Part A MCQ (30%)

($5pts$) **Problem 1**

Evaluate the improper integral $L = \displaystyle\int_2^{\infty} \frac{dx}{x \ln^2 x}$.

$(a) \quad L = \ln 2$

$(b) \quad L = \frac{1}{\ln 2}$

$(c) \quad L = +\infty$

$(d) \quad L = 2$

$(e) \quad L = e^2$

$\underline{Answer\ is\ (b)}$

($5pts$) **Problem 2**

Evaluate the improper integral $A = \displaystyle\int_0^{1} \frac{x}{\sqrt{1-x^2}} dx$

$(a) \quad A = 1$

$(b)\ A = \ 2$

$(c) \quad A = -1$

$(d)\ A = \ \frac{1}{2}$

$(e) \quad A = -\infty$

$\underline{Answer\ is\ (a)}$

$(5pts)$ **Problem 3**
Let a_n be the sequence given by

$$\cos(\pi), \ \cos\left(\frac{\pi}{2}\right), \ \cos\left(\frac{\pi}{3}\right), \ \cos\left(\frac{\pi}{4}\right), \ ...$$

If $L = \lim_{n\to\infty} a_n$, then

(a) $L = 4$

(b) $L = 3$

(c) $L = 2$

(d) $L = 1$

(e) $L = 0$

$\underline{Answer\ is\ (d)}$

$(5pts)$ **Problem 4**
The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$$

(a) converges absolutely

(b) converges conditionally

(c) diverges

(d) is a convergent geometric series

(e) is a divergent telescoping series

$\underline{Answer\ is\ (b)}$

$(5pts)$ **Problem 5**

The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$ is

(a) $\sqrt{2}$

(b) $\frac{1}{2}$

(c) 1

(d) ∞

(e) 2

<u>*Answer is (e)*</u>

$(5pts)$ **Problem 6**

The coefficient of x^3 in Maclaurin series of the function equal to

(a) $\frac{-1}{3}$

(b) $\frac{-1}{6}$

(c) -1

(d) -6

(e) $\frac{-1}{2}$

$$f(x) = \sin(\pi - x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O\left(x^7\right)$$

<u>*Answer is (b)*</u>

Part B Written Questions (70%)

$(15pts)$**Problem 1**
Find the interval of convergence of the following power series

$$1.\ \sum_{n=0}^{\infty} \frac{(x+2)^n}{\sqrt{n}} \qquad\qquad 2.\ \sum_{n=0}^{\infty} \frac{x^n}{n^n}.$$

Solution

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$ is a power series centered at $a=-2$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(x+2)^n}\right| &= \lim_{n\to\infty}\left|\frac{(x+2)^n(x+2)}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(x+2)^n}\right| \\
&= \lim_{n\to\infty}\sqrt{\frac{n}{n+2}}\,|x+2| = |x+2|.
\end{aligned}$$

$$R = 1 \qquad [\ \textbf{2 points}\ ]$$

$$a-R = -2-1 = -3 \ \text{ and } a+R = -2+1 = -1.$$

- When $x=-3$, the series becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}} \text{ converges by A.S.T} \qquad [\ \textbf{2 points}\ ]$$

- When $x=-1$, the series becomes

$$\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}} \text{ diverges by p-test.} \qquad [\ \textbf{2 points}\ ]$$

$$\text{IC} = [-3,\ -1) \qquad [\ \textbf{2 points}\ ]$$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n^n}$ is a power series centered at $a=0$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{x^n}\right| &= \lim_{n\to\infty}\left|\frac{x^n x}{(n+1)^n(n+1)}\cdot\frac{n^n}{x^n}\right| \\
&= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1}|x| \\
&= \frac{1}{e}\cdot 0 = 0 \qquad [\ \textbf{3 points}\ ]
\end{aligned}$$

$$R=\infty \ \text{ and } \ \text{IC} = (-\infty,\ \infty). \qquad [\ \textbf{4 points}\ ]$$

$(10pts)$**Problem 2**
Show that the equation is separable and solve it.

$$y\frac{dy}{dx} - \left(1+y^2\right)x^2 = 0$$

Solution

$$\begin{aligned} y\frac{dy}{dx} &= \left(1+y^2\right)x^2, \\ \int \frac{y}{1+y^2}dy &= \int x^2 dx \quad [\ \textbf{5 points}\ ] \end{aligned}$$

$$\frac{1}{2}\ln\left(1+y^2\right) = \frac{x^3}{3} + C_1$$

$$\ln\left(1+y^2\right) = \frac{2x^3}{3} + C_2.$$

$$1+y^2 = e^{\frac{2x^3}{3}+C_2}$$

$$y^2 = e^{\frac{2x^3}{3}+C_2} - 1 \qquad [\ \textbf{5 points}\ ]$$

(15pts)**Problem 3**
Show that the differential equation is exact and solve the initial value problem.

$$\left(y - x^3\right) dx + \left(x + y^3\right) dy = 0, \qquad y(0) = \sqrt{2}$$

Solution

$$M = y - x^3 \text{ and } N = x + y^3.$$

$$M_y = 1, \qquad N_x = 1$$

$$M_y = N_x \Rightarrow \text{Equation is exact.} \qquad [\ \textbf{3 points}\ ]$$

$$\begin{cases} \frac{\partial f}{\partial x} = y - x^3 \\ \frac{\partial f}{\partial y} = x + y^3 \end{cases},$$

$$\frac{\partial f}{\partial x} = y - x^3 \Rightarrow f(x,y) = xy - \frac{x^4}{4} + C(y). \qquad [\ \textbf{4 points}\ ]$$

Now using this temporary expression of f,

$$\frac{\partial f}{\partial y} = x + y^3 \Leftrightarrow x + C'(y) = x + y^3,$$

$$C'(y) = y^3 \Rightarrow C(y) = \frac{y^4}{4}.$$

Thus,

$$f(x,y) = xy - \frac{x^4}{4} + \frac{y^4}{4} \qquad [\ \textbf{5 points}\ ]$$

and the solution is given by

$$xy - \frac{x^4}{4} + \frac{y^4}{4} = C.$$

Now using the initial condition, we obtain

$$y(0) = \sqrt{2} \Leftrightarrow C = \frac{\left(\sqrt{2}\right)^4}{4} = 1 \qquad [\ \textbf{3 points}\ ]$$

(15pts)**Problem 4**
Show that the equation is Bernoulli and solve it.

$$\frac{dy}{dx} = \frac{2y}{x} - x^2y^2$$

Solution
The given equation is equivalent to equivalent to

$$\frac{dy}{dx} - \frac{2y}{x} = -x^2y^2.$$

This is a Bernoulli equation with $n = 2$.
We have

$$y^{-2}\frac{dy}{dx} - \frac{2y^{-1}}{x} = -x^2.$$

Put

$$u = y^{1-n} = y^{-1} \qquad [\ \mathbf{3\ points}\ ]$$

$$\frac{du}{dx} = -\frac{dy}{dx}y^{-2},$$

$$y^{-2}\frac{dy}{dx} = -\frac{du}{dx}.$$

The equation becomes

$$-\frac{du}{dx} - \frac{2}{x}u = -x^2,$$

Equivalently,

$$\frac{du}{dx} + \frac{2}{x}u = x^2, \qquad [\ \mathbf{5\ points}\ ]$$

This is a linear equation with

$$P(x) = \frac{2}{x}.$$

$$\text{IF} = e^{\int P(x)dx} = e^{2\ln x} = x^2 \qquad (x > 0). \qquad [\ \mathbf{2\ points}\ ]$$

$$x^2u = \int x^4 dx,$$

$$x^2u = \frac{x^5}{5} + C.$$

$$u = \frac{x^3}{5} + \frac{C}{x^2},$$

$$y^{-1} = \frac{x^3}{5} + \frac{C}{x^2}. \qquad [\ \mathbf{3\ points}\ ]$$

Hence,

$$y = \frac{1}{\frac{x^3}{5} + \frac{C}{x^2}}. \qquad [\ \mathbf{2\ points}\ ]$$

$(15pts)$**Problem 5**
Show that the equation is homogeneous and solve it.

$$x^2\frac{dy}{dx} = xy - y^2$$

Solution

$$x^2\frac{dy}{dx} = xy - y^2$$

is equivalent to

$$\begin{aligned}\frac{dy}{dx} &= \frac{xy-y^2}{x^2} = \left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2 = F\left(\frac{y}{x}\right) \\ &\Rightarrow \quad \text{equation is homogeneous} \qquad [\ \textbf{5 points}\ ]\end{aligned}$$

Put $u = \dfrac{y}{x} \Rightarrow y = xu$ and $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$.
The equation becomes

$$u + x\frac{du}{dx} = u - u^2$$

$\Leftrightarrow$

$$x\frac{du}{dx} = -u^2$$

$$\frac{-du}{u^2} = \frac{dx}{x}$$

$$\int \frac{-du}{u^2} = \int \frac{dx}{x} \qquad [\ \textbf{5 points}\ ]$$

$$\frac{1}{u} = \ln|x| + C.$$

Now replacing u by $\dfrac{y}{x}$, we obtain

$$\frac{x}{y} = \ln|x| + C.$$

$$y = \frac{x}{\ln|x| + C}. \qquad [\ \textbf{5 points}\ ]$$