CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

Dr. QI WANG

Department of Computer Science and Engineering
Office: Room413, CoE South Tower
Email: wangqi@sustech.edu.cn

Euclidean Algorithm

- The Euclidean algorithm in pseudocode

ALGORITHM 1 The Euclidean Algorithm.

```
procedure gcd(a, b: positive integers)
x := a
y := b
while y ≠ 0
    r := x mod y
    x := y
    y := r
return x{gcd(a, b) is x}
```

The number of divisions required to find $\gcd(a, b)$ is $O(\log b)$, where $a \geq b$. (this will be proved later.)

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$x := a$
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while $y \neq 0$
  $r := x$ **mod** y
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The number of divisions required to find $\gcd(a, b)$ is $O(\log b)$, where $a \geq b$. (this will be proved later.)

Why ?

Euclidean Algorithm

- Key steps in the Euclidean algorithm

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r_0 &= r_1 q_1 + r_2 && 0 \le r_2 < r_1, \\
r_1 &= r_2 q_2 + r_3 && 0 \le r_3 < r_2, \\
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See [Theorem 1 p. 347].

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  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

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By induction, such a recurrence relation is uniquely determined by this recurrence relation, and k initial conditions $a_0, a_1, \ldots, a_{k-1}$.

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Examples

$$P_n = (1.11)P_{n-1}$$

$$f_n = f_{n-1} + f_{n-2}$$

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$$H_n = 2H_{n-1} + 1$$

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$H_n = 2H_{n-1} + 1$	NOT homogeneous
$B_n = nB_{n-1}$	coefficients are not constants

Solving Linear Recurrence Relations

- **Example** Consider the recurrence relation

  $a_n = 2a_{n-1} - a_{n-2}$,

  Which of the following are solutions?

  - $a_n = 3n$:

  - $a_n = 2^n$:

  - $a_n = 5$:

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  Which of the following are solutions?

  - ◇ $a_n = 3n$: YES
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  ◇ Bring $a_n = r^n$ back to the recurrence relation:

  $$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k},$$

  i.e.,

  $$r^{n-k}(r^k - c_1 r^{k-1} - \cdots - c_k) = 0$$

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$$\text{i.e.,}\quad \begin{aligned} r^n &= c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}, \\ r^{n-k}&(r^k - c_1 r^{k-1} - \cdots - c_k) = 0 \end{aligned}$$

◇ The solutions to the *characteristic equation* can yield an explicit formula for the sequence.

$$(r^k - c_1 r^{k-1} - \cdots - c_k) = 0$$

Recall: Problem IV

- **Fibonacci number**

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  Consider $x^n = x^{n-1} + x^{n-2}$, with $x \neq 0$. There are two different roots

$$\phi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2}$$

  Then F_n can be the form of $a\phi^n + b\psi^n$. By $F_0 = 0$ and $F_1 = 1$, we have $a + b = 0$ and $\phi a + \psi b = 1$, leading to $a = \frac{1}{\sqrt{5}}$, $b = -a$. Therefore,

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

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  Theorem If this CE has 2 roots $r_1 \neq r_2$, then the sequence $\{a_n\}$ is a solution of the recurrence relation if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n \geq 0$ and constants α_1, α_2.

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  See [Theorem 1 p. 515].

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  We get $\alpha_1 = 3$ and $\alpha_2 = -1$. Thus,

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  Example $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$

The Case of Degenerate Roots

- **Theorem** If the CE $r^2 - c_1 r - c_2 = 0$ has only 1 root r_0, then

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  Exercise.

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  $$a_0 = \alpha_1 = 1$$
  $$a_1 = 2\alpha_1 + 2\alpha_2 = 0$$

  We get $\alpha_1 = 1$ and $\alpha_2 = -1$. Thus,

  $$a_n = 2^n - n2^n$$

The Case of Degenerate Roots in General

- **Theorem** [Theorem 4, p.519] Suppose that there are t roots $r_1, \ldots, r_t$ with multiplicities $m_1, \ldots, m_t$. Then

$$a_n = \sum_{i=1}^{t} \left(\sum_{j=0}^{m_i - 1} \alpha_{i,j} n^j \right) r_i^n,$$

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  Example

  $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 1$, $a_1 = -2$, $a_2 = -1$

Linear Nonhomogeneous Recurrence Relations

- **Definition** A *linear nonhomogeneous relation* with constant coefficients may contain some terms $F(n)$ that depend only on n

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n).$$

The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ is called the *associated homogeneous recurrence relation*.

- **Theorem** If $a_n = p(n)$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = p(n) + h(n),$$

where $a_n = h(n)$ is any solution to the associated homogeneous recurrence relation

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  We get $c = -1$ and $d = -3/2$. Thus,
  $$p(n) = -n - 3/2$$

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Definition The *generating funciton* for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k$$

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- **Theorem** Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then

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Useful Generating Functions

$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$

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$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

- **Problem 1** How many solutions are there to the equation

$$x_1 + x_2 + x_3 = 17,$$

where x_1, x_2, x_3 are nonnegative integers?

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$$C(n + r - 1, r) = C(19, 17) = C(19, 2)$$

- **Problem 2** Find the number of solutions of

$$x_1 + x_2 + x_3 = 17,$$

where x_1, x_2, x_3 are nonnegative integers with $2 \le x_1 \le 5$, $3 \le x_2 \le 6$, $4 \le x_3 \le 7$.

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Using *generating functions*, the number is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$$

- **Problem 3** In how many ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

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The coefficient of x^8 in the expansion

$$(x^2 + x^3 + x^4)^3$$

- **Problem 4** Use generating functions to find the number of k-combinations of a set with n elements, $C(n, k)$.

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Each of the n elements in the set contributes the term $(1+x)$ to the generating function $f(x) = \sum_{k=0}^{n} a^k x^k$. Hence, $f(x) = (1+x)^n$.

Then by the binomial theorem, we have $a_k = \binom{n}{k}$.

Next Lecture

- relation ...