The Lagrangian is

$$
\begin{aligned}
\mathcal{L} = \mu_{1,0}V_{1,0} + \mu_{2,0}V_{2,0} + \sum_{t=0}^{\infty} E_t \Big\{ & \lambda_{1,t}\left(G_1(a_{1,t}, b_{1,t}) - c_{1,t}\right) + \\
& \lambda_{2,t}\left(G_2(a_{2,t}, b_{2,t}) - c_{2,t}\right) + \\
& \lambda_{3,t}\left(Y_1(k_{1,t}, 1) - a_{1,t} - a_{2,t} - i_{1,a,t} - i_{2,a,t}\right) + \\
& \lambda_{4,t}\left(Y_2(k_{2,t}, \xi) - b_{1,t} - b_{2,t} - i_{1,b,t} - i_{2,b,t}\right) + \\
& \lambda_{5,t}\left((1-\delta_1)k_{1,t} + H_1\left(i_{1,a,t}, i_{1,b,t}\right) - g'_{t+1}k_{1,t+1}\right) + \\
& \lambda_{6,t}\left((1-\delta_2)k_{2,t} + H_2\left(i_{2,a,t}, i_{2,b,t}\right) - g'_{t+1}k_{2,t+1}\right) + \\
& \Big\} \qquad (1)
\end{aligned}
$$

The FOC are

$$
\begin{aligned}
c_{1,t}: \quad & \mu_{1,0}\frac{\partial V_{1,0}}{\partial V_{1,t}}\frac{\partial V_{1,t}}{\partial c_{1,t}} &= \lambda_{1,t} \qquad (2) \\
a_{1,t}: \quad & \frac{\partial G_1(a_{1,t}, b_{1,t})}{\partial a_{1,t}}\lambda_{1,t} &= \lambda_{3,t} \qquad (3) \\
b_{1,t}: \quad & \frac{\partial G_1(a_{1,t}, b_{1,t})}{\partial b_{1,t}}\lambda_{1,t} &= \lambda_{4,t} \qquad (4) \\
i_{1,a,t}: \quad & \frac{\partial H_1(i_{1,a,t}, i_{1,b,t})}{\partial i_{1,a,t}}\lambda_{5,t} &= \lambda_{3,t} \qquad (5) \\
i_{1,b,t}: \quad & \frac{\partial H_1(i_{1,a,t}, i_{1,b,t})}{\partial i_{1,b,t}}\lambda_{5,t} &= \lambda_{4,t} \qquad (6) \\
c_{2,t}: \quad & \mu_{2,0}\frac{\partial V_{2,0}}{\partial V_{2,t}}\frac{\partial V_{2,t}}{\partial c_{2,t}} &= \lambda_{2,t} \qquad (7) \\
a_{2,t}: \quad & \frac{\partial G_2(a_{2,t}, b_{2,t})}{\partial a_{2,t}}\lambda_{2,t} &= \lambda_{3,t} \qquad (8) \\
b_{2,t}: \quad & \frac{\partial G_2(a_{2,t}, b_{2,t})}{\partial b_{2,t}}\lambda_{2,t} &= \lambda_{4,t} \qquad (9) \\
i_{2,a,t}: \quad & \frac{\partial H_2(i_{2,a,t}, i_{2,b,t})}{\partial i_{2,a,t}}\lambda_{6,t} &= \lambda_{3,t} \qquad (10) \\
i_{2,b,t}: \quad & \frac{\partial H_2(i_{2,a,t}, i_{2,b,t})}{\partial i_{2,b,t}}\lambda_{6,t} &= \lambda_{4,t} \qquad (11)
\end{aligned}
$$

The state variables
We can combine (2) and (3) to obtain

$$
\mu_{1,0}\frac{\partial V_{1,0}}{\partial V_{1,t}}\frac{\partial V_{1,t}}{\partial c_{1,t}}\frac{\partial G_1(a_{1,t}, b_{1,t})}{\partial a_{1,t}} = \lambda_{3,t}. \qquad (12)
$$

Similarly, (7) and (8) give

$$\mu_{2,0}\frac{\partial V_{2,0}}{\partial V_{2,t}}\frac{\partial V_{2,t}}{\partial c_{2,t}}\frac{\partial G_2(a_{2,t},b_{2,t})}{\partial a_{2,t}} = \lambda_{3,t}. \tag{13}$$

Together (12) and (13) give

$$\frac{\mu_{1,0}}{\mu_{2,0}} = \frac{\frac{\partial V_{2,0}}{\partial V_{2,t}}\frac{\partial V_{2,t}}{\partial c_{2,t}}\frac{\partial G_2(a_{2,t},b_{2,t})}{\partial a_{2,t}}}{\frac{\partial V_{1,0}}{\partial V_{1,t}}\frac{\partial V_{1,t}}{\partial c_{1,t}}\frac{\partial G_1(a_{1,t},b_{1,t})}{\partial a_{1,t}}}. \tag{14}$$

Now we define an ad hoc time $t$ Pareto weight:

$$\begin{aligned}
\mu_{i,t} &= \mu_{i,0}\frac{\partial V_{i,0}}{\partial V_{i,t}}\frac{\partial V_{i,t}}{\partial c_{i,t}}c_{i,t} \\
&= \mu_{i,t-1}\frac{\partial V_{i,t-1}}{\partial V_{i,t}}\frac{\partial V_{i,t}/\partial c_{i,t}}{\partial V_{i,t-1}/\partial c_{i,t-1}}\frac{c_{i,t}}{c_{i,t-1}} \qquad (15)\\
&= \mu_{i,t-1}M_t\frac{c_{i,t}}{c_{i,t-1}} \qquad (16)
\end{aligned}$$

where we have used the chain rule to write $\frac{\partial V_{i,0}}{\partial V_{i,t}} = \prod_{\tau=1}^{t}\frac{\partial V_{i,\tau-1}}{\partial V_{i,\tau}}$ and have used

$$\begin{aligned}
M_{i,t+1} &= \frac{\partial V_{i,t}}{\partial V_{i,t+1}}\frac{\partial V_{i,t+1}/\partial c_{i,t+1}}{\partial V_{i,t}/\partial c_{i,t}} \qquad (17)\\
&= \beta\left(\frac{c_{i,t+1}}{c_{i,t}}\right)^{\rho-1}\left(\frac{V_{i,t+1}}{E\left[V_{i,t+1}^{\alpha}\right]^{\alpha}}\right)^{\alpha-\rho} \qquad (18)
\end{aligned}$$

We can combine (15) with (14) to get

$$\frac{1}{c_{1,t}}\mu_{1,t}\frac{\partial G_{1,t}(a_{1,t},b_{1,t})}{\partial a_{1,t}} = \frac{1}{c_{2,t}}\mu_{2,t}\frac{\partial G_{2,t}(a_{2,t},b_{2,t})}{\partial a_{2,t}} \tag{19}$$

Define $S_t = \frac{\mu_{1,t}}{\mu_{2,t}}$ and write (19) as

$$S_t\frac{\partial G_{1,t}(a_{1,t},b_{1,t})}{\partial a_{1,t}}\frac{1}{c_{1,t}} = \frac{\partial G_{2,t}(a_{2,t},b_{2,t})}{\partial a_{2,t}}\frac{1}{c_{2,t}}. \tag{20}$$

We can also use (15) for both agents to write:

$$S_t = S_{t-1}\frac{M_{1,t}}{M_{2,t}}\frac{c_{1,t}/c_{1,t-1}}{c_{2,t}/c_{2,t-1}}. \tag{21}$$

Following the same steps we can write the marginal condition for good $b$ as

$$S_t \frac{\partial G_{1,t}(a_{1,t}, b_{1,t})}{\partial b_{1,t}} \frac{1}{c_{1,t}} = \frac{\partial G_{2,t}(a_{2,t}, b_{2,t})}{\partial b_{2,t}} \frac{1}{c_{2,t}}. \tag{22}$$

## 0.1 Investment terms

TODO: determine why They write C5 and C6 lagged one period. Do we need to do this also? If so, why. If not, why not?