Math 170E
Summer 2022
Midterm
07/08/2022
Time Limit: 24 Hours

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This exam contains 10 pages (including this cover page) and 5 questions. Total of points is 100. Make sure to write your answers in full detail, so that you may get the maximum possible partial points when applicable.

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

- *You may use all the definitions and propositions stated during lectures and the textbook, unless it is otherwise stated and/or you are trying to prove the proposition itself.*

- Please type the following statement in your handwriting, then sign and date below:

  "I hereby acknowledge that I am aware I may use my textbook, lecture notes and recordings during the exam and swear on my honor as a Bruin *all* the answers I present belong solely to me, in thought and in writing."

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Jiayu Li 07/08/2022.

1. (20 points) (a) (10 points) Consider a 10-card poker hand. A special type of hand that has three denominations repeated three times and the last denomination repeated once is called a *chill house*. For example

King of Diamonds, King of Hearts, King of Spades, 5 of Clubs, 5 of Hearts, 5 of Spades, 2 of Clubs, 2 of Diamonds, 2 of Spades, Jack of Hearts

is a chill house.

What is the probability that in a randomly dealt hand, where all $\binom{52}{10}$ hands are equally likely, we get a chill house? (You can leave your answer in a form with binomial coefficients.)

$$P(\text{get a chill house}) = \frac{{}_{52}C_1 \cdot {}_{12}C_3 \cdot {}_4C_3 \cdot {}_4C_3 \cdot {}_4C_3}{{}_{52}C_{10}} = \frac{\binom{52}{1}\cdot\binom{12}{3}\cdot\binom{4}{3}\cdot\binom{4}{3}\cdot\binom{4}{3}}{\binom{52}{10}}$$

$$= \frac{\frac{52!}{1\cdot 51!}\cdot\frac{12!}{3!\,9!}\cdot\left(\frac{4!}{3!}\right)^3}{\frac{52!}{10!\,42!}}$$

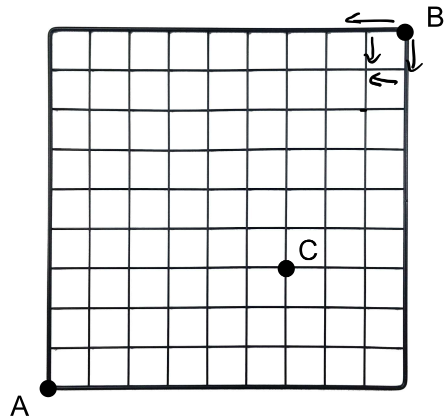
$$= \frac{52\cdot 220\cdot 4^3}{15,820,024,220}$$

$$= \frac{732,160}{15,820,026,220}$$

$$= 0.00004628$$

(b) (10 points) Consider the following 9×9 grid of cities. Assume that a travelling salesman named Tod Packer is to go from city B (top right corner) to city A (bottom left corner), while taking only either "left" (L) steps or "down" (D) steps, a total of 18 steps.

$$\underbrace{\underbrace{LLLL}_{9} \cdots \underbrace{DDDD}_{9}}_{18}$$



Assuming Tod chooses such a path at random (each possible path equally likely), what is the probability that he will go through city C?

(HINT: First, find the total number of paths. How many of each of the "L" and "D" steps does this person need to go from B to A? After that, restrict the paths to go through C. You can leave your answer in a form with binomial coefficients.)

$$\text{total number of paths} = \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot \cdots \cdot 10}{9!} = {}_{18}C_9 = \binom{18}{9}$$

$B \to C$ L 3   D 6 $\frac{9 \cdot 8 \cdot 7}{3!} = {}_9C_3$

$C \to A$ L 6   D 3 $\frac{9 \cdot 8 \cdot 7}{3!} = {}_9C_3$

$$\#\text{ go through } C \text{ of paths} = {}_9C_3 \cdot {}_9C_3$$

$$P(\text{go through } C) = \frac{{}_9C_3 \cdot {}_9C_3}{{}_{18}C_9} = \frac{\binom{9}{3} \cdot \binom{9}{3}}{\binom{18}{9}} = \frac{84 \cdot 84}{48620}$$

$$= 14.5\%.$$

2. (20 points) In the town of Goslar, police records show that 20% of all the crimes are violent and 80% of all the crimes are nonviolent. 90% of violent crimes are reported, whereas only 70% of nonviolent crimes are reported.

   (a) (10 points) What is the probability that a crime goes unreported?

Let R: the overall crime is reported

R^c: the overall crime is unreported.

and V: violent crimes, V^c: nonviolent crimes

$\Rightarrow P(V) = 20\%$, $P(V^c) = 80\%$

Then $P(R \mid V) = \dfrac{P(R \cap V)}{P(V)} = 90\%$

$P(R \mid V^c) = \dfrac{P(R \cap V^c)}{P(V^c)} = 70\%$

$$P(R) = P(R|V)P(V) + P(R|V^c)P(V^c)$$

So $P(R^c) = 1 - P(R)$

$= 1 - [P(R|V)P(V) + P(R|V^c)P(V^c)]$

$= 1 - [(0.9)\cdot(0.2) + (0.7)\cdot(0.8)]$

$= 1 - (0.18 + 0.56)$

$= 1 - 0.74$

$= 0.26$

(b) (10 points) Assume that a crime is reported. What is the probability that the crime is violent?

From part a), we know that the probability of ~~total~~ crime is reported is

$$P(R) = P(R|V)P(V) + P(R|V^c)P(V^c) = 0.74$$

So the probability that the crime is violent:

$$P(V|R) = \frac{P(V \cap R)}{P(R)}$$

$$= \frac{P(V)P(R|V)}{P(R|V)P(V) + P(R|V^c)P(V^c)}$$

$$= \frac{(0.2)\cdot(0.9)}{0.74}$$

$$= \frac{0.18}{0.74}$$

$$\approx 0.2432$$

$$= 24.32\%.$$

3. (20 points) We roll two fair and independent dice, d_1 and d_2. Let $X = \max(d_1, d_2)$, the maximum of these two dice.

   (a) (10 points) Let F be the cumulative distribution function of X. Write of F completely, as a piece-wise function, so that $F(x)$ is accounted for every $x \in \mathbb{R}$.

(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

we know that $X \in \{1, 2, 3, 4, 5, 6\}$

so CDF: $$F_X(x) = \begin{cases} 0, & \text{if } x < 1 \\ 1/36, & \text{if } 1 \le x < 2 \\ 1/12, & \text{if } 2 \le x < 3 \\ 5/36, & \text{if } 3 \le x < 4 \\ 7/36, & \text{if } 4 \le x < 5 \\ 1/4 & \text{if } 5 \le x < 6 \\ 11/36 & \text{if } x \ge 6 \end{cases}$$

(b) (10 points) Let $Y = \min(d_1, d_2)$. Are $E_a = \{X = a\}$ and $F_b = \{Y = b\}$ pairwise independent events? Either (i) find an example, a selection of a and b where these events are not independent, or (ii) show that no matter what a and b you may choose the events are independent.

$X = \max(a_1, a_2)$ $\quad$ $Y = \min(d_1, d_2)$

The following events are mutually independent:

$Y = 1$

$X = 1 \quad Y = 2, 3, 4, 5, 6$

$X = 2 \quad Y = 3, 4, 5, 6$

$X = 3 \quad Y = 4, 5, 6$

$X = 4 \quad Y = 5, 6$

$X = 5 \quad Y = 6$

$X = 6$

i) When we choose $a = 6$ or $b = 1$, these events are not independent.

Since the events listed above, so when we choose $a = 6$, $b = 1$. the case ii) does not exist.

4. (20 points) We are given the following probability distribution for x, the number of coffee breaks taken per day by coffee drinkers.

x	0	1	2	3	4	5
$f(x)$	0.27	0.38	0.16	0.12	0.05	0.02

(a) (10 points) Calculate the mean, μ, and variance, σ^2, for the number of coffee breaks per day.

suppose the random variable X has the space $S=\{0,1,2,3,4,5\}$

$$\mu=\sum_{x\in S} xf(x) = 0\cdot 0.27 + 1\cdot 0.38 + 2\cdot 0.16 + 3\cdot 0.12 + 4\cdot 0.05 + 5\cdot 0.02$$

$$= 1.36$$

$$\sigma^2 = Var(X) = \mathbb{E}[(X-\mu)^2]$$

$$= \sum_{x\in S}(x-\mu)^2 f(x)$$

$$= 0.27\cdot(0-1.36)^2 + 0.38\cdot(1-1.36)^2 + 0.16\cdot(2-1.36)^2 + 0.12\cdot(3-1.36)^2$$

$$+ 0.05\cdot(4-1.36)^2 + 0.02\cdot(5-1.36)^2$$

$$= 0.499392 + 0.049248 + 0.065536 + 0.322752$$
$$+ 0.34848 + 0.264992$$

$$= 1.5504$$

(b) (10 points) What's the probability that the number of coffee breaks falls within two standard deviations away from the mean, i.e., $(\mu - 2\sigma, \mu + 2\sigma)$?

From part a), $\sigma^2 = 1.5504$, $\mu = 1.36$

$\Rightarrow$ standard deviations

$$SD(X) = \sqrt{Var(X)} = \sigma = \sqrt{1.5504} = 1.245$$

$$(\mu - 2\sigma, \mu + 2\sigma) \Rightarrow (-1.13, 3.85)$$

since $X \in [0, 5]$ so we take $(0, 3.85)$

$$P(0 \sim 3.85) = P(0) + P(1) + P(2) + P(3) + P(3.85)$$

$$= 0.27 + 0.38 + 0.16 + 0.12 + 0.85 \cdot 5\%$$

$$= 0.93 + 0.0425$$

$$= 0.9725$$

$$= 97.25\%$$

5. (20 points) Let X be a Negative Binomial random variable with $\mathbb{E}[X] = 50$ and $Var(X) = 200$. Calculate $F(13) - F(11)$, where F is the CDF of X. (You can leave your answer in a form with binomial coefficients.)

$$\mathbb{E}[X] = \frac{r}{p} \qquad \Rightarrow \begin{cases} 50 = \frac{r}{p} \\ 200 = \frac{r(1-p)}{p^2} \end{cases}$$

$$Var(X) = \frac{r(1-p)}{p^2}$$

$$\Rightarrow \begin{cases} p = \frac{1}{5} \\ r = 10 \end{cases}$$

And $F(X) = \binom{x-1}{r-1} \cdot p^r \cdot (1-p)^{x-r}$

we have $F(13) = \binom{12}{9} \cdot \left(\frac{1}{5}\right)^{10} \left(\frac{4}{5}\right)^3$

$$F(11) = \binom{10}{9} \cdot \left(\frac{1}{5}\right)^{10} \left(\frac{4}{5}\right)^1$$

$$F(13) - F(11) = \binom{12}{9} \cdot \left(\frac{1}{5}\right)^{10} \left(\frac{4}{5}\right)^3 - \binom{10}{9} \cdot \left(\frac{1}{5}\right)^{10} \left(\frac{4}{5}\right)^1$$