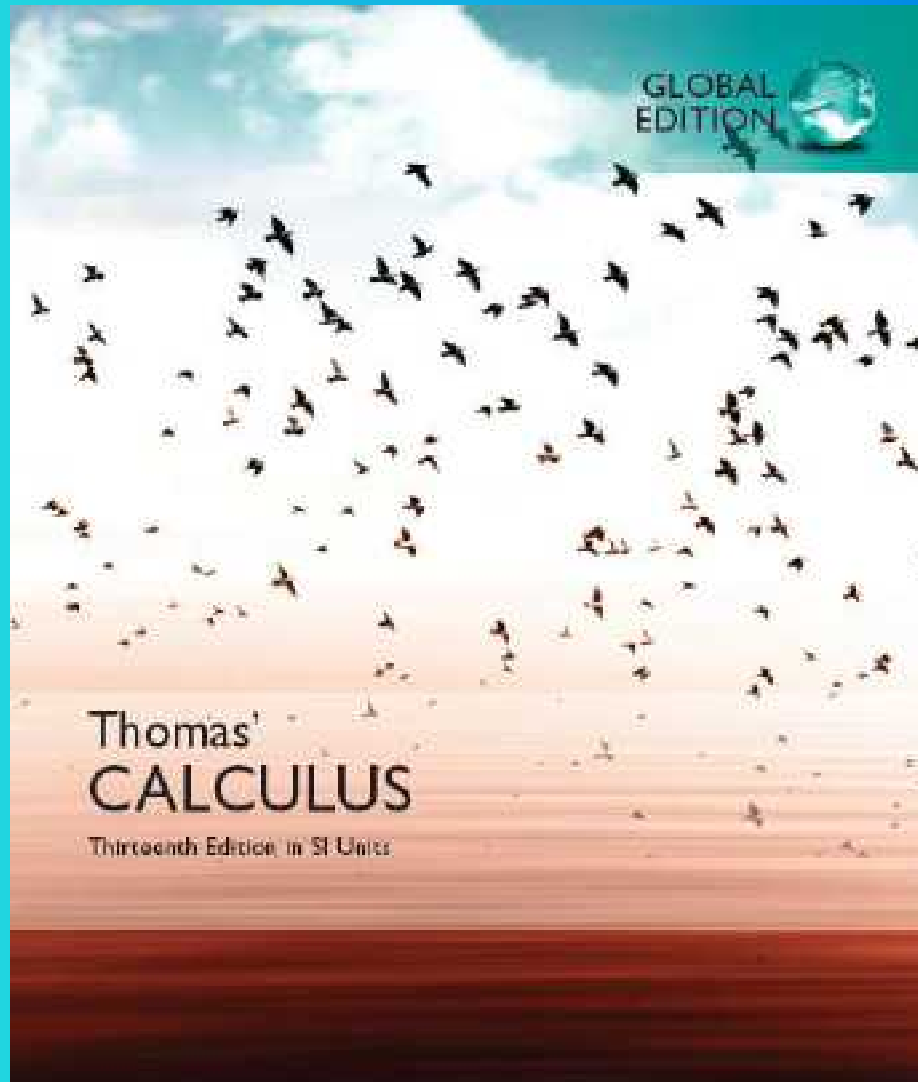
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Thirteenth Edition in SI Units

Chapter 2

Limits and Continuity

极限和连续性

2.1

Rates of Change and Tangents to Curves

变化率和曲线的切线

函数极限这个概念是由什么问题产生的呢？

Ex.1

已知自由落体运动的位移函数为

$y = 4.9t^2$,

(1)求在$t_0 = 1$时刻的速度；

(2)求在$t_0 = 2$时刻的速度.

平均速度

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

$$\text{Average speed: } \frac{\Delta y}{\Delta t} = \frac{4.9(t_0 + h)^2 - 4.9t_0^2}{h}$$

Length of time interval h	**Average speed over interval of length** h **starting at** $t_0 = 1$	**Average speed over interval of length** h **starting at** $t_0 = 2$
1	14.7	24.5
0.1	10.29	20.09
0.01	9.849	19.649
0.001	9.8049	19.6049
0.0001	9.80049 **→ 9.8**	19.60049 **→ 19.6**

函数在区间内的平均变化率

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

在x_1这点处的变化率？

$$h \to 0\text{时，}\frac{f(x_1+h) - f(x_1)}{h} \to ?$$

Ex.2 求曲线 $y = f(x)$

在点 $P(x_1, f(x_1))$ 处的切线斜率 .

曲线的切线？ → 割线变化的稳定位置

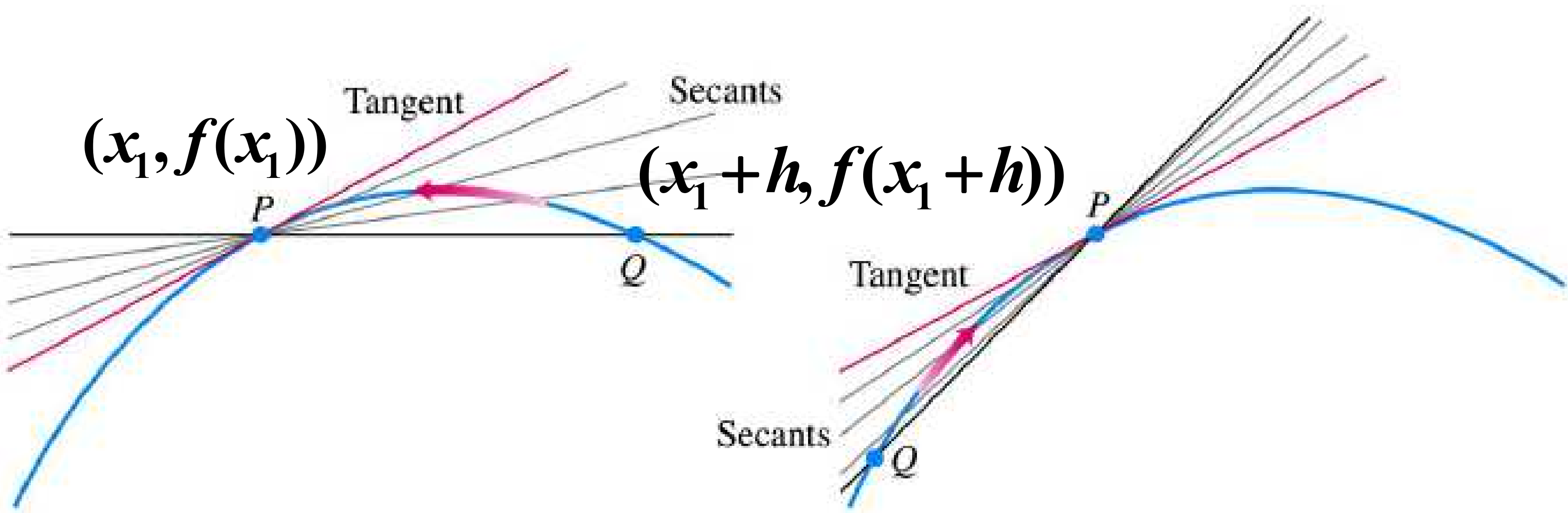


FIGURE 2.3 The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

曲线的割线的斜率

$$\frac{f(x_1+h)-f(x_1)}{h}$$

曲线在点P的切线的斜率？

$$h \to 0\text{时}, \frac{f(x_1+h)-f(x_1)}{h} \to ?$$

由于上面的例子可知，对函数需要研究

$$h \to 0\text{时}, \frac{f(x_1+h)-f(x_1)}{h} \to ?$$

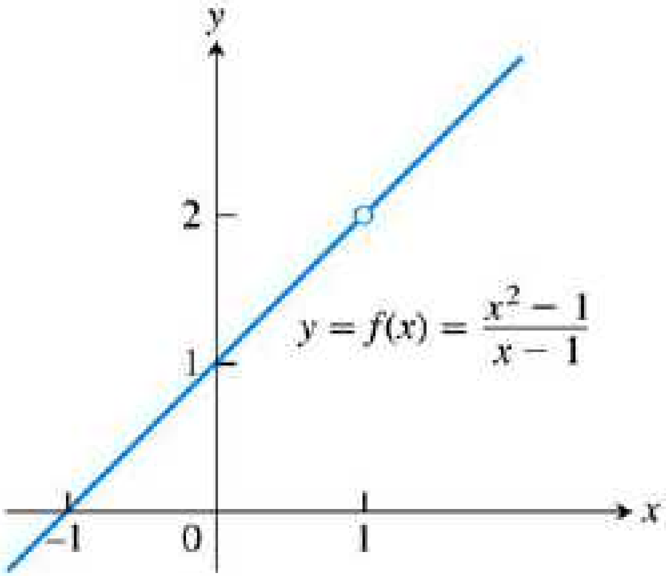
更一般地，对函数需要研究

$$x \to c\text{时}，f(x) \to ?$$

2.2

Limit of a Function and Limit Laws

函数的极限和极限法则



Ex.1

$$x \to 1\text{时，}f(x) = \frac{x^2 - 1}{x - 1} \to ?$$

$$x \to 1\text{时，}f(x) = \frac{x^2 - 1}{x - 1} \to 2$$

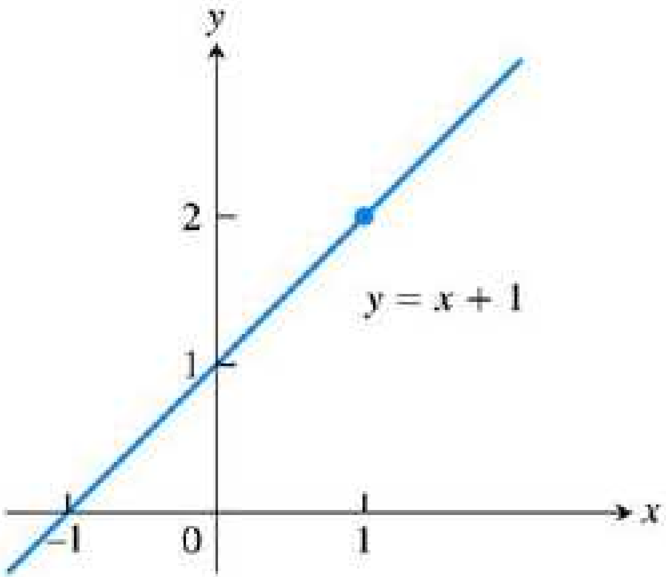
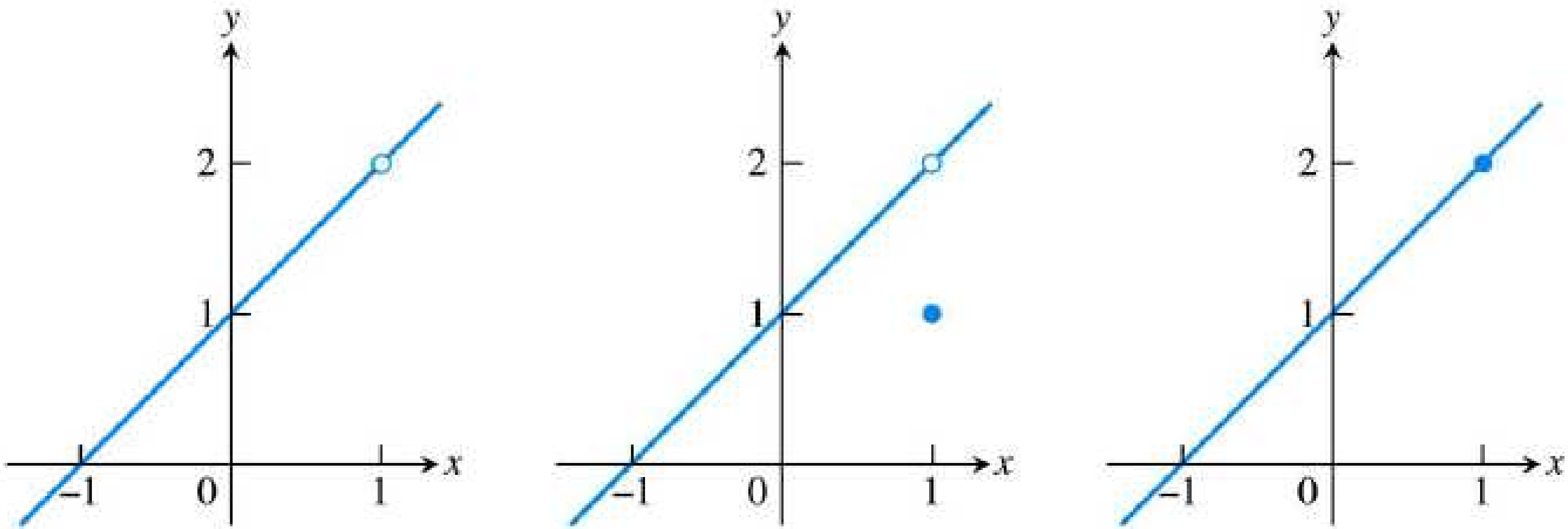


FIGURE 2.7 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

Ex.2 $x \to 1$时，$f(x)$, $g(x)$, $h(x) \to ?$



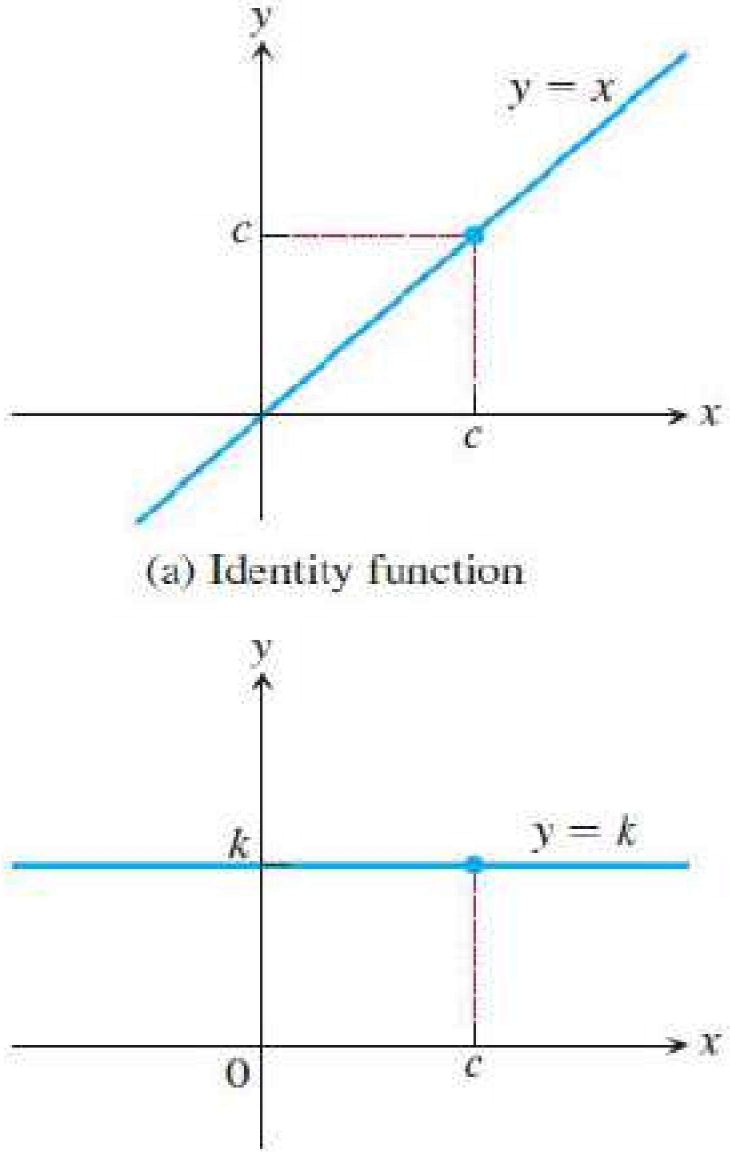
(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

Ex.3



(a) Identity function

(b) Constant function

FIGURE 2.9 The functions in Example 3 have limits at all points c.

$$\lim_{x \to c} x = c$$

$$\lim_{x \to c} k = k$$

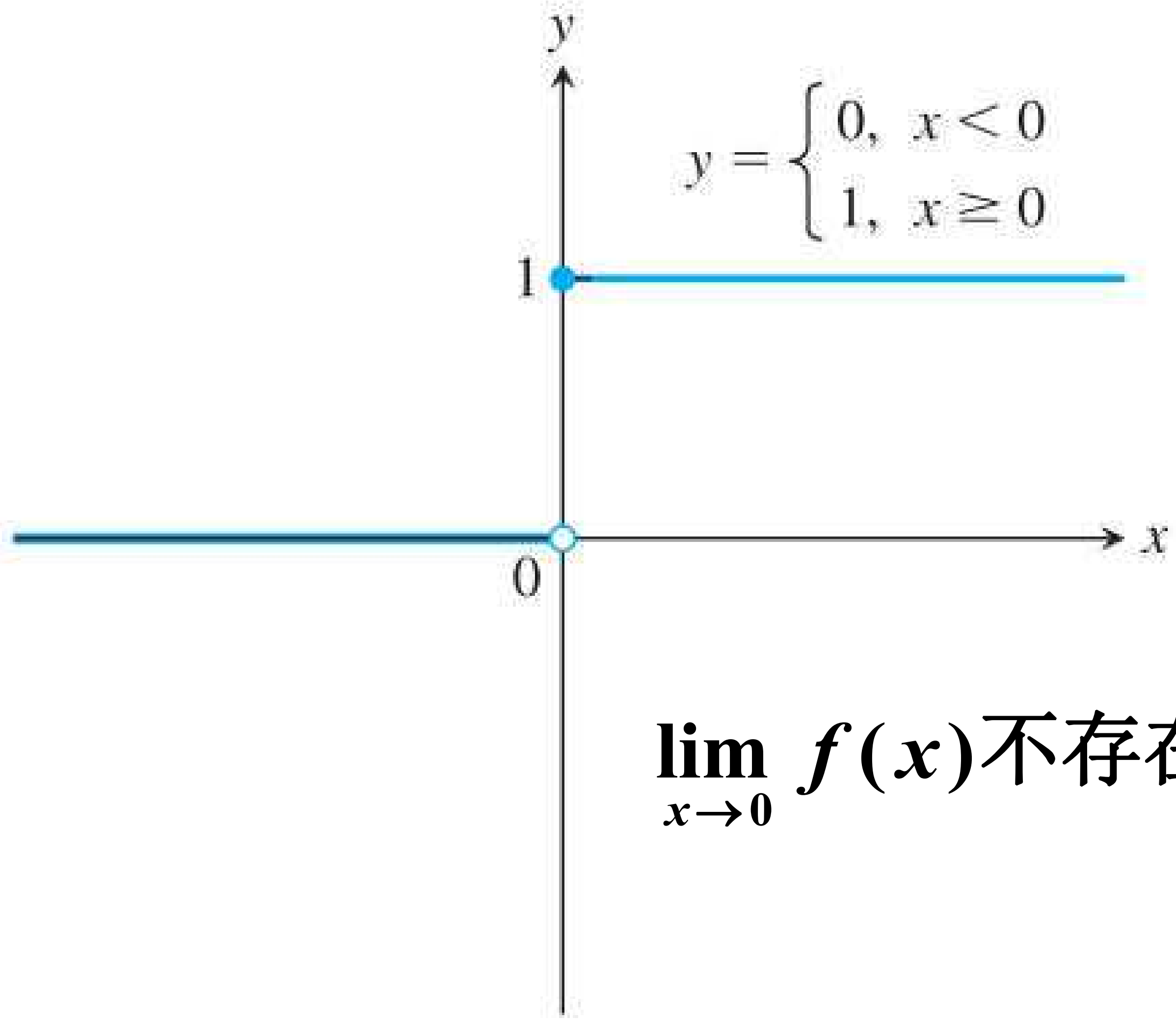
函数极限的描述性定义：

设函数$f(x)$在点$x=c$的左右附近有定义.
若只需x与c充分接近，$f(x)$的值可以与一个常数L任意接近，要多近有多 近，则称当x趋于c时，$f(x)$的极限是L,记为$\lim_{x\to c} f(x)=L$.

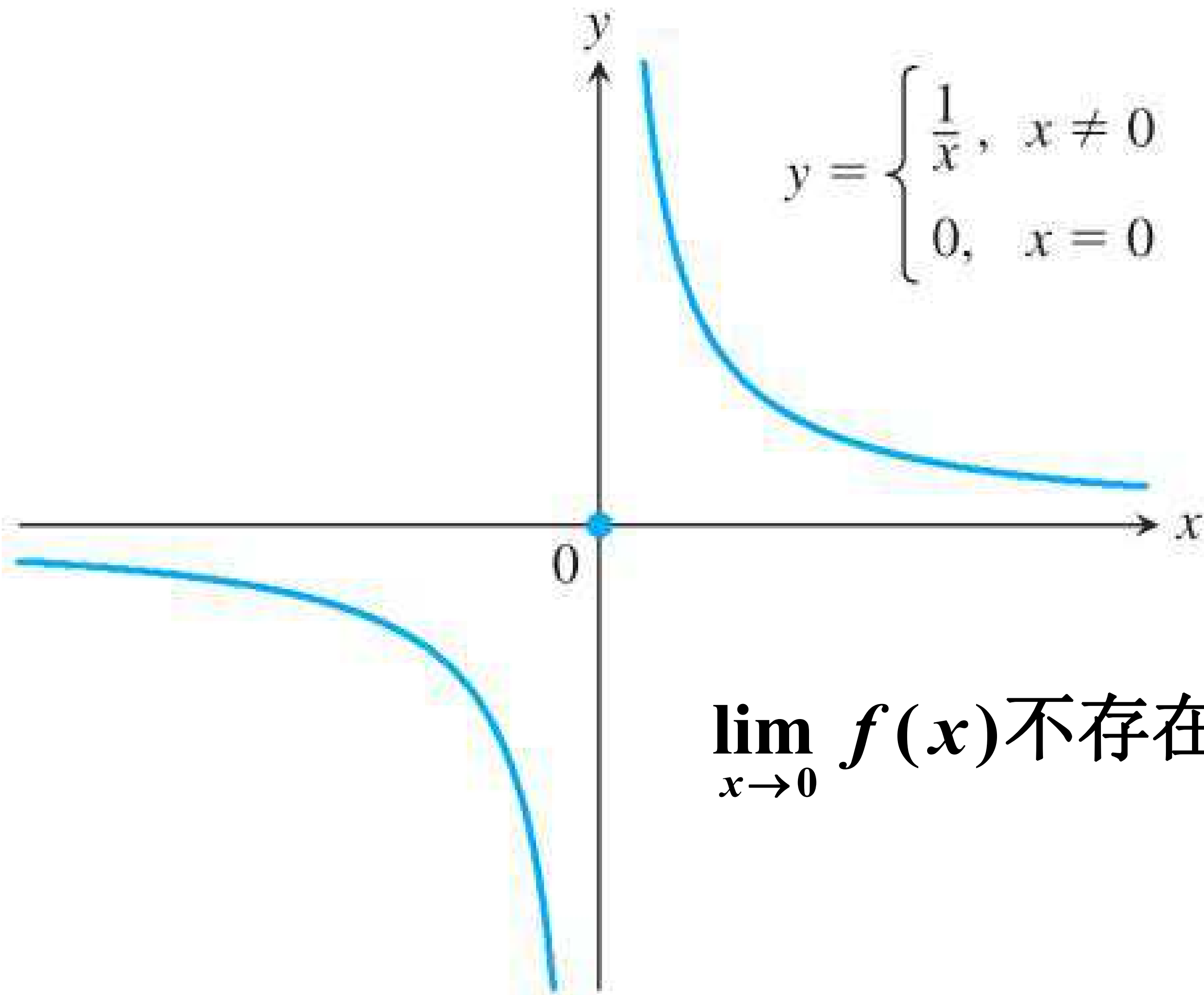
$\lim_{x\to c} f(x)=L$时$f(x)$在c处可以没有定义！

$\lim_{x\to c} x = c$ $\lim_{x\to c} k = k$

Ex.4



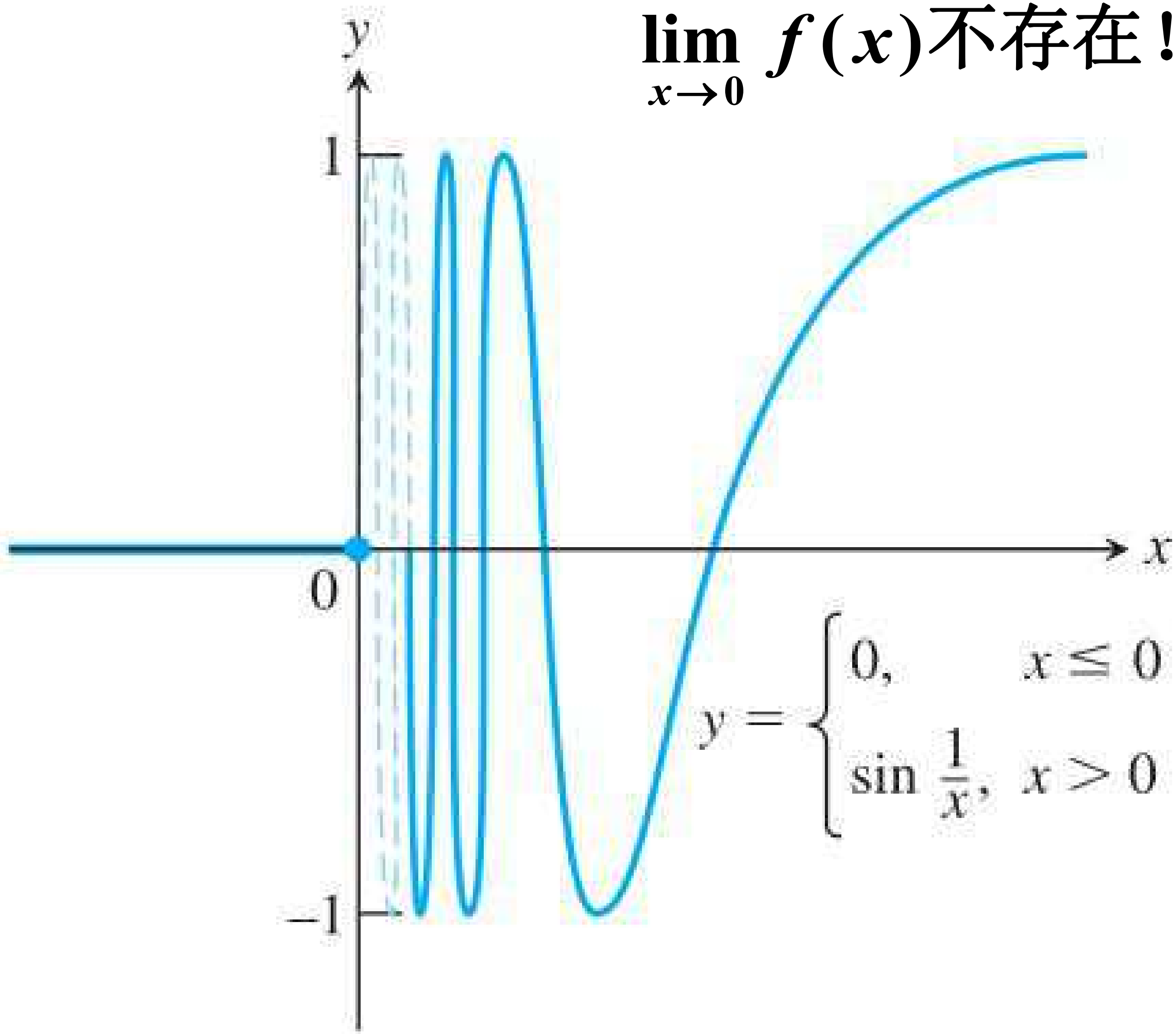
$\lim_{x \to 0} f(x)$不存在！

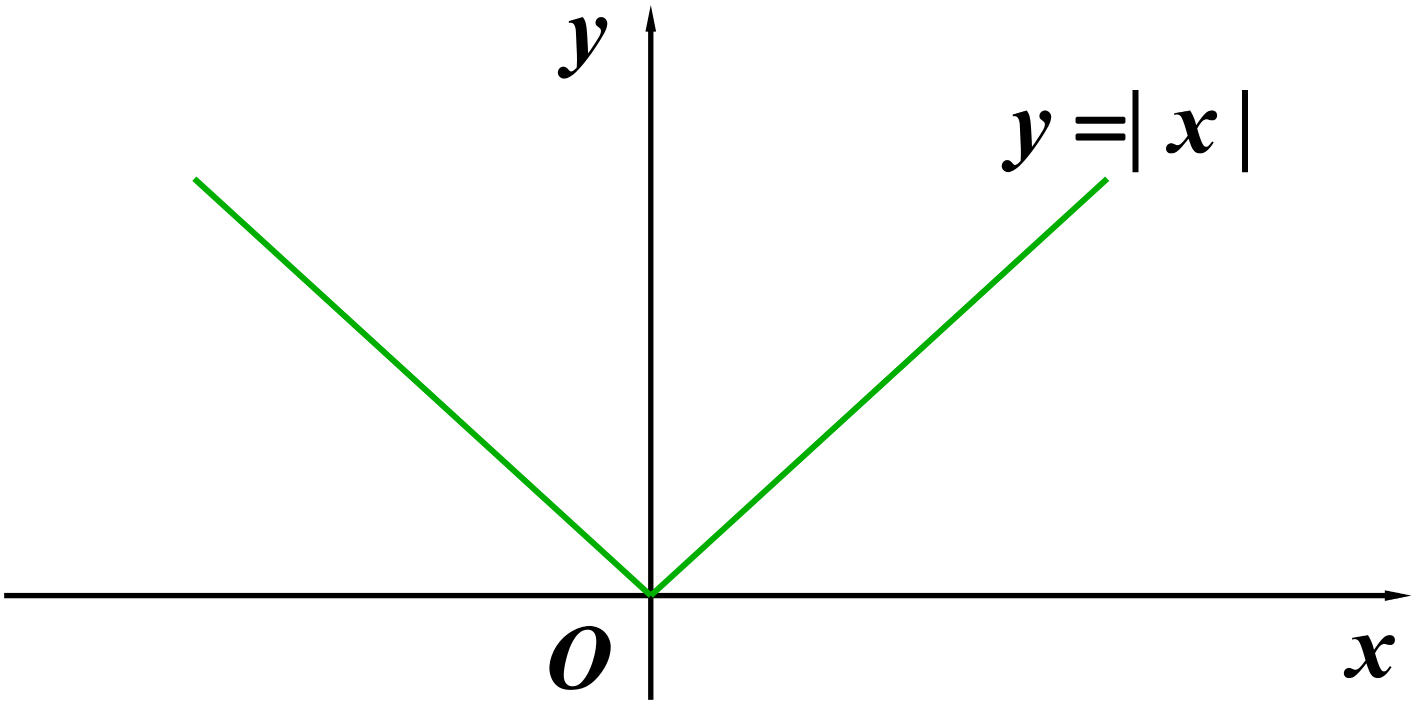


(b) $g(x)$

$$\lim_{x \to 0} f(x) \text{不存在！}$$

$$\lim_{x\to 0} f(x) \text{不存在！}$$





$$\lim_{x \to 0} |x| = 0$$

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

$$\lim_{x \to c} x^2 = c^2$$

1. *Sum Rule:* $\lim_{x \to c} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c} (f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$

$$\lim_{x \to 1} (2x^2 + x^3) = 3$$

4. *Product Rule:* $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

多项式的极限

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions 有理函数的极限

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Ex.5 $\lim_{x \to c}(x^3 + 4x^2 - 3) = c^3 + 4c^2 - 3.$

$$\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{c^4 + c^2 - 1}{c^2 + 5}.$$

$$\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{13}$$

Ex. 6 $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^2-x}$

solution $x\to 1$时,分子,分母的极限都是零.

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}=\lim_{x\to 1}\frac{(x+2)(x-1)}{x(x-1)}=\lim_{x\to 1}\frac{x+2}{x}$$

$$=3$$

Ex. 7 $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+100}-10}{x^2}$

$$=\lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}=\frac{1}{20}$$

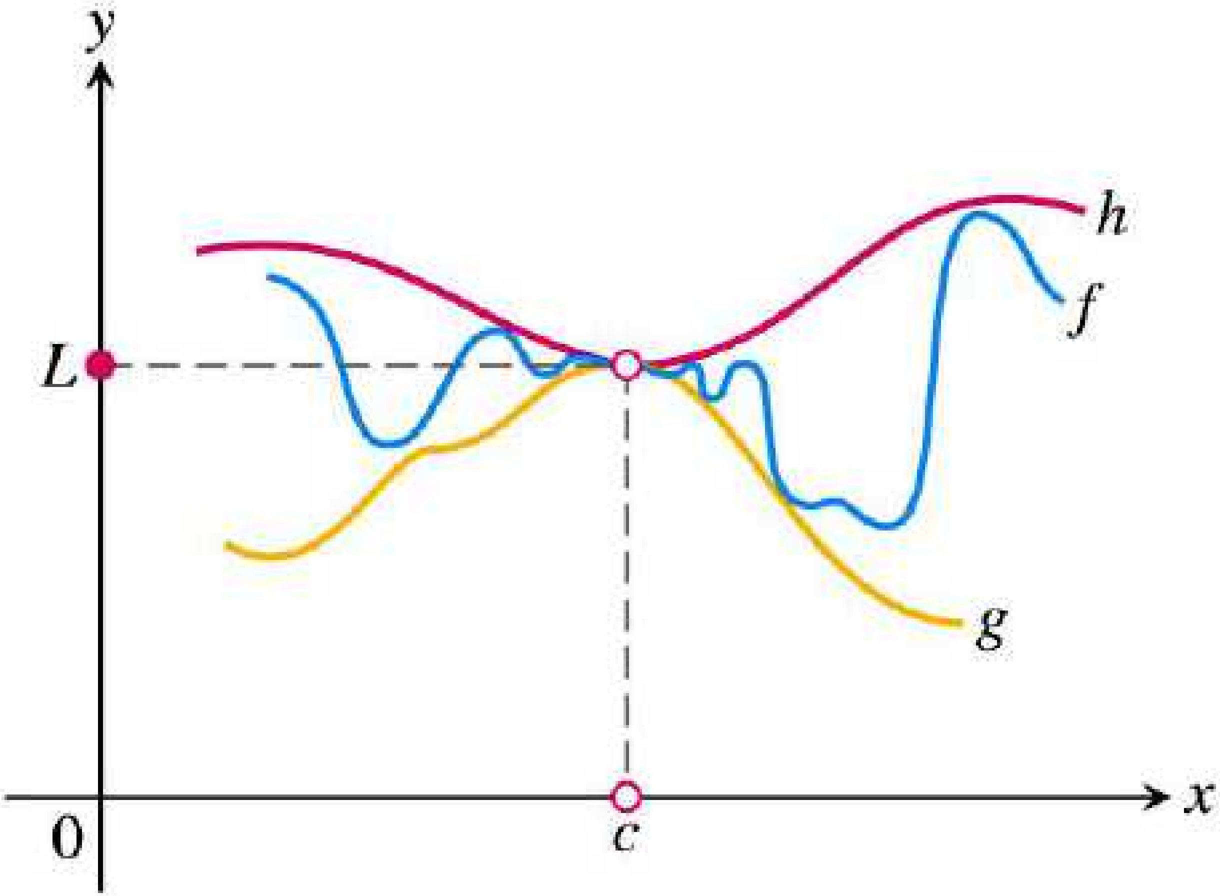


FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h.

夹逼定理

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Ex.8 $1 - \dfrac{x^2}{4} \le f(x) \le 1 + \dfrac{x^2}{2}$,求$\lim\limits_{x \to 0} f(x)$.

$$\lim_{x \to 0} f(x) = 1$$

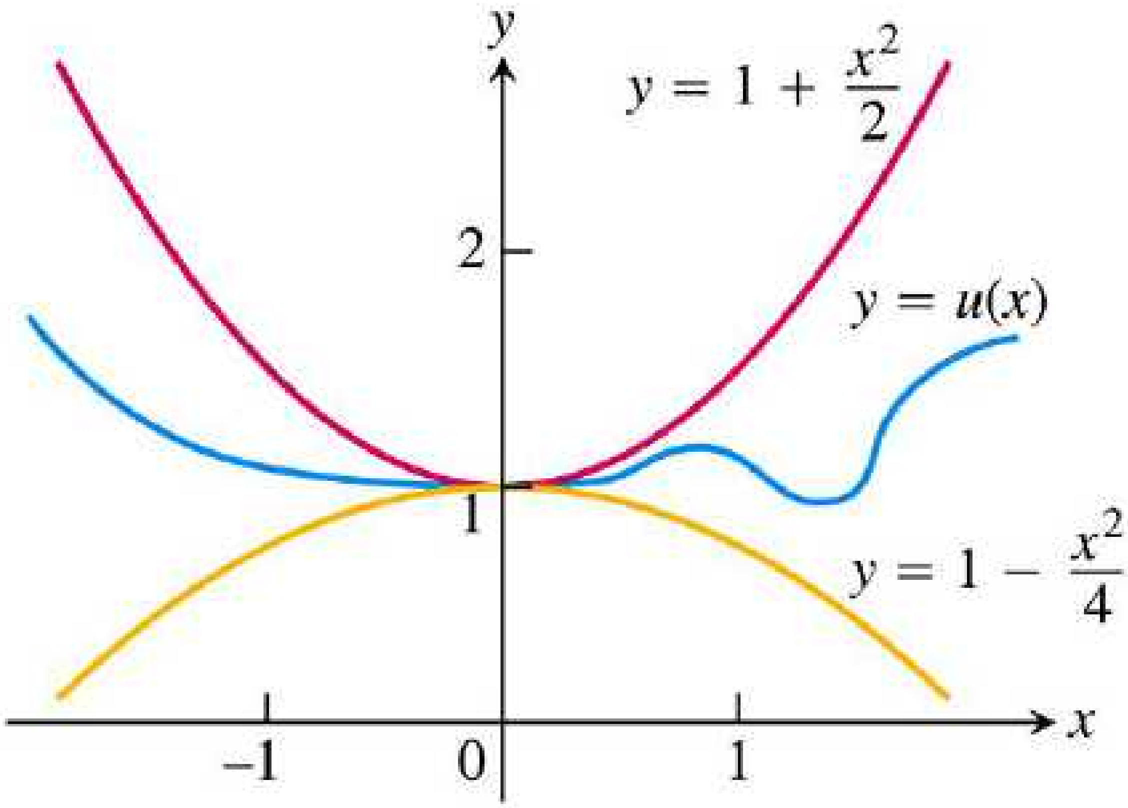


FIGURE 2.13 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$ (Example 10).

Ex. 9 证明 (a) $\lim_{x\to 0}\sin x = 0$ (b) $\lim_{x\to 0}\cos x = 1$

(c) $\lim_{x\to c}|f(x)| = 0 \Leftrightarrow \lim_{x\to c} f(x) = 0.$

证明： (a) $-|x| \le \sin x \le |x|$

(b) $0 \le 1-\cos x = 2\sin^2\dfrac{x}{2} \le \dfrac{x^2}{2}$

(c)“$\Rightarrow$”: $-|f(x)| \le f(x) \le |f(x)|.$

“$\Leftarrow$”: $|f(x)| = \sqrt{f^2(x)}.$

问题：$\lim_{x\to c}|f(x)| = |l| \Leftrightarrow \lim_{x\to c} f(x) = l$?

THEOREM 5 If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x).$$

作业：

2.1 5,9,

2.2 2,22,32,42,47,61,65,78

补充作业

1.计算极限 $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$ $\lim\limits_{x\to 1}\dfrac{x^4-1}{x^3-1}$

2.证明 若 $\lim\limits_{x\to 0} g(x)=0$,则 $\lim\limits_{x\to 0} g(x)\sin\dfrac{1}{x}=0$

3.找反例说明命题

"若 $\lim\limits_{x\to c}|f(x)|=|l|$,则 $\lim\limits_{x\to c} f(x)=l$。"

是错误的。

4.找反例说明命题

"若 $\lim\limits_{x\to c}[f(x)+g(x)]$存在,则 $\lim\limits_{x\to c} f(x)$和

$\lim\limits_{x\to c} g(x)$存在"是错误的。

2.3

The Precise Definition of a Limit

函数极限的精确定义

函数极限的描述性定义：

设函数$f(x)$在点$x=c$的左右附近有定义．若只需x与c充分接近，$f(x)$的值可以与一个常数L任意接近，要多近有多近，则称当x趋于c时，$f(x)$的极限是L, 记为$\lim\limits_{x\to c} f(x)=L$.

如何精确刻划上述“任意接近”“充分接近”?

对于任给 $\varepsilon>0$, 可使 $|f(x)-L|<\varepsilon$ ；只要x与c充分靠近

x与c需多近 ？ 应该与ε有关的！

存在一个与ε有关的$\delta>0$, 当$0<|x-c|<\delta$时，

可使$|f(x)-L|<\varepsilon$ 即可

Ex.1

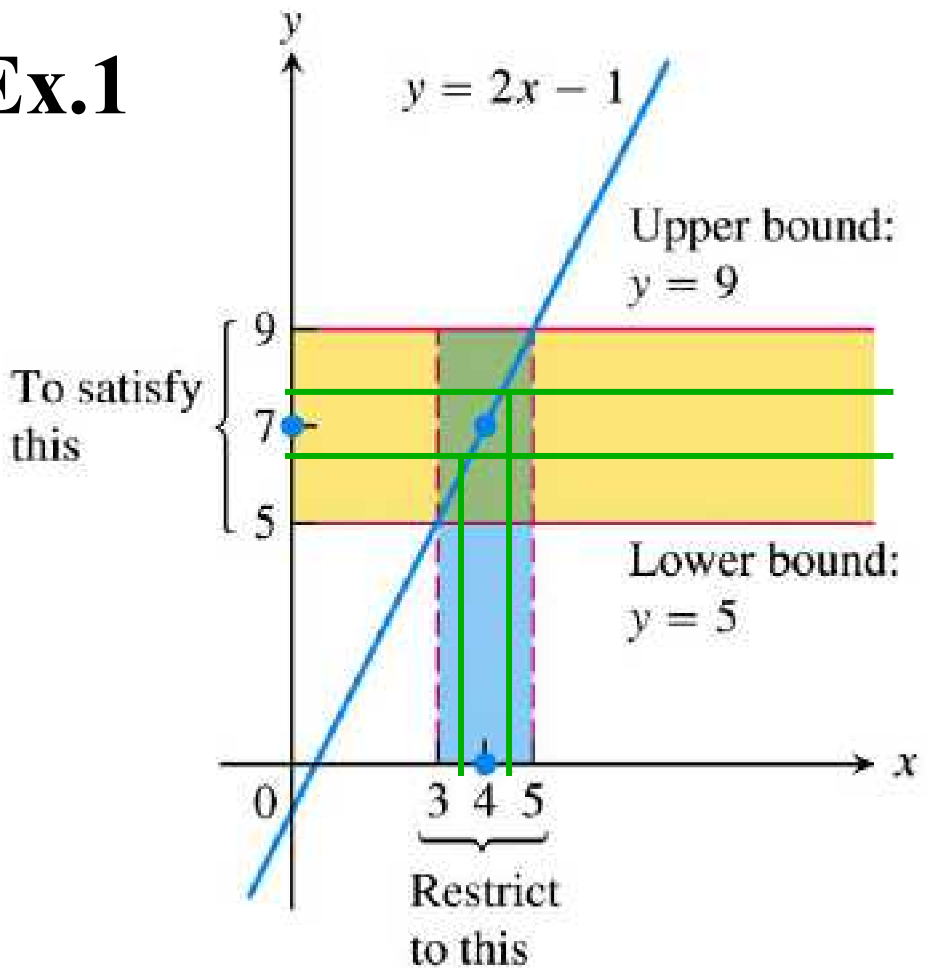


FIGURE 2.15 Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Example 1).

若要$|f(x)-L|<2$

即$|2x-1-7|<2$

只需$|x-4|<1$

若要$|f(x)-L|<1$

即$|2x-1-7|<1$

只需$|x-4|<\dfrac{1}{2}$.

若要$|f(x)-L|<\varepsilon$

即$|2x-1-7|<\varepsilon$

只需$|x-4|<\dfrac{\varepsilon}{2}$.

DEFINITION Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

注意： 1.函数极限与$f(x)$在点x_0是否有定义无关；

2.δ与任意给定的正数 ε有关.

3.δ不唯一，若存在一个，则有无穷多.

4.可以只考虑$0 < \varepsilon < 1$.

5.使$|f(x) - L| < 2\varepsilon$ 也可.

Ex.2 证明: $\lim_{x\to 1}(5x-3)=2.$

证 任给 $\varepsilon>0,$

要使 $|5x-3-2|<\varepsilon$, 只需 $|x-1|<\dfrac{\varepsilon}{5},$

取 $\delta=\dfrac{\varepsilon}{5},$ 当 $0<|x-1|<\delta$ 时,

有 $|5x-3-2|<\varepsilon.$

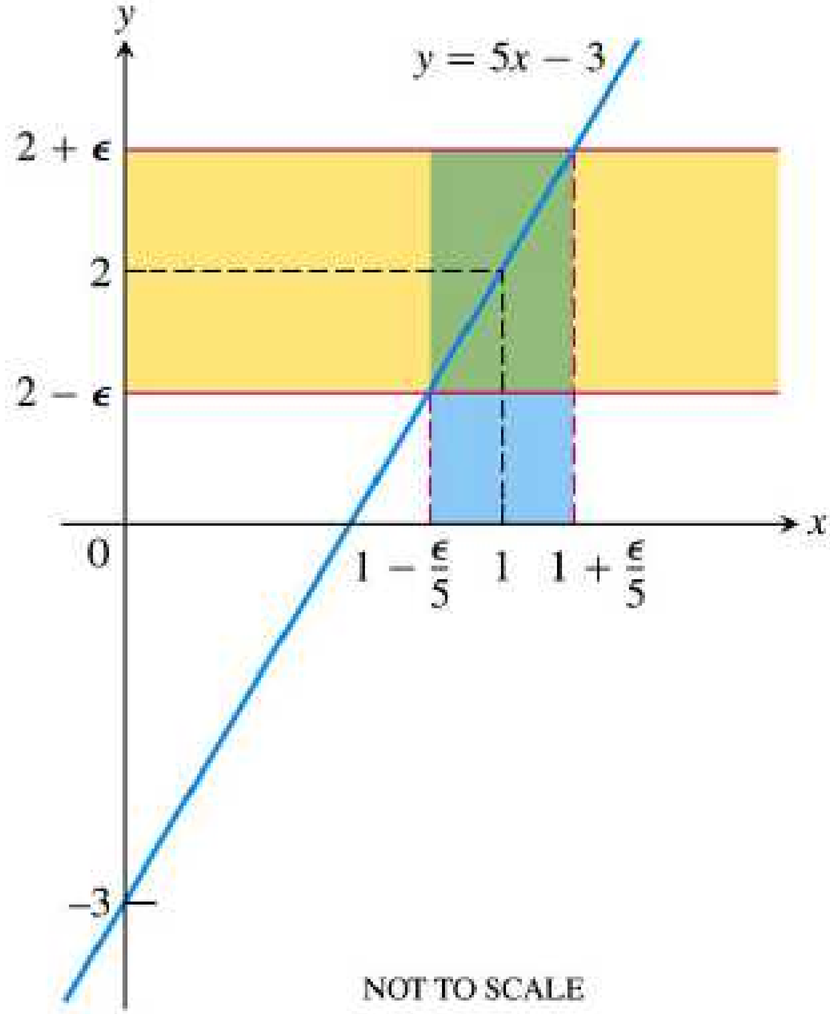


FIGURE 2.18 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

Ex.3(a) 证明 $\lim\limits_{x\to c} k = k$, ($k$为常数).

证 任给$\varepsilon > 0$, 任取$\delta > 0$, 当$0 < |x-c| < \delta$时,

$$|f(x)-L| = |k-k| = 0 < \varepsilon \text{成立.}$$

(b) 证明 $\lim\limits_{x\to c} x = c$.

证 $\because |f(x)-L| = |x-c|$, 任给$\varepsilon > 0$, 取$\delta = \varepsilon$,

当$0 < |x-c| < \delta = \varepsilon$时,

$$|f(x)-L| = |x-c| < \varepsilon \text{成立.}$$

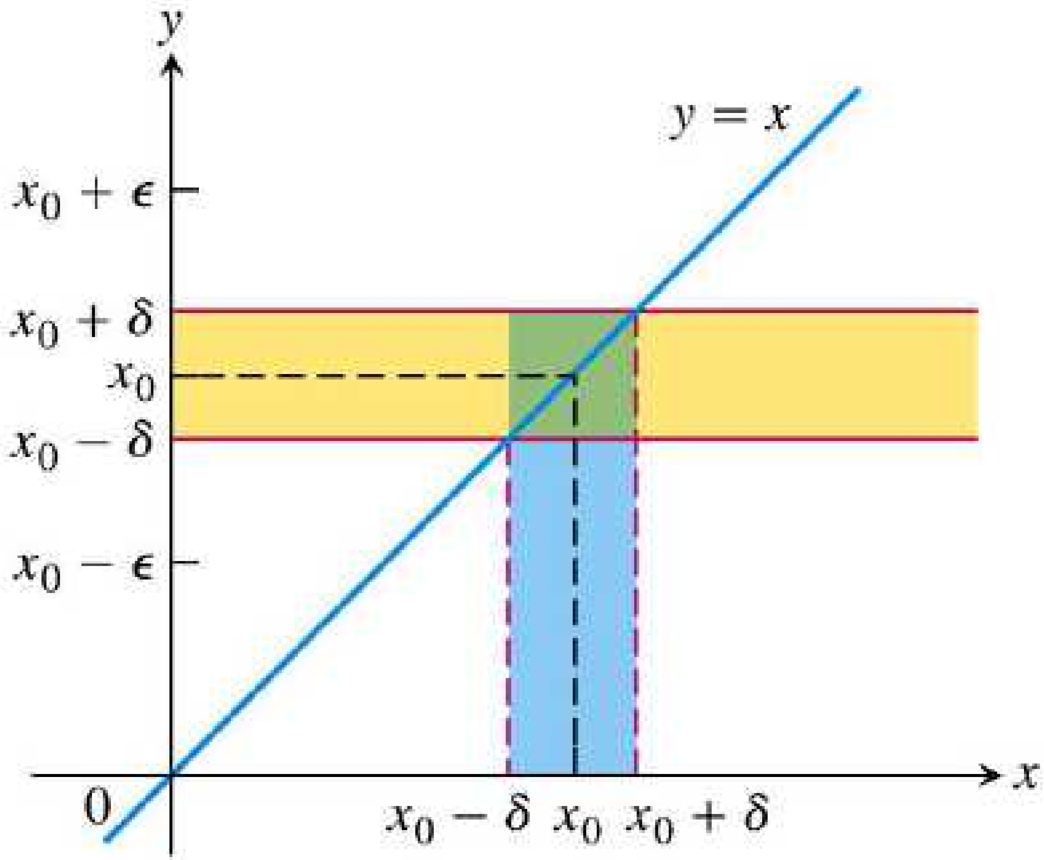


FIGURE 2.19 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \leq \epsilon$ (Example 3a).

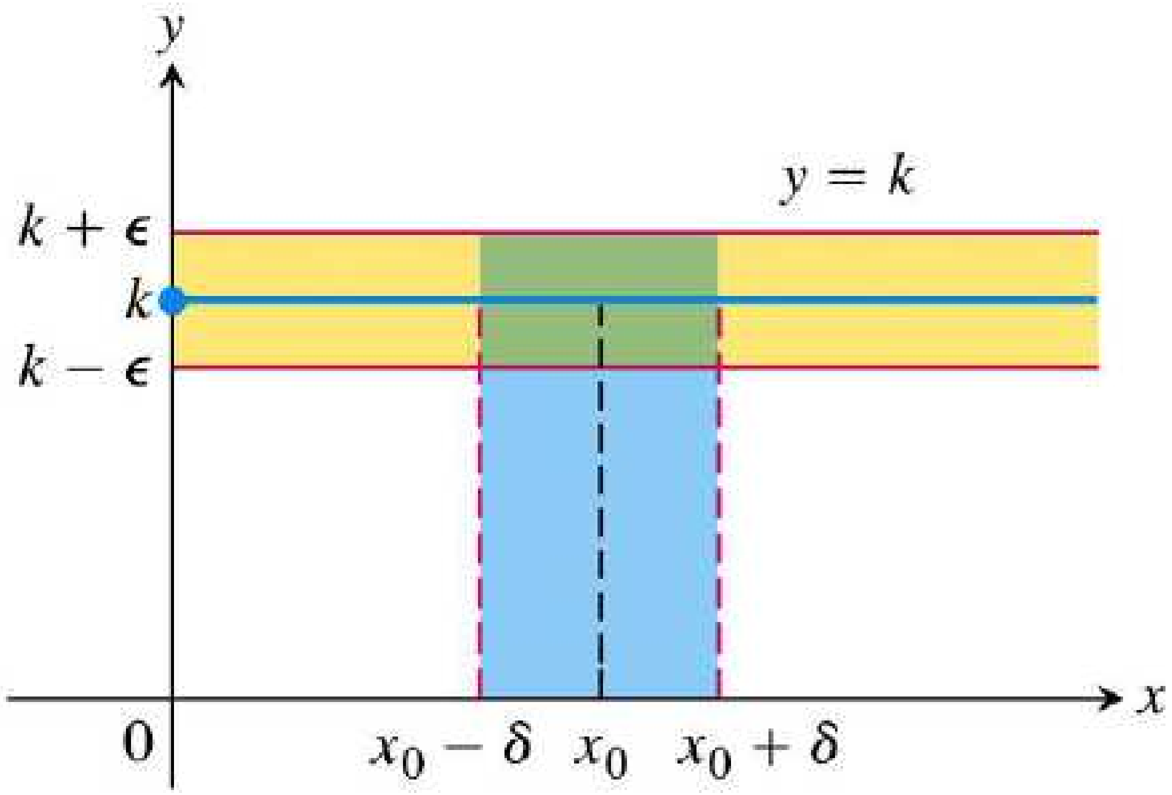


FIGURE 2.20 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 3b).

对于任给 $\varepsilon > 0$, 是否要找使 $|f(x) - L| < \varepsilon$ 成立的所有的 x?

Ex.4　Proof　$\lim\limits_{x\to 5}\sqrt{x-1} = 2.$

Proof.　任给 $\varepsilon > 0$,　要使 $\left|\sqrt{x-1} - 2\right| < \varepsilon$

只需 $5 - (4\varepsilon - \varepsilon^2) < x < 5 + (4\varepsilon + \varepsilon^2)$

只需 $-(4\varepsilon - \varepsilon^2) < x - 5 < (4\varepsilon + \varepsilon^2)$

取 $\delta = 4\varepsilon - \varepsilon^2$,　当 $0 < |x - 5| < \delta$ 时,

有 $\left|\sqrt{x-1} - 2\right| < \varepsilon.$

对称

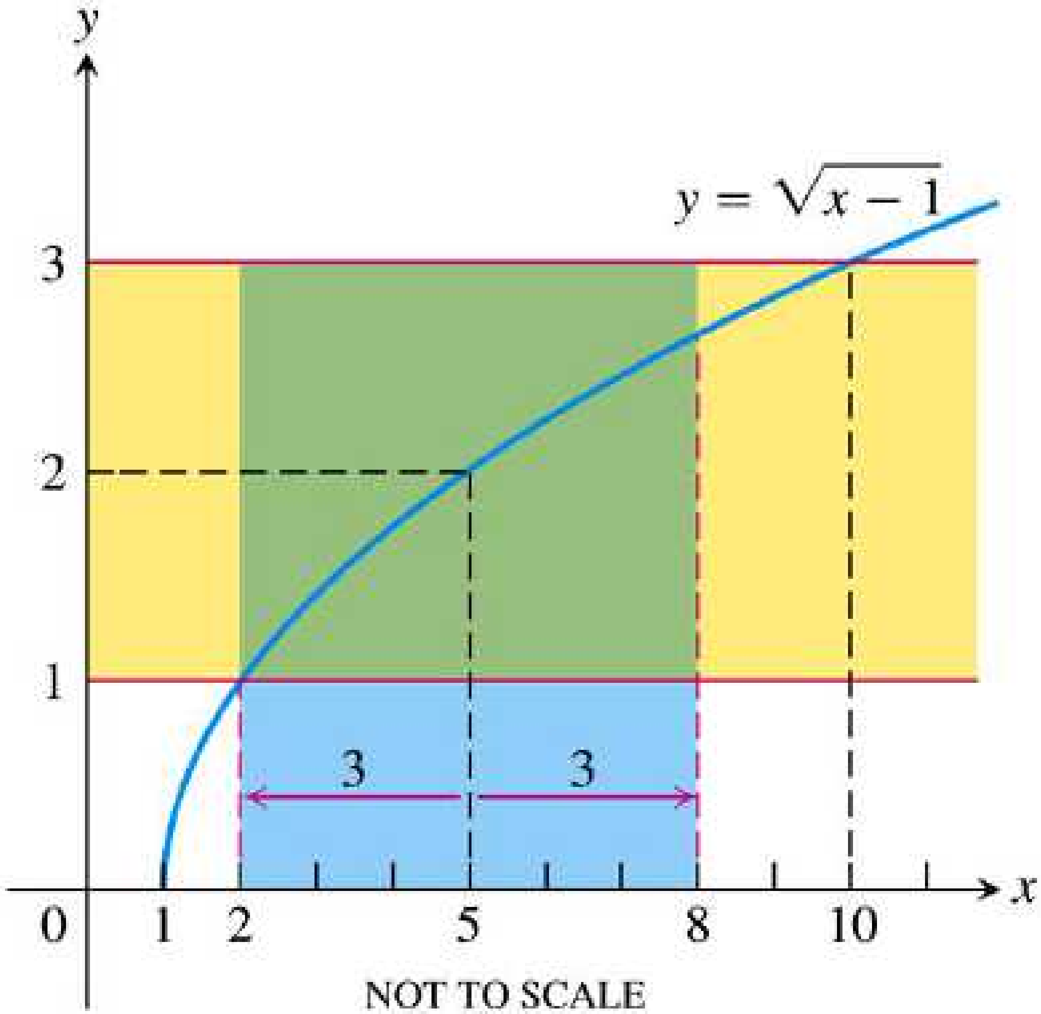


FIGURE 2.22 The function and intervals in Example 4.

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.

2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

Ex.5 Proof $\lim_{x\to 2} f(x) = 4,\ f(x) = \begin{cases} x^2, & x \neq 2, \\ 1, & x = 2. \end{cases}$

Proof. 任给$\varepsilon > 0$,

要使$|f(x) - 4| < \varepsilon \quad (x \neq 2)$, 即$|x^2 - 4| < \varepsilon$

只需$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$

取$\delta = \min\{2 - \sqrt{4-\varepsilon}, \sqrt{4+\varepsilon} - 2\}$,

当$0 < |x - 2| < \delta$时, 有$|f(x) - 4| < \varepsilon$.

用极限的精确定义证明

Ex.6 $\lim\limits_{x\to c} f(x)=L, \lim\limits_{x\to c} g(x)=M$,则

$$\lim_{x\to c}(f(x)\pm g(x))=L\pm M.$$

Ex.7 若 $\lim\limits_{x\to c} f(x)=L$,且 $L>0(<0)$,则 保号性

存在某 $0<|x-c|<\delta$,使 $f(x)>0(<0)$.

Ex.8 若 $\lim\limits_{x\to c} f(x)=L$,且 $\forall$ $f(x)\geq 0(\leq 0)$,则 $L\geq 0(\leq 0)$.

反证

Ex.9 若 $\lim\limits_{x\to c} f(x)=L, \lim\limits_{x\to c} g(x)=M$,

且 $f(x)\geq g(x)$,则 $L\geq M$.

2.4

One-Sided Limits

单侧极限

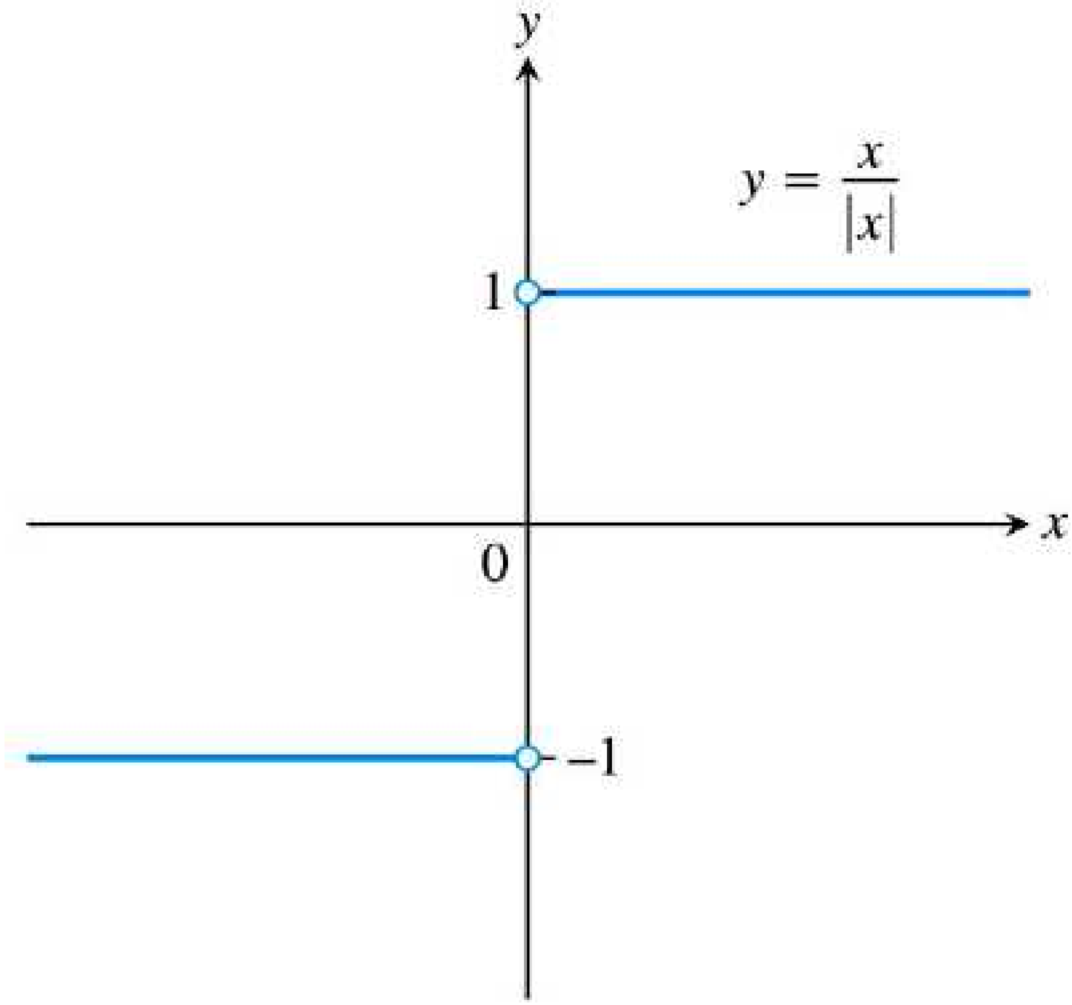
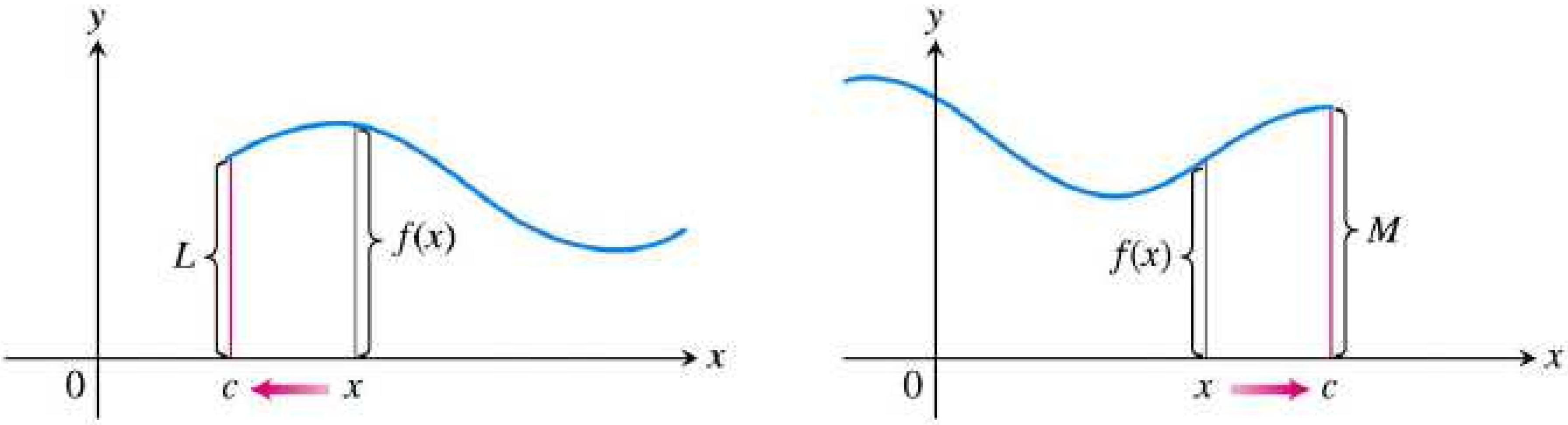


FIGURE 2.24 Different right-hand and left-hand limits at the origin.



(a) $\lim_{x \to c^+} f(x) = L$

(b) $\lim_{x \to c^-} f(x) = M$

FIGURE 2.25 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

Ex.1

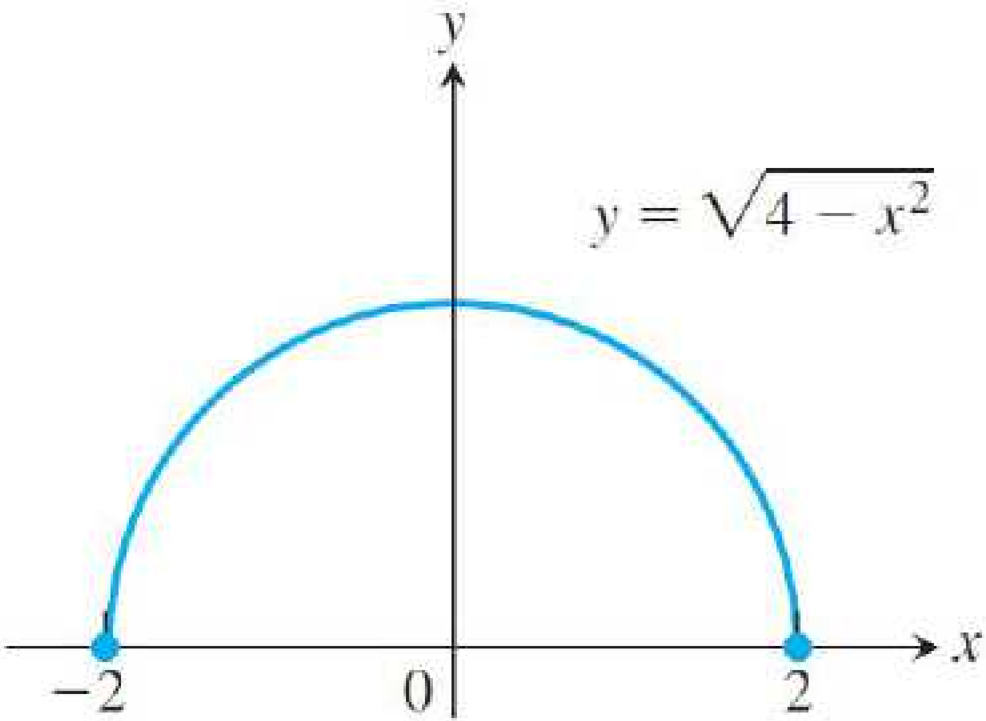


FIGURE 2.26 The function $f(x) = \sqrt{4 - x^2}$ has right-hand limit 0 at $x = -2$ and left-hand limit 0 at $x = 2$ (Example 1).

Ex.2

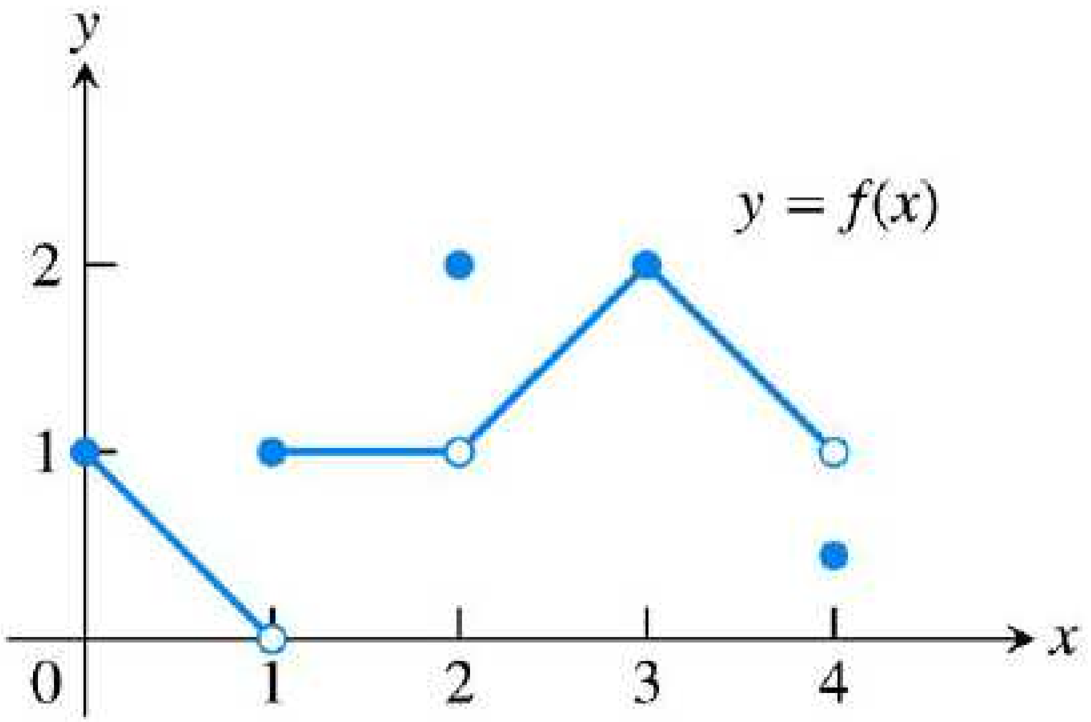


FIGURE 2.27 Graph of the function in Example 2.

DEFINITIONS We say that $f(x)$ has **right-hand limit L at c**, and write

$$\lim_{x \to c^+} f(x) = L \qquad \text{(see Figure 2.28)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c < x < c + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit L at c**, and write

$$\lim_{x \to c^-} f(x) = L \qquad \text{(see Figure 2.29)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c - \delta < x < c \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Ex.3 证明 $\lim\limits_{x\to 0^+}\sqrt{x}=0.$

$Pr\,oof.$ 任给 $\varepsilon>0,$

要使 $\left|\sqrt{x}-0\right|<\varepsilon$ 即 $\sqrt{x}<\varepsilon$

只需 $0<x<\varepsilon^2,$ 取 $\delta=\varepsilon^2,$

当 $0<x<\delta$ 时, 有 $\left|\sqrt{x}-0\right|<\varepsilon.$

Ex.4

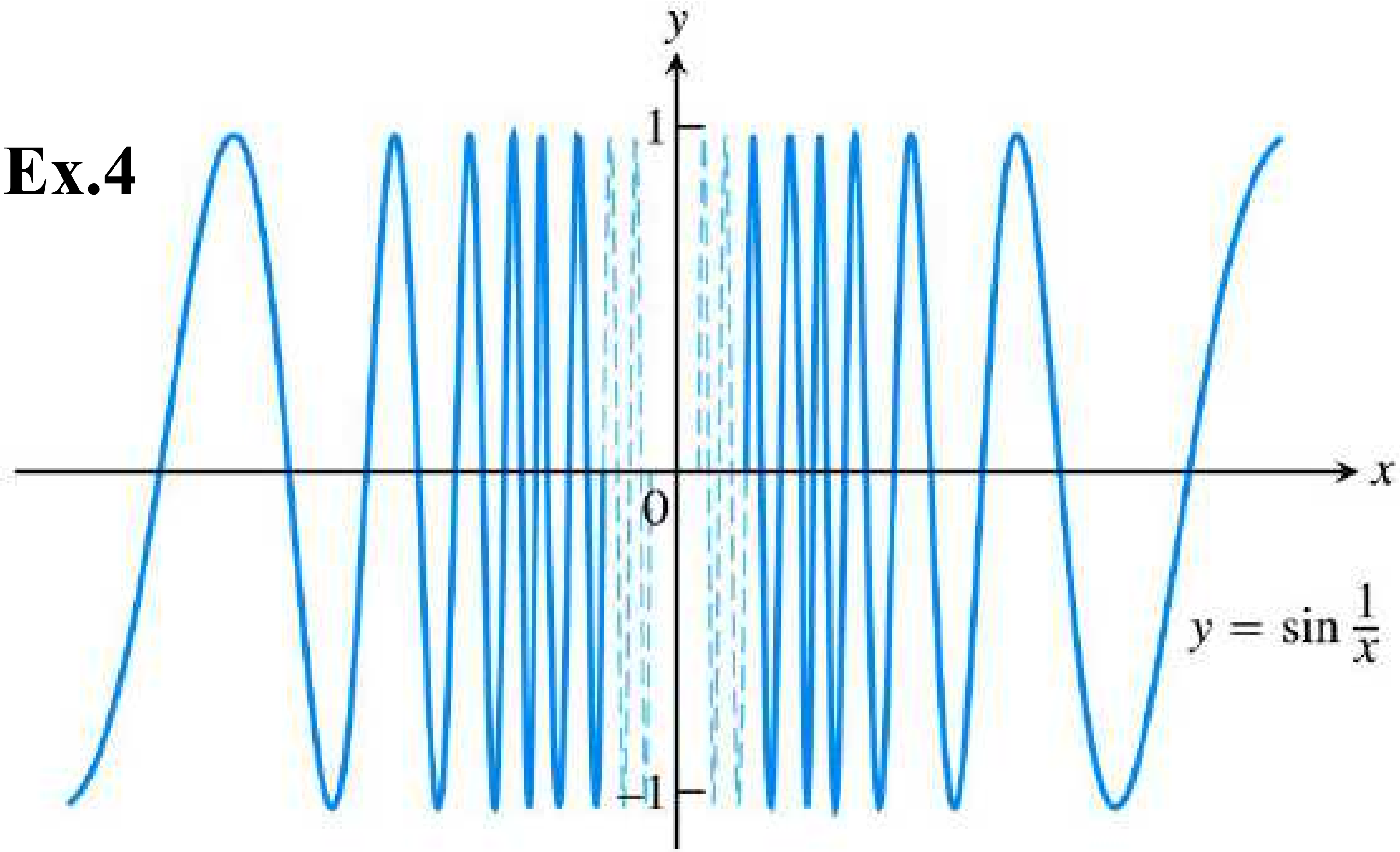


FIGURE 2.31 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4). The graph here omits values very near the y-axis.

Ex.5

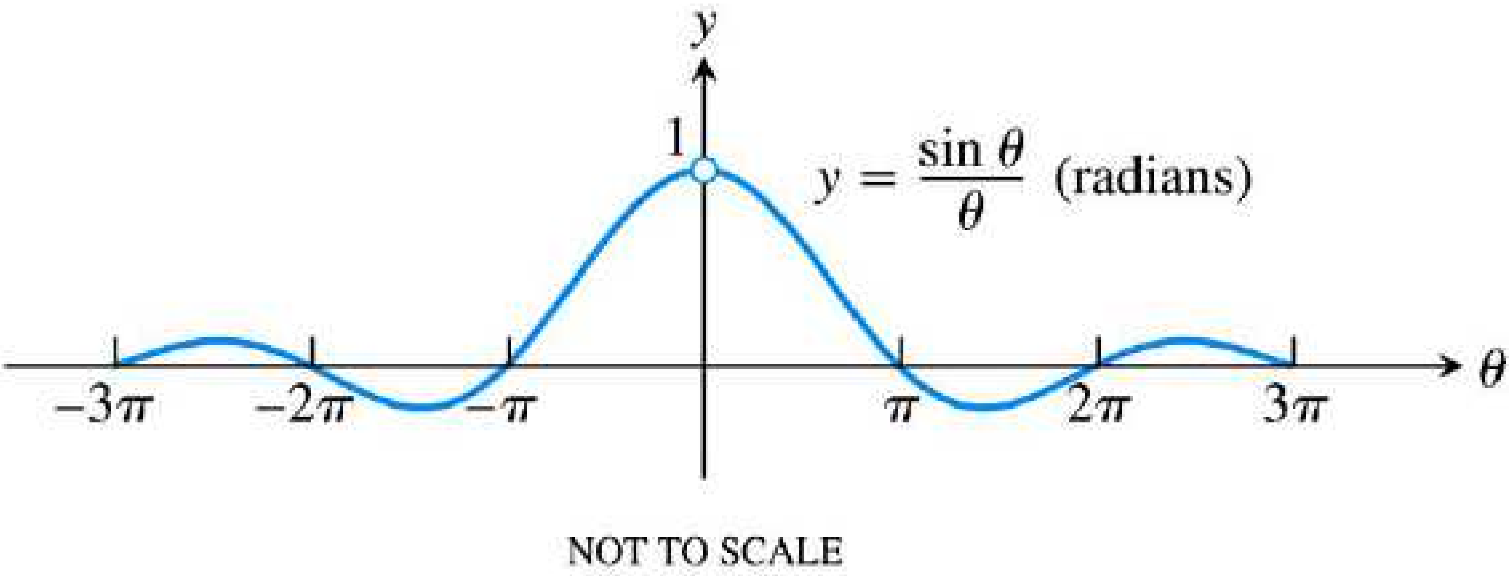


FIGURE 2.32 The graph of $f(\theta) = (\sin\theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

重要极限

THEOREM 7—Limit of the Ratio sin θ/θ as $\theta \to 0$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

Ex. 6 (*a*) Evaluate $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$.

解

$$\text{原式}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\frac{1}{2}\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}=\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}.$$

(*b*) Find $\lim\limits_{x\to\pi}\dfrac{\sin 2x}{5(\pi-x)}$.

解

$$\text{原式}=\lim_{u\to 0}\frac{-\sin 2u}{5u}\qquad (u=\pi-x)$$

$$=-\frac{2}{5}$$

Ex. 7 Evaluate $\lim\limits_{x \to 0} \dfrac{\tan x \sec 2x}{3x}$.

解 原式 $= \dfrac{1}{3} \lim\limits_{x \to 0} \dfrac{\sin x}{x} \cdot \dfrac{1}{\cos 2x} \cdot \dfrac{1}{\cos x}$

$= \dfrac{1}{3}$.

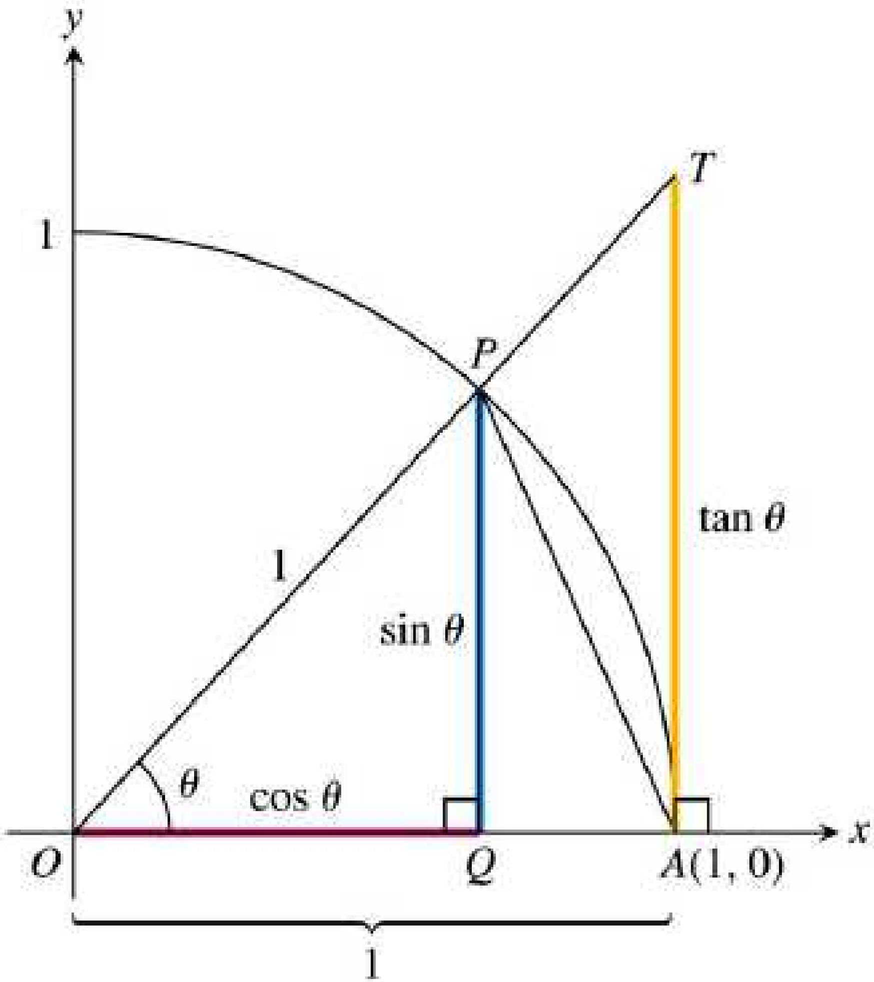


FIGURE 2.33 The figure for the proof of Theorem 7. By definition, $TA/OA = \tan\theta$, but $OA = 1$, so $TA = \tan\theta$.

当$0<\theta<\dfrac{\pi}{2}$时,

$$\frac{1}{2}\sin\theta<\frac{\theta}{2}<\frac{1}{2}\tan\theta,$$

$$1<\frac{\theta}{\sin\theta}<\frac{1}{\cos\theta},$$

$$\cos\theta<\frac{\sin\theta}{\theta}<1,$$

$$\therefore \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1,$$

作业：

- 2.3　41,49
- 2.4　1,5,9,15,17,19,25,34,41,45.

补充作业:

1.判断下列命题

(1)若 $\lim\limits_{x\to c} f(x)=l$,则存在某 $\delta>0$,使得$f(x)$在c的邻域$(c-\delta,c)\cup(c,c+\delta)$内有界 .

(2)若 $\lim\limits_{x\to c} f(x)=1$,则存在某 $\delta>0$,使得在 c的邻域$(c-\delta,c)\cup(c,c+\delta)$内$f(x)<\dfrac{3}{2}$.

(3)若$f(x)>0$,且 $\lim\limits_{x\to c} f(x)=l$,则$l>0$.

2.计算极限 $\lim\limits_{x\to 0}\dfrac{\tan 2x}{3x}$ $\lim\limits_{x\to 0}[\dfrac{1}{x}]\sin x$

3.若 $\lim\limits_{x\to 0^+} f(x)=l, \lim\limits_{x\to 0^-} f(x)=m$,问下列极限存在吗？若存在则求出 来。

$\lim\limits_{x\to 0} f(-x)$ $\lim\limits_{x\to 0+} f(x^2-x)$

$\lim\limits_{x\to 0^-}(2f(-x)+f(x^2))$

2.5

Continuity

Ex.1

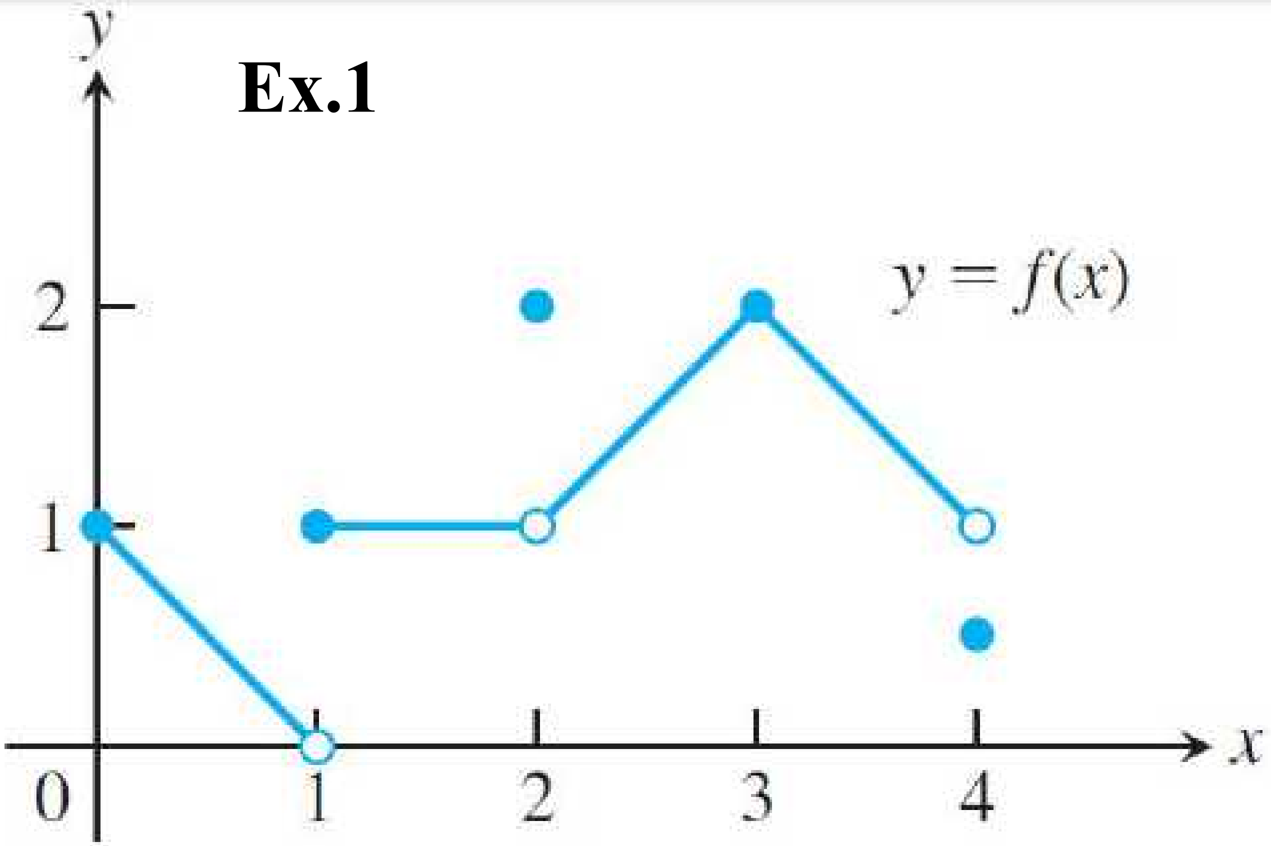


FIGURE 2.35 The function is not continuous at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

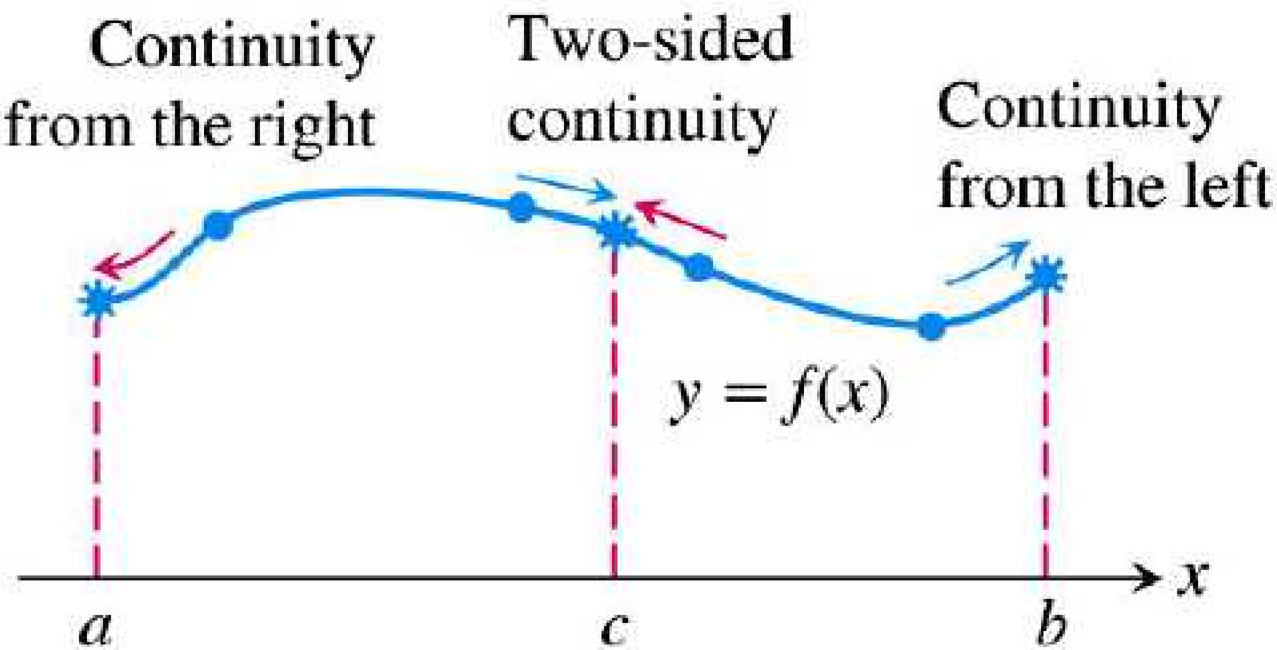


FIGURE 2.36 Continuity at points a, b, and c.

DEFINITIONS Let c be a real number on the x-axis.

The function f is **continuous at c** if

$$\lim_{x \to c} f(x) = f(c).$$

The function f is **right-continuous at c (or continuous from the right)** if

$$\lim_{x \to c^+} f(x) = f(c).$$

The function f is **left-continuous at c (or continuous from the left)** if

$$\lim_{x \to c^-} f(x) = f(c).$$

Ex.2

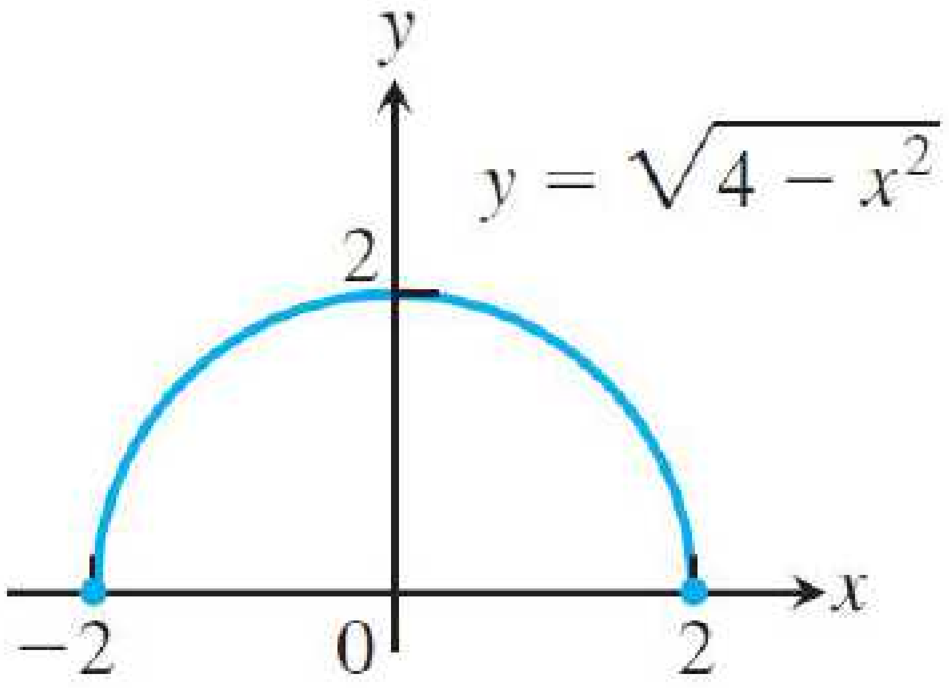


FIGURE 2.37 A function that is continuous over its domain (Example 2).

Ex.3

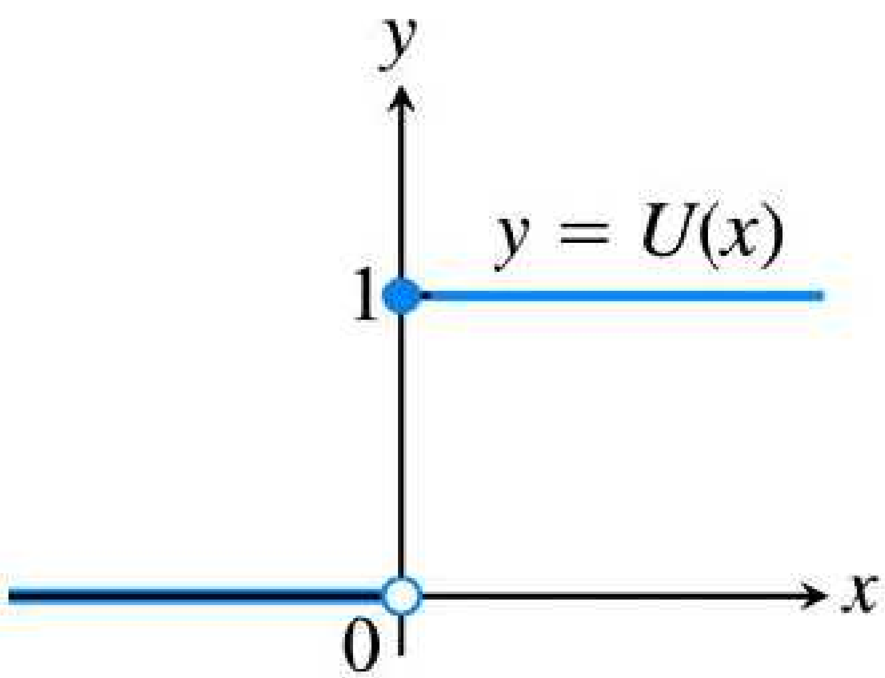


FIGURE 2.38 A function that has a jump discontinuity at the origin (Example 3).

Ex. 4 Show that $f(x)=\begin{cases} x\sin\dfrac{1}{x}, & x\neq 0, \\ 0, & x=0, \end{cases}$ 在$x=0$

is continuous.

证 $\because 0\leq |x\sin\dfrac{1}{x}|\leq |x|,\quad \therefore \lim\limits_{x\to 0} x\sin\dfrac{1}{x}=0,$

又 $f(0)=0,\qquad \lim\limits_{x\to 0} f(x)=f(0),$

由定义知

函数 $f(x)$在$x=0$处连续.

Ex5. 当a取何值时，

$$\text{函数 } f(x)=\begin{cases}\cos x, & x<0,\\ a+x, & x\ge 0,\end{cases} \text{ 在 } x=0 \text{处连续.}$$

解 $\because f(0)=a,$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\cos x=1,$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(a+x)=a,$$

要使 $f(0-0)=f(0+0)=f(0), \Rightarrow a=1,$

故当且仅当 $a=1$时，函数 $f(x)$在$x=0$处连续.

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

Ex.6

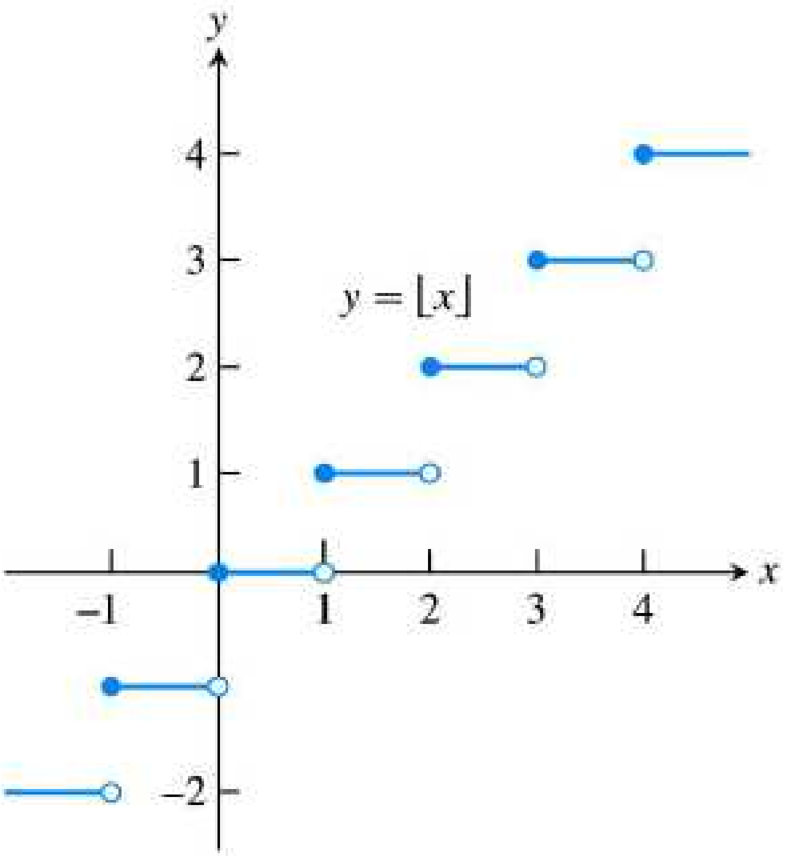


FIGURE 2.39 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

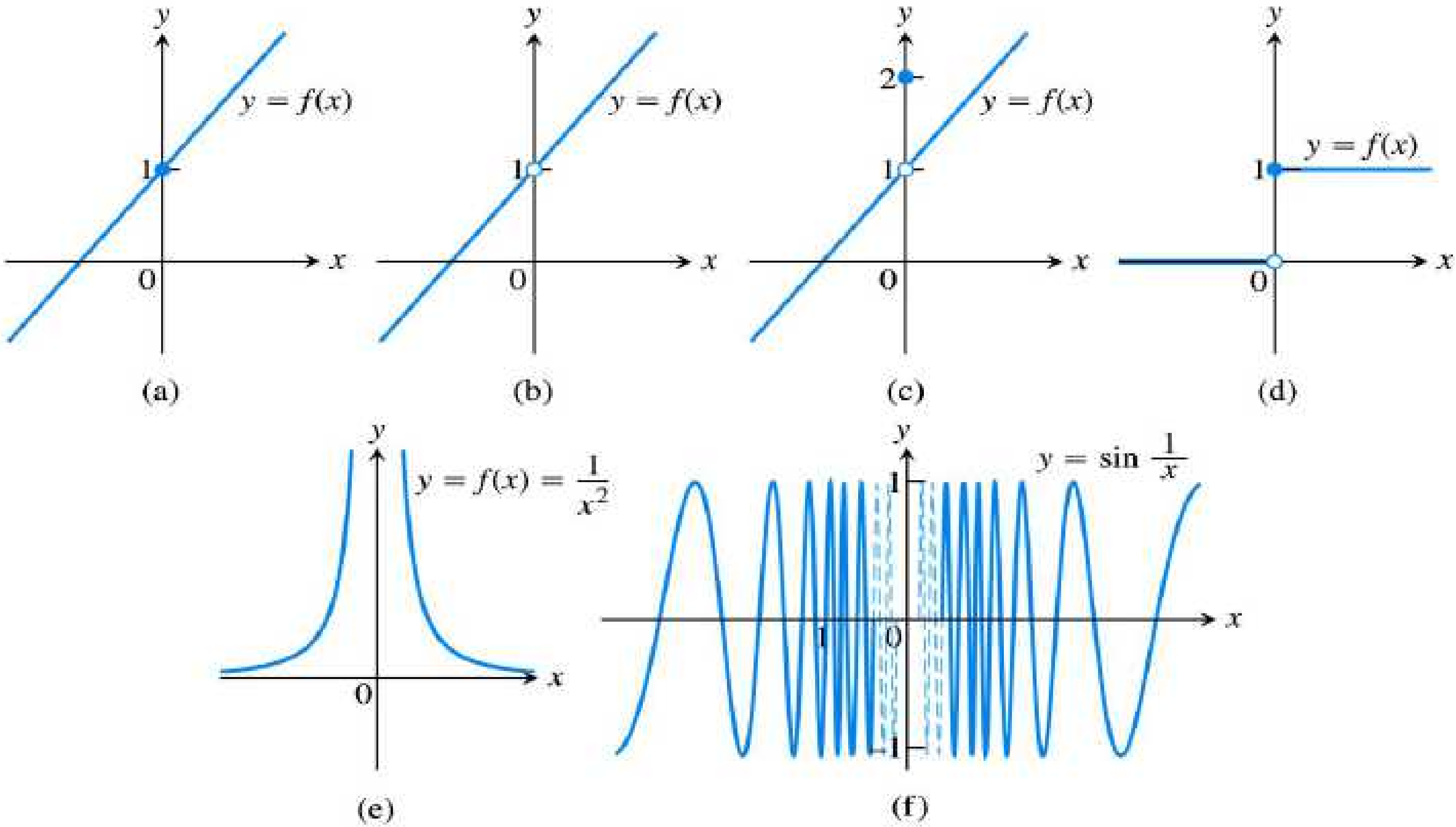


FIGURE 2.40 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Ex.7

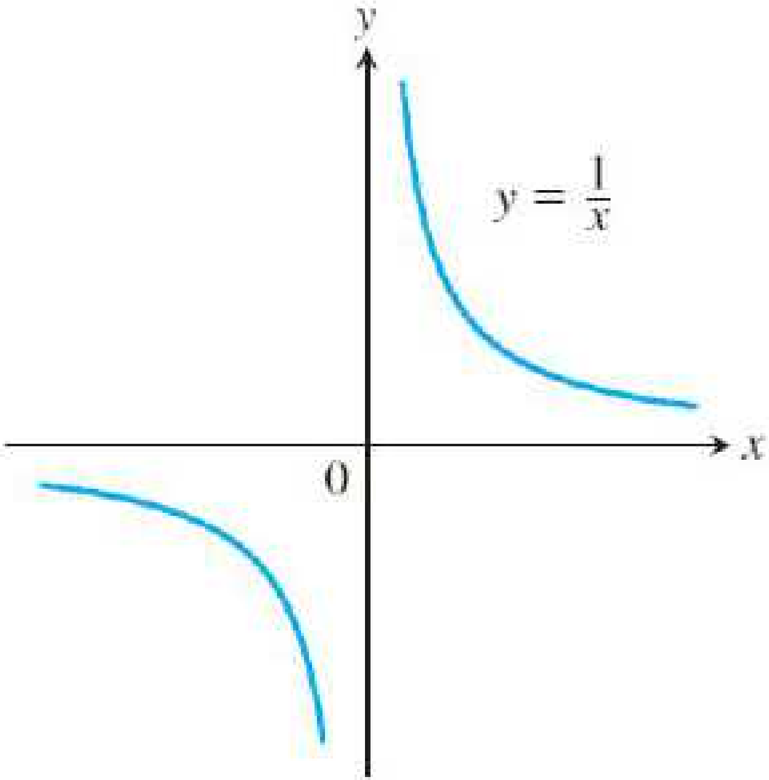


FIGURE 2.41 The function $y = 1/x$ is continuous over its natural domain. It has a point of discontinuity at the origin, so it is discontinuous on any interval containing $x = 0$ (Example 5).

Removable Discontinuity可去间断点

如果 $\lim\limits_{x\to x_0} f(x)$存在，但 $\lim\limits_{x\to x_0} f(x) \neq f(x_0)$, 则称点 x_0为函数 $f(x)$的可去间断点 .

Jump Discontinuity 跳跃间断点

如果 $f(x)$在点 x_0处左，右极限都存在，但 $f(x_0-0) \neq f(x_0+0)$, 则称点 x_0为函数$f(x)$的跳跃间断点 .

Infinite Discontinuity无穷间断点

如果$f(x_0-0)=\infty$或$f(x_0+0)=\infty$，则称点x_0为函数$f(x)$的无穷间断点．

Oscillating Discontinuity震荡间断点

如果当$x\to x_0$时，$f(x)$的值来回变化，则称点x_0为函数$f(x)$的震荡间断点．

间断点的分类:

间断点

- 第一类间断点:可去型,跳跃型.
- 第二类间断点:无穷型,振荡型.

连续函数与连续区间

在区间上每一点都连续的函数,叫做在该区间上的<u>连续函数</u>,或者说函数在该区间上连续.

如果函数在开区间 (a,b)内连续，并且在左端点 $x=a$处右连续，在右端点 $x=b$处左连续，则称函数 $f(x)$在闭区间 $[a,b]$上连续.

连续函数的图形是一条连续而不间断的曲线.

可知，函数 $f(x)=\dfrac{1}{x}$ 在 $x\neq 0$处处连续.

函数 $f(x)=x$ 在$(-\infty,+\infty)$处处连续.

函数 $f(x)=k$ 在$(-\infty,+\infty)$处处连续.

THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

Ex. 8 Show that

设 $P(x), Q(x)$ 是多项式，则

(1)它们在任何实数 c 处连续，

(2) $\dfrac{P(x)}{Q(x)}$ 在 c 处连续 $(Q(c) \neq 0)$.

Ex. 9 Show that

$f(x) = |x|$ 是连续函数，

$f(x) = \sin x, \cos x$ 是连续函数 .

$f(x) = \tan x, \cot x, \sec x, \csc x$ 在其定义域内连续 .

证明：$\lim\limits_{x \to x_0} \sin x = \sin x_0$.

$$0 \le |\sin x - \sin x_0| = \left|2\sin\frac{x - x_0}{2}\cos\frac{x + x_0}{2}\right| \le |x - x_0|$$

THEOREM 9—Composite of Continuous Functions If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

Ex.10 Show that the following functions are continuous on their natural domains.

$(a)\ y=\sqrt{x^2-2x-3}$

$(b)\ y=\sin\left(\dfrac{x^{\frac{2}{3}}}{1+x^4}\right)$

$(c)\ y=\left|\dfrac{x-2}{x^2-2}\right|$

$(d)\ y=\left|\dfrac{x\sin x}{x^2+2}\right|$

$(a)\ y=\sqrt{u},\quad u=x^2-2x-3,$

当$x\le -1, x\ge 3$时u连续且$u\ge 0$,

当$u\ge 0$时$y=\sqrt{u}$连续,

故$y=\sqrt{x^2-2x-3}$

当$x\le -1, x>3$时连续.

$(b)-\infty<x<+\infty,\ (c)x\ne\pm\sqrt{2}\quad (d)-\infty<x<+\infty$

THEOREM 10—Limits of Continuous Functions If g is continuous at the point b and $\lim_{x\to c} f(x) = b$, then

$$\lim_{x\to c} g(f(x)) = g(b) = g(\lim_{x\to c} f(x)).$$

证: 对于 $\forall \varepsilon > 0$, 由于 $\lim\limits_{u\to b} g(u) = g(b)$, 所以 $\exists \eta > 0$,

当 $|u-b| < \eta$ 时, 恒有 $|g(u)-g(b)| < \varepsilon$ 成立.

又 $\because \lim\limits_{x\to c} f(x) = b$,

对于 $\eta > 0$, $\exists \delta > 0$, 使当 $0 < |x-c| < \delta$ 时,

恒有 $|f(x)-b| < \eta$ 成立. $\therefore |g(f(x)) - g(b)| < \varepsilon$

$\therefore \lim\limits_{x\to c} g[f(x)] = g(b)$

the underlines are the defination of the limits of functions

若$\lim\limits_{x\to c} f(x)=b, \lim\limits_{u\to b} g(u)=g(b)$,则

$$\lim_{x\to c} g(f(x))=g(b)=\lim_{u\to b} g(u) \quad (u=f(x))$$

求极限的变量替换法则

$$\lim_{x\to c} g(f(x))=g(\lim_{x\to c} f(x))$$

连续函数求极限的法则

若$\lim\limits_{x\to c} f(x)=f(c), \lim\limits_{u\to b} g(u)=g(b)$,则

$$\lim_{x\to c} g(f(x))=g(f(c))$$

连续函数的复合函数是连续函数

Ex. 11 求下列极限

$$(a)\ \lim_{x\to\frac{\pi}{2}} \cos\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right) = -1.$$

$$(b)\ \lim_{x\to 0}\frac{\tan(\sin x)}{x} = \lim_{x\to 0}\frac{\tan(\cancel{\sin x})}{\cancel{\sin x}}\ u\ \frac{\sin x}{x} = 1.$$

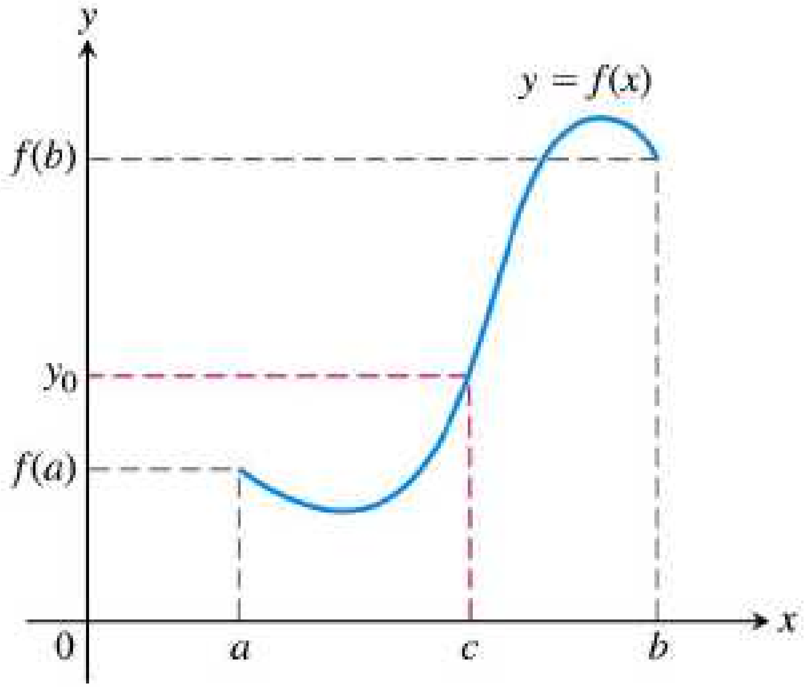
$$\frac{\sin u}{u\cos u}$$

$$(c)\ \lim_{x\to 0}\frac{\tan(2\sin x)}{x} = \lim_{x\to 0}\frac{\tan(2\sin x)}{2\sin x}\,\frac{2\sin x}{x} = 2.$$

闭区间上连续函数的性质

Intermediate Value Theorem 介值定理

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



推论(零点定理) 设函数 $f(x)$ 在闭区间 $[a,b]$ 上连续，且 $f(a)$ 与 $f(b)$ 异号(即 $f(a)\cdot f(b)<0$), 那末在开区间 (a,b) 内至少有函数 $f(x)$ 的一个零点, 即至少有一点 c $(a<c<b)$，使 $f(c)=0$.

（有界性）在闭区间上连续的函数一定在该区间上有界且在能取得它的最大值和最小值.

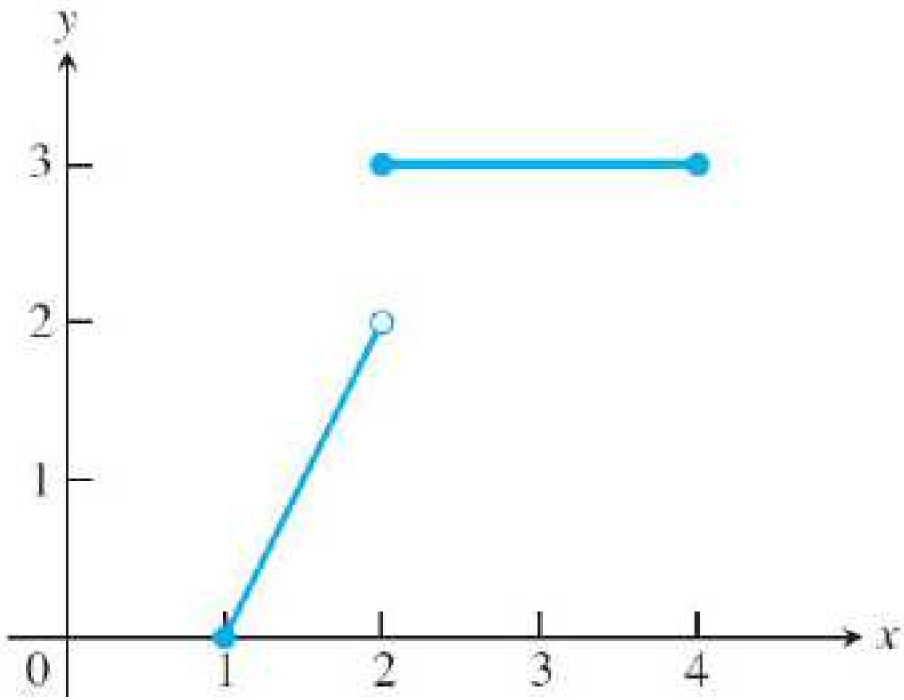
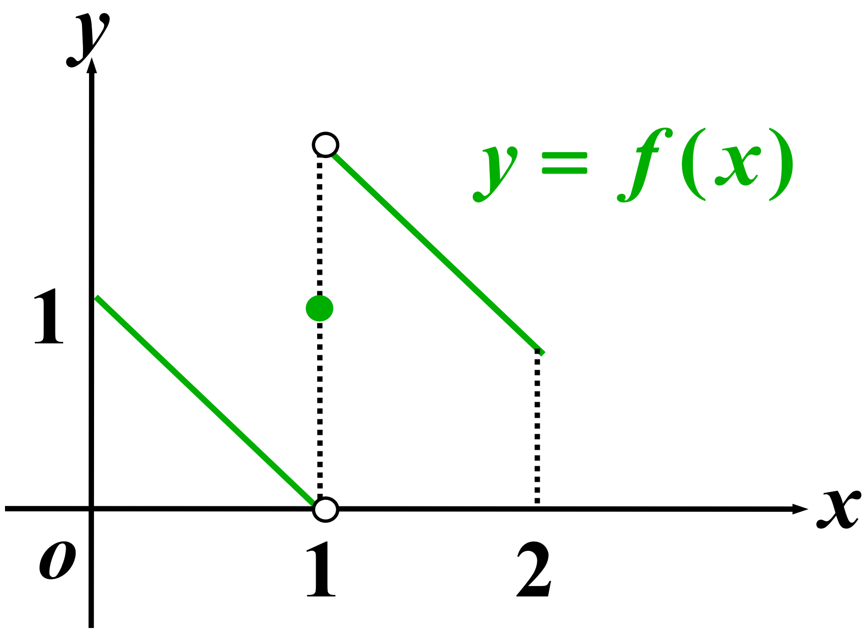


FIGURE 2.44 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.


y
y = f(x)
1
x
o
1
2

Ex. 12 证明方程在给定的区间内有实根：

$$x^3 - x - 1 = 0, \quad (1, 2).$$

证明：设 $f(x) = x^3 - x - 1$，在 $[1, 2]$ 连续 .

$f(1) = -1, \quad f(2) = 5,$

由连续函数的零点定理知，存在 $c \in (1,2)$，使 $f(c) = 0$.

Ex. 13 证明方程有实根：$\sqrt{2x+5} = 4 - x^2$.

证明：设 $f(x) = \sqrt{2x+5} + x^2$，在 $[-2.5, \infty)$ 连续 .

$f(0) = \sqrt{5}, \quad f(2) = 7, \quad \sqrt{5} < 4 < 7,$

由连续函数介值定理知，存在 $c \in (0,2)$，使 $f(c) = 4$.

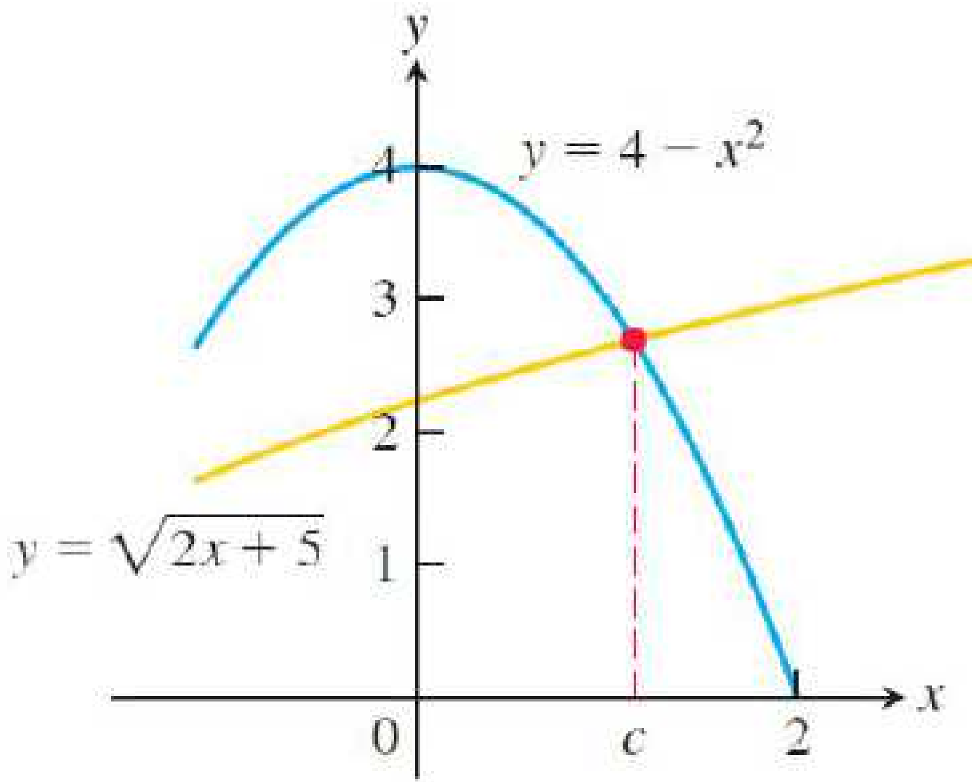


FIGURE 2.46 The curves $y = \sqrt{2x + 5}$ and $y = 4 - x^2$ have the same value at $x = c$ where $\sqrt{2x + 5} = 4 - x^2$ (Example 11).

A fixed point theorem Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \le f(x) \le 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$ (c is called a **fixed point** of f).

proof：Let $g(x) = f(x) - x$, continuous [0, 1].

$$g(0) = f(0) - 0 \ge 0, \quad g(1) = f(1) - 1 \le 0,$$

若 $g(0) = f(0) - 0 = 0$, or $g(1) = f(1) - 1 = 0$,

则结论成立。

若 $g(0) = f(0) - 0 > 0$, $g(1) = f(1) - 1 < 0$,

由连续函数的零点定理知，存在 $c \in (0,1)$, 使 $g(c) = 0$.

即 $f(c) - c = 0$.

补充定义使得可去间断点连续 Continuous Extension

$$f(x)=\frac{\sin x}{x}$$

$$F(x)=\begin{cases}\dfrac{\sin x}{x}, & x\neq 0,\\ 1, & x=0.\end{cases}$$

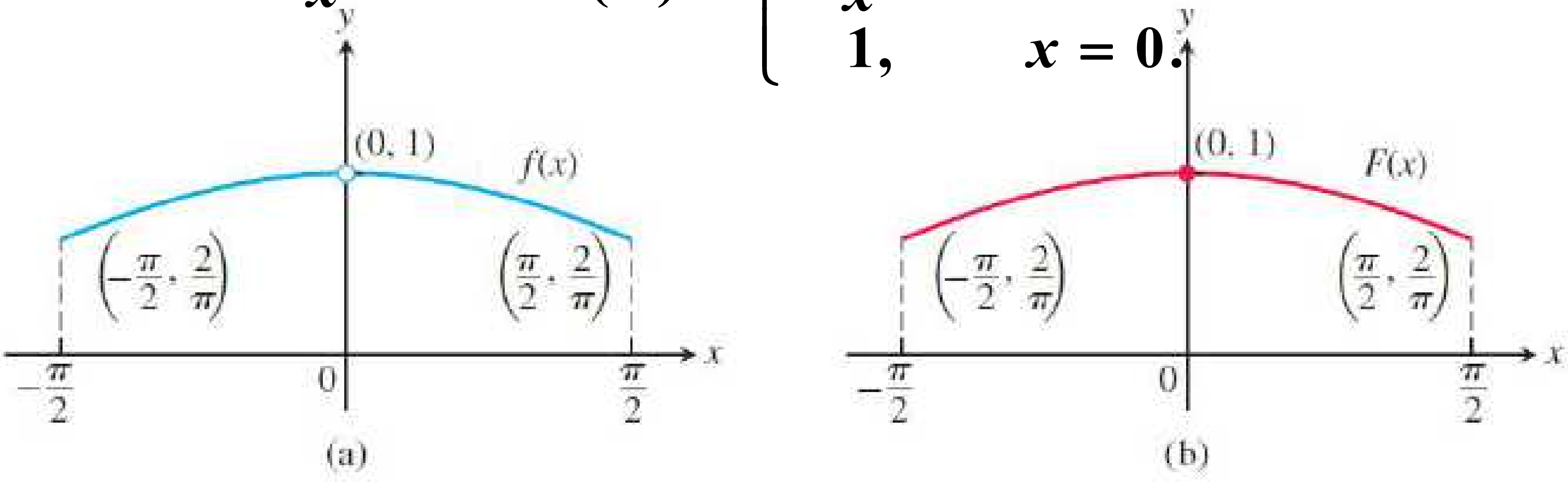


FIGURE 2.47 The graph (a) of $f(x) = (\sin x)/x$ for $-\pi/2 \le x \le \pi/2$ does not include the point (0, 1) because the function is not defined at $x = 0$. (b) We can remove the discontinuity from the graph by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x\to 0} f(x)$.

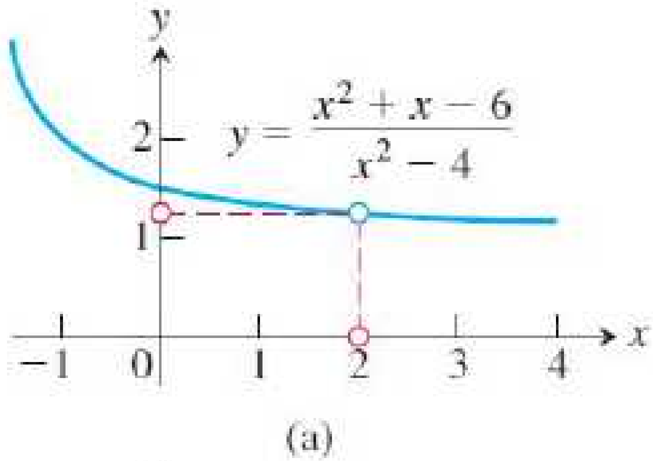
Ex. 14 Find the continuous extension for

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}.$$

Solution

$$f(x) = \frac{(x-2)(x+3)}{(x-2)(x+2)}. \quad \lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x+3}{x+2} = \frac{5}{4}$$

$$F(x) = \begin{cases} \dfrac{x+3}{x+2}, & x \neq 2, \\ \dfrac{5}{4}, & x = 2. \end{cases}$$



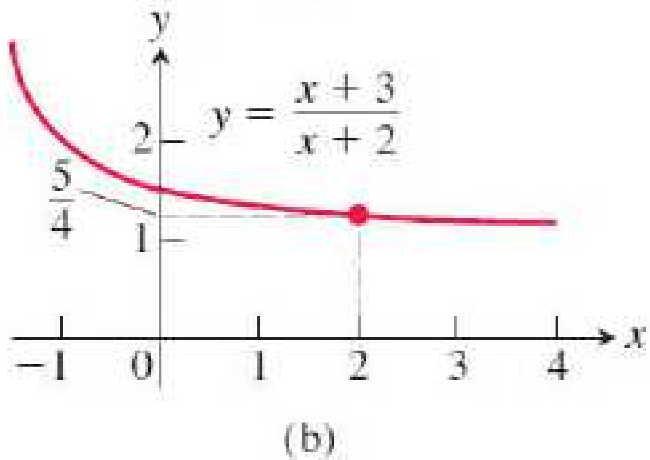


FIGURE 2.48 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 12).

作业

- 2.5　6,23,33,38,47,55,61,64,67

2.6

Limits Involving Infinity; Asymptotes of Graphs

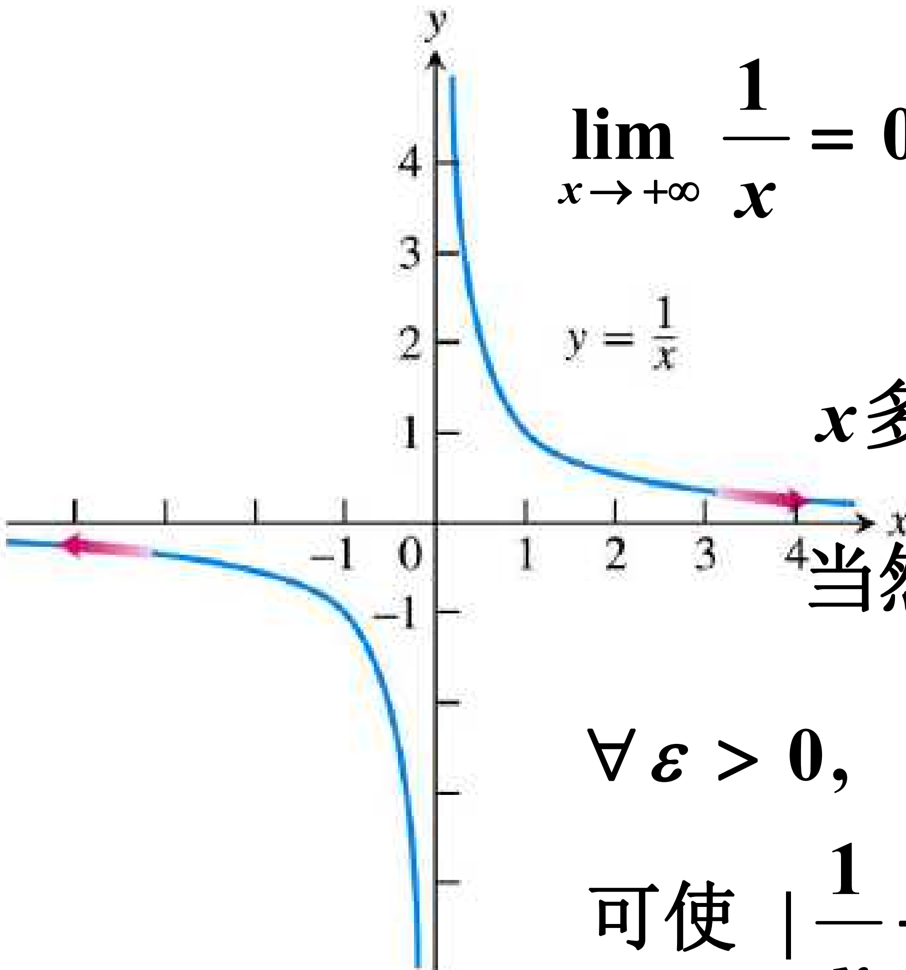


FIGURE 2.49 The graph of $y = 1/x$ approaches 0 as $x \to \infty$ or $x \to -\infty$.

$$\lim_{x \to +\infty} \frac{1}{x} = 0.$$

$\forall \varepsilon > 0$，当x充分大时，

可使 $\left|\frac{1}{x} - 0\right| < \varepsilon$，

x多大才算充分大？ $x > \frac{1}{\varepsilon}$

当然与 ε相关的，由 ε决定的。

$\forall \varepsilon > 0$，存在 $M > 0$，当 $x > M$时，

可使 $\left|\frac{1}{x} - 0\right| < \varepsilon$，

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

注：这里N, M不唯一，若存在，任何比它大的数都可作为N, M。

Ex. 1 用精确定义证明:$\lim\limits_{x\to-\infty}\dfrac{1}{x^k}=0.(k\in N^+)$

证 $\forall\varepsilon>0$, 要使 $\left|\dfrac{1}{x^k}-0\right|<\varepsilon$, 只要 $|x|>\dfrac{1}{\sqrt[k]{\varepsilon}}$,

取 $N=-\dfrac{1}{\sqrt[k]{\varepsilon}}$, 当$x<N$时, 就有 $\left|\dfrac{1}{x^k}-0\right|<\varepsilon$.

取 $M=\dfrac{1}{\sqrt[k]{\varepsilon}}$, 当$x>M$时,

$\lim\limits_{x\to-\infty}\dfrac{1}{x}=0.$ $\lim\limits_{x\to-\infty}\dfrac{1}{x^2}=0.$

$\lim\limits_{x\to\infty}\dfrac{1}{x^3}=0.$

THEOREM 12 All the limit laws in Theorem 1 are true when we replace $\lim_{x\to c}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

Ex. 2 Find

$$(a)\ \lim_{x\to+\infty}\left(5+\frac{2}{x}\right).$$

$$(b)\ \lim_{x\to-\infty}\frac{2\pi}{x^2}.$$

前面提出的所有极限的运算法则都正确，可以直接用。

Ex. 3 Find the limits:

$$(a)\ \lim_{x\to+\infty}\frac{5x^2+8x-3}{3x^2+2},(b)\ \lim_{x\to-\infty}\frac{11x+2}{2x^3-1}.$$

$$解(a)\ \lim_{x\to+\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to+\infty}\frac{5+\dfrac{8}{x}-\dfrac{3}{x^2}}{3+\dfrac{2}{x^2}}=\frac{5}{3}.$$

$$解(b)\ \lim_{x\to-\infty}\frac{11x+2}{2x^3-1}=\frac{\dfrac{11}{x^2}+\dfrac{2}{x^3}}{2-\dfrac{1}{x^3}}=0.$$

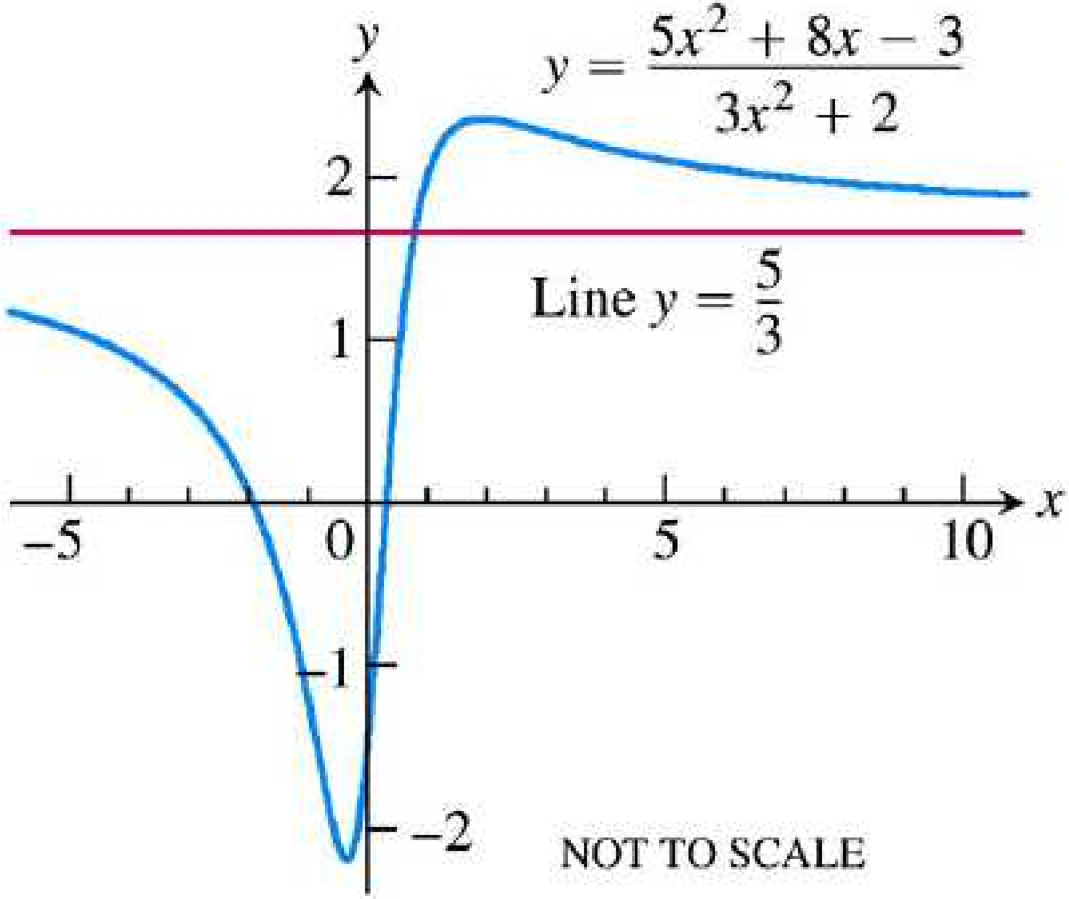


FIGURE 2.51 The graph of the function in Example 3a. The graph approaches the line $y = 5/3$ as $|x|$ increases.

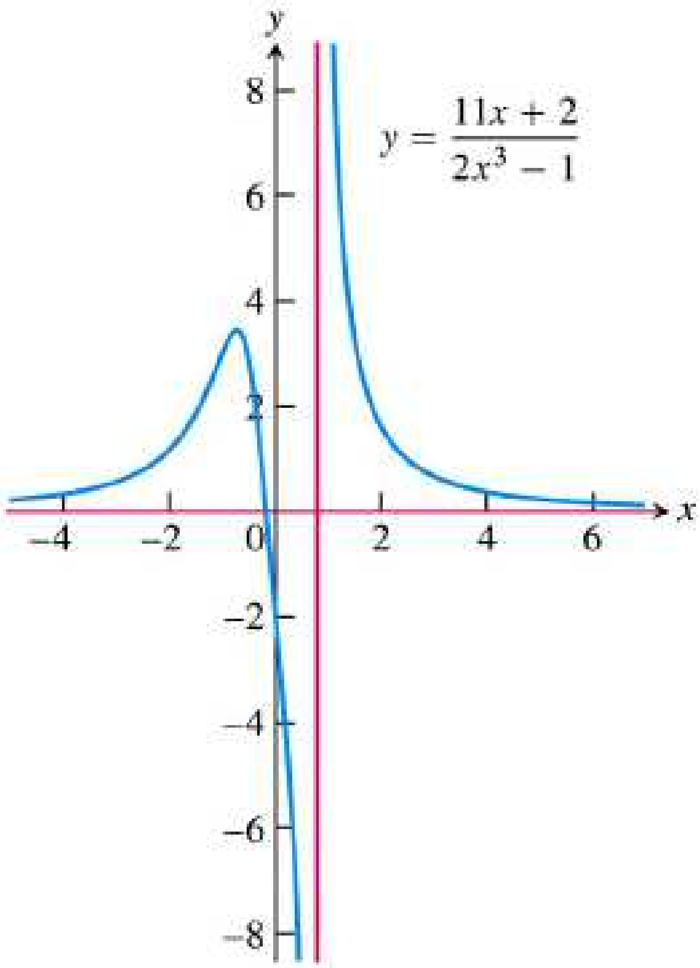


FIGURE 2.52 The graph of the function in Example 3b. The graph approaches the x-axis as $|x|$ increases.

Horizontal Asymptote

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b.$$

Ex. 4 Find horizontal asymptotes for

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^3 - 2}{x^3 + 1} = 1$$

$$y = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3 - 2}{-x^3 + 1} = -1$$

$$y = -1$$

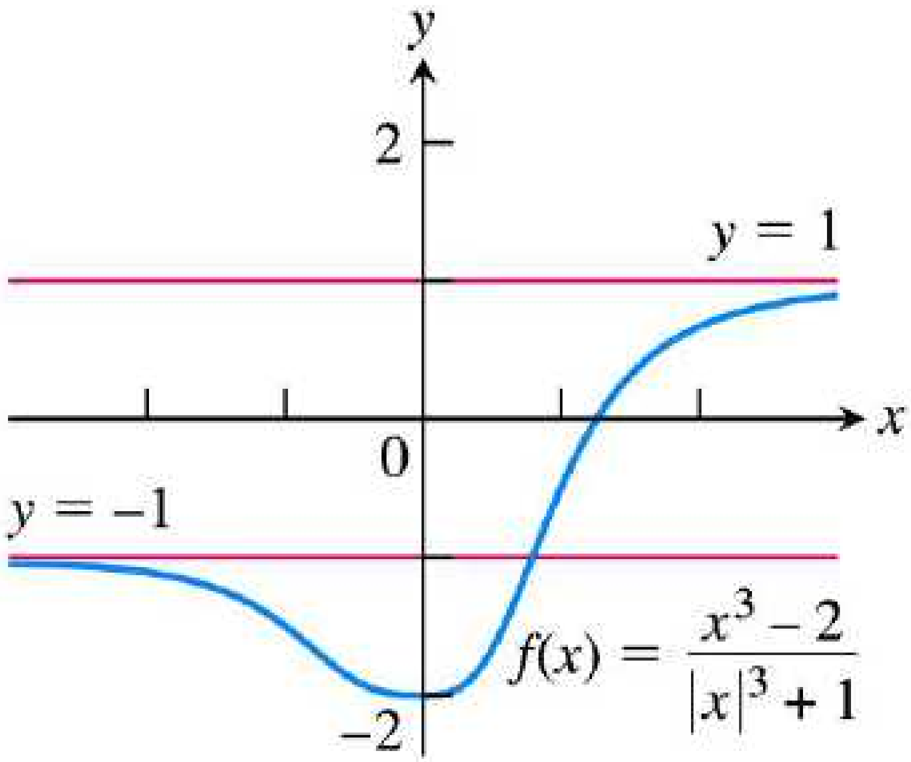


FIGURE 2.53 The graph of the function in Example 4 has two horizontal asymptotes.

Ex. 5 Find horizontal asymptotes for

$$(a)\ f(x) = \sin\left(\frac{1}{x}\right) \qquad y = 0$$

$$(b)\ f(x) = x\sin\left(\frac{1}{x}\right) \qquad y = 1$$

Ex. 6 Find horizontal asymptotes for

$$y = 2 + \frac{\sin x}{x} \qquad y = 2$$

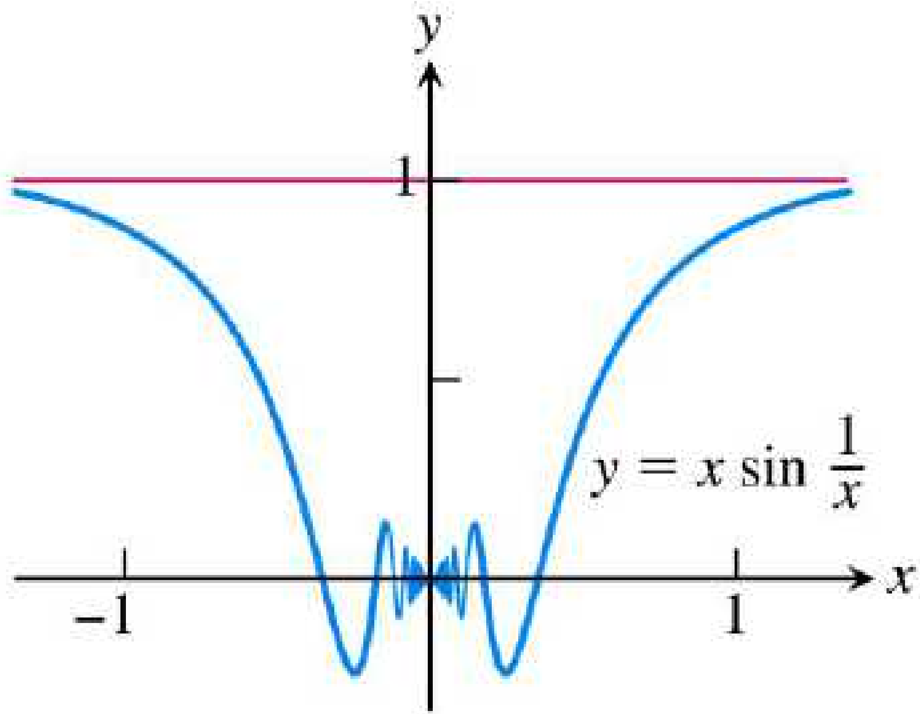


FIGURE 2.54 The line $y = 1$ is a horizontal asymptote of the function graphed here (Example 5b).

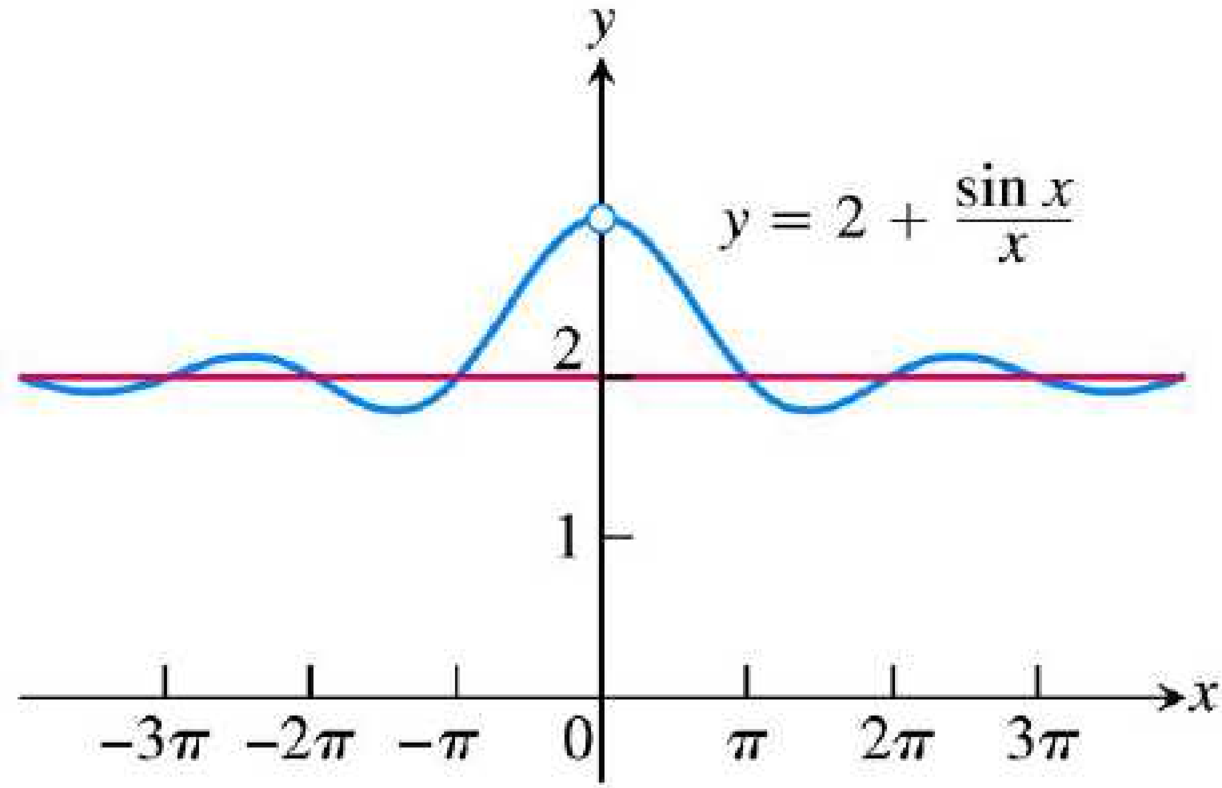


FIGURE 2.55 A curve may cross one of its asymptotes infinitely often (Example 6).

Ex. 7 Find $\lim\limits_{x\to+\infty}\dfrac{1}{x}\lfloor x\rfloor$

$$x-1<\lfloor x\rfloor\le x,\qquad 1-\frac{1}{x}<\frac{1}{x}\lfloor x\rfloor\le 1$$

Ex. 8 Find $\lim\limits_{x\to+\infty}(x-\sqrt{x^2+16})$

$$\lim_{x\to+\infty}(x-\sqrt{x^2+16})=\lim_{x\to+\infty}\frac{-16}{x+\sqrt{x^2+16}}$$

$$=\lim_{x\to+\infty}\frac{-\dfrac{16}{x}}{1+\sqrt{1+\dfrac{16}{x^2}}}=0$$

斜渐近线

若 $\lim\limits_{x\to+\infty}(f(x)-(ax+b))=0$, 或

$\lim\limits_{x\to-\infty}(f(x)-(ax+b))=0$，则称 $y=ax+b$

是$y=f(x)$的斜渐近线 $(a\neq 0)$.

按照定义，若 $f(x)=ax+b+g(x)$, 且

$\lim\limits_{x\to\pm\infty}g(x)=0$，则 $y=ax+b$

是$y=f(x)$的斜渐近线 $(a\neq 0)$.

Ex. 9 求斜渐近线 $f(x)=\dfrac{x^2-3}{2x-4}$

解 $f(x)=\dfrac{x^2-3}{2x-4}=\dfrac{x}{2}+1+\dfrac{1}{2x-4}$ 长除法

$\dfrac{1}{\infty}=0$

斜渐进线 $y=\dfrac{x}{2}+1.$

若 $\lim\limits_{x\to+\infty}(f(x)-(ax+b))=0,$

则 $\lim\limits_{x\to+\infty}(f(x)-ax)=b,$

即 $\lim\limits_{x\to+\infty}\dfrac{\frac{f(x)}{x}-a}{\frac{1}{x}}=b,$

则 $a=\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}.$

极限法

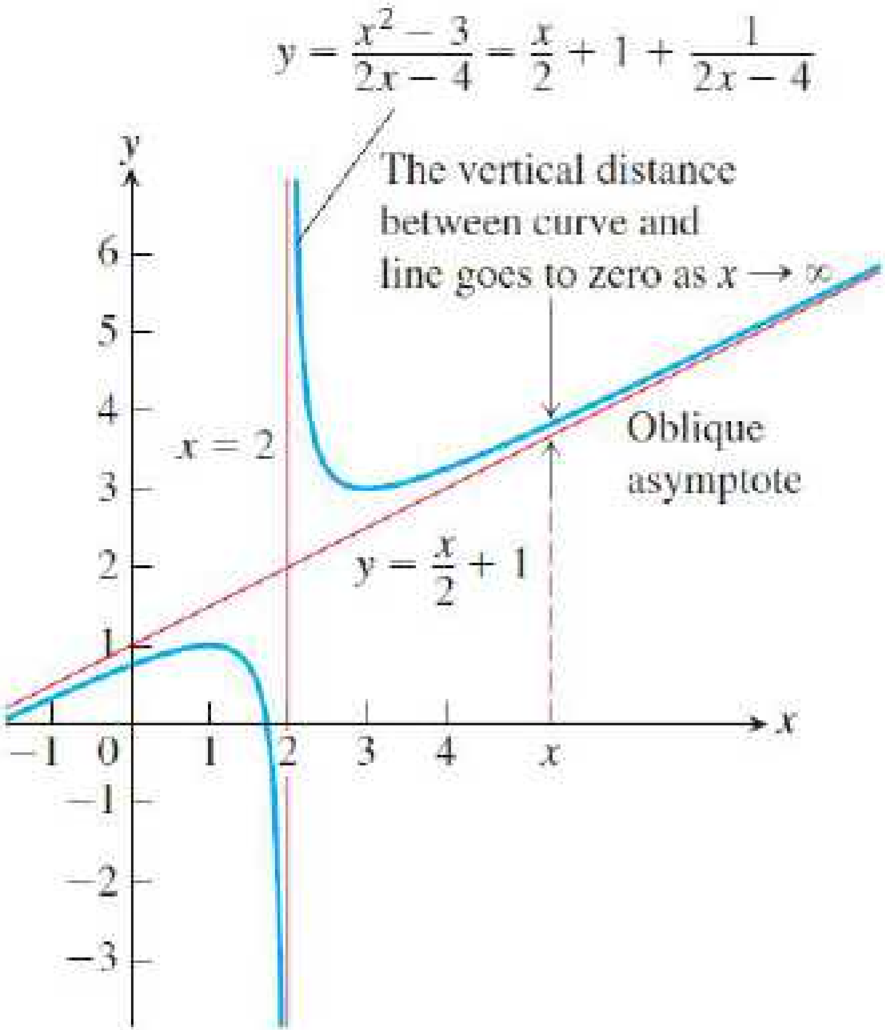


FIGURE 2.57 The graph of the function in Example 9 has an oblique asymptote.

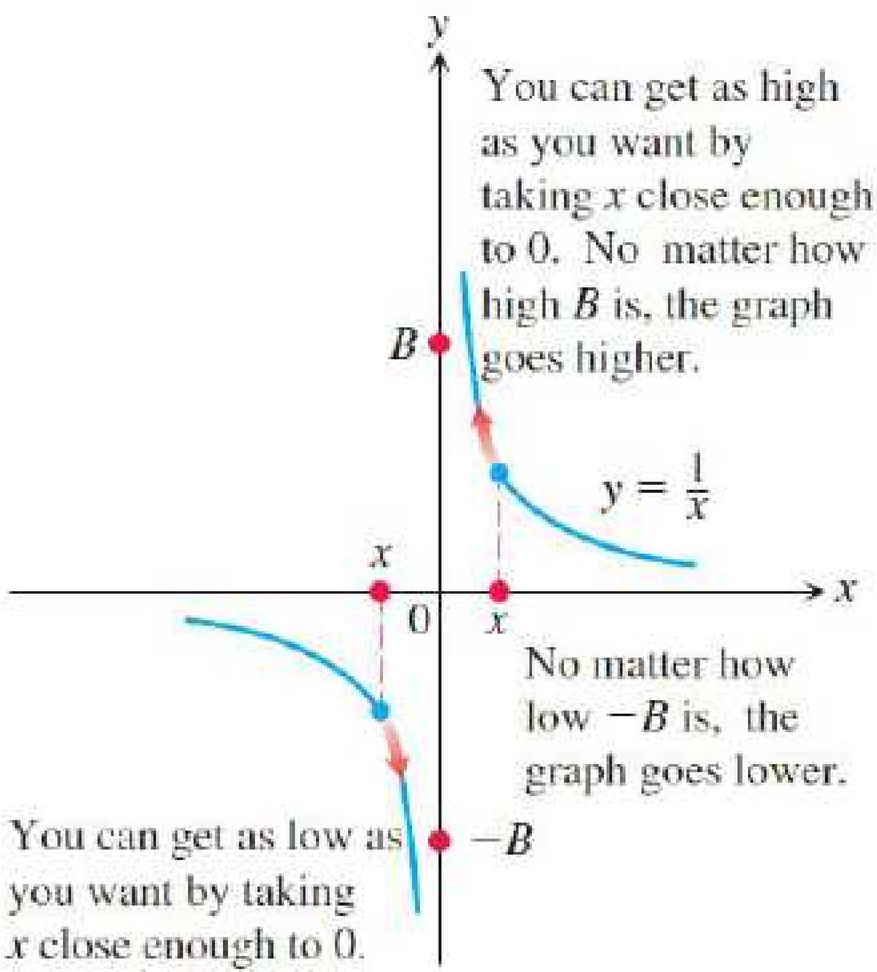


FIGURE 2.58 One-sided infinite limits:

$\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

$$\lim_{x\to c} f(x) = +\infty$$

任给$B > 0$,

当x充分靠近c时，

可使$f(x) > B$

存在$\delta > 0$,

当$0 < |x - c| < \delta$时,

可使$f(x) > B$

Infinity

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty, \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

DEFINITIONS

1. We say that **$f(x)$ approaches infinity as x approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) > B.$$

2. We say that **$f(x)$ approaches minus infinity as x approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) < -B.$$

Ex. 10 Find

$$\lim_{x \to 1^+} \frac{1}{x-1}$$

$$= +\infty$$

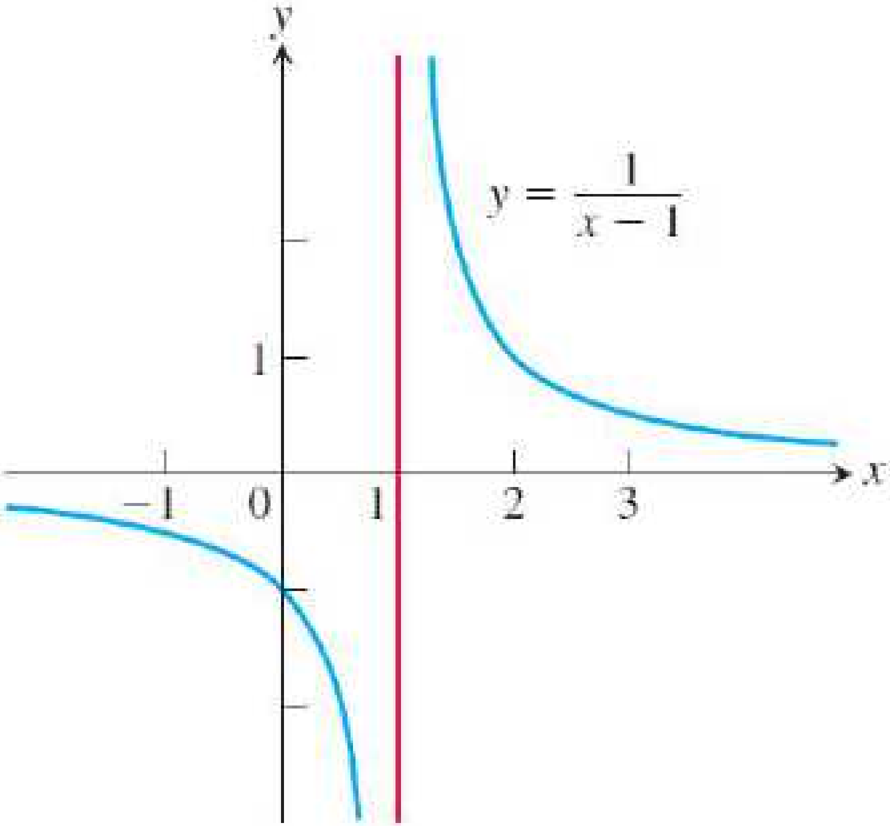


FIGURE 2.59 Near $x = 1$, the function $y = 1/(x-1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 10).

Ex. 11 Find

$$\lim_{x \to 0} \frac{1}{x^2} = +\infty$$

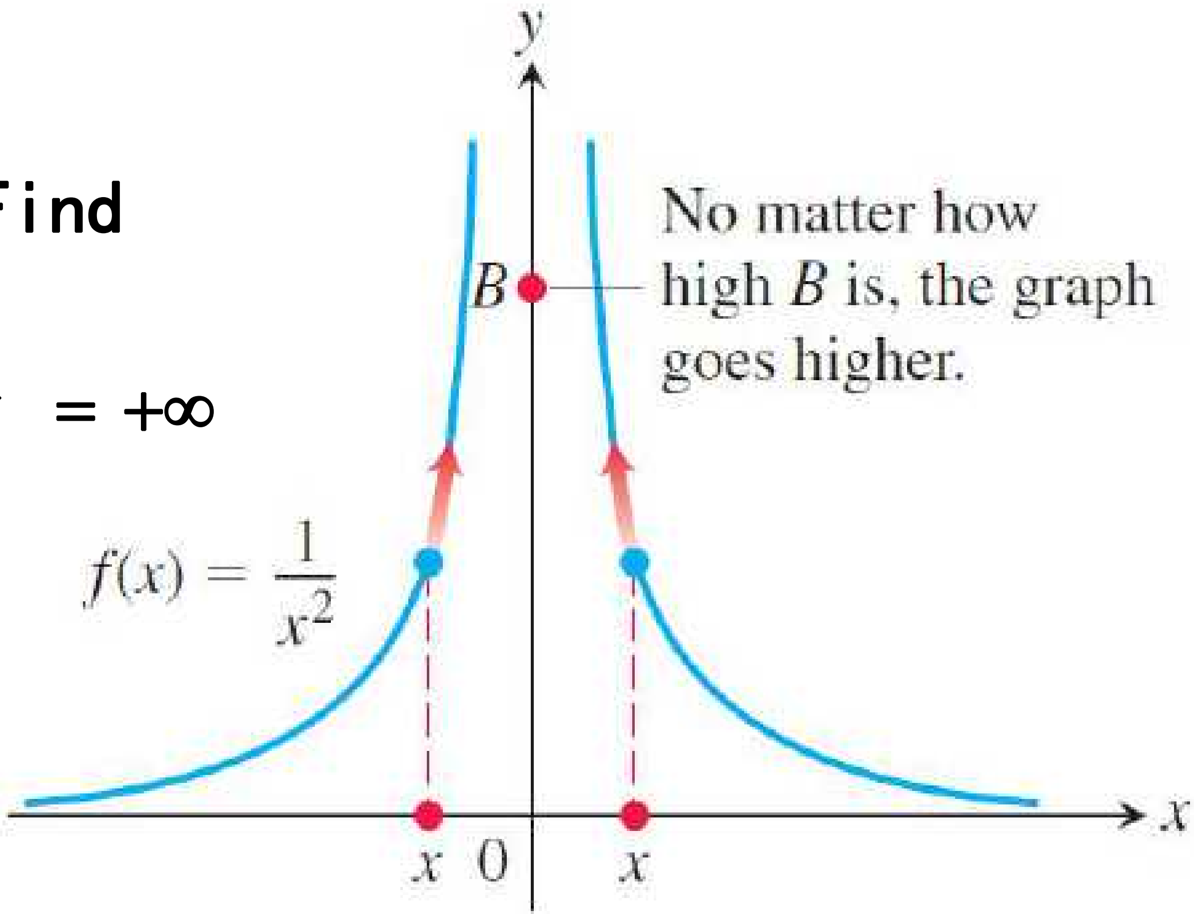


FIGURE 2.60 The graph of $f(x)$ in Example 11 approaches infinity as $x \to 0$.

$$\lim_{x\to 0^+}\frac{1}{x}=+\infty,\ \lim_{x\to 0^-}\frac{1}{x}=-\infty$$

Ex. 12 Find

$$(a)\lim_{x\to 2}\frac{(x-2)^2}{x^2-4}=0 \qquad (b)\lim_{x\to 2}\frac{x-2}{x^2-4}=\frac{1}{4}$$

$$(c)\lim_{x\to 2^+}\frac{x-3}{x^2-4}=-\infty \qquad (d)\lim_{x\to 2^-}\frac{x-3}{x^2-4}=+\infty$$

$$(e)\lim_{x\to 2}\frac{x-3}{x^2-4}\ \boxed{=\infty} \qquad (f)\lim_{x\to 2}\frac{2-x}{(x-2)^3}=-\infty$$

不存在！

Ex. 13 Find $\lim\limits_{x \to -\infty} \dfrac{2x^5 - 6x^4 + 1}{3x^2 + 2x - 7}$

解： $\lim\limits_{x \to -\infty} \dfrac{2x^5 - 6x^4 + 1}{3x^2 + 2x - 7} = \lim\limits_{x \to -\infty} \dfrac{2 - \dfrac{6}{x} + \dfrac{1}{x^5}}{\dfrac{3}{x^3} + \dfrac{2}{x^4} - \dfrac{7}{x^5}}$

$= -\infty$

Vertical Asymptote 铅直渐近线

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty.$$

Ex. 15 Find vertical asymptotes for $f(x)=\dfrac{x+3}{x+2}$

$$x=-2$$

$$\lim_{x\to -2^{+}}\frac{x+3}{x+2}=+\infty$$

$$\lim_{x\to -2^{-}}\frac{x+3}{x+2}=-\infty$$

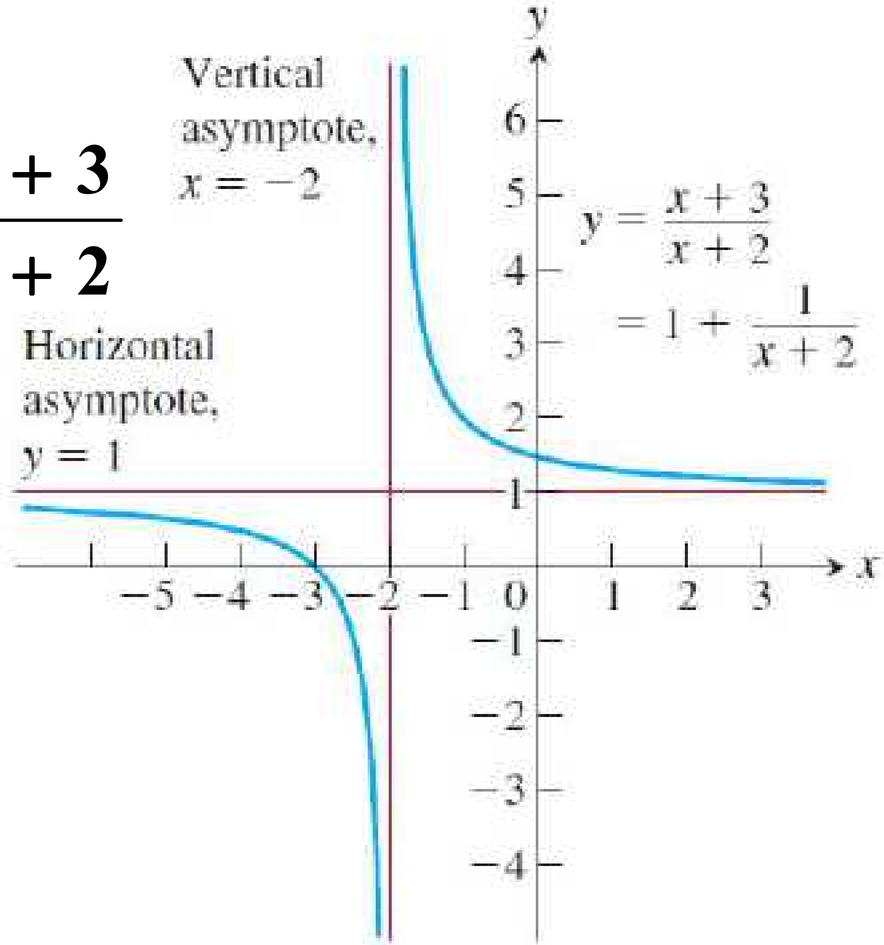


FIGURE 2.64 The lines $y=1$ and $x=-2$ are asymptotes of the curve in Example 15.

Ex. 16 Find vertical asymptotes for

$$f(x) = -\frac{8}{x^2 - 4}$$

$$x = \pm 2$$

$$\lim_{x \to 2^+} \frac{-8}{x^2 - 4} = -\infty$$

$$\lim_{x \to 2^-} \frac{-8}{x^2 - 4} = +\infty$$

$$\lim_{x \to -2^+} \frac{-8}{x^2 - 4} = +\infty$$

$$\lim_{x \to -2^-} \frac{-8}{x^2 - 4} = -\infty$$

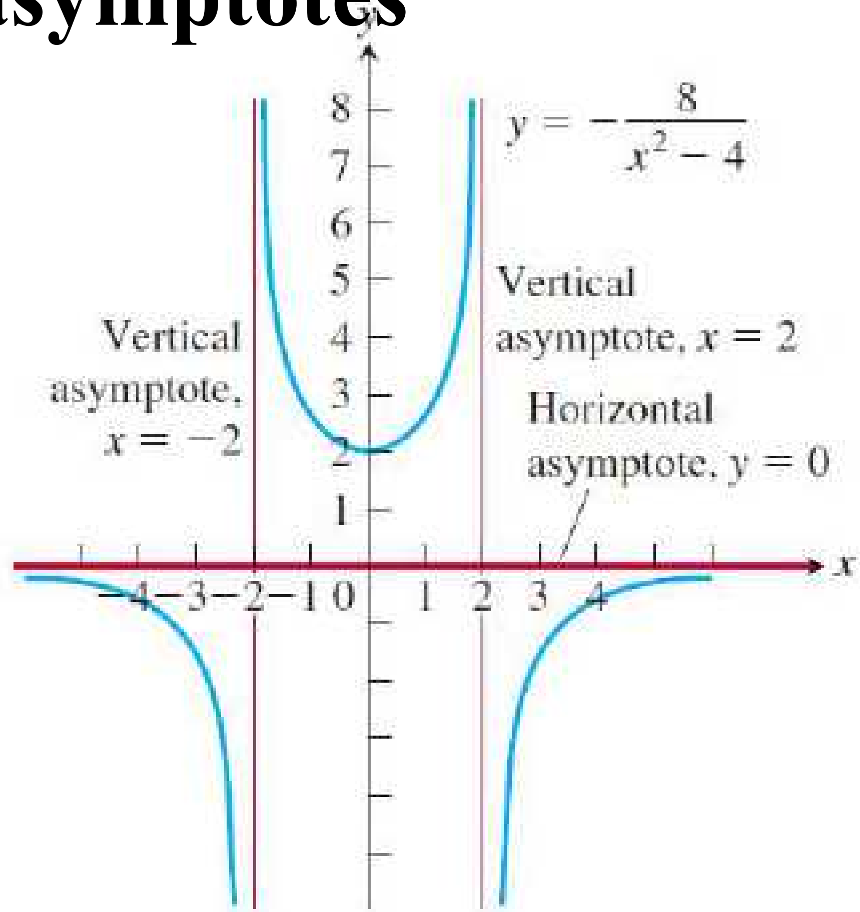


FIGURE 2.65 Graph of the function in Example 16. Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided.

Dominate Terms

Ex. 17 Find the dominate terms for $f(x)=\dfrac{x^2-3}{2x-4}$

$$f(x)=\frac{x}{2}+1+\frac{1}{2x-4}$$

$\dfrac{x}{2}+1$ is the dominate term.

Ex. 18 求统治项：

$$f(x) = 3x^4 - 2x^3 + 6$$

$$= 3x^4\left(1 - \frac{2}{3x} + \frac{2}{x^4}\right)$$

$3x^4$ is the dominate term.

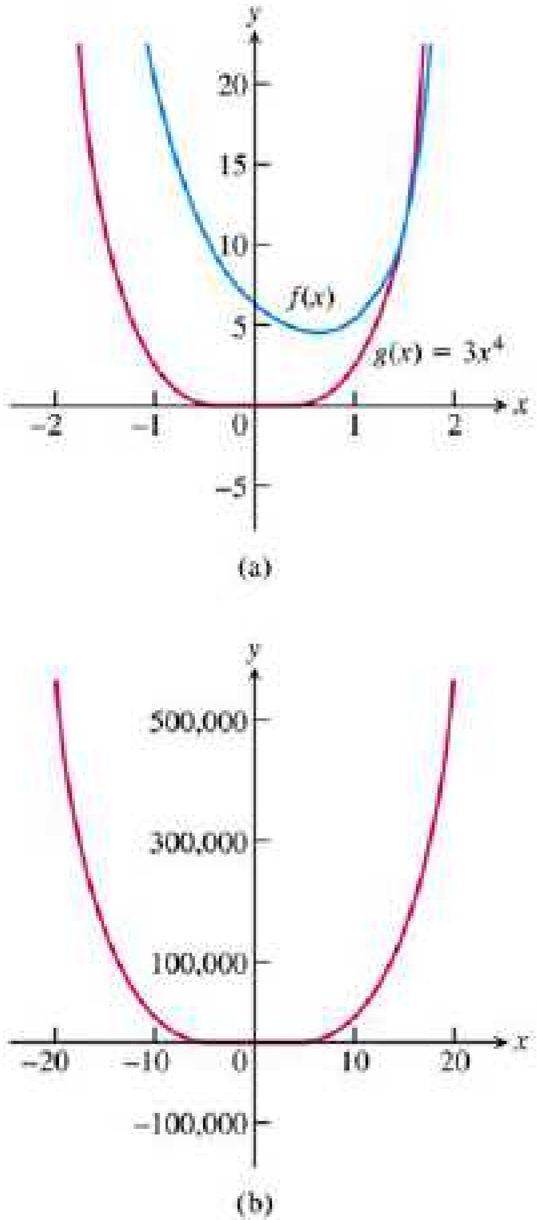


FIGURE 2.67 The graphs of f and g are (a) distinct for $|x|$ small, and (b) nearly identical for $|x|$ large (Example 18).