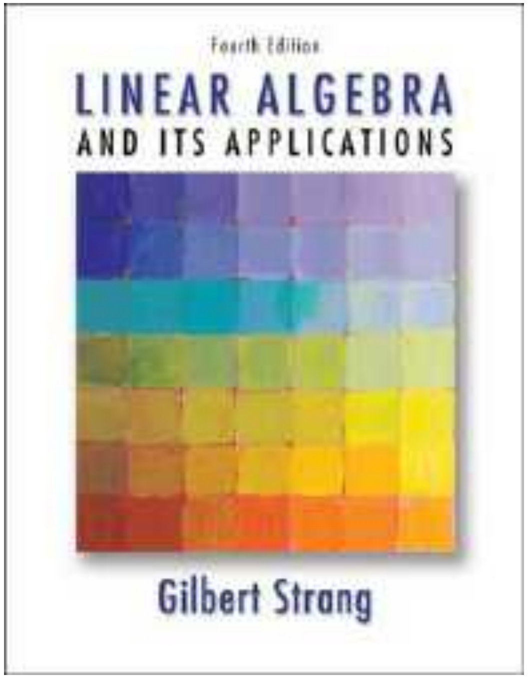
Linear Algebra



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2 Vector Spaces （向量空间）

2.4

THE FOUR FUNDAMENTAL SUBSPACES （四个基本子空间）

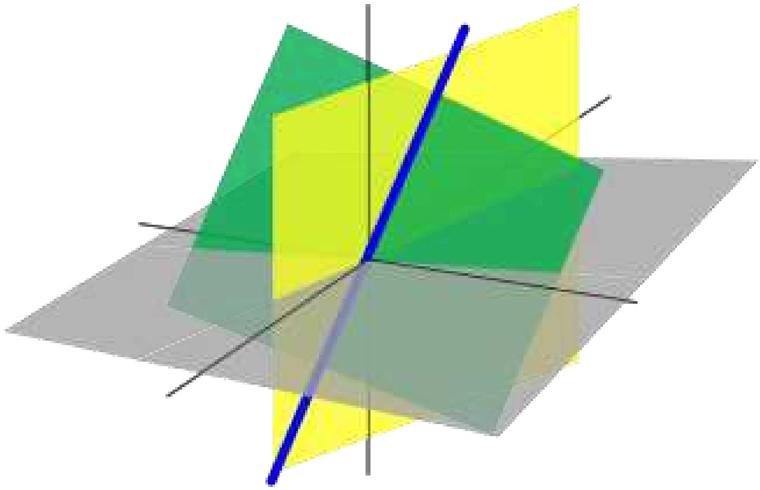
Column space

Row space

Nullspace

Left nullspace

Full Rank and Rank 1



The previous sections dealt with ***definitions***.

vector spaces
subspaces
spanned subspaces
spanning set
column spaces
nullspaces
free variables
basic / pivot variables
special solutions
particular solution
complete solution
rank
linearly dependent
linearly independent
basis
dimension
coordinate

Introduction

Subspaces can be described in *two ways*:

1. We may be given a set of vectors that span the space.
   - *Example*: The columns span the column space C(***A***).
   - may include useless vectors (dependent columns)
2. We may be told which conditions the vectors in the space must satisfy.
   - *Example*: The nullspace N(***A***) consists of all vectors that satisfy $\boldsymbol{Ax} = \boldsymbol{0}$.
   - may include repeated conditions (dependent rows)

We know what a basis is, but *how to find one*?

We need a systematic procedure to compute an explicit basis (基的显示表达).

THE FOUR FUNDAMENTAL SUBSPACES

Let $\boldsymbol{A}$ be an $(m\times n)$-matrix.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

There are 4 vector spaces related to $\boldsymbol{A}$:

C($\boldsymbol{A}$): *column space* of $\boldsymbol{A}$

列空间

C($\boldsymbol{A}^{\mathrm{T}}$): *row space* of $\boldsymbol{A}$

行空间

N($\boldsymbol{A}$): *nullspace* of $\boldsymbol{A}$

零空间

N($\boldsymbol{A}^{\mathrm{T}}$): *left nullspace* of $\boldsymbol{A}$

左零空间

Some equivalent notations and definitions:

$C(\boldsymbol{A})$ or $\mathrm{Col}(\boldsymbol{A})$: column space of $\boldsymbol{A}$

the subspace spanned by the columns of $\boldsymbol{A}$

$C(\boldsymbol{A}^{\mathrm{T}})$ or $R(\boldsymbol{A})$: row space of $\boldsymbol{A}$

the subspace spanned by the rows of $\boldsymbol{A}$

It is the column space of $\boldsymbol{A}^{\mathrm{T}}$.

$N(\boldsymbol{A})$ or $\mathrm{null}(\boldsymbol{A})$: nullspace of $\boldsymbol{A}$

i.e., $\{\boldsymbol{x}: \boldsymbol{x}\in\mathbf{R}^n, \boldsymbol{Ax} = \boldsymbol{0}\}$.

$N(\boldsymbol{A}^{\mathrm{T}})$ or $\mathrm{Lnull}(\boldsymbol{A})$: left nullspace of $\boldsymbol{A}$

It contains all vectors $\boldsymbol{y}\in\mathbf{R}^m$ such that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$, so it is the *nullspace* of $\boldsymbol{A}^{\mathrm{T}}$, i.e., $\{\boldsymbol{y}: \boldsymbol{y}\in\mathbf{R}^m, \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}\}$.

It can also be written as $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{0}$, so it is called the *left nullspace* of $\boldsymbol{A}$.

The nullspace $N(\boldsymbol{A})$ and row space $C(\boldsymbol{A}^{\mathrm{T}})$ are subspaces of $\mathbf{R}^n$.
The left nullspace $N(\boldsymbol{A}^{\mathrm{T}})$ and column space $C(\boldsymbol{A})$ are subspaces of $\mathbf{R}^m$.

Example 1 For a simple matrix like

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

The column space C($\boldsymbol{A}$) is the line through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and it is in $\mathbf{R}^2$.

The row space C($\boldsymbol{A}^{\mathrm{T}}$) is the line through $[1 \;\; 0 \;\; 0]^{\mathrm{T}}$, and it is in $\mathbf{R}^3$.

The nullspace is a plane in $\mathbf{R}^3$, N($\boldsymbol{A}$) contains $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

The left nullspace is a line in $\mathbf{R}^2$, N($\boldsymbol{A}^{\mathrm{T}}$) contains $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Remark: Note that all vectors here are expressed as column vectors. (Even the rows are transposed, and the row space of $\boldsymbol{A}$ is the *column* space of $\boldsymbol{A}^{\mathrm{T}}$.)

Example 2 For a non-zero 2×4 matrix

$$\boldsymbol{A}=\begin{bmatrix}0 & 1 & 2 & 3\\ 1 & 2 & 3 & 4\end{bmatrix},$$

Column space: $C(\boldsymbol{A})=\operatorname{Span}\left\{\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}3\\4\end{bmatrix}\right\}\subseteq\mathbf{R}^2,$

Row space: $C(\boldsymbol{A}^{\mathrm{T}})=\operatorname{Span}\left\{(0,1,2,3)^{\mathrm{T}},(1,2,3,4)^{\mathrm{T}}\right\}\subseteq\mathbf{R}^4.$

Obviously, both subspaces are of dimension equal to 2.

To find a basis of $N(\boldsymbol{A})$, we convert $\boldsymbol{A}$ into reduced row echelon form

$$\begin{bmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 3\end{bmatrix},$$

So $\boldsymbol{Ax}=\boldsymbol{0}$ has two free variables, say $x_3 = s$ and $x_4 = t$.

Then a general solution is of the form $(s + 2t, -2s-3t, s, t)^{\mathrm{T}}$, where $s, t \in \mathbf{R}$.

Thus $(1, -2, 1, 0)^{\mathrm{T}}$ and $(2, -3, 0, 1)^{\mathrm{T}}$ form a basis of $N(\boldsymbol{A})$.

Example 2 For a non-zero 2×4 matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix},$$

(Continued)
To obtain a basis of N($\boldsymbol{A}^{\mathrm{T}}$), we convert $\boldsymbol{A}^{\mathrm{T}}$ into row echelon form

$$\boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \quad \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

It follows that the left nullspace N($\boldsymbol{A}^{\mathrm{T}}$) only contains $\boldsymbol{0}$ (zero vector).

I. Row Spaces

Let $\boldsymbol{A}$ be a matrix, and $\boldsymbol{U}$ the row echelon form of $\boldsymbol{A}$. Then

the row spaces of $\boldsymbol{A}$ *and* $\boldsymbol{U}$ *are the same, i.e.,* $\mathrm{R}(\boldsymbol{A}) = \mathrm{R}(\boldsymbol{U})$.

This is because each row of $\boldsymbol{U}$ is a linear combination of the rows of $\boldsymbol{A}$.
On the other hand, since each elementary row operation is reversible, each row of $\boldsymbol{A}$ can be written as a linear combination of the rows of $\boldsymbol{U}$.

Thus the rows of $\boldsymbol{A}$ and $\boldsymbol{U}$ span the same vector space.

This leads to a method for finding a basis of the row space of a matrix $\boldsymbol{A}$, i.e., converting $\boldsymbol{A}$ into row echelon form $\boldsymbol{U}$ so that the non-zero rows of $\boldsymbol{U}$ form a basis of $\mathrm{R}(\boldsymbol{A})$, and the row space has dimension r (rank of $\boldsymbol{A}$ and $\boldsymbol{U}$).

For example,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \rightarrow \boldsymbol{U} = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

II. Nullspaces

Given an $(m \times n)$-matrix $\boldsymbol{A}$ with row echelon form $\boldsymbol{U}$ and reduced row echelon form $\boldsymbol{R}$, the system of linear equations $\boldsymbol{Ax} = \boldsymbol{0}$ has the same solutions with the system $\boldsymbol{Ux} = \boldsymbol{0}$ and $\boldsymbol{Rx} = \boldsymbol{0}$.
Thus the nullspace of $\boldsymbol{A}$ is the same as the nullspace of $\boldsymbol{U}$ (and $\boldsymbol{R}$).
Let r be the rank of $\boldsymbol{A}$, i.e., the number of non-zero rows of $\boldsymbol{U}$.
Then there are $n - r$ free variables in the system $\boldsymbol{Ux} = \boldsymbol{0}$, and the nullspace of $\boldsymbol{U}$ is of dimension $n - r$. We therefore obtain

Theorem 1 (*rank-nullity theorem*) *Let $\boldsymbol{A}$ be an $(m \times n)$-matrix. Then*

$$\text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) = n.$$

Similarly,

$$\text{rank}(\boldsymbol{A}^{\mathrm{T}}) + \text{nullity}(\boldsymbol{A}^{\mathrm{T}}) = m.$$

The nullspace is also called the *kernel* (核) of $\boldsymbol{A}$,
and its dimension $n - r$ is the *nullity* (零度) of $\boldsymbol{A}$.
nullity($\boldsymbol{A}$)=dimension(N($\boldsymbol{A}$)).

For example

$$\boldsymbol{A}=\begin{bmatrix}1 & 3 & 3 & 2\\ 2 & 6 & 9 & 7\\ -1 & -3 & 3 & 4\end{bmatrix}\rightarrow \boldsymbol{U}=\begin{bmatrix}1 & 3 & 3 & 2\\ 0 & 0 & 3 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\rightarrow \boldsymbol{R}=\begin{bmatrix}1 & 3 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

The "special solutions" are a basis—each free variable is given the value 1, while the other free variables are 0.

Free variables: x_2, x_4

Special solutions: $\begin{bmatrix}-3\\ 1\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}1\\ 0\\ -1\\ 1\end{bmatrix}$

are definitely independent.

$n = 4$

$r = 2$

The dimension of N($\boldsymbol{A}$) = $n - r$ =2

(nullity)

III. Column Spaces

We notice that, in Example 2, $\boldsymbol{A}=\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$,

$C(\boldsymbol{A}^{\mathrm{T}})$ and $C(\boldsymbol{A})$ have the same dimension.

Is it always true that row space and column space have the same dimension?

Example 3 A non-zero $2\times n$ matrix $\boldsymbol{A}=\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{bmatrix}$,

has row space and column space:

$C(\boldsymbol{A}^{\mathrm{T}}) = \mathrm{span}\{(a_1, a_2, \ldots, a_n)^{\mathrm{T}}, (b_1, b_2, \ldots, b_n)^{\mathrm{T}}\} \subseteq \mathbf{R}^n$,

$C(\boldsymbol{A}) = \mathrm{span}\{(a_1, b_1)^{\mathrm{T}}, (a_2, b_2)^{\mathrm{T}}, \ldots, (a_n, b_n)^{\mathrm{T}}\} \subseteq \mathbf{R}^2$.

Obviously, both are of dimension at most 2.

If $C(\boldsymbol{A}^T)$ is of dimension 1, then

$$(b_1, b_2, \ldots, b_n) = k(a_1, a_2, \ldots, a_n),$$

so $b_i = ka_i$ for all i.

Suppose $a_1 \neq 0$. Then for $a_i \neq 0$, we have

$$\frac{b_i}{a_i} = k = \frac{b_1}{a_1} \quad \text{and} \quad \frac{b_i}{b_1} = \frac{a_i}{a_1} = k_i$$

Thus, $(a_i, b_i) = k_i(a_1, b_1)$ for all i,

and hence $C(\boldsymbol{A})$ is of dimension 1.

Similarly, if $C(\boldsymbol{A})$ is of dimension 1, then $C(\boldsymbol{A}^T)$ is also of dimension 1.

So the row space and the column space of $\boldsymbol{A}$ have the same dimension.

This is actually true in the general case.

Theorem 2 *For any matrix, its row space and column space have the same dimension, which equals the rank of the matrix.*

For example,

$$\boldsymbol{A}=\begin{bmatrix}1 & 3 & 3 & 2\\ 2 & 6 & 9 & 7\\ -1 & -3 & 3 & 4\end{bmatrix}\to \boldsymbol{U}=\begin{bmatrix}1 & 3 & 3 & 2\\ 0 & 0 & 3 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}\to \boldsymbol{R}=\begin{bmatrix}1 & 3 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Note: the column spaces of $\boldsymbol{U}$ and $\boldsymbol{A}$ *are different* (just look at the matrices!)

but their dimensions are the same.

The first and third columns of $\boldsymbol{U}$ are a basis for its column space. They are the columns with pivots.

Furthermore, the same is true of the original $\boldsymbol{A}$—even though its columns are different.

The pivot columns of $\boldsymbol{A}$ are a basis for its column space.

Every linear dependence $\boldsymbol{Ax} = \boldsymbol{0}$ among the columns of $\boldsymbol{A}$ is matched by a dependence $\boldsymbol{Ux} = \boldsymbol{0}$ ($\boldsymbol{Rx} = \boldsymbol{0}$) among the columns of $\boldsymbol{U}$ ($\boldsymbol{R}$), with exactly the same coefficients. (行变换不改变列之间的相关性)
If a set of columns of $\boldsymbol{A}$ *is independent, then so are the corresponding columns of* $\boldsymbol{U}$ (and $\boldsymbol{R}$), *and vice versa*.

Theorem 2 is one of the most important theorems in linear algebra. It is often abbreviated as

"***row rank = column rank***." (行秩=列秩)

For example,

$$U = \begin{bmatrix} d_1 & * & * & * & * & * \\ 0 & 0 & 0 & d_2 & * & * \\ 0 & 0 & 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Given an $m \times n$ matrix $\boldsymbol{A}$ with row echelon form $\boldsymbol{U}$,

Dimension of $\mathrm{C}(\boldsymbol{U})$ $=$ Dimension of $\mathrm{C}(\boldsymbol{U}^{\mathrm{T}})$ $=$ rank$(\boldsymbol{U})$

ERO's do not change the linear dependence relations among the columns.

$\mathrm{C}(\boldsymbol{U}^{\mathrm{T}})$
||
$\mathrm{C}(\boldsymbol{A}^{\mathrm{T}})$

By definition

Dimension of $\mathrm{C}(\boldsymbol{A})$ $=$ Dimension of $\mathrm{C}(\boldsymbol{A}^{\mathrm{T}})$ $=$ rank$(\boldsymbol{A})$

Theorem. For any matrix, its row space and column space have the same dimension, which equals the rank of the matrix.

row rank= column rank

IV. Left Nullspaces

In Theorem 1 *Let* $\boldsymbol{A}$ *be an* $(m \times n)$*-matrix. Then*

$$\operatorname{rank}(\boldsymbol{A}) + \operatorname{nullity}(\boldsymbol{A}) = n.$$

Similarly,

$$\operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}) + \operatorname{nullity}(\boldsymbol{A}^{\mathrm{T}}) = m.$$

Now that $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}^{\mathrm{T}})$ (Theorem 2), we have

$$\operatorname{nullity}(\boldsymbol{A}^{\mathrm{T}}) = \operatorname{dimension}(\mathrm{N}(\boldsymbol{A}^{\mathrm{T}})) = m - r\ .$$

$\boldsymbol{C}(A) =$ column space of A; dimension r.

$\boldsymbol{N}(A) =$ nullspace of A; dimension $n - r$.

$\boldsymbol{C}(A^{\mathrm{T}}) =$ row space of A; dimension r.

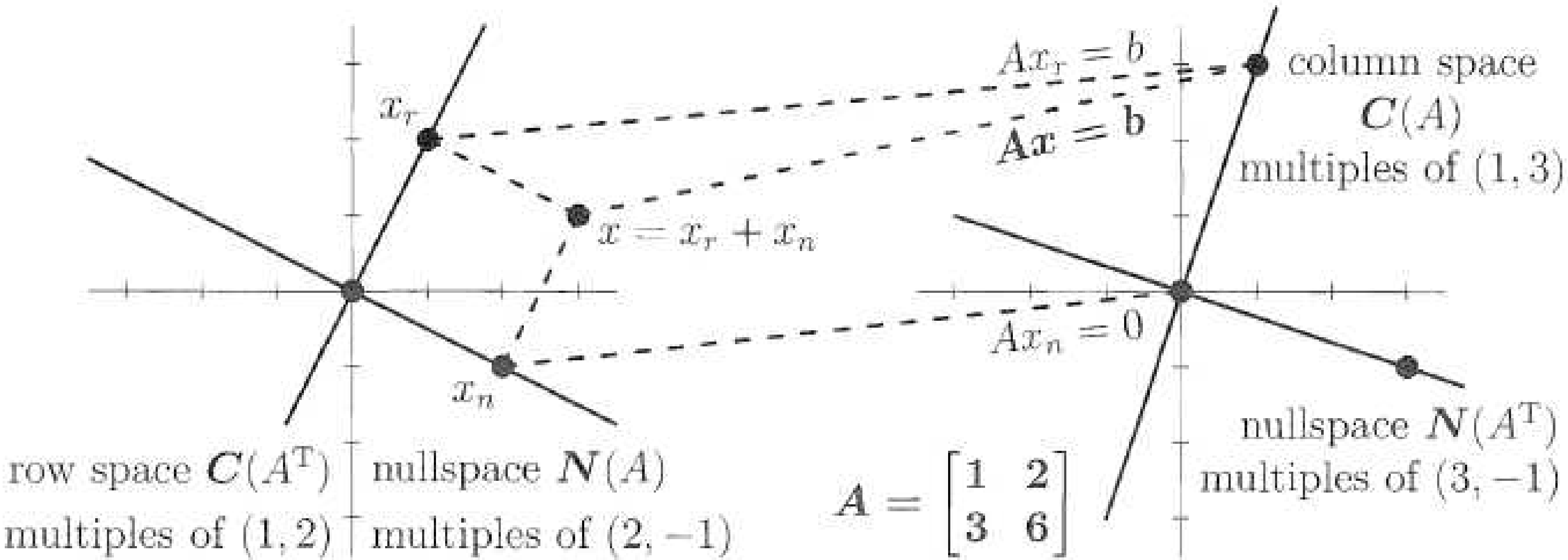
$\boldsymbol{N}(A^{\mathrm{T}}) =$ left nullspace of A; dimension $m - r$.

Fundamental Theorem of Linear Algebra (Part I)

Example 4 $\quad \boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad m = n = 2, \; r = 1.$ **Singular matrix**

Column space: $\text{Span}\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\}$ Nullspace: $\text{Span}\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\}$

Row space: $\text{Span}\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$ Left nullspace: $\text{Span}\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}\}$



Example 4
What if --

$$\boldsymbol{B} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix},$$

$m = n = 2,$
$r = 2.$

Invertible matrix

Column space: $Span\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix}\}$
$= \mathbf{R}^2$

Nullspace: $\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$

Row space: $Span\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix}\right\}$
$= \mathbf{R}^2$

Left nullspace: $\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$

V. Full Rank Matrices (满秩矩阵: extreme case)

We now consider some extremal cases. Let $\boldsymbol{A}$ be an $(m \times n)$-matrix.

• If rank($\boldsymbol{A}$) = m, then $\boldsymbol{A}$ is said to have **full row rank (行满秩)**.

• If rank($\boldsymbol{A}$) = n, then $\boldsymbol{A}$ is said to have **full column rank (列满秩)**.

For example,

The following matrix $\boldsymbol{A}$ has full row rank, and $\boldsymbol{B}$ has full column rank.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 5 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \\ 2 & 7 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

Definition 1 Let $\boldsymbol{A}$ be an $(m \times n)$-matrix.

• If there exists a matrix $\boldsymbol{C}$ such that $\boldsymbol{AC} = \boldsymbol{I}_m$, then we say $\boldsymbol{A}$ has ***right-inverse*** (右逆) $\boldsymbol{C}$.

• If there exists a matrix $\boldsymbol{B}$ such that $\boldsymbol{BA} = \boldsymbol{I}_n$, then we say $\boldsymbol{A}$ has ***left-inverse*** (左逆) $\boldsymbol{B}$.

- If $\boldsymbol{A}$ has a left-inverse ($\boldsymbol{BA} = \boldsymbol{I}$) and a right-inverse ($\boldsymbol{AC} = \boldsymbol{I}$), then the two inverses are equal: $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{AC}) = (\boldsymbol{BA})\boldsymbol{C} = \boldsymbol{C}$.
- The rank always satisfies $r \le \min(m, n)$.
- *Existence and Uniqueness (of solution to Ax=b)*
- **EXISTENCE**: **Full row rank** $r = m$. (possible only if $m \le n$)

$\boldsymbol{Ax} = \boldsymbol{b}$ has ***at least*** one solution $\boldsymbol{x}$ for every $\boldsymbol{b}$ if and only if the columns span $\mathbf{R}^m$.

Then $\boldsymbol{A}$ has an n by m ***right-inverse C*** such that $\boldsymbol{AC} = \boldsymbol{I}_m$ (m by m).

- **UNIQUENESS**: **Full column rank** $r = n$. (possible only if $m \ge n$)

$\boldsymbol{Ax} = \boldsymbol{b}$ has ***at most*** one solution $\boldsymbol{x}$ for every $\boldsymbol{b}$ if and only if the columns are linearly independent.

Then $\boldsymbol{A}$ has an n by m ***left-inverse B*** such that $\boldsymbol{BA} = \boldsymbol{I}_n$ (n by n).

- Only a square matrix can have both $r = m$ and $r = n$, and therefore-- Only a square matrix can achieve both existence and uniqueness of solution to $\boldsymbol{Ax}=\boldsymbol{b}$. Only a square matrix has a two-sided inverse.

How can we find a left-inverse, or a right-inverse for a matrix?

Lemma

Let $\boldsymbol{A}$ be an $(m \times n)$-matrix.

(1) If $\boldsymbol{A}$ has full row rank, and the first m columns are linearly independent, then, writing

$$\boldsymbol{A} = \left[\boldsymbol{A}_0 \mid \boldsymbol{X} \right],$$

the matrix

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{A}_0^{-1} \\ \hline \boldsymbol{0} \end{bmatrix}$$

is a right-inverse of $\boldsymbol{A}$.

(2) If A has full column rank, and the first n rows are linearly independent, then, writing

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_0 \\ \hline \boldsymbol{Y} \end{bmatrix},$$

the matrix

$$\boldsymbol{B} = \left[\boldsymbol{A}_0^{-1} \mid \boldsymbol{0} \right]$$

is a left-inverse of $\boldsymbol{A}$.

Example 5 Using this lemma to the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, it is easy to find a right-inverse of $\boldsymbol{A}$, and a left-inverse of $\boldsymbol{B}$.

$$\boldsymbol{A}=\begin{bmatrix}1 & 0 & 1 & 2\\ 0 & 2 & 2 & 5\end{bmatrix}, \qquad \boldsymbol{B}=\begin{bmatrix}2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 6\\ 2 & 7 & 3\\ 1 & 2 & 3\end{bmatrix}.$$

Remark: Generally, if $\boldsymbol{AA}^{\mathrm{T}}$ is invertible, then a right-inverse of $\boldsymbol{A}$ is

$$\boldsymbol{C}=\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{AA}^{\mathrm{T}})^{-1},$$

while if $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is invertible, then a left-inverse of $\boldsymbol{A}$ is

$$\boldsymbol{B}=(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}.$$

We shall study this stuff further later.

VI. Matrices of Rank 1(秩为1的矩阵: simplest case)

Finally, we make a simple observation about matrices of rank 1. (The rank is as small as possible, except for the zero matrix.)

Let $\boldsymbol{A}$ be an $(m \times n)$-matrix of rank 1.

For example,

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix}, \quad \text{has rank}(\boldsymbol{A})=1.$$

Then

$$\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}},$$

where $\boldsymbol{u}$ is a column vector (i.e., $(m \times 1)$-matrix), and $\boldsymbol{v}^{\mathrm{T}}$ is a row vector ($(1 \times n)$-matrix).

This is because

- the rows are all multiples of the same row $\boldsymbol{v}^{\mathrm{T}}$, and
- the columns are all multiples of the same column $\boldsymbol{u}$.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix}}_{\boldsymbol{u}} \underbrace{\begin{bmatrix} 2 & 1 & 1 \end{bmatrix}}_{\boldsymbol{v}^{\mathrm{T}}} = \begin{bmatrix} 2 \\ 4 \\ 8 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Every matrix of rank 1 has the simple form

$$\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} = \textit{column times row}.$$

The row space and column space are lines—the easiest case.

We remark that if $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$, then $\boldsymbol{A} = (c\boldsymbol{u})(c^{-1}\boldsymbol{v}^{\mathrm{T}})$, where c is a non-zero number.

Key words:

row space, column space, nullspace, left nullspace, rank, full rank

Homework

See Blackboard

Note: In Ex. 35, $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}$ are column vectors.