CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Applications of Number Theory in Cryptography

- Introduction
- Symmetric cryptography
- Asymmetric cryptography
- RSA Cryptosystem
- DLP and El Gamal cryptography
- Diffie-Hellman key exchange protocol
- Crytocurrency, e.g., bitcoin

Cryptography

- History of almost 4000 years (from 1900 B.C.)

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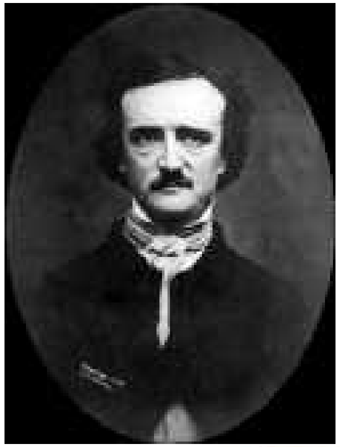
(secret) (writing)

The term was first used in *The Gold-Bug*, by Edgar Allan Poe (1809 - 1849).



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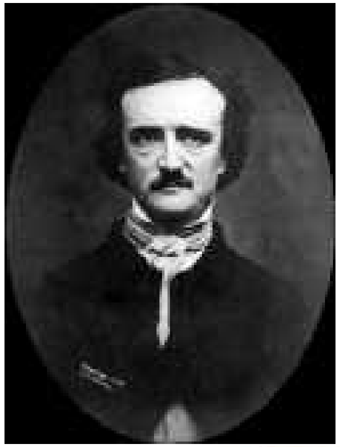


"Human ingenuity cannot concoct a cipher which human ingenuity cannot resolve." – 1841

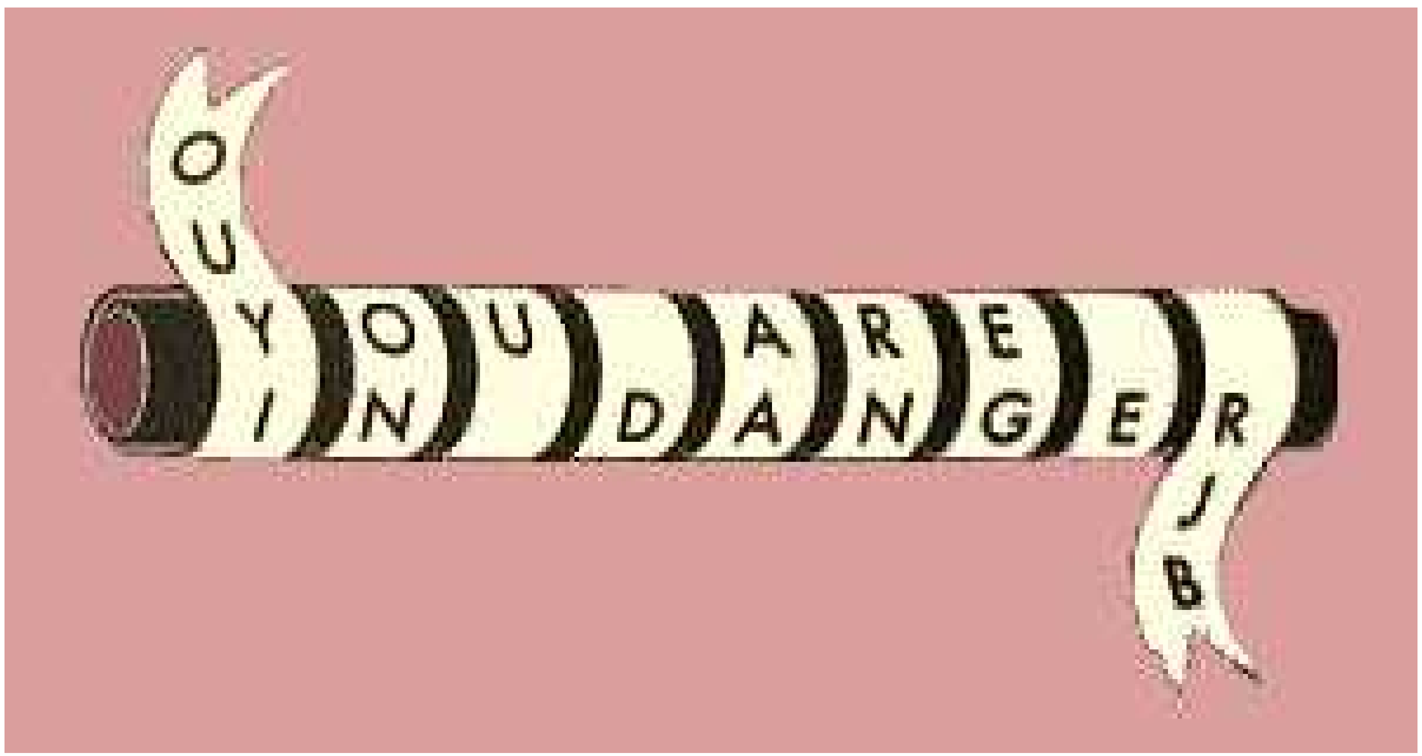
Cryptography

- One-sentence definition:

  "Cryptography is the practice and study of techniques for secure communication in the presence of third parties called *adversaries*." – Ronald L. Rivest

Some Examples

- In 405 BC, the Greek general LYSANDER OF SPARTA was sent a coded message written on the inside of a servant's belt.



Some Examples

- The Greeks also invented a cipher which changed letters to numbers. A form of this code was still being used during *World War I*.

	1	2	3	4	5
1	A	B	C	D	E
2	F	G	H	I/J	K
3	L	M	N	O	P
4	Q	R	S	T	U
5	V	W	X	Y	Z

Some Examples

- Caesar Cipher (after the name of JULIUS CAESAR)

VENI, VIDI, VICI

YHQL YLGL YLFL

Some Examples

- Morse Code: created by Samuel Morse in 1838

Morse Alphabet

A • —	O — — —
B — • • •	P • — — •
C — • — •	Q — — • —
D — • •	R • — •
E •	S • • •
F • • — •	T —
G — — •	U • • —
H • • • •	V • • • —
I • •	W • — —
J • — — —	X — • • —
K — • —	Y — • — —
L • — • •	Z — — • •
M — —	
N — •	

Full stop (.) • — • — • —
Break signal or fresh line — • • • —
Apostrophe (') • — — — — •
Hyphen (-) — • • • • —
Exclamation (!) — — • • — —
Interrogation (?) • • — — • •
Underline (———) • • — — • —
Parenthesis ( ) — • — — • —
Inverted commas (“ ”) • — • • — •

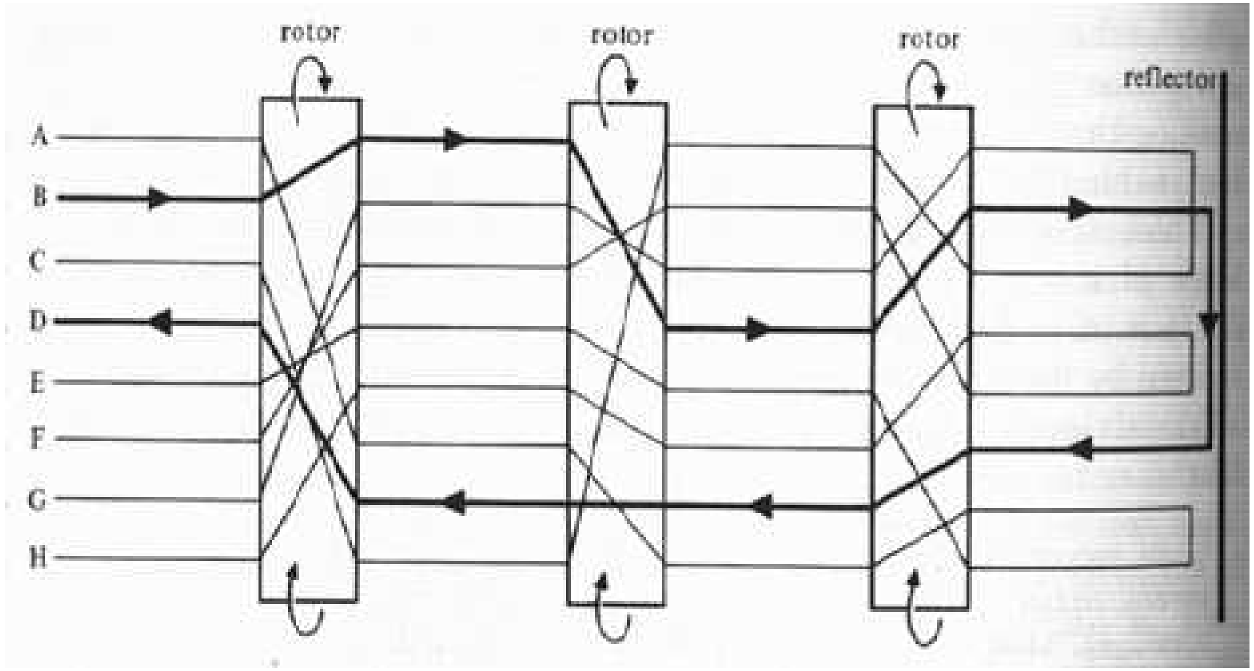
Some Examples

- Crytograms from the Chinese gold bars

http://www.iacr.org/misc/china/china.html

Some Examples

- Enigma, Germany coding machine in *World War II.*



Some Examples

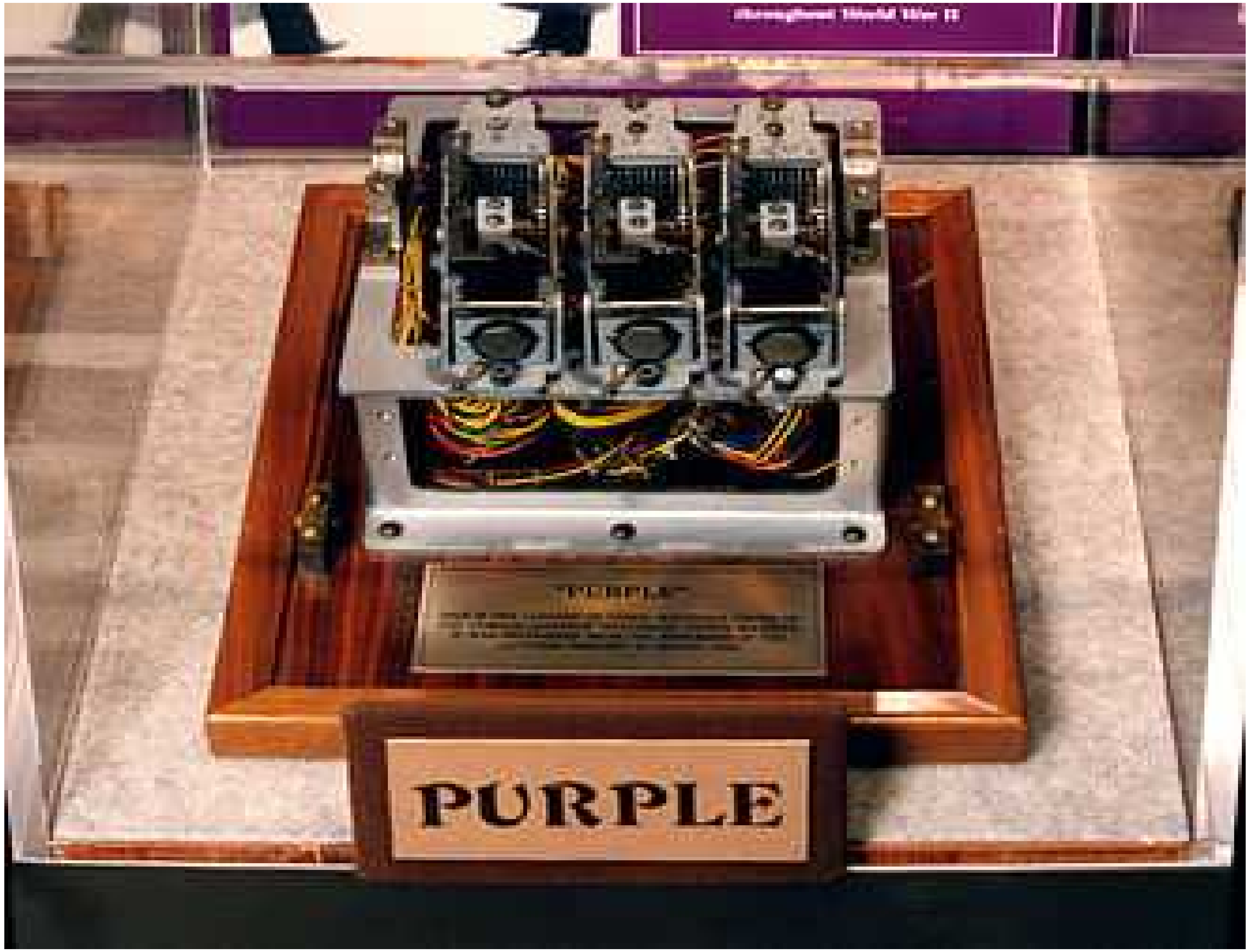
- Sigaba, used by U.S. during *World War II.*

Some Examples

- Japanese "Enigma" Rotor Cipher Machine

Some Examples

- Japanese Purple Machine (97-shiki obun inji-ki)



People Working in Breaking Codes

Alan Turing
(1912-1954)

Claude E. Shannon
(1916-2001)

Cryptography History

- History (until 1970's)

  "*Symmetric*" cryptography

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$\mathcal{Q}$: What if Bob could send Alice a "special key" useful only for encryption but no help for decryption?

Caesar Cipher

- Key: $k = 0, 1, \ldots, 25$

  Encryption: encode i as $(i + k) \bmod 26$

  Decryption: decode j as $(j - k) \bmod 26$

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  Key: 2
  ciphertext: UGPF TGKPHQTEGOGPV

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  Problem: only 26 possibilities for keys!

  Kerchoff's Principle (1883): System should be secure even if algorithms are known, as long as key is secret.

Substitution Cipher

- Key: table mapping each letter to another letter

A	B	C		Z
V	R	E		D

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However, substitution cipher is still insecure!

Key observation: can recover plaintext using *statistics* on *letter frequencies.*

Substitution Cipher

Table 1: Relative frequencies of the letters of the English language

Letter	Relative Frequency (%)	Letter	Relative Frequency (%)
a	8.167	n	6.749
b	1.492	o	7.507
c	2.782	p	1.929
d	4.253	q	0.095
e	12.702	r	5.987
f	2.228	s	6.327
g	2.015	t	9.056
h	6.094	u	2.758
i	6.966	v	0.978
j	0.153	w	2.360
k	0.772	x	0.150
l	4.025	y	1.974
m	2.406	z	0.074

Substitution Cipher

Table 2: Number of Diagraphs Expected in 2,000 Letters of English Text

th	-	50	at	-	25	st	-	20
er	-	40	en	-	25	io	-	18
on	-	39	es	-	25	le	-	18
an	-	38	of	-	25	is	-	17
re	-	36	or	-	25	ou	-	17
he	-	33	nt	-	24	ar	-	16
in	-	31	ea	-	22	as	-	16
ed	-	30	ti	-	22	de	-	16
ne	-	30	to	-	22	rt	-	16
ha	-	26	it	-	20	ve	-	16

Table 3: The 15 Most Common Trigraphs in the English Language

1	-	the	6	-	tio	11	-	edt
2	-	and	7	-	for	12	-	tis
3	-	tha	8	-	nde	13	-	oft
4	-	ent	9	-	has	14	-	sth
5	-	ion	10	-	nce	15	-	men

Substitution Cipher

- LIVITCSWPIYVEWHEVSRIQMXLEYVEOIEWHRXEXIPFE MVEWHKVSTYLXZIXLIKIIXPIJVSZEYPERRGERIMWQL MGLMXQERIWGPSRIHMXQEREKI

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I = e
L = h
X = t

LIVI = he?e

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HereUpOnLeGrandAroseWithAGraveAndStatelyAirAndBroug MeTheBeetleFromAGlassCaseInWhichItWasEnclosedIt-WasABe

- History (from 1976)
  - ◇ W. Diffie, M. Hellman, "New direction in cryptography", *IEEE Transactions on Information Theory, vol. 22, pp. 644-654, 1976.*

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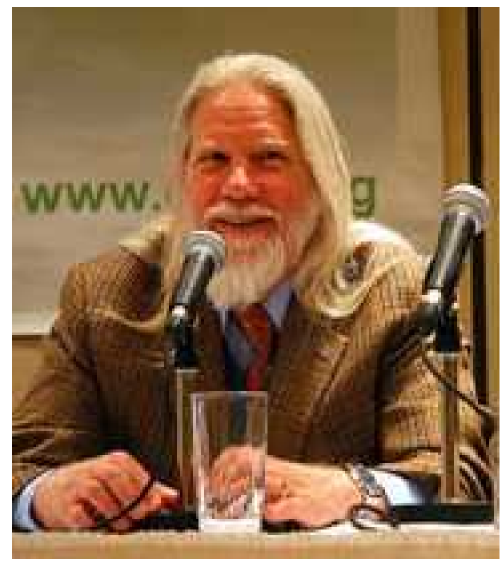
2015 **Turing Award**



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2015	Martin E. Hellman Whitfield Diffie	For fundamental contributions to **modern cryptography**. Diffie and Hellman's groundbreaking 1976 paper, "New Directions in Cryptography,"[39] introduced the ideas of public-key cryptography and digital signatures, which are the foundation for most regularly-used security protocols on the internet today.[40]

Public Key Cryptography

- Alice wants to send a message to Bob

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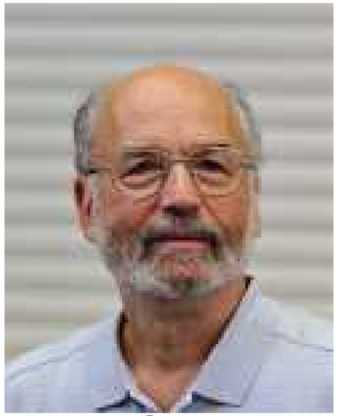
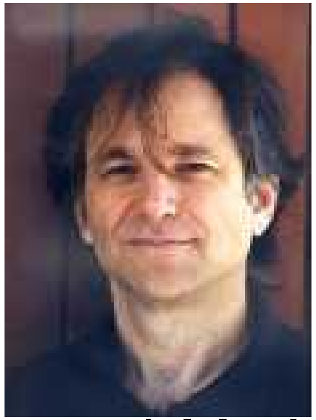

Adi Shamir


Leonard M. Adleman

R. Rivest, A. Shamir, L. Adleman, "A method for obtaining digital signatures and public-key cryptosystems", *Communications of the ACM*, vol. 21-2, pages 120-126, 1978.

RSA Public Key Cryptosystem

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Pick two large primes, p and q. Let $n = pq$, then $\phi(n) = (p-1)(q-1)$. Encryption and decryption keys e and d are selected such that

- $\gcd(e, \phi(n)) = 1$
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Theorem (*Correctness*) : Let p and q be two odd primes, and define $n = pq$. Let e be relatively prime to $\phi(n)$ and let d be the multiplicative inverse of e modulo $\phi(n)$. For each integer x such that $0 \le x < n$,

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$\mathcal{Q}$: How to prove this?

RSA Public Key Cryptosystem: Example

Parameters:	p	q	n	$\phi(n)$	e	d
	5	11	55	40	7	23

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Encryption: $M = 28, C = M^7 \bmod 55 = 52$

Decryption: $M = C^{23} \bmod 55 = 28$

RSA Public Key Cryptosystem: Parameters

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Comment: It is believed that determining $\phi(n)$ is equivalent to factoring n. Meanwhile, determining d given e and n, appears to be at least as time-consuming as the integer factoring problem.

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$\mathcal{Q}$: Consider the RSA system, where $n = pq$ is the modulus. Let (e, d) be a key pair for the RSA. Define

$$\lambda(n) = \mathrm{lcm}(p-1, q-1)$$

and compute $d' = e^{-1} \bmod \lambda(n)$. Will decryption using d' instead of d still work?

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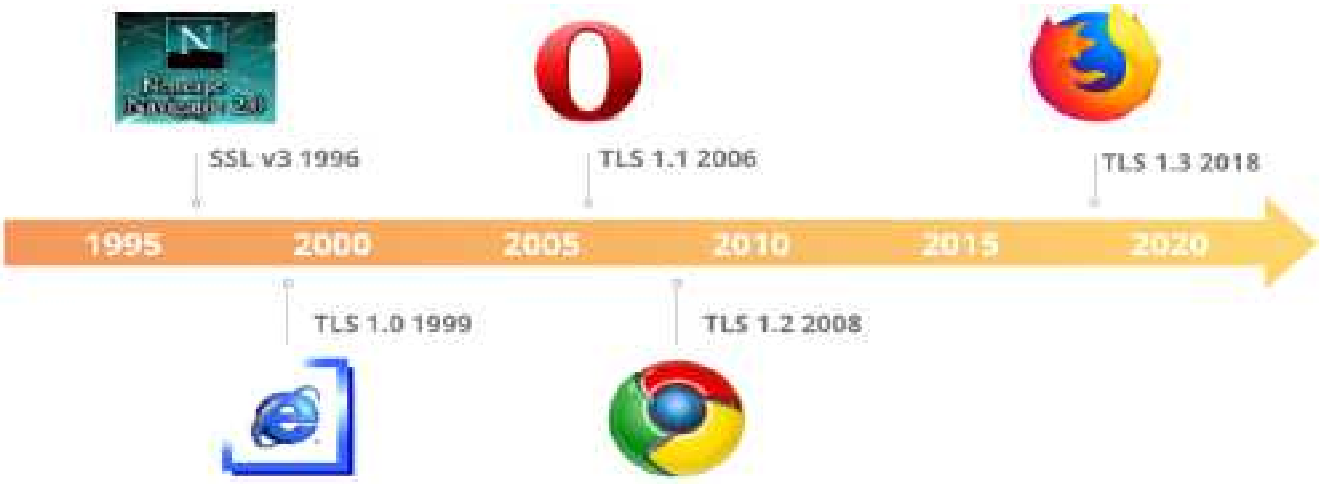

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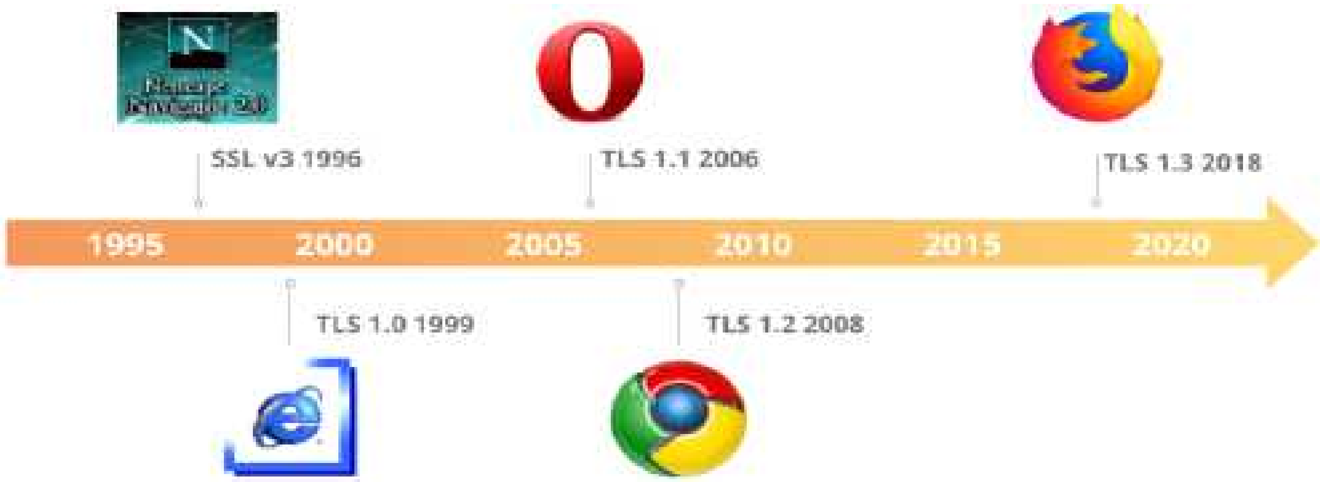

Key exchange/agreement and authentication

Algorithm	SSL 2.0	SSL 3.0	TLS 1.0	TLS 1.1	TLS 1.2	TLS 1.3
RSA	Yes	Yes	Yes	Yes	Yes	No
DH-RSA	No	Yes	Yes	Yes	Yes	No
DHE-RSA (forward secrecy)	No	Yes	Yes	Yes	Yes	Yes
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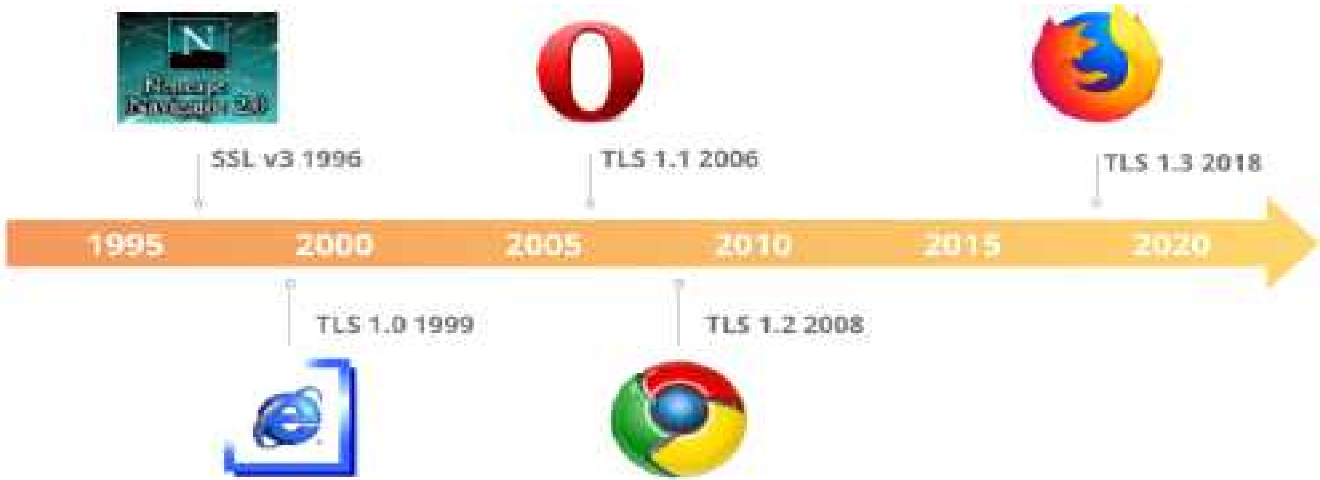

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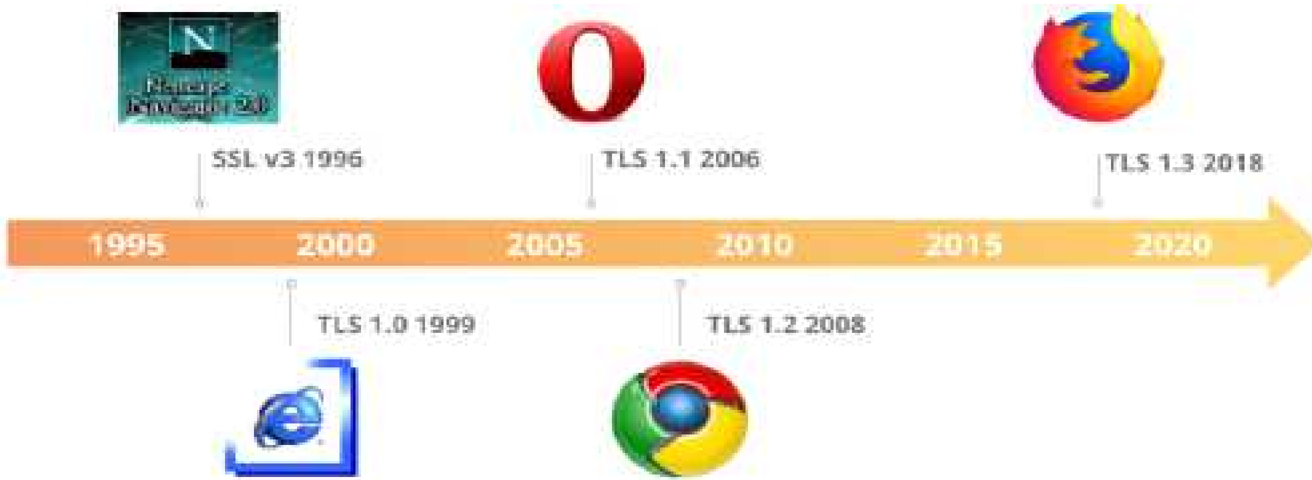

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Using RSA for Digital Signature

$S = M^d \bmod n$ (RSA **signature)**

$M = S^e \bmod n$ (RSA **verification)**

Why?

The Discrete Logrithm

- **The discrete logarithm** of an integer y to the base b is an integer x, such that

$$b^x \equiv y \bmod n.$$

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This is very hard!

El Gamal Encryption

- **Setup** Let p be a prime, and g be a generator of $\mathbb{Z}_p$. The private key x is an integer with $1 < x < p - 2$. Let $y = g^x \bmod p$. The public key for *El Gamal encryption* is (p, g, y).

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  El Gamal Encryption: Pick a random integer k from $\mathbb{Z}_{p-1}$,

  $$a = g^k \bmod p$$
  $$b = My^k \bmod p$$

  The ciphertext C consists of the pair (a, b).

  El Gamal Decryption:

  $$M = b(a^x)^{-1} \bmod p$$

Using El Gamal for Digital Signature

$$a = g^k \bmod p$$
$$b = k^{-1}(M - xa) \bmod (p-1)$$

(El Gamal **signature**)

$$y^a a^b \equiv g^M \pmod{p}$$

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$\mathcal{Q}$: How to verify it?

An Example

Choose $p = 2579$, $g = 2$, and $x = 765$. Hence $y = 2^{765} \bmod 2579 = 949$.

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Encryption: Let $M = 1299$ and choose a random $k = 853$,

$$\begin{aligned}(a, b) &= (g^k \bmod p, My^k \bmod p)\\ &= (2^{853} \bmod 2579, 1299 \cdot 949^{853} \bmod 2579)\\ &= (435, 2396).\end{aligned}$$

Decryption:

$$M = b(a^x)^{-1} \bmod p = 2396 \times (435^{765})^{-1} \bmod 2579 = 1299.$$

Question 1: Is it feasible to derive x from (p, g, y)?

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Security of the El Gamal Cryptosystem

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Attack 1: Use $M = by^{-k}$. However, k is randomly picked.

Attack 2: Use $M = b(a^x)^{-1} \bmod p$, but x is secret.

Diffie-Hellman Key Exchange Protocol

User A		User B
Generate random $X_A < p$ calculate $Y_A = \alpha^{X_A} \bmod p$ Calculate $k = (Y_B)^{X_A} \bmod p$	$\xrightarrow{Y_A}$ $\xleftarrow[Y_B]{}$	Generate random $X_B < p$ Calculate $Y_B = \alpha^{X_B} \bmod p$ Calculate $k = (Y_A)^{X_B} \bmod p$

Cryptography Wonders

- *Digital Signatures*. Electronically sign documents

  Zero-knowledge Proofs. Alice proves to Bob that she earns $< \$50k$ without Bob learning her income.

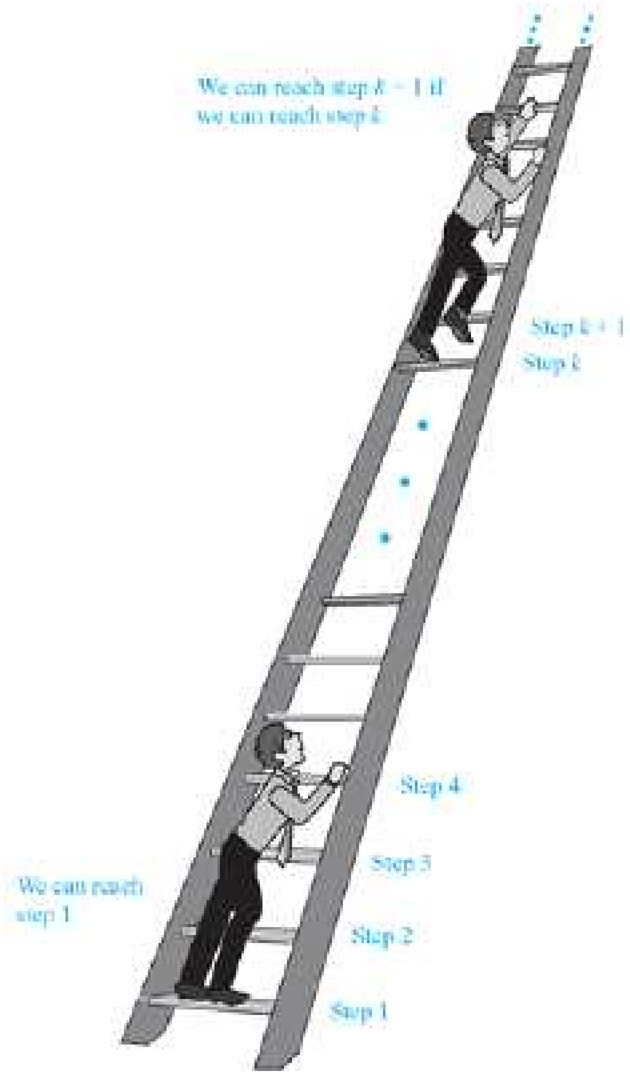
  Privacy-perserving data mining. Bob holds DB. Alice gets answer to one query, without Bob knowing what she asked.

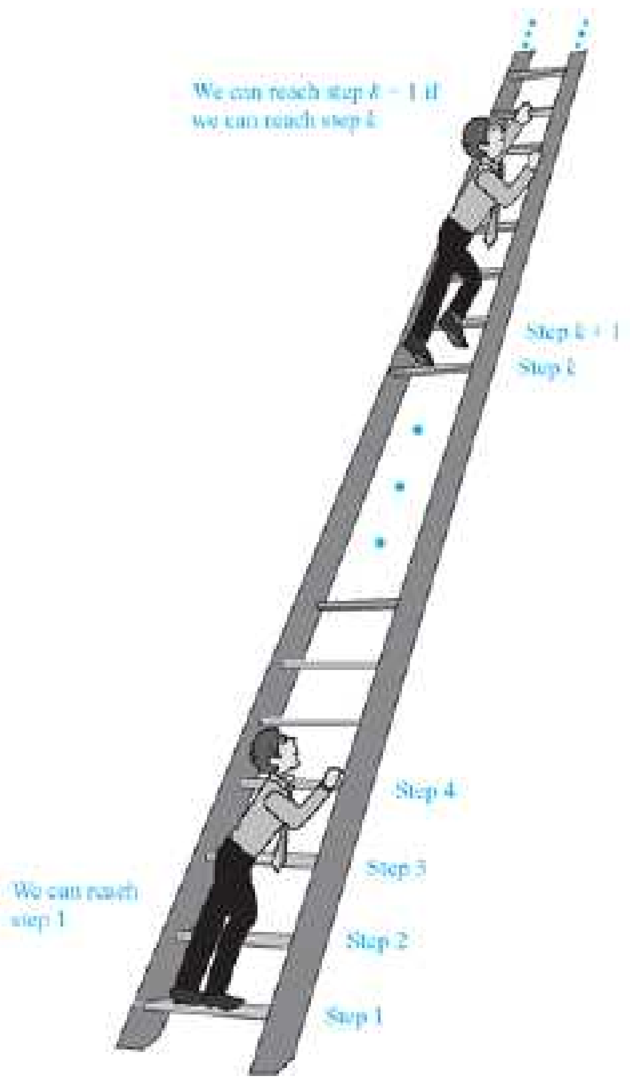
  Playing poker over the net. Alice, Bob, Carol and David can play Poker over the net without trusting each other or any central server. (*E-Voting*)

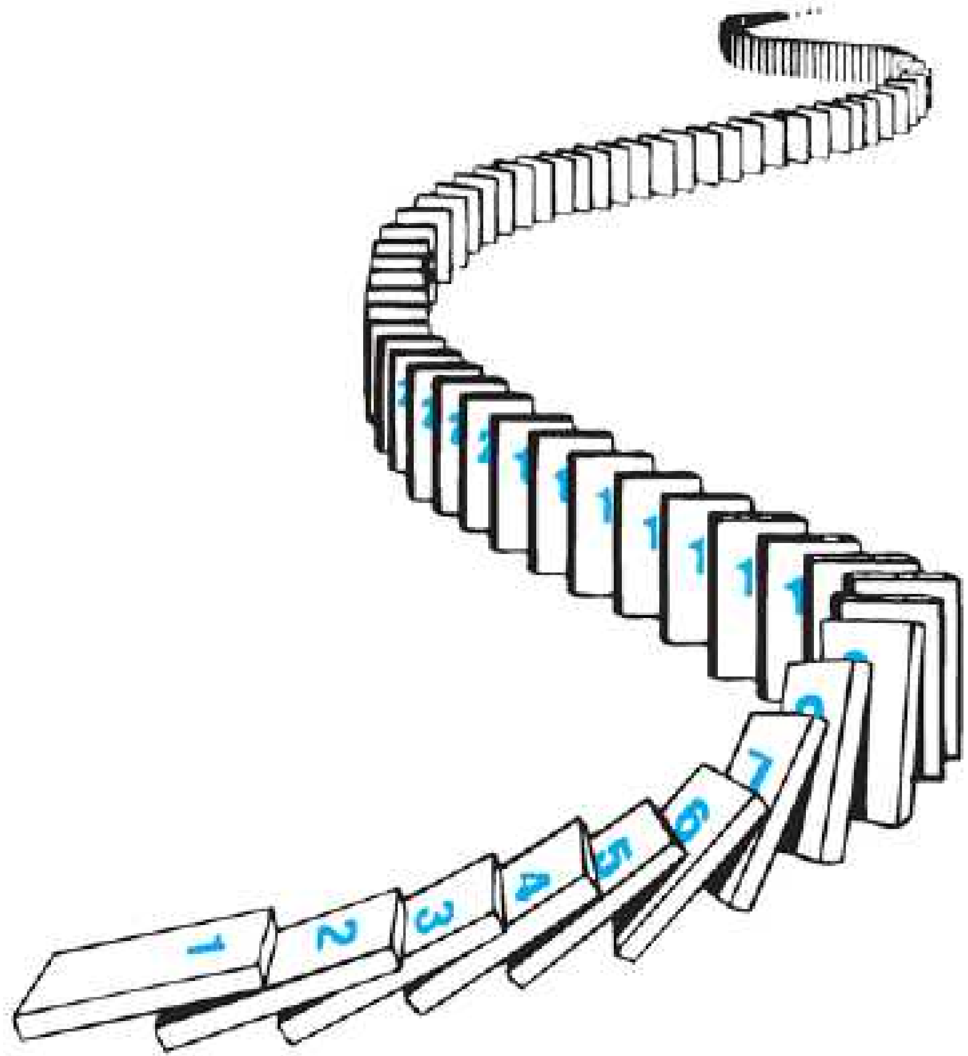
  Electronic Auctions. Can run auctions s.t. no one (even not seller) learns anything other than winning party and bid.

  Fully Homomorphic Encryption. Encrypt $E(m)$ in a way that allows to compute $E(f(m))$.

Mathematical Induction




We can reach step k + 1 if
we can reach step k
Step k + 1
Step k
Step 4
Step 3
We can reach
step 1
Step 2
Step 1



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  The *strong principle* can actually be derived from the *weak principle*.

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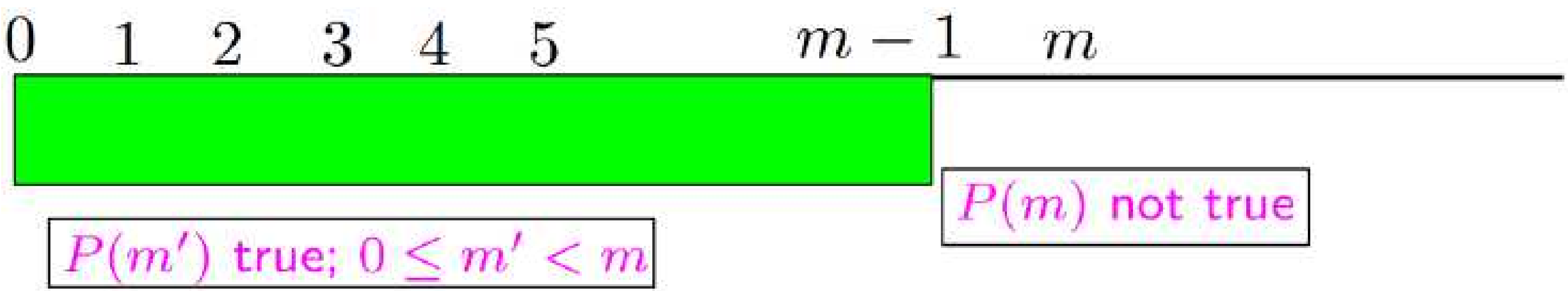

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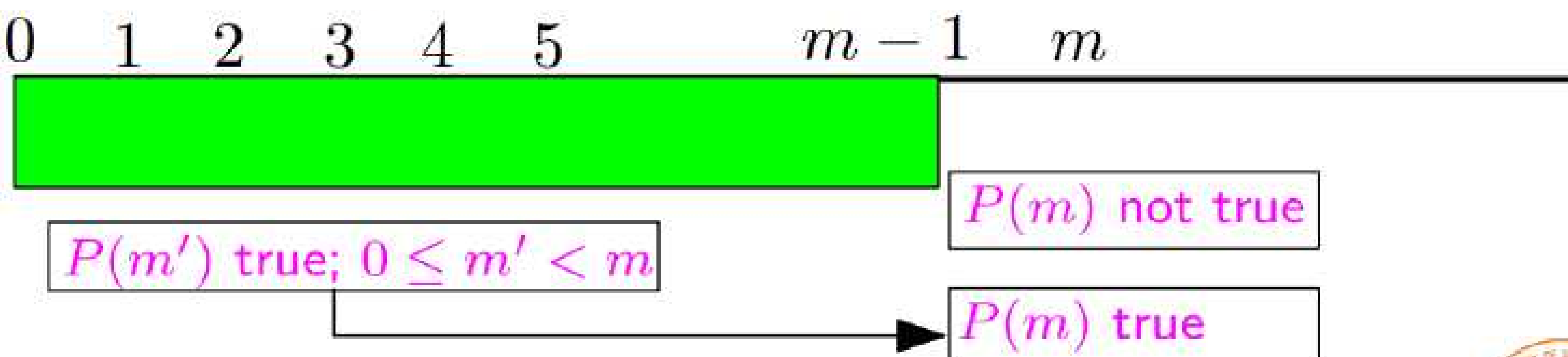

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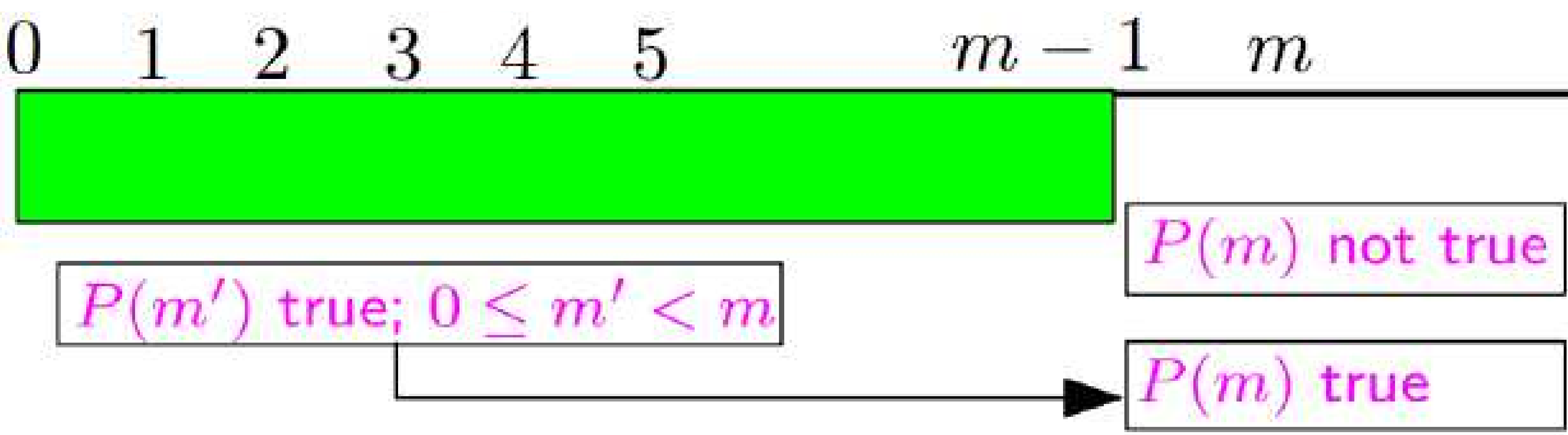

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  ◇ Therefore, $(*)$ holds for all positive integers n.

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  The key step was proving that

$$P(n-1) \rightarrow P(n)$$

  where $P(n)$ is the statement

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When a for all quantifier is false, there must be some n for which it is false. Let n be the smallest nonnegative integer for which $2^{n+1} \not\geq n^2 + 2$.

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  Then setting $i = n - 1$ gives
  $$2^{(n-1)+1} \geq (n-1)^2 + 2.$$

  or
  $$(*) \quad 2^n \geq n^2 - 2n + 1 + 2 = n^2 - 2n + 3$$

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  Since $P(n-1) \rightarrow P(n)$, we see that

  $P(0)$ implies $P(1)$, $P(1)$ implies $P(2)$, ...

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  (a) – *Basic Step* *Inductive Hypothesis*

  (b) – *Inductive Step* *Inductive Conclusion*

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By mathematical induction, $\forall n > 0,\ 2^{n+1} \geq n^2 + 2$.

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Inductive Conclusion

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  - ◇ Iterating gives us a proof of $P(n)$ for all n

Strong Induction

- **Principle** (*The Strong Principle of Mathematical Induction*)

  (a) If the statement $P(b)$ is true

  (b) for all $n > b$, the statement
  $P(b) \wedge P(b+1) \wedge \cdots \wedge P(n-1) \rightarrow P(n)$ is true.

  then $P(n)$ is true for all integers $n \geq b$.

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  ◇ Then, if n is not a prime power, it is a product of two smaller numbers, each of which is, by the inductive hypothesis, a power of a prime or a product of powers of primes.

  ◇ Thus, by the strong principle of mathematical induction, every positive integer is a power of a prime or a product of powers of primes.

Mathematical Induction

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- In reality, they are equivalent to each other in that the weak form is a special case of the strong form, and the strong form can be derived from the weak form.

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3. We conclude on the basis of the principle of mathematical induction that $P(n)$ is true for all $n \geq b$.

Next Lecture

- recurrence ...