Chapter 2

Point-set Topology

2.1 Topological spaces

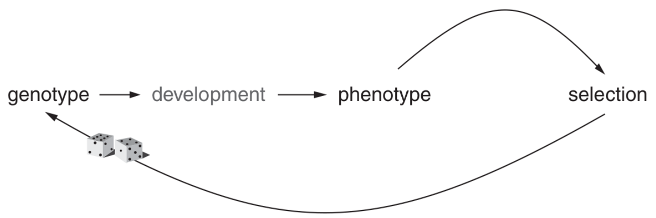
2.1.1 A motivating problem from biology

Phenotype and genotype are two fundamental concepts in the classical framework of evolution.

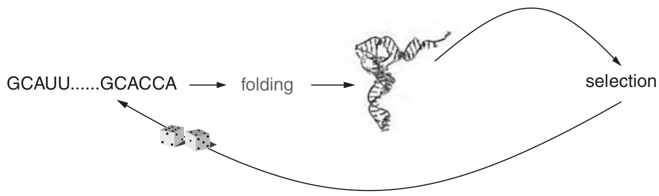
Definition 2.1. *Phenotype* refers to the physical, organizational, and behavioral expression of an organism during its lifetime while *genotype* refers to a heritable repository of information that instructs the production of molecules, whose interactions with the environment generate and maintain the phenotype.

Example 2.1. The collection of genes responsible for eye color in a particular individual is a genotype while the observable eye coloration in the individual is the corresponding phenotype.

The genotype-phenotype relationship is of great importance in biology in that evolution is driven by the selection of phenotypes that causes the amplification of their underlying genotypes and the production of novel phenotypes through genetic mutation.



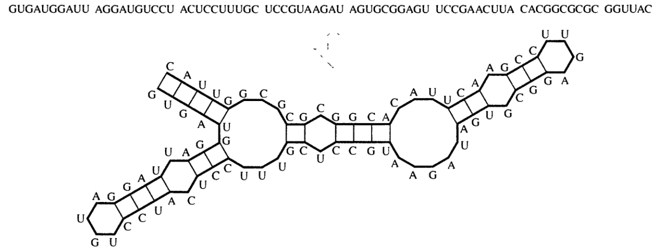
In phenotypic innovation, the heritable modification of a phenotype usually does not involve a direct intervention at the phenotypic level, but proceeds indirectly through changes at the genetic level during a number of processes known as development. While selection is clearly an important driving force of evolution, the dynamics of selection does not tell us much about how evolutionary innovations arise in the first place. A mutation is advantageous if it generates a phenotype favored by selection, but this definition reveals nothing about why or how that mutation could innovate the phenotype. Hence a model of genotype-phenotype relation is needed to illuminate how genetic changes map into phenotypic changes.



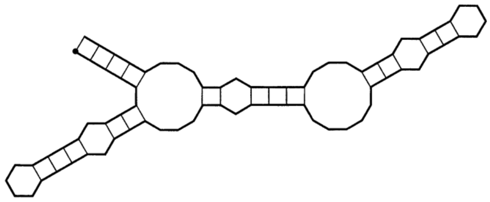
This subsection concerns such a model proposed by Fontana and Schuster [1998a,b] and Stadler et al. [2001] based on the shape of ribonucleic acid (RNA) sequences. Building blocks of strands of RNA are smaller molecules called nucleotides, which have four different types: guanine (G), cytosine (C), adenine (A) and uracil (U). This sequence of an RNA molecule functions as a genotype, since it can be directly replicated by suitable enzymes. Meanwhile, nonadjacent nucleotide pairs undergo additional (weaker) bonding, contorting the sequence into a more complicated three-dimensional structure. It is by this process that an RNA sequence always acquires a physical shape and this shape functions as the phenotype.

Definition 2.2. The *primary structure* of an RNA molecule is its unfolded nucleotide chain, often represented by a *genotype sequence* over the alphabet set $\{C, G, A, U\}$. The *secondary structure* or *bonding diagram* or *RNA shape* of an RNA molecule is an unlabeled diagram depicting the bonding that occurs in the resulting RNA molecule.

Example 2.2. In the plot below, a genotype sequence gets folded into a three-dimensional structure represented by the planar graph.



The following plot shows the bonding diagram, with the dot as the location of the first nucleotide in the sequence.



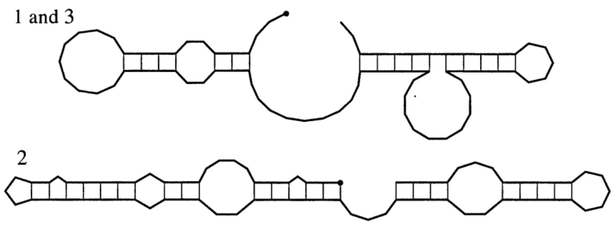
This RNA shape conveys biochemical behavior to an RNA molecule and is therefore subject to selection.

Question 2.3. How phenotype changes with genotype?

Example 2.3. A unique RNA shape can be assigned to each genotype sequence. This function is not injective since multiple genotype sequences may result in the same RNA shape. In the meantime, a single-entry change in the genotype sequence may completly alter the RNA shape. Consider the following genotype sequences.

1. GGGCAGUCUC CCGGCGUUUA AGGGAUCCUG AACUUCGUCG CUCCCAUCCA AUCAGUCCGC CUCACGGAUG GAGUUG
2. GGGCAGUCUC CCGGCGUUUA AGGAAUCCUG AACUUCGUCG CUCCCAUCCA AUCAGUCCGC CUCACGGAUG GAGUUG
3. GGGCAGUCUC CCGGCCUUUA AGGGAUCCUG AACUUCGUCG CUCCCAUCCA AUCAGUCCGC CUCACGGAUG GAGUUG

The corresponding bonding diagrams are as follow.

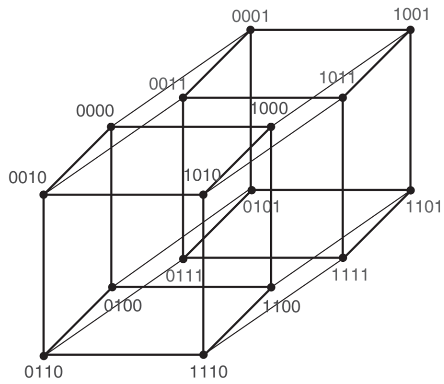


The bonding diagram are the same for sequences 1 and 3, which are identical except for the 16th entry. On the other hand, sequence 2 differs from sequence 1 in only the 24th entry, but their diagrams are very different.

Definition 2.4. A *point mutation* is a mutation from one genotype sequence to another by changing a single entry in the sequence. Two sequences are called *neighbors* if they can be converted to each other by a point mutation.

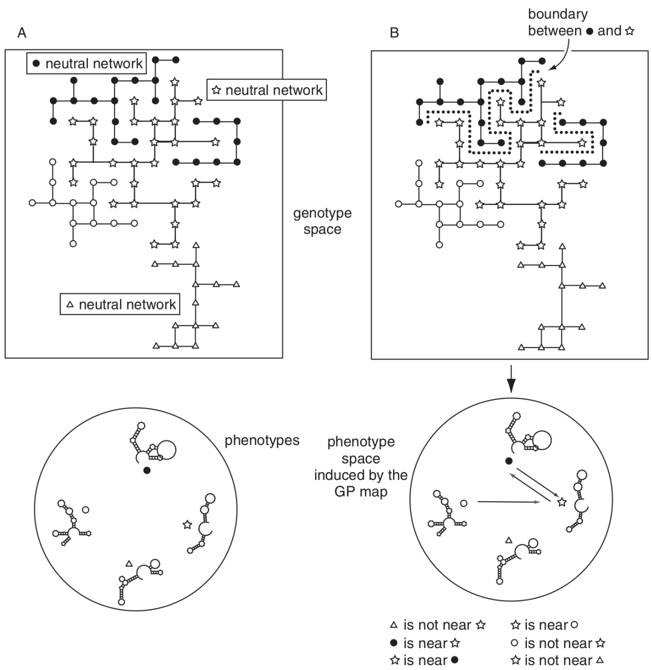
Definition 2.5. The *sequence space* of length n (proposed by Eigen [1971]) is a metric space of all genotype sequences of length n with the metric being *the distance between two sequences*, i.e., the smallest number of point mutations required to convert one sequence to the other.

Example 2.4. We show below a sequence space of length 4 over the binary alphabet $\{0, 1\}$.



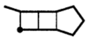
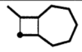
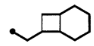
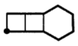
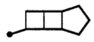
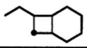
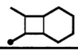
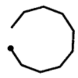
Definition 2.6. The *neutral network* of an RNA shape s, denoted by $N(s)$, is the set of all genotype sequences that result in s after folding and bonding.

Example 2.5. The possibility of changing the genotype while preserving the phenotype is a manifestation of a certain degree of phenotypic robustness toward genetic mutations. Meanwhile it is a key factor underlying the capacity of a system to evolve.



In the above plots, imagine a population with phenotype ‘star’ in an evolutionary situation where phenotype ‘triangle’ would be advantageous or desirable. But phenotype ‘triangle’ may not be accessible to phenotype ‘star’ in the vicinity of the population’s current location. However, due to the neutral network of ‘star,’ the population is not stuck, but can drift on that network into far away regions, vastly improving its chances of encountering the neutral network of ‘triangle.’ Therefore, neutral networks enable phenotypic innovation by permitting the accumulation of neutral mutations.

Example 2.6. Consider the sequence space GC_{10} of length 10 over the alphabet $\{G, C\}$. There 1024 possible sequences, and, after folding and bonding, they result in eight different RNA shapes, as shown below.

GC_{10}								
S_1		105	S_4		26	S_7		47
S_2		128	S_5		80			
S_3		137	S_6		70	S_8		431

The number to the right of a sequence S_i is $\#N(S_i)$.

Question 2.7. How do we capture and quantify the accessibility of one (favorable) phenotype from another (less favorable) by means of mutations in the sequence space? For any two phenotypes, is there always a directed path from one to the other?

Definition 2.8. A *phenotype space* is a set of RNA shapes on which a topology is defined to quantify proximity of RNA shapes.

Definition 2.9. The *mutation probability* of an RNA shape r to another RNA shape s is defined as

$$p_{r,s} := \frac{m_{r,s}}{m_{r,*}}, \tag{2.1}$$

where $m_{r,s}$ is the number of point mutations that change a sequence in $N(r)$ to a neighboring sequence in $N(s)$ and $m_{r,*}$ is the number of point mutations that change a sequence in $N(r)$ to a neighboring sequence in any other network.

Exercise 2.7. Show that the mutation probability cannot be a metric on the phenotype space.

Example 2.8 (Bubble sort)**.** To sort the sequence `51428`, the first pass of the algorithm goes as follows.

```
( 5 1 4 2 8 ) --> ( 1 5 4 2 8 )
( 1 5 4 2 8 ) --> ( 1 4 5 2 8 )
( 1 4 5 2 8 ) --> ( 1 4 2 5 8 )
( 1 4 2 5 8 ) --> ( 1 4 2 5 8 )
```

The second pass goes as follows.

```
( 1 4 2 5 8 ) --> ( 1 4 2 5 8 )
( 1 4 2 5 8 ) --> ( 1 2 4 5 8 )
( 1 2 4 5 8 ) --> ( 1 2 4 5 8 )
( 1 2 4 5 8 ) --> ( 1 2 4 5 8 )
```

Now, the array is already sorted, but the algorithm does not know if it is completed. The algorithm needs one whole pass without any swap to know it is sorted. The third pass goes as follows.

```
( 1 2 4 5 8 ) --> ( 1 2 4 5 8 )
( 1 2 4 5 8 ) --> ( 1 2 4 5 8 )
( 1 2 4 5 8 ) --> ( 1 2 4 5 8 )
( 1 2 4 5 8 ) --> ( 1 2 4 5 8 )
```

This algorithm is expressed in `C` as follows.

```
void bubble_sort(int a[]){
    int n = sizeof(a)/sizeof(a[0]);
    for (int j=0; j<n-1; j++)
        for (int i = 0; i<n-1-j; i++)
            if(a[i] > a[i+1])
                swap(a[i], a[i+1]);
}

void swap(int& b, int& c){
    int temp = b;
    b = c;
    c = temp;
}
```

As a limit of the above implementation, the program does not apply to the data type `char`, nor any other data type without an implicit conversion, even if the "less than" binary relation for such a data type is natural. You have to manually repeat the above program for each data type. An elegant solution is to use function template in `C++` as follows.

```
template<typename T>
void bubble_sort(T a[])
{
    int n = sizeof(a)/sizeof(a[0]);
    for (int i=0; i<n-1; i++)
        for (int j=0; j<n-1-i; j++)
        if (a[j] > a[j+1])
            swap<T>(a[j], a[j+1]);
}

template<typename T>
void swap(T& b, T& c){
    T temp = b;
    b = c;
    c = temp;
}
```

2.1.2 Generalizing continuous maps

Definition 2.10. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at* a iff

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ s.t. } |x - a| < \delta \ \Rightarrow\ |f(x) - f(a)| < \epsilon. \tag{2.2}$$

Definition 2.11. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *continuous at* $\mathbf{x} = a$ iff

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ s.t. } f(B(a, \delta)) \subset B(f(a), \epsilon), \tag{2.3}$$

where the n-dimensional open ball $B(p, r)$ is

$$B(p, r) = \{x \in \mathbb{R}^n : \|x - p\|_2 < r\}. \tag{2.4}$$

Definition 2.12. A function $f : X \to Y$ with $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ is *continuous at* $\mathbf{x} = a$ iff

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ s.t. } f(V_a) \subset U_a, \tag{2.5}$$

where the two sets associated with a are

$$V_a := B(a, \delta) \cap X, \qquad U_a := B(f(a), \epsilon) \cap Y. \tag{2.6}$$

Definition 2.13. A function f is *continuous* if it is continuous at every point of its domain.

Definition 2.14. A function $f : X \to Y$ with $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ is *continuous* iff

$$\forall U_a \in \gamma_Y, \exists V_a \in \gamma_X \text{ s.t. } f(V_a) \subset U_a, \tag{2.7}$$

where γ_X and γ_Y are sets of intersections of the open balls to X and Y, respectively,

$$\begin{aligned}\gamma_X &:= \{B(a,\delta) \cap X : a \in X, \delta \in \mathbb{R}^+\};\\ \gamma_Y &:= \{B(f(a),\epsilon) \cap Y : f(a) \in Y, \epsilon \in \mathbb{R}^+\}.\end{aligned}$$

Example 2.9. Is the function $x \mapsto \frac{1}{x}$ continuous? It depends on whether its domain includes the origin. But it is indeed continuous on domains such as $(0,1]$, $\mathbb{R} \setminus \{0\}$, and $[1,2]$. Note that definitions of the one-sided continuity in calculus are nicely incorporated in Definition 2.12.

Definition 2.15. A *basis of neighborhoods* (or a *basis*) on a set X is a collection $\mathcal{B}$ of subsets of X such that

- covering: $\cup\mathcal{B} = X$, and
- refining:

$$\forall U, V \in \mathcal{B}, \forall x \in U \cap V, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset (U \cap V).$$

Definition 2.16. For two sets X, Y with bases of neighborhoods $\mathcal{B}_X$, $\mathcal{B}_Y$, a surjective function $f : X \to Y$ is *continuous* iff

$$\forall U \in \mathcal{B}_Y \ \exists V \in \mathcal{B}_X \text{ s.t. } f(V) \subset U. \tag{2.8}$$

Lemma 2.17. If a surjective function $f : X \to Y$ is continuous in the sense of Definitions 2.12 and 2.13, then it is continuous in the sense of Definition 2.16.

Proof. By Definition 2.15, the following collections are bases of $X \subseteq \mathbb{R}^m$ and $Y = f(X) \subseteq \mathbb{R}^n$, respectively,

$$\begin{aligned}\mathcal{B}_X &= \{B(a,\delta) \cap X : a \in X, \delta > 0\};\\ \mathcal{B}_Y &= \{B(b,\epsilon) \cap Y : b \in Y, \epsilon > 0\}.\end{aligned}$$

The rest follows from Definitions 2.16 and 2.12. $\square$

Example 2.10. The *right rays*

$$\mathcal{B}_{RR} = \{\{x : x > s\} : s \in \mathbb{R}\} \tag{2.9}$$

form a basis of $\mathbb{R}$.

Exercise 2.11. Prove that the set of all right half-intervals in $\mathbb{R}$ is a basis of neighborhoods:

$$\mathcal{B} = \{[a,b) : a < b\}. \tag{2.10}$$

Example 2.12. A basis on $\mathbb{R}^2$ is the set of all quadrants

$$\mathcal{B}_q = \{Q(r,s) : r, s \in \mathbb{R}\}, \tag{2.11}$$

$$Q(r,s) = \{(x,y) \in \mathbb{R}^2 : x > r, y > s\}. \tag{2.12}$$

Exercise 2.13. Prove that the collection of all open squares in $\mathbb{R}^2$ is a basis of $\mathbb{R}^2$,

$$\mathcal{B}_s = \{S((a,b),d) : (a,b) \in \mathbb{R}^2, d > 0\},$$

where $S((a,b),d) = \{(x,y) : \max(|x-a|, |y-b|) < d\}$.

Exercise 2.14. Show that the *closed balls* $(r > 0)$

$$\bar{B}(p,r) = \{x \in \mathbb{R}^n : \|x-p\|_2 \le r\} \tag{2.13}$$

do not form a basis of $\mathbb{R}^n$. However, the following collection is indeed a basis:

$$\mathcal{B}_p = \{\bar{B}(a,r) : a \in \mathbb{R}^n, r \ge 0\}, \tag{2.14}$$

which is the union of all closed balls and all singleton sets.

2.1.3 Open sets: from bases to topologies

Definition 2.18. A subset U of X is *open* (with respect to a given basis of neighborhoods $\mathcal{B}$ of X) iff

$$\forall x \in U \ \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U. \tag{2.15}$$

Lemma 2.19. Each neighborhood in the basis $\mathcal{B}$ is open.

Proof. This follows from $B \subset B \in \mathcal{B}$ and Definition 2.18. $\square$

Exercise 2.15. What are the open subsets of $\mathbb{R}$ with respect to the right rays in (2.9)?

Lemma 2.20. The intersection of two open sets is open.

Proof. Let U_1 and U_2 be two open sets and fix a point $x \in U_1 \cap U_2$. By Definition 2.18, there exists $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. Then Definition 2.15 implies that there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$. Then the proof is completed by Definition 2.18 and x being arbitrary. $\square$

Lemma 2.21. The union of two open sets is open.

Lemma 2.22. The union of any collection of open sets is open.

Definition 2.23. *The topology of X generated by a basis* $\mathcal{B}$ is the collection $\mathcal{T}$ of all open subsets of X in the sense of Definition 2.18.

Definition 2.24. The *standard topology* is the topology generated by the *standard Euclidean basis*, which is the collection of all open balls in $X = \mathbb{R}^n$.

Theorem 2.25. The topology of X generated by a basis satisfies

- $\emptyset, X \in \mathcal{T}$;
- $\alpha \subset \mathcal{T} \ \Rightarrow \ \cup_{U\in\alpha} U \in \mathcal{T}$;
- $U, V \in \mathcal{T} \ \Rightarrow \ U \cap V \in \mathcal{T}$.

Proof. The first item follows from Definition 2.18. The others follows from Lemmas 2.20 and 2.22. $\square$

Example 2.16. The largest basis on a set X is the set of all subsets of X,

$$\mathcal{B}_d(X) = \{A \subset X\} = 2^X, \tag{2.16}$$

and the topology it generates is called *the discrete topology*, which coincides with the basis. This topology is more economically generated by the basis of all singletons,

$$\mathcal{B}_s(X) = \big\{\{x\} : x \in X\big\}. \tag{2.17}$$

The smallest basis on X is simply $\{X\}$ and the topology it generates is called the *trivial/anti-discrete/indiscrete topology* $\mathcal{T}_a = \{\emptyset, X\}$.

Exercise 2.17. Show that if U is open with respect to a basis $\mathcal{B}$, then $\mathcal{B} \cup \{U\}$ is also a basis.

2.1.4 Topological spaces: from topologies to bases

Definition 2.26. For an arbitrary set X, a collection $\mathcal{T}$ of subsets of X is called a *topology on* X iff it satisfies the following conditions,

(TPO-1) $\emptyset, X \in \mathcal{T}$;

(TPO-2) $\alpha \subset \mathcal{T} \Rightarrow \cup\alpha \in \mathcal{T}$;

(TPO-3) $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *topological space*. The elements of $\mathcal{T}$ are called *open sets*.

Corollary 2.27. The topology of X generated by a basis $\mathcal{B}$ as in Definition 2.23 is indeed a topology in the sense of Definition 2.26.

Proof. This follows directly from Theorem 2.25. □

Example 2.18. For each $n \in \mathbb{Z}$, define

$$B(n) = \begin{cases} \{n\} & \text{if } n \text{ is odd;} \\ \{n-1, n, n+1\} & \text{if } n \text{ is even.} \end{cases} \tag{2.18}$$

The topology generated by the basis $\mathcal{B} = \{B(n) : n \in \mathbb{Z}\}$ is called the *digital line topology* and we refer to $\mathbb{Z}$ with this topology as the *digital line*.

Theorem 2.28. A topology generated by a basis $\mathcal{B}$ equals the collection of all unions of elements of $\mathcal{B}$.

Proof. Given a collection of elements of $\mathcal{B}$, Lemma 2.19 states that each of them belongs to $\mathcal{T}$. Since $\mathcal{T}$ is a topology, (TPO-2) implies that all unions of these elements are also in $\mathcal{T}$. Conversely, given an open set $U \in \mathcal{T}$, we can choose for each $x \in U$ an element $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. Hence $U = \cup_{x \in U} B_x$ and this completes the proof. □

Corollary 2.29. Let $\mathcal{T}$ be a topology on X generated by the basis $\mathcal{B}$. Then every open set $U \in \mathcal{T}$ is a union of some basis neighborhoods in $\mathcal{B}$. (In particular, the empty set is the union of "empty collections" of elements of $\mathcal{B}$.)

Lemma 2.30. Let $(X, \mathcal{T})$ be a topological space. Suppose a collection of open sets $\mathcal{C} \subset \mathcal{T}$ satisfies

$$\forall U \in \mathcal{T}, \forall x \in U, \exists C \in \mathcal{C} \text{ s.t. } x \in C \subset U. \tag{2.19}$$

Then $\mathcal{C}$ is a basis for $\mathcal{T}$.

Proof. We first show that $\mathcal{C}$ is a basis. The covering relation holds trivially by setting $U = X$ in (2.19). As for the refining condition, let $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathcal{C}$. Since $C_1 \cap C_2$ is open, (2.19) implies that there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$. Hence $\mathcal{C}$ is a basis by Definition 2.15.

Then we show the topology $\mathcal{T}'$ generated by $\mathcal{C}$ equals $\mathcal{T}$. On one hand, for any $U \in \mathcal{T}$ and any $x \in U$, by (2.19) there exists $C \in \mathcal{C}$ such that $x \in C \subset U$. By Definitions 2.18 and 2.23, we have $U \in \mathcal{T}'$. On the other hand, it follows from Corollary 2.29 that any $W \in \mathcal{T}'$ is a union of elements of $\mathcal{C}$. Since each element of $\mathcal{C}$ is in $\mathcal{T}$, we have $W \in \mathcal{T}$. □

Example 2.19. The following countable collection

$$\mathcal{B} = \{(a, b) : a < b, a \text{ and } b \text{ are rational}\} \tag{2.20}$$

is a basis that generates the standard topology on $\mathbb{R}$.

Lemma 2.31. A collection of subsets of X is a topology on X if and only if it generates itself.

Proof. The necessity holds trivially since (TPO-1) implies the covering condition and (TPO-3) implies the refining condition. As for the sufficiency, suppose $U, V \in \mathcal{T}$. By Definition 2.18, $U \cup V$ is also open, hence $U \cup V \in \mathcal{T}$. This argument holds for the union of an arbitrary number of open sets. □

2.1.5 Generalized continuous maps

Definition 2.32. The *preimage of a set* $U \subset Y$ (or the *fiber over* U) under $f : X \to Y$ is

$$f^{-1}(U) := \{x \in X : f(x) \in U\}. \tag{2.21}$$

Exercise 2.20. Show that the operation f^{-1} preserves inclusions, unions, intersections, and differences of sets:

$$\begin{cases} B_0 \subseteq B_1 \Rightarrow f^{-1}(B_0) \subseteq f^{-1}(B_1), \\ f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1), \\ f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1), \\ f^{-1}(B_0 \setminus B_1) = f^{-1}(B_0) \setminus f^{-1}(B_1). \end{cases} \tag{2.22}$$

In comparison, f only preserves inclusions and unions:

$$\begin{cases} A_0 \subseteq A_1 \Rightarrow f(A_0) \subseteq f(A_1), \\ f(A_0 \cup A_1) = f(A_0) \cup f(A_1), \\ f(A_0 \cap A_1) \subseteq f(A_0) \cap f(A_1), \\ f(A_0 \setminus A_1) \supseteq f(A_0) \setminus f(A_1), \end{cases} \tag{2.23}$$

where the equalities in the last two equations holds if f is injective.

Lemma 2.33. For a map $f : X \to Y$, $A \subseteq X$, and $B \subseteq Y$, we have

$$A \subseteq f^{-1}(f(A)), \;\; f(f^{-1}(B)) \subseteq B, \tag{2.24}$$

where the first inclusion is an equality if f is injective and the second is an equaility if f is surjective or $B \subseteq f(X)$.

Proof. By (2.21), $a \in A$ implies $a \in f^{-1}(f(A))$. Conversely, $a \in f^{-1}(f(A))$ implies $f(a) \in f(A)$. f being injective dictates $a \in A$.

By (2.21), $b \in f(f^{-1}(B))$ implies $b \in B$. Furthermore, if f is surjective or $B \subseteq f(X)$, then for any $b \in B$ we have $f^{-1}(\{b\}) \neq \emptyset$ and thus

$$b \in f(f^{-1}(\{b\})) \subseteq f(f^{-1}(B)).$$

□

Definition 2.34 (The most general definition of continuous maps)**.** A function $f : X \to Y$ is *continuous* iff the preimage of each open set $U \subset Y$ is open in X.

Lemma 2.35. For a continuous function $f : X \to Y$ and an open subset $U \subset Y$, we have

$$f^{-1}(U) = \cup_{y \in U} V_y, \tag{2.25}$$

where the set V_y is a basis element of X containing some x such that $f(x) = y$ and $f(V_y) \subset U$.

Proof. If $x \in f^{-1}(U)$, then $y := f(x) \in U$ and by the covering condition V_y exists. Hence $f^{-1}(U) \subset \cup_{y \in U} V_y$. Conversely, any V_y is a subset of $f^{-1}(U)$ because of $f(V_y) \subset U$ and Definition 2.32. Hence $f^{-1}(U) \supset \cup_{y \in U} V_y$. □

Theorem 2.36. Definitions 2.16 and 2.34 are equivalent for surjective functions.

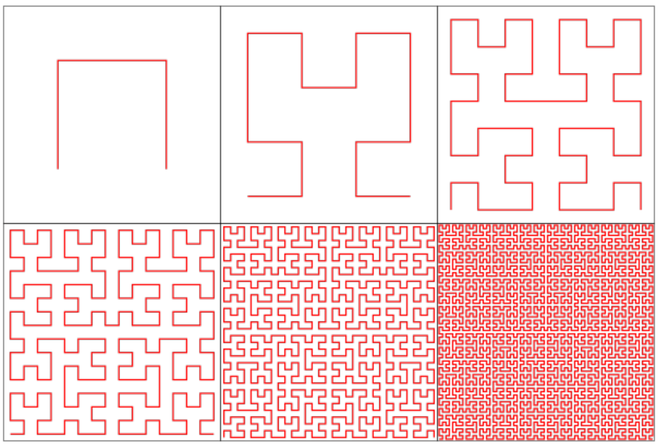
Proof. We first show that Definition 2.34 yields Definition 2.16. Consider $U \in \mathcal{B}_Y$. By Lemma 2.19, U is open and then Definition 2.34 implies that $f^{-1}(U)$ is open in X. The surjectivity of f implies that $f^{-1}(U)$ is not empty. Then by Definition 2.18 we have

$$\forall x \in f^{-1}(U), \exists V \in \mathcal{B}_X \text{ s.t. } x \in V \subset f^{-1}(U),$$

hence $f(V) \subset f(f^{-1}(U)) = U$.

Conversely, Definition 2.16 yields Definition 2.34: by Lemma 2.35 the preimage of any open subset U of Y can be expressed as the RHS of (2.25), and $f^{-1}(U)$ is open because of Lemma 2.22 and the fact that each V_y is open (by Lemma 2.19). □

Example 2.21. A continuous function is not necessarily "well behaved," as exemplified by the following space-filling *Hilbert curve*.



2.1.6 The subbasis topology

Definition 2.37. A *subbasis* $\mathcal{S}$ on X is a collection of subsets of X such that the covering condition in Definition 2.15 holds.

Example 2.22. The set of all open balls with their radii no less than a given $h > 0$, written $\mathcal{B}_h$, is a subbasis but not a basis.

Definition 2.38. *The topology of X generated by a subbasis* $\mathcal{S}$ is the collection $\mathcal{T}_{\mathcal{S}}$ of all unions of finite intersections of elements of $\mathcal{S}$.

Exercise 2.23. Show that the topology generated by a subbasis $\mathcal{S}$ as in Definition 2.38 is indeed a topology in the sense of Definition 2.26.

Proof. By Definition 2.38 and Theorem 2.28, we only need to show that all finite intersections of elements of $\mathcal{S}$ form a basis of X in the sense of Definition 2.15. The covering condition holds trivially. As for the covering condition, it suffices to note that the intersection of two sets of the form $B_1 = \cap_{i=1}^m S_i$ and $B_2 = \cap_{j=1}^n S_j$ is still a finite intersection of elements in $\mathcal{S}$. □

Exercise 2.24. Show $\mathcal{S} \subset \mathcal{T}_{\mathcal{S}}$. In other words, for the topology generated by a subbasis $\mathcal{S}$, every set in $\mathcal{S}$ is an open set in X.

Exercise 2.25. Show that if $\mathcal{T}$ is a topology on X containing $\mathcal{S}$, then $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}$.

Exercise 2.26. Assume that each $x \in X$ is contained in at most finitely many sets in $\mathcal{S}$ and let B_x be the intersection of the sets in $\mathcal{S}$ containing x. Show that

- the collection $\mathcal{B}_{\mathcal{S}} := \{B_x : x \in X\}$ is a basis for $\mathcal{T}_{\mathcal{S}}$;
- if $\mathcal{B}$ is a basis for $\mathcal{T}_{\mathcal{S}}$, then $\mathcal{B}_{\mathcal{S}} \subset \mathcal{B}$.

2.1.7 The topology of phenotype spaces

Example 2.27. Consider the sequence space GC_{10} in Example 2.6. Suppose for GC_{10} the value of $p_{i,j}$, i.e. the mutation probability from s_i to s_j as in Definition 2.9, is the number in the ith row and the jth column in the following table.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	—	0.13	0.15	0.08	0.07	0.09	0.04	0.44
s_2	0.11	—	0.15	0	0.11	0.18	0.05	0.40
s_3	0.12	0.15	—	0.03	0.09	0.06	0.05	0.50
s_4	0.29	0	0.14	—	0.07	0.09	0.06	0.35
s_5	0.08	0.15	0.12	0.02	—	0.08	0.08	0.47
s_6	0.12	0.29	0.09	0.03	0.09	—	0.03	0.35
s_7	0.08	0.12	0.13	0.03	0.13	0.05	—	0.46
s_8	0.18	0.19	0.24	0.04	0.15	0.11	0.09	—

The proximity of RNA shapes is based on the likelihood of a point mutation from one RNA shape to another. By Example 2.6, there are eight RNA shapes and hence it is reasonable to assume that $p_{i,j} > 1/7$ implies that the s_j phenotype is accessible to s_i. Hence we define

$$\forall i = 1, 2, \ldots, 8, \quad R_i := \{s_i\} \cup \left\{ s_j : p_{i,j} > \frac{1}{7} \right\}. \tag{2.26}$$

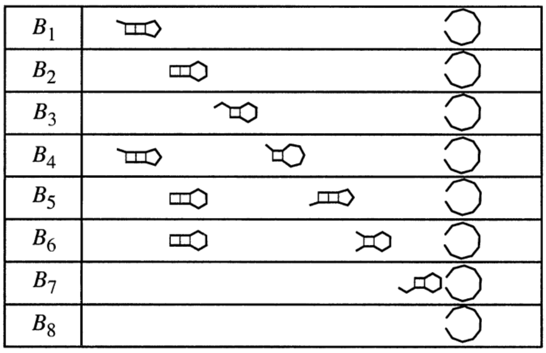
Each R_i is a row in the following table.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	✓		✓					✓
s_2		✓	✓			✓		✓
s_3		✓	✓					✓
s_4	✓			✓				✓
s_5		✓			✓			✓
s_6		✓				✓		✓
s_7							✓	✓
s_8	✓	✓	✓		✓			✓

The topology $\mathcal{T}_{1/7}$ on GC_{10} is defined as the topology generated by the subbasis

$$\mathcal{R}_{1/7} := \{R_i : i = 1, 2, \ldots, 8\}. \tag{2.27}$$

It can be shown that a topology on a finite set has a unique minimal basis that generates the topology. In this case, the basis is illustrated as follows.



Exercise 2.28. Change the threshold value in Example from 1/7 to 1/10 and repeat the entire process. What is the minimum value of q such that $\mathcal{T}_q$ on GC_{10} becomes the discrete topology?

2.1.8 Closed sets

Definition 2.39. A subset of X is called *closed* if its complement is open.

Example 2.29. The set

$$K = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \tag{2.28}$$

is neither open nor closed. In comparison, $K \cup \{0\}$ is closed.

Theorem 2.40. The set σ of all closed subsets of X satisfies the following conditions:

(TPC-1) $\emptyset, X \in \sigma$;

(TPC-2) $\alpha \subset \sigma \;\Rightarrow\; \cap\alpha \in \sigma$;

(TPC-3) $U, V \in \sigma \;\Rightarrow\; U \cup V \in \sigma$.

Example 2.30. The following example shows that infinite intersections of open sets might not be open and infinite unions of closed sets might not be closed.

$$\bigcap \left\{ \left(-\frac{1}{n}, \frac{1}{n} \right) : n = 1, 2, \ldots \right\} = \{0\};$$
$$\bigcup \left\{ \left[-1 + \frac{1}{n}, 1 - \frac{1}{n} \right] : n = 1, 2, \ldots \right\} = (-1, 1).$$

Lemma 2.41. A function $f : X \to Y$ is continuous if and only if the preimage of any closed set is closed.

Proof. By Definition 2.32, we have

$$f^{-1}(U) = f^{-1}(Y \setminus (Y \setminus U)) = X \setminus f^{-1}(Y \setminus U).$$

The rest follows from Definitions 2.34 and 2.39. □

Definition 2.42. The *graph of a function* $f : X \to Y$ is the set $\{(x, y) \in X \times Y : y = f(x)\}$.

Lemma 2.43. The graph of a continuous function $f : [a, b] \to \mathbb{R}$ is closed in the space $[a, b] \times \mathbb{R}$.

Exercise 2.31. Give an example of the graph of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ being not closed in $\mathbb{R}^2$. Give another example of the graph of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ being closed in $\mathbb{R}^2$.

Exercise 2.32. Let X be a topological space.

(a) For a continuous function $f : X \to \mathbb{R}$, show that the set $\{x \in X : f(x) = r\}$, i.e. the solution set of any equation with respect to f for some $r \in \mathbb{R}$, is closed.

(b) Show that this fails for a general continuous function $f : X \to Y$ where Y is an arbitrary topological space.

(c) What condition on Y would guarantee that the conclusion holds?

2.1.9 Interior–Frontier–Exterior

Definition 2.44. A point $x \in X$ is an *interior point* of A if there is a neighborhood W of x that lies entirely in A. The set of interior points of a set U is called its *interior* and denoted by $\mathrm{Int}(U)$.

Lemma 2.45. $\mathrm{Int}(A)$ is open for any A.

Proof. Exercise. □

Example 2.33. The interior of a closed ball is the corresponding open ball.

Definition 2.46. A point $x \in X$ is an *exterior point* of A if there is a neighborhood W of x that lies entirely in $X \setminus A$. The set of exterior points of a set U is called its *exterior* and denoted by $\mathrm{Ext}(U)$.

Example 2.34. The exterior of the set K in (2.28) is $\mathbb{R} \setminus K \setminus \{0\}$. Why not 0?

Definition 2.47. A point x is a *closure point* of A if each neighborhood of x contains some point in A.

Example 2.35. Any point in the set K in (2.28) is a closure point of K, so is 0.

Definition 2.48. A point x is an *accumulation point* (or a *limit point*) of A if each neighborhood of x contains some point $p \in A$ with $p \neq x$.

Example 2.36. The only accumulation point of the set K in (2.28) is 0.

Example 2.37. Each point in $\mathbb{R}$ is an accumulation point of $\mathbb{Q}$.

Definition 2.49. A point x in a set A is *isolated* if there exists a neighborhood of x such that x is the only point of A in this neighborhood.

Example 2.38. Every point of the set K in (2.28) is isolated.

Definition 2.50. A point x is a *frontier point* of a set A iff it is a closure point for both A and its complement. The set of all frontier points is called *the frontier* $\mathrm{Fr}(A)$ of A.

Theorem 2.51. For any set A in X, its interior, its frontier, and its exterior form a partition of X.

Proof. Consider an arbitrary point $a \in X$. If there exists a neighborhood $\mathcal{N}_a$ of a such that $\mathcal{N}_a \subset A$, then Definition 2.44 implies $a \in \mathrm{Int}(A)$. If $\mathcal{N}_a \subset X \setminus A$, then Definition 2.46 implies $a \in \mathrm{Ext}(A)$. Otherwise, for all neighborhoods of a we have $\mathcal{N}_a \not\subset A$ and $\mathcal{N}_a \not\subset X \setminus A$, which implies that any $\mathcal{N}_a$ contains points both from A and $X \setminus A$. The rest follows from Definition 2.50. □

Definition 2.52. The closure of A, written $\mathrm{Cl}(A)$ or $\overline{A}$, is the set of all closure points of A.

Lemma 2.53. $\mathrm{Int}(A) \subset A \subset \mathrm{Cl}(A)$.

Lemma 2.54. $\mathrm{Cl}(A) = \mathrm{Int}(A) \cup \mathrm{Fr}(A)$.

Theorem 2.55. The closure of a set A is the smallest closed set containing A:

$$\mathrm{Cl}(A) = \cap\{G : A \subset G, G \text{ is closed in } X\}. \tag{2.29}$$

Proof. Write $\alpha := \{G : A \subset G, G \text{ is closed in } X\}$ and $A^- := \cap\alpha$ and we need to show

- $A^- \subset \mathrm{Cl}(A)$;
- $A^- \supset \mathrm{Cl}(A)$.

We only prove the first part and leave the other as an exercise. Consider $x \notin \mathrm{Cl}(A)$. Then by Definitions 2.47 and 2.52 there exists an open neighborhood $\mathcal{N}_x$ of x such that $\mathcal{N}_x \cap A = \emptyset$. Hence the set $P := X \setminus \mathcal{N}_x$ contains A. P is also closed because $\mathcal{N}_x$ is open. Therefore $P \in \alpha$ and $x \notin A^-$. □

Exercise 2.39. Prove $\mathrm{Cl}(A \cap B) \subset \mathrm{Cl}(A) \cap \mathrm{Cl}(B)$. What if we have infinitely many sets?

Theorem 2.56. The interior of a set A is the largest open set contained in A,

$$\mathrm{Int}(A) = \cup\{U : U \subset A,\ U \text{ is open in } X\}. \tag{2.30}$$

Theorem 2.57. Let A' be the set of accumulation points of A. Then $\mathrm{Cl}(A) = A \cup A'$.

Proof. Suppose $x \in \mathrm{Cl}(A)$, If $x \in A$, then $x \in A \cup A'$ trivially holds. Otherwise $x \notin A$, Definition 2.52 dictates that its neighborhood must contain at lease one point in A. Hence Definition 2.44 yields $x \in A'$. In both cases we have $x \in A \cup A'$.

Conversely, suppose $x \in A \cup A'$. If $x \in A$, Lemma 2.53 implies $x \in \mathrm{Cl}(A)$. If $x \in A'$, x is an accumulation point of A and is thus a closure point A. □

Corollary 2.58. A subset of a topological space is closed if and only if it contains all of its accumulation points.

Proof. If A is a superset of A', the set of all accumulation points of A. We have $A \cup A' = A = \mathrm{Cl}(A)$ from Theorem 2.57. Definition 2.52 implies that A is closed.

Suppose A is closed, but there is an accumulation point x of A such that $x \notin A$. By Definition 2.48, in any neighborhood of x there exists a point $p \in A$ such that $p \neq x$; this contradicts the complement of A being open. □

2.1.10 Hausdorff spaces

Definition 2.59. Suppose X is a set with a basis of neighborhoods γ. Let $\{x_n : n = 1, 2, \ldots\}$ be a sequence of elements of X and $a \in X$. Then we say the sequence *converges* to a, written

$$\lim_{n\to\infty} x_n = a, \text{ or } x_n \to a \text{ as } n \to \infty,$$

iff

$$\forall U \in \gamma \text{ with } a \in U,\ \exists N \in \mathbb{N}^+ \text{ s.t. } n > N \ \Rightarrow\ x_n \in U. \tag{2.31}$$

Exercise 2.40. Prove that the definition remains equivalent if we replace "basis γ" with "topology $\mathcal{T}$."

Exercise 2.41. Show that if a sequence converges with respect to a basis γ, it also converges with respect to any basis equivalent to γ.

Theorem 2.60. Continuous functions preserve convergence, i.e., for a continuous $f : X \to Y$, $\lim_{n\to\infty} x_n = a$ implies $\lim_{n\to\infty} f(x_n) = f(a)$.

Proof. This follows from Definitions 2.59 and 2.16. □

Exercise 2.42. A sequence $\alpha = \{x_n : n = 1, 2, \ldots\}$ in a topological space X can be viewed as a subset of X, $A = \{x_n : n \in \mathbb{N}^+\}$. Compare the meanings of the closure points of A and the accumulation points of A. What about the limit of α?

Exercise 2.43. For metric topology, show that a function $f : X \to Y$ is continuous if and only if the function commutes with limits for any convergent sequence in X.

Example 2.44. When do we have $x_n \to a$ for discrete topology?

Example 2.45. When do we have $x_n \to a$ for anti-discrete topology?

Definition 2.61. A topological space $(X, \mathcal{T})$ is called a *Hausdorff space* iff

$$\forall a, b \in X,\ a \neq b, \exists U, V \in \mathcal{T} \text{ s.t. } a \in U, b \in V, U \cap V = \emptyset. \tag{2.32}$$

Lemma 2.62. Every subset of finite points in a Hausdorff space is closed.

Proof. By (TPC-3) in Theorem 2.40, it suffices to show that every singleton set is closed. Consider $X \setminus \{x_0\}$. For any $x \neq x_0$, Definition states that there exists $U \supset x$, $V \supset x_0$ such that $U \cap V = \emptyset$, hence $x_0 \notin U$ and $U \in X \setminus \{x_0\}$. Therefore $X \setminus \{x_0\}$ is open. □

Exercise 2.46. Does there exist a topological space X that is not Hausdorff but in which every finite point set is closed?

Definition 2.63. A topological space is called a *T1 space* iff every finite subset is closed in it.

Theorem 2.64. Let X be a T1 space and A a subset of X. A point x is an accumulation point of A if and only if every neighborhood of x intersects with infinitely many points of A.

Proof. The sufficiency follows directly from Definition 2.48. As for the necessity, suppose there exists a neighborhood U of x such that $(A \setminus \{x\}) \cap U = \{x_1, x_2, \ldots, x_m\}$. Then by Definition 2.63 we know

$$U \cap (X \setminus \{x_1, x_2, \ldots, x_m\}) = U \cap (X \setminus (A \setminus \{x\}))$$

is an open set containing x, yet it does not contain any points in A other than x. This contradicts the condition of x being an accumulation point of A. □

Theorem 2.65. A sequence of points in a Hausdorff space X converges to at most one point in X.

Proof. By Definition 2.59, a convergence to two points in X would be a contradiction to Definition 2.61. □

2.2 Continuous maps

2.2.1 The subspace/relative topology

Lemma 2.66. Consider a subset A of a topological space X. Suppose γ_X is a basis of neighborhoods of X. Then

$$\gamma_A := \{W \cap A : W \in \gamma_X\} \tag{2.33}$$

is a basis of neighborhoods of A.

Proof. The covering condition for A holds because the covering condition of X holds. As for the refining condition, for any $U, V \in \gamma_A$ and any $x \in U \cap V$, there exists $U', V' \in \gamma_X$ such that $U = U' \cap A$, $V = V' \cap A$, and $W' \subset U' \cap V'$ for some $W' \in \gamma_X$. Setting $W := W' \cap A$ and we have

$$x \in W \subset (U' \cap V') \cap A = (U' \cap A) \cap (V' \cap A) = U \cap V,$$

which completes the proof. □

Definition 2.67. The topology generated by γ_A in (2.33) is called the *relative topology* or *subspace topology* on A generated by the basis γ_X of X.

Lemma 2.68. Consider a subset A of a topological space X. Suppose $\mathcal{T}_X$ is a topology on X. Then

$$\mathcal{T}_A := \{W \cap A : W \in \mathcal{T}_X\} \tag{2.34}$$

is a topology on A.

Proof. For (TPO-1), we choose $W = \emptyset, A$. For (TPO-2),

$$\bigcup_{W \in \alpha} (W \cap A) = (\bigcup_{W \in \alpha} W) \cap A,$$

where $\bigcup_{W \in \alpha} W$ is a subset of X. For (TPO-3),

$$(U \cap A) \cap (V \cap A) = (U \cap V) \cap A,$$

where $U \cap V$ is a subset of X. □

Definition 2.69 (Subspace and subspace topology)**.** Given a topological space $(X, \mathcal{T})$ and a subset $A \subset X$, the topological space $(A, \mathcal{T}_A)$ is called a *subspace* of X and the topology $\mathcal{T}_A$ in (2.34) is called the *subspace topology or relative topology induced by* X.

Theorem 2.70. Let γ_X be a basis that generates the topology $\mathcal{T}_X$ on a topological space X. Then the subspace topology on A induced by $\mathcal{T}_X$ is equivalent to the subspace topology generated by γ_X. In other words, $\mathcal{T}_A$ is generated by γ_A.

$$\begin{tikzcd} \gamma_X \arrow[r, "\text{open}"] \arrow[d, "\cap A"] & \mathcal{T}_X \arrow[d, "\cap A"] \\ \gamma_A \arrow[r, "\text{open}"] & \mathcal{T}_A \end{tikzcd}$$

Proof. We first show that U is open with respect to (w.r.t.) γ_A for any given $U \in \mathcal{T}_A$. By Lemma 2.68, there exists $U' \in \mathcal{T}_X$ such that $U = U' \cap A$. The condition of γ_X being a basis of X yields

$$\forall y \in U',\ \exists B' \in \gamma_X \text{ s.t. } y \in B' \subset U',$$

which implies

$$\forall x \in U \subset U',\ \exists B := (B' \cap A) \in \gamma_A \text{ s.t. } x \in B \subset U.$$

It remains to show that any set U that is open w.r.t. γ_A is in $\mathcal{T}_A$, i.e., we need to find $U' \in \mathcal{T}_X$ such that $U = U' \cap A$. Since U is open w.r.t. γ_A, Definition 2.18 yields

$$\forall x \in U,\ \exists N_x \in \gamma_A \text{ s.t. } x \in N_x \subset U,$$

where $N_x = N'_x \cap A$ for some $N'_x \in X$. We then choose

$$U' := \bigcup_{x \in U} N'_x.$$

Theorem 2.28 implies that U' is open and $U = U' \cap A$. □

Lemma 2.71. Let A be a subspace of X. If U is open in A and A is open in X, then U is open in X.

Proof. Since U is open in A, Definition 2.69 yields

$$\exists U' \in X \text{ open s.t. } U = U' \cap A,$$

the rest of the proof follows from A being open in X. □

Lemma 2.72 (Closedness in a subspace)**.** Let A be a subspace of X. Then a set V is closed in A if and only if it equals the intersection of a closed subset of X with A.

Proof. Suppose V is closed in A. Then

$$\exists V' \subset A \text{ s.t. } V \cup V' = A, V' \in \mathcal{T}_A.$$

Since A is a subspace of X, we have from Definition 2.69

$$\exists U' \subset X, \text{ s.t. } V' = U' \cap A, U' \in \mathcal{T}_X.$$

Hence the set $U := X \setminus U'$ is closed in X and

$$\begin{aligned} A \cap U &= A \cap (X \setminus U') = A \setminus (X \setminus (X \setminus U')) \\ &= A \setminus U' = A \setminus (U' \cap A) = A \setminus V' = V. \end{aligned}$$

Conversely, suppose

$$\exists U \in X \text{ s.t. } (X \setminus U) \in \mathcal{T}_X, V = U \cap A.$$

Define $V' := (X \setminus U) \cap A$ and we know from Definition 2.69 that V' is open in A. The proof is then completed by

$$V \cup V' = (U \cap A) \cup ((X \setminus U) \cap A) = A,$$

where the last step follows from the condition $A \subset X$. □

Corollary 2.73 (Transitivity of relative closedness)**.** Let A be a subspace of X. If V is closed in A and A is closed in X, then V is closed in X.

Proof. This follows directly from Lemma 2.72 by using $V = V \cap A$. □

2.2.2 New maps from old ones

Theorem 2.74. The composition of continuous functions is continuous.

Proof. Suppose we have continuous functions $f = X \to Y$ and $g = Y \to Z$. Let $h = gf : X \to Z$ be their composition. Then for any open set $U \in Z$,

$$h^{-1}(U) = (gf)^{-1}(U) = f^{-1}(g^{-1}(U))$$

is open due to the continuity of g and f and Definition 2.34. □

Theorem 2.75. Suppose X is a topological space and $f, g : X \to \mathbb{R}$ are continuous functions. Then $f + g$, $f - g$, and $f \cdot g$ are continuous; f/g is also continuous if $g(x) \neq 0$ for all x.

Proof. By Theorem 2.108, the function $h : X \to \mathbb{R}^2$ given by $h(x) = (f(x), g(x))$ is continuous. We also know that the function $+ : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Hence the function $f + g = + \circ h$ is continuous. □

Definition 2.76. Let X be a topological space and A a subset of X. The *inclusion* $i_A : A \hookrightarrow X$ is given by

$$\forall x \in A, \qquad i_A(x) = x. \tag{2.35}$$

Definition 2.77. Let X and Y be topological spaces and A a subset of X. The *restriction of a function* $f : X \to Y$ to A is a function given as

$$\forall x \in A, \qquad f|_A(x) := f(x). \tag{2.36}$$

Theorem 2.78 (Restricting the domain)**.** Any restriction of a continuous function is continuous.

Proof. For any open set U in Y, we have $i_A^{-1}(U) = U \cap A$. The rest follows from the relative topology. □

Exercise 2.47. Let $i_A : A \hookrightarrow X$ be an inclusion. Suppose the set A is given a topology such that, for every topological space Y and every function $f : Y \to A$, f is continuous if and only if the composition $(i_A \circ f) : Y \to X$ is continuous. Prove that this topology of A is the same as the relative topology of A in X.

Lemma 2.79 (Restricting the range)**.** If $f : X \to Y$ is a continuous function, so is $g_f : X \to f(X)$ given by $g_f(x) := f(x)$ for all $x \in X$.

Proof. Of course the topology of $f(X)$ is understood as the subspace topology of Y. The rest follows from Definition 2.69. □

Lemma 2.80 (Expanding the range)**.** Let $f : X \to Y$ be a continuous function and Y a subspace of Z. Then the function $g : X \to Z$ given by $g(x) := f(x)$ for all $x \in X$ is continuous.

Proof. Write $g = i_Y \circ f$. □

Lemma 2.81 (Pasting lemma)**.** Let A, B be two closed subsets of a topological space X such that $X = A \cup B$. Suppose $f_A : A \to Y$ and $f_B : B \to Y$ are continuous functions

$$\forall x \in A \cap B, \qquad f_A = f_B. \tag{2.37}$$

Then the following function $f : X \to Y$ is continuous,

$$f(x) := \begin{cases} f_A(x) & \text{if } x \in A, \\ f_B(x) & \text{if } x \in B. \end{cases} \tag{2.38}$$

Proof. Define $W := f_A(A) \cup f_B(B)$. Then for any $V \subset Y$, (2.38) and the condition (2.37) yields

$$V = (V \cap W) \cup (V \setminus W) = (V \cap f_A(A)) \cup (V \cap f_B(B)) \cup (V \setminus W).$$

If V is closed in Y, then its preimage is

$$\begin{aligned} f^{-1}(V) &= f^{-1}(V \cap f_A(A)) \cup f^{-1}(V \cap f_B(B)) \\ &= f_A^{-1}(V \cap f_A(A)) \cup f_B^{-1}(V \cap f_B(B)) \\ &= g_A^{-1}(V \cap f_A(A)) \cup g_B^{-1}(V \cap f_B(B)), \end{aligned}$$

where g_A and g_B are defined in Lemma 2.79. We claim that $f^{-1}(V)$ is also closed in X with arguments as follows.

(i) Since V is closed, Lemma 2.72 implies that $V \cap f_A(A)$ and $V \cap f_B(B)$ are closed in $f_A(A)$ and $f_B(B)$, respectively.

(ii) By Lemma 2.79, both g_A and g_B are continuous. Hence the two sets to be unioned in the last line of the above equation are closed in A and B, respectively.

(iii) By Corollary 2.73, both sets in the last step are closed in X.

The rest of the proof follows from Lemma 2.41. □

Exercise 2.48. Show that Lemma 2.81 fails if A and B are not closed.

Exercise 2.49. Formulate the pasting lemma in terms of open sets and prove it.

Exercise 2.50. What is the counterpart of the pasting lemma in complex analysis?

Definition 2.82 (Expanding the domain)**.** For $A \subset X$ and a given function $f : A \to Y$, a function $F : X \to Y$ is called an *extension* of f if $F|_A = f$.

Exercise 2.51. State and prove the sufficient and necessary conditions for existence of the extension of a continuous function $f : (a, b) \to \mathbb{R}$ to another continuous function $F : (a, b] \to \mathbb{R}$.

Exercise 2.52. State and prove the sufficient and necessary condition for existence of a *linear* extension of a linear operator $f : A \to W$ to $f : V \to W$ where V, W are vector spaces and A is a subspace of V.

2.2.3 Homeomorphisms

Definition 2.83. A function $f : X \to Y$ between topological spaces X and Y is called a *homeomorphism* iff f is bijective and both f and f^{-1} are continuous. Then X and Y are said to be *homeomorphic* or *topologically equivalent*, written $X \approx Y$.

Lemma 2.84. If two spaces X and Y are homeomorphic, then

$$\forall a \in X,\ \exists b \in Y \text{ s.t. } X \setminus \{a\} \approx Y \setminus \{b\}. \tag{2.39}$$

Exercise 2.53. Show that the function $f : \{A, B\} \to \{C\}$ given by $f(A) = f(B) = C$ is continuous, but not a homeomorphism. Hence a necessary condition for homeomorphism is the number of connected components.

Example 2.54. Consider X the letter "T" and Y a line segment. They are not homeomorphic because removing the junction point in T would result in three pieces while removing any point in the line segment yields at most two connected components.

Exercise 2.55. Classify the following symbols of the standard computer keyboard by considering them as 1-dimensional topological spaces.

```
‘ 1 2 3 4 5 6 7 8 9 0 - =
q w e r t y u i o p [ ] \
a s d f g h j k l ; ’
z x c v b n m , . /
˜ ! @ # $ % ˆ & * ( ) _ +
Q W E R T Y U I O P { } |
A S D F G H J K L : "
Z X C V B N M < > ?
```

Exercise 2.56. Consider the identity function $f = \mathrm{Id}_X : (X, \mathcal{T}) \to (X, \kappa)$ where κ is the anti-discrete topology and $\mathcal{T}$ is not. Show that f^{-1} is not continuous and hence f is not a homeomorphism.

Exercise 2.57. Give an example of a continuous bijection $f : X \to Y$ that isn't a homeomorphism; this time both X and Y are subspaces of $\mathbb{R}^2$.

Exercise 2.58. For a continuous function $f : \mathbb{R} \to \mathbb{R}$, define $g : \mathbb{R} \to \mathbb{R}^2$ by $g(x) = (x, f(x))$. Prove that g is continuous and its image, the graph of f, is homeomorphic to $\mathbb{R}$.

Lemma 2.85. All closed intervals of a non-zero, finite length are homeomorphic.

Lemma 2.86. All open intervals, including infinite ones, are homeomorphic.

Proof. The tangent function gives you a homeomorphism between $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $(-\infty, +\infty)$. □

Lemma 2.87. An open interval is not homeomorphic to a closed interval (nor half-open).

Definition 2.88. The *n-sphere* is a subset in $\mathbb{R}^{n+1}$,

$$\mathbb{S}^n := \{\mathbf{x} \in \mathbb{R}^{n+1} : \|\mathbf{x}\| = 1\}. \tag{2.40}$$

Its north pole is denoted by $N = (0, 0, \cdots, 0, 1)$.

Definition 2.89. The *stereographic projection*

$$P : \mathbb{S}^n \setminus N \to \mathbb{R}^n$$

is given by

$$P(\mathbf{x}) := \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right). \tag{2.41}$$

Lemma 2.90. The stereographic projection is a homeomorphism with its inverse as

$$P^{-1}(\mathbf{y}) = \frac{1}{1 + \|\mathbf{y}\|^2} \left(2y_1, 2y_2, \ldots, 2y_n, \|\mathbf{y}\|^2 - 1 \right). \tag{2.42}$$

Exercise 2.59. Show that the 2-sphere and the hollow cube are homeomorphic by using the *radial projection* f,

$$f(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}. \tag{2.43}$$

Theorem 2.91. Homeomorphisms form an equivalence relation on the set of all topological spaces.

Proof. For a homeomorphism $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, we can define a function $f_{\mathcal{T}} : \mathcal{T}_X \to \mathcal{T}_Y$ by setting $f_{\mathcal{T}}(V) := f(V)$. It is easy to show that $f_{\mathcal{T}}$ is also a bijection from Definition 2.83. □

Definition 2.92. An *embedding of X in Y* is a function $f : X \to Y$ that maps X homeomorphically to the subspace $f(X)$ in Y.

Example 2.60. For an embedding $f : [0, 1] \to X$, its image is called an *arc* in X. For an embedding $f : \mathbb{S}^1 \to X$, its image is called a *simple closed curve* in X.