1 Envelope Method

Original problem is

$$J_t(k_t, x_t) = \max_{\chi_t} \left[(1-\beta)c_t^\rho + \beta\mu(J_{t+1}(k_{t+1}, x_{t+1})g_{t+1})^\rho\right]^{\frac{1}{\rho}}$$

Subject To

$$\begin{aligned}
c_t &= [\eta k_t^\nu + (1-\eta)]^{\frac{1}{\nu}} + (1-\delta)k_t - \chi_t \\
\chi_t &= k_{t+1}g_{t+1} \\
x_{t+1} &= Ax_t + B\bar{b}^{\frac{1}{2}}\varepsilon_{1,t+1} \\
\log(g_t) &= \log(z_t) - \log(z_{t-1}) = \log(\bar{g}) + x_t
\end{aligned}$$

We can take first order conditions to get

$$(1-\beta)c_t^{\rho-1} = \beta\mu(J_{t+1}g_{t+1})^{\rho-\alpha}E_t\left[(g_{t+1}J_{t+1})^{\alpha-1}\frac{\partial J_{t+1}}{\partial k_{t+1}}\right]$$

The envelope condition reveals

$$\frac{\partial J_t(k_t, x_t)}{\partial k_t} = J_t^{1-\rho}(1-\beta)c_t^{\rho-1}(\eta y_t^{1-\nu}k_t^{\nu-1} + 1 - \delta)$$

Notice for any pair (k_t, x_t) we can solve the envelope condition to get

$$\begin{aligned}
c_t &= \left(\frac{J_{k,t}(k_t, x_t)}{(1-\beta)J_t^{1-\rho}(\eta y_t^{1-\nu}k_t^{\nu-1} + 1 - \delta)}\right)^{\frac{1}{\rho-1}} \\
\rightarrow \chi_t &= y_t + (1-\delta)k_t - c_t \\
\chi_t &= y_t + (1-\delta)k_t - \left(\frac{J_{k,t}(k_t, x_t)}{(1-\beta)J_t^{1-\rho}(\eta y_t^{1-\nu}k_t^{\nu-1} + 1 - \delta)}\right)^{\frac{1}{\rho-1}}
\end{aligned}$$