

# *Linear Algebra*



Instructor: Jing YAO

## 5

# Eigenvalues and Eigenvectors (特征值与特征向量)

## 5.1

## EIGENVALUES AND EIGENVECTORS

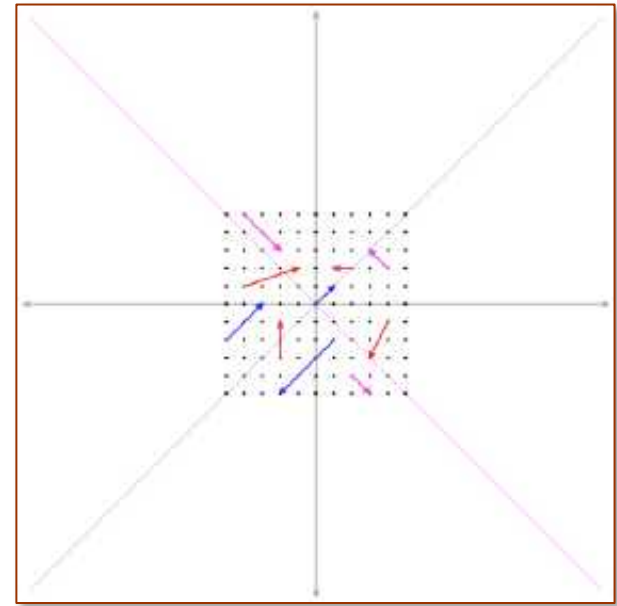
Introduction

Definition

Calculations

Properties

Application



# I. Introductory example - 人口流动问题



# 简化模型

❶ 假设:

从事**农业工作**的人员中每年有**四分之一**转为从事非农业工作,  
从事**非农业工作**的人员中每年有**六分之一**转为从事农业工作.



❷ 人口总数不变.

🔍 **预测多年之后劳动力从业情况的发展趋势.**

**分析** 设最初农业人员和非农业人员的数量分别为  $y_0, z_0$ ,

第 1 年末数量为  $y_1, z_1$ ,

$$\begin{cases} y_1 = \frac{3}{4}y_0 + \frac{1}{6}z_0, \\ z_1 = \frac{1}{4}y_0 + \frac{5}{6}z_0. \end{cases} \quad \longrightarrow \quad \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

**分析** 设最初农业人员和非农业人员的数量分别为  $y_0, z_0$ ,  
第 1 年末数量为  $y_1, z_1$ ,

$$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

矩阵  $A$

**分析** 设最初农业人员和非农业人员的数量分别为  $y_0, z_0$ ,  
第  $k$  年末数量为  $y_k, z_k$ ,

$$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

矩阵  $A$

如何计算  
方阵  $A$  的幂  
 $A^k$ ?

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} y_{k-1} \\ z_{k-1} \end{bmatrix} = \dots = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{5}{6} \end{bmatrix}^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \mathbf{A}^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

最简单的方阵  $Q$ :  $Q = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \quad Q^k = \begin{bmatrix} \lambda_1^k & \\ & \lambda_2^k \end{bmatrix}.$

有无可能?

设想

$$A = PQP^{-1}$$



$$A^k = PQ^kP^{-1}$$

(其中 $Q$ 为对角阵)

$$A^2 = AA = PQP^{-1}PQP^{-1} = PQ^2P^{-1}$$

$$A^3 = AAA = \cancel{PQP^{-1}}\cancel{PQP^{-1}}PQP^{-1} = PQ^3P^{-1}$$

$$AP = PQ$$



$$A[P_1 \mid P_2] = [P_1 \mid P_2] \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

(其中 $P_1, P_2$ 为2维列向量)

$$AP_1 = \lambda_1 P_1, \quad AP_2 = \lambda_2 P_2$$



$$[AP_1 \mid AP_2] = [\lambda_1 P_1 \mid \lambda_2 P_2]$$

$$A_{2 \times 2} \mathbf{x}_{2 \times 1} = \lambda \mathbf{x}_{2 \times 1}$$



## II. Eigenvalues and Eigenvectors – Definition & Calculation

**Definition 1** Let  $A$  be a *square matrix* of degree  $n$ .

If there exist a non-zero vector  $\mathbf{x}$  and a scalar  $\lambda$  such that

$$A \mathbf{x} = \lambda \mathbf{x},$$

then  $\lambda$  is called an **eigenvalue (特征值)** of  $A$ , and  $\mathbf{x}$  is called an **eigenvector (特征向量)**, corresponding to the eigenvalue  $\lambda$ .



*David Hilbert*  
1862-1943



德语 *eigen* (有特征的, 自身的, 个体的)



$A$  的特征值与特征向量有何**直观含义**?

$$R^n \rightarrow R^n$$

$$A : \mathbf{x} \rightarrow A\mathbf{x}$$

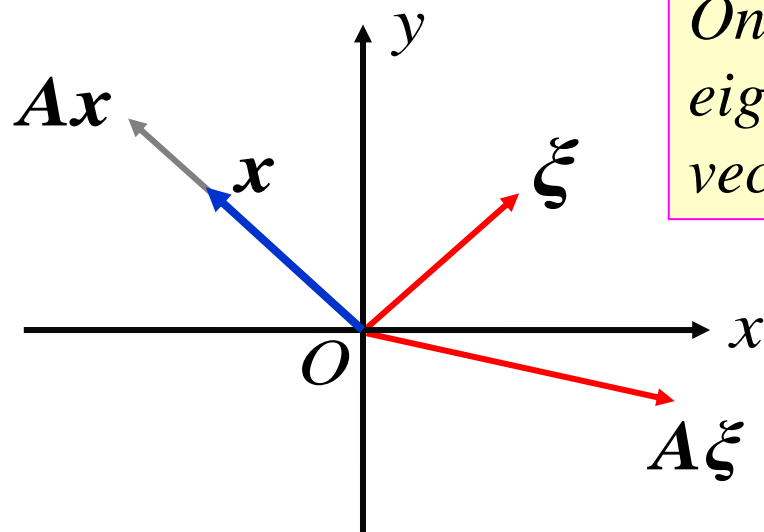
$$R^n \rightarrow R^n$$

$$\lambda : \mathbf{x} \rightarrow \lambda \mathbf{x}$$



给定方阵  $A \in \mathbf{R}^{n \times n}$ , 一般说来, 对于  $\xi \in \mathbf{R}^n$ , 向量  $A\xi$  与  $\xi$  不在 同一方向上.

但也可能 存在 向量  $x$ , 使得  $Ax$  在  $x$  的方向上.  $Ax$  is a multiple of  $x$ .

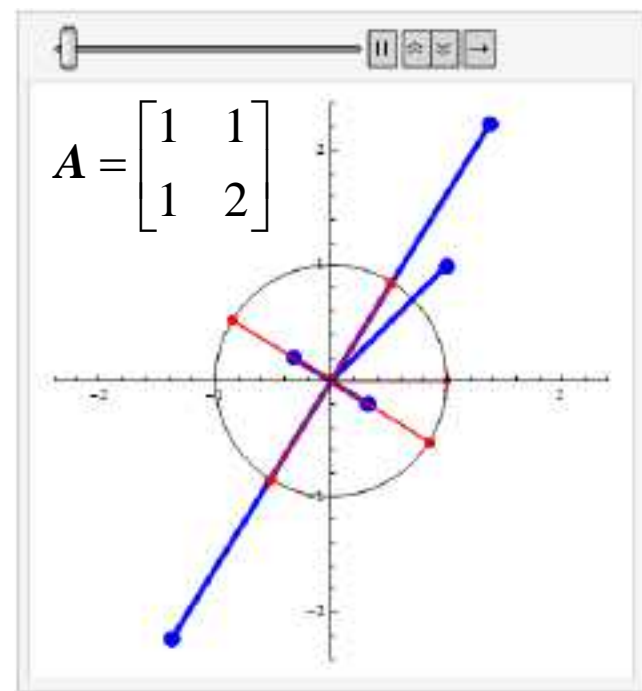


*Only certain special numbers are eigenvalues, and only certain special vectors are eigenvectors.*

$A$  的特征值与特征向量的直观含义:

特征向量: 与  $A$  左乘下的像 “共线”

特征值: “伸缩比例”



**For example,**  $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which shows that 4 is an eigenvalue of  $\mathbf{A}$ ,  $(1,1)^T$  is an eigenvector for 4 (an eigenvector corresponding to 4).

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{2}{3} \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix},$$

which tells us that  $-1$  is also an eigenvalue of  $\mathbf{A}$ , and  $(1, -2/3)^T$  is an eigenvector corresponding to  $-1$ .

$$A\mathbf{x} = \lambda \mathbf{x}$$

How to get?

**!dea**

Equivalently, an eigenvalue  $\lambda$  and a corresponding eigenvector  $\mathbf{x}$  satisfy:

$$(A - \lambda I) \mathbf{x} = \mathbf{0}.$$

Since  $\mathbf{x}$  is a *non-zero* vector, the matrix  $A - \lambda I$  is not invertible, and thus the determinant  $|A - \lambda I|$  equals 0.

**注:** 特征向量  $\mathbf{x}$  是非零向量, 是齐次线性方程组

$$(A - \lambda I) \mathbf{x} = \mathbf{0}$$

的非零解.  $\lambda$  应满足

$$|A - \lambda I| = 0.$$

(characteristic equation: 特征方程)

We note that the determinant  $|A - \lambda I|$  is a polynomial of degree  $n$  in  $\lambda$ , called the **characteristic polynomial** of  $A$ .

**Definition 2** Let  $A=[a_{ij}]_{n \times n}$  be a *square matrix* of degree  $n$ . Then

$$f(\lambda) = |A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

is called the **characteristic polynomial (特征多项式)** of  $A$ .

$n$  阶矩阵  $A$  的特征多项式是  $\lambda$  的  $n$  次多项式.

**Lemma 1** A scalar  $\lambda$  is an eigenvalue of  $A$  if and only if  $\lambda$  is a root of the characteristic polynomial.

( $\lambda$  是  $A$  的特征值, 当且仅当  $\lambda$  是特征多项式的根.)

$n$  阶矩阵  $A$  的特征多项式是  $\lambda$  的  $n$  次多项式.



Gauss  
1777-1855

**代数基本定理(Fundamental theorem of algebra)**  
在复数范围内每个  $n$  次复系数方程恰有  $n$  个根.

**注释:**  $n$  阶方阵  $A$  在复数范围内有  $n$  个特征值.

$n$  阶矩阵  $A$  的特征多项式在复数域上的  $n$  个根都是矩阵  $A$  的特征值, 其  $k$  重根叫做  $k$  重特征值.

当  $n \geq 5$  时, 特征多项式没有一般的求根公式.

(Galois and Abel proved that there can be no algebraic formula for the roots of a fifth-degree polynomial).

即使是三阶矩阵, 一般也难以求根.

**解决方法:** “计算方法 (数值分析)”



Galois 1811-1832



Abel 1802-1829

**Example 1** Let  $\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ . Then

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 4 - \lambda & -5 \\ 2 & -3 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 2.$$

The roots are  $\lambda_1 = -1$  and  $\lambda_2 = 2$ . Eigenvectors of  $\mathbf{A}$  can be obtained as follows.

For  $\lambda_1 = -1$ , we solve the system of linear equations  $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{x} = (\mathbf{A} + \mathbf{I})\mathbf{x} = \mathbf{0}$ , i.e.,

$$\begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It follows that  $\mathbf{x}_1 = k_1(1,1)^T$  ( $k_1 \in \mathbf{R}$ ,  $k_1 \neq 0$ ) are eigenvectors corresponding to the eigenvalue  $\lambda_1 = -1$ .

For  $\lambda_2 = 2$ , we solve the system of linear equations  $(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{x} = (\mathbf{A} - 2\mathbf{I})\mathbf{x} = \mathbf{0}$ , i.e.,

$$\begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It follows that  $\mathbf{x}_2 = k_2(5,2)^T$  ( $k_2 \in \mathbf{R}$ ,  $k_2 \neq 0$ ) are eigenvectors corresponding to the eigenvalue  $\lambda_2 = 2$ .

This example illustrates a method for finding eigenvalues and eigenvectors of a given matrix, which is important in the area of matrix theory and many applications.

*A Process for finding eigenvalues and eigenvectors of a matrix  $A$  :*

1. *Compute the determinant of  $A - \lambda I$ .*

With  $\lambda$  subtracted along the diagonal, this determinant is a polynomial of degree  $n$ . It starts with  $(-\lambda)^n$ .

2. *Find the roots of this polynomial.*

The  $n$  roots are the eigenvalues of  $A$ .

3. *For each eigenvalue solve the equation  $(A - \lambda I) \mathbf{x} = \mathbf{0}$ .*

Since the determinant is zero, there are solutions other than  $\mathbf{x} = \mathbf{0}$ . Those are the eigenvectors.



## 特征值与特征向量的求解 步骤

第一步 计算 $A$ 的特征多项式;

第二步 求出特征多项式的全部根, 即得 $A$ 的全部特征值;

第三步 将每一个特征值代入相应的线性方程组进行求解, 即得该特征值的特征向量.

$$A \rightarrow |A - \lambda I| = 0 \rightarrow (A - \lambda_i I)x = 0$$

求特征值 $\lambda_i$

求特征向量

**Remark:** 另一种等价求法

$$A \rightarrow |\lambda I - A| = 0 \rightarrow (\lambda_i I - A)x = 0$$

求特征值 $\lambda_i$

求特征向量

**Example 2** Everything is clear when  $\mathbf{A}$  is a *diagonal matrix*:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \text{ has } \lambda_1 = 3 \text{ with } \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\lambda_2 = 2 \text{ with } \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

On each eigenvector  $\mathbf{A}$  acts like:  $\mathbf{A}\mathbf{x}_1 = 3\mathbf{x}_1$  and  $\mathbf{A}\mathbf{x}_2 = 2\mathbf{x}_2$ .

Other vectors like  $\mathbf{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$  are mixtures  $\mathbf{x}_1 + 5\mathbf{x}_2$  of the two eigenvectors,

and  $\mathbf{A}$  acts like:  $\mathbf{A}(\mathbf{x}_1 + 5\mathbf{x}_2) = 3\mathbf{x}_1 + 10\mathbf{x}_2 = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$ .

This is  $\mathbf{A}\mathbf{x}$  for a typical vector  $\mathbf{x}$ —not an eigenvector. But the action of  $\mathbf{A}$  is determined by its eigenvectors and eigenvalues.

**Remark: 1.** The eigenvalues are on the main diagonal when  $A$  is *diagonal*.

( $n$  阶对角矩阵  $A$  的特征值是它的  $n$  个主对角元  $a_{11}, a_{22}, \dots, a_{nn}$ .)

This is true since the characteristic polynomial of  $A$  is

$$|A - \lambda I| = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda).$$

**2.** The eigenvalues are on the main diagonal when  $B$  is *triangular*.

( $n$  阶上(下)三角形矩阵  $B$  的特征值也是它的  $n$  个主对角元  $b_{11}, b_{22}, \dots, b_{nn}$ .)

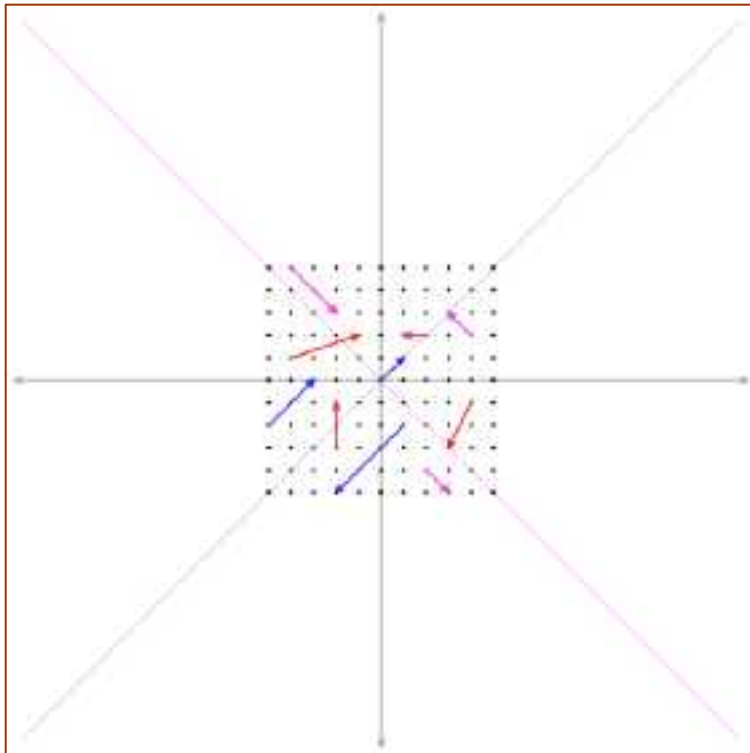
---

## Exercise

Find the eigenvalues and eigenvectors of  $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ .

**Example 3** Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$



The Eigenvectors

$$k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(k_1 \neq 0)$$

$$k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(k_2 \neq 0)$$

corresponding respectively to  
the Eigenvalues:

$$\lambda_1 = 1$$

$$\lambda_2 = 3$$

**Example 4** If a square matrix  $A$  satisfies  $A^2 = A$ .  
Show that the only possible eigenvalues of  $A$  are 0 or 1.

**Proof** Let  $\lambda$  be the eigenvalue of  $A$ , then  $A\mathbf{x} = \lambda\mathbf{x}$ . And

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda\lambda\mathbf{x} = \lambda^2\mathbf{x}$$

So  $(\lambda^2 - \lambda)\mathbf{x} = \mathbf{0}$ .

Since  $\mathbf{x} \neq \mathbf{0}$ , we have  $\lambda^2 - \lambda = 0$ , and  $\lambda = 0$  or  $\lambda = 1$ .

**For example,**

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

**Eigenvalues:**                      0                      1                      1, 0

**Remark:** The eigenvalues of a *projection matrix* are 1 or 0.  
(投影矩阵的特征值为0或1)

**Remarks:** Give a matrix  $A$  of degree  $n$ .

- $A$  has exactly  $n$  eigenvalues, some of them might be *repeated*.
- Some eigenvalues of  $A$  may be *complex* numbers; some matrices may have no *real* eigenvalue, for instance, rotation matrices.

$$A_1 = \begin{bmatrix} 0 & -2 & -2 \\ 2 & -4 & -2 \\ -2 & 2 & 0 \end{bmatrix} \text{ has eigenvalues: } \lambda_1=0, \lambda_2=-2 \text{ (二重特征值, } \\ \text{a root of multiplicity 2).}$$

$$|A_1 - \lambda I| = -\lambda(\lambda + 2)^2$$

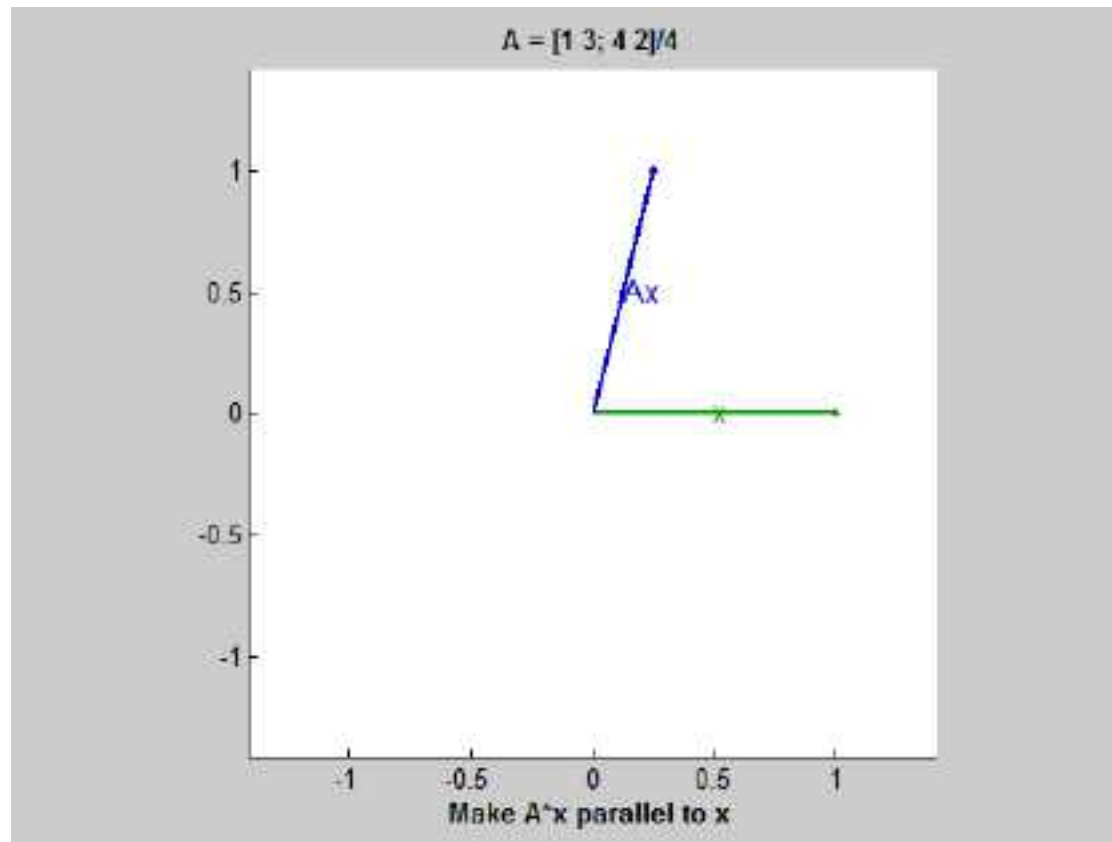
$$A_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \text{ has eigenvalues: } \lambda = \cos \theta \pm i \sin \theta.$$

$$|A_2 - \lambda I| = \lambda^2 - (2 \cos \theta)\lambda + 1$$

*Matlab demo: eigshow*

$$A = \frac{1}{4} \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\lambda_1 = -0.5, \\ \lambda_2 = 1.25.$$



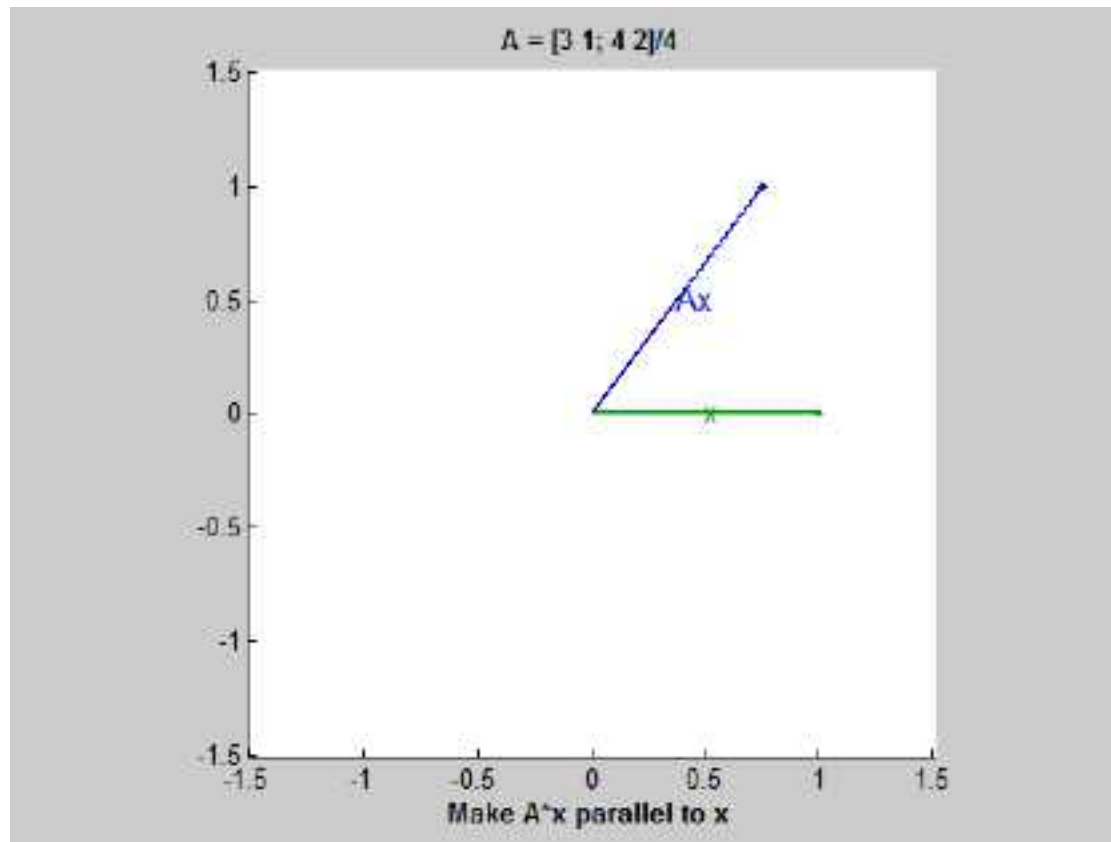
<https://blogs.mathworks.com/cleve/2013/07/08/eigshow-week-1/#f96996aa-ef86-4137-b343-07584baff36c>



*Matlab demo: eigshow*

$$A = \frac{1}{4} \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

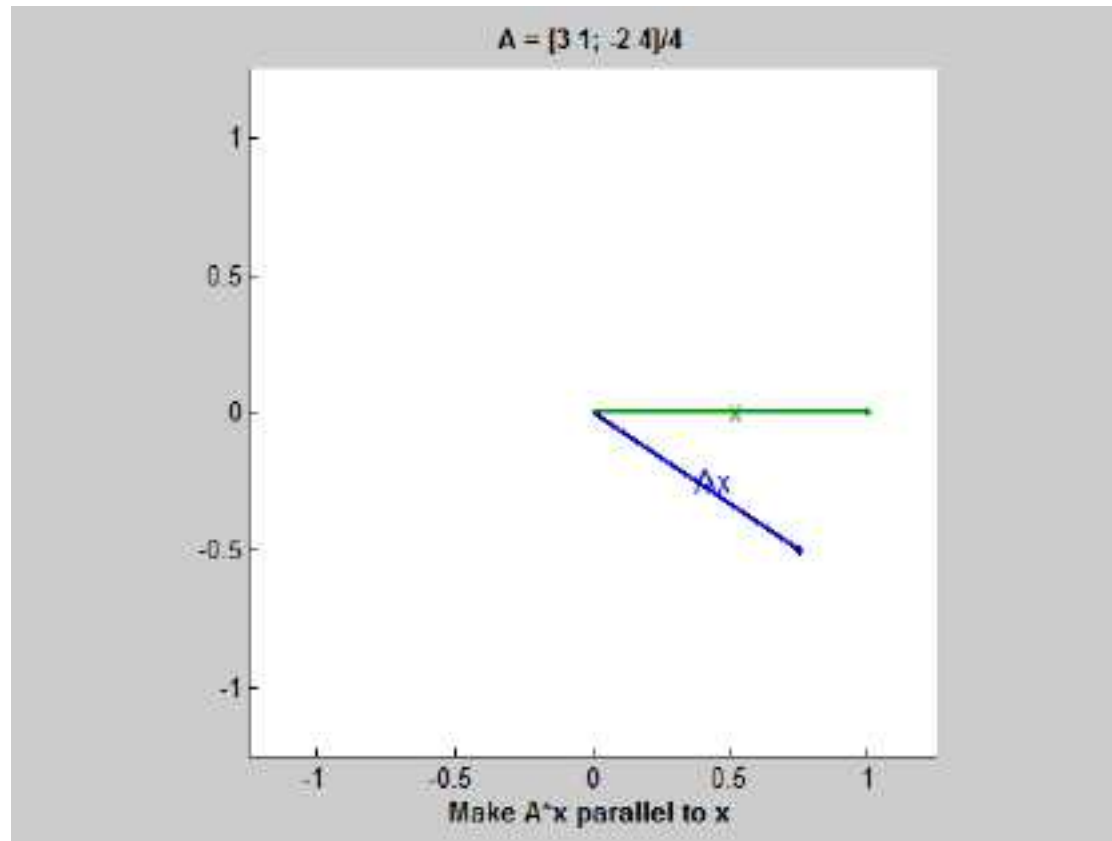
$$\lambda_1 = 1.14,$$
$$\lambda_2 = 0.11.$$



<https://blogs.mathworks.com/cleve/2013/07/08/eigshow-week-1/#f96996aa-ef86-4137-b343-07584baff36c>

*Matlab demo: eigshow*

$$A = \frac{1}{4} \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix} \quad \begin{aligned} \lambda_1 &= 0.8750 + 0.3307i, \\ \lambda_2 &= 0.8750 - 0.3307i. \end{aligned}$$



<https://blogs.mathworks.com/cleve/2013/07/08/eigshow-week-1/#f96996aa-ef86-4137-b343-07584baff36c>

# III. Eigenvalues and Eigenvectors – Properties

**Theorem 1** If  $\mathbf{x}_1, \mathbf{x}_2$  are two eigenvectors of  $A$  corresponding to the eigenvalue  $\lambda_0$ , then  $k_1\mathbf{x}_1 + k_2\mathbf{x}_2$  is also an eigenvector for  $\lambda_0$ , where  $k_1, k_2$  are any numbers that make  $k_1\mathbf{x}_1 + k_2\mathbf{x}_2 \neq \mathbf{0}$ .

**定理1** 若 $\mathbf{x}_1, \mathbf{x}_2$ 是 $A$ 属于 $\lambda_0$ 的两个的特征向量, 则 $k_1\mathbf{x}_1 + k_2\mathbf{x}_2$ 也是 $A$ 属于 $\lambda_0$ 的特征向量 (其中 $k_1, k_2$ 是任意常数, 但 $k_1\mathbf{x}_1 + k_2\mathbf{x}_2 \neq \mathbf{0}$ ).

**Proof**  $\mathbf{x}_1, \mathbf{x}_2$  are solutions to the following homogeneous system of linear equations:

$$(A - \lambda_0 I) \mathbf{x} = \mathbf{0},$$

So  $k_1\mathbf{x}_1 + k_2\mathbf{x}_2$  is also a solution.

Therefore, nonzero vector  $k_1\mathbf{x}_1 + k_2\mathbf{x}_2$  is also an eigenvector of  $A$  corresponding to the eigenvalue  $\lambda_0$ .

**Theorem 2** Let  $A = [a_{ij}]_{n \times n}$ , and  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the  $n$  eigenvalues of  $A$ .

Then

$$(1) \prod_{i=1}^n \lambda_i = |A| \text{ (i.e., } \det A),$$

$$(2) \sum_{i=1}^n \lambda_i = \sum_{i=1}^n a_{ii} = \text{trace}(A) \text{ (i.e., } \text{tr}(A)).$$

The sum of all diagonal entries of  $A$  is called the **trace of  $A$**  ( $A$  的迹).

\*Some properties about the traces of matrices:

$$\text{tr}(A_{n \times n} + B_{n \times n}) = \text{tr}(A_{n \times n}) + \text{tr}(B_{n \times n}), \quad \text{tr}(A_{m \times n} B_{n \times m}) = \text{tr}(B_{n \times m} A_{m \times n}).$$

## Proof

(1) Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of the square matrix  $\mathbf{A}$ .

Then the characteristic polynomial of  $\mathbf{A}$  can be expressed as

$$|\mathbf{A} - \lambda \mathbf{I}| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

Let  $\lambda = 0$  on both sides, we can immediately get

$$|\mathbf{A}| = \prod_{i=1}^n \lambda_i.$$

( make a clever choice of  $\lambda$  )

$$(2) \quad |\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix},$$

Among the expansion of the determinant, there is a term  $(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$ , and the terms  $\lambda^n, \lambda^{n-1}$  in the characteristic polynomial **only** come from this term.

$$\begin{aligned} \text{Therefore, } |\mathbf{A} - \lambda \mathbf{I}| &= (-\lambda)^n + (a_{11} + a_{22} + \cdots + a_{nn})(-\lambda)^{n-1} + \cdots + |\mathbf{A}|. \end{aligned}$$

On the other hand

$$\begin{aligned} |\mathbf{A} - \lambda \mathbf{I}| &= (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) \\ &= (-\lambda)^n + (\lambda_1 + \lambda_2 + \cdots + \lambda_n)(-\lambda)^{n-1} + \cdots + \prod_{i=1}^n \lambda_i, \end{aligned}$$

$$\text{Finally, } \sum_{i=1}^n \lambda_i = \sum_{i=1}^n a_{ii}. \quad \left( \text{find the coefficient of } (-\lambda)^{n-1} \text{ and compare} \right)$$

**Remark:** Zero is an eigenvalue of  $A$  if and only if  $A$  is not invertible.

矩阵 $A$ 可逆的充要条件是  $A$ 的任意一个特征值不等于零.

$A$ 为奇异(singular)矩阵的充要条件是 $A$ 至少有一个特征值等于零.

**Note:** A certain eigenvector of  $A$  cannot be corresponding to different eigenvalues. (  $A$ 的一个特征向量不能属于不同的特征值.)

If  $\mathbf{x}$  were an eigenvector of  $A$  corresponding to different eigenvalues  $\lambda_1, \lambda_2 (\lambda_1 \neq \lambda_2)$ ,

i.e.,  $A\mathbf{x} = \lambda_1\mathbf{x}$  and  $A\mathbf{x} = \lambda_2\mathbf{x}$ ,

then  $\lambda_1\mathbf{x} = \lambda_2\mathbf{x}$ , that is,  $(\lambda_1 - \lambda_2)\mathbf{x} = \mathbf{0}$ .

Since  $\lambda_1 - \lambda_2 \neq 0$ , then  $\mathbf{x} = \mathbf{0}$ , which is impossible for an eigenvector.



**Property 1** If  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of  $\mathbf{A}$ , then  $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$  are the eigenvalues of  $\mathbf{A}^k$ , where  $k$  is a positive integer, and  $k$  may equal  $-1$  if  $\mathbf{A}$  is invertible.

Moreover, if  $\mathbf{x}$  is an eigenvector of  $\mathbf{A}$  corresponding to  $\lambda$ , then  $\mathbf{x}$  is also an eigenvector of  $\mathbf{A}^k$  corresponding to  $\lambda^k$ .

**Proof.** Notice that, if  $\mathbf{x}$  is an eigenvector of  $\mathbf{A}$  corresponding to  $\lambda_i$ , then

$$\mathbf{A}^2 \mathbf{x} = \mathbf{A}(\mathbf{A}\mathbf{x}) = \mathbf{A}(\lambda_i \mathbf{x}) = \lambda_i \mathbf{A}\mathbf{x} = \lambda_i(\lambda_i \mathbf{x}) = \lambda_i^2 \mathbf{x},$$

The proposition then follows.

**性质1** 若 $\lambda$ 是 $\mathbf{A}$ 的特征值,  $\mathbf{x}$  是 $\mathbf{A}$ 的属于 $\lambda$  的特征向量. 则

- (1)  $k\lambda$  是 $k\mathbf{A}$ 的特征值 ( $k$ 为任意常数) ;
- (2)  $\lambda^m$ 是 $\mathbf{A}^m$ 的特征值;
- (3) 若 $\mathbf{A}$ 可逆, 则 $\lambda^{-1}$ 为 $\mathbf{A}^{-1}$ 的一个特征值;

且  $\mathbf{x}$  仍然是矩阵 $k\mathbf{A}$ ,  $\mathbf{A}^m$ 和 $\mathbf{A}^{-1}$ 的分别对应于特征值  $k\lambda$ ,  $\lambda^m$  和  $\lambda^{-1}$  的特征向量.

**Example 5** Suppose a  $3 \times 3$  matrix  $A$  has eigenvalues  $1, -1, 2$ . And  $B = A^3 - 5A^2$ . Find  $|B|$ .

**Solution** The eigenvalues of  $B$  are:  $1^3 - 5 \cdot 1^2 = -4$ ,  
 $(-1)^3 - 5 \cdot (-1)^2 = -6$ ,  $2^3 - 5 \cdot 2^2 = -12$ .

So  $|B| = (-4)(-6)(-12) = -288$ .

---

**Property 2** The matrices  $A$  and  $A^T$  have same eigenvalues.

(性质2 矩阵  $A$  和  $A^T$  的特征值相同.)

**Proof.**  $\det(A - \lambda I) = \det(A - \lambda I)^T$   
 $= \det(A^T - (\lambda I)^T) = \det(A^T - \lambda I)$ .

Thus  $A$  and  $A^T$  have the same characteristic polynomials.

The proposition then follows.

## 回顾：人口流动问题

Possible? Yes!

$$A = PQP^{-1}$$

$$A^k = PQ^k P^{-1} \quad (Q: \text{diagonal})$$

$$AP = PQ$$

$$A \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 1 \times \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{7}{12} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{5}{6} \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & \\ & \frac{7}{12} \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} y_k \\ z_k \end{bmatrix} &= \mathbf{A}^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \mathbf{P} \mathbf{Q}^k \mathbf{P}^{-1} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{5}(y_0 + z_0) + \frac{1}{5} \times \left(\frac{7}{12}\right)^k (3y_0 - 2z_0) \\ \frac{3}{5}(y_0 + z_0) + \frac{1}{5} \times \left(\frac{7}{12}\right)^k (2z_0 - 3y_0) \end{bmatrix} \rightarrow \begin{bmatrix} \frac{2}{5}(y_0 + z_0) \\ \frac{3}{5}(y_0 + z_0) \end{bmatrix}, \quad \text{当 } k \rightarrow \infty \text{ 时.}
\end{aligned}$$

农业与非农业人口比例的趋势为 2 : 3.



- ❶  $y_0$  与  $z_0$  的比例对稳定的趋势有多大影响?
- ❷ 稳定的趋势与  $\mathbf{A}$  的什么特征有联系?
- ❸ 如果人口总数是变化的, 如何建模?



$$A = \begin{bmatrix} 3 & 1 \\ 4 & 6 \\ 1 & 5 \\ 4 & 6 \end{bmatrix}.$$

$$= \mathbf{P} \mathbf{Q} \mathbf{P}^{-1}.$$

$$\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 7/12 \end{bmatrix}$$

对  $A_{n \times n}$ ,

- 不同的特征值对应的特征向量线性无关吗?
- 是否存在可逆矩阵  $P$  和对角矩阵  $Q$ , 使得

$$A = P Q P^{-1} ?$$

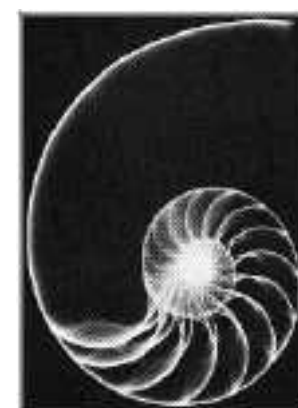
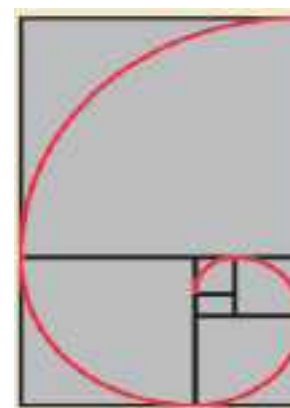
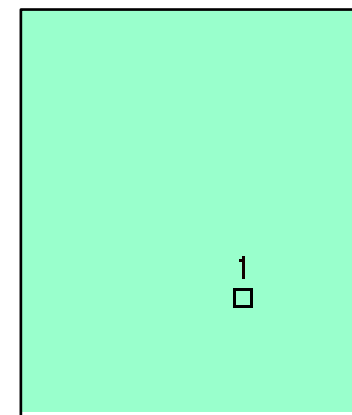
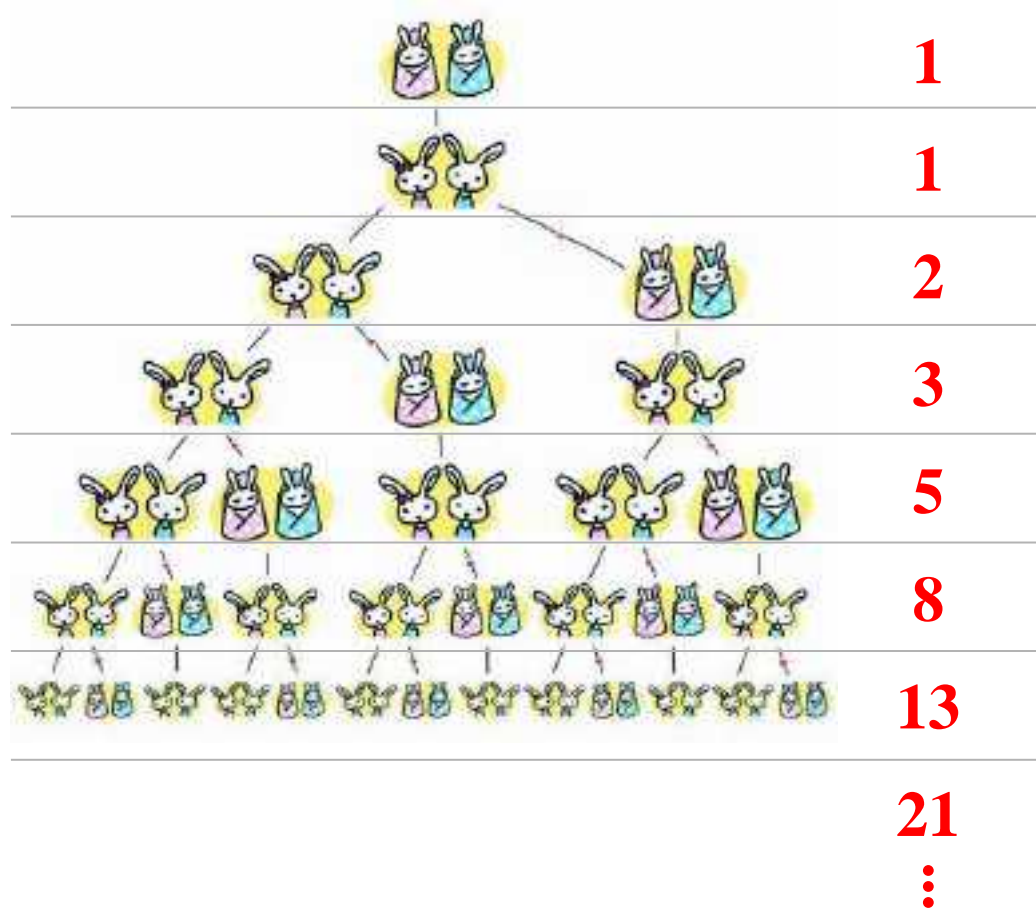
Come back later in 5.2:  
*Diagonalization* (对角化)  
*of a Matrix.*

# Another interesting application...

用矩阵方法求 *Fibonacci* 数列的通项公式



*Fibonacci*  
1175-1250



*Fibonacci*数列的递推关系为

$$F_0=0, F_1=1, F_{n+2}=F_{n+1}+F_n, \quad n=0,1,2,\dots$$

首先构造一组恒等式

$$\begin{cases} F_{n+2} = F_{n+1} + F_n, \\ F_{n+1} = F_{n+1}. \end{cases} \quad \Rightarrow \quad \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}}_{\mathbf{u}_n}.$$

$$\Rightarrow \quad \mathbf{u}_n = \mathbf{A}\mathbf{u}_{n-1} = \mathbf{A}^n \mathbf{u}_0.$$

$\mathbf{A}$ 的特征值为

$$\lambda_1 = \frac{1-\sqrt{5}}{2}, \lambda_2 = \frac{1+\sqrt{5}}{2}.$$

进而求出数列通项

$$F_n = \frac{\sqrt{5}}{5} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right].$$



## Key words:

*Definition*

*Calculation*

*Properties*

*Examples*

