The FOC are

$$
\begin{aligned}
c_{1,t}: \quad & \mu_{1,0}\frac{\partial V_{1,0}}{\partial V_{1,t}}\frac{\partial V_{1,t}}{\partial c_{1,t}} && = \lambda_{1,t} && (1)\\
a_{1,t}: \quad & \frac{\partial G_1(a_{1,t},b_{1,t})}{\partial a_{1,t}}\lambda_{1,t} && = \lambda_{3,t} && (2)\\
b_{1,t}: \quad & \frac{\partial G_1(a_{1,t},b_{1,t})}{\partial b_{1,t}}\lambda_{1,t} && = \lambda_{4,t} && (3)\\
i_{1,a,t}: \quad & \frac{\partial H_1(i_{1,a,t},i_{1,b,t})}{\partial i_{1,a,t}}\lambda_{5,t} && = \lambda_{3,t} && (4)\\
i_{1,b,t}: \quad & \frac{\partial H_1(i_{1,a,t},i_{1,b,t})}{\partial i_{1,b,t}}\lambda_{5,t} && = \lambda_{4,t} && (5)\\
c_{2,t}: \quad & \mu_{2,0}\frac{\partial V_{2,0}}{\partial V_{2,t}}\frac{\partial V_{2,t}}{\partial c_{2,t}} && = \lambda_{2,t} && (6)\\
a_{2,t}: \quad & \frac{\partial G_2(a_{2,t},b_{2,t})}{\partial a_{2,t}}\lambda_{2,t} && = \lambda_{3,t} && (7)\\
b_{2,t}: \quad & \frac{\partial G_2(a_{2,t},b_{2,t})}{\partial b_{2,t}}\lambda_{2,t} && = \lambda_{4,t} && (8)\\
i_{2,a,t}: \quad & \frac{\partial H_2(i_{2,a,t},i_{2,b,t})}{\partial i_{2,a,t}}\lambda_{6,t} && = \lambda_{3,t} && (9)\\
i_{2,b,t}: \quad & \frac{\partial H_2(i_{2,a,t},i_{2,b,t})}{\partial i_{2,b,t}}\lambda_{6,t} && = \lambda_{4,t} && (10)
\end{aligned}
$$

We can combine (1) and (2) to obtain

$$\mu_{1,0}\frac{\partial V_{1,0}}{\partial V_{1,t}}\frac{\partial V_{1,t}}{\partial c_{1,t}}\frac{\partial G_1(a_{1,t},b_{1,t})}{\partial a_{1,t}} = \lambda_{3,t}. \tag{11}$$

Similarly, (6) and (7) give

$$\mu_{2,0}\frac{\partial V_{2,0}}{\partial V_{2,t}}\frac{\partial V_{2,t}}{\partial c_{2,t}}\frac{\partial G_2(a_{2,t},b_{2,t})}{\partial a_{2,t}} = \lambda_{3,t}. \tag{12}$$

Together (11) and (12) give

$$\frac{\mu_{1,0}}{\mu_{2,0}} = \frac{\frac{\partial V_{2,0}}{\partial V_{2,t}}\frac{\partial V_{2,t}}{\partial c_{2,t}}\frac{\partial G_2(a_{2,t},b_{2,t})}{\partial a_{2,t}}}{\frac{\partial V_{1,0}}{\partial V_{1,t}}\frac{\partial V_{1,t}}{\partial c_{1,t}}\frac{\partial G_1(a_{1,t},b_{1,t})}{\partial a_{1,t}}}. \tag{13}$$

Now we define an ad hoc time t Pareto weight:

$$\begin{aligned}
\mu_{i,t} &= \mu_{i,0}\frac{\partial V_{i,0}}{\partial V_{i,t}}\frac{\partial V_{i,t}}{\partial c_{i,t}}c_{i,t} \\
&= \mu_{i,t-1}\frac{\partial V_{i,t-1}}{\partial V_{i,t}}\frac{\partial V_{i,t}/\partial c_{i,t}}{\partial V_{i,t-1}/\partial c_{i,t-1}}\frac{c_{i,t}}{c_{i,t-1}} &(14)\\
&= \mu_{i,t-1}M_t\frac{c_{i,t}}{c_{i,t-1}} &(15)
\end{aligned}$$

where we have used the chain rule to write $\frac{\partial V_{i,0}}{\partial V_{i,t}} = \prod_{\tau=1}^{t}\frac{\partial V_{i,\tau-1}}{\partial V_{i,\tau}}$ and have used

$$\begin{aligned}
M_{i,t+1} &= \frac{\partial V_{i,t}}{\partial V_{i,t+1}}\frac{\partial V_{i,t+1}/\partial c_{i,t+1}}{\partial V_{i,t}/\partial c_{i,t}} &(16)\\
&= \beta\left(\frac{c_{i,t+1}}{c_{i,t}}\right)^{\rho-1}\left(\frac{V_{i,t+1}}{E\left[V_{i,t+1}^{\alpha}\right]^{\alpha}}\right)^{\alpha-\rho} &(17)
\end{aligned}$$

We can combine (14) with (13) to get

$$\frac{1}{c_{1,t}}\mu_{1,t}\frac{\partial G_{1,t}(a_{1,t},b_{1,t})}{\partial a_{1,t}} = \frac{1}{c_{2,t}}\mu_{2,t}\frac{\partial G_{2,t}(a_{2,t},b_{2,t})}{\partial a_{2,t}} \tag{18}$$

Define $S_t = \frac{\mu_{1,t}}{\mu_{2,t}}$ and write (18) as

$$S_t\frac{\partial G_{1,t}(a_{1,t},b_{1,t})}{\partial a_{1,t}}\frac{1}{c_{1,t}} = \frac{\partial G_{2,t}(a_{2,t},b_{2,t})}{\partial a_{2,t}}\frac{1}{c_{2,t}}. \tag{19}$$

We can also use (14) for both agents to write:

$$S_t = S_{t-1}\frac{M_{1,t}}{M_{2,t}}\frac{c_{1,t}/c_{1,t-1}}{c_{2,t}/c_{2,t-1}}. \tag{20}$$

Following the same steps we can write the marginal condition for good b as

$$S_t\frac{\partial G_{1,t}(a_{1,t},b_{1,t})}{\partial b_{1,t}}\frac{1}{c_{1,t}} = \frac{\partial G_{2,t}(a_{2,t},b_{2,t})}{\partial b_{2,t}}\frac{1}{c_{2,t}}. \tag{21}$$

0.1 Investment terms

TODO: determine why They write C5 and C6 lagged one period. Do we need to do this also? If so, why. If not, why not?