# § Bernoulli Equation.

A first order differential equation of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$$

is called a Bernoulli equation.

If $n = 0$ or $n = 1$

the equation is linear.

* Assume that $n \neq 0$ and $n \neq 1$
* To solve the Bernoulli equation

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$$

Do the following:

1.) divide both sides by $y^n$ to get

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x)$$

2.) Next, you do the substitution

$$u = y^{1-n}$$

$$\frac{du}{dx} = (1-n)\,y^{1-n-1}\cdot\frac{dy}{dx} = (1-n)\,y^{-n}\frac{dy}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$$

$$\frac{1}{1-n}\frac{du}{dx} + P(x)u = Q(x)$$

$$\frac{du}{dx} + (1-n)P(x)u = Q(x)$$

This is a simple linear equation in $u$.

Exp'le Show that the equation is Bernoulli and solve the initial-value problem

$$x\frac{dy}{dx} + 5y = 2x^2y^4, \quad y(1) = 3.$$

$$\frac{dy}{dx} + \frac{5}{x}y = 2xy^4$$

The equation is Bernoulli with $n=4$.

$$y^{-4}\frac{dy}{dx} + \frac{5}{x}y^{-3} = 2x$$

Put $u = y^{-3}$.

$$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$y^{-4}\frac{dy}{dx} = -\frac{1}{3}\frac{du}{dx}.$$

The equation becomes

$$-\frac{1}{3}\frac{du}{dx} + \frac{5}{x}u = 2x$$

$$\frac{du}{dx} - \frac{15}{x}u = -6x$$

$x>0$

$$IF = e^{\int P(x)\,dx} = e^{\int -\frac{15}{x}dx} = e^{-15\ln x}$$

$$= e^{\ln x^{-15}} = x^{-15} = \frac{1}{x^{15}}$$

$$u = \frac{1}{IF}\int IF\, Q(x)\,dx$$

$$u = x^{15}\int \frac{1}{x^{15}}(-6x)\,dx$$

$$u = x^{15}\int -6x^{-14}\,dx$$

$$u = x^{15}\left[-6\frac{x^{-13}}{-13} + c\right]$$

$$= x^{15}\left(\frac{6}{13}\cdot x^{-13} + c\right)$$

$$u = \frac{6}{13}x^{2} + cx^{15}$$

$$y^{-3} = \frac{6}{13}x^{2} + cx^{15}$$

$$y = \frac{1}{\sqrt[3]{\frac{6}{13}x^{2} + cx^{15}}}$$

$$y(1) = 3 \; : \; x = 1, \quad y = 3$$

$$3 = \frac{1}{\sqrt[3]{\frac{6}{13} + c}}$$

$$\frac{1}{27} = \frac{6}{13} + c \longrightarrow c = \frac{1}{27} - \frac{6}{13} = \frac{-149}{351}$$

## § Sequences

* A sequence is a function defined on a set of natural numbers.

$$a(x) = 2x - 1 \quad \text{with } x \in \{1, 2, 3, 4, 5\}$$

$$a(n) = 2n - 1, \quad n = 1, 2, 3, 4, 5$$

$$a_n = 2n - 1, \quad n = 1, 2, 3, 4, 5$$

* Given a sequence $a_n$, $n = 1, 2, \ldots\ldots$

→ $a_1$ is called the first term of $a_n$

→ $a_2$ ———— second term of $a_n$

⋮

$a_n$ ———— the n-th term of $a_n$

Exple Find the first 4 terms of the sequence

$$a_n = \frac{\log n}{n+1}$$

$a_1 = \frac{\log 1}{1+1} = 0$ , $a_2 = \frac{\log 2}{3}$

$a_3 = \frac{\log 3}{4}$ , $a_4 = \frac{\log 4}{5}$.

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$a(x) = 2x - 1$ , $x \in \{1, 2, 3, 4\}$

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $a(x)$ | 1 | 3 | 5 | 7 |

$1, 3, 5, 7,$

$a_1$ $a_2$

(2) $4, 6, 8, 10,$

$2 + (n-1)(2) = \cancel{2} + 2n - \cancel{2} = 2n$

<u>Exple</u> Find the n-th term $a_n$ of the sequence

$\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}, \frac{11}{10}, \ldots\ldots$ $\frac{2n+1}{2n}$

$3, 5, 7, 9, 11, \ldots$

$3 + (n-1)(2) = 2n - 2 + 3 = 2n+1$

$\frac{1+2n}{2n}$

## § Arithmetic Sequences

A sequence $a_n$ is said to be arithmetic if $a_{n+1} - a_n = d$ constant

* $d$ is called the common difference.

If $a_n$ is arithmetic with first term $a_1$ and common difference $d$ then

$$a_n = a_1 + (n-1)d$$

## Geometric Sequences

A sequence $a_n$ is called geometric if

$$\frac{a_{n+1}}{a_n} = r \text{, a constant}$$

$r$ is called the common ratio of the geometric equation

* If $a_n$ is a geometric sequence with first term $a_1$ and common ratio $r$, then its n-th term is given by

$$a_n = a_1 r^{n-1}$$

Exple Find the n.th term of the sequence

$$1, \frac{3}{5}, \frac{9}{25}, \frac{27}{125}, \ldots\ldots$$

$$\frac{a_2}{a_1} = \frac{\frac{3}{5}}{1} \qquad \frac{a_3}{a_2} = \frac{\frac{9}{25}}{\frac{3}{5}} = \frac{3}{5} \qquad \frac{a_4}{a_3} = \frac{\frac{27}{125}}{\frac{9}{25}} = \frac{3}{5}$$

this is a geometric sequence with $a_1 = 1$ and $r = \frac{3}{5}$

$$a_n = (1)\left(\frac{3}{5}\right)^{n-1} = \left(\frac{3}{5}\right)^{n-1}.$$

## Recursive representation of sequences

A recursive representation is a representation that gives the first term(s) of the sequence along with a relationship between the remaining terms.

Exple

$$\begin{cases} a_1 = 2 \\ a_{n+1} = 3a_n - a_n^2 + 1 \end{cases}$$

Find $a_6$.

$$a_2 = 3a_1 - a_1^2 + 1$$

$$a_2 = 3(2) - 2^2 + 1 = 6 - 4 + 1 = 3$$

$$a_3 = 3a_2 - a_2^2 + 1$$

$$a_3 = 3(3) - 3^2 + 1 = 1$$

$$a_4 = 3a_3 - a_3^2 + 1$$

$$a_4 = 3(1) - 1 + 1 = 3$$

$a_5 = 1$, $a_6 = 3$.

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§ limit of sequences

Because sequences are function defined on set of natural numbers 1, 2, 3, 4, . . .

We will be interested in finding limit of $a_n$ when $n$ approches plus infinity.

$$\lim_{n \to +\infty} a_n .$$

* All properties for limit of functions are valid for limit of sequences.

Exple Evaluate the following limits

1.) $\displaystyle \lim_{n \to \infty} \frac{5n^3 + 4n - 6}{7 - 3n^3} = \lim_{n \to \infty} \frac{5\cancel{n^3}}{-3\cancel{n^3}} = \lim_{n \to \infty} \frac{-5}{3} = -\frac{5}{3}$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

2.) $\lim_{n \to \infty} n \sin \frac{4}{n} = \lim_{n \to \infty} \frac{\sin \frac{4}{n}}{\frac{1}{4} \frac{4}{n}} = \lim_{n \to \infty} 4 \left( \frac{\sin \frac{4}{n}}{\frac{4}{n}} \right) = 4$

3.) $\lim_{n \to \infty} \frac{n^2}{2^n - 1} = \lim_{x \to +\infty} \frac{x^2}{2^x - 1} = \lim_{x \to +\infty} \frac{2x}{(\ln 2) 2^x} = \lim_{x \to +\infty} \frac{2}{(\ln 2)^2 2^x} = 0$

4) $\lim_{n \to \infty} \frac{n!}{n^n}$ .

* We have the following result.

If $a_n = f(n)$ and $\lim_{x \to +\infty} f(x) = L$,

then $\lim_{n \to +\infty} a_n = L$

4.) $\lim_{n \to +\infty} \frac{n!}{n^n}$

The squeezing Theorem ( Sandwich Theorem)

If

$b_n \leq a_n \leq c_n$ for all $n$

and $\lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = L$ , then

$\lim_{n \to \infty} a_n = L$

As a consequence of this, we have $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

* $\lim_{n\to\infty} \frac{n!}{n^n}$.

$$\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot n \cdot \ldots \cdot n}$$

$$= \frac{1}{n} \cdot \frac{2 \cdot 3 \cdot \ldots n}{n \cdot n \cdots n}$$

$\to \frac{1}{n}$

$$0 \leq \frac{2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdots n} \leq 1$$

$$0 \leq \underbrace{\frac{1}{n} \cdot \frac{2 \cdot 3 \cdots n}{n \cdot \cdots n}} \leq \frac{1}{n}$$

$$0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$$

$\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to+\infty} \frac{n!}{n^n} = 0$

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## § Series

Consider the sequence

$$a_1, a_2, a_3, \ldots a_n, \ldots$$

Adding the terms of this sequence results in what is called a series.

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum_{k=1}^{\infty} a_k \quad \text{or} \quad \sum_{i=1}^{\infty} a_i$$

$$\int_1^{\infty} f(x)\,dx = \lim_{t \to \infty} \int_1^{t} f(x)\,dx$$

$$\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} \sum_{n=1}^{N} a_n$$

Given a series $\sum_{n=1}^{\infty} a_n$, the finite sum

$$\sum_{n=1}^{N} a_n$$

is called the n-th partial sum of the series. It is denoted by $S_N$

$$S_N = \sum_{n=1}^{N} a_n$$

$S_N$ is a sequence.

* Consider the series $\sum_{n=1}^{\infty} a_n$

If $\lim_{N \to \infty} S_N = \lim_{N \to \infty} \sum_{n=1}^{N} a_n = S$ exists,

then the series converges and its value is $S$

$$\sum_{n=1}^{\infty} a_n = S$$ Otherwise, we say that the series diverges.

Exp'le Determine convergence or divergence of the following series

1.) $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

2.) $$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$$

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1.) $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$$

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)}$$

$$\frac{1}{n(n+1)} = \frac{1+n-n}{n(n+1)} = \frac{n+1}{n(\cancel{n+1})} - \frac{\cancel{n}}{\cancel{n}(n+1)}$$

$$= \frac{1}{n} - \frac{1}{n+1}$$

$$S_N = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right)$$

$$= 1 - \frac{1}{N+1}$$

$$S_N = 1 - \frac{1}{N+1}$$

$$\lim_{N \to +\infty} S_N = \lim_{N \to +\infty} \left(1 - \frac{1}{N+1}\right) = 1$$

$\Rightarrow$ the series converges and its value is $S = 1$.

2.) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

$$S_N = \sum_{n=1}^{N} \ln\left(\frac{n}{n+1}\right)$$

$$= \sum_{n=1}^{N} \ln(n) - \ln(n+1)$$

$$= [\ln(1) - \cancel{\ln(2)}] + [\cancel{\ln 2} - \cancel{\ln 3}] + [\cancel{\ln 3} - \cancel{\ln 4}] + \dots + \cancel{\ln(N)} - \ln(N+1)$$

$$= \ln(1) - \ln(N+1)$$

$$= -\ln(N+1)$$

$$S_N = -\ln(N+1).$$

$$\lim_{N \to +\infty} S_N = \lim_{N \to \infty} -\ln(N+1) = -\infty$$

$\Rightarrow$ the series $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ diverges

* Series for which when you write out the term, you will get a telescoping phenomenon are called Telescoping series.

§ Geometric Series

$$a_n = a_1 r^{n-1}.$$

A series of the form

$$\sum_{n=1}^{\infty} a_1 r^{n-1} \quad \text{or} \quad \sum_{n=0}^{\infty} a_1 r^n$$

are called geometric series.

$$S_N = \sum_{n=1}^{N} a_1 r^{n-1}$$

$$S_N = a_1 + \cancel{a_1 r} + \cancel{a_1 r^2} + \cdots + \cancel{a_1 r^{N-2}} + \cancel{a_1 r^{N-1}}$$

$$r S_N = \cancel{a_1 r} + \cancel{a_1 r^2} + \cancel{a_1 r^3} + \cdots + \cancel{a_1 r^{N-1}} + a_1 r^N$$

$$S_N - r S_N = a_1 - a_1 r^N$$

$$(1-r) S_N = a_1 (1 - r^N)$$

$$\lim_{N \to} \left(\frac{1}{2}\right)^N$$

If $r \neq 1$

$$0$$

$$S_N = \frac{a_1 (1 - r^N)}{1 - r}$$

Now using the fact that

$$\lim_{N \to +\infty} r^N = \begin{cases} 0 & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| > 1 \end{cases}$$

Consider the geometric series $\sum_{n=1}^{\infty} a_1 r^{n-1}$.

* If $|r| < 1$, then the series converges and its sum is

$$S = \frac{a_1}{1-r}$$

* If $|r| \geq 1$, the series diverges.

Exple Determine convergence or divergence.

1.) $$\sum_{n=1}^{\infty} 4^{3-n} = \sum_{n=1}^{\infty} a_1 r^{n-1}$$

2.) $$\sum_{n=1}^{\infty} 5^{2n} \cdot 7^{1-n} = \sum_{n=1}^{\infty} a_1 r^{n-1}$$

$$\sum_{n=1}^{\infty} 4^{3-n} = \sum_{n=1}^{\infty} 4^{3} \cdot 4^{-n} = \sum_{n=1}^{\infty} 64\left(\frac{1}{4}\right)^{n}$$

$$= \sum_{n=1}^{\infty} 64\left(\frac{1}{4}\right)^{n-1+1} = \sum_{n=1}^{\infty} \underbrace{(64)\left(\frac{1}{4}\right)}_{a_1} \cdot \left(\frac{1}{4}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} 16\left(\frac{1}{4}\right)^{n-1}$$

geometric with $a_1 = 16$ and $r = \frac{1}{4}$

$|r| = \frac{1}{4} < 1 \Rightarrow$ the series converges and its sum is

$$S = \frac{a_1}{1-r} = \frac{16}{1-\frac{1}{4}} = \frac{16}{\frac{3}{4}} = 16 \cdot \frac{4}{3}$$

$$5^{2n} = (5^2)^n$$

$$= \frac{64}{3}.$$

2) $$\sum_{n=1}^{\infty} 5^{2n} 7^{1-n} = \sum_{n=1}^{\infty} 25^n \cdot \frac{1}{7^{n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{25^{n-1+1}}{7^{n-1}} = \sum_{n=1}^{\infty} 25 \cdot \left(\frac{25}{7}\right)^{n-1}$$

this is a geometric series with $r = \frac{25}{7}$ and $a_1 = 25$.

$$|r| = \left|\frac{25}{7}\right| = \frac{25}{7} > 1 \Rightarrow \text{Series diverges.}$$

* The divergence test.

Consider the series $\sum_{n=1}^{\infty} a_n$.

$\lim_{n \to \infty} a_n \neq 0$, then the series cannot converge. It diverges

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \lim_{N \to \infty} S_N = S \text{ exists}$$

$$S_N = \cancel{a_1} + \cancel{a_2} + \cdots + \cancel{a_{N-1}} + a_N$$

$$S_{N-1} = \cancel{a_1} + \cancel{a_2} + \cdots + \cancel{a_{N-1}}$$

$$S_N - S_{N-1} = a_N \qquad \lim_{N \to \infty} S_N = S$$

$$a_N = S_N - S_{N-1} \qquad \lim_{N \to \infty} S_{N-1} = S$$

$$\lim_{N\to\infty} a_N = \lim_{N\to\infty} S_N - \lim_{N\to\infty} S_{N-1} = S - S = 0$$

## Properties of Series

If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ converge

then we have

1.) $$\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n .$$

2.) $$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

Exple Find the sum of the series

$$\sum_{n=1}^{\infty} \left( \frac{1}{n^2+n} + 2^{3-n} \right)$$