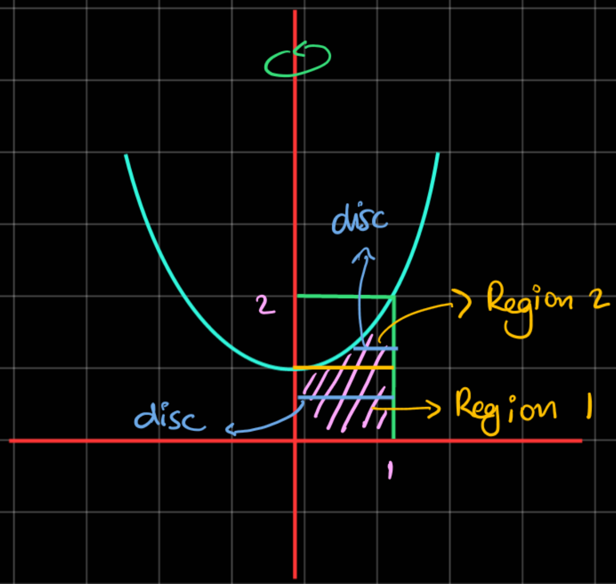
Example

Find the volume of the solid formed by rotating the region bounded by the graphs of $y = x^2+1$, $y=0$, $x=0$, and $x=1$ about the y axis

A) By the disc method

B) By the shell method



A)

$$V_1 = \int_0^1 (\pi \times 1^2)\, dy$$

$$V_2 = \int_1^2 \left(\pi \cdot 1^2 - \pi \left(\sqrt{y-1}\right)^2 \right) dy$$

$$V_1 = \pi$$

$$V_2 = \int_1^2 \left(\pi - \pi (y-1)\right) dy$$

$$= \pi \int_1^2 (1 - y + 1)\, dy$$

$$= \pi \int_1^2 (2-y)\, dy$$

$$= \pi \left[2y - \frac{y^2}{2} \right]_1^2$$

$$= \pi \left[4 - 2 - (2 - 0.5) \right]$$

$$= \pi \left[4 - 2 - 2 + 0.5 \right]$$

$$= \frac{\pi}{2}$$

$V = V_1 + V_2$

$$= \pi + \frac{\pi}{2}$$

$$= \frac{3\pi}{2}$$

B) $V = \int_0^1 2\pi x (x^2+1)\, dx$

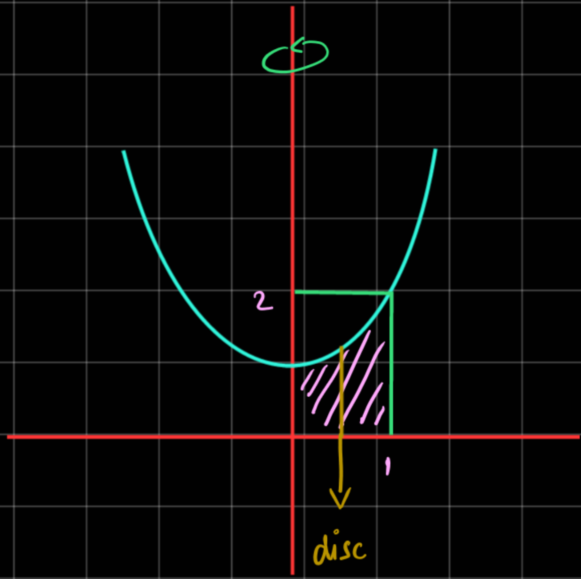
$$= 2\pi \int_0^1 (x^3 + x)\, dx$$

$$= 2\pi \left[\frac{x^4}{4} + \frac{x^2}{2} \right]_0^1$$

$$= 2\pi \left[\frac{1}{4} + \frac{1}{2} \right]$$

$$= \frac{\cancel{2}\pi \times 3}{\cancel{4}2}$$

$$= \frac{3\pi}{2}$$



Techniques of Integration

Integration by parts

$$\int u'v\, dx = uv - \int uv'\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

Example

Evaluate $\int x \sin x\, dx$

$u = x \qquad u' = 1$

$v' = \sin x \qquad v = -\cos x$

$$\int x \sin x \, dx = -x\cos x - \int -\cos x \, dx$$
$$= -x\cos x + \int \cos x \, dx$$
$$= -x\cos x + \sin x + C$$

Example

Evaluate the following integrals

A) $\int x^2 e^x \, dx$

$u = x^2 \qquad u' = 2x$

$v' = e^x \qquad v = e^x$

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$$
$$= x^2 e^x - 2\int x e^x \, dx$$

$u = x \qquad u' = 1$

$v' = e^x \qquad v = e^x$

$$\int x^2 e^x \, dx = x^2 e^x - 2\left[x e^x - \int e^x \, dx\right]$$
$$= x^2 e^x - 2x e^x + 2e^x + C$$

B) $\int x^3 \ln x \, dx$

$u = \ln x \qquad u' = 1/x$

$v' = x^3 \qquad v = \frac{x^4}{4}$

$$\int x^3 \ln x \, dx = \frac{x^4 \ln x}{4} - \int \frac{1}{x} \times \frac{x^4}{4} \, dx$$
$$= \frac{x^4 \ln x}{4} - \frac{1}{4}\int x^3 \, dx$$
$$= \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C$$
$$= \frac{x^4}{4}\left(\ln x - \frac{1}{4}\right) + C$$

c) $\int 1 \cdot \ln x \, dx$

$u = \ln x \qquad u' = 1/x$

$v' = 1 \qquad v = x$

$$\int \ln x \, dx = x \ln x - \int x \times \frac{1}{x} \, dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x$$

$$= x(\ln x - 1) + C$$

D) $\int e^x \sin x \, dx$

Let $A = \int e^x \sin x \, dx$

$u = e^x \qquad u' = e^x$

$v' = \sin x \qquad v = -\cos x$

$$A = -e^x \cos x - \int -e^x \cos x \, dx$$

$$= -e^x \cos x + \int e^x \cos x \, dx$$

$u = e^x \qquad u' = e^x$

$v' = \cos x \qquad v = \sin x$

$$A = -e^x \cos x + \left[e^x \sin x - \int e^x \sin x \, dx \right]$$

$$A = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$= -e^x \cos x + e^x \sin x - A$$

$$2A = e^x \sin x - e^x \cos x$$

$$A = \frac{1}{2}\left[e^x \sin x - e^x \cos x\right] + C$$

$$\therefore \int e^x \sin x \, dx = \frac{1}{2}\left[e^x \sin x - e^x \cos x\right] + C$$

E) $\int \sec^3 x \, dx$

$$A = \int \sec^2 x \sec x \, dx$$

$u = \sec x$ $\quad\quad$ $u' = \sec x \tan x$

$v' = \sec^2 x$ $\quad\quad$ $v = \tan x$

$$A = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$= \sec x \tan x + \ln|\sec x + \tan x| - A$$

$$2A = \sec x \tan x + \ln|\sec x + \tan x|$$

$$A = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C$$

$$\therefore \int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

Definite Integrals - Integration by Parts

$$\int_a^b u'v \, dx = uv\Big|_a^b - \int_a^b uv' \, dx$$

Example

Evaluate $\int_0^1 x \cos x \, dx$

$u = x \qquad u' = 1$

$v' = \cos x \qquad v = \sin x$

$x \sin x \Big|_0^1 - \int_0^1 \sin x$

$= [x \sin x]_0^1 - [-\cos x]_0^1$

$= \sin(1) - (-\cos 1 + \cos 0)$

$= \sin(1) + \cos(1) - 1$

$\int_1^2 x \ln x \, dx$

$u = \ln x \qquad u' = 1/x$

$v' = x \qquad v = \frac{x^2}{2}$

$\frac{x^2 \ln x}{2}\Big|_1^2 - \int_1^2 \frac{1}{\cancel{x}} \times \frac{x^{\cancel{2}}}{2} \, dx$

$= \frac{x^2 \ln x}{2} - \frac{1}{2}\int_1^2 x \, dx$

$= \frac{x^2}{2} \ln x \Big|_1^2 - \frac{1}{2} \frac{x^2}{2}\Big|_1^2$

$= 2 \ln 2 - \frac{1}{2}\left(2 - \frac{1}{2}\right)$

$= 2 \ln 2 - \frac{3}{4}$

Integration of Rational Functions by Partial Fraction Decomposition

$$\int \frac{P(x)}{Q(x)}\,dx$$ where both $P(x)$ and $Q(x)$ are polynomial functions

Case I : $\deg P(x) < \deg Q(x)$

Case II : $\deg P(x) \geq \deg Q(x)$

$$\frac{P(x)}{Q(x)} = f(x) + \frac{r(x)}{Q(x)}$$ where $\deg r(x) < \deg Q(x)$

$$\int \frac{P(x)}{Q(x)} = \int f(x)\,dx + \underbrace{\int \frac{r(x)}{Q(x)}\,dx}_{\text{Case I}}$$

Case I : $\deg P(x) < \deg Q(x)$

Rule 1

If $Q(x)$ can be decomposed as a product of distinct linear factors (none of which is repeated), then the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ is given by

$$\int \frac{P(x)}{Q(x)} = \int \frac{A_1}{x - c_1} + \int \frac{A_2}{x - c_2} \text{ ------- } \int \frac{A_n}{x - c_n}$$

$$Q(x) = a(x - c_1)(x - c_2) \text{ ---- } (x - c_n)$$

Example

Evaluate the integral of $\int \frac{dx}{x^2 - 5x + 6}$

$$x^2 - 5x + 6 = x(x-3) - 2(x-3)$$
$$= (x-3)(x-2)$$

$$\frac{1}{(x-3)(x-2)} = \frac{A_1}{(x-3)} + \frac{A_2}{(x-2)}$$

Shortcut Method $\Rightarrow$ FOR RULE 1 ONLY !!

for A_1 $\rightarrow$ Multiply both sides by A_1 denominator

$$\frac{x-\cancel{3}}{(\cancel{x-3})(x-2)} = \frac{A_1\cancel{(x-3)}}{\cancel{(x-3)}} + \frac{A_2(x-3)}{(x-2)}$$

$$\frac{1}{x-2} = A_1 + \frac{(x-3)A_2}{x-2}$$

$\rightarrow$ let $x = 3$

$$\frac{1}{3-2} = A_1 + 0$$

$$A_1 = 1$$

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for A_2 $\quad \dfrac{1}{x-3} = \dfrac{A_1(x-2)}{x-3} + A_2$

let $x = 2$

$$\underline{A_2 = -1}$$

$$\frac{1}{(x-3)(x-2)} = \frac{1}{x-3} - \frac{1}{x-2}$$

$$\int \frac{dx}{x^2-5x+6} = \int \frac{dx}{x-3} - \int \frac{dx}{x-2}$$

$$= \ln|x-3| - \ln|x-2| + C$$

$$= \ln\left|\frac{x-3}{x-2}\right| + C$$

General Method

$$\frac{1}{(x-3)(x-2)} = \frac{A_1}{x-3} + \frac{A_2}{x-2}$$

$$1 = A_1(x-2) + A_2(x-3)$$

at $x = 2$

$$-A_2 = 1$$

$$A_2 = -1$$

at $x = 3$

$$A_1 = 1$$

$1 = A_1(x-2) + A_2(x-3)$

$1 = A_1x - 2A_1 + A_2x - 3A_2$

$1 = (A_1 + A_2)x - 2A_1 - 3A_2$

$A_1 + A_2 = 0$

$A_1 = -A_2$

$-2A_1 - 3A_2 = 1$

$2A_2 - 3A_2 = 1$

$-A_2 = 1$

$A_2 = -1$

$\therefore A_1 = -(-1)$

$= 1$

Example

Evaluate $\int \frac{x+2}{x^3 - x^2 - 2x} dx$

$x^3 - x^2 - 2x$

$= x(x^2 - x - 2)$

$= x(x+1)(x-2)$

$$\frac{x+2}{x(x+1)(x-2)} = \frac{A_1}{x} + \frac{A_2}{x+1} + \frac{A_3}{x-2}$$

$$\frac{x+2}{(x+1)(x-2)} = \frac{x A_2}{x+1} + \frac{x A_3}{x-2} + A_1$$

let $x = 0$

$A_1 = -1$

$$\frac{x+2}{x(x-2)} = \frac{A_1(x+1)}{x} + A_2 + \frac{A_3(x+1)}{x-2}$$

let $x = -1$

$$\frac{1}{} = A_2$$

3

$A_2 = -1/3$

$$\frac{x+2}{x(x+1)} = \frac{A_1(x-2)}{x} + \frac{A_2(x-2)}{x+1} + A_3$$

let $x = 2$

$$\frac{4}{6} = A_3$$

$A_3 = 2/3$

$$\int \frac{x+2}{x^3 - x^2 - 2x}\, dx = -\int \frac{1}{x}\, dx + \frac{1}{3}\int \frac{1}{x+1}\, dx + \frac{2}{3}\int \frac{1}{x-2}\, dx$$

$$= -\ln|x| + \frac{1}{3}\ln|x+1| + \frac{2}{3}\ln|x-2| + C$$