1 Model

The unscaled model can be written as

$$\begin{aligned}
J(k_1, k_2, z_1, z_2, U) &= \max_{c_1, c_2, I_1, I_2, U'} [(1-\beta_1)c_1^{\rho_1} + \beta_1\mu(J')^{\rho}]^{\frac{1}{\rho_1}} \\
\text{subject to } & [(1-\beta_2)c_2^{\rho_2} + \beta_2\mu(U')^{\rho}]^{\frac{1}{\rho_2}} \\
c_1 + c_2 + I_1 + I_2 &= f_1(k_1, z_1) + f_2(k_2, z_2) \\
k_1' &= \Gamma_1(k_1, I_1) \\
k_2' &= \Gamma_2(k_2, I_2)
\end{aligned}$$

where

$$\begin{aligned}
f_1(k_1, z_1) &= [(1-\eta_1)k_1^{\nu_1} + \eta_1 z_1^{\nu_1}]]^{\frac{1}{\nu_1}} \\
f_2(k_2, z_2) &= [(1-\eta_2)k_2^{\nu_2} + \eta_2 z_2^{\nu_2}]]^{\frac{1}{\nu_2}} \\
\Gamma_1(k_1, I_1) &= [(1-\delta)k_1^{\gamma_1} + \delta I_1^{\gamma_1}]^{\frac{1}{\gamma_1}} \\
\Gamma_2(k_2, I_2) &= [(1-\delta)k_2^{\gamma_2} + \delta I_2^{\gamma_2}]^{\frac{1}{\gamma_2}}
\end{aligned}$$

Note: For $\gamma_i = 1$ and $I_i = 0$ that we get $k_{t+1} = (1-\delta)k_t$ which is the standard rate of depreciation.

If we then scale the model by z_1, we get[1]

$$\begin{aligned}
J(k_1, k_2, \xi, U) &= \max_{c_1, c_2, I_1, I_2, U'} [(1-\beta_1)c_1^{\rho_1} + \beta_1\mu(g'J')^{\rho}]^{\frac{1}{\rho_1}} \\
\text{subject to } & [(1-\beta_2)c_2^{\rho_2} + \beta_2\mu(g'U')^{\rho}]^{\frac{1}{\rho_2}} \quad (\lambda_P) \\
c_1 + c_2 + I_1 + I_2 &= f_1(k_1) + f_2(k_2, \xi) \quad (\lambda_{BC}) \\
g'k_1' &= \Gamma_1(k_1, I_1) \quad (\lambda_1) \\
g'k_2' &= \Gamma_2(k_2, I_2) \quad (\lambda_2)
\end{aligned}$$

2 First Order Conditions

$$\begin{aligned}
c_1 &: J^{1-\rho_1}(1-\beta_1)c_1^{\rho_1-1} + \lambda_{BC} = 0 \\
c_2 &: U^{1-\rho_2}(1-\beta_2)c_2^{\rho_2-1} + \lambda_{BC} = 0 \\
I_1 &: J^{1-\rho_1}\beta_1\mu_1(g'J')^{\rho-1}(...) + ... \\
I_2 &: U^{1-\rho_2}
\end{aligned}$$

[1] I'm going to use same variable names, but everything in site is scaled by z_1