

1 Напоминание: дифференциал

$$df(x) := f'(x)dx$$

$$d(f(g(x))) = f'(g(x))dg(x)$$

$$(f(g(x)))'dx = f'(g(x))g'(x)dx$$

$$d(\sin^3 x) = 3 \sin^2 x \cos x dx$$

$$d(\sin(x^3)) = \cos(x^3)3x^2 dx = \cos x^3 dx^3$$

$$d(\ln(x^2 + 3x + 4)) = \frac{d(x^2 + 3x + 4)}{x^2 + 3x + 4} = \frac{(2x + 3)dx}{x^2 + 3x + 4}$$

$$d(\sin(5x + 2)) = \cos(5x + 2) \cdot 5 \cdot dx$$

$$\cos x dx = d \sin x \text{ по определению } dx$$

$$3 \sin^2 x \cos x dx = 3 \sin^2 x d \sin x = d(\sin^3 x)$$

Не у всего можно взять интеграл: $\nexists f(x) : d(f(x)) = e^{x^2} dx$

2 Первообразная — антипроизводная

$$F' = f$$

F' — первообразная f , f — производная F

Первообразных много (с точностью до прибавления константы)

$$f(x) := \frac{1}{x}$$

$$F(x) = \ln|x| + C$$

Но еще есть другие первообразные:

$$F(x) = \begin{cases} \ln|x| + C_1, & x > 0 \\ \ln|x| + C_2, & x < 0 \end{cases}$$

В произвольной точке так поделить нельзя, т.к. разрыв и не будет производной

3 Неопределенный интеграл

$$\int f(x)dx := \{F : F' = f\}$$

$$\int f(x)dx := F'(x) + C$$

$$\int \frac{1 + 2x^2}{x^2(1 + x^2)} dx = \int \left(\frac{1 + x^2}{x^2(1 + x^2)} + \frac{x^2}{x^2(1 + x^2)} \right) = -x^{-1} + \arctg x + C$$

$$\int \frac{(1+x)^2}{x(1+x^2)} dx = \int \left(\frac{1}{x} + \frac{2}{1+x^2} \right) dx = \ln|x| + 2 \operatorname{arctg} x + C$$

$$\int \frac{dx}{\cos 2x + \sin^2 x} = \int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$$

$$\int \frac{(\arcsin x + \arccos x) dx}{\frac{\pi}{2}} = \frac{\pi}{2} x + C$$

$$\int \operatorname{tg}^3 x d \operatorname{tg} x = [t = \operatorname{tg} x] = \int t^3 dt = \frac{t^4}{4} + C = \frac{\operatorname{tg}^4 x}{4} + C$$

$$\int \frac{d(1+x^2)}{\sqrt{1+x^2}} = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{1+x^2}$$

$$\int \sqrt[5]{(3+8x)^6} dx = \frac{1}{8} \int (3+8x)^{\frac{6}{5}} d(3+8x) = \frac{1}{8} \frac{5}{11} (3+8x)^{\frac{11}{5}} + C$$

$$\int \frac{\operatorname{arctg}^2 x}{1+x^2} dx = \int \operatorname{arctg}^2 x d \operatorname{arctg} x = \int t^2 dt = \frac{t^3}{3} = \frac{\operatorname{arctg}^3 x}{3} + C$$

$$\int f(g(x)) \cdot g'(x) dx = [g(x) = t, dt = g'(x) dx] = \int f(t) dt$$

$$\begin{aligned} \int \frac{dx}{1 + \sqrt[3]{x+1}} &= [\sqrt[3]{x+1} = t, x = t^3 - 1, dx = 3t^2 dt] = \int \frac{3t^2 dt}{1+t} = \int \frac{3t^2 - 3 + 3}{1+t} dt = 3 \int (t-1 + \frac{1}{t+1}) dt = \\ &= 3(\frac{t^2}{2} - t + \ln(t+1)) \end{aligned}$$

$$\begin{aligned} \int \frac{\ln \operatorname{tg} x}{\sin x \cos x} dx &= \left[\operatorname{tg} x = t, dt = \frac{dx}{\cos^2 x} \right] = \int \frac{\ln t \cos^2 x dt}{\sin x \cos x} = \int \frac{\ln t}{t} dt = \int \ln t dt = [y = \ln t] = \frac{y^2}{2} + C = \\ &= \frac{(\ln \operatorname{tg} x)^2}{2} \end{aligned}$$

Проверка:

$$\left(\frac{(\ln \operatorname{tg} x)^2}{2} \right)' = \frac{1}{2} \cdot 2 \cdot \ln \operatorname{tg} x \cdot \frac{1}{\operatorname{tg} x \cos^2 x} = \ln \operatorname{tg} x \frac{1}{\sin x \cos x}$$

4 Интегрирование по частям

$$(uv)' = uv' + v'u$$

$$d(uv) = u dv + v du$$

$$u dv = d(uv) - v du$$

$$\int u dv = \int d(uv) - \int v du$$

$$\int u dv = uv - \int v du$$

u должна не “портиться” при дифф., а dv не “портится” при интегрировании.

$$\begin{aligned} \int e^{3x}(x+1)dx &= [u = x+1, dv = e^{3x}dx, du = dx, v = \frac{e^{3x}}{3}] = \int u dx = uv - \int v du = \\ &= (x+1)\frac{e^{3x}}{3} - \frac{1}{3} \int e^{3x}dx = (x+1)\frac{e^{3x}}{3} - \frac{1}{9}e^{3x}dx + C \end{aligned}$$

$$\begin{aligned} \int \sin x(x^2+x)dx &= [u = x^2+x, dv = \sin x dx, du = (2x+1)dx, v = -\cos x] = \\ &= uv - \int v du = -(x^2+x)\cos x + \int \cos x(2x+1)dx = [u = 2x+1, dv = \cos x dx, du = 2dx, v = \sin x] = \\ &= -(x^2+x)\cos x + ((2x+1)\sin x - \int 2dx \sin x) = -(x^2+x)\cos x + (2x+1)\sin x + 2\cos x \end{aligned}$$

$$\int \ln x dx = [u = \ln x, dv = dx, du = \frac{1}{x}dx, v = x] = x \ln x - \int dx = x \ln x - x + c$$

Проверка:

$$(x \ln x - x + c)' = \ln x + 1 - 1 = \ln x$$

$$\begin{aligned} \int \ln(x^2+1)dx &= [u = \ln(x^2+1), dv = dx, v = x, du = \frac{1}{x^2+1}2xdx] = \\ &= \ln(x^2+1)x - \int x \frac{1}{x^2+1}2xdx = x \ln(x^2+1) - 2 \int (1 - \frac{1}{x^2+1})dx = x \ln(x^2+1) - 2(x - \operatorname{arctg} x) + C \end{aligned}$$

$$\begin{aligned} \int e^x \sin 2x dx &= [u = e^x, dv = \sin 2x dx, du = e^x, v = \frac{-1}{2} \cos 2x] = \\ &= e^x \frac{-1}{2} \cos 2x + \int \frac{1}{2} e^x \cos 2x dx = -e^x \frac{1}{2} \cos 2x + \frac{1}{4} e^x \sin 2x - \frac{1}{4} \int e^x \sin 2x dx \end{aligned}$$

$$\frac{5}{4} \int e^x \sin 2x dx = e^x \left(-\frac{1}{2} \cos 2x + \frac{1}{4} \sin 2x \right)$$

$$\int e^x \sin 2x dx = \frac{2}{5} e^x \left(-\cos 2x + \frac{1}{2} \sin 2x \right) +$$