I.Linear Interpolation of $f(x) = \frac{1}{x}$ at $x_0 = 1$ and $x_1 = 2$

a.Determine $\xi(x)$ explicitly

Using Lagrange Formula, we get

$$p_1(x) = f_0 \frac{x - x_0}{x_0 - x_1} + f_1 \frac{x - x_1}{x_1 - x_0} = \frac{3 - x}{2}$$

Because $f''(x) = \frac{2}{x^3}$, it is obvious that $f''(\xi(x)) = \frac{2}{\xi^3(x)}$, then

$$f(x) - p_1(x) = \frac{1}{x} - \frac{3 - x}{2} = \frac{(x-1)(x-2)}{2x} = \frac{(x-1)(x-2)}{\xi^3(x)}$$

Hence, we know that $\xi(x) = \sqrt[3]{2x}$.

b.Find max$\xi(x)$, min$\xi(x)$ and max$f''(\xi(x))$

For $x \in [1, 2]$, we can conclude

$$max\xi(x) = \xi(2) = \sqrt[3]{4}$$
$$min\xi(x) = \xi(1) = \sqrt[3]{2}$$

From above, notice $f''(\xi(x)) = \frac{2}{\xi^3(x)}$, so

$$maxf''(\xi(x)) = max\frac{1}{2x} = \frac{1}{2}$$

II.Find $p \in \mathbb{P}^+_{2n}$ Satisfies Distinct Points on $\mathbb{R}$

We can modify the Lagrange Formula into

$$p(x) = \sum_{k=0}^{n} f_k l_k$$

where the

$$l_k = \prod_{i=0; i \neq k}^{n} \frac{(x - x_i)^2}{(x_k - x_i)^2}$$

and the degree of $p \leq 2n$, which implies $p(x) \in \mathbb{P}$. It is obvious that $l_k \geq 0$, since $(x - x_i)^2 \geq 0$ and $(x_k - x_i)^2 \geq 0$. Additionally, $f_k \geq 0$ given by the condition, therefore, $f_k l_k \geq 0$ so that $p(x) \geq 0$ has proved.

III.Consider $f(x) = e^x$

a.Prove Induction

If $n = 0$, namely, $f[t] = f(t) = e^t = \frac{(e-1)^n}{n!} e^t$, it is true obviously.
If this conclusion is true when $n = s - 1$, then we can know $\forall t \in \mathbb{R}$

$$f[t, t+1, \cdots, t+s-1] = \frac{(e-1)^{s-1}}{(s-1)!} e^t$$

Next, we can subsititute t with $t+1$, so we can get

$$f[t+1,t+2,\cdots,t+s]=\frac{(e-1)^{s-1}}{(s-1)!}e^{t+1}$$

Using therorem 3.14, we can draw the conlcusion

$$f[t,t+1,\cdots,t+s]=\frac{\frac{(e-1)^{s-1}}{(s-1)!}e^{t+1}-\frac{(e-1)^{s-1}}{(s-1)!}e^{t}}{s}=\frac{(e-1)^{s}}{s!}e^{t}$$

It implies when $n=s$ the formula is also true. Hence proved.

b.Determine ξ From the Above Two Equation

From above, take $t=0$ into the equation,

$$f[0,1,\cdots,n]=\frac{(e-1)^n}{n!}$$

When $x\in(0,n)$, $\frac{e^x}{n!}\in(\frac{1}{n!},\frac{e^n}{n!})$. Since e^x is continous function, we know $\exists\xi\in(0,n)$ such that

$$\frac{e^\xi}{n!}=\frac{(e-1)^n}{n!}$$

by Intermediate Value Theorem. Therefore,

$$x=n\ln(e-1)>\frac{n}{2}$$

which means that ξ always lays on the right of the midpoint $x=\frac{n}{2}$.

IV.Cosider $f(0)=5$, $f(1)=3$, $f(3)=5$, $f(4)=12$

a.Obtain $p_3(f;x)$ Using Newton Formula

According to the formula,

$$f[0]=5$$

$$f[0,1]=\frac{f_0}{x_0-x_1}+\frac{f_1}{x_1-x_0}=-2$$

$$f[0,1,3]=\frac{f_0}{(x_0-x_1)(x_0-x_2)}+\frac{f_1}{(x_1-x_0)(x_1-x_2)}+\frac{f_2}{(x_3-x_0)(x_3-x_2)}=\frac{5}{6}$$

$$f[0,1,3,4]=\frac{f_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}+\frac{f_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$
$$+\frac{f_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}+\frac{f_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}=\frac{1}{4}$$

Easily, we can get

$$p_3(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+a_3(x-x_0)(x-x_1)(x-x_2)$$

$$=\frac{x^3}{4}-\frac{9}{4}x+5$$

b.Find an Appropriate Value For the location x_{min} of the Minimum

When $f'(x) = \frac{3}{4}x^2 - \frac{9}{4} = 0$, we know $x = \sqrt{3}$. It implies

$$x_{min} = \sqrt{3}, minf(x) = 5 - \frac{3}{2}\sqrt{3}$$

V.Consider $f(x) = x^7$

a.Compute $f[0, 1, 1, 1, 2, 2]$

x	0	1	1	1	2	2
$f(x)$	0	1	1	1	128	128

Using therorem 3.14, then we can get the chart recursively

0	0					
1	1	1				
1	1	7	6			
1	1	7	21	15		
2	128	127	120	99	42	
2	128	448	321	201	102	30

which shows that $f[0, 1, 1, 1, 2, 2] = 30$.

b.Divided Difference of $f^{(5)}$ Evaluated at $\xi \in (0, 2)$. Determine ξ

According to corollary 3.17,

$$f[0, 1, 1, 1, 2, x] = \frac{1}{5!}f^{(5)}(\xi(x))$$

when $x = 2$, namely, $\frac{1}{5!} \cdot 2520 \cdot \xi^2 = 30$, so $\xi = \frac{\sqrt{70}}{7}$.

VI.f is defined on [0,3] with conditions

a.Estimate $f(2)$ Using Hermite Interpolation

x	0	1	1	3	3
$f(x)$	1	2	2	0	0

0	1				
1	2	1			
1	2	-1	-2		
3	0	-1	0	$\frac{2}{3}$	
3	0	0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{5}{36}$

Therefore, $p(x) = 1 + x - 2x(x-1) + \frac{2}{3}x(x-1)^2 - \frac{5}{36}x(x-1)^2(x-3)$, in particular, $p(2) = \frac{11}{18}$.

b.Estimate the Maximum Possible Error

According to therorem 3.27, in particular, when $x = 2$,

$$\epsilon \leq \frac{M}{60}$$

In general,

$$f(x)-p(x)=\frac{f^{(5)}(\xi)}{5!}x(x-1)^2(x-3)^2=\frac{f^{(5)}(\xi)}{5!}(x^5-8x^4+22x^3-24x^2+9x)$$

Take

$$h(x)=x^5-8x^4+22x^3-24x^2+9x$$

then,

$$h'(x)=(x-1)(x-3)(5x^2-12x+3)$$

After trivial calculation, the maximum of possible error

$$\epsilon=\max_{x\in(0,3)}|f(x)-g(x)|\leq\frac{M}{120}h(\frac{6+\sqrt{21}}{5})=\frac{204+14\sqrt{21}}{15625}M$$

VII.Prove Two Equation about Forward and Backward Difference

When $k=1$, then

$$\triangle f(x)=f(x+h)-f(x)=hf[x_0,x_1]$$

satisfies the conclusion. If the conclusion is true, when $k=s-1$, namely,

$$\triangle^{s-1}f(x)=(s-1)!h^{s-1}f[x_0,\cdots,x_{s-1}]$$

$$\triangle^{s-1}f(x+h)=(s-1)!h^{s-1}f[x_1,\cdots,x_s]$$

Then according to the definition,

$$\triangle^{s}f(x)=\triangle^{s-1}f(x+h)-\triangle^{s-1}f(x)$$

$$=(s-1)!h^{s-1}f[x_1,\cdots,x_s]-(s-1)!h^{s-1}f[x_0,\cdots,x_{s-1}]$$

$$=(s-1)!h^{s-1}f[x_1,\cdots,x_s]-(s-1)!h^{s-1}f[x_0,\cdots,x_{s-1}]$$

$$=(s-1)!h^{s-1}(f[x_1,\cdots,x_s]-f[x_0,\cdots,x_{s-1}])$$

According to therorem 3.14,

$$f[x_1,\cdots,x_s]-f[x_0,\cdots,x_{s-1}]=khf[x_0,\cdots,x_k]$$

So take it into the other formula,

$$\triangle^{s}f(x)=s!h^{s}f[x_0,\cdots,x_s])$$

The situation of backward difference is similar to the above, hence proved.

VIII.Prove equation and Expand It

a.prove equation

When $n = 0$, $LHS = \frac{\partial}{\partial x_0} f[x_0] = f'(x_0)$ and $RHS = f[x_0, x_0] = f'(x_0)$, hence the equation holds.
When $n = k$, suppose $\frac{\partial}{\partial x_0} = f[x_0, x_0, \cdots, x_k]$ is true,
Then, when $n = k + 1$,

$$LHS = \frac{\partial}{\partial x_0} \frac{f[x_1, x_2, \cdots, x_{k+1}] - f[x_0, x_1, \cdots, x_k]}{x_{k+1} - x_0} \tag{1}$$

$$= \frac{f[x_1, x_2, \cdots, x_{k+1}]}{(x_{k+1} - x_0)^2} - \frac{f[x_0, x_0, \cdots, x_k](x_{k+1} - x_0) + f[x_1, x_2, \cdots, x_k]}{(x_{k+1} - x_0)^2} \tag{2}$$

$$= \frac{f[x_1, x_2, \cdots, x_{k+1}] - f[x_0, x_1, \cdots, x_k]}{(x_{k+1} - x_0)^2} - \frac{f[x_0, x_0, \cdots, x_k]}{x_{k+1} - x_0} \tag{3}$$

$$= \frac{f[x_0, x_1, \cdots, x_{k+1}] - f[x_0, x_0, \cdots, x_k]}{x_{k+1} - x_0} \tag{4}$$

$$= f[x_0, x_0, x_1, \cdots, x_{k+1}] \tag{5}$$

$$= RHS \tag{6}$$

Hence proved.

b.expand it

The proof of the following equation is simialr to the above content that just exchange x_0 with x_i , then we can get

$$\frac{\partial}{\partial x_i} f[x_0, x_1, \cdots, x_i, \cdots, x_n] = f[x_i, x_0, x_1, \cdots, \cdots, x_i, \cdots, x_n] \tag{7}$$