I.A Min-Max Problem

We can define

$$x = g(t) = (\frac{b-a}{2})t + (\frac{a+b}{2}) \tag{1}$$
$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \tag{2}$$

where $t \in [-1, 1]$ and $x \in [a, b]$, then

$$f(x) = f(g(t)) = a_0g^n(t) + a_1g(t)^{n-1} + \cdots + a_{n-1}g(t) + a_n \tag{3}$$
$$= a_0[(\frac{b-a}{2})t + \frac{a+b}{2}]^n + a_1[(\frac{b-a}{2})t + \frac{a+b}{2}]^{n-1} + \cdots + a_{n-1}[(\frac{b-a}{2})t + \frac{a+b}{2}] + a_n \tag{4}$$
$$= b_0t^n + b_1t^{n-1} + \cdots + b_{n-1}t + b_n \tag{5}$$

where $b_0 = a_0[\frac{(b-a)}{2}]^n$, so

$$\min \max_{x\in[a,b]} |f(x)| = \min \max_{t\in[-1,1]} |b_0t^n + b_1t^{n-1} + \cdots + b_{n-1}t + b_n| \tag{6}$$
$$= |b_0| \min \max_{t\in[-1,1]} |t^n + \frac{b_1}{b_0}t^{n-1} + \cdots + \frac{b_{n-1}}{b_0}t + \frac{b_n}{b_0}| \tag{7}$$

By Corollary 3.33,

$$\max_{t\in[-1,1]} |t^n + \frac{b_1}{b_0}t^{n-1} + \cdots + \frac{b_{n-1}}{b_0}t + \frac{b_n}{b_0} \geq \frac{1}{2^{n-1}} \tag{8}$$

namely,

$$|b_0| \min \max_{t\in[-1,1]} |t^n + \frac{b_1}{b_0}t^{n-1} + \cdots + \frac{b_{n-1}}{b_0}t + \frac{b_n}{b_0}| = \frac{|b_0|}{2^{n-1}}$$

From above, we can draw the conclusion safely

$$\min \max_{x\in[a,b]} |f(x)| = |a_0|\frac{(b-a)^n}{2^{2n-1}}$$

II.Imitate the Proof of Cheybyshev Theorem

Assume $\exists p(x) \in \overline{\mathbb{P}}^n$, such that

$$\max_{x\in[-1,1]} p(x) < \max_{x\in[-1,1]} \frac{T_n(x)}{T_n(a)} \tag{9}$$

Following Definition 3.28, it is obvious that

$$\max_{x\in[-1,1]} T_n(x) = 1 \tag{10}$$
$$\min_{x\in[-1,1]} T_n(x) = 1 \tag{11}$$

Take (10) and (11) into (9), then we get

$$\max_{x\in[-1,1]} p(x) \leq \max_{x\in[-1,1]} |\frac{1}{T_n(a)}| \tag{12}$$

We can define

$$Q(x) = \frac{T_n(x)}{T_n(a)} - p(x) \tag{13}$$

In particular, the property of $Q(x)$ worth noticing is that

$$Q(a) = \frac{T_n(a)}{T_n(a)} - p(a) = 0$$

By Theorem 3.31,

$$Q(x_k') = \frac{(-1)^k}{T_n(a)} - p(x_k') \tag{14}$$

The sigh of the sequence $\{Q(x_i')\}_{i=1}^{n+1}$ is alternating, which means there are n zeros of $Q(x)$ at least on $[a, b]$. Besides, there is another zero of $Q(x)$ on $x = a > 1$. Therefore, the degree of $Q(x)$ is $n+1$ at least. Howerver, the degree of $T_n(x)$ and $p(x)$ is both n, which implies $Q(x) = 0$, otherwise, it contrasts with the corollary we just get. The assumption is flase. Hence prove.

III.Programming

b.Runge Phenomenon

As the degree of interpolating polynomial increases, $\max|f(x) - p(x)|$ is also increses rapidly, especially near the beginning and ending points.

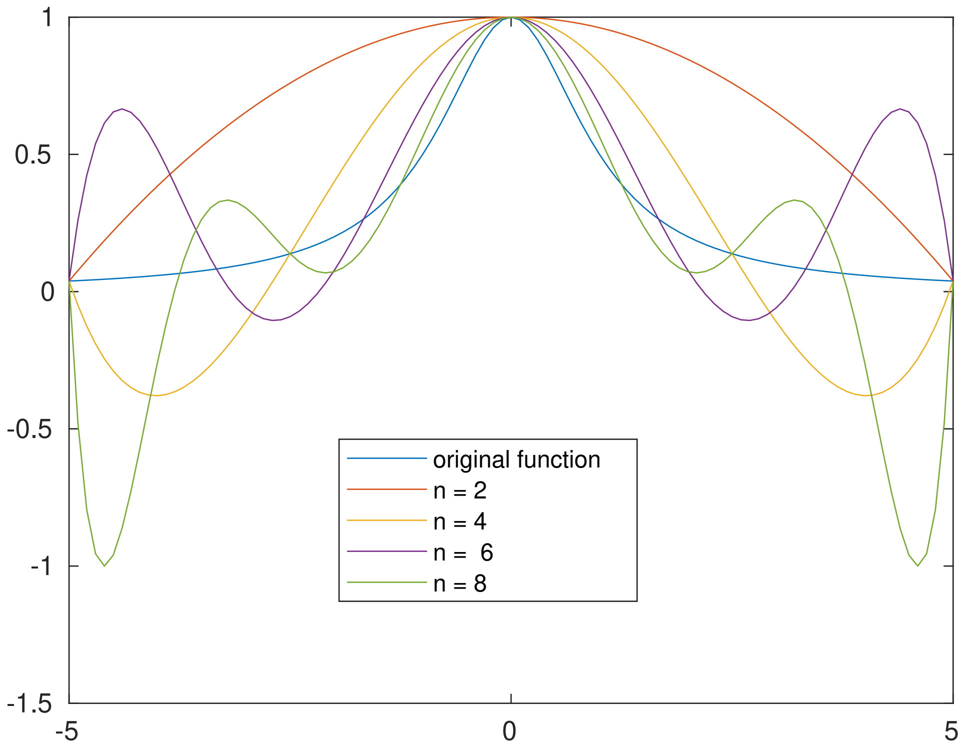

Figure 1: Newton Interpolation

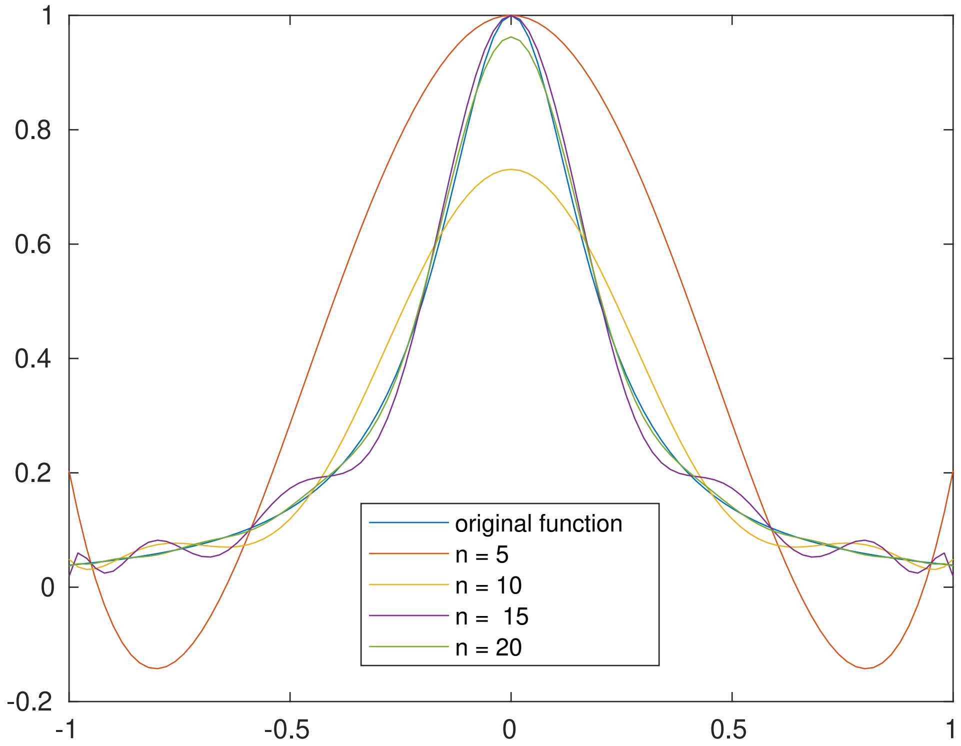

Figure 2: Chebyshev Interpolation

c.Chebyshev Interpolation

We can induce the following conclusion by Figure 2,

$$\lim_{n\to\infty} \sup_{x\in[-1,1]} |f(x) - p_n(x)| = 0 \tag{15}$$

which implies the $\{p_n(x)\}$ converge to $f(x)$ uniformly.