$$\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n$$

Exple Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2+n}+2^{3-n}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2+n}+\sum_{n=1}^{\infty}2^{3-n}$$

$$=1+8=9$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2+n}. \qquad S_N=\sum_{n=1}^{N}\frac{1}{n(n+1)}$$

$$S_N=\sum_{n=1}^{N}\frac{1+n-n}{n(n+1)}=\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\cdots+\left(\cancel{\frac{1}{N}}-\frac{1}{N+1}\right)$$

$$=1-\frac{1}{N+1}$$

$$S_N=1-\frac{1}{N+1}. \qquad \lim_{N\to+\infty}S_N=1$$

$$\sum_{n=1}^{\infty}2^{3-n}=\sum_{n=1}^{\infty}\frac{2^3}{2^n}=\sum_{n=1}^{\infty}8\cdot\left(\frac{1}{2}\right)^n$$

$$=\sum_{n=1}^{\infty}8\cdot\left(\frac{1}{2}\right)^{n-1+1}=\sum_{n=1}^{\infty}4\left(\frac{1}{2}\right)^{n-1}$$

geometric with $a_1=4$ and $r=\frac{1}{2}$

$|r|=\frac{1}{2}<1 \Rightarrow \sum_{n=1}^{\infty}2^{3-n}$ converges and

its sum is

$$\sum_{n=1}^{\infty}2^{3-n}=\frac{a_1}{1-r}=\frac{4}{1-\frac{1}{2}}=8$$

* The integral test

Consider the series

$$\sum_{n=1}^{\infty} a_n \quad \text{with } a_n = f(n)$$

where f is continuous, positive and decreasing on $[1, \infty)$

$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

Recall that

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

As a consequence of the integral test we have the following test called

The p-series test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

Exple Determine convergence or divergence of the following series

1-) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$, 2-) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ 3-) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

4-) $\sum_{n=1}^{\infty} \frac{7}{n}$. 5-) $\sum_{n=1}^{\infty} n e^{-n}$

1.) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. $a_n = \frac{n}{n^2+1} = f(n)$

where $f(x) = \frac{x}{x^2+1}$

on $[1, \infty)$

$f(x) = \frac{x}{x^2+1}$ is continous, positive, decreasing.

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \frac{1}{2}\int_1^t \frac{2x}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{2}\left(\ln(x^2+1)\Big|_1^t \right)$$

$$= \lim_{t\to\infty} \frac{1}{2}\left(\ln(t^2+1) - \ln 2 \right) = +\infty$$

divergs

$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ divergs.

2.) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$. $a_n = f(n)$ where

$f(x) = \frac{1}{x\ln x}$ with $x \in [2, \infty)$

On the intervol $[2, \infty)$,

$f(x) = \frac{1}{x\ln x}$ is continuous, positive, decreasing.

$$\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{t\to\infty} \int_2^t \frac{\frac{1}{x}}{\ln x}\,dx$$

$$= \lim_{t\to\infty} \left(\ln|\ln x| \Big|_2^t \right)$$

$$= \lim_{t\to\infty} \left(\ln|\ln t| - \ln\ln 2 \right) = +\infty$$

diverges. $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges

5) $\sum_{n=1}^{\infty} n e^{-n}$. $n e^{-n} = f(n)$, where

$f(x) = x e^{-x} = \frac{x}{e^x}$

on $[1, \infty)$, $f(x) = x e^{-x} = \frac{x}{e^x}$ is positive continuous and decreasing.

$u = x \quad u' = 1$

$v' = e^{-x} \quad v = -e^{-x}$

$$\int_1^{\infty} x e^{-x} dx = \lim_{t \to \infty} \int_1^t x e^{-x} dx$$

$$= \lim_{t \to \infty} \left[-x e^{-x} \Big|_1^t + \underbrace{\int_1^t e^{-x} dx}_{-e^{-x} \Big|_1^t} \right]$$

$$= \lim_{t \to \infty} \left(\underbrace{-\frac{t}{e^t}}_{\to 0} + e^{-1} - \underbrace{e^{-t}}_{\to 0} + e^{-1} \right)$$

$$= 2e^{-1} \text{ Converges}$$

$\sum_{n=1}^{\infty} n e^{-n}$ converges

3) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $\quad p = \frac{3}{2} > 1$

Converges by the p- series test

4) $\sum_{n=1}^{\infty} \frac{7}{n} = 7 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}$ $\quad p = 1$

diverges by the p- series test.

Estimating the Sum

Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = f(n)$ with f continuous, positive and decreasing.

Suppose that series converges and call it value S

$$S = \sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + \dots + a_n}_{S_n} + \underbrace{a_{n+1} + \dots}_{R_n}$$

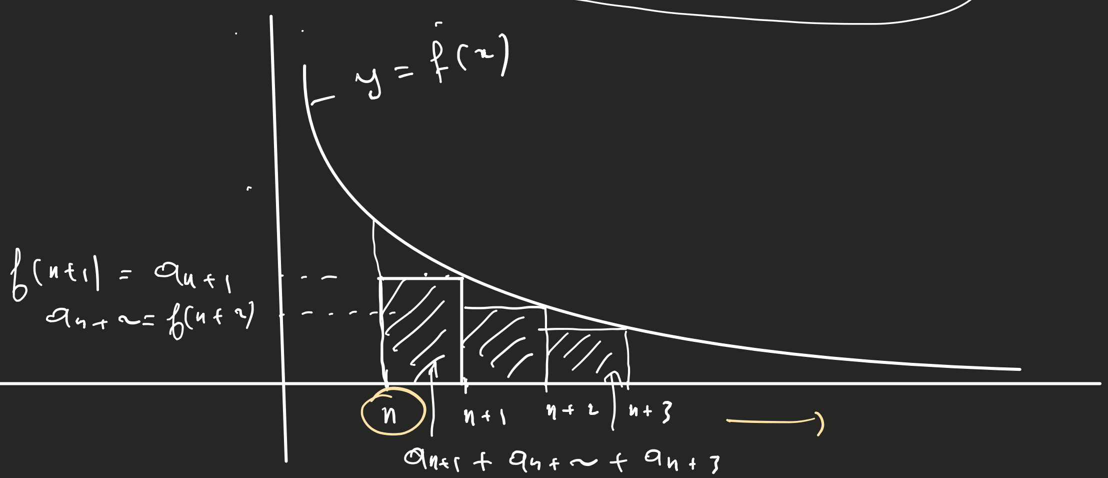
let $S_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^{n} a_k$

$$S = S_n + R_n \quad , \; R_n = a_{n+1} + a_{n+2} + \dots$$

$$S - S_n = R_n$$

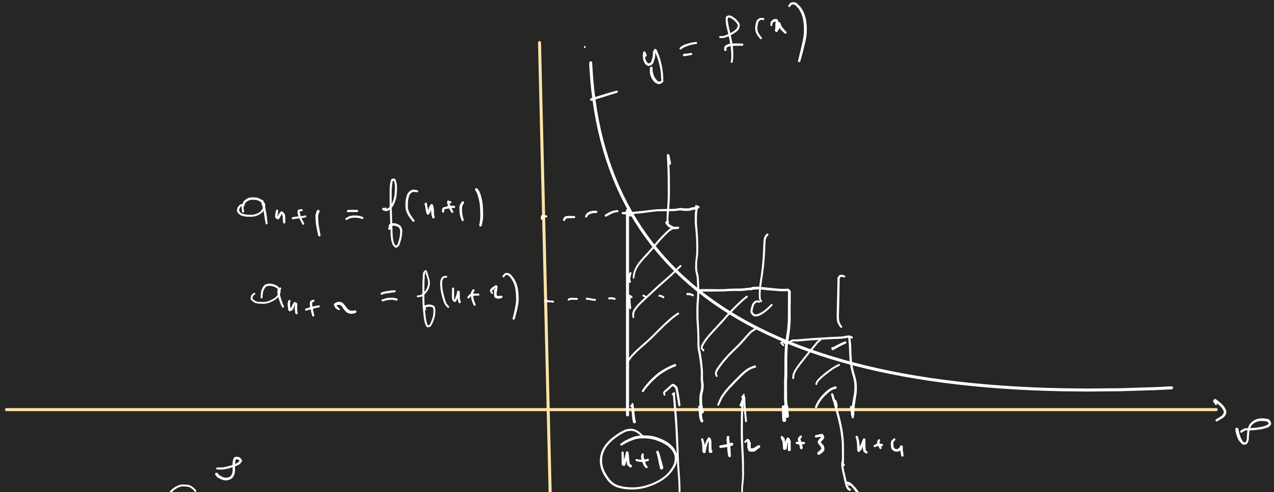
We have the following formula:

$$\int_{n+1}^{\infty} f(x)\,dx \le \underbrace{S - S_n}_{R_n = a_{n+1} + a_{n+2} + \dots} \le \int_{n}^{\infty} f(x)\,dx$$



$$\underbrace{a_{n+1} + a_{n+2} + \cdots +}_{} \leq \int_n^\infty f(x)\,dx$$

$$S - S_n \leq \int_n^\infty f(x)\,dx$$



$$\int_{n+1}^\infty f(x)\,dx \leq \underbrace{a_{n+1} + a_{n+2} + a_{n+3} \cdots}_{R_n = S - S_n}$$

$$\boxed{\int_{n+1}^\infty f(x)\,dx \leq \underbrace{S - S_n}_{R_n} \leq \int_n^\infty f(x)\,dx}$$

Exple A) Use $n = 10$ to estimate the sum of the series $\sum_{n=1}^\infty \frac{1}{n^3}$

B) How many terms are required to ensure that the sum is accurate to within 0.0005.

$$\sum_{n=1}^\infty \frac{1}{n^3}. \qquad \frac{1}{n^3} = f(n) \text{ where } f(x) = \frac{1}{x^3}.$$

$$\int_{11}^{\infty}\frac{dx}{x^3} \le S - S_{10} \le \int_{10}^{\infty}\frac{dx}{x^3} \to \frac{1}{200}$$

$$\int_{11}^{\infty}\frac{dx}{x^3} + S_{10} \le S \le \int_{10}^{\infty}\frac{dx}{x^3} + S_{10}$$

$$S_{10} = \sum_{k=1}^{10}\frac{1}{k^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots + \frac{1}{10^3} = 1.1975$$

$$\int_{11}^{\infty} x^{-3}\,dx = \lim_{t\to\infty}\int_{11}^{t} x^{-3}\,dx = \lim_{t\to\infty}\left(\frac{x^{-2}}{-2}\Big|_{11}^{t}\right)$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2t^2} + \frac{1}{2(121)}\right) = \frac{1}{242}$$

$$\frac{1}{242} + 1.1975 \le S \le \frac{1}{200} + 1.1975$$

$$1.201664 \le S \le 1.202532$$

2.)

$$R_n = S - S_n \le \int_{n}^{\infty}\frac{dx}{x^3} = \frac{1}{2n^2} \le 0.0005$$

$$\frac{1}{2n^2}$$

$$S - S_n \le \frac{1}{2n^2} \le 0.0005$$

$$\frac{1}{2n^2} \le 0.0005 \iff \frac{1}{n^2} \le 0.001 = 10^{-3}$$

$$n^2 \ge 1000, \quad n \ge \sqrt{1000} = 31.6$$

$n \geq 32$

§ The Alternating Series Test (A.S.T)

A series of the form

$$\sum_{n=1}^{\infty} (-1)^n b_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

where $b_n > 0$ is called an alternating series.

* The Alternating series test (A.S.T)

Any alternating series $\sum_{n=1}^{\infty} (-1)^n b_n$ where b_n satisfies the following

* $b_n > 0$ for $n \geq 1$
* b_n is decreasing ($b_{n+1} < b_n$)
* $\lim_{n \to \infty} b_n = 0$

will converge.

Exple Determine convergence or divergence

1.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ 2.) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(-3)^{n-1}}$

3.) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

1.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n}}$

$b_n = \frac{1}{\sqrt{n}}$.

$\begin{cases} * \ b_n = \frac{1}{\sqrt{n}} > 0 \\ * \ b_n \text{ is decreasing} \\ * \ \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \end{cases}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ Converges by A.S.T.

2.) $\sum_{n=1}^{\infty} \frac{n}{(-3)^{n-1}} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{3^{n-1}}$.

$\frac{n}{(-3)^{n-1}} = \frac{n}{(-1)^{n-1} \cdot 3^{n-1}} = \frac{(-1)^{n-1}\, n}{\underbrace{(-1)^{n-1}(-1)^{n-1}}\, 3^{n-1}} = \frac{(-1)^{n-1} n}{3^{n-1}}$

$b_n = \frac{n}{3^{n-1}}$

$\begin{cases} * \ b_n = \frac{n}{3^{n-1}} > 0 \\ * \ b_n \text{ is decreasing} \\ * \ \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{3^{n-1}} \end{cases}$

$= \lim_{x\to\infty} \frac{x}{3^{x-1}} = \lim_{x\to\infty} \frac{1}{(\ln 3)3^{x-1}}$

$= \frac{1}{\infty} = 0$.

CL: $\sum_{n=1}^{\infty} \frac{n}{(-3)^{n-1}}$ converges by A.S.T.

3.) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ $\qquad b_n = \frac{n^2}{n^3+1}$

$$\begin{cases} * \ b_n = \dfrac{n^2}{n^3+1} > 0 \\ * \ b_n = \dfrac{n^2}{n^3+1} \text{ is decreasing.} \\ * \ \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{n^2}{n^3+1} = \lim\limits_{n\to\infty} \dfrac{n^2}{n^3} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0 \end{cases}$$

The series converges by A.S.T.

Estimating the sum of an alternating Series.

Consider the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

where

$$\begin{cases} * \ b_n > 0 \quad \text{for } n \geq 1 \\ * \ b_n \text{ decreasing} \\ * \ \lim\limits_{n\to\infty} b_n = 0 \end{cases}$$

the series converges. let

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n.$$

In the approximation of S by

$S_n = \sum_{k=1}^{n} (-1)^{k-1} b_k$ the absolute error

$$|S - S_n| \leq b_{n+1}$$

$|\square| \leq 0$

$0 \leq \square \leq 0$

$\underline{n=10}$ $$|S - S_{10}| \le b_{11}$$

$$-b_{11} \le S - S_{10} \le b_{11}$$

$$S_{10} - b_{11} \le S \le S_{10} + b_{11}$$

A) Use $n=6$ to approximate the sum of the alternating series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+1}$$

B) How many terms are required to ensure that the sum is accurate to within 0.001.

A) $n=6$ $$|S - S_6| \le b_7$$

$$S_6 - b_7 \le S \le S_6 + b_7$$

$$S_6 = \sum_{k=1}^{6} \frac{(-1)^{k-1}}{k^2+1} = \frac{1}{2} - \frac{1}{5} + \frac{1}{10} - \frac{1}{17} + \frac{1}{26} - \frac{1}{37}$$
$$= 0.35261$$

$$b_7 = \frac{1}{50}$$

$$0.35261 - \frac{1}{50} \le S \le 0.35261 + \frac{1}{50}$$

$$0.3326 \le S \le 0.3726$$

B) $|\underbrace{S - S_n}_{R_n}| < b_{n+1} = \dfrac{1}{(n+1)^2+1} \le 10^{-3}$

$$\frac{1}{(n+1)^2+1} \le 10^{-3} \iff (n+1)^2+1 \geq 10^3$$

$$(n+1)^2 \geq 999$$

$$n+1 \geq \sqrt{999}$$

$$n \geq \sqrt{999} - 1 = 30.6$$

$$n \geq 31$$

* The limit Comparison test

Suppose $a_n > 0$ and $b_n > 0$

If $\lim_{n \to \infty} \dfrac{a_n}{b_n} = L \neq 0$ then

the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

both converge or both diverge.

Exple Determine Convergence or divergence

1.) $\sum_{n=1}^{\infty} \dfrac{2n}{n^3+4n+1}$

2.) $\sum_{n=1}^{\infty} \dfrac{\sqrt{n}}{n^2+7}$

3.) $\sum_{n=1}^{\infty} \dfrac{n\,2^n}{4n^3+1}$

1.) $\sum_{n=1}^{\infty} \frac{2n}{n^3+4n+1}$

$a_n = \frac{2n}{n^3+4n+1}$

$b_n = \frac{2n}{n^3} = \frac{2}{n^2}$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n^2}$ p-series $p = 2 > 1$

converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{2n}{n^3+4n+1}$ converges.

$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$

$\lim_{n\to\infty} \frac{\frac{2n}{n^3+4n+1}}{\frac{2n}{n^3}} = \lim_{n\to\infty} \frac{2n}{n^3+4n+1} \cdot \frac{n^3}{2n}$

$= \lim_{n\to\infty} \frac{2n^4}{2n(n^3+4n+1)} = 1$

2) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+7} \sim \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2}$

$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

p-series $p = \frac{3}{2} > 1$

converges

CL : $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+7}$ converges

3-) $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1} \sim \sum_{n=1}^{\infty} \frac{\cancel{n}2^n}{4n^{\cancel{3}\,2}} = \sum_{n=1}^{\infty} \frac{2^n}{4n^2}$

$$\lim_{n\to\infty} \frac{2^n}{4n^2} = \lim_{x\to\infty} \frac{2^x}{4x^2} = \lim_{x\to\infty} \frac{(\ln 2)2^x}{8x}$$

$$= \lim_{n\to\infty} \frac{(\ln 2)^2 2^x}{8} = +\infty \neq 0$$

$\sum_{n=1}^{\infty} \frac{2^n}{4n^2}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$ diverges.

$$\sum_{n} \frac{2n+\sqrt{n}}{4n^2+1} \sim \sum_{n} \frac{2n}{4n^{\cancel{2}}}$$

$\sum_{n} \frac{1}{2n}$ $p=1$

§ Absolute vs Conditional convergence

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* A series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

Absolute convergence implies normal Convergence

$$\sum_{n=1}^{\infty} |a_n| \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

* If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then we say that the series $\sum_{n=1}^{\infty} a_n$ converges conditionally

$$\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}} \qquad \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{diverges}}$$

Exple Determine whether the series converges or diverges and classify any convergence as conditional or absolute.

1.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)}}{9^n}$ 2.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+2}$

3.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

1.) $$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n(n+1)}}{9^n} \right| = \sum_{n=1}^{\infty} \frac{1}{9^n} = \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)^{n-1+1} = \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)\left(\frac{1}{9}\right)^{n-1}$$

$r = \frac{1}{9} \qquad |r| = \frac{1}{9} < 1 \Rightarrow$

the Series Converges.

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n(n+1}}{9^n}$ Converges absolutely.

2.) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+2}$?

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n}{n^2+2} \right| = \sum_{n=1}^{\infty} \frac{n}{n^2+2}$$

$$\sim \sum_{n=1}^{\infty} \frac{n}{n^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{p-series}$$

$p=1$

divergs

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n}{n^2+2} \right|$ divergs

$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+2}$. $\quad b_n = \frac{n}{n^2+2}$

* $b_n > 0$
* b_n decreasing
* $\lim_{n\to\infty} b_n =$

$\lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$

converges by A.S.T

$\Rightarrow$ Series Convergs conditionally.

3.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ p- series $p = 2 > 1$ convergs

$\Rightarrow$ Absolute convergence.

* **The Ratio Test.**

Consider the series $\sum_{n=1}^{\infty} a_n$.

$\rightarrow$ If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ convergs absolutely.

$\rightarrow$ If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ divergs

$\rightarrow$ If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ then the test cannot decide

Exple Determine convergence or divergence

1.) $\sum_{n=0}^{\infty} \frac{5^n}{n!}$

2.)

3.) $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$

4.)

1.) $\sum_{n=0}^{\infty} \frac{5^n}{n!}$? $\quad a_n = \frac{5^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{5^{n+1}}{(n+1)!}}{\frac{5^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{5^n} \cdot 5}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{5^n}} \right| = \lim_{n\to\infty} \left| \frac{5}{n+1} \right| = 0 < 1$$

$\sum_{n=0}^{\infty} \frac{5^n}{n!}$ convergs absolutely.

$\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ $\quad a_n = \frac{n^2 2^{n+1}}{3^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 \cdot 2^{n+2}}{3^{n+1}} \cdot \frac{3^n}{n^2 2^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2 \cdot \cancel{2^{n+1}} \cdot 2}{\cancel{3^n} \cdot 3} \cdot \frac{\cancel{3^n}}{n^2 \cancel{2^{n+1}}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2(n+1)^2}{3n^2} \right| = \lim_{n\to\infty} \frac{2}{3} \left(\frac{n+1}{n} \right)^2 = \frac{2}{3} < 1$$

$\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ Converges absolutely.

3-) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^n}{n!}$ $\qquad a_n = (-1)^n \frac{n^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-1)^n\cdot n^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(-1)^n}(-1)(n+1)^n\cancel{(n+1)}}{\cancel{(n+1)}\,\cancel{n!}}\cdot\frac{\cancel{n!}}{\cancel{(-1)^n}\cdot n^n}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$$

$\Rightarrow$ Series diverges.

$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$ $\qquad a_n = \frac{(-1)^n\sqrt{n}}{n+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\sqrt{n+1}}{n+2}\cdot\frac{n+1}{(-1)^n\sqrt{n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{\sqrt{n}}\cdot\frac{n+1}{n+2}\cdot\frac{\cancel{(-1)^n}(-1)}{\cancel{(-1)^n}}\right|$$

$$= \lim_{n\to\infty}\underbrace{\sqrt{\frac{n+1}{n}}}_{1}\cdot\underbrace{\frac{n+1}{n+2}}_{1} = 1$$

Not Conclusif