Numerical Analysis Homework #2

due 2019 MAR 31, 9:50 a.m.

1 Assignments

Caution:

- To get full credit, *you must write down sufficient intermediate steps*, only giving the final answer earns you no credit!
- Please make sure that your handwriting is recognizable, otherwise you only get partial credit for the recognizable part.

I. For $f \in \mathcal{C}^2[x_0, x_1]$ and $x \in (x_0, x_1)$, linear interpolation of f at x_0 and x_1 yields

$$f(x) - p_1(f; x) = \frac{f''(\xi(x))}{2}(x - x_0)(x - x_1).$$

Consider the case $f(x) = \frac{1}{x}$, $x_0 = 1$, $x_1 = 2$.

- Determine $\xi(x)$ explicitly.
- For $x \in [x_0, x_1]$, find $\max \xi(x)$, $\min \xi(x)$, and $\max f''(\xi(x))$.

II. Let $\mathcal{P}_m^+$ be the set of all polynomials of degree $\leq m$ that are non-negative on the real line,

$$\mathbb{P}_m^+ = \{p : p \in \mathbb{P}_m,\ \forall x \in \mathbb{R}, p(x) \geq 0\}.$$

Find $p \in \mathbb{P}_{2n}^+$ such that $p(x_i) = f_i$ for $i = 0, 1, \ldots, n$ where $f_i \geq 0$ and x_i are distinct points on $\mathbb{R}$.

III. Consider $f(x) = e^x$.

- Prove by induction that

$$\forall t \in \mathbb{R}, \qquad f[t, t+1, \ldots, t+n] = \frac{(e-1)^n}{n!}e^t.$$

- From Corollary 3.17 we know

$$\xi \in (0, n) \text{ s.t. } f[0, 1, \ldots, n] = \frac{1}{n!}f^{(n)}(\xi).$$

Determine ξ from the above two equations. Is ξ located to the left or to the right of the midpoint $n/2$?

IV. Consider $f(0) = 5$, $f(1) = 3$, $f(3) = 5$, $f(4) = 12$.

- Use the Newton formula to obtain $p_3(f; x)$;
- The data suggest that f has a minimum in $x \in (1, 3)$. Find an approximate value for the location $x_{\min}$ of the minimum.

V. Consider $f(x) = x^7$.

- Compute $f[0, 1, 1, 1, 2, 2]$.
- We know that this divided difference is expressible in terms of the 5th derivative of f evaluated at some $\xi \in (0, 2)$. Determine ξ.

VI. f is a function on $[0, 3]$ for which one knows that

$$f(0) = 1, f(1) = 2, f'(1) = -1, f(3) = f'(3) = 0.$$

- Estimate $f(2)$ using Hermite interpolation.
- Estimate the maximum possible error of the above answer if one knows, in addition, that $f \in \mathcal{C}^5[0, 3]$ and $|f^{(5)}(x)| \leq M$ on $[0, 3]$. Express the answer in terms of M.

VII. Define forward difference by

$$\Delta f(x) = f(x+h) - f(x),$$
$$\Delta^{k+1} f(x) = \Delta\Delta^k f(x) = \Delta^k f(x+h) - \Delta^k f(x)$$

and backward difference by

$$\nabla f(x) = f(x) - f(x-h),$$
$$\nabla^{k+1} f(x) = \nabla\nabla^k f(x) = \nabla^k f(x) - \nabla^k f(x-h).$$

Prove

$$\Delta^k f(x) = k!h^k f[x_0, x_1, \ldots, x_k],$$
$$\nabla^k f(x) = k!h^k f[x_0, x_{-1}, \ldots, x_{-k}],$$

where $x_j = x + jh$.

VIII. $*$ Assume f is differentiable at x_0. Prove

$$\frac{\partial}{\partial x_0} f[x_0, x_1, \ldots, x_n] = f[x_0, x_0, x_1, \ldots, x_n].$$

What about the partial derivative with respect to one of the other variables?

Problem II weighs 4 points, VIII weighs 5 points, and all other problems weigh 6 points each. The total point is thus 40. The last problem is for extra credit and you do not have to solve it. However, my graduate students who audit this class have to solve all problems.

2 Extra credits

Additional 10% credits will be given to you if you typeset your solutions in LaTeX. You are welcome to use the LaTeX template available on my webpage. You can also get partial extra credit for typesetting solutions of *some* problems.

Note: If you choose to typeset your solutions in LaTeX, you still need to turn in a hard copy in class. In addition, please upload your latex source (.tex) and supporting files in a single zip file (**format**: `YourName_Homework2.zip`) to the course email `NumApproximation@163.com`.