Linear algebra reminders

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Matrices

A matrix $A \in \mathbb{R}^{n \times p}$ has n rows and p columns

It can be seen as:

A table of numbers

A linear application: $A : \mathbb{R}^p \to \mathbb{R}^n$

For $x \in \mathbb{R}^p, \quad Ax \in \mathbb{R}^n$

Eigenvalues

When A is square (n = p), an eigenvalue of A is a scalar $\lambda \in \mathbb{R}$ such that for some $x \neq 0$:

$$Ax = \lambda x$$

x is called an eigenvector of A

Spectral theorem

When A is symmetric $A^\top = A$ there is an orthonormal basis of eigenvectors $x_1, \ldots, x_p$ associated to eigenvalues $\lambda_1 \leq \cdots \leq \lambda_p$ such that :

Orthogonality: For all $i \neq j, \langle x_i, x_j \rangle = 0$

Unit norm: For all i, $\|x_i\| = 1$

Eigenvalues: For all i, $Ax_i = \lambda_i x_i$

Spectral theorem, matrix formulation

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Q: define the p x p matrix $U = [x_1, \ldots, x_p]$
What can we say about U?

Spectral theorem, matrix formulation

Define the p x p matrix $U = [x_1, \dots, x_p]$

Orthogonality + unit norm: U is *orthogonal* :

$$UU^\top = U^\top U = I_p$$

Eigenvalues: $AU = \mathrm{diag}(\lambda_i)U$

$$A = U^\top \Lambda U$$

Singular value decomposition (SVD)

Now, A is no longer square : $A \in \mathbb{R}^{n \times p}, \;\; n \geq p$

There exist $U \in \mathbb{R}^{n \times p}, \;\; V \in \mathbb{R}^{p \times p}, \;\;$ and $0 \leq \sigma_1 \leq \cdots \leq \sigma_p$ such that

Orthogonality: $\quad U^\top U = V^\top V = I_p$

!! $UU^\top \neq I_n$

Decomposition: $\quad A = U \Sigma V^\top, \;\; \Sigma = \mathrm{diag}(\sigma_i)$

The σ_i are called the singular values of A

Link between spectral theorem and SVD

Question : how can we recover the SVD of A using the spectral theorem ?

Hint: you can use an analysis-synthesis reasoning. If $A = U\Sigma V^\top$ is the SVD of A, does V correspond to the eigenvalue decomposition of some matrix?

Matrix norms

There are many ways to define norms on matrices.

Frobenius: $$\|A\|_F = \sqrt{\sum_{i,j=1}^{n} A_{ij}^2}$$

Spectral: $$\|A\|_2 = \max_{\|x\|=1} \|Ax\|$$

Q: can you relate these quantities to the singular values of A?