I. Determine $p \in \mathbb{P}_3$

We can suppose

$$p(x) = a_1x^3 + a_2x^2 + a_3x$$

Because $s(x) \in \mathbb{C}^2[0, 2]$, then p and s derivatives are equal at $x = 1$

$$\begin{cases} a_1 + a_2 + a_3 = 1 \\ 3a_1 + 2a_2 + a_3 = -3 \\ 6a_1 + 2a_2 = 6 \end{cases}$$

We can get these coefficients easily, that is

$$\begin{cases} a_1 = 7 \\ a_2 = -18 \\ a_3 = 12 \end{cases}$$

Namely, $p(x) = 7x^3 - 18x^2 + 12x$, where $x \in [0, 1]$. Functiion $s(x)$ isn't a natrual cubic spline since $p''(0) = -36$.

II. Interpolating f on $[a, b]$ with a quadratic spline $s \in \mathbb{P}_2^1$

a.Why an additional condition is needed to determine s uniquely

We can suppose $p(x) = a_1x^2 + a_2x + a_3$ where $x \in [x_i, x_{i+1}]$ and $a_1 \neq 0$, $\forall i \in \{1, 2, \cdots, n-1\}$. There are three coefficients, howerever, only can get two equations

$$\begin{cases} a_1x_i^2 + a_2x_i + a_3 = f_i \\ a_1x_{i+1}^2 + a_2x_{i+1} + a_3 = f_{i+1} \end{cases}$$

according to given conditions. So s can't be determined uniquely.

b.determine p_i and m_i

Suppose $p_i(x) = a_1(x - x_i)^2 + a_2(x - x_i) + f_i$ where $x \in [x_i, x_{i+1}]$, then

$$\begin{cases} a_1(x_{i+1} - x_i)^2 + a_2(x_{i+1} - x_i) + f_i = f_{i+1} \\ a_2 = m_i \end{cases}$$

We cat get the explicit coefficient by solving linear equations

$$p_i(x) = \frac{f[x_i, x_{i+1}] - m_i}{x_{i+1} - x_i}(x - x_i)^2 + m_i(x - x_i) + f_i$$

c.show how m_i can be computed

The the first-order derivative of $p_i(x)$ is

$$p_i'(x) = 2a_1(x - x_i) + a_2$$

If $m_i = p_i'(x_i)$, we can know the answer recursively

$$m_{i+1} = m_i + 2\frac{f[x_i, x_{i+1}] - m_i}{x_{i+1} - x_i}$$

III. Determine $s_2(x)$ on $[-1, 1]$ and how to chose c such that $s(1) = -1$

Suppose $s_2(x) = a_1x^3 + a_2x^2 + a_3x + (1 + c)$. It is obvious that $s_1'(0) = 3c$, $s_1''(0) = 6c$ and $s_2''(1) = 0$, then

$$\begin{cases} a_1 = -c \\ a_2 = 3c \\ a_3 = 3c \end{cases}$$

since $s(x) \in \mathbb{P}_3^2$. Namely, $s_2(x) = -cx^3 + 3cx^2 + 3cx + (1 + c)$. If $s(1) = -1$, it implies $c = -\frac{1}{3}$.

IV.Consider $f(x) = \cos\left(\frac{\pi}{2}x\right)$ with $x \in [-1, 1]$

a.determine natrual cubic spline on knots $-1, 0, 1$

Suppose $p_1(x) = a_1(x+1)^3 + a_2(x+1)^2 + a_3(x+1)$, where $x \in [-1, 0]$. Since p_1 is a natrual cubic spline, $p_1''(-1) = 0$, $p_1(0) = f(0)$, then take them into p_1 and we get

$$\begin{cases} a_1 + a_2 + a_3 = 1 \\ a_2 = 0 \end{cases}$$

So we can rewrite p_1 as $p_1(x) = a_1(x+1)^3 + (1-a_1)(x+1)$, where $x \in [-1, 0]$. Similarly, on $[0, 1]$, suppose

$$p_2(x) = b_1(x-1)^3 + b_2(x-1)^2 + b_3(x-1)$$

Through $p_2''(1) = 0$ and $p_2(0) = 1$, we can get

$$\begin{cases} b_2 = 0 \\ b_1 + b_2 + b_3 = -1 \end{cases}$$

Namely, $p_2(x) = b_1(x-1)^3 - (b_1+1)(x-1)$.

Function $s(x)$ must satisfy $p_1'(0) = p_2'(0)$ and $p_1''(0) = p_2''(0)$, so we can know

$$\begin{cases} a_1 + 1 = b_1 \\ a_1 = -b_1 \end{cases}$$

Consequently, $a_1 = -\frac{1}{2}$ and $b_1 = \frac{1}{2}$. Take them into $s(x)$,

$$\text{s(x)=}\begin{cases} -\dfrac{1}{2}(x+1)^3 + \dfrac{3}{2}(x+1), x \in [-1, 0] \\ \dfrac{1}{2}(x-1)^3 - \dfrac{3}{2}(x-1), x \in [0, 1] \end{cases}$$

b.verify natural cubic splines have the minimal total bending energy

We can get the second-order derivatives of $s(x)$ from the discussion above

$$s''(x) = \begin{cases} -3(x+1), x \in [-1, 0] \\ 3(x-1), x \in [0, 1] \end{cases}$$

So it is obvious that

$$\int_{-1}^{1} [s''(x)]^2 \mathrm{d}x = 6 \tag{1}$$

(1) When $g(x) = -x^2 + 1$, $g''(x) = -2$, we can conclude

$$\int_{-1}^{1} [s''(x)]^2 \mathrm{d}x < \int_{-1}^{1} [g''(x)]^2 \mathrm{d}x = 8 \tag{2}$$

(2) When $g(x) = \cos\left(\frac{\pi}{2}x\right)$, $g''(x) = -\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}x\right)$, we can conclude

$$\int_{-1}^{1} [s''(x)]^2 \mathrm{d}x < \int_{-1}^{1} [g''(x)]^2 \mathrm{d}x = \frac{\pi^4}{16} \tag{3}$$

V.The quadratic B-spline $B_i^2(x)$

a.derive the expression of $B_i^2(x)$

According to definition 4.28 and example 4.7,

$$B_i^1(x) = \begin{cases} \dfrac{x - t_{i-1}}{t_i - t_{i-1}}, x \in (t_{i-1}, t_i] \\ \dfrac{t_{i+1} - x}{t_{i+1} - t_i}, x \in (t_i, t_{i+1}] \end{cases}$$

$$B_{i+1}^1(x) = \begin{cases} \frac{x-t_i}{t_{i+1}-t_i}, x \in (t_i, t_{i+1}] \\ \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}}, x \in (t_{i+1}, t_{i+2}] \end{cases}$$

$$B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} B_i^1(x) + \frac{t_{i+2}-x}{t_{i+2}-t_i} B_{i+1}^1(x) \tag{4}$$

So we can easily get the answer,

$$B_i^2(x) = \begin{cases} \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})}, & x \in (t_{i-1}, t_i] \\ \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(x-t_i)(t_{i+2}-x)}{(t_{i+2}-t_i)(t_{i+1}-t_i)}, & x \in (t_i, t_{i+1}] \\ \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}, & x \in (t_{i+1}, t_{i+2}] \\ 0, & otherwise \end{cases}$$

b.verify $\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(x)$ is continuous at t_i and t_{i+1}

From the explicit expression of $B_i^2(x)$, we can get its derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(x) = \begin{cases} \frac{2x-2t_{i-1}}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})}, & x \in (t_{i-1}, t_i] \\ \frac{-2x+t_{i-1}+t_{i+1}}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{-2x+t_i+t_{i+2}}{(t_{i+2}-t_i)(t_{i+1}-t_i)}, & x \in (t_i, t_{i+1}] \\ \frac{2x-2t_{i+2}}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}, & x \in (t_{i+1}, t_{i+2}] \\ 0, & otherwise \end{cases}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(t_i) = \frac{2t_i-2t_{i-1}}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} = \frac{-2t_i+t_{i-1}+t_{i+1}}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{-t_i+t_{i+2}}{(t_{i+2}-t_i)(t_{i+1}-t_i)} = \frac{2}{t_{i+1}-t_{i-1}} \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(t_{i+1}) = \frac{t_{i-1}-t_{i+1}}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{-2t_{i+1}+t_i+t_{i+2}}{(t_{i+2}-t_i)(t_{i+1}-t_i)} = \frac{2t_{i+1}-2t_{i+2}}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} = \frac{2}{t_i-t_{i+2}} \tag{6}$$

Hence, $\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(x)$ is continuous at $x = t_i$ and $x = t_{i+1}$.

c.show that only one $x^* \in (t_{i-1}, t_{i+1})$ satisfies $\frac{d}{dx} B_i^2(x^*) = 0$

Notice that $\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(t_{i-1}) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(x)$ increases on $(t_{i-1}, t_i]$ strictly, so there is no zero of $\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(x)$ in this interval.

As for $(t_i, t_{i+1}]$, $\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(x)$ decreases on this interval strictly. Besides, $\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(t_i) = \frac{2}{t_{i+1}-t_{i-1}} > 0$ and $\frac{2}{t_i-t_{i+2}<0}$,

which indicates that $x^* = \frac{t_{i+1}t_{i+2}-t_{i-1}t_i}{t_{i+2}+t_{i+1}-t_{i-1}-t_i}$ is the unique zero of $\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(x) = 0$.

d.show $B_i^2(x) \in [0, 1)$

From the discussion aboove, we know $\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(x)$ increases on $[t_{i-1}, x^*]$ and decreases on $[x^*, t_{i+2}]$, so

$$\min B_i^2(x) = \min \{B_i^2(t_i), B_i^2(t_{i+2})\} = 0 \tag{7}$$

$$\max B_i^2(x) = B_i^2(x^*) = \frac{t_{i+2}-t_{i-1}}{t_{i+2}+t_{i+1}-t_{i-1}-t_i} < \frac{t_{i+2}-t_{i-1}}{t_{i+2}-t_{t-1}} = 1 \tag{8}$$

Hence, $0 \le B_i^2(x) \le B_i^2(x^*) < 1$, namely , $B_i^2(x) \in [0, 1)$ has proved.

e.Plot $B_1^2(x)$ for $t_i = i$

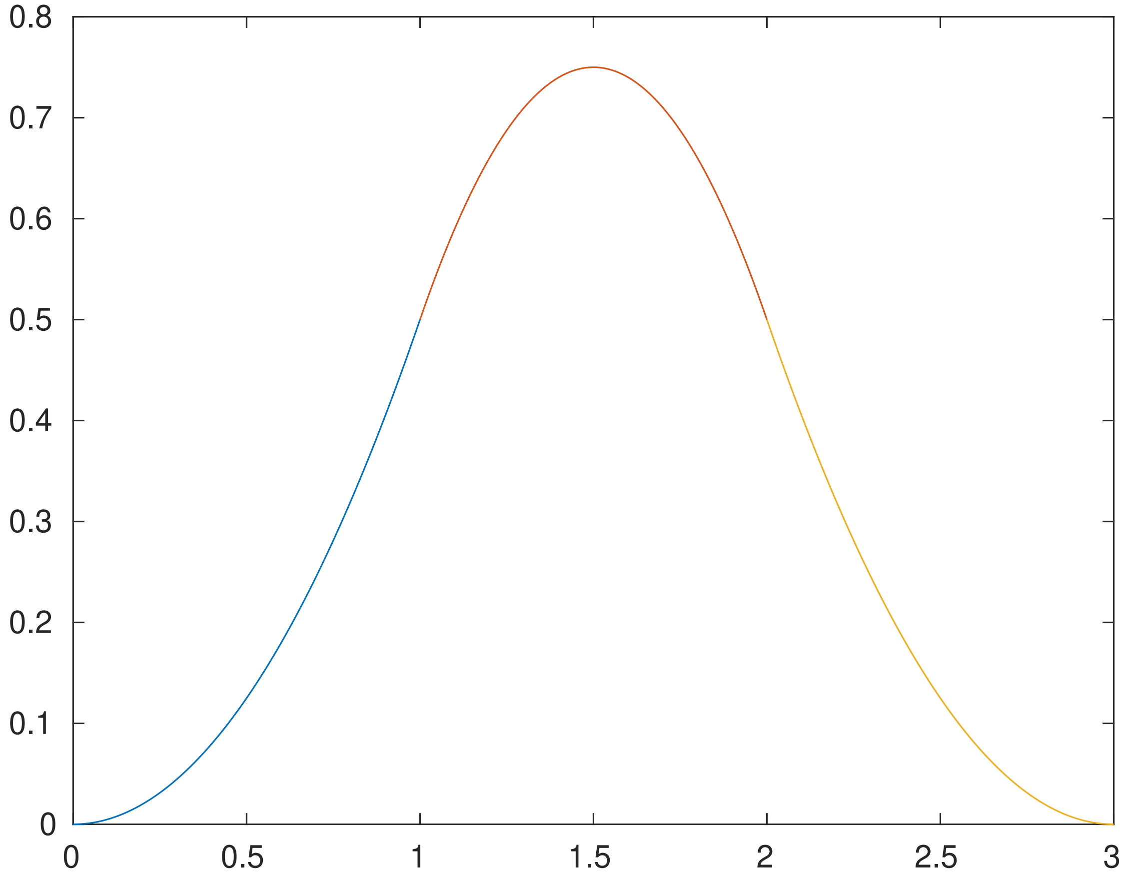

Figure 1: $B_1^2(x)$

VI.Verify Theorem 4.23 in case of $n = 2$

Definition 4.18 and the Example 4.10 yield

$$B_1^2(x) = \beta(x) + \gamma(x) \tag{9}$$
$$B_i^1(x) = (t_{i+1} - t_{t-1})[t_{i-1}, t_i, t_{i+1}](t-x)_+ \tag{10}$$
$$B_{i+1}^1(x) = (t_{i+2} - t_t)[t_i, t_i, t_{i+2}](t-x)_+ \tag{11}$$

where,

$$\beta(x) = \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} B_i^1(x) = [t_i, t_{i+1}](t-x)_+^1 - [t_{i-1}, t_i, t_{i+1}](t-x)_+^2 \tag{12}$$
$$\gamma(x) = \frac{t_{n+2} - x}{t_{n+2} - t_i} B_{i+1}^1(x) = [t_i, t_{i+1}, t_{i+2}](t-x)_+^2 - [t_i, t_{i+1}](t-x)_+^1 \tag{13}$$

Therefore,

$$B_1^2(x) = [t_i, t_{i+1}, t_{i+2}](t-x)_+^2 - [t_{i-1}, t_i, t_{i+1}](t-x)_+^2 \tag{14}$$
$$= (t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 \tag{15}$$

Hence proved.

Programming

From the picture below, Runge phenomenon doesn't appear when we utilize cubic splines to approch the original function $f(x) = \frac{1}{1+25x^2}$ Besides, the errors at each sampling points are as following as Figure 3 and Figure 4. To be

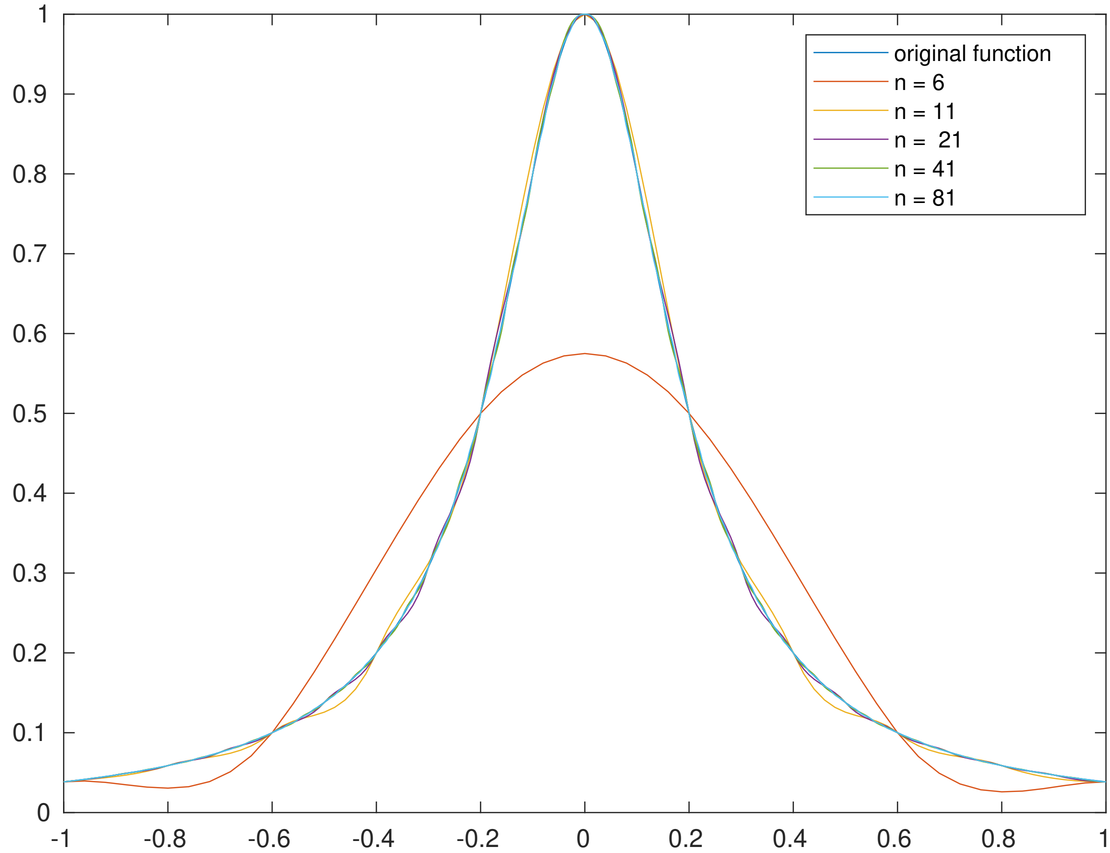

Figure 2: cubic splines

specific, error varies extremely depending on N, so Figure 3 is overall situation and Figure 4 is the situation near $x = 0$. These figures will present the max-norm of interpoloting errors clearly.

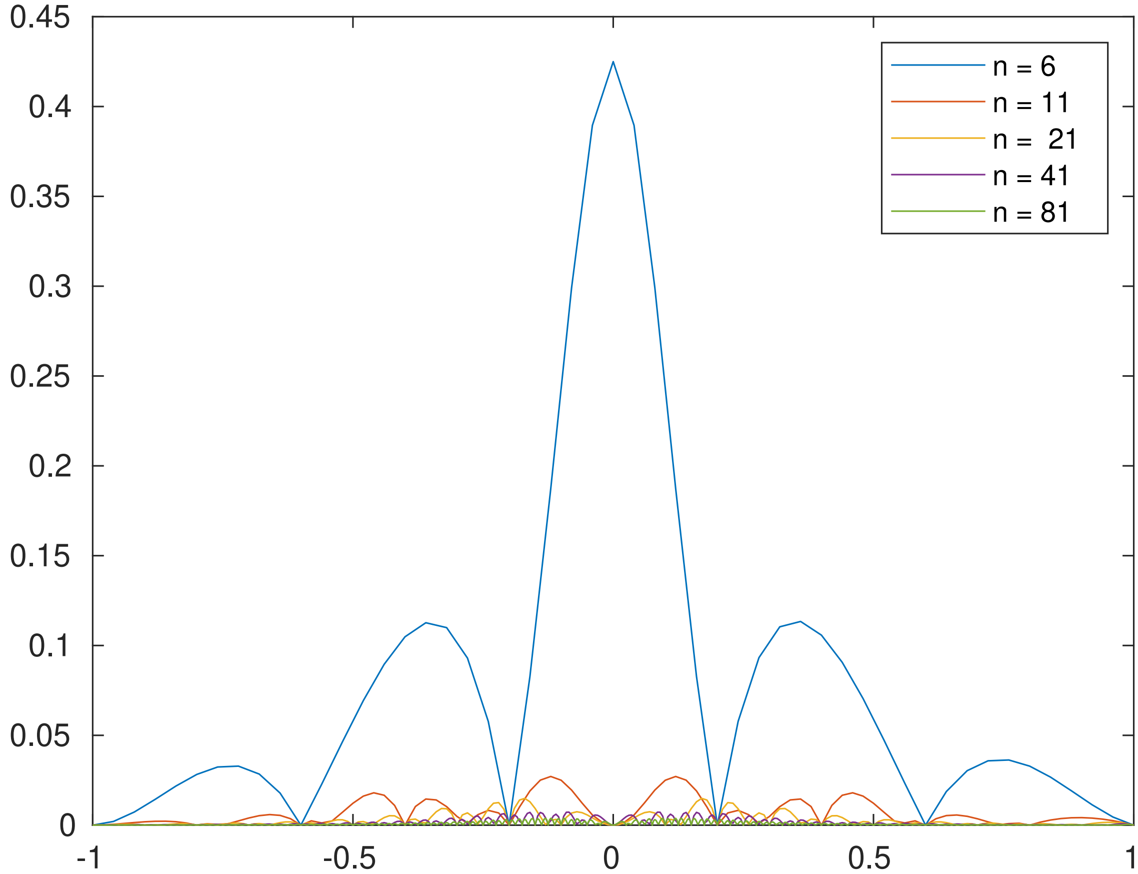

Figure 3: overall errors

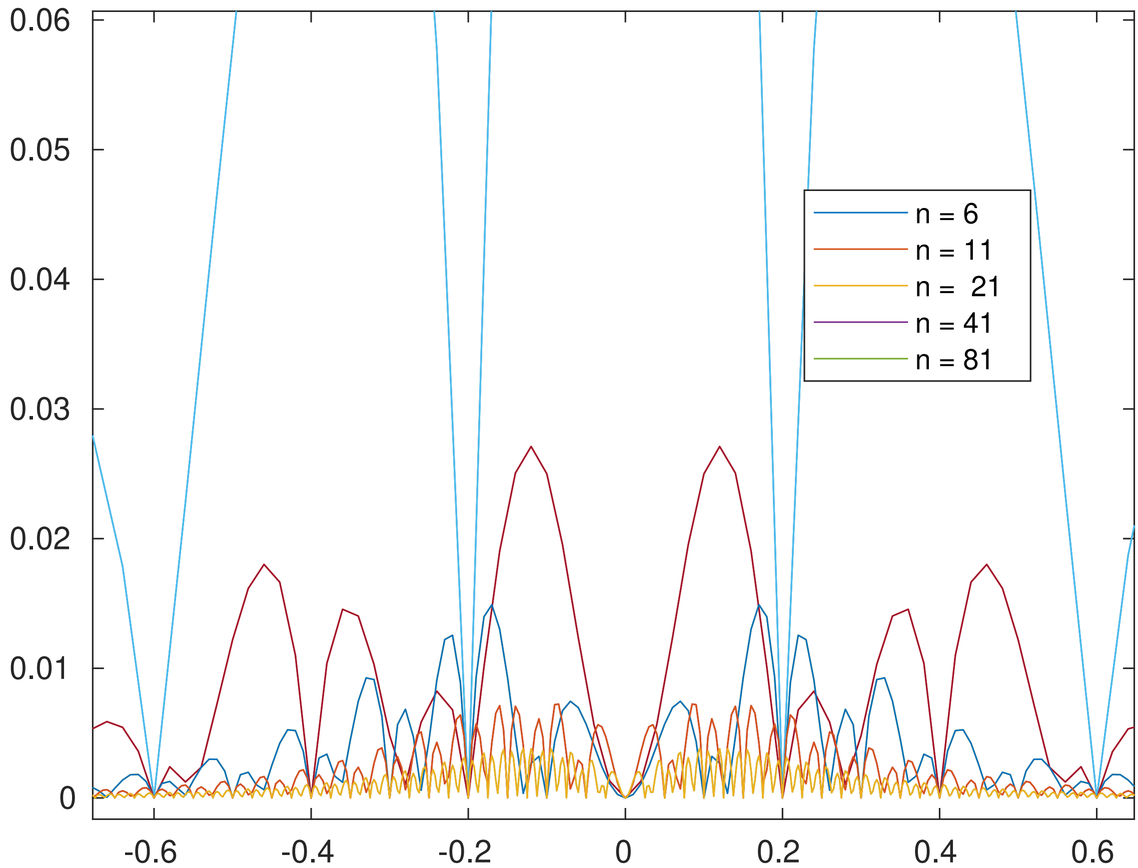

Figure 4: local errors