THIS EXAMINATION CONTENT IS STRICTLY CONFIDENTIAL Students must comply with requirements stated in the Examination Policy & Procedures	
Student Number:	
First Name:	
Family Name:	
Date of Examination: (DD/MM/YY)	
Subject Code:	**Math 142**
Subject Title:	**Essentials of Engineering Mathematics**
Time Permitted to Write Exam:	2 Hours
Total Number of Questions:	**9 (9 written questions)**
Total Number of Pages *(including this page):*	**10**

INSTRUCTIONS TO STUDENTS FOR THE EXAM

1. Please note that subject lecturer/tutor will be unavailable during exams. *If there is a doubt in any of the exam questions i.e. problem solving etc. students should proceed by assuming values etc. Students should mention their assumption on the question paper.*
2. Answers must be written (and drawn) in black or blue ink
3. Any mistakes must be crossed out. Whitener and ink erasers must not be used.
4. Answer ALL/ 9 questions. The marks for each question are shown next to each question.
5. Total marks: 100. This Exam is worth 40% of your final marks for MATH 142.

Problem 1 (12 points)

(**6pts**) A) Suppose a nuclear power plant generate heat at a rate of $R(t) = 5000e^{-0.01t}$ megawatts/hour, where t is measured in hours since the start of the day. The power plant operates indefinitely. What is the total heat energy generated by the power plant? i.e. $\int_0^\infty R(t)dt$.

Solution

$$\begin{aligned} \text{Total heat} &= \int_0^\infty 5000e^{-0.01t}dt \\ &= 5000\int_0^\infty e^{-0.01t}dt \end{aligned}$$

$$\begin{aligned} \int_0^\infty e^{-0.01t}dt &= \lim_{T\to\infty}\int_0^T e^{-0.01t}dt \qquad (\textbf{3pts}) \\ &= \lim_{T\to\infty}\left[\frac{1}{-0.01}e^{-0.01t}\right]_0^T \\ &= \lim_{T\to\infty}\left[-100\left(e^{-0.01T}-1\right)\right] \\ &= -100\,(0-1) = 100 \end{aligned}$$

$$\text{Total heat } = (5000)\,(100) = 500000\ MW \qquad (\textbf{3pts})$$

(**6pts**) B) Suppose a nuclear power plant generate heat at a rate of $R(t) = \dfrac{5000}{\sqrt{t-1}}$ megawatts/hour, where t is measured in hours since the start of the day. The power plant operates from 1 hour to 24 hours. What is the total heat energy generated by the power plant over the 23-hour period? i.e. $\int_1^{24} R(t)dt$.

Solution

$$\begin{aligned} \text{Total heat} &= \int_1^{24}\frac{5000}{\sqrt{t-1}}dt \\ &= 5000\int_1^{24}\frac{dt}{\sqrt{t-1}} \end{aligned}$$

$$\begin{aligned} \int_1^{24}\frac{dt}{\sqrt{t-1}} &= \lim_{T\to 1^+}\int_T^{24}(t-1)^{-1/2}\,dt \qquad (\textbf{3pts}) \\ &= \lim_{T\to 1^+}\left[2\sqrt{t-1}\right]_T^{24} \\ &= \lim_{T\to 1^+}\left(2\sqrt{23}-2\sqrt{T-1}\right) \\ &= 2\sqrt{23} \end{aligned}$$

$$\begin{aligned} \text{Total heat} &= (5000)\left(2\sqrt{23}\right) \\ &= 10\,000\sqrt{23} = 47958 \quad MW \qquad (\textbf{3pts}) \end{aligned}$$

Problem 2 (12 points)

Show that the equation is separable and solve the initial value problem

$$\left(1+y^2\right)x^2dx - ydy = 0, \qquad y(0)=1$$

Solution

$$\left(1+y^2\right)x^2dx - ydy = 0$$

$\Leftrightarrow$

$$\frac{dy}{dx} = \frac{\left(1+y^2\right)x^2}{y} = f(x)g(y) \Rightarrow \text{ the equation is separable.} \quad (\mathbf{4pts})$$

We have

$$\frac{y}{1+y^2}dy = x^2dx \Rightarrow \int \frac{y}{1+y^2}dy = \int x^2dx$$

$\Leftrightarrow$

$$\frac{1}{2}\int \frac{2y}{1+y^2}dy = \int x^2dx$$

$\Leftrightarrow$

$$\frac{1}{2}\ln\left(1+y^2\right) = \frac{x^3}{3} + C_1$$

$\Leftrightarrow$

$$\ln\left(1+y^2\right) = \frac{2x^3}{3} + C_2, \qquad C_2 = 2C_1$$

$\Leftrightarrow$

$$1+y^2 = e^{C_2}e^{\frac{2x^3}{3}}$$

$\Leftrightarrow$

$$y^2 = Ce^{\frac{2x^3}{3}} - 1, \quad \text{where } C = e^{C_2}. \qquad (\mathbf{4pts})$$

Now $y(0)=1$ gives

$$C - 1 = 1 \Leftrightarrow C = 2. \qquad (\mathbf{4pts})$$

The solution is given by

$$y^2 = 2e^{\frac{2x^3}{3}} - 1.$$

Problem 3 (10 points)
Show that the equation is linear and solve it.

$$\frac{dy}{dx} = x - y$$

Solution

$$\frac{dy}{dx} = x - y \Leftrightarrow \frac{dy}{dx} + y = x.$$

This is of the form

$$\frac{dy}{dx} + P(x)y = Q(x), \text{ where } P(x) = 1 \text{ and } Q(x) = x. \qquad (\mathbf{3pts})$$

This shows that the equation is linear.

$$IF = e^{\int 1dx} = e^x. \qquad (\mathbf{3pts})$$

The solution is given by

$$y = \frac{1}{IF}\left(\int IFxdx\right)$$

$$y = e^{-x}\int xe^x dx.$$

Using integration by parts, we have

$$\int xe^x dx = e^x(x-1) + C.$$

Hence,

$$y = e^{-x}\left[e^x(x-1) + C\right]$$

$\Leftrightarrow$

$$y = x - 1 + Ce^{-x} \qquad (\mathbf{4pts})$$

Problem 4 (12 points)
Show that the equation is exact and solve the initial value problem

$$(4xy+1)\,dx + \left(2x^2+\cos y\right)dy = 0, \qquad y(1)=0.$$

Solution
Put

$$M = 4xy+1, \text{ and } N = 2x^2+\cos y$$

$$\frac{\partial M}{\partial y} = 4x \quad \text{and} \quad \frac{\partial N}{\partial x} = 4x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{ the equation is exact.} \qquad (\mathbf{4pts})$$

We now need to find f such that

$$\begin{cases} f_x = 4xy+1 \\ f_y = 2x^2+\cos y \end{cases}$$

$$f_x = 4xy+1 \Rightarrow f = \int (4xy+1)\,dx + C(y)$$

$$f = 2x^2y + x + C(y). \qquad (\mathbf{3pts})$$

Now $f_y = 2x^2+\cos y$ gives

$$2x^2 + C'(y) = 2x^2+\cos y$$

$\Leftrightarrow$

$$C'(y) = \cos y \Rightarrow C(y) = \sin y.$$

Thus

$$f(x,y) = 2x^2y + x + \sin y. \qquad (\mathbf{3pts})$$

The general solution is

$$2x^2y + x + \sin y = C.$$

$$y(1) = 0 \text{ gives } C = 1. \qquad (\mathbf{2pts})$$

The solution is

$$2x^2y + x + \sin y = 1.$$

Problem 5 (12 points)
Show that the equation is homogeneous and solve it.

$$(x-y)\,dx + xdy = 0.$$

Solution

$$\begin{aligned}
(x-y)\,dx + xdy &= 0 \\
&\Leftrightarrow \frac{dy}{dx} = \frac{y-x}{x} \\
&\Leftrightarrow \frac{dy}{dx} = \frac{y}{x} - 1 = F(\frac{y}{x})
\end{aligned}$$

The equation is homogeneous. (**4pts**)

Put

$$u = \frac{y}{x} \Leftrightarrow y = xu$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

The equation becomes

$$u + x\frac{du}{dx} = u - 1$$

$\Leftrightarrow$

$$du = \frac{-dx}{x} \qquad (\textbf{4pts})$$

$$u = -\ln|x| + C.$$

Equivalently,

$$\frac{y}{x} = -\ln|x| + C.$$

$$y = -x\ln|x| + Cx \qquad (\textbf{4pts})$$

Problem 6 (10 points)
Find $\lim_{n\to\infty} a_n$.

$$1.\ a_n = 2n\sin\left(\frac{1}{n}\right), \qquad 2.\ a_n = \frac{\cos(2n)}{2^n}$$

Solution
1.

$$\begin{aligned}\lim_{n\to\infty} 2n\sin\left(\frac{2}{n}\right) &= \lim_{n\to\infty} 2\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \\ &= 2(1) = 2 \qquad (\mathbf{5pts})\end{aligned}$$

2.

$$-1 \le \cos(2n) \le 1$$
$$\frac{-1}{2^n} \le \frac{\cos(2n)}{2^n} \le \frac{1}{2^n}$$

Since

$$\lim_{n\to\infty}\left(\frac{-1}{2^n}\right) = \lim_{n\to\infty}\left(\frac{1}{2^n}\right) = 0,$$

$$\lim_{n\to\infty}\frac{\cos(2n)}{2^n} = 0 \text{ by the squeezing Theorem.} \qquad (\mathbf{5pts})$$

Problem 7 (10 points)
Find the sum of the following series

$$1.\ \sum_{n=2}^{\infty} \frac{2}{n^2-1}, \qquad\qquad 2.\ \sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}}$$

Solution

1. $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = ?$

$$\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}$$

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{2}{n^2-1} &= \lim_{N\to\infty} \sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n+1}\right) \\
&= \lim_{N\to\infty} \left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{4}-\frac{1}{6}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)\right] \\
&= \lim_{N\to\infty} \left[1 + \frac{1}{2} - \frac{1}{n+1}\right] = 1 + \frac{1}{2} \\
&= \frac{3}{2} \qquad (\textbf{5pts})
\end{aligned}$$

2. $\sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}} = ?$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}} &= \sum_{n=1}^{\infty} \frac{2^{n-1+1}}{3^{n-1}} \\
&= \sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1} \\
&= \frac{2}{1-\frac{2}{3}} = 6 \qquad (\textbf{5pts})
\end{aligned}$$

Problem 8 (12 points)
Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{2n}.$$

Solution
This is a power series centered at $a = 3$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(x-3)^{n+1}}{2n+2}}{\frac{(-1)^n(x-3)^n}{2n}} \right| &= \lim_{n\to\infty} \left| \frac{-(-1)^n (x-3)^n (x-3)}{2n+2} \times \frac{2n}{(-1)^n (x-3)^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{2n}{2n+2} (x-3) \right| = |x-3|
\end{aligned}$$

$$R = 1 \qquad (\textbf{4pts})$$

$$a - R = 3 - 1 = 2$$

$$a + R = 3 + 1 = 4$$

When $x = 2$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n}{2n} = \sum_{n=0}^{\infty} \frac{1}{2n} \text{ diverges by the p-test with } p = 1. \ (\textbf{3pts})$$

When $x = 4$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n} \text{ converges by the alternating series test.} \qquad (\textbf{3pts})$$

$$\text{Interval of convergence } = (2,\ 4] \qquad (\textbf{2pts})$$

Problem 9 (10 points)
Find a power series representation for the function

$$f(x) = \frac{x}{9+x^2}$$

and determine the interval of convergence.

Solution

$$\begin{aligned}
\frac{x}{9+x^2} &= \frac{x}{9\left(1+\frac{x^2}{9}\right)} \\
&= \frac{x}{9}\frac{1}{\left(1-\left[-\frac{x^2}{9}\right]\right)} \\
&= \frac{x}{9}\sum_{n=0}^{\infty}\left[-\frac{x^2}{9}\right]^n, \quad \text{for } \left|\left[-\left(\frac{x}{3}\right)^2\right]\right| < 1 \\
&= \frac{x}{9}\sum_{n=0}^{\infty}(-1)^n\,\frac{x^{2n}}{9^n}, \quad \text{for } |x| < 3 \\
&= \sum_{n=0}^{\infty}(-1)^n\,\frac{x^{2n+1}}{9^{n+1}}, \quad \text{for } |x| < 3 \qquad (\mathbf{6pts})+(\mathbf{4pts})
\end{aligned}$$