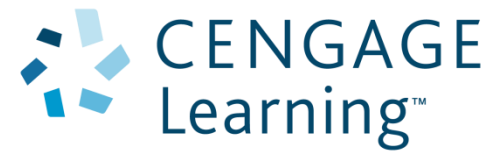


Chapter 32

Inductance



Inductance

Self-inductance

- A time-varying current in a circuit produces an induced emf opposing the emf that initially set up the time-varying current.
 - Basis of the electrical circuit element called an *inductor*
- Energy is stored in the magnetic field of an inductor.
- There is an energy density associated with the magnetic field.

Mutual induction

- An emf is induced in a coil as a result of a changing magnetic flux produced by a second coil.

Circuits may contain inductors as well as resistors and capacitors.

Joseph Henry

1797 – 1878

American physicist

First director of the Smithsonian

First president of the Academy of Natural Science

Improved design of electromagnet

Constructed one of the first motors

Discovered self-inductance

- Didn't publish his results

Unit of inductance is named in his honor



Some Terminology

Use *emf* and *current* when they are caused by batteries or other sources.

Use *induced emf* and *induced current* when they are caused by changing magnetic fields.

When dealing with problems in electromagnetism, it is important to distinguish between the two situations.

Self-Inductance

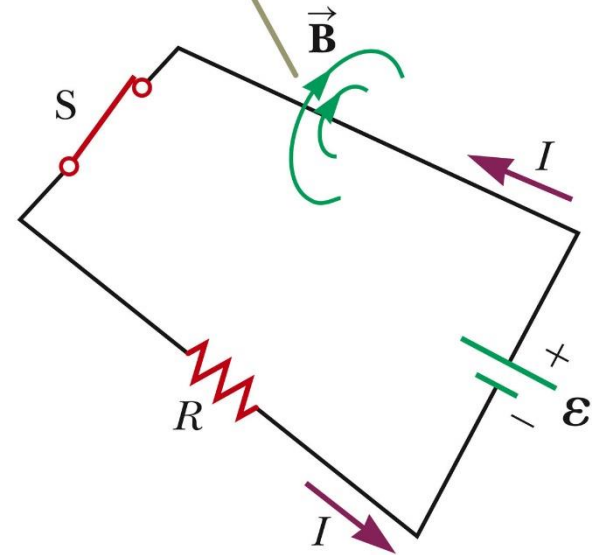
When the switch is closed, the current does not immediately reach its maximum value.

Faraday's law of electromagnetic induction can be used to describe the effect.

As the current increases with time, the magnetic flux through the circuit loop due to this current also increases with time.

This increasing flux creates an induced emf in the circuit.

After the switch is closed, the current produces a magnetic flux through the area enclosed by the loop. As the current increases toward its equilibrium value, this magnetic flux changes in time and induces an emf in the loop.



Self-Inductance, cont.

The direction of the induced emf is such that it would cause an induced current in the loop which would establish a magnetic field opposing the change in the original magnetic field.

The direction of the induced emf is opposite the direction of the emf of the battery.

This results in a gradual rather than instantaneous increase in the current to its final equilibrium value.

This effect is called **self-inductance**.

- Because the changing flux through the circuit and the resultant induced emf arise from the circuit itself.

The emf ε_L is called a **self-induced emf**.

Self-Inductance, Equations

An induced emf is always proportional to the time rate of change of the current.

- The emf is proportional to the flux, which is proportional to the field and the field is proportional to the current.

$$\varepsilon_L = -L \frac{dI}{dt}$$

L is a constant of proportionality called the **inductance** of the coil.

- It depends on the geometry of the coil and other physical characteristics.

Inductance of a Coil

A closely spaced coil of N turns carrying current i has an induced emf of

$$\mathcal{E}_L = -N d\Phi_B/dt.$$

Which leads to an inductance of

$$L = \frac{N\Phi_B}{i}$$

$$L = -\frac{\mathcal{E}_L}{di/dt}$$

The inductance is a measure of the opposition to a *change* in current.

The resistance is a measure of the opposition to current.

Inductance Units

The SI unit of inductance is the **henry** (H)

$$1\text{H} = 1 \frac{\text{V} \cdot \text{s}}{\text{A}}$$

Named for Joseph Henry

Inductance of a Solenoid, Example

Consider a uniformly wound solenoid having N turns and length ℓ . Assume ℓ is much longer than the radius of the windings and the core of the solenoid is air.

(A) Find the inductance of the solenoid.

$$\Phi_B = BA = \mu_0 n i A = \mu_0 \frac{N}{\ell} i A \quad n = N/\ell \text{ is the number of turns per unit length}$$

$$L = \frac{N\Phi_B}{i} = \mu_0 \frac{N^2}{\ell} A \quad \text{Or by using } N = n\ell \Rightarrow L = \mu_0 \frac{(n\ell)^2}{\ell} A = \mu_0 n^2 A \ell = \mu_0 n^2 V$$

where $V = A\ell$ is the interior volume of the solenoid.

(B) Calculate the inductance of the solenoid if it contains 300 turns, its length is 25.0 cm, and its cross-sectional area is 4.00 cm^2 .

$$L = (4\pi \times 10^{-7} \text{ T} \cdot \text{m/A}) \frac{300^2}{25.0 \times 10^{-2} \text{ m}} (4.00 \times 10^{-4} \text{ m}^2) \\ = 1.81 \times 10^{-4} \text{ T} \cdot \text{m}^2/\text{A} = 0.181 \text{ mH}$$

(C) Calculate the self-induced emf in the solenoid if the current it carries decreases at the rate of 50.0 A/s .

$$\mathcal{E}_L = -L \frac{di}{dt} = -(1.81 \times 10^{-4} \text{ H})(-50.0 \text{ A/s}) \\ = 9.05 \text{ mV}$$

RL Circuit, Introduction

A circuit element that has a large self-inductance is called an **inductor**.

The circuit symbol is 

We assume the self-inductance of the rest of the circuit is negligible compared to the inductor.

- However, even without a coil, a circuit will have some self-inductance.

Effect of an Inductor in a Circuit

The inductance results in a back emf.

Therefore, the inductor in a circuit opposes changes in current in that circuit.

- The inductor attempts to keep the current the same way it was before the change occurred.
- The inductor can cause the circuit to be “sluggish” as it reacts to changes in the voltage.

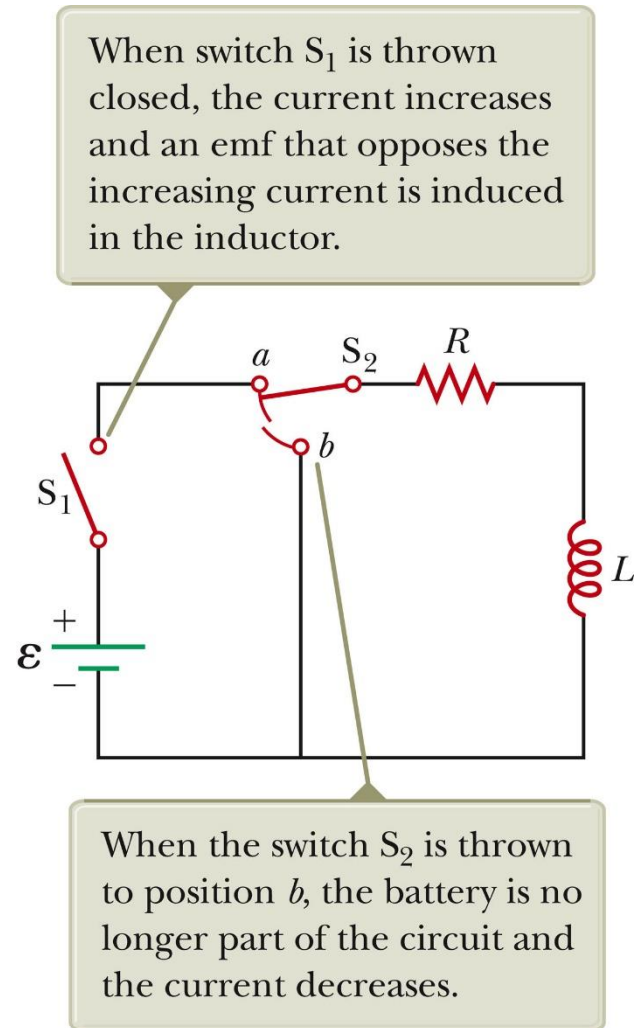
RL Circuit, Analysis

An RL circuit contains an inductor and a resistor.

Assume S_2 is connected to a

When switch S_1 is closed (at time $t = 0$), the current begins to increase.

At the same time, a back emf is induced in the inductor that opposes the original increasing current.



RL Circuit, Analysis, cont.

Applying Kirchhoff's loop rule to the previous circuit in the clockwise direction gives

$$\mathcal{E} - iR - L \frac{di}{dt} = 0$$

Looking at the current, we find

$$i = \frac{\mathcal{E}}{R} (1 - e^{-Rt/L})$$

RL Circuit, Analysis, Final

The inductor affects the current exponentially.

The current does not instantly increase to its final equilibrium value.

If there is no inductor, the exponential term goes to zero and the current would instantaneously reach its maximum value as expected.

RL Circuit, Time Constant

The expression for the current can also be expressed in terms of the time constant, τ , of the circuit.

$$i = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau})$$

- where $\tau = L / R$

Physically, τ is the time required for the current to reach 63.2% of its maximum value \mathcal{E}/R .

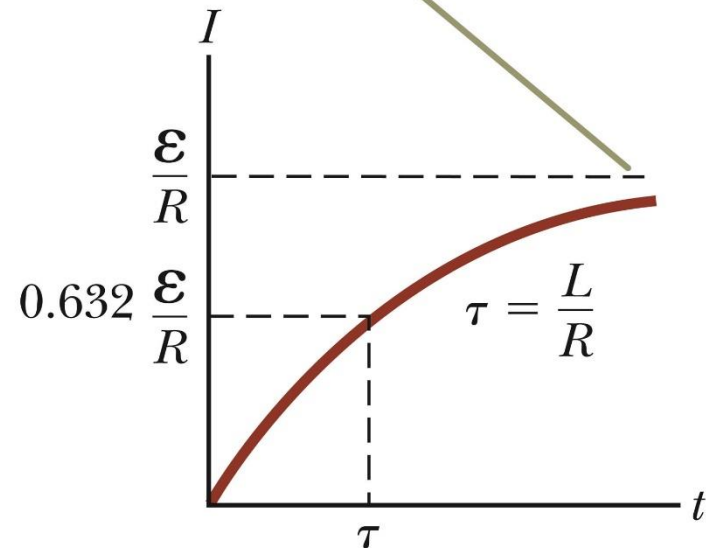
RL Circuit, Current-Time Graph, Charging

The equilibrium value of the current is \mathcal{E}/R and is reached as t approaches infinity.

The current initially increases very rapidly.

The current then gradually approaches the equilibrium value.

After switch S_1 is thrown closed at $t = 0$, the current increases toward its maximum value \mathcal{E}/R .



RL Circuit, Current-Time Graph

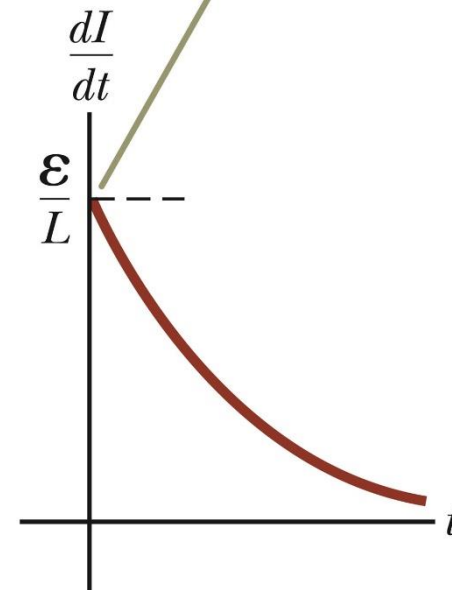
The time rate of change of the current is a maximum at $t = 0$.

It falls off exponentially as t approaches infinity.

In general,

$$\frac{di}{dt} = \frac{\mathcal{E}}{L} e^{-t/\tau}$$

The time rate of change of current is a maximum at $t = 0$, which is the instant at which switch S_1 is thrown closed.



RL Circuit Without A Battery

Now set S_2 to position b

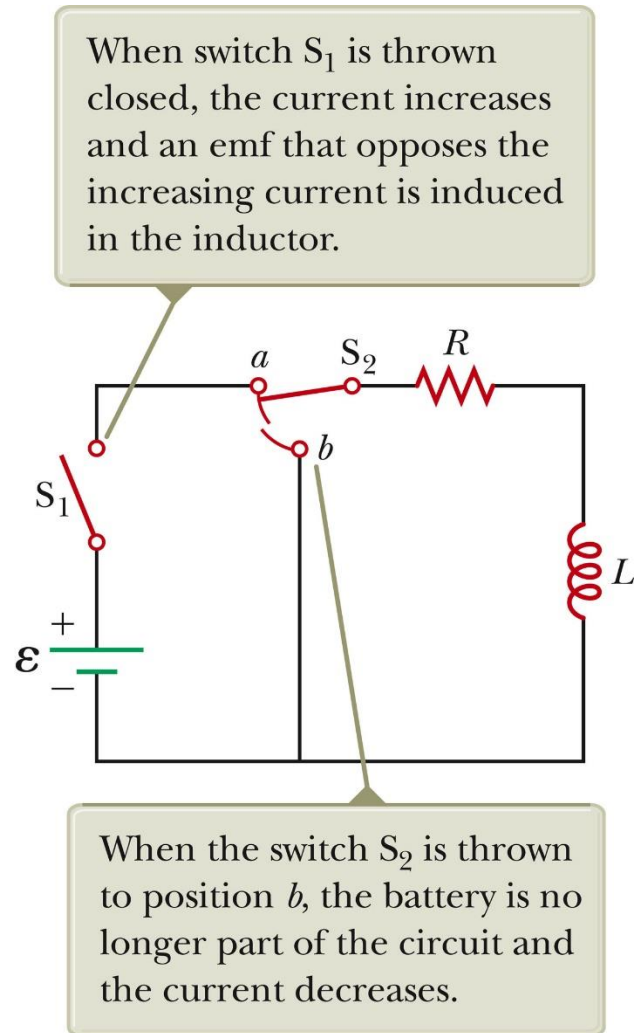
The circuit now contains just the right hand loop .

The battery has been eliminated $\mathcal{E} = 0$.

The expression for the current becomes

$$i = \frac{\mathcal{E}}{R} e^{-t/\tau} = I_i e^{-t/\tau}$$

If the circuit did not contain an inductor, the current would immediately decrease to zero when the battery is removed. When the inductor is present, it opposes the decrease in the current and causes the current to decrease exponentially.



Time Constant of an RL Circuit, Example

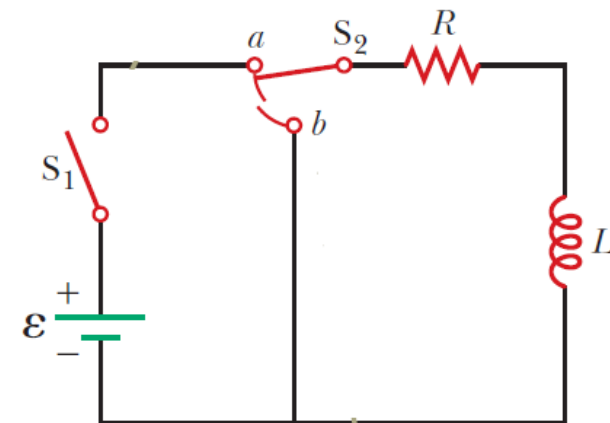
Consider the circuit in Figure 32.2 again. Suppose the circuit elements have the following values: $\mathcal{E} = 12.0 \text{ V}$, $R = 6.00 \Omega$, and $L = 30.0 \text{ mH}$.

(A) Find the time constant of the circuit.

$$\tau = \frac{L}{R} = \frac{30.0 \times 10^{-3} \text{ H}}{6.00 \Omega} = 5.00 \text{ ms}$$

(B) Switch S_2 is at position a , and switch S_1 is thrown closed at $t = 0$. Calculate the current in the circuit at $t = 2.00 \text{ ms}$.

$$\begin{aligned} i &= \frac{\mathcal{E}}{R} (1 - e^{-t/\tau}) = \frac{12.0 \text{ V}}{6.00 \Omega} (1 - e^{-2.00 \text{ ms}/5.00 \text{ ms}}) = 2.00 \text{ A} (1 - e^{-0.400}) \\ &= 0.659 \text{ A} \end{aligned}$$



Energy in a Magnetic Field

In a circuit with an inductor, the battery must supply more energy than in a circuit without an inductor.

Part of the energy supplied by the battery appears as internal energy in the resistor.

The remaining energy is stored in the magnetic field of the inductor.

Energy in a Magnetic Field, cont.

Looking at this energy (in terms of rate)

$$I\mathcal{E} = I^2 R + LI \frac{dI}{dt}$$

- $I\mathcal{E}$ is the rate at which energy is being supplied by the battery.
- $I^2 R$ is the rate at which the energy is being delivered to the resistor.
- Therefore, $LI (dI/dt)$ must be the rate at which the energy is being stored in the magnetic field.

Energy in a Magnetic Field, final

Let U denote the energy stored in the inductor at any time.

The rate at which the energy is stored is

$$\frac{dU}{dt} = LI \frac{dI}{dt}$$

To find the total energy, integrate and

$$U = L \int_0^I I \, dI = \frac{1}{2} LI^2$$

Energy Density of a Magnetic Field

Given $U = \frac{1}{2} L I^2$ and assume (for simplicity) a solenoid with $L = \mu_0 n^2 V$

$$U = \frac{1}{2} \mu_0 n^2 V \left(\frac{B}{\mu_0 n} \right)^2 = \frac{B^2}{2\mu_0} V$$

Since V is the volume of the solenoid, the magnetic energy density, u_B is

$$u_B = \frac{U}{V} = \frac{B^2}{2\mu_0}$$

This applies to any region in which a magnetic field exists (not just the solenoid).

Energy Storage Summary

A resistor, inductor and capacitor all store energy through different mechanisms.

- Charged capacitor
 - Stores energy as electric potential energy
- Inductor
 - When it carries a current, stores energy as magnetic potential energy
- Resistor
 - Energy delivered is transformed into internal energy

Example: The Coaxial Cable

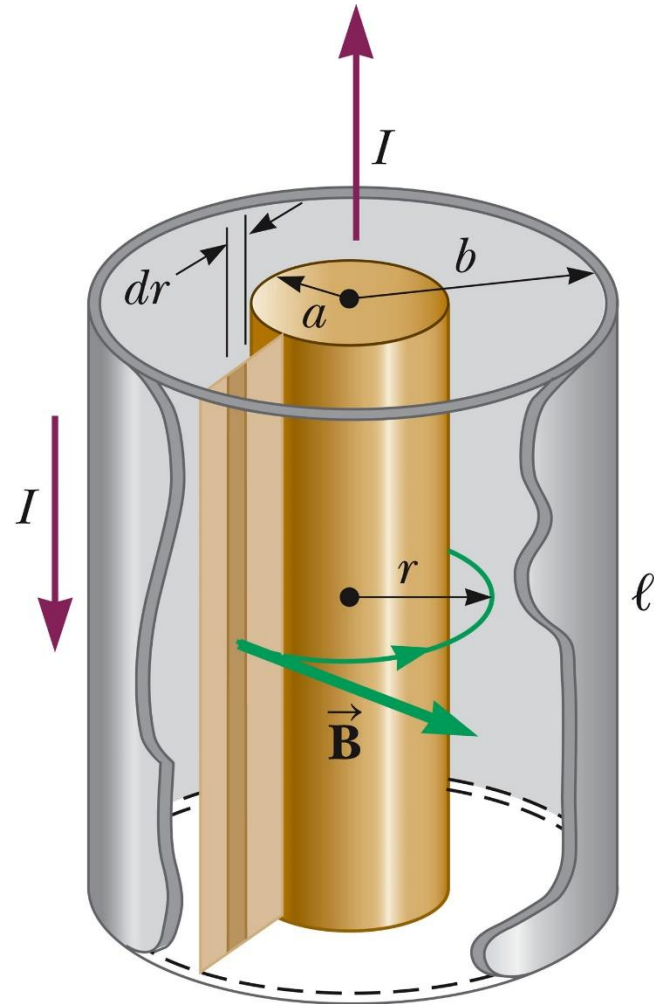
Calculate L of a length ℓ for the cable

The total flux is

$$\begin{aligned}\Phi_B &= \int B \, dA = \int_a^b \frac{\mu_o I}{2\pi r} \ell \, dr \\ &= \frac{\mu_o I \ell}{2\pi} \ln\left(\frac{b}{a}\right)\end{aligned}$$

Therefore, L is

$$L = \frac{\Phi_B}{I} = \frac{\mu_o \ell}{2\pi} \ln\left(\frac{b}{a}\right)$$



Mutual Inductance

The magnetic flux through the area enclosed by a circuit often varies with time because of time-varying currents in nearby circuits.

This process is known as *mutual induction* because it depends on the interaction of two circuits.

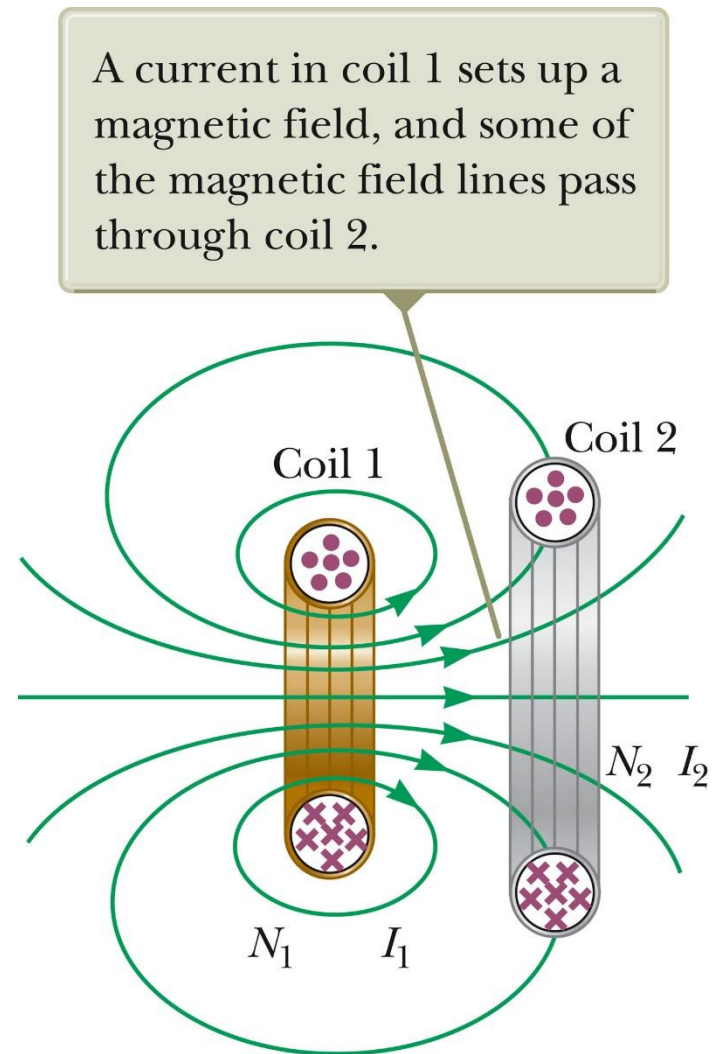
Mutual Inductance, cont.

The current in coil 1 sets up a magnetic field.

Some of the magnetic field lines pass through coil 2.

Coil 1 has a current I_1 and N_1 turns.

Coil 2 has N_2 turns.



Mutual Inductance, final

The magnetic flux caused by the current in coil 1 and passing through coil 2 is represented Φ_{12} .

The **mutual inductance** M_{12} of coil 2 with respect to coil 1 is

$$M_{12} \equiv \frac{N_2 \Phi_{12}}{I_1}$$

Mutual inductance depends on the geometry of both circuits and on their orientation with respect to each other.

As the circuit separation distance increases, the mutual inductance decreases because the flux linking the circuits decreases.

Induced emf in Mutual Inductance

If current i_1 varies with time, the emf induced by coil 1 in coil 2 is

$$\mathcal{E}_2 = -N_2 \frac{d\Phi_{12}}{dt} = -N_2 \frac{d}{dt} \left(\frac{M_{12}i_1}{N_2} \right) = -M_{12} \frac{di_1}{dt}$$

If the current is in coil 2, there is a mutual inductance M_{21} .

If current 2 varies with time, the emf induced by coil 2 in coil 1 is

$$\mathcal{E}_1 = -M_{21} \frac{di_2}{dt}$$

Induced emf in Mutual Inductance, cont.

In mutual induction, the emf induced in one coil is always proportional to the rate at which the current in the other coil is changing.

The mutual inductance in one coil is equal to the mutual inductance in the other coil.

- $M_{12} = M_{21} = M$

The induced emf's can be expressed as

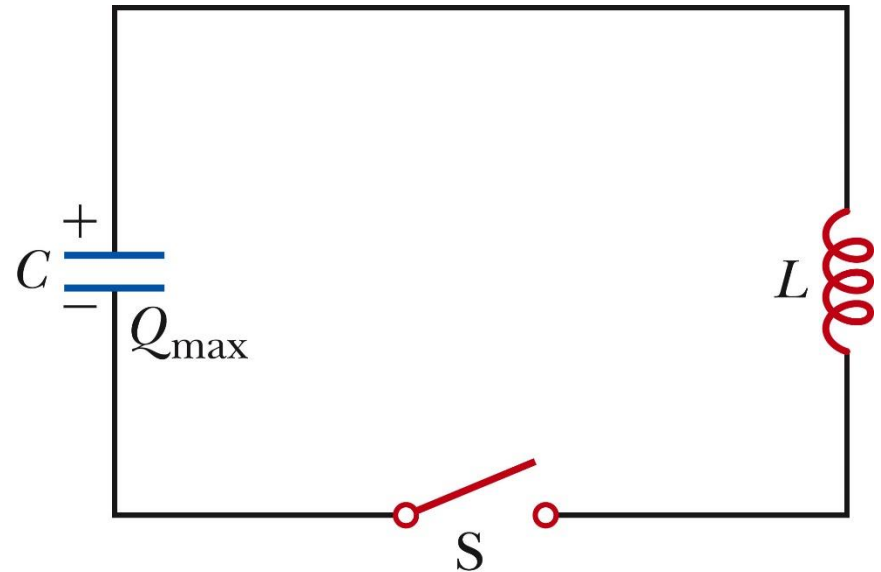
$$\mathcal{E}_2 = -M \frac{di_1}{dt} \quad \text{and} \quad \mathcal{E}_1 = -M \frac{di_2}{dt}$$

LC Circuits

A capacitor is connected to an inductor in an LC circuit.

Assume the capacitor is initially charged and then the switch is closed.

Assume no resistance and no energy losses to radiation.



Oscillations in an LC Circuit

Under the previous conditions, the current in the circuit and the charge on the capacitor oscillate between maximum positive and negative values.

With zero resistance, no energy is transformed into internal energy.

Ideally, the oscillations in the circuit persist indefinitely.

- The idealizations are no resistance and no radiation.

The capacitor is fully charged.

- The energy U in the circuit is stored in the electric field of the capacitor.
- The energy is equal to $Q_{\text{max}}^2 / 2C$.
- The current in the circuit is zero.
- No energy is stored in the inductor.

The switch is closed.

Oscillations in an LC Circuit, cont.

The current is equal to the rate at which the charge changes on the capacitor.

- As the capacitor discharges, the energy stored in the electric field decreases.
- Since there is now a current, some energy is stored in the magnetic field of the inductor.
- Energy is transferred from the electric field to the magnetic field.

Eventually, the capacitor becomes fully discharged.

- It stores no energy.
- All of the energy is stored in the magnetic field of the inductor.
- The current reaches its maximum value.

The current now decreases in magnitude, recharging the capacitor with its plates having opposite their initial polarity.

Oscillations in an LC Circuit, final

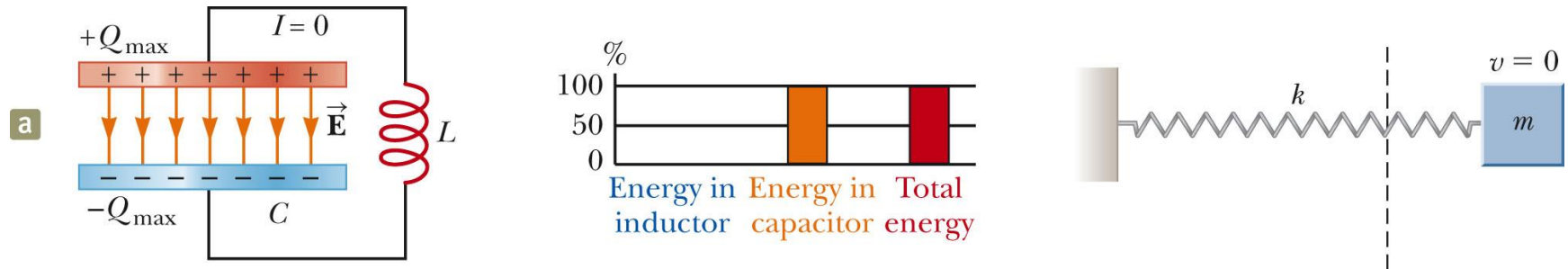
The capacitor becomes fully charged and the cycle repeats.

The energy continues to oscillate between the inductor and the capacitor.

The total energy stored in the LC circuit remains constant in time and equals.

$$U = U_E + U_B = \frac{q^2}{2C} + \frac{1}{2}Li^2 = \frac{Q_{\max}^2}{2C}$$

LC Circuit Analogy to Spring-Mass System, 1

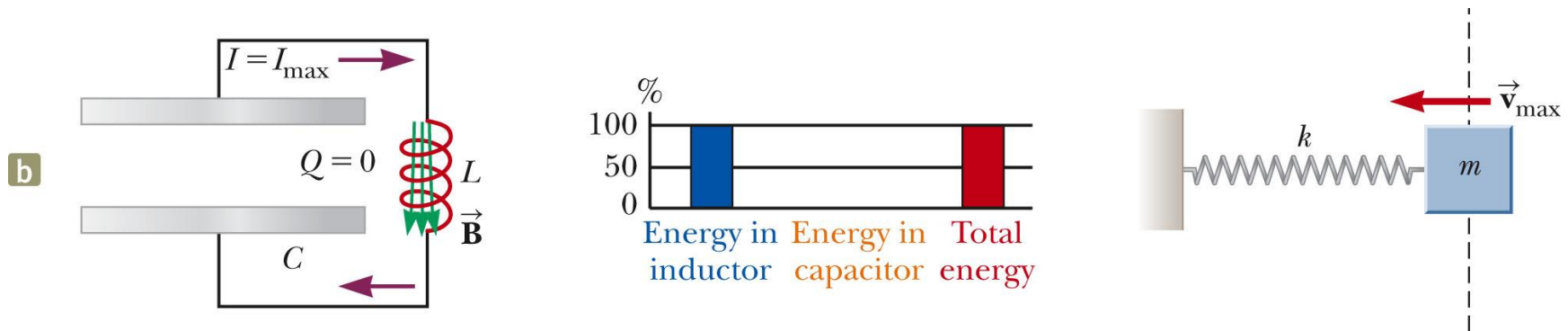


The potential energy $\frac{1}{2}kx^2$ stored in the spring is analogous to the electric potential energy $(Q_{\max})^2/(2C)$ stored in the capacitor.

All the energy is stored in the capacitor at $t = 0$.

This is analogous to the spring stretched to its amplitude.

LC Circuit Analogy to Spring-Mass System, 2



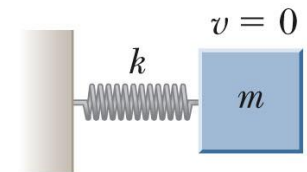
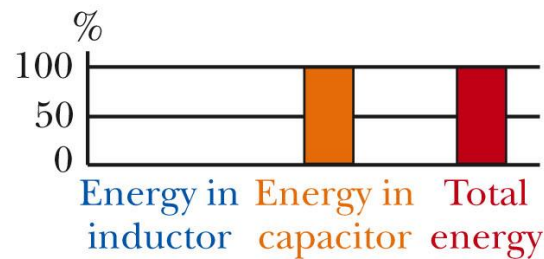
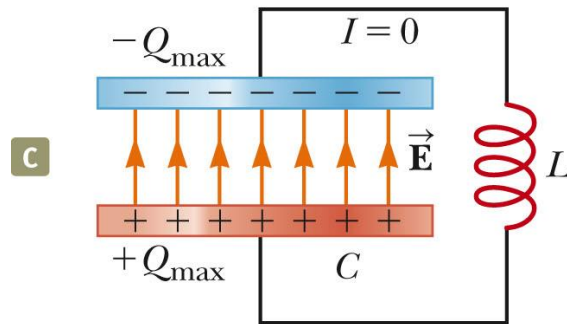
The kinetic energy ($\frac{1}{2} mv^2$) of the spring is analogous to the magnetic energy ($\frac{1}{2} L I^2$) stored in the inductor.

At $t = \frac{1}{4} T$, all the energy is stored as magnetic energy in the inductor.

The maximum current occurs in the circuit.

This is analogous to the mass at equilibrium.

LC Circuit Analogy to Spring-Mass System, 3

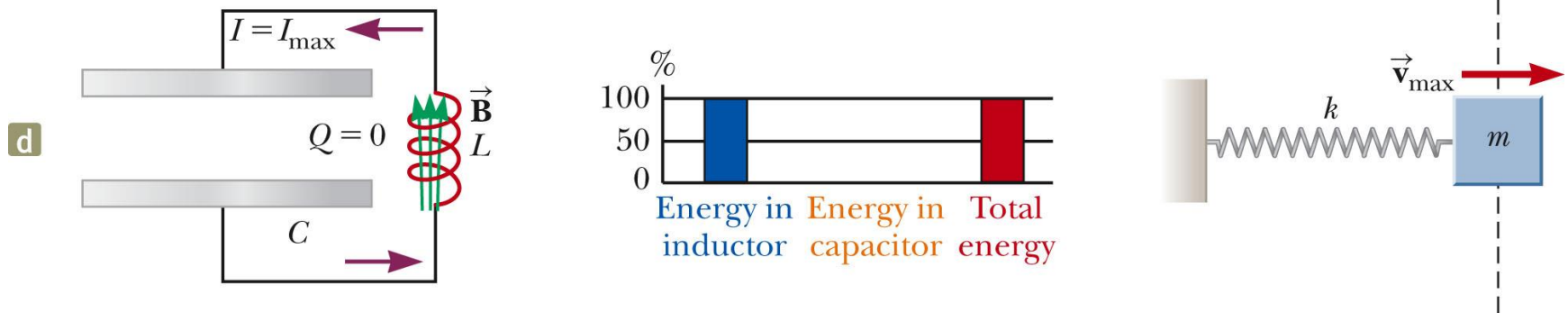


At $t = \frac{1}{2}T$, the energy in the circuit is completely stored in the capacitor.

The polarity of the capacitor is reversed.

This is analogous to the spring stretched to $-A$.

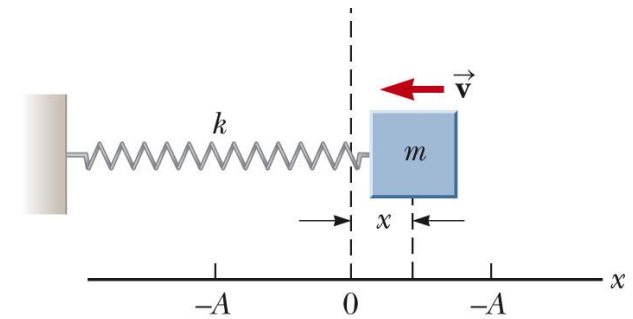
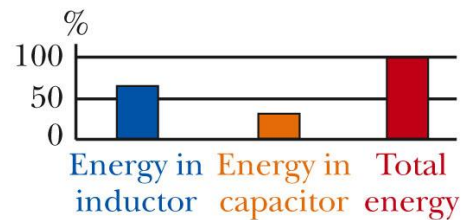
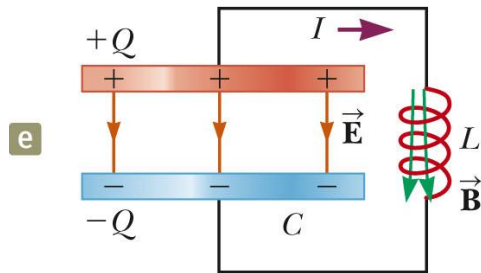
LC Circuit Analogy to Spring-Mass System, 4



At $t = \frac{3}{4} T$, the energy is again stored in the magnetic field of the inductor.

This is analogous to the mass again reaching the equilibrium position.

LC Circuit Analogy to Spring-Mass System, 5



At $t = T$, the cycle is completed

The conditions return to those identical to the initial conditions.

At other points in the cycle, energy is shared between the electric and magnetic fields.

Time Functions of an LC Circuit

In an LC circuit, charge can be expressed as a function of time.

- $Q = Q_{\max} \cos (\omega t + \varphi)$
- This is for an ideal LC circuit

The angular frequency, ω , of the circuit depends on the inductance and the capacitance.

- It is the natural frequency of oscillation of the circuit.

$$\omega = \frac{1}{\sqrt{LC}}$$

Time Functions of an LC Circuit, cont.

The current can be expressed as a function of time:

$$i = \frac{dq}{dt} = -\omega Q_{\max} \sin(\omega t + \phi)$$

The total energy can be expressed as a function of time:

$$U = U_E + U_B = \frac{Q_{\max}^2}{2C} \cos^2 \omega t + \frac{1}{2} L I_{\max}^2 \sin^2 \omega t$$

Charge and Current in an LC Circuit

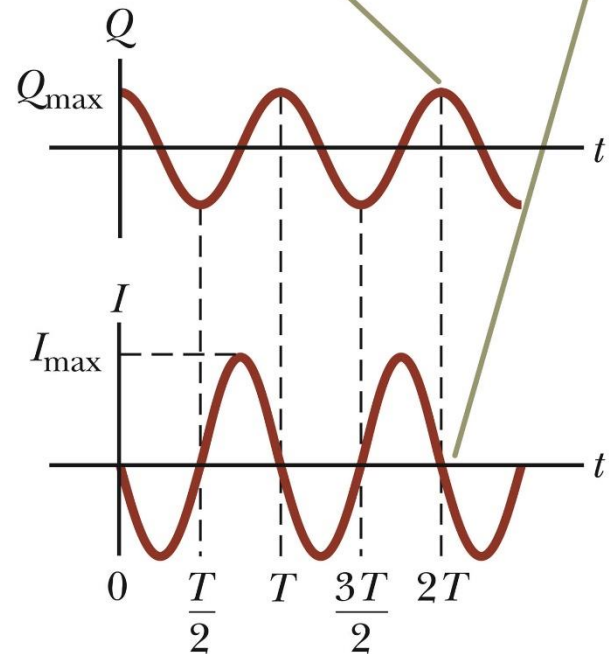
The charge on the capacitor oscillates between Q_{\max} and $-Q_{\max}$.

The current in the inductor oscillates between I_{\max} and $-I_{\max}$.

Q and I are 90° out of phase with each other

- So when Q is a maximum, I is zero, etc.

The charge Q and the current I are 90° out of phase with each other.



Energy in an LC Circuit – Graphs

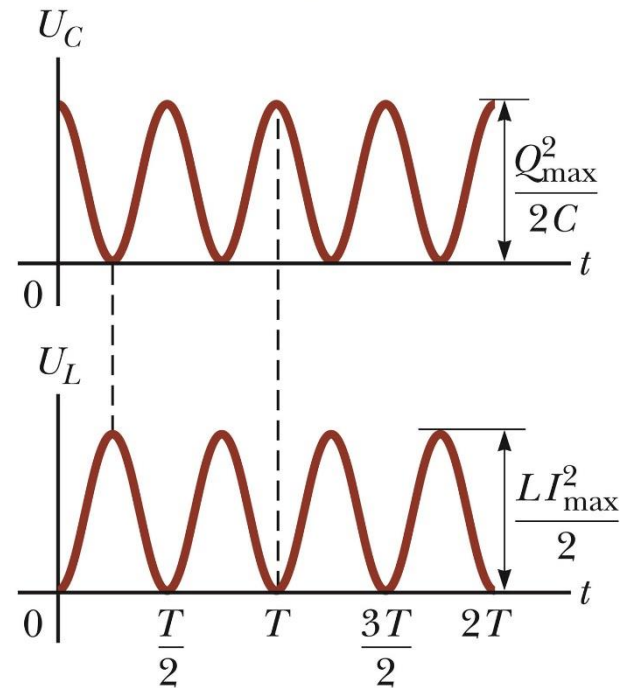
The energy continually oscillates between the energy stored in the electric and magnetic fields.

When the total energy is stored in one field, the energy stored in the other field is zero.

The amplitudes of the two graphs in must be equal because the maximum energy stored in the capacitor (when $I = 0$) must equal the maximum energy stored in the inductor (when $q = 0$).

$$\frac{Q_{\max}^2}{2C} = \frac{LI_{\max}^2}{2}$$

The sum of the two curves is a constant and is equal to the total energy stored in the circuit.



Notes About Real LC Circuits

In actual circuits, there is always some resistance.

Therefore, there is some energy transformed to internal energy.

Radiation is also inevitable in this type of circuit.

The total energy in the circuit continuously decreases as a result of these processes.

Oscillations in an LC Circuit, example

In Figure 32.14, the battery has an emf of 12.0 V, the inductance is 2.81 mH, and the capacitance is 9.00 pF. The switch has been set to position *a* for a long time so that the capacitor is charged. The switch is then thrown to position *b*, removing the battery from the circuit and connecting the capacitor directly across the inductor.

(A) Find the frequency of oscillation of the circuit.

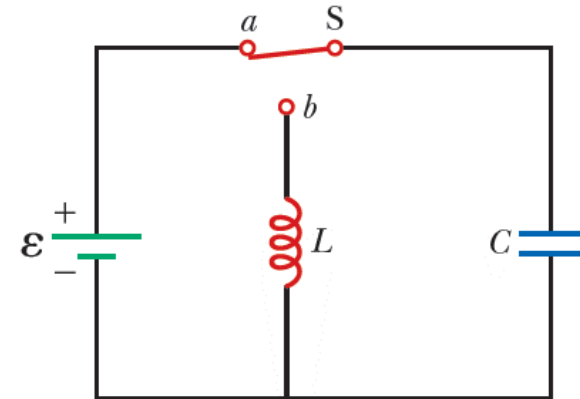
$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$f = \frac{1}{2\pi[(2.81 \times 10^{-3} \text{ H})(9.00 \times 10^{-12} \text{ F})]^{1/2}} = 1.00 \times 10^6 \text{ Hz}$$

(B) What are the maximum values of charge on the capacitor and current in the circuit?

$$Q_{\max} = C \Delta V = (9.00 \times 10^{-12} \text{ F})(12.0 \text{ V}) = 1.08 \times 10^{-10} \text{ C}$$

$$\begin{aligned} I_{\max} &= \omega Q_{\max} = 2\pi f Q_{\max} = (2\pi \times 10^6 \text{ s}^{-1})(1.08 \times 10^{-10} \text{ C}) \\ &= 6.79 \times 10^{-4} \text{ A} \end{aligned}$$

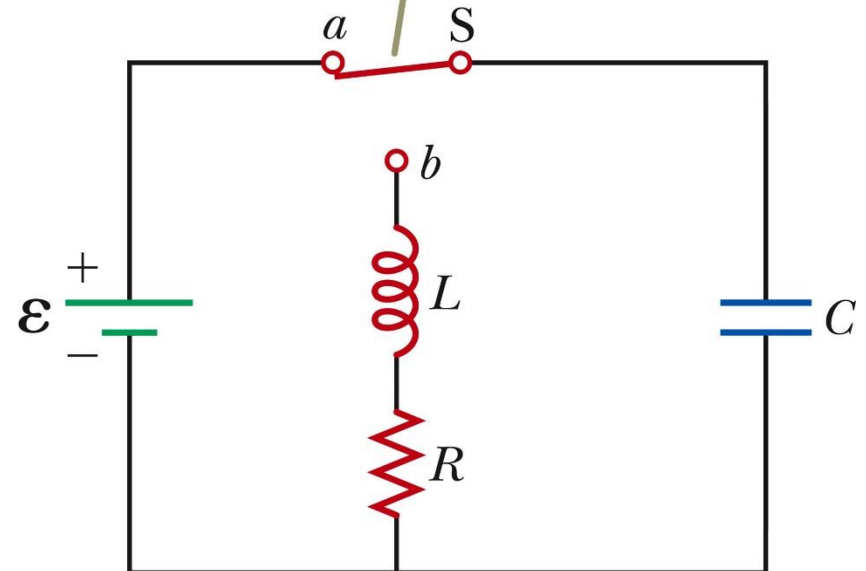


The RLC Circuit

A circuit containing a resistor, an inductor and a capacitor is called an RLC Circuit.

Assume the resistor represents the total resistance of the circuit.

The switch is set first to position a , and the capacitor is charged. The switch is then thrown to position b .



RLC Circuit, Analysis

The total energy is not constant, since there is a transformation to internal energy in the resistor at the rate of $dU/dt = -I^2 R$.

- Radiation losses are still ignored

The circuit's operation can be expressed as

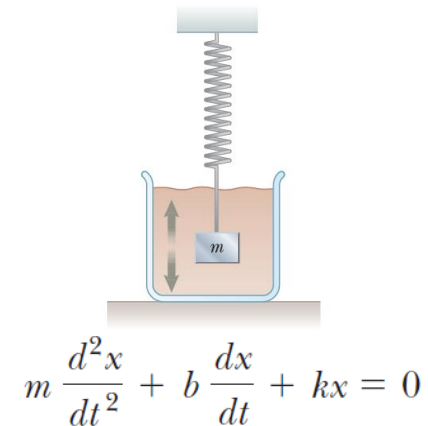
$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0$$

RLC Circuit Compared to Damped Oscillators

The RLC circuit is analogous to a damped harmonic oscillator.

When $R = 0$

- The circuit reduces to an LC circuit and is equivalent to no damping in a mechanical oscillator.



When R is small:

- The RLC circuit is analogous to light damping in a mechanical oscillator.

$$q = Q_{\max} e^{-Rt/2L} \cos \omega_d t$$

- ω_d is the angular frequency of oscillation for the circuit and $\omega_d = \left[\frac{1}{LC} - \left(\frac{R}{2L} \right)^2 \right]^{1/2}$

When $R \ll \sqrt{4L/C}$ (so that the second term in the brackets is much smaller than the first), the frequency ω_d of the damped oscillator is close to that of the undamped oscillator, $1/\sqrt{LC}$

RLC Circuit Compared to Damped Oscillators, cont.

When R is very large, the oscillations damp out very rapidly.

There is a critical value of R above which no oscillations occur.

$$R_C = \sqrt{4L/C}$$

If $R = R_C$, the circuit is said to be *critically damped*.

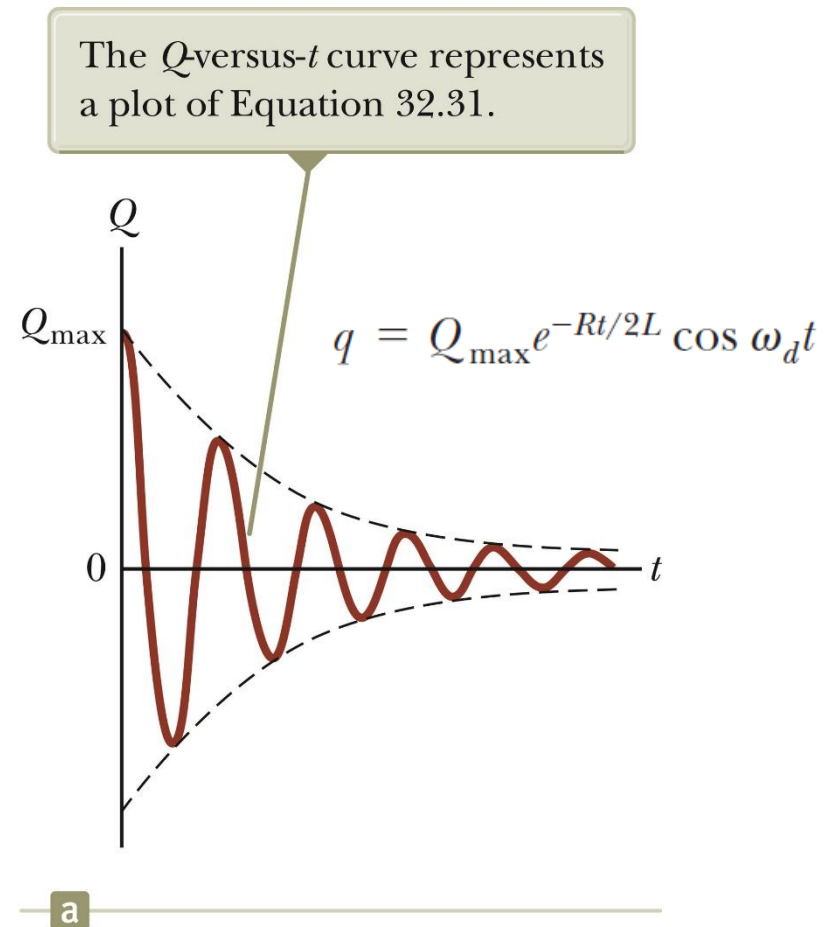
When $R > R_C$, the circuit is said to be *overdamped*.

Damped RLC Circuit, Graph

The maximum value of Q decreases after each oscillation.

- $R < R_C$

This is analogous to the amplitude of a damped spring-mass system.



Summary: Analogies Between Electrical and Mechanic Systems

TABLE 32.1 *Analogies Between Electrical and Mechanical Systems*

Electric Circuit		One-Dimensional Mechanical System
Charge	$Q \leftrightarrow x$	Position
Current	$I \leftrightarrow v_x$	Velocity
Potential difference	$\Delta V \leftrightarrow F_x$	Force
Resistance	$R \leftrightarrow b$	Viscous damping coefficient
Capacitance	$C \leftrightarrow 1/k$	(k = spring constant)
Inductance	$L \leftrightarrow m$	Mass
Current = time derivative of charge	$I = \frac{dQ}{dt} \leftrightarrow v_x = \frac{dx}{dt}$	Velocity = time derivative of position
Rate of change of current = second time derivative of charge	$\frac{dI}{dt} = \frac{d^2Q}{dt^2} \leftrightarrow a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$	Acceleration = second time derivative of position
Energy in inductor	$U_L = \frac{1}{2}LI^2 \leftrightarrow K = \frac{1}{2}mv^2$	Kinetic energy of moving object
Energy in capacitor	$U_C = \frac{1}{2}\frac{Q^2}{C} \leftrightarrow U = \frac{1}{2}kx^2$	Potential energy stored in a spring
Rate of energy loss due to resistance	$I^2R \leftrightarrow bv^2$	Rate of energy loss due to friction
RLC circuit	$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \leftrightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$	Damped object on a spring