Chapter 24

Gauss's Law

Gauss' Law

Gauss' Law can be used as an alternative procedure for calculating electric fields.

Gauss' Law is based on the inverse-square behavior of the electric force between point charges.

It is convenient for calculating the electric field of highly symmetric charge distributions.

Gauss' Law is important in understanding and verifying the properties of conductors in electrostatic equilibrium.

Electric Flux

Electric flux is the product of the magnitude of the electric field and the surface area, *A*, perpendicular to the field.

$\Phi_E = EA$

Units: $\mathrm{N \cdot m^2 / C}$

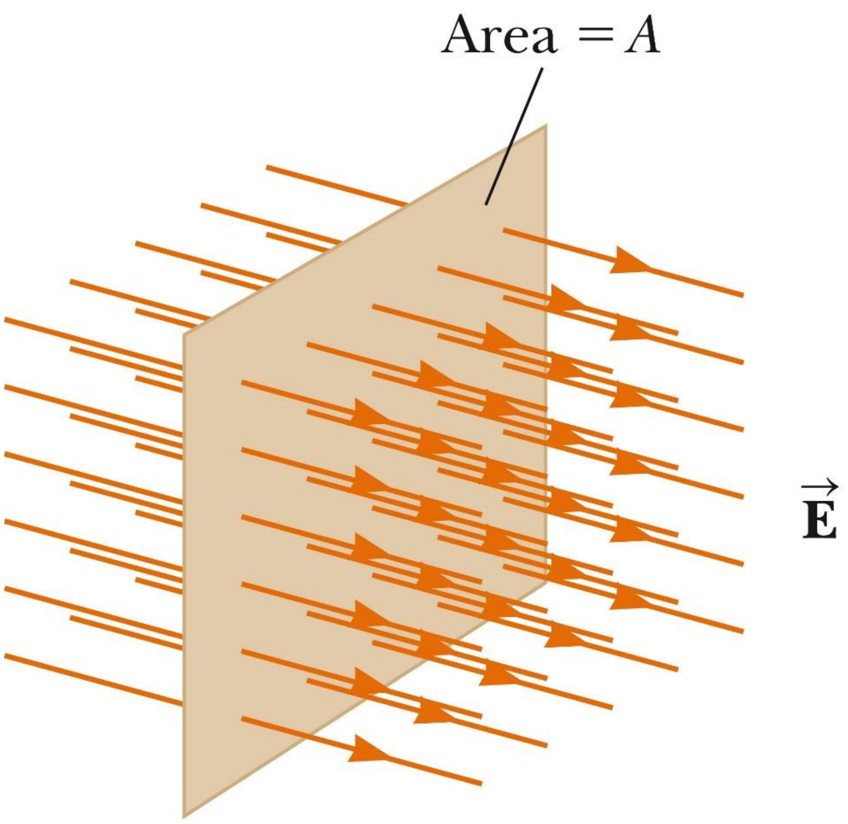


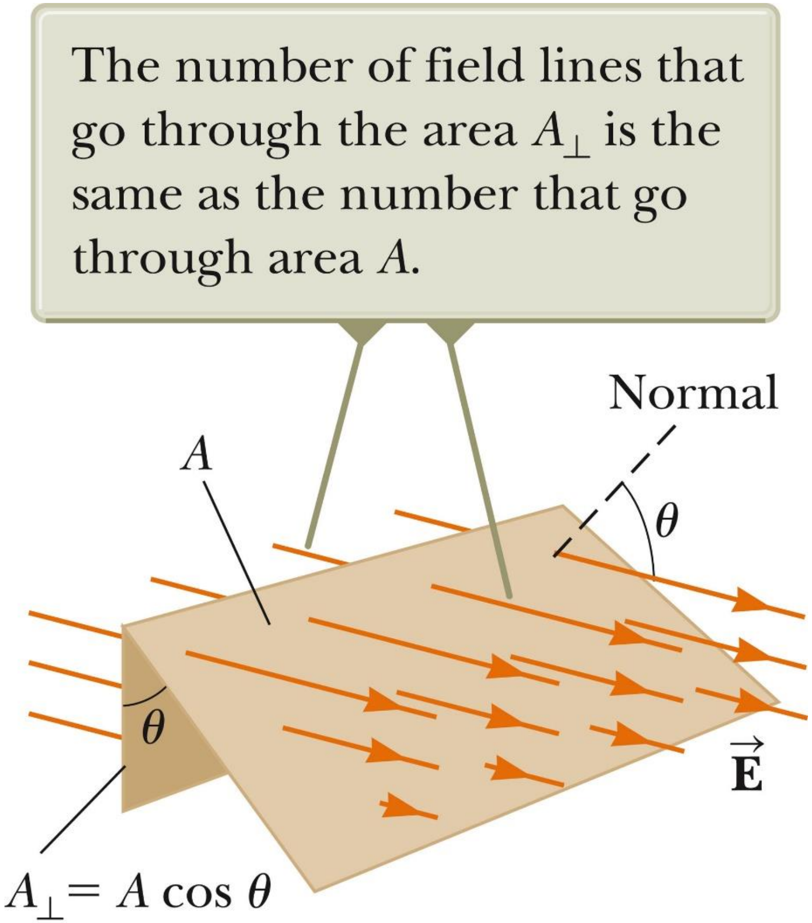
Figure 24.1 Field lines representing a uniform electric field penetrating a plane of area perpendicular to the field.

Electric Flux, General Area

The electric flux is proportional to the number of electric field lines penetrating some surface.

The field lines may make some angle θ with the perpendicular to the surface.

Then $\Phi_E = EA \cos\theta$



Electric Flux, Interpreting the Equation

The flux is a maximum when the surface is perpendicular to the field.

- $\theta = 0^\circ$

The flux is zero when the surface is parallel to the field.

- $\theta = 90^\circ$

If the field varies over the surface, $\Phi = EA \cos \theta$ is valid for only a small element of the area (over which the field is approximately constant).

We can also interpret the angle as that between the electric field vector and the normal to the surface. In this case, the product $E \cos \theta$ is the component of the electric field perpendicular to the surface.

The flux through the surface can then be written $\Phi_E = (E \cos \theta) A = E_n A$, where we use E_n as the component of the electric field normal to the surface.

Electric Flux, General

In the more general case, where electric field may vary over a large surface, look at a small area element.

$$\Delta\Phi_E = E_i \Delta A_i \cos\theta_i = \vec{\mathbf{E}}_i \cdot \Delta\vec{\mathbf{A}}_i$$

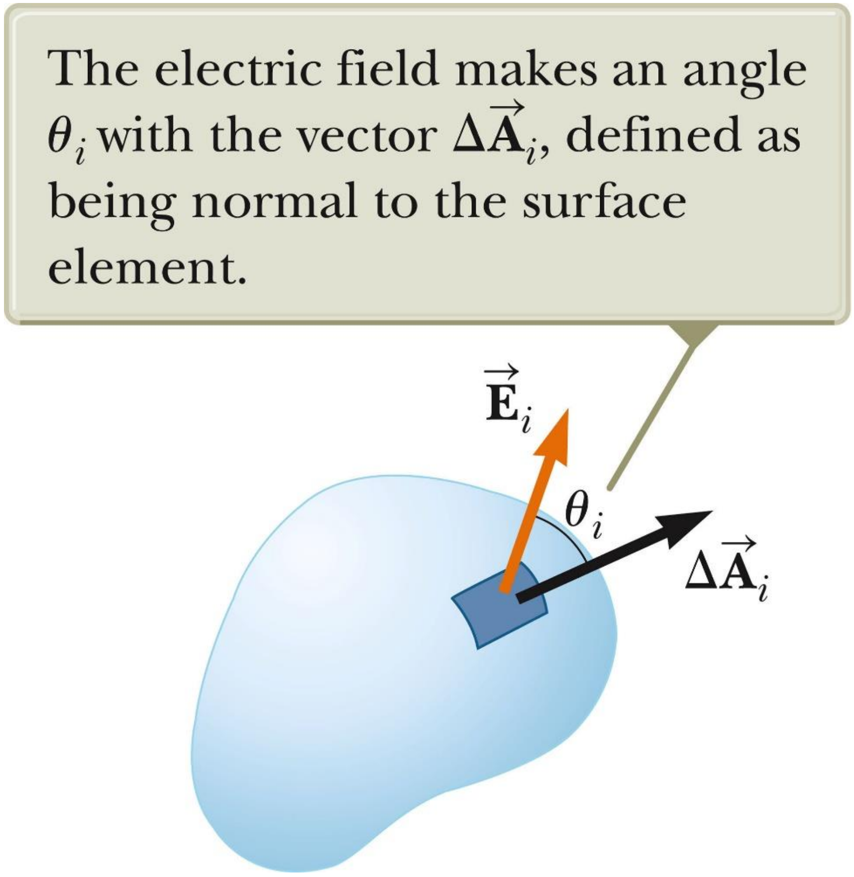
In general, this becomes

$$\Phi_E = \lim_{\Delta A_i \to 0} \sum E_i \cdot \Delta A_i$$

$$\Phi_E = \int_{\text{surface}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

- The surface integral means the integral must be evaluated over the surface in question.

In general, the value of the flux will depend both on the field pattern and on the surface.



Scalar Product: $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} \equiv AB\cos\theta$

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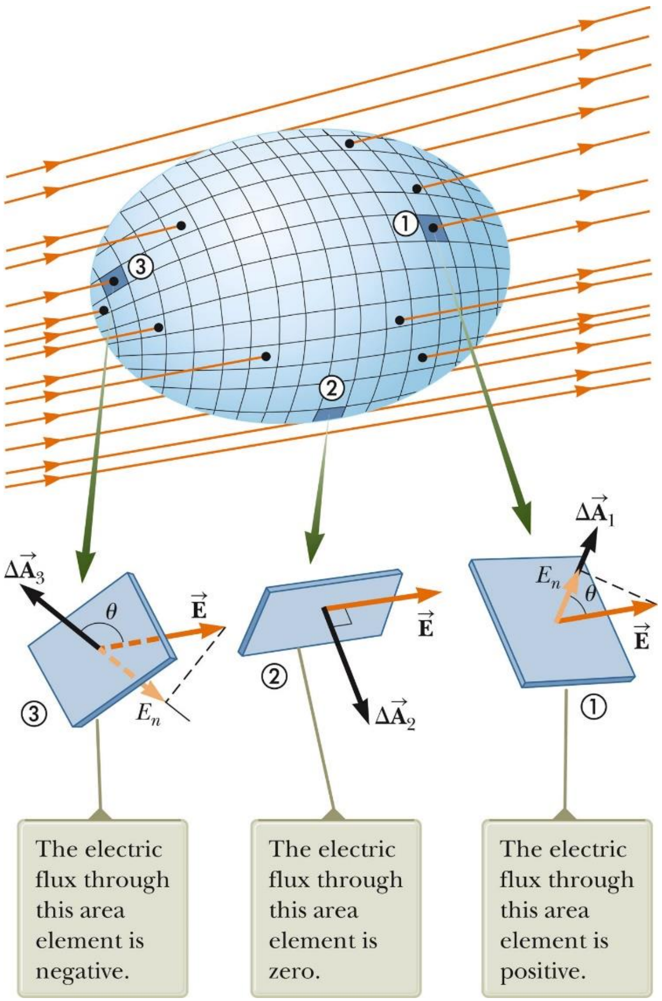
Electric Flux, Closed Surface

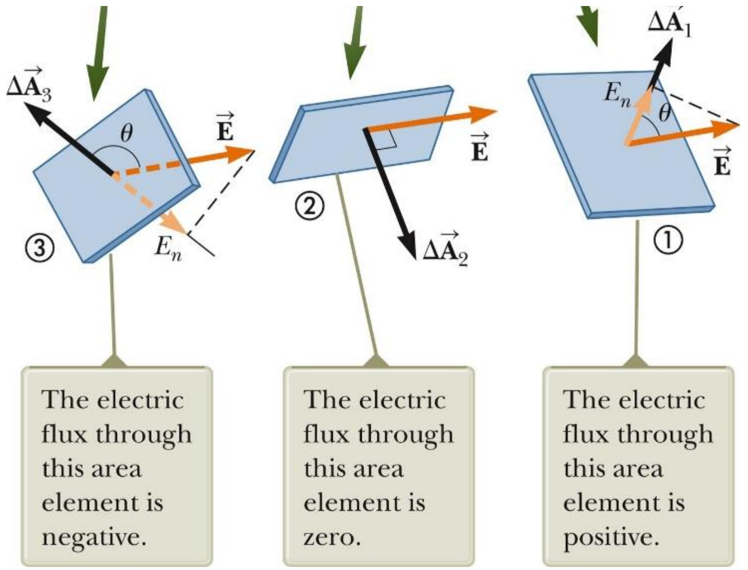
Assume a closed surface

The vectors $\Delta\vec{\mathbf{A}}_i$ point in different directions.

- At each point, they are perpendicular to the surface.
- By convention, they point outward.



Flux Through Closed Surface, cont.



At (1), the field lines are crossing the surface from the inside to the outside; $\theta < 90^o$, Φ is positive.

At (2), the field lines graze surface; $\theta = 90^o$, $\Phi = 0$

At (3), the field lines are crossing the surface from the outside to the inside; $180^o > \theta > 90^o$, Φ is negative.

Flux Through Closed Surface, final

The ***net*** flux through the surface is proportional to the net number of lines leaving the surface.

- This net number of lines is the number of lines leaving the surface minus the number entering the surface.

If E_n is the component of the field perpendicular to the surface, then

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E_n \, dA$$

- The integral is over a closed surface.

If more lines are leaving than entering, the net flux is positive.

If more lines are entering than leaving, the net flux is negative.

Flux Through a Cube, Example

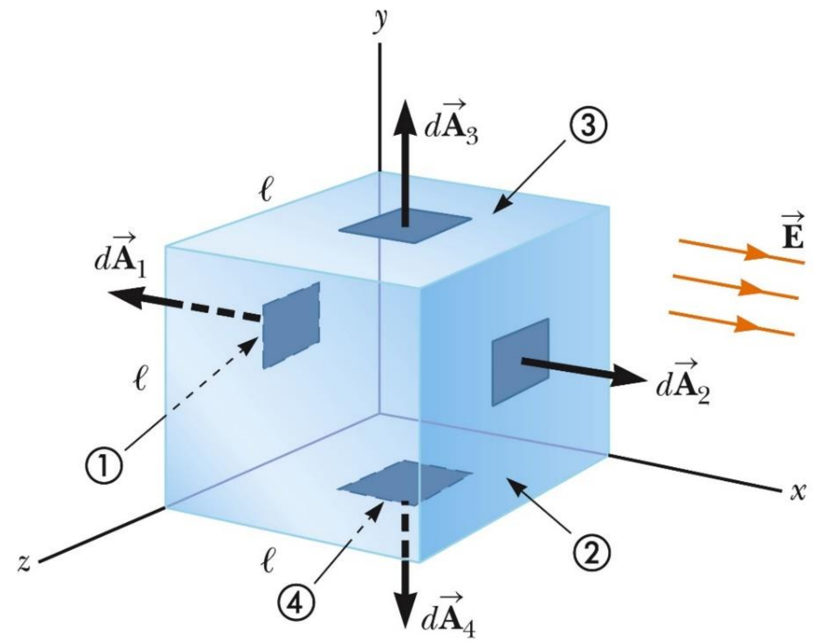
The field lines pass through two surfaces perpendicularly and are parallel to the other four surfaces.

For face 1, $\Phi = -E\ell^2$

For face 2, $\Phi = E\ell^2$

For the other sides, $\Phi = 0$

Therefore, $\Phi_{total} = 0$



Write the integrals for the net flux through faces ① and ②:

$$\Phi_E = \int_1 \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} + \int_2 \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

For face ①, $\vec{\mathbf{E}}$ is constant and directed inward but $d\vec{\mathbf{A}}_1$ is directed outward ($\theta = 180°$). Find the flux through this face:

$$\int_1 \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \int_1 E(\cos 180°)\, dA = -E\int_1 dA = -EA = -E\ell^2$$

For face ②, $\vec{\mathbf{E}}$ is constant and outward and in the same direction as $d\vec{\mathbf{A}}_2$ ($\theta = 0°$). Find the flux through this face:

$$\int_2 \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \int_2 E(\cos 0°)\, dA = E\int_2 dA = +EA = E\ell^2$$

Find the net flux by adding the flux over all six faces:

$$\Phi_E = -E\ell^2 + E\ell^2 + 0 + 0 + 0 + 0 = \boxed{0}$$

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Karl Friedrich Gauss

1777 – 1855

Made contributions in

- Electromagnetism
- Number theory
- Statistics
- Non-Euclidean geometry
- Cometary orbital mechanics
- A founder of the German Magnetic Union
  - Studies the Earth's magnetic field

Gauss's Law, Introduction

Gauss's law is an expression of the general relationship between the net electric flux through a closed surface and the charge enclosed by the surface.

- The closed surface is often called a *gaussian surface.*

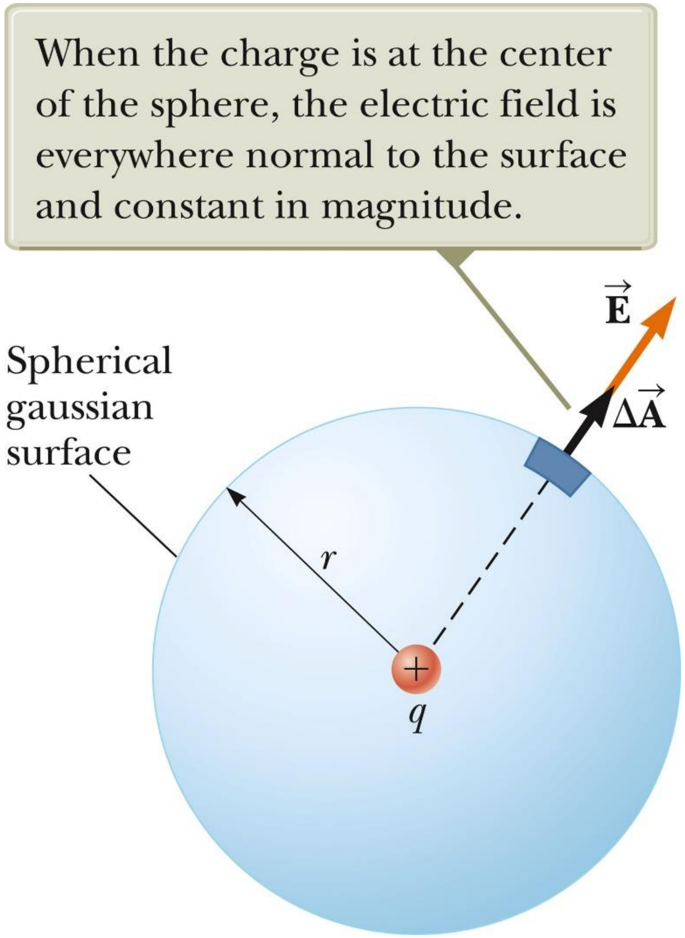
Gauss's law is of fundamental importance in the study of electric fields.

Gauss's Law – General

A positive point charge, q, is located at the center of a sphere of radius r.

The magnitude of the electric field everywhere on the surface of the sphere is

$E = k_e q / r^2$



Gauss's Law – General, cont.

The field lines are directed radially outward and are perpendicular to the surface at every point.

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E\, dA = E \oint dA$$

This will be the net flux through the gaussian surface, the sphere of radius *r.*

We know $E = k_e q/r^2$ and $A_{sphere} = 4\pi r^2$,

$$\Phi_E = k_e \frac{q}{r^2}(4\pi r^2) = 4\pi k_e q \quad , \quad k_e = 1/4\pi\epsilon_0$$

$$\Phi_E = \frac{q}{\epsilon_0}$$

Gauss's Law – General, notes

The net flux through any closed surface surrounding a point charge, *q*, is given by q/ε_o and is independent of the shape of that surface.

The net electric flux through a closed surface that surrounds no charge is zero.

Since the electric field due to many charges is the vector sum of the electric fields produced by the individual charges, the flux through any closed surface can be expressed as

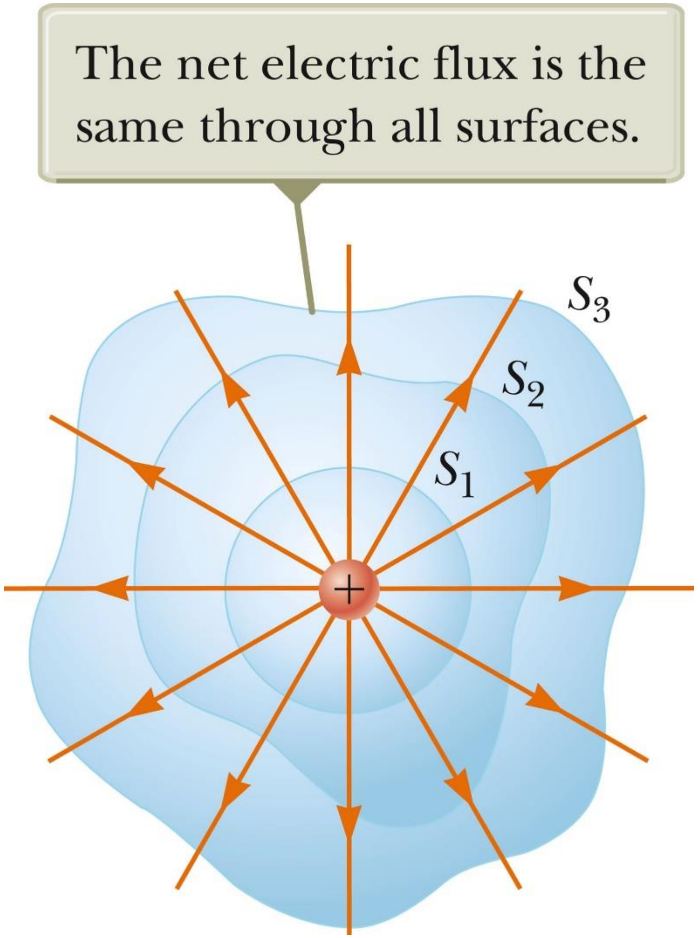
$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint (\vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 + \cdots) \cdot d\vec{\mathbf{A}}$$

Gaussian Surface, Example

Closed surfaces of various shapes can surround the charge.

- Only S_1 is spherical

Verifies the net flux through any closed surface surrounding a point charge q is given by q/ε_o and is independent of the shape of the surface.

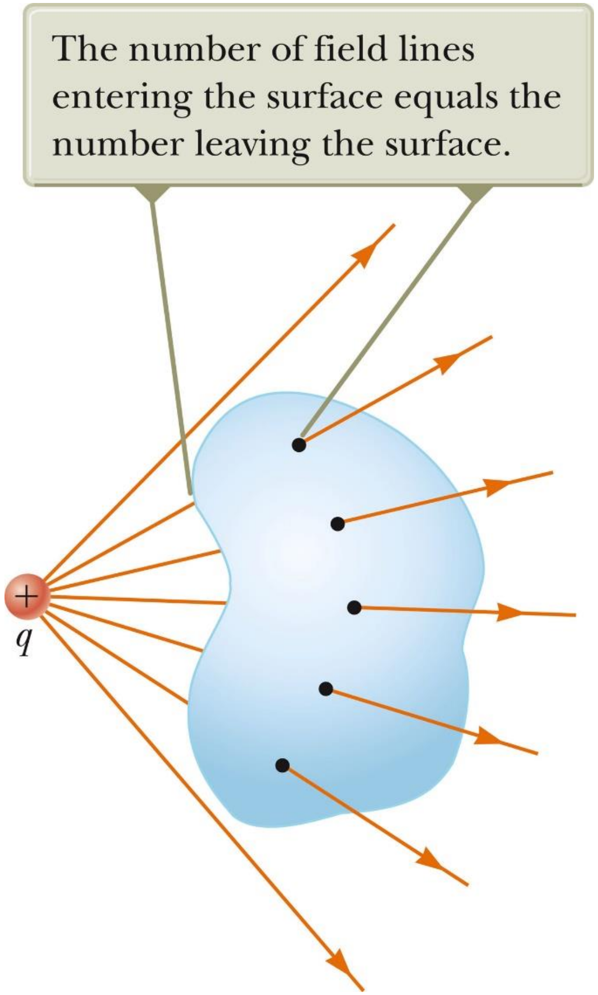


Gaussian Surface, Example 2

The charge is *outside* the closed surface with an arbitrary shape.

Any field line entering the surface leaves at another point.

Verifies the electric flux through a closed surface that surrounds no charge is zero.



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Gaussian Surface, Example 3

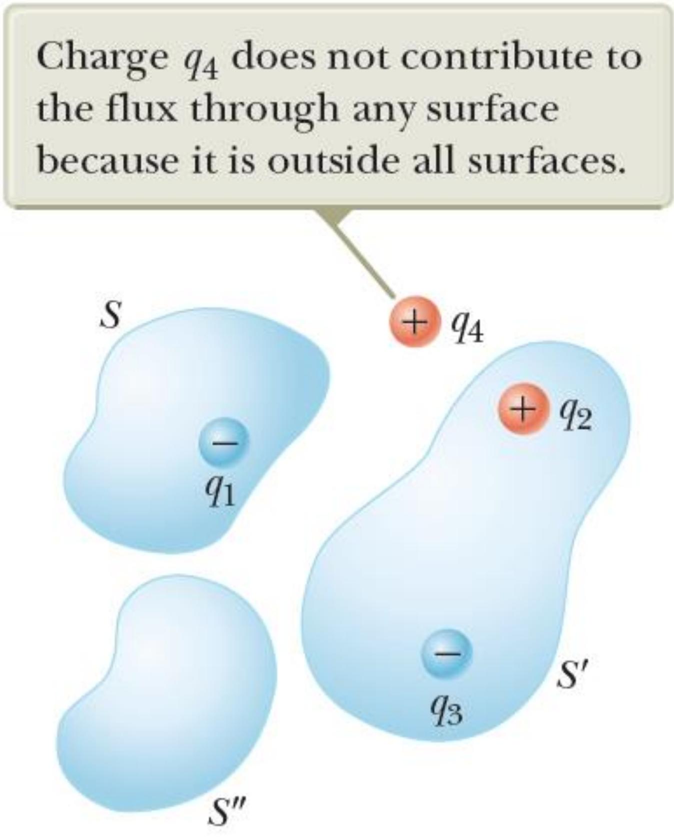
The surface *S* surrounds only one charge, *q*1; hence, the net flux through *S* is q_1/ϵ_0. The flux through *S* due to charges *q*2, *q*3, and *q*4 outside it is zero because each electric field line from these charges that enters *S* at one point leaves it at another.

The surface *S'* surrounds charges *q*2 and *q*3; hence, the net flux through it is

$$(q_2 + q_3)/\epsilon_0$$

Finally, the net flux through surface *S''* is zero because there is no charge inside this surface. That is, *all* the electric field lines that enter *S''* at one point leave at another.

Charge *q*4 does not contribute to the net flux through any of the surfaces.



Gauss's Law – Final

The mathematical form of Gauss's law states

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{in}}}{\epsilon_0}$$

- q_{in} is the net charge inside the surface.

$\vec{\mathbf{E}}$ represents the electric field at any point on the surface.

- $\vec{\mathbf{E}}$ is the *total electric field* and may have contributions from charges both inside and outside of the surface.

Although Gauss's law can, in theory, be solved to find $\vec{\mathbf{E}}$ for any charge configuration, in practice it is limited to symmetric situations.

Gauss's Law – Final, Conceptual Example

A spherical gaussian surface surrounds a point charge *q.* Describe what happens to the total flux through the surface if **(A)** the charge is tripled, **(B)** the radius of the sphere is doubled, **(C)** the surface is changed to a cube, and **(D)** the charge is moved to another location inside the surface.

(A) The flux through the surface is tripled because flux is proportional to the amount of charge inside the surface.

(B) The flux does not change because all electric field lines from the charge pass through the sphere, regardless of its radius.

(C) The flux does not change when the shape of the gaussian surface changes because all electric field lines from the charge pass through the surface, regardless of its shape.

(D) The flux does not change when the charge is moved to another location inside that surface because Gauss's law refers to the total charge enclosed, regardless of where the charge is located inside the surface.

Applying Gauss's Law

To use Gauss's law, you want to choose a gaussian surface over which the surface integral can be simplified and the electric field determined.

Take advantage of symmetry.

Remember, the gaussian surface is a surface you choose, it does not have to coincide with a real surface.

Conditions for a Gaussian Surface

Try to choose a surface that satisfies one or more of these conditions:

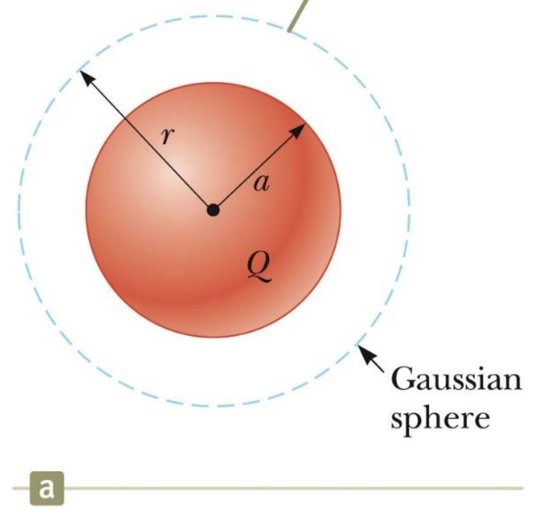
1. The value of the electric field can be argued from symmetry to be constant over the surface.
2. The dot product of $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$ can be expressed as a simple algebraic product EdA because $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ are parallel.
3. The dot product is 0 because $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ are perpendicular.
4. The field is zero over the portion of the surface.

If the charge distribution does not have sufficient symmetry such that a gaussian surface that satisfies these conditions can be found, Gauss' law is not useful for determining the electric field for that charge distribution.

Field Due to a Spherically Symmetric Charge Distribution

Select a sphere as the gaussian surface. For this choice, condition (2) is satisfied everywhere on the surface.

For points outside the sphere, a large, spherical gaussian surface is drawn concentric with the sphere.



Replace $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$ in Gauss's law with $E\,dA$:

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E\,dA = \frac{Q}{\epsilon_0}$$

By symmetry, E has the same value everywhere on the surface, which satisfies condition (1), so we can remove E from the integral:

$$\oint E\,dA = E\oint dA = E(4\pi r^2) = \frac{Q}{\epsilon_0}$$

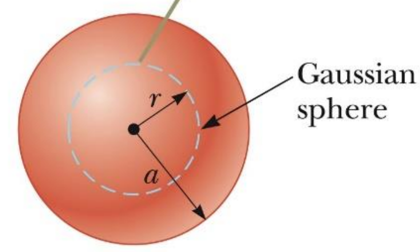
Solve for E:

$$(1)\quad E = \frac{Q}{4\pi\epsilon_0 r^2} = \boxed{k_e \frac{Q}{r^2} \quad (\text{for } r > a)}$$

Spherically Symmetric, cont.

Select a sphere as the gaussian surface, r < a. Let V' be the volume of this smaller sphere. To apply Gauss's law in this situation, recognize that the charge q_{in} within the gaussian surface of volume V' is less than Q.

For points inside the sphere, a spherical gaussian surface smaller than the sphere is drawn.



Calculate q_{in} by using $q_{\text{in}} = \rho V'$:

$$q_{\text{in}} = \rho V' = \rho\left(\tfrac{4}{3}\pi r^3\right)$$

Notice that conditions (1) and (2) are satisfied everywhere on the gaussian surface in Figure 24.10b. Apply Gauss's law in the region $r < a$:

$$\oint E\,dA = E\oint dA = E(4\pi r^2) = \frac{q_{\text{in}}}{\epsilon_0}$$

Solve for E and substitute for q_{in}:

$$E = \frac{q_{\text{in}}}{4\pi\epsilon_0 r^2} = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{4\pi\epsilon_0 r^2} = \frac{\rho}{3\epsilon_0}r$$

Substitute $\rho = Q/\frac{4}{3}\pi a^3$ and $\epsilon_0 = 1/4\pi k_e$:

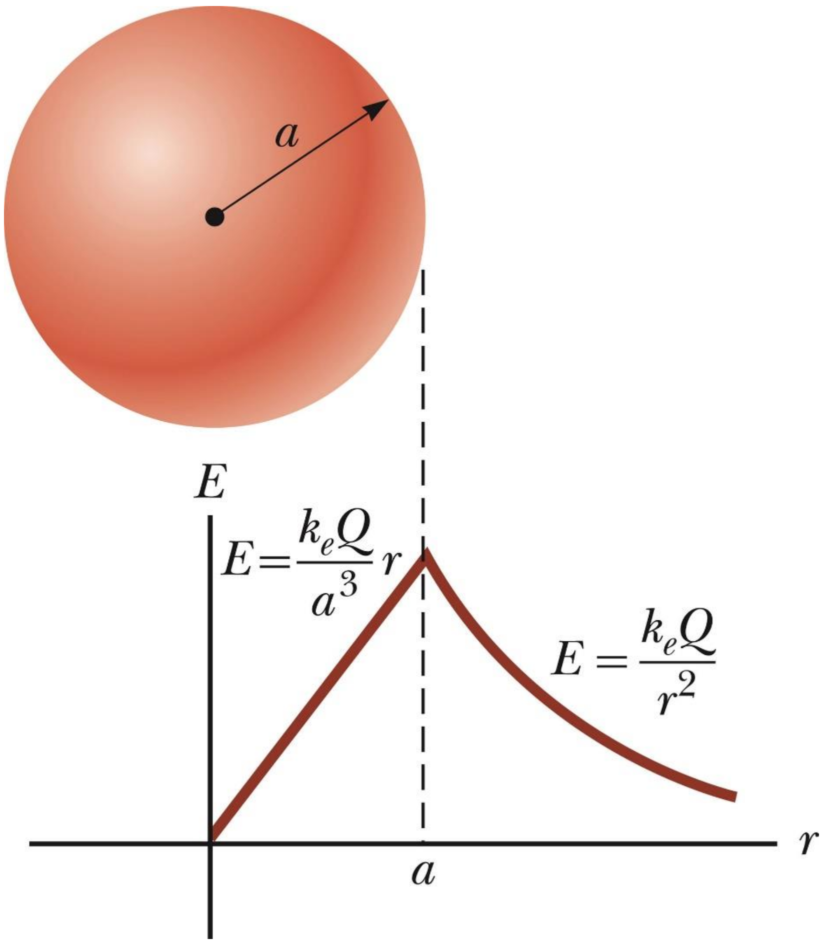
$$(2)\quad E = \frac{Q/\frac{4}{3}\pi a^3}{3(1/4\pi k_e)}r = \boxed{k_e\frac{Q}{a^3}r \quad (\text{for } r < a)}$$

Spherically Symmetric Distribution, final

Inside the sphere, E varies linearly with r

- $E \to 0$ as $r \to 0$

The field outside the sphere is equivalent to that of a point charge located at the center of the sphere.



Field at a Distance from a Line of Charge

The flux through the ends of the gaussian cylinder is zero because $\vec{\mathbf{E}}$ is parallel to these surfaces.

Select a cylindrical charge distribution .

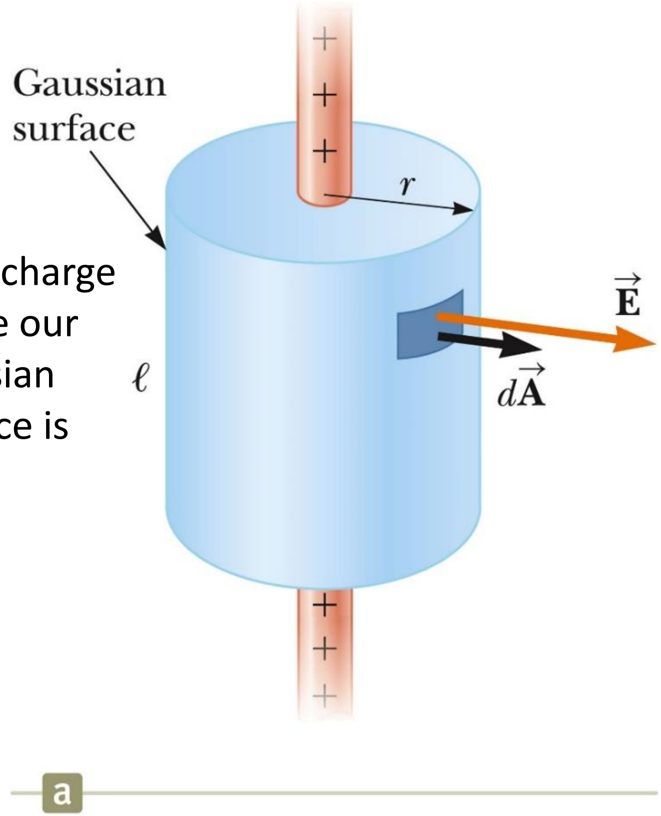
- The cylinder has a radius of *r* and a length of *ℓ.*

$\vec{\mathbf{E}}$ is constant in magnitude and perpendicular to the surface at every point on the curved part of the surface.

Use Gauss's law to find the field.

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E\oint dA = EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\lambda \ell}{\epsilon_0}$$

Total charge inside our gaussian surface is $\lambda\ell$



Substitute the area $A = 2\pi r\ell$ the curved surface:

$$E(2\pi r\ell) = \frac{\lambda\ell}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = \boxed{2k_e \frac{\lambda}{r}}$$

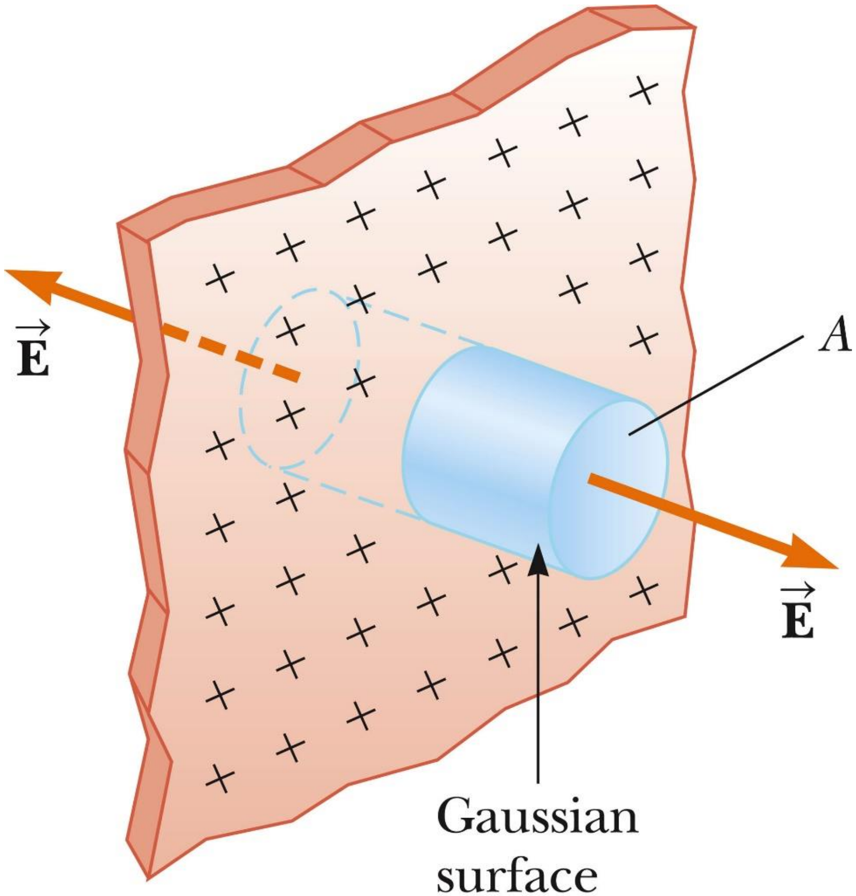
Field Due to a Plane of Charge

$\vec{\mathbf{E}}$ must be perpendicular to the plane and must have the same magnitude at all points equidistant from the plane.

Choose a small cylinder whose axis is perpendicular to the plane for the gaussian surface.

$\vec{\mathbf{E}}$ is parallel to the curved surface and there is no contribution to the surface area from this curved part of the cylinder.

The flux through each end of the cylinder is *EA* and so the total flux is 2*EA.*



Field Due to a Plane of Charge, final

The total charge in the surface is *σA.*

Applying Gauss's law:

$$\Phi_E = 2EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$E = \boxed{\frac{\sigma}{2\epsilon_0}}$$

where σ is surface charge density.

Note, this does not depend on *r.*

Therefore, the field is uniform everywhere.

Properties of a Conductor in Electrostatic Equilibrium

When there is no net motion of charge within a conductor, the conductor is said to be in **electrostatic equilibrium.** A conductor in electrostatic equilibrium has the following properties:

1. The electric field is zero everywhere inside the conductor whether the conductor is solid or hollow
2. If the conductor is isolated and carries a charge, the charge resides on its surface.
3. The electric field at a point just outside a charged conductor is perpendicular to the surface and has a magnitude of σ/ε_o.
   - σ is the surface charge density at that point.
4. On an irregularly shaped conductor, the surface charge density is greatest at locations where the radius of curvature is the smallest.

We verify the first three properties in the discussion that follows.

Property 1: $\text{Field}_{\text{inside}} = 0$

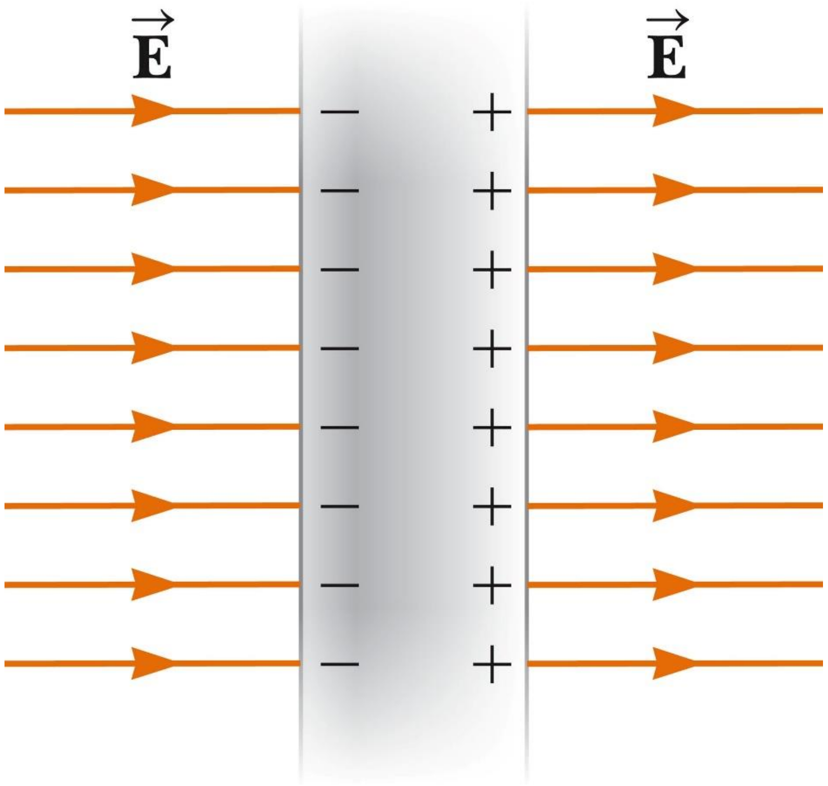
Consider a conducting slab in an external field.

If the field inside the conductor were not zero, free electrons in the conductor would experience an electrical force.

These electrons would accelerate.

This motion of electrons, however, would mean that the conductor is not in electrostatic equilibrium.

Therefore, the existence of electrostatic equilibrium is consistent only with a zero field in the conductor.



The electric field inside the conductor *must* be zero, assuming electrostatic equilibrium exists.

Property 1: $\text{Field}_{\text{inside}} = 0$, cont.

Before the external field is applied, free electrons are uniformly distributed throughout the conductor.

When the external field is applied, the free electrons accelerate to the left (see Figure), causing a plane of negative charge to accumulate on the left surface.

The movement of electrons to the left results in a plane of positive charge on the right surface.

These planes of charge create an additional electric field inside the conductor that opposes the external field. As the electrons move, the surface charge densities on the left and right surfaces increase until the magnitude of the internal field equals that of the external field, resulting in a net field of zero inside the conductor.

This redistribution takes about 10^{-16} s and can be considered instantaneous.

If the conductor is hollow, the electric field inside the conductor is also zero.

- Either the points in the conductor or in the cavity within the conductor can be considered.

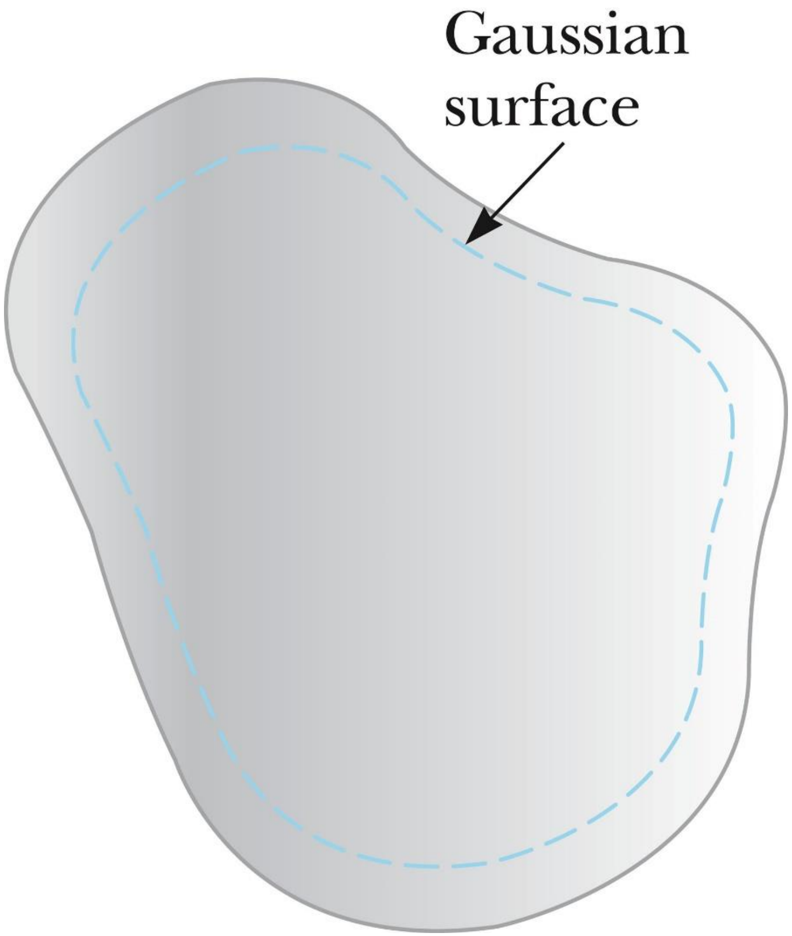
Property 2: Charge Resides on the Surface

Choose a gaussian surface inside but close to the actual surface.

The electric field inside is zero (property 1).

There is no net flux through the gaussian surface.

Because the gaussian surface can be as close to the actual surface as desired, there can be no charge inside the surface.



Property 2: Charge Resides on the Surface, cont.

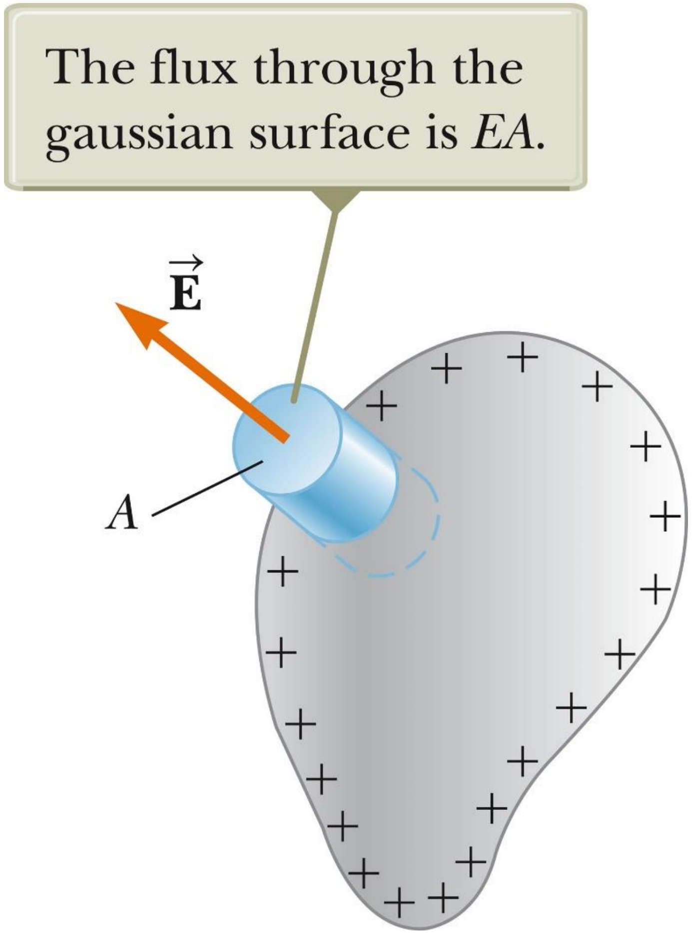
Since no net charge can be inside the surface, any net charge must reside ***on*** the surface.

Gauss's law does not indicate the distribution of these charges, only that it must be on the surface of the conductor.

Property 3: Field's Magnitude and Direction

Choose a cylinder as the gaussian surface.

If the field vector $\vec{\mathbf{E}}$ had a component parallel to the conductor's surface, free electrons would experience an electric force and move along the surface; in such a case, the conductor would not be in equilibrium. Therefore, the field vector must be perpendicular to the surface.



Property 3: Field's Magnitude and Direction, cont.

The net flux through the gaussian surface is through only the flat face outside the conductor.

- The field here is perpendicular to the surface.
- There is no flux through curved part of the gaussian surface because $\vec{\mathbf{E}}$ is parallel to the surface.
- There is no flux through the flat face of the cylinder inside the conductor because here $\vec{\mathbf{E}} = 0$.

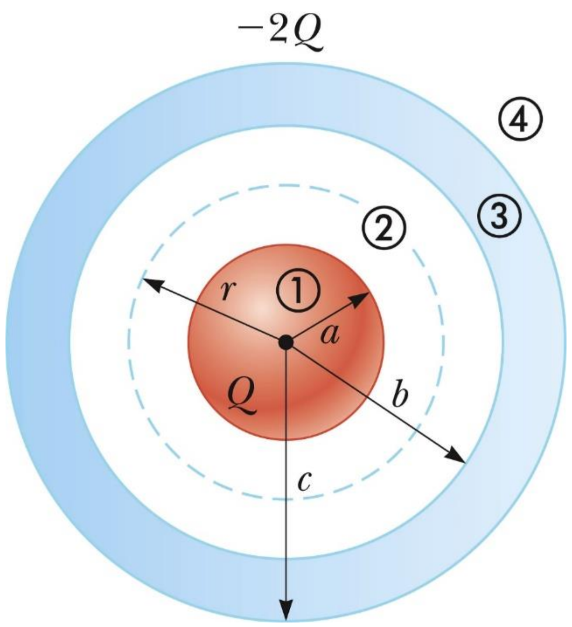
The net flux through the Gaussian surface is equal to that through only the flat face outside the conductor, where the field is perpendicular to the gaussian surface.

Applying Gauss's law

$$\Phi_E = \oint E\,dA = EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0} \quad \Longrightarrow \quad E = \frac{\sigma}{\epsilon_0}$$

Sphere and Shell Example

A solid insulating sphere of radius a carries a net positive charge Q uniformly distributed throughout its volume. A conducting spherical shell of inner radius b and outer radius c is concentric with the solid sphere and carries a net charge $-2Q$. Using Gauss's law, find the electric field in the regions labeled ①, ②, ③, and ④ in Figure 24.19 and the charge distribution on the shell when the entire system is in electrostatic equilibrium.



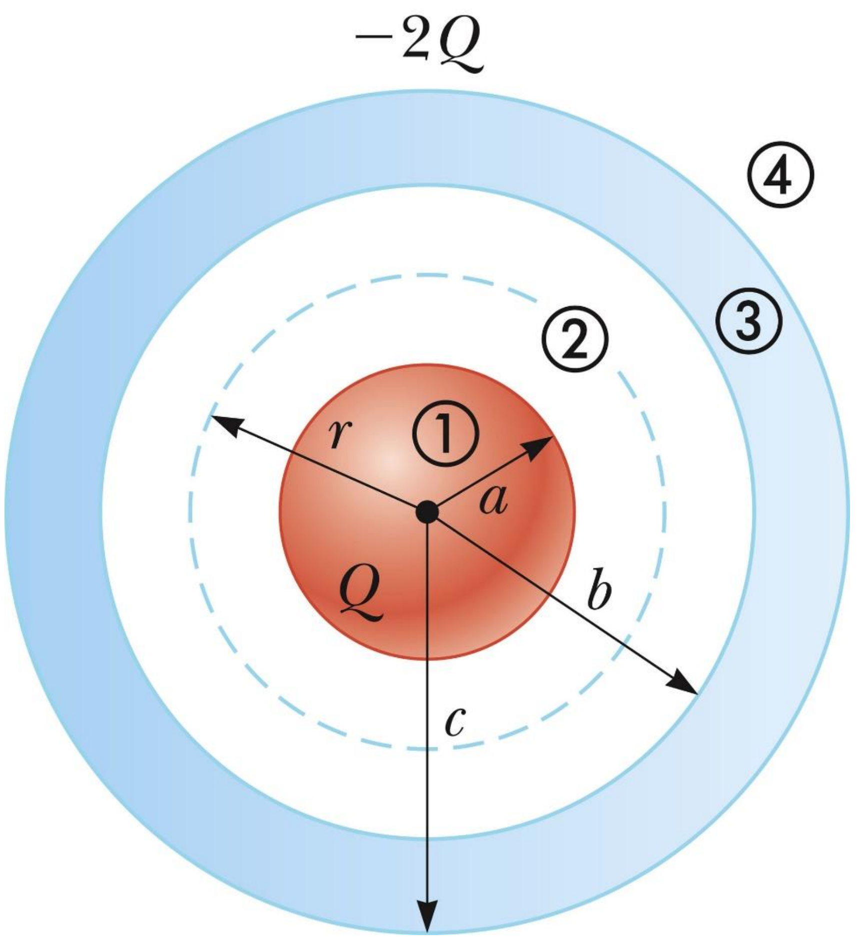
Sphere and Shell Example

Conceptualize

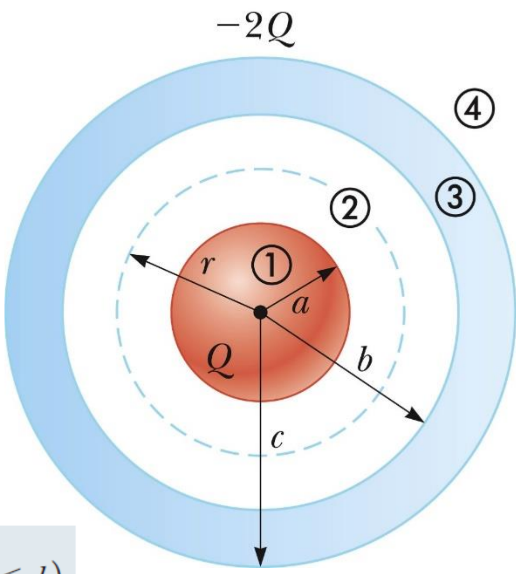
- Similar to the sphere example
- Now a charged sphere is surrounded by a shell

Categorize

- System has spherical symmetry
- Gauss' Law can be applied



Sphere and Shell Example, cont.



Analyze In region ②—between the surface of the solid sphere and the inner surface of the shell—we construct a spherical gaussian surface of radius r, where $a < r < b$, noting that the charge inside this surface is $+Q$ (the charge on the solid sphere). Because of the spherical symmetry, the electric field lines must be directed radially outward and be constant in magnitude on the gaussian surface.

The charge on the conducting shell creates zero electric field in the region $r < b$, so the shell has no effect on the field in region ② due to the sphere. Therefore, write an expression for the field in region ② as that due to the sphere from part (A) of Example 24.3:

$$E_2 = \boxed{k_e \frac{Q}{r^2} \quad (\text{for } a < r < b)}$$

Sphere and Shell Example, 3

Because the conducting shell creates zero field inside itself, it also has no effect on the field inside the sphere. Therefore, write an expression for the field in region ① as that due to the sphere from part (B) of Example 24.3:

$$E_1 = \boxed{k_e \frac{Q}{a^3} r \quad (\text{for } r < a)}$$

In region ④, where $r > c$, construct a spherical gaussian surface; this surface surrounds a total charge $q_{\text{in}} = Q + (-2Q) = -Q$. Therefore, model the charge distribution as a sphere with charge $-Q$ and write an expression for the field in region ④ from part (A) of Example 24.3:

$$E_4 = \boxed{-k_e \frac{Q}{r^2} \quad (\text{for } r > c)}$$

In region ③, the electric field must be zero because the spherical shell is a conductor in equilibrium:

$$E_3 = \boxed{0 \quad (\text{for } b < r < c)}$$

Construct a gaussian surface of radius r in region ③, where $b < r < c$, and note that q_{in} must be zero because $E_3 = 0$. Find the amount of charge q_{inner} on the inner surface of the shell:

$$q_{\text{in}} = q_{\text{sphere}} + q_{\text{inner}}$$
$$q_{\text{inner}} = q_{\text{in}} - q_{\text{sphere}} = 0 - Q = -Q$$

Sphere and Shell Example

Finalize The charge on the inner surface of the spherical shell must be $-Q$ to cancel the charge $+Q$ on the solid sphere and give zero electric field in the material of the shell. Because the net charge on the shell is $-2Q$, its outer surface must carry a charge $-Q$.