2-3 Ch.0

Ex 0.10. (a) $\exists p, q\ (p \neq q) \in \mathbb{P}$, s.t. p and q are even.

(b) $\exists a, b, c \in \mathbb{Z}$, s.t. $(ab)c \neq a(bc)$

(c) $\forall N$, $\exists n \in N^+$ which satisfies $n > N$, s.t. $\forall p, q \in \mathbb{P}$,

$\exists a_i$ which satisfies $a_i \neq p+q$ and $a_i \neq a_j\ (i \neq j)$

and a_i is even $(i = 1, 2, \cdots, n)$.

Ex 0.28. ① $\because$ $\forall \varepsilon > 0$, take $\delta = \frac{\varepsilon a^3}{4}$

$\therefore$ $\forall x, y \in (a, +\infty)$ which satisfy $|x-y| < \delta$, then

$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{1}{x} - \frac{1}{y}\right|\left|\frac{1}{x} + \frac{1}{y}\right| < \left|\frac{y-x}{xy}\right| \cdot \frac{2}{a} < \frac{2\delta}{a^3}$

$= \frac{2}{a^3} \cdot \frac{\varepsilon a^3}{4} < \frac{\varepsilon}{2} < \varepsilon$. $\square$

② if $a = 0$, $\exists \varepsilon = 1$, $\forall \delta > 0$, take $\forall x, y \in (0, +\infty)$

which satisfy $0 < x, y < \min\{1, \delta\}$. and $|x-y| < \delta$.

$\because$ $|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right|\left|\frac{1}{x} + \frac{1}{y}\right| > \frac{|y-x|}{xy} \cdot \frac{2}{\delta} > \frac{\delta}{xy} \cdot \frac{2}{\delta}$

$= \frac{2}{xy} > 2 > \varepsilon$.

$\therefore$ when $a = 0$, f is not uniformly continuous. $\square$

Ex 0.37. $\because$ It is obvious that $d(x, y) = d(y, x) > 0$.

$\therefore$ the following part is trying to prove $d(x, y) \leq d(x, z) + d(z, y)$

Define $x = \{x_n\}$, $y = \{y_n\}$, $z = \{z_n\}$

$\because \frac{|x_j-y_j|}{1+|x_j-y_j|} = 1-\frac{1}{1+|x_j-y_j|} \le 1-\frac{1}{1+|x_j-z_j|+|z_j-y_j|} = \frac{|x_j-z_j|+|z_j-y_j|}{1+|x_j-z_j|+|z_j-y_j|}$

$\le \frac{|x_j-z_j|}{1+|x_j-z_j|} + \frac{|z_j-y_j|}{1+|z_j-y_j|}$

$\therefore d(x,z)+d(z,y)-d(x,y) = \sum_{j=1}^{\infty}\frac{1}{2^j}\left(\frac{|x_j-z_j|}{1+|x_j-z_j|}+\frac{|z_j-y_j|}{1+|z_j-y_j|}-\frac{|x_j-y_j|}{1+|x_j-y_j|}\right) \ge 0.$

then, d is a metric on X!

Ex 0.38. (c) Define $z=(\alpha_i)$, $\because$ $d(x,y)=d(y,x)\ge 0$ is obvious

$\therefore$ we need to prove $d(x,z)+d(z,y)\ge d(x,y)$.

$\because$ according to Minkowski inequality,

$\left(\sum_{k=1}^{\infty}|\xi_k-\alpha_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty}|\alpha_k-\eta_k|^p\right)^{\frac{1}{p}} \ge \left(\sum_{k=1}^{\infty}|\xi_k-\eta_k|^p\right)^{\frac{1}{p}}$

$\therefore$ $d(x,z)+d(z,y)\ge d(x,y)$

(a) If $\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{\frac{1}{p}}=0$ or $\left(\sum_{k=1}^{\infty}|y_k|^q\right)^{\frac{1}{q}}=0$, then the inequality is true obviously.

If $\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{\frac{1}{p}}$ and $\left(\sum_{k=1}^{\infty}|y_k|^q\right)^{\frac{1}{q}}$ aren't equal to zero, then we can define $z_k=\frac{x_k}{\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{\frac{1}{p}}}$ and $w_k=\frac{y_k}{\left(\sum_{k=1}^{\infty}|y_k|^q\right)^{\frac{1}{q}}}$, according to Lemma 0.61, we know that

$\sum_{k=1}^{\infty}|z_kw_k| \le \sum_{k=1}^{\infty}\left(\frac{|z_k|^p}{p}+\frac{|w_k|^q}{q}\right).$

Notice that $\sum_{k=1}^{\infty}\left(\frac{|z_k|^p}{p}+\frac{|w_k|^q}{q}\right) = \sum_{k=1}^{\infty}\left(\frac{|x_k|^p}{p\left(\sum_{k=1}^{\infty}|x_k|^p\right)}+\frac{|y_k|^q}{q\left(\sum_{k=1}^{\infty}|y_k|^q\right)}\right) \le 1.$

So $\sum_{k=1}^{\infty}|z_kw_k|\le 1$, in other word,

$\sum_{k=1}^{\infty}|x_ky_k| \le \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{\infty}|y_k|^q\right)^{\frac{1}{q}}.$ □

(b) We know that $\sum_{n=1}^{k}(x_n+y_n)^p = \sum_{n=1}^{k} x_n(x_n+y_n)^{p-1} + \sum_{n=1}^{k} y_n(x_n+y_n)^{p-1}$

We can define $t=\frac{p}{p-1}$, so $\sum_{n=1}^{k} x_n(x_n+y_n)^{p-1} + \sum_{n=1}^{k} y_n(x_n+y_n)^{p-1} \le$
$\left(\left(\sum_{n=1}^{k} x_n^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{k} y_n^p\right)^{\frac{1}{p}}\right)\left(\sum_{n=1}^{k}(x_n+y_n)^p\right)^{\frac{1}{t}} \Big/ \left(\sum_{n=1}^{k}(x_n+y_n)^p\right)^{\frac{1}{t}}$,

Now we get $\sum_{n=1}^{k}(x_n+y_n)^{\frac{1}{p}} \le \left(\sum_{n=1}^{k} x_n^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{k} y_n^p\right)^{\frac{1}{p}}$. $\square$

Ex 0.56. Additivity. According to Def 0.87 (3) and (5).

$\because \langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle = \overline{\langle w, u+v\rangle} = \overline{\langle w,u\rangle} + \overline{\langle w,v\rangle}$

$= \overline{\langle w,u\rangle + \langle w,v\rangle}$, $\forall u,v,w \in V$

$\therefore \langle w, u+v\rangle = \langle w,u\rangle + \langle w,v\rangle$. Hence additivity proved.

Homogeneity. $\because \langle av, w\rangle = a\langle v,w\rangle = a\overline{\langle w,v\rangle} = \overline{a\langle w,v\rangle}$

$= \overline{\langle w, av\rangle}$ $\therefore \langle w, av\rangle = a\langle w,v\rangle$ Hence proved.

~~Ex 0.62~~ $\because \|u+v\|^2 + \|u-v\|^2 = \left(\sum_{i=1}^{n}|u_i+v_i|^p\right)^{\frac{2}{p}} + \left(\sum_{i=1}^{n}|u_i-v_i|^p\right)^{\frac{2}{p}}$

$\therefore$ ~~according~~ to basic Inequality,

$\left(\sum_{i=1}^{n}|u_i+v_i|^p\right)^{\frac{2}{p}} + \left(\sum_{i=1}^{n}|u_i-v_i|^p\right)^{\frac{2}{p}} \ge 2\left(\sum_{i=1}^{n}|u_i+v_i|^p \cdot \sum_{i=1}^{n}|u_i-v_i|^p\right)^{\frac{1}{p}}$

$\because$ we can ~~assume~~ $\frac{2}{p}=k$, ~~according~~ to Holder Inequality,

$\left(\sum_{i=1}^{n}\left[|u_i+v_i|^{\frac{1}{k}}\right]^{kp}\right)^{\frac{1}{kp}} \cdot \left(\sum_{i=1}^{n}\left[|u_i-v_i|^{\frac{1}{k}}\right]^{kp}\right)^{\frac{1}{kp}} \ge \sum_{i=1}^{n}(u_i^2-v_i^2)^{\frac{1}{k}} = \sum_{i=1}^{n}(u_i^2-v_i^2)^{\frac{2}{p}}$

Also, $\because \left(\sum_{i=1}^{n}|u_i|^p\right)^{\frac{2}{p}} + \left(\sum_{i=1}^{n}|v_i|^p\right)^{\frac{2}{p}} = \left(\sum_{i=1}^{n}|u_i|^p\right)^{\frac{2}{p}} + \left(\sum_{i=1}^{n}|-v_i|^p\right)^{\frac{2}{p}}$

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Ex 0.62. Consider an example $u = (1, 0, \cdots, 0)$, $v = (0, 1, 0, \cdots, 0)$

then $\|u\| = 1$, $\|v\| = 1$, $\|u+v\| = 2^{\frac{1}{p}}$, $\|u-v\| = 2^{\frac{1}{p}}$

so if $2\|u\|^2 + 2\|v\|^2 = \|u+v\|^2 + \|u-v\|^2$ is true,

then $p=2$, in other word, when $p \neq 2$, it is a counterexample

Hence proved.