

The Lagrangian is

$$\begin{aligned} \mathcal{L} = \mu_{1,0}V_{1,0} + \mu_{2,0}V_{2,0} + \sum_{t=0}^{\infty} E_t \Bigg\{ & \lambda_{1,t} (G_1(a_{1,t}, b_{1,t}) - c_{1,t}) + \\ & \lambda_{2,t} (G_2(a_{2,t}, b_{2,t}) - c_{2,t}) + \\ & \lambda_{3,t} (Y_1(k_{1,t}, 1) - a_{1,t} - a_{2,t} - i_{1,a,t} - i_{2,a,t}) + \\ & \lambda_{4,t} (Y_2(k_{2,t}, \xi) - b_{1,t} - b_{2,t} - i_{1,b,t} - i_{2,b,t}) + \\ & \lambda_{5,t} ((1 - \delta_1)k_{1,t} + H_1(i_{1,a,t}, i_{1,b,t}) - g_{t+1}k_{1,t+1}) + \\ & \lambda_{6,t} ((1 - \delta_2)k_{2,t} + H_2(i_{2,a,t}, i_{2,b,t}) - g_{t+1}k_{2,t+1}) + \\ & \Bigg\} \end{aligned} \quad (1)$$

The FOC are

$$c_{1,t} : \mu_{1,0} \frac{\partial V_{1,0}}{\partial V_{1,t}} \frac{\partial V_{1,t}}{\partial c_{1,t}} = \lambda_{1,t} \quad (2)$$

$$a_{1,t} : \frac{\partial G_1(a_{1,t}, b_{1,t})}{\partial a_{1,t}} \lambda_{1,t} = \lambda_{3,t} \quad (3)$$

$$b_{1,t} : \frac{\partial G_1(a_{1,t}, b_{1,t})}{\partial b_{1,t}} \lambda_{1,t} = \lambda_{4,t} \quad (4)$$

$$i_{1,a,t} : \frac{\partial H_1(i_{1,a,t}, i_{1,b,t})}{\partial i_{1,a,t}} \lambda_{5,t} = \lambda_{3,t} \quad (5)$$

$$i_{1,b,t} : \frac{\partial H_1(i_{1,a,t}, i_{1,b,t})}{\partial i_{1,b,t}} \lambda_{5,t} = \lambda_{4,t} \quad (6)$$

$$c_{2,t} : \mu_{2,0} \frac{\partial V_{2,0}}{\partial V_{2,t}} \frac{\partial V_{2,t}}{\partial c_{2,t}} = \lambda_{2,t} \quad (7)$$

$$a_{2,t} : \frac{\partial G_2(a_{2,t}, b_{2,t})}{\partial a_{2,t}} \lambda_{2,t} = \lambda_{3,t} \quad (8)$$

$$b_{2,t} : \frac{\partial G_2(a_{2,t}, b_{2,t})}{\partial b_{2,t}} \lambda_{2,t} = \lambda_{4,t} \quad (9)$$

$$i_{2,a,t} : \frac{\partial H_2(i_{2,a,t}, i_{2,b,t})}{\partial i_{2,a,t}} \lambda_{6,t} = \lambda_{3,t} \quad (10)$$

$$i_{2,b,t} : \frac{\partial H_2(i_{2,a,t}, i_{2,b,t})}{\partial i_{2,b,t}} \lambda_{6,t} = \lambda_{4,t} \quad (11)$$

The state variables

We can combine (2) and (3) to obtain

$$\mu_{1,0} \frac{\partial V_{1,0}}{\partial V_{1,t}} \frac{\partial V_{1,t}}{\partial c_{1,t}} \frac{\partial G_1(a_{1,t}, b_{1,t})}{\partial a_{1,t}} = \lambda_{3,t}. \quad (12)$$

Similarly, (7) and (8) give

$$\mu_{2,0} \frac{\partial V_{2,0}}{\partial V_{2,t}} \frac{\partial V_{2,t}}{\partial c_{2,t}} \frac{\partial G_2(a_{2,t}, b_{2,t})}{\partial a_{2,t}} = \lambda_{3,t}. \quad (13)$$

Together (12) and (13) give

$$\frac{\mu_{1,0}}{\mu_{2,0}} = \frac{\frac{\partial V_{2,0}}{\partial V_{2,t}} \frac{\partial V_{2,t}}{\partial c_{2,t}} \frac{\partial G_2(a_{2,t}, b_{2,t})}{\partial a_{2,t}}}{\frac{\partial V_{1,0}}{\partial V_{1,t}} \frac{\partial V_{1,t}}{\partial c_{1,t}} \frac{\partial G_1(a_{1,t}, b_{1,t})}{\partial a_{1,t}}}. \quad (14)$$

Now we define an ad hoc time  $t$  Pareto weight:

$$\begin{aligned} \mu_{i,t} &= \mu_{i,0} \frac{\partial V_{i,0}}{\partial V_{i,t}} \frac{\partial V_{i,t}}{\partial c_{i,t}} c_{i,t} \\ &= \mu_{i,t-1} \frac{\partial V_{i,t-1}}{\partial V_{i,t}} \frac{\partial V_{i,t}/\partial c_{i,t}}{\partial V_{i,t-1}/\partial c_{i,t-1}} \frac{c_{i,t}}{c_{i,t-1}} \end{aligned} \quad (15)$$

$$= \mu_{i,t-1} M_t \frac{c_{i,t}}{c_{i,t-1}} \quad (16)$$

where we have used the chain rule to write  $\frac{\partial V_{i,0}}{\partial V_{i,t}} = \prod_{\tau=1}^t \frac{\partial V_{i,\tau-1}}{\partial V_{i,\tau}}$  and have used

$$M_{i,t+1} = \frac{\partial V_{i,t}}{\partial V_{i,t+1}} \frac{\partial V_{i,t+1}/\partial c_{i,t+1}}{\partial V_{i,t}/\partial c_{i,t}} \quad (17)$$

$$= \beta \left( \frac{c_{i,t+1}}{c_{i,t}} \right)^{\rho-1} \left( \frac{V_{i,t+1}}{E[V_{i,t+1}^\alpha]^\alpha} \right)^{\alpha-\rho} \quad (18)$$

We can combine (15) with (14) to get

$$\frac{1}{c_{1,t}} \mu_{1,t} \frac{\partial G_{1,t}(a_{1,t}, b_{1,t})}{\partial a_{1,t}} = \frac{1}{c_{2,t}} \mu_{2,t} \frac{\partial G_{2,t}(a_{2,t}, b_{2,t})}{\partial a_{2,t}} \quad (19)$$

Define  $S_t = \frac{\mu_{1,t}}{\mu_{2,t}}$  and write (19) as

$$S_t \frac{\partial G_{1,t}(a_{1,t}, b_{1,t})}{\partial a_{1,t}} \frac{1}{c_{1,t}} = \frac{\partial G_{2,t}(a_{2,t}, b_{2,t})}{\partial a_{2,t}} \frac{1}{c_{2,t}}. \quad (20)$$

We can also use (15) for both agents to write:

$$S_t = S_{t-1} \frac{M_{1,t}}{M_{2,t}} \frac{c_{1,t}/c_{1,t-1}}{c_{2,t}/c_{2,t-1}}. \quad (21)$$

Following the same steps we can write the marginal condition for good  $b$  as

$$S_t \frac{\partial G_{1,t}(a_{1,t}, b_{1,t})}{\partial b_{1,t}} \frac{1}{c_{1,t}} = \frac{\partial G_{2,t}(a_{2,t}, b_{2,t})}{\partial b_{2,t}} \frac{1}{c_{2,t}}. \quad (22)$$

### 0.1 Investment terms

TODO: determine why They write C5 and C6 lagged one period. Do we need to do this also? If so, why. If not, why not?