Bilinear forms

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Definition 1. *Consider vector space V.V. A positive definite, symmetric bilinear mapping $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ is called an inner product on V.*

- Symmetric
$$(\forall x, y \in V)(\langle x, y\rangle = \langle y, x\rangle)$$
- Positive definite
$$(\forall x \in V/\left\{0\right\})\left(\langle x, x\rangle > 0\right) \wedge \left(\langle 0, 0\rangle = 0\right)$$
- Bilinear
$$\begin{aligned}&\forall x, y, z \in V, \lambda \in \mathbb{R}\\&\langle \lambda x + y, z\rangle = \lambda\langle x, z\rangle + \langle y, z\rangle\\&\langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, z\rangle\end{aligned}$$

Lemma 1. *A function $\beta(x, y) : V \times V \to \mathbb{R}$ that can be written in the form $\beta(x, y) = x^T Ay$ where A is a square matrix has the property of bilinearity. We call such function bilinear form.*

Proof. We can directly show that this is the case.

$$\begin{aligned}&\beta(\lambda x + y, z) = (\lambda x + y)^T Az\\&= \lambda x^T Az + y^T Az\\&= \lambda\beta(x, z) + \beta(y, z).\end{aligned}$$

Also,

$$\begin{aligned}&\beta(x, \lambda y + z) = x^T A(\lambda y + z)\\&= \lambda x^T Ay + x^T Az\\&= \lambda\beta(x, y) + \beta(x, z).\end{aligned}$$

$\square$

Lemma 2. *A bilinar form is symmetric if and only if matrix A is symmetric.*

Proof. Since the result of the function is a real number, transposing it doesn't change it.

$$\begin{aligned} \beta(x, y) &= x^T A y = \left(x^T A y\right)^T \\ &= y^T A^T x. \end{aligned}$$

We can see that if $A^T = A$ the above can be written as:

$$\beta(x, y) = y^T A x = \beta(y, x).$$

$\square$

Definition 2. *Matrix A is positive definite if for any $x \in V \;\; \{0\}$ $x^T A x > 0$.*

Theorem 3. *Matrix A is positive definite if its eigenvalues are positive. Its eigenvalues are found as the solutions to the equation:*

$$det(A - \lambda I) = 0$$

In summary, a bilinear form is:

- *Symmetric*: if $A^T = A$.
- *Bilinear*: always
- *Positive definite*: if the solutions to $\det(A - \lambda I) = 0$ are positive [1].

If all of the above are satisfied then it's an inner product.

[1] It's easy to see that if x=0 the result of $x^T A x = 0$ is true for any matrix A.