UNIVERSITY
OF WOLLONGONG
IN DUBAI

Faculty of Engineering
and Information Sciences

Tutorial 1-Math142

Differential Equations

Solve ① $\boxed{\frac{dy}{dx} - 4y = 32x^2}$ (I)

linear ODE $\frac{dy}{dx} + P(x)y = \varphi(x)$

$P(x) = -4 \quad \varphi(x) = 32x^2$

$\mu(x) = e^{\int p(x)\,dx} \Rightarrow \mu(x) = e^{\int -4\,dx}$
$= e^{-4x}$

$(I) \xrightarrow{e^{-4x}} e^{-4x}\frac{dy}{dx} - 4e^{-4x}y = 32x^2 e^{-4x}$

$\frac{d}{dx}\left(e^{-4x}\cdot y\right) = 32x^2 e^{-4x}$

$\int d\left(e^{-4x}\cdot y\right) = \int 32x^2 e^{-4x}\,dx$

D		I
		e^{-4x}
$32x^2$	+	$-\frac{1}{4}e^{-4x}$
$64x$	−	$\frac{1}{16}e^{-4x}$
64	+	$-\frac{1}{64}e^{-4x}$
0		

$e^{-4x}\cdot y = -8x^2 e^{-4x} - 4xe^{-4x} - e^{-4x} + C$

$$\boxed{y = -8x^2 - 4x - 1 + Ce^{4x}}$$

(2) $2xy^3\,dx - (1-x^2)\,dy = 0$

Show this equation is separable and solve it.

We have to show that $\frac{dy}{dx} = f(x,y)$
$= h(x)\cdot g(y)$

$$2xy^3\,dx = (1-x^2)\,dy$$

$$2xy^3 = (1-x^2)\frac{dy}{dx} \rightarrow \frac{dy}{dx} = \frac{2xy^3}{1-x^2}$$

$$\frac{dy}{dx} = \frac{2x}{1-x^2}\cdot y^3 = h(x)\cdot g(y) \quad \text{separable}$$

$$\int \frac{dy}{y^3} = \int \frac{2x}{1-x^2}\,dx \Rightarrow \int y^{-3}\,dy = -1\int \frac{-2x}{1-x^2}\,dx$$

$$\frac{y^{-2}}{-2} = -1\cdot \ln|1-x^2| + c_1$$

$$y^{-2} = 2\ln|1-x^2| - 2c_1$$

$$y^{-2} = 2\ln|1-x^2| + C$$

$$\frac{1}{y^2} = \frac{2\ln|1-x^2|+C}{1}$$

$$\boxed{y^2 = \frac{1}{2\ln|1-x^2|+C}}$$

③ $\dfrac{dy}{dx} + \dfrac{2}{x}y = 2x^2y^2 \quad \ldots (I)$

Solve the Bernoulli equation

$$\frac{dy}{dx} + P(x)y = \varphi(x)y^n$$

$\boxed{n=2}$

$$(I) \xrightarrow{y^{-2}} \boxed{y^{-2}\frac{dy}{dx} + \frac{2}{x}y^{-1} = 2x^2} \quad (II)$$

let $V = y^{1-n} \Rightarrow V = y^{-1} \Rightarrow \dfrac{dV}{dx} = (-1)y^{-2}\cdot\dfrac{dy}{dx}$

$$(II) \xrightarrow{(-1)} -y^{-2}\frac{dy}{dx} - \frac{2}{x}y^{-1} = -2x^2$$

$$\boxed{\frac{dV}{dx} - \frac{2}{x}V = -2x^2} \quad (III)$$

linear equation in V

$$\frac{dV}{dx} + P(x)V = \varphi(x)$$

$$P(x) = -\frac{2}{x}$$

$$\mu(x) = e^{\int -\frac{2}{x} dx}$$

$$\mu(x) = e^{-2\ln x} = \cancel{e^{\ln x^{-2}}} \qquad (x>0)$$

$$\mu(x) = x^{-2}$$

$$(III) \xrightarrow{x^{-2}} \underbrace{x^{-2}\frac{dv}{dx} - 2x^{-3}v} = -2$$

$$\frac{d}{dx}\left(x^{-2}\cdot v\right) = -2 \Rightarrow \int d\left(x^{-2}\cdot v\right) = \int -2\, dx$$

$$x^{-2}\cdot v = -2x + C$$

$$v = -2x^3 + Cx^2$$

$$y^{-1} = -2x^3 + Cx^2$$

$$\frac{1}{y} = \frac{-2x^3 + Cx^2}{1}$$

$$y = \frac{1}{-2x^3 + Cx^2}$$

(4) $(y^3 + 4e^x y)\,dx + (2e^x + 3y^2)\,dy = 0 \quad (I)$

First show that the equation is not exact, then solve it by finding the integrating factor.

$M(x,y) = y^3 + 4e^x y \qquad N(x,y) = 2e^x + 3y^2$

$\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial x}$ No!

$\frac{\partial M}{\partial y} = 3y^2 + 4e^x \qquad \frac{\partial N}{\partial x} = 2e^x$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3y^2 + 4e^x - 2e^x}{2e^x + 3y^2}$$

$$= \frac{3y^2 + 2e^x}{3y^2 + 2e^x} = 1$$

$= f(x)$

$$\mu(x) = e^{\int f(x)\,dx} = e^{\int 1\,dx} = e^{x}$$

$$(I) \xrightarrow{e^{x}}$$

$$\underbrace{(y^3 e^{x} + 4e^{2x} y)}_{M'(x,y)}dx + \underbrace{(2e^{2x} + 3y^2 e^{x})}_{N'(x,y)}dx$$

$$\frac{\partial f}{\partial x} = M'(x,y) = y^3 e^{x} + 4e^{2x} y$$

$$f(x,y) = \int (y^3 e^{x} + 4e^{2x} y)\,dx + h(y)$$

$$= y^3 e^{x} + 2e^{2x} \cdot y + h(y)$$

$$\frac{\partial f}{\partial y} = 3y^2 e^{x} + 2e^{2x} + h'(y) = N'(x,y)$$

$$3y^2 e^{x} + 2e^{2x} + h'(y) = 2e^{2x} + 3y^2 e^{x}$$

$$h'(y) = 0$$

$$h(y) = c_1$$

$$f(x,y) = y^3 e^x + 2e^{2x} \cdot y + c_1$$

$$f(x,y) = c_2$$

$$y^3 e^x + 2e^{2x} \cdot y + c_1 = c_2$$

$$\therefore \boxed{y^3 e^x + 2e^{2x} \cdot y = c}$$

(5) Solve the following equation by computing the integrating factor

$$\boxed{\frac{dy}{dx} + y\tan x + \sin x = 0}$$

$$\frac{dy}{dx} + (\tan x)y = -\sin x \qquad (I)$$

$$\frac{dy}{dx} + P(x)y = \varphi(x)$$

$$P(x) = \tan x \Rightarrow \mu(x) = e^{\int \tan x \, dx}$$

$$\mu(x) = \cancel{e}^{\ln|\sec x|}$$

$$= \sec x$$

$$\int \tan x \, dx = -\ln|\cos x|$$

$$= -\ln\left|\frac{1}{\sec x}\right|$$

$$= \ln|\sec x|$$

$$(I) \xrightarrow{\sec x} \underbrace{\sec x \frac{dy}{dx} + \sec x \tan x \, y}_{} = -\sec x \sin x$$

$$\frac{d}{dx}(\sec x \cdot y) = -\sec x \sin x$$

$$\int d(\sec x \cdot y) = \int -\sin x \sec x \, dx$$

$$\sec x \cdot y = -\int \tan x \, dx$$

$$\sec x \cdot y = -\left(-\ln(\cos x)\right) + C$$

$$\sec x \cdot y = \ln(\cos x) + C$$

$$\sec x \cdot y = -\ln(\sec x) + C$$

$$\boxed{y = -\frac{\ln(\sec x)}{\sec x} + \frac{C}{\sec x}}$$

(6) Solve the following homogeneous equation.

$$(x^2-3y^2)\,dx+2xy\,dy=0$$

$$\frac{dy}{dx}=f(x,y)=F\left(\frac{y}{x}\right)$$

$$2xy\,dy=-(x^2-3y^2)\,dx$$

$$\frac{dy}{dx}=-\frac{(x^2-3y^2)}{2xy}$$

$$=\frac{3y^2-x^2}{2xy}$$

$$=\frac{3y^2}{2xy}-\frac{x^2}{2xy}=\frac{3y}{2x}-\frac{x}{2y}$$

$$= \frac{1}{2}\left(\frac{3y}{x} - \frac{x}{y}\right)$$

$$= \frac{1}{2}\left(\frac{3y}{x} - \left(\frac{y}{x}\right)^{-1}\right)$$

$$= F\left(\frac{y}{x}\right) \quad \text{Hom.}$$

Let $v = \frac{y}{x} \Rightarrow y = xv$

$$\frac{dy}{dx} = (1)\cdot v + \frac{dv}{dx}\cdot x = \boxed{v + x\cdot\frac{dv}{dx}}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{3y}{x} - \left(\frac{y}{x}\right)^{-1}\right)$$

$$v + x\cdot\frac{dv}{dx} = \frac{1}{2}\left(3v - v^{-1}\right)$$ separable

$$v + x\frac{dv}{dx} = \frac{1}{2}\left(3v - v^{-1}\right)$$

$$x\frac{dv}{dx} = \frac{3}{2}v - \frac{1}{2}v^{-1} - v$$

$$x\frac{dv}{dx} = \frac{1}{2}v - \frac{1}{2}v^{-1}$$

$$x\frac{dv}{dx} = \frac{1}{2}\left(v - \frac{1}{v}\right) = \frac{1}{2}\left(\frac{v^2-1}{v}\right)$$

$$x\frac{dv}{dx} = \frac{v^2-1}{2v} \Rightarrow \frac{dv}{\frac{v^2-1}{2v}} = \frac{dx}{x}$$

$$\int\frac{2v}{v^2-1}\,dv = \int\frac{dx}{x} \Rightarrow \ln\left|v^2-1\right| = \ln|x| + C$$

$$\boxed{\ln\left|\frac{y^2}{x^2}-1\right| = \ln|x| + C}$$

7) show that the following equation is exact and solve it.

$$\left(\sqrt{\frac{y}{x}} + \cos x\right) dx + \left(\sqrt{\frac{x}{y}} + \sin y\right) dy = 0$$

$$M(x,y) = \sqrt{\frac{y}{x}} + \cos x \qquad N(x,y) = \sqrt{\frac{x}{y}} + \sin y$$

$$= \frac{\sqrt{y}}{\sqrt{x}} + \cos x \qquad = \frac{\sqrt{x}}{\sqrt{y}} + \sin y$$

$$= \frac{1}{\sqrt{x}} \cdot \sqrt{y} + \cos x \qquad = \frac{1}{\sqrt{y}} \cdot \sqrt{x} + \sin y$$

$$\frac{\partial M}{\partial y} = \frac{1}{2\sqrt{x} \cdot \sqrt{y}} \qquad \frac{\partial N}{\partial x} = \frac{1}{2\sqrt{y} \cdot \sqrt{x}}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{Exact !}$$

$$\frac{\partial f}{\partial x} = \frac{\sqrt{y}}{\sqrt{x}} + \cos x$$

$$f(x,y) = \int \left(\frac{\sqrt{y}}{\sqrt{x}} + \cos x \right) dx + h(y)$$

$$= \int \left(2 \cdot \frac{\sqrt{y}}{2\sqrt{x}} + \cos x \right) dx + h(y)$$

$$f(x,y) = 2\sqrt{y} \cdot \sqrt{x} + \sin x + h(y)$$

$$\frac{\partial f}{\partial y} = \cancel{2} \cdot \frac{1}{\cancel{2}\sqrt{y}} \cdot \sqrt{x} + 0 + h'(y)$$

$$\cancel{\frac{\sqrt{x}}{\sqrt{y}}} + h'(y) = \cancel{\frac{\sqrt{x}}{\sqrt{y}}} + \sin y$$

$$h'(y) = \sin y$$

$h(y) = -\cos y$

$$\therefore f(x,y) = 2\sqrt{xy} + \sin x + h(y)$$

$$\boxed{2\sqrt{xy} + \sin x - \cos y = C}$$

(8) Solve the following linear equation

$$(x^3 - y)\,dx + x\,dy = 0$$