Last time: $X \sim \mathcal{N}(\mu, \sigma^2)$ σ known

Null hypothesis $\mu = \mu_0$

Alternative hypothesis $\mu = \mu_1$

Say $\mu_1 > \mu_0$ choose $\mu_* \in (\mu_0, \mu_1)$ → rejection threshold.

$\mu_0 \quad \mu_* \quad \mu_1$

Test: reject H_0 in favor of H_1 if $\bar{X} \geq \mu_*$

otherwise accept (don't reject H_0)

significance level : critical region

$$\alpha = \mathbb{P}(\underbrace{\bar{X} \geq \mu_*, H_0}_{\text{type I error}}) \quad \text{for } \mu_* = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

$$\varepsilon = z_\alpha \frac{\sigma}{\sqrt{n}}$$

Larger $\bar{X}$ favor H_1 over H_0

since $\mu_1 > \mu_0$

Variant : $\mu_1 < \mu_0$

$\mu_1 \quad \mu_* \quad \mu_0$

$\bar{X} \qquad \bar{X}$

smaller $\bar{X}$ favor H_1 over H_0

since $\mu_1 < \mu_0$

$-\mu_0 \quad -\mu_* \quad -\mu_1$

Test: reject H_0 in favor of H_1 if $\boxed{\bar{X} \leq \mu_*} = \mu_0 - \varepsilon$

otherwise accept (don't reject) H_0

$$\alpha = \mathbb{P}(\underbrace{\bar{X} - \mu_0 \leq -\varepsilon \,;\, H_0}_{\text{type I error}})$$

$$\mu_* = \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}$$

$$\varepsilon = z_\alpha \frac{\sigma}{\sqrt{n}}$$

$$= \mathbb{P}(\underbrace{\bar{X} \leq \mu_*}_{\text{critical region}} ; H_1)$$

$$= \mathbb{P}(-\bar{X} \geq -\mu_* ; H_0) = \alpha$$

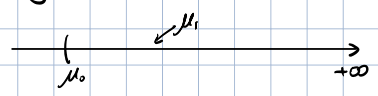
when $-\mu_* = -\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$, $\mu_* = \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}$

Composite Alternative Hypothesis ①

Null : $H_0 : \mu = \mu_0$

Alternative : $H_1 : \mu > \mu_0$

– May be viewed as $\{\mu = \mu_1 : \mu_1 \in (\mu_0, +\infty)\}$



Choose $\varepsilon > 0$.

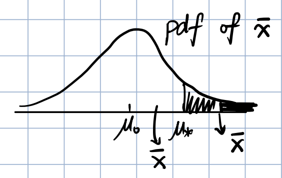
Test: reject H_0 in favor of H_1 if $\bar{x} \geq \mu_0 + \varepsilon = \mu_*$

$\bar{x} - \mu_0 \geq \varepsilon$

significance level

$\alpha = \mathbb{P}(\bar{X} \geq \mu_0 + \varepsilon \,;\, H_0)$ for $\varepsilon = z_\alpha \frac{\sigma}{\sqrt{n}}$

$= \mathbb{P}(\bar{X} - \mu_0 \geq \varepsilon \,;\, H_0)$



- Note $\bar{x} \geq \mu_*$ (critical region = sample outcomes for which Reject)

$\Longleftrightarrow \underbrace{\mathbb{P}(\bar{X} \geq \bar{x} \,;\, H_0)}_{\text{p-value of } \bar{x}} \leq \mathbb{P}(\bar{X} \geq \mu_* \,;\, H_0) = \alpha$

"probability getting a result that's as extreme as the given $\bar{x}$" under the null H_0.

Reformulate the test as:

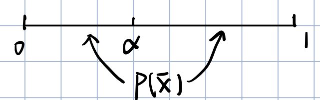
Reject H_0 if (p-value of $\bar{x}$) $\leq \alpha$

- We can view the p-value as a stastic of the sample

$X_1, \dots, X_n$ → $\boxed{\text{experiment}}$ → $x_1, x_2, \dots, x_n$

$\bar{X}$ $\qquad$ $\bar{x}$

p-value of $\bar{X} = P(\bar{X})$ $\qquad$ p-value of $\bar{x}$



$\mathbb{P}(p(\bar{X}) \leq \alpha \,;\, H_0) = \mathbb{P}(\text{Reject } H_0 \,;\, H_0)$

$= \mathbb{P}(\text{type I error}) = \alpha$

- $P(\bar{X}) \sim \text{unif}(0,1)$ under H_0

If H_0 is true, p-value is uniformly distributed in $(0,1)$

$\mathbb{P}(\text{p-value} \leq 0.05) = 0.05 = 5\%$

– If H_0 is true, then 5% of experiments will reject H_0 for significance level 0.05

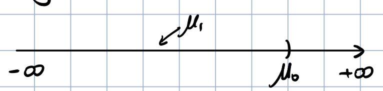
1% of experiments will reject H_0 for significance level 0.01

Composite Alternative Hypothesis ②

Null : $H_0 : \mu = \mu_0$

Alternative : $H_1 : \mu < \mu_0$

— May be viewed as $\{\mu = \mu_1 : \mu_1 \in (-\infty, \mu_0)\}$



choose $\varepsilon > 0$.

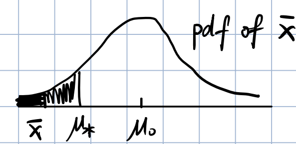
Test: reject H_0 in favor of H_1 if $\bar{X} < \mu_0 + \varepsilon = \mu_*$

$\bar{X} - \mu_0 < -\varepsilon$

significance level

$\alpha = \mathbb{P}(\bar{X} \le \mu_0 + \varepsilon ; H_0)$ for $\varepsilon = z_\alpha \frac{\sigma}{\sqrt{n}}$

$= \mathbb{P}(\bar{X} - \mu_0 \le -\varepsilon ; H_0)$



- Note $\bar{x} \le \mu_*$ (critical region = sample outcomes for which Reject)

  $\Leftrightarrow \underbrace{\mathbb{P}(\bar{X} \le \bar{x} ; H_0)}_{\text{P-value of } \bar{x}} \le \mathbb{P}(\bar{X} \le \mu_* ; H_0) = \alpha$

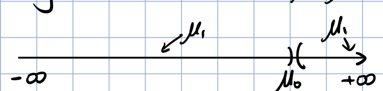
2-sided

Composite Alternative Hypothesis ③

Null : $H_0 : \mu = \mu_0$

Alternative : $H_1 : \mu \neq \mu_0$

— May be viewed as $\{\mu = \mu_1 : \mu_1 \in (-\infty, \mu_0) \cup (\mu_0, +\infty)\}$

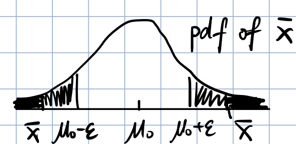


choose $\varepsilon > 0$.

Test: reject H_0 in favor of H_1 if $|\bar{X} - \mu_0| \geq \varepsilon$

significance level

$\alpha = \mathbb{P}(|\bar{X} - \mu_0| \geq \varepsilon ; H_0)$ for $\varepsilon = z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$



- Note $\bar{x} \le \mu_*$ (critical region = sample outcomes for which Reject)

  $\Leftrightarrow \underbrace{\mathbb{P}(|\bar{X} - \mu_0| \geq |\bar{x} - \mu_0| ; H_0)}_{\text{P-value of } \bar{x}} \le \mathbb{P}(|\bar{X} - \mu_0| \geq \varepsilon ; H_0) = \alpha$

Test against (2-sided) composite alternative		(2-sided) confidence interval for μ. $[\bar{x}-\varepsilon, \bar{x}+\varepsilon]$
$\mu \neq \mu_0$: reject if $\lvert\bar{x}-\mu_0\rvert \geq \varepsilon$ $\alpha = \mathbb{P}(\lvert\bar{x}-\mu_0\rvert \geq \varepsilon ; \mu = \mu_0)$	$\varepsilon = z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$	$\mathbb{P}(\mu \in [\bar{x}-\varepsilon, \bar{x}+\varepsilon]) = 1-\alpha$ $= \mathbb{P}(\lvert\bar{x}-\mu\rvert \leq \varepsilon) = 1-\alpha$
$\mu > \mu_0$: reject if $\bar{x}-\mu_0 \geq \varepsilon$ $\alpha = \mathbb{P}(\bar{x}-\mu_0 \geq \varepsilon ; \mu = \mu_0)$	$\varepsilon = z_{\alpha} \frac{\sigma}{\sqrt{n}}$	$\mathbb{P}(\bar{x}-\mu \leq \varepsilon) = 1-\alpha$ $\quad \bar{x}-\varepsilon \leq \mu \quad [\bar{x}-\varepsilon, +\infty)$
$\mu < \mu_0$: reject if $\bar{x}-\mu_0 \leq -\varepsilon$ $\alpha = \mathbb{P}(\bar{x}-\mu_0 \leq -\varepsilon ; \mu = \mu_0)$	$\varepsilon = z_{\alpha} \frac{\sigma}{\sqrt{n}}$	$\mathbb{P}(\bar{x}-\mu \geq -\varepsilon) = 1-\alpha$ $\quad \bar{x}+\varepsilon \geq \mu \quad (-\infty, \bar{x}+\varepsilon]$

Population variance σ^2 unknown,

Replace with $\varepsilon = t_{\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}}$ 2-sided $\mu \neq \mu_0$

s^2 = sample var

$t_{\frac{\alpha}{2}}(n-1)$ t-distribution n-1 degree of free.

$\varepsilon = t_{\alpha}(n-1)\frac{s}{\sqrt{n}}$

1-sided

$\mu > \mu_0$, $\mu < \mu_0$

11/09 Wed.

Apply to test of the equality of 2 means

Set-up: X, Y RV's, mean μ_X, μ_Y, variance σ_X^2, σ_Y^2.

Null hypothesis: $H_0: \mu_X = \mu_Y \quad \mu_X - \mu_Y = 0$

Alternative hypothesis: $H_1: \mu_X \neq \mu_Y \quad \mu_X - \mu_Y \neq 0$

Test statistic $\bar{X} - \bar{Y}$ difference of sample means.

Choose $\varepsilon > 0$ "rejection threshold"

Test: Reject H_0 in favor of H_1 if $|\bar{X} - \bar{Y}| \geq \varepsilon$.
otherwise accept (don't reject) H_0.

Significance level

$\alpha = \mathbb{P}(\text{type I error})$

$= \mathbb{P}(\text{reject } H_0 ; H_0)$

$= \mathbb{P}(\underbrace{|\bar{X} - \bar{Y}| \geq \varepsilon}_{\text{critical region}} ; \mu_X = \mu_Y)$

"probability of getting result as extreme as the rejection threshold ε decided beforehand"

P-value for $\bar{x} - \bar{y}$

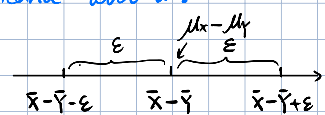
$\mathbb{P}(|\bar{X} - \bar{Y}| \geq |\bar{x} - \bar{y}| ; H_0)$

this is a statistic of the sample.

"probability of getting result as extreme as the observed difference $\bar{x} - \bar{y}$"

$\boxed{\text{Test: reject } H_0 \text{ if } (\text{p-value of } \bar{x} - \bar{y}) \leq \alpha}$

? How should ε be chosen to achieve a given significance level α?



$\alpha = \mathbb{P}(|\bar{X} - \bar{Y}| \geq \varepsilon ; H_0)$

$1 - \alpha = 1 - \mathbb{P}(|\bar{X} - \bar{Y}| \geq \varepsilon ; H_0)$

$= \mathbb{P}(|\bar{X} - \bar{Y}| \leq \varepsilon ; \mu_X - \mu_Y = 0)$

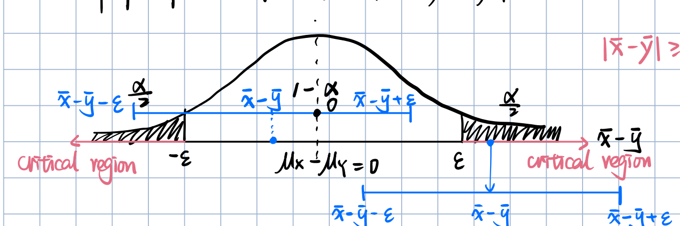
$= \mathbb{P}(|(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)| \leq \varepsilon ; \mu_X - \mu_Y = 0)$

$= \mathbb{P}(\mu_X - \mu_Y \in \underbrace{[\bar{X} - \bar{Y} - \varepsilon, \bar{X} - \bar{Y} + \varepsilon]}_{\text{confidence interval w/ confidence level } 1-\alpha} ; \mu_X - \mu_Y = 0)$

We should choose ε to be the size ε of a $100(1-\alpha)\%$ CI for $\mu_X - \mu_Y$.

pdf of $\bar{X} - \bar{Y}$ under $H_0: \mu_X = \mu_Y$

$\mathbb{E}(\bar{X} - \bar{Y}) = \mathbb{E}\bar{X} - \mathbb{E}\bar{Y} = \mu_X - \mu_Y = 0$



$|\bar{x} - \bar{y}| \geq \varepsilon$ iff $\boxed{0 \in [\bar{x} - \bar{y} - \varepsilon, \bar{x} - \bar{y} + \varepsilon]}$

$\boxed{\begin{array}{l}\text{Test: reject } H_0: \mu_X = \mu_Y \\ \quad \text{in favor of } H_1: \mu_X \neq \mu_Y \\ \quad \text{if the } 100(1-\alpha)\% \text{ CI for } \mu_X - \mu_Y \\ \quad \text{does not contain } 0\end{array}}$

Example: pair of RV (X, Y) unknown distribution

Know: $D = X - Y \sim N(\mu_D, \sigma_D^2)$, σ_D unknown

10 samples $(2,3)$, $(2,4)$, $(5,1)$, $(2,1)$, $(3,4)$

$(1,0)$, $(4,2)$, $(2,2)$, $(4,1)$, $(1,2)$.

Test $H_0: \mu_D = 0$ $\quad (\mu_X - \mu_Y = 0)$

against $H_1: \mu_D \neq 0$ $\quad (\mu_X - \mu_Y \neq 0)$ at significance level $\alpha = 0.05$

Sol 1 Use definition

Test statistic $\bar{D} = \frac{1}{n}(D_1 + \cdots + D_n)$

choose $\varepsilon > 0$, rejection threshold

"Reject H_0 if $|\bar{D}| \geq \varepsilon$"

$\alpha = \mathbb{P}(\text{reject } H_0; H_0)$

$= \mathbb{P}(|\bar{D}| \geq \varepsilon; \mu_D = 0)$

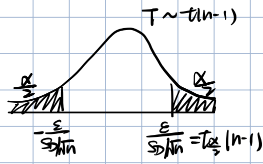
know: $T = \dfrac{\bar{D} - \mu_D}{S_D/\sqrt{n}} \sim t(n-1)$

S_D^2 is sample variance for D.

under H_0, $\mu_D = 0$.

$\alpha = \mathbb{P}\left(\left|\frac{\bar{D}}{S_D/\sqrt{n}}\right| \geq \frac{\varepsilon}{S_D/\sqrt{n}}; \mu_D = 0\right)$

$= \mathbb{P}\left(|T| \geq \frac{\varepsilon}{S_D/\sqrt{n}}\right)$



so $\dfrac{\varepsilon}{S_D/\sqrt{n}} = t_{\frac{\alpha}{2}}(n-1)$

$\varepsilon = t_{\frac{\alpha}{2}}(n-1)\dfrac{S_D}{\sqrt{n}}$

$\boxed{\text{Test: Reject } H_0 \text{ if } |\bar{D}| \geq \varepsilon = t_{\frac{\alpha}{2}}(n-1)\frac{S_D}{\sqrt{n}}}$

"Paired t-test"

$d_i = X_i - Y_i$

$-1, -2, 4, 1, -1, 1, 2, 0, 3, -1$

$\bar{d} = 0.6$

$V_D = \mathbb{E}_{\hat{p}}\hat{X}^2 - (\mathbb{E}_{\hat{p}}\hat{X})^2 = \frac{1}{10}(1^2 + 2^2 + \cdots + 3^2 + 1^2) - \bar{d}^2 = 3.8 - 0.6^2 = 3.44$

Variance of emperical distribution for D.

$S_D^2 = \frac{n}{n-1}V_D = \frac{10}{9} \times 3.44 \approx 3.822$, $\quad S_D = 1.955$

For $\alpha = 0.05$, $n = 10$ $\quad t_{\frac{\alpha}{2}}(n-1) = 2.262$, $\quad \frac{S_D}{\sqrt{n}} = 0.618$.

Reject H_0 if $|\bar{d}| \geq \varepsilon = 2.262 \times 0.618 = 1.398$

Since $|\bar{d}| = 0.6 < 1.398$ Accept H_0, doesn't reject

Sol 2 Compute p-value of $\bar{d}$

$\mathbb{P}(|\bar{D}| > |\bar{d}|; H_0) = \mathbb{P}(|\bar{D}| > 0.6; \mu_D = 0)$

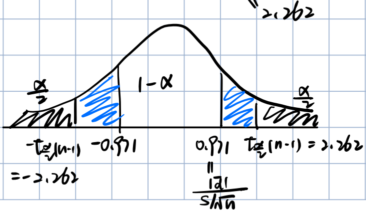
$= \mathbb{P}\left(\left|\frac{\bar{D}}{S/\sqrt{n}}\right| > \frac{0.6}{S/\sqrt{n}}; \mu_D = 0\right)$

$= \mathbb{P}\left(|T| > \frac{0.6}{0.618}\right)$

$= \mathbb{P}(|T| > 0.971)$

Test: Reject H_0 if p-value $\leq \alpha$

$\alpha = \mathbb{P}(|T| > t_{\frac{\alpha}{2}}(n-1))$, $\quad t_{\frac{\alpha}{2}}(n-1) = 2.262$



Since $0.971 < 2.262$

$\mathbb{P}(T > 0.971) > \mathbb{P}(T > 2.262) = \alpha = 0.05$

we accept H_0 (don't reject)

$\boxed{\text{Reject } \frac{|\bar{d}|}{S/\sqrt{n}} < t_{\frac{\alpha}{2}}(n-1) \quad |\bar{d}| < t_{\frac{\alpha}{2}}(n-1)\cdot\frac{S}{\sqrt{n}} = \varepsilon.}$

Sol 3 Compute 95% CI for μ_D.

Recall: that's $[\bar{D}-\varepsilon, \bar{D}+\varepsilon]$, $\varepsilon = t_{\frac{\alpha}{2}}(n-1) \cdot \frac{S}{\sqrt{n}} = 1.398$

So it's $[0.6 - 1.398, 0.6 + 1.398] = [-0.798, 1.998]$

Since it contains 0, accept H_0 (doesn't reject).