1 Preliminaries

Skipped due to triviality.

2 Categories

2.1 Basic definitions

1. Prove that sets (as objects) and injective functions (as arrows) form a category with functional composition as the composition operation c.

   Solution. Take id_A to be $x \mapsto x$, then $\mathrm{id}_A \circ f = f$ and $g \circ \mathrm{id}_A = g$ is trivial. The last thing to check is that $g \circ f$ is injective, that is, whenever $s \neq s'$, then $g(f(s)) \neq g(f(s'))$. By injectivity of f, we have $f(s) \neq f(s')$ and by injectivity of g we have $g(f(s)) \neq g(f(s'))$. ☐

2. Do the same as Exercise 1 for sets and surjective functions.

   Solution. Let $f : A \to B, g : B \to C$ be injective functions. Then $f(A) = B, g(B) = C \Rightarrow g(f(A)) = C$. ☐

3. Show that composition of relations (2.1.14) is associative.

   Solution. Let α, β, γ be relations from A to B, from B to C and from C to D.

$$\begin{aligned}\alpha \circ \beta \circ \gamma &= \{(a, c) \mid \exists b : (a, b) \in \alpha, (b, c) \in \beta\} \circ \gamma \\ &= \{(a, d) \mid \exists b, c : (a, b) \in \alpha, (b, c) \in \beta, (c, d) \in \gamma\} \\ &= \alpha \circ (\beta \circ \gamma)\end{aligned}$$

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4. Prove the following for any arrow $u : A \to A$ of a category $\mathcal{C}$. It follows from these facts that C−3 and C−4 of 2.1.3. characterize the identity arrows of a category.

   (a) If $g \circ u = g$ for every object B of $\mathcal{C}$ and arrow $g : A \to B$, then $u = \mathrm{id}_A$.

   (b) If $u \circ h = h$ for every object C of $\mathcal{C}$ and arrow $h : C \to A$, then $u = \mathrm{id}_A$.

   Solution.

   (a) $\mathrm{id}_A \circ u \stackrel{\text{def}}{=} u$, but also $\mathrm{id}_A \circ u = \mathrm{id}_A$ by assumption. $\Rightarrow u = \mathrm{id}_A$.

   (b) $u \circ \mathrm{id}_A \stackrel{\text{def}}{=} u$, but also $u \circ \mathrm{id}_A = \mathrm{id}_A$ by assumption. $\Rightarrow u = \mathrm{id}_A$.

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2.2 Functional programming languages

1. $\texttt{nonzero} : \texttt{NAT} \to \texttt{BOOLEAN}$, subject to equations $\texttt{nonzero} \circ \texttt{succ} = \texttt{false}$ and $\texttt{nonzero} \circ \texttt{succ} = \texttt{true}$.

2.3 Mathematical structures as categories

1. For which sets A is $F(A)$ a commutative monoid?

   Solution. $F(A)$ is always a monoid, so the only property to check is commutativity. If $A = \{\}$, then $F(A) = \{\}$ and is vacuously commutative. If $A = \{a\}$, then $F(A) = \{(), (a), (a, a), \dots\}$ and is commutative. Otherwise, if A has at least two elements, a and b, $(a)(b) = (a, b)$, but $(b)(a) = (b, a) \neq (a, b)$, therefore it is not commutative. All in all, $|A| \leq 1 \Leftrightarrow F(A)$ is commutative. $\square$

2. Prove that for each object A in a category $\mathcal{C}$, $\hom(A, A)$ is a monoid with composition of arrows as the operation.

   Solution. Take id_A as the identity element. Then $\mathrm{id}_A \circ f = f \circ \mathrm{id}_A = f$ by definition of id. $\hom(A, A)$ is closed under composition. $\square$

3. Prove that a semigroup has at most one identity element.

   Solution. Let e_1, e_2 be identity elements. Then $e_2 = e_1 e_2 = e_1$, so the identity elements are equal. This is very similar to exercise 2.1.4. $\square$