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Persistence of generalised density functions

Master Thesis

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Abstract

Topological data analysis (TDA) often uses density-like functions defined on the ambient space $\mathbb{R}^d$ to infer the underlying topological structure of a dataset $P \subseteq \mathbb{R}^d$. The persistent homology of the sublevelset or superlevelset filtrations induced by these functions captures multi-scale topological features. A key desirable property is *stability*: small perturbations in the dataset result in similarly small changes in the persistence diagrams. A classic example is the nearest-neighbour distance function $f_{\text{dist}}(P, x) := \min_{p \in P} d(x, p)$, whose sublevel sets are homotopy equivalent to the Čech complex [26], for which the bottleneck distance between persistence diagrams $\text{Dgm}(f_P)$ and $\text{Dgm}(f_Q)$ is bounded by the Hausdorff distance $d_H(P, Q)$ between datasets [8]:

$$d_b(\text{Dgm}(f_P), \text{Dgm}(f_Q)) \leq d_H(P, Q).$$

This thesis investigates *generalised density functions*, which are functions $f : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$, where $\mathcal{P}(\mathbb{R}^d)$ denotes the set of all finite subsets of $\mathbb{R}^d$, and the conditions under which they satisfy a stability property of the form

$$d_b(\text{Dgm}(f_P), \text{Dgm}(f_Q)) \leq c \cdot d_H(P, Q)$$

for some finite constant c.

The primary contributions of this work are stability theorems for several classes of generalised density functions. Specifically, we prove stability bounds for:

- Several generalisations of the nearest-neighbour distance function of the form $f(P, x) = \min_{p \in P} h(x, p)$.
- Functions that are Lipschitz continuous with respect to the Hausdorff distance on the space of point clouds.
- Morse functions that satisfy a Lipschitz-like condition.

Beyond these core stability results, we explore the properties of the space of stable functions and investigate how common operations, such as addition and taking minima, affect stability. We identify conditions under which stability is preserved under these operations and provide counterexamples demonstrating cases where it is not.

Our findings unify and extend existing stability results, offering practical guidance for the selection and design of generalised density functions for topological data analysis.

Contents

Chapter 1

Introduction

Topological Data Analysis (TDA) provides a robust framework for understanding the shape of data [6]. Its primary tool, *persistent homology*, extracts topological features across different scales [2] and is traditionally applied to a finite point cloud $P \subseteq \mathbb{R}^d$ through filtrations of simplicial complexes built on top of P, such as the Vietoris–Rips or Čech complexes [9]. An alternative (though sometimes equivalent) approach involves defining a real-valued function $\mathbb{R}^d \to \mathbb{R}$ that encodes the data's geometry, and then computing the persistent homology of the function's sublevel (or superlevel) set filtration [12]. The resulting persistent homology captures topological features such as connected components, loops, and voids, and encodes them in a *persistence diagram*, which summarises their birth and death across the filtration [9].

A classical and widely studied example is the *nearest-neighbour distance function*,

$$f(P, x) = \min_{p \in P} d(x, p), \tag{1.1}$$

where d is a metric on $\mathbb{R}^d$. The sublevel sets of this function are homotopy equivalent to the Čech filtration [26], a fundamental construction in TDA. This function satisfies a *stability* property [8]:

$$d_b(\mathrm{Dgm}(f_P), \mathrm{Dgm}(f_Q)) \leq c \cdot d_H(P, Q), \tag{1.2}$$

where d_b is the bottleneck distance between persistence diagrams and d_H is the Hausdorff distance between point clouds. Stability is not merely a theoretical nicety; it is the foundation that provides reliability of TDA [9]. In real-world applications, where data is inevitably noisy or subject to measurement error, an unstable method could produce drastically different topological summaries from minor, inconsequential variations in the input. A stable GDF ensures that the extracted topological features are genuine reflections of the data's underlying structure, rather than artifacts of noise or sampling.

While the nearest-neighbour function is stable, it is just one of many possible density-like functions for TDA [1, 16, 24]. In practice, alternative functions may offer computational advantages [15, 4], better capture intrinsic structure [1], or incorporate domain knowledge [18]. For example, the DTM filtration [1] modifies the nearest-neighbour function to make it more robust to noise and outliers. This motivates the study of *generalised density functions* (GDFs), which are functions of the form

$$f(P, x) : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}. \tag{1.3}$$

For a given point cloud P, such a function defines a real-valued function $f_P : \mathbb{R}^d \to \mathbb{R}$, whose sublevel set filtration can be used to analyze the topological features of P. We say f is *c-stable* if for all finite point clouds $P, Q \subseteq \mathbb{R}^d$ we have

$$d_b(\mathrm{Dgm}(f_P), \mathrm{Dgm}(f_Q)) \leq c \cdot d_H(P, Q), \tag{1.4}$$

where the *stability constant* c measures the sensitivity of the persistence diagrams to perturbations in the data.

We limit ourselves to the case where P and Q are finite sets, as known stability results for GDFs often hold only in this case, because otherwise the functions f_P often are not tame, and, consequently, the persistence diagrams $\mathrm{Dgm}(f_P)$ are not q-tame.

We also only consider the space $\mathbb{R}^d$ instead of a general topological space for the sake of simplicity, as the various theorems used in this thesis impose different conditions on the underlying topological space. The euclidean space $\mathbb{R}^d$ was chosen as the lowest common denominator, although many results in this thesis hold for more general spaces. An inquisitive reader is invited to track down the most general conditions for the various novel theorems by keeping track of the theorems used in the proofs.

In this thesis, we primarily investigate which GDFs are stable. Existing stability results, such as those for the Čech filtration and its weighted variants, provide a starting point. This thesis aims to create a more comprehensive framework by identifying broader conditions that ensure stability, thereby unifying and extending these results. Concretely, we prove stability of the following classes of functions:

1. We begin by examining natural extensions of the nearest-neighbour function, specifically of the form $f(P, x) = \min_{p \in P} h(d(p, x))$, where h is a monotone Lipschitz continuous function. This is a modification of the Čech complex which allows the balls to grow at non-uniform rates, although it still requires the growth rate to be the same for all points $p \in P$.

2. This is further extended to $f(P, x) = \min_{p \in P} h(p, x)$, where $h(p, x)$ is a Lipschitz function with respect to p. This allows for more general shapes, as well as different growth rates for different points p.

3. We then investigate a more direct condition: functions $f(P, x)$ that are Lipschitz with respect to point clouds *(PC-Lipschitz)*, where $\mathcal{P}(\mathbb{R}^d)$ is equipped with the Hausdorff distance, and the space of real-valued functions $f_P : \mathbb{R}^d \to \mathbb{R}$ is equipped with the L_∞-norm. This class includes the previous two classes as special cases, and allows for more general functions that are still stable, but may not be represented as a minimum over a set of functions.

4. Finally, recognising that full Lipschitz continuity might be too strong, we consider Morse functions where Lipschitz-like conditions are imposed only on critical values. This aims to capture stability for functions whose may not be Lipschitz continuous globally, but are nonetheless stable.

Beyond these classes, we also study when stability is preserved under common operations. For example, if two functions are each stable, under what conditions is their pointwise minimum or average also stable? We identify sufficient conditions and provide concrete counterexamples demonstrating cases where stability fails. This analysis shows how more complex stable functions can be constructed from simpler ones, and provides insight into the structure of the space of stable functions.

Finally, we examine the topological and algebraic properties of the space of c-stable functions itself. We analyze properties such as convexity, contractibility, and closedness, as well as establish a lattice structure on the set of c-PC-Lipschitz functions.

Taken together, our results contribute to the theoretical foundations of TDA by deepening the understanding of stability for generalised density functions. They also provide practical guidance for designing stable filtrations.

Chapter 2

Background

This chapter reviews the mathematical foundations for stability of generalised density functions. We briefly review the core concepts of how real-valued functions induce filtrations for persistent homology and then focus on established stability theorems that serve as a foundation and motivation for this work. We assume the reader has a working knowledge of topological data analysis, mainly persistent homology. Otherwise, a comprehensive introduction to the subject is provided in the textbook by Herbert Edelsbrunner and John Harer [12].

At the heart of this thesis lies the concept of a generalised density function (GDF).

Definition 2.1 (Generalised density function) *A generalised density function is a map $f : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$, where $\mathcal{P}(\mathbb{R}^d)$ denotes the collection of all finite subsets of $\mathbb{R}^d$. For a given point cloud $P \in \mathcal{P}(\mathbb{R}^d)$, a GDF f induces a real-valued function $f_P : \mathbb{R}^d \to \mathbb{R}$ defined by $f_P(x) := f(P, x)$.*

These functions f_P are used to construct *sublevel set filtrations*.

Definition 2.2 (Sublevel set filtration) *Given a function $f_P : \mathbb{R}^d \to \mathbb{R}$, its sublevel set at a value $a \in \mathbb{R}$ is*

$$f_P^{-1}(-\infty, a] = \{x \in \mathbb{R}^d \mid f_P(x) \leq a\}. \tag{2.1}$$

The sublevel set filtration of g is the nested family of sets $\{f_P^{-1}(-\infty, a]\}_{a \in \mathbb{R}}$.

Alternatively, superlevel set filtrations can be used, which are defined by $\{f_P^{-1}[a, +\infty)\}_{a \in \mathbb{R}}$. The choice between sublevel and superlevel set filtrations is a matter of convention, as the superlevel set filtration of f_P is exactly the sublevel set filtration of the function $-f_P$, and vice versa. For the sake of consistency, we will use sublevel set filtrations throughout this thesis.

Applying a homology functor H_k to a filtration $\{f_P^{-1}(-\infty, a]\}_{a \in \mathbb{R}}$ yields a persistence module, which tracks the evolution of topological features as

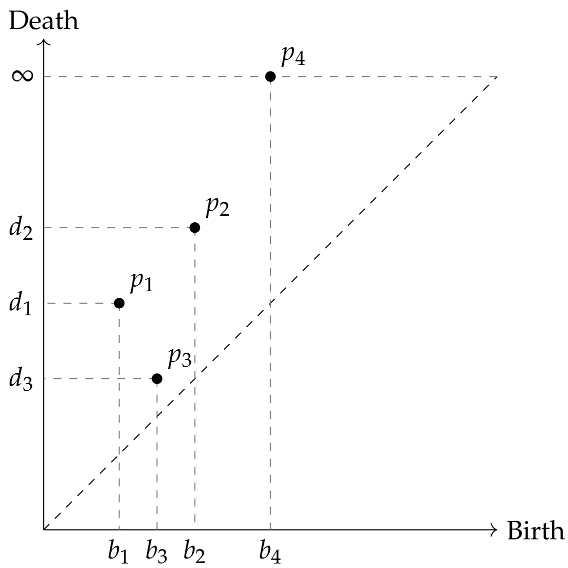


Figure 2.1: An example of a persistence diagram. A point p_i is born at b_i and dies at d_i. The point p_4 does not die, so its death time is ∞. The dashed line indicates the diagonal Δ, where $b_i = d_i$.

the parameter a varies [12]. The *persistence diagram*, denoted $\mathrm{Dgm}_k(f_P)$, is a multiset of points (b_i, d_i) in the extended plane $\overline{\mathbb{R}}^2$, where b_i represents the birth time and d_i the death time of a k-dimensional feature. An example of a persistence diagram is shown in Figure 2.1.

2.1 Tameness

Many established stability results and the original findings of this thesis rely on the involved persistence modules being *q-tame*, a condition ensuring the finiteness of the rank for maps induced by filtrations.

Definition 2.3 (q-tame persistence module, [7]) *A persistence module $\mathbb{M}$ with vector spaces M_a and linear maps $m_a^b : M_a \to M_b$ is* q-tame *if m_a^b is of finite rank for all $a < b$.*

q-tameness is a generalisation of the following notion:

Definition 2.4 (Pointwise finite-dimensional persistence module) *A persistence module $\mathbb{M}$ is* pointwise finite-dimensional (PFD) *if for every $a \in \mathbb{R}$, the vector space M_a is finite-dimensional.*

Lemma 2.5 ([7]) *If a persistence module is PFD, then it is q-tame.*

The following theorems can be used to prove q-tameness of persistence modules induced by sublevel set filtrations.

Theorem 2.6 ([5]) *Let X be a realisation of a finite complex, and let $f : X \to \mathbb{R}$ be a continuous function. Then the persistence module induced by sublevel sets of f is q-tame.*

While this theorem is powerful, it is of limited use in this thesis, as we consider functions $f_P : \mathbb{R}^d \to \mathbb{R}$, and no triangulation of $\mathbb{R}^d$ is finite, as $\mathbb{R}^d$ is not compact. This condition is relaxed in the following theorem:

Theorem 2.7 ([7], Corollary 3.34) *Let X be a realisation of a* locally *finite complex, and let $f : X \to \mathbb{R}$ be a continuous function which is bounded below and the preimage $f^{-1}(K)$ of every compact set $K \subseteq \mathbb{R}$ is compact. Then the persistence module induced by sublevel sets of f is q-tame.*

This theorem is applicable to $\mathbb{R}^d$, as it admits a locally finite triangulation [14]. The condition that the preimage of every compact set is compact, also known as *properness*, is required to remove pathological functions such as the sine function, whose sublevel sets $\sin^{-1}(-\infty, a]$ have infinite-dimensional 0-homology for all $a \in (-1, 1)$.

2.2 Quantifying stability

To formalize the notion of stability, we need metrics to compare both point clouds and persistence diagrams. A common metric for point clouds is the *Hausdorff distance*, which is also used in the definition of stability of the nearest-neighbour distance function.

Definition 2.8 (Hausdorff distance) *Given two non-empty sets $P, Q \subseteq \mathbb{R}^d$, where $\mathbb{R}^d$ is a metric space, the Hausdorff distance between P and Q is defined as*

$$d_H(P,Q) = \max\left\{\sup_{p\in P}\inf_{q\in Q} d(p,q), \sup_{q\in Q}\inf_{p\in P} d(q,p)\right\}. \tag{2.2}$$

Intuitively, the Hausdorff distance measures the "maximum mismatch" between the two sets, capturing the largest distance between a point in one set and its closest neighbour in the other set.

Definition 2.9 (Bottleneck distance) *Given two persistence diagrams D_1 and D_2, the bottleneck distance between them is defined as*

$$d_b(D_1, D_2) = \inf_{\pi} \sup_{(b,d)\in D_1} \|(b,d) - \pi(b,d)\|_\infty, \tag{2.3}$$

where the infimum is taken over all bijections π between D_1 and D_2, allowing for the possibility of matching points to the diagonal $\Delta = \{(x,x) \mid x \in \overline{\mathbb{R}}\}$.

The bottleneck distance measures the cost of the "least expensive" matching between the points of the two diagrams, where the cost is defined as the longest edge in the matching.

With these metrics defined, we can formally define stability for a GDF.

Definition 2.10 *A generalised density function $f : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ is c-*stable *if for all non-empty finite point clouds $P, Q \subseteq \mathbb{R}^d$ and all homology dimensions $k \geq 0$, we have*

$$d_b(\mathrm{Dgm}_k(f_P), \mathrm{Dgm}_k(f_Q)) \leq c \cdot d_H(P, Q). \tag{2.4}$$

If no such finite c exists, we say that f is unstable *or ∞-stable.*

2.3 Known stability results

This section reviews the known stability results for GDFs and simplicial filtrations that can be represented as GDFs.

2.3.1 Čech filtration

As mentioned in Chapter 1, the Čech filtration can be represented as the sublevel set filtration of the nearest-neighbour distance function $f_{\text{dist}}(P, x) := \min_{p \in P} d(x, p)$. The sublevel sets $f_P^{-1}(-\infty, a]$ are precisely $\bigcup_{p \in P} B(p, a)$, the union of closed balls of radius a centered at the points of P. The nerve of this union of balls defines the Čech complex. By the Nerve theorem [3, 19], if P is finite, the Čech complex is homotopy equivalent to the union of balls.

The stability of this construction is a well-known result:

Theorem 2.11 (Stability of the Čech filtration, [8]) *Let $f_P(x) := \min_{p \in P} d(x, p)$. Then for all finite $P, Q \subseteq \mathcal{P}(\mathbb{R}^d)$ and any dimension $k \geq 0$, we have*

$$d_b(\mathrm{Dgm}_k(f_P), \mathrm{Dgm}_k(f_Q)) \leq d_H(P, Q). \tag{2.5}$$

Using the terminology of this thesis, we say the nearest-neighbour distance function is 1-stable.

2.3.2 Weighted Čech filtration

The weighted Čech filtration is a generalisation of the Čech filtration used to define the DTM-filtration [1].

Definition 2.12 (Weighted Čech filtration) *Let $q \in [1, \infty)$, $P \subseteq \mathbb{R}^d$ and $g : P \to \mathbb{R}_{\geq 0}$. For every $p \in P, a \in \mathbb{R}^+$, we define the function $r_p(a)$ to be:*

$$r_p^{(q)}(a) = \begin{cases} -\infty, & \text{if } a < g(p), \\ (a^q - g(p)^q)^{1/q}, & \text{otherwise.} \end{cases} \tag{2.6}$$

For $q = \infty$, we also define

$$r_p^{(q)}(a) = \begin{cases} -\infty, & \text{if } a < g(p), \\ a, & \text{otherwise.} \end{cases} \tag{2.7}$$

$g(p)$ acts as a weight for each point $p \in P$, influencing the radius $r_p^{(q)}(a)$ of the ball $B(p, r_p^{(q)}(a))$. Concretely, the weighted Čech filtration is defined as

$$V_q^a[P, g] = \bigcup_{p \in P} B(p, r_p^{(q)}(a)), \tag{2.8}$$

where the balls are closed, and the ball of radius $-\infty$ is the empty set.

The weighted Čech filtration is a strict generalisation of the Čech filtration, as the Čech filtration is the special case where $g(p) = 0$ for all $p \in P$.

Similarly to the Čech filtration, this filtration can be represented as a GDF of the form

$$f_{g,q}(P, x) = \min_{p \in P} (d(x, p)^q + g(p)^q)^{1/q}, \tag{2.9}$$

where similarly to the L_∞-norm, when $q = \infty$, the term $(d(x, p)^q + g(p)^q)^{1/q}$ is replaced by $\max(d(x, p), g(p))$. A proof of the equivalence of this GDF and the weighted Čech filtration is given in Appendix A.

If g is Lipschitz continuous, then the weighted Čech filtration is guaranteed to be stable:

Theorem 2.13 (Stability of the weighted Čech filtration, [1]) *Let $P, Q \subseteq \mathbb{R}^d$ be compact and $g : P \cup Q \to \mathbb{R}^+$ be a Lipschitz continuous function with Lipschitz constant c. Then for any dimension $k \geq 0$, we have*

$$d_b(\mathrm{Dgm}_k(V_q^a[P, g]), \mathrm{Dgm}_k(V_q^a[Q, g])) \leq (1 + c^q)^{1/q} \cdot d_H(P, Q). \tag{2.10}$$

This theorem immediately implies that the corresponding GDF is $(1 + c^q)^{1/q}$-stable.

2.3.3 Stability of L_∞-norm-bounded functions

This fundamental result relates the bottleneck distance between persistence diagrams of two functions to the L_∞-distance between the functions themselves.

Theorem 2.14 (Stability of persistence diagrams, [10]) *Let X be a triangulable space, and $g, h : X \to \mathbb{R}$ be two continuous tame functions. Then for all $k \geq 0$, the persistence diagrams of g and h satisfy*

$$d_b(\mathrm{Dgm}_k(g), \mathrm{Dgm}_k(h)) \leq \|g - h\|_\infty. \tag{2.11}$$

Remark 2.15 *This theorem also holds when g and h are not necessarily tame, as long as the resulting persistence modules are q-tame [7].*

Importantly, this theorem is applicable to the case $X = \mathbb{R}^d$, as the space $\mathbb{R}^d$ is triangulable [23]. We can easily adapt this theorem to the case of GDFs:

$$d_b(\mathrm{Dgm}_k(f_P), \mathrm{Dgm}_k(f_Q)) \leq \|f_P - f_Q\|_\infty. \tag{2.12}$$

This theorem directly implies that if a GDF f satisfies $\|f_P - f_Q\|_\infty \leq c \cdot d_H(P, Q)$ for some constant c, then f is c-stable. This provides a powerful tool for proving stability of GDFs. As an example, we prove 1-stability of nearest neighbour distance function using this theorem. To do so, we first need to show the following lemma:

Lemma 2.16 *The persistence modules induced by the nearest-neighbour distance function $f_{\mathrm{dist}}(P, x)$ are q-tame.*

Proof The Čech filtration $\check{C}(P)$ is a filtration of a finite simplicial complex if P is finite, and its persistence modules $H_k(\check{C}(P))$ are thus pointwise finite-dimensional. The sublevel set filtration induced by f_{dist} is homotopy equivalent to $\check{C}(P)$ [26], which implies that the persistence modules corresponding to f_{dist} are also PFD, which implies q-tameness by Lemma 2.5. □

Now we can prove the stability of the nearest-neighbour distance function:

Proof Let $P, Q \subseteq \mathbb{R}^d$ be two finite point clouds with Hausdorff distance $d = d_H(P, Q)$. Let x be an arbitrary point in $\mathbb{R}^d$ with the nearest point in P denoted by p. This implies that $f_{\mathrm{dist},P}(x) = d(x, p)$. By the definition of the Hausdorff distance, there exists a point $q' \in Q$ such that $d(p, q') \leq d$. By the triangle inequality, we have

$$d(x, q') \leq d(x, p) + d(p, q') \leq d(x, p) + d. \tag{2.13}$$

and thus:

$$f_{\mathrm{dist},Q}(x) = \min_{q \in Q} d(x, q) \leq d(x, q') \leq d(x, p) + d = f_{\mathrm{dist},P}(x) + d. \tag{2.14}$$

We can show that $f_{\mathrm{dist},P}(x) \leq f_{\mathrm{dist},Q}(x) + d$ by the same reasoning. Thus, we have

$$\|f_{\mathrm{dist},P} - f_{\mathrm{dist},Q}\|_\infty \leq d, \tag{2.15}$$

which implies that f_{dist} is 1-stable by Theorem 2.14 and Lemma 2.16. □

2.3.4 Stability of functions with interleaved persistence modules

An alternative approach to show stability is to use the *interleaving distance*, which bounds the bottleneck distance not only from above, but also from below.

Definition 2.17 (Interleaving of persistence modules) *Two persistence modules $\mathbb{M}$ and $\mathbb{N}$ with M_a, N_a as their respective vector spaces and $m_a^{a'}, n_a^{a'}$ as their homomorphisms are* ε-interleaved *for $\varepsilon \geq 0$ if there exist families of maps $\varphi_a : M_a \to N_{a+\varepsilon}$ and $\psi_a : N_a \to N_{a+\varepsilon}$ such that the following diagrams commute for all $a < a'$:*

$$
\begin{tikzcd}
M_a \arrow[rr, "m_a^{a'}"] \arrow[dr, "\varphi_a"'] & & M_{a'} \arrow[dr, "\varphi_{a'}"] & \\
& N_{a+\varepsilon} \arrow[rr, "n_{a+\varepsilon}^{a'+\varepsilon}"'] & & N_{a'+\varepsilon}
\end{tikzcd}
\qquad
\begin{tikzcd}
M_a \arrow[rr, "m_a^{a+2\varepsilon}"] \arrow[dr, "\varphi_a"'] & & M_{a+2\varepsilon} \\
& N_{a+\varepsilon} \arrow[ur, "\psi_{a+\varepsilon}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& M_{a+\varepsilon} \arrow[rr, "n_{a+\varepsilon}^{a'+\varepsilon}"] & & M_{a'+\varepsilon} \\
N_a \arrow[rr, "n_a^{a'}"'] \arrow[ur, "\psi_a"] & & N_{a'} \arrow[ur, "\psi_{a'}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& M_{a+\varepsilon} \arrow[dr, "\varphi_{a+\varepsilon}"] & \\
N_a \arrow[rr, "n_a^{a+2\varepsilon}"'] \arrow[ur, "\psi_a"] & & N_{a+2\varepsilon}
\end{tikzcd}
$$

Note that $\mathbb{M}$ and $\mathbb{N}$ are 0-interleaved if and only if they are isomorphic [26]. The triangular diagrams on the right collapse to diagrams that show that φ_t and ψ_t are inverses of each other, while the trapezoidal diagrams on the left ensure they commute with the linear maps m and n.

Definition 2.18 (Interleaving distance) *The interleaving distance between two persistence modules $\mathbb{M}$ and $\mathbb{N}$ is given by:*

$$d_i(\mathbb{M}, \mathbb{N}) = \inf\{\varepsilon \mid \mathbb{M} \textit{ and } \mathbb{N} \textit{ are } \varepsilon\textit{-interleaved}\}. \tag{2.16}$$

Finally, the interleaving distance and the bottleneck distance are equal as long as the persistence modules are q-tame:

Theorem 2.19 (Isometry theorem, [7]) *Let $\mathbb{M}, \mathbb{N}$ be q-tame persistence modules. Then*

$$d_i(\mathbb{M}, \mathbb{N}) = d_b(\mathrm{Dgm}(\mathbb{M}), \mathrm{Dgm}(\mathbb{N})). \tag{2.17}$$

Additionally, we can define interleaving for filtrations:

Definition 2.20 (Interleaving of filtrations) *Let $\mathcal{F}, \mathcal{G}$ be filtrations over $\mathbb{R}$, that is, $\mathbb{R}$-indexed sequence of nested subspaces of some topological space X. $\mathcal{F}$ and $\mathcal{G}$ are* ε-interleaved *if there exist families of maps $\varphi_a : F_a \to G_{a+\varepsilon}$ and $\psi_a : G_a \to F_{a+\varepsilon}$ such that the same diagrams as in Definition 2.17 commute up to homotopy.*

The interleaving distance between two filtrations is defined the same way as for persistence modules. The following lemma shows that the former bounds the latter from above:

Lemma 2.21 ([26]) *For any two filtrations $\mathcal{F}$ and $\mathcal{G}$ over $\mathbb{R}$, we have*

$$d_i(H_p\mathcal{F}, H_p\mathcal{G}) \leq d_i(\mathcal{F}, \mathcal{G}). \tag{2.18}$$

With these results in hand, we can apply the interleaving approach to once again prove 1-stability of the Čech filtration.

Proof ([26]) Let X be a metric topological space, and $P, Q \subseteq X$ be two finite point clouds with Hausdorff distance $d = d_H(P, Q)$. Recall that the Čech complex $\check{C}^r(P)$ at scale r is homotopy equivalent to the union of the balls centered at points of P, formally

$$\check{C}^r(P) \simeq \bigcup_{p \in P} B(p, r). \tag{2.19}$$

Let $x \in B(p, r)$. By the definition of the Hausdorff distance, there exists a point $q \in Q$ such that $d(p, q) \leq d$. Therefore, by the triangle inequality we have

$$d(x, q) \leq d(x, p) + d(p, q) \leq r + d, \tag{2.20}$$

implying $x \in B(q, r + d)$. As x was arbitrary, this provides an inclusion

$$\bigcup_{p \in P} B(p, r) \subseteq \bigcup_{q \in Q} B(q, r + d). \tag{2.21}$$

By symmetric reasoning, we have the same with the roles of P and Q reversed. With these three facts, we see that the following diagram commutes up to homotopy, as all maps are either inclusions or homotopies:

$$\begin{tikzcd}
\check{C}^r(P) \arrow[r, hook] \arrow[d, leftrightarrow, "\simeq"] & \check{C}^{r+d}(P) \arrow[r, hook] \arrow[d, leftrightarrow, "\simeq"] & \check{C}^{r+2d}(P) \arrow[d, leftrightarrow, "\simeq"] \\
\bigcup_{p \in P} B(p, r) \arrow[r, hook] \arrow[dr, hook] & \bigcup_{p \in P} B(p, r + d) \arrow[r, hook] \arrow[dr, hook] & \bigcup_{p \in P} B(p, r + 2d) \\
\bigcup_{q \in Q} B(q, r) \arrow[r, hook] \arrow[ur, hook] & \bigcup_{q \in Q} B(q, r + d) \arrow[r, hook] \arrow[ur, hook] & \bigcup_{q \in Q} B(q, r + 2d) \\
\check{C}^r(Q) \arrow[r, hook] \arrow[u, leftrightarrow, "\simeq"] & \check{C}^{r+d}(Q) \arrow[r, hook] \arrow[u, leftrightarrow, "\simeq"] & \check{C}^{r+2d}(Q) \arrow[u, leftrightarrow, "\simeq"]
\end{tikzcd} \tag{2.22}$$

All diagrams that are needed to show that $\check{C}^r$ and $\check{C}^r(Q)$ are d-interleaved are included in this diagram, and thus we have

$$d_i(\check{C}(P), \check{C}(Q)) \leq d. \tag{2.23}$$

Applying Lemma 2.16 and Theorem 2.19, as well as Lemma 2.21, we have

$$\begin{aligned} d_b(\mathrm{Dgm}_k(\check{C}(P)), \mathrm{Dgm}_k(\check{C}(Q))) &\leq d_i(H_k(\check{C}(P)), H_k(\check{C}(Q))) && (2.24)\\ &\leq d_i(\check{C}(P), \check{C}(Q)) && (2.25)\\ &\leq d, && (2.26)\end{aligned}$$

which is precisely the statement of stability of the Čech filtration. □

Chapter 3

Stable Generalised Density Functions

Chapter 2 reviewed foundational stability results for GDFs. This chapter examines the central question of this thesis: *for which GDFs $f(P, x)$ can we guarantee that the sublevel set filtration of f_P is stable?* We explore several classes of GDFs for which stability can be guaranteed, ranging from simple cases to more complex constructions. We also highlight examples of functions that are not stable.

3.1 0-stable generalised density functions

The simplest class of stable GDFs are those whose persistence diagrams are invariant under changes to the input point cloud P.

Definition 3.1 (Point cloud independent GDF) *A GDF $f(P, x)$ is* point cloud independent *if for any two point clouds $P, Q \subseteq \mathbb{R}^d$, we have $f(P, x) = f(Q, x)$ for all $x \in \mathbb{R}^d$. Equivalently, $f_P = f_Q$ as functions $\mathbb{R}^d \to \mathbb{R}$.*

Theorem 3.2 *A point cloud independent GDF $f : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ is 0-stable.*

Proof If $f(P, x)$ is point cloud independent, then for any two finite point clouds $P, Q \subseteq \mathbb{R}^d$, we have identical sublevel set filtrations $\{f_P^{-1}(-\infty, a]\}_a$ and $\{f_Q^{-1}(-\infty, a]\}_a$. This implies that their persistence modules and persistence diagrams are also identical. As the bottleneck distance between identical persistence diagrams is zero [12], we have

$$d_b(\mathrm{Dgm}(f_P), \mathrm{Dgm}(f_Q)) = 0 \leq 0 \cdot d_H(P, Q), \tag{3.1}$$

satisfying the condition for 0-stability. $\square$

However, point cloud independence is a sufficient, but not necessary, condition for 0-stability. Some GDFs whose values depend on P can still yield identical persistence diagrams for all P, thus being 0-stable.

Example 3.3 *Consider the GDF $f : \mathcal{P}(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ defined by $f(P, x) = x + g(P)$, where $g : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is an arbitrary function. The sublevel sets $f_P^{-1}(-\infty, a]$ are exactly the intervals $(-\infty, a - g(P)]$, which always have exactly one connected component that persists indefinitely. Thus, the dimension 0 persistence diagram of f_P contains a single point born at $-\infty$ that never dies, and the PD is empty in higher dimensions. Thus, for any two finite point clouds $P, Q \subseteq \mathbb{R}$, the persistence diagrams $\mathrm{Dgm}(f_P)$ and $\mathrm{Dgm}(f_Q)$ are identical, and we have 0-stability without independence on the point cloud. This example also trivially generalises to $\mathbb{R}^d$ where f defines a hyperplane.*

3.2 Unstable generalised density functions

To form an intuition for stability, we construct a simple GDF that is not c-stable for any $c \in \mathbb{R}^+$, i.e. *unstable*.

Example 3.4 *Consider the GDF $f : \mathcal{P}(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ defined by*

$$f(P, x) = \begin{cases} 0, & \text{if } x \in P, \\ 1, & \text{otherwise.} \end{cases} \tag{3.2}$$

Consider two point clouds $P = \{\mathbf{0}\}$ and $Q = \{\mathbf{0}, \varepsilon \cdot \mathbf{1}\}$, where $\mathbf{0} \in \mathbb{R}^d$ denotes the vector of zeros and $\mathbf{1} \in \mathbb{R}^d$ is the vector of ones, and $\varepsilon > 0$ is a small positive number. The Hausdorff distance between P and Q is $d_H(P, Q) = d(\mathbf{0}, \varepsilon \cdot \mathbf{1}) = \varepsilon$. As shown in Figure 3.1, the persistence diagrams have bottleneck distance of at least $\frac{1}{\sqrt{2}}$. As $\varepsilon \to 0$, the bottleneck distance remains constant, while the Hausdorff distance tends to zero, which implies that f is not c-stable for any $c \in \mathbb{R}^+$.

While the example above is discontinuous, discontinuity is not required for instability. The following example has the same 0-dimensional persistence diagrams for $P = \{\mathbf{0}\}$ and $Q = \{\mathbf{0}, \varepsilon \cdot \mathbf{1}\}$ as the function in (3.2), but is continuous:

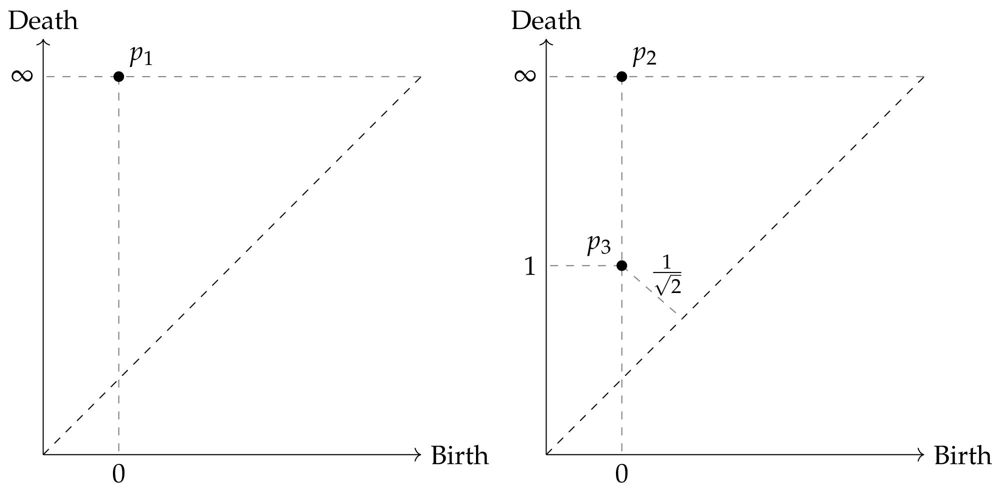
$$f(P, x) = 2 \cdot D(P) \cdot \min_{p \in P} d(x, p), \tag{3.3}$$

where $D(P)$ is the diameter of the point cloud P.

3.3 Point cloud Lipschitz generalised density functions

A powerful sufficient condition for stability is a form of Lipschitz continuity with respect to the point cloud:

Definition 3.5 (Point cloud Lipschitz (PC-Lipschitz) GDF) *A GDF f is c-*PC Lipschitz *if, when considered as a map that assigns to each finite non-empty point cloud $P \subseteq \mathbb{R}^d$ a function $f_P : \mathbb{R}^d \to \mathbb{R}$, it is Lipschitz continuous with constant c, where the space of finite point clouds is equipped with the Hausdorff distance,*



(a) The 0-dimensional PD of f_P. The point p_1 corresponds to the single connected component of the sublevel set $f_P^{-1}(-\infty, a]$, which includes only $\mathbf{0}$ for $a \in [0, 1)$ and $\mathbb{R}^d$ for $a \geq 1$.

(b) The 0-dimensional PD of f_Q. The point p_2 corresponds to a connected component of the sublevel set $f_Q^{-1}(-\infty, a]$, which includes only $\mathbf{0}$ for $a \in [0, 1)$ and $\mathbb{R}^d$ for $a \geq 1$, while the point p_3 corresponds to the connected component that contains $\varepsilon \cdot \mathbf{1}$ for $a \in [0, 1)$.

Figure 3.1: Persistence diagrams in dimension 0 of f_P and f_Q for the unstable GDF example. When matching points between the diagrams, the optimal matching connects p_1 to p_2 with cost 0 and p_3 to the diagonal with cost $\frac{1}{\sqrt{2}}$.

and the space of real-valued functions $f_P : \mathbb{R}^d \to \mathbb{R}$ is equipped with the L_∞-norm. Formally, for any non-empty finite point clouds $P, Q \subseteq \mathbb{R}^d$, we have

$$\|f_P - f_Q\|_\infty \leq c \cdot d_H(P, Q). \tag{3.4}$$

Lemma 3.6 *If f is a c-PC Lipschitz GDF and for all finite $P \subseteq \mathbb{R}^d$, the induced persistence modules are q-tame, then f is c-stable.*

Proof Let f be a c-PC Lipschitz GDF. By definition, for any non-empty two finite point clouds $P, Q \subseteq \mathbb{R}^d$, we have

$$\|f_P - f_Q\|_\infty \leq c \cdot d_H(P, Q). \tag{3.5}$$

Since the persistence modules induced by f_P and f_Q are q-tame, by Theorem 2.14, the bottleneck distance between their persistence diagrams is bounded by the L_∞-norm of the difference $f_P - f_Q$. Combining these two inequalities, we obtain:

$$d_b(\mathrm{Dgm}(f_P), \mathrm{Dgm}(f_Q)) \leq \|f_P - f_Q\|_\infty \leq c \cdot d_H(P, Q), \tag{3.6}$$

which shows that f is c-stable. $\square$

While this lemma is a straightforward consequence of Theorem 2.14, it provides a versatile tool for establishing stability, as demonstrated in the following sections.

3.4 Weighted Čech filtration revisited

The stability of the weighted Čech filtration (and thus DTM-filtrations) was stated in Theorem 2.13. We provide a more direct proof of this result than in the original paper [1] by showing that the corresponding GDF is PC-Lipschitz.

Theorem 3.7 *The GDF*

$$f_{g,q}(P,x) = \min_{p\in P}(d(x,p)^q + g(p)^q)^{1/q} \tag{3.7}$$

for the weighted Čech filtration is d-stable with $d = (1+c^q)^{1/q}$, where c is the Lipschitz constant of the weight function $g : \mathbb{R}^d \to \mathbb{R}^+$.

Proof Let $P, Q \subseteq \mathbb{R}^d$ be two finite point clouds with Hausdorff distance $d_H(P,Q)$. Similarly to the proof of Čech stability using Theorem 2.14, fix $x \in \mathbb{R}^d$ and let $p \in P$ be the point where the minimum for $(d(x,p)^q + g(p)^q)^{1/q}$ is attained. This implies that

$$f_{g,q}(P,x) = (d(x,p)^q + g(p)^q)^{1/q}. \tag{3.8}$$

By the definition of Hausdorff distance, there exists a point $q' \in Q$ such that $d(p,q') \leq d_H(P,Q)$, which implies that

$$d(x,q') \leq d(x,p) + d(p,q') \leq d(x,p) + d_H(P,Q). \tag{3.9}$$

By monotonicity of (3.8) with respect to $d(x,p)$ and $g(p)$, we have

$$\begin{aligned}
f_{g,q}(Q,x) &= \min_{q''\in Q}(d(x,q'')^q + g(q'')^q)^{1/q} && (3.10)\\
&\leq (d(x,q')^q + g(q')^q)^{1/q} && (3.11)\\
&\leq ((d(x,p)+d_H(P,Q))^q + (g(p) + c\cdot d_H(P,Q))^q)^{1/q} && (3.12)\\
&= \|(d(x,p)+d_H(P,Q), g(p)+c\cdot d_H(P,Q))\|_q. && (3.13)
\end{aligned}$$

Comparing the values of $f_{g,q}(P,x)$ and $f_{g,q}(Q,x)$, we have

$$\begin{aligned}
&f_{g,q}(Q,x) - f_{g,q}(P,x)\\
&\leq \|(d(x,p)+d_H(P,Q), g(p)+c\cdot d_H(P,Q))\|_q - \|(d(x,p),g(p))\|_q && (3.14)\\
&\leq \|(d(x,p)+d_H(P,Q)-d(x,p), g(p)+c\cdot d_H(P,Q)-g(p))\|_q && (3.15)\\
&= \|(d_H(P,Q), c\cdot d_H(P,Q))\|_q, && (3.16)
\end{aligned}$$

and by the same reasoning with P and Q swapped, we have

$$f_{g,q}(P,x) - f_{g,q}(Q,x) \leq \|(d_H(P,Q), c\cdot d_H(P,Q))\|_q, \tag{3.17}$$

which combined gives us

$$\begin{aligned}
\|f_{g,q}(P,\cdot) - f_{g,q}(Q,\cdot)\|_\infty &= \sup_{x\in\mathbb{R}^d} |f_{g,q}(P,x) - f_{g,q}(Q,x)| && (3.18)\\
&\le \|(d_H(P,Q), c\cdot d_H(P,Q))\|_q && (3.19)\\
&= d_H(P,Q)\cdot \|(1,c)\|_q && (3.20)\\
&= d_H(P,Q)\cdot (1+c^q)^{1/q}, && (3.21)
\end{aligned}$$

which is exactly the condition for d-PC Lipschitz continuity with $d = (1 + c^q)^{1/q}$.

For finite point clouds P, the weighted Čech filtration induces pointwise finite-dimensional persistence modules ([1], Proposition 3.1), which are by Lemma 2.5 q-tame. By Lemma 3.6, the weighted Čech filtration is d-stable with $d = (1 + c^q)^{1/q}$. □

3.5 Generalised Čech filtrations

The nearest-neighbour distance function $f_{\text{dist}}(P,x) = \min_{p\in P} d(x,p)$ can be seen as taking a minimum over a set of functions $h_p(x) = d(x,p)$, each centered around a point $p \in P$. We can generalise this to $f(P,x) = \min_{p\in P} h(x,p)$, where $h(x,p) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a more general "kernel" function. We refer to GDFs of this form as defining *generalised Čech filtrations*. The stability of such GDFs depends on the properties of the kernel h.

3.5.1 Isotropic, monotone and Lipschitz kernels

Borrowing terminology from kernel methods, we call a kernel *isotropic* if it depends only on the distance between its arguments [13].

Definition 3.8 (Isotropic kernel) *A kernel $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is* isotropic *if there exists a function $k : \mathbb{R}_+ \to \mathbb{R}$ such that $h(x,p) = k(d(x,p))$ for all $x, p \in \mathbb{R}^d$.*

Such kernels correspond to generalised Čech filtrations where the radius of the balls grows with the same rate for each point $p \in P$, but that rate may not be uniform.

A generalised Čech filtration is stable if the kernel h is isotropic, monotone and Lipschitz continuous:

Theorem 3.9 *Let $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a kernel that is isotropic, such that k is monotone increasing and c-Lipschitz continuous. Then the generalised Čech filtration using h is c-stable.*

Proof We want to show that the corresponding GDF f is c-PC Lipschitz. Let $P, Q \subseteq \mathbb{R}^d$ be two finite point clouds. Fix a point $x \in \mathbb{R}^d$ and $p \in P$ be such that $f(P, x) = k(d(x, p))$. There exists $q' \in Q$ such that $d(p, q') \leq d_H(P, Q)$ and

$$d(x, q') \leq d(x, p) + d(p, q') \leq d(x, p) + d_H(P, Q). \tag{3.22}$$

Since k is monotone and c-Lipschitz, we have:

$$\begin{aligned} f(Q, x) &= \min_{q \in Q} k(d(x, q)) && (3.23) \\ &\leq k(d(x, q')) && (3.24) \\ &\leq k(d(x, p) + d_H(P, Q)) && (3.25) \\ &\leq k(d(x, p)) + c \cdot d_H(P, Q) && (3.26) \\ &= f(P, x) + c \cdot d_H(P, Q), && (3.27) \end{aligned}$$

and by the same reasoning with P and Q swapped, we have

$$f(P, x) \leq f(Q, x) + c \cdot d_H(P, Q). \tag{3.28}$$

Combining these inequalities, we obtain

$$\begin{aligned} \|f(P, \cdot) - f(Q, \cdot)\|_\infty &= \sup_{x \in \mathbb{R}^d} |f(P, x) - f(Q, x)| && (3.29) \\ &\leq c \cdot d_H(P, Q). && (3.30) \end{aligned}$$

The induced persistence modules of $f(P, x)$ are q-tame by the same reasoning as in the proof of Lemma 2.5. Therefore, by Lemma 3.6, the generalised Čech filtration using h is c-stable. □

3.5.2 Kernels Lipschitz in p

We generalise further by considering $f(P, x) = \min_{p \in P} h(x, p)$ where the kernel $h(x, p)$ is c-Lipschitz with respect to its second argument p, for any fixed x:

$$|h(x, p_1) - h(x, p_2)| \leq c \cdot d(p_1, p_2). \tag{3.31}$$

Such kernels correspond to generalised Čech filtrations where the standard process of growing balls around points $p \in P$ is modified significantly; the sublevel sets $f_P^{-1}(-\infty, a]$ may not be symmetric, connected, grow at the same rate for different $p \in P$ or even include the point p itself. The only restriction imposed by Lipschitzness of the kernel is that as a point p moves, the shape changes in a controlled manner. Somewhat surprisingly, even this weak condition coupled with q-tameness is sufficient for stability.

Theorem 3.10 *Let $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a kernel that is c-Lipschitz continuous with respect to its second argument. Then the generalised Čech filtration using h is c-stable if the induced persistence modules are q-tame.*

Proof Fix $x \in \mathbb{R}^d$ and $p \in P$. Then we have $q' \in Q$ such that $d(p, q') \leq d_H(P, Q)$. By Lipschitzness of h, we have

$$|h(x, q') - h(x, p)| \leq c \cdot d(q', p) \leq c \cdot d_H(P, Q) \tag{3.32}$$
$$f(Q, x) \leq h(x, q') \leq c \cdot d_H(P, Q) + h(x, p), \tag{3.33}$$

and as this holds for any $p \in P$, we have

$$f(Q, x) \leq c \cdot d_H(P, Q) + f(P, x) \tag{3.34}$$
$$f(Q, x) - f(P, x) \leq c \cdot d_H(P, Q). \tag{3.35}$$

By the same reasoning, we have the same with P and Q swapped, and combining the two inequalities, we obtain

$$|f(P, x) - f(Q, x)| \leq c \cdot d_H(P, Q) \tag{3.36}$$

for any x, which by Lemma 3.6 implies that f is c-stable. □

This theorem generalises Theorem 3.9, as any isotropic kernel $h(x, p) = k(d(x, p))$ with k being c-Lipschitz automatically is c-Lipschitz with respect to p. This follows because the distance function is 1-Lipschitz, and composing it with a c-Lipschitz function yields a c-Lipschitz function [27].

The q-tameness condition for the induced persistence modules, while crucial, is not always straightforward to verify directly for a given kernel h. In the following lemma, we provide sufficient conditions for this.

Lemma 3.11 *Let $f(P, x) = \min_{p \in P} h(x, p)$. If, for every $p \in P$, the function $h_p(x) := h(x, p)$ is continuous in x, bounded below and proper (i.e., $h_p^{-1}(K)$ is compact for every compact $K \subseteq \mathbb{R}$), then $f_P(x)$ is continuous, bounded below and proper. Consequently, by Theorem 2.7, the induced persistence modules are q-tame.*

Proof To use Theorem 2.7, we need to show for f_P:

1. Continuity: $f_P(x)$ is the minimum of a finite set of continuous functions $h_p(x)$, thus f_P is continuous.
2. Bounded below: If each $h_p(x) \geq M_p$ for some M_p, then $f_P(x) = \min_p h_p(x) \geq \min_p M_p$, so f_P is bounded below.
3. Properness: A function g is proper if and only if for every sequence $\{x_i\}$ such that $x_i \to \infty$, we have $g(x_i) \to \infty$ ([17], Proposition 2.17). As every $h_p(x)$ is proper, then $p \in P$, we have $h_p(x_i) \to \infty$. Since $f_P(x_i) = \min_p h_p(x_i)$, it follows that $f_P(x_i) \to \infty$ as well. Thus, f_P is proper.

With f_P continuous, bounded below, and proper, Theorem 2.7 guarantees that the induced persistence modules are q-tame. □

3.5.3 Kernels depending on the point cloud

We further generalise by allowing the kernel itself to depend on the point cloud P:

$$f(P, x) = \min_{p \in P} h(x, p, P). \tag{3.37}$$

Example 3.12 *Consider $h(x, p, P)$ that is the pairwise Mahalanobis distance between x and p, defined by locally estimated covariance of the point cloud P [21]. Let $N_k(p, P)$ be the set of k nearest neighbours of p in P. The covariance estimate Σ_p using k nearest neighbours of p is given by*

$$\Sigma_p = \frac{1}{k} \sum_{q \in N_k(p,P)} (q - p)(q - p)^\top. \tag{3.38}$$

We would like to invert Σ_p, but it may be numerically unstable or not invertible if k is too small or if some points in $N_k(p, P)$ are close to each other. To avoid this, we add a small positive regularisation term εI, where I is the identity matrix. The Mahalanobis distance is then defined as

$$h(x, p, P) = \sqrt{(x - p)^\top (\Sigma_p + \varepsilon I)^{-1} (x - p)}. \tag{3.39}$$

Geometrically, this GDF defines a generalised Čech filtration where the sublevel sets $f_P^{-1}(-\infty, a]$ are unions of ellipsoids centered at p with each ellipsoid having axes aligned with the eigenvectors of $\Sigma_p + \varepsilon I$ and lengths proportional to the square roots of the eigenvalues of $\Sigma_p + \varepsilon I$. The regularisation term ensures that the ellipsoids are not degenerate. The standard Čech filtration can be seen as a special case of this GDF with $(\Sigma_p + \varepsilon I)$ replaced by the identity matrix.

We prove two theorems that guarantee stability of such GDFs.

Theorem 3.13 *Let $h : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ be a kernel that is c-Lipschitz continuous with respect to its second argument (the point P) and d-Lipschitz continuous with respect to its third argument (the point cloud P, using d_H on $\mathcal{P}(\mathbb{R}^d)$):*

$$|h(x, p_1, P) - h(x, p_2, P)| \le c \cdot d(p_1, p_2), \tag{3.40}$$
$$|h(x, p, P_1) - h(x, p, P_2)| \le d \cdot d_H(P_1, P_2). \tag{3.41}$$

Then the GDF $f(P, x) = \min_{p \in P} h(x, p, P)$ is $(c + d)$-stable if the induced persistence modules are q-tame.

Proof Fix $x \in \mathbb{R}^d$ and $p \in P$. Then we have $q' \in Q$ such that $d(p, q') \le d_H(P, Q)$. By Lipschitzness of h, we have

$$|h(x, p, P) - h(x, q', Q)| \tag{3.42}$$
$$\le |h(x, p, P) - h(x, p, Q)| + |h(x, p, Q) - h(x, q', Q)| \tag{3.43}$$
$$\le d \cdot d_H(P, Q) + c \cdot d(p, q'), \tag{3.44}$$

and as this holds for any $p \in P$, we have

$$
\begin{aligned}
f(Q,x) - f(P,x) &\leq h(x,q',Q) - f(P,x) && (3.45)\\
&\leq h(x,q',Q) - h(x,p,P) && (3.46)\\
&\leq (c+d) \cdot d_H(P,Q). && (3.47)
\end{aligned}
$$

By the same reasoning, we have the same inequality with P and Q swapped, and combining the two inequalities, we obtain

$$
|f(P,x) - f(Q,x)| \leq (c+d) \cdot d_H(P,Q). \tag{3.48}
$$

By Lemma 3.6, this implies that f is $(c+d)$-PC Lipschitz. □

Using this theorem, we show in Appendix B that the GDF from Example 3.12 is stable.

Theorem 3.14 *Let $h : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ be a kernel that is c-Lipschitz continuous with respect to its second argument and d-Lipschitz continuous with respect to every element in P, i.e.:*

$$
\begin{aligned}
|h(x,p_1,P) - h(x,p_2,P)| &\leq c \cdot d(p_1,p_2), && (3.49)\\
|h(x,p,P \cup \{p_1\}) - h(x,p,P \cup \{p_2\})| &\leq d \cdot d(p_1,p_2) && (3.50)\\
|h(x,p,\{p_1\}) - h(x,p,\{p_2\})| &\leq d \cdot d(p_1,p_2). && (3.51)
\end{aligned}
$$

Then the generalised Čech filtration using h is $(c + d \cdot \max(|P|,|Q|))$-stable if the induced persistence modules are q-tame.

Proof If $|P| = |Q|$, we can match every point $p_i \in P$ with a point $q_i \in Q$ such that $d(p_i,q_i) \leq d_H(P,Q)$, and thus we have

$$
\begin{aligned}
&|h(x,p,\{p_i\}_{i=1}^n) - h(x,p,\{q_i\}_{i=1}^n)| && (3.52)\\
&= |h(x,p,\{p_i\}_{i=1}^{n-1} \cup \{p_n\}) - h(x,p,\{p_i\}_{i=1}^{n-1} \cup \{q_n\})| && (3.53)\\
&\leq |h(x,p,\{p_i\}_{i=1}^{n-1}) - h(x,p,\{q_i\}_{i=1}^{n-1})| + d \cdot d_H(P,Q) && (3.54)\\
&\leq |h(x,p,\{p_i\}_{i=1}^{n-2}) - h(x,p,\{q_i\}_{i=1}^{n-2})| + 2 \cdot d \cdot d_H(P,Q) && (3.55)\\
&\leq \dots && (3.56)\\
&\leq |P| \cdot d \cdot d_H(P,Q). && (3.57)
\end{aligned}
$$

If $|P| < |Q|$, we can add points to P that are arbitrarily close to existing points in P such that the Hausdorff distance $d_H(P,Q)$ changes arbitrarily little, and $f(P,x)$ also changes arbitrarily little. The same is true with P and Q swapped. Thus, we can assume that $|P| = |Q|$ without loss of generality. This means that

$$
|h(x,p,P) - h(x,p,Q)| \leq d \cdot \max(|P|,|Q|) \cdot d_H(P,Q). \tag{3.58}
$$

Following the same reasoning as in the previous proof, we have

$$|h(x,p,P) - h(x,q',Q)| \tag{3.59}$$
$$\leq |h(x,p,P) - h(x,p,Q)| + |h(x,p,Q) - h(x,q',Q)| \tag{3.60}$$
$$\leq d \cdot \max(|P|,|Q|) \cdot d_H(P,Q) + c \cdot d_H(P,Q). \tag{3.61}$$

□

The conditions on the kernel for the induced persistence modules to be q-tame are exactly the same as the ones in Lemma 3.11.

3.6 Morse functions

The PC-Lipschitz condition is a strong global condition, which may not always hold for some stable GDFs.

Example 3.15 *Consider a Čech-like GDF*

$$f_P(x) = \min_{p \in P} d(x,p)^2, \tag{3.62}$$

where the balls grow faster the larger they are. While this function is not Lipschitz continuous on the whole space $\mathbb{R}^d$, it is Lipschitz continuous on some subset of $\mathbb{R}^d$ that covers the point cloud P. We can intuit that Lipschitz continuity everywhere is too strong of a condition, as it does not matter how the function behaves far away from P.

Example 3.16 *Consider a GDF of the form*

$$f_P(x) = \|x - g(P)\|, \tag{3.63}$$

where $g : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ is an arbitrary function of the point cloud P. This GDF is not Lipschitz continuous for many choices of g, but it is stable for any g, as it only has a single critical value at zero, which results in a single point $(0,\infty)$ in the 0-dimensional persistence diagram and no points in higher dimensions.

Motivated by the two examples above, we can relax the PC-Lipschitz condition by restricting ourselves to *Morse functions*, that is, smooth functions $f_P : \mathbb{R}^d \to \mathbb{R}$ whose critical points $x \in \mathrm{Crit}(f_P)$ (where the gradient vanishes) are non-degenerate, i.e., the Hessian matrix $H(f_P)(x)$ is non-singular [20].

Theorem 3.17 *Let $f : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ be a GDF such that for any finite point clouds $P, Q \subseteq \mathbb{R}^d$, the functions f_P and f_Q are Morse, and all sublevel sets $f_P^{-1}(-\infty, a]$ and $f_Q^{-1}(-\infty, a]$ are compact. Furthermore, let there exist a bijection $\pi : \mathrm{Crit}(f_P) \to \mathrm{Crit}(f_Q)$ of critical points of f_P and f_Q such that the index of x is equal to the index of $\pi(x)$ for all $x \in \mathrm{Crit}(f_P) \cup \mathrm{Crit}(f_Q)$, and the following condition holds:*

$$\forall x \in \mathrm{Crit}(f_P) : |f_P(x) - f_Q(\pi(x))| \leq c \cdot d_H(P,Q). \tag{3.64}$$

Then the GDF f is c-stable, if the induced persistence modules are q-tame.

Proof As f_P is Morse and its sublevel sets are compact, each sublevel set $f_P^{-1}(-\infty, a]$ has the homotopy type of a finite CW-complex, with one cell of dimension λ for each critical point of index λ in $f_P^{-1}(-\infty, a]$ ([22], Theorem 3.5 and the remark following it). Let us denote such a CW-complex by $C_P(a)$. The previous statement can be written concisely as

$$\forall a \in \mathbb{R} \;\; C_P(a) \simeq f_P^{-1}(-\infty, a]. \tag{3.65}$$

Let $d = c \cdot d_H(P, Q)$. We now show that $C_P(a)$ as a set is included in $C_Q(a+d)$. Consider a cell e^λ in $C_P(a)$. This cell corresponds to a critical point $x \in \mathrm{Crit}(f_P)$ of index λ such that $f_P(x) \leq a$. By the condition (3.64), we have

$$f_Q(\pi(x)) \leq f_P(x) + d \leq a + d. \tag{3.66}$$

Thus, we have a critical point $\pi(x) \in \mathrm{Crit}(f_Q)$ also of index λ such that $f_Q(\pi(x)) \leq a + d$. This means that the cell e^λ in $C_P(a)$ corresponds to a cell of the same dimensionality in $C_Q(a+d)$. Since this holds for any cell in $C_P(a)$, we have

$$C_P(a) \subseteq C_Q(a+d). \tag{3.67}$$

By the same reasoning, we can show that $C_Q(a)$ is included in $C_P(a+d)$. We also trivially have that $C_P(a) \subseteq C_P(a')$ for any $a' \geq a$ and the same for C_Q.

Similarly to the proof of stability of the Čech filtration using interleaving, the following diagram commutes up to homotopy, as all maps are either inclusions or homotopies:

$$\begin{tikzcd}
f_P^{-1}(-\infty, a] \arrow[r, hook] \arrow[d, leftrightarrow, "\simeq"] & f_P^{-1}(-\infty, a+d] \arrow[r, hook] \arrow[d, leftrightarrow, "\simeq"] & f_P^{-1}(-\infty, a+2d] \arrow[d, leftrightarrow, "\simeq"] \\
C_P(a) \arrow[r, hook] \arrow[dr, hook] & C_P(a+d) \arrow[r, hook] \arrow[dr, hook] & C_P(a+2d) \\
C_Q(a) \arrow[r, hook] \arrow[ur, hook] & C_Q(a+d) \arrow[r, hook] \arrow[ur, hook] & C_Q(a+2d) \\
f_Q^{-1}(-\infty, a] \arrow[r, hook] \arrow[u, leftrightarrow, "\simeq"] & f_Q^{-1}(-\infty, a+d] \arrow[r, hook] \arrow[u, leftrightarrow, "\simeq"] & f_Q^{-1}(-\infty, a+2d] \arrow[u, leftrightarrow, "\simeq"]
\end{tikzcd} \tag{3.68}$$

Thus we have

$$d_i(\{f_P^{-1}(-\infty, a]\}_a, \{f_Q^{-1}(-\infty, a]\}_a) \leq d, \tag{3.69}$$

and similarly to the proof of stability of the Čech filtration, we have that the bottleneck distance between the persistence diagrams is bounded by d from above. $\square$

It should be noted that this theorem is not a strict generalisation of previous theorems, as there are GDFs that are PC-Lipschitz but which are not Morse.

For conditions that guarantee q-tameness of induced persistence modules of Morse functions, we refer the reader to Theorem 8.1.4 and Corollary 8.1.12 of Maximilian Schmahl's PhD thesis [25].

Chapter 4

Operations on stable functions

Having identified several classes of stable Generalized Density Functions (GDFs) in Chapter 3, we now turn to a natural subsequent question: *how does stability behave under common operations?* If we combine stable GDFs, for instance, by pointwise addition or by taking their minimum, is the resulting GDF also stable? If so, can we bound its stability constant in terms of the constants of the original functions? This chapter explores these questions, identifying conditions under which stability is preserved and providing counterexamples where it is not. Understanding these properties is crucial for constructing more complex stable GDFs from simpler ones and for gaining deeper insight into the structure of the space of stable functions.

4.1 Affine transformations

We begin with the simplest operations: adding a constant to a stable GDF and multiplying it by a constant.

Theorem 4.1 (Stability under constant addition) *Let $f(P,x)$ be a c-stable GDF with q-tame induced persistence modules. Then, for any constant $z \in \mathbb{R}$, the GDF $g(P,x) = f(P,x) + z$ is also c-stable with q-tame induced persistence modules.*

Proof First we related the sublevel sets of g_P and f_P. For any $a \in \mathbb{R}$:

$$\begin{aligned} g_P^{-1}(-\infty, a] &= \{x \in \mathbb{R}^d \mid g_P(x) \leq a\} && (4.1)\\ &= \{x \in \mathbb{R}^d \mid f_P(x) + z \leq a\} && (4.2)\\ &= \{x \in \mathbb{R}^d \mid f_P(x) \leq a - z\} && (4.3)\\ &= f_P^{-1}(-\infty, a - z]. && (4.4) \end{aligned}$$

This means the sublevel set filtration $\{g_P^{-1}(-\infty, a]\}_{a\in\mathbb{R}}$ is identical to the filtration $\{f_P^{-1}(-\infty, b]\}_{b\in\mathbb{R}}$ but with the indices shifted by z, i.e., $b = a - z$.

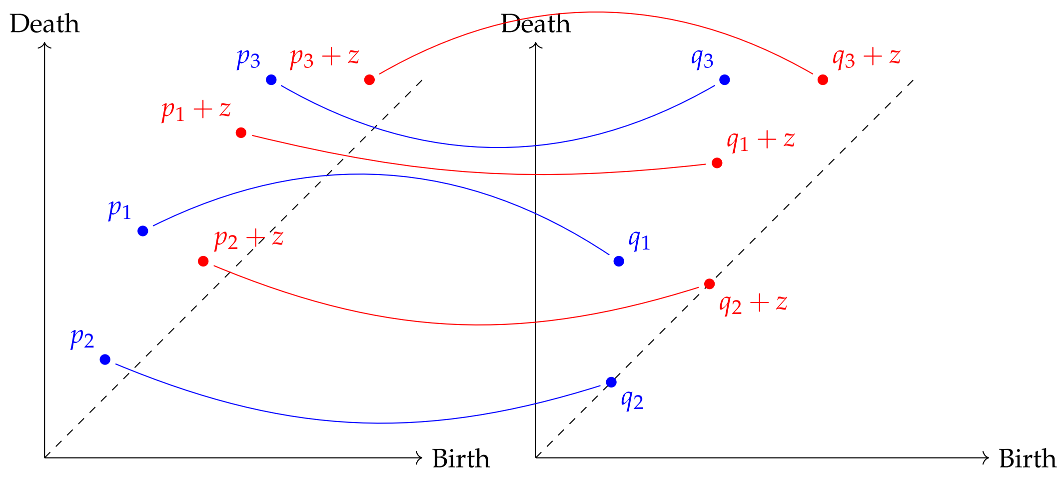


Figure 4.1: The persistence diagrams of f_P (left, blue), f_Q (right, blue), and g_P (left, red), g_Q (right, red) where $g_P = f_P + z$ and $g_Q = f_Q + z$. A matching between the diagrams of f_P and f_Q is shown in blue, and a matching between the diagrams of g_P and g_Q is shown in red.

Consequently, if a topological feature is born at b and dies at d in the filtration of f_P, a corresponding feature in the filtration of g_P will be born at $b + z$ and die at $d + z$, and vice versa. Thus, the persistence diagrams of f_P and g_P are related by a constant shift along both axes:

$$\mathrm{Dgm}_k(g_P) = \{(b + z, d + z) \mid (b, d) \in \mathrm{Dgm}_k(f_P)\}. \tag{4.5}$$

Next, we address q-tameness. Let $\mathbb{F} = (\{F_a\}, \{\mathfrak{f}_a^{a'}\})$ be a persistence module induced by f_P and $\mathbb{G} = (\{G_a\}, \{\mathfrak{g}_a^{a'}\})$ be that induced by g_P. From the relationship between the sublevel sets, we have $G_a = F_{a-z}$ and $\mathfrak{g}_a^{a'} = \mathfrak{f}_{a-z}^{a'-z}$. Since $\mathbb{F}$ is q-tame by assumption, $\mathfrak{f}_a^{a'}$ is of finite rank for $b < b'$. This directly implies that $\mathfrak{g}_a^{a'}$ is also of finite rank for all $a < a'$, so $\mathbb{G}$ is also q-tame.

Finally, we prove c-stability for g. Fix point clouds P and Q. By stability of f, we have that

$$d_b(\mathrm{Dgm}_k(f_P), \mathrm{Dgm}_k(f_Q)) \leq c \cdot d_H(P, Q). \tag{4.6}$$

Let π_i be a sequence of matchings between $\mathrm{Dgm}_k(f_P) \cup \Delta$ and $\mathrm{Dgm}_k(f_Q) \cup \Delta$ that achieves this bound. For each i, we can construct a matching π_i' between $\mathrm{Dgm}_k(g_P) \cup \Delta$ and $\mathrm{Dgm}_k(g_Q) \cup \Delta$ by shifting the points in π_i by z:

$$\pi_i'(b + z, d + z) = (b' + z, d' + z) \text{ where } (b', d') = \pi_i(b, d). \tag{4.7}$$

The cost of this matching is the same as that of π_i:

$$\|(b + z, d + z) - \pi_i'(b + z, d + z)\|_\infty = \|(b, d) - (b', d')\|_\infty, \tag{4.8}$$

thus π'_i is a matching between $\mathrm{Dgm}_k(g_P)$ and $\mathrm{Dgm}_k(g_Q)$ with the same cost as π_i. An example of this is shown in Figure 4.1. Therefore, we have:

$$d_b(\mathrm{Dgm}_k(g_P), \mathrm{Dgm}_k(g_Q)) \leq c \cdot d_H(P, Q). \tag{4.9}$$

Since this holds for arbitrary P and Q, g is c-stable. □

Theorem 4.2 (Stability under positive constant multiplication) *Let $f(P,x)$ be a c-stable GDF with q-tame induced persistence modules. Then for any constant $\alpha \geq 0$, the function $g(P,x) = \alpha f(P,x)$ is also a $c\alpha$-stable GDF with q-tame induced persistence modules.*

Proof We first consider the case when $\alpha = 0$. In this case, $g(P,x) = 0$ for all $x \in \mathbb{R}^d$, which is 0-stable by Theorem 3.2. Its induced persistence modules are pointwise finite dimensional, hence q-tame.

Now, let $\alpha > 0$. The argument mirrors that of Theorem 4.1. The sublevel sets of g and of f are related by:

$$g_P^{-1}(-\infty, a] = \{x \in \mathbb{R}^d \mid \alpha f_P(x) \leq a\} = f_P^{-1}(-\infty, a/\alpha]. \tag{4.10}$$

This implies that the persistence module $\mathbb{G}$ induced by g_P is the same as the persistence module $\mathbb{F}$ induced by f_P up to reindexing where a is replaced with $a \cdot \alpha$, i.e. $F_a = G_{a \cdot \alpha}$ and the maps $\mathfrak{f}_a^{a'}$ and $\mathfrak{g}_{\alpha \cdot a}^{\alpha \cdot a'}$ are identical. The q-tameness of $\mathbb{F}$ implies the q-tameness of $\mathbb{G}$.

A point $(b,d) \in \mathrm{Dgm}_k(f_P)$ corresponds to $(\alpha b, \alpha d) \in \mathrm{Dgm}_k(g_P)$ and vice versa. By the same argument as in the proof of Theorem 4.1, we can construct a matching $\pi'_i(b \cdot \alpha, d \cdot \alpha) = (b' \cdot \alpha, d' \cdot \alpha)$ where $(b', d') = \pi_i(b,d)$, which gives a bound on the bottleneck distance between the persistence diagrams of g_P and g_Q:

$$d_b(\mathrm{Dgm}_k(g_P), \mathrm{Dgm}_k(g_Q)) \leq c \cdot \alpha \cdot d_H(P, Q). \tag{4.11}$$

□

We can extend this theorem to negative constants using the following lemma:

Lemma 4.3 (Stability under negation) *Let $f(P,x)$ be a c-stable GDF with q-tame induced persistence modules. Then the function $g(P,x) = -f(P,x)$ is also a c-stable GDF with q-tame induced persistence modules.*

Proof This is a direct consequence of the symmetry theorem by David Cohen-Steiner et al. [11], which states that for a real-valued function f on a d-manifold the persistence diagrams $\mathrm{Dgm}_k(f)$ and $\mathrm{Dgm}_{d-k}(-f)$ are reflections of each other across the minor diagonal. Given that $\mathbb{R}^d$ is a d-manifold and reflection preserves bottleneck distance, the lemma follows immediately. □

Combining Theorem 4.2 and Lemma 4.3, we obtain the following corollary:

Corollary 4.4 (Stability under constant multiplication) *Let $f(P,x)$ be a c-stable GDF with q-tame induced persistence modules. Then for any constant $\alpha \in \mathbb{R}$, the function $g(P,x) = \alpha f(P,x)$ is also a $c|\alpha|$-stable GDF with q-tame induced persistence modules.*

4.2 Addition of stable functions

Having examined affine transformations, we now explore the effect of adding two stable GDFs. Let f and g be GDFs with stability constants c_f and c_g. What can be said about the stability constant of their sum $h(P,x) = f(P,x) + g(P,x)$? A simple case offers initial intuition: if $g = f$, then $h = f + f = 2 \cdot f$. By Theorem 4.1, h is $(2 \cdot c_f)$-stable. One would naturally expect that the stability constant of $h = f + g$ is then bounded by $c_f + c_g$, but this is not always the case, as the following example demonstrates.

Example 4.5 (Sum of stable GDFs may not be stable) *Consider GDFs on $\mathbb{R}$ ($d = 1$). Fix a constant $a > 0$, let*

$$f(P,x) = x + a \cdot \min P \quad \text{and} \quad g(P,x) = -x + a \cdot \min P. \tag{4.12}$$

As described in Example 3.3, both GDFs are 0-stable.

The sum of f and g is:

$$h(P,x) = f(P,x) + g(P,x) = 2a \cdot \min P. \tag{4.13}$$

The function h_P is constant for a fixed P. Its 0-dimensional persistence diagram has exactly one point $(2a \cdot \min P, \infty)$, representing the single connected component $\mathbb{R}$ itself, born at $2a \cdot \min P$ and never dying.

Let $P = \{1\}, Q = \{2\}$ with $d_H(P,Q) = 1$. The 0-dimensional persistence diagrams of h are $\mathrm{Dgm}_0(h_P) = \{(2a,\infty)\}$ and $\mathrm{Dgm}_0(h_Q) = \{(4a,\infty)\}$. In the optimal matching π between these diagrams, the point $(2a,\infty)$ must be matched to $(4a,\infty)$, as matching either of them to the diagonal Δ would result in an infinite cost. The cost of π is thus:

$$\|(2a,\infty) - (4a,\infty)\|_\infty = 2a. \tag{4.14}$$

Therefore, the bottleneck distance between the diagrams is $2a$, and for h to be c-stable, we need:

$$2a \leq c \cdot d_H(P,Q) = c. \tag{4.15}$$

This means that for any $c \in [0,\infty)$, we can choose $a = c$ such that the stability constant of h is larger than c. Thus, the sum of two 0-stable GDFs is not necessarily c-stable for any $c \in [0,\infty)$.

Despite this counterexample for general stable GDFs, stability under addition holds for the more restrictive class of PC-Lipschitz GDFs.

Theorem 4.6 (Sum of PC-Lipschitz GDFs) *Let $f(P,x)$ and $g(P,x)$ be a c_f-PC-Lipschitz GDF and a c_g-PC-Lipschitz GDF. Then their sum $h(P,x) = f(P,x) + g(P,x)$ is a $(c_f + c_g)$-PC-Lipschitz GDF and is thus $(c_f + c_g)$-stable if the induced persistence modules of h are q-tame.*

Proof Let $P, Q \subseteq \mathbb{R}^d$ be finite point clouds. By subadditivity of the ∞-norm and the definition of PC-Lipschitz continuity, we have:

$$\begin{aligned} \|h_P - h_Q\|_\infty &= \|(f_P + g_P) - (f_Q + g_Q)\|_\infty && (4.16)\\ &\leq \|f_P - f_Q\|_\infty + \|g_P - g_Q\|_\infty && (4.17)\\ &\leq c_f \cdot d_H(P,Q) + c_g \cdot d_H(P,Q) && (4.18)\\ &= (c_f + c_g) \cdot d_H(P,Q), && (4.19)\end{aligned}$$

which shows that h is $(c_f + c_g)$PC-Lipschitz. By Lemma 3.6, h is $(c_f + c_g)$-stable if its induced persistence modules are q-tame. □

4.3 Minimum and maximum of stable functions

Similar to addition, the pointwise minimum of two PC-Lipschitz GDFs is also PC-Lipschitz, and thus stable.

Theorem 4.7 (Minimum of PC-Lipschitz GDFs) *Let $f(P,x)$ and $g(P,x)$ be a c_f-PC-Lipschitz GDF and a c_g-PC-Lipschitz GDF. Then the function $h(P,x) = \min\{f(P,x), g(P,x)\}$ is a $\max\{c_f, c_g\}$-PC-Lipschitz GDF and is thus $\max\{c_f, c_g\}$-stable if the induced persistence modules of h are q-tame.*

Proof This result follows directly from a standard property of Lipschitz functions: the pointwise minimum (or maximum) of two Lipschitz functions $F_1, F_2 : X \to Y$ with Lipschitz constants L_1, L_2 is itself Lipschitz with constant $\max\{L_1, L_2\}$ ([27], Proposition 1.5.5). □

However, this property does not hold for general stable GDFs, as shown in the following counterexample.

Example 4.8 (Minimum of 0-stable GDFs can be unstable) *Consider GDFs on $\mathbb{R}$ ($d = 1$). Fix a constant $a \in \mathbb{R}$ and let*

$$f(P,x) = a \cdot |x - \min p| \quad \text{and} \quad g(P,x) = a \cdot |x - \max p|. \tag{4.20}$$

The 0-dimensional persistence diagrams for these GDFs are $\mathrm{Dgm}_0(f_P) = \{(0,\infty)\}$ and $\mathrm{Dgm}_0(g_P) = \{(0,\infty)\}$ for any P, and thus are both 0-stable.

Consider $P = \{-1, 1\}$ and $Q = \{0\}$ with $d_H(P,Q) = 1$. As we can see in Figure 4.2, the 0-dimensional persistence diagrams of h_P and h_Q are

$$\begin{aligned} \mathrm{Dgm}_0(h_P) &= \{(0,a), (0,\infty)\}, && (4.21)\\ \mathrm{Dgm}_0(h_Q) &= \{(0,\infty)\}. && (4.22)\end{aligned}$$

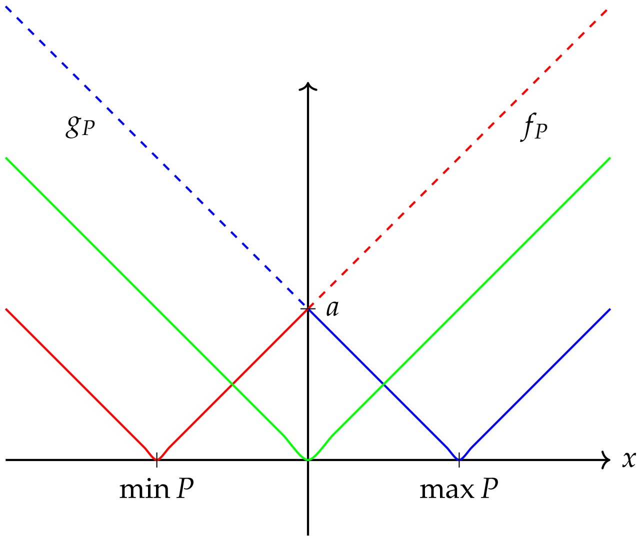


Figure 4.2: The GDFs f_P (red), g_P (blue) and f_Q (green) for point clouds $P = \{-1, 1\}$ and $Q = \{0\}$. The minimum h_P of the f_P and g_P is shown in solid lines.

The cost of matching $(0, a)$ to Δ is $\frac{a}{2}$, so the bottleneck distance between the diagrams is $\frac{a}{2}$. Thus, for every $c \in [0, \infty)$ we can choose $a = 2c$ such that the stability constant of $h(P, x) = f(P, x) + g(P, x)$ is c. Thus, the minimum of two 0-stable GDFs is not necessarily c-stable for any $c \in [0, \infty)$.

TODO: connect this theorem to the results above, something about how we can trade the Lipschitz requirement for some other ones

Theorem 4.9 *Let $f(P, x)$ and $g(P, x)$ be a c_f-stable GDF and a c_g-stable GDF with q-tame induced persistence modules $\mathbb{F}_P$ and $\mathbb{G}_P$ for all P. Let the sublevel sets of f and g be denoted as $f(P)_a = f_P^{-1}(-\infty, a]$ and $g(P)_a = g_P^{-1}(-\infty, a]$. Let also*

$$d_i(\mathbb{F}_P, \mathbb{F}_Q) = d_i(\{f(P)_a\}, \{f(Q)_a\}) \tag{4.23}$$
$$d_i(\mathbb{G}_P, \mathbb{G}_Q) = d_i(\{g(P)_a\}, \{g(Q)_a\}), \tag{4.24}$$

i.e. the interleaving maps for the persistence modules are induced by maps between filtrations of sublevel sets. Let those maps be denoted $\varphi_a^f : f(P)_a \to f(Q)_{a+c_f \cdot d_H(P,Q)}$, $\psi_a^f : f(Q)_a \to f(P)_{a+c_f \cdot d_H(P,Q)}$, and the same for g. Finally, we require three conditions:

1. $f(P, x) = g(P, x) \implies \varphi_a^f(x) = \varphi_a^g(x)$ *and* $\psi_a^f(x) = \psi_a^g(x)$ *for all* $a \geq f(P, x)$.
2. f_P *and* g_P *are both continuous for all* P.
3. $f(P, x) \leq g(P, x) \implies f(Q, x) \leq g(Q, x)$

Then the function $h(P, x) = \min\{f(P, x), g(P, x)\}$ is a $\max\{c_f, c_g\}$-stable GDF with q-tame induced persistence modules.

Proof TODO: □

The results in this section concerning the pointwise minimum can be extended to the pointwise maximum. This is because $\max(A, B) = -\min(-A, -B)$, and negation preserves stability by Lemma 4.3.

Chapter 5

The space of stable functions

TODO: This section is very much a draft and requires a lot of finishing and rewriting to flow properly. Some results are not stated yet.

Let $\mathcal{S}_c$ be the set of all c-stable functions and $\mathcal{L}_c$ be the set of all c-PC-Lipschitz functions. Per Lemma 3.6, we have $\mathcal{L}_c \subset \mathcal{S}_c$. TODO: example to show that the subset is proper. We first investigate $\mathcal{L}_c$, then $\mathcal{S}_c$ and finally the relationship between the two.

5.1 The space of c-PC-Lipschitz functions

Theorem 5.1 *$\mathcal{L}_c$ forms a distributive lattice with respect to the operations of pointwise minimum and pointwise maximum. This lattice is not bounded neither from above nor from below.*

Proof Both minimum and maximum are closed in $\mathcal{L}_c$ by Theorem 4.7. Distributivity is well-known TODO: reference. To show that the lattice is not bounded, suppose f is a top element. $f + 1$ is also in $\mathcal{L}_c$ by Theorem 4.1 and $\max(f, f + 1) = f + 1 \neq f$, thus f cannot be a top element. We can similarly show that there is no bottom element by considering $f - 1$. □

As the sum of two c-PC-Lipschitz functions is only $(2c)$-PC-Lipschitz, the space $\mathcal{L}_c$ is not closed under addition and thus does not form a vector space with addition and scalar multiplication. A natural attempt to form a vector space is to consider the average instead of the sum, but that does not play well with scalar multiplication.

The space $\mathcal{L} = \bigcup_{c \in \mathbb{R}^+} \mathcal{L}_c$ is a vector space, but is not of much interest, as practical applications do not concern themselves with merely stable functions, but rather with c-stable functions for some fixed c. We refer an interested reader to the book by Weaver [27] for a comprehensive treatment of the space of Lipschitz functions.

Theorem 5.2 *$\mathcal{L}_c$ is convex.*

Proof Let $f, g \in \mathcal{L}_c$ and $\alpha \in [0, 1]$. We need to show that $\alpha f + (1 - \alpha)g \in \mathcal{L}_c$. By Theorem 4.2, we have $\alpha f \in \mathcal{L}_{\alpha c}$ and $(1 - \alpha)g \in \mathcal{L}_{(1-\alpha)c}$.

Finally, by Theorem 4.6,

$$\alpha f + (1 - \alpha)g \in \mathcal{L}_{\alpha c + (1-\alpha)c} = \mathcal{L}_c. \tag{5.1}$$

□

Corollary 5.3 *Let $\mathcal{F}$ be a topological vector space of functions containing $\mathcal{L}_c$, i.e. addition and scalar multiplication are continuous operations. Then $\mathcal{L}_c$ is a closed contractible subset of $\mathcal{F}$. TODO: reference*

5.2 The space of c-stable functions

As shown in various examples in Chapter 4, the $\mathcal{S}_c$ is not closed under addition or pointwise maximum or minimum. Nonetheless, we can still use scalar multiplication to prove contractibility:

Theorem 5.4 *The set $\mathcal{S}_c \subseteq \mathcal{F}$ is path-connected in any topology on $\mathcal{F}$ where scalar multiplication is continuous.*

Proof Let $f, g \in \mathcal{S}_c$. Let $h_t(P, x)$ be defined for $t \in [0, 1]$ as

$$h_t(P, x) := \begin{cases} (1 - 2t)f(P, x), & t \in [0, 0.5] \\ (2t - 1)g(P, x), & t \in (0.5, 1] \end{cases} \tag{5.2}$$

Then h_t defines a path in $\mathcal{F}$ connecting f and g. As $1 - 2t \leq 1$ for $t \in [0, 0.5]$ and $2t - 1 \leq 1$ for $t \in (0.5, 1]$, we have $h_t(P, x) \in \mathcal{S}_c$ for all $t \in [0, 1]$ by Theorem 4.2. Thus, the path h_t is entirely contained in $\mathcal{S}_c$. □

Corollary 5.5 *The set $\mathcal{S}_c \subseteq \mathcal{F}$ is contractible in any topology on $\mathcal{F}$ where scalar multiplication is continuous.*

Theorem 5.6 *The singleton set of the function $g(P, x) = 0$ is a strong deformation retract of $\mathcal{S}_c$ in any topology on $\mathcal{F}$ where scalar multiplication is continuous.*

Proof Let $f \in \mathcal{S}_c$. We can define a homotopy $H_t(f)$ for $t \in [0, 1]$ as

$$H_t(f)(P, x) := (1 - t)f(P, x) \tag{5.3}$$

Then $H_0(f) = f$ and $H_1(f) = 0$. As $H_t(g) = g$, this deformation retraction is strong. □

Corollary 5.7 *The set $\mathcal{S}_c \subseteq \mathcal{F}$ is contractible in any topology on $\mathcal{F}$ where scalar multiplication is continuous.*

Theorem 5.8 *The set $\mathcal{S}_c$ is closed in the topology of pointwise convergence.*

Proof Let $f^n \to f$ be a converging sequence of functions in $\mathcal{S}_c$. We need to show that $f \in \mathcal{S}_c$. We have:

$$\|f_P - f_Q\|_\infty \leq \|f_P - f_P^n\|_\infty + \|f_P^n - f_Q^n\|_\infty + \|f_Q^n - f_Q\|_\infty \tag{5.4}$$

$$\leq \|f_P - f_P^n\|_\infty + c \cdot d_H(P, Q) + \|f_Q^n - f_Q\|_\infty, \tag{5.5}$$

which converges to $c \cdot d_H(P, Q)$ and thus f is c-stable. □

5.3 The relationship between $\mathcal{S}_c$ and $\mathcal{L}_c$

As $\mathcal{S}_c$ is much more well behaved than $\mathcal{L}_c$, it would be desirable to have $\mathcal{S}_c$ dense in $\mathcal{L}_c$. However, as both $\mathcal{S}_c$ is closed in any TVS, $\mathcal{S}_c$ is unfortunately not dense in $\mathcal{L}_c$ in such topologies. Even stronger, $\mathcal{S}_c$ is "of measure zero" in $\mathcal{L}_c$ in the sense that it is shy.

Theorem 5.9 *The set $\mathcal{S}_c$ is shy in $\mathcal{L}_c$.*

Proof TODO: Consider arbitrary $f \in \mathcal{S}_c$ and its neighbourhood N. Find a function $g \in N$ such that $g\mathcal{L}_s \setminus \mathcal{S}_c$ by adding a small non-Lipschitz bump to f. □

Chapter 6

Conclusion

TODO:

Appendix A

Weighted Čech complex can be represented as a GDF

Lemma A.1 *For any point cloud $P \subseteq \mathbb{R}^d$, a function $g : P \to \mathbb{R}_{\geq 0}$, and a parameter $q \in [1, \infty)$, the weighted Čech filtration $V_q^a[P, g]$ is equal to the sublevel set of the generalised density function*

$$f_{g,q}(P, x) = \min_{p \in P} (d(x, p)^q + g(p)^q)^{1/q}. \tag{A.1}$$

Proof Fix P, g, q, and a. A point $x \in \mathbb{R}^d$ being included in $V_q^a[P, g]$ is equivalent to the condition

$$\exists p \in P : d(x, p) \leq r_p^{(q)}(a). \tag{A.2}$$

Suppose $a \geq g(p)$. Then, we have two cases:

1. $q = \infty$. Then, the condition is equivalent to

$$\exists p \in P : d(x, p) \leq a. \tag{A.3}$$

   The proposed GDF takes value

$$f_{g,q}(P, x) = \min_{p \in P} \max(d(x, p), g(p)), \tag{A.4}$$

   and the sublevel set $f^{-1}_{g,q,P}(-\infty, a]$ is precisely

$$\{x \in \mathbb{R}^d \mid \exists p \in P : \max(d(x, p), g(p)) \leq a\}, \tag{A.5}$$

   which, due to $a \geq g(p)$, is the set

$$\{x \in \mathbb{R}^d \mid \exists p \in P : d(x, p) \leq a\}, \tag{A.6}$$

   which is the same as $V_\infty^a[P, g]$.

2. $q < \infty$. Then, the condition is equivalent to

$$\exists p \in P : d(x,p) \leq (a^q - g(p)^q)^{1/q} \tag{A.7}$$

The proposed GDF takes value

$$f_{g,q}(P,x) = \min_{p \in P} (d(x,p)^q + g(p)^q)^{1/q}, \tag{A.8}$$

and the sublevel set $f^{-1}_{g,q,P}(-\infty, a]$ is precisely

$$\{x \in \mathbb{R}^d \mid \exists p \in P : (d(x,p)^q + g(p)^q)^{1/q} \leq a\}, \tag{A.9}$$

which can be seen to be the same with trivial algebraic manipulations:

$$\begin{aligned}
(d(x,p)^q + g(p)^q)^{1/q} &\leq a && \text{(A.10)}\\
d(x,p)^q + g(p)^q &\leq a^q && \text{(A.11)}\\
d(x,p)^q &\leq a^q - g(p)^q && \text{(A.12)}\\
d(x,p) &\leq (a^q - g(p)^q)^{1/q} && \text{(A.13)}
\end{aligned}$$

Now suppose $a < g(p)$. Then $r_p^{(q)}(a) = -\infty$, and thus the balls contain no points. As $(d(x,p)^q + g(p)^q)^{1/q} \geq g(p) > a$, the sublevel set is also empty.$\square$

Appendix B

The generalised Čech complex using the Mahalanobis distance kernel is stable

TODO:

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