1 Найти второе приближение Пикара для системы: $x(t)' = t + x, y(t)' = xyt$ и задачи Коши $x(0) = 1, y(0) = -1$

$$x_0 \equiv 1, y_0 \equiv -1$$

$$\begin{aligned} x_1 &= x_0 + \int_{t_0}^{t} (\xi + x_0(\xi)) d\xi \\ &= 1 + \int_0^t (\xi + 1) d\xi \\ &= 1 + \left(\frac{\xi^2}{2} + \xi \right) \Bigg|_0^t \\ &= 1 + \frac{t^2}{2} + t \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + \int_{t_0}^{t} (\xi \cdot x_0(\xi) \cdot y_0(\xi)) d\xi \\ &= -1 + \int_0^t -\xi d\xi \\ &= -1 + \frac{-t^2}{2} \end{aligned}$$

$$\begin{aligned} x_2 &= x_0 + \int_{t_0}^{t} (\xi + x_1(\xi)) d\xi \\ &= 1 + \int_0^t \left(\xi + 1 + \frac{\xi^2}{2} + \xi \right) d\xi \\ &= 1 + \left(\xi^2 + \xi + \frac{\xi^3}{6} \right) \Bigg|_0^t \\ &= 1 + t^2 + t + \frac{t^3}{6} \end{aligned}$$

$$\begin{aligned} y_2 &= y_0 + \int_{t_0}^{t} (\xi \cdot x_1(\xi) \cdot y_1(\xi)) d\xi \\ &= -1 + \int_0^t \xi \cdot \left(1 + \frac{\xi^2}{2} + \xi \right) \cdot \left(-1 + \frac{-\xi^2}{2} \right) d\xi \end{aligned}$$

$$
\begin{aligned}
&= -1 - \int_0^t \left(\xi + \frac{\xi^3}{2} + \xi^2\right) \cdot \left(1 + \frac{\xi^2}{2}\right) d\xi \\
&= -1 - \int_0^t \left(\xi + \frac{\xi^3}{2} + \xi^2 + \frac{\xi^3}{2} + \frac{\xi^5}{4} + \frac{\xi^4}{2}\right) d\xi \\
&= -1 - \frac{\xi^2}{2} - \frac{\xi^4}{4} - \frac{\xi^3}{3} - \frac{\xi^6}{24} - \frac{\xi^5}{10}
\end{aligned}
$$

2 Найти второе приближение Пикара для уравнения $y'' - y' \sin x - x^2 = 0, y(0) = 1, y'(0) = 0$

Пусть $z = y'$. Тогда $z' = z \sin x + x^2$

$$z_0 \equiv 0$$

$$
\begin{aligned}
z_1 &= z_0 + \int_{x_0}^x (z_0 \sin \xi + \xi^2) d\xi \\
&= \frac{\xi^3}{3}\Big|_0^x \\
&= \frac{x^3}{3}
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= y_0 + \int_0^x z_1 d\xi \\
&= 1 + \frac{\xi^4}{12}\Big|_0^x \\
&= 1 + \frac{x^4}{12}
\end{aligned}
$$

$$
\begin{aligned}
z_2 &= z_0 + \int_{x_0}^x (z_1 \sin \xi + \xi^2) d\xi \\
&= \int_0^x \left(\frac{\xi^3}{3} \sin \xi + \xi^2\right) d\xi \\
&= \frac{x^3}{3} + \frac{1}{3} \int_0^x \frac{\xi^3}{3} \sin \xi d\xi
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{x^3}{3} + \frac{1}{3}\left(-\xi^3 \cos x\Big|_0^x + 3\int_0^x x^2 \cos\xi d\xi\right) \\
&= \frac{x^3}{3}(1-\cos x) + \int_0^x x^2 \cos\xi d\xi \\
&= \frac{x^3}{3}(1-\cos x) + \xi^2 \sin\xi\Big|_0^x - \int 2\xi \sin\xi d\xi \\
&= \frac{x^3}{3}(1-\cos x) + x^2 \sin x - \int 2\xi \sin\xi d\xi \\
&= \frac{x^3}{3}(1-\cos x) + x^2 \sin x + 2x\cos x - 2\int \cos\xi d\xi \\
&= \frac{x^3}{3}(1-\cos x) + x^2 \sin x + 2x\cos x - 2\sin\xi\Big|_0^x \\
&= \frac{x^3}{3}(1-\cos x) + x^2 \sin x + 2x\cos x - 2\sin x \\
&= \frac{1}{3}x^3 - \frac{1}{3}(x^3-6x)\cos x + (x^2-2)\sin x
\end{aligned}
$$

3 $(x+y+1)dx + (2x+2y-1)dy = 0$

$$
\begin{aligned}
t &:= x+y \\
(t+1)dx + (2t-1)dy &= 0 \\
(t+1) + (2t-1)(t'-1) &= 0 \\
-t+2+(2t-1)t' &= 0 \\
t' &= \frac{t-2}{2t-1} \\
dt &= dx\frac{t-2}{2t-1} \\
\frac{2t-1}{t-2}dt &= dx \\
\int \frac{2t-1}{t-2}dt &= \int dx \\
\int \frac{2t-4+3}{t-2}dt &= x \\
2t + \int \frac{3}{t-2}dt &= x \\
2t + 3\ln|t-2| + C &= x \\
x + 2y + 3\ln|x+y-2| + C &= 0
\end{aligned}
$$

4 Решить уравнение $x^2(yy''-y'^2)+xyy' = (2xy'-3y)\sqrt{x^3}$. Особое решение привести к виду $y(x) = a \cdot x^b \cdot \ln(cx)$, где c - константа.

$y \equiv 0$ — решение

$x \equiv 0$ — решение

$$
\begin{aligned}
x^2(yy''-y'^2)+xyy' &= (2xy'-3y)\sqrt{x^3} \\
x^2\left(\frac{y'}{y}\right)' + x\frac{y'}{y} &= \left(2x\frac{y'}{y^2} - \frac{3}{y}\right)\sqrt{x^3} \\
x^2\left(\frac{y'}{y}\right)' + x\frac{y'}{y} &= \frac{y}{x^2}\left(\frac{2x^3y'}{y^3} - \frac{3x^2y}{y^3}\right)\sqrt{x^3} \\
x^2\left(\frac{y'}{y}\right)' + x\frac{y'}{y} &= -\frac{y}{x^2}\left(\frac{x^3}{y^2}\right)'\sqrt{x^3} \\
t := \frac{y'}{y} &\quad a := \frac{x^3}{y^2} \\
x^2t' + xt &= -\frac{y}{x^2}a'\sqrt{x^3} \\
xt' + t &= -\frac{y}{\sqrt{x^3}}a' \\
xt' + t &= -\frac{a'}{\sqrt{a}} \\
(xt)' &= -\frac{a'}{\sqrt{a}} \\
\frac{1}{2}(xt)' &= -\frac{a'}{2\sqrt{a}} \\
\frac{1}{2}(xt)' &= -\left(\sqrt{a}\right)' \\
\frac{xt}{2} &= -\sqrt{a} + C \\
\frac{xy'}{2y} &= -\sqrt{\frac{x^3}{y^2}} + C \\
\frac{xy'}{2} &= -\sqrt{x^3} + Cy \\
y' &= -2\sqrt{x} + C\frac{y}{x}
\end{aligned}
$$

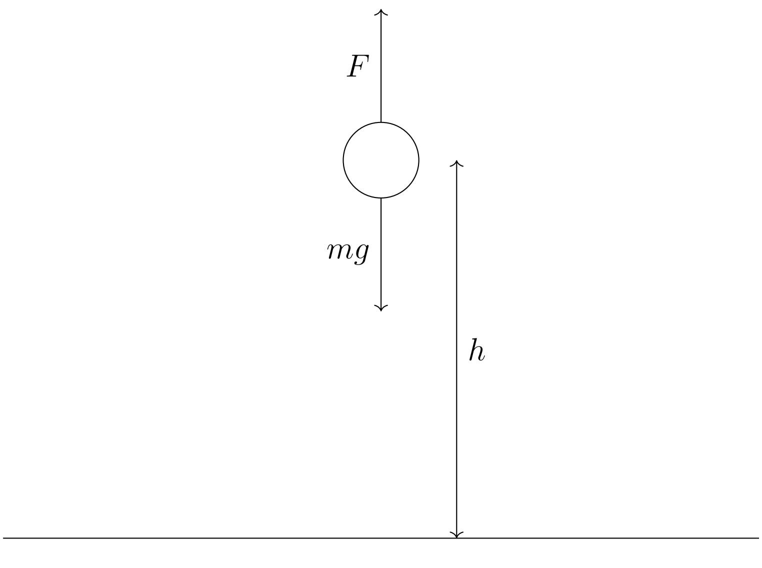
Это линейное уравнение при $p = \dfrac{C}{x}, q = -2\sqrt{x}$

$$
\begin{aligned}
y &= \left(\int e^{-\int p} q\right) e^{\int p} \\
&= -2\left(\int e^{-C\int x^{-1}dx} \sqrt{x}dx\right) e^{C\int x^{-1}dx} \\
&= -2\left(\int x^{-C}\sqrt{x}dx\right) x^{C} \\
&= -2\left(\int x^{-C+1/2}dx\right) x^{C}
\end{aligned}
$$

Разберем случай $C = 1/2$. Тогда $y = -2x^{1/2}(\ln x + C) = -2x^{1/2}\ln(cx)$

$$
y = -2\left(\frac{x^{-C+3/2}}{3/2} + C_1\right) x^{C}
$$

5 Вычислить время падения мяча 0,4 кг с высоты 16,3 метра с учётом сопротивления воздуха. Сопротивление воздуха пропорционально квадрату скорости и равно 0,48 г при скорости 1 м/сек.



Подставим $F\Big|_{v=1} = 4.8 \cdot 10^{-4}$ кгс $= 4.8 \cdot 10^{-4} g$ Н:

$$
F = kv^2
$$

$$4.8 \cdot 10^{-4} g = k$$

$$\begin{aligned}
-F + mg &= ma \\
-kv^2 + mg &= mv' \\
-\frac{k}{m}v^2 + g &= v' \\
t &= \int \frac{dv}{g - \frac{k}{m}v^2} \\
&= -\frac{\sqrt{m}}{2\sqrt{kg}} \ln\left|\frac{kv - \sqrt{gkm}}{kv + \sqrt{gkm}}\right| + C
\end{aligned}$$

Подставим $v(0) = 0$:

$$\begin{aligned}
0 &= \ln 1 + C \\
0 &= C
\end{aligned}$$

$$\begin{aligned}
t &= -\frac{\sqrt{m}}{2\sqrt{kg}} \ln\left|\frac{kv - \sqrt{gkm}}{kv + \sqrt{gkm}}\right| \\
-\frac{2t\sqrt{kg}}{\sqrt{m}} &= \ln\left|\frac{kv - \sqrt{gkm}}{kv + \sqrt{gkm}}\right| \\
\exp\left(-\frac{2t\sqrt{kg}}{\sqrt{m}}\right) &= \left|\frac{kv - \sqrt{gkm}}{kv + \sqrt{gkm}}\right| \\
(kv + \sqrt{gkm})\exp\left(-\frac{2t\sqrt{kg}}{\sqrt{m}}\right) &= \left|kv - \sqrt{gkm}\right| \\
\pm(kv + \sqrt{gkm})\exp\left(-\frac{2t\sqrt{kg}}{\sqrt{m}}\right) &= kv - \sqrt{gkm} \\
\sqrt{gkm}\left(1 \mp \exp\left(-\frac{2t\sqrt{kg}}{\sqrt{m}}\right)\right) &= kv\left(1 \pm \exp\left(-\frac{2t\sqrt{kg}}{\sqrt{m}}\right)\right) \\
\frac{\sqrt{gm}\left(1 \mp \exp\left(-\frac{2t\sqrt{kg}}{\sqrt{m}}\right)\right)}{\sqrt{k}\left(1 \pm \exp\left(-\frac{2t\sqrt{kg}}{\sqrt{m}}\right)\right)} &= v
\end{aligned}$$

$$\int_0^t v dt = h$$

$$\int_0^t \frac{\sqrt{gm}\left(1 \mp \exp\left(-\frac{2t\sqrt{kg}}{\sqrt{m}}\right)\right)}{\sqrt{k}\left(1 \pm \exp\left(-\frac{2t\sqrt{kg}}{\sqrt{m}}\right)\right)} dt = h$$

$$\frac{\sqrt{gm}}{\sqrt{k}} \int_0^t \frac{1 \mp \exp\left(-\frac{2t\sqrt{kg}}{\sqrt{m}}\right)}{1 \pm \exp\left(-\frac{2t\sqrt{kg}}{\sqrt{m}}\right)} dt = h$$

$$\frac{\sqrt{gm}}{\sqrt{k}} \int_0^t -\frac{\pm \exp\left(-\frac{2t\sqrt{kg}}{\sqrt{m}}\right) - 1}{1 \pm \exp\left(-\frac{2t\sqrt{kg}}{\sqrt{m}}\right)} dt = h$$

$$\frac{\sqrt{gm}}{\sqrt{k}} \left(t + \left(2\frac{\ln\left|1 \pm \exp\left(\frac{2t\sqrt{kg}}{\sqrt{m}}\right)\right|}{\frac{2\sqrt{kg}}{\sqrt{m}}}\right)\Bigg|_0^t\right) = h$$

$$\frac{\sqrt{gm}}{\sqrt{k}} \left(t + \frac{\ln\left|1 \pm \exp\left(\frac{t\sqrt{kg}}{\sqrt{m}}\right)\right|}{\frac{\sqrt{kg}}{\sqrt{m}}} - \frac{\ln|1 \pm 1|}{\frac{\sqrt{kg}}{\sqrt{m}}}\right) = h$$

$$\frac{\sqrt{gm}}{\sqrt{k}} \left(t + \frac{\ln\left|1 + \exp\left(\frac{t\sqrt{kg}}{\sqrt{m}}\right)\right|}{\frac{\sqrt{kg}}{\sqrt{m}}} - \frac{\ln|2|}{\frac{\sqrt{kg}}{\sqrt{m}}}\right) = h$$

$$\frac{m}{k}\left(\frac{t\sqrt{kg}}{\sqrt{m}} + \ln\left|1 + \exp\left(\frac{t\sqrt{kg}}{\sqrt{m}}\right)\right| - \ln 2\right) = h$$

$$\frac{m}{k}\left(\frac{t\sqrt{kg}}{\sqrt{m}} + \ln\left(\frac{1 + \exp\left(\frac{t\sqrt{kg}}{\sqrt{m}}\right)}{2}\right)\right) = h$$

$$\frac{0.4}{9.6 \cdot 10^{-4} g} \ln\left(\frac{1 + \exp\left(\frac{tg\sqrt{4.8 \cdot 10^{-4}}}{\sqrt{0.4}}\right)}{2}\right) + t\frac{\sqrt{0.4g}}{\sqrt{4.8 \cdot 10^{-4} g}} = h$$

$$\frac{1}{24 \cdot 10^{-4} g} \ln\left(\frac{1 + \exp\left(tg\sqrt{12 \cdot 10^{-4}}\right)}{2}\right) + t\frac{1}{\sqrt{12 \cdot 10^{-4}}} = h$$

$$t \approx 1.87$$