1 Preliminaries

Skipped due to triviality.

2 Categories

2.1 Basic definitions

Exercise 2.1.1. Prove that sets (as objects) and injective functions (as arrows) form a category with functional composition as the composition operation c.

Solution. Take id_A to be $x \mapsto x$, then $\mathrm{id}_A \circ f = f$ and $g \circ \mathrm{id}_A = g$ is trivial. The last thing to check is that $g \circ f$ is injective, that is, whenever $s \neq s'$, then $g(f(s)) \neq g(f(s'))$. By injectivity of f, we have $f(s) \neq f(s')$ and by injectivity of g we have $g(f(s)) \neq g(f(s'))$. □

Exercise 2.1.2. Do the same as Exercise 1 for sets and surjective functions.

Solution. Let $f : A \to B, g : B \to C$ be injective functions. Then $f(A) = B, g(B) = C \Rightarrow g(f(A)) = C$. □

Exercise 2.1.3. Show that composition of relations (2.1.14) is associative.

Solution. Let α, β, γ be relations from A to B, from B to C and from C to D.

$$\begin{aligned} \alpha \circ \beta \circ \gamma &= \{(a, c) \mid \exists b : (a, b) \in \alpha, (b, c) \in \beta\} \circ \gamma \\ &= \{(a, d) \mid \exists b, c : (a, b) \in \alpha, (b, c) \in \beta, (c, d) \in \gamma\} \\ &= \alpha \circ (\beta \circ \gamma) \end{aligned}$$

□

Exercise 2.1.4. Prove the following for any arrow $u : A \to A$ of a category $\mathcal{C}$. It follows from these facts that C−3 and C−4 of 2.1.3. characterize the identity arrows of a category.

1. If $g \circ u = g$ for every object B of $\mathcal{C}$ and arrow $g : A \to B$, then $u = \mathrm{id}_A$.
2. If $u \circ h = h$ for every object C of $\mathcal{C}$ and arrow $h : C \to A$, then $u = \mathrm{id}_A$.

Solution.

1. $\mathrm{id}_A \circ u \overset{\text{def}}{=} u$, but also $\mathrm{id}_A \circ u = \mathrm{id}_A$ by assumption. $\Rightarrow u = \mathrm{id}_A$.
2. $u \circ \mathrm{id}_A \overset{\text{def}}{=} u$, but also $u \circ \mathrm{id}_A = \mathrm{id}_A$ by assumption. $\Rightarrow u = \mathrm{id}_A$.

□

2.2 Functional programming languages

Exercise 2.2.1. $\texttt{nonzero} : \texttt{NAT} \to \texttt{BOOLEAN}$, subject to equations $\texttt{nonzero} \circ \texttt{succ} = \texttt{false}$ and $\texttt{nonzero} \circ \texttt{succ} = \texttt{true}$.

2.3 Mathematical structures as categories

Exercise 2.3.1. For which sets A is $F(A)$ a commutative monoid?

Solution. $F(A)$ is always a monoid, so the only property to check is commutativity. If $A = \{\}$, then $F(A) = \{\}$ and is vacuously commutative. If $A = \{a\}$, then $F(A) = \{(), (a), (a, a), \dots\}$ and is commutative. Otherwise, if A has at least two elements, a and b, $(a)(b) = (a, b)$, but $(b)(a) = (b, a) \neq (a, b)$, therefore it is not commutative. All in all, $|A| \leq 1 \Leftrightarrow F(A)$ is commutative. ☐

Exercise 2.3.2. Prove that for each object A in a category $\mathcal{C}$, $\hom(A, A)$ is a monoid with composition of arrows as the operation.

Solution. Take id_A as the identity element. Then $\mathrm{id}_A \circ f = f \circ \mathrm{id}_A = f$ by definition of id. $\hom(A, A)$ is closed under composition. ☐

Exercise 2.3.3. Prove that a semigroup has at most one identity element.

Solution. Let e_1, e_2 be identity elements. Then $e_2 = e_1 e_2 = e_1$, so the identity elements are equal. This is very similar to exercise 2.1.4. ☐

2.4 Categories of sets with structure

Exercise 2.4.1. Let (S, α) and (T, β) be sets with relations on them. A **homomorphism** from (S, α) to (T, β) is a function $f : S \to T$ with the property that if $x\alpha y$ in S then $f(x)\beta f(y)$ in T.

1. Show that sets with relations and homomorphisms between them form a category.
2. Show that if (S, α) and (T, β) are both posets, then $f : S \to T$ is a homomorphism of relations if and only if it is a monotone map.

Solution.

1. Take the identity map as id_A. It is obviously a homomorphism. $\mathrm{id}_A \circ f = f \circ \mathrm{id}_A = f$ holds. Transitivity also holds:

$$x\alpha y \Rightarrow f(x)\beta f(y) \Rightarrow g(f(x))\gamma g(f(y))$$

2. By definition.

□

Exercise 2.4.2. Show that (strict) ω-complete partial orders and (strict) continuous functions form a category.

Solution. Take the identity map as id_A, which is clearly (strict) continuous. $\mathrm{id}_A \circ f = f \circ \mathrm{id}_A = f$ clearly holds. Composition $g \circ f$ of continuous functions is continuous:

$$s = \sup \mathcal{C} \Rightarrow f(s) = \sup f(\mathcal{C}) \Rightarrow g(f(s)) = \sup g(f(\mathcal{C}))$$

Same holds for strictness:

$$g(f(\bot)) = g(\bot) = \bot$$

□

Exercise 2.4.3. Let $\mathbb{R}^+$ be the set of nonnegative real numbers. Show that the poset $(\mathbb{R}^+, \leq)$ is not an ω-CPO.

Solution. Let $\mathcal{C}$ be $(1, 2, \dots)$. Suppose it has a supremum s. $\lfloor s \rfloor + 1 \in \mathcal{C}$ and is greater than s, therefore s is not the supremum. □

Exercise 2.4.4. Show that for every set S, the poset $(\mathcal{P}(S), \subseteq)$ is a strict ω-CPO.

Solution. Let $\mathcal{C}$ be $(s_1, s_2 \dots)$. Let $S' = \bigcup_{i=1}^{\infty}$. By definition $\forall i \;\; s_i \subseteq S'$ and if any other S'' has this property, then $S' \subseteq S''$ because otherwise $\exists s \in S'' : s \notin S'$ which implies that $\exists i : s_i \not\subseteq S''$. The bottom is $\bot$. □

Exercise 2.4.5. Give an example of ω-CPOs with a monote map between them that is not continuous.

Solution. Let the underlying set be $\mathbb{Z} \cup \{a, b\}$, and the order be the standart $\leq$ on integers, and $\forall i \in \mathbb{Z} \;\; i \leq a \leq b$. Then let f be:

$$f(x) = \begin{cases} x, & x \in \mathbb{Z} \\ b, & \text{otherwise} \end{cases}$$

f is monotone since on $\mathbb{Z}$ it is the indentity map, and $b = f(a) \leq f(b) = b$. It is not continuous because the chain $\mathcal{C}$ defined as $(0, 1, \dots)$ has a as its supremum, but $\sup f(\mathcal{C}) = \sup \mathcal{C} = a \neq f(a) = b$. □

Exercise 2.4.6. Let $g : \mathbb{N} \to \mathbb{N}$ be tge function such that $f(n) = 2^n$. Exhibit g as the least fixed point of a continuous function $\psi : \mathscr{P} \to \mathscr{P}$.

Solution. $\psi(h)(n) = \begin{cases} 1 & n = 0 \\ 2 * h(n-1) & n > 0 \end{cases}$ □

Exercise 2.4.7. Exhibit the Fibonacci function as the least fixed point of a continuous function from an ω-CPO to itself.

Solution. $\psi(h)(n) = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ h(n-1) + h(n-2) & n > 1 \end{cases}$ □