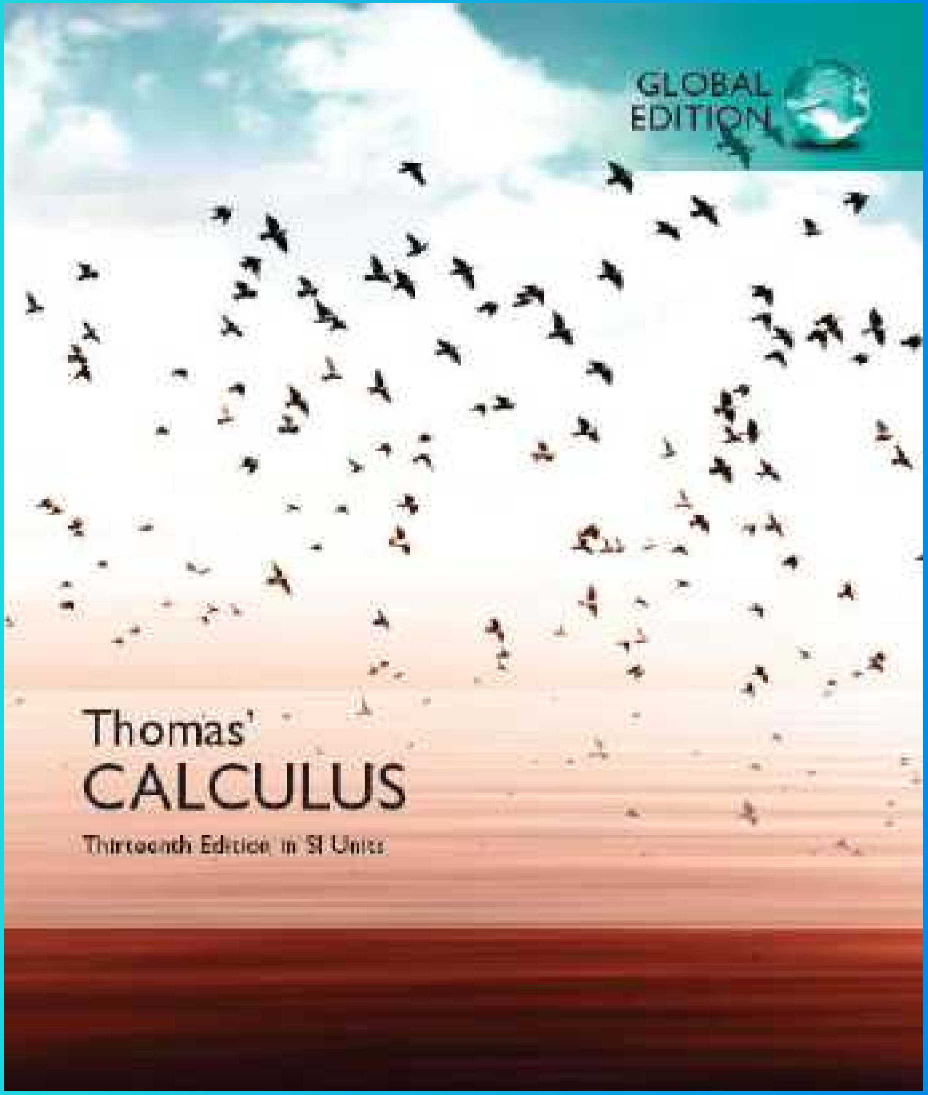

GLOBAL
EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 9

First-Order Differential Equations
一阶微分方程

9.1

Solutions, Slope Fields, and Euler's Method

解，斜率场和欧拉方法

General First-Order Differential Equations and Solutions

A **first-order differential equation** is an equation

一阶微分方程

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. The equation is of *first order* because it involves only the first derivative dy/dx (and not higher-order derivatives). We point out that the equations

$$y' = f(x, y) \qquad \text{and} \qquad \frac{d}{dx}y = f(x, y)$$

are equivalent to Equation (1) and all three forms will be used interchangeably in the text.

微分方程的解

A **solution** of Equation (1) $\frac{dy}{dx} = f(x, y)$

a differentiable function $y = y(x)$ such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

The **general solution** 通解

a solution that contains all possible solutions.

always contains an arbitrary constant,

方程（1）的通解：函数族，含有一个任意常数的解.

EXAMPLE 1 Show that every member of the family of functions

$$y = \frac{C}{x} + 2$$

is a solution of the first-order differential equation

$$\frac{dy}{dx} = \frac{1}{x}(2 - y)$$

on the interval $(0, \infty)$, where C is any constant.

Solution Differentiating $y = C/x + 2$ gives

$$\frac{dy}{dx} = C\frac{d}{dx}\left(\frac{1}{x}\right) + 0 = -\frac{C}{x^2}.$$

$$\frac{1}{x}\left[2 - \left(\frac{C}{x} + 2\right)\right] = \frac{1}{x}\left(-\frac{C}{x}\right) = -\frac{C}{x^2}.$$

The particular solution **特解**

the solution $y = y(x)$ satisfying the initial condition $y(x_0) = y_0$

A first-order initial value problem **一阶初值问题**

$$y' = f(x, y) \quad y(x_0) = y_0.$$

EXAMPLE 2 Show that the function

$$y = (x + 1) - \frac{1}{3}e^x$$

is a solution to the first-order initial value problem

$$\frac{dy}{dx} = y - x, \qquad y(0) = \frac{2}{3}.$$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\left(x + 1 - \frac{1}{3}e^x\right) = 1 - \frac{1}{3}e^x.$$

$$y - x = (x + 1) - \frac{1}{3}e^x - x = 1 - \frac{1}{3}e^x.$$

$$y(0) = \left[(x + 1) - \frac{1}{3}e^x\right]_{x=0} = 1 - \frac{1}{3} = \frac{2}{3}.$$

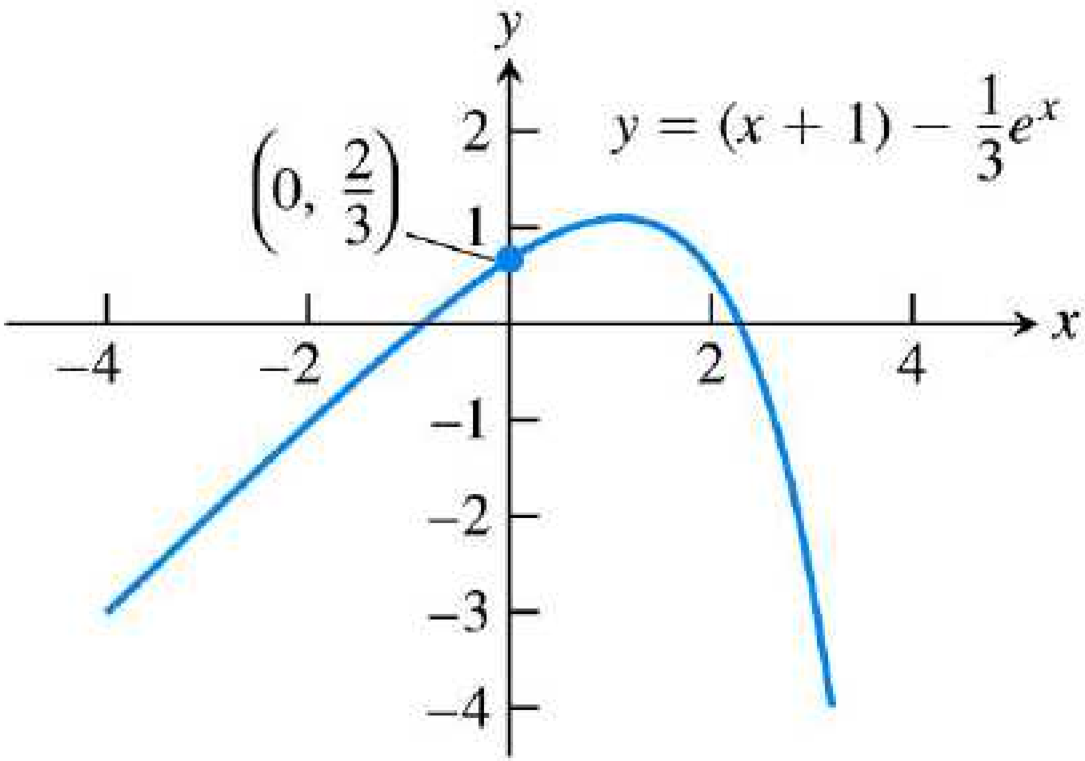


FIGURE 9.1 Graph of the solution to the initial value problem in Example 2.

斜率场

Slope Fields: Viewing Solution Curves

$$\frac{dy}{dx} = f(x, y)$$ $$\text{Riccati}: y' = x^2 + y^2$$

它的解曲线上的任何一点(x, y)处的斜率 $= f(x, y)$可在该点画一个很短的有向线段，来表示曲线的变化情况.例如，

$$\frac{dy}{dx} = y - x$$

在点$(0,1),(1,2),(2,3),(3,4),\cdots$处的斜率都是1

在点$(0,2),(1,3),(2,4),(3,5),\cdots$处的斜率都是2

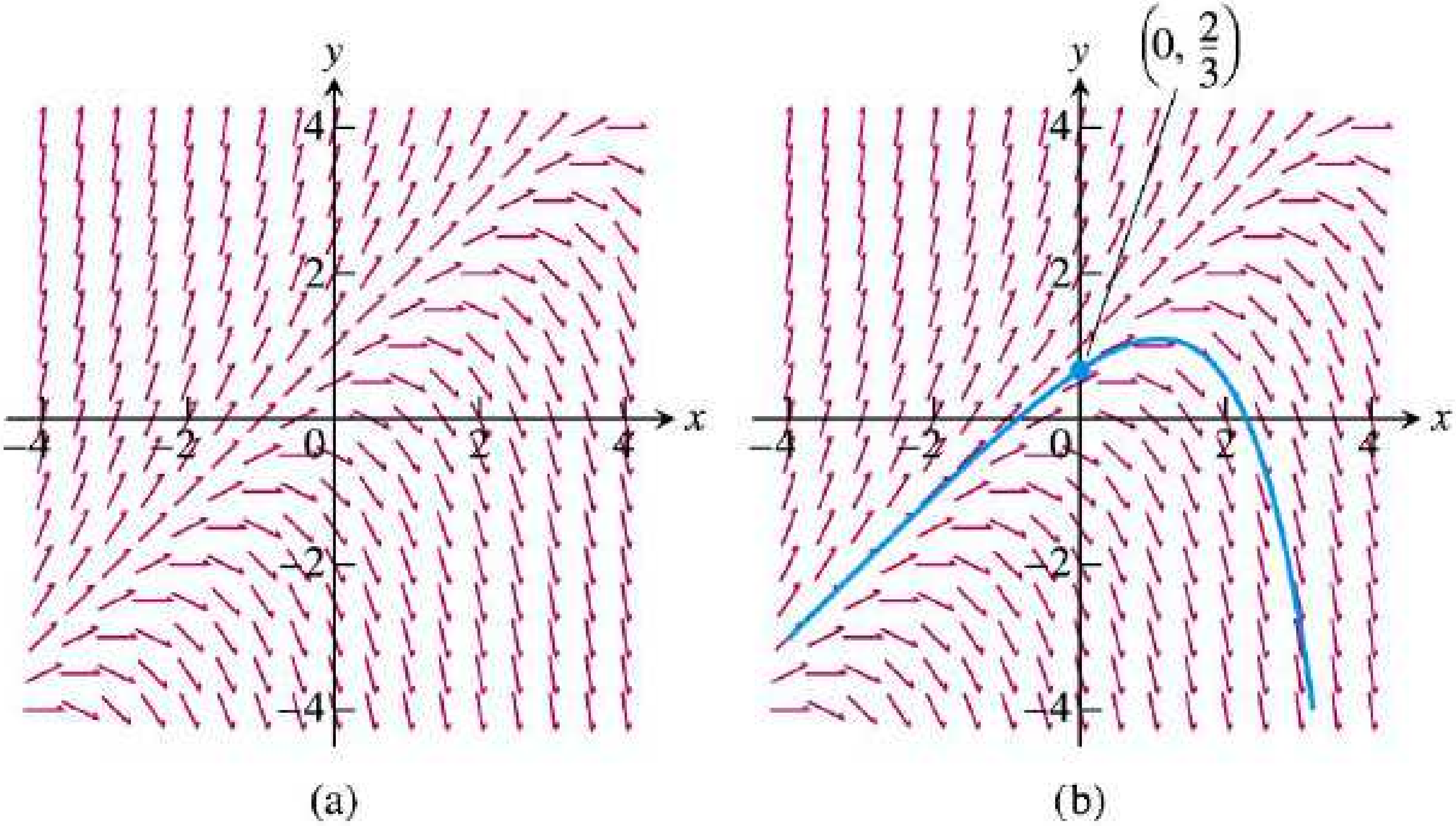
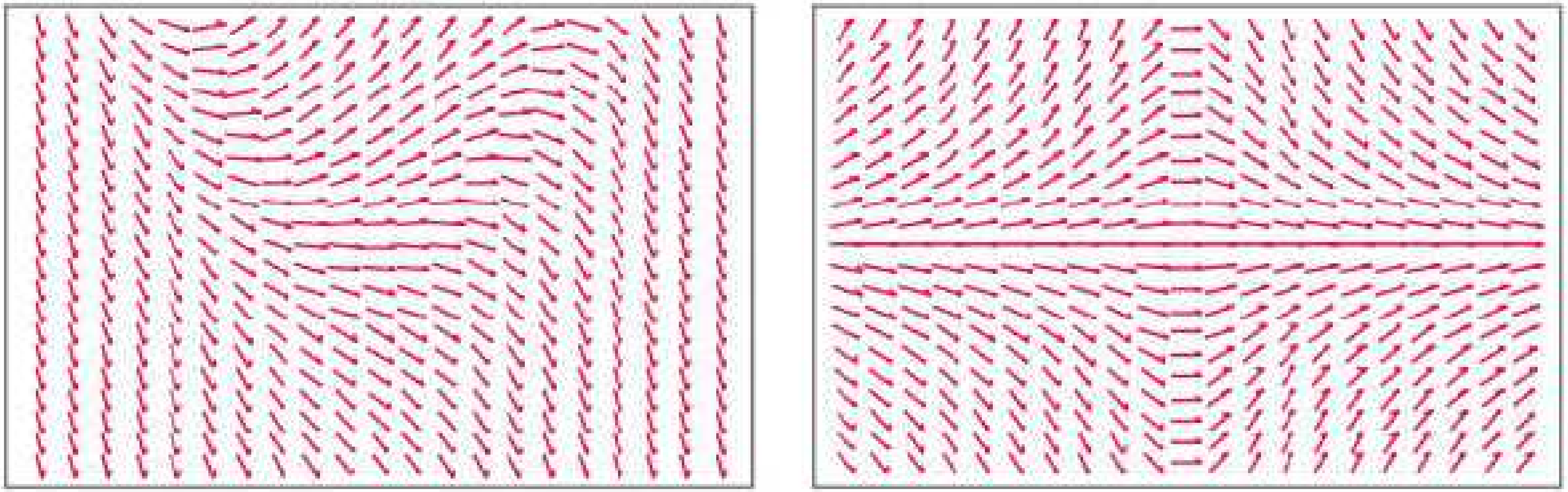
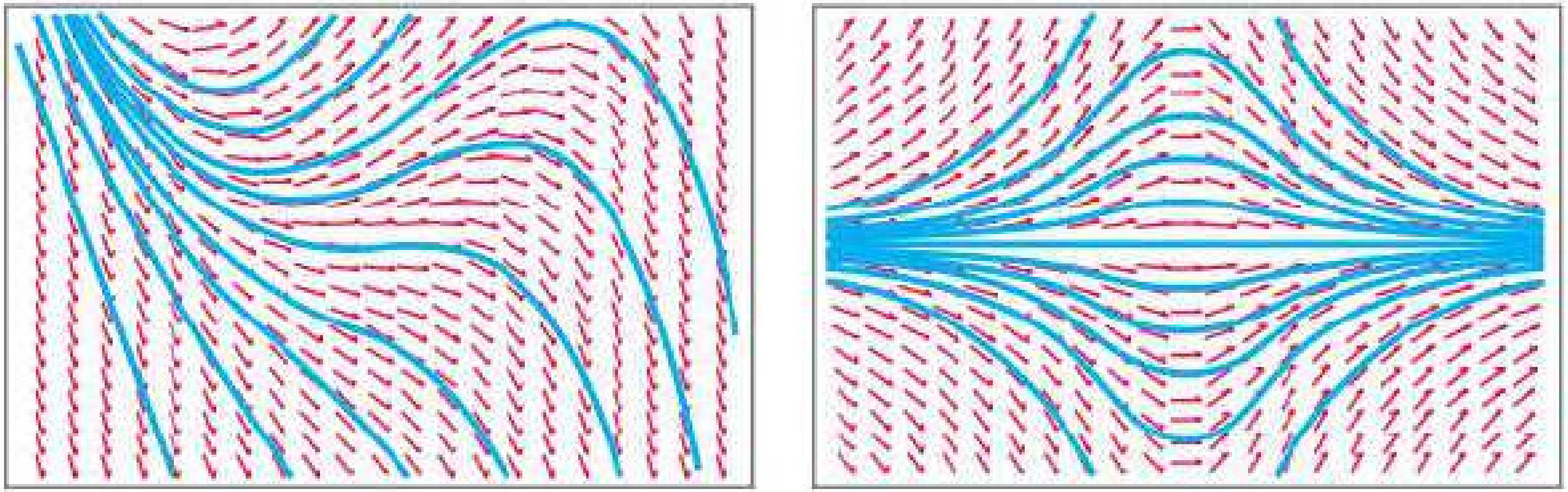


FIGURE 9.2 (a) Slope field for $\frac{dy}{dx} = y - x$. (b) The particular solution curve through the point $\left(0, \frac{2}{3}\right)$ (Example 2).



(a) $y' = y - x^2$

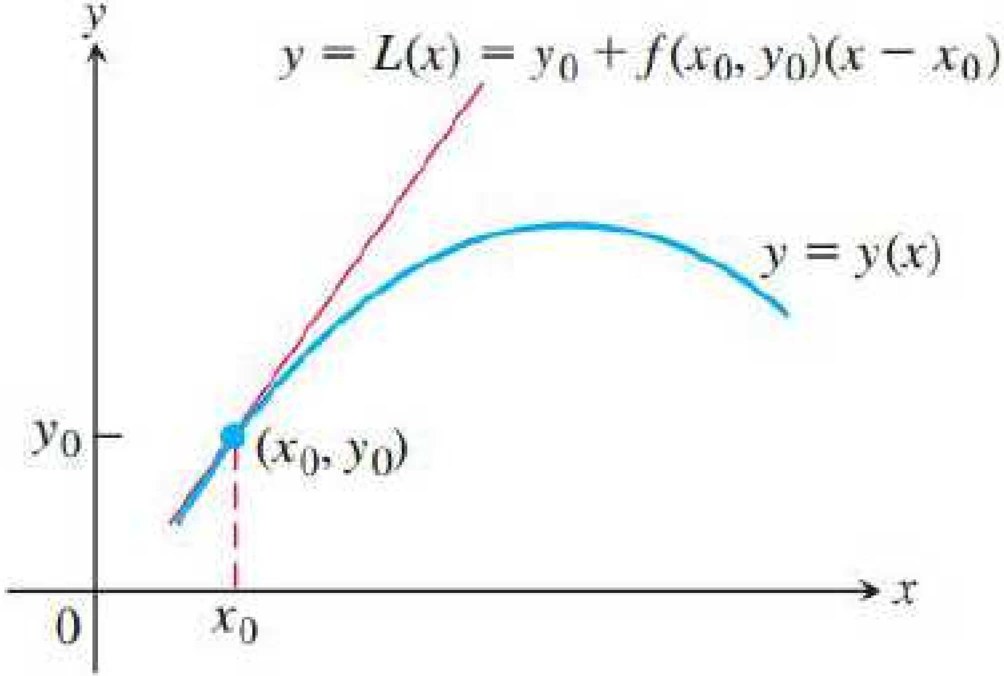
(b) $y' = -\dfrac{2xy}{1 + x^2}$



Euler's Method 欧拉方法

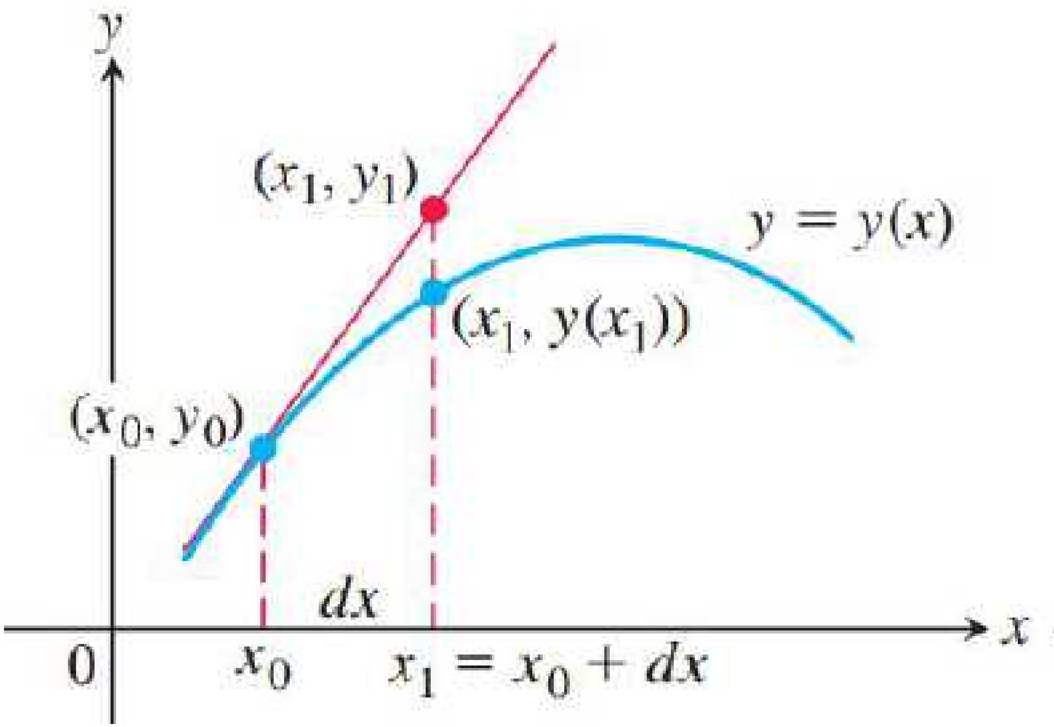
Given a differential equation $dy/dx = f(x, y)$ and an initial condition $y(x_0) = y_0$, can approximate the solution $y = y(x)$ by its linearization

$$L(x) = y(x_0) + y'(x_0)(x - x_0) \qquad \text{or} \qquad L(x) = y_0 + f(x_0, y_0)(x - x_0).$$



$$L(x) = y_0 + f(x_0, y_0)(x - x_0).$$

$$x_1 = x_0 + dx. \qquad y_1 = L(x_1) = y_0 + f(x_0, y_0)\,dx$$



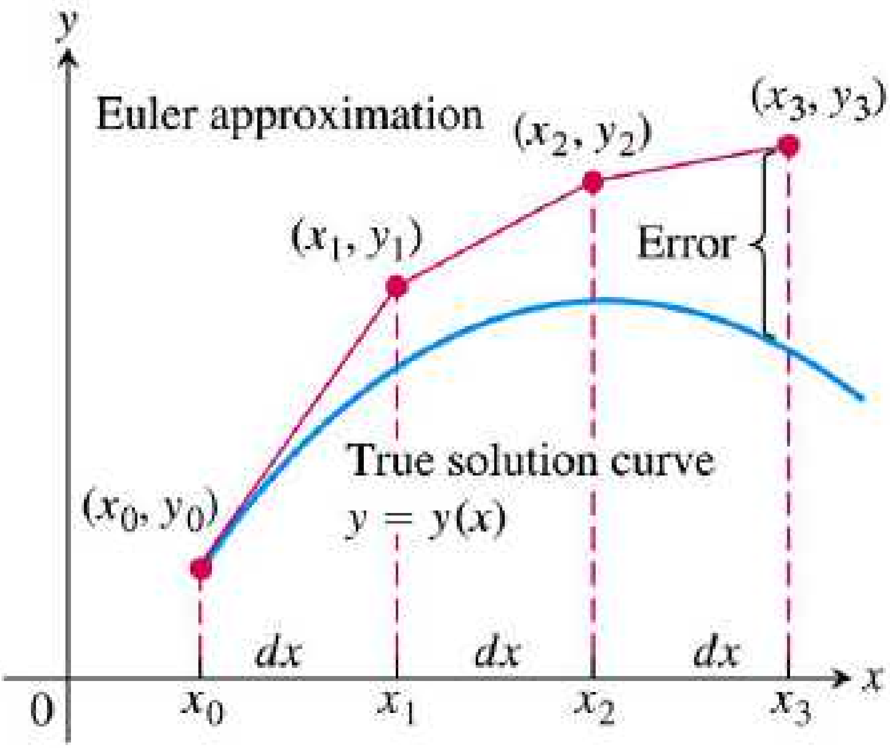


FIGURE 9.6 Three steps in the Euler approximation to the solution of the initial value problem $y' = f(x, y), y(x_0) = y_0$. As we take more steps, the errors involved usually accumulate, but not in the exaggerated way shown here.

$$L(x) = y_0 + f(x_0, y_0)(x - x_0).$$

$$y_1 = L(x_1) = y_0 + f(x_0, y_0)\,dx$$

$$L_1(x) = y_1 + f(x_1, y_1)(x - x_1)$$

$$x_2 = x_1 + dx$$

$$y_2 = y_1 + f(x_1, y_1)\,dx.$$

$$L_2(x) = y_2 + f(x_2, y_2)(x - x_2)$$

$$x_3 = x_2 + dx$$

$$y_3 = y_2 + f(x_2, y_2)\,dx,$$

$$
\begin{aligned}
x_1 &= x_0 + dx \\
x_2 &= x_1 + dx \\
&\vdots \\
x_n &= x_{n-1} + dx.
\end{aligned}
\qquad
\begin{aligned}
y_1 &= y_0 + f(x_0, y_0)\, dx \\
y_2 &= y_1 + f(x_1, y_1)\, dx \\
&\vdots \\
y_n &= y_{n-1} + f(x_{n-1}, y_{n-1})\, dx.
\end{aligned}
$$

Euler's Method

EXAMPLE 3 Find the first three approximations y_1, y_2, y_3 using Euler's method for the initial value problem

$$y' = 1 + y, \qquad y(0) = 1,$$

starting at $x_0 = 0$ with $dx = 0.1$.

Solution We have the starting values $x_0 = 0$ and $y_0 = 1$.

$$x_1 = x_0 + dx = 0.1,$$

$$x_2 = x_0 + 2\,dx = 0.2, \text{ and } x_3 = x_0 + 3\,dx = 0.3.$$

First:
$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)\,dx \\ &= y_0 + (1 + y_0)\,dx \\ &= 1 + (1 + 1)(0.1) = 1.2 \end{aligned}$$

Second:

$$y_2 = y_1 + f(x_1, y_1)\,dx$$
$$= y_1 + (1 + y_1)\,dx$$
$$= 1.2 + (1 + 1.2)(0.1) = 1.42$$

Third:

$$y_3 = y_2 + f(x_2, y_2)\,dx$$
$$= y_2 + (1 + y_2)\,dx$$
$$= 1.42 + (1 + 1.42)(0.1) = 1.662$$

The number of steps n can be as large as we like, but errors can accumulate

EXAMPLE 4 Use Euler's method to solve

$$y' = 1 + y, \qquad y(0) = 1,$$

on the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking **(a)** $dx = 0.1$ **(b)** $dx = 0.05$.

Solution $x_0 = 0, y_0 = 1, f(x, y) = 1 + y$

(a) We used a computer to generate the approximate values

$$x_1 = x_0 + dx = 0.1 \qquad y_1 = 1 + (1 + 1) \times 0.1 = 1.2$$

$$x_2 = x_1 + dx = 0.2 \qquad y = 1.2 + (1 + 1.2) \times 0.1 = 1.42$$

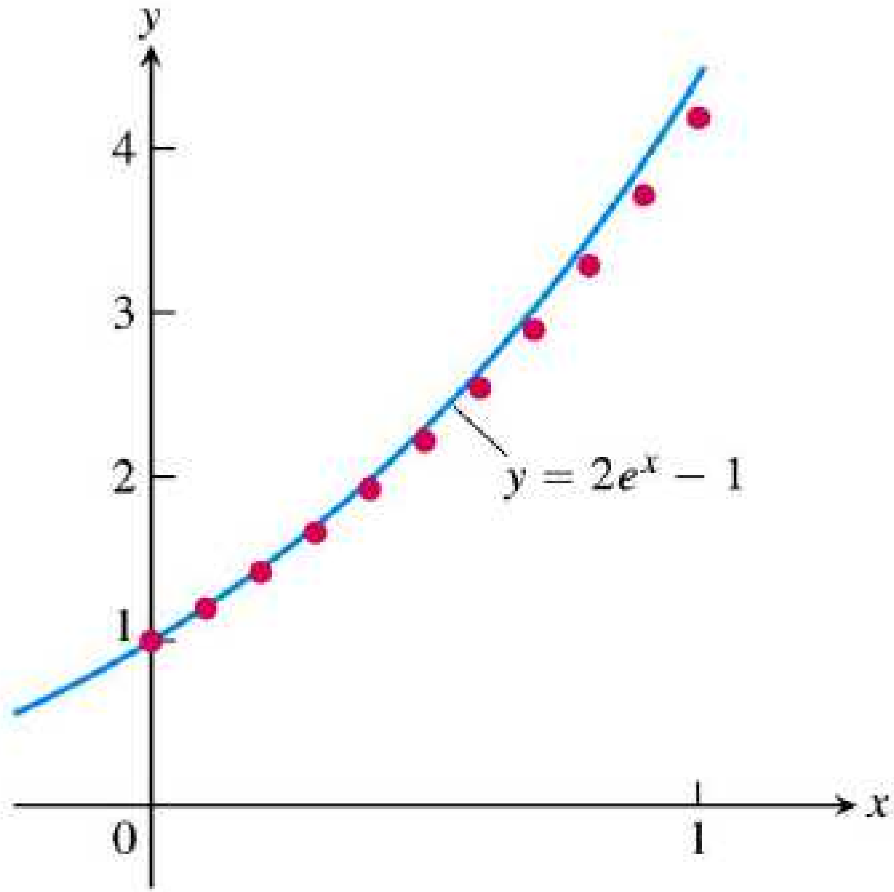
$$\vdots \qquad\qquad \vdots$$

$$x_n = x_{n-1} + dx. = 1 \qquad y_n = y_{n-1} + f(x_{n-1}, y_{n-1})\, dx.$$

TABLE 9.1 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.1$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.1	1.2	1.2103	0.0103
0.2	1.42	1.4428	0.0228
0.3	1.662	1.6997	0.0377
0.4	1.9282	1.9836	0.0554
0.5	2.2210	2.2974	0.0764
0.6	2.5431	2.6442	0.1011
0.7	2.8974	3.0275	0.1301
0.8	3.2872	3.4511	0.1639
0.9	3.7159	3.9192	0.2033
1.0	4.1875	4.4366	0.2491

$x = 1$ (after 10 steps), the error is about 5.6%



$$y = 2e^x - 1.$$

FIGURE 9.7 The graph of $y = 2e^x - 1$ superimposed on a scatterplot of the Euler approximations shown in Table 9.1 (Example 4).

(b) $dx = 0.05.$

One way to try to reduce the error is to decrease the step size.

$$
\begin{aligned}
x_1 &= x_0 + dx \\
x_2 &= x_1 + dx \\
&\vdots \\
x_n &= x_{n-1} + dx.
\end{aligned}
\qquad
\begin{aligned}
y_1 &= y_0 + f(x_0, y_0)\,dx \\
y_2 &= y_1 + f(x_1, y_1)\,dx \\
&\vdots \\
y_n &= y_{n-1} + f(x_{n-1}, y_{n-1})\,dx.
\end{aligned}
$$

TABLE 9.2 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.05$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.05	1.1	1.1025	0.0025
0.10	1.205	1.2103	0.0053
0.15	1.3153	1.3237	0.0084
0.20	1.4310	1.4428	0.0118
0.25	1.5526	1.5681	0.0155
0.30	1.6802	1.6997	0.0195
0.35	1.8142	1.8381	0.0239
0.40	1.9549	1.9836	0.0287

0.45	2.1027	2.1366	0.0340
0.50	2.2578	2.2974	0.0397
0.55	2.4207	2.4665	0.0458
0.60	2.5917	2.6442	0.0525
0.65	2.7713	2.8311	0.0598
0.70	2.9599	3.0275	0.0676
0.75	3.1579	3.2340	0.0761
0.80	3.3657	3.4511	0.0853
0.85	3.5840	3.6793	0.0953
0.90	3.8132	3.9192	0.1060
0.95	4.0539	4.1714	0.1175
1.00	4.3066	4.4366	0.1300

about 2.9%.

9.2

First-Order Linear Equations
一阶线性方程

A first-order **linear** differential equation

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad \text{in standard form}$$

the exponential growth/decay equation $dy/dx = ky$:

$$\frac{dy}{dx} - ky = 0,$$

$P(x) = -k \qquad Q(x) = 0.$

linear

y and its derivative dy/dx occur only to the first power,

EXAMPLE 1 Put the following equation in standard form:

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution

$$\frac{dy}{dx} = x + \frac{3}{x}y$$

$$\frac{dy}{dx} - \frac{3}{x}y = x$$

$P(x) = -3/x \qquad Q(x) = x$

Solving Linear Equations

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (1)$$

$$y' + \frac{1}{x}y = \frac{\sin x}{x} \qquad xy' + y = \sin x \qquad (xy)' = \sin x$$

$$xy = -\cos x + C$$

$$v(x)\frac{dy}{dx} + v'(x) \quad y = v(x)Q(x)$$

such that $P(x)v(x) = v'(x)$

$$\frac{dv}{dx} = Pv$$

$v(x)$ an **integrating factor** for Equation (1)

$$\frac{d}{dx}(v(x)\cdot y) = v(x)Q(x)$$

$$v(x)\cdot y = \int v(x)Q(x)\,dx \qquad y = \frac{1}{v(x)}\int v(x)Q(x)\,dx$$

$$\frac{dv}{dx} = Pv \qquad \frac{dv}{v} = P\,dx \qquad \int \frac{dv}{v} = \int P\,dx$$

$$\ln v = \int P\,dx \qquad e^{\ln v} = e^{\int P\,dx} \qquad \underline{v = e^{\int P\,dx}}$$

$$y = \frac{1}{v(x)} \int v(x)Q(x)\,dx$$

$$\underline{y = e^{-\int P(x)dx}\left(\int e^{\int P(x)dx} Q(x)dx + C\right)}$$

To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\int P(x)\,dx}$ and integrate both sides.

积分因子 $v(x) = e^{\int P(x)dx}$,

EXAMPLE 2 Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution First we put the equation in standard form (Example 1):

$$\frac{dy}{dx} - \frac{3}{x}y = x,$$

$$\begin{aligned} v(x) = e^{\int P(x)\,dx} &= e^{\int (-3/x)\,dx} \\ &= e^{-3\ln|x|} \\ &= e^{-3\ln x} \\ &= e^{\ln x^{-3}} = \frac{1}{x^3}. \end{aligned}$$

$$\frac{1}{x^3}\cdot\left(\frac{dy}{dx}-\frac{3}{x}y\right)=\frac{1}{x^3}\cdot x \qquad \frac{1}{x^3}\frac{dy}{dx}-\frac{3}{x^4}y=\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{x^3}y\right)=\frac{1}{x^2} \qquad \frac{1}{x^3}y=\int\frac{1}{x^2}\,dx$$

$$\frac{1}{x^3}y=-\frac{1}{x}+C.$$

$$y=x^3\left(-\frac{1}{x}+C\right)=-x^2+Cx^3,\qquad x>0.$$

EXAMPLE 3 Find the particular solution of

$$3xy' - y = \ln x + 1, \qquad x > 0,$$

satisfying $y(1) = -2$.

Solution With $x > 0$, we write the equation in standard form:

$$y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}.$$

$$v = e^{\int -dx/3x} = e^{(-1/3)\ln x} = x^{-1/3}. \qquad x > 0$$

$$(x^{-\frac{1}{3}}y)' = x^{-\frac{1}{3}}\left(\frac{\ln x + 1}{3x}\right) \qquad x^{-1/3}y = \frac{1}{3}\int (\ln x + 1)x^{-4/3}\,dx.$$

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) + \int x^{-4/3}\,dx + C.$$

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) + \int x^{-4/3}\,dx + C.$$

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) - 3x^{-1/3} + C$$

$$y = -(\ln x + 4) + Cx^{1/3}.$$

When $x = 1$ and $y = -2$ this last equation becomes

$$-2 = -(0 + 4) + C, \qquad C = 2.$$

$$y = 2x^{1/3} - \ln x - 4.$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dy}{dx} + P(x)y = 0$$

$Q(x) = 0$

$$\frac{dy}{y} = -P(x)\,dx$$

一阶线性齐次方程

Separating the variables

$$\ln y = -\int P(x)dx + \ln C$$

$$y = Ce^{-\int P(x)dx}$$

$$y = e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)dx + C\right)$$

$$= Ce^{-\int P(x)dx} + e^{-\int P(x)dx}\int e^{\int P(x)dx}Q(x)dx$$

通解解构

非齐次方程的通解 = 齐次方程的通解 + 非齐次方程的特解

$$\frac{dy}{dx} + P(x)y = y^2 Q(x)$$

$$\frac{dy}{dx} + P(x)y = y^n Q(x)$$

$$y^{-2}\frac{dy}{dx} + P(x)y^{-1} = Q(x)$$

$$-\frac{d(y^{-1})}{dx} + P(x)y^{-1} = Q(x)$$

$$\frac{d(y^{-1})}{dx} - P(x)y^{-1} = -Q(x)$$

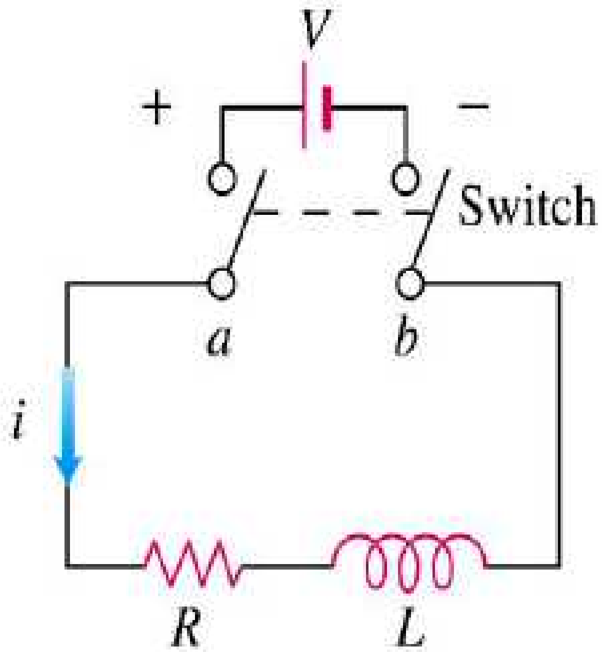
$$\frac{dz}{dx} - P(x)z = -Q(x)$$

9.3

Applications

电路问题

EXAMPLE 4 The switch in the RL circuit in Figure 9.8 is closed at time $t = 0$. How will the current flow as a function of time?



$$L\frac{di}{dt} + Ri = V,$$

Solution

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L},$$

$$i = 0 \text{ when } t = 0,$$

$$i = e^{-\int \frac{R}{L}dt}\left(\int \frac{V}{L}e^{\int \frac{R}{L}dt}dt + C\right)$$

$$i = \frac{V}{R} - \frac{V}{R}e^{-(R/L)t}.$$

$$\lim_{t\to\infty} i = \lim_{t\to\infty}\left(\frac{V}{R} - \frac{V}{R}e^{-(R/L)t}\right) = \frac{V}{R} - \frac{V}{R}\cdot 0 = \frac{V}{R}.$$

FIGURE 9.8 The RL ci[rcuit in] Example 4.

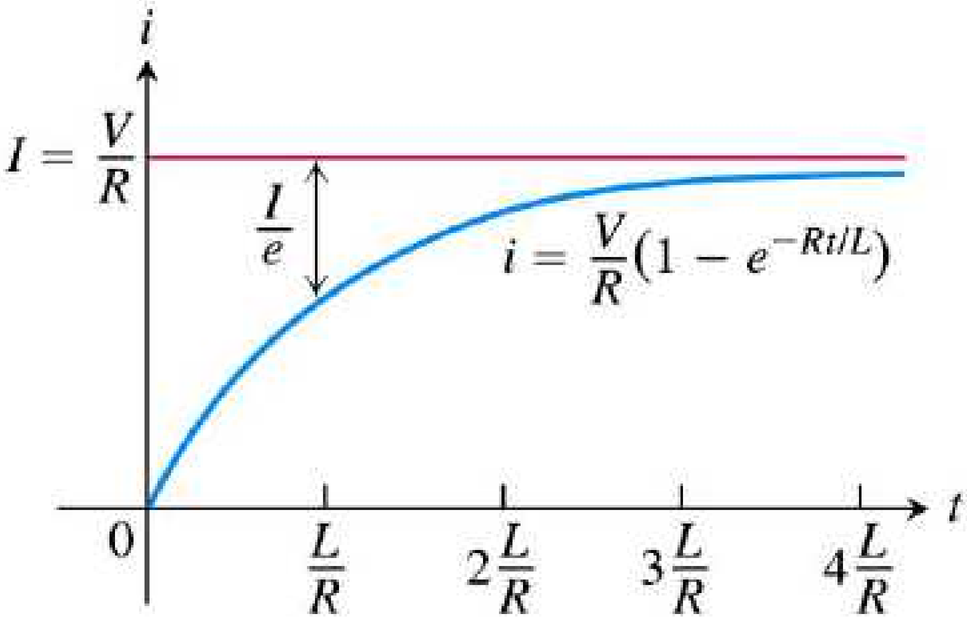


FIGURE 9.9 The growth of the current in the *RL* circuit in Example 4. I is the current's steady-state value. The number $t = L/R$ is the time constant of the circuit. The current gets to within 5% of its steady-state value in 3 time constants (Exercise 27).

Motion with Resistance Proportional to Velocity 滑行问题

the resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. $\textbf{Force} = -\boldsymbol{k} \cdot \boldsymbol{v}$

$$\text{Force} = \text{mass} \times \text{acceleration} = m\frac{dv}{dt}.$$

$$m\frac{dv}{dt} = -kv \qquad \text{or} \qquad \frac{dv}{dt} = -\frac{k}{m}v \qquad (k > 0).$$

$$v = v_0 \text{ at } t = 0$$

$$\boxed{v = v_0 e^{-(k/m)t}.} \qquad \frac{ds}{dt} = v_0 e^{-(k/m)t}, \qquad s(0) = 0.$$

$$s = -\frac{v_0 m}{k} e^{-(k/m)t} + C. \qquad C = \frac{v_0 m}{k}.$$

$$s(t) = -\frac{v_0 m}{k} e^{-(k/m)t} + \frac{v_0 m}{k} = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right).$$

$$\lim_{t\to\infty} s(t) = \lim_{t\to\infty} \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right)$$

$$= \frac{v_0 m}{k}(1 - 0) = \frac{v_0 m}{k}.$$

$$\text{Distance coasted} = \frac{v_0 m}{k}.$$

最大滑行距离

EXAMPLE 1 For a 192-lb ice skater, the k in Equation (1) is about $1/3$ slug/sec and $m = 192/32 = 6$ slugs. How long will it take the skater to coast from 11 ft/sec (7.5 mph) to 1 ft/sec? How far will the skater coast before coming to a complete stop?

Solution

$$v = v_0 e^{-(k/m)t}.$$

$$11e^{-t/18} = 1$$

$$e^{-t/18} = 1/11$$

$$-t/18 = \ln(1/11) = -\ln 11$$

$$t = 18 \ln 11 \approx 43 \text{ sec.}$$

$$\text{Distance coasted} = \frac{v_0 m}{k} = \frac{11 \cdot 6}{1/3}$$

$$= 198 \text{ ft.}$$

人口模型的修改

Inaccuracy of the Exponential Population Growth Model

we modeled population growth with the Law of Exponential Change:

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0$$

where P is the population at time t, $k > 0$

notice

$$\frac{dP/dt}{P} = k$$

the **relative growth rate**.

TABLE 9.3 World population (midyear)

$$\frac{dP/dt}{P} = k$$

Year	Population (millions)	$\Delta P/P$
1980	4454	$76/4454 \approx 0.0171$
1981	4530	$80/4530 \approx 0.0177$
1982	4610	$80/4610 \approx 0.0174$
1983	4690	$80/4690 \approx 0.0171$
1984	4770	$81/4770 \approx 0.0170$
1985	4851	$82/4851 \approx 0.0169$
1986	4933	$85/4933 \approx 0.0172$
1987	5018	$87/5018 \approx 0.0173$
1988	5105	$85/5105 \approx 0.0167$
1989	5190	

Source: U.S. Bureau of the Census (Sept., 2007): www.census.gov/ipc/www/idb.

$$\frac{dP}{dt} = 0.017P, \qquad P(0) = 4454.$$

The solution to this initial value problem

$$P = 4454e^{0.017t}.$$

2008 (so $t = 28$). 7169 million,

6707 million from the U.S. Bureau of the Census.

0.012 since 1987

$$\frac{dP}{dt} = r(M - P)P$$

logistic Population Growth

Orthogonal Trajectories 正交轨问题

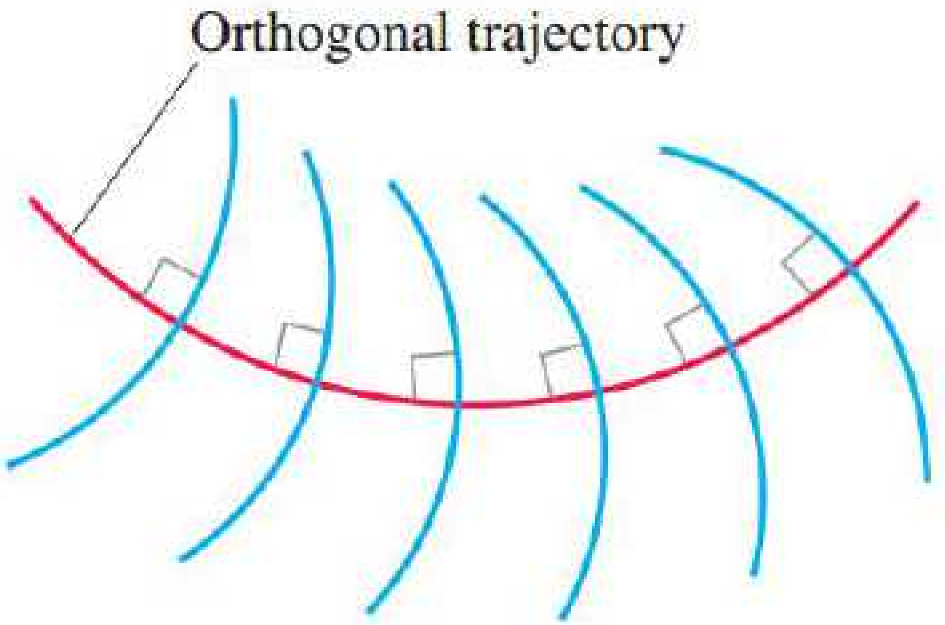
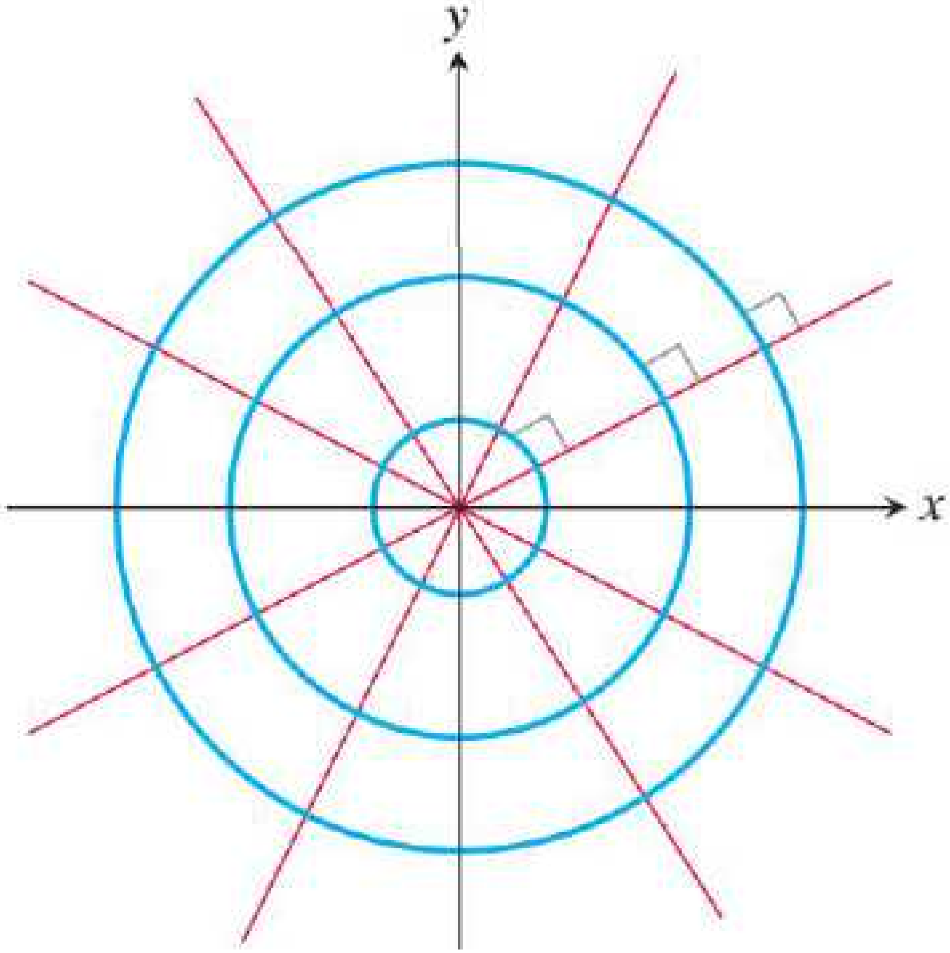


FIGURE 9.11 An orthogonal trajectory intersects the family of curves at right angles, or orthogonally.

an orthogonal trajectory of the family of circles $x^2 + y^2 = a^2$,



EXAMPLE 2 Find the orthogonal trajectories of the family of curves $xy = a$, where $a \neq 0$ is an arbitrary constant.

Solution The curves $xy = a$ form a family of hyperbolas

Differentiating $xy = a$ implicitly gives

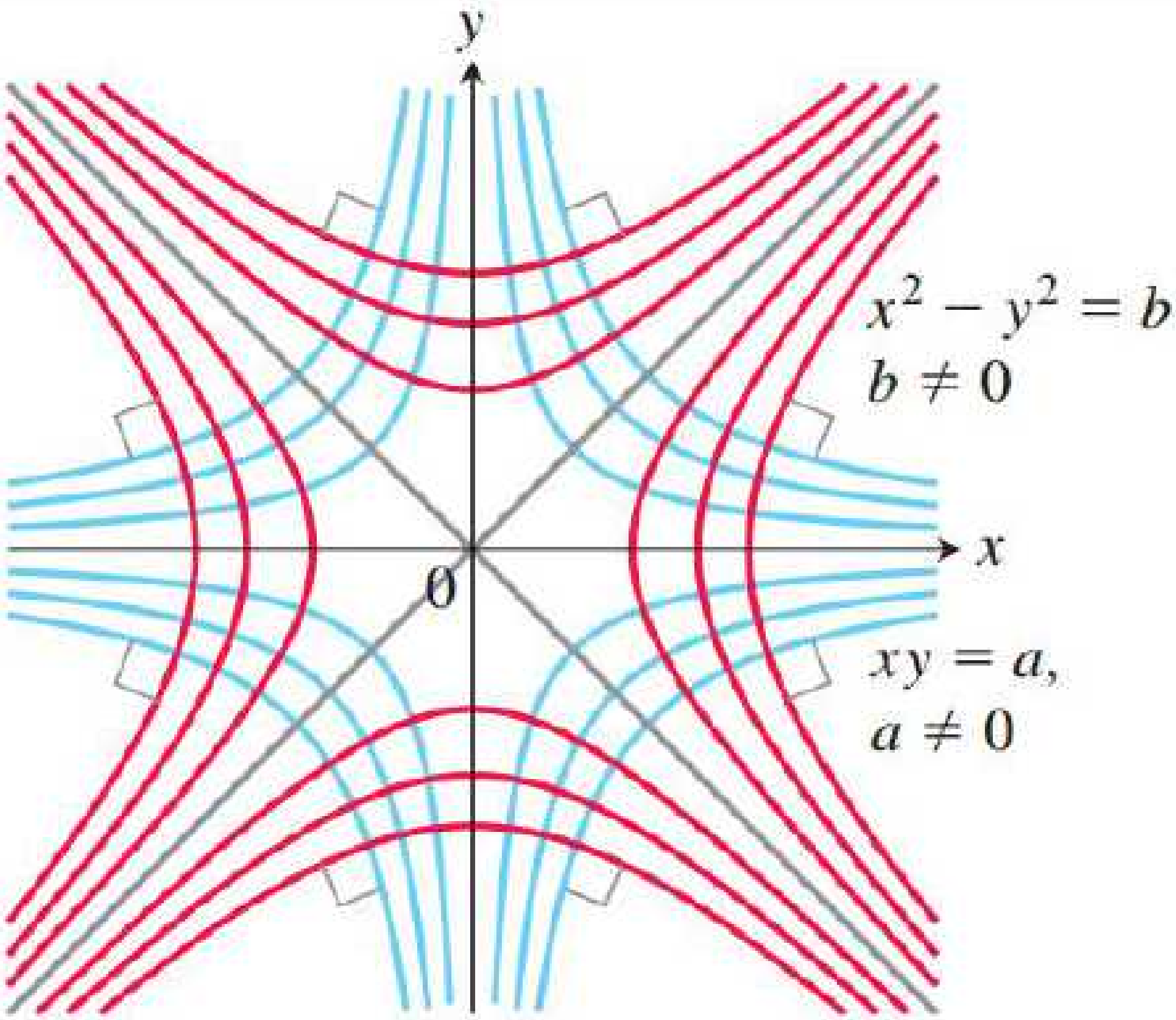
$$x\frac{dy}{dx} + y = 0 \qquad \frac{dy}{dx} = -\frac{y}{x}.$$

Therefore, the orthogonal trajectories must satisfy $\dfrac{dy}{dx} = \dfrac{x}{y}.$

$$y\,dy = x\,dx$$

$$\int y\,dy = \int x\,dx \qquad \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

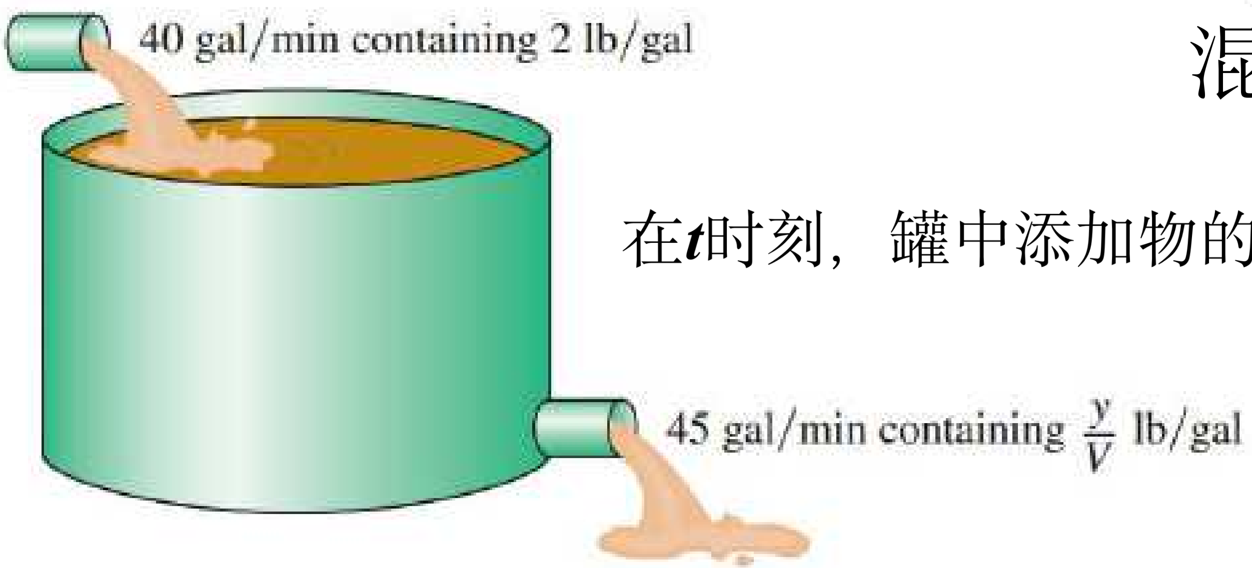
$$y^2 - x^2 = b,$$



y
x
0
$x^2 - y^2 = b$
$b \neq 0$
$xy = a,$
$a \neq 0$

EXAMPLE 3 In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins?

If $y(t)$ is the amount of chemical in the container at time t



混合问题

在t时刻，罐中添加物的浓度: $\dfrac{y(t)}{V(t)}$

Mixture Problems

If $y(t)$ is the amount of chemical in the container at time t

$V(t)$ is the total volume liquid in the container at time t,

then the departure rate of the chemical at time t is

$$\text{Departure rate} = \frac{y(t)}{V(t)} \cdot (\text{outflow rate})$$

$$= \begin{pmatrix} \text{concentration in} \\ \text{container at time } t \end{pmatrix} \cdot (\text{outflow rate}).$$

$$\boxed{\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)} \cdot (\text{outflow rate}).}$$

Solution Let y be the amount (in pounds) of additive in the tank at time t.

$y = 100$ when $t = 0$.

$$\boxed{\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)} \cdot (\text{outflow rate}).}$$

$$\text{Rate in} = \left(2\frac{\text{lb}}{\text{gal}}\right)\left(40\frac{\text{gal}}{\text{min}}\right) = 80\frac{\text{lb}}{\text{min}}.$$

$$V(t) = 2000\text{ gal} + \left(40\frac{\text{gal}}{\text{min}} - 45\frac{\text{gal}}{\text{min}}\right)(t\text{ min}) = (2000 - 5t)\text{ gal}.$$

$$\text{Rate out} = \frac{y(t)}{V(t)} \cdot \text{outflow rate} = \left(\frac{y}{2000 - 5t}\right)45$$

$$= \frac{45y}{2000 - 5t}\frac{\text{lb}}{\text{min}}.$$

$$\frac{dy}{dt} = 80 - \frac{45y}{2000 - 5t}$$

$$\frac{dy}{dt} + \frac{45}{2000 - 5t}y = 80.$$

Thus, $P(t) = 45/(2000 - 5t)$ and $Q(t) = 80.$

$$v(t) = e^{\int P\,dt} = e^{\int \frac{45}{2000-5t}dt} = e^{-9\ln(2000-5t)} = (2000 - 5t)^{-9}.$$

$$\frac{d}{dt}\left[(2000 - 5t)^{-9}y\right] = 80(2000 - 5t)^{-9}$$

$$(2000 - 5t)^{-9}y = \int 80(2000 - 5t)^{-9}\,dt$$

$$= 80 \cdot \frac{(2000 - 5t)^{-8}}{(-8)(-5)} + C.$$

The general solution is $y = 2(2000 - 5t) + C(2000 - 5t)^9.$

Because $y = 100$ when $t = 0$, we can determine the value of C:

$$C = -\frac{3900}{(2000)^9}.$$

The particular solution $y = 2(2000 - 5t) - \frac{3900}{(2000)^9}(2000 - 5t)^9.$

The amount of additive in the tank 20 min after

$$y(20) = 2[2000 - 5(20)] - \frac{3900}{(2000)^9}[2000 - 5(20)]^9 \approx 1342 \text{ lb}.$$

Salt mixture A tank initially contains 100 gal of brine in which 50 lb of salt are dissolved. A brine containing 2 lb/gal of salt runs into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring and flows out of the tank at the rate of 4 gal/min.

Let y be the amount (in pounds) of salt in the tank at time t.

a. At what rate (pounds per minute) does salt enter the tank at time t?

b. What is the volume of brine in the tank at time t?

c. At what rate (pounds per minute) does salt leave the tank at time t?

d. Write down and solve the initial value problem describing the mixing process.

e. Find the concentration of salt in the tank 25 min after the process starts.

10 lb/min

$(100 + t)$ gal

$$4\left(\frac{y}{100 + t}\right) \text{lb/min}$$

$$\frac{dy}{dt} = 10 - \frac{4y}{100 + t}, \quad y(0) = 50,$$

$$y = 2(100 + t) + \frac{C}{(100 + t)^4}$$

$$y = 2(100 + t) - \frac{15 \times 10^9}{(100 + t)^4}$$

$$\text{Concentration} = \frac{y(25)}{\text{amt. brine in tank}} = \frac{188.6}{125} \approx 1.5 \text{ lb/gal}$$

9.4

Graphical Solutions of Autonomous Equations

自治微分方程的图解

Equilibrium Values and Phase Lines

平衡值与相线

the ability to discern physical behavior from graphs is a powerful tool in understanding real-world

When we differentiate implicitly the equation

$$\frac{1}{5}\ln(5y - 15) = x + 1,$$

we obtain

$$\frac{1}{5}\left(\frac{5}{5y - 15}\right)\frac{dy}{dx} = 1.$$

$$y' = 5(y - 3).$$

$y = 3$ 是微分方程的一个解曲线

dy/dx is a function of y only is called an **autonomous** 自治

DEFINITION If $dy/dx = g(y)$ is an autonomous differential equation, then the values of y for which $dy/dx = 0$ are called **equilibrium values** or **rest points**.

注意: The emphasis is on the value of y where $dy/dx = 0$, not the value of x, as we studied in Chapter 4.

the equilibrium values for the autonomous differential equation

$$\frac{dy}{dx} = (y + 1)(y - 2)$$

are $y = -1$ and $y = 2$.

平衡值有什么用呢？

可以将方程不同类型的解曲线分开。

a **phase line** for the equation

EXAMPLE 1 Draw a phase line for the equation

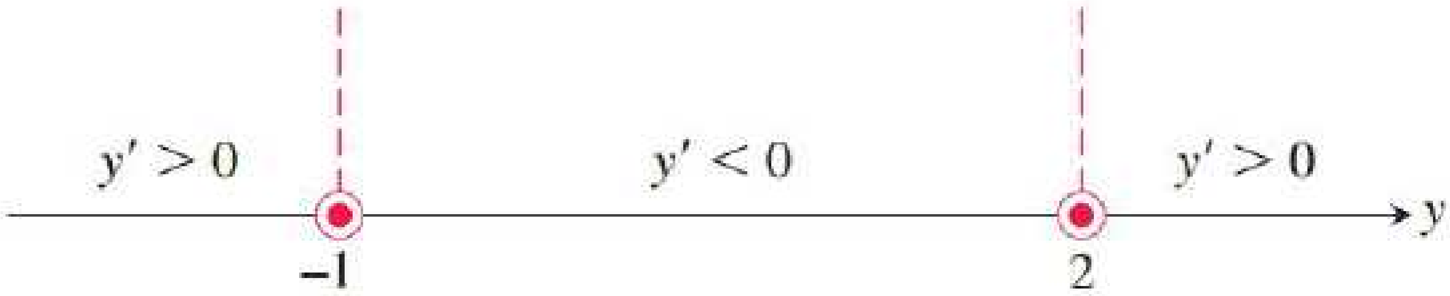
$$\frac{dy}{dx} = (y + 1)(y - 2)$$

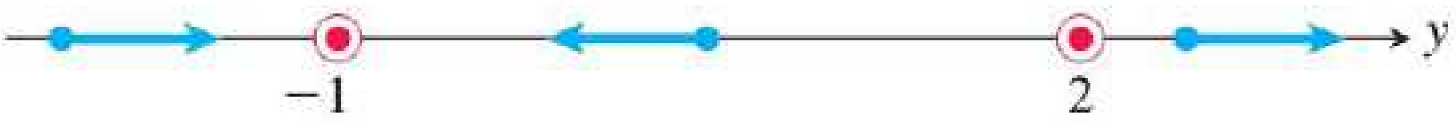
and use it to sketch solutions to the equation.

Solution

1. *Draw a number line for y and mark the equilibrium values* $dy/dx = 0$.

2. *Identify and label the intervals where* $y' > 0$ *and* $y' < 0$.



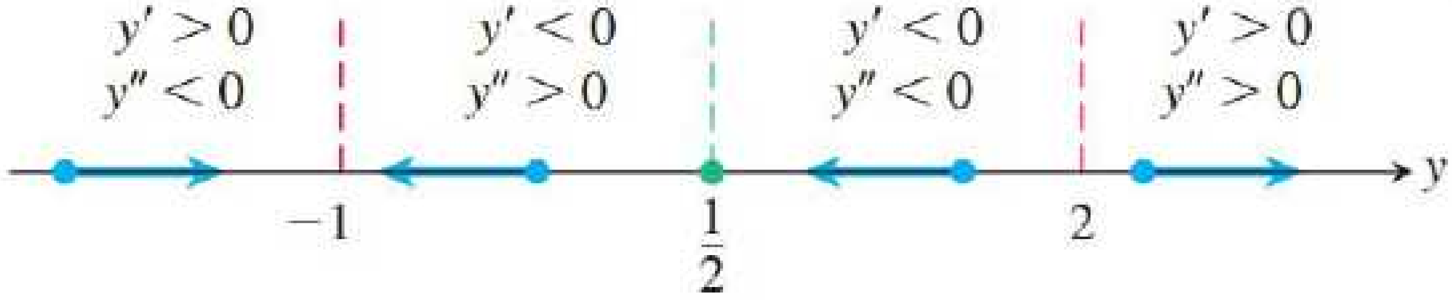


3. *Calculate* y'' *and mark the intervals where* $y'' > 0$ *and* $y'' < 0$.

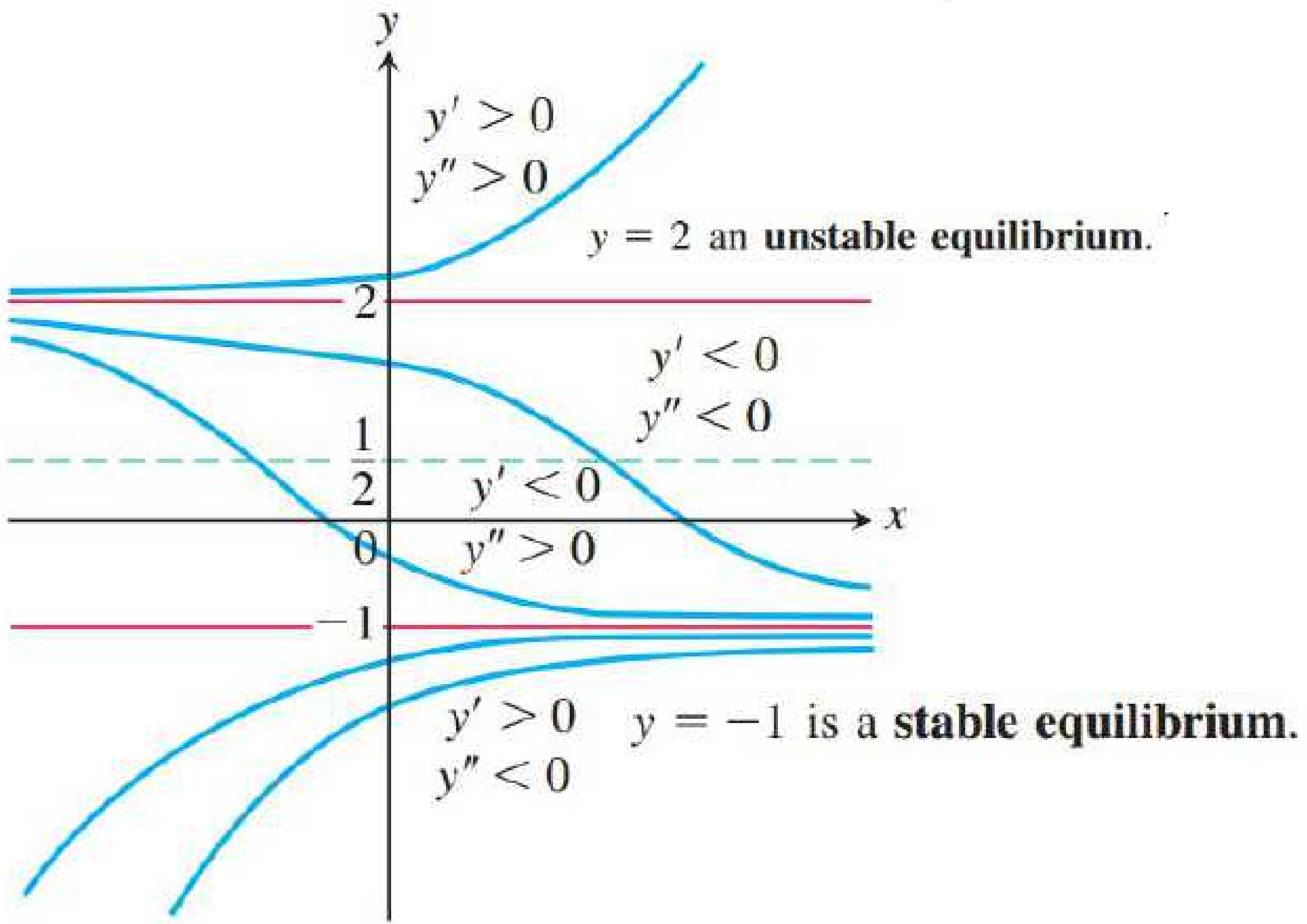
$$\frac{dy}{dx} = (y + 1)(y - 2)$$

$$y'' = (2y - 1)y'$$

$$y'' = (2y - 1)(y + 1)(y - 2).$$



4. *Sketch an assortment of solution curves in the xy-plane.*



y
y' > 0
y'' > 0
y = 2 an unstable equilibrium.
2
y' < 0
y'' < 0
1/2
y' < 0
y'' > 0
x
0
−1
y' > 0
y'' < 0
y = −1 is a stable equilibrium.

Newton's Law of Cooling

H is the temperature of an object at time t

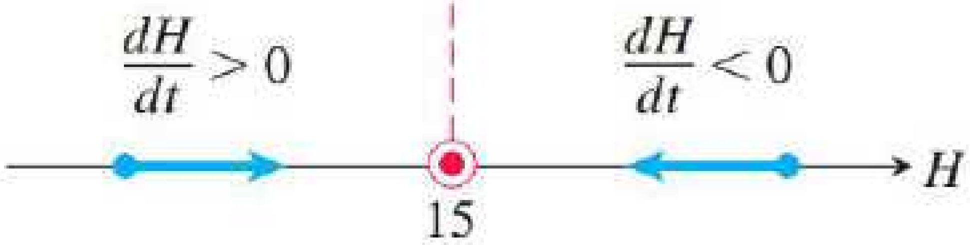
$$\frac{dH}{dt} = -k(H - 15) \qquad k > 0$$

the surrounding medium temperature of 15°C.

Since $dH/dt = 0$ at $H = 15$. 15°C is an equilibrium value.

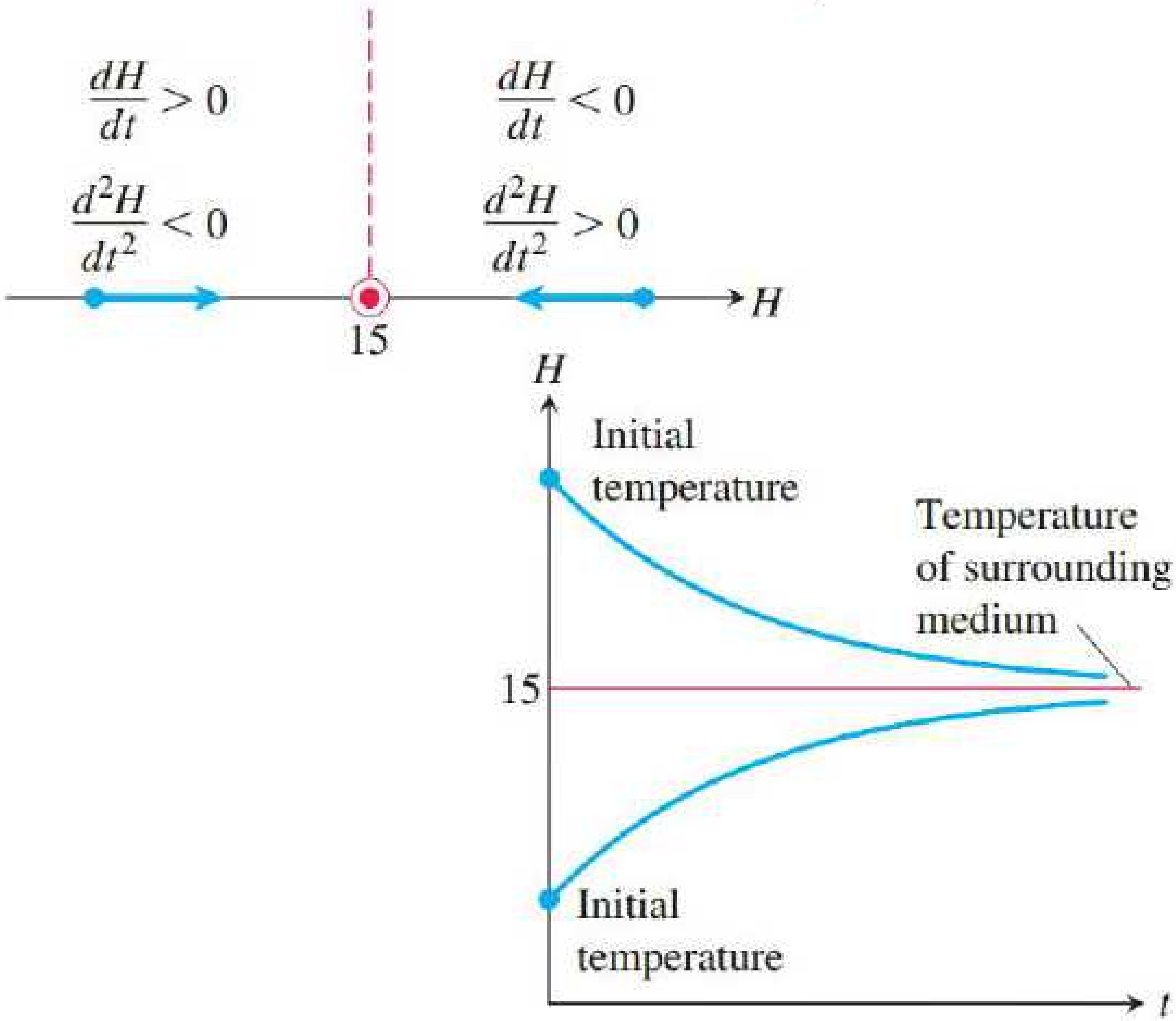
$H > 15$, $dH/dt < 0$.

$H < 15$, $dH/dt > 0$.



$$\frac{d^2H}{dt^2} = -k\frac{dH}{dt}.$$

d^2H/dt^2 is positive when $dH/dt < 0$ and negative when $dH/dt > 0$.



$\frac{dH}{dt} > 0$
$\frac{d^2H}{dt^2} < 0$
$\frac{dH}{dt} < 0$
$\frac{d^2H}{dt^2} > 0$
15
H
H
Initial temperature
Temperature of surrounding medium
15
Initial temperature
t

A Falling Body Encountering Resistance

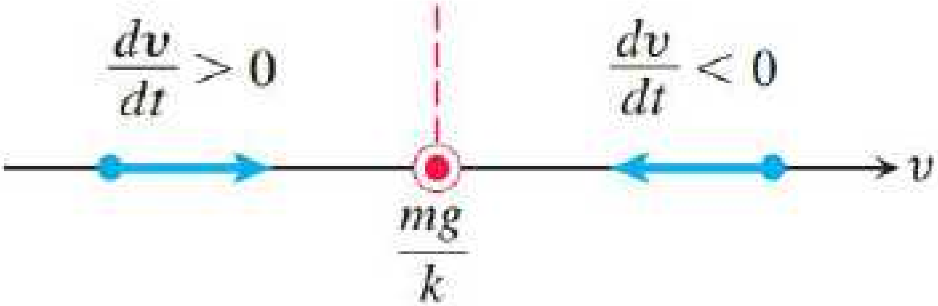
$F = ma,$ In free fall, $F_p = mg,$

A more realistic model of free fall would include air resistance, F_r

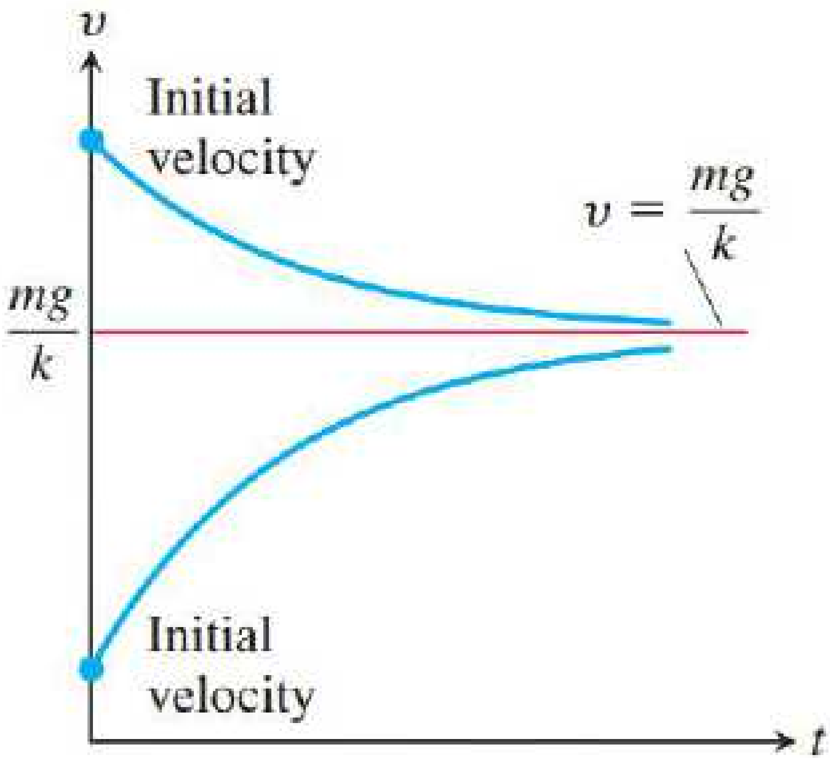
F_r is approximately proportional to the body's velocity.

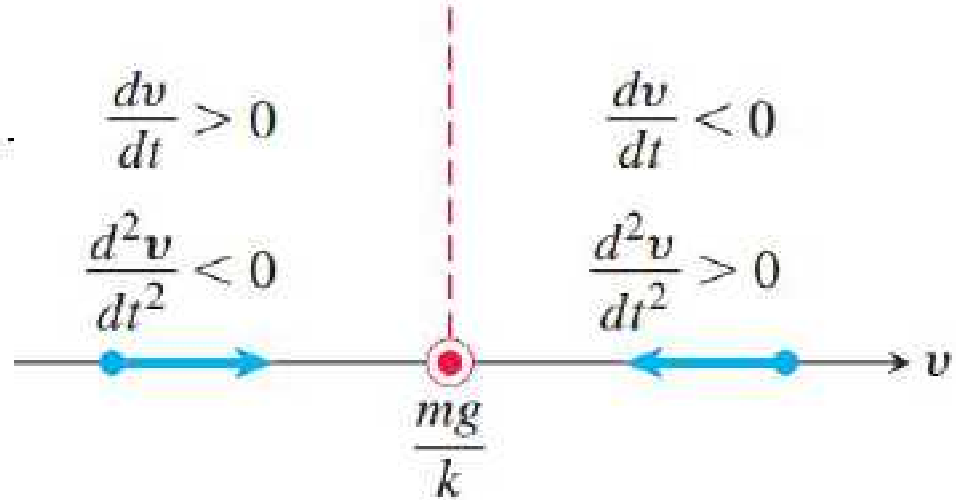
$$m\frac{dv}{dt} = mg - kv \qquad \frac{dv}{dt} = g - \frac{k}{m}v.$$

The equilibrium point, $v = \dfrac{mg}{k}.$



$$\frac{d^2v}{dt^2} = \frac{d}{dt}\left(g - \frac{k}{m}v\right) = -\frac{k}{m}\frac{dv}{dt}.$$





Logistic Population Growth

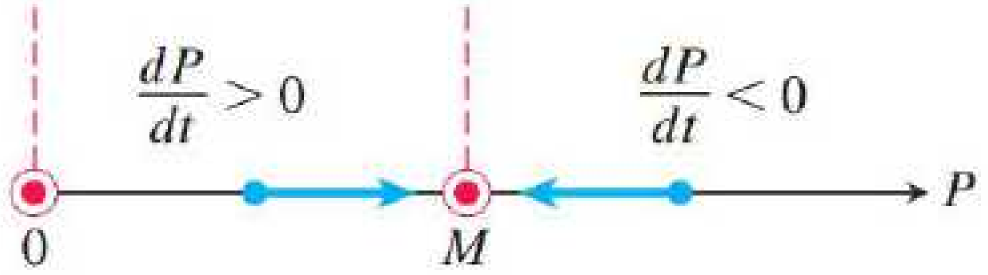
$$\frac{dP}{dt} = kP,$$

$k > 0$ is the birth rate minus the death rate per individual per unit time.

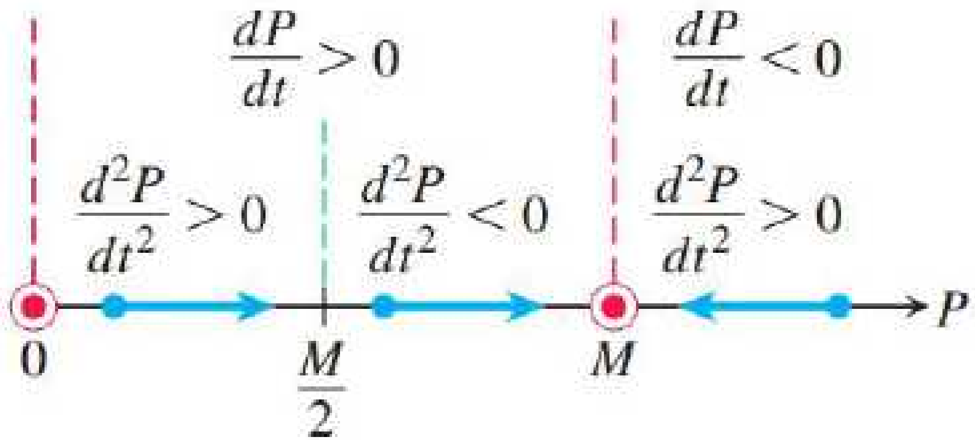
$$k = r(M - P),$$

$$\frac{dP}{dt} = r(M - P)P = rMP - rP^2. \qquad \textbf{logistic growth.}$$

The equilibrium values are $P = M$ and $P = 0$,



$$
\begin{aligned}
\frac{d^2P}{dt^2} &= \frac{d}{dt}(rMP - rP^2) \\
&= rM\frac{dP}{dt} - 2rP\frac{dP}{dt} \\
&= r(M - 2P)\frac{dP}{dt}.
\end{aligned}
$$



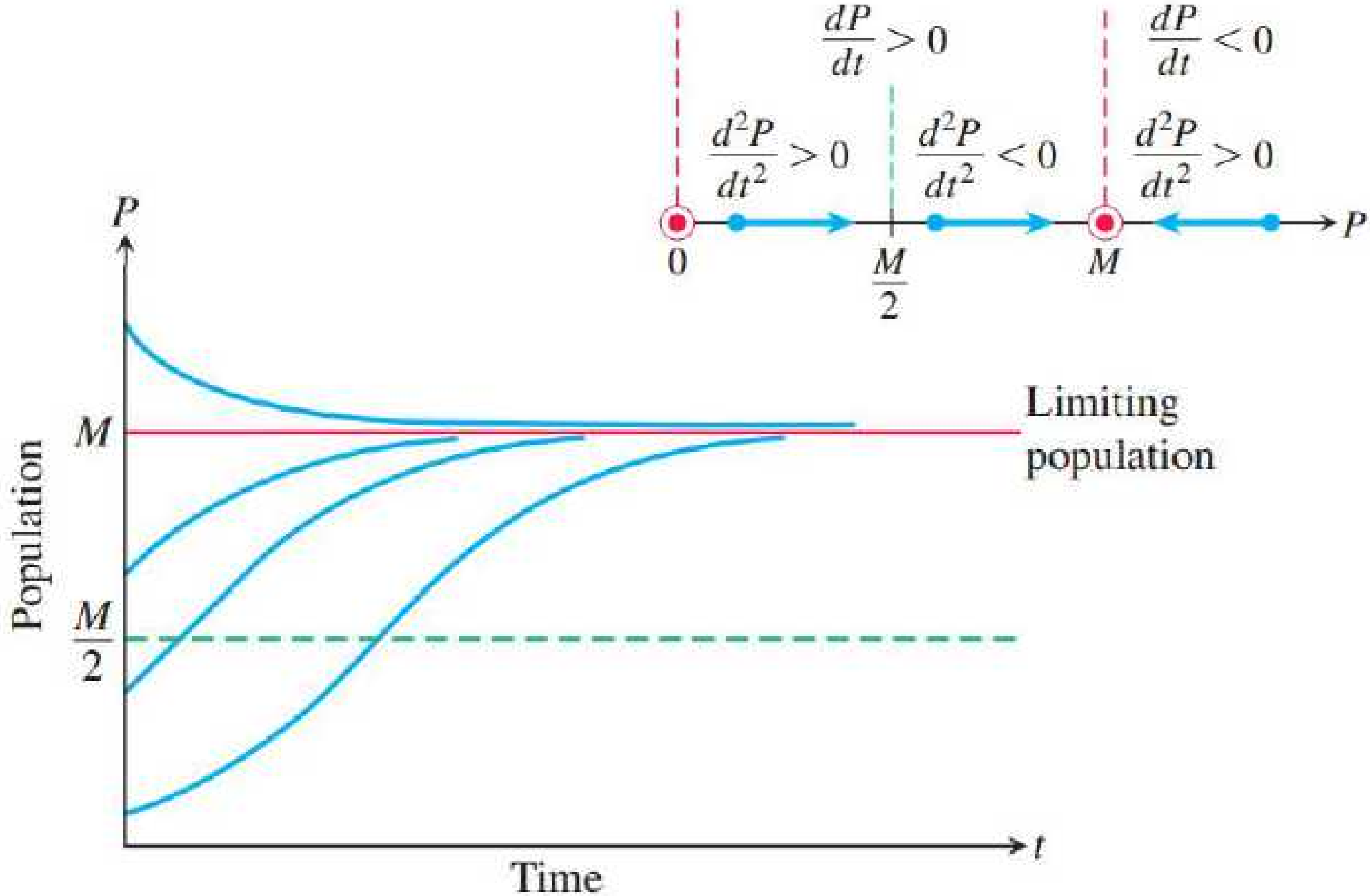


FIGURE 9.25 Population curves for logistic growth.

9.5

Systems of Equations and Phase Planes

Phase Planes

$$\frac{dx}{dt} = F(x, y),$$

$$\frac{dy}{dt} = G(x, y).$$

dx/dt and dy/dt do not depend on the independent variable time t,

Such a system of equations is called **autonomous**

solutions $x(t)$ or $y(t)$ satisfies both equations

plotting the points $(x(t), y(t))$ in the xy-plane

Therefore the solution functions define a solution curve called a **trajectory** of the system.

The xy-plane itself is referred to as the **phase plane**.

A Competitive-Hunter Model

let $x(t)$ represent the number of trout

$y(t)$ the number of bass living in the pond at time t.

$$\frac{dx}{dt} = (a - by)x,$$

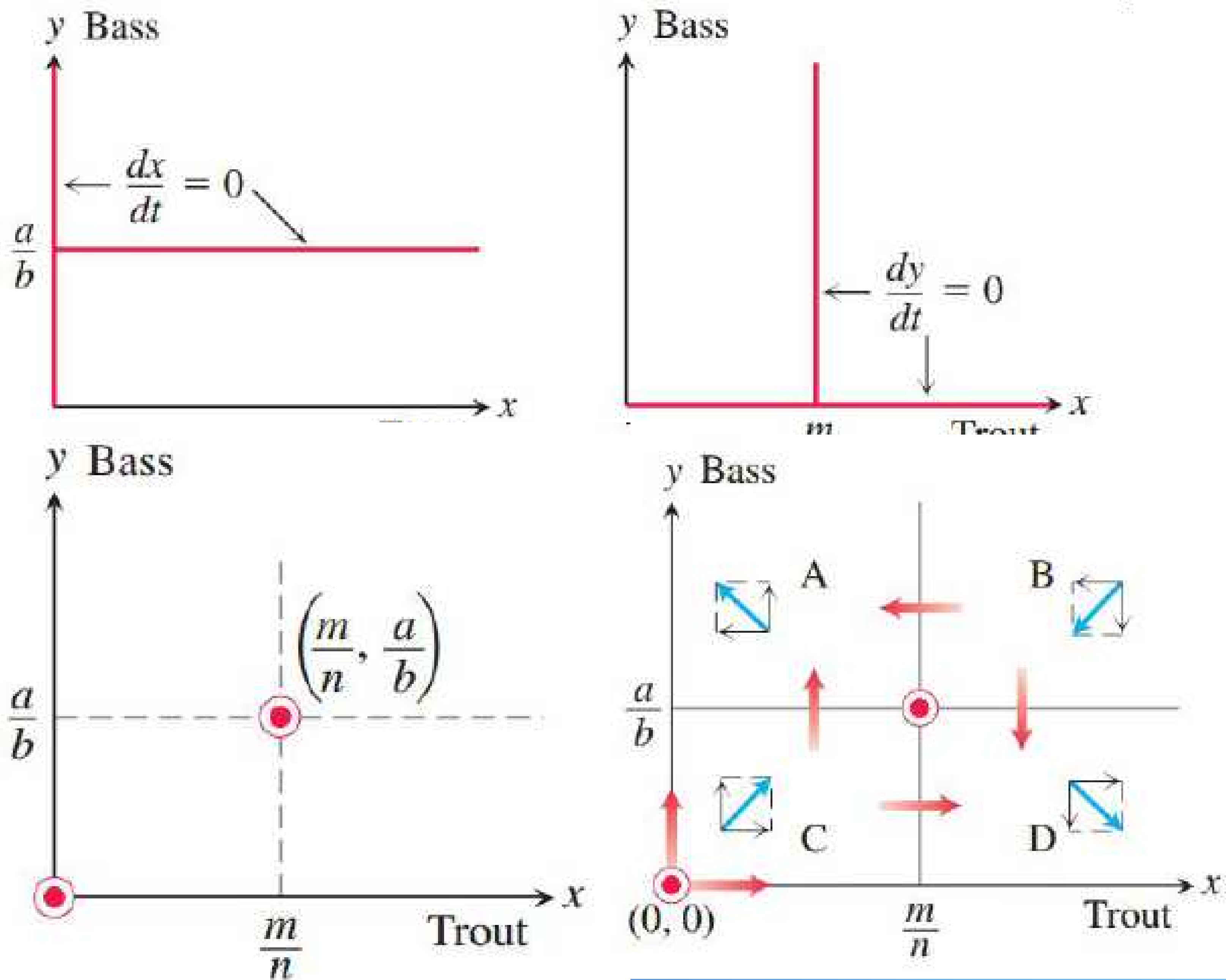
$$\frac{dy}{dt} = (m - nx)y.$$

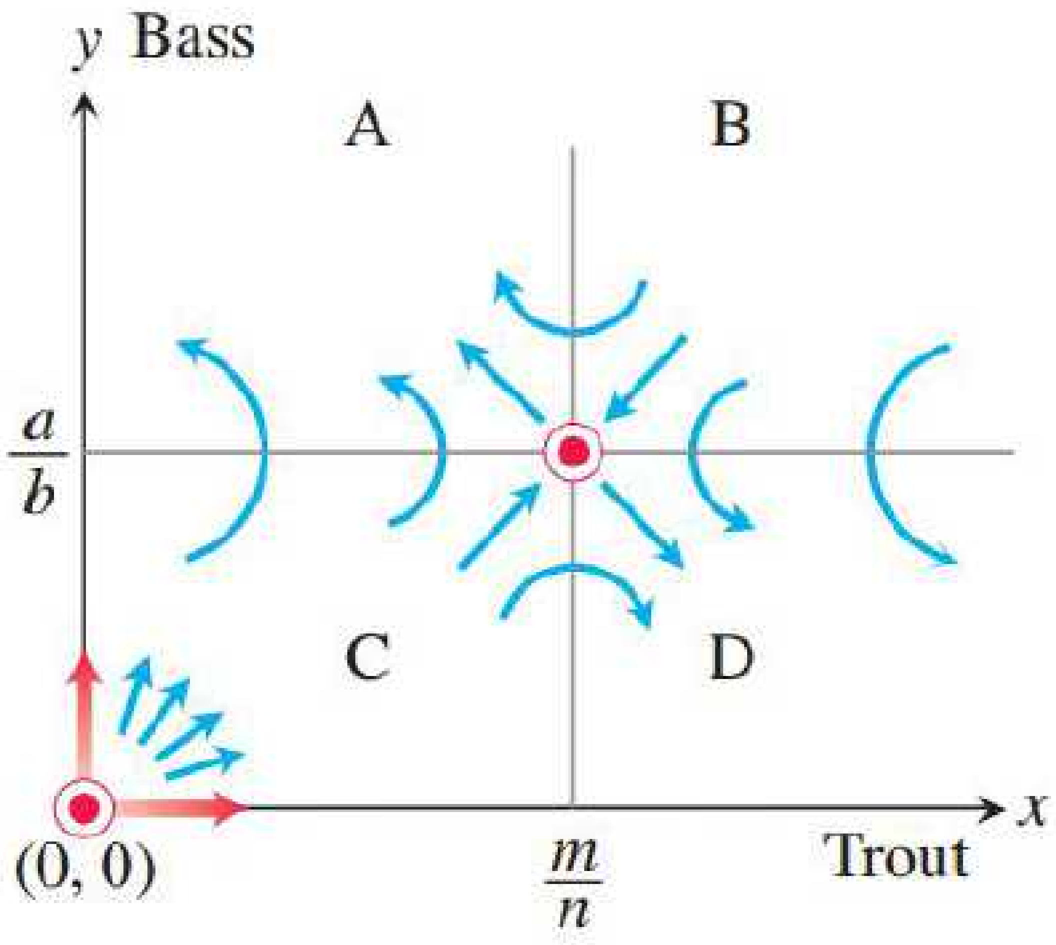
a, b, m, n are positive constants.

$$(a - by)x = 0,$$
$$(m - nx)y = 0.$$

$(x, y) = (0, 0)$ and $(x, y) = (m/n, a/b)$.

these (x, y) values, called **equilibrium** or **rest points**,


y Bass
$\frac{dx}{dt} = 0$
$\frac{a}{b}$
x
y Bass
$\frac{dy}{dt} = 0$
x
y Bass
$\left(\frac{m}{n}, \frac{a}{b}\right)$
$\frac{a}{b}$
$\frac{m}{n}$
Trout
x
y Bass
A
B
C
D
$\frac{a}{b}$
(0, 0)
$\frac{m}{n}$
Trout
x


y Bass
A
B
C
D
a/b
(0, 0)
m/n
Trout
x

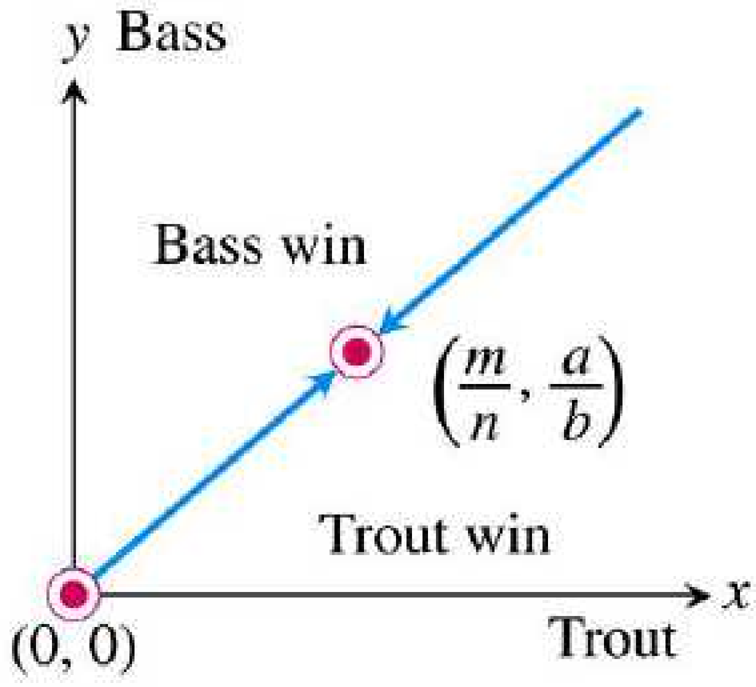


FIGURE 9.31 Qualitative results of analyzing the competitive-hunter model. There are exactly two trajectories approaching the point $(m/n, a/b)$.