Mandatory assignment 1
MAT-MEK4270
Tasks 1.2.3 and 1.2.4

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1 Exact solution

1.1 Introduction

We aim to show that

$$u(t, x, y) = e^{\imath(k_x x + k_y y - \omega t)} \tag{1}$$

satisfies the standard 2D wave equation given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

1.2 Solution Strategy

1.2.1 Step 1: Differentiate $u(x, y, t)$

The second derivatives with respect to t, x, and y are computed as follows:

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 e^{\imath(k_x x + k_y y - \omega t)},$$

$$\frac{\partial^2 u}{\partial x^2} = -k_x^2 e^{\imath(k_x x + k_y y - \omega t)},$$

$$\frac{\partial^2 u}{\partial y^2} = -k_y^2 e^{\imath(k_x x + k_y y - \omega t)}.$$

1.2.2 Step 2: Substitute into the Wave Equation

Substituting these into the standard wave equation, we find that

$$\omega^2 e^{\imath(k_x x + k_y y - \omega t)} = c^2 (k_x^2 + k_y^2) e^{\imath(k_x x + k_y y - \omega t)}.$$

This simplifies to

$$\omega^2 = c^2 (k_x^2 + k_y^2).$$

1.2.3 Step 3: Conclusion

Therefore, we conclude that $u(t,x,y) = e^{\imath(k_x x + k_y y - \omega t)}$ satisfies the standard 2D wave equation.

2 Dispersion coefficient

2.1 Introduction

Now, assuming that $m_x = m_y$ such that $k_x = k_y = k$. A discrete version of will then read

$$u_{ij}^n = e^{\imath(kh(i+j) - \tilde{\omega} n \Delta t)} \tag{2}$$

where $\tilde{\omega}$ is a numerical dispersion coefficient, i.e., the numerical approximation of the exact ω.

Want to insert the equation into the discretized 2d wave equation and show that for CFL number $C = 1/\sqrt{2}$ we get $\tilde{\omega} = \omega$.

The discretized 2d wave equation is as follows

$$\begin{aligned} &\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} \\ =& c^2 \left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2} \right) \end{aligned} \tag{3}$$

Let's first look at the LHS. We then have

$$\begin{aligned} &\frac{e^{-\imath kh(i+j)}}{\Delta t} \left(e^{-\imath \tilde{\omega}(n+1)\Delta t} - 2e^{-\imath \tilde{\omega} n \Delta t} + e^{-\imath \tilde{\omega}(n-1)\Delta t} \right) \\ =& \frac{e^{-\imath kh(i+j)}}{\Delta t} \left(e^{-\imath \tilde{\omega} n \Delta t} e^{-\imath \tilde{\omega} \Delta t} - 2e^{-\imath \tilde{\omega} n \Delta t} + e^{-\imath \tilde{\omega} n \Delta t} e^{\imath \tilde{\omega} \Delta t} \right) \\ =& \frac{e^{\imath(kh(i+j) - \tilde{\omega}\Delta t)}}{\Delta t} \left(e^{-\imath \tilde{\omega} n \Delta t} - 2 + e^{-\imath \tilde{\omega} n \Delta t} \right). \end{aligned}$$

And for the RHS we get

$$\begin{aligned} &2\frac{c^2}{h^2} e^{-\imath \tilde{\omega} n \Delta t} \left(e^{\imath kh(i+j+1)} - 2e^{\imath kh(i+j)} + e^{\imath kh(i+j-1)} \right) \\ &= 2\frac{c^2}{h^2} e^{\imath(kh(i+j) - \tilde{\omega} n \Delta t)} \left(e^{\imath kh} - 2 + e^{-\imath kh} \right). \end{aligned}$$

If we now set LHS = RHS we get

$$\begin{aligned}
\frac{e^{\imath(kh(i+j)-\tilde{\omega}\Delta t)}}{\Delta t}\left(e^{-\imath\tilde{\omega}n\Delta t}-2+e^{-\imath\tilde{\omega}n\Delta t}\right) &= 2\frac{c^2}{h^2}e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t)}\left(e^{\imath kh}-2+e^{-\imath kh}\right)\\
e^{-\imath\tilde{\omega}n\Delta t}-2+e^{-\imath\tilde{\omega}n\Delta t} &= 2C^2\left(e^{\imath kh}-2+e^{-\imath kh}\right)\\
\cos\left(\tilde{\omega}\Delta t\right) &= \cos\left(kh\right)\\
\tilde{\omega}\Delta t &= kh\\
\tilde{\omega} &= \frac{kh}{\Delta t}
\end{aligned}$$

Now, we have that the analytical dispersion coefficient ω is

$$\omega^2 = c^2(k_x^2+k_y^2)$$

rewriting c^2 using the CFL number and its condition we get

$$\text{CFL} = \frac{1}{\sqrt{2}} = \frac{c\Delta t}{h} \Rightarrow c = \frac{h}{\Delta t\sqrt{2}}.$$

Thus, by setting $k_x = k_y = k$ we get

$$\omega^2 = \frac{h^2}{2\Delta t^2}2k^2 \Rightarrow \omega = \frac{kh}{\Delta t} = \tilde{\omega}.$$