I. Prove theorem 6.4 by assuming $\forall x \in (a, b)$ weight function $\rho(x) > 0$

a. prove $L^2_\rho[a, b]$ is a vector space

$\forall u, v, w \in L^2_\rho[a, b]$ and $\forall a, b \in \mathbb{F}$ we can get following results easily

$$u + v = v + u \tag{1}$$
$$(u + v) + w = u + (v + w) \tag{2}$$
$$(ab)u = a(bu) \tag{3}$$
$$0 + u = u \tag{4}$$
$$1u = u \tag{5}$$
$$(a + b)u = au + bv \tag{6}$$
$$a(u + v) = au + av \tag{7}$$

where $0 \in L^2_\rho[a, b]$ and $1 \in \mathbb{F}$. Besides, $\forall u$, there exists $-u \in L^2_\rho[a, b]$ such that $u + v = 0$, which satisfy all property of vector space. Hence proved.

b. prove $L^2_\rho[a, b]$ is inner product space

$\forall u, v, w \in L^2_\rho[a, b]$ and $\forall a \in \mathbb{F}$ assuming $\rho(x) > 0$, we can get following results easily

$$< v, v >= \int_a^b \rho(x)v(x)\overline{v(x)}\mathrm{d}x \geq 0 \tag{8}$$
$$\tag{9}$$

Because $\rho(x) > 0$, $< v, v >= 0$ iff $v(x) = 0$.

$$< u + v, w >= \int_a^b \rho(x)(u(x) + v(x))\overline{w(x)}\mathrm{d}x = \int_a^b \rho(x)v(x)\overline{w(x)}\mathrm{d}x + \int_a^b \rho(x)u(x)\overline{w(x)}\mathrm{d}x =< u, w > + < v, w > \tag{10}$$

$$< av, w >= \int_a^b a\rho(x)v(x)\overline{w(x)}\mathrm{d}x = a\int_a^b \rho(x)v(x)\overline{w(x)}\mathrm{d}x = a < v, w > \tag{11}$$

$$< v, w >= \int_a^b \rho(x)v(x)\overline{w(x)} = \int_a^b \rho(x)\overline{\overline{v(x)}w(x)}\mathrm{d}x = \overline{< w, v >} \tag{12}$$

Hence $L^2_\rho[a, b]$ is an inner product space.

c. prove $L^2_\rho[a, b]$ is norm space

From the definition of norm, we can know that $\forall v \in L^2_\rho[a, b]$

$$||v||_2 = (\int_a^b \rho(x)|v(x)|^2\mathrm{d}x)^{\frac{1}{2}} = (\int_a^b \rho(x)v(x)\overline{v(x)}\mathrm{d}x)^{\frac{1}{2}} = \sqrt{< v, v >} \tag{13}$$

Hence proved.

II. Consider Chebyshev polynimials of the first kind

a. Show that they are orthogonal on $[-1, 1]$

Because Chebyshev polinomials have form of $T_n(x) = \cos(n \arccos x)$, we can deduce

$$< T_n, T_m >= \int_a^b \frac{\cos(n \arccos x) \cos(m \arccos x)}{\sqrt{1 - x^2}}\mathrm{d}x \tag{14}$$

Then we can take $x = \cos\theta$ into the above equation as following, where $\theta \in [0, \pi]$

$$< T_n, T_m >= -\int_0^\pi \cos(m\theta)\cos(n\theta)\mathrm{d}\theta = -\int_0^\pi \frac{\cos(m + n)\theta + \cos(m - n)\theta}{2} = 0 \tag{15}$$

Hence proved.

b. Normalize the first three Chebyshev polynomials

The first three Chebyshev polynomials are as following

$$\begin{cases} u_1(x) = 1 \\ u_2(x) = x \\ u_3(x) = 2x^2 - 1 \end{cases}$$

We can deduce as following steps. Firstly,

$$v_1 = u_1 \tag{16}$$

$$u_1^* = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{\pi}} \tag{17}$$

Secondly,

$$v_2 = u_2 = x \tag{18}$$

$$u_2^* = \frac{v_2}{||v_2||} = \sqrt{\frac{2}{\pi}}x \tag{19}$$

Thirdly,

$$v_3 = u_3 = 2x^2 - 1 \tag{20}$$

$$||v_3|| = \sqrt{\int_{-1}^{1} \frac{(2x^2-1)^2}{\sqrt{1-x^2}} \mathrm{d}x} = \sqrt{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\sin^2\theta - 1)^2 \mathrm{d}\theta} = \sqrt{\frac{\pi}{2}} \tag{21}$$

$$u_3^* = \frac{v_3}{||v_3||} = \sqrt{\frac{2}{\pi}}(2x^2 - 1) \tag{22}$$

III. Least-square approxiamation of a continuous function

a. $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ with Fourier expansion

We select orthonormal polynomials in $\mathbb{P}_2$ as $u_1^* = \frac{1}{\sqrt{\pi}}$, $u_2 = \sqrt{\frac{2}{\pi}}x$ and $u_3 = \sqrt{\frac{2}{\pi}}(2x^2-1)$, then we can deduce that

$$< y, u_1^* >= \int_{-1}^{1} \frac{1}{\sqrt{\pi}} 2\mathrm{d}x = \frac{2}{\sqrt{\pi}} \tag{23}$$

$$< y, u_2^* >= \int_{-1}^{1} \sqrt{\frac{2}{\pi}} x\mathrm{d}x = 0 \tag{24}$$

$$< y, u_3^* >= \int_{-1}^{1} \sqrt{\frac{2}{\pi}}(2x^2-1)\mathrm{d}x = -\frac{2}{3}\sqrt{\frac{2}{\pi}} \tag{25}$$

Therefore, the quadratic approxiamation of y is $\varphi(x) = -\frac{8}{3\pi}x^2 + \frac{10}{3\pi}$.

b. $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ with normal equation

We select linearly independent a set of basis as $u_1 = 1$, $u_2 = x$ and $u_3 = x^2$, then

$$G(u_1, u_2, u_3) = \begin{pmatrix} \langle u_1, u_1 \rangle & \langle u_1, u_2 \rangle & \langle u_1, u_3 \rangle \\ \langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle & \langle u_2, u_3 \rangle \\ \langle u_3, u_1 \rangle & \langle u_3, u_2 \rangle & \langle u_3, u_3 \rangle \end{pmatrix} = \begin{pmatrix} \pi & 0 & \frac{\pi}{2} \\ 0 & \frac{\pi}{2} & 0 \\ \frac{\pi}{2} & 0 & \frac{3\pi}{8} \end{pmatrix}$$

$$c = \begin{pmatrix} \langle y, u_1 \rangle & \langle y, u_2 \rangle & \langle y, u_3 \rangle \end{pmatrix} = \begin{pmatrix} 2 & 0 & \frac{2}{3} \end{pmatrix}$$

As a result, we can get cofficents matrix by solving $Ga^T = c^T$

$$a = \begin{pmatrix} \frac{10}{3\pi} & 0 & -\frac{8}{3\pi} \end{pmatrix}$$

Therefore, the quadratic approxiamation of y is $\varphi(x) = -\frac{8}{3\pi}x^2 + \frac{10}{3\pi}$.

IV. Discrete least square via orthonormal polynomials

a. Construct orthonormal polynomials by the Gram-Schmidt process

The set of basis is as following

$$\begin{cases} u_1(x) = 1 \\ u_2(x) = x \\ u_3(x) = x^2 \end{cases}$$

We can deduce as following steps. Firstly,

$$v_1 = u_1 = 1 \tag{26}$$
$$||v_1|| = 2\sqrt{3} \approx 3.46 \tag{27}$$
$$u_1^* = \frac{v_1}{||v_1||} = \frac{\sqrt{3}}{6} \approx 0.29 \tag{28}$$

Secondly,

$$v_2 = u_2 - < u_2, u_1^* > u_1^* = x - \frac{13}{2} \tag{29}$$
$$||v_2|| = \sqrt{143} \approx 11.96 \tag{30}$$
$$u_2^* = \frac{v_2}{||v_2||} = \frac{x}{\sqrt{143}} - \frac{\sqrt{143}}{22} \approx \frac{x}{11.96} - 0.54 \tag{31}$$

Thirdly,

$$v_3 = u_3 - \langle u_3, u_2^* \rangle u_2^* - \langle u_3, u_1^* \rangle u_1^* \approx x^2 - 13x + 30.3 \tag{32}$$
$$||v_3|| \approx 36.53 \tag{33}$$
$$u_3^* = \frac{v_3}{||v_3||} = \frac{1}{36.53}x^2 - \frac{13}{36.53}x + \frac{30.3}{36.53} \tag{34}$$

b. Find the best approxiamation $\widehat{\varphi} = \sum_{i=0}^{2} a_i x^i$

$$G(u_1^*, u_2^*, u_3^*) = \begin{pmatrix} \langle u_1^*, u_1^* \rangle & \langle u_1^*, u_2^* \rangle & \langle u_1^*, u_3^* \rangle \\ \langle u_2^*, u_1^* \rangle & \langle u_2^*, u_2^* \rangle & \langle u_2^*, u_3^* \rangle \\ \langle u_3^*, u_1^* \rangle & \langle u_3^*, u_2^* \rangle & \langle u_3^*, u_3^* \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$c = \begin{pmatrix} \langle y, u_1^* \rangle & \langle y, u_2^* \rangle & \langle y, u_3^* \rangle \end{pmatrix} = \begin{pmatrix} 481.98 & 55.03 & 328.84 \end{pmatrix}$$

Then the normal equation yield

$$a = c = \begin{pmatrix} 481.98 & 55.03 & 328.84 \end{pmatrix}$$

Hence,

$$\varphi(x) = 328.84u_3^* + 55.03u_2^* + 481.98u_1^* = 9.01x^2 - 112.42x + 382.82 \tag{35}$$

which is very similar to the answer in the note.

c. Suppose there are other tables of sales. Which calculations can be reused?

The orthonormal polynomials and Gram matrix can be reused. But we need to recalculate

$$c = \begin{pmatrix} \langle y, u_1^* \rangle & \langle y, u_2^* \rangle & \langle y, u_3^* \rangle \end{pmatrix}$$

The biggest advantage of orthonormal polynomials is that you can get the coefficents a as soon as you can get matrix c without solving equation $a = G^{-1}c$, because we have already known that G^{-1} is an identity matrix.

Programming

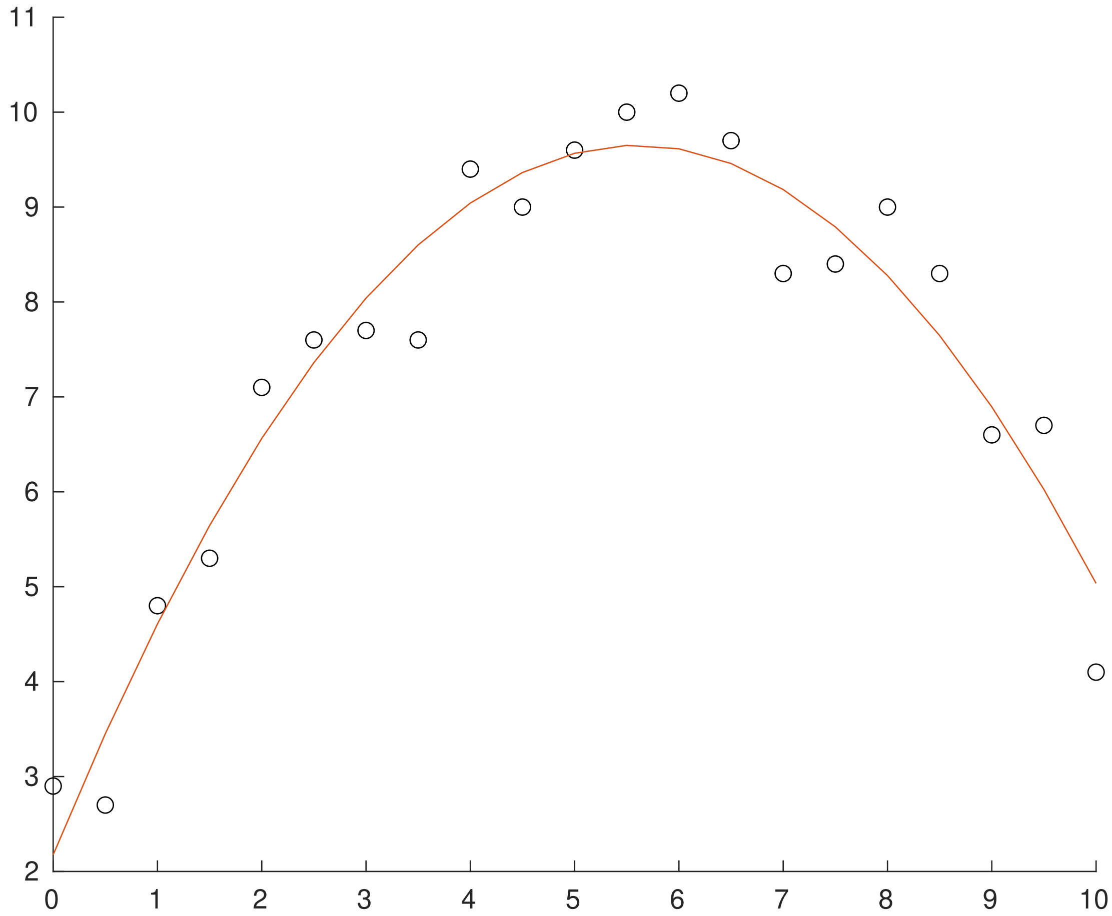

Figure 1: Discrete Least Square via normal equations

The best approximation I find is $\varphi(x) = -0.238444 * x^2 + 2.67041 * x + 2.17572$.