I.Width of Interval During Bisecting Process

I-a.Width at nth step

We can Define h_n as the width at the nth step. The width of interval will be reduced to half of the original each step according to the definition of bisection. So $\mathrm{h}_n = \frac{2}{2^n} = \frac{1}{2^{n-1}}$

I-b.Maximun possible distance

If the program exits the loop at the nth step, then the maximum distance between the root and the midpoint of interval is $\mathrm{h}_n = \frac{1}{2^n}$.

II.Prove the Inequation about Step and Relative Error

We can define the midpoint of the interval as X_n, and the root as X, and the actual relative error as η.Then, suppose X_n lays in [a_n, b_n] after n times of bisection. It is obvious that the width of [a_n, b_n] is $\frac{b_0-a_0}{2^n}$ according to the first question. Notice that

$$\eta = \frac{X_n - X}{X} \leq \frac{|b_n - a_n|}{2X} = \frac{b_0 - a_0}{2^{n+1}X} \leq \frac{b_0 - a_0}{2^{n+1}a_0}$$

Because $\eta \leq \epsilon$ must be satisfied, it implies that

$$\frac{b_0 - a_0}{2^{n+1}a_0} \leq \epsilon$$

Then, apply logarithm to both sides, we can get

$$n \geq \frac{\log(b_0 - a_0) - \log\epsilon - \log a_0}{\log 2} - 1$$

as required.

III.Perform Four Iterations of Newton's Method

Step	x	f(x)
0	-1	-3
1	-0.8125	-0.46582
2	-0.770804	-0.0201379
3	-768832	-4.437084$\times 10^{-5}$
4	-0.768828	-2.07412$\times 10^{-10}$

IV.A Variation of Newton Method

I think we can't find out constant C and s such that $e_{n+1} = Ce_n^s$. For example, take $f(x) = x^2$, iterating from $x_0 = 1$. Since $x_1 = \frac{1}{2}$, $x_2 = \frac{3}{8}$, $x_3 = \frac{39}{128}$ and the root $r = 0$, $e_0 = 1$, $e_1 = \frac{1}{2}$, $e_2 = \frac{3}{8}$, $e_3 = \frac{39}{128}$.

$$e_1 = Ce_0^s \Longrightarrow C = \frac{1}{2}$$

$$e_2 = Ce_1^s \Longrightarrow s = \log_2\frac{4}{3}$$

But,

$$e_3 \neq \frac{1}{2} e_2^{\log_2 \frac{4}{3}}$$

Hence, I find a counterexample.

V.Whether the Iteration Is Convergent or Not

Define $f(x) = \arctan x$, then transform iterating formula into $x_{n+1} = f(x_n)$. It is easy to prove that $x > \arctan x$ when $x > 0$, in another word, $0 < x_{n+1} < x_n$ when $x_1 > 0$. So sequence $\{\mathrm{x}_n\}$ is monotonic and bounded. Using theorem 2.11, the sequence above is convergent. The situation on $x_1 < 0$ is similar to $x_1 > 0$ and sequence $\{x_n\}$ is constant when $x_1 = 0$. Therefore, the sequence is convergent on $(-\frac{\pi}{2}, \frac{\pi}{2})$.Hence proved.

VI.Convergent Value of A Sequence

We can conclude the iterating formula that is $x_{n+1} = \frac{1}{p+x_n}$ where $x_1 = \frac{1}{p}$ and $p > 1$. Define $g(x) = \frac{1}{p+x}$ where $p > 1$, so $x_{n+1} = g(x_n)$. Funtion g is a contractive mapping on $(0,\infty)$, because

$$|g(x) - g(y)| = |\frac{1}{x+p} - \frac{1}{y+p}| = |\frac{x-y}{(x+p)(y+p)}| \leq \frac{|x-y|}{p^2} = \lambda|x-y|$$

where $\lambda = \frac{1}{p^2} < 1$ and $\forall x, y \in (0, \infty)$. Then according to theorem 2.29, there $\exists\alpha$ as a fixed point of $g(x)$ that is

$$\alpha = \frac{\sqrt{p^2+4} - p}{2}$$

Furthermore, $|x_n - \alpha| \leq \frac{\lambda^n}{1-\lambda}|x_1 - x_0|$. It implies that x_n is a series converging to α, in another word, the value of

$$x = \frac{1}{p + \frac{1}{p+\frac{1}{p+\dots}}} = \lim_{n\to\infty} x_n = \alpha = \frac{\sqrt{p^2+4} - p}{2} \tag{1}$$

Hence we get the answer.

VII.What happens in II if $a_0 < 0 < b_0$

We must transform the inequation as

$$\eta = \frac{X_n - X}{X} \leq \frac{b_n - a_n}{2X} = \frac{b_n - a_n}{2^{n+1}X} \leq \frac{b_0 - a_0}{2^{n+1}|X|}$$

since we don't know the relation between $|a_0|$ and $|X|$. If $\eta \leq \epsilon$ is always true, then we can draw a conclusion

$$\eta \leq \frac{b_0 - 1_0}{2^{n+1}|X|} \leq \epsilon$$

Therefore obtaining

$$n \geq \frac{\log b_0 - a_0 - \log \epsilon - \log |X|}{\log 2} - 1$$

It is not an appropriate estimateion for the number of steps, because it can be a huge number when the actual value is near the origin.