1 Model

The unscaled model can be written as

$$
\begin{aligned}
J(k_1, k_2, z_1, z_2, U) &= \max_{c_1, c_2, I_1, I_2, U'} \left[(1-\beta_1)c_1^{\rho_1} + \beta_1 \mu (J')^{\rho}\right]^{\frac{1}{\rho_1}} \\
\text{subject to } &\left[(1-\beta_2)c_2^{\rho_2} + \beta_2 \mu (U')^{\rho}\right]^{\frac{1}{\rho_2}} \geq U \\
c_1 + c_2 + I_1 + I_2 &= f_1(k_1, z_1) + f_2(k_2, z_2) \\
k_1' &= \Gamma_1(k_1, I_1) \\
k_2' &= \Gamma_2(k_2, I_2)
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(k_1, z_1) &= \left[(1-\eta_1)k_1^{\nu_1} + \eta_1 z_1^{\nu_1}]\right]^{\frac{1}{\nu_1}} \\
f_2(k_2, z_2) &= \left[(1-\eta_2)k_2^{\nu_2} + \eta_2 z_2^{\nu_2}]\right]^{\frac{1}{\nu_2}} \\
\Gamma_1(k_1, I_1) &= \left[(1-\delta)k_1^{\gamma_1} + \delta I_1^{\gamma_1}\right]^{\frac{1}{\gamma_1}} \\
\Gamma_2(k_2, I_2) &= \left[(1-\delta)k_2^{\gamma_2} + \delta I_2^{\gamma_2}\right]^{\frac{1}{\gamma_2}}
\end{aligned}
$$

Note: For $\gamma_i = 1$ and $I_i = 0$ that we get $k_{t+1} = (1-\delta)k_t$ which is the standard rate of depreciation.

If we then scale the model by z_1 and define $\xi := \frac{z_2}{z_1}$, we get[1]

$$
\begin{aligned}
J(k_1, k_2, \xi, U) &= \max_{c_1, c_2, I_1, I_2, U'} \left[(1-\beta_1)c_1^{\rho_1} + \beta_1 \mu (g' J')^{\rho}\right]^{\frac{1}{\rho_1}} \\
\text{subject to } &\left[(1-\beta_2)c_2^{\rho_2} + \beta_2 \mu (g' U')^{\rho}\right]^{\frac{1}{\rho_2}} \geq U \quad (\lambda_P) \\
c_1 + c_2 + I_1 + I_2 &= f_1(k_1) + f_2(k_2, \xi) \quad (\lambda_{BC}) \\
g' k_1' &= \Gamma_1(k_1, I_1) \quad (\lambda_1) \\
g' k_2' &= \Gamma_2(k_2, I_2) \quad (\lambda_2)
\end{aligned}
$$

[1] I'm going to use same variable names, but everything in sight is scaled by z_1

2 First Order Conditions

$$
\begin{aligned}
c_1 &: J^{1-\rho_1}(1-\beta_1)c_1^{\rho_1-1} - \lambda_{BC} = 0 \\
c_2 &: \lambda_p U^{1-\rho_2}(1-\beta_2)c_2^{\rho_2-1} - \lambda_{BC} = 0 \\
I_1 &: J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} E\left[(g'J')^{\alpha_1-1}\frac{\partial\Gamma_1}{\partial I_1}J'_{k_1}\right] - \lambda_{BC} = 0 \\
I_2 &: J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} E\left[(g'J')^{\alpha_1-1}\frac{\partial\Gamma_2}{\partial I_2}J'_{k_2}\right] - \lambda_{BC} = 0 \\
U'(\text{state}') &: \lambda_p U^{1-\rho_2}\beta_2 E\left[(g'U')^{\alpha_2}\right]^{\frac{\rho_2-\alpha_2}{\alpha_2}} g'^{\alpha_2}U'^{\alpha_2-1} + J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} g'^{\alpha_1}J'^{\alpha_1-1}J'_u = 0
\end{aligned}
$$

3 Envelope Condition

$$
\begin{aligned}
J_U &= -\lambda_p \\
J_{k1} &= \frac{\partial f_1}{\partial k_1}\lambda_{BC} + J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} E\left[(g'J')^{\alpha_1-1}\frac{\partial\Gamma_1}{\partial k_1}J'_{k_1}\right] \\
J_{k2} &= \frac{\partial f_2}{\partial k_2}\lambda_{BC} + J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} E\left[(g'J')^{\alpha_1-1}\frac{\partial\Gamma_2}{\partial k_2}J'_{k_2}\right]
\end{aligned}
$$

4 ECM

We will use the FOC in c_1 and c_2 together with the envelope for J_{k_1} and J_{k_2} to solve for the optimal c_1, c_2 in closed form (note we can pull $\frac{\partial\Gamma_i}{\partial k_i}$ out of the expectaiton):

$$
c_1 = \left(\frac{J_{k_1} - J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} \frac{\partial\Gamma_1}{\partial k_1} E\left[(g'J')^{\alpha_1-1}J'_{k_1}\right]}{J^{1-\rho_1}\frac{\partial f_1}{\partial k_1}(1-\beta_1)}\right)^{\frac{1}{\rho_1-1}} \tag{1}
$$

$$
c_2 = \left(\frac{J_{k_2} - J^{1-\rho_1}\beta_1 E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}} \frac{\partial\Gamma_2}{\partial k_2} E\left[(g'J')^{\alpha_1-1}J'_{k_2}\right]}{\lambda_p U^{1-\rho_2}\frac{\partial f_2}{\partial k_2}(1-\beta_2)}\right)^{\frac{1}{\rho_2-1}} \tag{2}
$$

$$
\tag{3}
$$

Or we could use FOC c_1 and env J_{k_1} to get the same c_1 as above and use FOC c_1 with FOC c_2 to get c_2:

$$
c_2 = \left(\frac{\lambda_p U^{1-\rho_2}(1-\beta_2)}{J^{1-\rho_1}(1-\beta_1)c_1^{\rho_1-1}}\right)^{\frac{1}{\rho_2-1}}
$$

The we can combine the I_1 and I_2 FOC to obtain

$$E\left[(g'J')^{\alpha_1-1}\frac{\partial\Gamma_1}{\partial I_1}J'_{k_1}\right]=E\left[(g'J')^{\alpha_1-1}\frac{\partial\Gamma_2}{\partial I_2}J'_{k_2}\right] \tag{4}$$

$$\frac{\partial\Gamma_1}{\partial I_1}E\left[(g'J')^{\alpha_1-1}J'_{k_1}\right]=\frac{\partial\Gamma_2}{\partial I_2}E\left[(g'J')^{\alpha_1-1}J'_{k_2}\right] \tag{5}$$

$$\tag{6}$$

We can use this equation to solve for either I_1 or I_2 and use the budget constraint to get the other one.

Now we are just left with the FOC wrt $U'(\text{state}')$ for choosing U' state by state for the next period. There are at least two approaches to doing this:

4.1 Lars' Style

Define

$$\begin{aligned}a&=\frac{E\left[(g'U')^{\alpha_2}\right]^{\frac{\rho_2-\alpha_2}{\alpha_2}}}{E\left[(g'J')^{\alpha_1}\right]^{\frac{\rho_1-\alpha_1}{\alpha_1}}}\\&=\frac{\mu_2(g'U')^{\rho_2-\alpha_2}}{\mu_1(g'J')^{\rho_1-\alpha_1}}\end{aligned}$$

Then write the U' FOC as

$$\lambda_p\frac{U^{1-\rho_2}}{J^{1-\rho_1}}\frac{\beta_2}{\beta_1}ag'^{\alpha_2-\alpha_1}=\frac{J'^{\alpha_1-1}}{U'^{\alpha_1-1}}(-J'_u)$$

We can then start with a guess for a, use it to solve for U' at each possible state tomorrow, update the value of a and iterate until convergence.

This this nice because it allows us to solve a series of univariate optimization problems and enables us to use uber-robust bracketing methods to compute t

4.2 System

An alternative to this approach is to solve for I_1, U'(state') as a system for every point in the current period state space. The approach to doing this would be

- Start with a guess for I_1 and U'(state')
- Use (1) to solve for c_1, c_2
- Given c_1, c_2, I_1, use the budget constraint to solve for I_2.
- Use (4) and the FOC for U' to provide the solver a system of $1+n_{\text{state}'}$ residuals for I_1, U'(state')

- After the solver converges, use the same routines called internally to produce the optimal value of c_1, c_2, I_2.

There are some advantages to this approach:

- We can leverage automatic differentiation routines to provide analytical derivatives to the solver. This means we can leverage derivative information in the solver and hopefully achieve faster than quadratic convergence (which is the best we can do with our current bracketing methods `brent`)
- It is simpler to avoid duplicating computation. We can simply compute things like expectations and interpolands once and re-use them in solving for c_1, c_2, I_2 and in computing the residuals.
- We avoid the complications associated with the additional variable a

There might be some dis-advantages also:

- We can't use super robust bracketing methods. like we did with the sequence of univariate problems