CS201: Discrete Math for Computer Science
2021 Fall Semester Written Assignment # 6
Due: Dec. 29th, 2021, please submit at the beginning of class

Q.1 Let G be a simple graph. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.

Q.2 The complement of a simple graph $G = (V, E)$ is the graph $(V, \{(x, y) : x, y \in V, x \neq y\} \setminus E)$. A graph is *self-complementary* if it is isomorphic to its complement.

(a) Prove that no simple graph with two or three vertices is self-complementary, without enumerating all isomorphisms of such simple graphs.

(b) Find examples of self-complementary simple graphs with 4 and 5 vertices.

Q.3 Let G be a *simple* graph with n vertices. Show that if the degree of any vertex of G is $\geq (n - 1)/2$, then G must be connected.

Q.4 Let $n \geq 5$ be an integer. Consider the graph G_n whose vertices are the sets $\{a, b\}$, where $a, b \in \{1, \ldots, n\}$ and $a \neq b$, and whose adjacency rule is *disjointness*, that is, $\{a, b\}$ is adjacent to $\{a', b'\}$ whenever $\{a, b\} \cap \{a', b'\} = \emptyset$.

(a) Draw G_5.

(b) Find the degree of each vertex in G_n.

Q.5 Suppose that G is a graph on a finite set of n vertices. Prove that if G is disconnected, then its complement is connected.

Q.6 In an n-player *round-robin tournament*, every pair of distinct players compete in a single game. Assume that every game has a winner – there are no ties. The results of such a tournament can then be represented with a *tournament directed graph* where the vertices correspond to players and there is an edge $x \to y$ iff x beats y in their game.

(a) Explain whey a tournament directed graph cannot have cycles of length 1 or 2.

(b) Is the "beats" relation for a tournament graph always/sometimes/never: antisymmetric? reflexive? irreflexive? transitive?

(c) Show that a tournament graph represents a total ordering iff there are no cycles of length 3.

Q.7 Let G be a connected simple graph. Show that if an edge in a connected graph is not traversed by any simple cycle, then this edge is a *cut edge*.

Q.8 Given a graph $G = (V, E)$, an edge $e \in E$ is said to be a *bridge* if the graph $G' = (V, E \setminus \{e\})$ has more connected components than G. Prove that if all vertex degrees in a graph G are even, then G has no bridge.

Q.9 Let G be a connected graph, with the vertex set V. The *distance* between two vertices u and v, denoted by $dist(u, v)$, is defined as the *minimal* length of a path from u to v. Show that $dist(u, v)$ is a metric, i.e., the following properties hold for any $u, v, w \in V$:

(i) $dist(u, v) \geq 0$ and $dist(u, v) = 0$ if and only if $u = v$.

(ii) $dist(u, v) = dist(v, u)$.

(iii) $dist(u, v) \leq dist(u, w) + dist(w, v)$.

Q.10 Show that isomorphism of simple graphs is an equivalence relation.

Q.11 Suppose that G_1 and H_1 are isomorphic and that G_1 and H_2 are isomorphic. Prove or disprove that $G_1 \cup G_2$ and $H_1 \cup H_2$ are isomorphic.

Q.12 Given a graph G, its *line graph* $L(G)$ is defined as follows: every edge of G corresponds to a unique vertex of $L(G)$; any two vertices of $L(G)$ are adjacent if and only if their corresponding edges of G share a common endpoint. Prove that if G is regular (all vertices have the same degree) and connected, then $L(G)$ has an Euler circuit.

Q.13 Show that if G is simple graph with at least 11 vertices, then either G or its complement graph $\overline{G}$, the complement of G, is nonplanar.

Q.14 Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)v - (10/3)$ if $v \geq 4$.

Q.15 The **distance** between two distinct vertices v_1 and v_2 of a connected simple graph is the length (number of edges) of the shortest path between v_1 and v_2. The **radius** of a graph is the *minimum* over all vertices v of the maximum distance from v to another vertex. The **diameter** of a graph is the maximum distance between two distinct vertices. Find the radius and diameter of

(1) K_6

(2) $K_{4,5}$

(3) Q_3

(4) C_6

Q.16 Let G be a graph in which all vertices have degree at least d. Prove that G contains a path of length d.

Q.17 Let n be a positive integer. Construct a **connected** graph with $2n$ vertices, such that there are *exactly* **two** vertices of degree i for each $i = 1, 2, \ldots, n$. (You can sketch some pictures, but your graph has to be described by a concise adjacency rule. Remember to prove that your graph is connected.)

Q.18 Consider the two graphs G and H. Answer the following three questions, and explain your answers.

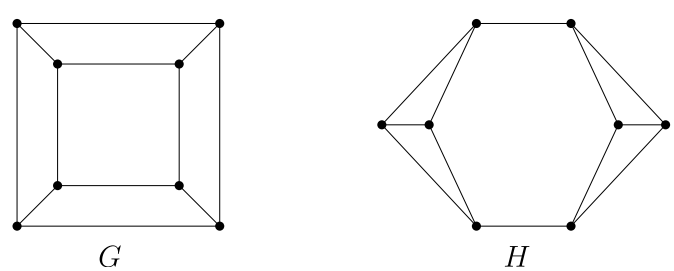

Figure 1: Q.18

(1) Which of the two graphs is/are *bipartite*?

(2) Are the two graphs *isomorphic* to each other?

(3) Which of the two graphs has/have an *Euler circuit*?

Q.19 Prove that $G = (V, E)$ is a tree if and only if $|V| = |E| + 1$ and G has no cycles.

Q.20 The **rooted Fibonacci trees** T_n are defined recursively in the following way. T_1 and T_2 are both the rooted tree consisting of a single vertex, and for $n = 3, 4, \ldots$, the rooted tree T_n is constructed from a root with T_{n-1} as its left subtree and T_{n-2} as its right subtree. How many vertices, leaves, and internal vertices does the rooted Fibonacci tree T_n have, where n is a positive integer? What is its height?

Q.21

What is the value of each of these postfix expressions?

(a) $5\ 2\ 1\ -\ -\ 3\ 1\ 4\ +\ +\ *$

(b) $9\ 3\ /\ 5\ +\ 7\ 2\ -\ *$

Q.22

Use Prim's algorithm to find a minimum spanning tree for the given weighted graph.

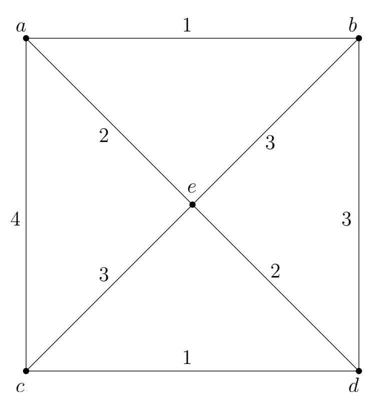

Figure 2: Q.22

Q.23

Use Kruskal's algorithm to find a minimum spanning tree for the weighted graph in Q.22.