1 $2y' = x + \ln y'$

$$\begin{aligned}
2y' &= x + \ln y' \\
y' &:= t \\
2t &= x + \ln t \\
2t - \ln t &= x \\
\frac{dy}{dx} &= t \\
dy &= tdx \\
\int dy &= \int tdx \\
y &= \int td(2t - \ln t) \\
y &= \int t\left(2dt - \frac{dt}{t}\right) \\
y &= t^2 - t + C
\end{aligned}$$

Ответ: $\begin{cases} x = 2t - \ln t \\ y = t^2 - t + C \end{cases}$

2 $4y = x^2 + y'^2$

$$\begin{aligned}
4y &= x^2 + y'^2 \\
t &:= y' \\
4y &= x^2 + t^2 \\
4t &= 2x + 2tt' \\
2t &= x + tt' \\
\frac{2t}{x} &= 1 + \frac{tt'}{x} \\
a &:= \frac{t}{x} \quad t' = a'x + a \\
2a &= 1 + a(a'x + a) \\
2a &= 1 + a'ax + a^2 \\
\frac{-1 + 2a - a^2}{ax} &= a' \\
\frac{1}{x} &= \frac{aa'}{-1 + 2a - a^2}
\end{aligned}$$

$$\begin{aligned}
\frac{dx}{x} &= \frac{ada}{-1+2a-a^2} \\
\int \frac{dx}{x} &= \int \frac{ada}{-1+2a-a^2} \\
\ln|x| + C &= -\ln|a-1| + \frac{1}{a-1} \qquad (1) \\
\ln|x| + C &= -\ln\left|\frac{t}{x} - 1\right| + \frac{1}{\frac{t}{x} - 1} \\
\ln|x| + C &= -\ln\left|\frac{\sqrt{4y-x^2}}{x} - 1\right| + \frac{1}{\frac{\sqrt{4y-x^2}}{x} - 1}
\end{aligned}$$

(1): $a = 1$ — особое решение, рассмотрим этот случай:

$$\begin{aligned}
1 &= \frac{t}{x} \\
t &= x \\
\pm\sqrt{4y-x^2} &= x \\
4y - x^2 &= \pm x^2 \\
y &= \pm\frac{x^2}{2}
\end{aligned}$$

3 $y' = \operatorname{tg}(y-2x)$

$$\begin{aligned}
y' &= \operatorname{tg}(y-2x) \\
\operatorname{arctg}(y') &= y - 2x + \pi k \\
t &:= y' \\
\operatorname{arctg}(t) &= y - 2x + \pi k \\
t = y' &= (\operatorname{arctg} t + 2x - \pi k)' \\
&= \frac{1}{1+t^2} t' + 2 \\
&= \frac{1}{1+t^2} \frac{dt}{dx} + 2 \\
dx &= \frac{dt}{(1+t^2)(t-2)} \\
\int dx &= \int \frac{dt}{(1+t^2)(t-2)}
\end{aligned}$$

$$x = \int \frac{dt}{(1+t^2)(t-2)}$$

$$\begin{aligned}
\int \frac{dt}{(t-2)(t^2+1)} &= \int \frac{dt}{5(t-2)} - \int \frac{t+2}{5(t^2+1)}dt \\
&= \frac{\ln|t-2|}{5} - \int \frac{t}{5(t^2+1)}dt - \int \frac{2}{5(t^2+1)}dt \\
&= \frac{\ln|t-2|}{5} - \frac{\ln|t^2+1|}{10} - \frac{2}{5}\operatorname{arctg} t + C
\end{aligned}$$

$$\begin{aligned}
dy &= tdx \\
y &= \int tdx \\
&= \int \frac{tdt}{(1+t^2)(t-2)} \\
&= \int \frac{2dt}{5(t-2)} - \int \frac{2t-1}{5(t^2+1)}dt \\
&= \frac{2}{5}\ln|t-2| - \int \frac{2t}{5(t^2+1)}dt + \int \frac{1}{5(t^2+1)}dt \\
&= \frac{2}{5}\ln|t-2| - \frac{1}{5}\ln|x^2+1| + \frac{\operatorname{arctg} t}{5} + C_1 \\
y &= 2x + \operatorname{arctg} t + C_1
\end{aligned}$$

4 $yy' + xyy'' = x(y')^2 + x^3$

5 $2xy' - y = y'\ln(yy')$

$$\begin{aligned}
2xy' - y &= y'\ln(yy') \\
t &:= y' \\
2xt - y &= t\ln(yt) \\
x &= \frac{t\ln(yt) + y}{2t} \\
x &= \frac{y}{2t} + \frac{\ln y}{2} + \frac{\ln t}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{dx}{dy} &= \frac{1}{2y} + \frac{1}{2t} + \left(\frac{1}{2t} - \frac{y}{2t^2}\right)\frac{dt}{dy} \\
\frac{dx}{dy} &= \frac{t+y}{2yt} + \left(\frac{t-y}{2t^2}\right)\frac{dt}{dy} \\
\frac{1}{t} &= \frac{t+y}{2yt} + \left(\frac{t-y}{2t^2}\right)\frac{dt}{dy} \\
\frac{2y-t-y}{2yt} &= \left(\frac{t-y}{2t^2}\right)\frac{dt}{dy} \\
\frac{y-t}{y} &= \left(\frac{t-y}{t}\right)\frac{dt}{dy} \qquad (1) \\
-\frac{1}{y} &= \frac{1}{t}\frac{dt}{dy} \\
-\frac{dy}{y} &= \frac{dt}{t} \\
-\ln|y| &= \ln|t| + C \\
y &= \pm\frac{e^C}{t} \\
y' &= \pm\frac{e^C}{y} \\
\pm 2x\frac{e^C}{y} - y &= \pm\frac{e^C}{y}\ln(\pm e^C) \\
2x\frac{e^C}{y} - y &= \frac{e^C}{y}\ln(e^C) \\
(2x - C)\frac{e^C}{y} - y &= 0
\end{aligned}$$

(1): рассмотрим $t = y$:

$$\begin{aligned}
2xy - y &= y\ln(y^2) \\
2xy - y &= 2y\ln(y) \qquad (2) \\
2x - 1 &= 2\ln(y) \\
e^{\frac{2x-1}{2}} &= y \\
\frac{e^x}{\sqrt{e}} &= y
\end{aligned}$$

(2): вертикальная прямая $x(y) = \text{const}$ не подходит.

Ответ: $y = \frac{e^x}{\sqrt{e}}$ или $\begin{cases} x = \pm\frac{e^C}{2t^2} - \frac{C}{2} \\ y = \pm\frac{e^C}{t} \end{cases}$

6 $y = 2xy' + y^2(y')^3$

$$\begin{aligned}
y &= 2xy' + y^2y'^3 && (1)\\
x &= \frac{y - y^2y'^3}{2y'}\\
x &= \frac{y}{2y'} - \frac{y^2y'^2}{2}\\
t &:= y'\\
x &= \frac{y}{2t} - \frac{y^2t^2}{2}\\
\frac{dx}{dy} &= \frac{1}{2t} - \frac{2yt^2}{2} - \left(-\frac{y}{2t^2} + \frac{2y^2t}{2}\right)\frac{dt}{dy}\\
\frac{1}{t} &= \frac{1}{2t} - yt^2 + \left(-\frac{y}{2t^2} - y^2t\right)\frac{dt}{dy}\\
\frac{1}{2t} + yt^2 &= \left(-\frac{y}{2t^2} - y^2t\right)\frac{dt}{dy}\\
\frac{1}{2t} + yt^2 &= \frac{y}{t}\left(-\frac{1}{2t} - yt^2\right)\frac{dt}{dy} && (2)\\
1 &= -\frac{y}{t}\frac{dt}{dy}\\
\frac{dy}{y} &= -\frac{dt}{t}\\
\ln|y| &= -\ln|t| + C\\
y &= \frac{\pm e^C}{t}\\
\frac{\pm e^C}{y} &= t\\
y &= 2x\frac{\pm e^C}{y} + \frac{\pm e^{3C}}{y}
\end{aligned}$$

(1): $y \equiv 0$ подходит.

(2): $\triangleleft \frac{1}{2t} = yt^2$:

$$
\begin{aligned}
\frac{1}{2t} &= yt^2 \\
\frac{1}{2t^3} &= y \\
\frac{1}{2t^3} &= 2xt + \frac{1}{2t^6}t^3 \\
\frac{1}{2t^3} &= 2xt + \frac{1}{2t^3} \\
0 &= 2xt
\end{aligned}
$$

7 $y + xy' = 4\sqrt{y'}$

Это уравнение Лагранжа

$f(y') = -y', g(y') = 4\sqrt{y'}$

$$
\begin{aligned}
y &= 4\sqrt{y'} - xy' \\
y &= 4\sqrt{t} - xt \\
t &= \frac{2}{\sqrt{t}}t' - xt' - t \\
2t &= \frac{2}{\sqrt{t}}t' - xt' \\
2t &= t'\left(\frac{2}{\sqrt{t}} - x\right) \\
2t &= \frac{1}{x'}\left(\frac{2}{\sqrt{t}} - x\right) \qquad (1) \\
2t &= \frac{1}{x'}\frac{2 - x\sqrt{t}}{\sqrt{t}} \\
\frac{2t\sqrt{t}}{2 - x\sqrt{t}} &= \frac{1}{x'} \\
\frac{2 - x\sqrt{t}}{2t\sqrt{t}} &= x' \\
\frac{1}{t\sqrt{t}} - \frac{x}{2t} &= x' \\
\frac{1}{t} - \frac{x\sqrt{t}}{2t} &= x'\sqrt{t} \\
\frac{1}{t} &= x'\sqrt{t} + \frac{x\sqrt{t}}{2t}
\end{aligned}
$$

$$\begin{aligned}
\frac{1}{t} &= x'\sqrt{t} + \frac{x}{2\sqrt{t}} \\
\frac{1}{t} &= x'\sqrt{t} + x(\sqrt{t})' \\
\frac{1}{t} &= (x\sqrt{t})' \\
\int \frac{dt}{t} &= \int d(x\sqrt{t})' \\
\ln|t| + C &= x\sqrt{t} \\
\frac{\ln|t| + C}{\sqrt{t}} &= x
\end{aligned}$$

(1): производная берется по t.

Ответ: $\begin{cases} x = \frac{\ln|t|+C}{\sqrt{t}} \\ y = -t\frac{\ln|t|+C}{\sqrt{t}} + 4\sqrt{t} \end{cases}$

8 Найти кривую, касательная к которой отсекает на осях координат такие отрезки, что сумма величин, обратных квадратам длин этих отрезков, равна 1.

$$\begin{aligned}
\frac{1}{\left(\frac{xy' - y}{y'}\right)^2} + \frac{1}{(y - xy')^2} &= 1 \\
\frac{y'^2}{(xy' - y)^2} + \frac{1}{(y - xy')^2} &= 1 \\
y'^2 + 1 &= (y - xy')^2 \\
y^2 - 2yxy' - (xy')^2 - y'^2 - 1 &= 0 \\
y &= xy' \pm \sqrt{1 + y'^2} \\
y' &= xy'' + y' \pm \frac{y'y''}{\sqrt{1 + y'^2}} \\
0 &= xy'' \pm \frac{y'y''}{\sqrt{1 + y'^2}} \\
0 &= y''\left(x \pm \frac{y'}{\sqrt{1 + y'^2}}\right)
\end{aligned}$$

1 случай: $y'' = 0 \Rightarrow y' = C \Rightarrow y = C_1 + Cx, C_1 = \pm\sqrt{C^2+1}$

2 случай: $x \mp \frac{y'}{\sqrt{1+y'^2}} = 0$

$$\begin{aligned} y &= xy' \pm \sqrt{1+y'^2} \\ y &= \mp\frac{y'}{\sqrt{1+y'^2}}y' \pm \sqrt{1+y'^2} \\ y\sqrt{1+y'^2} &= \pm 1 \\ y^2(1+y'^2) &= 1 \\ y'^2 &= \frac{1}{y^2} - 1 \\ y' &= \sqrt{\frac{1}{y^2}-1} \\ x &= \int \frac{dy}{\sqrt{\frac{1}{y^2}-1}} \\ x &= -y\sqrt{\frac{1}{y^2}-1} + C \end{aligned}$$

Ответ: $\begin{cases} y = \pm\sqrt{C^2+1} + Cx \\ x = -y\sqrt{\frac{1}{y^2}-1} + C \end{cases}$ — это разные случаи

9 Модель войны, изменение численности армий: $x' = -by$; $y' = -ax$. a,b - мощность армий y,x соответственно. Описать результат военных действий на фазовом пространстве.

$$\begin{cases} x' = -by & (1) \\ y' = -ax & (2) \end{cases}$$

$$x = \frac{y'}{-a}$$

$$x' = \frac{y''}{-a} \tag{3}$$

$$\frac{y''}{-a} = -by \tag{4}$$

$$y'' = aby$$

$$t := y' \quad y'' = t't$$

$$
\begin{aligned}
t't &= aby \\
t &= \pm\sqrt{aby^2 + C} \\
y' &= \pm\sqrt{aby^2 + C} \\
x &= \int \frac{dy}{\pm\sqrt{aby^2 + C}} \\
z := \sqrt{\frac{ab}{C}}y \quad dy &= \frac{\sqrt{C}}{\sqrt{ab}}dz \\
x &= \int \frac{\sqrt{C}dz}{\pm\sqrt{ab}\sqrt{Cz^2 + C}} \\
&= \frac{1}{\sqrt{ab}} \int \frac{dz}{\pm\sqrt{z^2 + 1}} \\
&= \frac{\ln|\sqrt{z^2 + 1} + z|}{\sqrt{ab}} + C \\
&= \frac{\ln|\sqrt{\frac{ab}{C}y^2 + 1} + \sqrt{\frac{ab}{C}}y|}{\sqrt{ab}} + C_1
\end{aligned}
$$

(4): подставили (3) в (1).