Horizontal & Vertical Tangent (Parametric)

$y = f(x)$

Horizontal tangent
$f'(x) = 0$

Vertical tangent
$f'(x) = \pm \infty$

Consider a set of parametric equations given by
$x = f(t)$ and $y = g(t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

If $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

then the curve given by $x = f(t)$ and $y = g(t)$ has a horizontal tangent at t.

If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

then the curve given by $x = f(t)$ and $y = g(t)$ has a vertical tangent at t.

Example

A curve C is defined by the parametric equations
$x = t^2$ and $y = t^3 - 3t$

1. Show that the curve has two tangents at point $(3, 0)$ and find the equations
2. Find the points on C where the tangent is horizontal or vertical.
3. Determine where the curve is concave up or concave down.
4. Sketch the curve C.

1. $x = t^2$ $\quad y = t^3 - 3t$

$(3, 0) \longrightarrow \begin{cases} x = 3 \\ y = 0 \end{cases}$

$t^2 = 3$ $\quad t^3 - 3t = 0$

$t = \pm\sqrt{3}$ $\quad t(t^2 - 3) = 0$

$t = 0$ OR $t = \pm\sqrt{3}$

↳ Rejected as it doesn't satisfy other equation

$\therefore \underline{t = \pm\sqrt{3}}$

Since there are two distinct values of t,
there will be two tangents

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}$$

@ $t = \sqrt{3}$

$$\frac{dy}{dx} = \frac{6}{2\sqrt{3}} = \underline{\sqrt{3}}$$

@ $t = -\sqrt{3}$

$$\frac{dy}{dx} = \frac{6}{-2\sqrt{3}} = \underline{-\sqrt{3}}$$

Equation of tangent lines at $(3, 0)$

$y = \sqrt{3}(x - 3) = x\sqrt{3} - 3\sqrt{3}$

$y = -\sqrt{3}(x - 3) = 3\sqrt{3} - x\sqrt{3}$

2. Horizontal Tangent

$\frac{dy}{dt} = 0 \quad \frac{dx}{dt} \neq 0$

$3t^2 - 3 = 0 \qquad\qquad 2t = 0$

$3t^2 = 3 \qquad\qquad t = 0$

$t^2 = 1$

$t = \pm 1$

at $t = 1$

$x = (1)^2 \qquad y = t^3 - 3t$

$x = 1 \qquad y = (1)^3 - 3(1)$

$= -2$

at $t = -1$

$x = (-1)^2 \qquad y = (-1)^3 - 3(-1)$

$x = 1 \qquad = -1 + 3$

$= 2$

$\therefore$ C has horizontal tangents at points $(1, -2)$ and $(1, 2)$

Vertical tangent

$\frac{dx}{dt} = 0 \quad \frac{dy}{dt} \neq 0$

$2t = 0 \qquad\qquad 3t^2 - 3 = 0$

$t = 0 \qquad\qquad t^2 =$

$t = \pm 1$

@ $t = 0$

$x = 0$

$y = 0$

$\therefore$ C has a vertical tangent at $(0,0)$

3. $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$

$$= \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t}$$

$$= \frac{\frac{(6t)(2t)-(3t^2-3)(2)}{(2t)^2}}{2t}$$

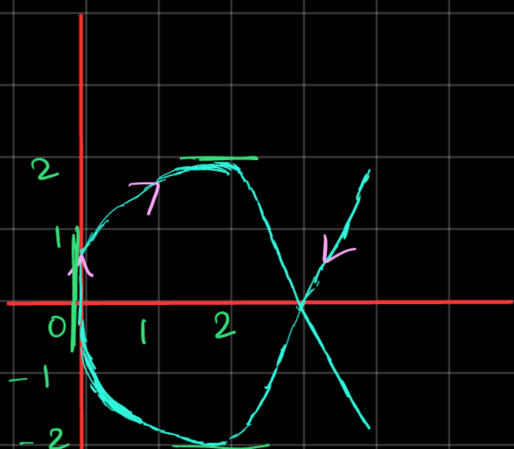
$$= \frac{\frac{12t^2-6t^2+6}{4t^2}}{2t}$$

$$= \frac{6t^2+6}{8t^3}$$

$$= \frac{3t^2+3}{4t^3}$$

* If $t > 0 \longrightarrow$ concave up
* If $t < 0 \longrightarrow$ concave down

4.



Arc Length (Parametric)

If a curve C is given by

$x = f(t)$ and $y = g(t)$ $\quad a \leq t \leq b$

where both f' and g' are continuous

then the arc length of the curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example

Find the arc length of the parametrized curve given by

$x = 5\cos(t) - \cos(5t)$ $\qquad y = 5\sin(t) - \sin(5t)$

$0 \leq t \leq \frac{\pi}{2}$

$$\frac{dx}{dt} = -5\sin t + 5\sin(5t)$$

$$\frac{dy}{dt} = 5\cos t - 5\cos(5t)$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 25\sin^2 5t + 25\sin^2 t - 50\sin(t)\sin(5t)$$
$$+ 25\cos^2 5t + 25\cos^2 t - 50\cos(t)\cos(5t)$$

$$= 25 + 25 - 50(\sin t \sin 5t + \cos t \cos 5t)$$

$\hookrightarrow \cos(a-b) = \cos a \cos b - \sin a \sin b$

$$= 50 - 50\cos(-4t)$$

$$= 50 - 50\cos 4t$$

$$= 50(1 - \cos 4t) \longrightarrow \cos 2\alpha = 1 - 2\sin^2\alpha$$

$$= 50(1 - 1 + 2\sin^2 2t)$$

$$= 100 \sin^2 2t$$

$$L = \int_0^{\pi/2} \sqrt{100 \sin^2 2t} \; dt$$

$$= \int_0^{\pi/2} |10 \sin 2t| \; dt$$

$$= \int_0^{\pi/2} 10 \sin 2t \; dt$$

$$= -5 \left[\cos 2t\right]_0^{\pi/2}$$

$$= -5 (\cos \pi - \cos 0)$$

$$= -5 (-1 - 1)$$

$$= -5 (-2)$$

$$= \underline{10}$$

Area of Surface of Revolution

If a smooth curve is given by

$x = f(t)$ and $y = g(t)$ $\quad a \leq t \leq b$

where both f' and g' are continuous and the curve does not cross itself, then the area of the surface of revolution obtained is given by

1. $$S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

If rotation about x-axis, $g(t) > 0$

2. $$S = 2\pi \int^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 \quad \left(\frac{dy}{dt}\right)^2} \; dt$$

$$\int_a \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If rotation about y-axis, $f(t) > 0$

Example

Find the area obtained by rotating the curve about the x-axis

$x = 3\cos t$ and $y = 3\sin t$ $\quad 0 \leq t \leq \pi/3$

$$S = 2\pi \int_0^{\pi/3} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt$$

$$= 2\pi \int_0^{\pi/3} \sin t \sqrt{9}\, dt$$

$$= 18\pi \int_0^{\pi/3} \sin t\, dt$$

$$= -18\pi \left[\cos t\right]_0^{\pi/3}$$

$$= -18\pi \left[\frac{1}{2} - 1\right]$$

$$= \frac{18\pi}{2}$$

$$= 9\pi$$

Polar Coordinates

In the polar coordinate system, a point P is represented by the pair (r, θ) where

r is the distance from the origin to the point P.

θ is the angle, counterclockwise, from the +ve x-axis to the line segment joining the origin and P.

Remark

$$\boxed{\text{If } r > 0, \text{ then } (-r, \theta) = (r, \theta + \pi)}$$

Example

Plot the following points whose polar coordinates are

$\left(3, \frac{5\pi}{4}\right)$ $(2, 3\pi)$ $\left(\frac{1}{2}, -\pi/3\right)$ $(-4, 3\pi/4)$