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Math 174E HW2
Due April 10, 2023

$m=4$

4.11. A bank quotes an interest rate of 7% per annum with quarterly compounding. What is the equivalent rate with (a) continuous compounding and (b) annual compounding?

(a) $R_m = 7\%$ $\quad m=4$

$$\Rightarrow R_c = m\ln\left(1+\frac{R_m}{m}\right) = 4\cdot\ln\left(1+\frac{0.07}{4}\right) = 0.0694 = 6.94\% \quad (\text{p.a.})$$

Thus, the continuous compounding is 6.94% per annum.

(b) $m_1 = 4$, $R_1 = 7\%$, $m_2 = 1$

annual compounding:

$$R_2 = 1\cdot\left[\left(1+\frac{0.07}{4}\right)^4 - 1\right] = 0.0719 = 7.19\%.$$

Thus, the annual compounding is 7.19%

4.19. A deposit account pays 4% per annum with continuous compounding, but interest is actually paid quarterly. How much interest will be paid each quarter on a $10,000 deposit?

continuous compounding $R_c = 4\%$

interest paid quarterly : $m = 4$

equivalent rate with quarterly compounding is : $R_m = m(e^{R_c/m} - 1)$

$$R_m = 4(e^{\frac{0.04}{4}} - 1)$$

$$= 4(e^{0.01} - 1)$$

$$= 0.0402$$

$$\text{Interest} = 10{,}000 \times \left(\frac{0.0402}{4}\right) = 100.5$$

Thus, a $10,000 deposit paid $100.5 interest each quarter.

4.21. A 3-year bond provides a coupon of 8% semiannually and has a cash price of 104. What is the bond's yield?

3-year : 36 months.

Since cash price is \$104, is also its market value.

Let suppose that a 3-year bond with a principal of \$100 provides a coupon of 8% semiannually.

coupon: $100 \times \frac{0.08}{2} = 4$ per semiannual.

And \$104 in 36 months.

Let y is bond's yield, expressed with continuous compounding, it must be true that

$$4e^{-y\times 0.5} + 4e^{-y\times 1} + 4e^{-y\times 1.5} + 4e^{-y\times 2} + 4e^{-y\times 2.5} + 104e^{-y\times 3} = 104$$

$$\Rightarrow e^{-y\times 0.5} + e^{-y} + e^{-y\times 1.5} + e^{-y\times 2} + e^{-y\times 2.5} + 26e^{-y\times 3} = 26$$

$\Rightarrow$ By calculator, we can get $y \approx 0.0641 = 6.41\%$

Thus, the bond's yield is 6.41%.

4.28. A 5-year bond with a yield of 7% (continuously compounded) pays an 8% coupon at the end of each year.

a. What is the bond's price?
b. What is the bond's duration?
c. Use the duration to calculate the effect on the bond's price of a 0.2% decrease in its yield.
d. Recalculate the bond's price on the basis of a 6.8% per annum yield and verify that the result is in agreement with your answer to (c).

a. Let bond face value (or cash flow) = \$100

yield $y = 7\% = 0.07$, $t = 5$-year

coupon rate = 8%, coupon = 8% × 100 = \$8 per annum.

Bond Price $B = \sum_{i=1}^{n} C_i \cdot e^{-y \cdot t_i}$

$= 8e^{-0.07 \cdot 1} + 8e^{-0.07 \cdot 2} + 8e^{-0.07 \cdot 3} + 8e^{-0.07 \cdot 4} + 108e^{-0.07 \cdot 5}$

$= 103.05$

Thus, the bond's price is \$103.5.

b. Duration: $D = \dfrac{\sum_{i=1}^{n} t_i \cdot C_i \cdot e^{-y \cdot t_i}}{B}$

$\Rightarrow D = \frac{1}{103.5}\left[1 \times 8e^{-0.07 \cdot 1} + 2 \times 8e^{-0.07 \cdot 2} + 3 \times 8e^{-0.07 \cdot 3} + 4 \times 8e^{-0.07 \cdot 4} + 5 \times 8e^{-0.07 \cdot 5}\right]$

$= 4.3235$

Thus, the bond's duration is 4.3235 years.

c. Since the effect on the bond's price of a 0.2% decrease in its yield.

$\Rightarrow \Delta y = -0.2\% = -0.002$

Then $\Delta B = -BD\Delta y$

$= 103.05 \times 4.3235 \times 0.2\% = 0.89$.

$B = 103.05 + 0.89 = 103.94$

Thus, the bond's price rise to \$103.94.

d. Yield $y = 6.8\%$

$B = \sum_{i=1}^{n} C_i \cdot e^{-y \cdot t_i}$

$= 8e^{-0.068 \cdot 1} + 8e^{-0.068 \cdot 2} + 8e^{-0.068 \cdot 3} + 8e^{-0.068 \cdot 4} + 108e^{-0.068 \cdot 5}$

$= 103.946$

≈ 103.95

A 5-year bond with a yield of 6.8%, bond face value is \$100, which the bond's price is very close to my answer (c).

4.35. Portfolio A consists of a 1-year zero-coupon bond with a face value of \$2,000 and a 10-year zero-coupon bond with a face value of \$6,000. Portfolio B consists of a 5.95-year zero-coupon bond with a face value of \$5,000. The current yield on all bonds is 10% per annum.

a. Show that both portfolios have the same duration.
b. Show that the percentage changes in the values of the two portfolios for a 0.1% per annum increase in yields are the same.
c. What are the percentage changes in the values of the two portfolios for a 5% per annum increase in yields?

a. Proof: $y = 10\%$ (p.a.)

portfolio A: 1yr: \$2000 10yrs: \$6000

portfolio B: 5.95 yrs: \$5000

Duration of portfolio A:

$$\frac{1 \times 2000e^{-0.1\times1} + 10 \times 6000e^{-0.1\times10}}{2000e^{-0.1\times1} + 6000e^{-0.1\times10}} = 5.95 \text{ yrs}$$

Thus, the duration of portfolio A is same as portfolio.

b. Proof: Portfolio A:

$$\text{Total vaule} = 2000 \cdot e^{-0.1\times1} + 6000 \cdot e^{-0.1\times10} = 4016.95$$

Since $\Delta y = 0.1\% = 0.001$, the value of portfolio A we get:

$$2000 \cdot e^{-0.101\times1} + 6000 \cdot e^{-0.101\times10} = 3993.18$$

$$\text{percentage change} = \frac{3993.18 - 4016.95}{4016.95} = \frac{-23.77}{4016.95} = -0.0059 = -0.59\%$$

Portfolio B:

$$\text{Total value} = 5000 \cdot e^{-0.1\times5.95} = 2757.81$$

Since $\Delta y = 0.1\% = 0.001$, the value of portfolio B we get:

$$5000 \cdot e^{-0.101\times5.95} = 2741.45$$

$$\text{percentage change} = \frac{2741.45 - 2757.81}{2757.81} = \frac{-16.36}{2757.81} = -0.0059 = -0.59\%$$

Thus, the portfolio A and portfolio B has same percentage decrease in yield of 0.59%.

c. Also, $\Delta y = 5\% = 0.05$

$$\text{Total value of portfolio A} = 2000 \cdot e^{-0.15\times1} + 6000 \cdot e^{-0.15\times10} = 3060.20$$

$$\text{percentage change of A} = \frac{4016.95 - 3060.20}{4016.95} = \frac{956.75}{4016.95} = 0.2382 = 23.82\%$$

$$\text{Total value of portfolio B} = 5000 \cdot e^{-0.15\times5.95} = 2048.15$$

$$\text{percentage change of B} = \frac{2757.81 - 2048.15}{2757.81} = \frac{709.66}{2757.81} = 0.2573 = 25.73\%$$

Thus, for a 5% per annum increase in yield, the percentage change in the value of portfolio A is 23.82%; the percentage change in the value of portfolio B is 25.73%.