Math 170E
Summer 2022
Final Exam
07/29/2022
Time Limit: 24 Hours

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This exam contains 15 pages (including this cover page) and 8 questions. Total of points is 100. Make sure to write your answers in full detail, so that you may get the maximum possible partial points when applicable.

Question	Points	Score
1	12	
2	18	
3	10	
4	5	
5	15	
6	15	
7	15	
8	10	
Total:	100	

- *In order to get the maximum points, you must justify each statement you are making.*

- Please type the following statement in your handwriting, then sign and date below:

  "I hereby acknowledge that I am aware I may use my textbook, lecture notes and recordings during the exam and swear on my honor as a Bruin *all* the answers I present belong solely to me, in thought and in writing."

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Jiayu Li 07/29/2022.

1. (12 points) Balin and Dwalin are two brothers, playing a game with a slightly crooked coin, with $P(H) = 0.6$ and $P(T) = 0.4$. The coin is tossed 10 times (each toss is independent from others) and in any turn it shows heads, it is tossed one more time. The brothers are counting the cases where *the coin is tossed twice and the second toss, too, is heads.*

   Here is an example scenario:

$$\begin{array}{cccccccccc} T & H & T & T & T & T & T & T & T & H \\ & T & & & & & & & & H \end{array}$$

   The count here will be 1, as there are only two cases where the coin showed heads and in only one such case, the second toss was also heads. Thus, only the 10th turn is counted.

   Let X be the counted number of such double heads. What is the support, pmf, expected value and variance of X?

$P(H) = 0.6 \quad P(T) = 0.4$

$P(HT) = 0.6 \cdot 0.4 = 0.24 \qquad P(HH) = 0.6 \cdot 0.6 = 0.36$

Let X be the counted number of such double heads.

$X \sim \text{Bernoulli}(n, p)$ with $0 \le n \le 10$, $p = 0.36$.

$\Rightarrow$ pmf: $f(x) = \binom{10}{x} 0.36^x (1-0.36)^{10-x}$

$= \binom{10}{x} 0.36^x \, 0.64^{10-x}$

$E(X) = np = 10 \times 0.36 = 3.6$

$Var(X) = np(1-p) = 10 \times 0.36 \times (1 - 0.36) = 2.304.$

2. (18 points) Let N be the number of times we will toss a fair die, where $N \in \mathbb{Z}^+$ and $P(N = k) = 1/2^k$ for any $k \in \mathbb{Z}^+$. Let S be the sum of all the throws of the die.

   For example, say N turned out to be 4, then we toss the die 4 times. Say the outcomes are $6, 1, 1, 3$, then $S = 6 + 1 + 1 + 3 = 11$.

   (a) (6 points) Calculate $P(S = 5 \mid N = 3)$.

$P(S=5 \mid N=3)$

$5 = 1 + 1 + 3$

$5 = 1 + 2 + 2$

$5 = 2 + 1 + 2$

$5 = 2 + 2 + 1$

$5 = 3 + 1 + 1$

$5 = 1 + 3 + 1$

6 cases out of $6 \times 6 \times 6 = 216$ cases.

$P(N=3) = \frac{1}{2^3} = \frac{1}{8}$ $\quad$ $P(S=5 \mid N=3) = \frac{1}{36}$

$$P(S=5 \mid N=3) = \frac{P(S=5 \cap N=3)}{P(N=3)} = \frac{\frac{1}{36} \times \frac{1}{8}}{\frac{1}{8}} = \frac{1}{36}$$

(b) (12 points) Calculate $P(N = 2 \mid S = 5)$.

$$P(N=2 \mid S=5) = \frac{P(N=2 \cap S=5)}{P(S=5)}$$

5

$P(N=1) = \frac{1}{2}$ $P(S=5 \mid N=1) = \frac{1}{6}$

1+4, 4+1, 2+3, 3+2

$P(N=2) = \frac{1}{2^2} = \frac{1}{4}$ $P(S=5 \mid N=2) = \frac{4}{6^2} = \frac{1}{9}$

$P(N=3) = \frac{1}{2^3} = \frac{1}{8}$ $P(S=5 \mid N=3) = \frac{1}{36}$

1+1+1+2, 1+1+2+1, 1+2+1+1, 2+1+1+1

$P(N=4) = \frac{1}{2^4} = \frac{1}{16}$ $P(S=5 \mid N=4) = \frac{4}{6^4} = \frac{1}{324}$

1+1+1+1+1

$P(N=5) = \frac{1}{2^5} = \frac{1}{32}$ $P(S=5 \mid N=5) = \frac{1}{6^5} = \frac{1}{7776}$

Using the probability from above.

$$P(S=5) = \frac{1}{2} \times \frac{1}{6} + \frac{1}{4} \times \frac{1}{9} + \frac{1}{8} \times \frac{1}{36} + \frac{1}{16} \times \frac{1}{324} + \frac{1}{32} \times \frac{1}{7776}$$
$$= 0.11478$$

$$\Rightarrow P(N=2 \mid S=5) = \frac{\frac{1}{4} \times \frac{1}{9}}{0.11478} = 0.242$$

3. (10 points) Let X be a random variable with support $S_X = [-6, 3]$ and pdf $f(x) = \frac{1}{81}x^2$ for $x \in S_X$, zero otherwise.

   Consider the random variable $Y = \max(X, 0)$. Calculate the CDF of Y, $F_Y(y)$, where y can be any real number, including those not in the support of Y.

We have pdf $f(X) = \begin{cases} \frac{1}{81}X^2 & X \in [-6, 3] \\ 0 & \text{otherwise.} \end{cases}$

Since r.v. $Y = \max(X, 0)$, then $Y = \begin{cases} 0 & X \in [-6, 0] \\ X & X \in (0, 3] \end{cases}$

$P(Y < 0) = 0$

$P(Y = 0) = P(-6 \le X \le 0) = \int_{-6}^{0} \frac{1}{81} x^2 dx = \frac{1}{243} x^3 \Big|_{-6}^{0} = \frac{8}{9}$

$P(0 < Y < 3) = P(-6 < X \le 3) = \int_{-6}^{0} \frac{1}{81} x^2 dx + \int_{0}^{y} \frac{1}{81} x^2 dx$

$= \frac{8}{9} + \frac{1}{243} y^3$

$\Rightarrow$ CDF of Y, $F_Y = \begin{cases} 0 & Y < 0 \\ \frac{8}{9} & Y = 0 \\ \frac{8}{9} + \frac{y^3}{243} & 0 < Y < 3 \\ 1 & Y \ge 3 \end{cases}$

4. (5 points) Let $W \sim exp(\theta)$, where $\theta = 5$ (parametrization with θ, not λ). Find the 42nd percentile of W, $\pi_{0.42}$.

$W \sim exp(\theta)$ $\theta = 5$ find 42nd percentile $\pi_{0.42}$

$$\Rightarrow f(w) = \begin{cases} \frac{1}{\theta} e^{-\frac{w}{\theta}} & w > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(w) = \int_{-\infty}^{w} f(w)\,dw = \begin{cases} \int_0^w \frac{1}{\theta} e^{-\frac{w}{\theta}}\,dw = 1 - e^{-\frac{w}{\theta}} & , \theta = 5 \\ 0 & \text{otherwise} \end{cases}$$

$\because F(w) = 0.42$, $\theta = 5$

$$\Rightarrow \quad 1 - e^{-\frac{w}{5}} = 0.42 \Rightarrow w = -5 \ln 0.58 \approx 2.7236.$$

5. (15 points) Consider two independent Bernoulli r.v., U and V, both with probability of success 1/2. Let $X = U + V$ and $Y = |U - V|$.

   (a) (5 points) Calculate the covariance of X and Y, $\sigma_{X,Y}$.

Since U and V are two Independent Bernoulli r.v. .

Then $P(U=0) \cdot P(V=0) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(U=0) \cdot P(V=1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(U=1) \cdot P(V=1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(U=1) \cdot P(V=0) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Let $X = U + V$ and $Y = |U - V|$

U	V	X	Y
0	0	0	0
1	1	2	0
1	0	1	1
0	1	1	1

$P(X=0, Y=0) = P(U=0, V=0) = \frac{1}{4}$

$P(X=2, Y=0) = P(U=1, V=1) = \frac{1}{4}$

$P(X=1, Y=1) = P(U=1, V=0) + P(U=0, V=1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

By use $Cov(X,Y) = E(XY) - E(X)E(Y)$

$\Rightarrow E(XY) = 0 \cdot 0 \cdot \frac{1}{4} + 2 \cdot 0 \cdot \frac{1}{4} + 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$

$E(X) = 0 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} = 1$

$E(Y) = 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

Thus $\sigma_{X,Y} = Cov(X,Y) = \frac{1}{2} - 1 \cdot \frac{1}{2} = 0$.

(b) (5 points) Are X and Y independent? Justify your answer.

From part a), since $\sigma_{X,Y} = 0$, i.e., X and Y are not related each other.

And $P(X=0, Y=0) = \frac{1}{4} \neq 0$

$P(X=0) = \frac{1}{2}$ and $P(Y=0) = \frac{1}{2} + \frac{1}{2} = \frac{1}{4}$.

Thus, X and Y are not independent each other.

(c) (5 points) Find the random variable expressed as the conditional expectation of Y given X, i.e., $\mathbb{E}[Y|X]$. If it has a "named" distribution, you must state it. Otherwise support and pdf is enough.

From part (a) list,

as $X=0$, $E(Y|X=0) = 0$

as $X=1$, $E(Y|X=1) = 1$

as $X=2$, $E(Y|X=2) = 0$

6. (15 points) Consider two independent, exponential random variables $X, Y \sim exp(1)$. Let $U = X + Y$ and $V = X/(X + Y)$.

   (a) (5 points) Calculate the joint pdf of U and V.

we have $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

$$f(X,Y) = f(X)f(Y) = e^{-(X+Y)}$$

$$\begin{cases} U = X+Y \\ V = \frac{X}{X+Y} \end{cases} \Rightarrow \quad \begin{matrix} X = UV \\ Y = U - UV \end{matrix} \qquad \begin{matrix} U = X+Y > 0 \\ V = \frac{X+Y-Y}{X+Y} = 1 - \frac{Y}{X+Y} \end{matrix}$$

$$\Rightarrow 0 < V < 1$$

$$f(uv, u-uv) = e^{-u}$$

Then $\frac{\partial^2 F(u,v)}{\partial x \, \partial y} = f(x) \Rightarrow \frac{\partial^2 F(u,v)}{\partial u \, \partial v} \cdot \frac{\partial u \partial v}{\partial x \partial y} = f(x,y)$

$$\Rightarrow F(u,v) = f(x,y)\left[\frac{\partial x \cdot \partial y}{\partial u \cdot \partial v} - \frac{\partial x \, \partial y}{\partial v \, \partial u}\right]$$

$$= e^{-u}\left[-uv - u(1-v)\right]$$

$$= -ue^{-u} \quad \text{for } u > 0.$$

(b) (5 points) Identify the distribution of U. If it has a "named" distribution, you must state it. Otherwise support and pdf is enough. (HINT: You may refer to the front of the textbook with list of distributions.)

$F(u) = \int_v F(u,v)\,dv = \int_0^1 ue^{-u}\,dv = ue^{-u} \cdot v\Big|_0^1 = ue^{-u} \quad (U = X+Y)$

refer to the front of the textbook,

comparing with continuous distributions of Gamma,

$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \quad 0 < x < \infty$

$M(t) = \frac{1}{(1-\theta t)^{\alpha}}, \quad t < \frac{1}{\theta}$

$\mu = \alpha\theta, \quad \sigma^2 = \alpha\theta^2$

we have the similar as Gamma,

Gamma $(\theta = 1, \alpha = 2)$

$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}$.

(c) (5 points) Identify the distribution of V.If it has a "named" distribution, you must state it. Otherwise support and pdf is enough.(HINT: You may refer to the front of the textbook with list of distributions.)

$$F(v) = \int_u F(u,v)\,du = \int_0^\infty ue^{-u}\,du = -(u+1)e^{-u}\Big|_0^\infty = 1$$

$$F(v) = 1 \quad 0 \le v \le 1 \qquad \left(V = \frac{X}{X+Y}\right)$$

refer to the front of the textbook,

comparing with continuous distributions of Uniform.

$$U(a,b) \quad f(x) = \frac{1}{b-a}, \quad a \le x \le b$$

$$-\infty < a < b < \infty \quad M(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}, \quad t \neq 0; \quad M(0) = 1$$

$$\mu = \frac{a+b}{2}, \quad \sigma^2 = \frac{(b-a)^2}{12}$$

we have the similar as Uniform

$$U(0,1)$$

$$f(x) = \frac{1}{1-0} = 1.$$

7. (15 points) Let $X \sim \text{Binomial}(30, 0.6)$.

   (a) (5 points) Using the Central Limit Theorem (CLT), approximate the probability that $P(X \geq 20)$, using continuity correction.

$X \sim \text{Binomial}(30, 0.6)$

$\mu = E(X) = 30 \times 0.6 = 18$ $\quad \sigma^2 = Var(X) = 30 \times 0.6 \times 0.4 = 7.2$

with cc. $P(X \geq 20) = P(X > 19.5)$

$$= P\left(\frac{X-18}{\sqrt{7.2}} > \frac{19.5-18}{\sqrt{7.2}}\right)$$

$$\overset{CLT}{\approx} P(Z > 0.56)$$

$$= 1 - P(Z \leq 0.56)$$

$$= 1 - \Phi(0.56)$$

$$= 1 - 0.71226$$

$$= 0.28774.$$

(b) (5 points) Using CLT, approximate the probability that $P(X = 18)$, using continuity correction.

$$P(X=18) = P(17.5 < X < 18.5)$$

$$= P\left(\frac{17.5-18}{\sqrt{7.2}} < \frac{X-18}{\sqrt{7.2}} < \frac{18.5-18}{\sqrt{7.2}}\right)$$

$$\overset{CLT}{\approx} P\left(\frac{-0.5}{\sqrt{7.2}} < Z < \frac{0.5}{\sqrt{7.2}}\right)$$

$$= \phi(0.186) - \phi(-0.186)$$

$$= 2\phi(0.186) - 1 = 2 \times 0.57377 - 1$$

$$= 0.14754.$$

(c) (5 points) Calculate $P(X = 18)$ exactly and compare to part(b).

$$P(X=18) = \binom{30}{18}(0.6)^{18}(1-0.6)^{30-18}$$

$$= \binom{30}{18}(0.6)^{18}(0.4)^{12}$$

$$= 86493225 \cdot 0.6^{18} \cdot 0.4^{12}$$

$$= 0.147375$$

8. (10 points) Recall the random variable X in Question 3. $S_X = [-6, 3]$ and $f(x) = x^2/81$ for $x \in S_X$.

   (a) (5 points) Using Chebyshev's Inequality, provide an upperbound to the probability

$$P(X^2 + \frac{15X}{2} + 14 > 8.9375).$$

we have $f(X) = \begin{cases} \frac{1}{81}X^2 & x \in [-6, 3] \\ 0 & \text{otherwise.} \end{cases}$

$$E(X) = \int_{-6}^{3} x f(x)\,dx = \int_{-6}^{3} \frac{1}{81}x^3\,dx = \frac{1}{81}\cdot\frac{x^4}{4}\Big|_{-6}^{3} = -\frac{15}{4}$$

$$E(X^2) = \int_{-6}^{3} x^2 f(x)\,dx = \int_{-6}^{3} \frac{1}{81}x^4\,dx = \frac{1}{81}\cdot\frac{x^5}{5}\Big|_{-6}^{3} = \frac{99}{5}$$

$$\sigma^2 = E(X^2) - E^2(X) = \frac{99}{5} - \left(-\frac{15}{4}\right)^2 = \frac{459}{80} = 5.7375.$$

$P\{|X - E(X)| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}$ by Chebyshev's Inequality.

$$P\left(X^2 + \frac{15}{2}X + 14 > 8.9375\right)$$

$$= P\left(X^2 + \frac{15}{2}X + \left(\frac{15}{4}\right)^2 - \left(\frac{15}{4}\right)^2 + 14 > \frac{143}{16}\right)$$

$$= P\left(\left(X + \frac{15}{4}\right)^2 > \frac{143}{16} + \frac{225}{16} - 14\right)$$

$$= P\{|X - E(X)|^2 > 3^2\}$$

$$= P\{|X - E(X)| > 3\} \leq \frac{5.7375}{9} = 0.6375.$$

(b) (5 points) Calculate this probability exactly and compare to the bound found in part (a). How informative is this bound?

From part a),

$$P\{|X+\frac{15}{4}|>3\} = P\{X+\frac{15}{4}<-3\} + P\{X+\frac{15}{4}>3\}$$

$$= P\{X<-6.75\} + P\{X>-0.75\}$$

$$= 0 + \int_{-0.75}^{3} \frac{x^2}{8!}\,dx$$

$$= \frac{1}{8!}\cdot\frac{x^3}{3}\Big|_{-0.75}^{3}$$

$$= \frac{65}{576}$$

$$\approx 0.11285$$