1 Model

The unscaled model can be written as

$$
\begin{aligned}
J(k_1,k_2,z_1,z_2,U) &= \max_{c_1,c_2,I_1,I_2,U'} \left[(1-\beta_1)c_1^{\rho_1} + \beta_1\mu(J')^{\rho}\right]^{\frac{1}{\rho_1}} \\
\text{subject to } &\left[(1-\beta_2)c_2^{\rho_2} + \beta_2\mu(U')^{\rho}\right]^{\frac{1}{\rho_2}} \geq U \\
c_1+c_2+I_1+I_2 &= f_1(k_1,z_1) + f_2(k_2,z_2) \\
k_1' &= \Gamma_1(k_1,I_1) \\
k_2' &= \Gamma_2(k_2,I_2)
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(k_1,z_1) &= \left[(1-\eta_1)k_1^{\nu_1} + \eta_1 z_1^{\nu_1}]\right]^{\frac{1}{\nu_1}} \\
f_2(k_2,z_2) &= \left[(1-\eta_2)k_2^{\nu_2} + \eta_2 z_2^{\nu_2}]\right]^{\frac{1}{\nu_2}} \\
\Gamma_1(k_1,I_1) &= \left[(1-\delta)k_1^{\gamma_1} + \delta I_1^{\gamma_1}\right]^{\frac{1}{\gamma_1}} \\
\Gamma_2(k_2,I_2) &= \left[(1-\delta)k_2^{\gamma_2} + \delta I_2^{\gamma_2}\right]^{\frac{1}{\gamma_2}}
\end{aligned}
$$

Note: For $\gamma_i = 1$ and $I_i = 0$ that we get $k_{t+1} = (1-\delta)k_t$ which is the standard rate of depreciation.

If we then scale the model by z_1 and define $\xi := \frac{z_2}{z_1}$, we get[1]

$$
\begin{aligned}
J(k_1,k_2,\xi,U) &= \max_{c_1,c_2,I_1,I_2,U'} \left[(1-\beta_1)c_1^{\rho_1} + \beta_1\mu(g'J')^{\rho}\right]^{\frac{1}{\rho_1}} \\
\text{subject to } &\left[(1-\beta_2)c_2^{\rho_2} + \beta_2\mu(g'U')^{\rho}\right]^{\frac{1}{\rho_2}} \geq U \quad (\lambda_P) \\
c_1+c_2+I_1+I_2 &= f_1(k_1) + f_2(k_2,\xi) \quad (\lambda_{BC}) \\
g'k_1' &= \Gamma_1(k_1,I_1) \quad (\lambda_1) \\
g'k_2' &= \Gamma_2(k_2,I_2) \quad (\lambda_2)
\end{aligned}
$$

[1] I'm going to use same variable names, but everything in sight is scaled by z_1

2 First Order Conditions

$$
\begin{aligned}
c_1 &: J^{1-\rho_1}(1-\beta_1)c_1^{\rho_1-1} - \lambda_{BC} = 0\\
c_2 &: \lambda_p U^{1-\rho_2}(1-\beta_2)c_2^{\rho_2-1} - \lambda_{BC} = 0\\
I_1 &: J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} E\left[(g'J')^{\alpha_1-1}\frac{\partial \Gamma_1}{\partial I_1}J'_{k_1}\right] - \lambda_{BC} = 0\\
I_2 &: J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} E\left[(g'J')^{\alpha_1-1}\frac{\partial \Gamma_2}{\partial I_2}J'_{k_2}\right] - \lambda_{BC} = 0\\
U'(\text{state}') &: \lambda_p U^{1-\rho_2}\beta_2 E\left[(g'U')^{\alpha_2}\right]^{\frac{\rho_2-\alpha_2}{\alpha_2}} g'^{\alpha_2}U'^{\alpha_2-1} + J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} g'^{\alpha_1}J'^{\alpha_1-1}J'_u = 0
\end{aligned}
$$

3 Envelope Condition

$$
\begin{aligned}
J_U &= -\lambda_p\\
J_{k1} &= \frac{\partial f_1}{\partial k_1}\lambda_{BC} + J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} E\left[(g'J')^{\alpha_1-1}\frac{\partial \Gamma_1}{\partial k_1}J'_{k_1}\right]\\
J_{k2} &= \frac{\partial f_2}{\partial k_2}\lambda_{BC} + J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} E\left[(g'J')^{\alpha_1-1}\frac{\partial \Gamma_2}{\partial k_2}J'_{k_2}\right]
\end{aligned}
$$

4 ECM

We will use the FOC in c_1 and c_2 together with the envelope for J_{k_1} and J_{k_2} to solve for the optimal c_1, c_2 in closed form:

$$
\begin{aligned}
c_1 &= \left(\frac{J_{k_1} - J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} E\left[(g'J')^{\alpha_1-1}\frac{\partial \Gamma_1}{\partial k_1}J'_{k_1}\right]}{J^{1-\rho_1}\frac{\partial f_1}{\partial k_1}(1-\beta_1)}\right)^{\frac{1}{\rho_1-1}}\\
c_2 &= \left(\frac{J_{k_2} - J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} E\left[(g'J')^{\alpha_1-1}\frac{\partial \Gamma_2}{\partial k_2}J'_{k_2}\right]}{\lambda_p U^{1-\rho_2}\frac{\partial f_2}{\partial k_2}(1-\beta_2)}\right)^{\frac{1}{\rho_2-1}}
\end{aligned}
$$

Or we could use FOC c_1 and env J_{k_1} to get the same c_1 as above and use FOC c_1 with FOC c_2 to get c_2:

$$
c_2 = \left(\frac{\lambda_p U^{1-\rho_2}(1-\beta_2)}{J^{1-\rho_1}(1-\beta_1)c_1^{\rho_1-1}}\right)^{\frac{1}{\rho_2-1}}
$$