Моделирование двойного маятника

Михайлов Максим, Кулагин Ярослав, Винников Глеб

4 января 2021 г.

Обозначения:

- x_1, y_1, x_2, y_2 — координаты центров масс звеньев
- m_1, m_2 — массы звеньев
- l_1, l_2 — длины звеньев
- θ_1, θ_2 — угол отклонения от вертикали

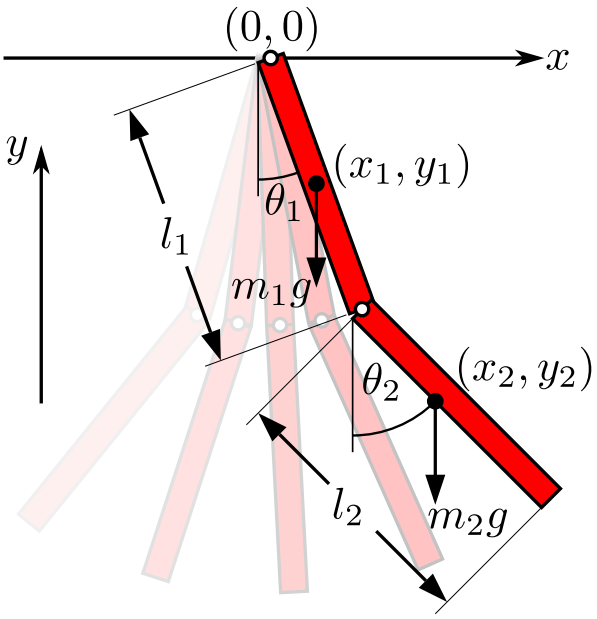

Рис. 1: Двойной маятник

Выразим координаты центров:

$$\begin{cases} x_1 = l_1 \sin\theta_1 & (1) \\ y_1 = -l_1 \cos\theta_1 & (2) \\ x_2 = x_1 + l_2 \sin\theta_2 & (3) \\ y_2 = y_2 - l_2 \cos\theta_2 & (4) \end{cases}$$

Возьмём первую и вторую производную этих выражений:

$$\begin{cases} x_1' = l_1\theta_1' \cos\theta_1 & (5) \\ y_1' = l_1\theta_1' \sin\theta_1 & (6) \\ x_2' = x_1' + \theta_2' l_2 \cos\theta_2 & (7) \\ y_2' = y_2' + \theta_2' l_2 \sin\theta_2 & (8) \end{cases}$$

$$\begin{cases} x_1'' = -l_1\theta_1'^2 \sin\theta_1 + l_1\theta_1'' \cos\theta_1 & (9) \\ y_1'' = l_1\theta_1'^2 \cos\theta_1 + l_1\theta_1'' \sin\theta_1 & (10) \\ x_2'' = x_1'' - \theta_2'^2 l_2 \sin\theta_2 + \theta_2'' l_2 \cos\theta_2 & (11) \\ y_2'' = y_1'' + \theta_2'^2 l_2 \cos\theta_2 + \theta_2'' l_2 \sin\theta_2 & (12) \end{cases}$$

По второму закону Ньютона *(в проекциях)* для верхнего звена:

$$\begin{cases} m_1 x_1'' = -T_1 \sin\theta_1 + T_2 \sin\theta_2 & (13) \\ m_1 y_1'' = T_1 \cos\theta_1 - T_2 \cos\theta_2 - m_1 g & (14) \end{cases}$$

Аналогично для нижнего звена:

$$\begin{cases} m_2 x_2'' = -T_2 \sin\theta_2 & (15) \\ m_2 y_2'' = T_2 \cos\theta_2 - m_2 g & (16) \end{cases}$$

Выразим $T_2 \sin\theta_2$ и $T_2 \cos\theta_2$ из (15), (16) и подставим в (13), (14)

$$\begin{cases} m_1 x_1'' = -T_1 \sin\theta_1 - m_2 x_2'' & (17) \\ m_1 y_1'' = T_1 \cos\theta_1 - m_2 y_2'' - m_2 g - m_1 g & (18) \end{cases}$$

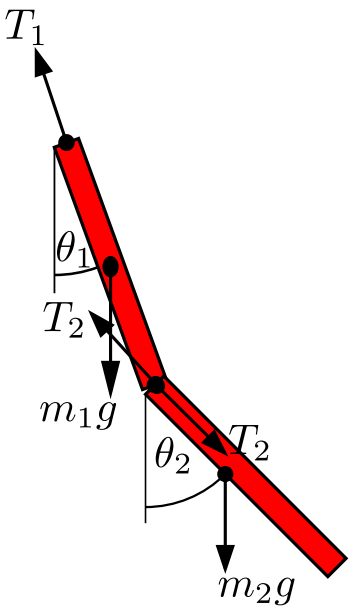

Рис. 2: Силы в двойном маятнике

Домножим (17) на $\cos\theta_1$, (18) на $\sin\theta_1$:

$$\begin{cases} m_1 x_1'' \cos\theta_1 = -T_1 \sin\theta_1 \cos\theta_1 - m_2 x_2'' \cos\theta_1 \\ m_1 y_1'' \sin\theta_1 = T_1 \cos\theta_1 \sin\theta_1 - \sin\theta_1 (m_2 y_2'' + m_2 g + m_1 g) \end{cases} \quad (19)$$

$$\begin{cases} T_1 \sin\theta_1 \cos\theta_1 = -\cos\theta_1 (m_1 x_1'' + m_2 x_2'') \\ T_1 \sin\theta_1 \cos\theta_1 = \sin\theta_1 (m_1 y_1'' + m_2 y_2'' + m_2 g + m_1 g) \end{cases} \quad (20)$$

$$\sin\theta_1 (m_1 y_1'' + m_2 y_2'' + m_2 g + m_1 g) = -\cos\theta_1 (m_1 x_1'' + m_2 x_2'') \quad (21)$$

Домножим (15) на $\cos\theta_2$ и (16) на $\sin\theta_2$:

$$\begin{cases} m_2 x_2'' \cos\theta_2 = -T_2 \sin\theta_2 \cos\theta_2 \\ m_2 y_2'' \sin\theta_2 = T_2 \sin\theta_2 \cos\theta_2 - m_2 g \sin\theta_2 \end{cases} \quad (22)$$

Приравняем правые части (22):

$$\sin\theta_2 (m_2 y_2'' + m_2 g) = -\cos\theta_2 (m_2 x_2'') \quad (23)$$

$$\sin\theta_2 (y_2'' + g) = -\cos\theta_2 (x_2'') \quad (24)$$

Подставим y_2'' и x_2'' из (11) и (12):

$$\sin\theta_2(y_1'' + \theta_2'^2 l_2\cos\theta_2 + \theta_2'' l_2\sin\theta_2 + g) = -\cos\theta_2(x_1'' - \theta_2'^2 l_2\sin\theta_2 + \theta_2'' l_2\cos\theta_2) \tag{25}$$

Подставим y_1'' и x_1'' из (9) и (10):

$$\begin{aligned}\sin\theta_2(l_1\theta_1'^2\cos\theta_1 + l_1\theta_1''\sin\theta_1 + \theta_2'^2 l_2\cos\theta_2 + \theta_2'' l_2\sin\theta_2 + g) = \\ -\cos\theta_2(-l_1\theta_1'^2\sin\theta_1 + l_1\theta_1''\cos\theta_1 - \theta_2'^2 l_2\sin\theta_2 + \theta_2'' l_2\cos\theta_2)\end{aligned} \tag{26}$$

$$\theta_1'^2 l_1(\sin\theta_2\cos\theta_1 - \sin\theta_1\cos\theta_2) + \theta_1'' l_1(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2) + \theta_2'' l_2 + \sin\theta_2 g = 0 \tag{27}$$

$$\theta_2'' = -\frac{l_1}{l_2}(\theta_1'^2(\sin\theta_2\cos\theta_1 - \sin\theta_1\cos\theta_2) + \theta_1''(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2)) - \sin\theta_2\frac{g}{l} \tag{28}$$

$$\theta_2'' = \frac{l_1}{l_2}(\theta_1'^2\sin(\theta_1 - \theta_2) - \theta_1''\cos(\theta_2 - \theta_1)) - \sin\theta_2\frac{g}{l_2} \tag{29}$$

Подставим в (21) y_2'' и x_2'' из (11) и (12)

$$\begin{aligned}\sin\theta_1(m_1 y_1'' + m_2(y_1'' + \theta_2'^2 l_2\cos\theta_2 + \theta_2'' l_2\sin\theta_2) + m_2 g + m_1 g) = \\ -\cos\theta_1(m_1 x_1'' + m_2(x_1'' - \theta_2'^2 l_2\sin\theta_2 + \theta_2'' l_2\cos\theta_2))\end{aligned} \tag{30}$$

Подставим y_1'' и x_1'' из (9) и (10):

$$\begin{aligned}\sin\theta_1((m_1+m_2)(l_1\theta_1'^2\cos\theta_1 + l_1\theta_1''\sin\theta_1) + m_2(\theta_2'^2 l_2\cos\theta_2 + \theta_2'' l_2\sin\theta_2) + m_2 g + m_1 g) = \\ -\cos\theta_1((m_1+m_2)(-l_1\theta_1'^2\sin\theta_1 + l_1\theta_1''\cos\theta_1) + m_2(-\theta_2'^2 l_2\sin\theta_2 + \theta_2'' l_2\cos\theta_2))\end{aligned}$$

$$\begin{aligned}\theta_1'' l_1(m_1+m_2) + \theta_1''\cdot 0 + \theta_2'' l_2 m_2(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2) + \\ \theta_2'^2 l_2 m_2(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1) + \sin\theta_1 g(m_1+m_2) = 0\end{aligned}$$

$$\theta_1'' l_1(m_1+m_2) + \theta_2'' l_2 m_2\cos(\theta_2-\theta_1) - \theta_2'^2 l_2 m_2\cos(\theta_1+\theta_2) + \sin\theta_1 g(m_1+m_2) = 0$$

$$\theta_1'' = \theta_2'^2\frac{l_2 m_2}{l_1(m_1+m_2)}\cos(\theta_1+\theta_2) - \theta_2''\frac{l_2 m_2}{l_1(m_1+m_2)}\cos(\theta_2-\theta_1) - \sin\theta_1\frac{g}{l_1} \tag{31}$$

Подставим (31) в (29):

$$\theta_2'' = \frac{l_1}{l_2}(\theta_1'^2\sin(\theta_1-\theta_2) - \left(\theta_2'^2\frac{l_2 m_2}{l_1(m_1+m_2)}\cos(\theta_1+\theta_2) - \theta_2''\frac{l_2 m_2}{l_1(m_1+m_2)}\cos(\theta_2-\theta_1) - \sin\theta_1\frac{g}{l_1}\right)\cdot \cos(\theta_2-\theta_1)) - \sin\theta_2\frac{g}{l_2}$$

$$\theta_2'' = \frac{l_1}{l_2}\theta_1'^2\sin(\theta_1-\theta_2) - \left(\theta_2'^2\frac{m_2}{m_1+m_2}\cos(\theta_1+\theta_2) - \theta_2''\frac{m_2}{m_1+m_2}\cos(\theta_2-\theta_1) - \sin\theta_1\frac{g}{l_2}\right)\cdot \cos(\theta_2-\theta_1)) - \sin\theta_2\frac{g}{l_2}$$

$$\theta_2'' = \frac{l_1}{l_2}\theta_1'^2 \sin(\theta_1 - \theta_2) - \theta_2'^2 \frac{m_2}{m_1 + m_2}\frac{1}{2}(\cos(2\theta_1) + \cos(2\theta_2)) - \theta_2'' \frac{m_2}{m_1 + m_2}\cos^2(\theta_2 - \theta_1) - \cos(\theta_2 - \theta_1)\sin\theta_1 \frac{g}{l_2} - \sin\theta_2 \frac{g}{l_2}$$

$$\theta_2'' \frac{2m_1 + 2m_2 - m_2\cos^2(\theta_2 - \theta_1)}{m_1 + m_2} = \frac{l_1}{l_2}\theta_1'^2 \sin(\theta_1 - \theta_2) - \theta_2'^2 \frac{m_2}{m_1 + m_2}\frac{1}{2}(\cos(2\theta_1) + \cos(2\theta_2)) - \cos(\theta_2 - \theta_1)\sin\theta_1 \frac{g}{l_2} - \sin\theta_2 \frac{g}{l}$$

$$\theta_2'' \frac{2m_1 + m_2 - m_2\cos(2\theta_2 - 2\theta_1)}{2m_1 + 2m_2} = \frac{l_1}{l_2}\theta_1'^2 \sin(\theta_1 - \theta_2) - \theta_2'^2 \frac{m_2}{m_1 + m_2}\frac{1}{2}(\cos(2\theta_1) + \cos(2\theta_2)) - \cos(\theta_2 - \theta_1)\sin\theta_1 \frac{g}{l_2} - \sin\theta_2 \frac{g}{l}$$

$$\theta_2'' \frac{2m_1 + m_2 - m_2\cos(2\theta_2 - 2\theta_1)}{2m_1 + 2m_2} = \frac{l_1^2(m_1 + m_2)\theta_1'^2 \sin(\theta_1 - \theta_2) - m_2\theta_2'^2 l_1 l_2 \sin(\theta_2 - \theta_1)\cos(\theta_2 - \theta_1)}{l_2 l_1 (m_1 + m_2)} + \frac{l_1 \sin\theta_1 \cos(\theta_1 - \theta_2) g(m_1 + m_2) - l_1(m_1 + m_2)\sin\theta_2 g}{l_2 l_1 (m_1 + m_2)}$$

$$\theta_2'' \frac{2m_1 + m_2 - m_2\cos(2\theta_2 - 2\theta_1)}{2} = \frac{l_1(m_1 + m_2)\theta_1'^2 \sin(\theta_1 - \theta_2) - m_2\theta_2'^2 l_2 \sin(\theta_2 - \theta_1)\cos(\theta_2 - \theta_1)}{l_2} + \frac{\sin\theta_1 \cos(\theta_1 - \theta_2) g(m_1 + m_2) - (m_1 + m_2)\sin\theta_2 g}{l_2}$$

$$\theta_2'' = 2\frac{l_1(m_1 + m_2)\theta_1'^2 \sin(\theta_1 - \theta_2) - m_2\theta_2'^2 l_2 \sin(\theta_2 - \theta_1)\cos(\theta_2 - \theta_1)}{l_2(2m_1 + m_2 - m_2\cos(2\theta_2 - 2\theta_1))} + 2\frac{\sin\theta_1 \cos(\theta_1 - \theta_2) g(m_1 + m_2) - (m_1 + m_2)\sin\theta_2 g}{l_2(2m_1 + m_2 - m_2\cos(2\theta_2 - 2\theta_1))}$$

$$\theta_2'' = 2\sin(\theta_1 - \theta_2)\frac{l_1(m_1 + m_2)\theta_1'^2 + m_2\theta_2'^2 l_2 \cos(\theta_1 - \theta_2) + g(m_1 + m_2)\cos\theta_1}{l_2(2m_1 + m_2 - m_2\cos(2\theta_2 - 2\theta_1))} \tag{32}$$

$$\theta_1'' = \frac{m_2\theta_2'^2 l_2 \sin(\theta_2 - \theta_1) - m_2(l_1(\theta_1'^2 \sin(\theta_1 - \theta_2) - \theta_1'' \cos(\theta_1 - \theta_2)) - \sin\theta_2 g)\cos(\theta_2 - \theta_1)}{l_1(m_1 + m_2)} + \sin\theta_1 \frac{g}{l_1}$$

$$\theta_1'' - \frac{m_2\theta_1''\cos(\theta_1-\theta_2)}{m_1+m_2} = \frac{m_2\theta_2'^2 l_2\sin(\theta_2-\theta_1) - m_2(l_1\theta_1'^2\sin(\theta_1-\theta_2) - \sin\theta_2 g)\cos(\theta_2-\theta_1)}{l_1(m_1+m_2)} + \sin\theta_1\frac{g}{l_1}$$

$$\begin{aligned}\theta_1'' &= \frac{2m_2\sin(\theta_2-\theta_1)(\theta_2' l_2 + \theta_1'^2 l_1\cos(\theta_1-\theta_2)) + m_2\sin(2\theta_2-\theta_1)g + g\sin\theta_1(m_2-2m_1-2m_2)}{l_1(2m_1+m_2-m_2\cos(2\theta_2-2\theta_1))} \\ &= \frac{2m_2\sin(\theta_2-\theta_1)(\theta_2' l_2 + \theta_1'^2 l_1\cos(\theta_1-\theta_2)) + m_2\sin(2\theta_2-\theta_1)g - g\sin\theta_1(2m_1+m_2)}{l_1(2m_1+m_2-m_2\cos(2\theta_2-2\theta_1))}\end{aligned} \tag{33}$$

Дифуры (33) и (32) решаются методом Рунге-Кутты, если ввести функции $w_1(t) = \theta_1'(t)$ и $w_2(t) = \theta_2'(t)$ *(угловые скорости)*.