Chapter 3

Polynomial Interpolation

Definition 3.1. *Interpolation* constructs new data points within the range of a discrete set of known data points, usually by generating an *interpolating function* whose graph goes through all known data points.

Example 3.1. The interpolating function may be piecewise constant, piecewise linear, polynomial, spline, or other non-polynomial functions.

3.1 The Vandermonde determinant

Definition 3.2. For $n+1$ given points $x_0, x_1, \ldots, x_n \in \mathbb{R}$, the associated *Vandermonde matrix* $V \in \mathbb{R}^{(n+1)\times(n+1)}$ is

$$V(x_0, x_1, \ldots, x_n) = \begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix}. \tag{3.1}$$

Lemma 3.3. The determinant of a Vandermonde matrix can be expressed as

$$\det V(x_0, x_1, \ldots, x_n) = \prod_{i>j}(x_i - x_j). \tag{3.2}$$

Proof. Consider the function

$$\begin{aligned} U(x) &= \det V(x_0, x_1, \ldots, x_{n-1}, x) \\ &= \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^n \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix}. \end{aligned} \tag{3.3}$$

Clearly, $U(x) \in \mathbb{P}_n$ and it vanishes at $x_0, x_1, \ldots, x_{n-1}$ since inserting these values in place of x yields two identical rows in the determinant. It follows that

$$U(x_0, x_1, \ldots, x_{n-1}, x) = A \prod_{i=0}^{n-1}(x - x_i),$$

where A depends only on $x_0, x_1, \ldots, x_{n-1}$. Meanwhile, the expansion of $U(x)$ in (3.3) by minors of its last row implies that the coefficient of x^n is $U(x_0, x_1, \ldots, x_{n-1})$. Hence we have

$$U(x_0, x_1, \ldots, x_{n-1}, x) = U(x_0, x_1, \ldots, x_{n-1}) \prod_{i=0}^{n-1}(x - x_i),$$

and consequently the recursion

$$U(x_0, x_1, \ldots, x_{n-1}, x_n) = U(x_0, x_1, \ldots, x_{n-1}) \prod_{i=0}^{n-1}(x_n - x_i).$$

An induction based on $U(x_0, x_1) = x_1 - x_0$ yields (3.2). □

Theorem 3.4. Given distinct points $x_0, x_1, \ldots, x_n \in \mathbb{C}$ and corresponding values $f_0, f_1, \ldots, f_n \in \mathbb{C}$. Denote by $\mathbb{P}_n$ the class of polynomials of degree at most n. There exists a unique polynomial $p_n(x) \in \mathbb{P}_n$ such that

$$\forall i = 0, 1, \ldots, n, \qquad p_n(x_i) = f_i. \tag{3.4}$$

Proof. Set up a polynomial $\sum_{i=0}^n a_i x^i$ with $n+1$ undetermined coefficients a_i. The condition (3.4) leads to the system of $n+1$ equations:

$$a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_n x_i^n = f_i,$$

where $i = 0, 1, \ldots, n$. By Lemma 3.3, the determinant of the system is $\prod_{i>j}(x_i - x_j)$. The proof is completed by the distinctness of the points and Cramer's rule. □

3.2 The Cauchy remainder

Theorem 3.5 (Generalized Rolle)**.** Let $n \geq 2$. Suppose that $f \in \mathcal{C}^{n-1}[a,b]$ and $f^{(n)}(x)$ exists at each point of (a,b). Suppose that $f(x_0) = f(x_1) = \cdots = f(x_n) = 0$ for $a \leq x_0 < x_1 < \cdots < x_n \leq b$. Then there is a point $\xi \in (x_0, x_n)$ such that $f^{(n)}(\xi) = 0$.

Proof. Applying Rolle's theorem (Theorem 0.34) on the n intervals (x_i, x_{i+1}) yields n points ζ_i where $f'(\zeta_i) = 0$. Consider $f', f'', \ldots, f^{(n-1)}$ as new functions. Repeatedly applying the above arguments completes the proof. □

Theorem 3.6 (Cauchy remainder of polynomial interpolation)**.** Let $f \in \mathcal{C}^n[a,b]$ and suppose that $f^{(n+1)}(x)$ exists at each point of (a,b). Let $p_n(f;x)$ denote the unique polynomial in $\mathbb{P}_n$ that coincides with f at $x_0, x_1, \ldots, x_n$. Define

$$R_n(f;x) := f(x) - p_n(f;x) \tag{3.5}$$

as the *Cauchy remainder of the polynomial interpolation.* If $a \le x_0 < x_1 < \cdots < x_n \le b$, then there exists some $\xi \in (a, b)$ such that

$$R_n(f;x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x - x_i) \tag{3.6}$$

where the value of ξ depends on $x, x_0, x_1, \ldots, x_n$, and f.

Proof. Since $f(x_k) = p_n(f;x_k)$, the remainder $R_n(f;x)$ vanishes at x_k's. Fix $x \ne x_0, x_1, \ldots, x_n$ and define

$$K(x) = \frac{f(x) - p_n(f;x)}{\prod_{i=0}^{n}(x - x_i)}$$

and a function of t

$$W(t) = f(t) - p_n(f;t) - K(x)\prod_{i=0}^{n}(t - x_i).$$

The function $W(t)$ vanishes at $t = x_0, x_1, \ldots, x_n$. In addition $W(x) = 0$. By Theorem 3.5, $W^{(n+1)}(\xi) = 0$ for some $\xi \in (a, b)$, i.e.

$$0 = W^{(n+1)}(\xi) = f^{(n+1)}(\xi) - (n+1)!K(x).$$

Hence $K(x) = f^{(n+1)}(\xi)/(n+1)!$ and (3.6) holds. □

Corollary 3.7. Suppose $f(x) \in \mathcal{C}^{n+1}[a,b]$. Then

$$|R_n(f;x)| \le \frac{M_{n+1}}{(n+1)!}\prod_{i=0}^{n}|x - x_i| < \frac{M_{n+1}}{(n+1)!}(b-a)^{n+1}, \tag{3.7}$$

where $M_{n+1} = \max_{x\in[a,b]}\left|f^{(n+1)}(x)\right|$.

Example 3.2. A value for arcsin(0.5335) is obtained by interpolating linearly between the values for $x = 0.5330$ and $x = 0.5340$. Estimate the error committed.

Let $f(x) = \arcsin(x)$. Then

$$f''(x) = x(1-x^2)^{-\frac{3}{2}}, \qquad f'''(x) = (1+2x^2)(1-x^2)^{-\frac{5}{2}}.$$

Since the third derivative is positive over $[0.5330, 0.5340]$. The maximum value of f'' occurs at 0.5340. By Corollary 3.7 we have $|R_1| \le 4.42 \times 10^{-7}$. The true error is about 1.10×10^{-7}.

3.3 The Lagrange formula

Definition 3.8. To interpolate given values $f_0, f_1, \ldots, f_n$ at distinct points $x_0, x_1, \ldots, x_n$, the *Lagrange formula* is

$$p_n(x) = \sum_{k=0}^{n} f_k \ell_k(x), \tag{3.8}$$

where the *fundamental polynomial for pointwise interpolation* (or *elementary Lagrange interpolation polynomial*) $\ell_k(x)$ is

$$\ell_k(x) = \prod_{i\ne k; i=0}^{n}\frac{x - x_i}{x_k - x_i}. \tag{3.9}$$

In particular, for $n = 0$, $\ell_0 = 1$.

Example 3.3. For $i = 0, 1, 2$, we are given $x_i = 1, 2, 4$ and $f(x_i) = 8, 1, 5$, respectively. The Lagrangian formula generates $p_2(x) = 3x^2 - 16x + 21$.

Lemma 3.9. Define a symmetric polynomial

$$\pi_n(x) = \begin{cases} 1, & n = 0; \\ \prod_{i=0}^{n-1}(x - x_i), & n > 0. \end{cases} \tag{3.10}$$

Then for $n > 0$ the fundamental polynomial for pointwise interpolation can be expressed as

$$\forall x \ne x_k, \qquad \ell_k(x) = \frac{\pi_{n+1}(x)}{(x - x_k)\pi'_{n+1}(x_k)}. \tag{3.11}$$

Proof. By the chain rule, $\pi'_{n+1}(x)$ is the summation of $n+1$ terms, each of which is a product of n terms. When x is replaced with x_k, all of the $n+1$ terms vanish except one. □

Lemma 3.10 (Cauchy relations)**.** The fundamental polynomials $\ell_k(x)$ satisfy the Cauchy relations as follows.

$$\sum_{k=0}^{n}\ell_k(x) \equiv 1 \tag{3.12}$$

$$\forall j = 1, \ldots, n, \qquad \sum_{k=0}^{n}(x_k - x)^j \ell_k(x) \equiv 0 \tag{3.13}$$

Proof. By Theorems 3.4 and 3.6, for each $q(x) \in \mathbb{P}_n$ we have $p_n(q;x) \equiv q(x)$. Interpolating the constant function $f(x) \equiv 1$ with the Lagrange formula yields (3.12). Similarly, (3.13) can be proved by interpolating the polynomial $q(u) = (u - x)^j$ for each $j = 1, \ldots, n$ with the Lagrange formula. □

3.4 The Newton formula

Definition 3.11 (Divided difference and the Newton formula)**.** The *Newton formula* for interpolating the values $f_0, f_1, \ldots, f_n$ at distinct points $x_0, x_1, \ldots, x_n$ is

$$p_n(x) = \sum_{k=0}^{n} a_k \pi_k(x), \tag{3.14}$$

where π_k is defined in (3.10) and the *kth divided difference* a_k is defined as the coefficient of x^k in $p_k(f;x)$ and is denoted by $f[x_0, x_1, \ldots, x_k]$ or $[x_0, x_1, \ldots, x_k]f$. In particular, $f[x_0] = f(x_0)$.

Corollary 3.12. Suppose $(i_0, i_1, i_2, \ldots, i_k)$ is a permutation of $(0, 1, 2, \ldots, k)$. Then

$$f[x_0, x_1, \ldots, x_k] = f[x_{i_0}, x_{i_1}, \ldots, x_{i_k}]. \tag{3.15}$$

Proof. The interpolating polynomial does not depend on the numbering of the interpolating nodes. The rest of the proof follows from the uniqueness of the interpolating polynomial in Theorem 3.4. □

Corollary 3.13. The kth divided difference can be expressed as

$$f[x_0, x_1, \ldots, x_k] = \sum_{i=0}^{k} \frac{f_i}{\prod_{j\neq i; j=0}^{k}(x_i - x_j)} = \sum_{i=0}^{k} \frac{f_i}{\pi'_{k+1}(x_i)}, \tag{3.16}$$

where $\pi_{k+1}(x)$ is defined in (3.10).

Proof. The uniqueness of interpolating polynomials in Theorem 3.4 implies that the two polynomials in (3.8) and (3.14) are the same. Then the first equality follows from (3.9) and Definition 3.11, while the second equality follows from Lemma 3.9. □

Theorem 3.14. Divided differences satisfy the recursion

$$f[x_0, x_1, \ldots, x_k] = \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_0}. \tag{3.17}$$

Proof. By Definition 3.11, $f[x_1, x_2, \ldots, x_k]$ is the coefficient of x^{k-1} in a degree-$(k-1)$ interpolating polynomial, say, $P_2(x)$. Similarly, let $P_1(x)$ be the interpolating polynomial whose coefficient of x^{k-1} is $f[x_0, x_1, \ldots, x_{k-1}]$. Construct a polynomial

$$P(x) = P_1(x) + \frac{x - x_0}{x_k - x_0}\left(P_2(x) - P_1(x)\right).$$

Clearly $P(x_0) = P_1(x_0)$. Furthermore, the interpolation condition implies $P_2(x_i) = P_1(x_i)$ for $i = 1, 2, \ldots, k-1$. Hence $P(x_i) = P_1(x_i)$ for $i = 1, 2, \ldots, k-1$. Lastly, $P(x_k) = P_2(x_k)$. Therefore, $P(x)$ as above is the interpolating polynomial for given values at the $k+1$ points. In particular, the term $f[x_0, x_1, \cdots, x_k]x^k$ in $P(x)$ is contained in $\frac{x}{x_k - x_0}(P_2(x) - P_1(x))$. The rest follows from the definitions of and the kth divided difference. □

Definition 3.15. The kth divided difference ($k \in \mathbb{N}^+$) on the *table of divided differences*

$$\begin{array}{c|llll}
x_0 & f[x_0] & & & \\
x_1 & f[x_1] & f[x_0, x_1] & & \\
x_2 & f[x_2] & f[x_1, x_2] & f[x_0, x_1, x_2] & \\
x_3 & f[x_3] & f[x_2, x_3] & f[x_1, x_2, x_3] & f[x_0, x_1, x_2, x_3] \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

is calculated as the difference of the entry immediately to the left and the one above it, divided by the difference of the x-value horizontal to the left and the one corresponding to the f-value found by going diagonally up.

Example 3.4. Derive the interpolating polynomial via the Newton formula for the function f with given values as follows. Then estimate $f(\frac{3}{2})$.

x	0	1	2	3
$f(x)$	6	−3	−6	9

By Definition 3.15, we can construct the following table of divided difference,

$$\begin{array}{c|llll}
0 & 6 & & & \\
1 & -3 & -9 & & \\
2 & -6 & -3 & 3 & \\
3 & 9 & 15 & 9 & 2
\end{array} \tag{3.18}$$

By Definition 3.11, the interpolating polynomial is generated from the main diagonal and the first column of the above table as follows.

$$p_3 = 6 - 9x + 3x(x-1) + 2x(x-1)(x-2). \tag{3.19}$$

Hence $f(\frac{3}{2}) \approx p_3(\frac{3}{2}) = -6$.

Exercise 3.5. Redo Example 3.3 with the Newton formula.

Theorem 3.16. For distinct points $x_0, x_1, \ldots, x_n$ and an arbitrary x, we have

$$\begin{aligned} f(x) = f[x_0] &+ f[x_0, x_1](x - x_0) + \cdots \\ &+ f[x_0, x_1, \cdots, x_n]\prod_{i=0}^{n-1}(x - x_i) \\ &+ f[x_0, x_1, \cdots, x_n, x]\prod_{i=0}^{n}(x - x_i). \end{aligned} \tag{3.20}$$

Proof. Take another point $z \neq x_i$. The Newton formula applied to $x_0, x_1, \ldots, x_n, z$ yields an interpolating polynomial

$$\begin{aligned} Q(x) = f[x_0] &+ f[x_0, x_1](x - x_0) + \cdots \\ &+ f[x_0, x_1, \cdots, x_n]\prod_{i=0}^{n-1}(x - x_i) \\ &+ f[x_0, x_1, \cdots, x_n, z]\prod_{i=0}^{n}(x - x_i). \end{aligned}$$

The interpolation condition $Q(z) = f(z)$ yields

$$\begin{aligned} f(z) = Q(z) = f[x_0] &+ f[x_0, x_1](z - x_0) + \cdots \\ &+ f[x_0, x_1, \cdots, x_n]\prod_{i=0}^{n-1}(z - x_i) \\ &+ f[x_0, x_1, \cdots, x_n, z]\prod_{i=0}^{n}(z - x_i). \end{aligned}$$

Replacing the dummy variable z with x yields (3.20).

The above argument assumes $x \neq x_i$. Now consider the case of $x = x_j$ for some fixed j. Rewrite (3.20) as $f(x) = p_n(f; x) + R(x)$ where $R(x)$ is clearly the last term in (3.20). We need to show

$$\forall j = 0, 1, \cdots, n, \qquad p_n(f; x_j) + R(x_j) - f(x_j) = 0,$$

which clearly holds because $R(x_j) = 0$ and the interpolation condition at x_j dictates $p_n(f; x_j) = f(x_j)$. □

Corollary 3.17. Suppose $f \in \mathcal{C}^n[a, b]$ and $f^{(n+1)}(x)$ exists at each point of (a, b). If $a = x_0 < x_1 < \cdots < x_n = b$ and $x \in [a, b]$, then

$$f[x_0, x_1, \cdots, x_n, x] = \frac{1}{(n+1)!} f^{(n+1)}(\xi(x)) \tag{3.21}$$

where ξ depends on x and $\xi(x) \in (a, b)$.

Proof. This follows from Theorems 3.16 and 3.6. □

Corollary 3.18. If $x_0 < x_1 < \cdots < x_n$ and $f \in \mathcal{C}^n[x_0, x_n]$, we have

$$\lim_{x_n \to x_0} f[x_0, x_1, \cdots, x_n] = \frac{1}{n!} f^{(n)}(x_0). \tag{3.22}$$

Proof. Set $x = x_{n+1}$ in Corollary 3.17, replace $n+1$ by n, and we have $\xi \to x_0$ as $x_n \to x_0$ since each $x_i \to x_0$. □

Definition 3.19. For $n \in \mathbb{N}^+$, the nth *forward difference* associated with a sequence of values $\{f_0, f_1, \ldots\}$ is

$$\begin{aligned} \Delta f_i &= f_{i+1} - f_i, \\ \Delta^{n+1} f_i &= \Delta\Delta^n f_i = \Delta^n f_{i+1} - \Delta^n f_i, \end{aligned} \tag{3.23}$$

and the nth *backward difference* is

$$\begin{aligned} \nabla f_i &= f_i - f_{i-1}, \\ \nabla^{n+1} f_i &= \nabla\nabla^n f_i = \nabla^n f_i - \nabla^n f_{i-1}. \end{aligned} \tag{3.24}$$

Theorem 3.20. The forward difference and backward difference are related as

$$\forall n \in \mathbb{N}^+, \qquad \Delta^n f_i = \nabla^n f_{i+n}. \tag{3.25}$$

Proof. An easy induction. □

Theorem 3.21. The forward difference can be expressed explicitly as

$$\Delta^n f_i = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f_{i+k}. \tag{3.26}$$

Proof. For $n = 1$, (3.26) reduces to $\Delta f_i = f_{i+1} - f_i$. The rest of the proof is an induction utilizing the identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}. \tag{3.27}$$

Suppose (3.26) holds. For the inductive step, we have

$$\begin{aligned} \Delta^{n+1} f_i =& \Delta\Delta^n f_i = \Delta\left(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f_{i+k}\right) \\ =& \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f_{i+k+1} - \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f_{i+k} \\ =& \sum_{k=1}^{n} (-1)^{n+1-k} \binom{n}{k-1} f_{i+k} + f_{i+n+1} \\ & + \sum_{k=1}^{n} (-1)^{n+1-k} \binom{n}{k} f_{i+k} + (-1)^{n+1} f_i \\ =& \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} f_{i+k}, \end{aligned}$$

where the second line follows from (3.23), the third line from splitting one term out of each sum and replacing the dummy variable in the first sum, and the fourth line from (3.27) and the fact that $(-1)^{n+1} f_i$ and f_{i+n+1} contribute to the first and last terms, respectively. □

Theorem 3.22. On a grid $x_i = x_0 + ih$ with uniform spacing h, the sequence of values $f_i = f(x_i)$ satisfies

$$\forall n \in \mathbb{N}^+, \qquad f[x_0, x_1, \ldots, x_n] = \frac{\Delta^n f_0}{n! h^n}. \tag{3.28}$$

Proof. Of course (3.28) can be proven by induction. Here we provide a more informative proof. For $\pi_{n+1}(x)$ defined in (3.10), we have $\pi'(x_k) = \prod_{i=0, i\neq k}^{n} (x_k - x_i)$. It follows from $x_k - x_i = (k-i)h$ that

$$\pi'(x_k) = \prod_{i=0, i\neq k}^{n} (k-i)h = h^n k!(n-k)!(-1)^{n-k}. \tag{3.29}$$

Then we have

$$\begin{aligned} f[x_0, x_1, \ldots, x_n] &= \sum_{k=0}^{n} \frac{f_k}{\pi'(x_k)} = \sum_{k=0}^{n} \frac{(-1)^{n-k} f_k}{h^n k!(n-k)!} \\ &= \frac{1}{h^n n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f_k = \frac{\Delta^n f_0}{h^n n!}, \end{aligned}$$

where the first step follows from Corollary 3.13, the second from (3.29), and the last from Theorem 3.21. □

Theorem 3.23 (Newton's forward difference formula)**.** Suppose $p_n(f; x) \in \mathbb{P}_n$ interpolates $f(x)$ on a uniform grid $x_i = x_0 + ih$ at $x_0, x_1, \ldots, x_n$ with $f_i = f(x_i)$. Then

$$\forall s \in \mathbb{R}, \qquad p_n(f; x_0 + sh) = \sum_{k=0}^{n} \binom{s}{k} \Delta^k f_0, \tag{3.30}$$

where $\Delta^0 f_0 = f_0$ and

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}. \tag{3.31}$$

Proof. Set $f(x) = p_n(f; x)$ in Theorem 3.16, apply Theorem 3.22, and we have

$$p(x) = f_0 + \sum_{k=1}^{n} \frac{\Delta^k f_0}{k! h^k} \prod_{i=0}^{k-1} (x - x_i);$$

the remainder is zero because any $(n+1)$th divided difference applied to a degree n polynomial is zero. The proof is completed by $x = x_0 + sh$, $x_i = x_0 + ih$, and (3.31). □

3.5 The Neville-Aitken algorithm

Theorem 3.24. Denote $p_0^{[i]} = f(x_i)$ for $i = 0, 1, \ldots, n$. For all $k = 0, 1, \ldots, n-1$ and $i = 0, 1, \ldots, n-k-1$, define

$$p_{k+1}^{[i]}(x) = \frac{(x - x_i) p_k^{[i+1]}(x) - (x - x_{i+k+1}) p_k^{[i]}(x)}{x_{i+k+1} - x_i}. \tag{3.32}$$

Then each $p_k^{[i]}$ is the interpolating polynomial for the function f at the points $x_i, x_{i+1}, \ldots, x_{i+k}$. In particular, $p_n^{[0]}$ is the interpolating polynomial of degree n for the function f at the points $x_0, x_1, \ldots, x_n$.

Proof. The induction basis clearly holds for $k = 0$ because of the definition $p_0^{[i]} = f(x_i)$. Suppose that $p_k^{[i]}$ is the interpolating polynomial of degree k for the function f at the points $x_i, x_{i+1}, \ldots, x_{i+k}$. Then we have

$$\forall j = i+1, i+2, \ldots, i+k, \qquad p_k^{[i+1]}(x_j) = p_k^{[i]}(x_j) = f(x_j),$$

which, together with (3.32), implies

$$\forall j = i+1, i+2, \ldots, i+k, \qquad p_{k+1}^{[i]}(x_j) = f(x_j).$$

In addition, (3.32) and the induction hypothesis yield

$$p_{k+1}^{[i]}(x_i) = p_k^{[i]}(x_i) = f(x_i),$$
$$p_{k+1}^{[i]}(x_{i+k+1}) = p_k^{[i+1]}(x_{i+k+1}) = f(x_{i+k+1}).$$

The proof is completed by the last three equations and the uniqueness of interpolating polynomials. □

Example 3.6. To estimate $f(x)$ for $x = \frac{3}{2}$ directly from the table in Example 3.4, we construct a table by repeating (3.32) with $x_i = i$ for $i = 0, 1, 2, 3$.

i	$x - x_i$	$f(x_i)$	$p_1^{[i]}(x)$	$p_2^{[i]}(x)$	$p_3^{[i]}(x)$
0	$\frac{3}{2}$	6	$-\frac{15}{2}$	$-\frac{21}{4}$	-6
1	$\frac{1}{2}$	-3	$-\frac{9}{2}$	$-\frac{27}{4}$	
2	$-\frac{1}{2}$	-6	$-\frac{27}{2}$		
3	$-\frac{3}{2}$	9			

(3.33)

The result is the same as that in Example 3.4. In contrast, the calculation and layout of the two tables are distinct.

3.6 Hermite interpolation

Definition 3.25. Given distinct points $x_0, x_1, \ldots, x_k$ in $[a, b]$, non-negative integers $m_0, m_1, \ldots, m_k$, and a function $f \in \mathcal{C}^M[a, b]$ where $M = \max_i m_i$, the *Hermite interpolation problem* seeks to find a polynomial p of the lowest degree such that

$$\forall i = 0, 1, \ldots, k,\ \forall \mu = 0, 1, \cdots, m_i,\ \ p^{(\mu)}(x_i) = f_i^{(\mu)}, \quad (3.34)$$

where $f_i^{(\mu)} = f^{(\mu)}(x_i)$ is the value of the μth derivative of f at x_i; in particular, $f_i^{(0)} = f(x_i)$.

Definition 3.26. The nth divided difference at $n+1$ "confluent" (i.e. identical) points is defined as

$$f[x_0, x_0, \cdots, x_0] = \frac{1}{n!} f^{(n)}(x_0), \quad (3.35)$$

where x_0 is repeated $n+1$ times on the left-hand side.

Theorem 3.27. For the Hermite interpolation problem in Definition 3.25, denote $N = k + \sum_i m_i$. Denote by $p_N(f; x)$ the unique element of $\mathbb{P}_N$ for which (3.34) holds. Suppose $f^{(N+1)}(x)$ exists in (a, b). Then

$$f(x) - p_N(f; x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{i=0}^{k} (x - x_i)^{m_i+1}. \quad (3.36)$$

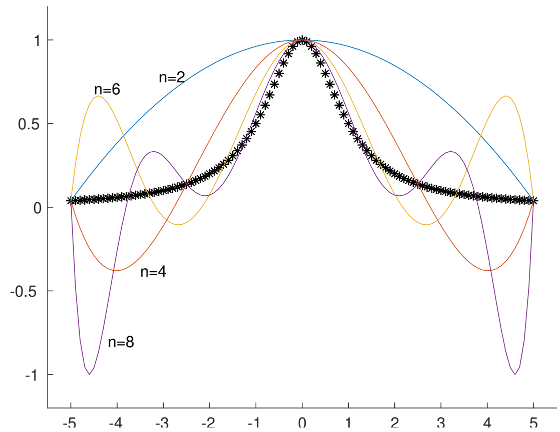
Proof. The proof is similar to that of Theorem 3.6. Pay attention to the difference caused by the multiple roots of the polynomial $\prod_{i=0}^{k}(x - x_i)^{m_i+1}$. □

3.7 The Chebyshev polynomials

Example 3.7 (Runge phenomenon)**.** The points $x_0, x_1, \ldots, x_n$ in Theorem 3.4 are usually given *a priori*, e.g., as uniformly distributed over the interval $[x_0, x_n]$. As n increases, the degree of the interpolating polynomial also increases. Ideally we would like to have

$$\forall f \in \mathcal{C}[x_0, x_n], \forall x \in [x_0, x_n], \quad \lim_{n \to +\infty} p_n(f; x) = f(x). \quad (3.37)$$

However, this is not true for polynomial interpolation on equally spaced points. The famous Runge's example illustrates the violent oscillations at the end of the interval.



The above plot is created by interpolating

$$f(x) = \frac{1}{1 + x^2} \quad (3.38)$$

on $x_i = -5 + 10\frac{i}{n}$, $i = 0, 1, \ldots, n$ with $n = 2, 4, 6, 8$.

Definition 3.28. The *Chebyshev polynomial* of degree n of the first kind is a polynomial $T_n : [-1, 1] \to \mathbb{R}$,

$$T_n(x) = \cos(n \arccos x). \quad (3.39)$$

Theorem 3.29.

$$\forall n \in \mathbb{N}^+, \qquad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \quad (3.40)$$

Proof. By trigonometric identities, we have

$$\cos(n+1)\theta = \cos n\theta \cos\theta - \sin n\theta \sin\theta,$$
$$\cos(n-1)\theta = \cos n\theta \cos\theta + \sin n\theta \sin\theta.$$

Adding up the two equations and setting $\cos\theta = x$ complete the proof. □

Corollary 3.30. The coefficient of x^n in T_n is 2^{n-1} for each $n > 0$.

Proof. Use (3.40) and $T_1 = x$ in an induction. □

Theorem 3.31. $T_n(x)$ has simple zeros at the n points

$$x_k = \cos \frac{2k-1}{2n}\pi, \tag{3.41}$$

where $k = 1, 2, \ldots, n$. For $x \in [-1,1]$ and $n \in \mathbb{N}^+$, $T_n(x)$ has extreme values at the $n+1$ points

$$x'_k = \cos \frac{k}{n}\pi, \qquad k = 0, 1, \ldots, n, \tag{3.42}$$

where it assumes the alternating values $(-1)^k$.

Proof. (3.39) and (3.41) yield

$$T_n(x_k) = \cos\left(n \arccos(\cos \frac{2k-1}{2n}\pi)\right) = \cos\left(\frac{2k-1}{2}\pi\right) = 0.$$

Differentiate (3.39) and we have

$$T'_n(x) = \frac{n}{\sqrt{1-x^2}} \sin(n \arccos x).$$

Then each x_k must be a simple zero since

$$T'_n(x_k) = \frac{n}{\sqrt{1-x_k^2}} \sin\left(\frac{2k-1}{2}\pi\right) \neq 0.$$

In contrast, $\forall k = 1, 2, \ldots, n-1$,

$$T'_n(x'_k) = n\left(1 - \cos^2 \frac{k\pi}{n}\right)^{-\frac{1}{2}} \sin(k\pi) = 0;$$

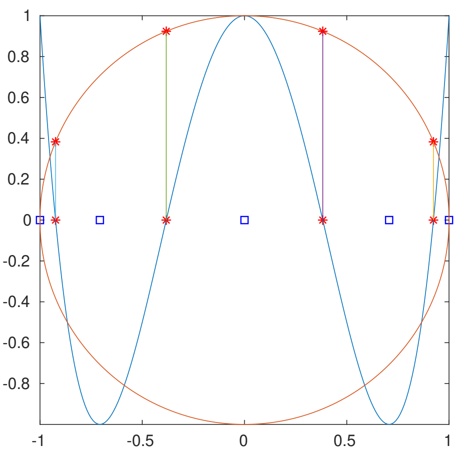
$$T''_n(x) = \frac{n^2 \cos(n \arccos(x))}{x^2-1} + \frac{n\,x \sin(n \arccos(x))}{(1-x^2)^{3/2}};$$

$$T''_n(x'_k) \neq 0.$$

Hence a Taylor expansion of T_n yields

$$T_n(x'_k + \delta) = T_n(x'_k) + \frac{1}{2}T''_n(x'_k)\delta^2 + O(\delta^3),$$

and T_n must attain local extremes at each x'_k. For $k = 0, 1, \ldots, n$, $T_n(x'_k)$ attains its extreme values at x'_k since $T_n(x'_0) = 1$, $T_n(x'_1) = -1$, ..., and by (3.39) we have $|T_n(x)| \le 1$. Clearly these are the only extrema of $T_n(x)$ on $[-1,1]$. □



Exercise 3.8. Write a program to reproduce the above plot.

Theorem 3.32 (Chebyshev)**.** Denote by $\tilde{\mathbb{P}}_n$ the class of all polynomials of degree $n \in \mathbb{N}^+$ with leading coefficient 1. Then

$$\forall p \in \tilde{\mathbb{P}}_n, \qquad \max_{x\in[-1,1]} \left|\frac{T_n(x)}{2^{n-1}}\right| \le \max_{x\in[-1,1]} |p(x)|. \tag{3.43}$$

Proof. By Theorem 3.31, $T_n(x)$ assumes its extrema $n+1$ times at the points x'_k defined in (3.42). Suppose (3.43) does not hold. Then Theorem 3.31 implies that

$$\exists p \in \tilde{\mathbb{P}}_n \text{ s.t. } \max_{x\in[-1,1]} |p(x)| < \frac{1}{2^{n-1}}. \tag{3.44}$$

Consider the polynomial $Q(x) = \frac{1}{2^{n-1}}T_n(x) - p(x)$.

$$Q(x'_k) = \frac{(-1)^k}{2^{n-1}} - p(x'_k), \qquad k = 0, 1, \ldots, n.$$

By (3.44), $Q(x)$ has alternating signs at these $n+1$ points. Hence $Q(x)$ must have n zeros. However, by the construction of $Q(x)$, the degree of $Q(x)$ is at most $n-1$. Therefore, $Q(x) \equiv 0$ and $p(x) = \frac{1}{2^{n-1}}T_n(x)$, which implies $\max|p(x)| = \frac{1}{2^{n-1}}$. This is a contradiction to (3.44). □

Corollary 3.33. For $n \in \mathbb{N}^+$, we have

$$\max_{x\in[-1,1]} |x^n + a_1x^{n-1} + \cdots + a_n| \ge \frac{1}{2^{n-1}}. \tag{3.45}$$

Corollary 3.34. Suppose polynomial interpolation is performed for f on the $n+1$ zeros of $T_{n+1}(x)$ as in Theorem 3.31. The Cauchy remainder in Theorem 3.6 satisfies

$$|R_n(f;x)| \le \frac{1}{2^n(n+1)!} \max_{x\in[-1,1]} \left|f^{(n+1)}(x)\right|. \tag{3.46}$$

Proof. Theorem 3.6, Corollary 3.30, and Theorem 3.31 yield

$$|R_n(f;x)| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!} \left|\prod_{i=0}^{n}(x-x_i)\right| = \frac{|f^{(n+1)}(\xi)|}{2^n(n+1)!}|T_{n+1}|.$$

Definition 3.28 completes the proof as $|T_{n+1}| \le 1$. □

Theorem 3.35 (Weierstrass approximation)**.** Every continuous function $f : [a,b] \to \mathbb{R}$ can be uniformly approximated as closely as desired by a polynomial function. More precisely, let $\mathbb{P}_n$ denote the polynomials of degree no more than n. Then we have

$$\begin{aligned}&\forall f \in \mathcal{C}[a,b],\ \forall \epsilon > 0,\ \exists N \in \mathbb{N}^+ \text{ s.t. } \forall n > N,\\ &\qquad \exists p_n \in \mathbb{P}_n \text{ s.t. } \forall x \in [a,b],\ \|p_n - f\| < \epsilon.\end{aligned} \tag{3.47}$$

Proof. Not required. □

3.8 Problems

3.8.1 Theoretical questions

I. For $f \in \mathcal{C}^2[x_0, x_1]$ and $x \in (x_0, x_1)$, linear interpolation of f at x_0 and x_1 yields

$$f(x) - p_1(f; x) = \frac{f''(\xi(x))}{2}(x - x_0)(x - x_1).$$

Consider the case $f(x) = \frac{1}{x}$, $x_0 = 1$, $x_1 = 2$.

- Determine $\xi(x)$ explicitly.
- For $x \in [x_0, x_1]$, find $\max \xi(x)$, $\min \xi(x)$, and $\max f''(\xi(x))$.

II. Let $\mathbb{P}_m^+$ be the set of all polynomials of degree $\le m$ that are non-negative on the real line,

$$\mathbb{P}_m^+ = \{p : p \in \mathbb{P}_m,\ \forall x \in \mathbb{R},\ p(x) \ge 0\}.$$

Find $p \in \mathbb{P}_{2n}^+$ such that $p(x_i) = f_i$ for $i = 0, 1, \ldots, n$ where $f_i \ge 0$ and x_i are distinct points on $\mathbb{R}$.

III. Consider $f(x) = e^x$.

- Prove by induction that

$$\forall t \in \mathbb{R}, \qquad f[t, t+1, \ldots, t+n] = \frac{(e-1)^n}{n!} e^t.$$

- From Corollary 3.17 we know

$$\exists \xi \in (0, n) \text{ s.t. } f[0, 1, \ldots, n] = \frac{1}{n!} f^{(n)}(\xi).$$

Determine ξ from the above two equations. Is ξ located to the left or to the right of the midpoint $n/2$?

IV. Consider $f(0) = 5$, $f(1) = 3$, $f(3) = 5$, $f(4) = 12$.

- Use the Newton formula to obtain $p_3(f; x)$;
- The data suggest that f has a minimum in $x \in (1, 3)$. Find an approximate value for the location $x_{\min}$ of the minimum.

V. Consider $f(x) = x^7$.

- Compute $f[0, 1, 1, 1, 2, 2]$.
- We know that this divided difference is expressible in terms of the 5th derivative of f evaluated at some $\xi \in (0, 2)$. Determine ξ.

VI. f is a function on $[0, 3]$ for which one knows that

$$f(0) = 1, f(1) = 2, f'(1) = -1, f(3) = f'(3) = 0.$$

- Estimate $f(2)$ using Hermite interpolation.
- Estimate the maximum possible error of the above answer if one knows, in addition, that $f \in \mathcal{C}^5[0, 3]$ and $|f^{(5)}(x)| \le M$ on $[0, 3]$. Express the answer in terms of M.

VII. Define forward difference by

$$\Delta f(x) = f(x+h) - f(x),$$
$$\Delta^{k+1} f(x) = \Delta\Delta^k f(x) = \Delta^k f(x+h) - \Delta^k f(x)$$

and backward difference by

$$\nabla f(x) = f(x) - f(x-h),$$
$$\nabla^{k+1} f(x) = \nabla\nabla^k f(x) = \nabla^k f(x) - \nabla^k f(x-h).$$

Prove

$$\Delta^k f(x) = k! h^k f[x_0, x_1, \ldots, x_k],$$
$$\nabla^k f(x) = k! h^k f[x_0, x_{-1}, \ldots, x_{-k}],$$

where $x_j = x + jh$.

VIII. Assume f is differentiable at x_0. Prove

$$\frac{\partial}{\partial x_0} f[x_0, x_1, \ldots, x_n] = f[x_0, x_0, x_1, \ldots, x_n].$$

What about the partial derivative with respect to one of the other variables?

IX. A min-max problem.
For $n \in \mathbb{N}^+$, determine

$$\min \max_{x \in [a,b]} |a_0 x^n + a_1 x^{n-1} + \cdots + a_n|,$$

where the minimum is taken over all $a_i \in \mathbb{R}$ and $a_0 \ne 0$.

X. Imitate the proof of Chebyshev Theorem.
Let $a > 1$ and denote $\mathbb{P}_n^a = \{p \in \mathbb{P}_n : p(a) = 1\}$. Define

$$\hat{p}_n(x) = \frac{T_n(x)}{T_n(a)},$$

where T_n is the Chebyshev polynomial of degree n. Clearly $\hat{p}_n(x) \in \mathbb{P}_n^a$. Define the *max-norm* of a function $f : \mathbb{R} \to \mathbb{R}$ as

$$\|f\|_\infty = \max_{x \in [-1,1]} |f(x)|.$$

Prove

$$\forall p \in \mathbb{P}_n^a, \qquad \|\hat{p}_n(x)\|_\infty \le \|p\|_\infty.$$

3.8.2 Programming assignments

A. Implement the Newton formula in a subroutine that produces the value of the interpolation polynomial $p_n(f; x_0, x_1, \ldots, x_n; x)$ at any real x, where $n \in \mathbb{N}^+$, x_i's are distinct, and f is a function assumed to be available in the form of a subroutine.

B. Run your routine on the function

$$f(x) = \frac{1}{1 + x^2}$$

for $x \in [-5, 5]$ using $x_i = -5 + 10\frac{i}{n}$, $i = 0, 1, \ldots, n$, and $n = 2, 4, 6, 8$. Plot the polynomials against the exact function to reproduce the plot in the notes that illustrate the Runge phenomenon.

C. Reuse your subroutine of Newton interpolation to perform Chebyshev interpolation for the function

$$f(x) = \frac{1}{1 + 25x^2}$$

for $x \in [-1, 1]$ on the zeros of Chebyshev polynomials T_n with $n = 5, 10, 15, 20$. Clearly the Runge function $f(x)$ is a scaled version of the function in B. Plot the interpolating polynomials against the exact function to observe that the Chebyshev interpolation is free of the wide oscillations in the previous assignment.