*About notation* Without loss of generality when I write $x_t$ I really mean $x(s^t)$. When the history dependence is potentially unclear, I will it out in longer form.

The Lagrangian is

$$
\begin{aligned}
\mathcal{L} = \mu_{1,0} V_{1,0}\left(c_{1,0}, V_{1,1}\right) \qquad & + \mu_{2,0} V_{2,0}\left(c_{2,0}, V_{2,1}\right) \\
& \sum_{t=0}^{\infty}\sum_{s^t}\Bigg\{ \lambda_1(s^t)\left(Y_1\left(\chi_{1,t}/g_t, 1\right) - a_{1,t} - a_{2,t} - i_{1,a,t} - i_{2,a,t}\right) + \\
& \lambda_2(s^t)\left(Y_2\left(\chi_{2,t}/g_t, \xi_t\right) - b_{1,t} - b_{2,t} - i_{1,b,t} - i_{2,b,t}\right) + \\
& \lambda_3(s^t)\left((1-\delta_1)\frac{\chi_{1,t-1}}{g_{t-1}} + H_1\left(i_{1,a,t-1}, i_{1,b,t-1}\right) - \chi_{1,t}\right) + \\
& \lambda_4(s^t)\left((1-\delta_2)\frac{\chi_{2,t-1}}{g_{t-1}} + H_2\left(i_{2,a,t-1}, i_{2,b,t-1}\right) - \chi_{2,t}\right)\Bigg\},
\end{aligned}
\tag{1}
$$

where $c_{i,t} = G_i(a_{i,t}, b_{i,t})$ and $\chi_{i,t} = g(s^t)k_i(s^t)$ for $i = 1, 2$ and $t = 0, 1, \ldots, \infty$.

The control variables are: $\{a_n(s^t), b_n(s^t), i_{n,a}(s^t), i_{n,b}(s^t), \chi_n(s^t)\}$ for $t = 0, 1, \ldots, \infty$, $n \in 1, 2$, and all histories of states $s^t$. The FONC are

$$a_1(s^t): \quad \mu_{1,0}\frac{\partial V_{1,0}}{\partial V_{1,t}}\frac{\partial V_{1,t}}{\partial c_{1,t}}\frac{\partial c_{1,t}}{\partial a_{t,1}} = \lambda_1(s^t) \tag{2}$$

$$b_1(s^t): \quad \mu_{1,0}\frac{\partial V_{1,0}}{\partial V_{1,t}}\frac{\partial V_{1,t}}{\partial c_{1,t}}\frac{\partial c_{1,t}}{\partial b_{t,1}} = \lambda_2(s^t) \tag{3}$$

$$i_{1,a}(s^t): \quad \sum_{s^{t+1}}\lambda_3(s^{t+1})\frac{\partial H_1(i_{1,a}(s^t), i_{1,b}(s^t)))}{\partial i_{1,a}(s^t)} = \lambda_1(s^t) \tag{4}$$

$$i_{1,b}(s^t): \quad \sum_{s^{t+1}}\lambda_3(s^{t+1})\frac{\partial H_1(i_{1,a}(s^t), i_{1,b}(s^t)))}{\partial i_{1,b}(s^t)} = \lambda_2(s^t) \tag{5}$$

$$\chi_1(s^t): \quad \lambda_1(s^t)\frac{\partial Y_1(\chi_{1,t}/g_t, 1)}{\partial(\chi_{1,t}/g_t)}\frac{1}{g_t} + \sum_{s^{t+1}}\frac{(1-\delta_1)}{g(s^{t+1}|s^t)}\lambda_3(s^{t+1}) = \lambda_3(s^t) \tag{6}$$

$$a_2(s^t): \quad \mu_{2,0}\frac{\partial V_{2,0}}{\partial V_{2,t}}\frac{\partial V_{2,t}}{\partial c_{2,t}}\frac{\partial c_{2,t}}{\partial a_{t,2}} = \lambda_1(s^t) \tag{7}$$

$$a_2(s^t): \quad \mu_{2,0}\frac{\partial V_{2,0}}{\partial V_{2,t}}\frac{\partial V_{2,t}}{\partial c_{2,t}}\frac{\partial c_{2,t}}{\partial b_{t,2}} = \lambda_2(s^t) \tag{8}$$

$$i_{2,a}(s^t): \quad \sum_{s^{t+1}}\lambda_4(s^{t+1})\frac{\partial H_2(i_{4,a}(s^t), i_{4,b}(s^t)))}{\partial i_{4,a}(s^t)} = \lambda_1(s^t) \tag{9}$$

$$i_{2,b}(s^t): \quad \sum_{s^{t+1}}\lambda_4(s^{t+1})\frac{\partial H_2(i_{4,a}(s^t), i_{4,b}(s^t)))}{\partial i_{4,b}(s^t)} = \lambda_2(s^t) \tag{10}$$

$$\chi_2(s^t): \quad \lambda_2(s^t)\frac{\partial Y_2(\chi_{2,t}/g_t, 1)}{\partial(\chi_{2,t}/g_t)}\frac{1}{g_t} + \sum_{s^{t+1}}\frac{(1-\delta_1)}{g(s^{t+1}|s^t)}\lambda_4(s^{t+1}) = \lambda_4(s^t) \tag{11}$$

Before proceeding, we will gather some tools we will use while analyzing the above conditions. First, notice that by the chain rule we have

$$\frac{\partial V_{i,0}}{\partial V_{i,t}} = \prod_{\tau=1}^{t} \frac{\partial V_{i,\tau-1}}{\partial V_{i,\tau}}. \tag{12}$$

The notice that we can write the SDF for agent $i$ as

$$\begin{aligned} M_{i,t+1} &= \frac{\partial V_{i,t}}{\partial V_{i,t+1}} \frac{\partial V_{i,t+1}/\partial c_{i,t+1}}{\partial V_{i,t}/\partial c_{i,t}} \\ &= \beta \left(\frac{C_{t+1}}{C_t}\right)^{\rho-1} \left(\frac{V_{i,t+1}}{E\left[V_{i,t+1}^{\alpha}\right]^{1/\alpha}}\right)^{\alpha-\rho}. \end{aligned}$$

We will also write stochastic discount factor in $a$ and $b$ units as follows:
We will make use of the time $t$ Pareto weight, which we define as

$$\begin{aligned} \mu_{i,t} &= \mu_{i,0} \frac{\partial V_{i,0}}{\partial V_{i,t}} \frac{\partial V_{i,t}}{\partial c_{i,t}} \\ &= \mu_{i,0} \prod_{\tau=1}^{t} \frac{\partial V_{i,\tau-1}}{\partial V_{i,\tau}} \frac{\partial V_{i,t}}{\partial c_{i,t}} \\ &= \mu_{i,t-1} \frac{\partial V_{i,t}}{\partial V_{i,t+1}} \frac{\partial V_{i,t+1}/\partial c_{i,t+1}}{\partial V_{i,t}/\partial c_{i,t}} \\ &= \mu_{i,t-1} M_{i,t} \end{aligned} \tag{13}$$