Homework00, Homework01 习题课

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Contents

题目

Rewrite each of the following statements and its *negation* into *logical statements* using symbols, quantifiers, and formulas.

a. The only even prime is 2.
   Let $\mathbb{P}$ denote the set of all prime numbers.

$$\mathcal{G} = (\mathbb{P} \cap 2\mathbb{Z} = \{2\});$$
$$\neg\mathcal{G} = (\mathbb{P} \cap 2\mathbb{Z} \neq \{2\}).$$

注意表达出“only”。

题目

Rewrite each of the following statements and its *negation* into *logical statements* using symbols, quantifiers, and formulas.

b. Multiplication of integers is associative.

$$\mathcal{G} = (\forall x, y, z \in \mathbb{Z},\ x(yz) = (xy)z);$$
$$\neg\mathcal{G} = (\exists x, y, z \in \mathbb{Z} \text{ s.t. } x(yz) \neq (xy)z).$$

注意"associative" 指结合律。

题目

Rewrite each of the following statements and its *negation* into *logical statements* using symbols, quantifiers, and formulas.

c. Goldbach's conjecture has at most a finite number of counterexamples.
Let $\mathbb{E} = 2\mathbb{N}^+ \setminus \{2\}$.
Let $\mathbb{P}$ denote the set of all prime numbers.

$$\begin{aligned}\mathcal{G} =& (\exists n \in \mathbb{N}^+ \text{ s.t. } \forall x \in \mathbb{E}, \\ & \quad (x > n) \Rightarrow (\exists p, q \in \mathbb{P} \text{ s.t. } x = p + q)); \\ \neg\mathcal{G} =& (\forall n \in \mathbb{N}^+, \exists x \in \mathbb{E} \text{ s.t.} \\ & \quad (x > n) \Rightarrow (\forall p, q \in \mathbb{P},\ x \neq p + q)).\end{aligned}$$

II

题目

On (a,∞), $f(x)=\frac{1}{x^2}$ is uniformly continuous if $a>0$ and is not so if $a=0$.

a>0 直接计算得

$$|f(x)-f(y)|=|x-y|\left(\frac{1}{x^2y}+\frac{1}{xy^2}\right).$$

由 $a<x$, $a<y$ 得 $|f(x)-f(y)|\leq\frac{2}{a^3}|x-y|$, 所以，$\forall\epsilon>0$ 取 $|x-y|\leq\delta<\frac{a^3\epsilon}{2}$ 即可说明 $f(x)$ 一致连续。

II

a=0 用数学语言表达命题:

The function $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, +\infty)$ iff

$$\exists \epsilon > 0 \text{ s.t. } \forall \delta > 0 \ \exists x, y \in (0, +\infty) \text{ s.t. } |x - y| < \delta \Rightarrow |f(x) - f(y)| \geq \epsilon.$$

我们证明一个更强的命题:

$$\forall \epsilon > 0 \text{ s.t. } \forall \delta > 0 \ \exists x, y \in (0, +\infty) \text{ s.t. } |x - y| < \delta \Rightarrow |f(x) - f(y)| \geq \epsilon.$$

II

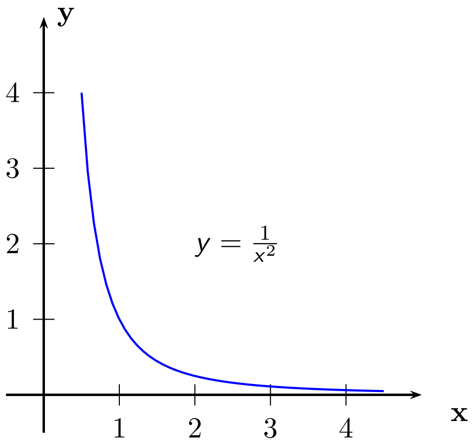
- $\delta > \frac{1}{2\sqrt{\epsilon}}$: 选择 $x = \frac{1}{2\sqrt{\epsilon}}$, $y < x$, $y = \frac{x}{2}$, 那么

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{4}{x^2}\right| = \left|\frac{3}{x^2}\right| = 12\epsilon > \epsilon;$$

- $\delta \leq \frac{1}{2\sqrt{\epsilon}}$: 选择 $x = \delta$, $y < x$, $y = \frac{x}{2}$, 那么

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{4}{x^2}\right| = \left|\frac{3}{x^2}\right| = \frac{3}{\delta^2} \geq 12\epsilon > \epsilon.$$

II



III

题目

Let $\mathcal{X}$ be the set of all bounded and unbounded sequences of complex numbers. Show that the following is a metric on $\mathcal{X}$,

$$d(x,y)=\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{|\xi_j-\eta_j|}{1+|\xi_j-\eta_j|},$$

where $x=(\xi_j)$ and $y=(\eta_j)$.

III

(0) $d(x, y)$ is a function, which means to prove $d(x, y) < +\infty$.

(1) $d(x, y) \geq 0$.

(2) $(x = y) \Leftrightarrow (d(x, y) = 0)$.

(3) $d(x, y) = d(y, x)$.

(4) $d(x, z) \leq d(x, y) + d(y, z)$.

注意不要忘记 (0), 以及 $(d(x, y) = 0) \Rightarrow (x = y)$ 的证明。

III

- 证明 $\forall x, y \in \mathcal{X}$ $s.t$ $d(x,y) < +\infty$.

$$\sum_{j=1}^{N} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \sum_{j=1}^{N} \frac{1}{2^j},$$

让 $N \to +\infty$，即得 $d(x,y) \leq 1$，所以 $d(x,y)$ 是一个单调有界序列的极限。

- 证明 $(d(x,y) = 0) \Rightarrow (x = y)$。

$$\forall j \in \mathbb{N}^+, \ \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq d(x,y) = 0.$$

由此得 $\forall j \in \mathbb{N}^+$, $|\xi_j - \eta_j| = 0$, 所以 $x = y$。

III

- **证明** $\forall x, y, z \in \mathcal{X}$ $s.t$ $d(x,y) \leq d(x,z) + d(y,z)$.
  The function $f(t) = \frac{t}{1+t}$ is monotonically increasing in $[0, +\infty)$.
  Then $|a+b| \leq |a| + |b|$ implies

$$f(|a+b|) \leq f(|a| + |b|),$$

and therefore

$$\begin{aligned} \frac{|a+b|}{1+|a+b|} &\leq \frac{|a|+|b|}{1+|a|+|b|} = \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|} \\ &\leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}. \end{aligned}$$

III

Set $a = \xi_j - \eta_j$ and $b = \eta_j - \gamma_j$, sum over $j = 1, 2, \ldots, N$, and we have

$$\begin{aligned}
\sum_{j=1}^{N} \frac{1}{2^j} \frac{|\xi_j - \gamma_j|}{1 + |\xi_j - \gamma_j|} \leq & \sum_{j=1}^{N} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} + \sum_{j=1}^{N} \frac{1}{2^j} \frac{|\eta_j - \gamma_j|}{1 + |\eta_j - \gamma_j|} \\
\leq & \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} + \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\eta_j - \gamma_j|}{1 + |\eta_j - \gamma_j|} \\
= & d(x, y) + d(y, z).
\end{aligned}$$

Letting $N \to +\infty$ yields $d(x, z) \leq d(x, y) + d(y, z)$.

IV

题目

Prove that the following is indeed a metric on $\mathcal{X}$,

$$d(x, y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p \right)^{1/p},$$

where $\mathcal{X} = \left\{ (\xi_j)_{j=1}^{\infty} : \xi_j \in \mathbb{C}; \sum_{j=1}^{\infty} |\xi_j|^p < +\infty \right\}$;

IV

(0) $d(x,y)$ is a function, which means to prove $d(x,y) < +\infty$.

(1) $d(x,y) \geq 0$.

(2) $(x = y) \Leftrightarrow (d(x,y) = 0)$.

(3) $d(x,y) = d(y,x)$.

(4) $d(x,z) \leq d(x,y) + d(y,z)$.

注意不要忘记 (0), 以及 $(d(x,y) = 0) \Rightarrow (x = y)$ 的证明；在 (4) 的证明中，*Hölder inequality* 要求 $p > 1$，因此在 *Minkowski inequality* 不等式的证明中 $p = 1$ 要单独讨论。

IV

- **证明** $\forall x, y \in \mathcal{X}\ s.t\ d(x,y) < +\infty$.
  $\forall N \in \mathbb{N}^+$,

$$\left(\sum_{j=1}^{N} |\xi_j - \eta_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^{N} (|\xi_j| + |\eta_j|)^p\right)^{1/p}$$

$$\leq \left(\sum_{j=1}^{N} 2^p \max(|\xi_j|^p, |\eta_j|^p)\right)^{1/p} \leq 2\left(\sum_{j=1}^{N} (|\xi_j|^p + |\eta_j|^p)\right)^{1/p}$$

$$\leq 2\left(\sum_{j=1}^{\infty} |\xi_j|^p + \sum_{j=1}^{\infty} |\eta_j|^p\right)^{1/p}.$$

IV

Let $N \to +\infty$, use the fact that $x = (\xi_j) \in \mathcal{X}, y = (\eta_j) \in \mathcal{X}$, and we have

$$d(x, y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p \right)^{1/p} \leq 2 \left(\sum_{j=1}^{\infty} |\xi_j|^p + \sum_{j=1}^{\infty} |\eta_j|^p \right)^{1/p} < +\infty.$$

IV

- 证明 $(d(x,y)=0) \Rightarrow (x=y)$。

$$\forall k \in N^+,\ |\xi_k - \eta_k| \le \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p\right)^{1/p} = d(x,y) = 0$$

  所以 $\forall k \in N^+, \xi_k = \eta_k$，由此得 $x = y$。

- 证明 $\forall x, y, z \in \mathcal{X}\ s.t.\ d(x,y) \le d(x,z) + d(y,z)$.
  See Section 1.2-3 of the book by Kreyszig [1989].

V

题目

Deduce *additivity* and *conjugate homogeneity* in the second slot of the *inner product* $\langle \mathbf{u}, \mathbf{v} \rangle$.

(1). $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.

(2). $\langle \mathbf{u}, a\mathbf{v} \rangle = \bar{a} \langle \mathbf{u}, \mathbf{v} \rangle$.

注意 LaTeX 中内积的括号 $\langle \cdot, \cdot \rangle$ 可以由“\langle, \rangle”输入。

V

Additivity in the second slot:

$$\begin{aligned}\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}, \langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle &= \overline{\langle \mathbf{v} + \mathbf{w}, \mathbf{u} \rangle} \\ &= \overline{\langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle} \\ &= \overline{\langle \mathbf{v}, \mathbf{u} \rangle} + \overline{\langle \mathbf{w}, \mathbf{u} \rangle} \\ &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle .\end{aligned}$$

Conjugate homogeneity in the second slot:

$$\forall a \in \mathbb{F}, \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}, \langle \mathbf{u}, a\mathbf{v} \rangle = \overline{\langle a\mathbf{v}, \mathbf{u} \rangle} = \overline{a \langle \mathbf{v}, \mathbf{u} \rangle} = \overline{a}\overline{\langle \mathbf{v}, \mathbf{u} \rangle} = \overline{a} \langle \mathbf{u}, \mathbf{v} \rangle .$$

VI

题目

In the case of Euclidean ℓ_p norms in $\mathbb{R}^n (n \geq 2)$, show that the parallelogram law holds if and only if $p = 2$.

注意题目中的命题为，

$$(\forall n \geq 2) \Rightarrow ((p = 2) \Leftrightarrow (\ell_p \text{satisfies the parallelogram law}));$$

我们分别证明

$$(\forall n \geq 2) \Rightarrow ((p = 2) \Rightarrow (\ell_p \text{satisfies the parallelogram law})) \tag{1}$$

$$(\forall n \geq 2) \Rightarrow ((p = 2) \Leftarrow (\ell_p \text{satisfies the parallelogram law})) \tag{2}$$

VI

Parallelogram law

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2.$$

(1) If $p = 2$, let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, we have

$$2\|x\|_2^2 + 2\|y\|_2^2 = 2\sum_{i=1}^{n} |x_i|^2 + 2\sum_{i=1}^{n} |y_i|^2.$$

VI

$$
\begin{aligned}
\|x+y\|_2^2+\|x-y\|_2^2 &= \sum_{i=1}^{n}|x_i+y_i|^2+\sum_{i=1}^{n}|x_i-y_i|^2 \\
&= \sum_{i=1}^{n}(x_i+y_i)(\overline{x_i}+\overline{y_i})+\sum_{i=1}^{n}(x_i-y_i)(\overline{x_i}-\overline{y_i}) \\
&= \sum_{i=1}^{n}(|x_i|^2+x_i\overline{y_i}+y_i\overline{x_i}+|y_i|^2)+\sum_{i=1}^{n}(|x_i|^2-x_i\overline{y_i}-y_i\overline{x_i}+|y_i|^2) \\
&= 2\sum_{i=1}^{n}|x_i|^2+2\sum_{i=1}^{n}|y_i|^2.
\end{aligned}
$$

Thus, we obtain the parallelogram law

$$
\|x+y\|_2^2+\|x-y\|_2^2=2\|x\|_2^2+2\|y\|_2^2.
$$

VI

(2) 使用反证法，我们举反例证明

$$(\forall n \geq 2) \Rightarrow (\neg(p=2) \Rightarrow \neg(\ell_p \text{satisfies the parallelogram law}))$$

选择

$$x = \mathbf{e}_1,\ y = \mathbf{e}_2,$$

那么，

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 2 \cdot 2^{2/p} \neq 4 = 2\|x\|_p^2 + 2\|y\|_p^2, \text{ if } p \neq \infty;$$
$$\|x+y\|_\infty^2 + \|x-y\|_\infty^2 = 2 \neq 4 = 2\|x\|_\infty^2 + 2\|y\|_\infty^2.$$

证明完成。

VII

题目

Prove that a norm is induced by some inner product $\langle\cdot,\cdot\rangle$ if and only if the parallelogram law holds for every pair of $\mathbf{u},\mathbf{v}\in\mathcal{V}$.

- 必要性：如果范数 $\|\cdot\|$ 是由内积 $\langle\cdot,\cdot\rangle$ 诱导的，使用定理 0.94 得

$$\|\mathbf{u}-\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\langle\mathbf{u},\mathbf{v}\rangle,\quad\forall\mathbf{u},\mathbf{v}\in\mathcal{V}.\tag{3}$$

VII

替换 $\mathbf{v}$ with $-\mathbf{v}$ in (3) 那么,

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle, \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}, \tag{4}$$

将 (3) 和 (4) 相加，得平行四边形公式

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2, \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}.$$

VII

题目

Prove that a norm is induced by some inner product $\langle \cdot, \cdot \rangle$ if and only if the parallelogram law holds for every pair of $\mathbf{u}, \mathbf{v} \in \mathcal{V}$.

- 充分性：参考《实变函数论与泛函分析（下册)》，第六章，夏道行编。

VIII

题目

Tell a story about determinants from the viewpoint of problem-driven abstraction.

写得比较好的同学的参考:

考虑这样一个问题: 现有 n 个向量 $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$, 这些向量可以在 $\mathbb{R}^n$ 中张成一个超平形体. 我们考虑如何求得这个超平形体的有向体积. 首先我们给出这个超平形体的体积的定义: 根据体积的概念, 可以将超平形体的体积定义为一个满足以下条件的体积函数 $\delta : \mathbb{R}^n \to \mathbb{R}$.

VIII

(SVP-1). I 是 $n \times n$ 的单位阵，则 $\delta(I) = 1$.

(SVP-2). 当 $i \neq j$ 时有 $\mathbf{v}_i = \mathbf{v}_j$，则 $\delta(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = 0$.

(SVP-3). 体积函数 δ 是线性的，即 $\forall j = 1, \ldots, n, \forall c \in \mathbb{R}$

$$
\begin{aligned}
&\delta(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v} + c\mathbf{w}, \mathbf{v}_{j+1}, \mathbf{v}_n) \\
&= \delta(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n) + c\delta(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{w}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n).
\end{aligned}
$$

有了体积的定义之后，我们尝试是否可以通过体积函数的性质推导出体积函数的表达形式。

VIII

由于 $\forall \mathbf{v}_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,n}) = \sum_{k=1}^{n} v_{i,k}\mathbf{e}_k$，再根据体积函数的线性性，我们可以推导出

$$\delta(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \sum_{i_1, i_2, \ldots, i_n = 1}^{n} v_{i_1,1} v_{i_2,2} \cdots v_{i_n,n} \delta(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_n}). \tag{5}$$

显然，$\delta(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_n})$ 代表由 $\mathbb{R}^n$ 中标准正交基张成超平形体的有向体积，且 $|\delta(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_n})| = 1$。

VIII

需要讨论的是 $\delta(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_n})$ 的符号. 通过条件 (SVP-2) 和 (SVP-3), 可知 $\delta(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_n})$ 的符号和 $\mathbf{e}_{i_k}$ 的排序有关. 与 $\delta(I)$ 相比, $\delta(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_n})$ 只是将正交基的顺序重新作了排序。我们将交换其中两个基底的操作抽象为一个调换 τ, 由体积函数的定义可以推导出当对 $\delta(I)$ 做偶数次调换时, 不改变 $\delta(I)$ 的符号; 当作奇数次调换时, 将得到 $-\delta(I)$. 对于复杂的排序操作可以抽象为一个置换 σ. 这个置换可以分解为偶数个或者奇数个 τ, 因此可以定义置换 σ 的符号, 并且所有对 $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ 的置换构成一个循环群 S_n。

VIII

所以

$$\delta(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_n}) = \delta(\sigma I) = \mathrm{sgn}(\sigma)\delta(I) = \mathrm{sgn}(\sigma).$$

因此, 等式 (5) 可以写为

$$\delta(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} v_{\sigma(i),i},$$

根据行列式的 Leibniz 公式, 可以得到 $\delta(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \det(A)$, 其中矩阵 A 是以 $\mathbf{v}_i$ 作为第 i 列的 n 阶方阵. 所以我们发现一个事实: 行列式就是体积函数的唯一表示形式。

VIII

反之, 一个 n 阶方阵 A 的行列式的几何含义就是由这个矩阵的 n 个列向量在 $\mathbb{R}^n$ 空间中张成的有向体积. 而这个体积方向是通过计算行列式得到的正负来判断的.

Remark:

1. 行列式的概念是有向体积概念的数学抽象, 对于高维空间的有向体积, 通过几何方法计算是非常复杂繁琐的, 而通过行列式的计算避免了维度的复杂性, 不仅可以得到体积的大小, 还可以获得有向体积的方向.
2. 行列式的几何意义就是其列向量在 $\mathbb{R}^n$ 空间上张成的有向体积. 此外, 他还可以表示作用在一个有向体积的矩阵变换对原来体积的改变量.

VIII

3. 由于行列式表达有向体积, 所以
   (1). $\det(I) = 1$ 对应几何意义下单位正交基张成的有向体积为 1.
   (2). 矩阵 A 奇异 $\Leftrightarrow$ 构成矩阵 A 的列向量 $\mathbf{v}_j$ 线性相关 $\Leftrightarrow \det(A) = 0$, 对应几何意义下当 $i \neq j$ 时有 $\mathbf{v}_i = \mathbf{v}_j$, 则 $\delta(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = 0$.
   (3). $\det(kA) = k\det(A)$ 对应几何意义下将某一变量作伸缩, 从几何意义下容易得知 $\delta(\mathbf{v}_1, \ldots, k\mathbf{v}_j, \ldots, \mathbf{v}_n) = k\delta(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$.
4. 有向体积的方向是通过计算行列式的正负来判断的.
5. 调换 τ 是一个置换 σ, 所有置换 σ 构成一个循环群, 循环群是一个群.

VIII

题目

Tell a story about determinants from the viewpoint of problem-driven abstraction.

写得比较好的同学的参考:

The concept of a determinant helps us solve problems which occurred a lot of times. In the early days, determinant was linked to systems of linear equations and its name followed from the fact that it determines whether the system has a unique solution. Some uses for the determinant, from finding eigenvalues and eigenvectors, determining whether a set of vectors are linearly independent or not, to handling the coordinates in multiple integrals, come from the geometric meaning of the determinant.

VIII

Geometrically, the determinant of a matrix can be thought of as the volume of the parallelotope formed by the vectors that are columns of that matrix. Say the volume of the parallelotope formed by $\mathbf{v}_1, \dots, \mathbf{v}_n$ is denoted by $\delta(\mathbf{v}_1, \dots, \mathbf{v}_n)$.
Here come some properties.

1. $\delta(I) = 1$ because the volume of a unit cube is 1.
2. $\delta(\mathbf{v}_1, \dots, \mathbf{v}_n) = 0$ if $\mathbf{v}_i = \mathbf{v}_j$ for some $i \neq j$, because that corresponds to the parallelotope being flat.
3. $\delta(\mathbf{v}_1, \dots, a\mathbf{v}_i, \dots, \mathbf{v}_n) = a\delta(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_n)$ since doubling the length of any of the sides doubles the volume.

VIII

Here come some properties.

4. $\delta(\mathbf{v}_1, \ldots, \mathbf{u}_i + \mathbf{v}_i, \ldots, \mathbf{v}_n) = \delta(\mathbf{v}_1, \ldots, \mathbf{u}_i, \ldots, \mathbf{v}_n) + \delta(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_n)$, since cutting and pasting a parallelotope maintain its volume.
5. $\delta(\mathbf{v}_1, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_j \ldots, \mathbf{v}_n) = -\delta(\mathbf{v}_1, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_n)$ because from a geometric viewpoint, when you exchange two directions, you turn the parallelotope inside-out. Swapping vectors is the same as a reflection over the $(n-1)$-dimensional vector space between them.

VIII

Notice that:

1. The sign of the signed volume depends on the given order of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$;
2. The set of all $n \times n$ matrices with determinant not equal to zero is a group with matrix multiplication;
3. The signed volume is the determinant of a square matrix. The cofactor is the determinant of some submatrix.

题目

Consider the bisection method starting with the initial interval $[1.5, 3.5]$. In the following questions "the interval" refers to the bisection interval whose width changes across different loops.

- What is the width of the interval at the *n*th step?
- What is the maximum possible distance between the root *r* and the midpoint of the interval?

注意题目中说明“In the following questions, the interval refers to the bisection interval whose width changes across different loops”。

a. By Algorithm 2.5, we have

step	width
0	$w_0 = 3.5 - 1.5 = 2$
1	$w_1 = \frac{w_0}{2} = 1$
2	$w_2 = \frac{w_0}{2^2} = \frac{1}{2}$
$\vdots$	$\vdots$
n	$w_n = \frac{w_0}{2^n} = 2^{1-n}$

Thus, the interval width at the nth step is $\left(\frac{1}{2}\right)^{n-1}$.

b. Let c_n be the midpoint of the interval $[a_n, b_n]$ at the nth step of the bisection method. Since $|c_n - a_n| = \frac{1}{2}(\frac{1}{2})^{n-1}$ and $|c_n - b_n| = \frac{1}{2}(\frac{1}{2})^{n-1}$, the maximum distance between the root r and c_n is

$$\begin{aligned}|r - c_n| &< \frac{1}{2}|a_n - b_n| \\ &= \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} \\ &= \left(\frac{1}{2}\right)^{n}.\end{aligned}$$

II

题目

In using the bisection algorithm with its initial interval as $[a_0, b_0]$ we want to determine the root with its *relative* error no greater than ϵ. Assume $a_0 > 0$. Prove that the number of steps n must satisfy

$$n \geq \frac{\log(b_0 - a_0) - \log \epsilon - \log a_0}{\log 2} - 1.$$

II

易知，n 次迭代后相对误差 $\frac{|r-c_n|}{|r|}$ 在 $r=a_0$ 时取得最大值，所以只需当 $r=a_0$ 时使得 $\frac{|a_0-c_n|}{a_0}\leq\epsilon$ 即可。由二分法收敛定理知：

$$|a_0-c_n|=\frac{1}{2^{n+1}}(b_0-a_0),$$

所以有，

$$\begin{aligned}\frac{1}{2^{n+1}}\frac{(b_0-a_0)}{a_0}\leq\epsilon\Rightarrow\frac{(b_0-a_0)}{\epsilon a_0}\leq 2^{n+1}\\ &\Rightarrow\log(b_0-a_0)-\log(a_0)-\log(\epsilon)\leq(n+1)\log(2)\\ &\Rightarrow n\geq\frac{\log(b_0-a_0)-\log(a_0)-\log(\epsilon)}{\log(2)}-1.\end{aligned}$$

III

题目

Perform four iterations of Newton's method for the polynomial equation $p(x) = 4x^3 - 2x^2 + 3 = 0$ with the starting point $x_0 = -1$. Use a hand calculator and organize results of the iterations in a table.

III

By Algorithm 2.10, we have

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} = x_n - \frac{4x_n^3 - 2x_n^2 + 3}{12x_n^2 - 4x_n}.$$

$n+1$	n	$x_n - \frac{p(x_n)}{p'(x_n)}$		x_{n+1}
1	0	$-1 - \frac{4(-1)^3-2(-1)^2+3}{12(-1)^2-4(-1)}$	$=$	-0.812500
2	1	$-0.8125 - \frac{4(-0.8125)^3-2(-0.8125)^2+3}{12(-0.8125)^2-4(-0.8125)}$	$\approx$	-0.770804
3	1	$-0.7708 - \frac{4(-0.7708)^3-2(-0.7708)^2+3}{12(-0.7708)^2-4(-0.7708)}$	$\approx$	-0.768832
4	3	$-0.7688 - \frac{4(-0.7688)^3-2(-0.7688)^2+3}{12(-0.7688)^2-4(-0.7688)}$	$\approx$	-0.768828

IV

题目

Consider a variation of Newton's method in which only the derivative at x_0 is used,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$

Find C and s such that

$$e_{n+1} = Ce_n^s.$$

C 应为 C_n。

IV

用 α 表示函数 $f(x)$ 的一个根，由泰勒定理知在 x_n 和根 α 之间存在 ξ_n，使得

$$f(x_n) = f(\alpha) + e_n f^{'}(\xi_n),$$

因此，$f(x_n) = e_n f^{'}(\xi_n)$，将上式带入即得

$$e_{n+1} = x_{n+1} - \alpha = x_n - \alpha - \frac{f(x_n)}{f'(x_0)} = e_n - \frac{f(x_n)}{f'(x_0)} = e_n\left[1 - \frac{f'(\xi_n)}{f'(x_0)}\right].$$

由此 $C_n = \left[1 - \frac{f'(\xi_n)}{f'(x_0)}\right]$，$s = 1$.

V

题目

Within $(-\frac{\pi}{2}, \frac{\pi}{2})$, will the iteration $x_{n+1} = \tan^{-1} x_n$ converge?

注意 $\tan^{-1}(\cdot)$ 指 $\arctan(\cdot)$，及不能通过直接求导的方式说明 $\arctan(\cdot)$ 在区间 $[-\frac{\pi}{2}, \frac{\pi}{2}]$ 上为压缩映射。

V

Consider $h(x) = \tan^{-1}(x) - x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Since $h'(x) = \frac{1}{1+x^2} - 1 \leq 0$, $h(x)$ is monotonically decreasing, and $h(0) = 0$, we obtain that

$$\begin{cases} 0 \leq tan^{-1}(x) \leq x, & \text{if } x > 0; \\ 0 = tan^{-1}(x) = x, & \text{if } x = 0; \\ 0 \geq tan^{-1}(x) \geq x, & \text{if } x < 0; \end{cases}$$

- $x_0 > 0$:
  By the monotonic sequence theorem, the sequence $x_{n+1} = \tan^{-1}(x_n)$ is convergent so that we can write $\lim_{n\to\infty} x_n = c \in \mathbb{R}$. Then

  $$\lim_{n\to\infty}(x_{n+1} - \tan^{-1}(x_n)) = c - \tan^{-1} c = 0$$

  implies $c = 0$ because the equation $y = \tan^{-1} y$ only has one root.
- $x_0 < 0$: Similar as $x_0 > 0$.
- $x_0 = 0$: $\{x_n\}$ is a constant sequence.

VI

题目

Let $p > 1$. What is the value of the following continued fraction?

$$x = \frac{1}{p + \frac{1}{p + \frac{1}{p + \dots}}};$$

Prove that the sequence of values converges.

VI

- 记 $x_0=\frac{1}{p}$，$x_{n+1}=\frac{1}{p+x_n}$，我们寻找序列 $\{x_n\}$ 的极限，由 $p>1$，$x_0>0$ 知，$\forall n>0$，$x_n\in[0,1]$，在区间 $[0,1]$ 上考虑函数 $f(x)=\frac{1}{x+p}$，得
$$f^{'}(x)=\frac{-1}{(x+p)^2},$$
当 $x\in[0,1]$ 时，有 $|f^{'}(x)|\le\frac{1}{p^2}<1$，所以 $f(x)$ 为压缩映射，因此序列 $\{x_n\}$ 收敛，记极限为 c。
- 为求得序列的极限，使迭代式 $x_{n+1}=\frac{1}{p+x_n}$ 的 n 趋于 $+\infty$，解得 $c=\frac{-p\pm\sqrt{p^2+4}}{2}$，舍去小于 0 的解，得 $c=\frac{-p+\sqrt{p^2+4}}{2}$.

VII

题目

What happens in problem II if $a_0 < 0 < b_0$? Derive an inequality of the number of steps similar to that in problem II. In this case, is the relative error still an appropriate measure?

VII

易知，n 次迭代后绝对误差 $|r - c_n|$ 在 $r = a_0$ 时取得最大值，所以只需当 $r = a_0$ 时使得 $|a_0 - c_n| \leq \epsilon$ 即可。由二分法收敛定理知：

$$|a_0 - c_n| = \frac{1}{2^{n+1}}(b_0 - a_0),$$

所以有，

$$\begin{aligned}
&\frac{1}{2^{n+1}}(b_0 - a_0) \leq \epsilon \\
\Rightarrow & 2^{n+1} \geq \frac{b_0 - a_0}{\epsilon} \\
\Rightarrow & (n+1)\log(2) \geq \log(b_0 - a_0) - \log(\epsilon) \\
\Rightarrow & n \geq \frac{\log(b_0 - a_0) - \log(\epsilon)}{\log(2)} - 1.
\end{aligned}$$

VIII

题目

Consider solving $f(x) = 0$ by Newton's method with the starting point x_0 close to a root of multiplicity k. We assume that $f \in \mathcal{C}^{k+1}$.

(a) How can a multiple zero be detected by examining the behavior of the points $(x_n, f(x_n))$?

(b) Prove that if r is a zero of multiplicity k of the function f, then quadratic convergence in Newton's iteration will be restored by making this modification:

$$x_{n+1} = x_n - k\frac{f(x_n)}{f'(x_n)}.$$

答案暂时不确定。

IX

题目

Analysis of the secant method for a root of multiplicity k by assuming that it converges.

- Prove that if α is a zero of multiplicity $k > 1$ of the function f, the secant method only has linear convergence.
- Use the same argument to show that the convergence rate of the secant method is $\frac{\sqrt{5}+1}{2}$.

答案暂时不确定。

评分标准

1. 结果是否正确；
2. 有无 Makefile；
3. 用于求根的函数、最大迭代次数和精度是否都可作为变量输入；
4. 是否有单独的测试程序；
5. 代码是否严格，如 *const* 的使用等；

注：由于问题 A.d 的输入函数在区间内不连续，所以正确答案是拒绝它作为输入。

UML 类图

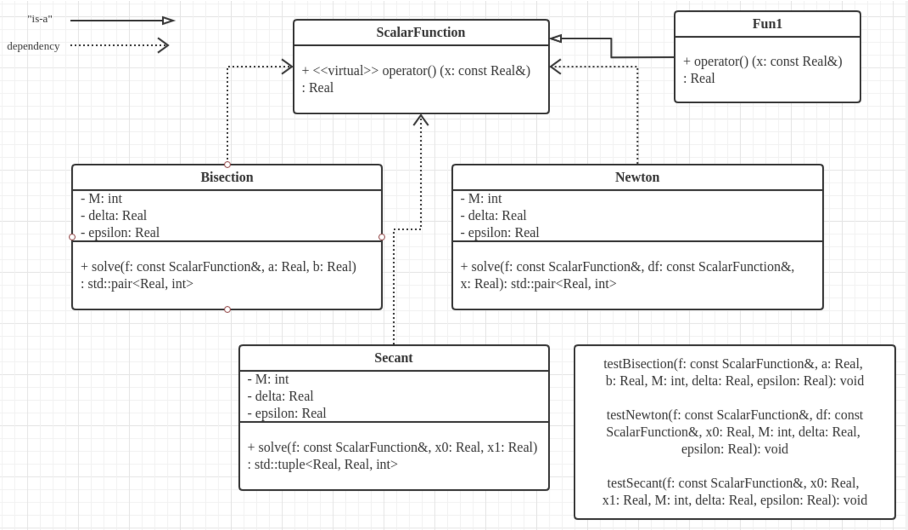


图: The overview design