Summary: Decidability, Rice's Theorem, and Language Properties

A concise reference for proving (un)decidability

Blatt 4 – Pflichtaufgabe 1

1. Key Concepts

Definition 0.1 (Turing Machine Language)**.** Given a Turing machine M over alphabet Σ, its *language* is

$$\mathcal{L}(M) \;=\; \{w \in \Sigma^* \mid M \text{ accepts } w\}.$$

Definition 0.2 (Property of a Language)**.** A *property* P of languages is a predicate on subsets of Σ^*, e.g. "L is empty", "L is regular", "$L \leq_m HP$", etc.

2. Rice's Theorem

Theorem 0.3 (Rice's Theorem)**.** *Every non-trivial property of semi-decidable languages is undecidable.*

- **Non-triviality:** $\exists L_1, L_2 \subseteq \Sigma^*$ such that $P(L_1) = \text{true}$ and $P(L_2) = \text{false}$.
- **Semi-decidable languages:** Languages accepted by some TM (RE).
- **Conclusion:** If P is non-trivial and refers only to $\mathcal{L}(M)$, then the language

  $$L_P \;=\; \{\langle M\rangle \mid P(\mathcal{L}(M))\}$$

  is undecidable.

3. Monotonicity of Properties

Definition 0.4 (Monotonic Property)**.** A property P of languages is *monotonic* if for all $L_1 \subseteq L_2$,

$$P(L_1) = \text{true} \implies P(L_2) = \text{true}.$$

Otherwise P is *non-monotonic.*

Typical counterexample:

$$L_1 = HP, \quad L_2 = \Sigma^*, \quad L_1 \subseteq L_2, \quad P(L_1) = \text{true},\; P(L_2) = \text{false} \implies P \text{ is non-monotonic.}$$

4. Proof Strategy

1. **Identify the property P.**
   Does it concern $\mathcal{L}(M)$ (accepted strings) or machine internals?
2. **Check applicability of Rice's Theorem.**
   P must be a property of the language $\mathcal{L}(M)$.
3. **Prove non-triviality.**
   Exhibit two TMs:

- M_1 with $\mathcal{L}(M_1) = \emptyset$ (or any L_1 where $P(L_1) = \text{false}$).
- M_2 with $\mathcal{L}(M_2) = HP$ (or any L_2 where $P(L_2) = \text{true}$).

4. **Conclude undecidability.**
   By Rice's Theorem, $\{\langle M \rangle \mid P(\mathcal{L}(M))\}$ is undecidable.
5. **(Optional) Test monotonicity.**
   Use extremal cases ($\emptyset, \Sigma^*$ or HP, Σ^*).

5. Useful Extremal Languages

5. Useful Extremal Languages

$\emptyset$: The empty language.
Decidable (there is a TM that immediately rejects every input).

Σ^*: The full language (all possible words).
Decidable (TM that accepts every input).

Regular Language: e.g. $L = \{a^*\}$ over alphabet $\{a\}$.
Decidable (can be decided by a finite automaton).

Context-Free Language: e.g. $L = \{a^n b^n \mid n \geq 0\}$.
Decidable (can be decided by a push-down automaton).

Decidable Language: Any language for which there is a TM that halts on every input with the correct yes/no answer.
Includes all regular and context-free languages.

Recursively Enumerable but Undecidable Language: For example the Halting Problem HP.
Recognizable but not decidable.

Non–Recursively Enumerable Language: For example the complement $\overline{HP}$.
Not even recognizable (no TM accepts exactly those inputs).

Pflichtaufgabe 1 (Blatt 4)

Wende den Satz von Rice auf folgende Sprachen an, um (Un-)Entscheidbarkeit zu beweisen. Prüfe bei Unentscheidbarkeit außerdem, ob die Eigenschaft nicht-monoton ist.

(a) Sprache $\mathcal{L}_1$

$$\mathcal{L}_1 = \{w \in \{0,1\}^* \mid \mathcal{L}(M_w) \leq_m HP\}$$

Lösung:

(i) *Spracheigenschaft:* Reduzierbarkeit $\leq_m HP$ bezieht sich auf $\mathcal{L}(M_w)$.

(ii) *Nicht-trivialität:*

- Positiv: M_2 mit $\mathcal{L}(M_2) = HP \Rightarrow HP \leq_m HP$.
- Negativ: M_1 mit $\mathcal{L}(M_1) = \overline{HP} \Rightarrow \overline{HP} \not\leq_m HP$.

(iii) Nach Rice's Theorem ist $\mathcal{L}_1$ *nicht entscheidbar.*

(iv) *Nicht-Monotonie:*
Betrachte $L_1 = HP \subseteq L_2 = \Sigma^*$. Dann

$$P(L_1) = \text{``}L_1 \leq_m HP\text{''} = \text{true}, \quad P(L_2) = \text{``}L_2 \leq_m HP\text{''} = \text{false}.$$

Damit ist die Eigenschaft nicht-monoton.

(b) Sprache $\mathcal{L}_2$

$$\mathcal{L}_2 = \{w \in \{0,1\}^* \mid P(\mathcal{L}(M_w))\}$$

Hier sei P die Eigenschaft „(ein Beispiel:) die Sprache ist nicht entscheidbar“. **Lösung:**

(i) *Spracheigenschaft:* P bezieht sich auf $\mathcal{L}(M_w)$.

(ii) *Nicht-trivialität:*

- Positiv: M_2 mit $\mathcal{L}(M_2) = HP \Rightarrow P$ gilt.
- Negativ: M_1 mit $\mathcal{L}(M_1) = \emptyset \Rightarrow P$ gilt nicht.

(iii) Nach Rice's Theorem ist $\mathcal{L}_2$ *nicht entscheidbar.*

(iv) *Nicht-Monotonie:* Mit $L_1 = HP \subseteq L_2 = \Sigma^*$ gilt $P(L_1) = \text{true}$, $P(L_2) = \text{false}$.