Probability and Statistics
Tutorial 9

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Outline

Review

1. Expectation
   - Discrete Case:
     - (DEF) $EX = \sum_{i=1}^{\infty} n_i P(X = n_i)$.
     - (Property) $Eh(X) = \sum_{i=1}^{\infty} h(n_i) P(X = n_i)$.
   - Continuous Case:
     - (DEF) $EX = \int_{-\infty}^{\infty} x f_X(x) dx$.
     - (Property) $Eh(X) = \int_{-\infty}^{\infty} h(x) f_X(x) dx$.
     - (2-dim case)$E(h(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) f_{X,Y}(x, y) dxdy$
   - Properties
     - (Linearity) $E(aX + bY) = aE(X) + bE(Y)$.
     - $E(aX + b) = aE(X) + b$.
     - If $X \geq Y$, then $E(X) \geq E(Y)$.
     - If $b \geq X \geq a$, then $b \geq E(X) \geq a$.
     - If X and Y are independent, then $E(XY) = E(X)E(Y)$. (Warning: the converse is not true in general)
     - If $A \subset \Omega$, then $E(1_A) = P(A)$.
     - If $A \subset \mathbb{R}$, then $E(1_A(X)) = P(X \in A)$.

2. Variance

- (DEF) $Var(X) = E(X - E(X))^2$
- $Var(X) = E(X^2) - (E(X))^2$
- $E(X^2) \geq (E(X))^2$ that is, $Var(X) \geq 0$.
- $Var(aX + b) = a^2 Var(X)$.
- $Var(X) = 0$ if and only if $X = c$ a.e.
- $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$.
- If X, Y are independent, then $Var(X + Y) = Var(X) + Var(Y)$.
- $E(X) = min_c\{E[(X - c)^2]\}$.

Review

3. Several Inequalities
   - $|E(X)| \leq E|X|$.
   - $(E(X))^2 \leq E(X)^2$.
   - For any nonnegative random variable X, we have $P(X \geq t) \leq \frac{E(X)}{t}$.
   - (The above implies) for any r.v. X, we have $P(|X| \geq t) \leq \frac{E(|X|)}{t}$.
   - For any random variable X, we have $P(|X - E(X)| \geq t) \leq \frac{Var(X)}{t^2}$.
   - $Var(X) = E(X - EX)^2 \leq E(X - c)^2$ for any c.

Homework

6. 令 X 是连续型随机变量，具有概率密度函数为 $f(x)=2x, 0\leqslant x\leqslant 1$。

   a. 计算 $E(X)$。

   b. 令 $Y=X^2$，计算 Y 的概率质量函数，并由其计算 $E(Y)$。

   c. 利用 4.1.1 节的定理 4.1.1.1 计算 $E(X^2)$，并与 b 中的答案进行比较。

   d. 根据 4.2 节方差的定义计算 $\mathrm{Var}(x)$，同时利用 4.2 节的定理 4.2.2 计算 $\mathrm{Var}(x)$。

Homework

Solution

6. Solution.

(1) $\mathbb{E}(X) = \int_0^1 x \cdot 2x \, dx = \frac{2}{3}$.

(2) $f_Y(y) = \begin{cases} |(\sqrt{y})'| \, 2\sqrt{y} = 1, & y \in (0,1) \\ 0, & \text{other} \end{cases}$

$\mathbb{E}(Y) = \frac{1}{2}$.

(3) $\mathbb{E}(X^2) = \int_0^1 x^2 \cdot 2x \, dx = \frac{1}{2}$.

(4) $Var(X) = \frac{1}{2} - (\frac{2}{3})^2 = \frac{1}{18}$.

Homework

16. 假设有两种彩票，每种彩票有 n 个可能的数字和相同的奖金，根据期望收入判断下列购买方式哪种更好：从一种彩票中购买两张，或从两种彩票中各买一张？

Homework

Solution

15. Solution. Let $m =$ 奖金的数额.

$X_i =$ earning of i-th plan, $i=1,2$.

$$\mathbb{E}(X_1) = m \cdot \frac{n-1}{C_n^2} = \frac{2m}{n}$$

$$\mathbb{E}(X_2) = m \cdot \frac{1}{n} \cdot \frac{n-1}{n} + m \cdot \frac{n-1}{n} \cdot \frac{1}{n} + 2m \frac{1}{n^2}$$

$$= \frac{2m(n-1) + 2m}{n^2} = \frac{2m}{n}$$

Then, $\mathbb{E}(X_1) = \mathbb{E}(X_2)$.

Homework

20. 计算 $E[1/(X+1)]$，其中 X 是泊松随机变量.

Solution

20. Solution.

$$\mathbb{E}\left(\frac{1}{X+1}\right) = \sum_{k=0}^{\infty} \frac{1}{k+1} e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \frac{1}{\lambda} \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{k+1}}{(k+1)!}$$

$$= \frac{1}{\lambda} (1 - e^{-\lambda}).$$

Homework

21. 随机正方形的边长是 $[0, 1]$ 上的均匀随机变量. 计算正方形的期望面积.

Homework

Solution

21. Solution. $X \sim U(0,1)$

$$\mathbb{E}(S) = \mathbb{E}(X^2) = \int_0^1 x^2 dx = \frac{1}{3}.$$

Homework

31. 令 X 均匀分布在区间 $[1,2]$ 上. 计算 $E(1/X)$. $E(1/X) = 1/E(X)$ 吗?

Homework

Solution

31. Solution. $\mathbb{E}X = \frac{3}{2}$.

$$\mathbb{E}\left(\frac{1}{X}\right) = \int_1^2 \frac{1}{x}\,dx = \ln 2 \neq \frac{1}{\mathbb{E}X}.$$

Homework

1. 设随机变量X的概率密度为

$$f(x)=\begin{cases} e^{-x}, & x>0, \\ 0, & x\leqslant 0. \end{cases}$$

求 (1) $Y=2X$; (2) $Y=e^{-2X}$ 的数学期望.

Homework

Solution

Solution.

(1) $\mathbb{E}Y = \mathbb{E}(2X) = 2\mathbb{E}X = 2$

(2) $\mathbb{E}Y = \mathbb{E}(e^{-2X}) = \int_0^{\infty} e^{-2x} e^{-x} dx$

$= \frac{1}{3}$.

Homework

2. 设随机变量(X,Y)的概率密度为

$$f(x,y)=\begin{cases}12y^2, & 0\leqslant y\leqslant x\leqslant 1,\\ 0, & \text{其他},\end{cases}$$

求 $E(X)$, $E(Y)$, $E(XY)$, $E(X^2+Y^2)$.

Homework

Solution

Solution.

$$① \ \mathbb{E}X = \int_0^1 \int_0^x x \cdot 12y^2 \, dy \, dx = \frac{4}{5}$$

$$② \ \mathbb{E}Y = \int_0^1 \int_0^x y \cdot 12y^2 \, dy \, dx = \frac{3}{5}$$

$$③ \ \mathbb{E}(XY) = \int_0^1 \int_0^x (xy) \cdot 12y^2 \, dy \, dx = \frac{1}{2}$$

$$④ \ \mathbb{E}(X^2+Y^2) = \int_0^1 \int_0^x (x^2+y^2) \, 12y^2 \, dy \, dx = \frac{16}{15}$$

Homework

49. X 和 Y 是取自数量 μ 的两个独立测量值. $E(X)=E(Y)=\mu$, σ_X 和 σ_Y 不相等. 利用加权平均组合两个测度值, 即

$$Z = \alpha X + (1-\alpha)Y$$

其中 α 是个标量, 且 $0 \leq \alpha \leq 1$.

a. 证明 $E(Z)=\mu$

b. 根据 σ_X 和 σ_Y, 寻找最小化 $\mathrm{Var}(Z)$ 的 α.

c. 在何种情况下, 使用平均数 $(X+Y)/2$ 优于单独使用 X 或 Y?

Homework

Solution

49. Solution.

a. $\mathbb{E}(Z) = \alpha\,\mathbb{E}(X) + (1-\alpha)\,\mathbb{E}(Y) = \mu.$

b. $Var(Z) = \alpha^2 \sigma_X^2 + (1-\alpha)^2 \sigma_Y^2$

$$= (\sigma_X^2 + \sigma_Y^2)\alpha^2 - 2\sigma_Y^2\alpha + \sigma_Y^2$$

$$\alpha_{min} = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$$

c. When $\alpha = \frac{1}{2}$, we have

$$Var(Z) = \frac{1}{4}(\sigma_X^2 + \sigma_Y^2).$$

If $Var(Z) \le \min(\sigma_X^2, \sigma_Y^2)$, then

$$\begin{cases} \sigma_Y^2 \le 3\sigma_X^2 \\ \sigma_X^2 \le 3\sigma_Y^2 \end{cases}.$$

Homework

50. 假设 $X_i\ (i=1,\cdots,n)$，是独立的随机变量，具有 $E(X_i)=\mu, \mathrm{Var}(X_i)=\sigma^2$. 令 $\overline{X}=n^{-1}\sum_{i=1}^{n}X_i$，证明 $E[\overline{X}]=\mu, \mathrm{Var}(\overline{X})=\sigma^2/n$

Solution

50. Solution.

① $\mathbb{E}(\overline{X}) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i) = \mu.$

② $Var(\overline{X}) = \frac{1}{n^2}\sum_{i=1}^{n}Var(X_i) = \frac{\sigma^2}{n}.$

Homework

55. 令 $T=\sum_{k=1}^{n} kX_k$，其中 X_k 为独立的随机变量，具有均值 μ，方差 σ^2。计算 $E(T)$ 和 $\mathrm{Var}(T)$。

Homework

Solution

55. Solution.

① $\mathbb{E}(T) = \sum_{k=1}^{n} k\,\mathbb{E}(X_k) = \mu \sum_{k=1}^{n} k = \frac{1}{2} n(n+1)\mu.$

② $\mathrm{Var}(T) = \sum_{k=1}^{n} k^2\,\mathrm{Var}(X_k) = \sigma^2 \sum_{k=1}^{n} k^2$

$= \frac{1}{6}(2n+1)(n+1)n\sigma^2.$

Homework

1. 设X, Y是互相独立的随机变量，且有$E(X)=3$, $E(Y)=1$, $D(X)=4$, $D(Y)=9$. 令$Z=5X-2Y+15$, 求 $E(Z)$ 和 $D(Z)$.

Homework

Solution

Solution.

① $\mathbb{E}Z = 5\mathbb{E}X - 2\mathbb{E}Y + 15 = 28$

② $Var(Z) = 25\,Var(X) + 4\,Var(Y) = 136$

Homework

2. 设随机变量X_1, X_2, X_3, X_4互相独立，且有$E(X_i) = 2i$，$D(X_i) = 5 - i$，其中$i = 1, 2, 3, 4$。令$Z = 2X_1 - X_2 + 3X_3 - \frac{1}{2}X_4$，求 $E(Z)$ 和 $D(Z)$。

Homework

Solution

Solution.

① $\mathbb{E}(Z) = 2\mathbb{E}X_1 - \mathbb{E}X_2 + 3\mathbb{E}X_3 - \frac{1}{2}\mathbb{E}X_4$

$= 14$

② $D(Z) = 4D(X_1) + D(X_2) + 9D(X_3) + \frac{1}{4}D(X_4)$

$= \frac{149}{4}.$

Supplement Exercises

Exercise 1

8. 证明：如果 X 为离散型随机变量，且取值正整数，那么 $E(X)=\sum_{k=1}^{\infty} P(X \geqslant k)$. 利用此结论计算几何随机变量的期望值.

Supplement Exercises

Solution

Proof. $RHS = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \mathbb{P}(X=j)$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{j} \mathbb{P}(X=j) = \sum_{j=1}^{\infty} j\,\mathbb{P}(X=j)$$

$$= \mathbb{E}X. \quad \square$$

Exercise 2

13. 如果 X 是非负的连续型随机变量，证明：

$$E(X) = \int_0^{\infty} [1 - F(x)] \mathrm{d}x$$

应用这个结论计算指数分布的均值.

Supplement Exercises

Solution

Proof. $$\text{RHS} = \int_0^\infty \mathbb{P}(X > x)\,dx$$

$$= \int_0^\infty \int_x^\infty f_X(u)\,du\,dx$$

$$= \int_0^\infty \int_0^u f_X(u)\,dx\,du = \int_0^\infty u f_X(u)\,du$$

$$= \mathbb{E}X \quad \square$$

Supplement Exercises

Exercise 3

16. 如果 X 是连续型随机变量，密度函数关于某个点 ξ 对称，证明：只要 $E(X)$ 存在，就有 $E(X)=\xi$.

Solution

16. Proof. $\mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx$

$\stackrel{y=x-\xi}{=\!=\!=} \int_{-\infty}^{+\infty} (y+\xi) f_X(y+\xi)\,dy$

$= \xi \int_{-\infty}^{+\infty} f_X(y+\xi)\,dy \quad (f_X(y+\xi) = f_X(-y+\xi))$

$= \xi \int_{-\infty}^{+\infty} f_X(x)\,dx = \xi. \quad \square$

Exercise 4

9. 试证:对任意的常数 $c \neq E(X)$,有

$$\mathrm{Var}(X) = E(X - E(X))^2 < E(X - c)^2.$$

Supplement Exercises

Solution

证

$$E(X-E(X))^2 = E[(X-c)-(E(X)-c)]^2 = E(X-c)^2-(E(X)-c)^2,$$

由于 $c \neq E(X)$，所以 $(E(X)-c)^2 > 0$，由此得

$$\mathrm{Var}(X) = E(X-E(X))^2 < E(X-c)^2.$$

Exercise 5

12. 从一个装有 m 个白球，n 个黑球的袋中进行有返回地摸球，直到摸到白球时停止. 试求取出黑球数的期望.

Solution

解 令 X 为取到白球时已取出的黑球数，则 $Y=X+1$ 服从几何分布 $Ge(m/(n+m))$。所以 $E(Y)=(n+m)/m=n/m+1$，由此得 $E(X)=E(Y)-1=n/m$。

Exercise 6

20. 设随机变量 $X \sim b(n,p)$，试证明：

$$E\left(\frac{1}{X+1}\right)=\frac{1-(1-p)^{n+1}}{(n+1)p}.$$

Solution

证

$$E\left(\frac{1}{X+1}\right)=\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}p^k(1-p)^{n-k}$$

$$=\sum_{k=0}^{n}\frac{n!}{(k+1)!\,(n-k)!}p^k(1-p)^{n-k}$$

$$=\frac{1}{(n+1)p}\sum_{k=0}^{n}\frac{(n+1)!}{(k+1)!\,[n+1-(k+1)]!}p^{k+1}(1-p)^{n+1-(k+1)}$$

$$(\text{令 } k'=k+1)=\frac{1}{(n+1)p}\sum_{k'=1}^{n+1}\binom{n+1}{k'}p^{k'}(1-p)^{n+1-k'}$$

$$=\frac{1}{(n+1)p}\left[1-(1-p)^{n+1}\right].$$

Exercise 7

21. 掷一枚不均匀硬币，一直掷到正、反面都出现为止. 记出现正面的概率为 $p(0 < p < 1)$，试求平均抛掷次数.

Supplement Exercises

Solution

解 记 X 为直到正、反面都出现时的抛掷次数，则 X 可取值 $2,3,\cdots$，且有

$$P(X=k)=(1-p)^{k-1}p+p^{k-1}(1-p),\quad k=2,3,\cdots$$

可以验证，这是一个分布列. 由此得 X 的数学期望为

$$E(X)=\sum_{k=2}^{+\infty}k[(1-p)^{k-1}p+p^{k-1}(1-p)]$$

$$=\sum_{k=1}^{+\infty}k(1-p)^{k-1}p-p+\sum_{k=1}^{+\infty}kp^{k-1}(1-p)-(1-p)$$

$$=\frac{1}{p}+\frac{1}{1-p}-1=\frac{1}{p(1-p)}-1.$$

Exercise 8

30. 设随机变量 $X \sim N(\mu, \sigma^2)$，求 $E|X-\mu|$.

Supplement Exercises

Solution

解　利用变换 $t=(x-\mu)/\sigma$ 及偶函数性质可得

$$E|X-\mu|=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}|x-\mu|\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx$$

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$$=\sigma\sqrt{\frac{2}{\pi}}\int_0^{+\infty}\exp\left\{-\frac{t^2}{2}\right\}d\left(\frac{t^2}{2}\right)=\sigma\sqrt{\frac{2}{\pi}}.$$

Exercise 9

2. 求掷 n 颗骰子出现点数之和的数学期望与方差.

Supplement Exercises

Solution

解 记 X_i 为第 i 颗骰子出现的点数，$i=1,2,\cdots,n$. 则 $X_1,X_2,\cdots,X_n$ 独立同分布，其共同的分布列为

X_i	1	2	3	4	5	6
P	1/6	1/6	1/6	1/6	1/6	1/6

所以

$$E(X_i)=\frac{1}{6}(1+2+3+4+5+6)=\frac{7}{2};$$

$$\mathrm{Var}(X_i)=\frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2)-\frac{49}{4}=\frac{35}{12}.$$

由此得

$$E\left(\sum_{i=1}^{n}X_i\right)=\sum_{i=1}^{n}E(X_i)=\frac{7n}{2};\quad \mathrm{Var}\left(\sum_{i=1}^{n}X_i\right)=\sum_{i=1}^{n}\mathrm{Var}(X_i)=\frac{35n}{12}.$$

Supplement Exercises

Exercise 10

3. 从数字 $0,1,\cdots,n$ 中任取两个不同的数字,求这两个数字之差的绝对值的数学期望.

Solution

解　记 X 与 Y 分别为第 1 次和第 2 次取出的数字，则

$$P(X=i, Y=j)=\frac{1}{(n+1)n}, \quad i,j=0,1,\cdots,n, \quad i \neq j.$$

所以

$$E|X-Y|=\frac{1}{(n+1)n}\sum_{i=0}^{n}\left\{\sum_{j=0}^{i}(i-j)+\sum_{j=i+1}^{n}(j-i)\right\}$$

$$=\frac{1}{(n+1)n}\sum_{i=0}^{n}\left\{\frac{i(i+1)}{2}+\frac{(n-i)[(n-i)+1]}{2}\right\}$$

$$=\frac{1}{(n+1)n}\sum_{i=0}^{n}\left\{i^2+\frac{n(n+1)}{2}-in\right\}$$

$$=\frac{2n+1}{6}+\frac{n+1}{2}-\frac{n}{2}=\frac{n+2}{3}.$$

Supplement Exercises

Exercise 11

4. 设在区间(0,1)上随机地取 n 个点,求相距最远的两点间的距离的数学期望.

Supplement Exercises

Solution

解 解法一：分别记此n个点为$X_1,X_2,\cdots,X_n$，则$X_1,X_2,\cdots,X_n$相互独立，且都服从区间$(0,1)$上的均匀分布$U(0,1)$。我们的目的是求

$$E(\max\{X_1,X_2,\cdots,X_n\}-\min\{X_1,X_2,\cdots,X_n\}).$$

而$Z=\max\{X_1,X_2,\cdots,X_n\}$和$Y=\min\{X_1,X_2,\cdots,X_n\}$的概率密度函数分别为

$$p_Z(z)=\begin{cases} nz^{n-1}, & 0<z<1,\\ 0, & \text{其他}.\end{cases}\qquad p_Y(y)=\begin{cases} n(1-y)^{n-1}, & 0<y<1,\\ 0, & \text{其他}.\end{cases}$$

又因为

$$E(Z)=\int_0^1 nz^{n-1}z\,dz=\frac{n}{n+1},\quad E(Y)=\int_0^1 ny(1-y)^{n-1}dy=\frac{1}{n+1},$$

所以

$$E(\max\{X_1,X_2,\cdots,X_n\}-\min\{X_1,X_2,\cdots,X_n\})=\frac{n}{n+1}-\frac{1}{n+1}=\frac{n-1}{n+1}.$$

Solution

解法二：n 个点把区间 $(0,1)$ 分成 $n+1$ 段，它们的长度依次记为 $Y_1, Y_2, \cdots, Y_{n+1}$. 因为此 n 个点是随机取的，所以 $Y_1, Y_2, \cdots, Y_{n+1}$ 具有相同的分布，从而有相同的数学期望. 而 $Y_1 + Y_2 + \cdots + Y_{n+1} = 1$，因此

$$E(Y_1) = E(Y_2) = \cdots = E(Y_{n+1}) = \frac{1}{n+1}.$$

而相距最远的两点间的距离为 $Y_2 + Y_3 + \cdots + Y_n$，因此所求期望为

$$E(Y_2 + Y_3 + \cdots + Y_n) = \frac{n-1}{n+1}.$$

Supplement Exercises

Exercise 12

6. 设随机变量(X,Y)的联合分布列为

X \ Y	0	1
0	0.1	0.15
1	0.25	0.2
2	0.15	0.15

Supplement Exercises

Solution

解

$$E(Z) = 0.1\sin 0 + 0.15\sin\frac{\pi}{2} + 0.25\sin\frac{\pi}{2} + 0.2\sin\pi + 0.15\sin\pi + 0.15\sin\frac{3\pi}{2}$$

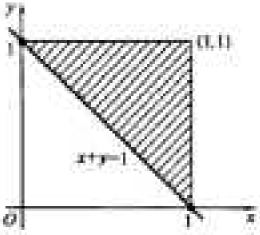
$$= 0.15\times 1 + 0.25\times 1 + 0.15\times(-1) = 0.25.$$

Supplement Exercises

Exercise 13

7. 随机变量(X,Y)服从以点$(0,1)$，$(1,0)$，$(1,1)$为顶点的三角形区域上的均匀分布，试求$E(X+Y)$和$\mathrm{Var}(X+Y)$.

解　记此三角形区域为D(如图 3.15 阴影部分).



Supplement Exercises

Solution

由于 X 与 Y 不独立，所以先计算

$$E(XY)=\int_0^1\int_{1-x}^1 2xy\,dy\,dx=\frac{5}{12}$$

由此得

$$\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)=\frac{5}{12}-\frac{4}{9}=-\frac{1}{36}\quad（负相关）$$

最后得

$$E(X+Y)=\frac{2}{3}+\frac{2}{3}=\frac{4}{3},$$

$$\mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)+2\mathrm{Cov}(X,Y)=\frac{1}{18}+\frac{1}{18}-\frac{2}{36}=\frac{1}{18}$$

Supplement Exercises

Exercise 14

8. 设 X,Y 均为 $(0,1)$ 上独立的均匀随机变量，试证：

$$E(|X-Y|^{\alpha})=\frac{2}{(\alpha+1)(\alpha+2)},\alpha>0.$$

Supplement Exercises

Solution

证　因为(X,Y)的联合密度函数为

$$p(x,y)=\begin{cases}1, & 0<x,y<1,\\ 0 & 其他.\end{cases}$$

所以

$$\begin{aligned}E|X-Y|^{\alpha} &= \int_0^1\int_0^1 |x-y|^{\alpha}\,\mathrm{d}x\mathrm{d}y\\ &= \int_0^1\int_0^y (y-x)^{\alpha}\,\mathrm{d}x\mathrm{d}y + \int_0^1\int_y^1 (x-y)^{\alpha}\,\mathrm{d}x\mathrm{d}y\\ &= \frac{1}{(\alpha+1)(\alpha+2)} + \frac{1}{(\alpha+1)(\alpha+2)}\\ &= \frac{2}{(\alpha+1)(\alpha+2)}.\end{aligned}$$

Supplement Exercises

Exercise 15

EXAMPLE 1.3(A) ***The Matching Problem.*** At a party n people put their hats in the center of a room where the hats are mixed together Each person then randomly selects one We are interested in the mean and variance of X—the number that select their own hat

Supplement Exercises

Solution

To solve, we use the representation

$$X = X_1 + X_2 + \cdots + X_n,$$

where

$$X_i = \begin{cases} 1 & \text{if the } i\text{th person selects his or her own hat} \\ 0 & \text{otherwise} \end{cases}$$

Now, as the ith person is equally likely to select any of the n hats, it follows that $P\{X_i = 1\} = 1/n$, and so

$$E[X_i] = 1/n,$$

$$\text{Var}(X_i) = \frac{1}{n}\left(1 - \frac{1}{n}\right) = \frac{n-1}{n^2}$$

Solution

Also

$$\mathrm{Cov}(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j].$$

Now,

$$X_i X_j = \begin{cases} 1 & \text{if the } i\text{th and } j\text{th party goers both select their own hats} \\ 0 & \text{otherwise,} \end{cases}$$

and thus

$$\begin{aligned} E[X_i X_j] &= P\{X_i = 1, X_j = 1\} \\ &= P\{X_i = 1\}P\{X_j = 1 \mid X_i = 1\} \\ &= \frac{1}{n}\frac{1}{n-1} \end{aligned}$$

Supplement Exercises

Solution

Hence,

$$\mathrm{Cov}(X_i, X_j) = \frac{1}{n(n-1)} - \left(\frac{1}{n}\right)^2 = \frac{1}{n^2(n-1)}$$

Therefore, from (1.3.3) and (1.3.4),

$$E[X] = 1$$

and

$$\begin{aligned}\mathrm{Var}(X) &= \frac{n-1}{n} + 2\binom{n}{2}\frac{1}{n^2(n-1)} \\ &= 1.\end{aligned}$$

Exercise 16

26. 单位棒断裂成两段. 计算较长一段与较短一段长度的期望比.

Solution

Solution. $\underline{x \,|\, 1-x}$, $X \sim U(0,1)$

$$Y = \max\{X, 1-X\}$$

$$\mathbb{P}(Y \leq y) = \begin{cases} 0, & y \leq \frac{1}{2} \\ \mathbb{P}(Y \in [1-y, y]) = 2y-1, & y \in (\frac{1}{2}, 1) \\ 1, & y \geq 1 \end{cases}$$

Then, $f_Y(y) = \begin{cases} 2, & y \in (\frac{1}{2}, 1) \\ 0, & \text{other}. \end{cases}$

$\mathbb{E}Y = \frac{3}{4}$. $\mathbb{E}[1-Y] = \frac{1}{4}$.

Hence, $\mathbb{E}Y / \mathbb{E}[1-Y] = 3$.

Thank you!