§ Improper Integrals

* $\int f(x)\,dx \longrightarrow$ indefinite integral

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

* If f is continuous on $[a, b]$

then

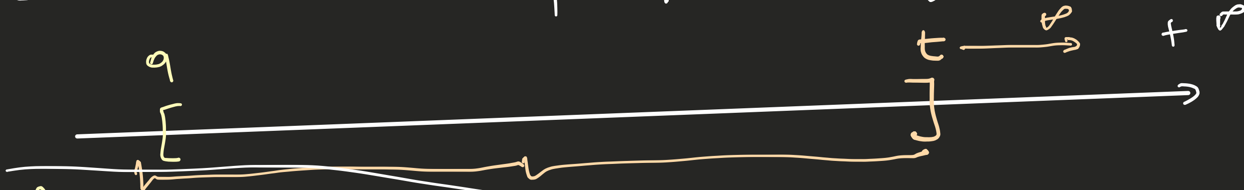
$\int_a^b f(x)\,dx \longrightarrow$ definite integral.

Improper integral of type I.

If f is continuous on $[a, +\infty)$, then the integral

$$\int_a^\infty f(x)\,dx$$

is called an improper integral of type I.



$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

* If the limit exists, then we say that

the improper integral converges.

otherwise it diverges.

Exple Determine convergence or divergence of the following improper integrals.

1-) $\int_1^{\infty} x e^{-x^2} dx$ 2-) $\int_2^{\infty} \frac{1}{x \ln x} dx$

1-) $\int_1^{\infty} x e^{-x^2} dx = \lim_{t \to \infty} \underbrace{\int_1^t x e^{-x^2} dx} = \lim_{t \to \infty} -\frac{1}{2}\left(\frac{1}{e^{t^2}} - \frac{1}{e}\right) = \frac{1}{2e}$

$\underbrace{\int_a^b u' e^u dx = e^u \Big|_a^b}$

$\frac{1}{-2}\int_1^t -2x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} \Big|_1^t$

$= -\frac{1}{2}\left(e^{-t^2} - e^{-1}\right)$

$= -\frac{1}{2}\left(\frac{1}{e^{t^2}} - \frac{1}{e}\right)$

CL : the improper integral $\int_1^{\infty} x e^{-x^2} dx$ converges and its value is $\frac{1}{2e}$

2) . $\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{t \to \infty} \underbrace{\int_2^t \frac{dx}{x \ln x}}$.

$\int_a^b \frac{u'}{u} dx = \ln|u| \Big|_a^b$

$$\int_2^t \frac{dx}{x\ln x} = \int_2^t \frac{\frac{1}{x}}{\ln x}\,dx$$

$$= \ln|\ln x|\Big|_2^t$$

$$= \ln|\ln t| - \ln\ln 2$$

$$\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{t\to\infty}\left[\ln|\ln t| - \ln\ln 2\right] = +\infty$$

diverges

* If f is continuous on $(-\infty\ b]$

then the integral

$$\int_{-\infty}^{b} f(x)\,dx \quad \text{is an improper integral}$$

of type I.

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

t

<u>Exple</u> Evaluate

$$\int_{-\infty}^{-1} \frac{dx}{x^2}$$

$u = -x$

$du = -dx$

$$\int_{+\infty}^{1} -\frac{du}{u^2} = \int_1^{\infty} \frac{du}{u^2}$$

$$\int_{-\infty}^{-2} \frac{dx}{x}$$

$$\int_{-\infty}^{-1} \frac{dx}{x^2} = \lim_{t \to -\infty} \int_t^{-1} x^{-2} = \lim_{t \to -\infty} \frac{x^{-2+1}}{-2+1} \Big|_t^{-1}$$

$$= \lim_{t \to -\infty} -\frac{1}{x} \Big|_t^{-1} = \lim_{t \to -\infty} \left(+1 + \frac{1}{t} \right) = 1$$

$\displaystyle\int_{-\infty}^{-1} \frac{dx}{x^2}$ convergs to 1.

$*$ $$\int_{-\infty}^{-2} \frac{dx}{x} = \lim_{t \to -\infty} \int_t^{-2} \frac{dx}{x} = \lim_{t \to -\infty} \left(\ln|x| \Big|_t^{-2} \right)$$

$$= \lim_{t \to -\infty} \left(\ln 2 - \ln|t| \right)$$

$$= \ln 2 - \ln|-\infty|$$

$$= \ln - \ln \infty = -\infty$$

divergs.

$*$ If f is continuous on $(-\infty, \infty)$
then $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ is an improper integral of type I.

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{a} f(x)\,dx}_{\mathrm{I}} + \underbrace{\int_{a}^{\infty} f(x)\,dx}_{\mathrm{I}}$$

For this to converge, both

$$\int_{-\infty}^{a} f(x)\,dx \quad \text{and} \quad \int_{a}^{\infty} f(x)\,dx$$

must converge.

<u>Exple</u> Evaluate

a) $\displaystyle\int_{-\infty}^{\infty} e^{x}\,dx$, b) $\displaystyle\int_{-\infty}^{\infty} \sin x\,dx$.

a) $$\int_{-\infty}^{\infty} e^{x}\,dx = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{x}\,dx$$

$$\int_{-\infty}^{0} e^{x}\,dx = \lim_{t\to-\infty} \int_{t}^{0} e^{x}\,dx = \lim_{t\to-\infty} e^{x}\Big|_{t}^{0}$$

$$= \lim_{t\to-\infty} \left(e^{0} - e^{t}\right)$$

$$\lim_{t\to-\infty} e^{t} = e^{-\infty} = \frac{1}{e^{\infty}} = 0$$

$$= 1 - 0 = 1$$

$$\int_{0}^{\infty} e^{x}\,dx = \lim_{t\to+\infty} \int_{0}^{t} e^{x}\,dx = \lim_{t\to\infty} \left(e^{x}\Big|_{0}^{t}\right)$$

$$= \lim_{t\to\infty} \left(e^{t} - 1\right) = +\infty$$

diverges

$$\int_{-\infty}^{\infty} \sin x \, dx = \lim_{t \to +\infty} \underbrace{\int_{-t}^{t} \sin x \, dx}_{0}$$

$$= \lim_{t \to +\infty} 0 = 0$$

* The p-test for improper integral of type I

If $a > 0$, then

$$\int_a^{\infty} \frac{dx}{x^p} = \begin{cases} \text{Converges if } p > 1 \\ \text{diverges if } p < 1 \end{cases}$$

Exple Determine Convergence or divergence.

$$\underbrace{\int_2^{\infty} \frac{dx}{x\sqrt{x}}}, \quad \underbrace{\int_1^{\infty} \frac{dx}{\sqrt{x}}}, \quad \underbrace{\int_{12}^{\infty} \frac{dx}{x^3}}$$

$p = 3 > 1$ converges.

$$\int_2^{\infty} \frac{dx}{x^{3/2}}$$

$p = \frac{3}{2} > 1$, converges

$$\int_1^{\infty} \frac{dx}{x^{\frac{1}{2}}}$$

$p = \frac{1}{2} < 1$

diverges

§ Improper integrals of type II

If f is continuous on $[a, b)$

then the integral

$$\int_a^b f(x)\,dx$$

is an improper integral of type II

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

If the limit exists, then the improper integral converges.

otherwise it diverges.

Exple Evaluate

a) $\displaystyle\int_1^2 \frac{dx}{(x-2)^2}$ b) $\displaystyle\int_1^2 \frac{dx}{\sqrt{2-x}}$.

a) $$\int_1^2 \frac{dx}{(x-2)^2} = \lim_{t\to 2^-} \int_1^t (x-2)^{-2}\,dx$$

$$= \lim_{t\to 2^-} \left(-\frac{1}{x-2}\Big|_1^t\right) = \lim_{t\to 2^-} \left[-\frac{1}{t-2} + (-1)\right]$$

$$= \pm\infty$$

diverges

b) $$\int_1^2 \frac{dx}{\sqrt{2-x}} = \lim_{t\to 2^-} \int_1^t \frac{dx}{\sqrt{2-x}}$$

$$= \lim_{t\to 2^-} (-1)\int_1^t (-1)(2-x)^{-\frac{1}{2}}\,dx$$

$$= \lim_{t\to 2^-} (-1)\,\frac{(2-x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\Bigg|_1^t$$

$$= \lim_{t\to 2^-} \left(-2\sqrt{2-x}\,\Big|_1^t\right)$$

$$= \lim_{t\to 2^-} \left(-2\sqrt{2-t}+2\right) = 0+2 = 2$$

Converges

* If f is continuous on $(a, b]$

then the integral

$$\int_a^b f(x)\,dx$$

is again an improper integral of type 2.

$a \leftarrow t \qquad b$

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

If the limit exists, it converges. otherwise it diverges.

<u>Exple</u> Evaluate

a) $\displaystyle\int_1^3 \frac{dx}{|x-1|}$ b) $\displaystyle\int_0^1 \ln x\,dx$

a) $\displaystyle\int_1^3 \frac{dx}{|x-1|}$

$1 < x \leq 3$

$x - 1 > 0$

$|x-1| = x-1$

$$= \int_{1}^3 \frac{dx}{x-1} = \lim_{t \to 1^+} \int_t^3 \frac{dx}{x-1}$$

$$= \lim_{t \to 1^+} \left(\ln|x-1| \Big|_t^3 \right)$$

$$= \lim_{t \to 1^+} \left(\ln 2 - \ln|t-1| \right)$$

$$= \ln 2 + \infty = +\infty$$

divergs

b) $\displaystyle\int_0^1 \ln x\,dx = \lim_{t \to 0^+} \int_t^1 \ln x\,dx$

$$= \lim_{t \to 0^+} \left(x\ln x - x \right)\Big|_t^1$$

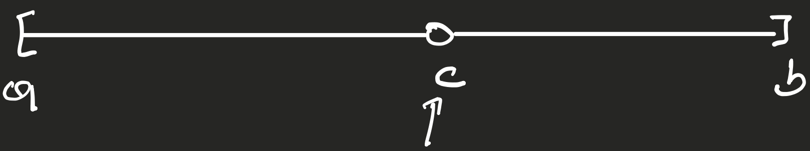
$$= \lim_{t \to 0^+} \left(-1 - \left(t\ln t - t \right) \right) = -1 - 0 = -1$$

converges

$$\lim_{t \to 0^+} t\ln t = \lim_{t \to 0^+} \frac{\ln t}{\frac{1}{t}} = \lim_{t \to 0^+} \frac{\frac{1}{\cancel{t}}}{-\frac{1}{t^{\cancel{2}}}}$$

$$= \lim_{t \to 0^+} -t = 0$$

* If f is continuous on $[a, b]$ and discontinuous at c in (a, b)



the integral

$$\int_a^b f(x)\,dx$$

is an improper integral of type II.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

For this to converge, both

$$\int_a^c f(x)\,dx \quad \text{and} \quad \int_c^b f(x)\,dx$$

must converge.

* <u>P- test for improper integrals of type II</u>

If $a>0$

$$\int_0^a \frac{dx}{x^p} = \begin{cases} \text{Converges if } p<1 \\ \text{diverges if } p \geq 1 \end{cases}$$

let $L = \int_{-2}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-2}^{1} = -1-\frac{1}{2} \quad -\frac{3}{2}$

(a) $L = -\frac{3}{2}$

(b) $L =$

(c) L does not exist ✓

(d) $L =$

e) None of these

$$\int_{-2}^{1} \frac{dx}{x^2} = \int_{-2}^{0} \frac{dx}{x^2} + \underbrace{\int_{0}^{1} \frac{dx}{x^2}}_{\to \text{ diverges}}$$

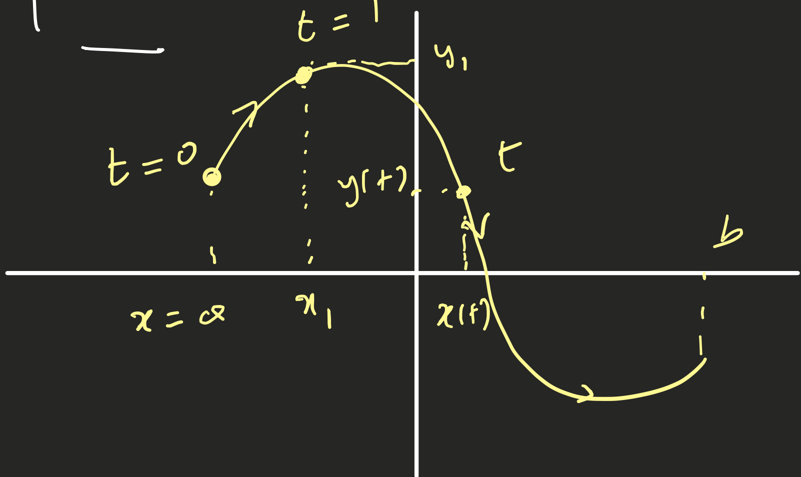
$x = x_0 + at = f(t)$

$y = y_0 + bt = g(t)$

$z = z_0 + ct = h(t)$

Parametric Equations

$y = f(x)$



$x = f(t)$, $y = g(t)$

$x = f(t)$ and $y = g(t)$, $a \le t \le b$

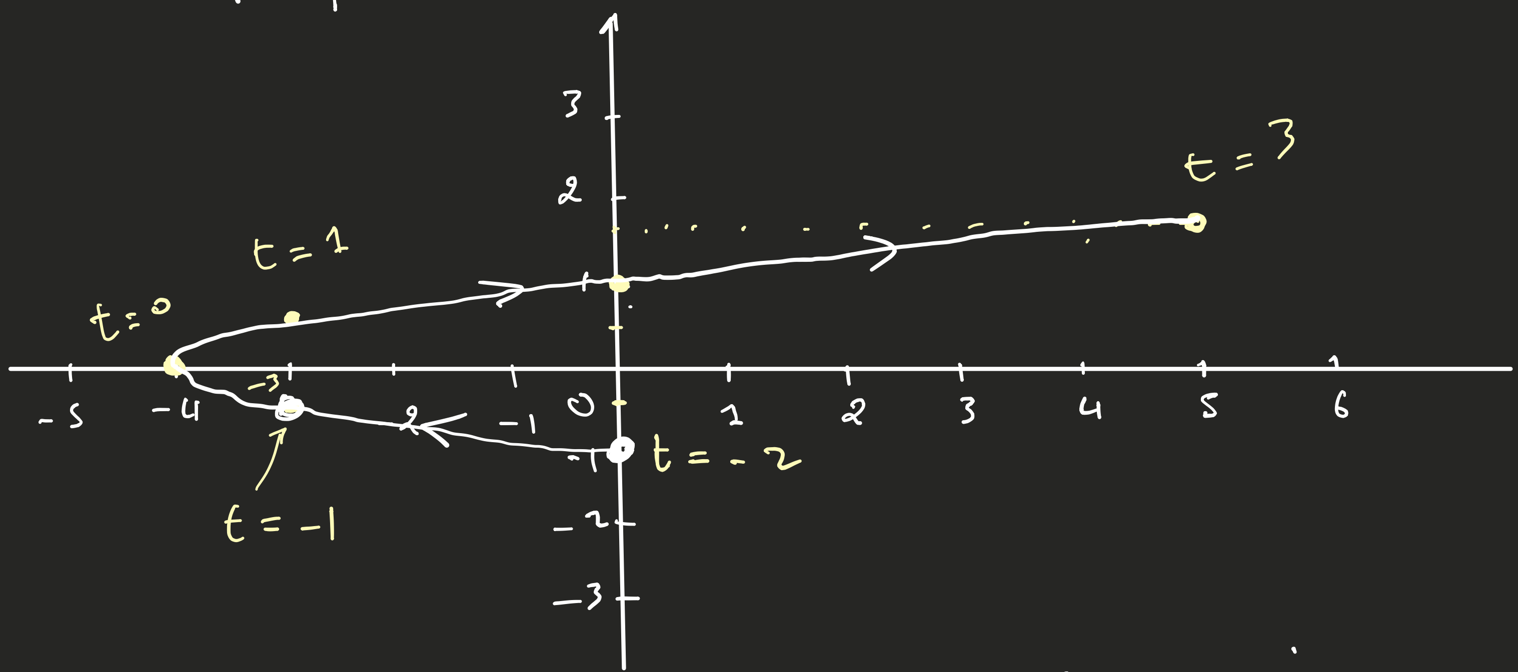
are called parametric equations.

t is called the parameter

Exple Sketch the curve described by the parametric equations

$x = t^2 - 4$ and $y = \frac{t}{2}$, $-2 \le t \le 3$

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

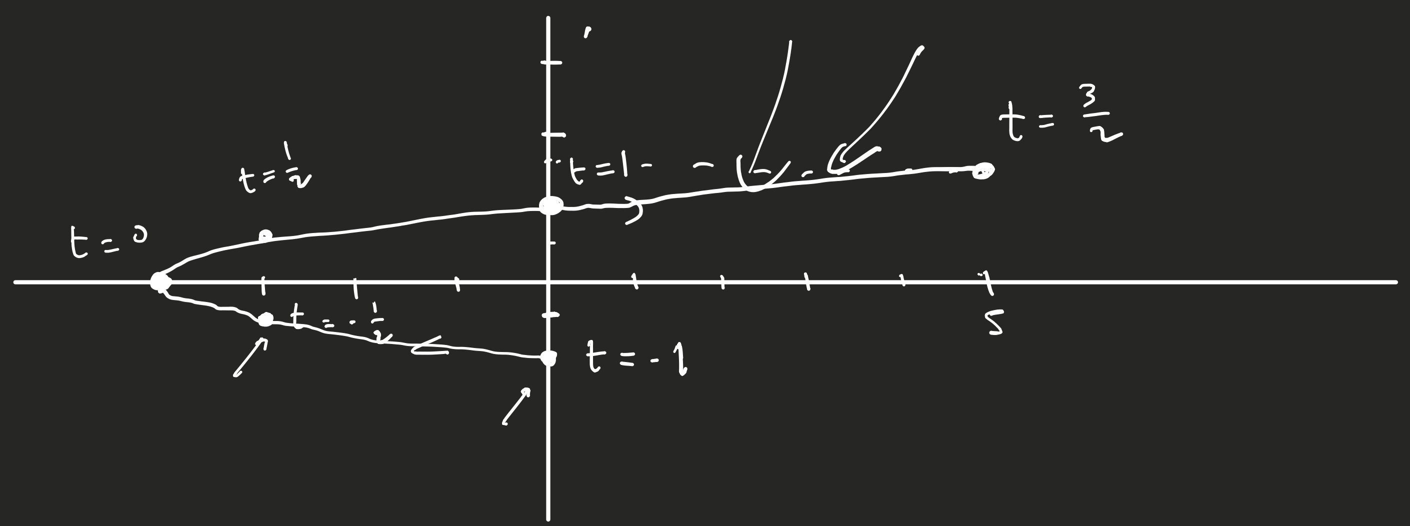


* The graph of a set of parametric equations is called a parametrized curve

Exple sketch the parametrized curve given by

$$x = 4t^2 - 4 \quad \text{and} \quad y = t \, , \quad -1 \le t \le \frac{3}{2} .$$

t	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$3/2$
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$3/2$



* From the examples above, we observe that a given parametrized curve may have several different parametric equations.

* Eliminating the parameter t to find the corresponding cartesian or rectangular equation ($y = f(x)$)

Consider the parametric equation

$x = f(t)$ and $y = g(t)$.

To eliminate the parameter t, do the following:

1.) Out of the two equations choose the one that is easier to solve for t and solve for t.

2.) plug the expression of t into the other equation to obtain the corresponding rectangular equation.

Exple Eliminate the parameter and sketch the corresponding parametrized curve.

1.)

2)

3.)

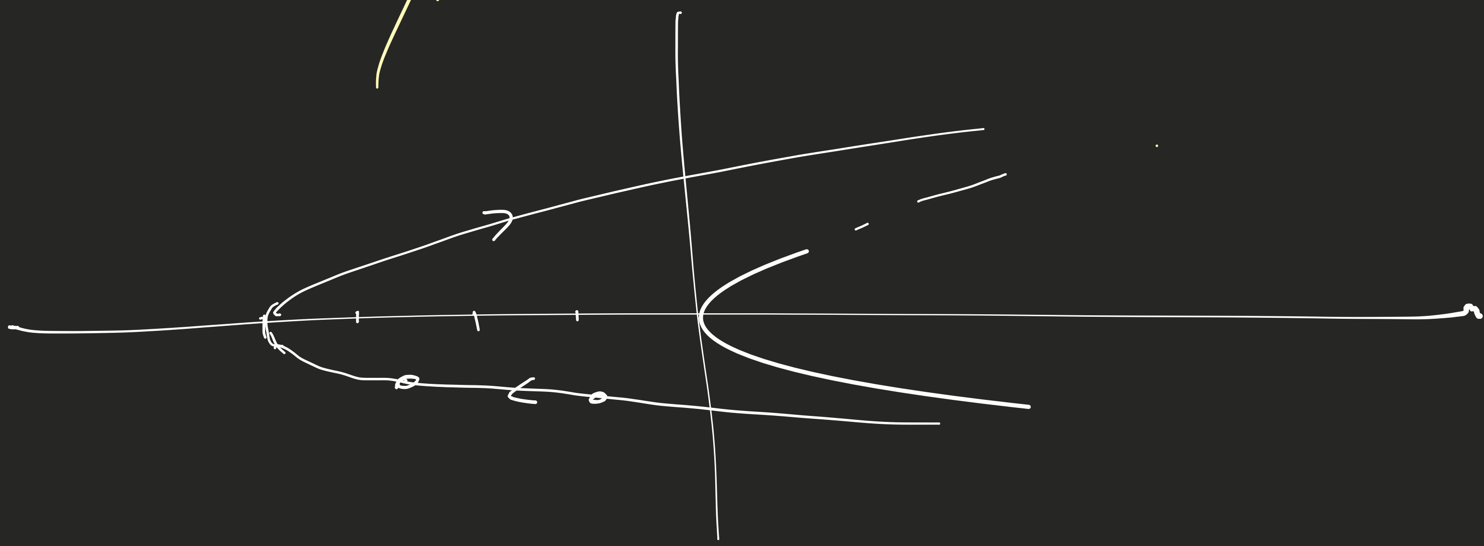
4)

$x = t^2 - 4$, $\underbrace{y = \frac{t}{2}}$, $-2 \le t \le 3$

$t = 2y$

$x = (2y)^2 - 4 = 4y^2 - 4$

$$\boxed{x = 4y^2 - 4} \qquad -1 \leq y \leq \frac{3}{2}$$



$x = \underbrace{4t^2 - 4}_{1}$, $y = t$, $-1 \leq t \leq \frac{3}{2}$

$t = y$

$x = 4y^2 - 4 \qquad -1 \leq y \leq \frac{3}{2}$

$x = \dfrac{1}{\sqrt{t+1}}$, $y = \dfrac{t}{t+1}$, $t > -1$

$x^2 = \dfrac{1}{t+1} \longleftrightarrow t + 1 = \dfrac{1}{x^2}$

$t = \dfrac{1}{x^2} - 1$

$y = \dfrac{t}{t+1} = t \cdot \dfrac{1}{t+1} = \left(\dfrac{1}{x^2} - 1\right) x^2$

$y = 1 - x^2$, $x > 0$

$-1,$

$t > -1$

t	0	3
x	1	$1/2$
y	0	$3/4$

$x = 3\cos\theta$, $y = 3\sin\theta$, $0 \le \theta \le 2\pi$

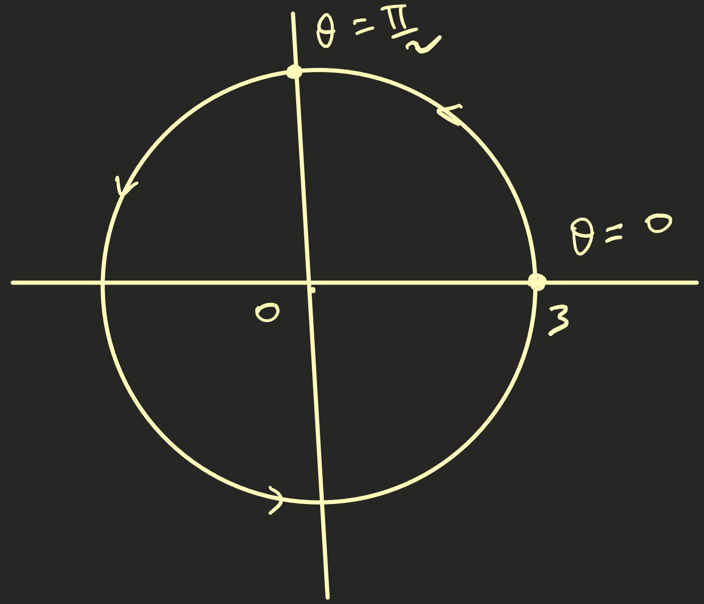
$\frac{x}{3} = \cos\theta$ $\frac{y}{3} = \sin\theta$

$\frac{x^2}{9} = \cos^2\theta$ $\frac{y^2}{9} = \sin^2\theta$

$\frac{x^2}{9} + \frac{y^2}{9} = \cos^2\theta + \sin^2\theta = 1$

$\frac{x^2}{9} + \frac{y^2}{9} = 1 \quad \Leftarrow$

$x^2 + y^2 = 9$



§ calculus with parametric equations

$$x = f(t) \quad , \quad y = g(t)$$

$$\frac{dy}{dx} = ?$$

$$y = F(x)$$

$$g(t) = \underbrace{F(f(t))}$$

$$g'(t) = \underbrace{F'(f(t))} \cdot f'(t)$$

$$g'(t) = F'(x)\, f'(t)$$

$$F'(x) = \frac{g'(t)}{f'(t)} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Let C be the parametrized curve given by

$x = f(t)$ and $y = g(t)$.

The slope of the curve at (x, y) is given

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$ with $\frac{dx}{dt} \neq 0$.

$$\frac{d\,\square}{dx} = \frac{\frac{d\,\square}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

Exple Consider the curve given by

$x = \sqrt{t}$ and $y = \frac{1}{4}(t^2 - 4)$, $t \geq 0$

a) Find the equation of the tangent line to the curve at the point $(2, 3)$

b) Find the second derivative at the point $(2,3)$

a) $y = m(x-2) + 3$

$m =$ slope at $(2,3)$.

$$\text{slope} = \frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{t}{2}}{\frac{1}{2\sqrt{t}}} = \frac{t}{2}\cdot 2\sqrt{t} = t\sqrt{t} = t^{3/2}$$

$\frac{dy}{dx}$ at $(2,3)$

We need to find the t that corresponds to the point $(2,3)$, with $x = 2$, $y = 3$.

$x = \sqrt{t}$

$y = \frac{1}{4}(t^2 - 4)$

$x = 2$, $y = 3$

$$\begin{cases} \sqrt{t} = 2 \\ \frac{1}{4}(t^2-4) = 3 \end{cases} \Longleftrightarrow \begin{cases} t = 4 \\ t = 4 \text{ or } t = -4 \end{cases}$$

$\frac{1}{4}(t^2-4) = 3$

$t^2 - 4 = 12$

$t^2 = 16$

$t = 4$ or $t = -4$

$t = 4$

$(2,3) \longleftrightarrow t = 4$

slope at $(2,3)$ is $m = 4\sqrt{4} = 8$

the equation of the tangent line is

$$y = 8(x-2)+3$$

$$y = 8x - 16 + 3$$

$$y = 8x - 13$$

b) $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(t^{\frac{3}{2}}\right)}{\frac{d}{dt}\left(t^{\frac{1}{2}}\right)}$$

$$= \frac{\frac{3}{2}t^{\frac{1}{2}}}{\frac{1}{2\sqrt{t}}} = \frac{3}{\cancel{2}}\sqrt{t}\cdot\cancel{2}\sqrt{t} = 3t$$

$$\frac{d^2y}{dx^2} = 3t$$

$$\frac{d^2y}{dx^2} \text{ at } (2,3) = 3(4) = 12.$$