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Math 174E HW7
Due May 26, 2023

20.17. A European call option on a certain stock has a strike price of \$30, a time to maturity of 1 year, and an implied volatility of 30%. A European put option on the same stock has a strike price of \$30, a time to maturity of 1 year, and an implied volatility of 33%. a. What is the arbitrage opportunity open to a trader? b. Does the arbitrage work only when the lognormal assumption underlying Black–Scholes–Merton holds? Explain carefully the reasons for your answer.

a. We have a European call option with $K = \$30$, $T = 1yr$, $\sigma = 30\%$.
and a European put option with $K = \$30$, $T = 1yr$, $\sigma = 33\%$.
Based on the put-call parity, European put and call options should have same implied volatility at same strike price:

$$P + S_0 e^{-qT} = C + Ke^{-rT}$$

Since implied volatility $33\% > 30\%$, but there are same underlying asset. Then the price of a call option is lower than the price of a put option. To exploit this arbitrage opportunity, the trader can do the following:

1. Sell the put option
2. Buy the call option
3. Short the stock.

b. The arbitrage opportunity described above does not rely on the lognormal assumption underlying the Black-Scholes-Merton model. The lognormal assumption assumes that the distribution of stock prices follows a lognormal distribution. However, in this case, the discrepancy in option prices arises due to differences in implied volatility, not the assumption about the distrubtion of stock prices. As long as the implied volatility differs, the arbitrage opportunity exists, regardless of the underlying assumptions of the Black-Scholes-Merton model. The put-call parity holds true for all option pricing model.

20.26. Consider a European call and a European put with the same strike price and time to maturity. Show that they change in value by the same amount when the volatility increases from a level σ_1 to a new level σ_2 within a short period of time. (*Hint*: Use put–call parity.)

For volatility $= \sigma_1$:

Suppose that c_1 and p_1 are call option and put option prices respectively

For volatility $= \sigma_2$:

Suppose that c_2 and p_2 are call option and put option prices respectively

Because put-call parity holds for the Black-Scholes-Merton model.

According to the put-call parity, we have

$$p_1 + S_0 e^{-qT} = c_1 + Ke^{-rT}$$

and

$$p_2 + S_0 e^{-qT} = c_2 + Ke^{-rT}$$

subsctracting these two equations, we get

$$p_1 - p_2 = c_1 - c_2$$

This shows that the pricing error when the Black-Scholes-Merton model is used to price a European put option should be exactly the same strike price and maturity.

29.3. Use the Black's model to value a 1-year European put option on a 10-year bond. Assume that the current cash price of the bond is $125, the strike price is $110, the 1-year risk-free interest rate is 10% per annum, the bond's forward price volatility is 8% per annum, and the present value of the coupons to be paid during the life of the option is $10.

According to the Black's model, the put option given by

$$p = P(0,T)[KN(-d_2) - F_B N(-d_1)]$$
$$= e^{-rT}[KN(-d_2) - F_B N(-d_1)]$$

We have $T = 1yr$, $B_0 = \$125$, $K = \$110$, $r = 10\% = 0.1$, $\sigma_B = 8\% = 0.08$

$I = \$10$.

$$P(0,T) = e^{-rT} = e^{-0.1\times 1}$$

$$F_0 = \frac{B-I}{P(0,T)} = (125-10)\times e^{0.1} = \$127.09$$

$$d_1 = \frac{\ln(F_0/K) + \sigma_B^2 T/2}{\sigma_B\sqrt{T}} = \frac{\ln(127.09/110) + 0.08^2\times 1/2}{0.08\times 1}$$

$$= 1.845$$

$$d_2 = d_1 - \sigma_B\sqrt{T} = 1.845 - 0.08$$
$$= 1.765$$

For the value of the put option is

$$p = e^{-rT}[KN(-d_2) - F_B N(-d_1)]$$
$$= e^{-0.1\times 1}[110\cdot N(-1.765) - 127.09 N(-1.845)]$$

$(N(-1.765) = 0.038782$

$N(-1.845) = 0.032519)$

$$= e^{-0.1}\cdot(110\times 0.03878 - 127.09\times 0.032519)$$
$$= \$0.12$$

Thus, in this case, the value of the put option is $0.12.

29.5. Calculate the price of an option that caps the 3-month rate, observed in 15 months for the 15–18 month period, at 13% R_K (quoted with quarterly compounding) on a principal amount of $1,000. The forward interest rate for the period in question is 12% per annum F_K (quoted with quarterly compounding), the 18-month risk-free interest rate (continuously compounded) is 11.5% per annum, r and the volatility of the forward rate is 12% per annum. σ_K

We have $L = \$1{,}000$, $t_k = 15\text{-month} = 1.25 \text{ yrs}$, $t_{k+1} = 18\text{-month} = 1.5 \text{ yrs}$,

$R_K = 13\% = 0.13$, $F_K = 12\% = 0.12$, $r = 11.5\% = 0.115$, $\sigma_K = 12\% = 0.12$

$\Rightarrow \delta_K = t_{k+1} - t_k = 3 \text{ months} = 0.25 \text{ yrs}.$

$$d_1 = \frac{\ln(F_K/R_K) + \sigma_K^2 t_k/2}{\sigma_K \sqrt{t_k}} = \frac{\ln(0.12/0.13) + 0.12^2 \cdot 1.25/2}{0.12 \cdot \sqrt{1.25}}$$

$$= -0.5295$$

$$d_2 = d_1 - \sigma_K \sqrt{t_k} = -0.5295 - 0.12 \cdot \sqrt{1.25}$$

$$= -0.6637$$

According to the Black's model, we have the value of the caplet is

$$L \delta_k P(0, t_{k+1}) [F_K N(d_1) - R_K N(d_2)]$$

then $P(0, t_{k+1}) = e^{-rT} = e^{-0.115 \times 1.5} = 0.8416$

$\Rightarrow$ $1000 \times 0.25 \times 0.8416 \times [0.12 \cdot N(-0.5295) - 0.13 \cdot N(-0.6637)]$ $\quad (N(-0.5295) = 0.2984,$ $N(-0.6637) = 0.2537)$

$= 210.4 \times (0.12 \times 0.2984 - 0.13 \times 0.2537)$

$= \$0.59$

Therefore, the value of this option is $0.59.

29.1. A company caps a 3-month floating rate at 2% per annum. The principal amount is $20 million. On a reset date, the floating rate is 4% per annum. What payment would this lead to under the cap? When would the payment be made?

T = 3-month = 0.25 yrs.
principal amount = $20 million
caps rate = 2% p.a.

Now, we can calculate the payment:

$$0.25 \times \$20 \text{ million} \times 2\% = \$100,000$$

Therefore, the payment under the cap would be $100,000.

As for when the payment would be made, it depends on the terms of the agreement or contract. Typically, such payments are made at the end of the designated period, which in this case is 3 months later.