evaluate

$$\int \frac{8x^3 + 13x}{(x^2+2)^2}\,dx$$

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{A_1x + B_1}{x^2+2} + \frac{A_2x+B_2}{(x^2+2)^2}$$

$$8x^3 + 13x = (A_1x + B_1)(x^2+2) + A_2x + B_2$$

$$= A_1(x^3+2x) + B_1(x^2+2) + A_2x + B_2$$

$$= A_1x^3 + B_1x^2 + (2A_1 + A_2)x + 2B_1 + B_2$$

$$\begin{cases} A_1 = 8 \\ B_1 = 0 \\ 2A_1 + A_2 = 13 \\ 2B_1 + B_2 = 0 \end{cases}$$

$B_1 = 0$

$A_1 = 8$

$B_2 = 0$

$A_2 = -3$

$$\int \frac{8x^3+13x}{(x^2+2)^2}\,dx = \int\left(\frac{8x}{x^2+2} - \frac{3x}{(x^2+2)^2}\right)dx$$

$$= 4\int \frac{2x}{x^2+2}\,dx - \frac{3}{2}\int 2x\,(x^2+2)^{-2}\,dx$$

$$= 4\ln(x^2+2) - \frac{3}{2}\cdot\frac{(x^2+2)^{-2+1}}{-2+1} + C$$

$$= 4\ln(x^2+2) + \frac{3}{2}\,\frac{1}{x^2+2} + C$$

<u>CASE 2</u> $\deg P(x) \geq \deg Q(x)$.

In this case, we can perform the long division to get

$$\frac{P(x)}{Q(x)} = f(x) + \frac{r(x)}{Q(x)} \quad \text{with } \deg r(x) < \deg Q(x)$$

$$\int \frac{P(x)}{Q(x)}\,dx = \int f(x)\,dx + \underbrace{\int \frac{r(x)}{Q(x)}\,dx}_{\text{CASE 1}}$$

<u>Exple</u> Evaluate

$$\int \frac{5x^3+3x^2-2}{x^2-x}\,dx.$$

$$\begin{array}{r} 5x+8 \\ x^2-x \overline{\big)\, 5x^3+3x^2-2} \\ 5x^3-5x^2+0 \\ \hline 0+8x^2-2 \\ 8x^2-8x \\ \hline 0+8x-2 \end{array}$$

$\uparrow$ $r(x)$

$$\frac{5x^3+3x^2-2}{x^2-x} = 5x+8+\frac{8x-2}{x^2-x}$$

$$= 5x+8 + 2\left(\frac{4x-1}{x(x-1)}\right)$$

$\uparrow$

$$\frac{4x-1}{x(x-1)} = \frac{A}{x}+\frac{B}{x-1}$$

$$* \quad \frac{4x-1}{x-1} = A + \frac{Bx}{x-1} \qquad x=0$$

$$A = 1$$

$$* \quad \frac{4x-1}{x} = \frac{A(x-1)}{x} + B \qquad x=1$$

$$B = 3$$

$$\int \frac{5x^3+3x^2-2}{x^2-x}\,dx = \int \left[5x+8+2\left(\frac{1}{x}+\frac{3}{x-1}\right)\right] dx$$

$$= \frac{5x^2}{2} + 8x + 2\ln|x| + 6\ln|x-1| + C$$

$\int$ Trigonometric substitution

$\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$, $a > 0$

$$\int \frac{dx}{\sqrt{4 + x^2}} \qquad \int \frac{x\,dx}{\sqrt{4 + x^2}}$$

$$\frac{1}{2}\int 2x\,(4+x^2)^{-\frac{1}{2}}\,dx \qquad \sqrt{\theta^2} = |\theta|$$

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* **Integrals involving $\sqrt{a^2 - x^2}$, $a > 0$**

For integrals involving $\sqrt{a^2 - x^2}$, you always do the substitution

$$x = a\sin\theta \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)}$$

$$= \sqrt{a^2\cos^2\theta} = \sqrt{(a\cos\theta)^2} = a|\cos\theta|$$

$$= a\cos\theta$$

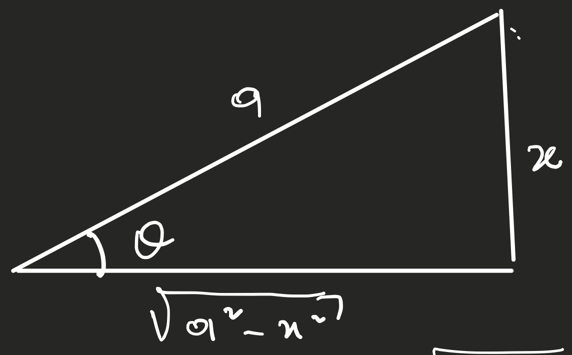
$x = a\sin\theta \iff \sin\theta = \dfrac{x}{a}$

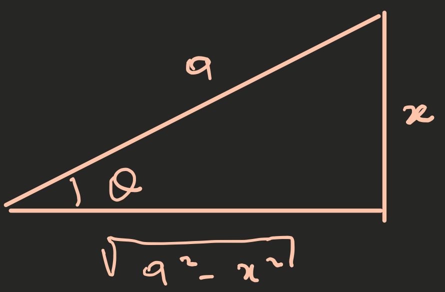
$\sqrt{a^2-x^2}$:

$x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = a\cos\theta\, d\theta$

$\sqrt{a^2-x^2} = a\cos\theta$





$\cos\theta = \dfrac{\sqrt{a^2-x^2}}{a}$

Exple Evaluate the integral

$$\int \frac{dx}{x^2\sqrt{4-x^2}}$$

$a = 2$. Put $x = 2\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = 2\cos\theta\, d\theta$ $\qquad \sin\theta = \dfrac{x}{2}$

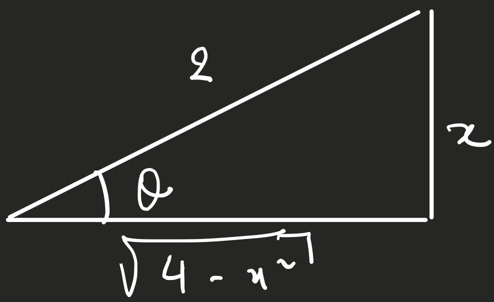
$\sqrt{4-x^2} = 2\cos\theta$

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{\cancel{2\cos\theta}\, d\theta}{4\sin^2\theta\,(\cancel{2\cos\theta})}$$

$$= \frac{1}{4}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

$$\cot\theta = \frac{\sqrt{4-x^2}}{x}.$$



$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

* **Integrals involving $\sqrt{a^2+x^2}$, $a>0$.**

For integrals involving $\sqrt{a^2+x^2}$, you always do the substitution

$$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dx = a\sec^2\theta\, d\theta$$

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)}$$

$$= \sqrt{a^2\sec^2\theta}$$

$$= \sqrt{(a\sec\theta)^2}$$

$$= a|\sec\theta|$$

$$= a\sec\theta$$

* $\sqrt{a^2+x^2}$:

$x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$\sqrt{a^2+x^2} = a\sec\theta$

$$\tan\theta = \frac{x}{a}$$

Exple Evaluate

$$\int \frac{dx}{(x^2+1)\sqrt{x^2+1}} = \frac{x}{\sqrt{1+x^2}} + C$$

$a = 1$.

$$x = \tan\theta \quad , \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

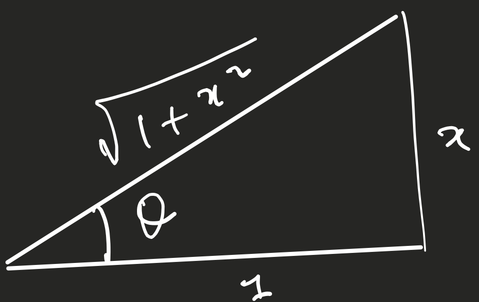
$$dx = \sec^2\theta \, d\theta$$

$$\sqrt{1+x^2} = \sec\theta .$$

$$\int \frac{dx}{\underbrace{(1+x^2)}\underbrace{\sqrt{1+x^2}}} = \int \frac{\cancel{\sec^2}\theta \, d\theta}{\cancel{\sec^2}\theta \cdot \sec\theta}$$

$$\tan\theta = \frac{x}{1}$$

$$= \int \frac{d\theta}{\sec\theta} = \int \cos\theta \, d\theta$$



$$= \sin\theta + C$$

$$\sin\theta = \frac{x}{\sqrt{1+x^2}} .$$

* For integrals involving $\sqrt{x^2-a^2}$, $\underline{a>0}$

you always do the substitution ?

$x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)}$$

$$= \sqrt{(a\tan\theta)^2} = a|\tan\theta| \qquad 0 \le \theta < \frac{\pi}{2}$$

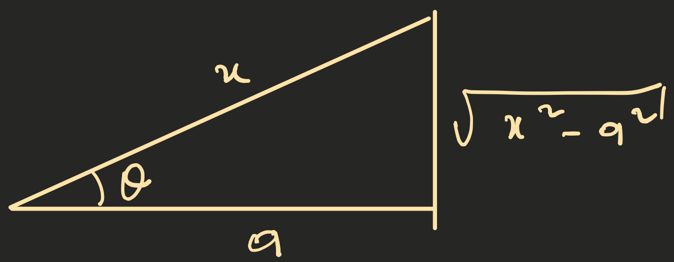
$$= a\tan\theta$$

$\sqrt{x^2-a^2}$:

$x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

$$dx = a\sec\theta\tan\theta\, d\theta$$

$$\sqrt{x^2-a^2} = a\tan\theta$$



$x = a\sec\theta$

$\sec\theta = \frac{x}{a}$

$\cos\theta = \frac{a}{x}$

Exple Evaluate

$$\int \frac{\sqrt{x^2-2}}{x}\, dx .$$

$a = \sqrt{2}$.

$x = \sqrt{2}\sec\theta$

$0 \le \theta < \frac{\pi}{2}$, $\pi \le \theta < \frac{3\pi}{2}$.

$$dx = \sqrt{2}\sec\theta\tan\theta\, d\theta$$

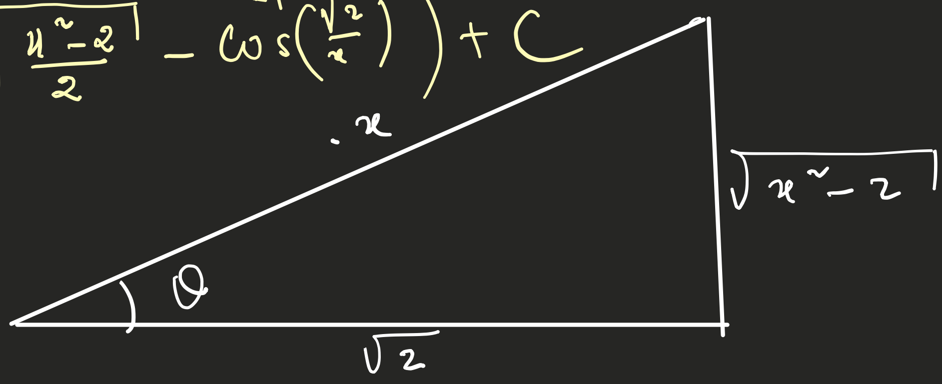
$$\sqrt{x^2-2} = \sqrt{2}\tan\theta$$

$$\int \frac{\sqrt{x^2-2}}{x}\,dx = \int \frac{\cancel{\sqrt{2}}\tan\theta \cdot \sqrt{2}\,\cancel{\sec\theta}\tan\theta\,d\theta}{\cancel{\sqrt{2}}\,\cancel{\sec\theta}}$$

$$= \sqrt{2}\int \tan^2\theta\,d\theta = \sqrt{2}\int (\sec^2\theta - 1)\,d\theta$$

$$= \sqrt{2}\,(\tan\theta - \theta) + C$$

$$= \sqrt{2}\left(\sqrt{\frac{x^2-2}{2}} - \cos^{-1}\left(\frac{\sqrt{2}}{x}\right)\right) + C$$



$$x = \sqrt{2}\sec\theta \iff \sec\theta = \frac{x}{\sqrt{2}} \iff \underbrace{\cos\theta = \frac{\sqrt{2}}{x}}_{\theta = \cos^{-1}\left(\frac{\sqrt{2}}{x}\right)}$$

$$\tan\theta = \frac{\sqrt{x^2-2}}{\sqrt{2}} = \underbrace{\sqrt{\frac{x^2-2}{2}}}$$

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§ Rationalization of integrals

Exple Evaluate the following integrals.

a) $\displaystyle\int \frac{\sqrt{x}}{2+x}\,dx$

c)

a) $\int \frac{\sqrt{x}}{2+x}\,dx$

Put $u = \sqrt{x} \rightarrow u^2 = x$

$2u\,du = dx$

$$\int \frac{\sqrt{x}}{2+x}\,dx = \int \frac{u\,2u\,du}{2+u^2}$$

$$= 2\int \frac{u^2}{2+u^2}\,du.$$

$$= 2\int \frac{u^2+2-2}{2+u^2}\,du$$

$$= 2\int \left(\frac{u^2+2}{u^2+2} - \frac{2}{u^2+2}\right)du$$

$$= 2\int \left(1 - \frac{2}{u^2+2}\right)du$$

$$= 2\left[u - \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)\right] + C$$

$$= 2u - \frac{4}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= 2\sqrt{x} - \frac{4}{\sqrt{2}} \tan^{-1}\sqrt{\frac{x}{2}} + C$$

b) $\displaystyle\int \frac{dx}{x + x\sqrt{x}}$ Put $u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$$\int \frac{dx}{x + x\sqrt{x}} = \int \frac{2\cancel{u}\,du}{u^{\cancel{2}} + u^{\cancel{3}\,2}}$$

$$= 2\int \frac{du}{u + u^2} = 2\int \frac{du}{u(1+u)}$$

$$\frac{1}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u} = \frac{1}{u} - \frac{1}{1+u}$$

$A = ?$ $\quad \frac{1}{1+u} = A + \frac{Bu}{1+u}$: $u = 0$

$A = 1$

$B\,?$ $\quad \frac{1}{u} = \frac{A(1+u)}{u} + B$: $u = -1$

$B = -1$

$$2\int \frac{du}{u(1+u)} = 2\int \left(\frac{1}{u} - \frac{1}{1+u}\right) du$$

$$= 2(\ln|u| - \ln|1+u|) + C$$

$u = \sqrt{x}$

$$= 2 \ln \left| \frac{u}{1+u} \right| + C$$

$$= \ln \left(\frac{u}{1+u} \right)^2 + C$$

$$= \ln \left(\frac{\sqrt{x}}{1+\sqrt{x}} \right)^2 + C$$

$$\int \frac{(1)\,du}{u(u+1)} = \int \frac{(u+1-u)\,du}{u(u+1)}$$

$$= \int \left(\frac{\cancel{u+1}}{u(\cancel{u+1})} - \frac{\cancel{u}}{\cancel{u}(u+1)} \right) du$$

$$= \int \left(\frac{1}{u} - \frac{1}{u+1} \right) du$$

$$\int \frac{dx}{\sqrt{x} + \sqrt[4]{x}}$$ Put $u = \sqrt[4]{x}$

$u^4 = x \longleftrightarrow \sqrt{x} = u^2$

$4u^3\,du = dx$.

$$\int \frac{dx}{\sqrt{x} + \sqrt[4]{x}} = \int \frac{4u^3\,du}{u^2 + u} = 4 \int \frac{u^{\cancel{3}\,2}}{\cancel{u}(u+1)}\,du$$

$$= 4 \int \frac{u^2}{u+1}\,du = 4 \int \frac{u^2 - 1 + 1}{u+1}\,du$$

$$= 4\int \left(\frac{u^2-1}{u+1} + \frac{1}{u+1} \right) du$$

$$= 4\int \left(\frac{(u-1)\cancel{(u+1)}}{\cancel{u+1}} + \frac{1}{u+1} \right) du$$

$$u = \sqrt[4]{x}$$

$$= 4\int \left(u-1 + \frac{1}{u+1} \right) du$$

$$= 4\left(\frac{u^2}{2} - u + \ln|u+1| \right) + C$$

$$= 2u^2 - 4u + \ln(1+u)^4 + C$$

$$= 2\sqrt{x} - 4\sqrt[4]{x} + \ln(1+\sqrt[4]{x})^4 + C$$

$$\int \cos(x^2)\,dx, \quad \int \sin(x^2)\,dx$$

$$\int e^{x^2}\,dx$$