南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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First-Order Linear Recurrences

- A recurrence of the form $T(n) = f(n)T(n-1) + g(n)$ is called a *first-order linear recurrence*.

  ◇ First Order because it only depends upon going back one step, i.e., $T(n-1)$

  If it depends upon $T(n-2)$, it would be a second-order recurrence, e.g., $T(n) = T(n-1) + 2T(n-2)$.

  ◇ Linear because $T(n-1)$ only appears to the first power.

  Something like $T(n) = (T(n-1))^2 + 3$ would be a non-linear first-order recurrence relation.

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  When $f(n)$ is a constant, say r, the general solution is almost as easy as we derived before. Iterating the recurrence gives

$$\begin{aligned}
T(n) &= rT(n-1) + g(n) \\
&= r(rT(n-2) + g(n-1)) + g(n) \\
&= r^2T(n-2) + rg(n-1) + g(n) \\
&= r^3T(n-3) + r^2g(n-2) + rg(n-1) + g(n) \\
&\vdots \\
&= r^nT(0) + \sum_{i=0}^{n-1} r^i g(n-i)
\end{aligned}$$

First-Order Linear Recurrences

- **Theorem** For any positive constants a and r, and any function g defined on nonnegative integers, the solution to the first-order linear recurrence

  $$T(n) = \begin{cases} rT(n-1) + g(n) & \text{if } n > 0 \\ a & \text{if } n = 0 \end{cases}$$

  is

  $$T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i).$$

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Proof by induction

Examples

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$$\begin{aligned} T(n) &= 6 \cdot 4^n + \sum_{i=1}^{n} 4^{n-i} \cdot 2^i \\ &= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} 4^{-i} \cdot 2^i \\ &= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} (\frac{1}{2})^i \\ &= 6 \cdot 4^n + (1 - \frac{1}{2^n}) \cdot 4^n \\ &= 7 \cdot 4^n - 2^n. \end{aligned}$$

Examples

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Theorem. For any real number $x \neq 1$,

$$\sum_{i=1}^{n} i x^i = \frac{n x^{n+2} - (n+1) x^{n+1} + x}{(1-x)^2}.$$

Examples

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T(n) &= 10 \cdot 3^n + \sum_{i=1}^{n} 3^{n-i} \cdot i \\
&= 10 \cdot 3^n + 3^n \sum_{i=1}^{n} i \cdot 3^{-i} \\
&= 10 \cdot 3^n + 3^n \left(-\frac{3}{2}(n+1)3^{-(n+1)} - \frac{3}{4}3^{-(n+1)} + \frac{3}{4} \right. \\
&= \frac{43}{4}3^n - \frac{n+1}{2} - \frac{1}{4}.
\end{aligned}$$

Growth Rates of Solutions to Recurrences

- Divide and conquer algorithms

- Iterating recurrences

- Three different behaviors

Divide and conquer algorithms

- We just analyzed recurrences of the form

$$T(n) = \begin{cases} b & \text{if } n = 0 \\ r \cdot T(n-1) + a & \text{if } n > 0 \end{cases}$$

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These corresponded to the analysis of recursive algorithms in which a problem of size n is solved by recursively solving a problem of size $n-1$.

$$T(n) = r^n b + a \frac{1 - r^n}{1 - r}$$

We will now look at recurrences of the form

$$T(n) = \begin{cases} \text{something given} & \text{if } n \leq n_0 \\ r \cdot T(n/m) + a & \text{if } n > n_0 \end{cases}$$

Binary Search

- Someone has chosen a number x between 1 and n. We need to discover x.

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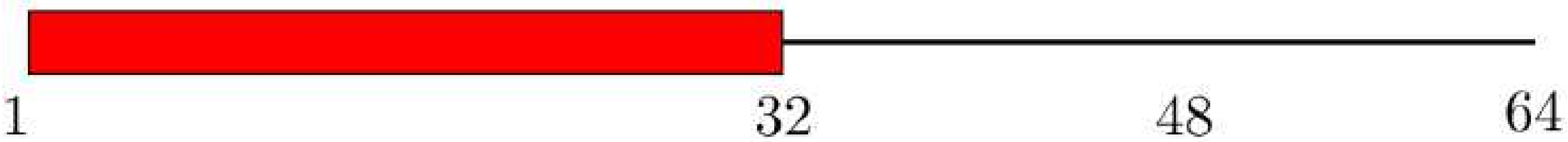
  Our strategy will be to always ask greater than questions, at each step halving our search range, until the range only contains one number, when we ask a final equal to question.

Binary Search Example

Binary Search Example

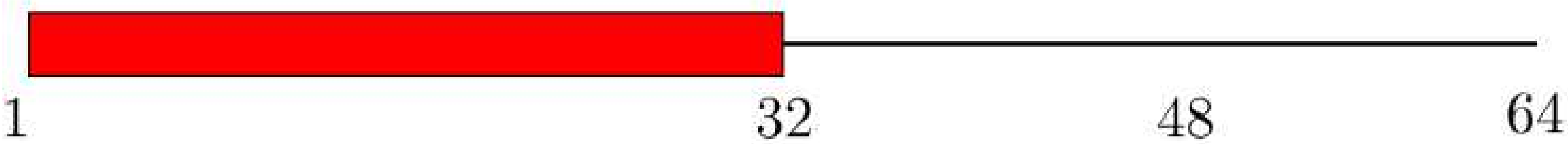
Is $x > 32$?

Binary Search Example



Is $x > 32$? Answer: Yes

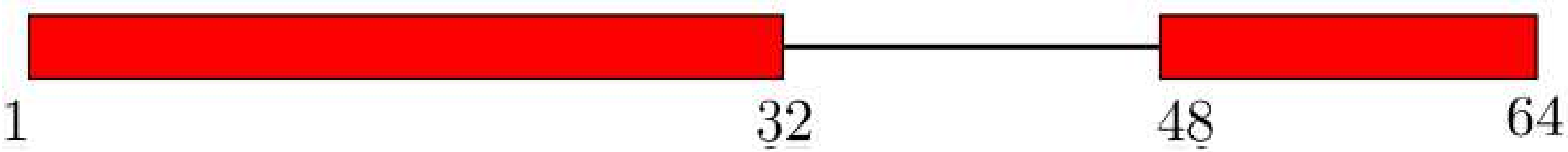
Binary Search Example



Is $x > 32$? Answer: Yes

Is $x > 48$?

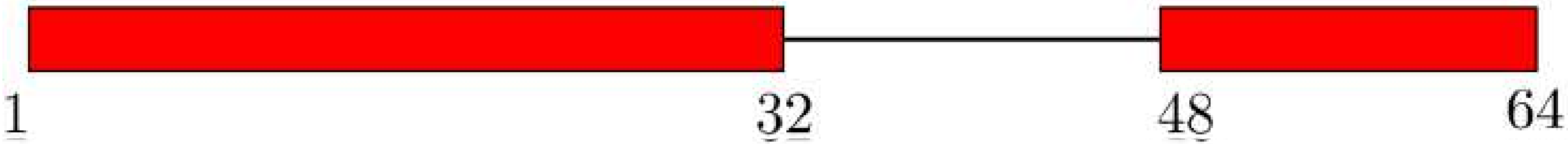
Binary Search Example



Is $x > 32$? Answer: Yes

Is $x > 48$? Answer: No

Binary Search Example



Is $x > 32$?	Answer: Yes
Is $x > 48$?	Answer: No
Is $x > 40$?	

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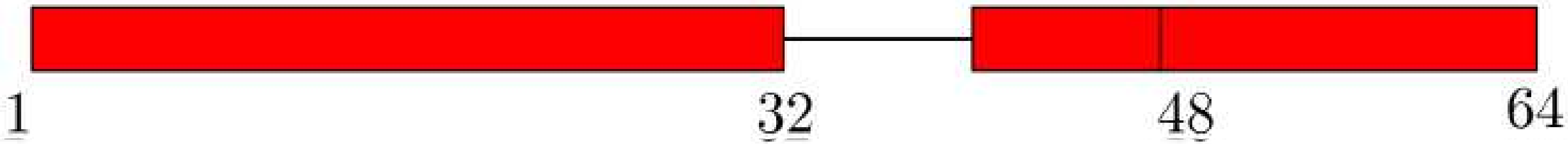

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Is $x > 40$?	Answer: No

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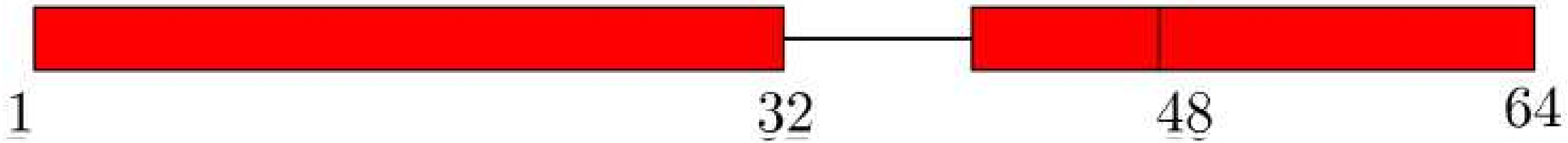

Is $x > 32$?	Answer: Yes
Is $x > 48$?	Answer: No
Is $x > 40$?	Answer: No
Is $x > 36$?	

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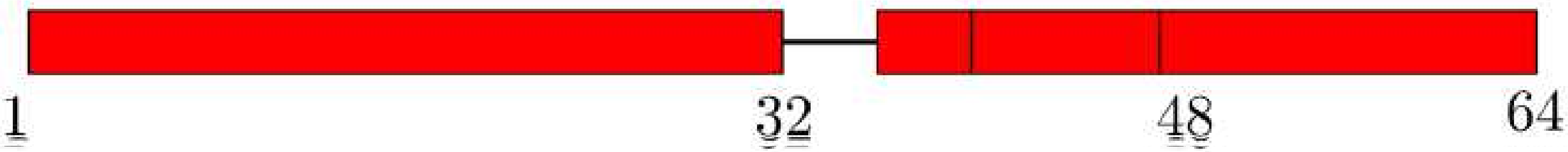

Is $x > 32$?	Answer: Yes
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Is $x > 40$?	Answer: No
Is $x > 36$?	Answer: No

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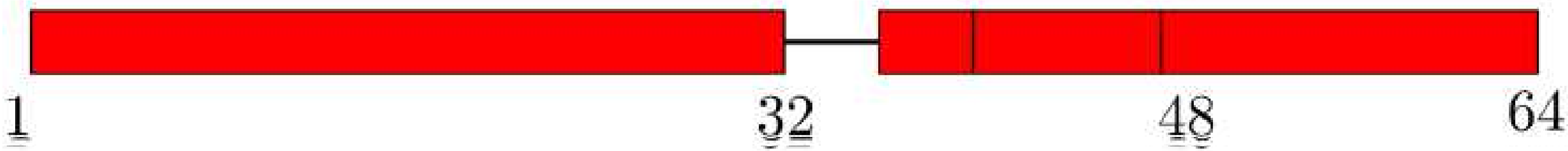

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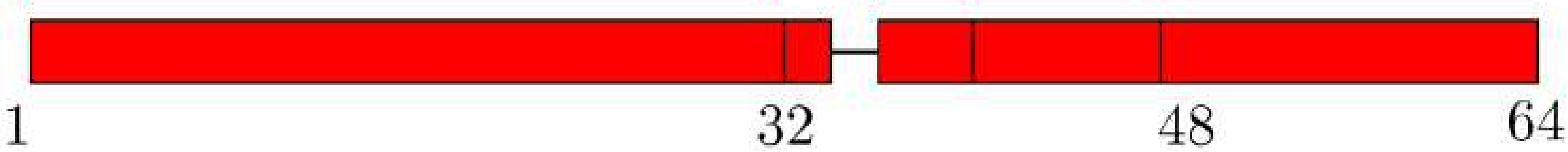

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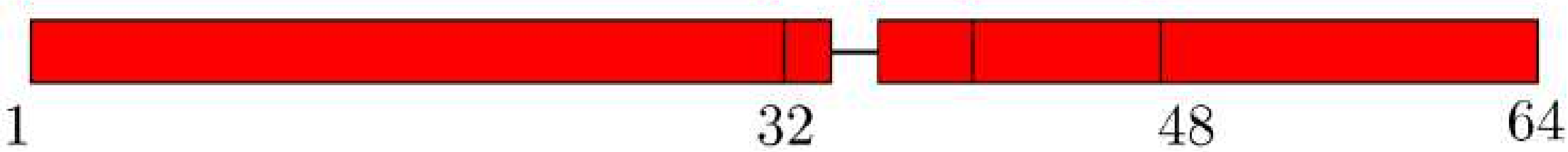

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Is $x > 34$?	Answer: Yes
Is $x > 35$?	

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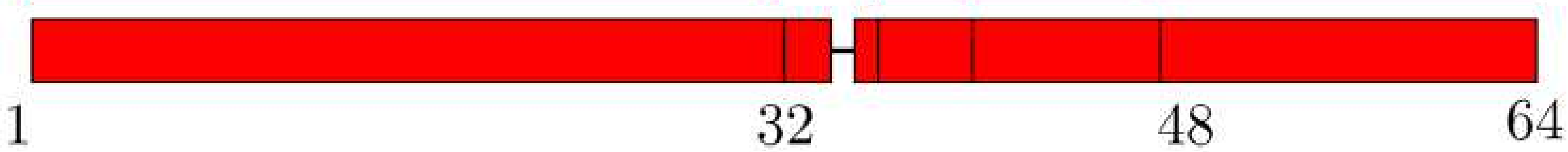

Is $x > 32$?	Answer: Yes
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Is $x > 34$?	Answer: Yes
Is $x > 35$?	Answer: No

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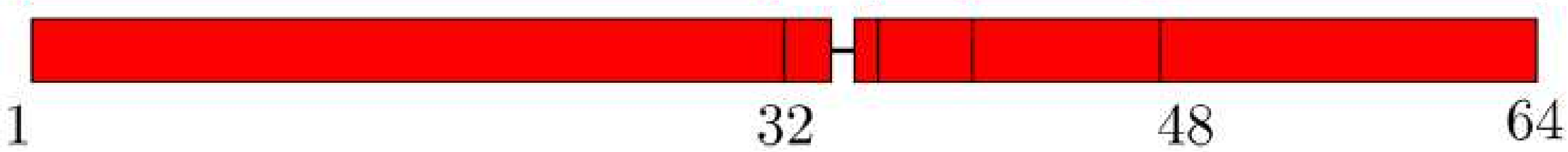

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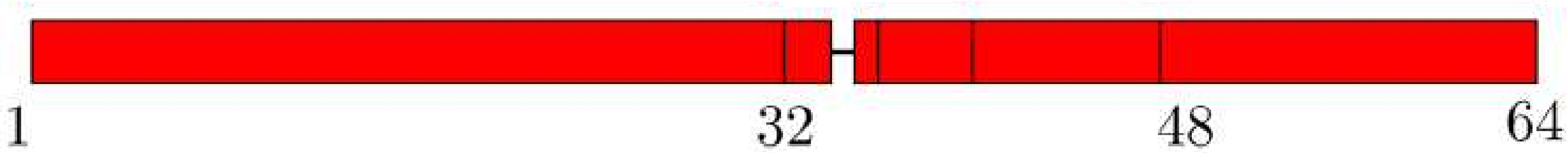

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Is $x > 35$?	Answer: No
Is $x = 35$?	Answer: BINGO!

Binary Search Example

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  Note: When n is a power of 2, $T(n)$, the number of questions in a binary search on $[1, n]$, satisfies

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

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  This can also be proved inductively, similar to the tower of Hanoi recurrence.

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first step

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Base case (1 item): $T(1) = 1$ to ask: “Is the number k?”

Binary Search Example

- $$(*) \qquad T(n) = \begin{cases} C_1 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + C_2 & \text{if } n \geq 2 \end{cases}$$

  For simplicity, we will (usually) assume that n is a power of 2 (or sometimes 3 or 4) and also often that constants such as C_1, C_2 are 1. This will let us replace a recurrence such as (*) by one such as (**).

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  In practice, the solution of (*) will be very close to that of (**) (this can be proved mathematically). Hence, we can restrict attention to (**).

Growth Rates of Solutions to Recurrences

- Divide and conquer algorithms
- Iterating recurrences
- Three different behaviors

Iterating Recurrences: Example 1

- $$(*) \qquad T(n) = \begin{cases} T(1) & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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This corresponds to solving a problem of size n, by

(i) solving 2 subproblems of size $n/2$ and
(ii) doing n units of additional work

or using $T(1)$ work for "bottom" case of $n = 1$

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  In the course "Analysis of Algorithms", this is exactly how Mergesort works.

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We now see how to solve (*) by algebraically iterating the recurrence.

Iterating Recurrences: Example 1

- Algebraically iterating the recurrence

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

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  Assume that n is a power of 2

$$T(n) = 2T\left(\frac{n}{2}\right) + n \qquad = 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$

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$$\vdots \qquad \vdots$$

$$= 2^i T\left(\frac{n}{2^i}\right) + in$$

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$$
\begin{aligned}
T(n) &= 2T\left(\tfrac{n}{2}\right) + n & &= 2\left(2T\left(\tfrac{n}{4}\right) + \tfrac{n}{2}\right) + n \\
&= 4T\left(\tfrac{n}{4}\right) + 2n & &= 4\left(2T\left(\tfrac{n}{8}\right) + \tfrac{n}{4}\right) + 2n \\
&= 8T\left(\tfrac{n}{8}\right) + 3n \\
&\ \ \vdots \qquad \vdots \\
&= 2^i T\left(\tfrac{n}{2^i}\right) + in \\
&\ \ \vdots \qquad \vdots \\
&= 2^{\log_2 n} T\left(\tfrac{n}{2^{\log_2 n}}\right) + (\log_2 n)n
\end{aligned}
$$

End when $i = \log_2 n$

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$$= 2^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + (\log_2 n)n$$

$$= nT(1) + n\log_2 n$$

Iterating Recurrences: Example 1

- We just iterated the recurrence to derive that the solution to

$$(*) \qquad T(n) = \begin{cases} T(1) & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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Note: Technically, we still need to use **induction** to prove that our solution is correct. Practically, we never explicitly perform this step, since it is obvious how the induction would work.

Iterating Recurrences: Example 2

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$$T(n) = T\left(\frac{n}{2}\right) + 1 \qquad = \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1$$

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$$T(n) = T\left(\frac{n}{2}\right) + 1 \qquad = \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1$$

$$= T\left(\frac{n}{2^2}\right) + 2$$

Iterating Recurrences: Example 2

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + 1 & &= \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1 \\ &= T\left(\frac{n}{2^2}\right) + 2 & &= \left(T\left(\frac{n}{2^3}\right) + 1\right) + 2 \end{aligned}$$

Iterating Recurrences: Example 2

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= T\left(\tfrac{n}{2}\right) + 1 & &= \left(T\left(\tfrac{n}{2^2}\right) + 1\right) + 1 \\ &= T\left(\tfrac{n}{2^2}\right) + 2 & &= \left(T\left(\tfrac{n}{2^3}\right) + 1\right) + 2 \\ &= T\left(\tfrac{n}{2^3}\right) + 3 \end{aligned}$$

Iterating Recurrences: Example 2

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + 1 & &= \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1 \\ &= T\left(\frac{n}{2^2}\right) + 2 & &= \left(T\left(\frac{n}{2^3}\right) + 1\right) + 2 \\ &= T\left(\frac{n}{2^3}\right) + 3 \\ &\ \ \vdots \qquad\quad \vdots \\ &= T\left(\frac{n}{2^i}\right) + i \end{aligned}$$

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$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + 1 & &= \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1 \\ &= T\left(\frac{n}{2^2}\right) + 2 & &= \left(T\left(\frac{n}{2^3}\right) + 1\right) + 2 \\ &= T\left(\frac{n}{2^3}\right) + 3 \\ &\vdots \qquad \vdots \\ &= T\left(\frac{n}{2^i}\right) + i \\ &\vdots \qquad \vdots \\ &= T\left(\frac{n}{2^{\log_2 n}}\right) + \log_2 n \end{aligned}$$

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$$= T\left(\frac{n}{2^2}\right) + 2 \qquad = \left(T\left(\frac{n}{2^3}\right) + 1\right) + 2$$

$$= T\left(\frac{n}{2^3}\right) + 3$$

$$\vdots \qquad \vdots$$

$$= T\left(\frac{n}{2^i}\right) + i$$

$$\vdots \qquad \vdots$$

$$= T\left(\frac{n}{2^{\log_2 n}}\right) + \log_2 n \;\; = 1 + \log_2 n$$

Iterating Recurrences: Example 3

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + n \\ &= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n \end{aligned}$$

Iterating Recurrences: Example 3

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$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + n \\ &= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n \\ &= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n \end{aligned}$$

Iterating Recurrences: Example 3

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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Iterating Recurrences: Example 3

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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Iterating Recurrences: Example 3

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned}
T(n) &= T\left(\frac{n}{2}\right) + n \\
&= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n \\
&= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n \\
\vdots \quad & \quad \vdots \\
&= T\left(\frac{n}{2^i}\right) + \frac{n}{2^{i-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
\vdots \quad & \quad \vdots \\
&= T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{2^{\log_2 n - 1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&= 1 + 2 + 2^2 + \cdots + \frac{n}{2^2} + \frac{n}{2} + n
\end{aligned}$$

Iterating Recurrences: Example 3

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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T(n) &= T\left(\frac{n}{2}\right) + n \\
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&\ \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^i}\right) + \frac{n}{2^{i-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&\ \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{2^{\log_2 n-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&= 1 + 2 + 2^2 + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \qquad = \Theta(n)
\end{aligned}$$

Iterating Recurrences: Example 4

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

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Iterating Recurrences: Example 4

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

$$T(n) = 3T\left(\frac{n}{3}\right) + n \qquad = 3\left(3T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right) + n$$

Iterating Recurrences: Example 4

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$$= 3^2T\left(\frac{n}{3^2}\right) + 2n$$

Iterating Recurrences: Example 4

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

$$\begin{aligned} T(n) = 3T\left(\tfrac{n}{3}\right) + n \quad & = 3\left(3T\left(\tfrac{n}{3^2}\right) + \tfrac{n}{3}\right) + n \\ = 3^2T\left(\tfrac{n}{3^2}\right) + 2n \quad & = 3^2\left(3T\left(\tfrac{n}{3^3}\right) + \tfrac{n}{3^2}\right) + 2n \end{aligned}$$

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$$\begin{aligned} T(n) &= 3T\left(\frac{n}{3}\right) + n & &= 3\left(3T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right) + n \\ &= 3^2 T\left(\frac{n}{3^2}\right) + 2n & &= 3^2\left(3T\left(\frac{n}{3^3}\right) + \frac{n}{3^2}\right) + 2n \\ &= 3^3 T\left(\frac{n}{3^3}\right) + 3n \end{aligned}$$

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&= 3^3T\left(\tfrac{n}{3^3}\right) + 3n & & \\
&\vdots \qquad \vdots & & \\
&= 3^iT\left(\tfrac{n}{3^i}\right) + in & & \\
&\vdots \qquad \vdots & & \\
&= 3^{\log_3 n}T\left(\tfrac{n}{3^{\log_3 n}}\right) + n\log_3 n & &= n + n\log_3 n
\end{aligned}$$

Iterating Recurrences: Example 5

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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Growth Rates of Solutions to Recurrences

- Divide and conquer algorithms
- Iteration recurrences
- Three different behaviors

Three Different Behaviors

- Compare the iteration for the recurrences

$$T(n) = 2T(n/2) + n$$

$$T(n) = T(n/2) + n$$

$$T(n) = 4T(n/2) + n$$

Three Different Behaviors

- Compare the iteration for the recurrences

$$T(n) = 2T(n/2) + n$$

$$T(n) = T(n/2) + n$$

$$T(n) = 4T(n/2) + n$$

◇ all three recurrences iterate $\log_2 n$ times

◇ in each case, size of subproblem in next iteration is half the size in the preceding iteration level

Three Different Behaviors

- **Theorem** Suppose that we have a recurrence of the form
$$T(n) = aT(n/2) + n,$$
where a is a positive integer and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

  1. If $a < 2$, then $T(n) = \Theta(n)$.
  2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
  3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$

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  Proof

  We already proved Case 1 when $a = 1$ in Example 3.
  (will not prove it for $1 < a < 2$)
  We already proved Case 2 in Example 1.
  We will now prove Case 3.

Iterating Recurrences

- $T(n) = aT(n/2) + n$, where $a > 2$. Assume that $n = 2^i$.

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$$T(n) = a^i T\left(\frac{n}{2^i}\right) + \left(\frac{a^{i-1}}{2^{i-1}} + \frac{a^{i-2}}{2^{i-2}} + \cdots \frac{a}{2} + 1\right) n$$

Iterating Recurrences

- $T(n) = aT(n/2) + n$, where $a > 2$. Assume that $n = 2^i$.

  Iterating as in Example 5 gives

  $$T(n) = a^i T\left(\frac{n}{2^i}\right) + \left(\frac{a^{i-1}}{2^{i-1}} + \frac{a^{i-2}}{2^{i-2}} + \cdots \frac{a}{2} + 1\right) n$$

  $$T(n) = a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

  Work at
  "bottom"

  Iterated
  Work

Total work

- The total work is

$$T(n) = a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

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Since $a > 2$, the geometric series is Θ of the largest term.

$$n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i = n \frac{1 - (a/2)^{\log_2 n}}{1 - a/2} = n\Theta((a/2)^{\log_2 n - 1})$$

Total work

- n times the largest term in the geometric series is

$$n\left(\frac{a}{2}\right)^{\log_2 n-1} = \frac{2}{a}\cdot\frac{n\cdot a^{\log_2 n}}{2^{\log_2 n}} = \frac{2}{a}\cdot\frac{n\cdot a^{\log_2 n}}{n} = \frac{2}{a}\cdot a^{\log_2 n}$$

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Notice that

$$a^{\log_2 n} = \left(2^{\log_2 a}\right)^{\log_2 n} = \left(2^{\log_2 n}\right)^{\log_2 a} = n^{\log_2 a}$$

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Notice that

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So the total work is

$$a^{\log_2 n}T(1) + n\sum_{i=0}^{\log_2 n-1}\left(\frac{a}{2}\right)^i$$

Total work

- n times the largest term in the geometric series is

$$n\left(\frac{a}{2}\right)^{\log_2 n-1} = \frac{2}{a}\cdot\frac{n\cdot a^{\log_2 n}}{2^{\log_2 n}} = \frac{2}{a}\cdot\frac{n\cdot a^{\log_2 n}}{n} = \frac{2}{a}\cdot a^{\log_2 n}$$

Notice that

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So the total work is

$$a^{\log_2 n}T(1) + n\sum_{i=0}^{\log_2 n-1}\left(\frac{a}{2}\right)^i$$

$$\Theta\left(n^{\log_2 a}\right) \qquad\qquad \Theta\left(n^{\log_2 a}\right)$$

Example 5 Recap

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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$a = 4$, so the Theorem says that

$$T(n) = \Theta\left(n^{\log_2 a}\right) = \Theta\left(n^{\log_2 4}\right) = \Theta(n^2)$$

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This matches with the exact answer of $2n^2 - n$.

Three Different Behaviors

- **Theorem** Suppose that we have a recurrence of the form
$$T(n) = aT(n/2) + n,$$
where a is a positive integer and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

  1. If $a < 2$, then $T(n) = \Theta(n)$.
  2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
  3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$

The Master Theorem

- **Theorem** Suppose that we have a recurrence of the form
  $$T(n) = aT(n/b) + cn^d,$$
  where a is a positive integer, $b \geq 1$, c, d are real numbers with c positive and d nonnegative, and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

  1. If $a < b^d$, then $T(n) = \Theta(n^d)$.
  2. If $a = b^d$, then $T(n) = \Theta(n^d \log n)$.
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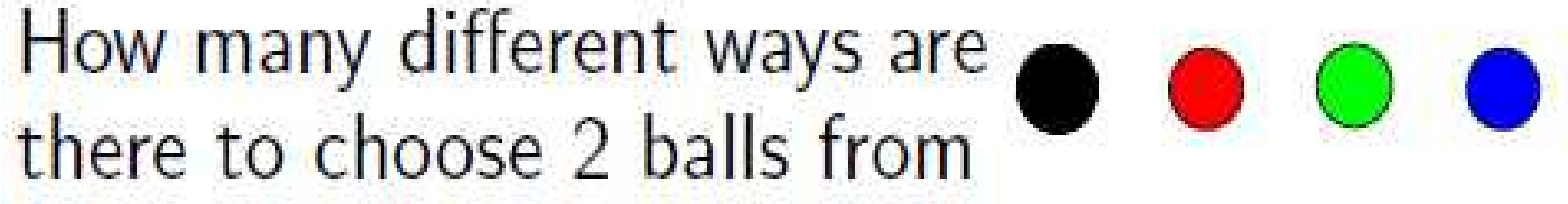

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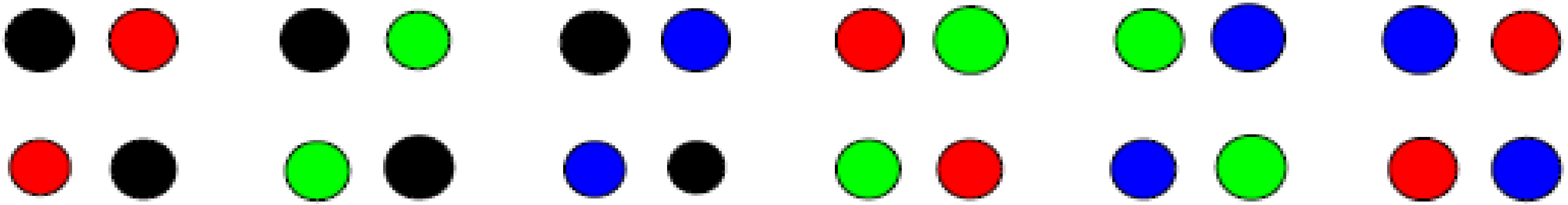

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  – simplify the solution by decomposing the problem

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- *the Sum Rule*

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In an auditorium, the seats are labeled by a letter and numbers in between 1 to 50 (e.g., $A23$). What is the total number of seats?

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Example

In an auditorium, the seats are labeled by a letter and numbers in between 1 to 50 (e.g., $A23$). What is the total number of seats?

We may either list all or use the product rule.

$$26 \times 50 = 1300$$

The Product Rule

- **Product Rule**: If a count of elements can be broken down into a sequence of dependent counts where the first count yields n_1 elements, the second n_2 elements, and kth count n_k elements, then the total number of elements is

$$n = n_1 \cdot n_2 \cdot \cdots \cdot n_k$$

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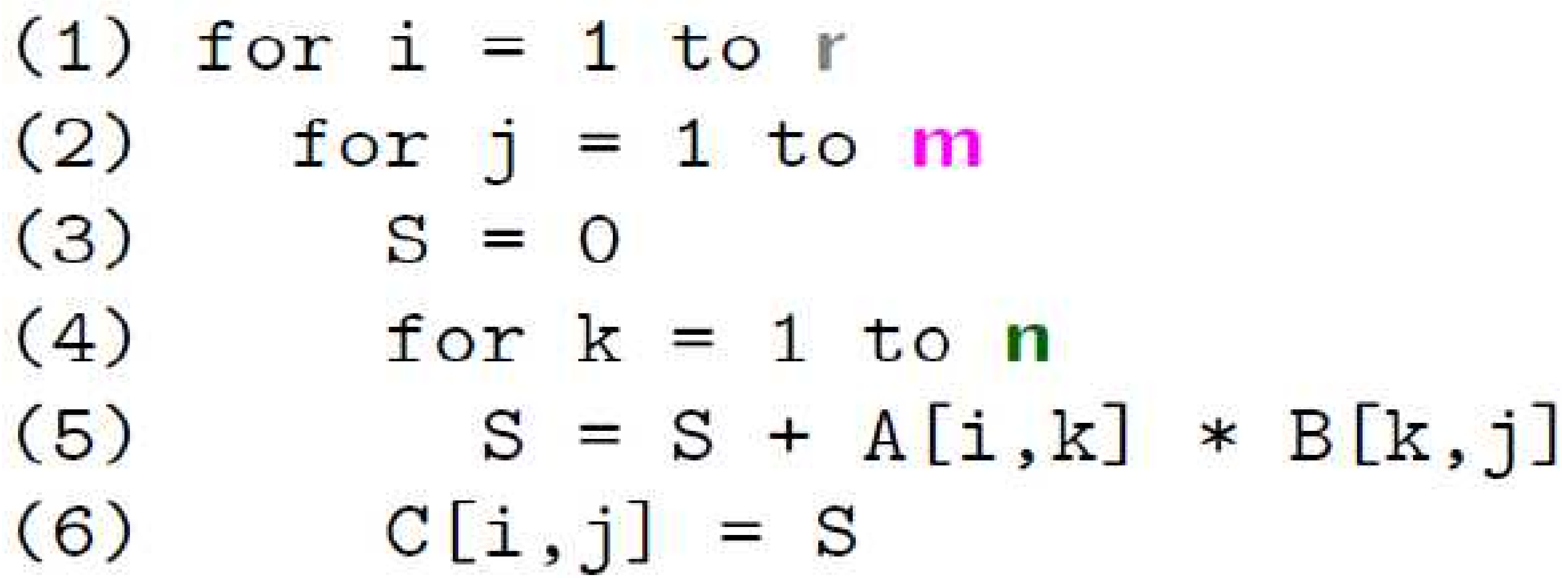
  How many one-to-one functions are there from a set with m elements to a set with n elements?

  How many onto functions?

The Product Rule

- The following loop is a part of program computing the product of two matrices.



```
for i = 1 to r
    for j = 1 to m
        S = 0
        for k = 1 to n
            S = S + A[i,k] * B[k,j]
        C[i,j] = S
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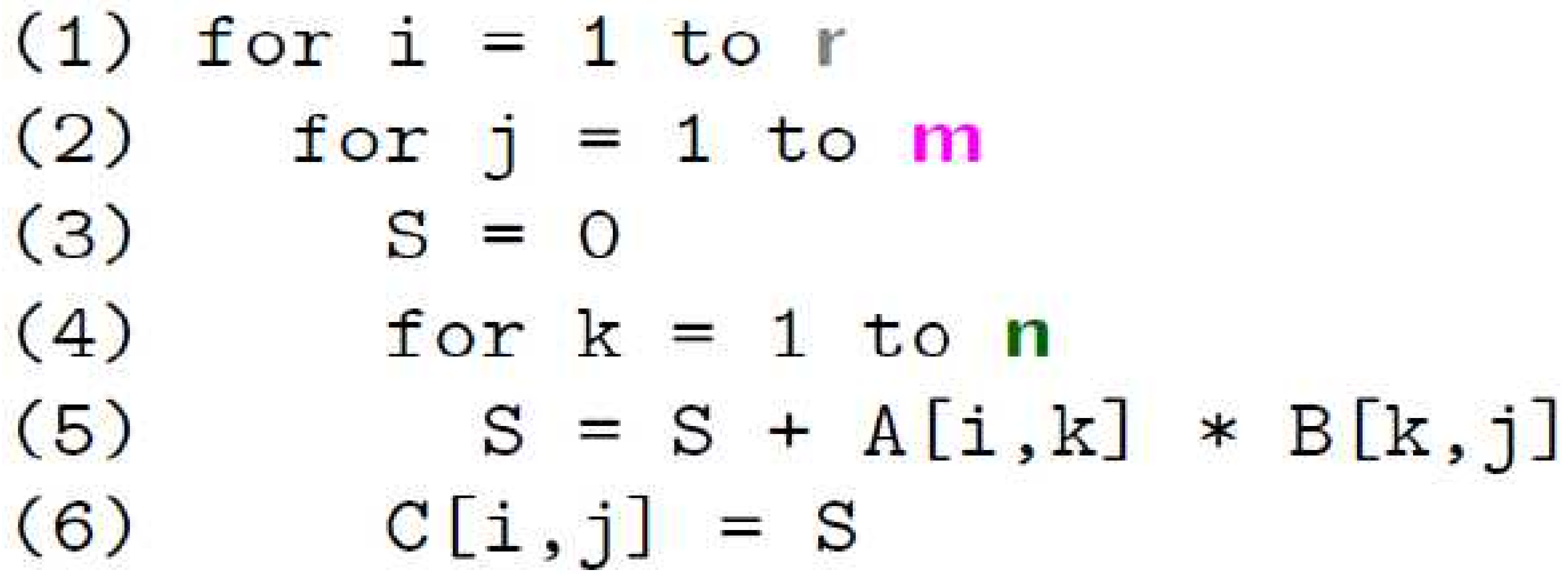

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How many multiplications (in terms of r, m, n) does this program carry out in total among all iterations of line 5?

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You need to travel from city A to B. You may either fly, take a train, or a bus. There are 12 different flights, 5 different trains and 10 buses. How many options do you have to get from A to B?

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We may use the sum rule.

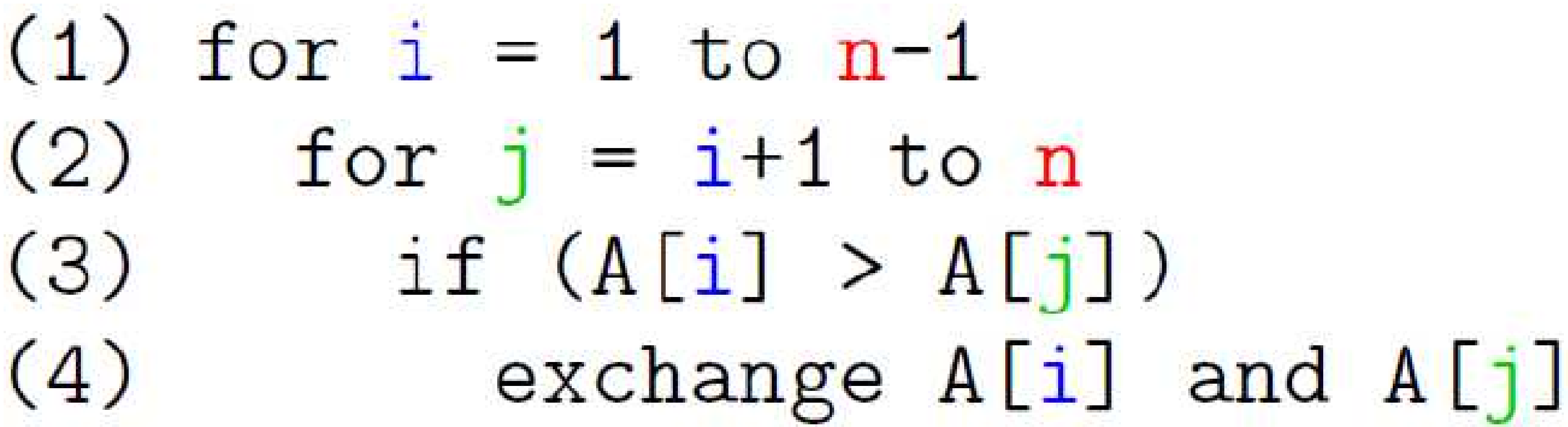
$$12 + 5 + 10$$

The Sum Rule

- **Sum Rule**: If a count of elements can be broken down into a set of independent counts where the first count yields n_1 elements, the second n_2 elements, and kth count n_k elements, then the total number of elements is
$$n = n_1 + n_2 + \cdots + n_k$$

The Sum Rule

- The following loop is from selection sort.



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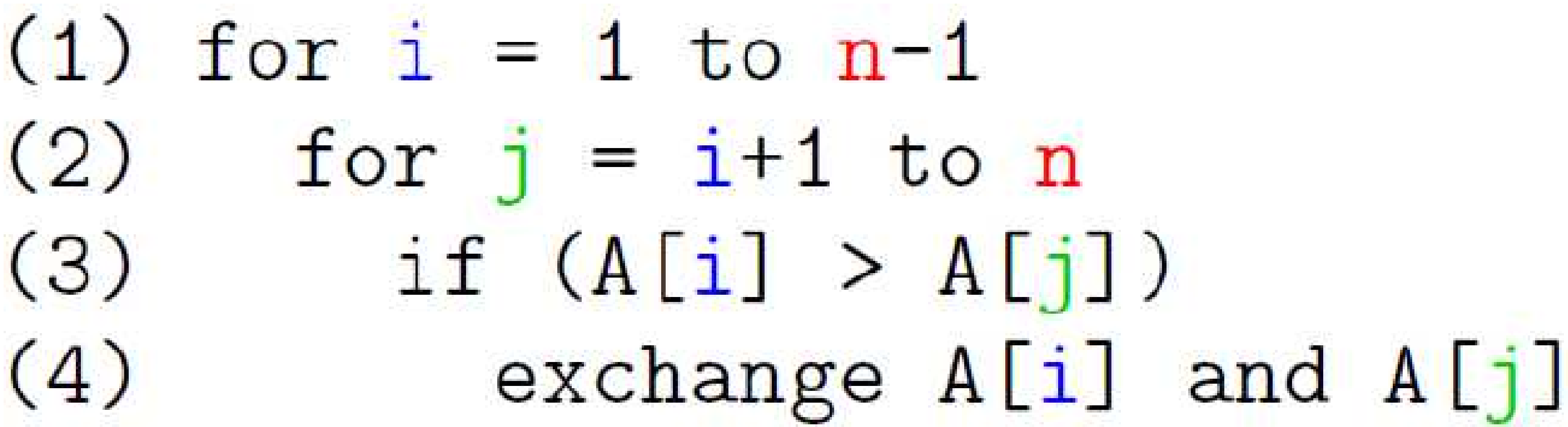

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How many comparisons (in terms of *n*) does this program carry out in total among all iterations of line 3?

More Complex Counting

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Example

Each password is 6 to 8 characters long, where each character is an lowercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

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$$P = P_6 + P_7 + P_8$$

Tree Diagrams

- A *tree* is a structure that consists of a root, branches and leaves.

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  What is the number of bit strings of length 4 that do not have two consecutive 1's?

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  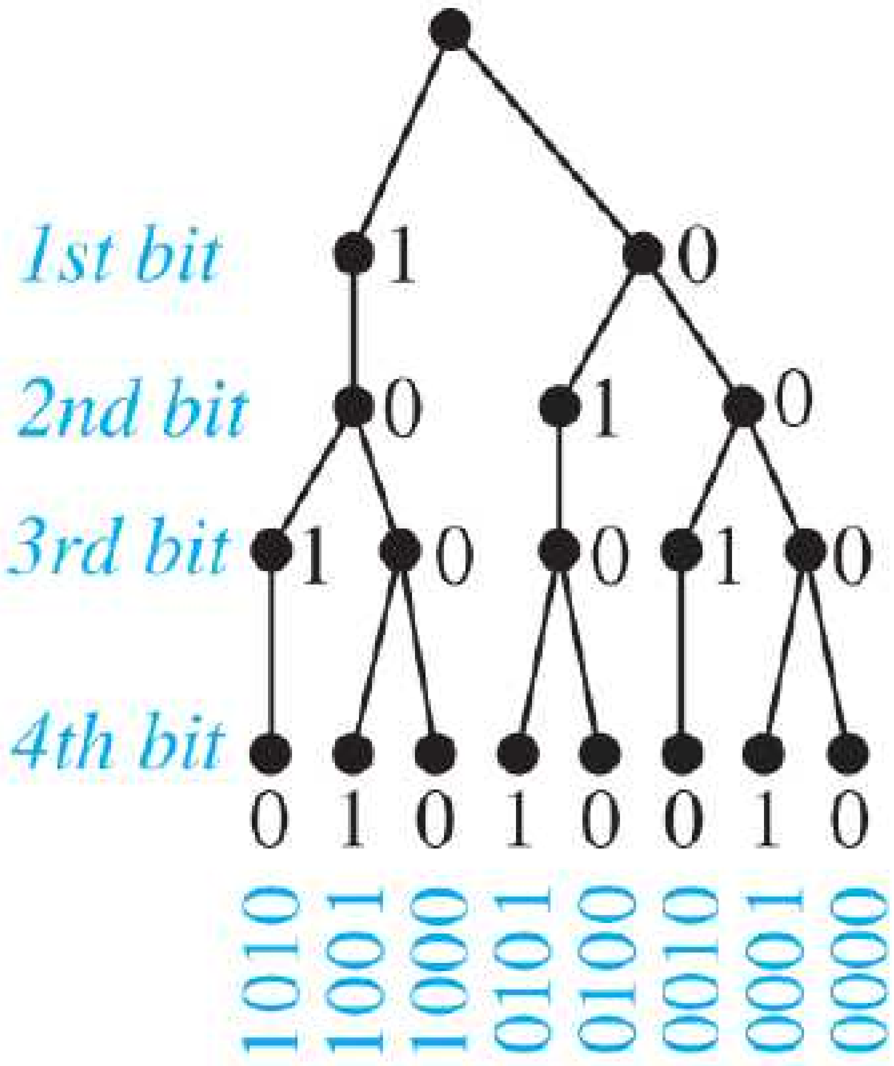

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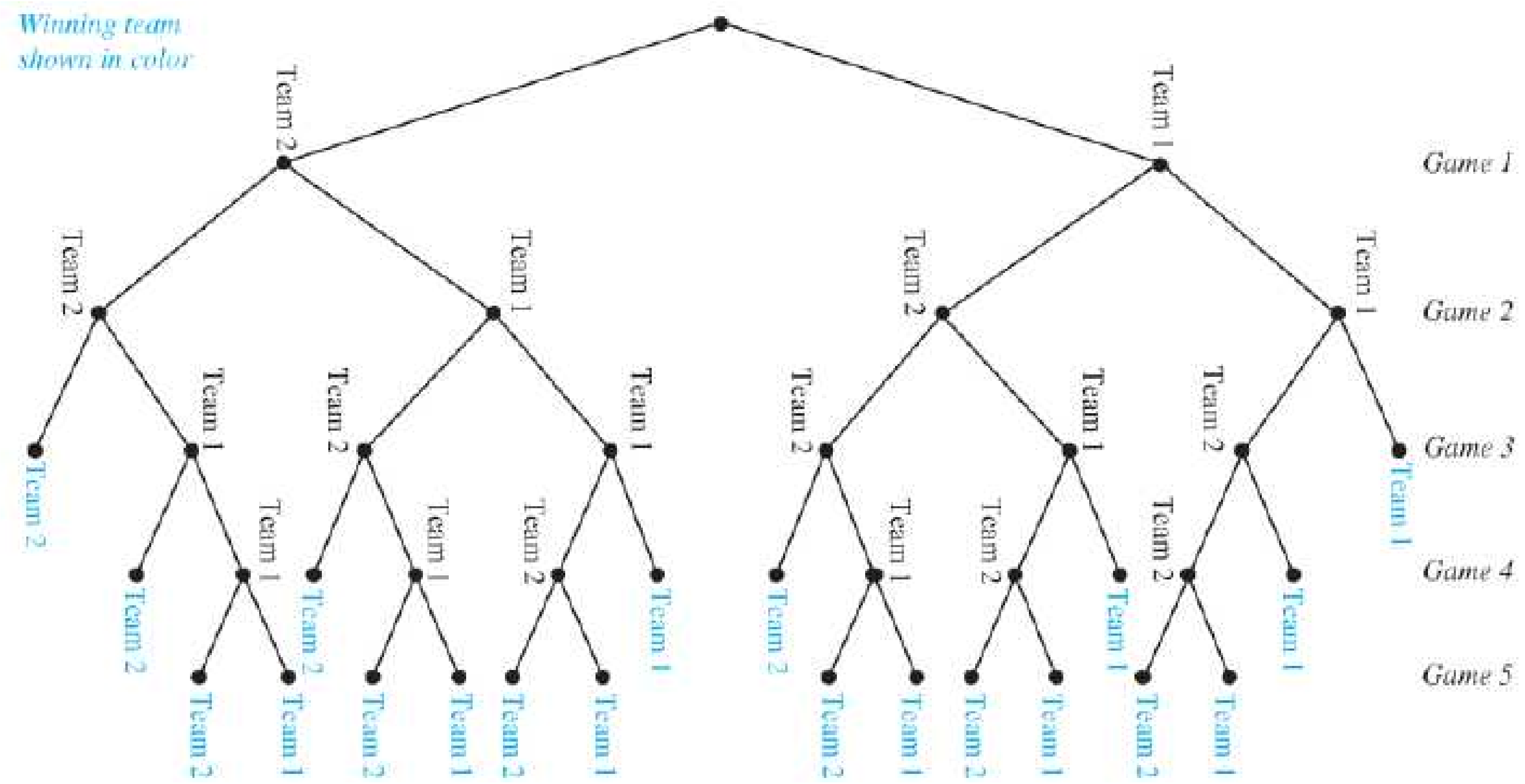

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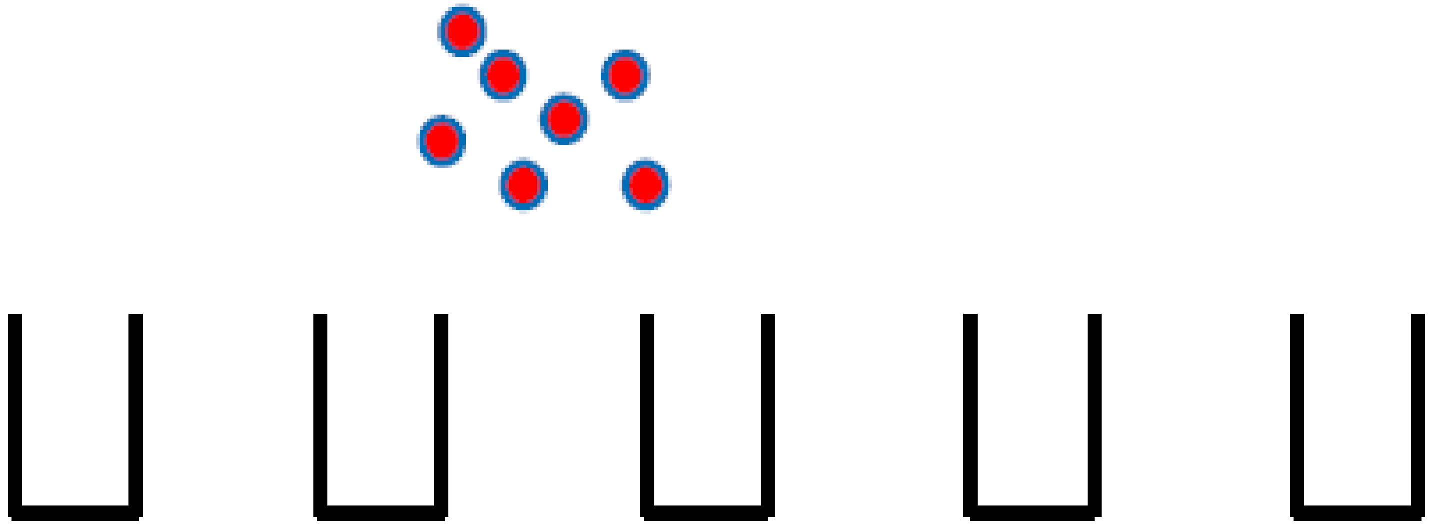

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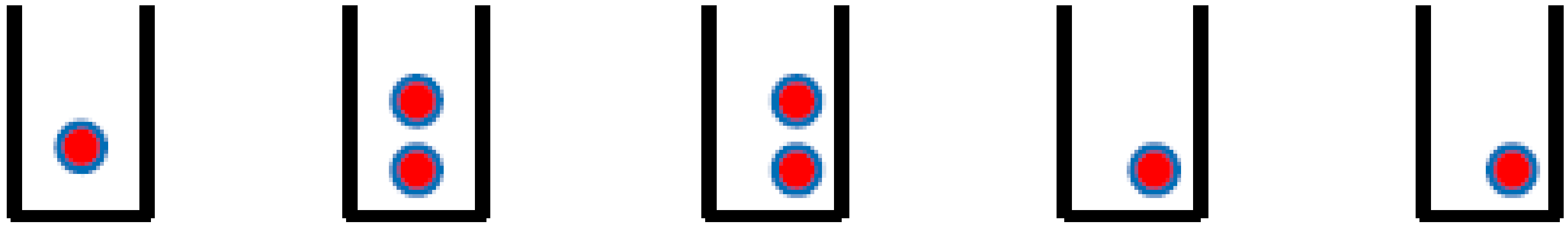

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  Proof by contradiction

  Example

  Assume that there are 367 students. Are there any two people who have the same birthday?

  There are 5 bins and 12 objects. Then there must be a bin with at least 3 objects. Why?

Generalized Pigeonhole Principle

- If N objects are placed into k bins, then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

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Example

Assume there are 100 students. How many of them were born in the same month?

Bijections and Permutations

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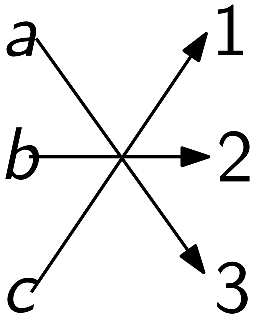

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  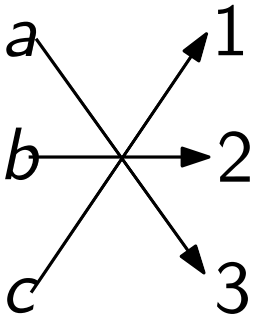

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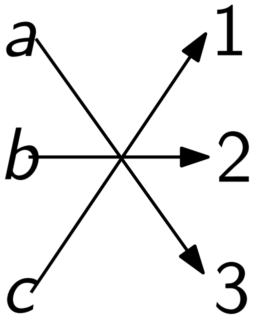

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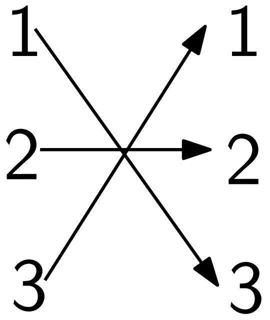

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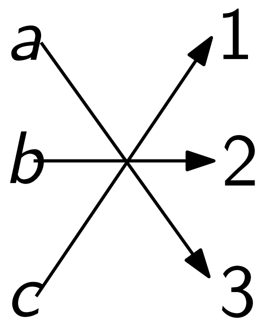

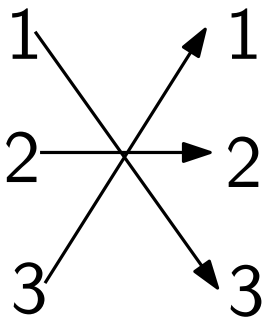

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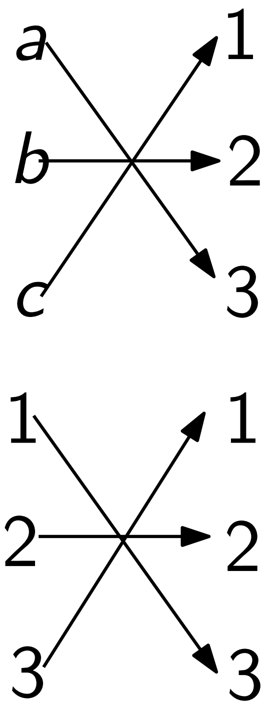
Thus,
the left and right sides must have the same size.



The Bijection Principle

- The following loop is a part of program to determine the number of triangles formed by *n* points in the plane.

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trianglecount = 0
  for i = 1 to n
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Among all iterations of line 5, what is the total number of times this line checks three points to see if they are collinear?

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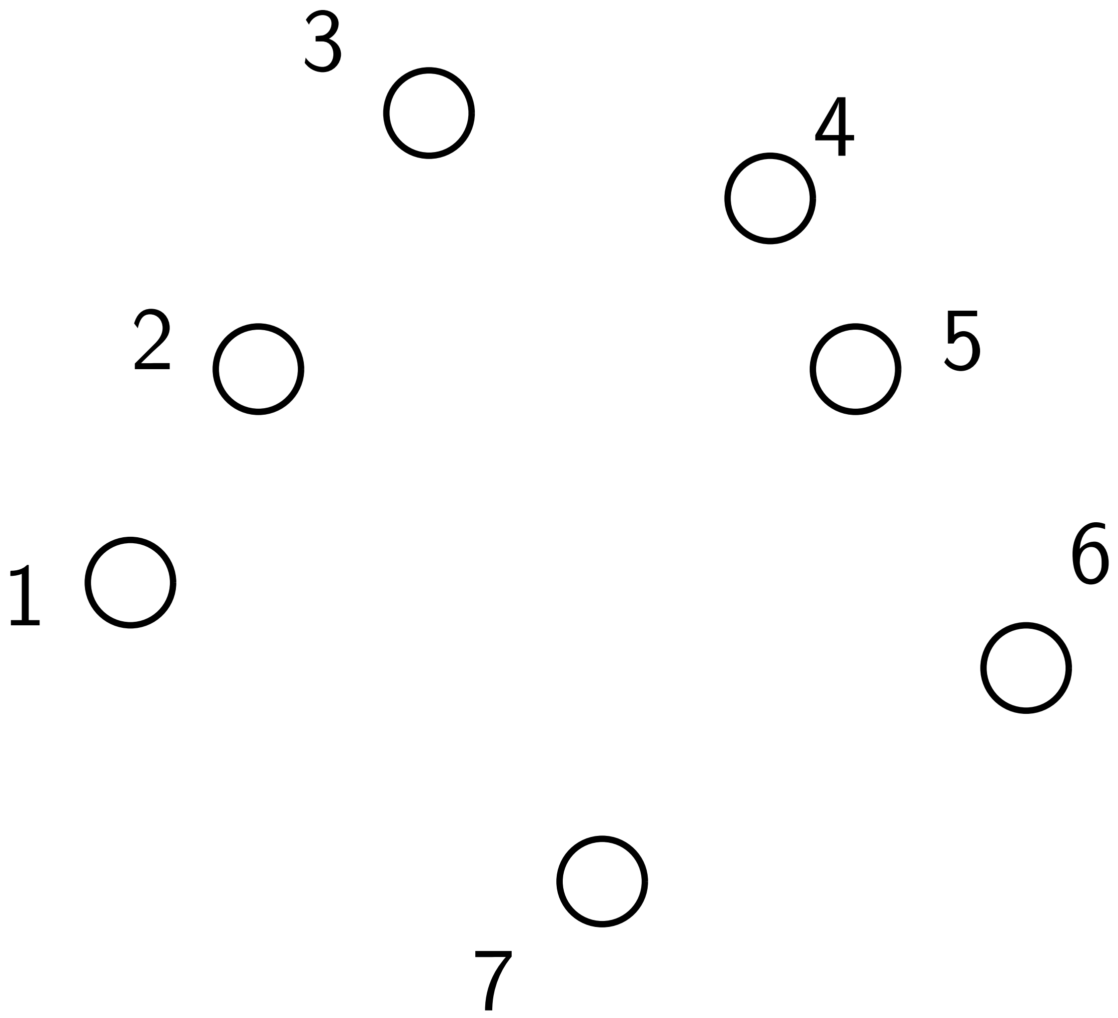

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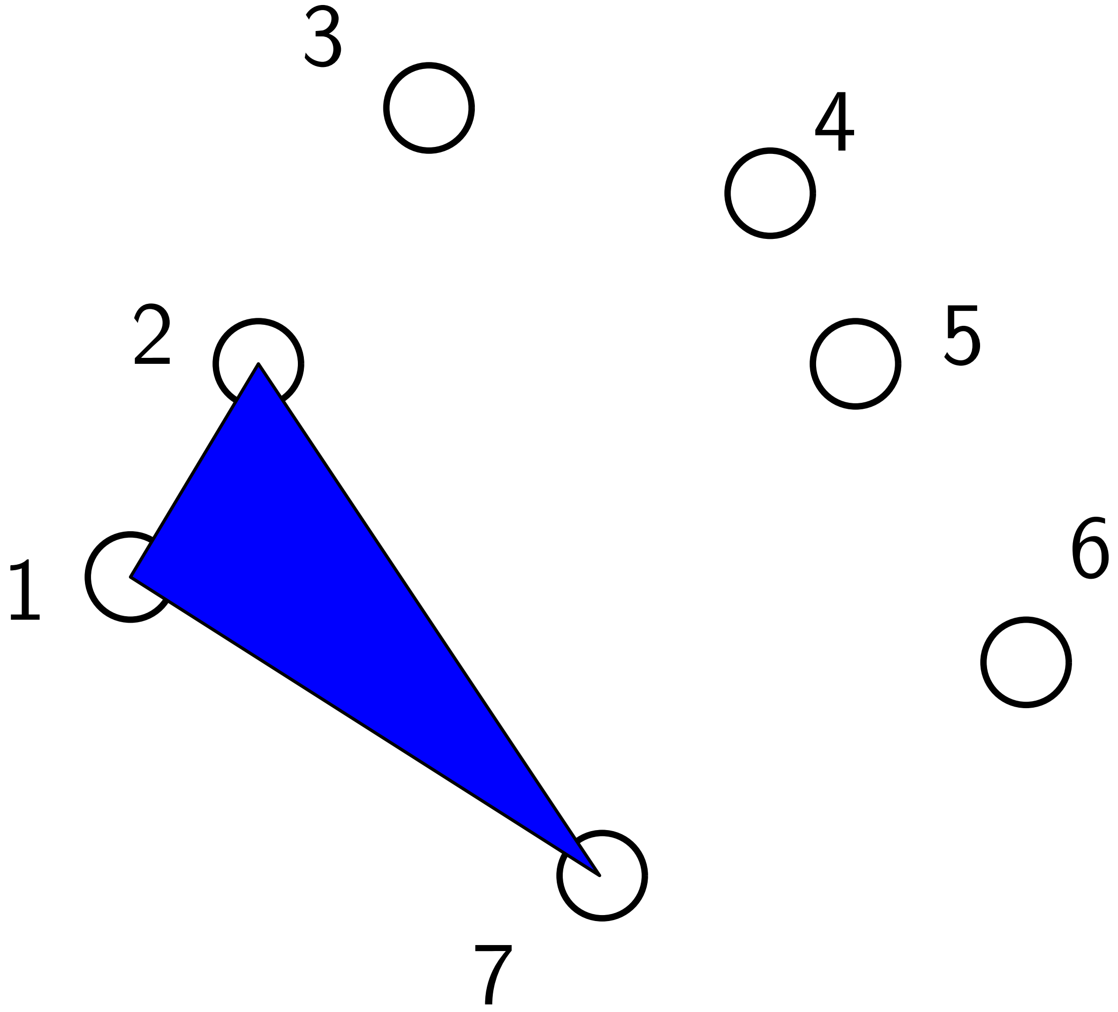

$1 - 2 - 7$: yes

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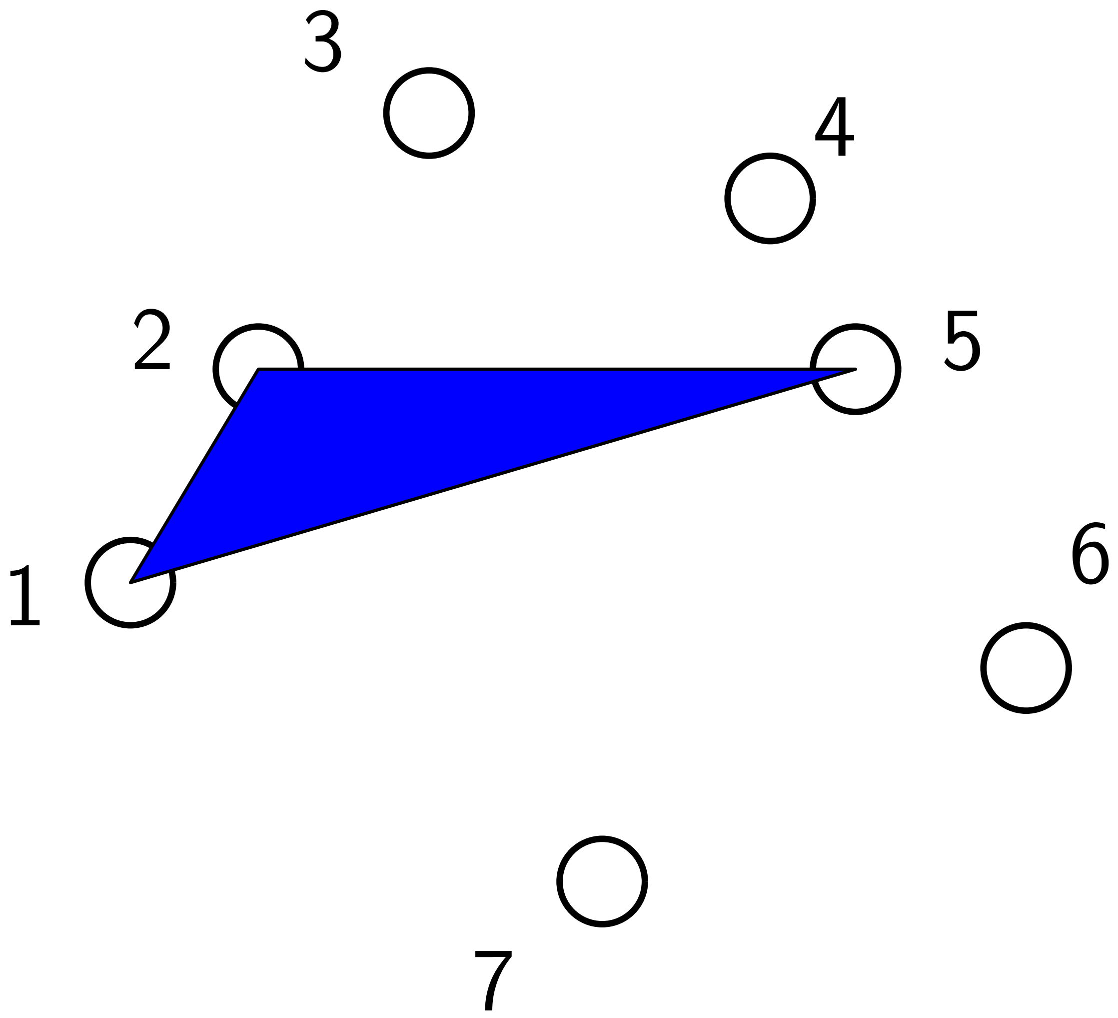

1 – 2 – 7: yes

1 – 2 – 5: yes

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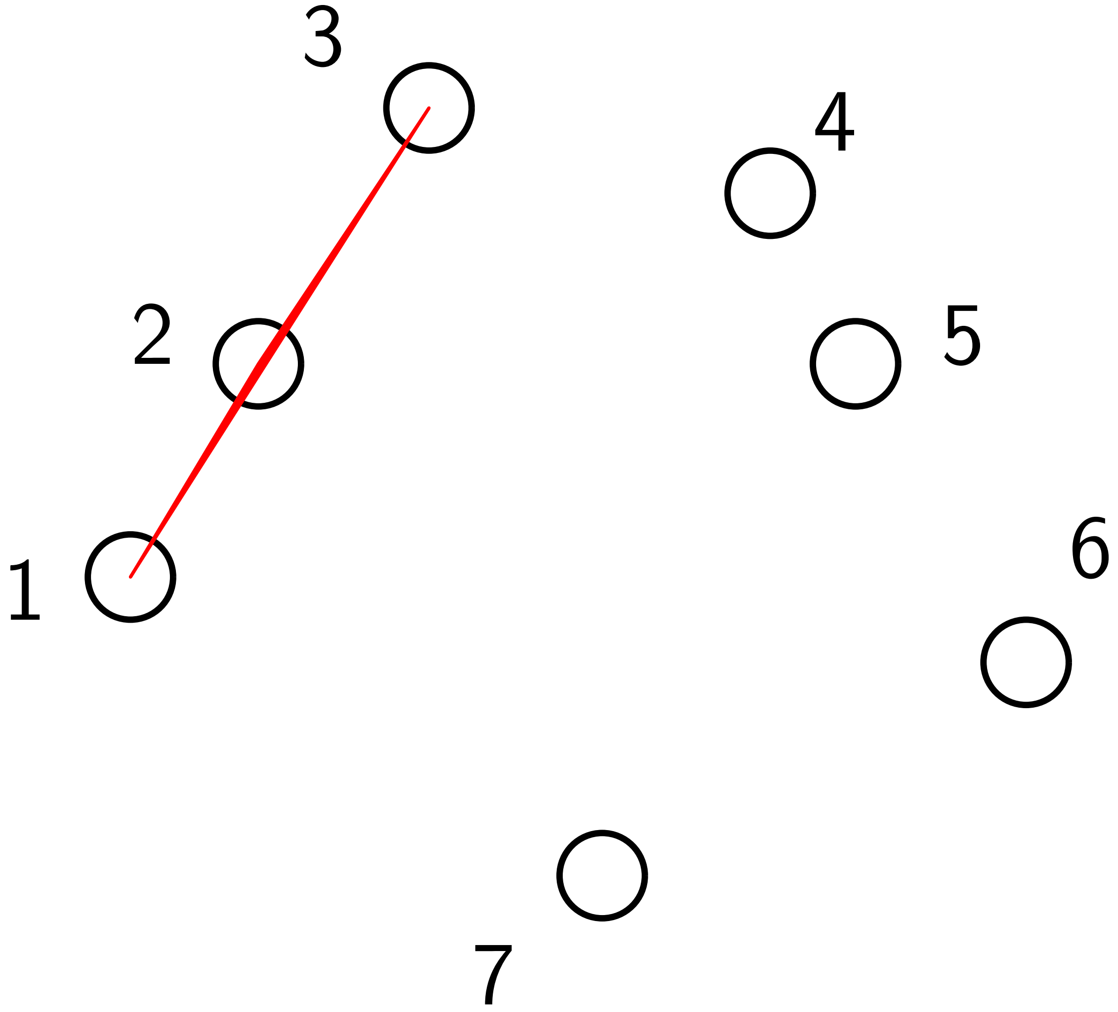

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$1 - 2 - 3$: no

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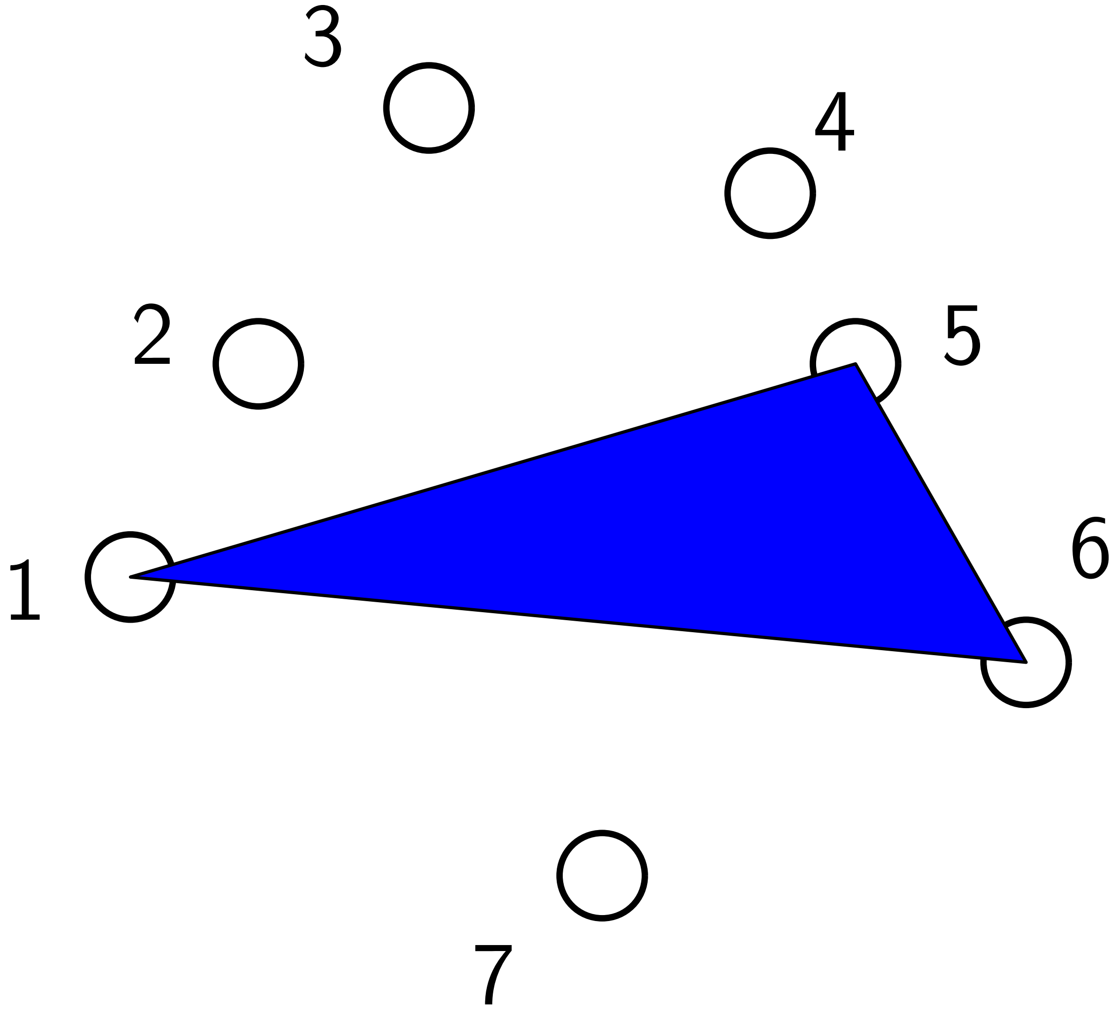

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1 – 5 – 6: yes

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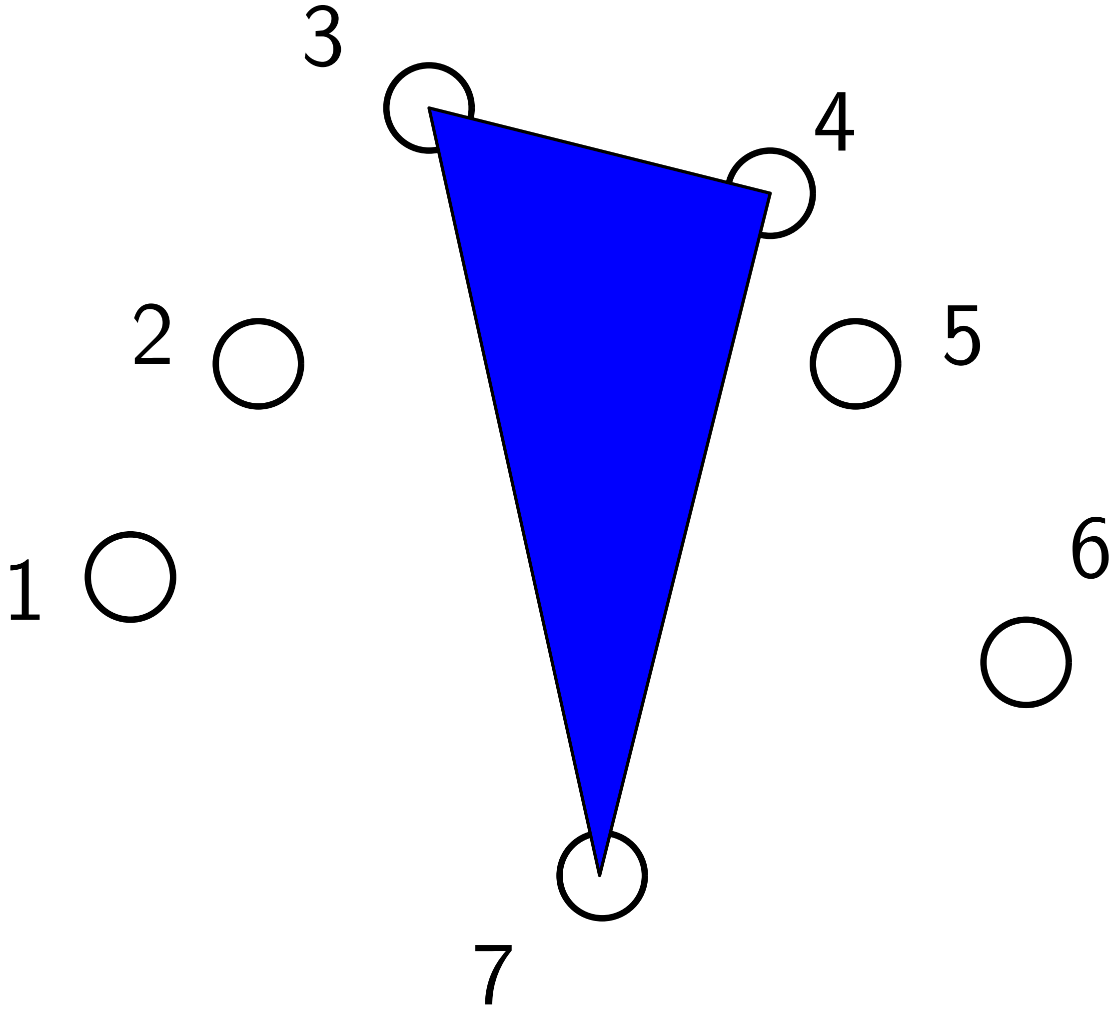

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$1 - 2 - 3$: no

$1 - 5 - 6$: yes

$3 - 4 - 7$: yes

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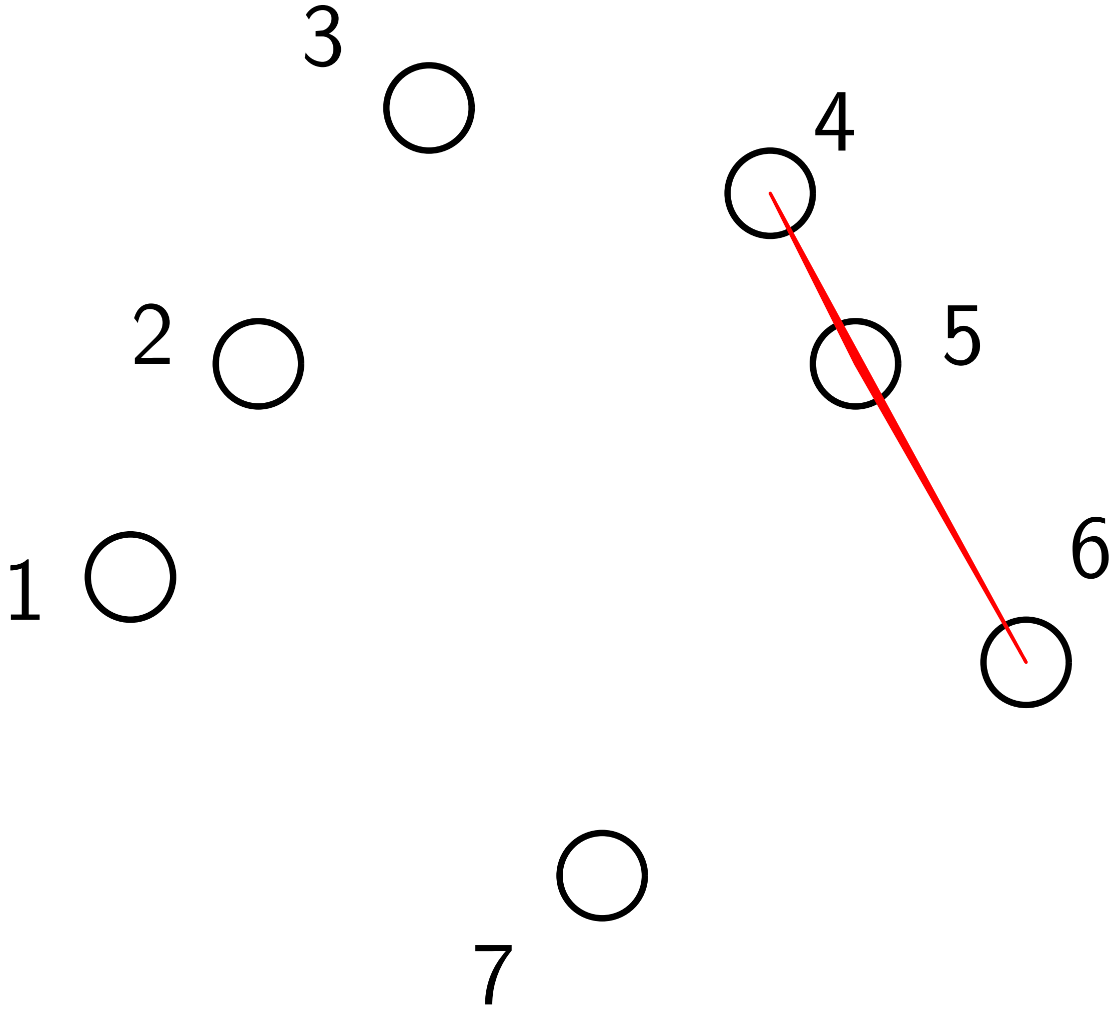

1 – 2 – 7: yes

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1 – 2 – 3: no

1 – 5 – 6: yes

3 – 4 – 7: yes

4 – 5 – 6: no

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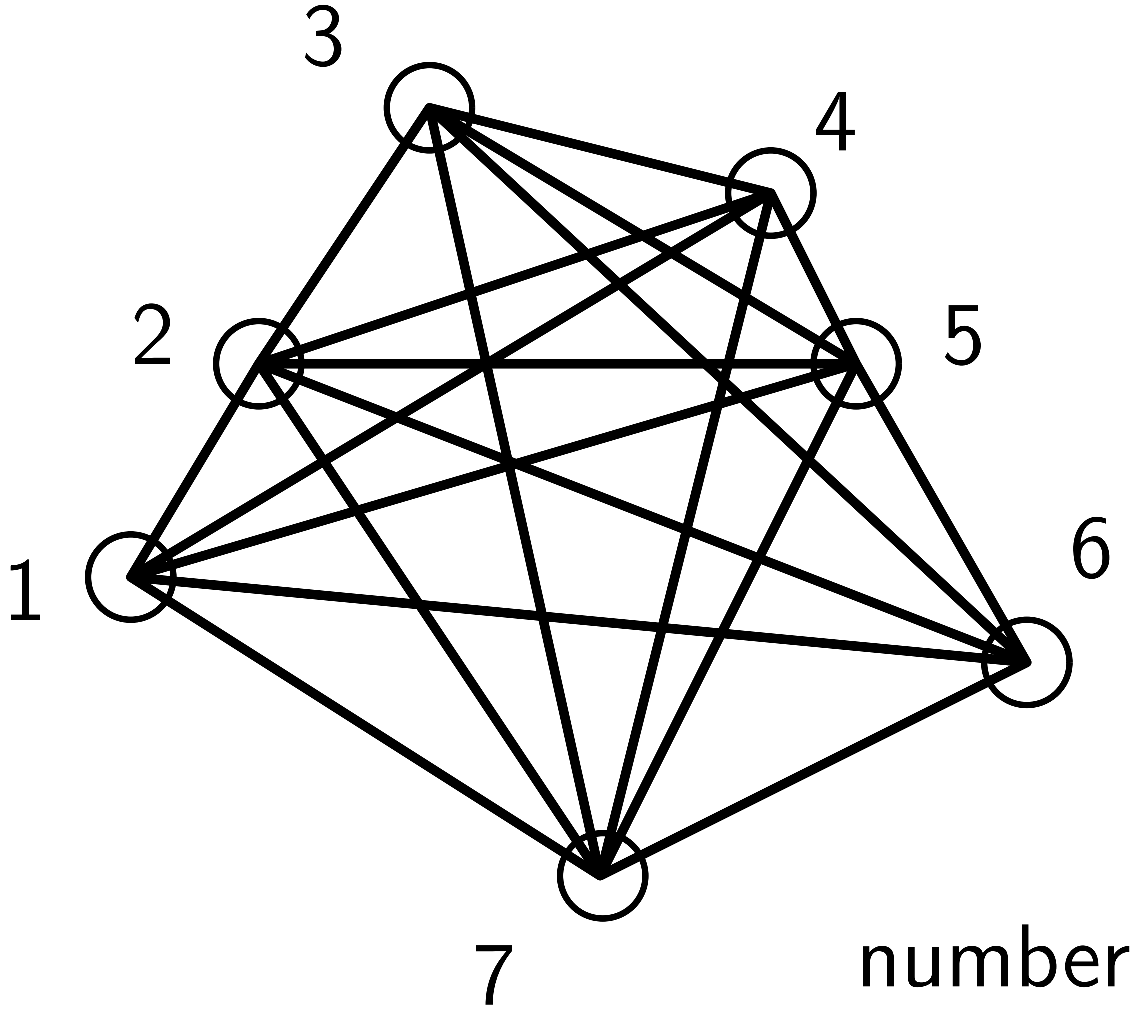

1 – 2 – 7: yes

1 – 2 – 5: yes

1 – 2 – 3: no

1 – 5 – 6: yes

3 – 4 – 7: yes

4 – 5 – 6: no

number of triangles: 33

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A loop

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A loop embedded in a loop

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A loop embedded in a loop embedded in another loop.

Second loop begins with $j = i + 1$ and j increases up to n.
Third loop begins with $k = j + 1$ and k increases up to n.

Counting Triangles

```
triangleCount = 0
  for i = 1 to n
    for j = i+1 to n
      for k = j+1 to n
        if points i, j, k are not collinear
          triangleCount = triangleCount + 1
```

A loop embedded in a loop embedded in another loop.

Second loop begins with $j = i + 1$ and j increases up to n.
Third loop begins with $k = j + 1$ and k increases up to n.

Thus each triple i, j, k with $i < j < k$ is examined exactly once.

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Thus each triple i, j, k with $i < j < k$ is examined exactly once.

For example, if $n = 4$, then triples (i, j, k) used by algorithm are (1,2,3), (1,2,4), (1,3,4), and (2,3,4).

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f is a bijection because

f is one-to-one

if $(i, j, k) \neq (i', j', k') \Rightarrow f(\,(i, j, k)\,) \neq f(\,(i', j', k'))$

f is onto

if γ is a 3-element subset then it can be written as $\gamma = \{i, j, k\}$ where $i < j < k$ so $f(\,(i, j, k)\,) = \gamma$.

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  We actually already saw that $|X| = |Y| = \binom{n}{2}$

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  Currently, we started with the problem of counting the # of increasing triples and changed it to the problem of counting the # of 3-element sets from $\{1, 2, \ldots, n\}$

Next Lecture

- recurrence ...