Numerical Analysis Homework #8
due 2020 JUN 02, 9:50 a.m.

1 Assignments

Caution:

- To get full credit, *you must write down sufficient intermediate steps*, only giving the final answer earns you no credit!

- Please make sure that your handwriting is recognizable, otherwise you only get partial credit for the recognizable part.

I. Prove Theorem 6.3 by assuming that $\forall x \in (a, b)$ the weight function $\rho(x) > 0$.

II. Consider the Chebyshev polynomials of the first kind.

   (a) Show that they are orthogonal on $[-1, 1]$ with respect to the inner product in Theorem 6.3 with the weight function $\rho(x) = \frac{1}{\sqrt{1-x^2}}$.

   (b) Normalize the first three Chebyshev polynomials to arrive at an orthonormal system.

III. Least-square approximation of a continuous function. Approximate the circular arc given by the equation $y(x) = \sqrt{1 - x^2}$ for $x \in [-1, 1]$ by a quadratic polynomial with respect to the inner product in Theorem 6.3.

   (a) $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ with Fourier expansion,

   (b) $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ with normal equations,

IV. Discrete least square via orthonormal polynomials. Consider the example on the table of sales record in the notes.

   (a) Starting from the independent list $(1, x, x^2)$, construct orthonormal polynomials by the Gram-Schmidt process using

$$\langle u(t), v(t) \rangle = \sum_{i=1}^{N} \rho(t_i) u(t_i) v(t_i) \qquad (1)$$

   as the inner product with $N = 12$ and $\rho(x) = 1$.

   (b) Find the best approximation $\hat{\varphi} = \sum_{i=0}^{2} a_i x^i$ such that $\|y - \hat{\varphi}\| \le \|y - \sum_{i=0}^{2} b_i x^i\|$ for all $b_i \in \mathbb{R}$. Verify that $\hat{\varphi}$ is the same as that of the example on the table of sales record in the notes.

   (c) Suppose there are other tables of sales record in the same format as that in the example . Values of N and x_i's are the same, but the values of y_i's are different. Which of the above calculations can be reused? Which cannot be reused? What advantage of orthonormal polynomials over normal equations does this reuse imply?

The first three problems weigh 6 points each while the last problem weighs 12 points.

2 C++ programming

Write a `C++` function to perform discrete least square via normal equations. Your subroutine should take two arrays x and y as the input and output three coefficients a_0, a_1, a_2 that determines a quadratic polynomial as the best fitting polynomial in the sense of least squares with the weight function $\rho = 1$.

Run your subroutine on the following data.

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0
y	2.9	2.7	4.8	5.3	7.1	7.6	7.7
x	3.5	4.0	4.5	5.0	5.5	6.0	6.5
y	7.6	9.4	9.0	9.6	10.0	10.2	9.7
x	7.0	7.5	8.0	8.5	9.0	9.5	10.0
y	8.3	8.4	9.0	8.3	6.6	6.7	4.1

This programming assignment weighs 10 points. Thus the total point of this homework is thus 40.

3 Extra credits

Additional 10% credits will be given to you if you typeset your solutions in LaTeX. You are welcome to use the LaTeX template available on my webpage. You can also get partial extra credit for typesetting solutions of *some* problems.

Note: If you choose to typeset your solutions in LaTeX, you still need to turn in a hard copy in class. In addition, please upload your latex source (.tex), supporting files, and `C++` program in a single zip file (**format**: `YourName_Homework8.zip`) to the course email `NumApproximation@163.com`.