4. $\lim_{n\to\infty} \frac{n!}{n^n} \longrightarrow$ Not a real number

∴ This cannot be solved normally

Sandwich / Squeezing Theorem (for sequences)

If

$$b_n \leq a_n \leq c_n \quad \forall n$$

and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$$

then

$$\lim_{n\to\infty} a_n = L$$

As a consequence of this, we have

if $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

$$\underset{\downarrow\, 0}{-|a_n|} \leq \underset{\downarrow\, 0}{a_n} \leq \underset{\downarrow\, 0}{|a_n|}$$

4. $\lim_{n\to\infty} \frac{n!}{n^n}$

$$\frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdot 4\cdots\cdots n}{n\cdot n\cdot n\cdot n\cdots\cdots n}$$

$$= \frac{1}{n}\cdot\frac{\boxed{2\cdot 3\cdot 4}\cdots n}{n\cdot n\cdot n\cdots n}$$

much smaller than n

$$0 \leq \frac{2\cdot 3\cdot 4\cdots n}{n\cdot n\cdot n\cdots n} \leq 1$$

$$0 \leq \frac{1}{n}\cdot\frac{2\cdot 3\cdot 4\cdots n}{n\cdot n\cdot n\cdots n} \leq \frac{1}{n}$$

$$\frac{n!}{n^n}$$

$$\lim_{n\to\infty} 0 = 0 \qquad \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

$\therefore$ Using Sandwich Theorem

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Series

Consider the sequence

$a_1, a_2, a_3, \ldots \ldots a_n, \ldots$

Adding the terms of this sequence results in what is called a series.

Series is the sum of the terms of a sequence

$$S = a_1 + a_2 + a_3 + \ldots\ldots + a_n + \ldots$$

$$= \sum_{n=1}^{\infty} a_n \text{ or } \sum_{k=1}^{\infty} a_k \text{ or } \sum_{i=1}^{\infty} a_i$$

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to+\infty} \sum_{n=1}^{N} a_n$$

Given a series $\sum_{n=1}^{\infty} a_n$, the finite sum

$$\sum_{n=1}^{N} a_n$$

is called the n-th partial sum of the series.

It is denoted by S_N

$$S_N = \sum^{N} a_n$$

$n=1$

S_N is a sequence.

Convergence / Divergence

Consider the series $\sum_{n=1}^{\infty} a_n$

If

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = S$$

exists, then the series converges and its value is S. Otherwise, we say that the series diverges.

$$\sum_{n=1}^{\infty} a_n = S$$

Example

Determine convergence or divergence of the following series.

1. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)}$$

$$\frac{1}{n(n+1)} = \frac{1+n-n}{n(n+1)}$$

$$= \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)}$$

$$= \frac{1}{n} - \frac{1}{n+1}$$

$$S_N = \sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$\rightarrow$ Telescopic Phenomenon

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right)$$

$$= 1 - \frac{1}{N+1}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right)$$

$$= \lim_{N\to\infty} (1 - 0)$$

$$= 1$$

$\therefore$ Series converges and its value is $S = 1$.

2. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

$$S_N = \sum_{n=1}^{N} \ln\left(\frac{n}{n+1}\right)$$

$$= \sum_{n=1}^{N} \ln(n) - \ln(n+1)$$

Telescopic Phenomenon

$$= [\ln(1) - \cancel{\ln(2)}] + [\cancel{\ln(2)} - \cancel{\ln(3)}] + [\cancel{\ln(3)} - \cancel{\ln(4)}] + \cdots [\cancel{\ln(N)} - \ln(N+1)]$$

$$= \ln(1) - \ln(N+1)$$

$$= -\ln(N+1)$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} -\ln(N+1)$$

$$= \lim_{N\to\infty} -\ln(\infty)$$

$$= \underline{-\infty}$$

$\therefore$ Diverges

Series for which when you write out the terms, you will get a telescoping phenomenon, are called Telescoping Series.

> Telescoping Series & Geometric series are the only two series for which convergence or divergence can be determined AND find the sum (for converging). For others, we can only determine convergence or divergence.

Geometric Series

Geometric Sequence $\Rightarrow$ $a_n r^{n-1}$

A series of the form

$$\sum_{n=1}^{\infty} a_1 r^{n-1} \quad \text{or} \quad \sum_{n=0}^{\infty} a_1 r^n$$

is called geometric series.

$$S_N = \sum_{n=1}^{N} a_1 r^{n-1}$$

$$S_N = a_1 + \cancel{a_1 r} + \cancel{a_1 r^2} + \cancel{a_1 r^3} + \cancel{\cdots} + \cancel{a_1 r^{N-2}} + \cancel{a_1 r^{N-1}}$$

$$r S_N = \cancel{a_1 r} + \cancel{a_1 r^2} + \cancel{a_1 r^3} + \cancel{a_1 r^4} + \cancel{\cdots} + \cancel{a_1 r^{N-1}} + a_1 r^N$$

$$S_N - r S_N = a_1 - a_1 r^N$$

$$S_N (1 - r) = a_1 (1 - r^N)$$

If $r \neq 1$

$$S_N = \frac{a_1 (1 - r^N)}{1 - r}$$

Now, using the fact that

$$\lim_{N \to \infty} r^N = \begin{cases} 0 & \text{if } |r| < 1 \end{cases}$$

diverges if $|r| > 1$

Geometric Series

Consider the geometric series $\sum_{n=1}^{\infty} a_1 r^{n-1}$

If $|r| < 1$, then the series converges and its sum is

$$S_N = \frac{a_1}{1-r}$$

If $|r| > 1$, then the series diverges

$$\sum_{n=1}^{\infty} 3\left(\frac{5}{7}\right)^n \implies 3\left(\frac{5}{7}\right)^{n-1+1} = \left(\frac{3\times 5}{7}\right)\left(\frac{5}{7}\right)^{n-1}$$

$r = \frac{5}{7}$

$|r| < 1 \therefore$ Converges

$a_1 \neq 3$

$a_1 = 15/7$

$r = 5/7$

$$S_N = \frac{15/7}{1-5/7} = \frac{15}{2}$$

$$\sum_{n=0}^{\infty} 3\left(\frac{5}{7}\right)^n \implies a = 3,\ r = 5/7$$

$$S_N = \frac{3}{2/7} = \frac{21}{2}$$

Example

Determine convergence or divergence of the following geometric series.

1. $\sum_{n=1}^{\infty} 4^{3-n}$

$$= \sum_{n=1}^{\infty} 4^{3-n}$$

$$= \sum_{n=1}^{\infty} 4^3 \times 4^{-n}$$

$$= \sum_{n=1}^{\infty} 64 \times \left(\frac{1}{4}\right)^n$$

$$= \sum_{n=1}^{\infty} 64 \times \left(\frac{1}{4}\right)^{n-1+1}$$

$$= \sum_{n=1}^{\infty} \frac{64}{4} \times \left(\frac{1}{4}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} 16 \left(\frac{1}{4}\right)^{n-1}$$

$$S = \frac{16}{1 - \frac{1}{4}}$$

$$= \frac{64}{3}$$

2. $\sum_{n=1}^{\infty} 5^{2n} \cdot 7^{1-n}$

$$= \sum_{n=1}^{\infty} 5^{2n} \cdot 7 \cdot 7^{-n}$$

$$= \sum_{n=1}^{\infty} 7 \cdot \left(\frac{5^2}{7}\right)^{n+1-1}$$

$$= \sum_{n=1}^{\infty} \left(\frac{7 \times 25}{7}\right) \left(\frac{25}{7}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} 25 \left(\frac{25}{7}\right)^{n-1}$$

$|r| > 1$

$\therefore$ Diverges

Divergence Test

Consider the series $\sum_{n=1}^{\infty} a_n$

If

$\lim_{n\to\infty} a_n \neq 0$

then the series diverges

Proof ↑ (Not required)

$\sum_{n=1}^{\infty} a_n$ converges $\Longleftrightarrow$ $\lim_{N\to\infty} S_N = S$ exists

$S_N = a_1 + a_2 + a_3 + \dots + a_N$

$S_{N-1} = a_1 + a_2 + a_3 + \dots + a_{N-1}$

$S_N - S_{N-1} = a_N$

$\lim_{N\to\infty} S_N = S$

$a_N = \underset{\downarrow\, S}{S_N} - \underset{\downarrow\, S}{S_{N-1}}$

$\lim_{N\to\infty} S_{N-1} = S$

$\lim_{N\to\infty} a_N = \lim_{N\to\infty} S_N - \lim_{N\to\infty} S_{N-1}$

$= S - S$

$= \underline{0}$

$\therefore$ If $\lim_{n\to\infty} a_n = 0$

then series converges

else series diverges

Properties of Series

If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ converge, then we have the following

1. $\sum^{\infty} ca_n = c\sum^{\infty} a_n$

$$\sum_{n=1} ca_n \qquad \sum_{n=1}$$

2. $$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

Example

Find the sum of the series.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2+n} + 2^{3-n} \right)$$

Telescoping

Geometric