Bilinear forms

Jovana Savic

February 20, 2021

Definition 1. *Consider vector space $V \times V$. A positive definite, symmetric bilinear mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is called an inner product on V.*

- Symmetric
$$(\forall x, y \in V)(\langle x, y \rangle = \langle y, x \rangle)$$
- Positive definite
$$(\forall x \in V / \{0\}) (\langle x, x \rangle > 0) \wedge (\langle 0, 0 \rangle = 0)$$
- Bilinear
$$\begin{aligned} &\forall x, y, z \in V, \lambda \in \mathbb{R} \\ &\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle \\ &\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle \end{aligned}$$

Lemma 1. *A function $\beta(x, y) : V \times V \to \mathbb{R}$ that can be written in the form $\beta(x, y) = x^T A y$ where A is a square matrix has the property of bilinearity. We call such function bilinear form.*

Proof. We can directly show that this is the case.

$$\begin{aligned} &\beta(\lambda x + y, z) = (\lambda x + y)^T A z \\ &= \lambda x^T A z + y^T A z \\ &= \lambda \beta(x, z) + \beta(y, z). \end{aligned}$$

Also,

$$\begin{aligned} &\beta(x, \lambda y + z) = x^T A (\lambda y + z) \\ &= \lambda x^T A y + x^T A z \\ &= \lambda \beta(x, y) + \beta(x, z). \end{aligned}$$

$\square$

Lemma 2. *A bilinar form is symmetric if and only if matrix A is symmetric.*

Proof. Since the result of the function is a real number, transposing it doesn't change it.

$$\beta(x, y) = x^T Ay = \left(x^T Ay\right)^T$$
$$= y^T A^T x.$$

We can see that if $A^T = A$ the above can be written as:

$$\beta(x, y) = y^T Ax = \beta(y, x).$$

□

Definition 2. *Matrix A is positive definite if for any* $x \in V/\left\{0\right\}$ $x^T Ax > 0$.

Theorem 3. *Matrix A is positive definite if its eigenvalues are positive. Its eigenvalues are found as the solutions to the equation:*

$$det\left(A - \lambda I\right) = 0$$

In summary, a bilinear form is:

- *Symmetric*: if $A^T = A$.
- *Bilinear*: always
- *Positive definite*: if the solutions to $\det\left(A - \lambda I\right) = 0$ are positive [1].

If all of the above are satisfied then it's an inner product.

[1]It's easy to see that if x=0 the result of $x^T Ax = 0$ is true for any matrix A.