Chapter 5: Law of Large Numbers and Limit Theorem

1. The Law of Large Numbers (大数定律)

2. The Central Limit Theorem (中心极限定理)

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Background of The Law of Large Numbers

Question: How does the concept of "probability" evolve ?

Random Trials → Statistical data → Statistical regularity → Frequency stability → Probability

?

Frequency stability (频率稳定性):

Assume that the number of times event A happens during n repeated independent trials is n_A. When $n \to \infty$,

$$\xi_n = \frac{n_A}{n} \longrightarrow p$$

Random variable

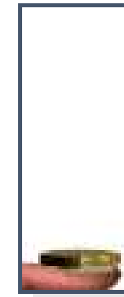
Frequency

n-independent Bernoulli trials

Probability $P(A)$

Question: How to define the limit (极限) $\lim_{n\to\infty} \xi_n = p$?

Question: What is the strict mathematical description of "Frequency stability" (频率稳定性) ?



Example: Coin tossing trials Toss a coin n times, let

$$A = \{\text{Head}\}$$

$$n_A = \text{The total number of A in n trials}$$

The frequency of A is

$$\xi_n = \frac{n_A}{n} \ (n = 1, 2, \cdots)$$

18th-19th century famous coin tossing tests

Tester	n	n_A	ξ_n
Buffon	4048	2048	0.5069
De Morgan	2048	1061	0.5181
Pearson	12000	6019	0.5016
Pearson	24000	12012	0.5005

Frequency stability (频率稳定性): $\xi_n \to 0.5 \ (n \to \infty)$

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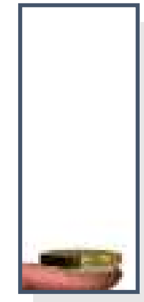


① $\{\xi_n\}_{n=1}^{\infty}$ is a list of random variables (随机变量列).

Simulation trials of n=4048

Trial results: ● ○ ○ ● ● ○ ○ ○ ● ○ ● ● ○ ● ○ ○ …… ●

$\xi_n = \dfrac{n_A}{n}$: $\frac{1}{1}$ $\frac{1}{2}$ $\frac{1}{3}$ $\frac{2}{4}$ $\frac{3}{5}$ $\frac{3}{6}$ $\frac{3}{7}$ $\frac{3}{8}$ $\frac{4}{9}$ $\frac{4}{10}$ $\frac{5}{11}$ $\frac{6}{12}$ $\frac{6}{13}$ $\frac{7}{14}$ $\frac{7}{15}$ $\frac{7}{16}$ …… $\frac{2021}{4048}$

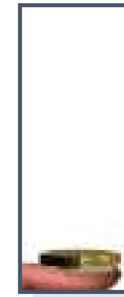
Is it possible to have the following results?

● ● ● ● ● ● ● ● ● ● ● ● ● ● ● ● …… ●

○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ …… ○

② $\{\xi_n = \xi_n(\omega)\}_{n=1}^{\infty}$ is a list of functions (函数列) defined on the sample space Ω.

Example: Coin tossing trials Toss a coin n times, let



$$A=\{\text{Head}\}$$

$$n_A=\text{The total number of A in n trials}$$

The frequency of A is

$$\xi_n=\frac{n_A}{n}\ (n=1,2,\cdots)$$

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Review the convergence (收敛性) of a function list in calculus

If functions $f(x), f_n(x), (n=1,2,\dots)$ are defined on the interval (a,b).

Then $f_n(x)$ converges to $f(x)$ means that for $\forall x\in(a,b)$:

$$\lim_{n\to\infty} f_n(x)=f(x)$$

Pointwise convergence (逐点收敛)

For a list of random variables (随机变量列), do we have

$$\lim_{n\to\infty}\xi_n(\omega)=p\ \ (\forall\omega\in\Omega)$$

?

Definition: Assume that $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ is a list of random variables. If for $\forall \varepsilon > 0$:

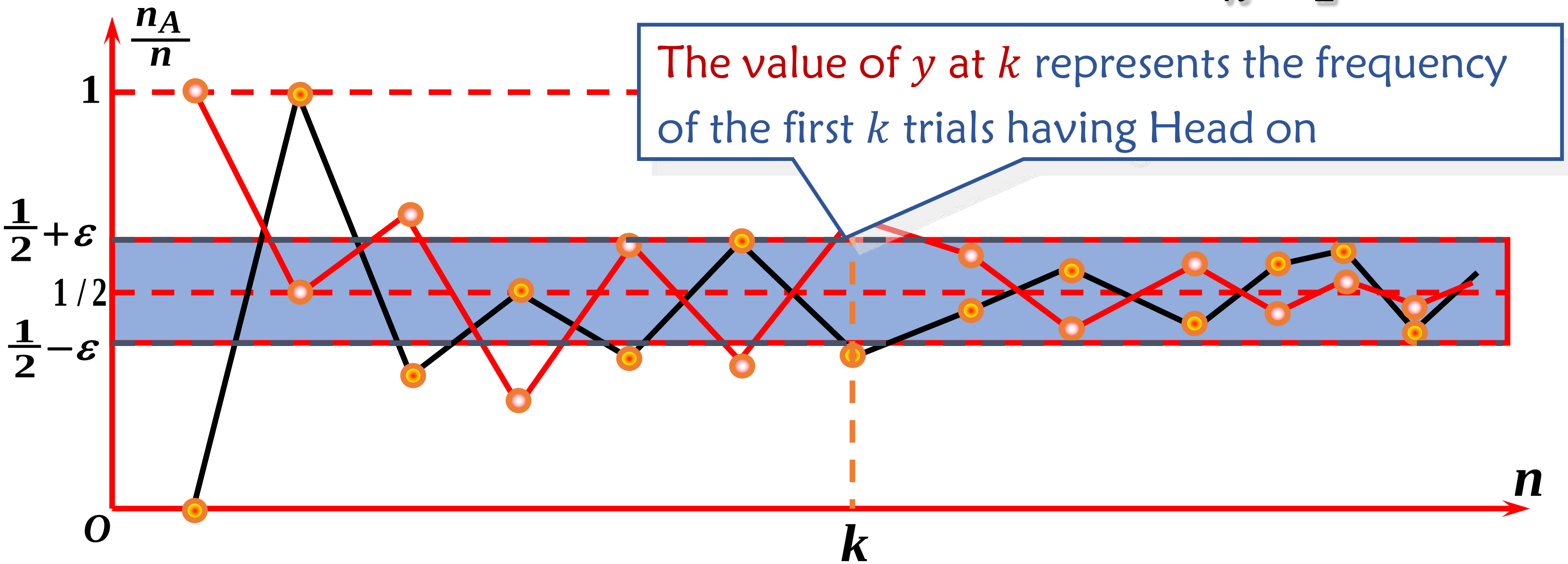
$$\lim_{n\to\infty} P\{|\xi_n - \xi| \geq \varepsilon\} = 0$$

Then $\{\xi_n\}$ is said to converge in probability (依概率收敛) to ξ, denoted as $\xi_n \xrightarrow{P} \xi$.

① $\lim_{n\to\infty} P\{|\xi_n - \xi| \geq \varepsilon\} = 0 \Leftrightarrow \lim_{n\to\infty} P\{|\xi_n - \xi| < \varepsilon\} = 1$

② The meaning of $\xi_n \xrightarrow{P} \xi$: as n increases, the probability that the absolute error (绝对误差) $|\xi_n - \xi|$ being large will become smaller.

③ The frequency stability of a coin tossing trials is $\xi_n = \dfrac{n_A}{n} \xrightarrow{P} \dfrac{1}{2}$.



Theorem 1: Bernoulli's law of large numbers (伯努利大数定律)

Assume that n_A is the number of times event A happens in n trials, and $P(A)=p$.
Then for $\forall\varepsilon>0$,

$$\lim_{n\to\infty} P\left\{\left|\frac{n_A}{n}-p\right|\geq\varepsilon\right\}=0.$$

How to prove ?

Analysis Let

$$X_i=\begin{cases}1, & A \text{ happened at the } i\text{th trial}\\ 0, & A \text{ didn't happend at the } i\text{th trail}\end{cases},\quad i=1,2,\cdots$$

Then ① $X_1,X_2,\ldots,X_n$ are independent.

② $E(X_i)=p, D(X_i)=p(1-p), (i=1,2,\ldots)$

③ $\dfrac{n_A}{n}=\dfrac{1}{n}\sum_{i=1}^{n}X_i \quad (n=1,2,\cdots)$

$\{X_n\}$ is a list of independent random variables (随机变量列); $\{X_i\}$ has the same mathematical expectation and variance.

Therefore

$$\lim_{n\to\infty} P\left\{\left|\frac{n_A}{n}-p\right|\geq\varepsilon\right\}=0. \iff \lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}X_i-p\right|\geq\varepsilon\right\}=0.$$

Theorem 1: Bernoulli's law of large numbers (伯努利大数定律)

Assume that n_A is the number of times event A happens in n trials, and $P(A) = p$. Then for $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} P\left\{\left|\frac{n_A}{n} - p\right| \geq \varepsilon\right\} = 0.$$

Theorem 2: Chebyshev's law of large numbers (切比雪夫大数定律)

Assume that $\{X_n\}$ is a list of independent random variables and they have the same mathematical expectation and variance:

$$E(X_i) = \mu, D(X_i) = \sigma^2, i = 1, 2, \cdots$$

Then for $\forall \varepsilon > 0$:

$$\lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon\right\} = 0.$$

Recall the Chebyshev's inequality

Assume that the variance of the random variable ξ exists, then for $\forall \varepsilon > 0$:

$$P\{|\xi - E(\xi)| \geq \varepsilon\} \leq \frac{D(\xi)}{\varepsilon^2}$$

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Note Both Bernoulli's LLN and Chebyshev's LLN require the **same variance** of a random variable list. It can be replaced by "identically distributed".

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Theorem 3: Khinchine's law of large numbers (辛钦大数定律)

Assume that $\{X_n\}$ is an independent and identically distributed r.v. list, and $E(X_1) \triangleq \mu$ exists. Then $\{X_n\}$ follows the law of large numbers. That is, for $\forall \varepsilon > 0$:

$$\lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| < \varepsilon\right\} = 1 \quad \text{or} \quad \lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon\right\} = 0.$$

The significance of the Law of Large Numbers

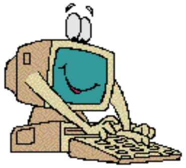
① It gives the strict mathematical explanation for "frequency stability" (频率稳定性)

$$\lim_{n\to\infty} P\{|\xi_n - \xi| \geq \varepsilon\} = 0 \Leftrightarrow \lim_{n\to\infty} P\{|\xi_n - \xi| < \varepsilon\} = 1$$

② It provides a method to find out the probability of an event by using trials .

③ It is one of the major theoretical basis for parameter estimation in mathematical statistics.

④ It is the major mathematical theory basis of the Monte Carlo method (蒙特卡洛法).

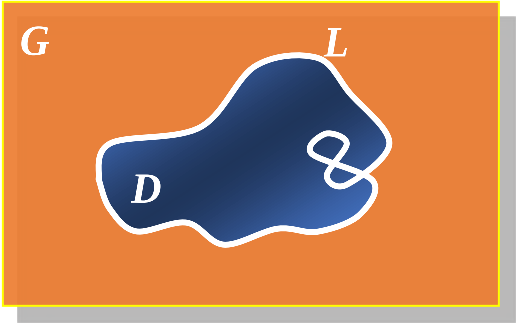


Application of The Law of Large Numbers
--The Monte Carlo Method
(蒙特卡洛法)

The Monte Carlo Method (or random computational simulation method, computational simulation method) is one of the most important tools for science and engineering.

The principle of the Monte Carlo Method is the Law of Large numbers.

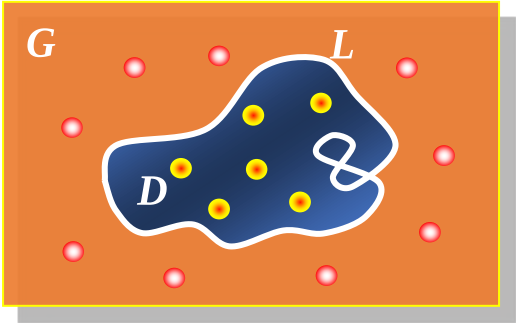
Example: There is a rectangle G. Assume that its area is 1. Draw a closed curve L within the rectangle. What is the area |D| in the enclosed area by L?



Example: There is a rectangle G. Assume that its area is 1. Draw a closed curve L within the rectangle. What is the area |D| in the enclosed area by L?

Answer: Use a computer to generate random points (random variables) which are independent, and each random variable follows the uniform distribution on G:

$$(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$$



Denote event A as $A = \{\text{a randomly generated point falls into } D\}$ and let n_A be the number of times event A happens among $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$.

According to the Bernoulli's Law of Large Numbers

$$\frac{n_A}{n} \xrightarrow{P} P(A) = \frac{\text{The area of } D}{\text{The area of } G} = |D|$$

Thus, if n is large enough, the area of D is $|D| \approx \frac{n_A}{n}$.

This is called the Monte Carlo method (蒙特卡洛法)

Why is called "The Law (定律) of Large Numbers" not "The Theorem (定理) of Large Numbers" ?

At the early age of **Probability Theory**, the mathematical definition of probability was not clear. Therefore, there was no theoretical basis to understand the concept of **convergence in probability**, the phenomenon that frequency "tending to" probability was considered as the results from **a large number of trials**, just like the laws in physics.

Since an axiomatic system (公理化体系) of probability was established and the law of large numbers can be strictly proved as a **theorem**, then the Law of Large Numbers has been also named as the Theorem of Large Numbers.

谢谢大家