Optim example: Solve HJB equation

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June 20, 2017

1 Problem overview

This document explains the mathematical problem solved by the Optim example at `https://gist.github.com/anriseth/d27f51d2e72a937f5752bb9fc0c0fdbb`.

The example file finds the function the function $v : [0, T] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ that solves the Hamilton-Jacobi-Bellman equation given by

$$v_t + \sup_{a \in A} \left\{ \frac{\gamma^2}{2} q(a)^2 v_{xx} - q(a) v_x + a q(a) \right\} = 0, \tag{1}$$

$$v(T, x) = -Cx, \tag{2}$$

$$v(t, 0) = 0. \tag{3}$$

In the file, $A = [0, 1]$, $q(a) = 1 - a$, $C = 1$, and $\gamma = 5 \times 10^{-2}$. The PDE is discretised and solved backwards in time using a Semi-Lagrangian scheme for diffusion problems, as described in [1]. We use Optim to solve the optimisation problem in this nonlinear PDE, and conceptually what we do at a given time t, is to find $v(t - \Delta t, x)$ from the optimisation

$$v(t - \Delta t, x) = v(t, x) + \Delta t \sup_{a \in A} \left\{ \frac{\gamma^2}{2} q(a)^2 v_{xx} - q(a) v_x + a q(a) \right\}. \tag{4}$$

Thus, for each time step and each point on the spatial grid, we must solve a constrained optimisation problem. Optim works very well for this purpose.

The file in the URL above approximates $v(t, x)$ and prints that, as well as the control function $a(t, x)$, given by

$$a(t, x) = \arg\max_{a \in A} \left\{ \frac{\gamma^2}{2} q(a)^2 v_{xx} - q(a) v_x + a q(a) \right\}. \tag{5}$$

1.1 Problem motivation

The PDE above arises from a stochastic optimal control problem. We wish to dynamically set the price of a product, in order to maximise the expected profit accrued from the revenue over a period $[0, T]$ minus the cost of handling unsold stock at time T. Let $X^\alpha(t)$ denote the amount of stock at time t, following a pricing strategy $\alpha(t)$. Given an expected demand function $q(a)$, and a Brownian motion $W(t)$, the stock evolves according to the SDE

$$dX^\alpha(t) = -q(\alpha(t))(dt + \gamma dW(t)), \qquad \text{when } X^\alpha(t) > 0. \tag{6}$$

We wish to maximise profit by finding pricing strategies that decides the price at time t, given the amount of stock left at this time, $X^\alpha(t)$. Call $\mathcal{A}$ the set of processes of the form $\alpha(t) = a(t, X^\alpha(t))$ for measurable functions $a(t,x)$ taking values in A. Let $C > 0$ denote the cost per unit unsold stock, and let $T^s \leq T$ be the hitting time for $X^\alpha(t) = 0$. The stochastic optimal control problem is thus to find an $\alpha(t)$ that solves

$$\max_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^{T^s} \alpha(t) q(\alpha(t)) dt - C X^\alpha(T)\right]. \tag{7}$$

This is an infinite-dimensional optimisation problem, but it turns out that one can find α from the function $a(t,x)$ in (5). For a reference, see [2].

Final note. The model for the uncertainty in the evolution of stock in (6) does not really make sense, as it allows negative sales with probability close to 0.5 over small timesteps.

References

[1] Kristian Debrabant and Espen Jakobsen. Semi-lagrangian schemes for linear and fully non-linear diffusion equations. *Mathematics of Computation*, 82(283):1433–1462, 2013.

[2] Huyên Pham. *Continuous-time stochastic control and optimization with financial applications*, volume 61. Springer Science & Business Media, 2009.