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Math 174E HW5
Due May 12, 2023

15.15. A stock price is currently \$40. Assume that the expected return (μ) from the stock is 15% and that its volatility (σ) is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a 2-year period?

We have $\mu = 15\%$, $\sigma = 25\%$, $T = 2$

Let's define the continuously compounded rate of return per annum between times 0 and T as x,

So the probability distribution:

$$x \sim \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right)$$

$$\Rightarrow x \sim \phi\left(0.15 - \frac{0.25^2}{2}, \frac{0.25^2}{2}\right)$$

$$\Rightarrow x \sim \phi(0.11875, 0.03125)$$

with mean $0.15 - \frac{0.25^2}{2} = 0.11875$ p.a.

with standard deviation $\sqrt{\frac{0.25^2}{2}} = 0.17678$ p.a.

Thus, the expected value of the return is 11.875% p.a.

And the standard deviation is 17.678% p.a.

15.36. Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows:

$$30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0, 32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2$$

Estimate the stock price volatility. What is the standard error of your estimate?

Computation of volatility

Week i	Closing stock price (dollars), S_i	Price relative S_i/S_{i-1}	Weekly return $u_i = \ln(S_i/S_{i-1})$
1	30.2		
2	32.0	1.0596	0.05789
3	31.1	0.971875	-0.02853
4	30.1	0.967846	-0.03268
5	30.2	1.003322	0.003316
6	30.3	1.003311	0.003306
7	30.6	1.009901	0.00985
8	33.0	1.07843	0.075506
9	32.9	0.99697	-0.003035
10	33.0	1.00304	0.003035
11	33.5	1.01515	0.015038
12	33.5	1	0
13	33.7	1.00597	0.00595
14	33.5	0.994065	-0.00595
15	33.2	0.991045	-0.008995

In this case, $n=14$, so that

$$\sum_{i=1}^{n} u_i = 0.05789 - 0.02853 - 0.03268 + \cdots - 0.008995$$
$$= 0.0947$$

and $$\sum_{i=1}^{n} u_i^2 = 0.05789^2 + 0.02853^2 + 0.03268^2 + \cdots + 0.008995^2$$
$$= 0.01145$$

and the estimate of the standard deviation of the weekly return is

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} u_i\right)^2}$$
$$= \sqrt{\frac{1}{13}\cdot 0.01145 - \frac{1}{14\cdot 13}\cdot (0.0947)^2}$$
$$= 0.02884 \quad \text{or} \quad 2.884\%$$

Assuming that there are 52 trading weeks per year, $\tau = \frac{1}{52}$ and the data given an estimate for the volatility per annum of

$$0.02884\sqrt{52} = 0.20797 \quad \text{or} \quad 20.797\%$$

The standard error of this estimate is

$$\frac{\hat{\sigma}}{\sqrt{2n}} = \frac{0.20797}{\sqrt{2\times 14}} = 0.0393 \quad \text{or} \quad 3.93\% \text{ per annum.}$$

Therefore, the stock price volatility is 20.797% per annum, the standard error of this estimate is 3.93% per annum.

15.26. Show that the Black–Scholes–Merton formulas for call and put options satisfy put–call parity.

<u>Proof</u>: The Black-Scholes-Merton formulas for the prices of European call and put options are:

call: $C = S_0 N(d_1) - Ke^{-rT} N(d_2)$ and

put: $P = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$

we can get

$$C + Ke^{-rT} = S_0 N(d_1) - Ke^{-rT} N(d_2) + Ke^{-rT}$$
$$= S_0 N(d_1) + Ke^{-rT} [1 - N(d_2)]$$

Since $N(-d_2) = 1 - N(d_2)$, this is

$$C + Ke^{-rT} = S_0 N(d_1) + Ke^{-rT} N(-d_2)$$

Also:

$$P + S_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1) + S_0$$
$$= Ke^{-rT} N(-d_2) + S_0 [1 - N(-d_1)]$$

Since $N(d_1) = 1 - N(-d_1)$, this is

$$P + S_0 = Ke^{-rT} N(-d_2) + S_0 N(d_1)$$

$$\Rightarrow \quad C + Ke^{-rT} = P + S_0$$

The Black-Scholes-Merton equations are therefore consistent with put-call parity.

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15.16. A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is $38.

μ σ S_0

a. What is the probability that a European call option on the stock with an exercise price of $40 and a maturity date in 6 months will be exercised?

K

b. What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

$$\mu = 16\%, \quad \sigma = 35\%, \quad S_0 = \$38, \quad K = \$40, \quad T = 6 \text{ months} = 0.5 \text{ yrs}.$$

a. The probability that the call option will be exercised is the probability that $S_T \geq K$, where S_T is the stock price at time T.

So the stock price satisfies the Wiener Processes, which is lognormally distributed,

we have $\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \ \sigma^2 T\right]$.

$$\Rightarrow \ln S_T \sim \phi\left[\ln 38 + \left(0.16 - \frac{0.35^2}{2}\right) \cdot 0.5, \ 0.35^2 \cdot 0.5\right]$$

$$\Rightarrow \ln S_T \sim \phi(3.687, \ 0.06125)$$

The probability that $S_T \geq K$ is the same as the probability that $\ln S_T \geq \ln K$.

This is

$$1 - N\left(\frac{\ln K - \ln S}{\sigma\sqrt{T}}\right) = 1 - N\left(\frac{3.6889 - 3.687}{\sqrt{0.06125}}\right)$$

$$= 1 - N(0.008)$$

By search $N(0.008) = 0.50319$, so that

$$1 - N(0.008) = 1 - 0.50319 = 0.49681$$

Thus, the probability of exercising is 0.49681.

b. By part a.

Similarly, for put option, the probability that $S_T \leq K$ is the same as the probability that $\ln S_T \leq \ln K$,

So the probability of exercising is $1 - 0.49681 = 0.50319$.

15.24. A call option on a non-dividend-paying stock has a market price of $2\frac{1}{2}$. The stock price is \$15, the exercise price is \$13, the time to maturity is 3 months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

$C = \$2.5$, $S_0 = \$15$, $K = \$13$. $T = 3\text{ months} = 0.25\text{ yrs}$, $r = 5\%$ p.a.

For call option, we have formula:

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$\Rightarrow 15N(d_1) - 13e^{-0.05\cdot 0.25} N(d_2) = 2.5$$

where $d_1 = \dfrac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$

$d_2 = \dfrac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$ where σ is implied volatility.

$$\Rightarrow d_1 = \frac{\ln(15/13) + (0.05 + \sigma^2/2)\cdot 0.25}{\sigma\sqrt{0.25}}$$

$$d_2 = \frac{\ln(15/13) + (0.05 - \sigma^2/2)\cdot 0.25}{\sigma\sqrt{0.25}} = d_1 - \sigma\sqrt{0.25}$$

Substituting into C:

$$15\cdot N\left(\frac{\ln(15/13) + (0.05 + \sigma^2/2)\cdot 0.25}{\sigma\sqrt{0.25}}\right) - 13e^{-0.05\cdot 0.25}\cdot N\left(\frac{\ln(15/13) + (0.05 - \sigma^2/2)\cdot 0.25}{\sigma\sqrt{0.25}}\right) = 2.5$$

Let's guess volatility $\sigma = 0.2$ first.

$d_1 = 1.606 \quad d_2 = 1.506$

$$C = 15N(1.606) - 13e^{-0.05\cdot 0.25} N(1.506)$$
$$= 15\cdot 0.9515 - 12.8385\cdot 0.9406$$
$$= 2.2$$

then let's suppose $\sigma = 0.4$,

$d_1 = 0.878 \quad d_2 = 0.678$

$$C = 15\cdot N(0.878) - 13e^{-0.05\cdot 0.25}\cdot N(0.678)$$
$$= 15\cdot 0.8106 - 12.8385\cdot 0.7517$$
$$= 2.508$$

Finally, by iterations from programming,

we get the solution of $\sigma = 0.3964$.

Thus, the implied volatility of the given option is 39.64%.