Exercises 6

Exercise 1 (Galois Connection)**.** Remember that a Galois connection is defined by two monotonic functions $\alpha : C \to A$ and $\gamma : A \to C$ between partial orders $\leq$ on C and $\sqsubseteq$ on A, such that

$$\forall a, c. \quad \alpha(c) \sqsubseteq a \iff c \leq \gamma(a) \tag{$*$}$$

a) Show that the condition $(*)$ is equivalent to the conjunction of these two conditions:

$$\forall c. \quad c \leq \gamma(\alpha(c)) \tag{1}$$
$$\forall a. \quad \alpha(\gamma(a)) \sqsubseteq a \tag{2}$$

b) Let α and γ satisfy the condition of a Galois connection. Show that the following three conditions are equivalent:

1. $\alpha(\gamma(a)) = a$ for all a
2. α is a surjective function
3. γ is an injective function

c) State the condition for $c = \gamma(\alpha(c))$ to hold for all c. When C is the set of sets of concrete states and A is a domain of static analysis, is it more reasonable to expect that $c = \gamma(\alpha(c))$ or $\alpha(\gamma(a)) = a$ to be satisfied, and why?

Exercise 2 (lub and glb)**.** Let $(A, \sqsubseteq)$ be a partial order such that every set $S \subseteq A$ has the greatest lower bound.
Prove that then every set $S \subseteq A$ has the least upper bound, or show a counterexample.
What about the lattice with three elements $\{0, 1_a, 1_b\}$ *the relations* $0 \leq 1_a$ *and* $0 \leq 1_b$*?*

Exercise 3 (Lattices)**.** Consider algebraic structures with signature $(\vee, \wedge)$, each of arity 2, and satisfying the following axioms:

$x \vee y = y \vee x$	($\vee$-Com)	$x \wedge y = y \wedge x$	($\wedge$-Com)
$(x \vee y) \vee z = x \vee (y \vee z)$	($\vee$-Assoc)	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$	($\wedge$-Assoc)
$x \vee x = x$	($\vee$-Idem)	$x \wedge x = x$	($\wedge$-Idem)
$x \vee (x \wedge y) = x$	($\vee$-Abs)	$x \wedge (x \vee y) = x$	($\wedge$-Abs)

a) Show that for any x and y, $x \wedge y = x$ if and only if $x \vee y = y$.

b) Define $x \leq y$ by $x \wedge y = x$. Show that $\leq$ is a partial order relation.

c) Show that $\wedge$ and $\vee$ are respectively the binary *greatest lower bound* and *least upper bound* for $\leq$.

Exercise 4 (post)**.** let S be any set, $r \subseteq S \times S$ a binary relation and $I \subseteq S$. Define $post: 2^S \to 2^S$ by $post(X) = I \cup r[X]$. Prove that $post$ is monotonic. Does $post$ admit a least fixed point?

Exercise 5. Partitioning

a) Show that for any set S, $(2^S, \subseteq)$ is a lattice.

b) Consider a set $S = P \times V$. For each set $g \in 2^{P \times V}$, define $\bar{g} : P \to 2^V$ by $\bar{g}(p) = \{v \mid (p, v) \in g\}$. Show that the bar function $\bar{\cdot}$ defines a bijection between $2^{P \times V}$ and $P \to 2^V$.

c) Consider the set of all functions $P \to 2^V$. Define a lattice on this set that is isomorphic to $(2^{P \times V}, \subseteq)$.

Exercise 6. Let $f : [0, 1] \to [0, 1]$ such that

$$f(x) = \begin{cases} \frac{x^2}{4} + \frac{1}{10} & \text{if } x < \frac{1}{2} \\ \frac{1}{1+4^{-x}} & \text{if } x \geq \frac{1}{2} \end{cases}$$

Prove, using the fact that $([0, 1], \leq)$ is a complete lattice, that there exists $x \in [0, 1]$ such that $f(x) = x$.