Linear and Angular Momentum

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1 Impulse

So far we considered cases where a constant force is present. We were not concerned about the when the force is applied how long is applied and when it is going to be removed.

In real interacting world however a applied force varies with time. See figure 1. The figure shows the *profile/curve* of a force around the time it is applied to an object.

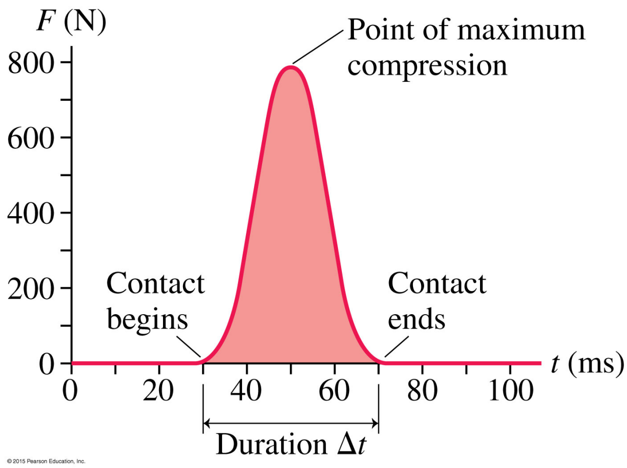


Fig 1. Consider kicking a soccer ball. The figure shows a possible profile of the force as a function of time as you kick the ball.

Assume before a time t_i the force is zero and after t_f the force is completely removed. During the interval $t_f - t_i$ the applied force goes from a zero at and before t_i to a maximum value and then returns to a zero value at and after time t_f. A quantity that distinguishes between different profiles of forces given an interval $t_f - t_i$ is called *Impulse*, denoted by J, and is defined as

$$J = \textbf{Area under the force curve}$$

Calculating the area under the force is not an easy task. However, one can prove that for a given curve and an interval Δt, there always exists an average force F_{avg} such that

$$J = F_{avg}\Delta t$$

2 Linear Momentum

Returning to the Newton's second law, we know that the force is by definition

$$\vec{F} = m\vec{a}$$

The $\vec{F}_{avg}$ defined above can also be accompanied with an average acceleration $\vec{a}_{avg}$ such that

$$\vec{a}_{avg} = \frac{\vec{F}_{avg}}{m},$$

and since the $\vec{a}_{avg} = \vec{v}_f - \vec{v}_i$, we have

$$\vec{F}_{avg}\Delta t = J = m\vec{v}_f - m\vec{v}_i.$$

This shows that the amount of impulse J depends only on two quantities $m\vec{v}_i$ and $m\vec{v}_f$ that are defined before and after the impact respectively, and *the impulse does not depend on the shape of the curve of* $\vec{F}$ *explicitly.*

$m\vec{v}$ has a special name and is called the *linear momentum*, denoted by $\vec{p}$:

$$\vec{p} = m\vec{v}.$$

Therefore

$$J = \Delta P$$

Also notice that the vector sign $\vec{p}$ assert the fact that the momentum just like force and velocity is a vector. Thus, specifying the momentum requires specification of the *magnitude* and *direction*.

2.1 Total (Linear) Momentum

Momentum is an *additive* property that is momentum of a system of particles is the addition of the momentum of each particle:

$$\vec{P}_{tot} = \sum_{e\in S} \vec{p}_e,$$

where the sum is over all the elements of a system S.

3 Conservation of Linear Momentum

A closed system is defined as a system of objects that are not in interaction with the outside world, or in other words this system does not experience any force from outside.

However, a closed system can contain objects that *internally* insert forces on each other. Consider two billiard balls on a pool table, it is fair to say that in the two dimensional plane of the table there is no external force applied to the two balls (ignoring friction). Thus, if the two balls are initially travelling with two velocities v_1 and v_2 after collision their velocities change to v_1' and v_2'. Since there is no external force on this system, the impulse J is zero, $J = \Delta P = 0$. That means the initial and final *total*- momentum of the system must be the same:

$$P_{tot} = m\vec{v_1} + m\vec{v_2} = m\vec{v_1'} + m\vec{v_2'} = P'_{tot}$$

arranged differently:

$$J_1 = m\vec{v_1'} - m\vec{v_1} = -(m\vec{v_2'} - m\vec{v_2}) = -J_2$$

That is the change in the impulse of ball one is captured by ball two.

This is an example of conservation of momentum which generally states that

For a closed system the total momentum is conserved.

3.1 In two dimension

Since you are familiar with the fact that momentum is vector variable, we know also that a vector can be decomposed to its components. So, a two dimensional collision is nothing by considering the conservation of the momentum separately for both components of the momentum.

4 Angular Momentum

A similar argument to the the discussion of impulse can be drawn to case where angular acceleration and torque are considered. We remember that the torque is defined as

$$\tau_{net} = I\alpha$$

. Since, $\alpha = \frac{\Delta\omega}{\Delta t}$ we can easily show that

$$\tau_{net}\Delta t = I\Delta\omega = I\omega_f - I\omega_i.$$

In analogy to what we had earlier in the linear case, we can define an *angular momentum* $L = I\omega$. Therefore,

$$\tau_n et \Delta t = \Delta L.$$

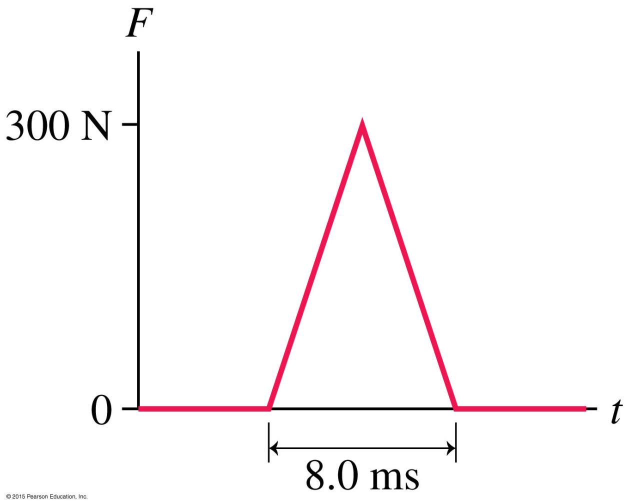
That is the effect of an external torque during a time interval Δt is to change the angular momentum of a system by an amount of ΔL.

5 Conservation of angular momentum

If there is no torque applied on a system the angular momentum does not change. As a result the angular momentum is a conserved quantities. See example 9. 10 in the book on page 272.

6 Examples

Example 1: Consider the following force-vs-time profile



Find the value of impulse J.
Answer: J is the area under the force curve. From the figure:

$$\begin{aligned} J &= \frac{1}{2} F_{max} \times \Delta t \\ &= \frac{1}{2} 300(8.0)(10^{-3}) \\ &= 1.2 \ \ [N.s] \end{aligned}$$

Example 2 : example 9.3 in the book page 261.
Example 3 : example 9.6 in the book page 266 – Conservation.