Abstract Interpretation Idea. Lattices and Fixpoints

Viktor Kunčak

Basic idea of abstract interpretation

Abstract interpretation is a way to infer properties of program computations.
Consider the assignment: $z = x + y$.

Interpreter:

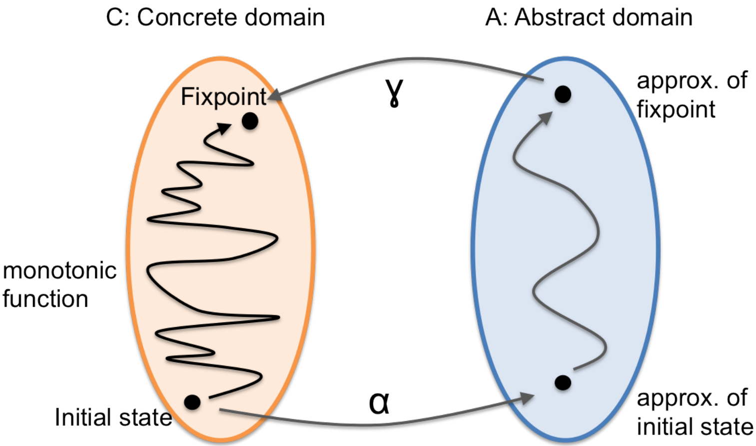
$$\begin{pmatrix} x : 10 \\ y : -2 \\ z : 3 \end{pmatrix} \xrightarrow{z=x+y} \begin{pmatrix} x : 10 \\ y : -2 \\ z : 8 \end{pmatrix}$$

Abstract interpreter:

$$\begin{pmatrix} x \in & [0, 10] \\ y \in & [-5, 5] \\ z \in & [0, 10] \end{pmatrix} \xrightarrow{z=x+y} \begin{pmatrix} x \in & [0, 10] \\ y \in & [-5, 5] \\ z \in & [-5, 15] \end{pmatrix}$$

Each abstract state represents a set of concrete states

Program Meaning is a Fixpoint. We Approximate It.



γ maps abstract states to concrete states

Programs as control-flow graphs

One possible corresponding control-flow graph is:

```
//a
i = 0;
    //b
while (i < 10) {
  //d
  if (i > 1)
    //e
    i = i + 3;
  else
    //f
    i = i + 2;
  //g
}
//c
```

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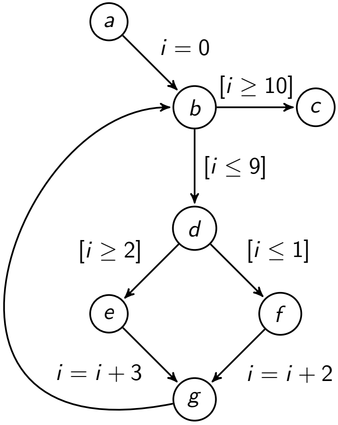

Sets of states at each program point

Suppose that

- program state is given by the value of the integer variable i
- initially, it is possible that i has any value

Compute the set of states at each vertex in the CFG.

```
//a
i = 0;
    //b
while (i < 10) {
  //d
  if (i > 1)
    //e
    i = i + 3;
  else
    //f
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  //g
}
//c
```

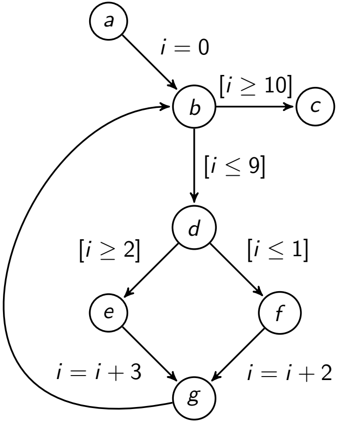

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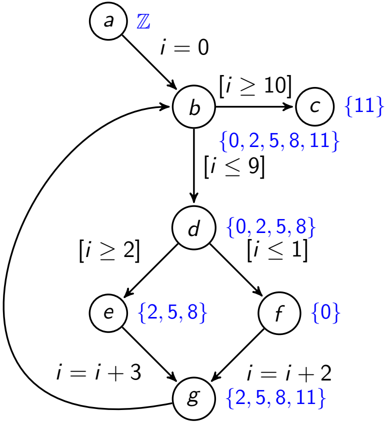


Sets of states at each program point

Running the Program
One way to describe the set of states for each program point: for each initial state, run the CFG with this state and insert the modified states at appropriate points.

Reachable States as A Set of Recursive Equations
If c is the label on the edge of the graph, let $\rho(c)$ denotes the relation between initial and final state that describes the meaning of statement. For example,

$$\rho(i = 0) = \{(i, i') \mid i' = 0\}$$
$$\rho(i = i + 2) = \{(i, i') \mid i' = i + 2\}$$
$$\rho(i = i + 3) = \{(i, i') \mid i' = i + 3\}$$
$$\rho([i < 10]) = \{(i, i') \mid i' = i \land i < 10\}$$

Sets of states at each program point

We will write $T(S, c)$ (transfer function) for the image of set S under relation $\rho(c)$. For example,

$$T(\{10, 15, 20\}, i = i + 2) = \{12, 17, 22\}$$

General definition can be given using the notion of strongest postcondition

$$T(S, c) = sp(S, \rho(c))$$

If [p] is a condition (assume(p), coming from 'if' or 'while') then

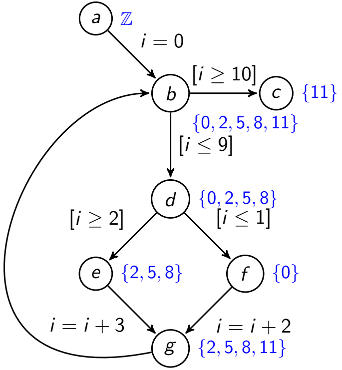
$$T(S, [p]) = \{x \in S \mid p\}$$

If an edge has no label, we denote it skip. So, $T(S, skip) = S$.

Reachable States as A Set of Recursive Equations

Now we can describe the meaning of our program using recursive equations:

$$
\begin{aligned}
S(a) &= \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
S(b) &= T(S(a), i = 0) \cup T(S(g), \mathit{skip}) \\
S(c) &= T(S(b), [\neg(i < 10)]) \\
S(d) &= T(S(b), [i < 10]) \\
S(e) &= T(S(d), [i > 1]) \\
S(f) &= T(S(d), [\neg(i > 1)]) \\
S(g) &= T(S(e), i = i + 3) \\
&\quad \cup T(S(f), i = i + 2)
\end{aligned}
$$



Our solution is the unique **least** solution of these equations. Can be computed by iterating starting from empty sets as initial solution.

The problem: These exact equations are as difficult to compute as running the program on all possible input states. Instead, we consider **approximate** descriptions of these sets of states.

A Large Analysis Domain: All Intervals of Integers

For every $L, U \in \mathbb{Z}$ interval:

$$\{x \mid L \leq x \wedge x \leq U\}$$

This domain has infinitely many elements, but is already an approximation of all possible sets of integers.

Smaller Domain: Finitely Many Intervals

We continue with the same example but instead of allowing to denote all possible sets, we will allow sets represented by expressions

$$[L, U]$$

which denote the set $\{x \mid L \leq x \wedge x \leq U\}$.
Example: $[0, 127]$ denotes integers between 0 and 127.

- L is the lower bound and U is the upper bound, with $L \leq U$.
- to ensure that we have only a few elements, we let

$$L, U \in \{\text{MININT}, -128, 1, 0, 1, 127, \text{MAXINT}\}$$

- $[\text{MININT}, \text{MAXINT}]$ denotes all possible integers, denote it $\top$
- instead of writing $[1, 0]$ and other empty sets, we will always write $\perp$

So, we only work with a finite number of sets $1 + \binom{7}{2} = 22$.
Denote the family of these sets by D (domain).

New Set of Recursive Equations

We want to write the same set of equations as before, but because we have only a finite number of sets, we must approximate. We approximate sets with possibly larger sets.

$$
\begin{aligned}
S^{\#}(a) &= \top \\
S^{\#}(b) &= T^{\#}(S^{\#}(a), i = 0) \\
&\quad \sqcup T^{\#}(S^{\#}(g), skip) \\
S^{\#}(c) &= T^{\#}(S^{\#}(b), [\neg(i < 10)]) \\
S^{\#}(d) &= T^{\#}(S^{\#}(b), [i < 10]) \\
S^{\#}(e) &= T^{\#}(S^{\#}(d), [i > 1]) \\
S^{\#}(f) &= T^{\#}(S^{\#}(d), [\neg(i > 1)]) \\
S^{\#}(g) &= T^{\#}(S^{\#}(e), i = i + 3) \\
&\quad \sqcup T^{\#}(S^{\#}(f), i = i + 2)
\end{aligned}
$$

- $S_1 \sqcup S_2$ denotes the approximation of $S_1 \cup S_2$: it is the set that contains both S_1 and S_2, that belongs to D, and is otherwise as small as possible. Here $[a, b] \sqcup [c, d] = [min(a, c), max(b, d)]$
- We use approximate functions $T^{\#}(S, c)$ that give a result in D.

Updating Sets

We solve the equations by starting in the initial state and repeatedly applying them.

- in the 'entry' point, we put $\top$, in all others we put $\bot$.

$$
\begin{aligned}
S^{\#}(a) &= \top \\
S^{\#}(b) &= T^{\#}(S^{\#}(a), i = 0) \\
&\quad \sqcup\ T^{\#}(S^{\#}(g), \mathit{skip}) \\
S^{\#}(c) &= T^{\#}(S^{\#}(b), [\neg(i < 10)]) \\
S^{\#}(d) &= T^{\#}(S^{\#}(b), [i < 10]) \\
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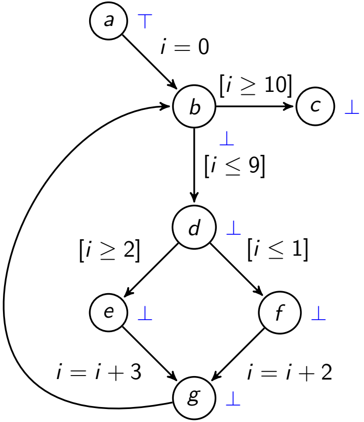

Updating Sets

Sets after a few iterations:

$$
\begin{aligned}
S^\#(a) &= \top \\
S^\#(b) &= T^\#(S^\#(a), i = 0) \\
&\quad \sqcup\ T^\#(S^\#(g), skip) \\
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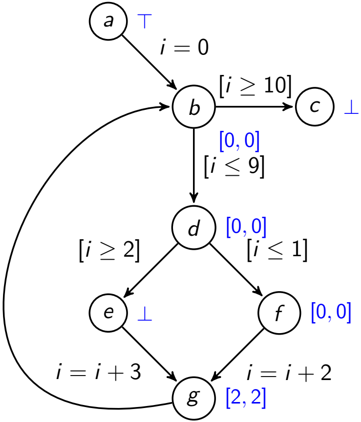

Updating Sets

Sets after a few more iterations:

$$
\begin{aligned}
S^\#(a) &= \top \\
S^\#(b) &= T^\#(S^\#(a), i = 0) \\
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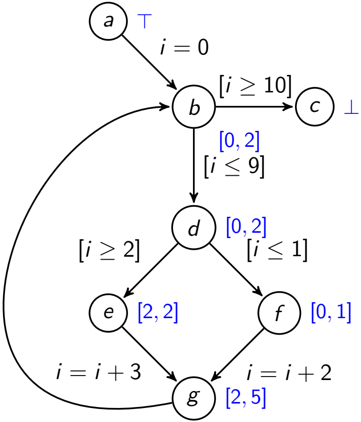

Fixpoint Found

Final values of sets:

$$
\begin{aligned}
S^\#(a) &= \top \\
S^\#(b) &= T^\#(S^\#(a), i = 0) \\
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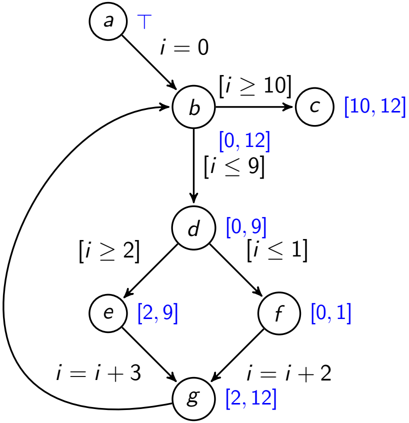
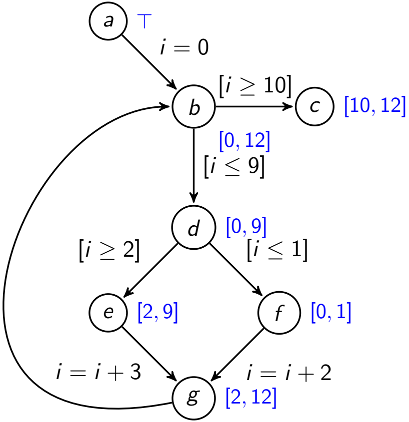


If we map intervals to sets, this is also solution of the original constraints.

Automatically Constructed Hoare Logic Proof

Final values of sets:

```
//a: true
i = 0;
      //b: 0 ≤ i ≤ 12
while (i < 10) {
  //d: 0 ≤ i ≤ 9
  if (i > 1)
    //e: 2 ≤ i ≤ 9
    i = i + 3;
  else
    //f: 0 ≤ i ≤ 1
    i = i + 2;
  //g: 2 ≤ i ≤ 12
}
//c: 10 ≤ i ≤ 12
```



This method constructed a sufficiently annotated program and ensured that all Hoare triples that were constructed hold

Proving through Fixpoints of Approximate Functions

Meaning of a program (e.g. a relation) is a least fixpoint of F.
Given specification s, the goal is to prove $\mathbf{lfp}(\mathbf{F}) \subseteq \mathbf{s}$

- if $F(s) \subseteq s$ then $lfp(F) \subseteq s$ and we are done
- $lfp(F) = \bigcup_{k \geq 0} F^k(\emptyset)$, but that is too hard to compute because it is infinite union unless, by some luck, $F^{n+1}(\emptyset) = F^n$ for some n

Instead, we search for an inductive strengthening of s: find s' such that:

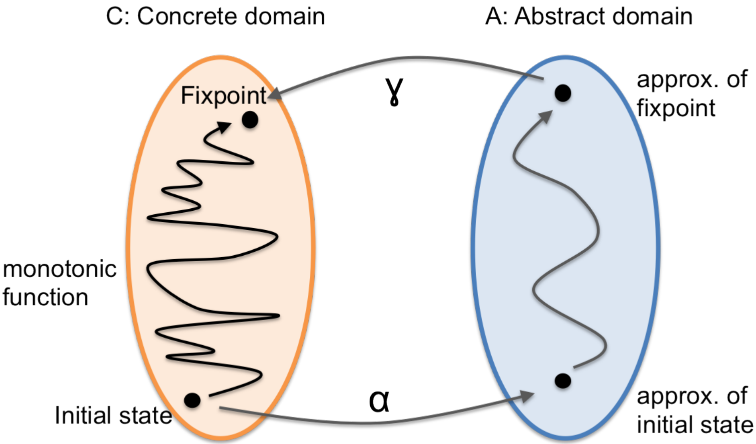
- $F(s') \subseteq s'$ (s' is inductive). If so, theorem says $lfp(F) \subseteq s'$
- $s' \subseteq s$ (s' implies the desired specification). Then $lfp(F) \subseteq s' \subseteq s$

How to find s'? Iterating F is hard, so we try some simpler function $F_\#$:

- suppose $F_\#$ is *approximation*: $F(r) \subseteq F_\#(r)$ for all r
- we can find s' such that: $F_\#(s') \subseteq s'$ (e.g. $s' = F_\#^{n+1}(\emptyset) = F_\#^n(\emptyset)$)

Then: $F(s') \subseteq F_\#(s') \subseteq s'$. So, if $s' \subseteq s$, we have know $lfp(F) \subseteq s'$.
Abstract interpretation: automatically construct $F_\#$ using F and s

Abstract Interpretation Big Picture



Abstract Domains are Partial Orders

Program semantics is given by certain sets (e.g. sets of reachable states).

- subset relation $\subseteq$: used to compare sets
- union of states: used to combine sets coming from different executions (e.g. if statement)

Our goal is to approximate such sets. We introduce a domain of elements $d \in D$ where each d represents a set.

- $\gamma(d)$ is a set of states. γ is called **concretization function**
- given d_1 and d_2, it could happen that there is **no element** d representing union

$$\gamma(d_1) \cup \gamma(d_2) = \gamma(d)$$

Instead, we use a set d that approximates union, and denote it $d_1 \sqcup d_2$

This leads us to review the theory of **partial orders** and **(semi)lattices**.

Partial Orders

Partial ordering relation is a binary relation $\leq$ that is reflexive, antisymmetric, and transitive, that is, the following properties hold for all x, y, z:

- $x \leq x$ (reflexivity)
- $x \leq y \land y \leq x \to x = y$ (antisymmetry)
- $x \leq y \land y \leq z \to x \leq z$ (transitivity)

If A is a set and $\leq$ a binary relation on A, we call the pair $(A, \leq)$ a **partial order**.

Given a partial ordering relation $\leq$, the corresponding **strict ordering relation** $x < y$ is defined by $x \leq y \land x \neq y$ and can be viewed as a shorthand for this conjunction.

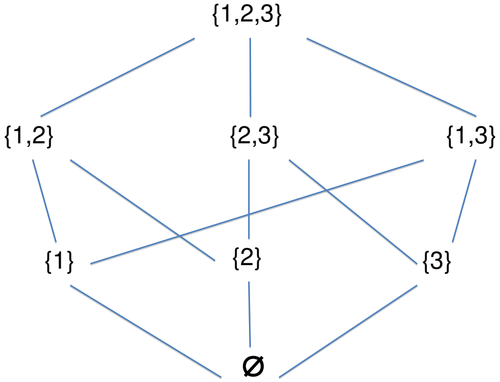
- Orders on integers, rationals, reals are all special cases of partial orders called *linear orders*.
- Given a set U, let A be any set of subsets of U, that is $A \subseteq 2^U$. Then $(A, \subseteq)$ is a partial order.

Example: Let $U = \{1, 2, 3\}$ and let $A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{2, 3\}, \{1, 2, 3\}\}$. Then $(A, \subseteq)$ is a partial order. We can draw it as a *Hasse diagram*.

Hasse diagram

presents the relation as a directed graph in a plane, such that

- the direction of edge is given by which nodes is drawn above
- transitive and reflexive edges are not represented (they can be derived)



Extreme Elements in Partial Orders

Given a partial order $(A, \leq)$ and a set $S \subseteq A$, we call an element $a \in A$

- **upper bound** of S if for all $a' \in S$ we have $a' \leq a$
- **lower bound** of S if for all $a' \in S$ we have $a \leq a'$
- **minimal element** of S if $a \in S$ and there is no element $a' \in S$ such that $a' < a$
- **maximal element** of S if $a \in S$ and there is no element $a' \in S$ such that $a < a'$
- **greatest element** of S if $a \in S$ and for all $a' \in S$ we have $a' \leq a$
- **least element** of S if $a \in S$ and for all $a' \in S$ we have $a \leq a'$
- **least upper bound** (lub, supremum, join, $\sqcup$) of S if a is the least element in the set of all upper bounds of S
- **greatest lower bound** (glb, infimum, meet, $\sqcap$) of S if a is the greatest element in the set of all lower bounds of S

Taking $S = A$ we obtain minimal, maximal, greatest, least elements for the entire partial order.

Extreme Elements in Partial Orders

Notes

- minimal element need not exist: $(0, 1)$ interval of rationals
- there may be multiple minimal elements: $\{\{a\}, \{b\}, \{a, b\}\}$
- if minimal element exists, it need not be least: above example
- there are no two distinct least elements for the same set
- least element is always glb and minimal
- if glb belongs to the set, then it is always least and minimal
- on a family of relations closed under $\cap$ and $\cup$, *glb* is $\cap$ and *lub* is $\cup$ for the partial order $\subseteq$; not all families of sets are closed; these are:
  - the set of all subsets
  - the family of open sets from topology

Least upper bound (lub, supremum, join, $\sqcup$)

Denoted $lub(S)$, least upper bound of S is an element M, if it exists, such that M is the least element of the set

$$U = \{x \mid x \text{ is upper bound on } S\}$$

In other words:

- M is an upper bound on S
- for every other upper bound M' on S, we have that $M \leq M'$

Note: this is the same definition as supremum in real analysis.

Least upper bound (glb, infimum, meet, $\sqcap$)

$a_1 \sqcup a_2$ denotes $lub(\{a_1, a_2\})$

$$(\dots(a_1 \sqcup a_2)\dots) \sqcup a_n \quad \text{is in fact } lub(\{a_1, \dots, a_n\})$$

So the operation is

- associative
- commutative
- idempotent

Real Analysis

Take as S the open interval of reals $(0, 1) = \{x \mid 0 < x < 1\}$
Then

- S has no maximal element
- S thus has no greatest element
- 2, 2.5, 3,... are all upper bounds on S
- $lub(S) = 1$

Execise: subsets of U

Consider

$$A = 2^U = \{S \mid S \subseteq U\} \qquad \text{and} \qquad (A, \subseteq)$$

Do these exist, and if so, what are they?

- $s_1 \subseteq S, s_2 \subseteq S, lub(\{s_1, s_2\}) =?$
- $lub(S) =?$

Partial order for the domain of intervals

Domain: $D = \{\bot\} \cup \{(L, U) \mid L \in \{-\infty\} \cup \mathbb{Z}, U \in \{+\infty\} \cup \mathbb{Z}$
such that $L \leq U$.

The associated set of elements is given by the function γ:

$$\gamma : D \to 2^{\mathbb{Z}}, \qquad \gamma((L, U)) = \{x \mid L \leq x \wedge x \leq U\}$$

Lub: for $d_1, d_2 \in D$, $d_1 \sqsubseteq d_2 \quad \leftrightarrow \quad \gamma(d_1) \subseteq \gamma(d_2)$
hence

$$(L_1, U_1) \sqsubseteq (L_2, U_2) \quad \leftrightarrow \quad L_2 \leq L_1 \wedge U_1 \leq U_2$$

$$\bot \sqsubseteq d \quad \forall d \in D$$

$$(L_1, U_1) \sqcup (L_2, U_2) = (min(L_1, L_2), max(U_1, U_2))$$

Remark on constructing orders using inverse images

Suppose $\gamma : D \to C$ where C is some collection of sets.
If we define relation $\sqsubseteq$ by:

$$d_1 \sqsubseteq d_2 \iff \gamma(d_1) \subseteq \gamma(d_2)$$

then

1. $\sqsubseteq$ is reflexive
2. $\sqsubseteq$ is transitive
3. $\sqsubseteq$ is antisymmetric if and only iff γ is injective

If $\sqsubseteq$ is not antisymmetric then we can define equivalence relation

$$d_1 \sim d_2 \iff \gamma(d_1) = \gamma(d_2)$$

and then take D' to be equivalence classes of such new set.
Example: suppose we defined intervals as all possible pairs of integers (L, U). Then there would be many representations of the empty set, all those intervals where $L > U$.

Lattices

Definition: A lattice is a partial order in which every two-element set has a least upper bound and a greatest lower bound.

Lemma: In a lattice every non-empty finite set has a lub ($\sqcup$) and glb ($\sqcap$).

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Lemma: In a lattice every non-empty finite set has a lub ($\sqcup$) and glb ($\sqcap$).

Proof: is by induction!
Case where the set S has three elements x,y and z:
Let $a = (x \sqcup y) \sqcup z$.
By definition of $\sqcup$ we have $z \sqsubseteq a$ and $x \sqcup y \sqsubseteq a$.
Then we have again by definition of $\sqcup$, $x \sqsubseteq x \sqcup y$ and $y \sqsubseteq x \sqcup y$. Thus by transitivity we have $x \sqsubseteq a$ and $y \sqsubseteq a$.
Thus we have $S \sqsubseteq a$ and a is an upper bound.
Now suppose that there exists a' such that $S \sqsubseteq a'$. We want $a \sqsubseteq a'$ (a least upper bound):
We have $x \sqsubseteq a'$ and $y \sqsubseteq a'$, thus $x \sqcup y \sqsubseteq a'$. But $z \sqsubseteq a'$, thus $((x \sqcup y) \sqcup z) \sqsubseteq a'$.
Thus a is the lub of our 3 elements set.

Examples of Lattices

Lemma: Every linear order is a lattice.

Example: Every bounded subset of the set of real numbers has a lub. This is an axiom of real numbers, the way they are defined (or constructed from rationals).

- If a lattice has least and greatest element, then every finite set (including empty set) has a lub and glb.
- This does not imply there are lub and glb for infinite sets.
  Example: In the oder $([0, 1), \leq)$ with standard ordering on reals is a lattice, the entire set has no lub. The set of all rationals of interval $[0, 10]$ is a lattice, but the set $\{x \mid 0 \leq x \wedge x^2 < 2\}$ has no lub.

Exercises

Prove the following:

1. $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
2. $\sqcup A \sqsubseteq \sqcap B \;\Leftrightarrow\; \forall x \in A. \forall y \in B. x \sqsubseteq y$
3. Let $(A, \sqsubseteq)$ be a partial order such that every set $S \subseteq A$ has the greatest lower bound. Prove that then every set $S \subseteq A$ has the least upper bound.

Constructing Partial Orders using Maps

Example: Let A be the set of all propositional formulas containing only variables p, q. For a formula $F \in A$ define

$$[F] = \{(u, v).\ u, v \in \{0, 1\} \wedge F \text{ is true for } p \mapsto u, q \mapsto v\}$$

i.e. $[F]$ denotes the set of assignments for which F is true. Note that $F \implies G$ is a tautology iff $[F] \subseteq [G]$. Define ordering on formulas A by

$$F \leq G \iff [F] \subseteq [G]$$

Is $\leq$ a partial order? Which laws does $\leq$ satisfy?

Constructing Partial Orders using Maps

Lemma: Let $(C, \leq)$ be an lattice and A a set. Let $\gamma : A \to C$ be an injective function. Define oder $x \sqsubseteq y$ on A by $\gamma(x) \leq \gamma(y)$. Then $(A, \sqsubseteq)$ is a partial order.

Note: even if $(C, \leq)$ had top and bottom element and was a lattice, the constructed order need not have top and bottom or be a lattice. For example, we take A to be a subset of A and define γ to be identity.

Lattices

Definition: A lattice is a partial order in which every two-element set has a least upper bound and a greatest lower bound (so, we have $\sqcap$ and $\sqcup$ as well-defined binary operations).
Lemma: In every lattice, $x \sqcup (x \sqcap y) = x$.

Lattices

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Lemma: In every lattice, $x \sqcup (x \sqcap y) = x$.

Proof:
We trivially have $x \sqsubseteq x \sqcup (x \sqcap y)$.
Let's prove that $x \sqcup (x \sqcap y) \sqsubseteq x$:
x is an upper bound of x and $x \sqcap y$, $x \sqcup (x \sqcap y)$ is the least upper bound of x and $x \sqcap y$, thus $x \sqcup (x \sqcap y) \sqsubseteq x$.
Definition: A lattice is *distributive* iff

$$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$$
$$x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$$

Lattice of all subsets of a set is distributive. Linear order is a distributive lattice.

Products of Lattices

Note: for $n = 2$ a function $f : \{1, 2\} \to (L_1 \cup L_2)$ with $f(1) \in L_1$, $f(2) \in L_2$ is isomorphic to an ordered pair $(f(1), f(2))$. We denote the product by $(L_1, \leq_1) \times (L_2, \leq_2)$.
Example: Let $R = \{a, b, c, d\}$ denote set of values. Let $A_1 = A_2 = 2^R$. Let

$$s_1 \leq_1 s_2 \iff s_1 \subseteq s_2$$

and let

$$t_1 \leq_2 t_2 \iff t_1 \supseteq t_2$$

Then we can define the product $(A_1, \leq_1) \times (A_2, \leq_2)$. In this product, $(s_1, t_1) \leq (s_2, t_2)$ iff: $s_1 \subseteq s_2$ and $t_1 \supseteq t_2$. The original partial orders were lattices, so the product is also a lattice. For example, we have

$$(\{a, b, c\}, \{a, b, d\}) \sqcap (\{b, c, d\}, \{c, d\}) = (\{b, c\}, \{a, b, c, d\})$$

Products of Lattices

Lattice elements can be combined into finite or infinite-dimensional vectors, and the result is again a lattice.
Lemma: Let $(A_1, \leq_1), \ldots, (A_n, \leq_n)$ be partial orders. Define $(L, \leq)$ by

$$A = \{f \mid f : \{1, \ldots, n\} \to (A_1 \cup \ldots \cup A_n) \text{ where } \forall i. f(i) \in A_i\}$$

For $f, g \in A$ define

$$f \leq g \iff \forall i. f(i) \leq_i g(i)$$

Then $(A, \leq)$ is a partial order. We denote $(A, \leq)$ by

$$\prod_{i=1}^{n} (L_i, \leq_i)$$

Moreover, if for each i, $(A_i, \leq_i)$ is a lattice, then $(A, \leq)$ is also a lattice.

Properties of $\sqcap S$ and $\sqcup S$

Consider a partial order $(A, \sqsubseteq)$.

- Suppose $S_1 \subseteq S_2 \subseteq A$ and $\sqcup S_1$ and $\sqcup S_2$ exist. In what relationship are these two elements?
- Suppose $S_1 \subseteq S_2 \subseteq A$ and $\sqcap S_1$ and $\sqcap S_2$ exist. In what relationship are these two elements?
- Suppose $\sqcup\emptyset$ exists. Describe this element.
- Suppose $\sqcap\emptyset$ exists. Describe this element.

Properties of $\sqcap S$ and $\sqcup S$

Consider a partial order $(A, \sqsubseteq)$.

- Suppose $S_1 \subseteq S_2 \subseteq A$ and $\sqcup S_1$ and $\sqcup S_2$ exist. In what relationship are these two elements?
- Suppose $S_1 \subseteq S_2 \subseteq A$ and $\sqcap S_1$ and $\sqcap S_2$ exist. In what relationship are these two elements?
- Suppose $\sqcup\emptyset$ exists. Describe this element.
- Suppose $\sqcap\emptyset$ exists. Describe this element.

$\sqcup\emptyset = \bot$ and $\sqcap\emptyset = \top$. This is because every element is an upper bound and a lower bound of $\emptyset$: $\forall x.\forall y \in \emptyset. y \sqsubseteq x$ is valid, as well as $\forall x.\forall y \in \emptyset. y \sqsupseteq x$.

Complete Semilattice is a Complete Lattice

If we have all $\sqcap$-s we then also have all $\sqcup$-s:
Theorem: Let $(A, \sqsubseteq)$ be a partial order such that every set $S \subseteq A$ has the greatest lower bound ($\sqcap$). Prove that then every set $S \subseteq A$ has the least upper bound ($\sqcup$).

Example: Application of the Previous Theorem

Let U be a set and $A \subseteq U \times U$ the set of all **equivalence relations** on this set. Consider the partial order $(A, \subseteq)$.

Lemma

If $I \subseteq A$ is a set of equivalence relations, then $\cap I$ is also an equivalence relation.

Consequence: Given $I \subseteq A$ there exists the least equivalence relation containing every relation from I (equivalence closure of relations in I).

Note: **congruence** is equivalence relation that agrees with some operations. For example, $x \sim x'$ and $y \sim y'$ implies $(x + y) \sim (x' + y')$. The analogous properties hold for congruence relations.

Complete Lattices

Definition: A **complete** lattice is a lattice where for every set S (including empty set and infinite sets) there exist $\sqcup S$ and $\sqcap S$.

Monotonic functions

Given two partial orders $(C, \leq)$ and $(A, \sqsubseteq)$, we call a function $\alpha : C \to A$ *monotonic* iff for all $x, y \in C$,

$$x \leq y \;\to\; \alpha(x) \sqsubseteq \alpha(y)$$

Fixpoints

Definition: Given a set A and a function $f : A \to A$ we say that $x \in A$ is a fixed point (fixpoint) of f if $f(x) = x$.

Definition: Let $(A, \leq)$ be a partial order, let $f : A \to A$ be a monotonic function on $(A, \leq)$, and let the set of its fixpoints be $S = \{x \mid f(x) = x\}$. If the least element of S exists, it is called the **least fixpoint**, if the greatest element of S exists, it is called the **greatest fixpoint**.

Fixpoints

Let $(A, \sqsubseteq)$ be a complete lattice and $G : A \to A$ a monotonic function.

Definition:
$\mathrm{Post} = \{x \mid G(x) \sqsubseteq x\}$ - the set of *postfix points* of G
(e.g. $\top$ is a postfix point)
$\mathrm{Pre} = \{x \mid x \sqsubseteq G(x)\}$ - the set of *prefix points* of G
$\mathrm{Fix} = \{x \mid G(x) = x\}$ - the set of *fixed points* of G.

Note that $\mathrm{Fix} \subseteq \mathrm{Post}$.

Tarski's fixed point theorem

Theorem: Let $a = \sqcap \mathrm{Post}$. Then a is the least element of Fix (dually, $\sqcup \mathrm{Pre}$ is the largest element of Fix).

Proof:

Let x range over elements of Post.

- applying monotonic G from $a \sqsubseteq x$ we get $G(a) \sqsubseteq G(x) \sqsubseteq x$
- so $G(a)$ is a lower bound on Post, but a is the greatest lower bound, so $G(a) \sqsubseteq a$
- therefore $a \in \mathrm{Post}$
- Post is closed under G, by monotonicity, so $G(a) \in \mathrm{Post}$
- a is a lower bound on Post, so $a \sqsubseteq G(a)$
- from $a \sqsubseteq G(a)$ and $G(a) \sqsubseteq a$ we have $a = G(a)$, so $a \in \mathrm{Fix}$
- a is a lower bound on Post so it is also a lower bound on a smaller set Fix

In fact, the set of all fixpoints Fix is a lattice itself.

Tarski's fixed point theorem

Tarski's Fixed Point theorem shows that in a complete lattice with a monotonic function G on this lattice, there is at least one fixed point of G, namely the least fixed point $\sqcap$Post.

- Tarski's theorem guarantees fixpoints in complete lattices, but the above proof does not say how to find them.
- How difficult it is to find fixpoints depends on the structure of the lattice.

Let G be a monotonic function on a lattice. Let $a_0 = \perp$ and $a_{n+1} = G(a_n)$. We obtain a sequence $\perp \sqsubseteq G(\perp) \sqsubseteq G^2(\perp) \sqsubseteq \cdots$. Let $a_* = \bigsqcup_{n \geq 0} a_n$.

Lemma: The value a_* is a prefix point.
Observation: a_* need not be a fixpoint (e.g. on lattice [0,1] of real numbers).

Omega continuity

Definition: A function G is ω-continuous if for every chain $x_0 \sqsubseteq x_1 \sqsubseteq \ldots \sqsubseteq x_n \sqsubseteq \ldots$ we have

$$G(\bigsqcup_{i \geq 0} x_i) = \bigsqcup_{i \geq 0} G(x_i)$$

Lemma: For an ω-continuous function G, the value $a_* = \bigsqcup_{n \geq 0} G^n(\bot)$ is the least fixpoint of G.

Iterating sequences and omega continuity

Lemma: For an ω-continuous function G, the value $a_* = \bigsqcup_{n\geq 0} G^n(\bot)$ is the least fixpoint of G.

Proof:

- By definition of ω-continuous we have $G(\bigsqcup_{n\geq 0} G^n(\bot)) = \bigsqcup_{n\geq 0} G^{n+1}(\bot) = \bigsqcup_{n\geq 1} G^n(\bot)$.
- But $\bigsqcup_{n\geq 0} G^n(\bot) = \bigsqcup_{n\geq 1} G^n(\bot) \sqcup \bot = \bigsqcup_{n\geq 1} G^n(\bot)$ because $\bot$ is the least element of the lattice.
- Thus $G(\bigsqcup_{n\geq 0} G^n(\bot)) = \bigsqcup_{n\geq 0} G^n(\bot)$ and a_* is a fixpoint.

Now let's prove it is the least. Let c be such that $G(c) = c$. We want $\bigsqcup_{n\geq 0} G^n(\bot) \sqsubseteq c$. This is equivalent to $\forall n \in \mathbb{N}. G^n(\bot) \sqsubseteq c$.
We can prove this by induction : $\bot \sqsubseteq c$ and if $G^n(\bot) \sqsubseteq c$, then by monotonicity of G and by definition of c we have $G^{n+1}(\bot) \sqsubseteq G(c) \sqsubseteq c$.

Iterating sequences and omega continuity

Lemma: For an ω-continuous function G, the value $a_* = \bigsqcup_{n \geq 0} G^n(\bot)$ is the least fixpoint of G.

When the function is not ω-continuous, then we obtain a_* as above (we jump over a discontinuity) and then continue iterating. We then take the limit of such sequence, and the limit of limits etc., ultimately we obtain the fixpoint.