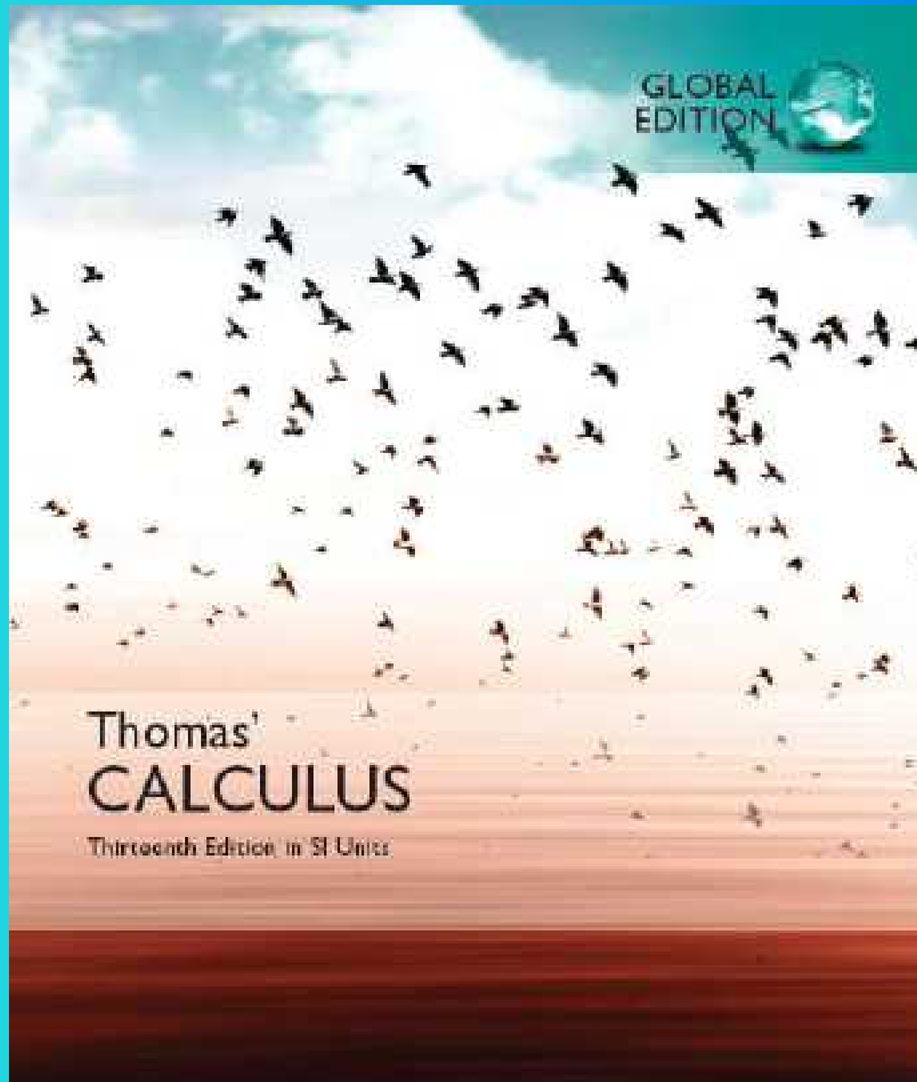
GLOBAL EDITION
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Chapter 6

Applications of Definite Integrals
定积分的应用

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6.1

Volumes Using Cross-Sections
用截面求体积

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平行截面的面积

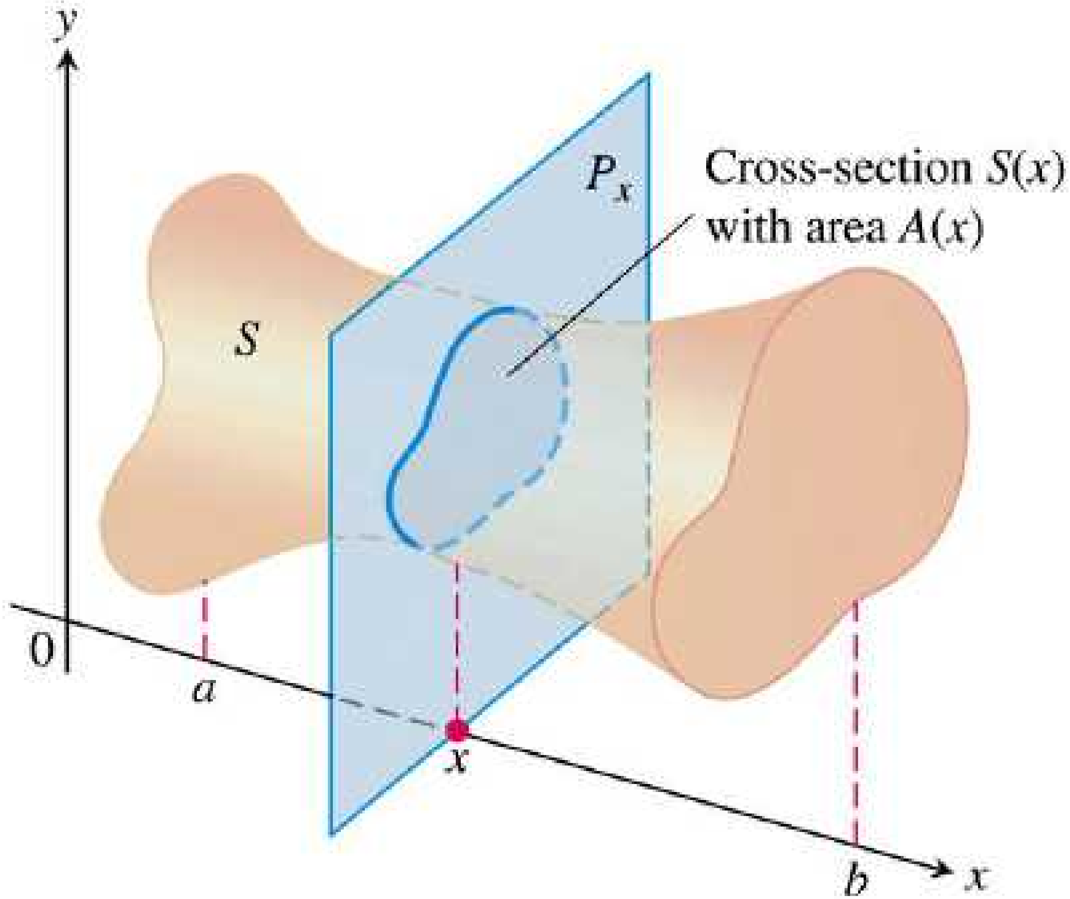

FIGURE 6.1 A cross-section $S(x)$ of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

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分割 $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

近似 $\Delta V_k \approx A(c_k)\Delta x_k$

求和 $V \approx \sum_{k=1}^{n} A(c_k)\Delta x_k$

取极限

$$V = \lim_{\|P\|\to 0} \sum_{k=1}^{n} A(c_k)\Delta x_k$$

$$V = \int_a^b A(x)dx$$

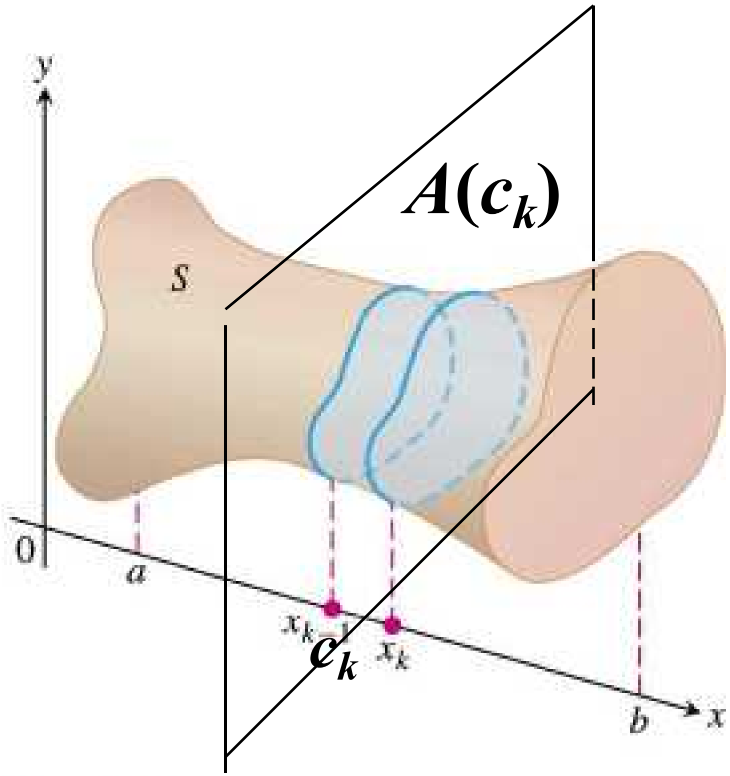

FIGURE 6.3 A typical thin slab in the solid S.

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DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\, dx.$$

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Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for $A(x)$*, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate $A(x)$* using the Fundamental Theorem.

1.画图.

2.确定积分区间$[a,b]$.

3.任取$x\in[a,b]$,过点x做x轴的垂面，求出截面面 积$A(x)$.

4.积分$\int_a^b A(x)dx$.

***Ex*.1** 正四棱锥体如图.求其体积.

Solution 1.画图，建立坐标系.

2.确定积分区间[0,3].

3.任取$x\in[0,3]$,过点x做x轴的垂面，

求截面面积$A(x)=x^2$.

4.积分$\int_0^3 x^2 dx=\frac{27}{3}=9$.

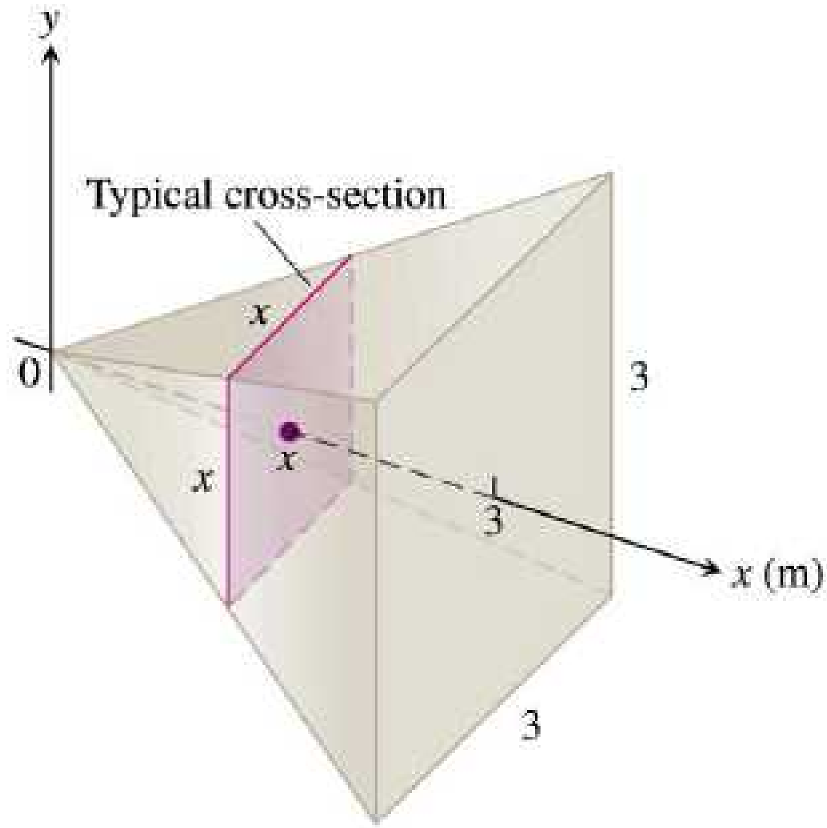

FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

Ex.2　木楔子从底半径为 3的圆柱形的木头上劈出，如图 .求其体积 .

Solution 1.画图，建立坐标系.

2.确定积分区间 [0,3].

3.任取$x\in[0,3]$,过点x做x轴的垂面，

求截面面积$A(x)=2x\sqrt{9-x^2}$.

4.积分$\int_0^3 2x\sqrt{9-x^2}\,dx=18$.

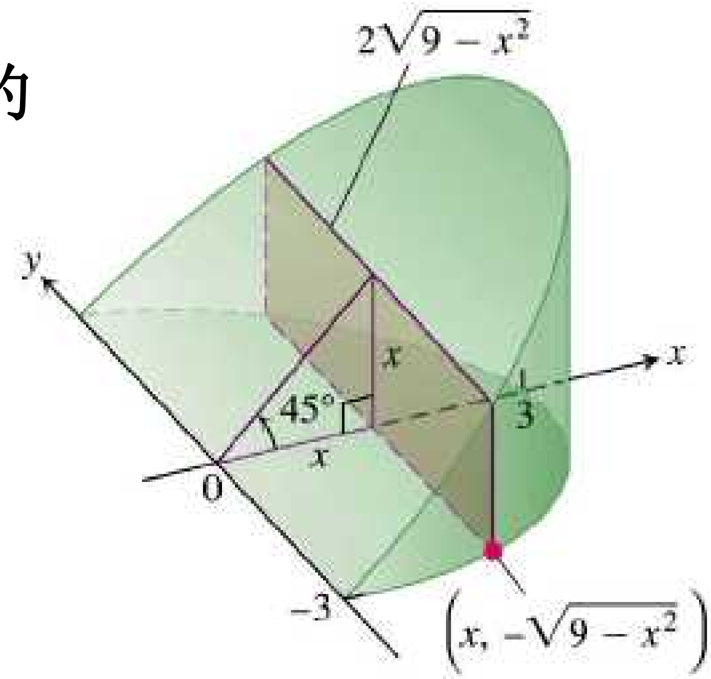

FIGURE 6.6 The wedge of Example 2, sliced perpendicular to the x-axis. The cross-sections are rectangles.

$Ex.3$ 说明Cavalieri' s 原理：两个立体若有相 同的高和相同的对应截面积 ，则这两个立体的体积 相等.

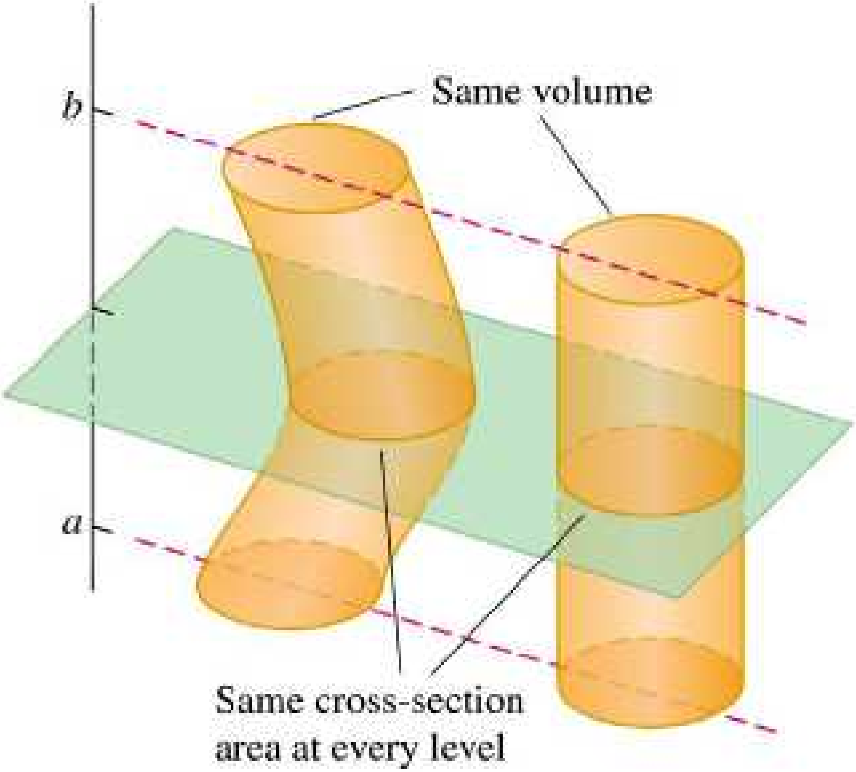

FIGURE 6.7 *Cavalieri's principle:* These solids have the same volume, which can be illustrated with stacks of coins.

Solution 1.建立坐标系.

2.确定积分区间 $[a,b]$.

3.任取$x \in [a,b]$, $A(x) = B(x)$.

4.$\int_a^b A(x)dx = \int_a^b B(x)dx$

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旋转体的体积

Solids of Revolution: The Disk Method 圆盘法

The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid like the one shown in Figure 6.8, we need only observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

So the definition of volume in this case gives

Volume by Disks for Rotation About the x-axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

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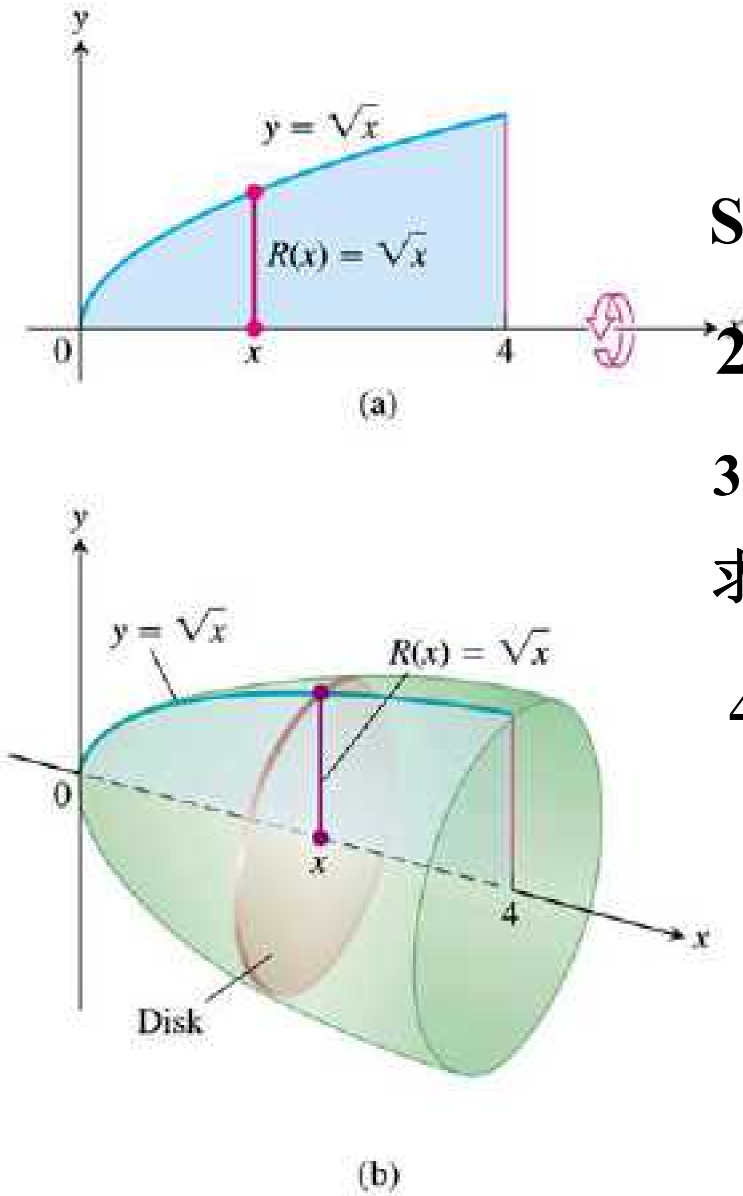

Ex.4　旋转体如图.求其体积.

Solution 1.画图，建立坐标系.

2.确定积分区间 [0,4].

3.任取$x\in[0,4]$,过点x做x轴的垂面，

求截面面积$A(x)=\pi(\sqrt{x})^2=\pi x$.

4.积分$\int_0^4 \pi x dx=8\pi$.

FIGURE 6.8 The region (a) and solid of revolution (b) in Example 4.

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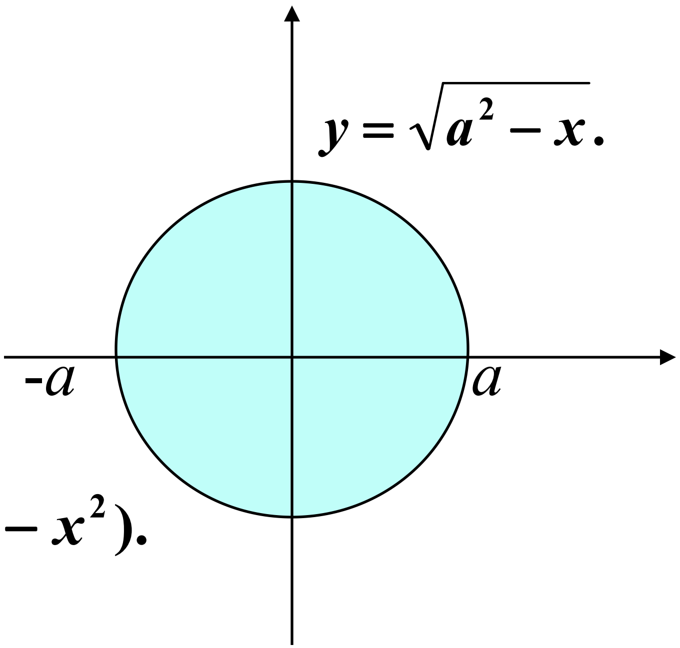
***Ex*.5** 证明半径为a的球体体积公式.

Solution 1.画图，建立坐标系.



2.确定积分区间 $[-a,a]$.

3.任取$x\in[-a,a]$,过点x做x轴的垂面,

求截面面积$A(x)=\pi(\sqrt{a^2-x^2})^2=\pi(a^2-x^2)$.

4.积分 $$V=\int_{-a}^{a}\pi(\sqrt{a^2-x^2})^2dx$$

$$=\int_{-a}^{a}\pi(a^2-x^2)dx$$

$$=\frac{4\pi a^3}{3}.$$

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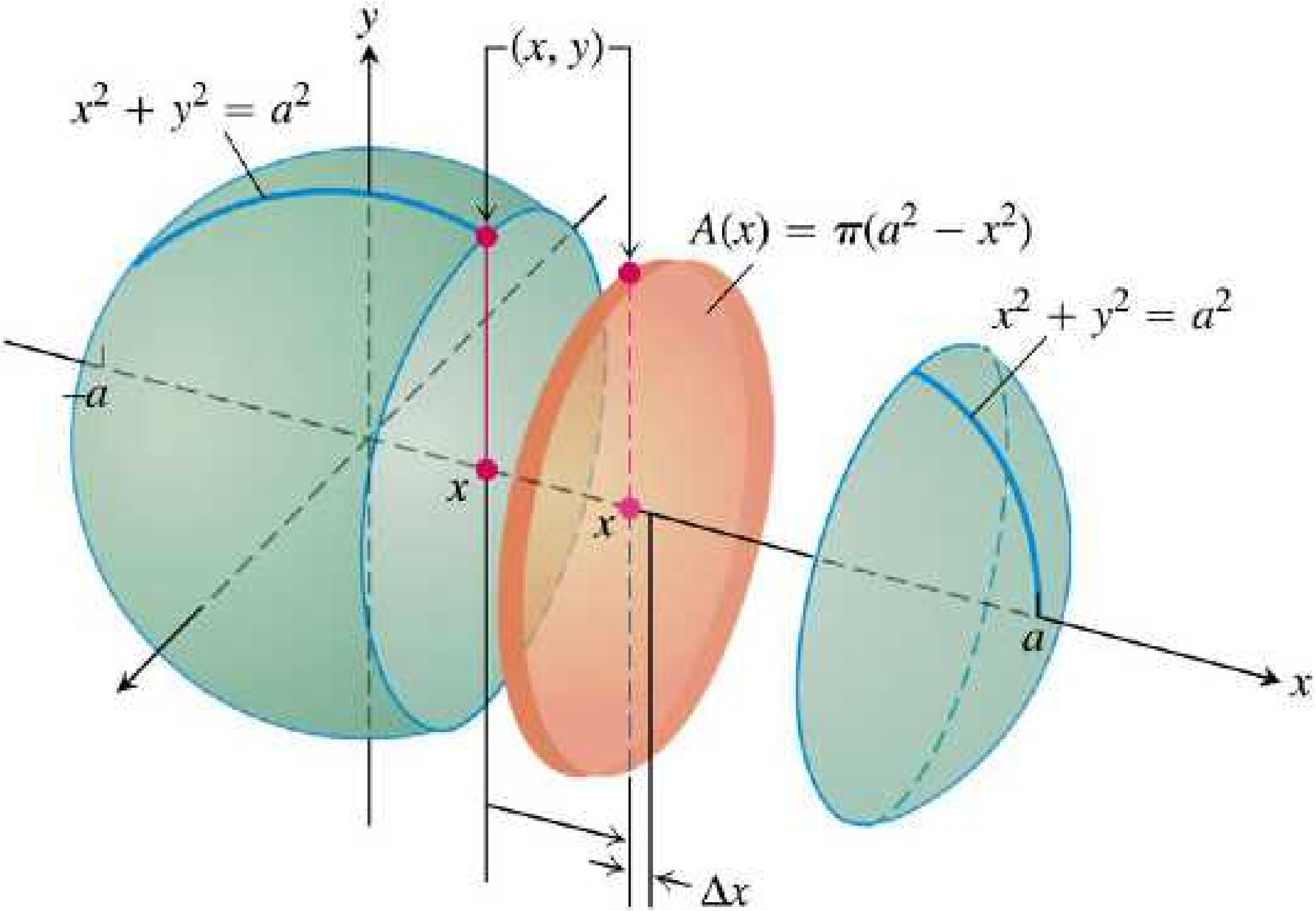

FIGURE 6.9 The sphere generated by rotating the circle $x^2 + y^2 = a^2$ about the x-axis. The radius is $R(x) = y = \sqrt{a^2 - x^2}$ (Example 5).

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$Ex.6$ **Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y=1, x=4$ about the line $y=1$.**

Solution :1.draw the figure.

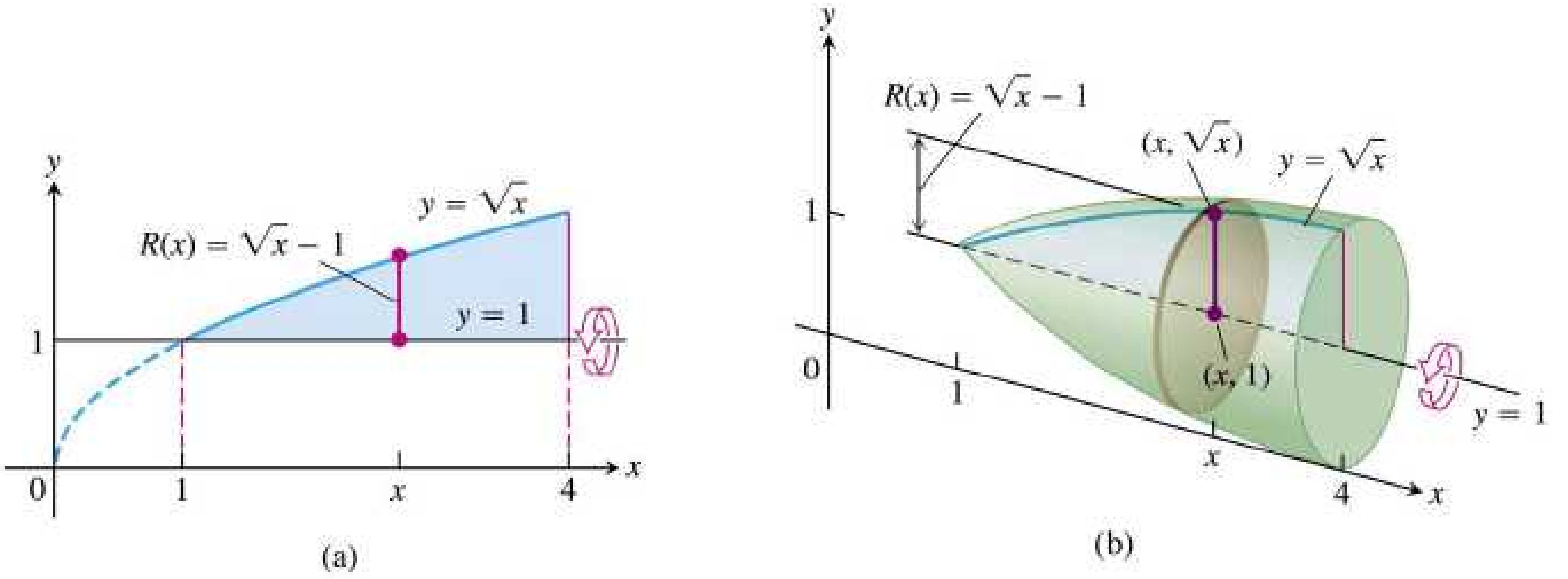

FIGURE 6.10 The region (a) and solid of revolution (b) in Example 6.

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2. Find the interval of the integral [1,4].

3. $\forall x \in [1,4]$, find the area of the typical cross-section

$$A(x) = \pi(\sqrt{x} - 1)^2 = \pi(x - 2\sqrt{x} + 1).$$

4. Integrate $\int_1^4 A(x)\mathrm{dx}$ to find the volume

$$V = \int_a^b \pi R^2(x)dx = \int_1^4 \pi(x - 2\sqrt{x} + 1)dx = \frac{7\pi}{6}.$$

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Volume by Disks for Rotation About the y-axis

$$V = \int_c^d A(y)\,dy = \int_c^d \pi\left[R(y)\right]^2\,dy.$$

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$Ex.7$ Find the volume of the solid generated by revolving the region between the y-axis and $xy = 2, \quad 1 \le y \le 4,$ about the y-axis.

Solution: 1. draw the figure.

2. Find the interval of the integral $[1,4]$.

3. $\forall y \in [1,4]$, find the area of the typical cross-section

$A(y) = \pi(2y^{-1})^2 = 4\pi y^{-2}.$

4. Integrate $\int_1^4 A(y)dy$ to find the volume

$$V = \int_a^b \pi R^2(y)dy = \int_1^4 4\pi y^{-2}dy = 3\pi.$$

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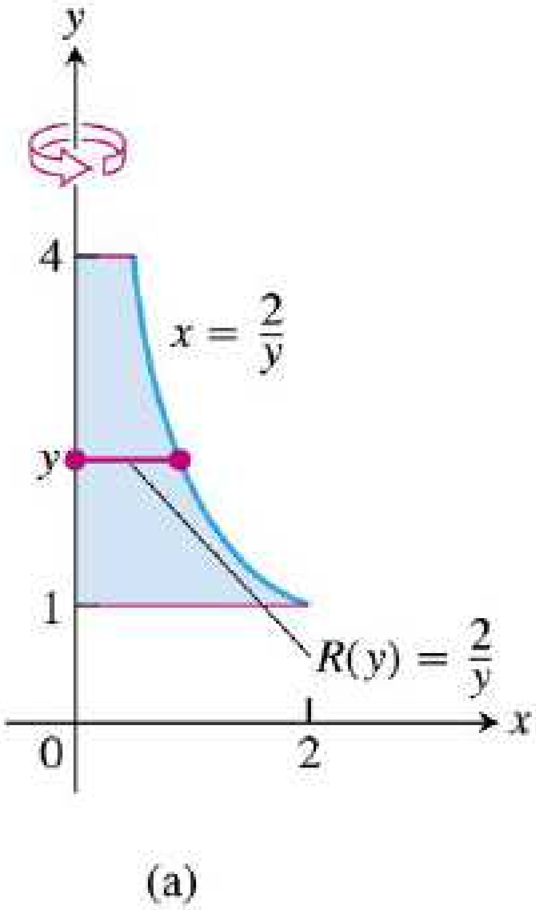

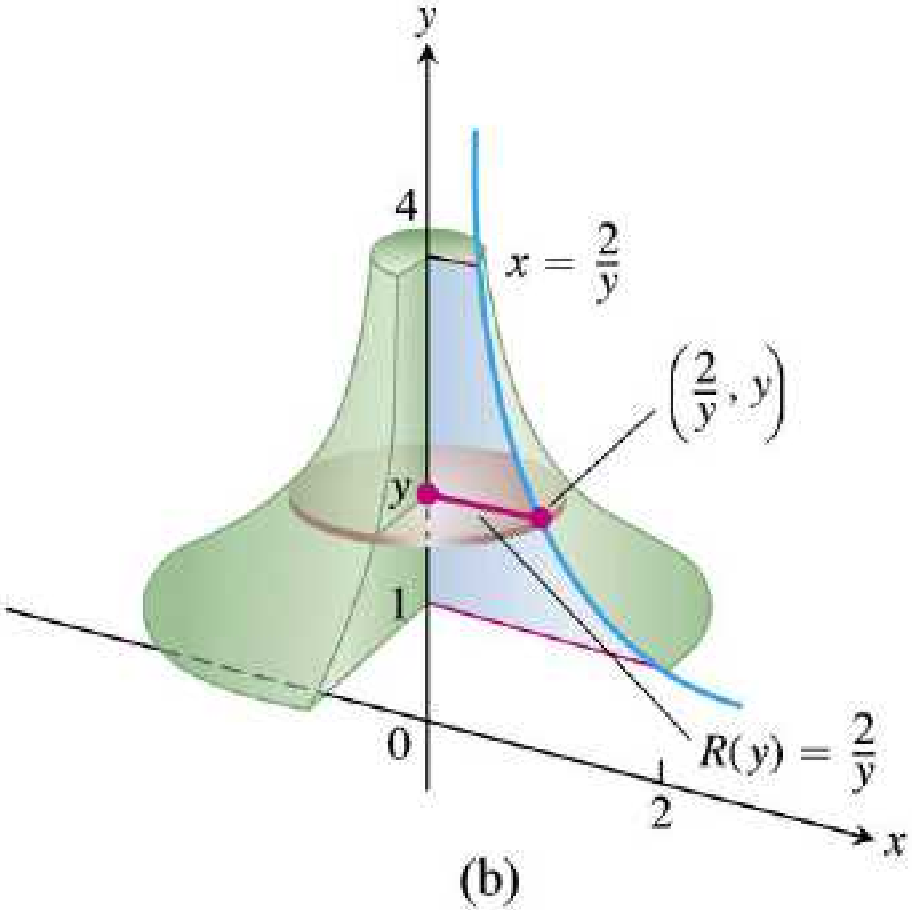

FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7.

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Ex.8 **Find the volume of the solid generated by revolving the region between $x = y^2 + 1$ and $x = 3$, about the line $x = 3$.**

Solution :1.draw the figure.

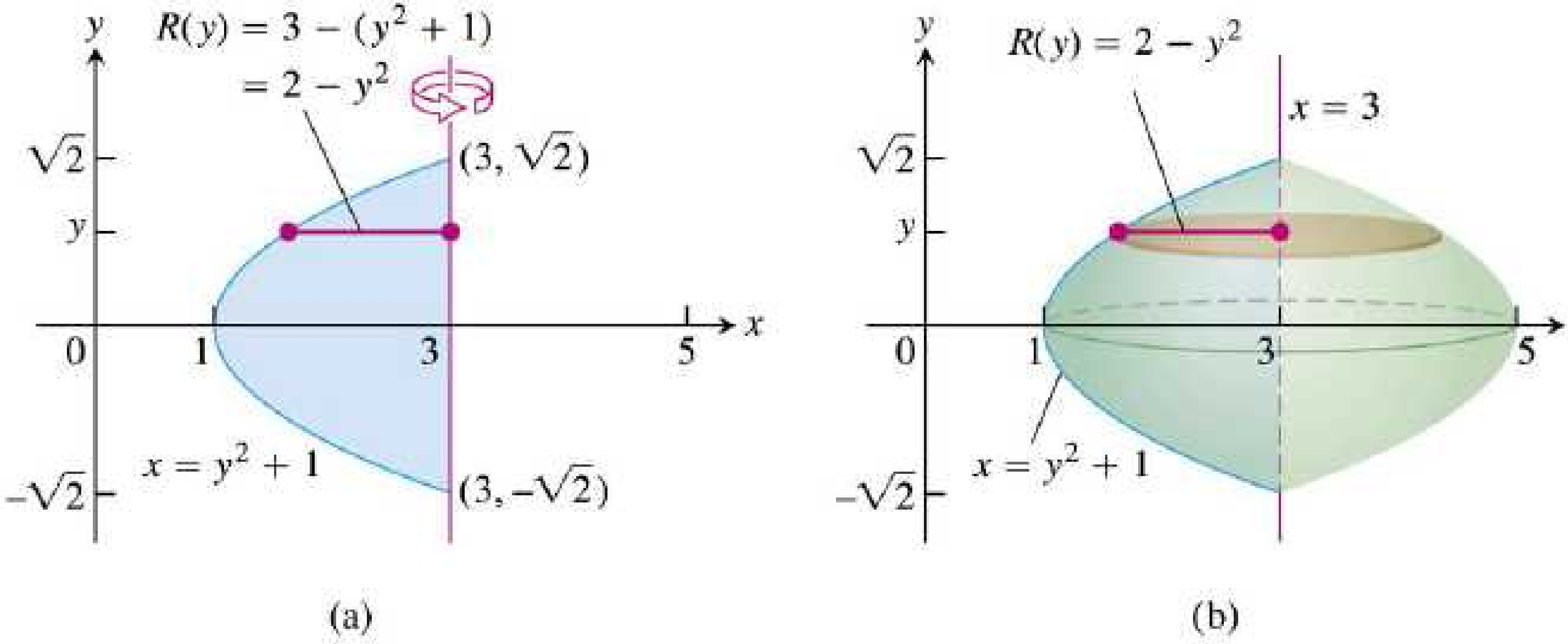

FIGURE 6.12 The region (a) and solid of revolution (b) in Example 8.

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2. Find the interval of the integral $[-\sqrt{2}, \sqrt{2}]$.

3. $\forall y \in [-\sqrt{2}, \sqrt{2}]$, find the area of the typical cross-section

$A(y) = \pi(2 - y^2)^2$.

4. Integrate $\int_{-\sqrt{2}}^{\sqrt{2}} A(y)dy$ to find the volume

$$V = \int_a^b \pi R^2(y)dy = \int_{-\sqrt{2}}^{\sqrt{2}} \pi(2 - y^2)^2 dy$$

$$= 2\int_0^{\sqrt{2}} \pi(4 - 4y^2 + y^4)dy = \frac{64\pi\sqrt{2}}{15}.$$

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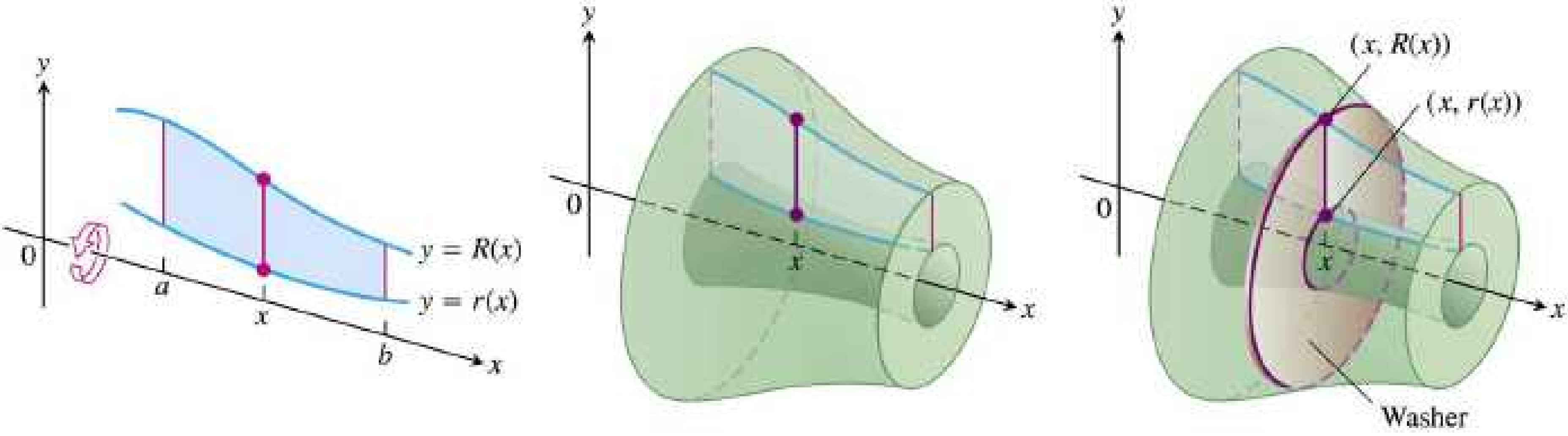

FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

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Solids of Revolution: The Washer Method 垫圈法

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are *washers* (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

$$\text{Outer radius:} \quad R(x)$$

$$\text{Inner radius:} \quad r(x)$$

The washer's area is

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$

Consequently, the definition of volume in this case gives

Volume by Washers for Rotation About the x-axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.$$

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$Ex.9$ **The region bounded by** $y = x^2 + 1$ **and** $y = 3 - x$ **is revolved about the x - axis to generate a solid . Find the volume of the solid .**

Solution :1.draw the figure.

2.Find the interval of the integral [-2,1].

3.$\forall x \in [-2,1]$**, find the area of the typical cross - section**

$$A(x) = \pi(3 - x)^2 - \pi(x^2 + 1)^2 .$$

4.Integrate $\int_{-2}^{1} A(x)\mathrm{d}x$ **to find the volume**

$$V = = \int_{-2}^{1} \pi[(3 - x)^2 - (x^2 + 1)^2]dx$$

$$= \int_{-2}^{1} \pi[8 - 6x - x^2 - x^4]dx = \frac{117\pi}{5}.$$

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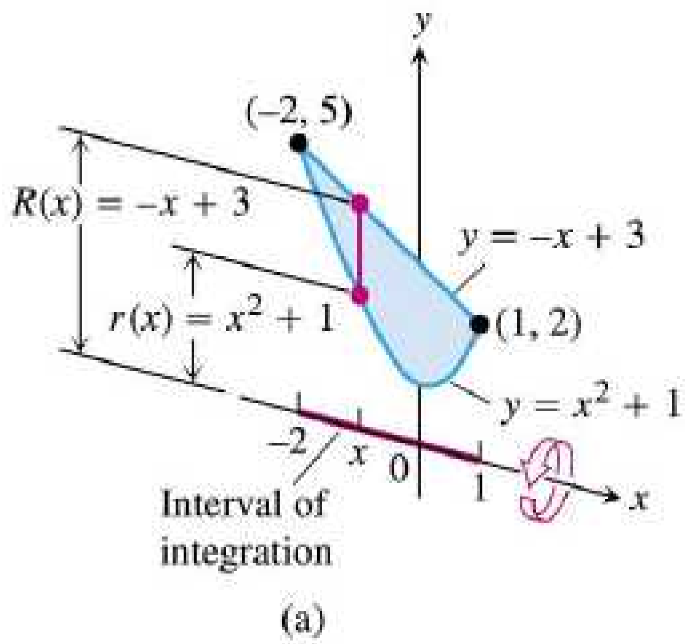

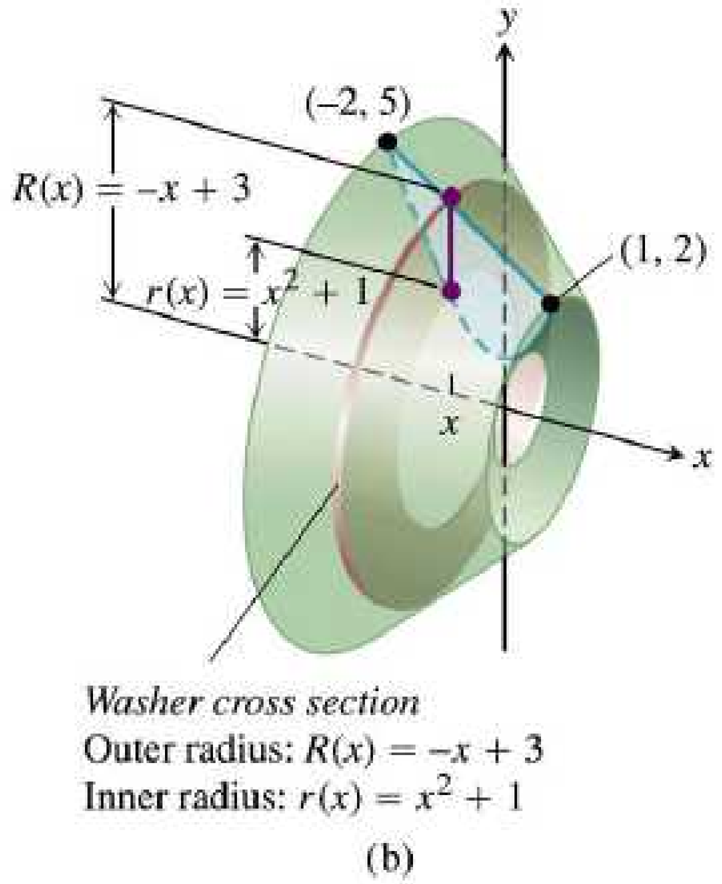

FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x-axis, the line segment generates a washer.

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$Ex.9$ The region bounded by $y = x^2$ and $y = 2x$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution: 1. draw the figure.

2. Find the interval of the integral $[0,4]$.

3. $\forall y \in [0,4]$, find the area of the typical cross-section $A(y) = \pi(\sqrt{y})^2 - \pi(\frac{y}{2})^2$.

4. Integrate $\int_0^4 A(y)dy$ to find the volume

$$V = \int_0^4 \pi[(\sqrt{y})^2 - (\frac{y}{2})^2]dy = \int_0^4 \pi[y - \frac{y^2}{4}]dy$$

$$= \frac{8\pi}{3}.$$

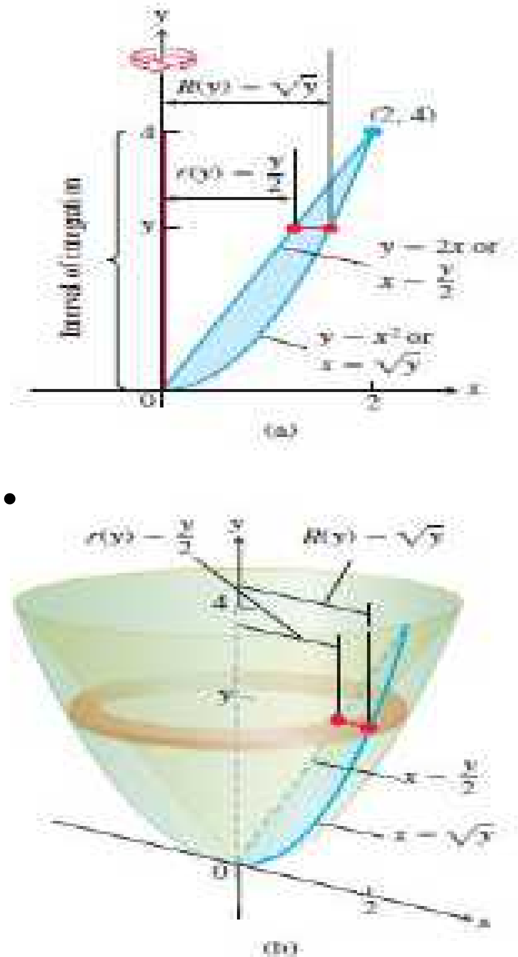

FIGURE 6.15 (a) The region being rotated about the y-axis, the washer radii, and limits of integration in Example 10. (b) The washer swept out by the line segment in part (a).

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6.2

Volumes Using Cylindrical Shells
用圆柱形壳求体积

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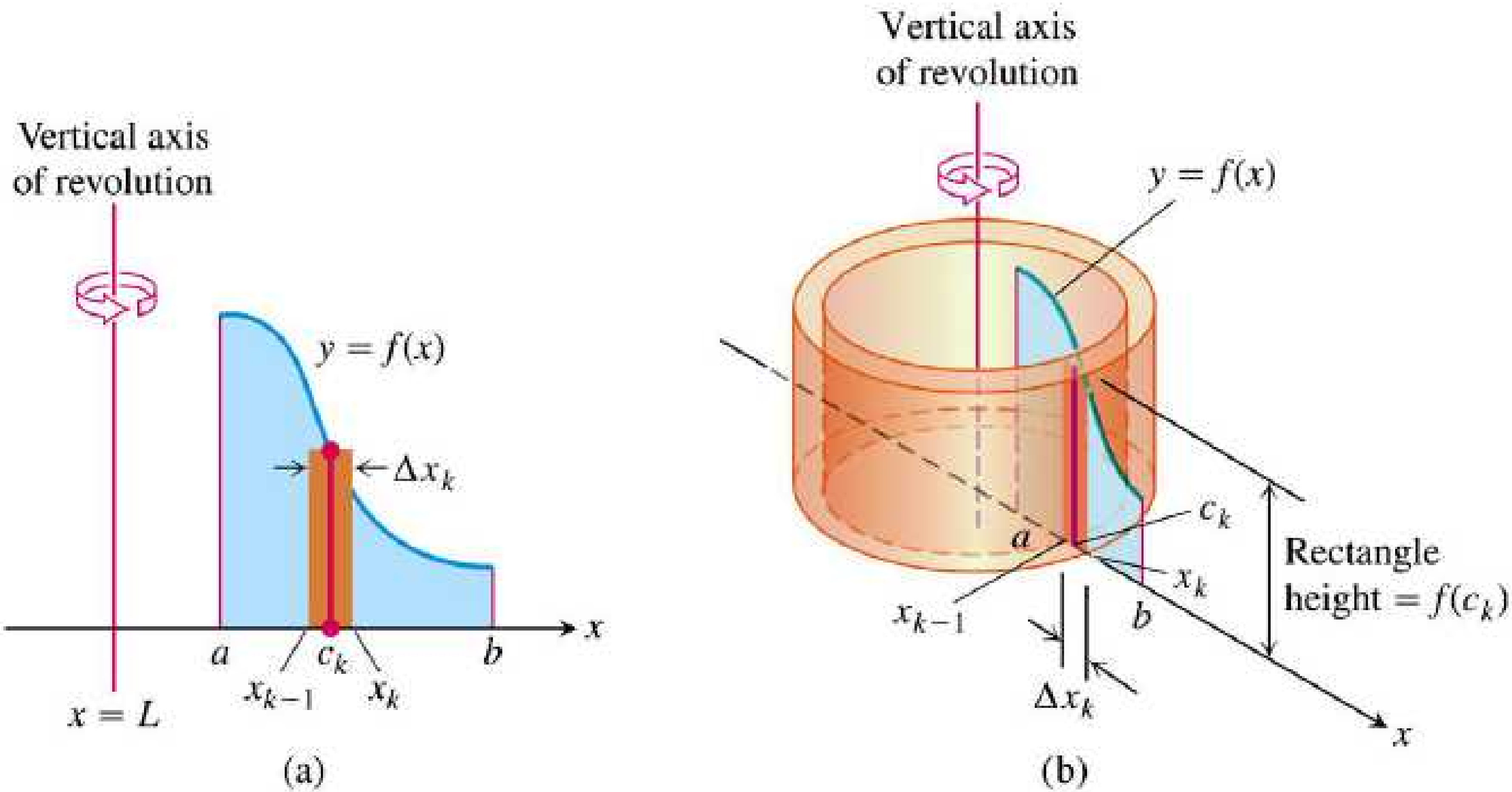

FIGURE 6.19 When the region shown in (a) is revolved about the vertical line $x = L$, a solid is produced which can be sliced into cylindrical shells. A typical shell is shown in (b).

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分割 $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

近似 $\Delta V_k \approx 2\pi c_k f(c_k)\Delta x_k$

求和 $V \approx \sum_{k=1}^{n} 2\pi c_k f(c_k)\Delta x_k$

取极限 $V = \lim_{\|P\|\to 0} \sum_{k=1}^{n} 2\pi c_k f(c_k)\Delta x_k$ $V = \int_a^b 2\pi x f(x) dx$

Solution steps: 1. draw the figure.

2. Find the interval of the integral $[a, \mathrm{b}]$.

3. $\forall x \in [a, b]$, find the volume of the typical shell,

$dV = 2\pi R(x) h(x) dx$

4. Integrate $\int_a^{\mathrm{b}} 2\pi \mathrm{R}(x)\mathrm{h}(x)\mathrm{d}x$ to find the volume

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Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0, L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx.$$

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$Ex.1$ The region bounded by x - axis and $y = 3x - x^2$ is revolved about the line $x = -1$ to generate a solid. Find the volume of the solid.

Solution: 1. draw the figure;

2. Find the interval of the integral $[0,3]$.

3. $\forall x \in [0,3]$, find the volume of the typical shell,

$$dV = 2\pi(x+1)(3x - x^2)dx$$

4. Integrate to find the volume

$$V = \int_0^3 2\pi(x+1)(3x - x^2)dx = \int_0^3 2\pi(3x + 2x^2 - x^3)dx = \frac{45\pi}{2}.$$

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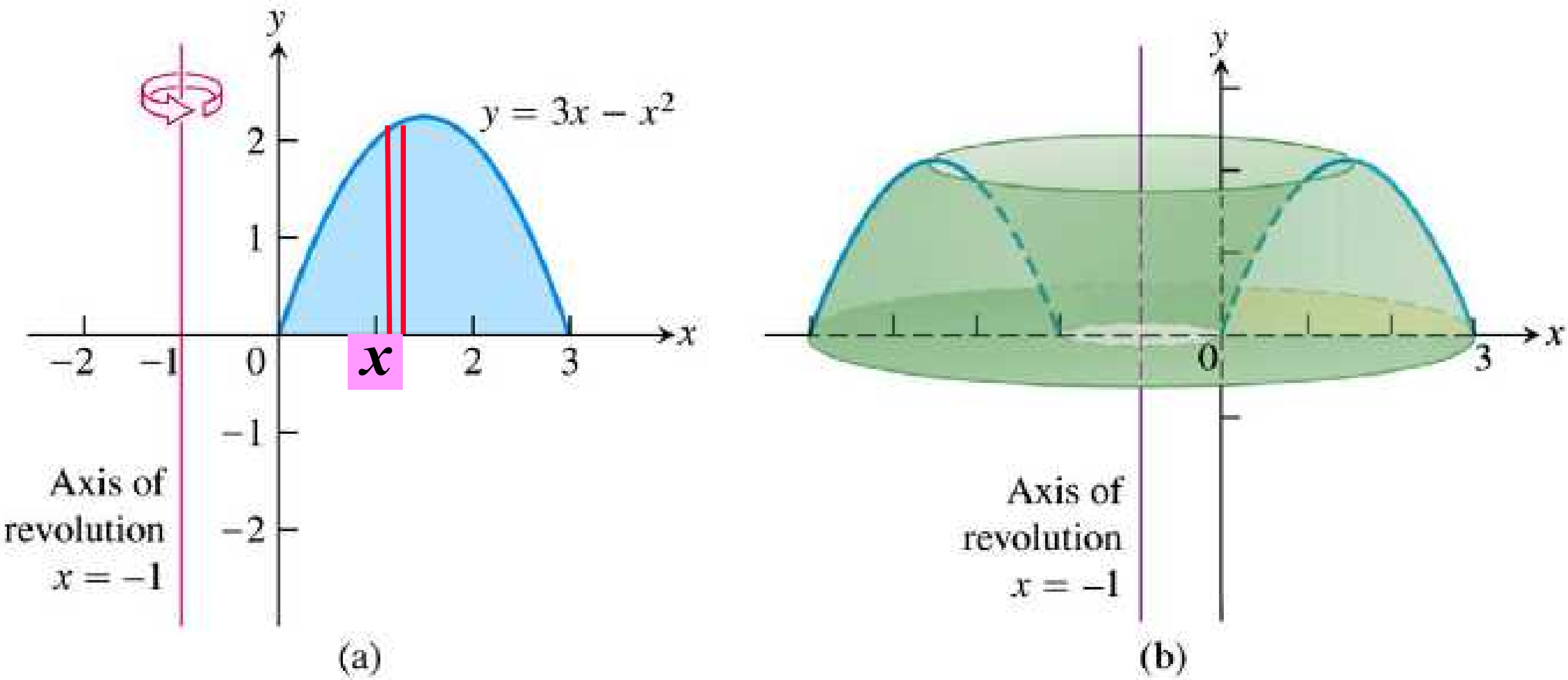

FIGURE 6.16 (a) The graph of the region in Example 1, before revolution. (b) The solid formed when the region in part (a) is revolved about the axis of revolution $x = -1$.

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$Ex.2$ The region bounded by $y=\sqrt{x}$, x-axis and $x=4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution: 1. draw the figure;

2. Find the interval of the integral [0,4].

3. $\forall x \in [0,4]$, find the volume of the typical shell,

$$dV = 2\pi x\sqrt{x}dx$$

4. Integrate to find the volume

$$V=\int_0^4 2\pi x\sqrt{x}dx=\frac{128\pi}{5}.$$

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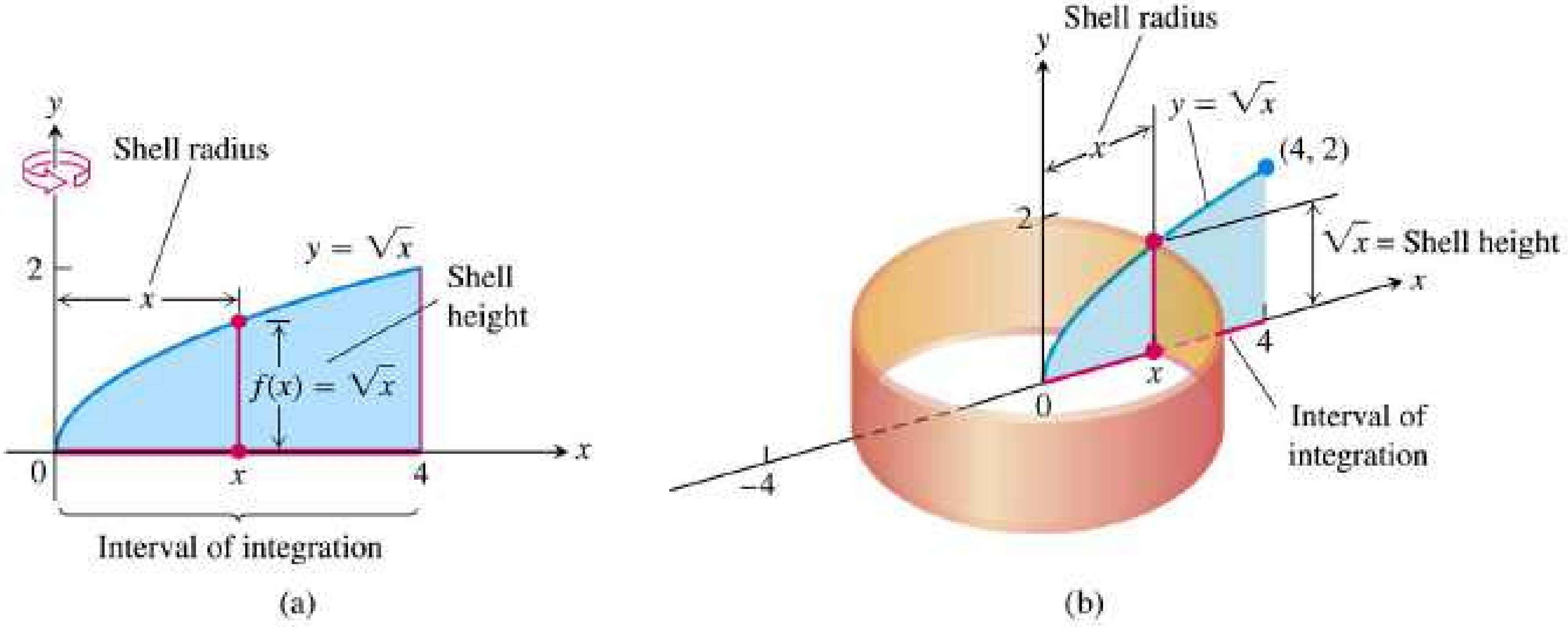

FIGURE 6.20 (a) The region, shell dimensions, and interval of integration in Example 2. (b) The shell swept out by the vertical segment in part (a) with a width Δx.

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$Ex.3$ The region bounded by $y=\sqrt{x}$, x-axis and $x=4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution: 1. draw the figure;

2. Find the interval of the integral $[0,2]$.

3. $\forall y \in [0,2]$, find the volume of the typical shell,

$$dV = 2\pi\ y\ (4-y^2)dy$$

4. Integrate to find the volume

$$V=\int_0^2 2\pi y(4-y^2)dy = \int_0^2 2\pi(4y-y^3)dy = 8\pi.$$

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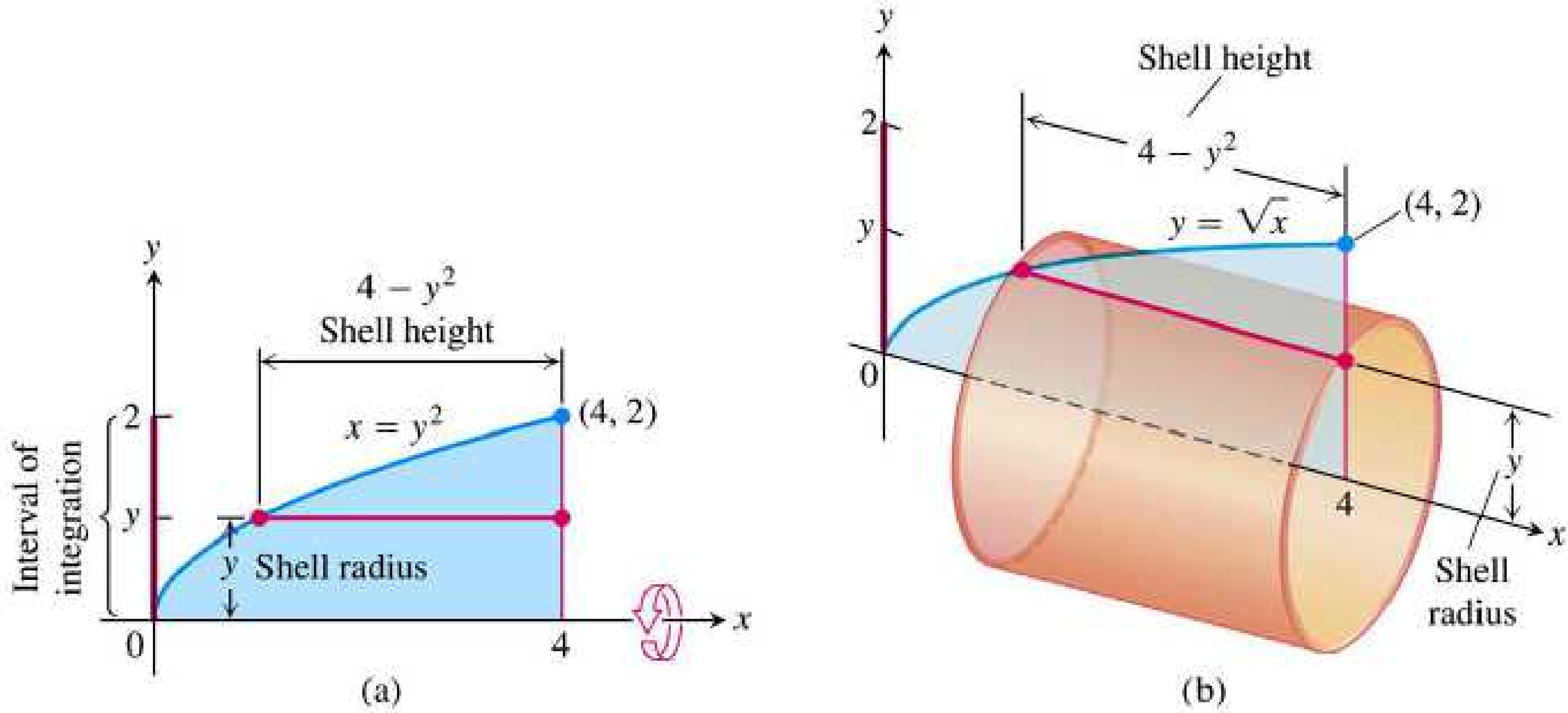

FIGURE 6.21 (a) The region, shell dimensions, and interval of integration in Example 3. (b) The shell swept out by the horizontal segment in part (a) with a width Δy.

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6.3

Arc Length

弧长

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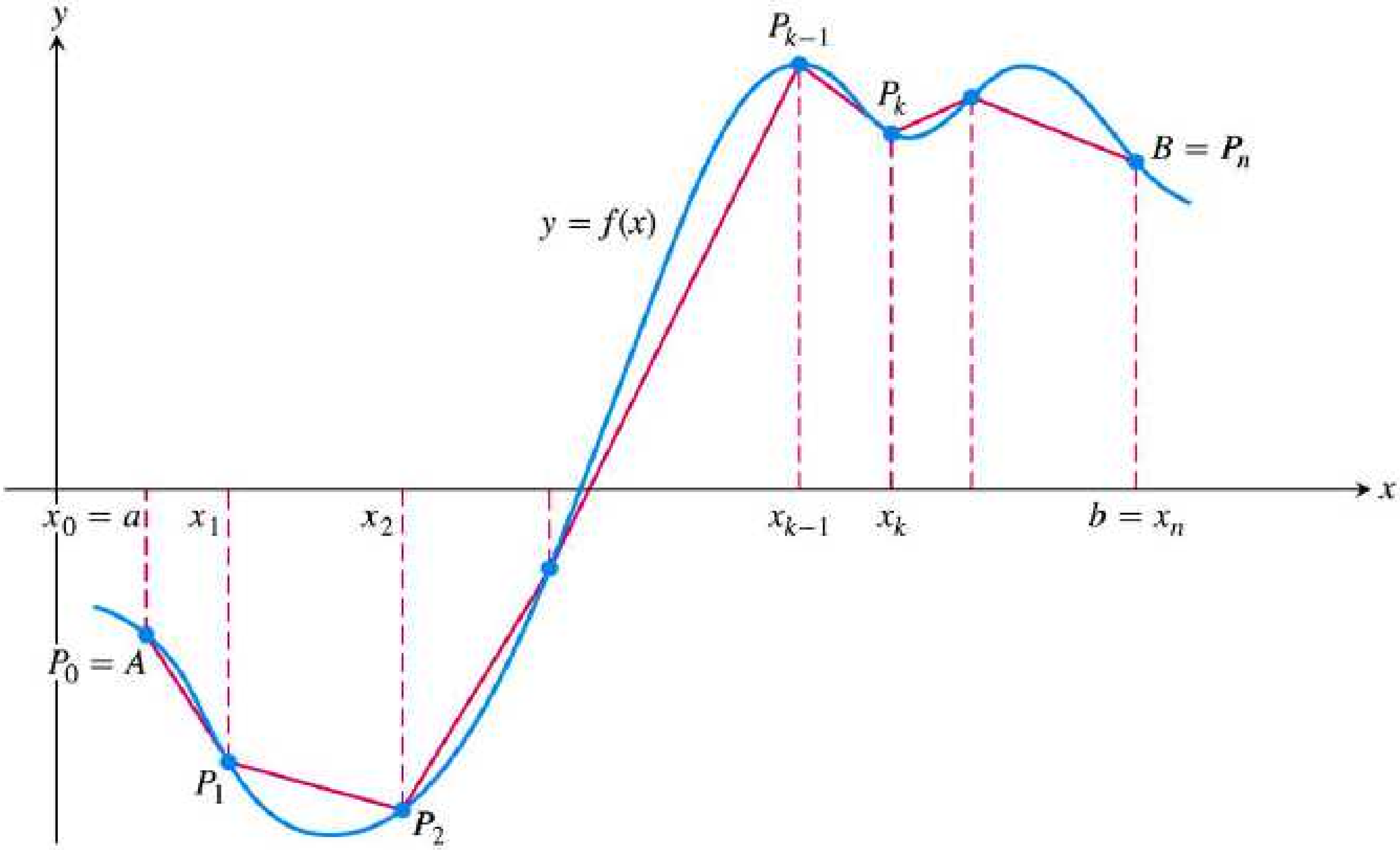

FIGURE 6.22 The length of the polygonal path $P_0P_1P_2 \cdots P_n$ approximates the length of the curve $y = f(x)$ from point A to point B.

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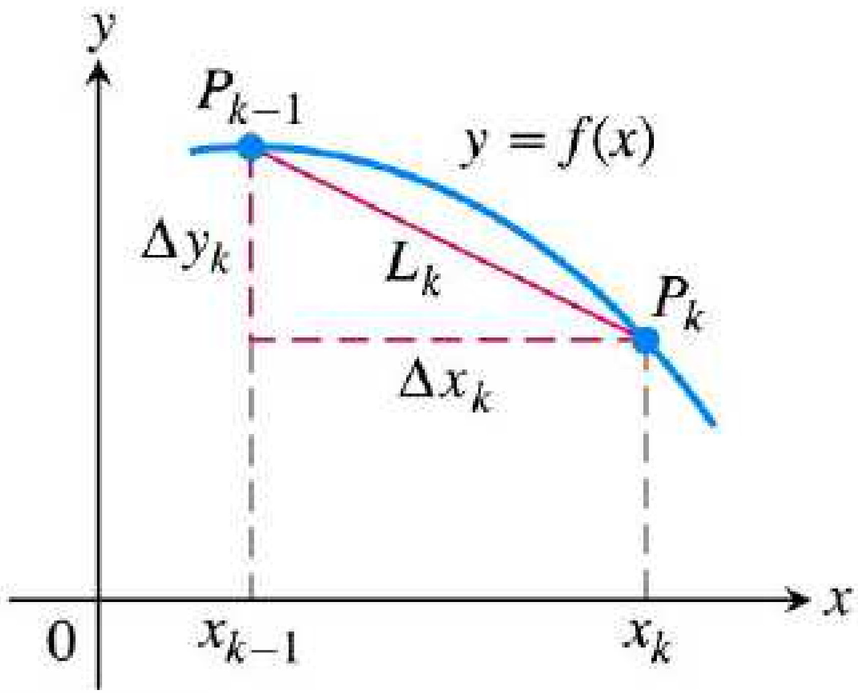

FIGURE 6.23 The arc $P_{k-1}P_k$ of the curve $y = f(x)$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

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分割 $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

近似 $L_k \approx \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$

求和 $L \approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (f'(c_k)\Delta x_k)^2}$

取极限

$$= \sum_{k=1}^{n} \sqrt{1 + (f'(c_k))^2}\,\Delta x_k$$

$$L = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \sqrt{1 + (f'(c_k))^2}\,\Delta x_k$$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

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DEFINITION If f' is continuous on $[a, b]$, then the **length** (**arc length**) of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{3}$$

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***Ex*.1** **Find the length of the curve**

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \le x \le 1.$$

Solution

$$L = \int_a^b \sqrt{1+(y')^2}\,dx$$

$$= \int_0^1 \sqrt{1+8x}\,dx$$

$$= \frac{13}{6}.$$

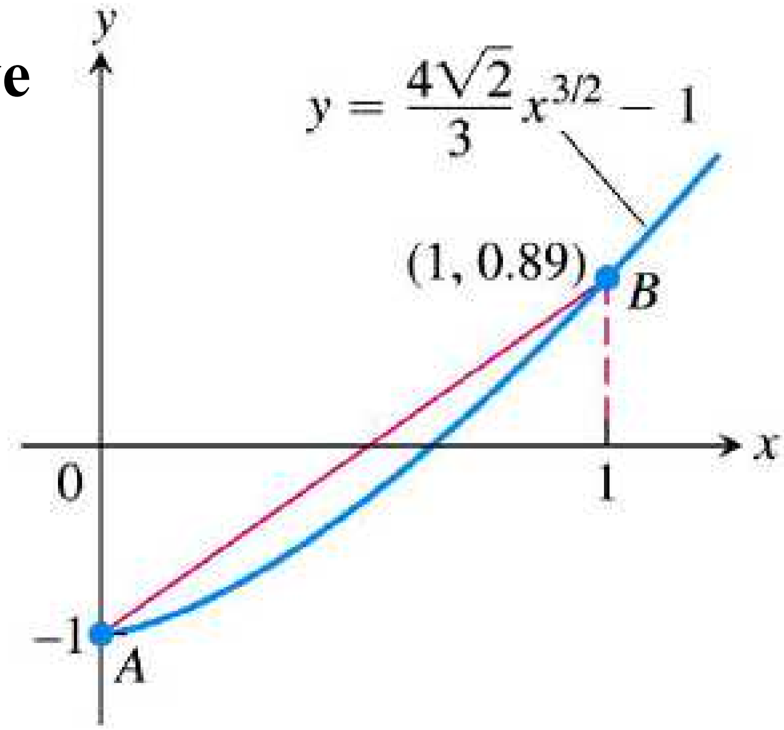

FIGURE 6.24 The length of the curve is slightly larger than the length of the line segment joining points *A* and *B* (Example 1).

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Ex.2 **Find the length of the curve**

$$y = \frac{x^3}{12} + \frac{1}{x}, \quad 1 \le x \le 4.$$

Solution

$$L = \int_a^b \sqrt{1 + (y')^2}\, dx$$

$$= \int_1^4 \sqrt{\frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}}\, dx$$

$$= \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$$

$$= 6.$$

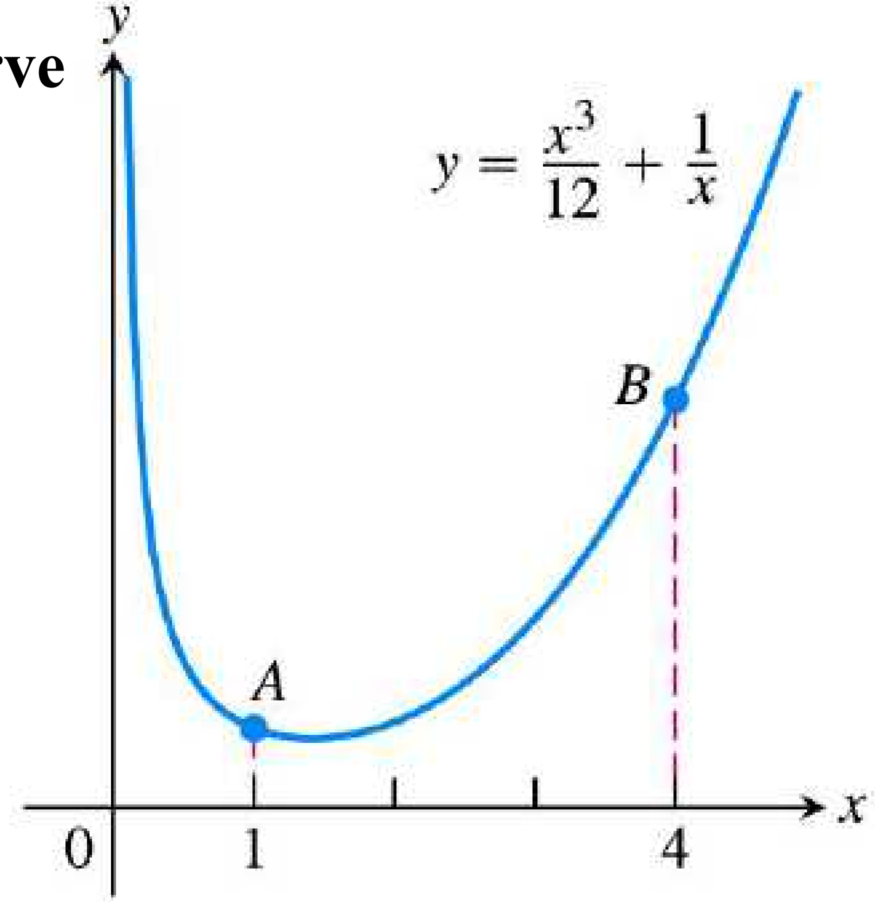

FIGURE 6.25 The curve in Example 2, where $A = (1, 13/12)$ and $B = (4, 67/12)$.

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有时曲线的方程写成 $x = g(y)$时更方便，则有

Formula for the Length of $x = g(y), c \le y \le d$

If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from $A = (g(c), c)$ to $B = (g(d), d)$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \qquad (4)$$

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Ex.3 **Find the length of the curve**

$$y=(\frac{x}{2})^{2/3}, \quad 0\le x\le 2.$$

Solution

$$x=2y^{3/2}, 0\le y\le 1,$$

$$x'=3y^{1/2}$$

$$L=\int_c^d\sqrt{1+(x')^2}\,dy$$

$$=\int_0^1\sqrt{1+9y}\,dy$$

$$=\frac{2}{27}(10\sqrt{10}-1).$$

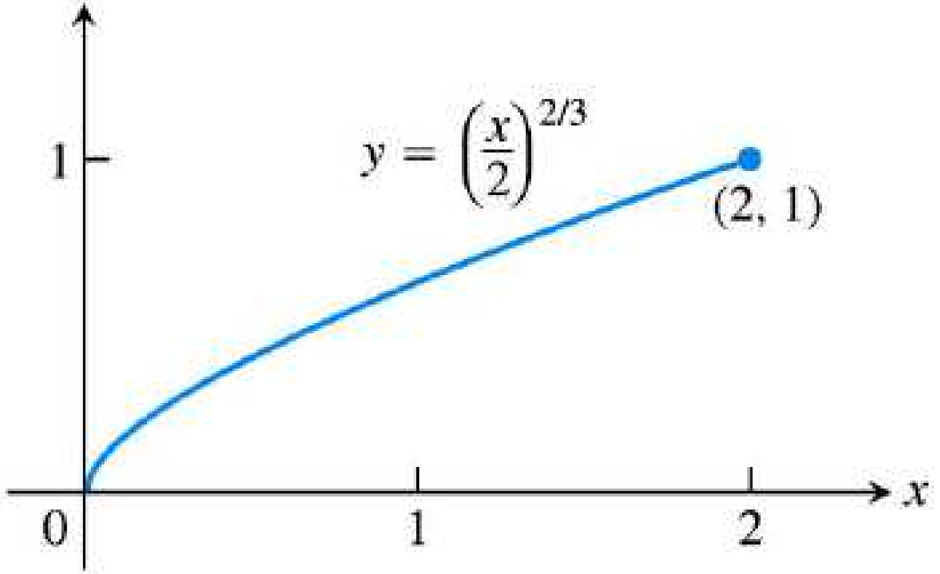

FIGURE 6.26 The graph of $y=(x/2)^{2/3}$ from $x=0$ to $x=2$ is also the graph of $x=2y^{3/2}$ from $y=0$ to $y=1$ (Example 3).

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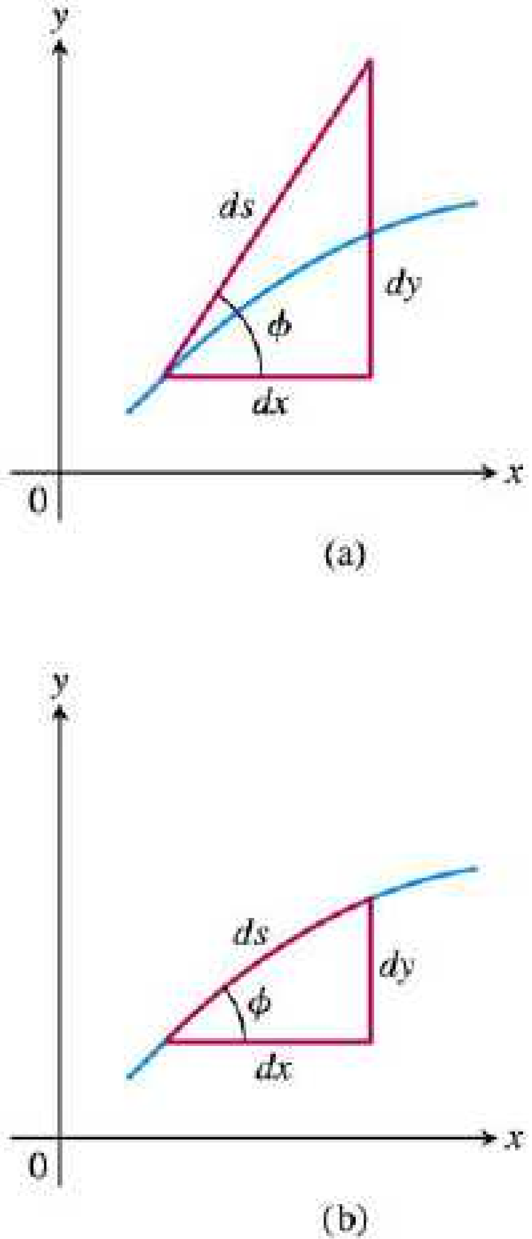

弧长微分

设$y=f(x)$在$[a,b]$上导数连续，则

$$s(x)=\int_a^x \sqrt{1+(f'(t))^2}\,dt$$

是$[a,x]$的弧长，

$$\frac{ds}{dx}=\sqrt{1+(f'(x))^2}$$

是在x处的弧长导数，

$$ds=\sqrt{1+(f'(x))^2}\,dx$$

是在x处的弧长微分.

FIGURE 6.27 Diagrams for remembering the equation $ds=\sqrt{dx^2+dy^2}$.

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$$L=\int_a^b \sqrt{1+(y')^2}\,dx \quad =\int_a^b \ ds \quad = s(x)\Big|_a^b$$

Ex.4 **Find the arc length function for the curve**

$$y=\frac{x^3}{12}+\frac{1}{x}, \qquad 1\le x\le 4.$$

Solution

$$s(x)=\int_1^x \sqrt{1+(y')^2}\,dx \ =\int_1^x \sqrt{\frac{x^4}{16}+\frac{1}{2}+\frac{1}{x^4}}\,dx$$

$$=\int_1^x \left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx \ =\frac{x^3}{12}-\frac{1}{x}+\frac{11}{12}. \qquad\qquad s(4)=6.$$

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6.4

Areas of Surfaces of Revolution

旋转面的面积

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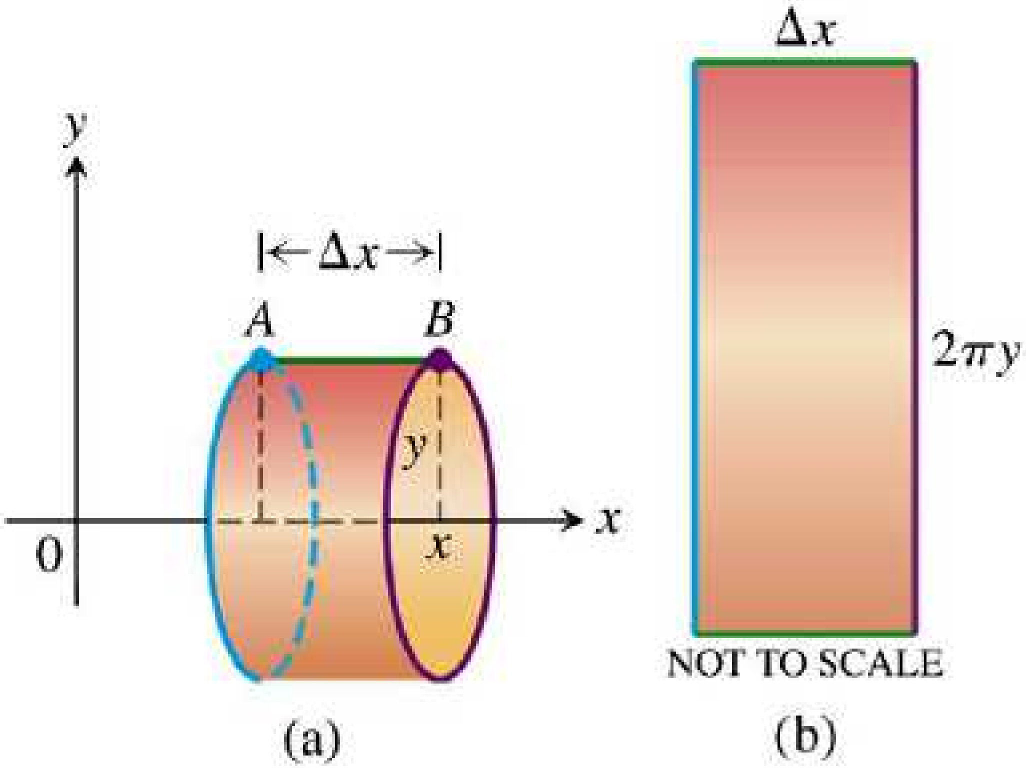

FIGURE 6.28 (a) A cylindrical surface generated by rotating the horizontal line segment AB of length Δx about the x-axis has area $2\pi y \Delta x$. (b) The cut and rolled-out cylindrical surface as a rectangle.

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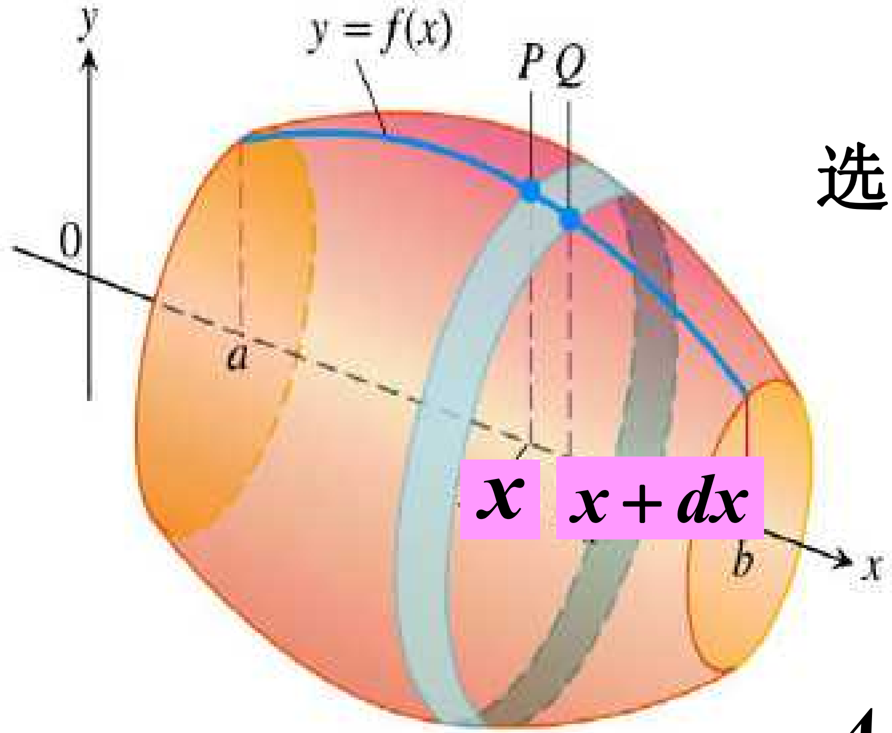

FIGURE 6.30 The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \leq x \leq b$, about the x-axis. The surface is a union of bands like the one swept out by the arc PQ.

选 x 为积分变量 $x \in [a, b]$

任取 $[x, x+dx] \subset [a, b]$

$$dA = 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$A = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

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绕x-轴的旋转面的面积公式

DEFINITION If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx. \qquad (3)$$

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$Ex.1$ **Find the area of the surface generated by revolving the curve** $y = 2\sqrt{x}, 1 \le x \le 2$, **about the x-axis.**

Solution

$$s = \int_a^b 2\pi y\sqrt{1+(y')^2}\,dx = \int_1^2 2\pi 2\sqrt{x}\,\frac{\sqrt{1+x}}{\sqrt{x}}\,dx$$

$$= 4\pi\int_1^2 \sqrt{1+x}\,dx = \frac{8\pi}{3}(3\sqrt{3}-2\sqrt{2}).$$

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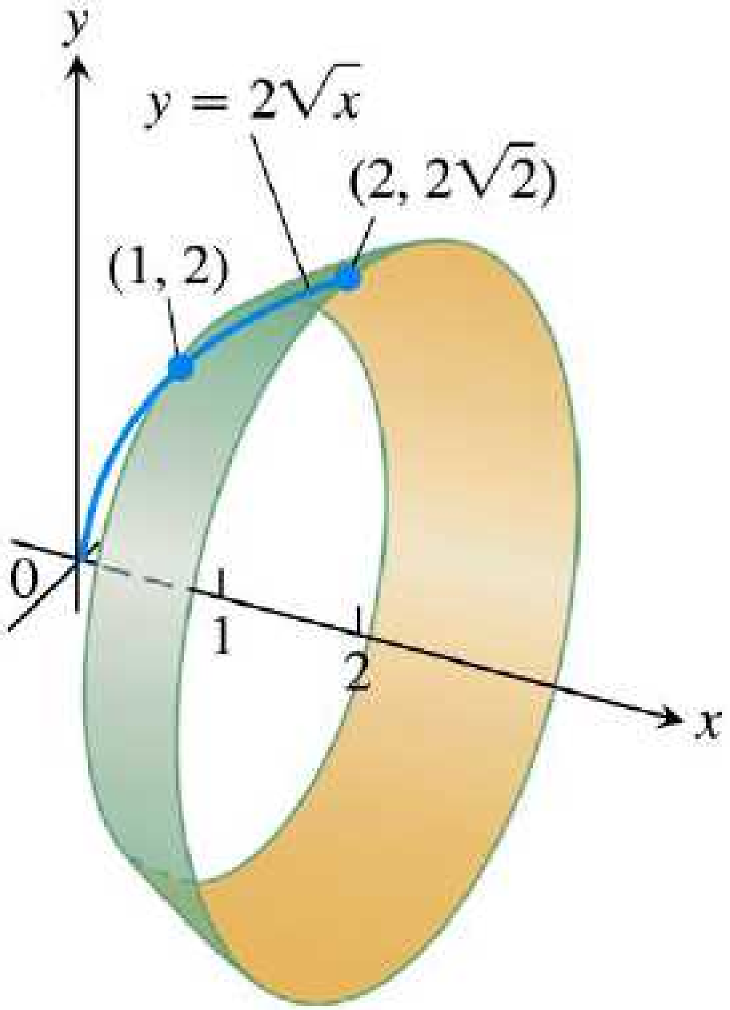

FIGURE 6.34 In Example 1 we calculate the area of this surface.

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绕y-轴的旋转面的面积公式

Surface Area for Revolution About the y-Axis

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

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$Ex.2$ **Find the area of the surface generated by revolving the curve $x = 1 - y, 0 \le y \le 1$, about the y-axis.**

Solution

$$s = \int_c^d 2\pi x\sqrt{1+(x')^2}\,dy = \int_0^1 2\pi(1-y)\sqrt{2}\,dy$$

$$= 2\sqrt{2}\pi\int_0^1 (1-y)\,dy = \pi\sqrt{2}.$$

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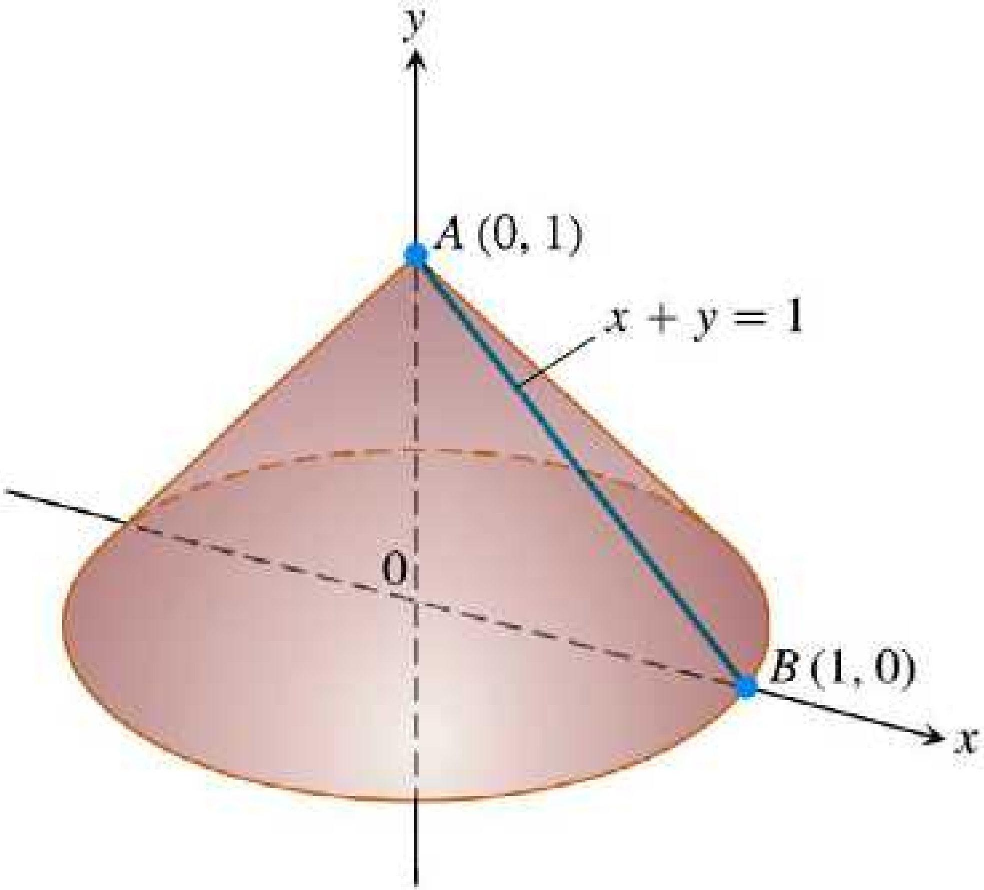

FIGURE 6.35 Revolving line segment AB about the y-axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

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6.5

Work and Fluid Forces
功和液体压力

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求变力作功举例

常力F对物体所作的功为 $W = F \cdot s$.

若物体在力 f的作用下，从 x-轴上的点 a移动到点 b，力是变化的 $f(x)$,求力所做的功 W.

选 x 为积分变量 $x \in [a, b]$

任取 $[x, x+dx] \subset [a, b]$

$$dW = f(x)dx$$

$$W = \int_a^b f(x)dx$$

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DEFINITION The **work** done by a variable force $F(x)$ in moving an object along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx. \tag{2}$$

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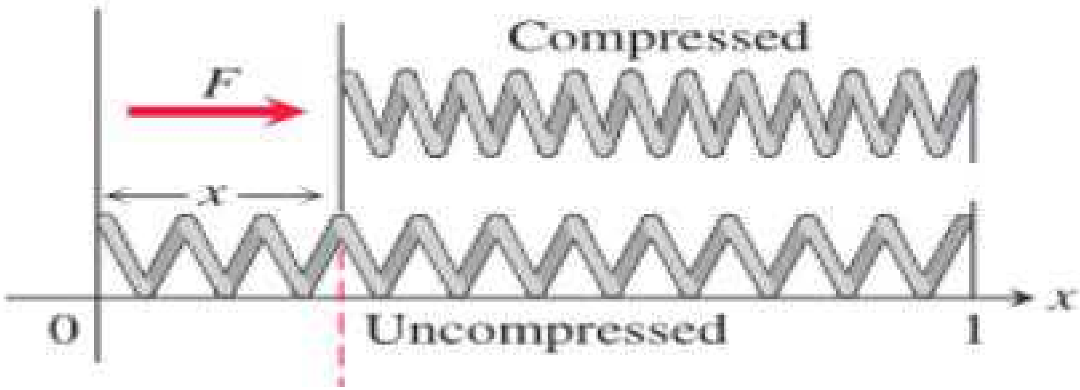
Ex .2 Find the work required to compress a spring from its natural lenth of 30cm to a lenth of 20cm if the force constant is k = 240N/m.

Solution 如图建立坐标系



由虎克定理知 $F(x) = kx = 240x,$

$$W = \int_0^{0.1} k\,x dx = \int_0^{0.1} 240x dx = 1.2(\mathrm{J}).$$

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Ex.3 A spring has a natural lenth of 1m. A force of 24N holds the spring stretched to a total lenth of 1.8m.

(a)Find the force constant.

(b)How much work will it take to stretch the spring 2m beyond its natural lenth?

(c)How far will a 45 - N force stretch the spring?

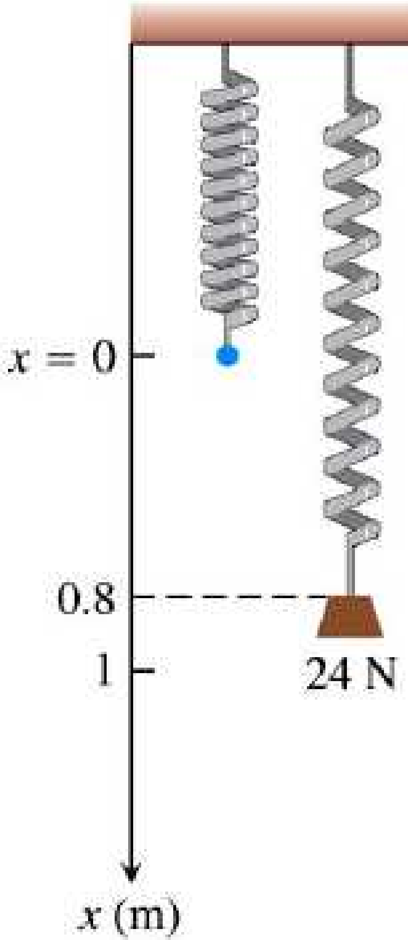

FIGURE 6.37 A 24-N weight stretches this spring 0.8 m beyond its unstressed length (Example 3).

Solution 如图建立坐标系

(a)由虎克定理知 $F(x) = kx,\ 24 = 0.8k$

$k = 30\text{N}/\text{m}$

(b)$W = \int_0^2 30x\,dx = 60(\text{J}).$

(c)$45 = 30x,\ x = 1.5\text{m}$

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Ex.4 A 2-kg bucket is lifted from the ground into the air by pulling in 6m of rope at a constant speed. The rope weighs 0.1kg/m. How much work was spent lifting the bucket and rope?

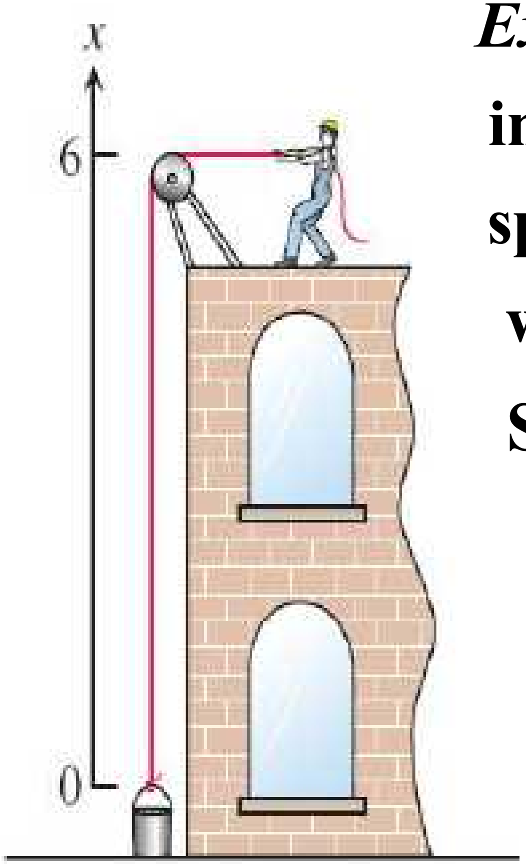

FIGURE 6.38 Lifting the bucket in Example 4.

Solution 如图建立坐标系

$$W_1 = 2\times 9.8\times 6 = 117.6(\mathrm{J}).$$

$$W_2 = \int_0^6 0.1\times 9.8\times (6-x)dx = 17.64(\mathrm{J}).$$

$$W = W_1 + W_2 = 135.24(\mathrm{J}).$$

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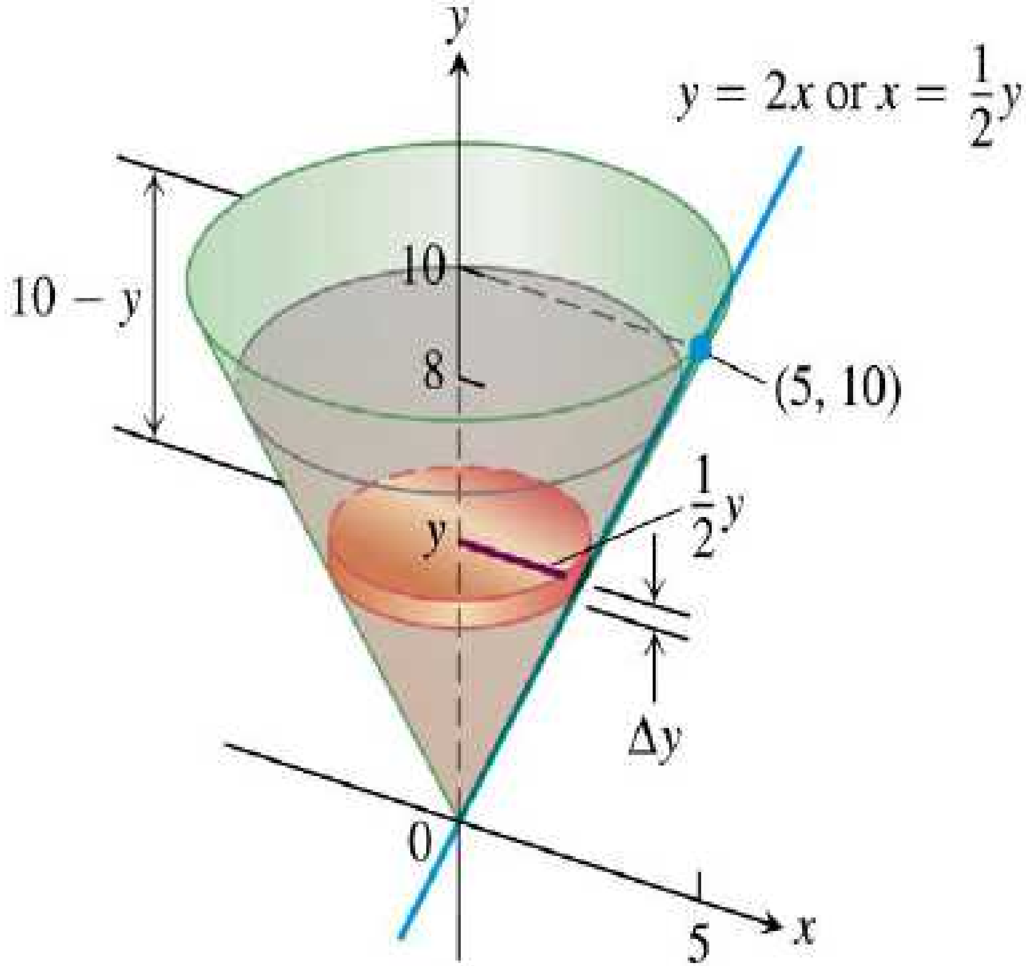

FIGURE 6.39 The olive oil and tank in Example 5.

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*Ex*5. The conical tank is filled to within 2m of the top with olive oil weighing 0.9g/cm^3 or 8820N/m^3.How much work does it take to pump the oil to the rim of the tank?

Solution 如图建立坐标系

取 y 为积分变量， $y \in [0,8]$

取任一小区间 $[y, y+dy]$,

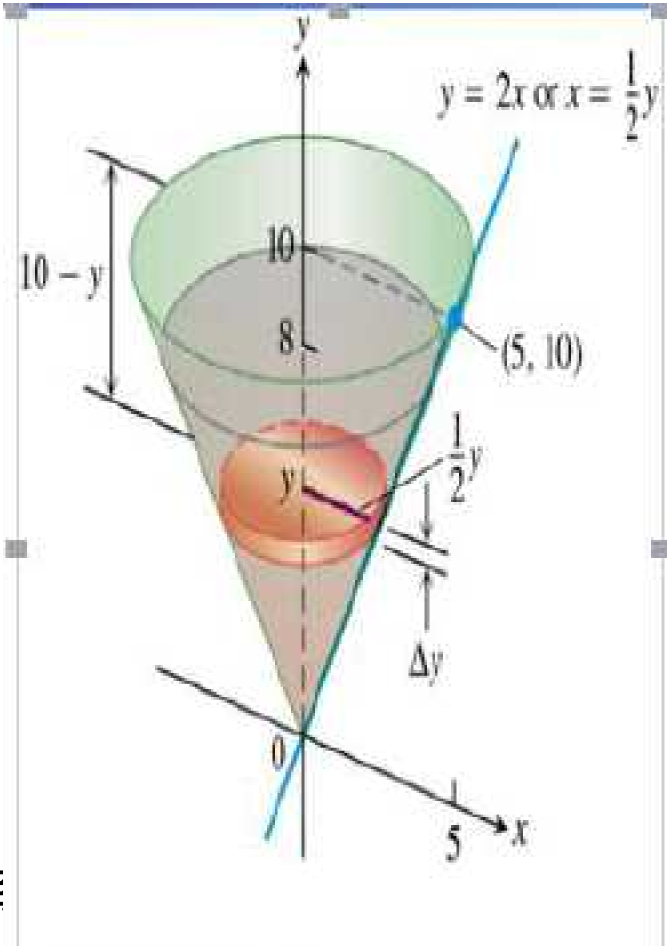
这一薄层油的重力为 $8820\cdot \pi(\frac{y}{2})^2 dy$

泵出这一薄层油的功为

$$dw = \frac{8820}{4}\pi \cdot y^2(10-y)\cdot dy,$$

$$w = 2205\pi\int_0^8 (10-y)y^2 \cdot dy$$

$$= 2205\pi\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^8 \approx 4728977 \text{ (焦}$$



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重力-密度

Weight-density

A fluid's weight-density w is its weight per unit volume. Typical values (N/m^3) are listed below.

Gasoline	6600
Mercury	133,000
Milk	10,100
Molasses	15,700
Olive oil	8820
Seawater	10,050
Freshwater	9800

The Pressure-Depth Equation

In a fluid that is standing still, the pressure p at depth h is the fluid's weight-density w times h:

$$p = wh. \qquad \text{压强（N/m}^2\text{）} \tag{4}$$

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求液体的压力

Fluid Force on a Constant-Depth Surface

$$F = pA = whA \tag{5}$$

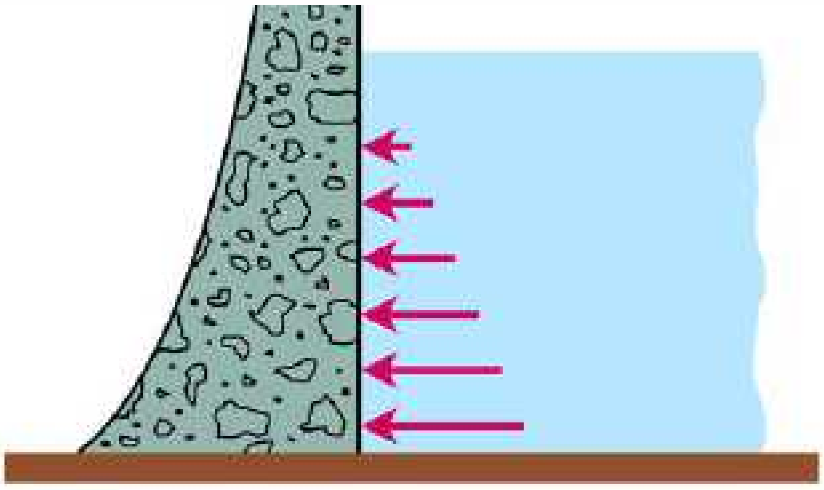

FIGURE 6.40 To withstand the increasing pressure, dams are built thicker as they go down.

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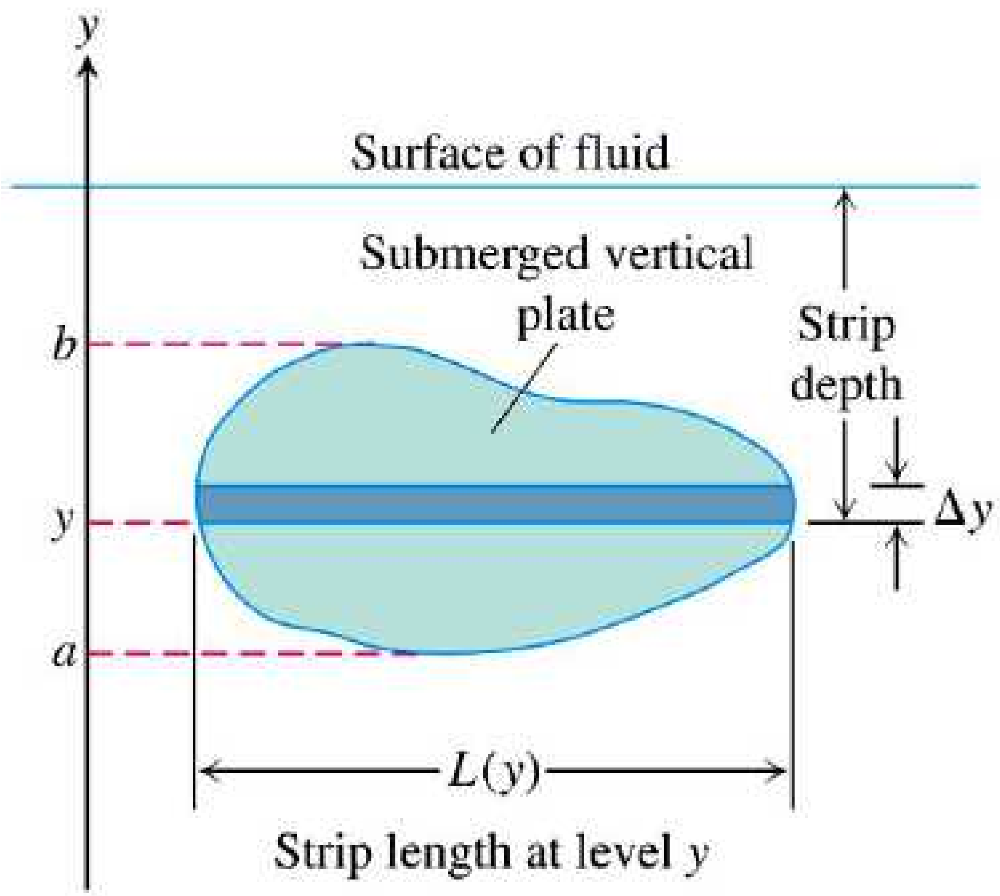

FIGURE 6.42 The force exerted by a fluid against one side of a thin, flat horizontal strip is about $\Delta F =$ pressure $\times$ area $=$ $w \times$ (strip depth) $\times L(y)\,\Delta y$.

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Fluid Force on a Constant-Depth Surface

$$F = pA = whA \tag{5}$$

$$F \approx \sum_{k=1}^{n} (w \cdot (\text{strip depth})_k \cdot L(y_k))\,\Delta y_k. \tag{6}$$

The Integral for Fluid Force Against a Vertical Flat Plate

Suppose that a plate submerged vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y-axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y. Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (\text{strip depth}) \cdot L(y)\, dy. \tag{7}$$

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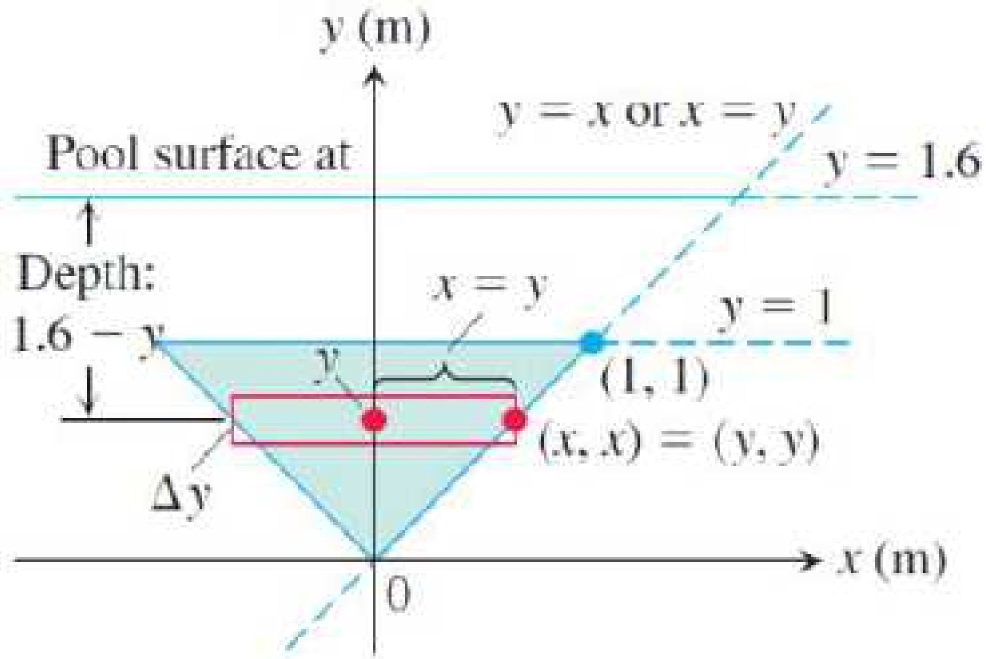
*Ex*5. A flat isosceles right - triangular plate with base 2m and height 1m is submerged vertically , base up, 0.6m below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.

Solution 如图建立坐标系



取 y 为积分变量, $y \in [0,1]$

取任一小区间 $[y, y+dy]$,

这一块板上受的压力为

$$9800 \cdot (1.6 - y) 2y dy$$

$$F = 19600 \int_0^1 (1.6y - y^2) dy \approx 9147(\text{N})$$

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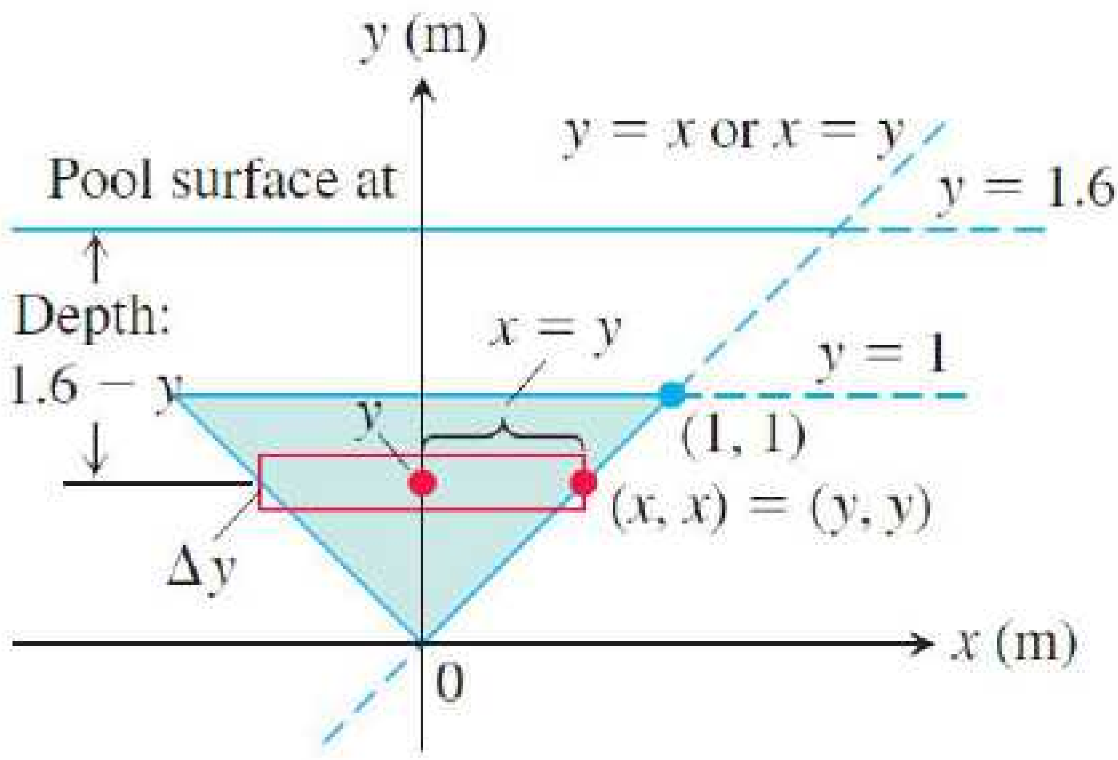

FIGURE 6.43 To find the force on one side of the submerged plate in Example 6, we can use a coordinate system like the one here.

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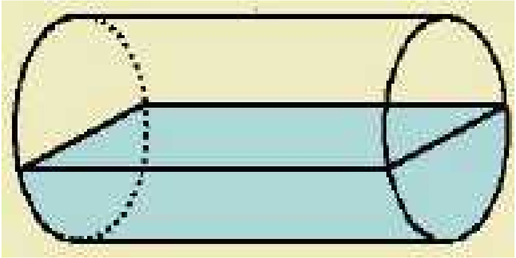
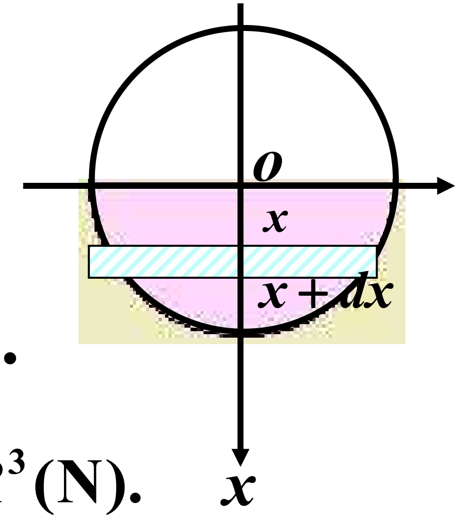
例　一个横放着的圆柱形水桶，桶内盛有半桶水，设桶的底半径为R，水的密度为$w=9800\text{N}/m^3$，计算桶的一端面上所受的压力.

解　在端面建立坐标系如图



取x为积分变量，　$x\in[0,R]$

取任一小区间$[x,x+dx]$



小矩形片上各处的压强近似相等　　$p=wx$,

小矩形片的面积为$2\sqrt{R^2-x^2}dx$.

$$P=\int_0^R 2wx\sqrt{R^2-x^2}dx \quad =\frac{2\times 9800}{3}R^3(\text{N}).$$

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6.6

Moments and Centers of Mass
矩和质心

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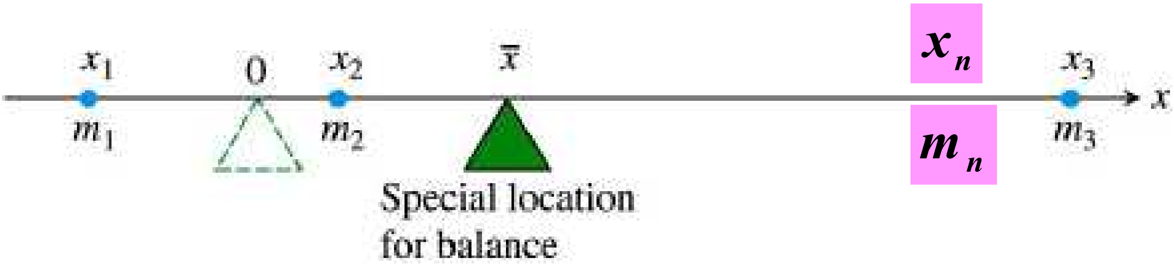
Masses Along a Line

We usually want to know where to place the fulcrum to make the system balance, that is, at what point $\bar{x}$ to place it to make the torques add to zero.



The torque of each mass about the fulcrum in this special location is

$$\text{Torque of } m_k \text{ about } \bar{x} = \begin{pmatrix}\text{signed distance} \\ \text{of } m_k \text{ from } \bar{x}\end{pmatrix}\begin{pmatrix}\text{downward} \\ \text{force}\end{pmatrix}$$

$$= (x_k - \bar{x})m_k g.$$

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整个系统产生的力矩：$\sum_{k=1}^{n} gm_k(x_k - \bar{x})$

如整个系统平衡：$\sum_{k=1}^{n} gm_k(x_k - \bar{x}) = 0$,

即$\sum_{k=1}^{n} m_k(x_k - \bar{x}) = 0$,

得质心坐标 $\bar{x} = \dfrac{\sum_{k=1}^{n} x_k m_k}{\sum_{k=1}^{n} m_k} = \dfrac{\text{系统相对于原点的矩}}{\text{系统的总质量}}$

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若一根细棍放置在x轴上$[a,b]$区间上，且线密度 $\rho(x)$

质心坐标 $\bar{x} = \dfrac{\sum_{k=1}^{n} x_k m_k}{\sum_{k=1}^{n} m_k}$

任取典型区间 $[x, x+dx] \subset [a, b]$

小段上的质量 $\rho(x)dx$，它关于原点的矩 $x\rho(x)dx$

细棍关于原点的矩 $\int_a^b x\rho(x)dx = \int_a^b xdm$，

整个细棍的质量 $\int_a^b \rho(x)dx = \int_a^b dm$，

$\therefore$ 细棍的质心坐标 $\bar{x} = \dfrac{\int_a^b x\rho(x)dx}{\int_a^b \rho(x)dx} = \dfrac{\int_a^b xdm}{\int_a^b dm}$。

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设平面质点系 $(x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)$，其上的质量依次为 $m_1, m_2, \cdots, m_n$，它的质心坐标 $(\bar{x}, \bar{y})$，

得 $$\bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k} = \frac{M_y}{M}, \qquad \bar{y} = \frac{\sum_{k=1}^{n} m_k y_k}{\sum_{k=1}^{n} m_k} = \frac{M_x}{M}.$$

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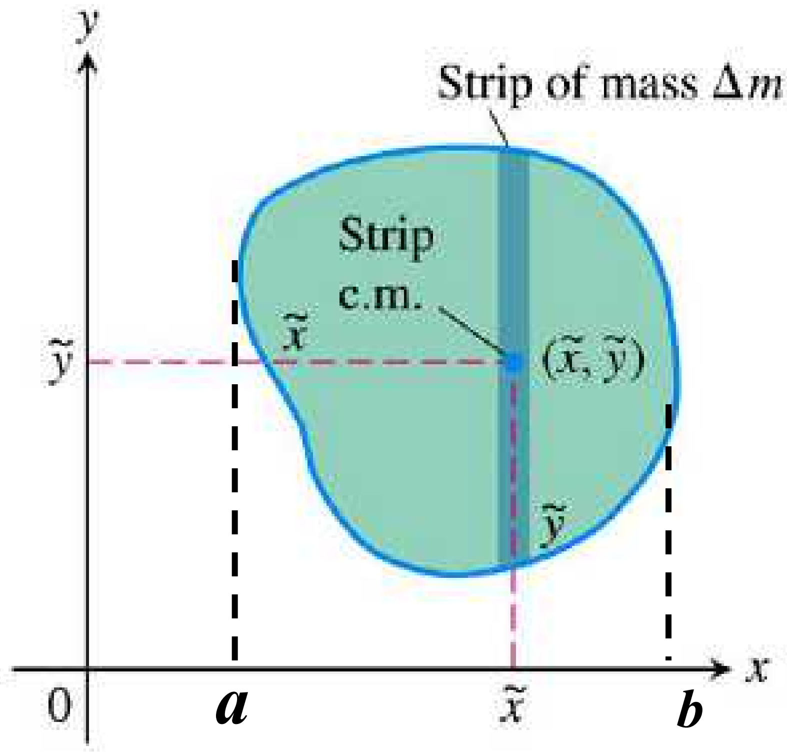

FIGURE 6.47 A plate cut into thin strips parallel to the y-axis. The moment exerted by a typical strip about each axis is the moment its mass Δm would exert if concentrated at the strip's center of mass $(\tilde{x}, \tilde{y})$.

设质量分布是均匀的

任取典型区间 $[x, x+dx] \subset [a, b]$

则这条细棍的质心 $(\tilde{x}, \tilde{y})$,

$\tilde{y}$在棍长一半的地方 .

将质量看成是集中在质 心$(\tilde{x}, \tilde{y})$

则细棍相对 y轴的矩 $\tilde{x}dm$,

整个薄片对 y轴的矩 $\int_a^b \tilde{x}dm$

整个薄片质心 x坐标$\bar{x} = \dfrac{\int_a^b \tilde{x}dm}{\int_a^b dm}$

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Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the *xy*-Plane

$$\text{Moment about the } x\text{-axis:} \quad M_x = \int \tilde{y}\, dm$$

$$\text{Moment about the } y\text{-axis:} \quad M_y = \int \tilde{x}\, dm$$

$$\text{Mass:} \quad M = \int dm \tag{5}$$

$$\text{Center of mass:} \quad \bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

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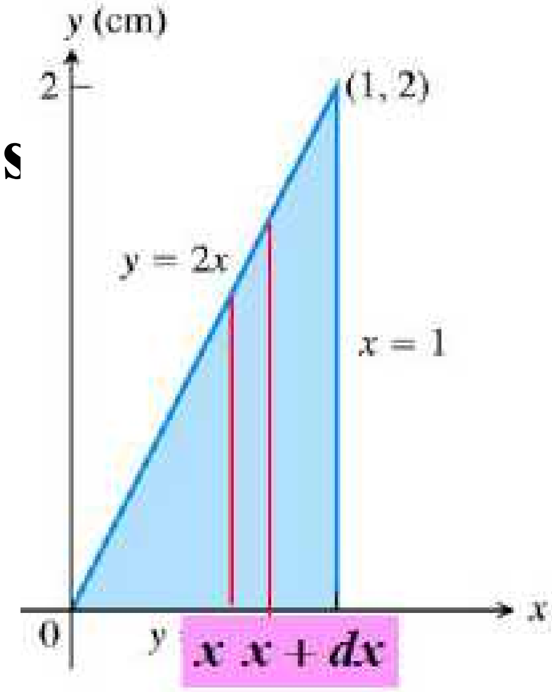
$Ex1.$ The triangular plate shown in the figure has a constant density of $\delta = 3\text{g/cm}^2$.Find

(a)the moment M_y about the y - axis.

(b)the mass M.

(c)the x - coodinate of the center of mass



Solution 如图建立坐标系

(a)取 x 为积分变量，$x \in [0,1]$

取任一小区间 $[x, x+dx]$,

这一条板的质心坐标 (x, x)

$$M_y = \int \tilde{x} dm \quad = \int_0^1 x \cdot 3 \cdot 2x dx = 6\int_0^1 x^2 dx \quad = 2(\text{g} \cdot \text{cm})$$

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(b)取 x 为积分变量，

$$M=\int dm$$

$$=\int_0^1 3\cdot 2x\,dx=6\int_0^1 x\,dx=3(g).$$

$$\text{(c)}\ \bar{x}=\frac{M_y}{M}=\frac{2}{3}(\text{cm})$$

$$\bar{y}=\frac{M_x}{M}=\frac{2}{3}$$

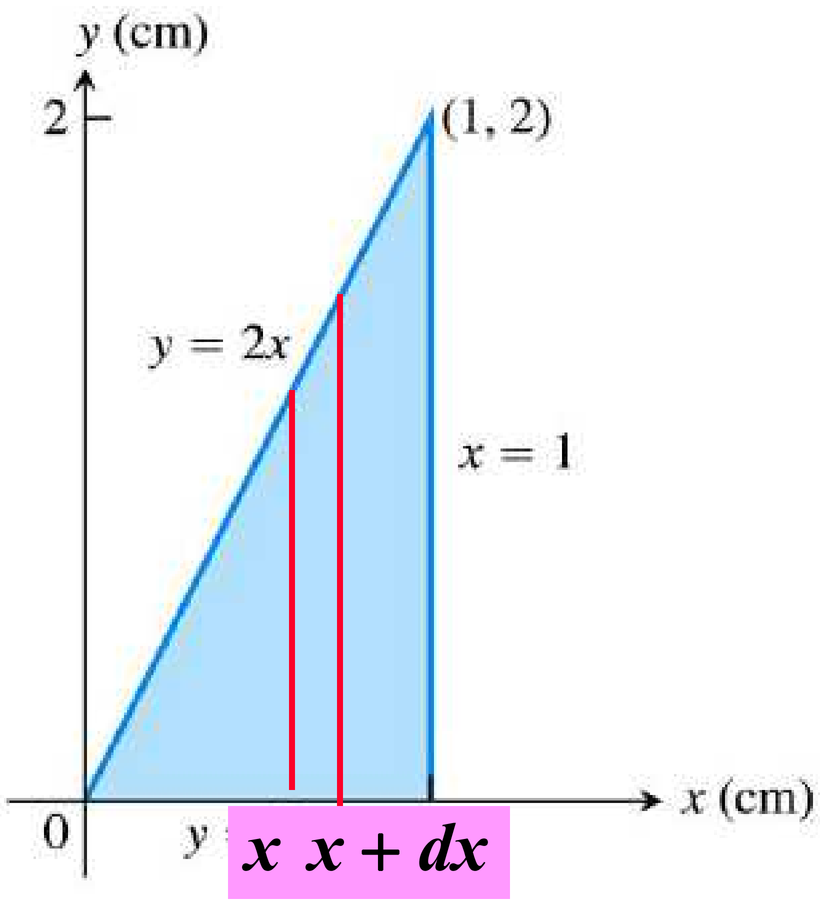

FIGURE 6.48 The plate in Example 1.

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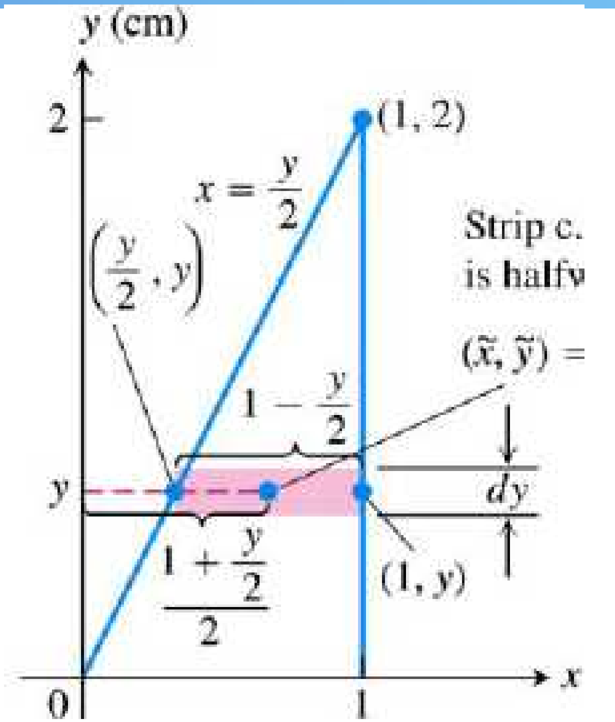
Solution

(a)取 y 为积分变量， $y\in[0,2]$

取任一小区间 $[y, y+dy]$

这一条板的质心坐标 $(\dfrac{2+y}{4}, y)$



$$M_y=\int \tilde{x}dm$$

$$=\int_0^2 \frac{2+y}{4}\cdot 3\cdot(1-\frac{y}{2})dy=\frac{3}{8}\int_0^2(4-y^2)dy=2(\mathrm{g}\cdot\mathrm{cm})$$

(b)
$$M=\int dm=\int_0^2 3\cdot(1-\frac{y}{2})dy=\frac{3}{2}\int_0^2(2-y)dy\ =3(g)$$

$$(c)\bar{x}=\frac{M_y}{M}=\ \frac{2}{3}(\mathrm{cm})$$

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*Ex*2. **Find The center of mass of a thin plate covering the region bounded above by $y = 4 - x^2$ and below by the x-axis. Assume the density of the plate at the point (x, y) is $\delta = 2x^2$.**

Solution 取 x 为积分变量，$x \in [-2,2]$

取任一小区间 $[x, x+dx]$，这一条板的质心坐标 $\left(x, \dfrac{4-x^2}{2}\right)$

$$M_x = \int \tilde{y}\,dm \qquad M = \int dm$$

$$= \int_{-2}^{2} \frac{4-x^2}{2} \cdot 2x^2 \cdot (4-x^2)dx = \int_{-2}^{2} x^2(4-x^2)^2 dx = \frac{2048}{105}$$

$$M = \int_{-2}^{2} 2x^2 \cdot (4-x^2)dx = \frac{256}{15} \qquad \bar{y} = \frac{M_x}{M} = \frac{8}{7}, \quad \bar{x} = 0.$$

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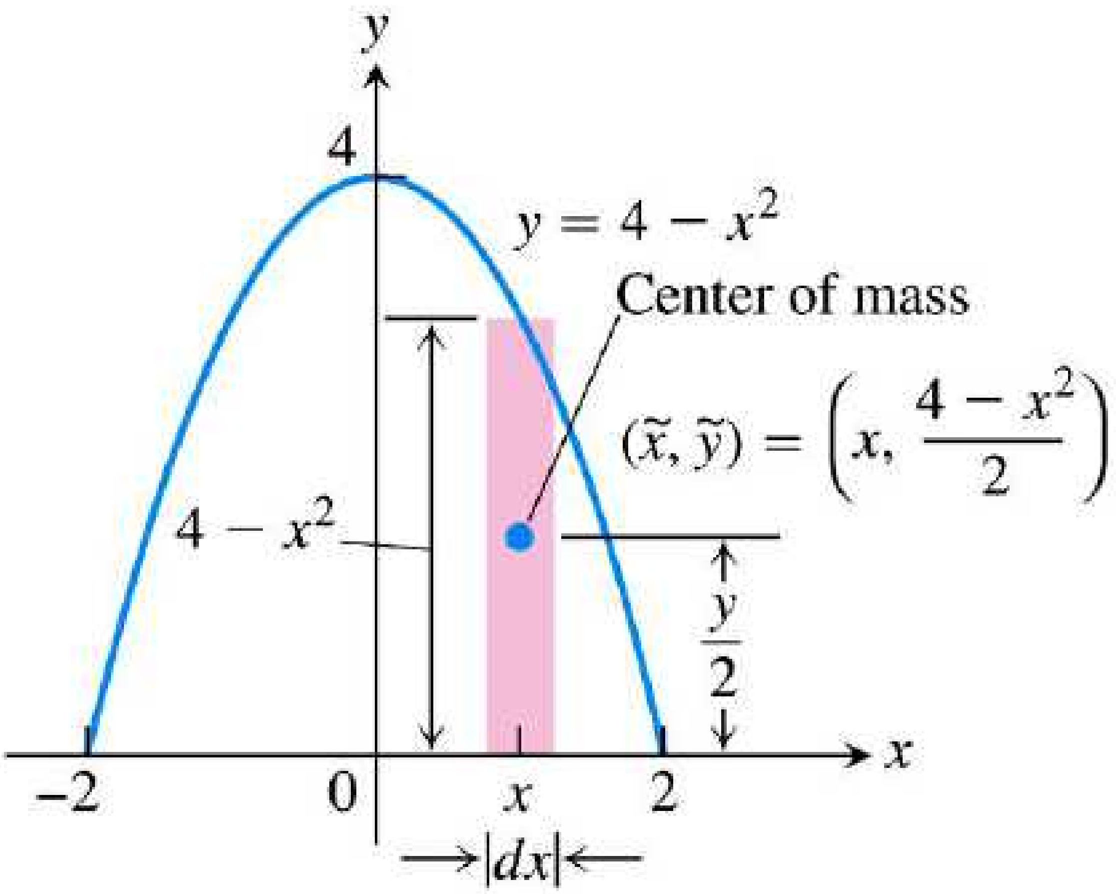

FIGURE 6.51 Modeling the plate in Example 2 with vertical strips.

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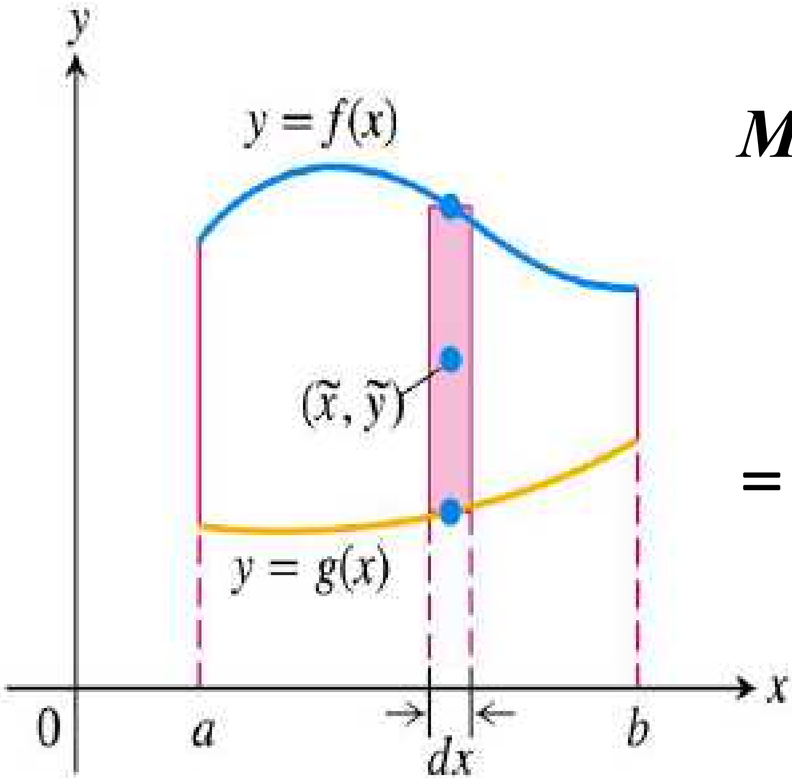

FIGURE 6.52 Modeling the plate bounded by two curves with vertical strips. The strip c.m. is halfway, so $\tilde{y} = \frac{1}{2}[f(x) + g(x)]$.

$$M_x = \int \tilde{y}\,dm$$

$$= \int \frac{f(x) + g(x)}{2}\,dm$$

$$= \int \frac{f(x) + g(x)}{2}(f(x) - g(x))\delta dx$$

$$= \int \frac{\delta}{2}(f^2(x) - g^2(x))dx$$

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$$\bar{x} = \frac{1}{M} \int_a^b \delta x \, [f(x) - g(x)] \, dx \qquad (6)$$

$$\bar{y} = \frac{1}{M} \int_a^b \frac{\delta}{2} [f^2(x) - g^2(x)] \, dx \qquad (7)$$

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$Ex3.$ **Find The center of mass for the thin plate bounded by** $g(x)=x/2$ **and** $f(x)=\sqrt{x}$ $\quad 0\le x\le 1$, **the density function of the plate at the point** (x,y) **is** $\delta(x)=x^2$.

Solution 取 x 为积分变量， $x\in[0,1]$ $\quad (x,\dfrac{\sqrt{x}+x/2}{2})$

取任一小区间$[x,x+dx]$, 这一条板的质心坐标

$$M_x=\int\tilde{y}dm \qquad M=\int dm$$

$$=\int_0^1\frac{\sqrt{x}+x/2}{2}\cdot x^2\cdot(\sqrt{x}-x/2)dx=\frac{1}{2}\int_0^1 x^2(x-\frac{x^2}{4})dx=\frac{1}{10}$$

$$M=\int_0^1 x^2\cdot(\sqrt{x}-x/2)dx=\frac{9}{56}$$

$$\boxed{\bar{y}=\frac{M_x}{M}=\frac{28}{45}.}$$

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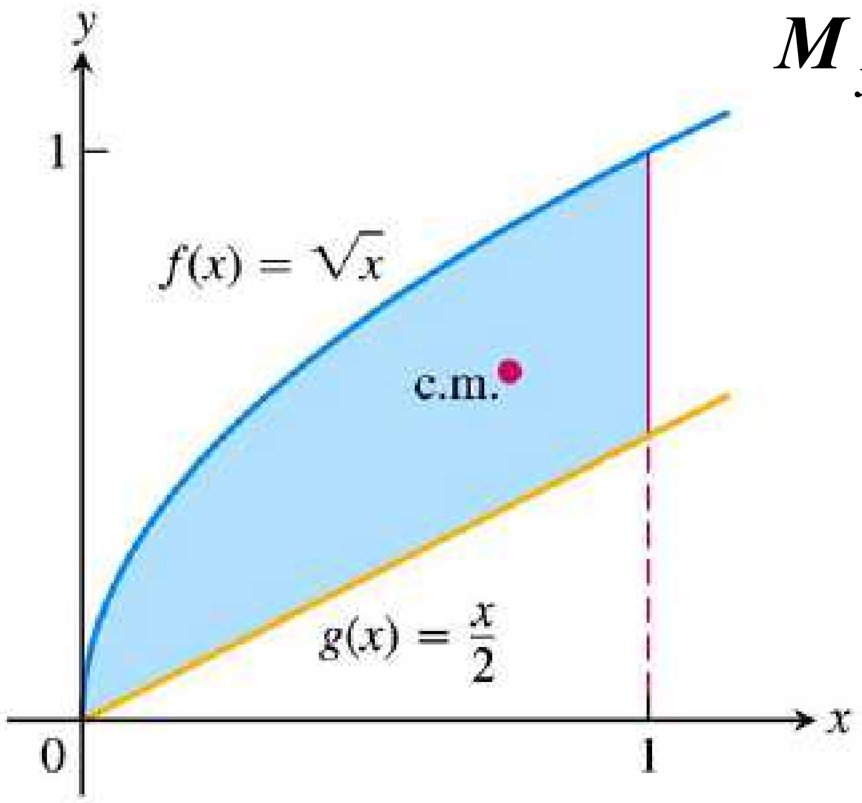

FIGURE 6.53 The region in Example 3.

$$M_y = \int \tilde{x}\,dm$$

$$= \int_0^1 x \cdot x^2 \cdot (\sqrt{x} - x/2)\,dx$$

$$= \int_0^1 x^3 (\sqrt{x} - \frac{x}{2})\,dx = \frac{11}{90}$$

$$\bar{x} = \frac{M_y}{M} = \frac{308}{405}.$$

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$$\bar{x} = \frac{M_y}{M} = \frac{\int \tilde{x}\,dm}{\int dm} = \frac{\int \tilde{x}\delta\,dA}{\int \delta\,dA} = \frac{\int \tilde{x}\,dA}{\int dA}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\int \tilde{y}\,dA}{\int dA}$$

几何中心----形心

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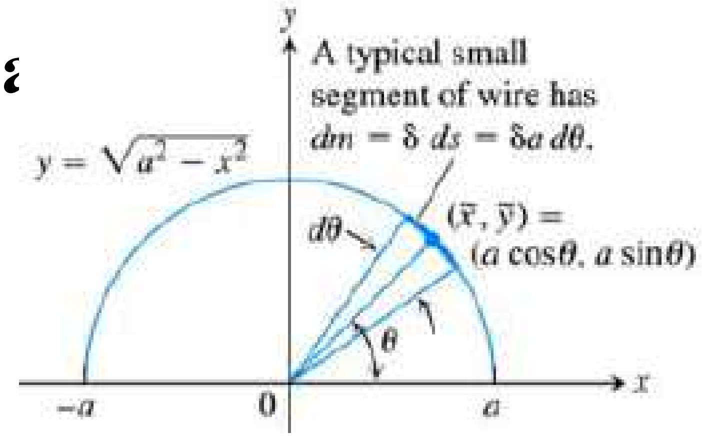
$Ex\,4.$ **Find The center of mass of a thin wire of constant density δ shaped like a semicircle of radius a.**



Solution 建立坐标系如图.

取 θ 为积分变量, $\theta \in [0, \pi]$

取任一小区间 $[\theta, \theta + d\theta]$, 这一段线的质心坐标

$(a\cos\theta, a\sin\theta)$

$$M_x = \int \tilde{y}\, dm$$

$$= \int_0^{\pi} a\sin\theta \cdot \delta a\, d\theta = a^2\delta \int_0^{\pi} \sin\theta\, d\theta = 2a^2\delta$$

$$M = \int dm = \int_0^{\pi} \delta a\, d\theta = \delta a \pi \qquad \bar{y} = \frac{2a}{\pi}.$$

形心$(0, \dfrac{2a}{\pi})$.

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取 x 为积分变量，$x \in [-a, a]$

取任一小区间 $[x, x+dx]$，

这一段线的质心坐标 $(x, \sqrt{a^2 - x^2})$

$$M_x = \int \tilde{y} dm = \int_{-a}^{a} \sqrt{a^2 - x^2} \delta \sqrt{1 + (y')^2} dx = a\delta \int_{-a}^{a} 1 dx$$

$$= 2a^2 \delta$$

$$M = \int dm = \pi a \delta \qquad \bar{y} = \frac{2a}{\pi}.$$

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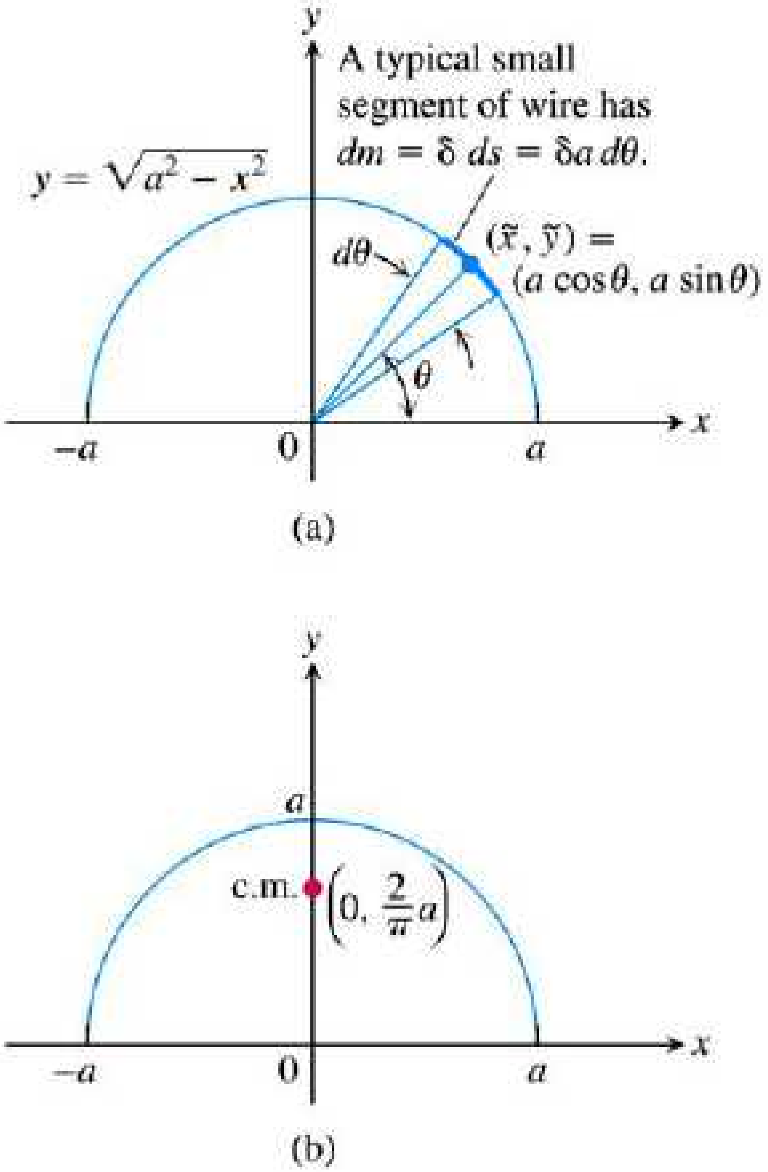

FIGURE 6.54 The semicircular wire in Example 4. (a) The dimensions and variables used in finding the center of mass. (b) The center of mass does not lie on the wire.

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如图建立坐标系

取 y 为积分变量，$y \in [a,b]$

取任一小区间 $[y, y+dy]$,

这一条板上受的压力为

$$w \cdot ydA$$

$$F = w\int_a^b ydA$$

$$\frac{\int_a^b ydA}{A} = \bar{y}, \int_a^b ydA = \bar{y}A,$$

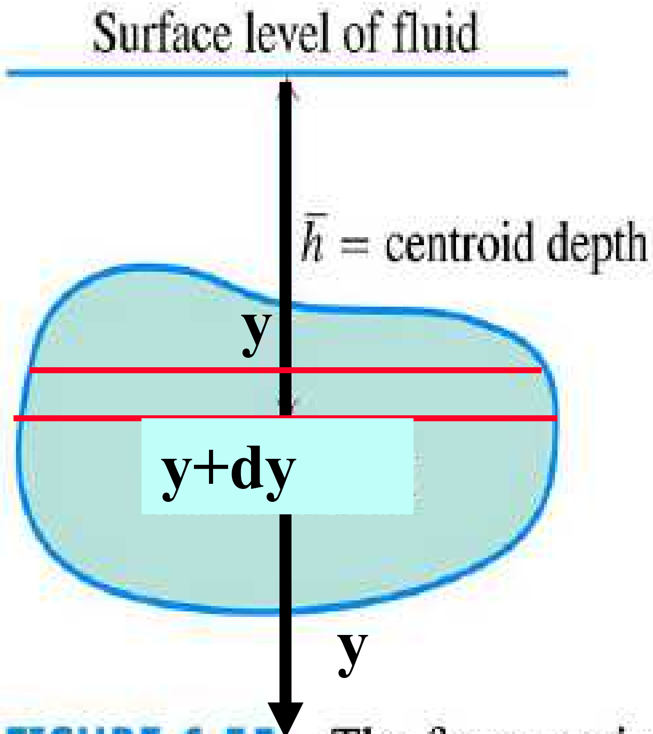

FIGURE 6.55 The force against one side of the plate is $w \cdot \bar{h} \cdot$ plate area.

$$F = w\bar{y}A$$

$\bar{y}$是薄板的几何中心在水下的深度

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Fluid Forces and Centroids

The force of a fluid of weight-density w against one side of a submerged flat vertical plate is the product of w, the distance $\bar{h}$ from the plate's centroid to the fluid surface, and the plate's area:

$$F = w\bar{h}A. \tag{8}$$

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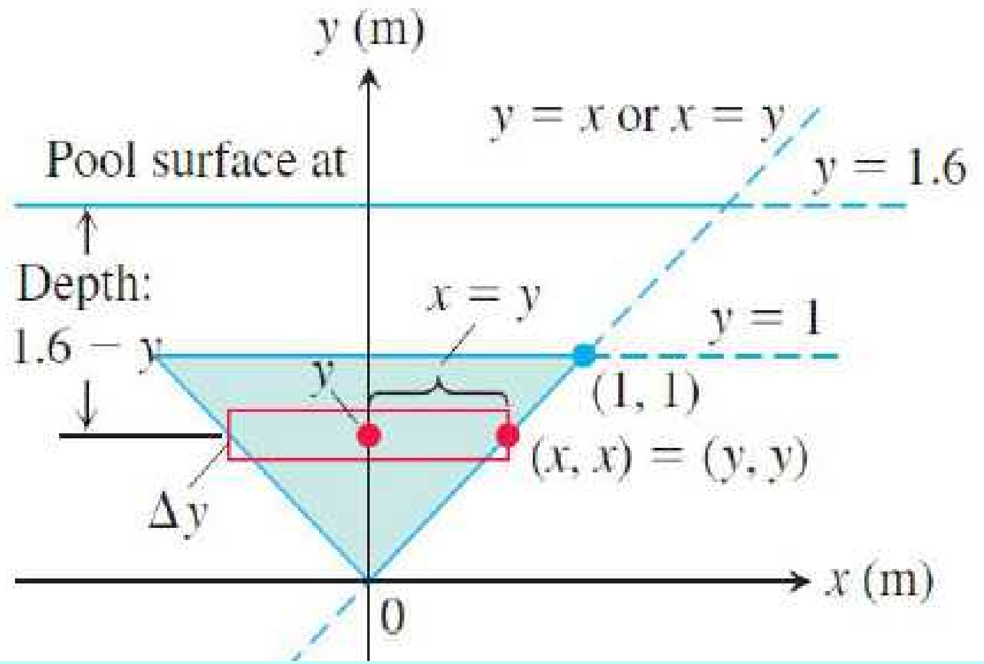
$$\bar{h} = \frac{1}{3} + 0.6 = \frac{14}{15}$$

$$F = 9800 \cdot \frac{14}{15} \cdot 1 \approx 9147(\mathrm{N})$$



***Ex*5.** **A flat isosceles right-triangular plate with base 2m and height 1m is submerged vertically, base up, 0.6m below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.**

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$$V = \int_c^d 2\pi \, y dA$$

$$\frac{\int_a^b y dA}{A} = \bar{y}, \int_a^b y dA = \bar{y} A,$$

$$V = \int_c^d 2\pi \, y dA = 2\pi A \bar{y}$$

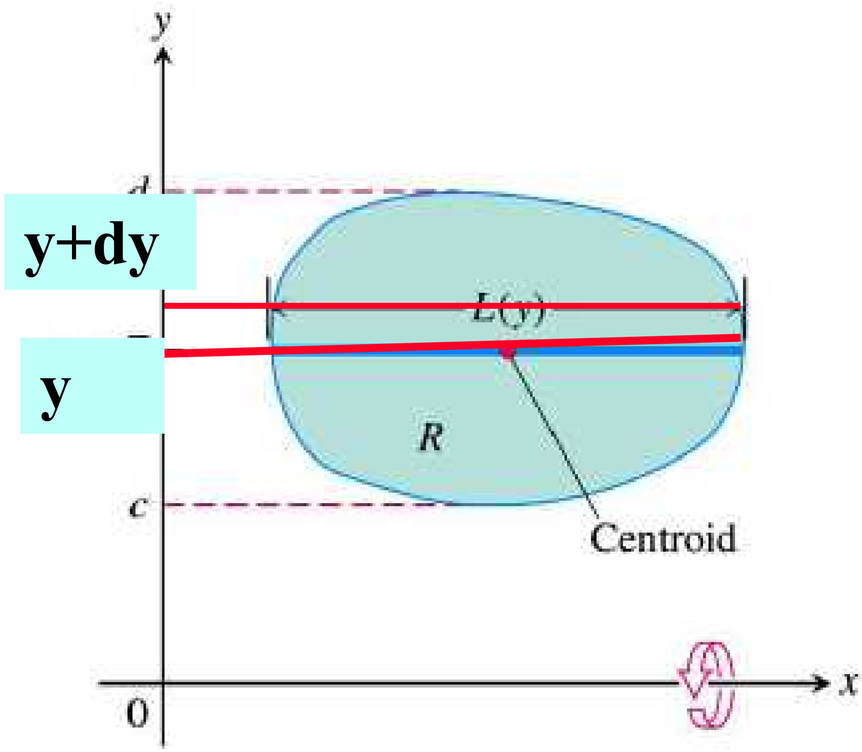

FIGURE 6.56 The region R is to be revolved (once) about the x-axis to generate a solid. A 1700-year-old theorem says that the solid's volume can be calculated by multiplying the region's area by the distance traveled by its centroid during the revolution.

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THEOREM 1 Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi\rho A. \tag{9}$$

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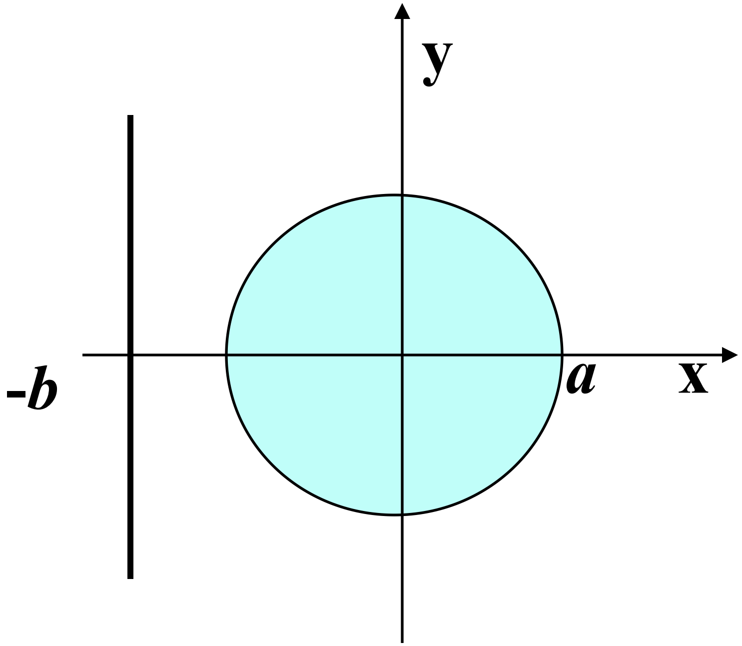
$Ex6.$ **Find The volum of the torus(doughnut) generated by revolving a circular disk of radius a about an axis in its plane at a distance $b \geq a$ from its center.**



Solution 建立坐标系如图.

$$V = 2\pi A\rho,$$

$$= 2\pi\pi a^2 b = 2a^2 b\pi^2.$$

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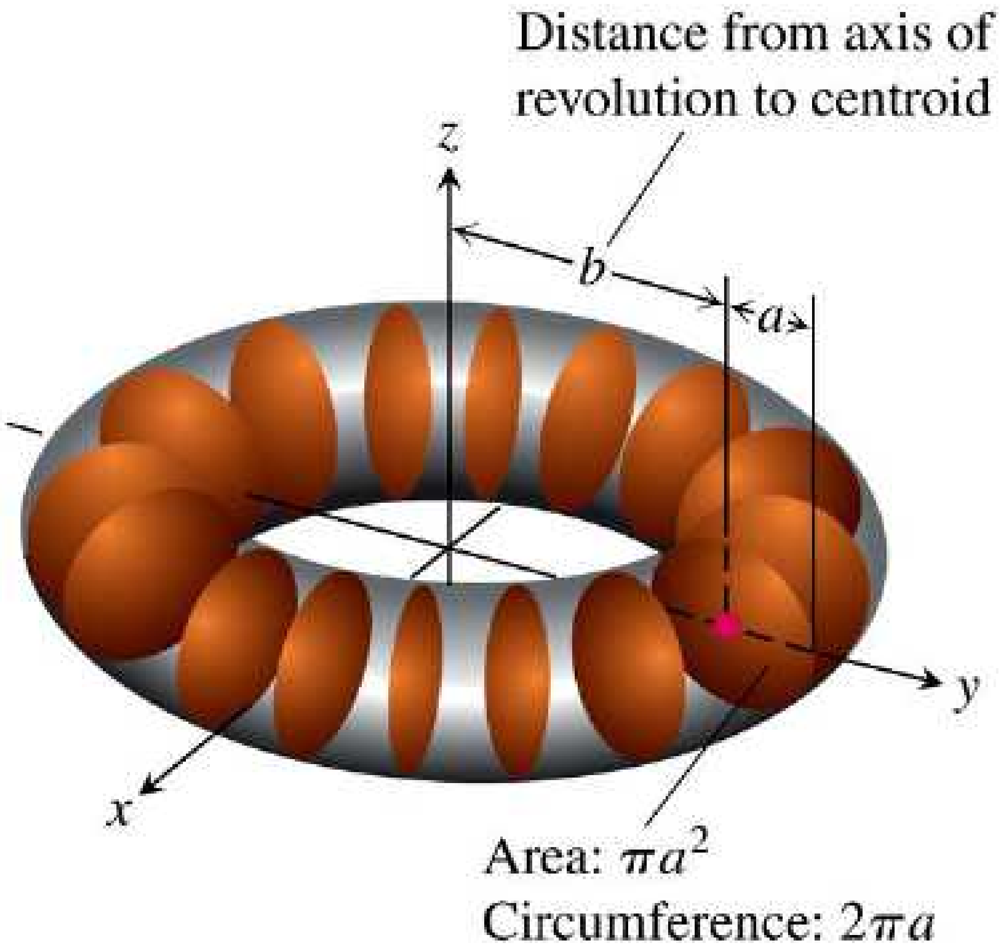

FIGURE 6.57 With Pappus's first theorem, we can find the volume of a torus without having to integrate (Example 6).

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$Ex6.$ Locate the centroid of a semicircular region of radius a.

Solution 建立坐标系如图.

$$V = 2\pi A\rho,$$

$$\rho = \frac{V}{2\pi A}$$

$$= \frac{\frac{4}{3}\pi a^3}{2\pi \frac{\pi a^2}{2}} = \frac{4a}{3\pi}.$$

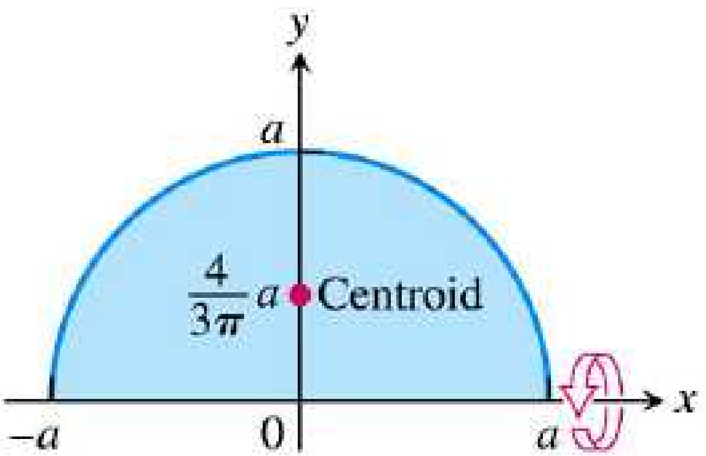

FIGURE 6.58 With Pappus's first theorem, we can locate the centroid of a semicircular region without having to integrate (Example 7).

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$$S = \int_a^b 2\pi\, y ds$$

$$\frac{\int_a^b y ds}{L} = \bar{y}, \int_a^b y dA = \bar{y} L,$$

$$S = \int_a^b 2\pi\, y ds = 2\pi \bar{y} L$$

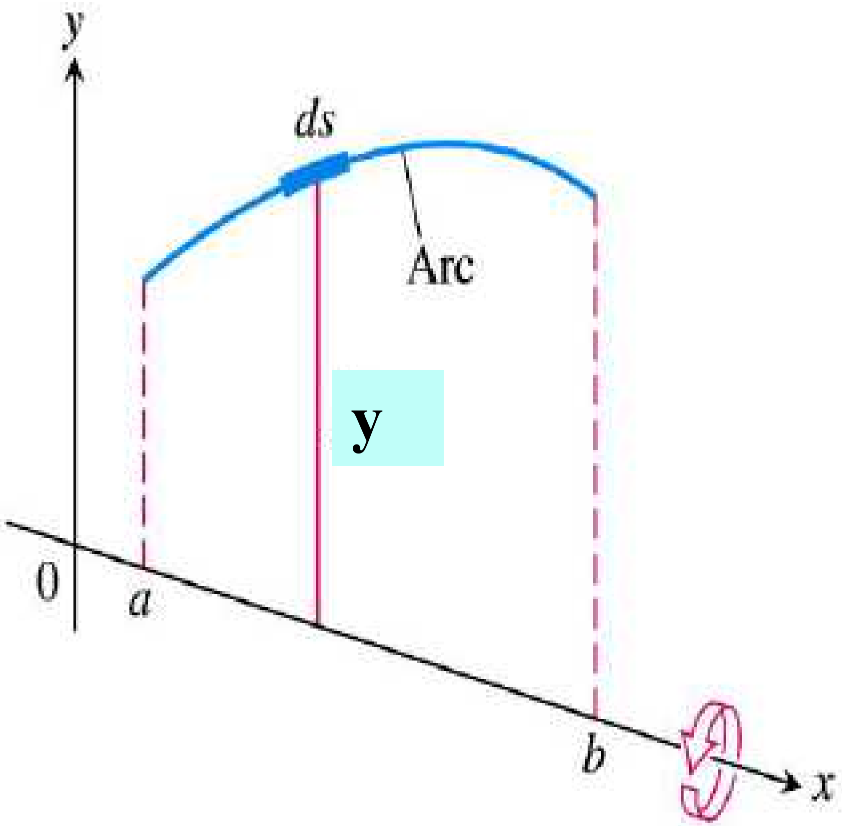

FIGURE 6.59 Figure for proving Pappus's Theorem for surface area. The arc length differential *ds* is given by Equation (6) in Section 6.3.

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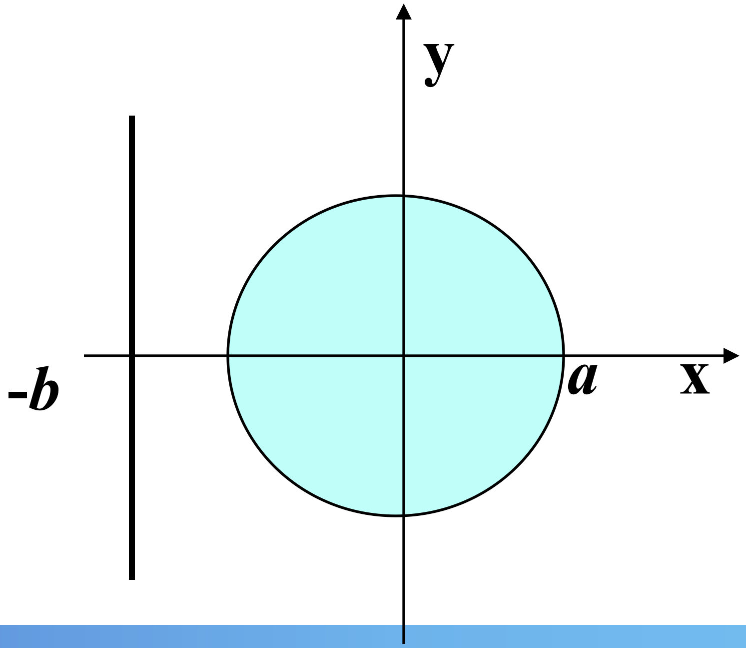
THEOREM 2 Pappus's Theorem for Surface Areas

If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L. \tag{11}$$

Ex8. Find the surface area of the torus(doughnut) in Example 6.

$$S = 2\pi\rho L$$
$$= 2\pi b 2\pi a$$
$$= 4ab\pi^2.$$



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