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Math 174E HW 4
Due April 28, 2023

11.14. What is a lower bound for the price of a 6-month call option on a non-dividend-paying stock when the stock price is $80, the strike price is $75, and the risk-free interest rate is 10% per annum?

$S_0 = \$80 \quad K = \$75 \quad r = 10\%$ (p.a.) $\quad T = 6$ months $= 0.5$ year

So a lower bound for the price of a 6-month European call option on a non-dividend-paying stock is

$$S_0 - Ke^{-rT} = 80 - 75e^{-0.1 \cdot 0.5}$$

$$= \$8.66$$

11.21. The price of an American call on a non-dividend-paying stock is $4. The stock price is $31, the strike price is $30, and the expiration date is in 3 months. The risk-free interest rate is 8%. Derive upper and lower bounds for the price of an American put on the same stock with the same strike price and expiration date.

$C = \$4 \quad S_0 = \$31 \quad K = \$30 \quad r = 8\% \quad T = 3$ months $= 0.25$ yr

Put-call parity holds only for European options. However, it is possible to derive some results for American option prices.

When there are no dividends, it can be shown

$$S_0 - K \le C - P \le S_0 - Ke^{-rT}$$

$$\Rightarrow \quad 31 - 30 \le 4 - P \le 31 - 30e^{-0.08 \cdot 0.25}$$

$$1 \le 4 - P \le 1.59$$

$$2.41 \le P \le 3$$

So the price of an American put lower bounds is $3, upper bounds is $2.41.

11.31. Suppose that c_1, c_2, and c_3 are the prices of European call options with strike prices K_1, K_2, and K_3, respectively, where $K_3 > K_2 > K_1$ and $K_3 - K_2 = K_2 - K_1$. All options have the same maturity. Show that

$$c_2 \le 0.5(c_1 + c_3)$$

(*Hint*: Consider a portfolio that is long one option with strike price K_1, long one option with strike price K_3, and short two options with strike price K_2.)

<u>Proof</u>: Suppose that C_1, C_2, C_3 are the prices of European call options with strike prices K_1, K_2, K_3.

So consider the lower bound for calls on Non-Dividend-Paying stocks,

we have:

$$C_1 \geq S_0 - K_1 e^{-rT} \Rightarrow K_1 \geq \frac{S_0 - C_1}{e^{-rT}} \quad ①$$

$$C_2 \geq S_0 - K_2 e^{-rT} \Rightarrow K_2 \geq \frac{S_0 - C_2}{e^{-rT}} \quad ②$$

$$C_3 \geq S_0 - K_3 e^{-rT} \Rightarrow K_3 \geq \frac{S_0 - C_3}{e^{-rT}} \quad ③$$

Since $K_3 > K_2 > K_1$ and $K_3 - K_2 = K_2 - K_1$

we have $K_2 = \frac{K_1 + K_3}{2}$

$$\Rightarrow K_2 = \frac{K_1 + K_3}{2} \geq \frac{S_0 - C_2}{e^{-rT}}$$

$$\Rightarrow \frac{C_1 + C_3}{2} \geq \frac{S_0 - C_2}{e^{-rT}}$$

by ② we have $C_2 \geq S_0 - K_2 e^{-rT}$,

which means $\frac{C_1 + C_3}{2} \geq C_2 \geq S_0 - K_2 e^{-rT}$ can satisfies the inequality holds.

Then consider the hint given,

case 1: $S_T \leq K_1$, the amount of portfolio $= 0$

case 2: $K_1 < S_T \leq K_2$, the amount of portfolio $= S_T - K$

case 3: $K_2 < S_T \leq K_3$, the amount of portfolio

$$= S_T - 2(S_T - K_2) - K_1 = 2K_2 - K_1 - S_T$$

case 4: $S_T > K_3$, the amount of portfolio

$$= \cancel{S_T} - 2(\cancel{S_T} - K_2) - K_1 + (\cancel{S_T} - K_3)$$

$$= 2K_2 - K_1 - K_3$$

Since the portfolio value must be greater than 0 such that

$$C_1 - 2C_2 + C_3 \geq 0$$

$$\Rightarrow \quad C_2 \leq \frac{C_1 + C_3}{2}$$

$\blacksquare$

13.11. A stock price is currently \$100. Over each of the next two 6-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a 1-year European call option with a strike price of \$100?

u d

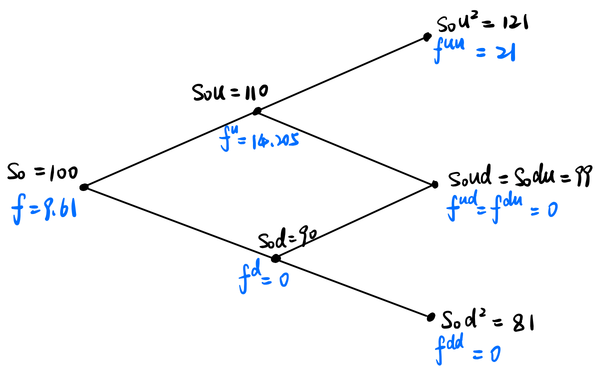
$S_0 = \$100 \quad K = \$100 \quad u = 1.1 \quad d = 0.9 \quad \Delta t = 0.5 \text{ yr} \quad r = 8\%$

risk - neutral probability:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.08 \cdot 0.5} - 0.9}{1.1 - 0.9}$$

$$= 0.70405$$

Illustration:



$S_0u = 100 \cdot 1.1 = 110$

$S_0u^2 = 100 \cdot 1.1^2 = 121$

$S_0d = 100 \cdot 0.9 = 90$

$S_0ud = S_0du = 100 \cdot 1.1 \cdot 0.9 = 99$

$S_0d^2 = 100 \cdot 0.9^2 = 81$

$$f^u = e^{-r\Delta t}(pf^{uu} + (1-p)f^{ud})$$
$$= e^{-0.08 \cdot 0.5}(0.70405 \cdot 21 + (1 - 0.70405) \cdot 0)$$
$$= 14.2053$$

$$f^d = e^{-r\Delta t}(pf^{du} + (1-p)f^{dd})$$
$$= e^{-0.08 \cdot 0.5}(0.70405 \cdot 0 + (1 - 0.70405) \cdot 0)$$
$$= 0$$

Then the value of a 1-year European call option with a strike price of \$100 is:

$$f = e^{-2r\Delta t}(p^2 \cdot f^{uu} + 2p(1-p) \cdot f^{ud} + (1-p)^2 f^{dd})$$
$$= e^{-2 \cdot 0.08 \cdot 0.5}(0.70405^2 \cdot 21 + 0 + 0)$$
$$= \$9.61$$

13.15. A stock price is currently \$80. It is known that at the end of 4 months it will be either \$75 or \$85. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a 4-month European put option with a strike price of \$80? Use no-arbitrage arguments.

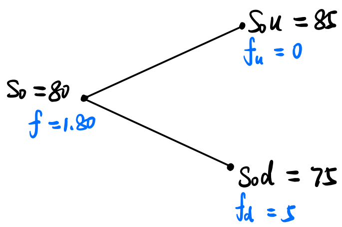
Implied the One-Step Binomial Model, suppose no-arbitrage

we have $S_0 = \$80$ $\quad S_0 u = 85 \Rightarrow u = 1.0625 \quad f_u = 0$

$S_0 d = 75 \Rightarrow d = 0.9375 \quad f_d = 5$

$r = 5\%$ (p.a.) $\quad T = 4\text{-month} = \frac{4}{12}$ yr

Illustration



Since $p = \dfrac{e^{rT} - d}{u - d} = \dfrac{e^{0.05 \cdot \frac{4}{12}} - 0.9375}{1.0625 - 0.9375} = 0.6345$

then $f = e^{-rT}[p f_u + (1-p) f_d]$

$= e^{-0.05 \cdot \frac{4}{12}}[0.6345 \cdot 0 + (1 - 0.6345) \cdot 5]$

$= \$1.80.$

Therefore, the value of a 4-month European put option with a strike price of \$80 is \$1.80.