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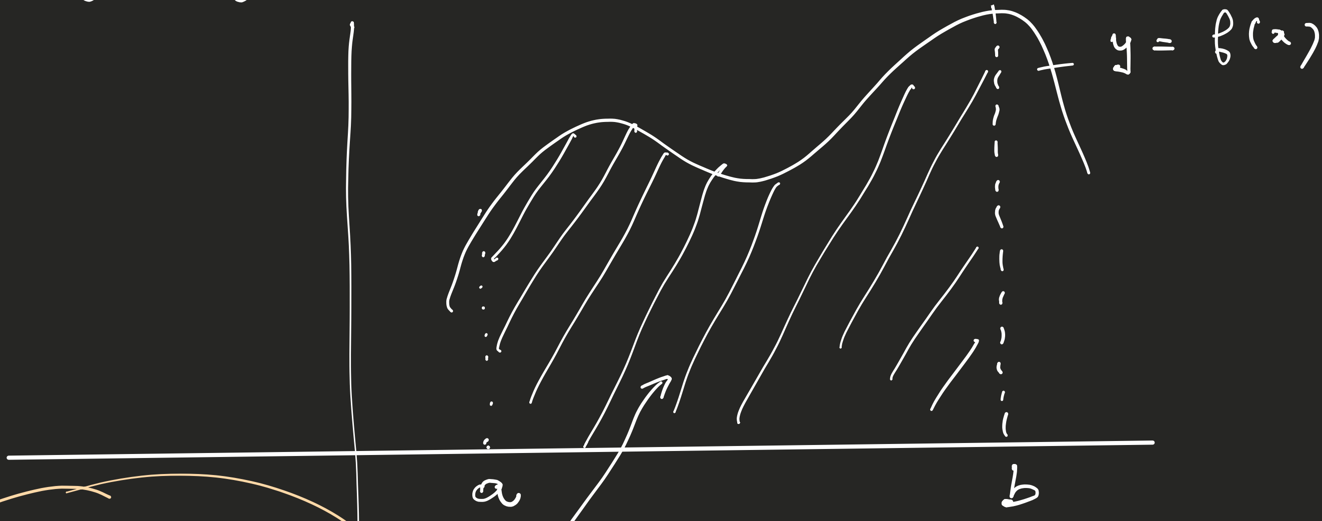
$$\int \cos^n x \sin^m x \, dx$$

§ Area between two curves

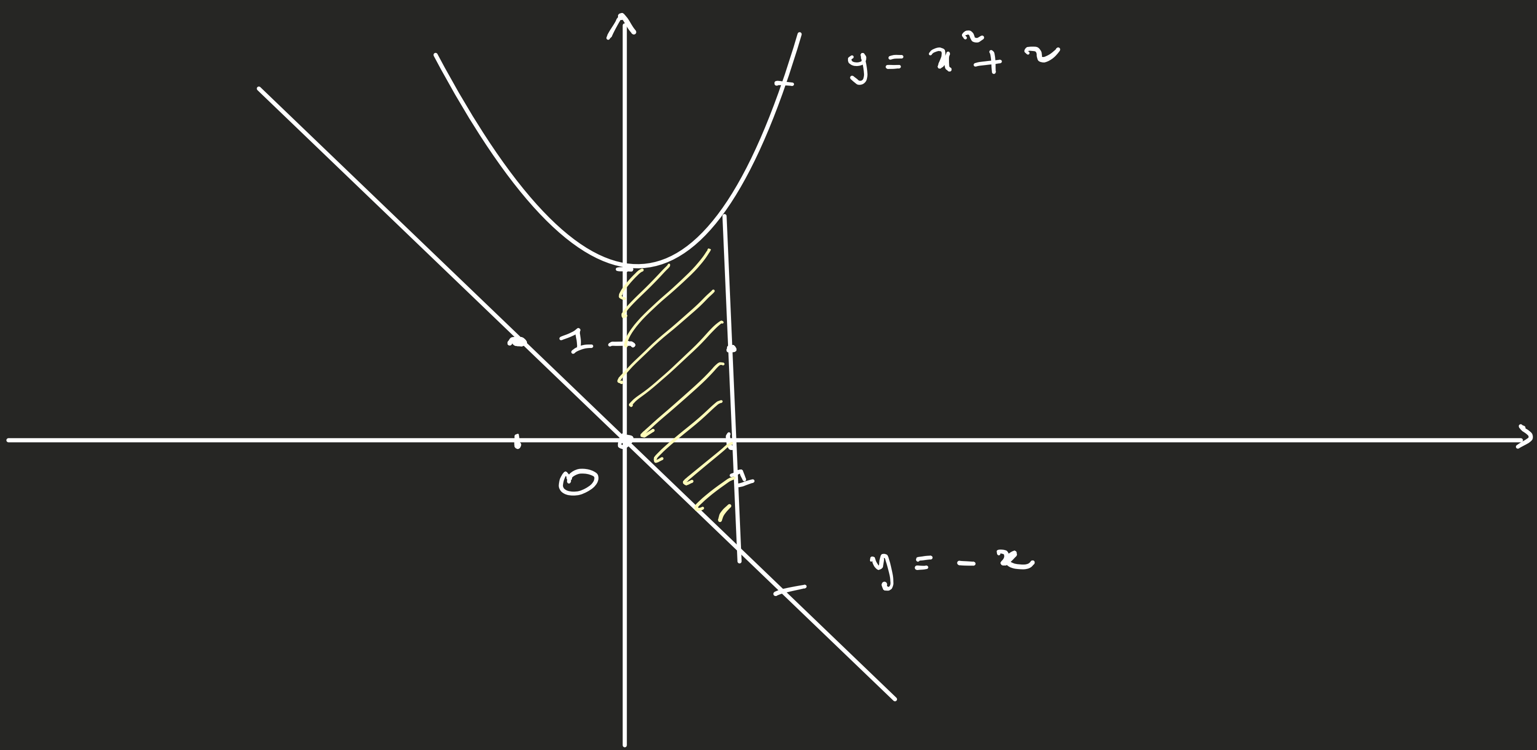
* If f is continuous on $[a, b]$



$$\int_a^b f(x)\, dx =$$

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$A = \int_a^b [f(x) - g(x)]\,dx$$



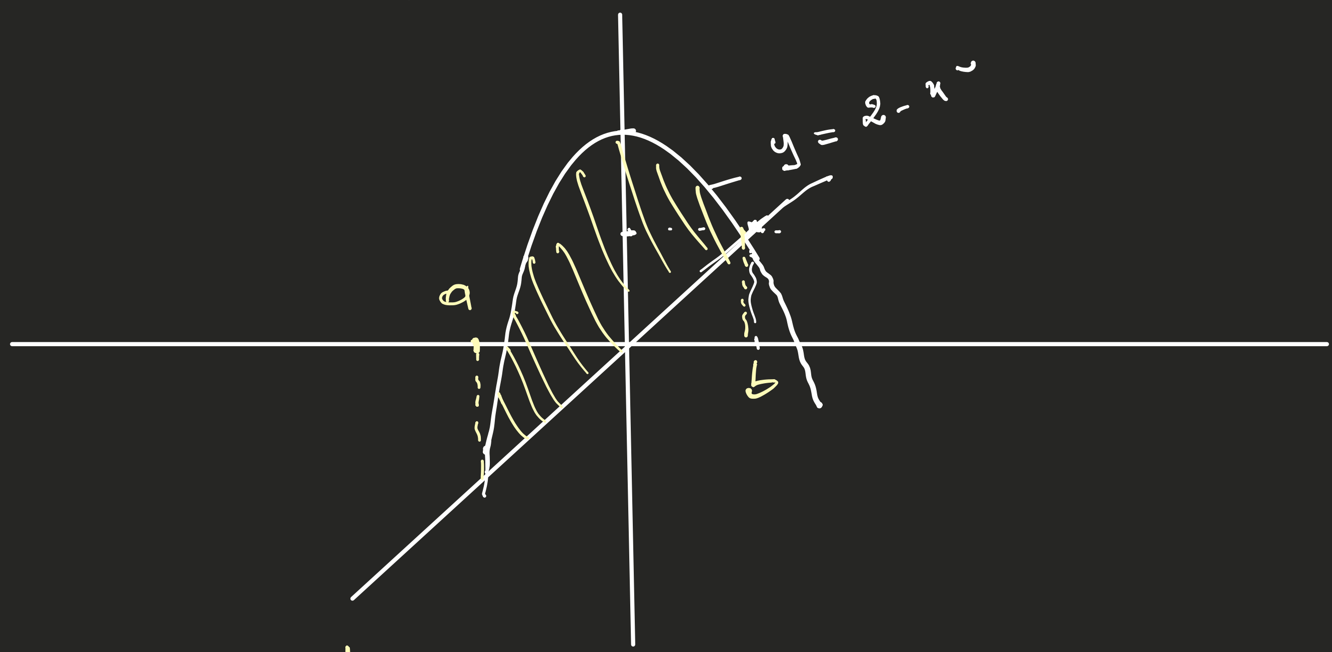
$$A = \int_0^1 [(x^2+2) - (-x)]\,dx$$

$$= \int_0^1 (x^2 + x + 2)\,dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_0^1 = \frac{1}{3} + \frac{1}{2} + 2 - (0)$$

$$= \frac{17}{6}$$

Exple Find the area bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.



$$A = \int_a^b [(2 - x^2) - (x)]\, dx$$

Finding a and b.

$$2 - x^2 = x$$

$$x^2 + x - 2 = 0$$

$$x = 1 \quad \text{or} \quad x = -2$$

$a = -2 \qquad b = 1$

$$A = \int_{-2}^{1} (2 - x^2 - x)\, dx = 2x - \frac{x^3}{3} - \frac{x^2}{2} \Big|_{-2}^{1} = \frac{9}{2}$$

Let f and g be continuous on $[a,b]$. the area bounded by the graphs of f and g between $x = a$ and $x = b$ is given by

$$A = \int_a^b \left| f(x) - g(x) \right| dx$$

$$= \int_a^b \left| g(x) - f(x) \right| dx$$

Exple Find the area of the region bounded by the graphs of

$y = x^2 + 2$, $y = -x$

$x = 0$ and $x = 1$.

$$A = \int_0^1 \left| -x - (x^2+2) \right| dx$$

$$= \int_0^1 \left| -x^2 - x - 2 \right| dx = \int_0^1 (x^2 + x + 2)\, dx$$

$-x^2 - x - 2$ $\quad a = -1, \quad b = -1, \quad c = -2$

$$b^2 - 4ac = 1 - 4(-1)(-2)$$

$$= 1 - 8 = -7 < 0$$

x	$-\infty$ $\qquad\qquad\qquad\qquad$ $+\infty$
$-x^2-x-2$	$-$
$\lvert -x^2-x-2\rvert$	$-(-x^2-x-2) = x^2+x+2$

<u>Ex'ple</u> Find the area between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

$$A = \int_a^b \left| \underbrace{(3x^3 - x^2 - 10x) - (-x^2+2x)} \right| dx$$

<u>Finding a and b</u>

$$3x^3 - x^2 - 10x = -x^2 + 2x$$

$$3x^3 \cancel{-x^2} \cancel{+x^2} - 10x - 2x = 0$$

$$3x^3 - 12x = 0$$

$$x(3x^2 - 12) = 0$$

$$\boxed{x = 0}$$

or

$$3x^2 - 12 = 0$$

$$x^2 = 4, \quad \boxed{x = \pm 2}$$

$0, -2, 2$

$a = -2, \quad b = 2$

$$A = \int_{-2}^{2} \left| 3x^3 - 12x \right| dx$$

$$= \int_{-2}^{2} \left| \underbrace{3x}\,\underbrace{(x^2-4)} \right| dx$$

x	$-\infty$		-2		0		2		$+\infty$
$3x$		$-$		$-$	0	$+$		$+$	
x^2-4		$+$	0	$-$		$-$	0	$+$	
$3x^3-12x$		$-$		$+$		$-$		$+$	
$\left|3x^3-12x\right|$				$3x^3-12x$		$-(3x^3-12x)$			

$$A = \int_{-2}^{2} \left| 3x^2 - 12x \right| dx = \int_{-2}^{0} (3x^3 - 12x)\,dx + \int_{0}^{2} -(3x^3-12x)\,dx$$

$$= \left. \frac{3x^4}{4} - 6x^2 \right|_{-2}^{0} + \left. -\frac{3x^4}{4} + 6x^2 \right|_{0}^{2} = 24$$

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* Integration of even and odd functions.

Even function. A function f is said to be even if for any x in the domain of f, $-x$ is also in the domain of f and $f(-x) = f(x)$

$$f(x) = x^2 \qquad f(x) = \cos x$$

$$f(-x) = \cos(-x) = \cos x = f(x)$$

* If f is even, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx ,$$

$$\int_{-2}^{2} |3x^3 - 12x|\,dx = 2\int_0^2 |3x^3 - 12x|\,dx$$

$$= 2\int_0^2 (-3x^3 + 12x)\,dx$$

$$2(12) = 24$$

Exple Find the area enclosed by the curve $y^2 = 2x + 6$ and $y = x - 1$.

$$y^2 = 2x + 6 \longrightarrow y = \pm\sqrt{2x+6} \quad ?$$

$$\downarrow \qquad\qquad\qquad\qquad y = x - 1$$

$$x = \frac{y^2 - 6}{2}, \qquad x = y + 1$$

$$A = \int_a^b \left| \frac{y^2 - 6}{2} - (y+1) \right| dy$$

$$\frac{y^2 - 6}{2} = y + 1$$

$$y^2 - 6 = 2y + 2$$

$$y^2 - 2y - 8 = 0 \qquad y = \underline{-2} \text{ or } y = \underline{4}$$

$$a = -2, \qquad b = 4.$$

$$A = \int_{-2}^{4} \left| \frac{y^2 - 6}{2} - (y+1) \right| dy$$

$$= \int_{-2}^{4} \left| \frac{y^2 - 6 - 2y - 2}{2} \right| dy$$

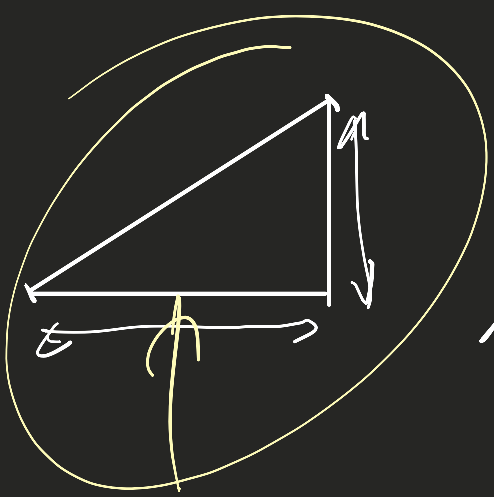
$$= \frac{1}{2} \int_{-2 \leftarrow}^{4 \leftarrow} \left| y^2 - 2y - 8 \right| dy$$

$$= \frac{1}{2} \int_{-2}^{4} -\left(y^2 - 2y - 8\right) dy$$

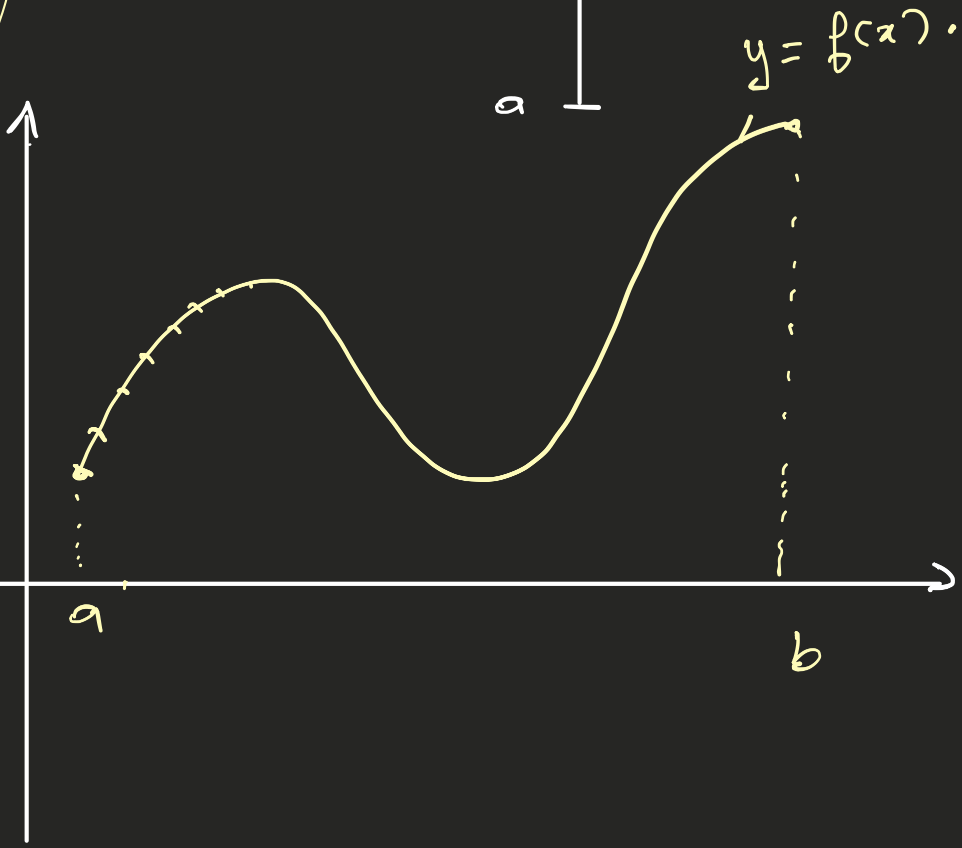
$$= -\frac{1}{2} \left(\left. \frac{y^3}{3} - y^2 - 8y \right|_{-2}^{4} \right) = 18$$

§ Arclength

$L = b - a$



$L = b - a$



* If f and f' are both continuous on $[a, b]$, then the length of the curve given by the graph of $y = f(x)$ is equal to

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

<u>Exple</u> Find the arclength of $y = \ln \cos x$ from $x = 0$ to $x = \frac{\pi}{4}$.

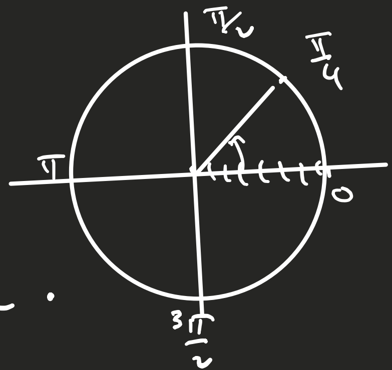
$x^2 = k$

$x = \pm\sqrt{k}$

$\sqrt{x^2} = |x|$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$\sqrt{\theta^2} = |\theta|$



$$\frac{dy}{dx} = -\frac{\sin x}{\cos x} = -\tan x.$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x.$$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\, dx = \int_0^{\frac{\pi}{4}} |\sec x|\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec x\, dx = \ln|\sec x + \tan x|\Big|_0^{\frac{\pi}{4}}$$

$$= \ln(\sqrt{2} + 1) - \ln(1 + 0)$$

$$= \ln(\sqrt{2} + 1) - \ln 1 = \ln(\sqrt{2} + 1).$$

Exple Find the arclength of

$$f(x) = \frac{x^3}{6} + \frac{1}{2x} \quad \text{from } x = 1 \text{ to } x = 2.$$

$$L = \int_1^2 \sqrt{1 + [f'(x)]^2}\, dx.$$

$$f'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$[f'(x)]^2 = \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2 = \frac{x^4}{4} - \cancel{2}\left(\frac{\cancel{x^2}}{\cancel{2}}\right)\left(\frac{1}{2\cancel{x^2}}\right) + \frac{1}{4x^4}$$

$$= \frac{x^4}{4}\left(-\frac{1}{2}\right) + \frac{1}{4x^4}$$

$$1 + [f'(x)]^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$$

$$= \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$$

$$L = \int_1^2 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\, dx = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$$

$$= \frac{x^3}{6} - \frac{1}{2x}\Big|_1^2 = \left(\frac{8}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{17}{12}.$$

Exple Find the arclength of the curve given by the equation $(y-1)^3 = x^2$ on the interval $[0, 8]$.

$$(y-1)^3 = x^2 \longrightarrow y-1 = x^{2/3}$$

$$y = 1 + x^{2/3} .$$

$$\frac{dy}{dx} = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3} x^{-\frac{1}{3}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4}{9} x^{-\frac{2}{3}}$$

$$L = \int_0^8 \sqrt{1 + \frac{4}{9 x^{2/3}}} \, dx$$

$$(y-1)^3 = x^2 \qquad 0 \le x \le 8$$

$$x = \sqrt{(y-1)^3} = (y-1)^{\frac{3}{2}}$$

$$\int_a^b u'(x)\, u(x)^n \, dx = \left. \frac{u(x)^{n+1}}{n+1} \right|_a^b$$

* $x = 0 \Rightarrow y = 1$
* $x = 8 \Rightarrow y = 5$

$$x = (y-1)^{\frac{3}{2}} , \qquad 1 \le y \le 5$$

$$L = \int_1^5 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$\frac{dx}{dy} = \frac{3}{2} (y-1)^{\frac{1}{2}}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{9}{4} (y-1)$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{9}{4} y - \frac{9}{4}$$

$$= \frac{9}{4} y - \frac{5}{4}$$

$$= \frac{1}{4} (9y - 5)$$

$$L = \int_1^5 \sqrt{\frac{1}{4}(9y-5)}\, dy$$

$$= \frac{1}{2\cdot 9}\int_1^5 9(9y-5)^{\frac{1}{2}}\, dy$$

$$= \frac{1}{18}\int_1^5 9(9y-5)^{\frac{1}{2}}\, dy$$

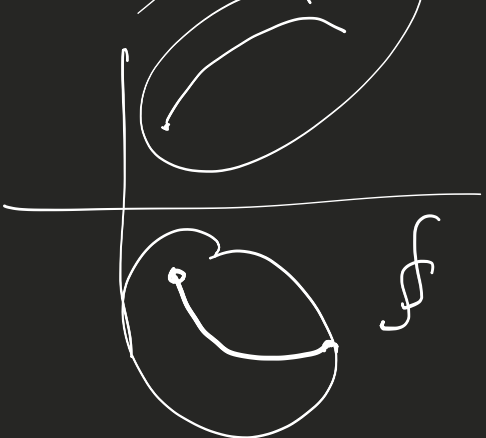
$$= \frac{1}{18}\, \frac{(9y-5)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_1^5$$

$$4^{\frac{3}{2}} = 4\sqrt{4}$$

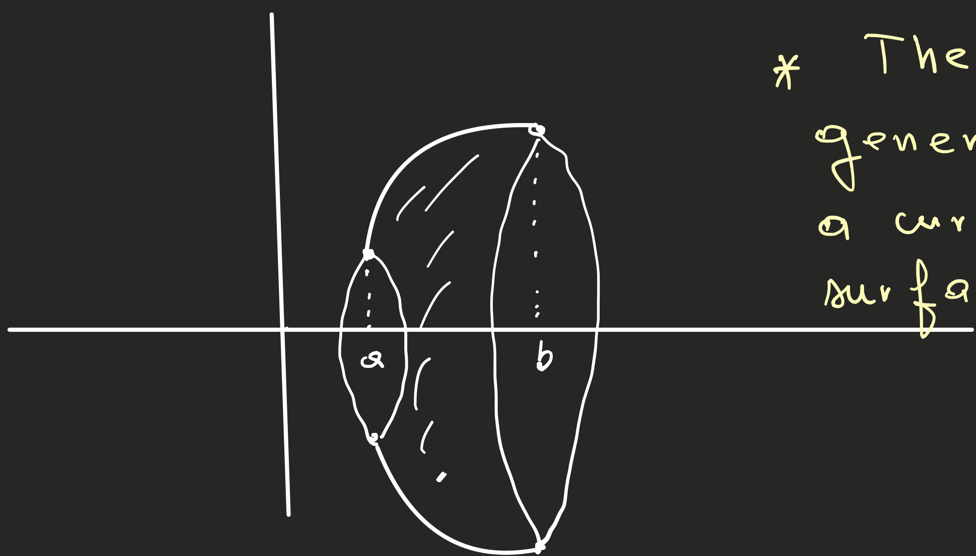
$$\sqrt{40} = \sqrt{4\cdot 10} = 2\sqrt{10}$$

$$= \frac{1}{18}\left(\frac{2}{3}\left(40^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)\right) = \frac{1}{27}\left(40\sqrt{40} - 4\sqrt{4}\right)$$

$$= \frac{1}{27}\left(80\sqrt{10} - 8\right).$$



§ Area of surface of Revolution



* The surface generated by rotating a curve is called a surface of revolution.

* If f and f' are both continuous on $[a,b]$ then the area S of the surface of revolution

obtained is given by:

A) $$S = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$$ if the rotation is about the x-axis with $\underbrace{f(x) \geq 0}$ on $[a,b]$.

B) $$S = 2\pi \int_a^b x\sqrt{1+[f'(x)]^2}\,dx$$ if the rotation is about the y-axis

Exple Find the area of the surface obtained by revolving the graph of

A) $f(x) = x^3$ on the interval $[0, 1]$ about the x-axis.

A) $$S = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$= 2\pi \int_0^1 x^3\sqrt{1+(3x^2)^2}\,dx$$

$$= \frac{2\pi}{36} \int_0^1 36x^3 (1+9x^4)^{\frac{1}{2}}\,dx$$

$$= \frac{\pi}{18} \left. \frac{(1+9x^4)^{3/2}}{3/2} \right|_0^1$$

$$= \frac{\pi}{18} \left(\frac{2}{3} (10\sqrt{10} - 1 \right)$$

$$= \frac{\pi}{27} (10\sqrt{10} - 1).$$

B) $f(x) = x^2$ on $[0, \sqrt{2}]$ about the y-axis

$$S = 2\pi \int_a^b x \sqrt{1+[f'(x)]^2}\, dx$$

$$= 2\pi \int_0^{\sqrt{2}} x \sqrt{1+(2x)^2}\, dx$$

$$= \frac{2\pi}{8} \int_0^{\sqrt{2}} 8x \, (\underline{1+4x^2})^{\frac{1}{2}} dx$$

$$= \frac{\pi}{4} \left. \frac{2}{3} (1+4x^2)^{\frac{3}{2}} \right|_0^{\sqrt{2}}$$

$$= \frac{\pi}{6} (9^{\frac{3}{2}} - 1) = \frac{\pi}{6} (9\sqrt{9} - 1)$$

$$= \frac{\pi}{6} (26) = \frac{13\pi}{3}.$$

§ Volume of Solid of Revolution

* The solid that is obtained by rotating a surface is called a solid of revolution.

* To find the volume of a solid of revolution we will two methods
    * The disc method
    * The shell method

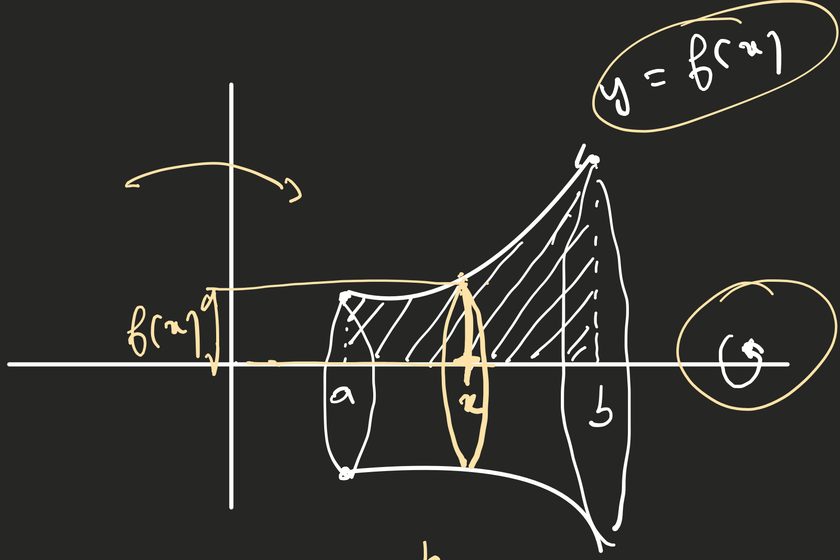
$$\text{Volume of solid of revolution} = \int_a^b \text{Area of slice } dx \text{ or } dy$$

* The disc method (Washer method).

* In the disc method the slice is always a disc and is chosen perpendicular to the axis of revolution

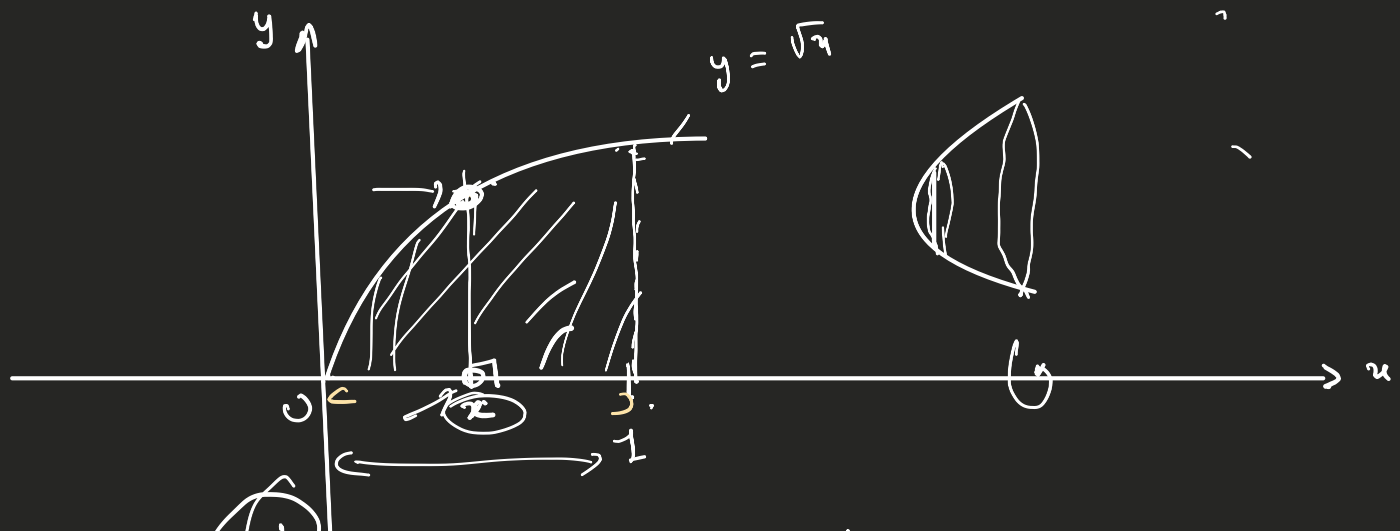
πr^2

πr^2



$$V = \int_a^b \pi [f(x)]^2 \, dx$$

Exple Find the Volume of the Solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$.



$$V = \int_0^1 \pi (\sqrt{x})^2 \, dx = \int_0^1 \pi x \, dx = \frac{\pi}{2}$$