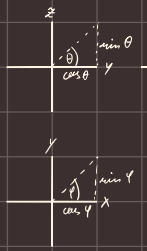
Trägheitsmoment Kugel



$$\begin{aligned} x &= r\cos\varphi\cos\theta \\ y &= r\sin\varphi\cos\theta \\ z &= r\sin\theta \end{aligned}$$

$$\begin{aligned}
dV &= \left|\det \frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}\right| dr\,d\varphi\,d\theta \\
&= \left|\det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{pmatrix}\right| dr\,d\varphi\,d\theta \\
&= \left|\det \begin{pmatrix} \cos\varphi\cos\theta & -r\sin\varphi\cos\theta & -r\cos\varphi\sin\theta \\ \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\theta & 0 & r\cos\theta \end{pmatrix}\right| dr\,d\varphi\,d\theta \\
&= \left|-r\sin\varphi\cos\theta\left(\sin\varphi\cos^2\theta r + r\sin\varphi\sin^2\theta\right) - r\cos\varphi\cos\theta\left(\cos\varphi\cos^2\theta r + r\cos\varphi\sin^2\theta\right)\right| dr\,d\varphi\,d\theta \\
&= \left|-r\sin\varphi\cos\theta\left(r\sin\varphi\left(\cos^2\theta+\sin^2\theta\right)\right) - r\cos\varphi\cos\theta\left(r\cos\varphi\left(\cos^2\theta+\sin^2\theta\right)\right)\right| dr\,d\varphi\,d\theta \\
&= \left|-r^2\sin^2\varphi\cos\theta - r^2\cos^2\varphi\cos\theta\right| dr\,d\varphi\,d\theta \\
&= \left|-r^2\cos\theta\left(\sin^2\varphi+\cos^2\varphi\right)\right| dr\,d\varphi\,d\theta \\
&= r^2\cos\theta\,dr\,d\varphi\,d\theta
\end{aligned}$$

$$\begin{aligned}
I &= \rho\int_V r_\perp^2\cos\theta\,dV \\
&= \rho\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^R r^4\cos^3\theta\,dr\,d\varphi\,d\theta \\
&= \rho\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\frac{1}{5}R^5\cos^3\theta\,d\varphi\,d\theta \\
&= \rho\int_{-\pi/2}^{\pi/2}2\pi\frac{1}{5}R^5\cos^3\theta\,d\theta \\
&= \rho\,2\pi\frac{1}{5}R^5\frac{4}{3} \\
&= \frac{2}{5}mR^2
\end{aligned}$$