CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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The Bijection Principle

- The following loop is a part of program to determine the number of triangles formed by n points in the plane.

```
triangleсount = 0
  for i = 1 to n
    for j = i+1 to n
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Among all iterations of line 5, what is the total number of times this line checks three points to see if they are collinear?

Counting Triangles

- 3 points form a triangle if and only if they are non collinear

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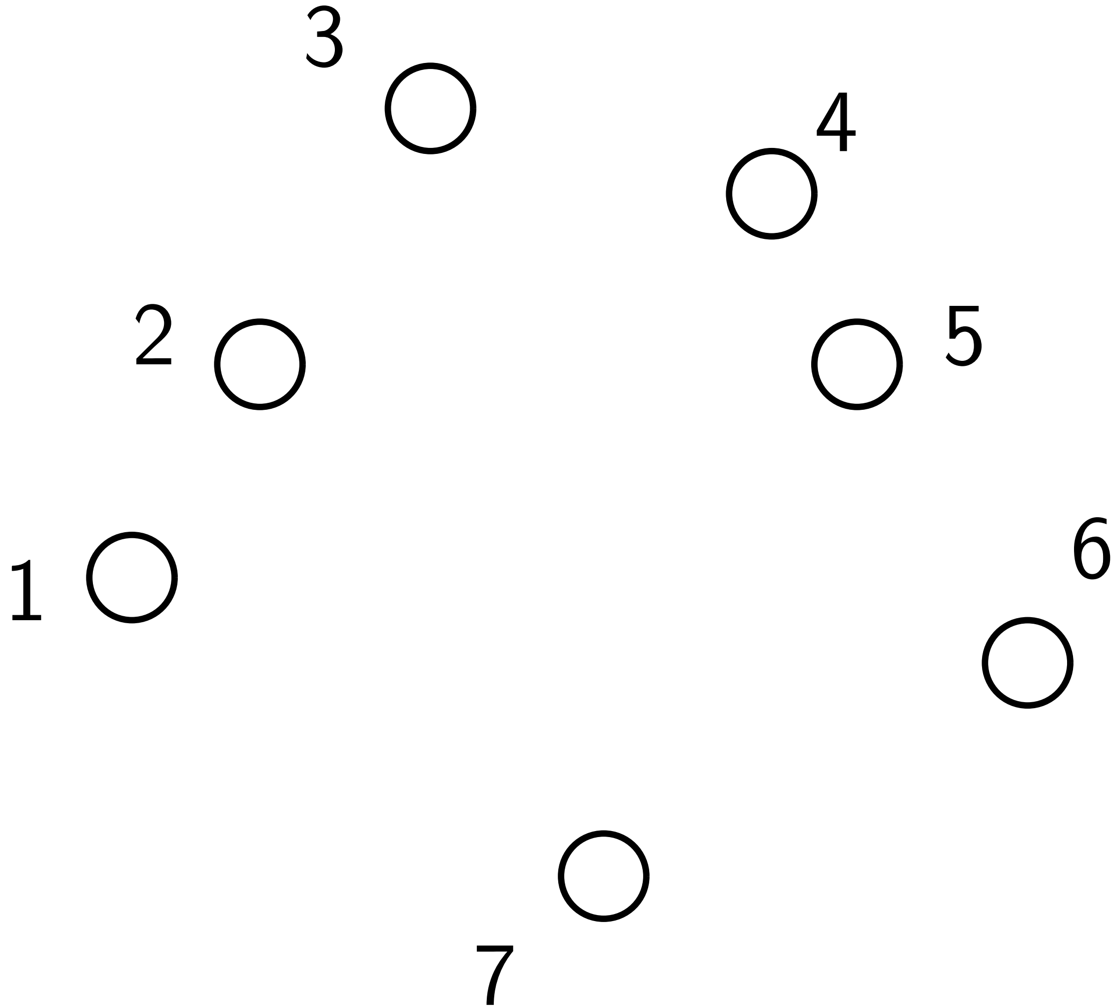

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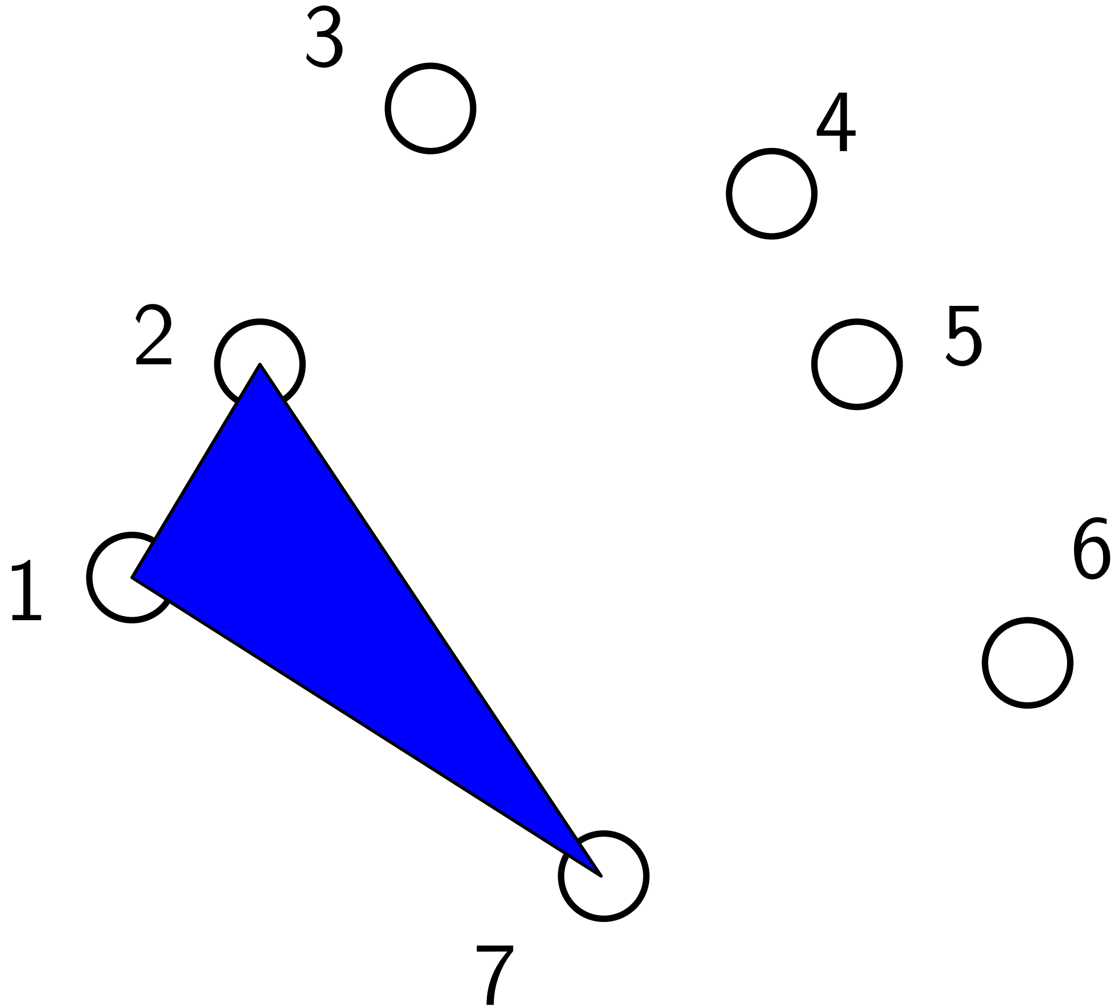

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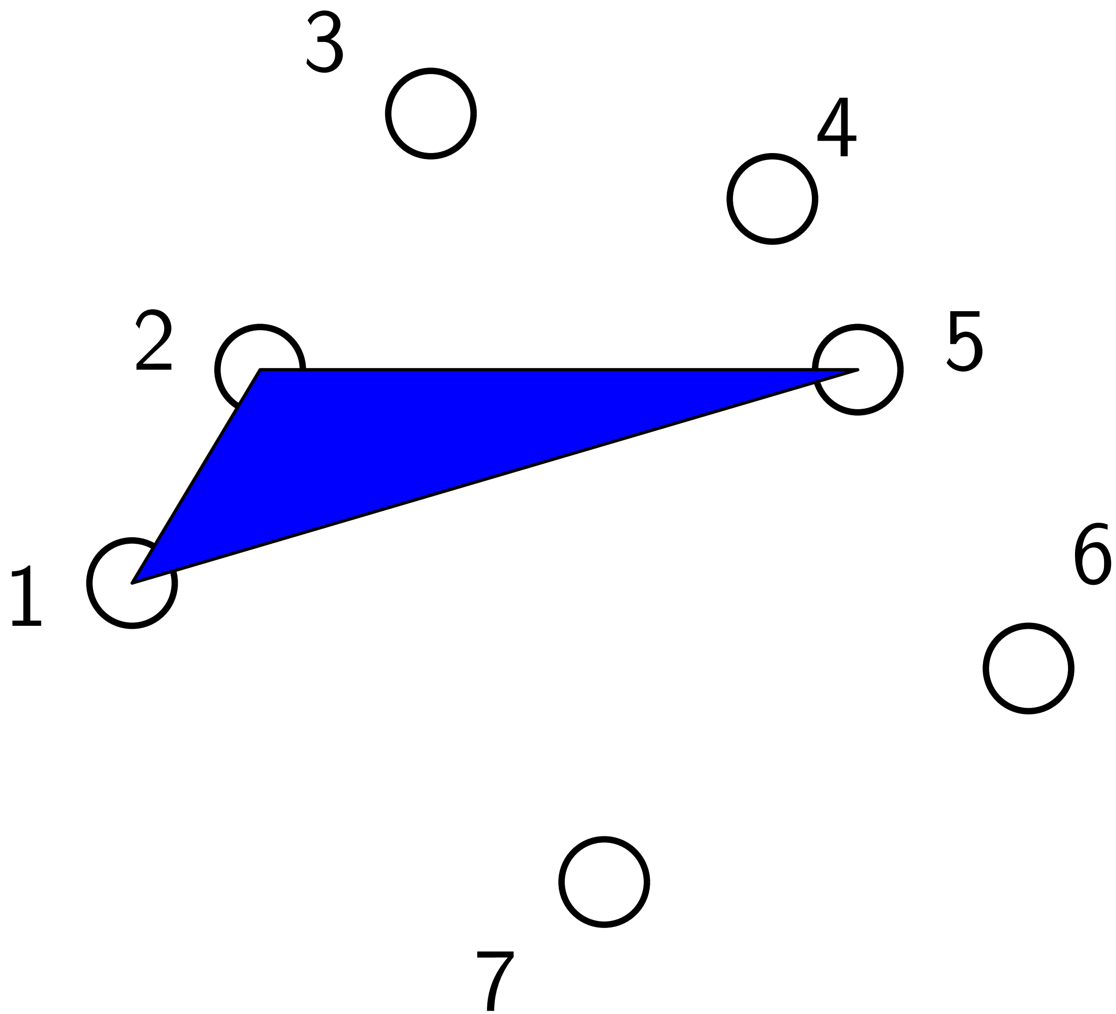

$1 - 2 - 7$: yes

$1 - 2 - 5$: yes

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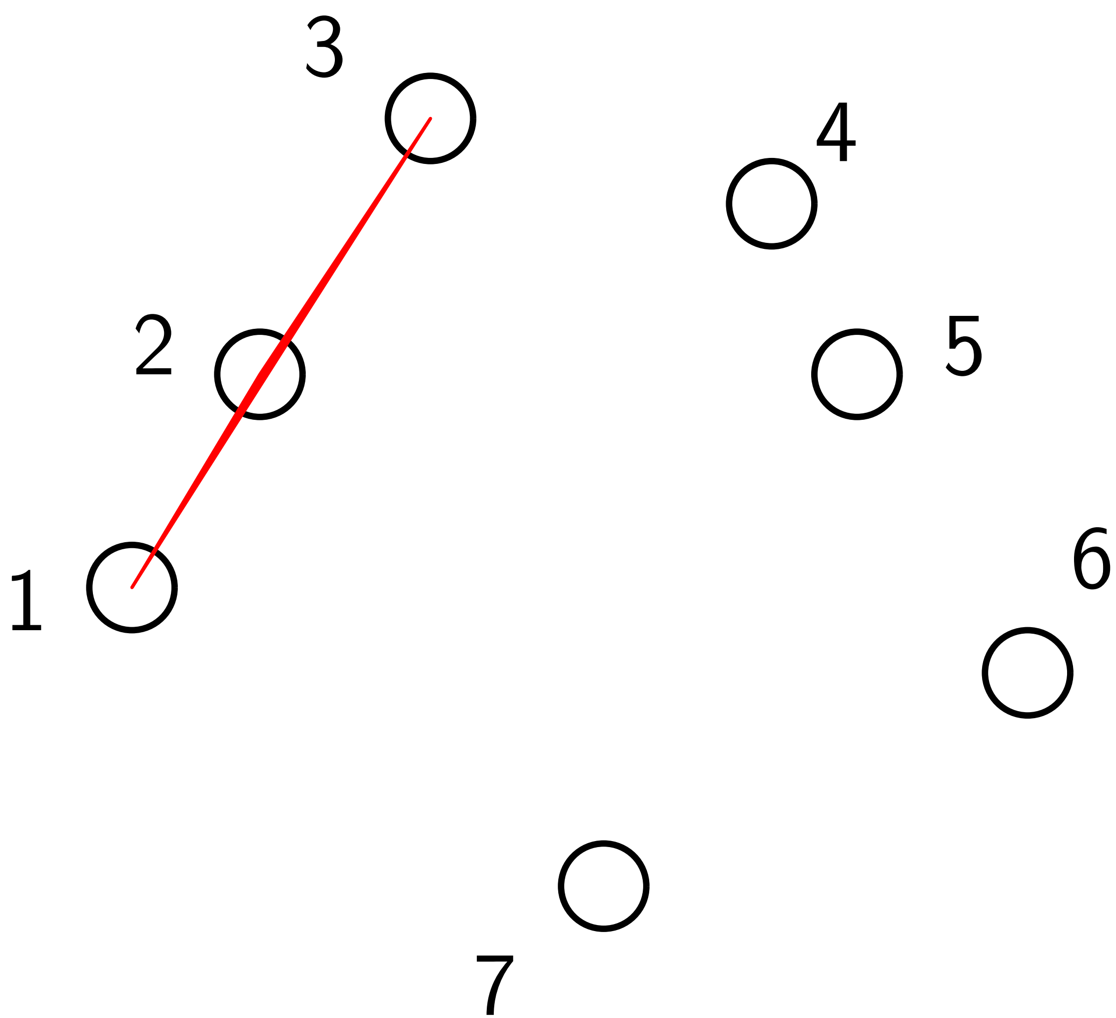

$1 - 2 - 7$: yes

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$1 - 2 - 3$: no

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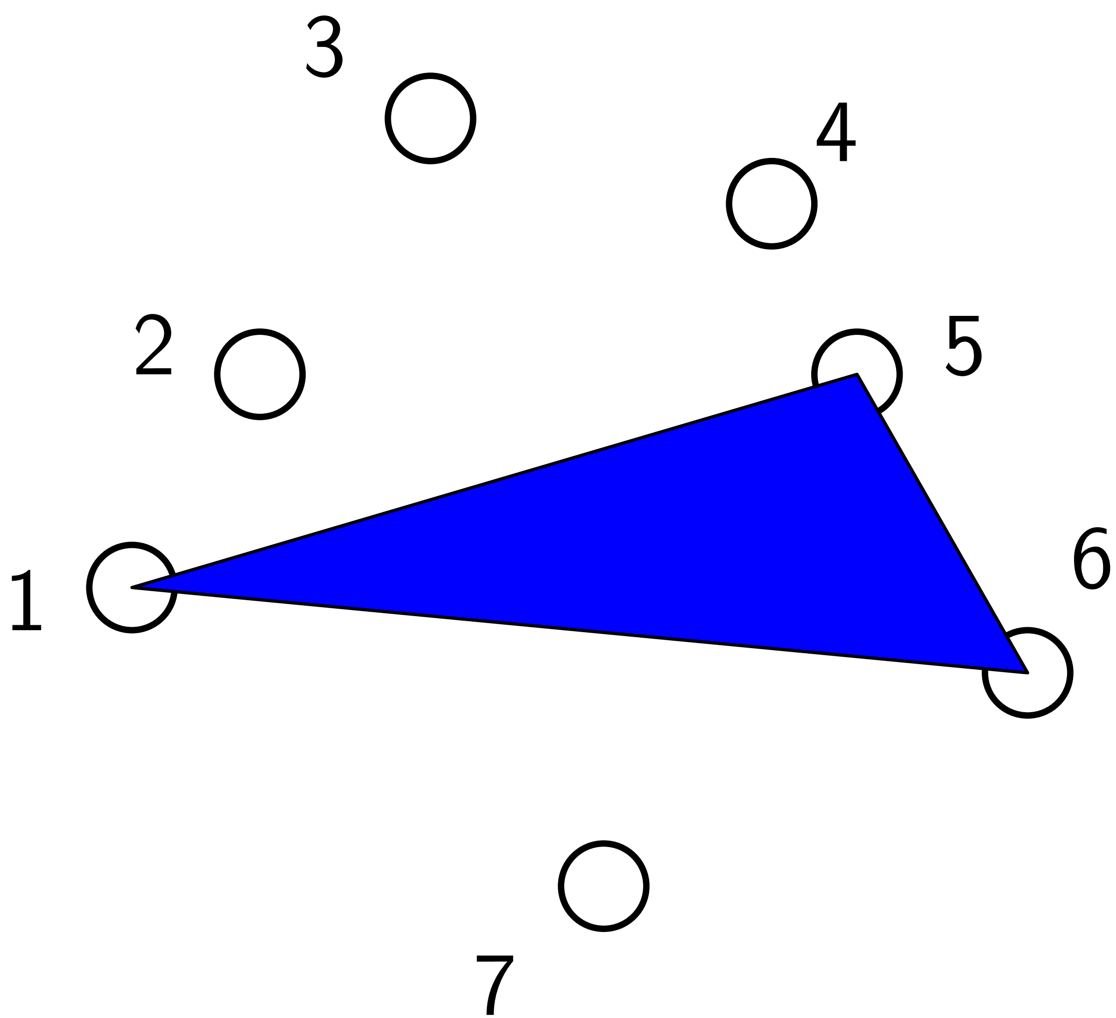

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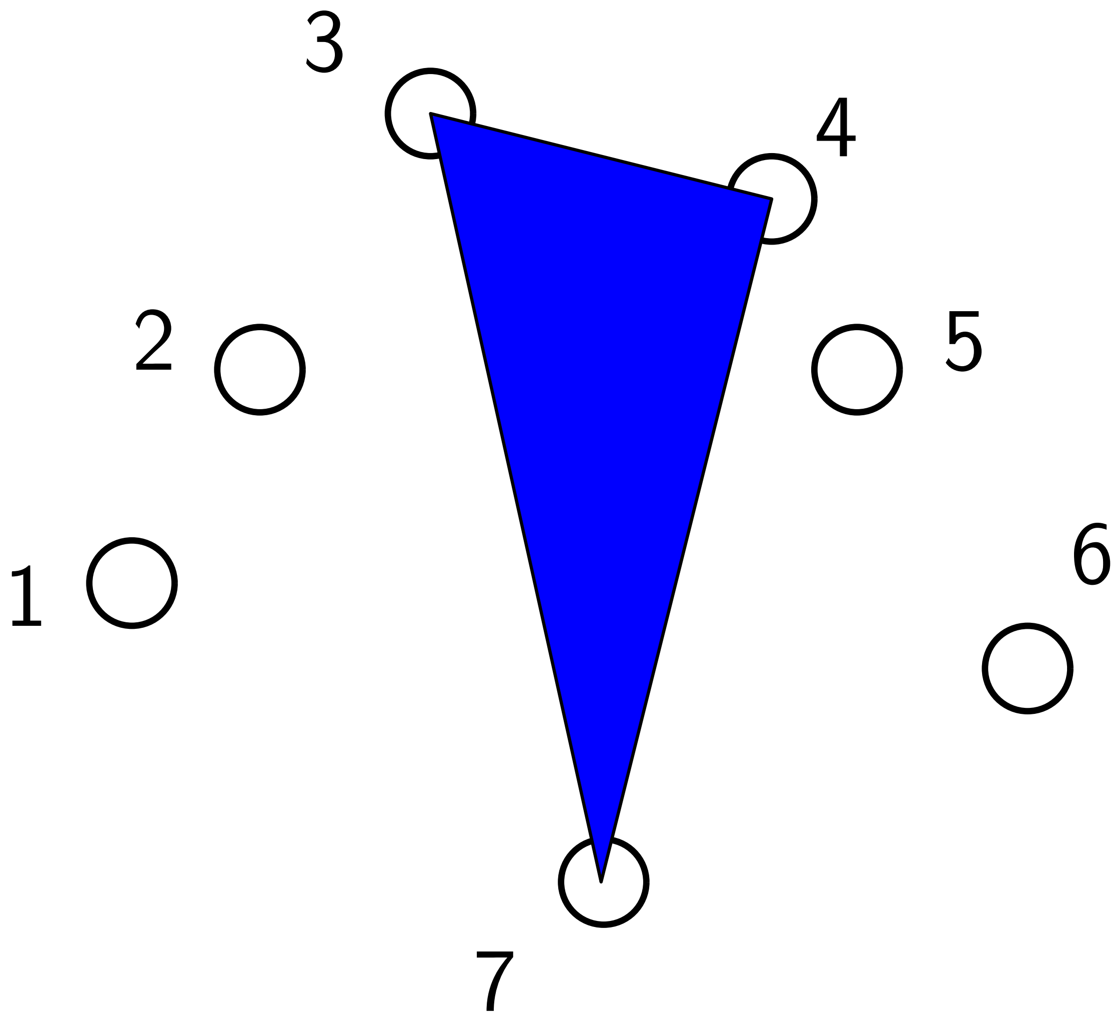

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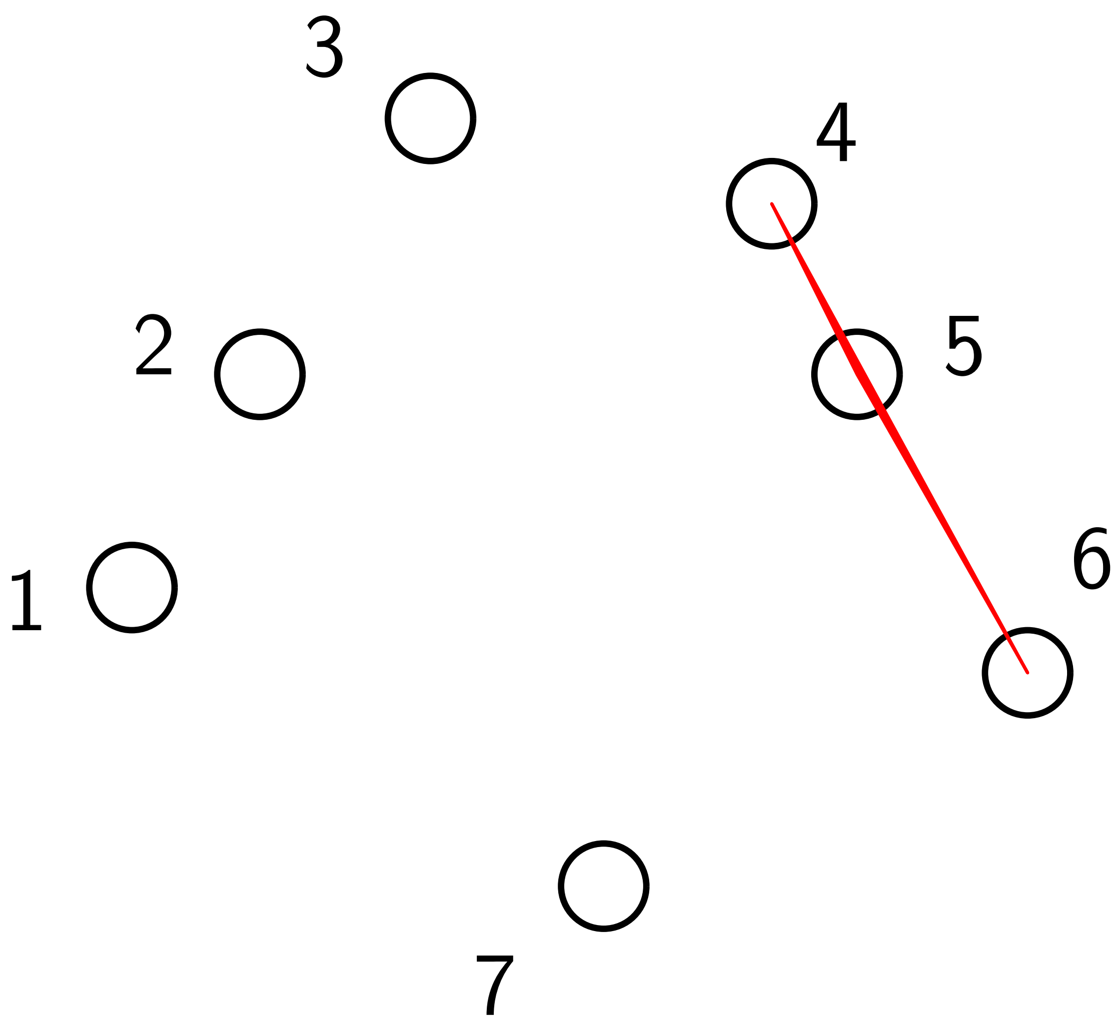

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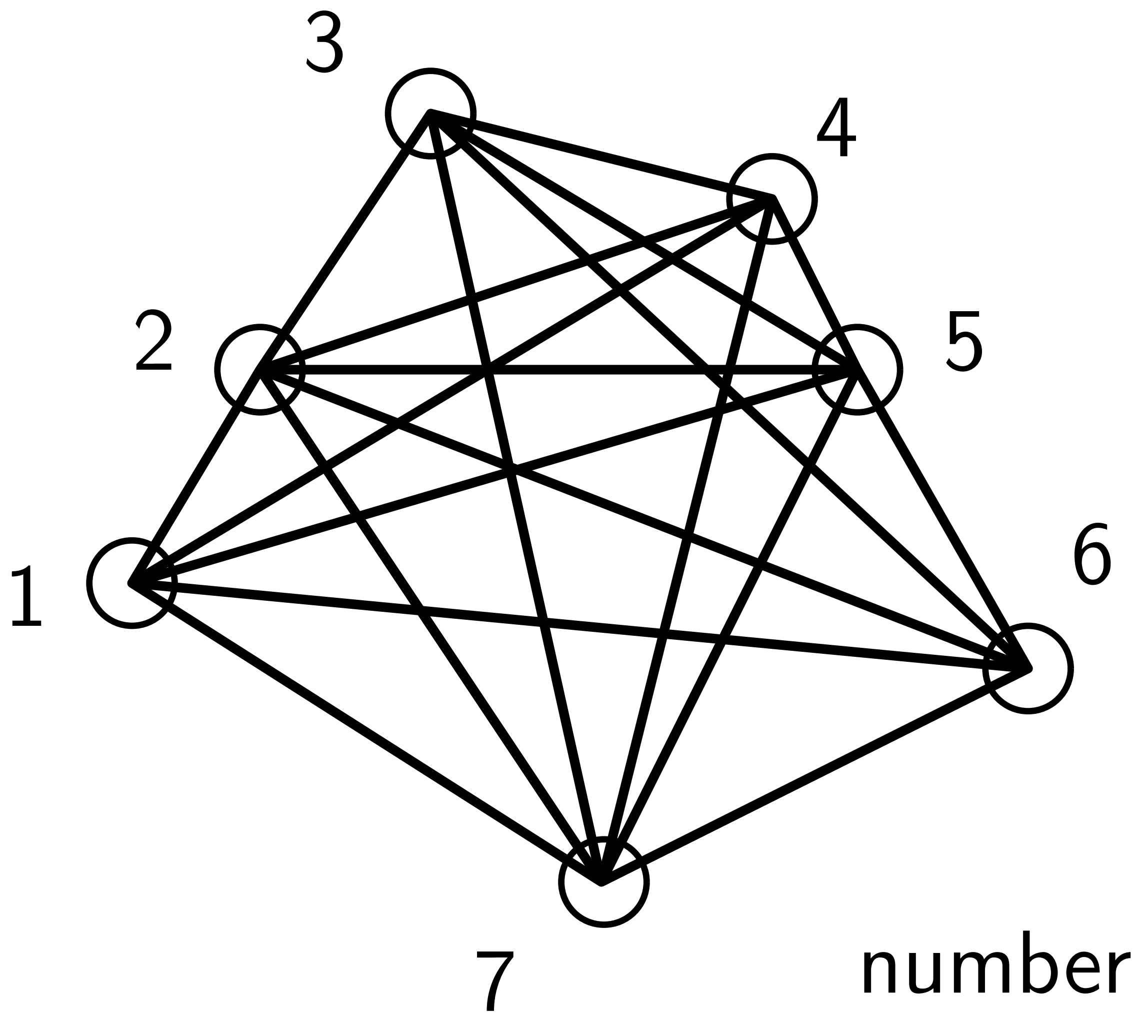

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number of triangles: 33

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A loop embedded in a loop

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Thus each triple i, j, k with $i < j < k$ is examined exactly once.

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Thus each triple i, j, k with $i < j < k$ is examined exactly once.

For example, if $n = 4$, then triples (i, j, k) used by algorithm are (1,2,3), (1,2,4), (1,3,4), and (2,3,4).

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f is a bijection because

f is one-to-one

if $(i, j, k) \neq (i', j', k') \Rightarrow f(\,(i, j, k)\,) \neq f(\,(i', j', k'))$

f is onto

if γ is a 3-element subset then it can be written as $\gamma = \{i, j, k\}$ where $i < j < k$ so $f(\,(i, j, k)\,) = \gamma$.

Counting Pairs

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  We actually already saw that $|X| = |Y| = \binom{n}{2}$

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  Currently, we started with the problem of counting the $\#$ of increasing triples and changed it to the problem of counting the $\#$ of 3-element sets from $\{1, 2, \dots, n\}$

Inclusion-Exclusion Principle

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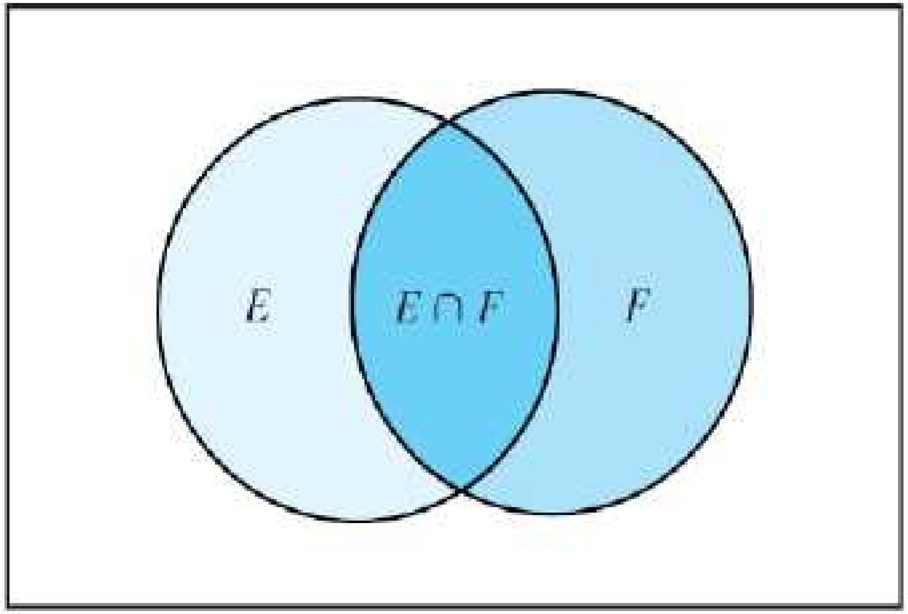
Inclusion-Exclusion Principle

- Two sets

  $|E \cup F| = |E| + |F| - |E \cap F|$



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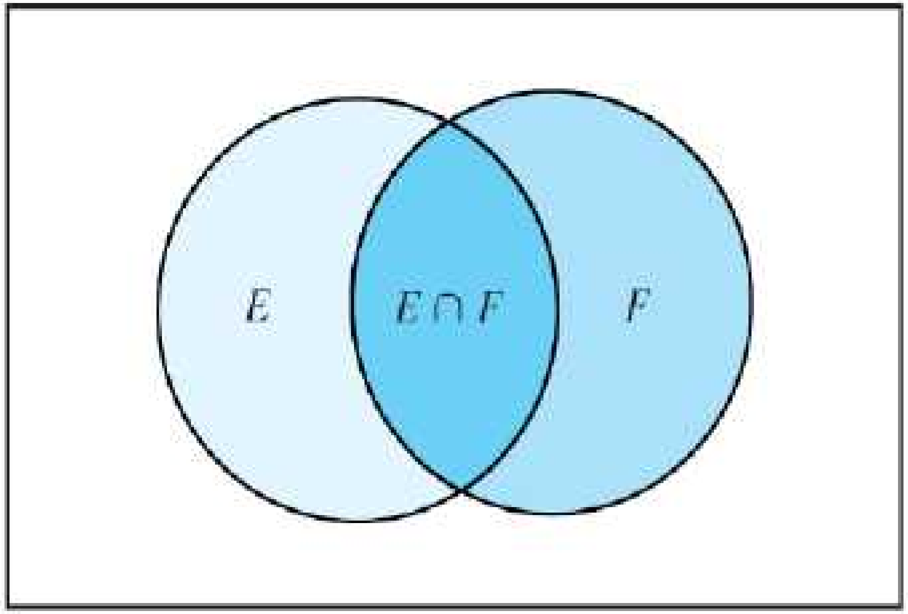
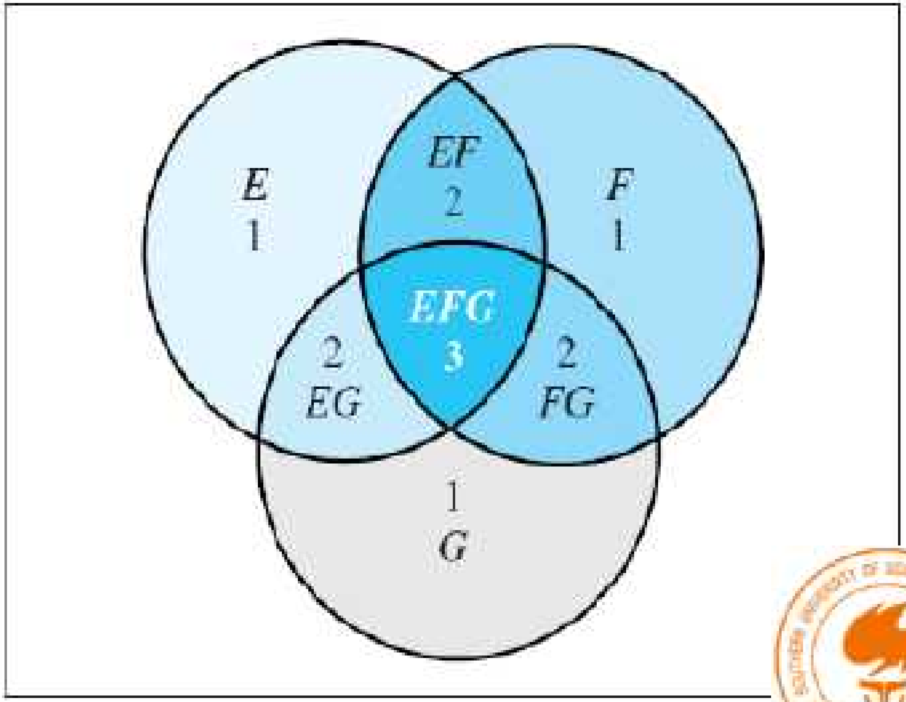


Three sets



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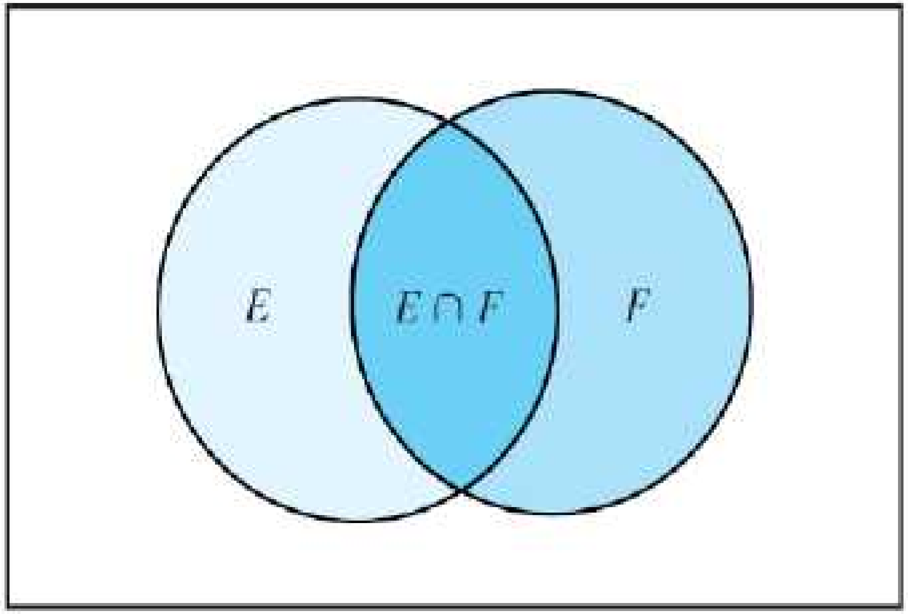
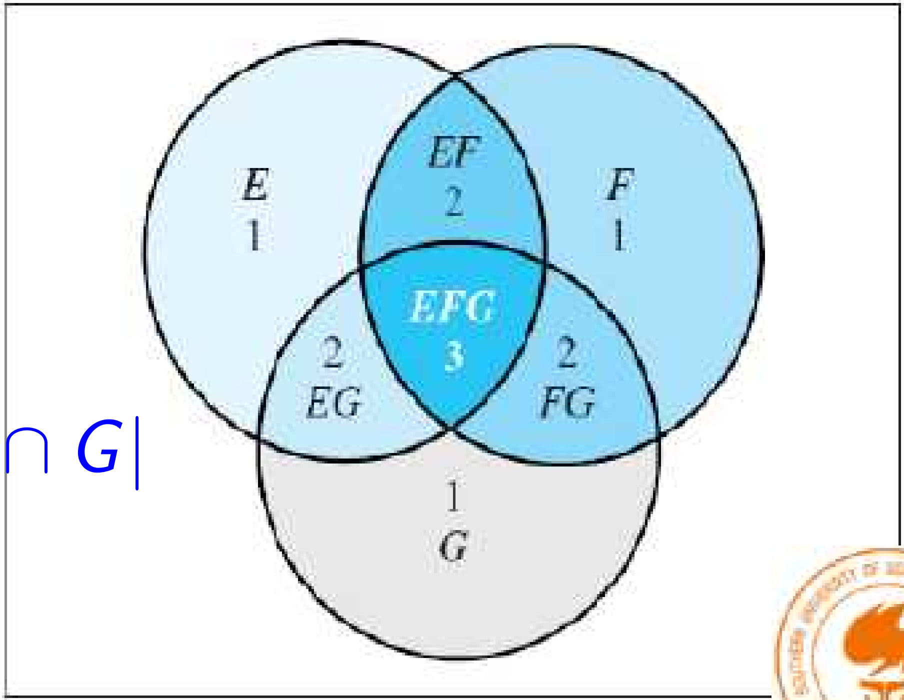


Three sets

$$\begin{aligned} &|E \cup F \cup G| \\ &= |E| + |F| + |G| \\ &\quad -|E \cap F| - |E \cap G| - |F \cap G| \\ &\quad +|E \cap F \cap G| \end{aligned}$$



Inclusion-Exclusion Principle

- $$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

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Base case $(n = 2)$

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Inductive Hypothesis

$$\left|\cup_{i=1}^{n-1} E_i\right| = \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n-1} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

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  Set $E = E_1 \cup \cdots \cup E_{n-1}$, and $F = E_n$.

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$$\left|\cup_{i=1}^{n} E_i\right| = \left|\cup_{i=1}^{n-1} E_i\right| + |E_n| - \left|\left(\cup_{i=1}^{n-1} E_i\right) \cap E_n\right|$$

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For the third term, by distributive law,

$$\left|(\cup_{i=1}^{n-1} E_i) \cap E_n\right| = \left|\cup_{i=1}^{n-1} (E_i \cap E_n)\right| = \left|\cup_{i=1}^{n-1} G_i\right|$$

where $G_i = E_i \cap E_n$.

Inclusion-Exclusion Principle

- So far

$$\left|\cup_{i=1}^{n} E_i\right| = \left|\cup_{i=1}^{n-1} E_i\right| + |E_n| - \left|\cup_{i=1}^{n-1} G_i\right|$$

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Note that (why?)

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Some discussion:
first summation sums $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that do not contain n
$|E_n|$ and second summation together sum $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that do contain n

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(a) How many onto functions are there from A to B?

(b) How many functions are there from A to B that map nothing to at least one element of B?

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  Set E_i – set of functions that map nothing to element i of B

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  (b) How many functions are there from A to B that map nothing to at least one element of B?

$$\#(a) + \#(b) = n^m$$

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Inclusion-Exclusion Principle

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$= \sum_{k=1}^{n}(-1)^{k+1} \binom{n}{k}(n-k)^m$

k-Element Permutations of a Set

- In how many ways can we choose **an ordered triple** of distinct elements from $\{1, 2, \ldots, n\}$?

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  Note that the case of $k = n$ is special;
  An n-element permutation of a set N of size $|N| = n$ is what we earlier simply called a permutation.

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Ex: When $n = 4$, there are $4 \times 3 \times 2 = 24$
3 -element permutations of $\{1, 2, 3, 4\}$

$$L = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243$$
$$312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432\}.$$

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Note: This type of "dictionary" ordering of tuples (assuming that we treat numbers the same as letters) is called a *lexicographic ordering* and is used quite often.

k-Element Permutations of a Set

- **Theorem** If N is a positive integer and k is an integer with $1 \leq k \leq n$, then there are
  $P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$
  k-element permutations with n distinct elements.

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  (# 3-element perms) = 6 $\times$ (# 3-element subsets)

  $$P(n, 3) = 3! \cdot C(n, 3)$$

Binomial Coefficient

- **Theorem** For integers n and k with $0 \le k \le n$, the number of k-element subsets of an n-element set is
$$\binom{n}{k} = C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}.$$

  This is the number of k-combinations of a set with n elements.

Some Properties of Binomial Coefficients

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of k-element subsets of an n-element set.

  $\binom{n}{0} = 1$ only one set of size 0.

  $\binom{n}{n} = 1$ only one set of size n.

  $\binom{n}{k} = \binom{n}{n-k}$ Obvious from equation. Can you think of a simple bijection that explains this?

Some Properties of Binomial Coefficients (cont.)

- $$\boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

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  Let P = set of all subsets of {1,2,. . . ,n}

  S_i = set of all i subsets of {1,2,. . . ,n}

  $$\Rightarrow |P| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i}$$

Some Properties of Binomial Coefficients (cont.)

- Let $L = L_1 L_2 \ldots L_n$ be a list of size n from $\{0, 1\}$

  If $\mathcal{L} = \text{set of all such lists} \Rightarrow \ |\mathcal{L}| = 2^n$

  There is a *bijection* between $\mathcal{L}$ and P so
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  Define the following function $f : \mathcal{L} \to P$

  If $L \in \mathcal{L}$ then $f(L)$ is the set $S \subseteq \{1, 2, \ldots, n\}$ defined by

$$i \in S \; \Leftrightarrow \; L_i = 1$$

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Ex: $n = 5$

$$f(10101) = \{1,3,5\}, \ f(11101) = \{1,2,3,5\}, \ f(00000) = \emptyset$$

Binomial Coefficients

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

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Sum of items on n-th row is 2^n

Pascal's Triangle

Take the table

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
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6	1	6	15	20	15	6	1

and shift each row slightly so that middle element is in middle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
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	1		5		10		10		5		1	
1		6		15		20		15		6		1

What is the next row in the table?

Pascal's Triangle

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Pascal's Triangle

$$
\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&1&&1&&&&&& \\
&&&&&1&&2&&1&&&&& \\
&&&&1&&3&&3&&1&&&& \\
&&&1&&4&&6&&4&&1&&& \\
&&1&&5&&10&&10&&5&&1&& \\
&1&&6&&15&&20&&15&&6&&1& \\
1&&7&&21&&35&&35&&21&&7&&1
\end{array}
$$

Pascal identity

Each (non-1) entry in Pascal's Triangle is the sum of the two entries directly above it (to left and to right).

Pascal's Triangle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
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  We will use a combinatorial proof.

A Combinatorial Proof

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$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

Therefore, each term (left and right) represents the number of subsets of a particular size chosen from an appropriately sized set.

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  Number of k-subsets of an $(n-1)$-element set.

  Try to use sum principle to explain relationship among these three terms.

  Example: $n = 5$, $k = 2$

  $$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

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Set S_1 of 2-subsets of S

$$\begin{aligned} S_1 = \{&\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \\ &\{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}. \end{aligned}$$

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Consider $S = \{A, B, C, D, E\}$.

Set S_1 of 2-subsets of S can be partitioned into 2 disjoint parts.

S_2 the 2-subsets that contain E and

S_3, the set of 2-subsets that do not contain E.

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To apply sum rule, partition S_1 into S_2 and S_3.

Let S_2 be set of k-element subsets that contain x_n.

Let S_3 be set of k-element subsets that don't contain x_n.

Blaise Pascal

Born 1623; Died 1662

French Mathematician

A Founder of Probability Theory

Inventor of one of the first mechanical calculating machines

Pascal Programming Language named for him

The Binomial Theorem

$$(x+y) = \binom{1}{0}x + \binom{1}{1}y$$

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$$\begin{aligned}(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\ &= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3\end{aligned}$$

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$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i.$$

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- Number of k-element subsets of an n-element set is called a **binomial coefficient** because of its role in the algebraic expansion of a binomial $(x+y)^n$.

The Binomial Theorem For any integer $n \geq 0$,

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}$$

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

Proof?

Application of the Binomial Theorem

- We may use the Binomial Theorem to prove

$$\boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

Labelling and Trinomial Coefficients

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  Show that if we have k_1 labels of one kind, e.g., red, k_2 labels of a second kind, e.g., blue, and $k_3 = n - k_1 - k_2$ labels of a third kind, then there are $\frac{n!}{k_1!k_2!k_3!}$ ways to apply these labels to n objects

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  What is the coefficient of $x^{k_1}y^{k_2}z^{k_3}$ in $(x+y+z)^n$?

Labelling and Trinomial Coefficients

- There are $\binom{n}{k_1}$ ways to choose the red items There are then $\binom{n-k_1}{k_2}$ ways to choose the blue items from the remaining $n - k_1$. The remaining k_3 items get labelled a third color.

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  Using the *product rule* the total number of labellings is

$$\begin{aligned}\binom{n}{k_1}\binom{n-k_1}{k_2} &= \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!} \\ &= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!k_3!}\end{aligned}$$

Labelling and Trinomial Coefficients

- When $k_1 + k_2 + k_3 = n$, we call

$$\frac{n!}{k_1!k_2!k_3!}$$

  a *trinomial coefficient* and denote it as

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  This will be very similar to the analysis of hashing n keys into a table of size 365.

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  $\#A_n + \#B_n = 365^n$

The Birthday Paradox

n	A_n	B_n	n	A_n	B_n
1	0.00000000	1.00000000	16	0.28360400	0.71639599
2	0.00273972	0.99726027	17	0.31500766	0.68499233
3	0.00820416	0.99179583	18	0.34691141	0.65308858
4	0.01635591	0.98364408	19	0.37911852	0.62088147
5	0.02713557	0.97286442	20	0.41143838	0.58856161
6	0.04046248	0.95953751	21	0.44368833	0.55631166
7	0.05623570	0.94376429	22	0.47569530	0.52430469
8	0.07433529	0.92566470	23	0.50729723	0.49270276
9	0.09462383	0.90537616	24	0.53834425	0.46165574
10	0.11694817	0.88305182	25	0.56869970	0.43130029
11	0.14114137	0.85885862	26	0.59824082	0.40175917
12	0.16702478	0.83297521	27	0.62685928	0.37314071
13	0.19441027	0.80558972	28	0.65446147	0.34553852
14	0.22310251	0.77689748	29	0.68096853	0.31903146
15	0.25290131	0.74709868	30	0.70631624	0.29368375

"Birthday" attacks

- Event A: at least two people in the room have the same birthday

  Event B: no two people in the room have the same birthday

  $\Pr[A] = 1 - \Pr[B]$

$$
\begin{aligned}
\Pr[B] &= \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdot \cdots \cdot \left(1 - \frac{n-1}{365}\right) \\
&= \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right).
\end{aligned}
$$

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  Recall that $p(n; H) := 1 - \prod_{i=1}^{n-1}(1 - \frac{i}{H})$

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  Let $n(p; H)$ be the smallest number of values we have to choose, such that the probability for finding a collision is at least p. By inverting the expression above, we have
  $$n(p; H) \approx \sqrt{2H \ln \frac{1}{1-p}}.$$

- The Euclidean algorithm in pseudocode

ALGORITHM 1 The Euclidean Algorithm.

procedure $gcd(a, b$: positive integers)
$x := a$
$y := b$
while $y \neq 0$
  $r := x$ **mod** y
  $x := y$
  $y := r$
return $x\{\gcd(a, b)$ is $x\}$

The number of divisions required to find $\gcd(a, b)$ is $O(\log b)$, where $a \geq b$. (this will be proved later.)

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Why ?

Euclidean Algorithm

- Key steps in the Euclidean algorithm

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\begin{aligned}
r_0 &= r_1 q_1 + r_2 && 0 \le r_2 < r_1, \\
r_1 &= r_2 q_2 + r_3 && 0 \le r_3 < r_2, \\
&\;\;\vdots \\
r_{n\text{-}2} &= r_{n\text{-}1} q_{n\text{-}1} + r_{\mathrm{n}} && 0 \le r_n < r_{n\text{-}1}, \\
r_{n\text{-}1} &= r_n q_n \,.
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Observation:

$r_{i+2} = r_i \bmod r_{i+1}$

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 & \cdot & \\
 & \cdot & \\
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Case (i): $r_{i+1} \le \frac{1}{2} r_i$: $r_{i+2} < r_{i+1} \le \frac{1}{2} r_i$.

Case (ii): $r_{i+1} > \frac{1}{2} r_i$: $r_{i+2} = r_i \bmod r_{i+1} = r_i - r_{i+1} < \frac{1}{2} r_i$.

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See [Theorem 1 p. 347].

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Next Lecture

- solving linear recurrence ...