

*About notation* Without loss of generality when I write  $x_t$  I really mean  $x(s^t)$ .  
When the history dependence is potentially unclear, I will it out in longer form.  
The Lagrangian is

$$\begin{aligned} \mathcal{L} = & \mu_{1,0} V_{1,0}(c_{1,0}, V_{1,1}) + \mu_{2,0} V_{2,0}(c_{2,0}, V_{2,1}) \\ & \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \lambda_1(s^t) (Y_1(\chi_{1,t}/g_t, 1) - a_{1,t} - a_{2,t} - i_{1,a,t} - i_{2,a,t}) + \right. \\ & \lambda_2(s^t) (Y_2(\chi_{2,t}/g_t, \xi_t) - b_{1,t} - b_{2,t} - i_{1,b,t} - i_{2,b,t}) + \\ & \lambda_3(s^t) \left( (1 - \delta_1) \frac{\chi_{1,t-1}}{g_{t-1}} + H_1(i_{1,a,t-1}, i_{1,b,t-1}) - \chi_{1,t} \right) + \\ & \left. \lambda_4(s^t) \left( (1 - \delta_2) \frac{\chi_{2,t-1}}{g_{t-1}} + H_2(i_{2,a,t-1}, i_{2,b,t-1}) - \chi_{2,t} \right) \right\}, \end{aligned} \quad (1)$$

where  $c_{i,t} = G_i(a_{i,t}, b_{i,t})$  and  $\chi_{i,t} = g(s^t)k_i(s^t)$  for  $i = 1, 2$  and  $t = 0, 1, \dots, \infty$ .  
The control variables are:  $\{a_n(s^t), b_n(s^t), i_{n,a}(s^t), i_{n,b}(s^t), \chi_n(s^t)\}$  for  $t = 0, 1, \dots, \infty$ ,  $n \in 1, 2$ , and all histories of states  $s^t$ . The FONC are

$$a_1(s^t) : \quad \mu_{1,0} \frac{\partial V_{1,0}}{\partial V_{1,t}} \frac{\partial V_{1,t}}{\partial c_{1,t}} \frac{\partial c_{1,t}}{\partial a_{1,t}} = \lambda_1(s^t) \quad (2)$$

$$b_1(s^t) : \quad \mu_{1,0} \frac{\partial V_{1,0}}{\partial V_{1,t}} \frac{\partial V_{1,t}}{\partial c_{1,t}} \frac{\partial c_{1,t}}{\partial b_{1,t}} = \lambda_2(s^t) \quad (3)$$

$$i_{1,a}(s^t) : \quad \sum_{s^{t+1}} \lambda_3(s^{t+1}) \frac{\partial H_1(i_{1,a}(s^t), i_{1,b}(s^t))}{\partial i_{1,a}(s^t)} = \lambda_1(s^t) \quad (4)$$

$$i_{1,b}(s^t) : \quad \sum_{s^{t+1}} \lambda_3(s^{t+1}) \frac{\partial H_1(i_{1,a}(s^t), i_{1,b}(s^t))}{\partial i_{1,b}(s^t)} = \lambda_2(s^t) \quad (5)$$

$$\chi_1(s^t) : \quad \lambda_1(s^t) \frac{\partial Y_1(\chi_{1,t}/g_t, 1)}{\partial (\chi_{1,t} g_t)} \frac{1}{g_t} + \sum_{s^{t+1}} \frac{(1 - \delta_1)}{g(s^{t+1}|s^t)} \lambda_3(s^{t+1}) = \lambda_3(s^t) \quad (6)$$

$$a_2(s^t) : \quad \mu_{2,0} \frac{\partial V_{2,0}}{\partial V_{2,t}} \frac{\partial V_{2,t}}{\partial c_{2,t}} \frac{\partial c_{2,t}}{\partial a_{2,t}} = \lambda_1(s^t) \quad (7)$$

$$b_2(s^t) : \quad \mu_{2,0} \frac{\partial V_{2,0}}{\partial V_{2,t}} \frac{\partial V_{2,t}}{\partial c_{2,t}} \frac{\partial c_{2,t}}{\partial b_{2,t}} = \lambda_2(s^t) \quad (8)$$

$$i_{2,a}(s^t) : \quad \sum_{s^{t+1}} \lambda_4(s^{t+1}) \frac{\partial H_2(i_{4,a}(s^t), i_{4,b}(s^t))}{\partial i_{4,a}(s^t)} = \lambda_1(s^t) \quad (9)$$

$$i_{2,b}(s^t) : \quad \sum_{s^{t+1}} \lambda_4(s^{t+1}) \frac{\partial H_2(i_{4,a}(s^t), i_{4,b}(s^t))}{\partial i_{4,b}(s^t)} = \lambda_2(s^t) \quad (10)$$

$$\chi_2(s^t) : \quad \lambda_2(s^t) \frac{\partial Y_2(\chi_{2,t}/g_t, 1)}{\partial (\chi_{2,t} g_t)} \frac{1}{g_t} + \sum_{s^{t+1}} \frac{(1 - \delta_1)}{g(s^{t+1}|s^t)} \lambda_4(s^{t+1}) = \lambda_4(s^t) \quad (11)$$

Before proceeding, we will gather some tools we will use while analyzing the above conditions. First, notice that by the chain rule we have

$$\frac{\partial V_{i,0}}{\partial V_{i,t}} = \prod_{\tau=1}^t \frac{\partial V_{i,\tau-1}}{\partial V_{i,\tau}}. \quad (12)$$

The notice that we can write the SDF for agent  $i$  as

$$\begin{aligned} M_{i,t+1} &= \frac{\partial V_{i,t}}{\partial V_{i,t+1}} \frac{\partial V_{i,t+1}/\partial c_{i,t+1}}{\partial V_{i,t}/\partial c_{i,t}} \\ &= \beta \left( \frac{C_{t+1}}{C_t} \right)^{\rho-1} \left( \frac{V_{i,t+1}}{E[V_{i,t+1}^\alpha]^{1/\alpha}} \right)^{\alpha-\rho}. \end{aligned}$$

We will also write stochastic discount

We will make use of the time  $t$  Pareto weight, which we define as

$$\begin{aligned} \mu_{i,t} &= \mu_{i,0} \frac{\partial V_{i,0}}{\partial V_{i,t}} \frac{\partial V_{i,t}}{\partial c_{i,t}} \\ &= \mu_{i,0} \prod_{\tau=1}^t \frac{\partial V_{i,\tau-1}}{\partial V_{i,\tau}} \frac{\partial V_{i,t}}{\partial c_{i,t}} \\ &= \mu_{i,t-1} \frac{\partial V_{i,t}}{\partial V_{i,t+1}} \frac{\partial V_{i,t+1}/\partial c_{i,t+1}}{\partial V_{i,t}/\partial c_{i,t}} \\ &= \mu_{i,t-1} M_{i,t} \end{aligned} \quad (13)$$