# I. Scaled Integral of B-splines

By theorem 4.25, we have

$$\frac{(n+1)B_i^n(x)}{t_{i+n}-t_{i-1}} = \mathrm{d}B_i^{n+1}(x) + \frac{(n+1)B_{i+1}^n(x)}{t_{i+n+1}-t_i}\mathrm{d}x \tag{1}$$

$$\frac{(n+1)B_{i+1}^n(x)}{t_{i+n+1}-t_i} = \mathrm{d}B_{i+1}^{n+1}(x) + \frac{(n+1)B_{i+2}^n(x)}{t_{i+n+2}-t_{i+1}}\mathrm{d}x \tag{2}$$

$$\cdots \tag{3}$$

$$\frac{(n+1)B_{i+n}^n(x)}{t_{i+2n}-t_{i+n-1}} = \mathrm{d}B_{i+n}^{n+1}(x) + \frac{(n+1)B_{i+n+1}^n(x)}{t_{i+2n+1}-t_{i+n}}\mathrm{d}x \tag{4}$$

So add up LHS and RHS together, we can get

$$\frac{(n+1)B_i^n(x)}{t_{i+n}-t_{i-1}} = \sum_{k=0}^{n}\mathrm{d}B_{i+k}^{n+1}(x) + \frac{(n+1)B_{i+n+1}^n(x)}{t_{i+2n+1}-t_{i+n}}\mathrm{d}x$$

Then apply $\int_{t_{i-1}}^{t_{i+n}}$ to both sides,

$$\int_{t_{i-1}}^{t_{i+n}}\frac{(n+1)B_i^n(x)}{t_{i+n}-t_{i-1}} = \sum_{k=0}^{n}\int_{t_{i-1}}^{t_{i+n}}\mathrm{d}B_{i+k}^{n+1}(x) + \int_{t_{i-1}}^{t_{i+n}}\frac{(n+1)B_{i+n+1}^n(x)}{t_{i+2n+1}-t_{i+n}}\mathrm{d}x \tag{5}$$

$$\tag{6}$$

According to theorem 4.23,

$$B_{i+k}^{n+1}(x) = [t_i,\cdots,t_n](t-x)_+^n - [t_{i-1},\cdots,t_{i+n-1}](t-x)_+^n \tag{7}$$

which implies

$$\sum_{k=0}^{n}B_{i+k}^{n+1}(x) = [t_{i+n},\cdots,t_{i+2n+1}](t-x)_+^{n+1} - [t_{i-1},\cdots,t_{i+n}](t-x)_+^{n+1} \tag{8}$$

$$\sum_{k=0}^{n}\int_{t_{i-1}}^{t_{i+n}}\mathrm{d}B_{i+k}^{n+1}(x) = -[t_{i-1},\cdots,t_{i+n}](t-x)_+^{n+1}\Big|_{t_{i-1}}^{t_{i+n}} = [t_{i-1},\cdots,t_{i+n}](t-t_{i-1})_+^{n+1} \tag{9}$$

$$\int_{t_{i-1}}^{t_{i+n}}\frac{(n+1)B_{i+n+1}^n(x)}{t_{i+2n+1}-t_{i+n}}\mathrm{d}x = \frac{(n+1)}{t_{i+2n+1}-t_{i+n}}\int_{t_{i-1}}^{t_{i+n}}B_{i+n+1}^n(x)\mathrm{d}x = 0 \tag{10}$$

$$\tag{11}$$

Besides, suppose $f(x) = (x-t_{i-1})$ , it is obvious that $f^{(n+1)}(x) = 1$, by corollary 3.17, there exiests $\xi \in (t_{i-1}, t_{i+n})$ such that

$$\frac{1}{(n+1)!}f^{(n+1)}(\xi) = [t_{i-1},\cdots,t_{i+n}]f = 1 \tag{12}$$

Because $x \in (t_{i-1}, t_{i+n})$ . Take (9) and (10) and (12) into (5) as

$$\frac{1}{t_{i+n}-t_{i-1}}\int_{t_{i-1}}^{t_{i+n}}B_i^n(x) = \frac{1}{(n+1)} \tag{13}$$

Hence the scaled integral of $B_i^n(x)$ is free from index $i$.

# II. Symmetric Polynomials

## II-a. Verify the theorem for $m=4$ and $n=2$

By definition 4.29, when $m=4$ and $n=2$ ,

$$\tau(x_i, x_{i+1}, x_{i+2}) = x_i x_{i+1} + x_{i+1}x_{i+2} + x_i x_{i+2} + x_i^2 + x_{i+1}^2 + x_{i+2}^2 \tag{14}$$

$$\tag{15}$$

By definition 3.15, we can get a table of divided difference

$$\begin{array}{c|cccc} x_i & x_i^4 & & \\ x_{i+1} & x_{i+1}^4 & x_{i+1}^3 + x_{i+1}^2 x_i + x_{i+1} x_i^2 + x_i^3 & \\ x_{i+2} & x_{i+2}^4 & x_{i+2}^3 + x_{i+2}^2 x_{i+1} + x_{i+2} x_{i+1}^2 + x_{i+1}^3 & [x_i, x_{i+1}, x_{i+2}]x^4 \end{array}$$

where

$$[x_i, x_{i+1}, x_{i+2}]x^4 \tag{16}$$
$$= \frac{(x_{i+2}^3 + x_{i+2}^2 x_{i+1} + x_{i+2} x_{i+1}^2 + x_{i+1}^3) - (x_{i+1}^3 + x_{i+1}^2 x_i + x_{i+1} x_i^2 + x_i^3)}{x_{i+2} - x_i} \tag{17}$$
$$= x_i x_{i+1} + x_{i+1} x_{i+2} + x_i x_{i+2} + x_i^2 + x_{i+1}^2 + x_{i+2}^2 \tag{18}$$
$$\tag{19}$$

Hence $\tau(x_i, x_{i+1}, x_{i+2}) = [x_i, x_{i+1}, x_{i+2}]x^4$ as theorem 4.34 shows.

## II-b. Prove the theorem by the lemma

According to the lemma 4.33, we have

$$\tau_{k+1}(x_1, \cdots, x_{n+1}) = \tau_{k+1}(x_2, \cdots, x_{n+1}) + x_1 \tau_k(x_1, \cdots, x_{n+1}) \tag{20}$$
$$\tau_{k+1}(x_1, \cdots, x_{n+1}) = \tau_{k+1}(x_1, \cdots, x_n) + x_{n+1} \tau_k(x_1, \cdots, x_{n+1}) \tag{21}$$

Therefore, (21)-(20) is

$$(x_{n+1} - x_1)\tau_k(x_1, \cdots, x_{n+1}) = \tau_{k+1}(x_2, \cdots, x_{n+1}) - \tau_{k+1}(x_1, \cdots, x_n) \tag{22}$$
$$\tau_k(x_1, \cdots, x_{n+1}) = \frac{\tau_{k+1}(x_2, \cdots, x_{n+1}) - \tau_{k+1}(x_1, \cdots, x_n)}{x_{n+1} - x_1} \tag{23}$$

For $n = 0$ , we know that $\tau_m(x_i) = [x_i]x^m$ . Next, suppose the following equations is ture, when $n < m$ ,

$$\tau_{m-n}(x_i, \cdots, x_n) = [x_i, \cdots, x_n]x^m$$

Then we can prove recursively with (23),

$$\tau_{m-n-1}(x_i, \cdots, x_{i+n+1}) \tag{24}$$
$$= \frac{\tau_{m-n}(x_{i+1}, \cdots, x_{i+n+1}) - \tau_{m-n}(x_i, \cdots, x_{i+n})}{x_{i+n+1} - x_i} \tag{25}$$
$$= \frac{[x_{i+1}, \cdots, x_{i+n+1}]x^m - [x_i, \cdots, x_{i+n}]x^m}{x_{i+n+1} - x_i} \tag{26}$$
$$= [x_i, \cdots, x_{i+n+1}]x^m \tag{27}$$

Namely, this concludsion also holds for $n + 1$. Hence proved.

# Programming

## b.plot quadratic and complete cubic B-splines aginst original function

The data used to plot are stored in the quacard-data.txt and cubcard-data.txt. Results are presented as Figure1 , Figure2 , Figure3 , Figure4 , Figure 5 , Figure 6.

## c.interpolation error

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| quadratic | 1.38778e-17 | 0.0014184 | 0 | 0.120241 | 1.11022e-16 | 0.00197562 | 0 |
| cubic | 0.000669568 | 0 | 0.0205289 | 1.11022e-16 | 0.0205289 | 0 | 0.000669568 |

As the chart shows that cubic interpolation is more close to the original function than quadratic interpolation overall, because the maximum of cubic interpolating errors is less than the quadratic. However, the quadratic is also precise near two end points. The some error are close to the machine procision because we use double to store numbers which occupies 64 bits. Error may come from the procedures solving the linear algebra problems and system abandons some overflowing bits.

## d.three plots of heart function

Results are presented as Figure7 , Figure8 , Figure9.

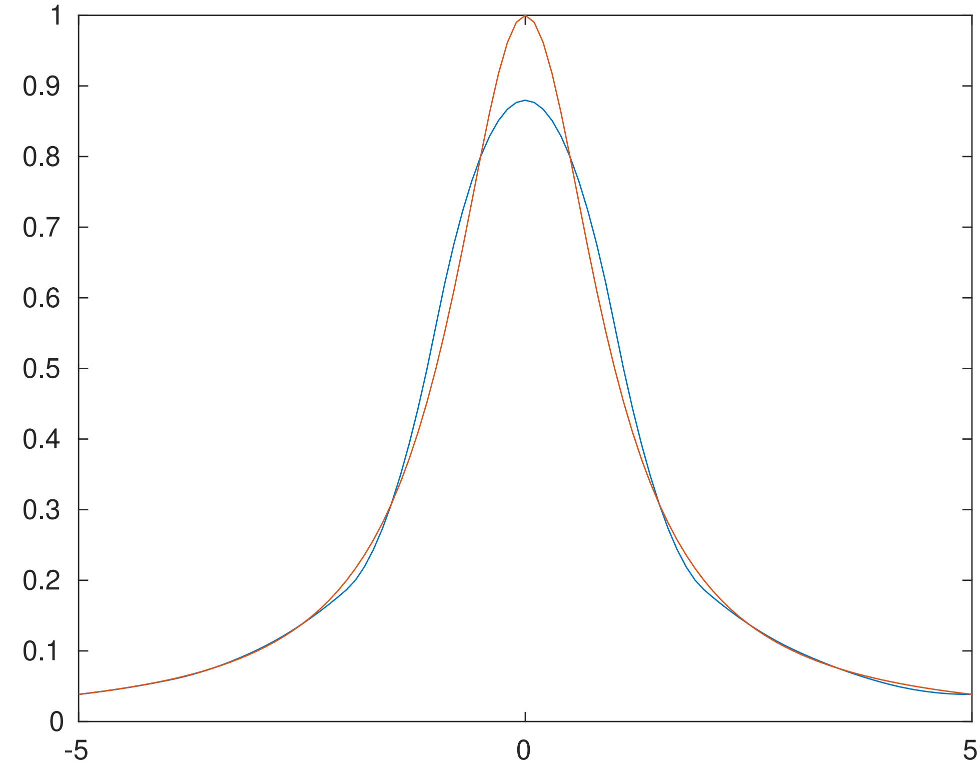


Figure 1: overall quadratic B-spline

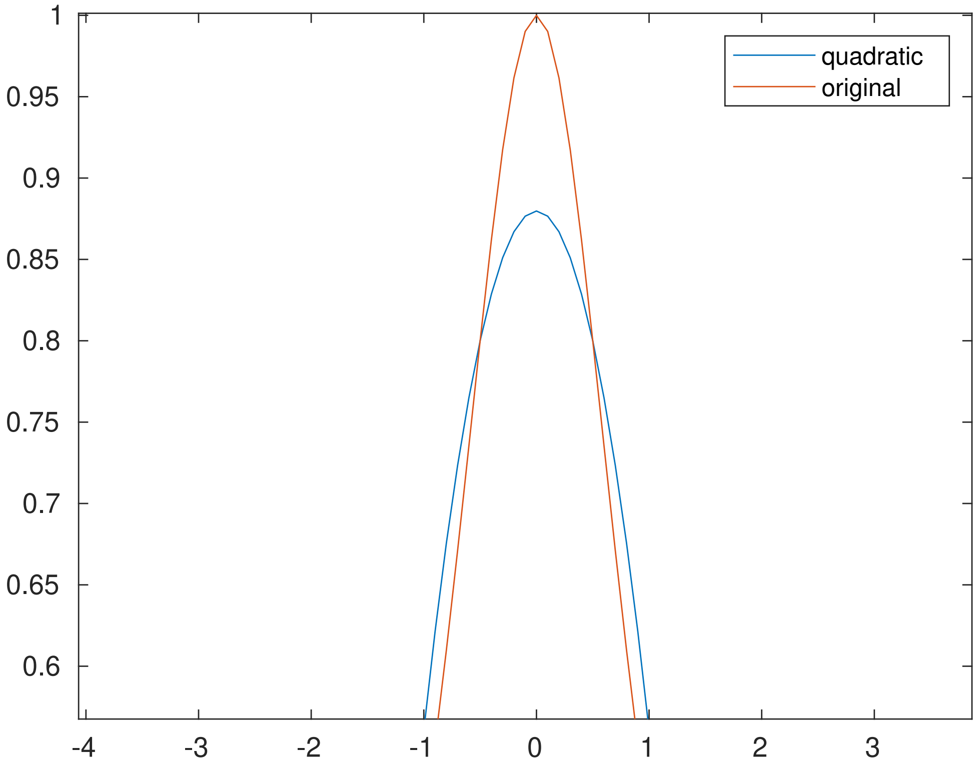


Figure 2: local quadratic B-spline

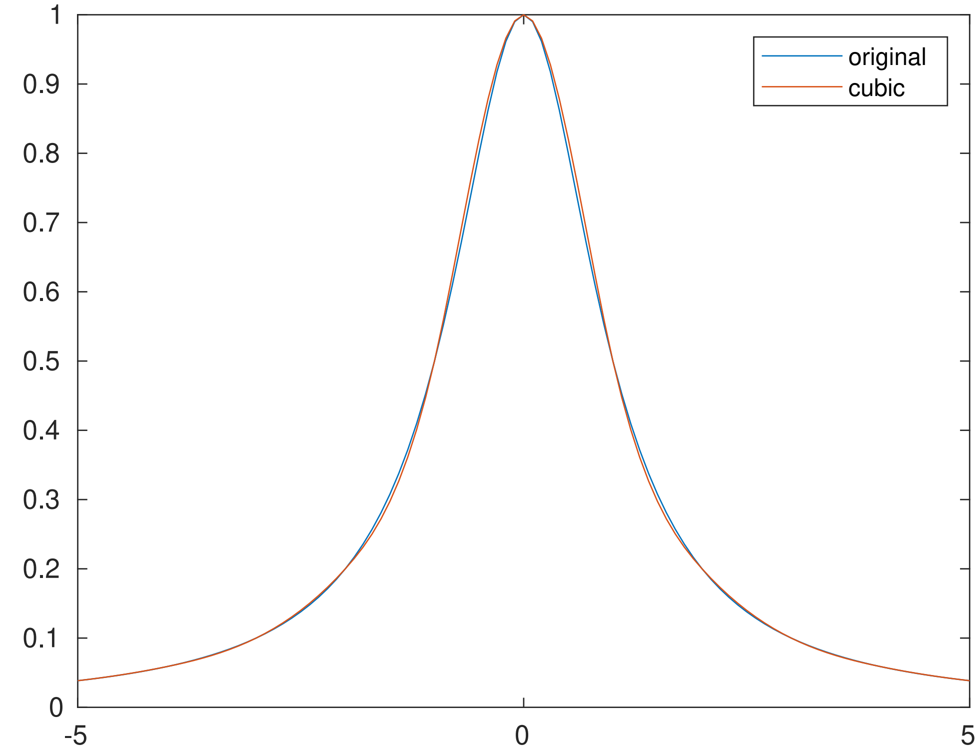


Figure 3: overall quadratic B-spline

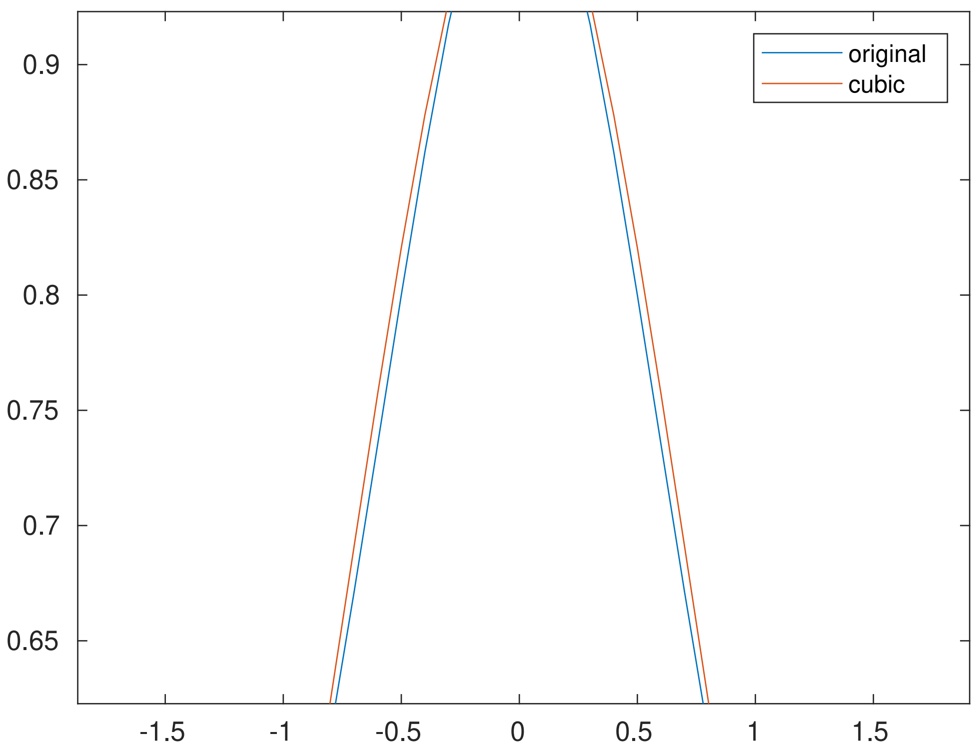


Figure 4: local quadratic B-spline

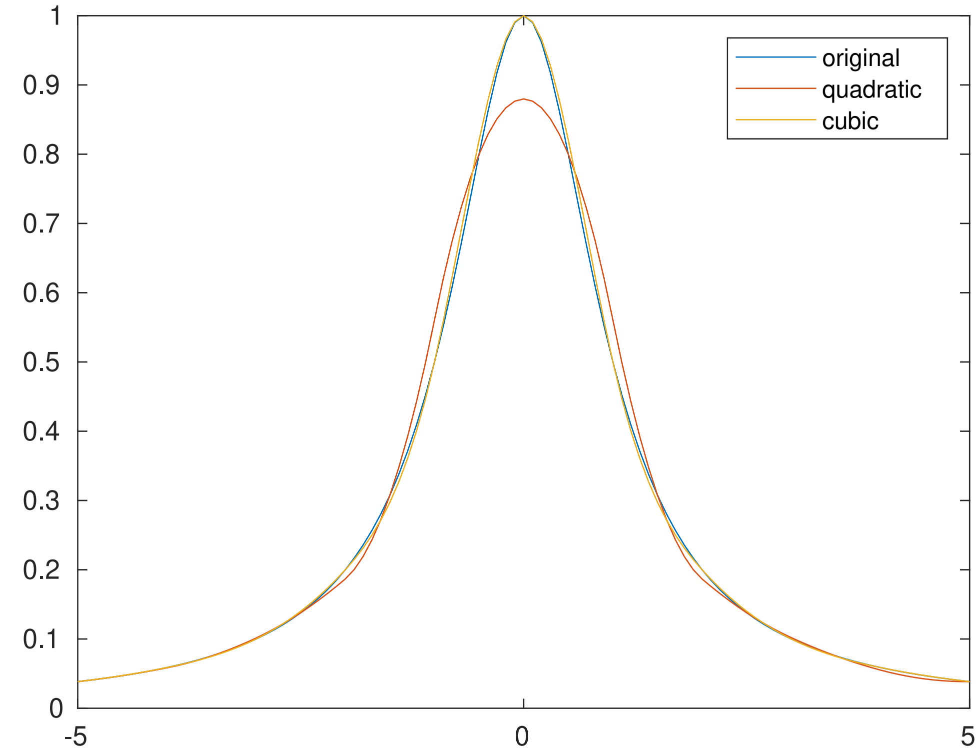


Figure 5: overall comparation

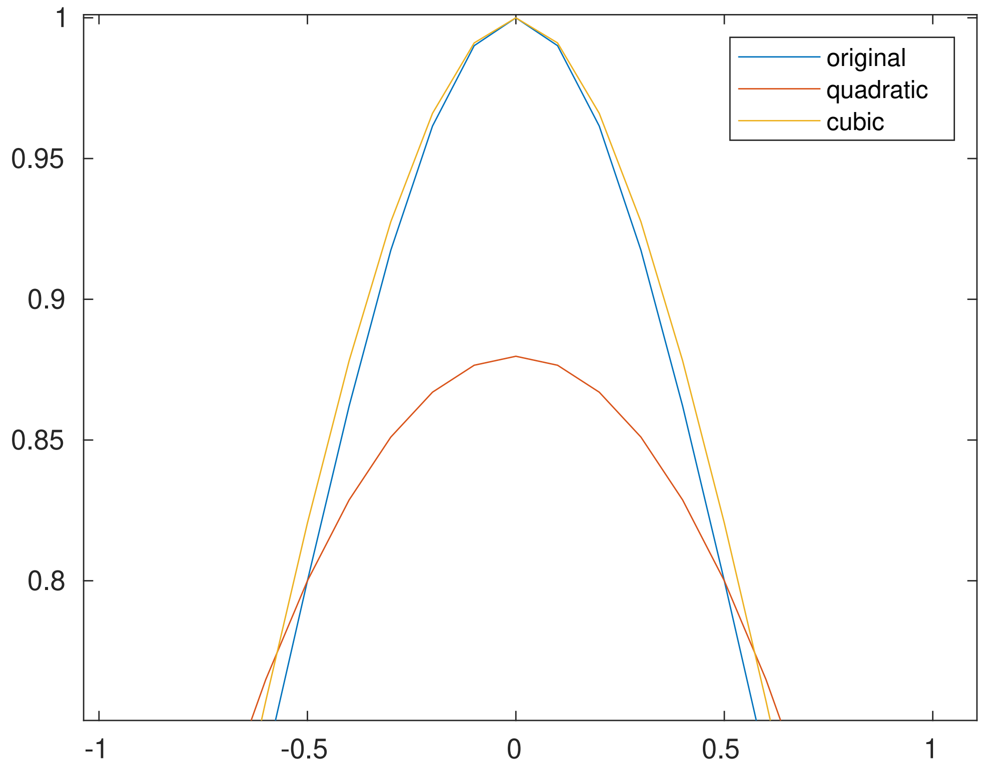


Figure 6: local comparation

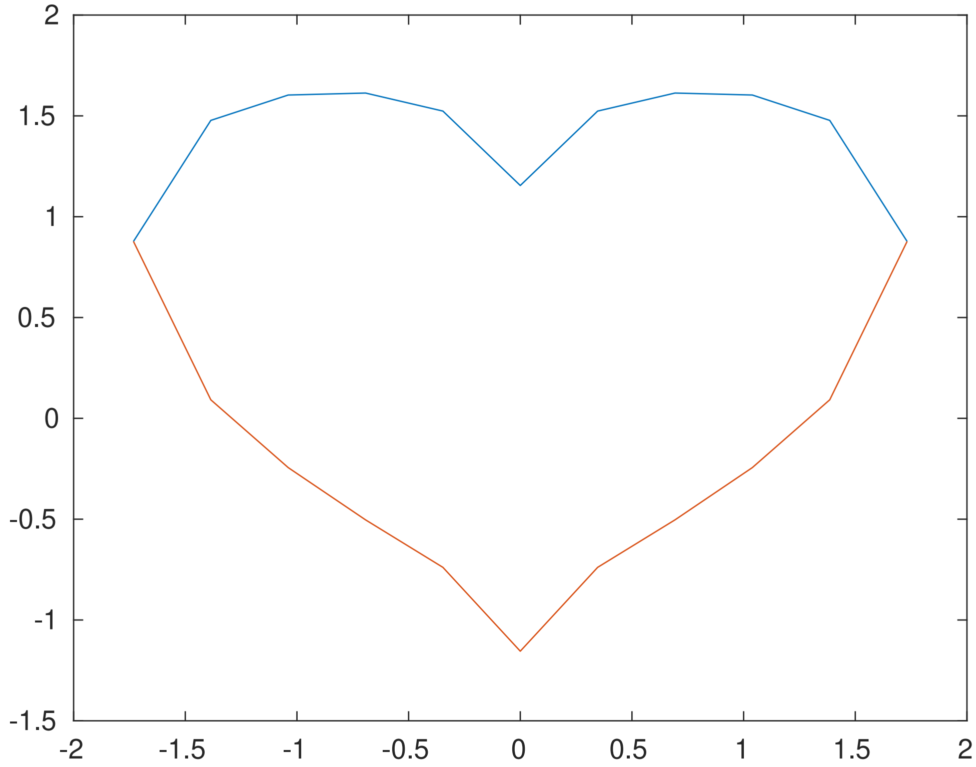


Figure 7: heart function for $n = 10$

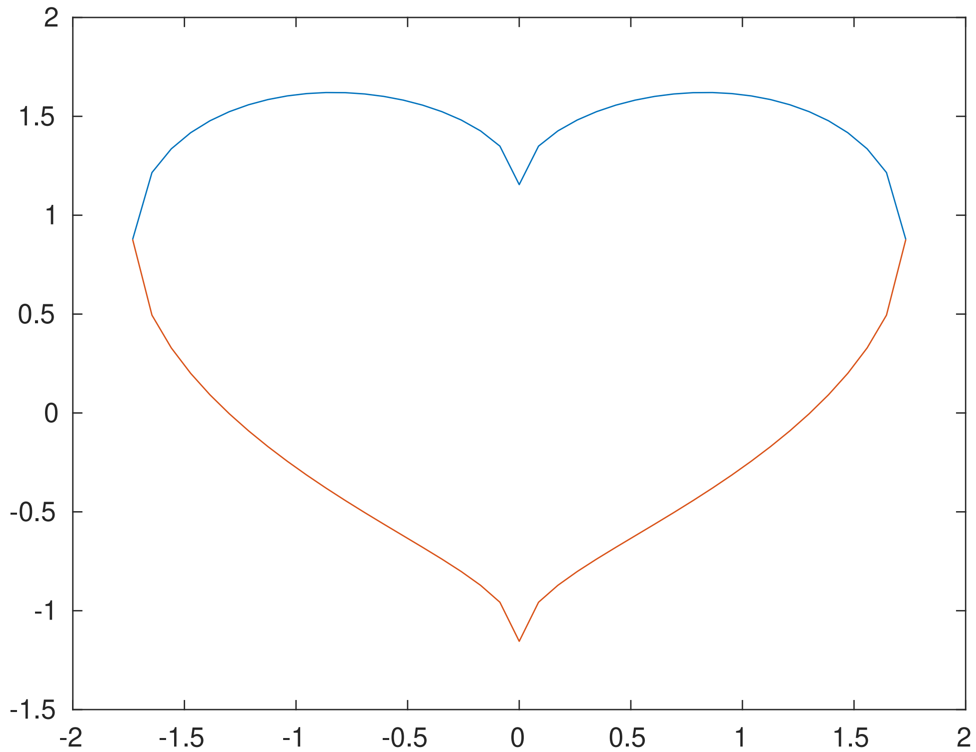


Figure 8: heart function for $n = 40$

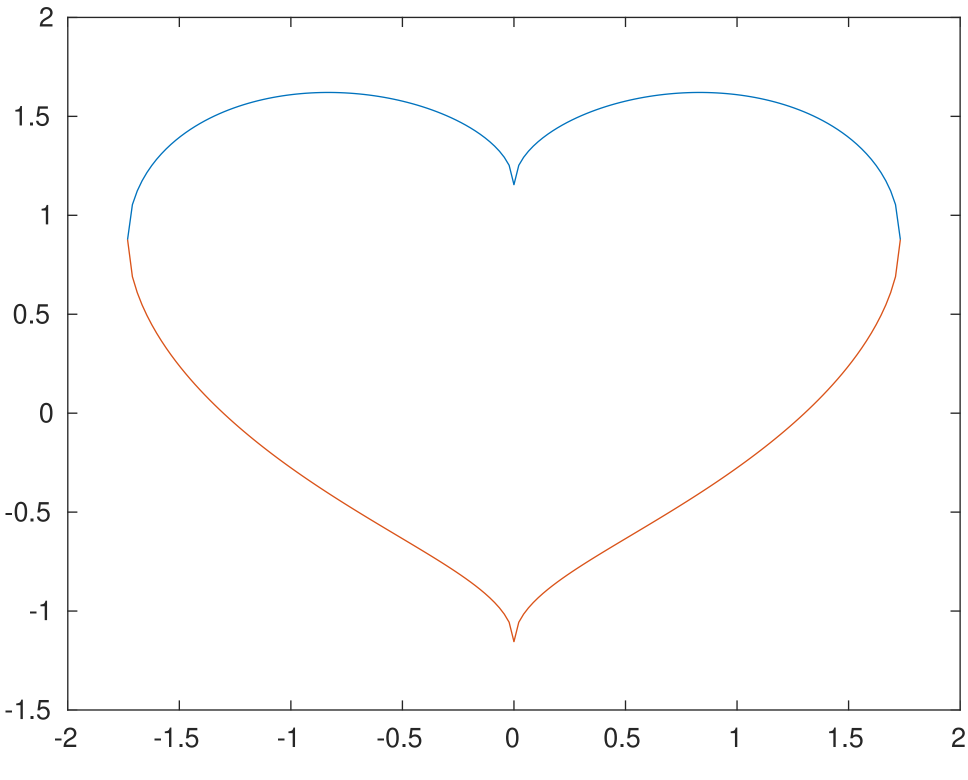


Figure 9: heart function for $n = 160$