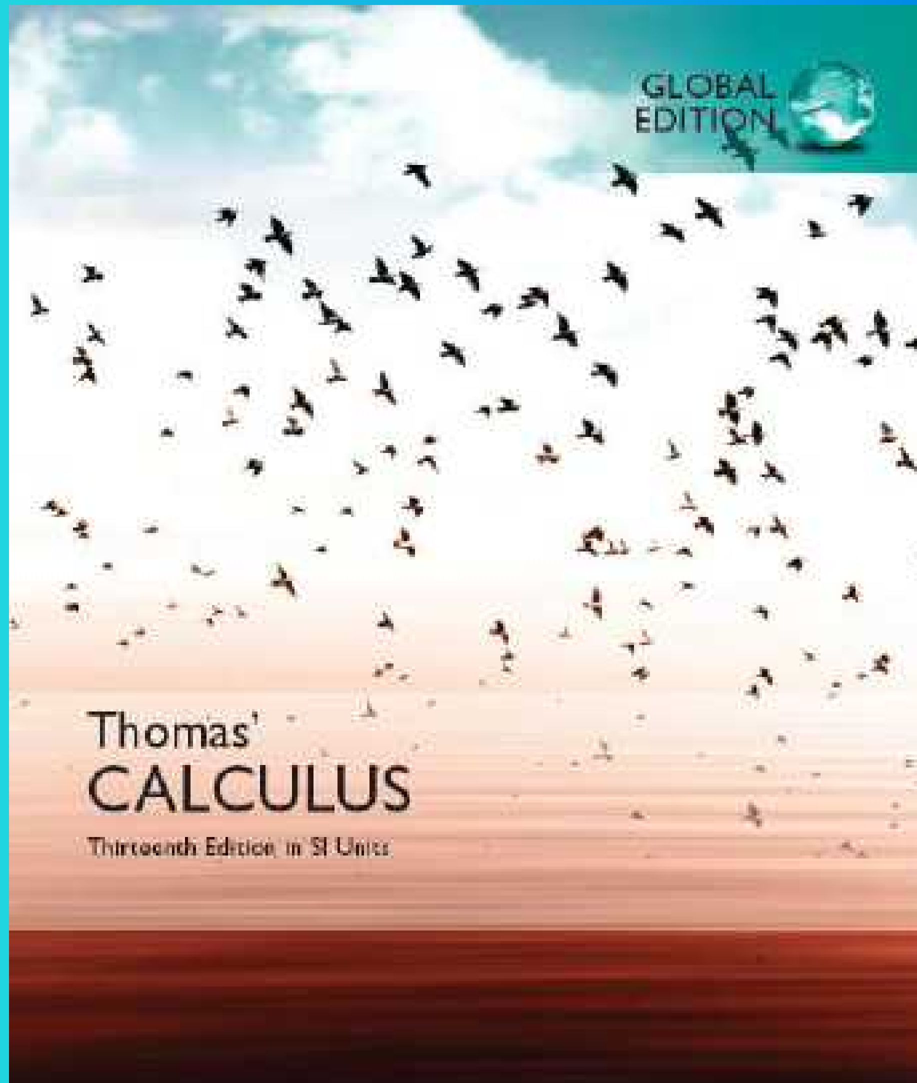

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 15

Multiple Integrals

重积分

15.1

Double and Iterated Integrals over Rectangles

矩形区域上的二重积分和累次积分

a function $f(x, y)$ defined on a rectangular region R,

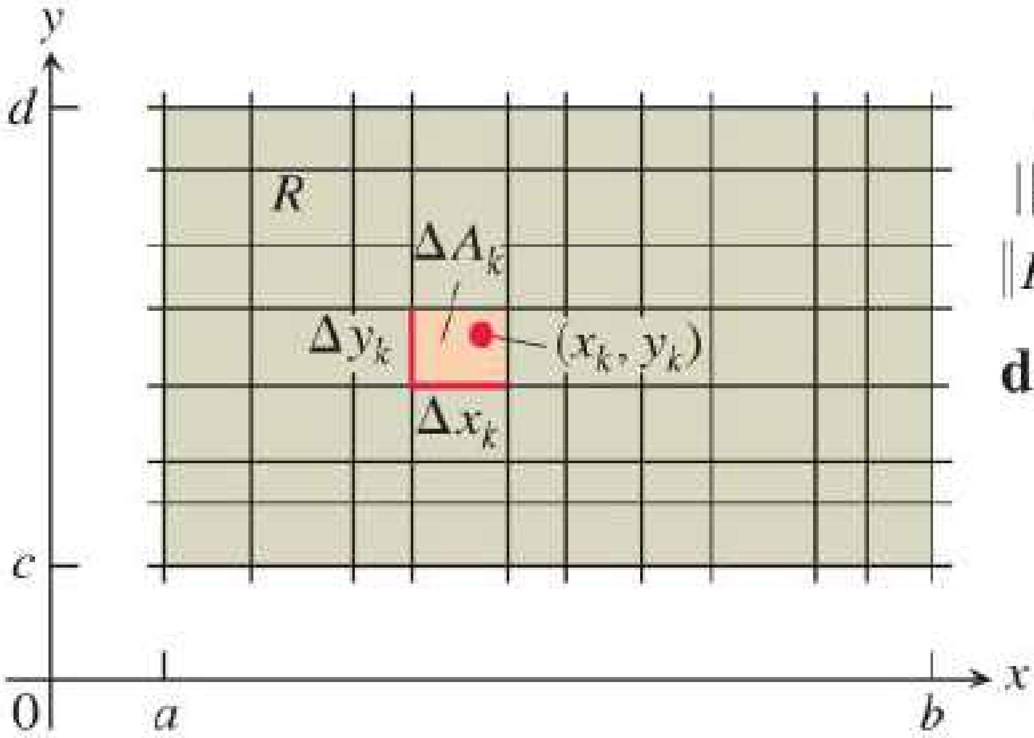
R: $\quad a \le x \le b, \quad c \le y \le d.$

$\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, where ΔA_k is the area of the kth small rectangle.

a point (x_k, y_k) in the kth small rectangle,

a Riemann sum over R,

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k.$$



$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k.$$

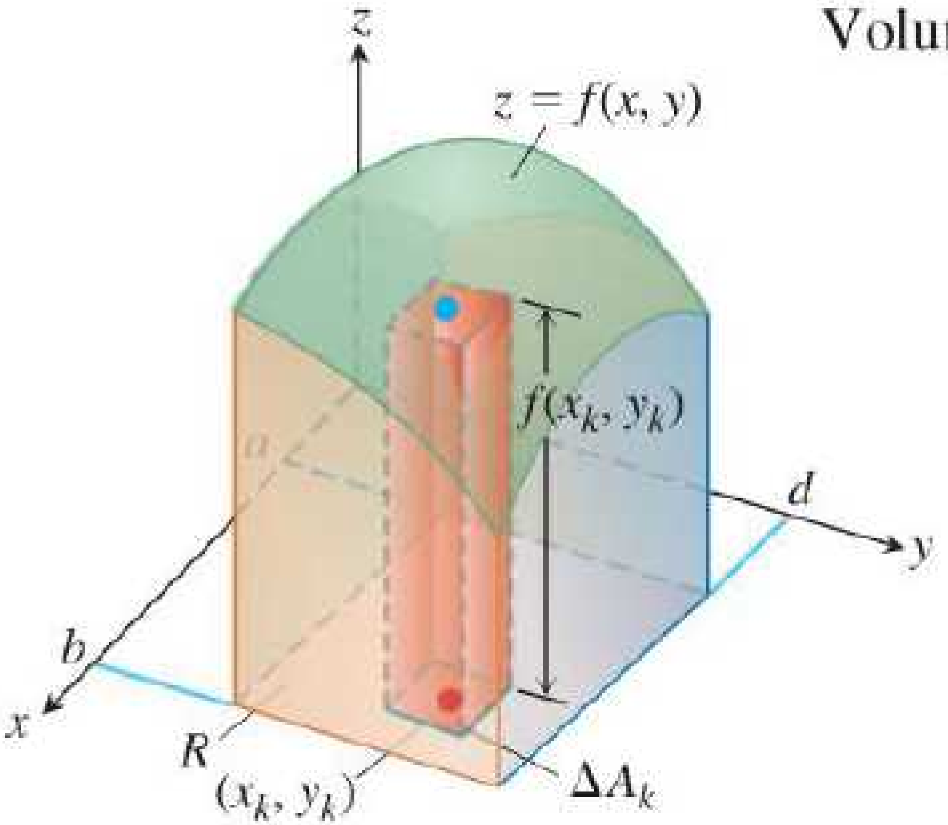
$\|P\|$, is the largest width or height

double integral of f over R,

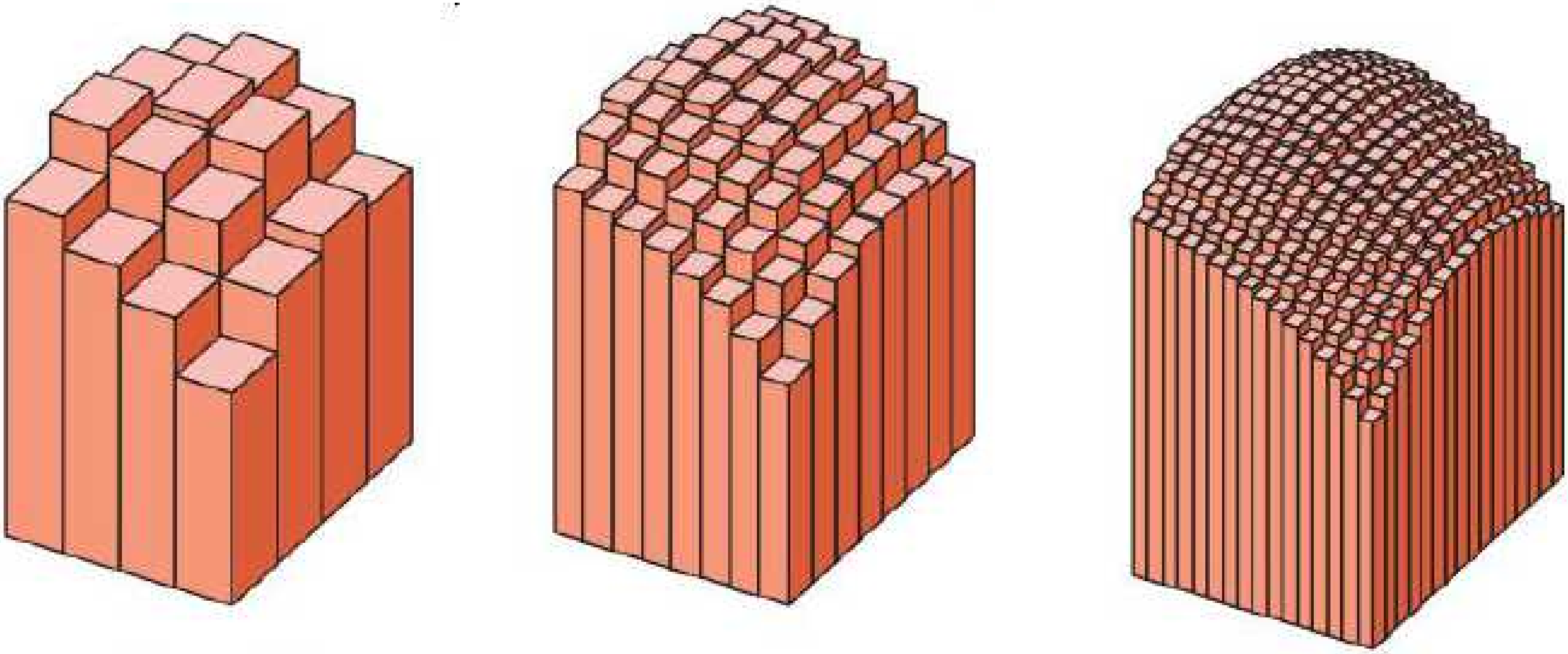
$$\iint_R f(x, y)\,dx\,dy.$$

$$\iint_R f(x,y)\,dx\,dy = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(x_k, y_k)\Delta A_k \qquad \iint_R f(x,y)\,dA$$

Double Integrals as Volumes

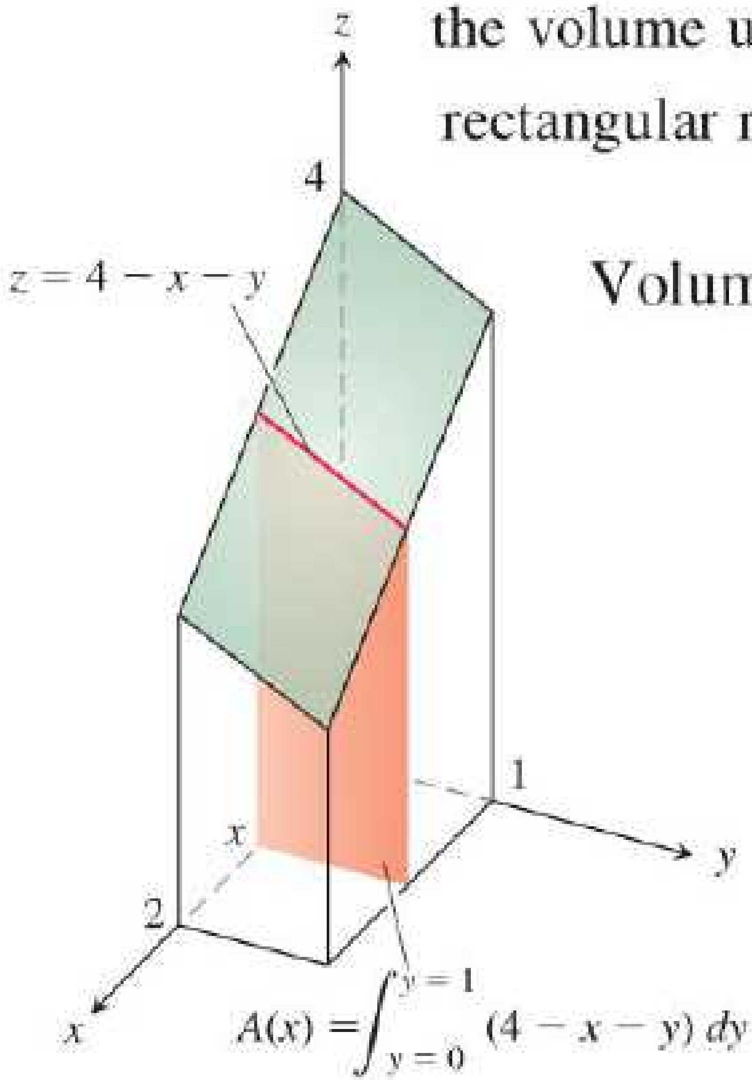


$$\text{Volume} = \lim_{n\to\infty} S_n = \iint_R f(x,y)\,dA,$$



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

Fubini's Theorem for Calculating Double Integrals



the volume under the plane $z = 4 - x - y$ over the rectangular region R: $0 \le x \le 2, 0 \le y \le 1$

$$\text{Volume} = \lim_{n\to\infty} S_n = \iint_R f(x, y)\, dA,$$

$$= \int_{x=0}^{x=2} A(x)\, dx$$

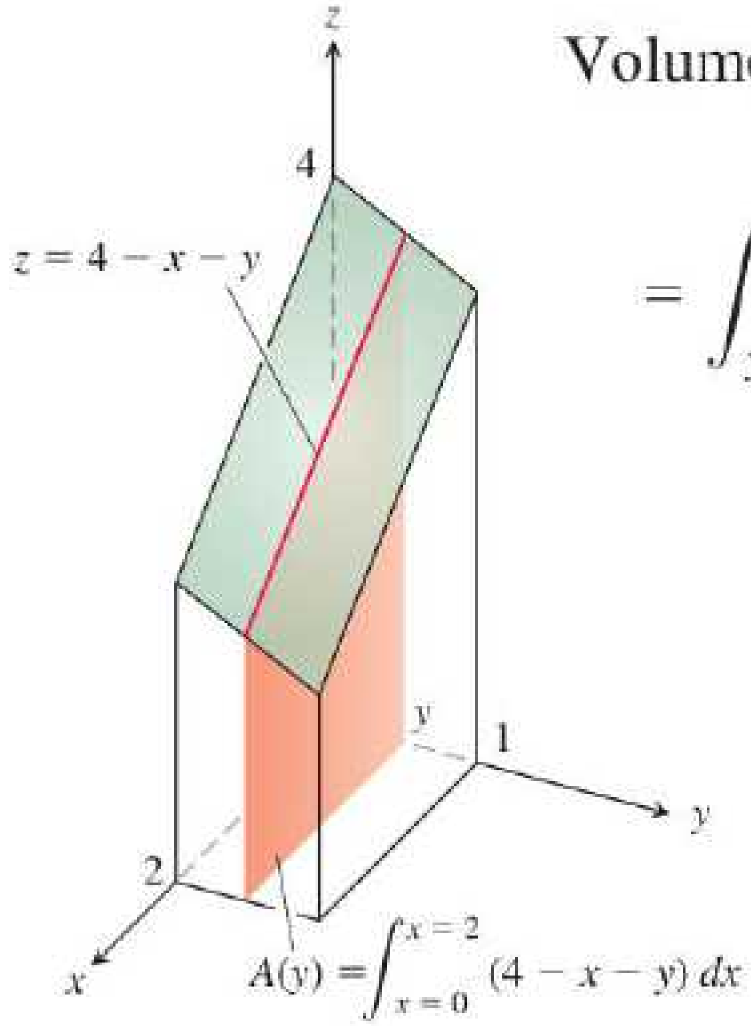
$$= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y)\, dy \right) dx$$

iterated or **repeated integral**,

$$= \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx = \left[\frac{7}{2}x - \frac{x^2}{2} \right]_0^2 = 5.$$

$$\text{Volume} = \int_0^1 \int_0^2 (4 - x - y)\, dx\, dy.$$

$$= \int_{y=0}^{y=1} (6 - 2y)\, dy = \left[6y - y^2 \right]_0^1 = 5,$$



THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region $R: a \leq x \leq b,\ c \leq y \leq d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

EXAMPLE 1 Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2 y \quad \text{and} \quad R: \quad 0 \leq x \leq 2, \quad -1 \leq y \leq 1.$$

Solution

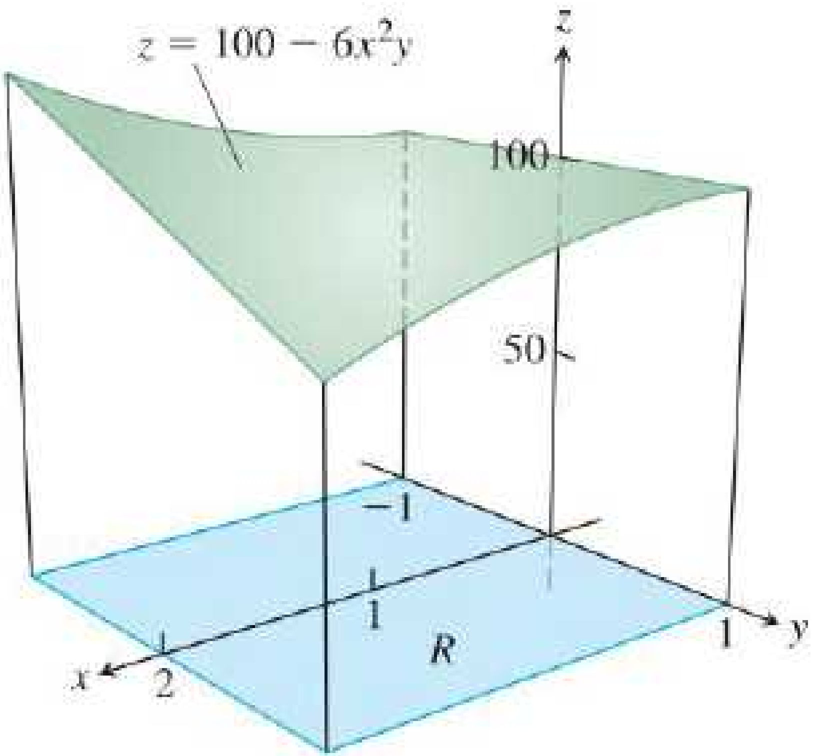
$$\iint_R f(x, y)\, dA = \int_{-1}^{1} \int_0^2 (100 - 6x^2 y)\, dx\, dy$$

$$= \int_{-1}^{1} (200 - 16y)\, dy = \Big[200y - 8y^2 \Big]_{-1}^{1} = 400.$$

$$\int_0^2 \int_{-1}^1 (100 - 6x^2y)\, dy\, dx =$$

$$= \int_0^2 \left[(100 - 3x^2) - (-100 - 3x^2) \right] dx$$

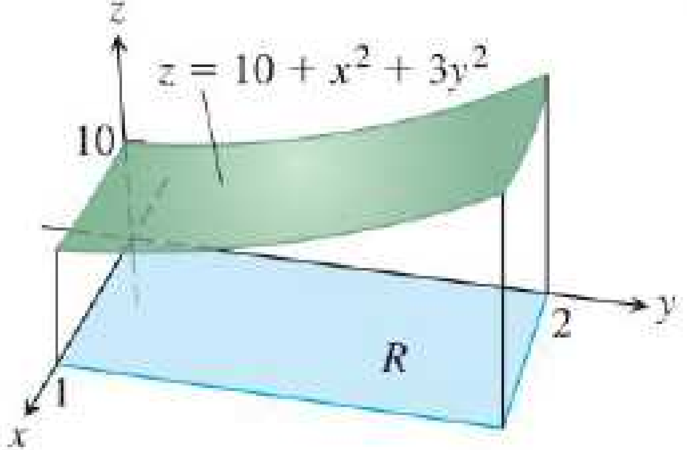
$$= \int_0^2 200\, dx = 400.$$



EXAMPLE 2

Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \le x \le 1$, $0 \le y \le 2$.

Solution



$$V = \iint (10 + x^2 + 3y^2)\, dA$$

$$= \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\, dy\, dx$$

$$= \int_0^1 \Big[10y + x^2 y + y^3 \Big]_{y=0}^{y=2} dx$$

$$= \int_0^1 (20 + 2x^2 + 8)\, dx = \Big[20x + \frac{2}{3}x^3 + 8x \Big]_0^1 = \frac{86}{3}.$$

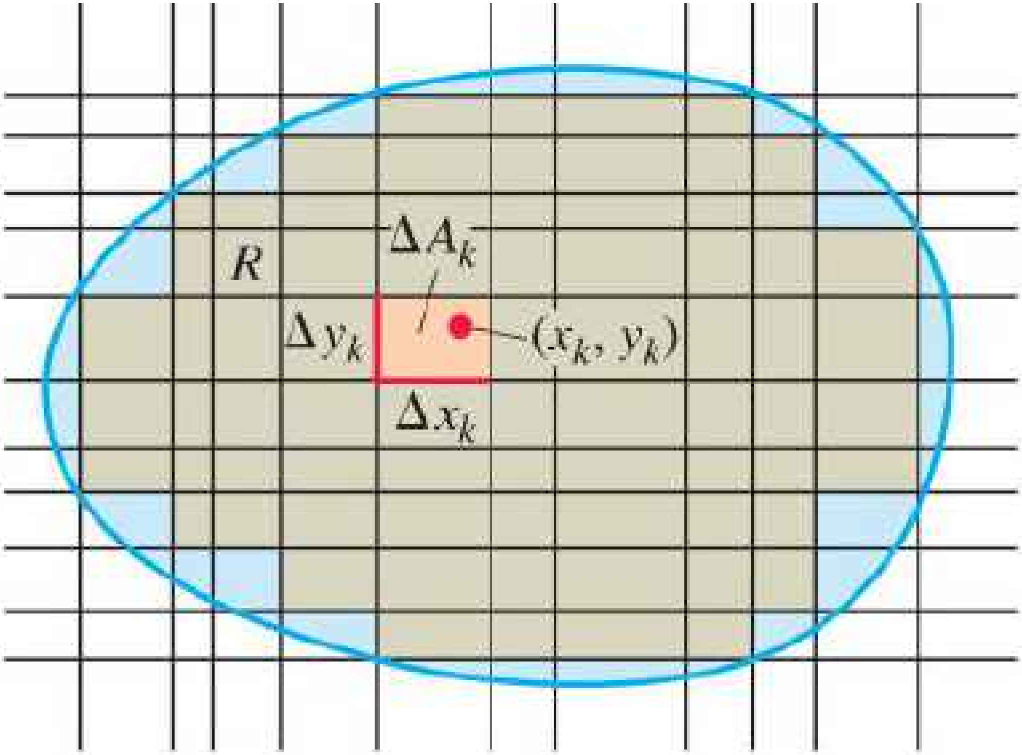
15.2

Double Integrals over General Regions

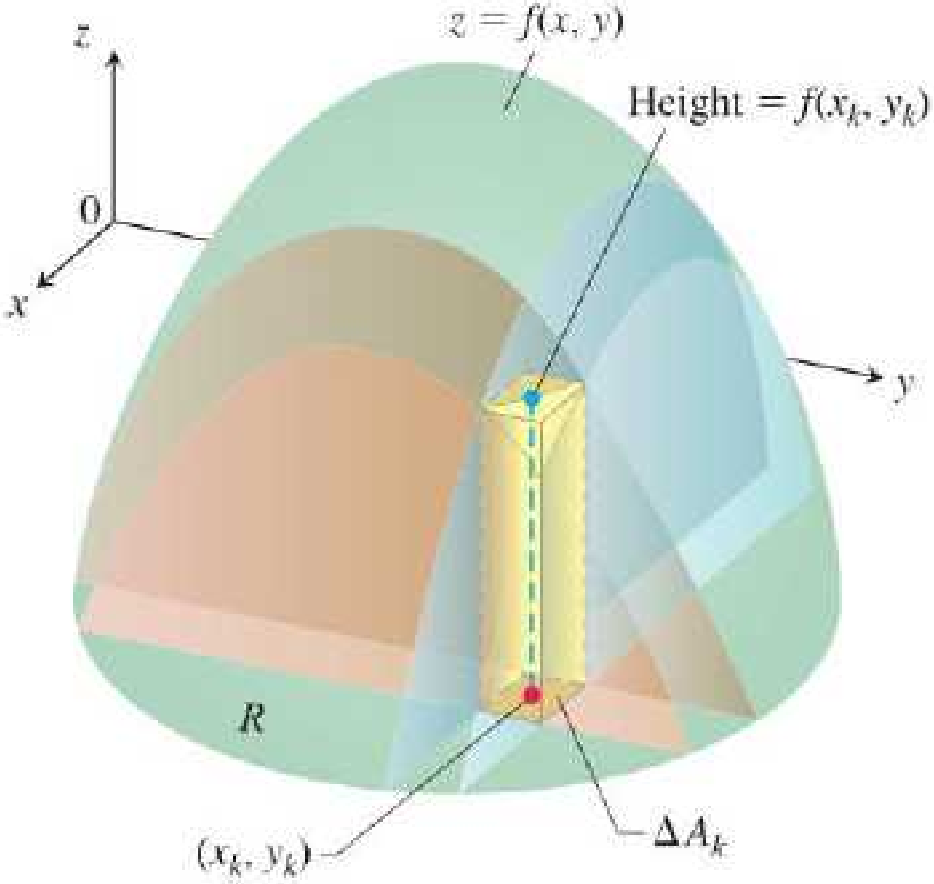
一般区域上的二重积分

Double Integrals over Bounded, Nonrectangular Regions

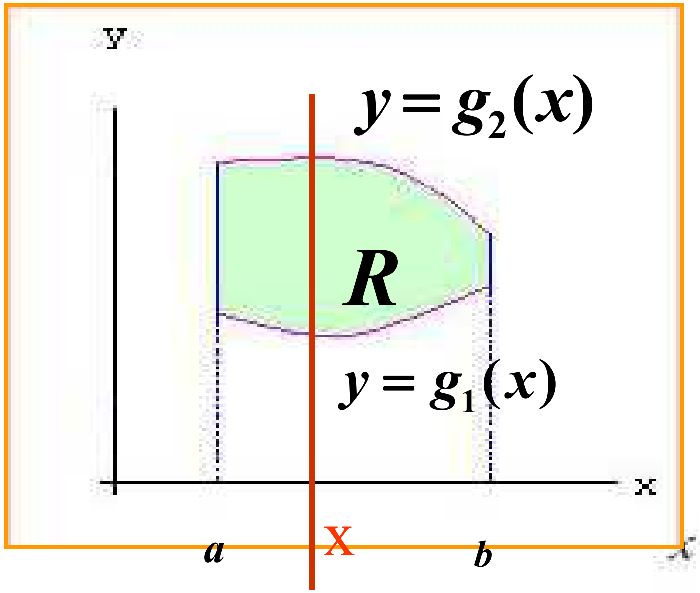
$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k = \iint_R f(x, y)\, dA.$$

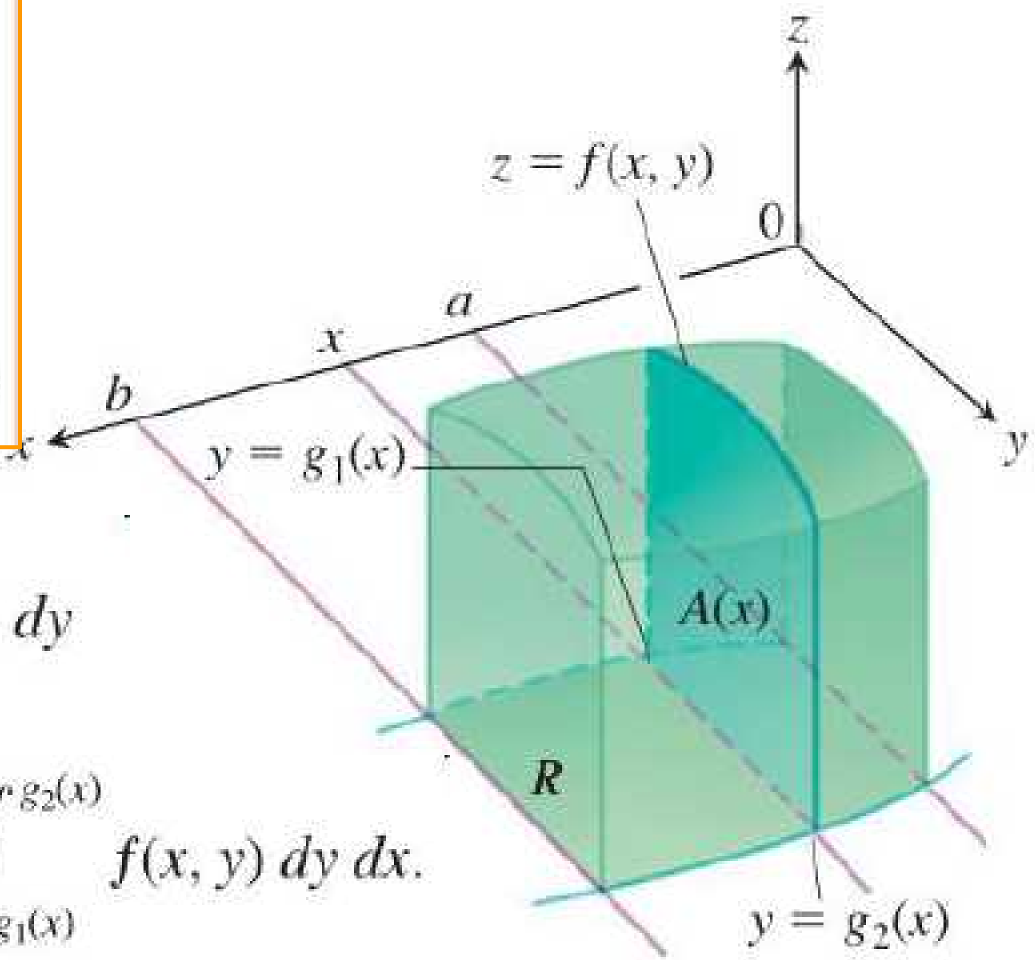


Volumes



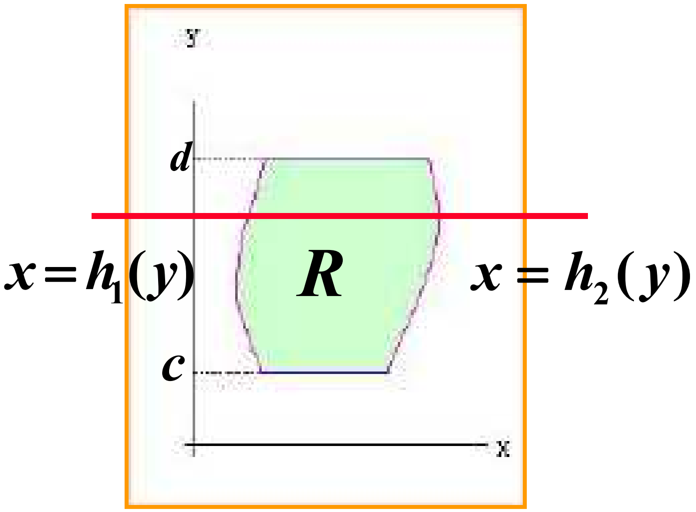
$$\text{Volume} = \lim \sum f(x_k, y_k)\,\Delta A_k = \iint_R f(x, y)\,dA$$





$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\, dy$$

$$V = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$



$$\text{Volume} = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

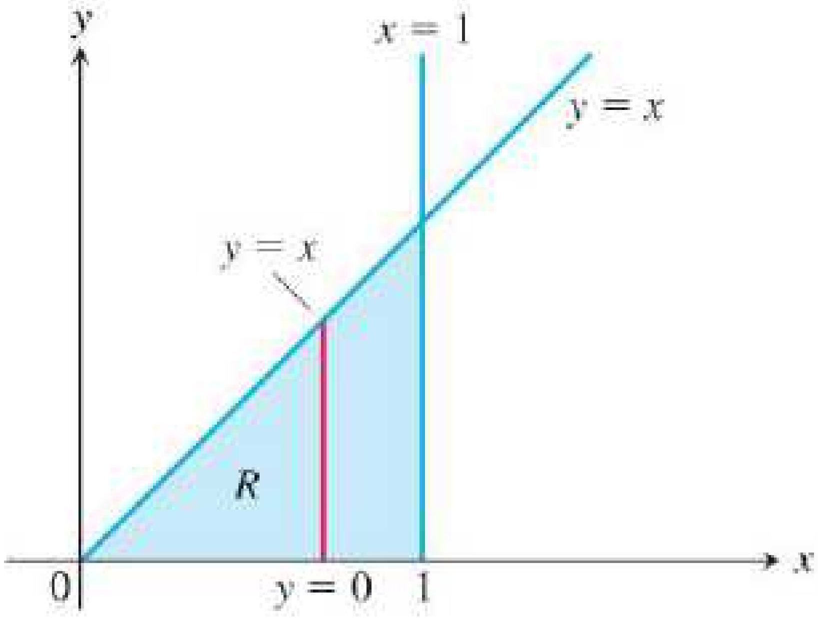
EXAMPLE 1

Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = f(x, y) = 3 - x - y$.

Solution

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx$$

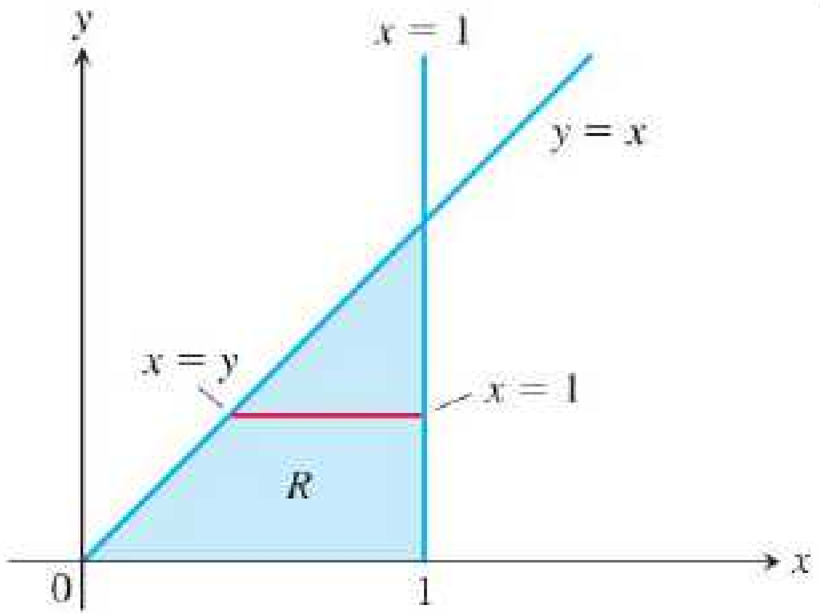
$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = 1.$$



$$V = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy$$

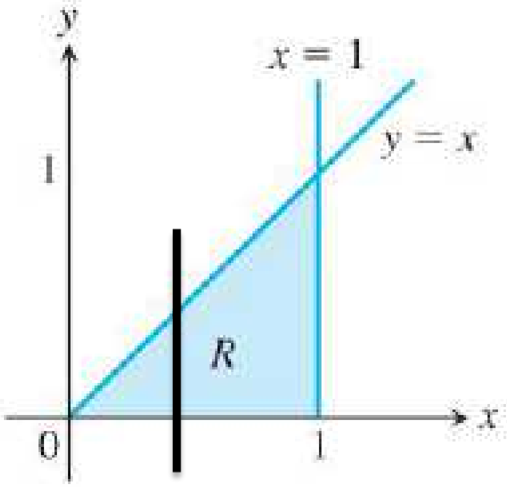
$$= 1.$$



EXAMPLE 2 Calculate

$$\iint_R \frac{\sin x}{x}\,dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, t
the line $y = x$, and the line $x = 1$.



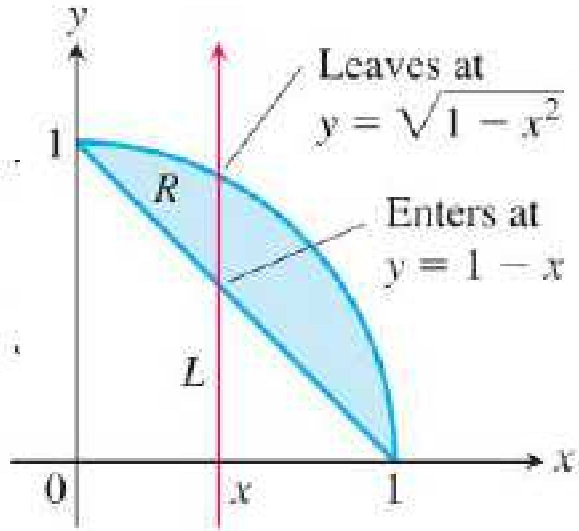
Solution $$\iint_R \frac{\sin x}{x}\,dA,$$

$$= \int_0^1 \left(\int_0^x \frac{\sin x}{x}\,dy \right) dx$$

$$= \int_0^1 \sin x\,dx = -\cos(1) + 1 \approx 0.46.$$

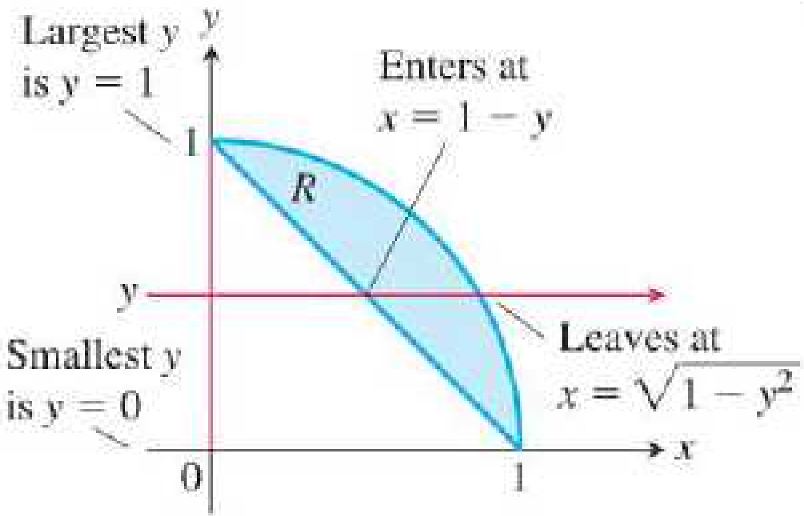
Using Vertical Cross-Sections

$$\iint_R f(x, y)\, dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y)\, dy\, dx.$$



Using Horizontal Cross-Sections

$$\iint_R f(x, y)\, dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy.$$



EXAMPLE 3 Sketch the region of integration for the integral

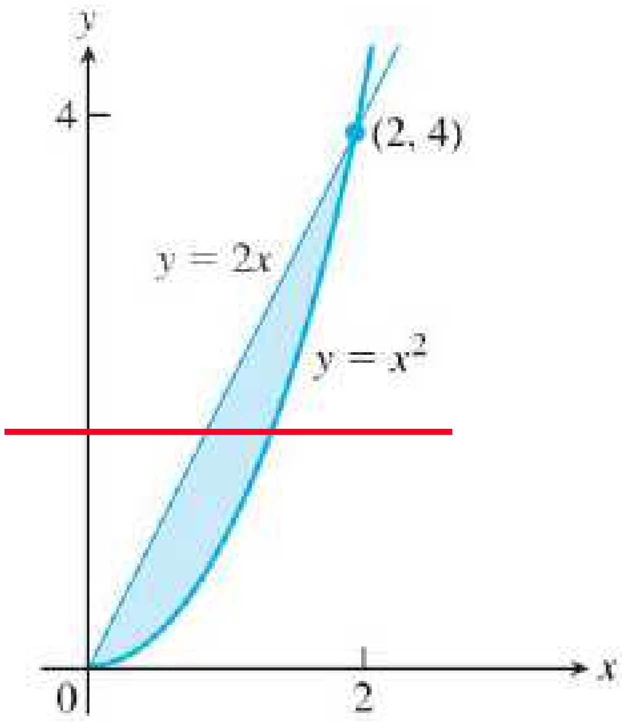
$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

and reverse the order of the integral.

Solution

$x^2 \leq y \leq 2x$ and $0 \leq x \leq 2$.

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$



Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c\iint_R f(x, y)\, dA$

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

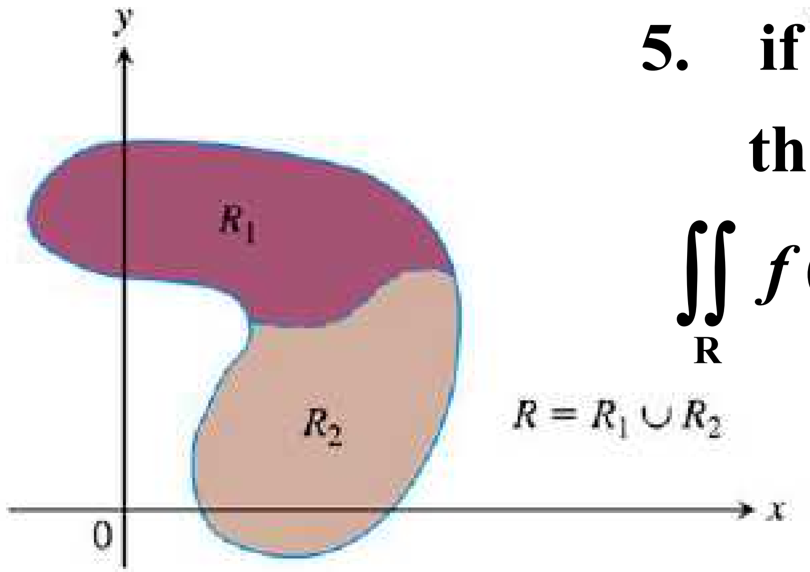
3. *Domination:*

(a) $\displaystyle\iint_R f(x, y)\, dA \geq 0 \quad \text{if} \quad f(x, y) \geq 0 \text{ on } R$

(b) $$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA \qquad \text{if} \qquad f(x, y) \geq g(x, y) \text{ on } R$$

4. *Additivity:* $$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

if R is the union of two nonoverlapping regions R_1 and R_2



5. **if f is continuous on the R, then there is $(c_1, c_2) \in R$ such that**

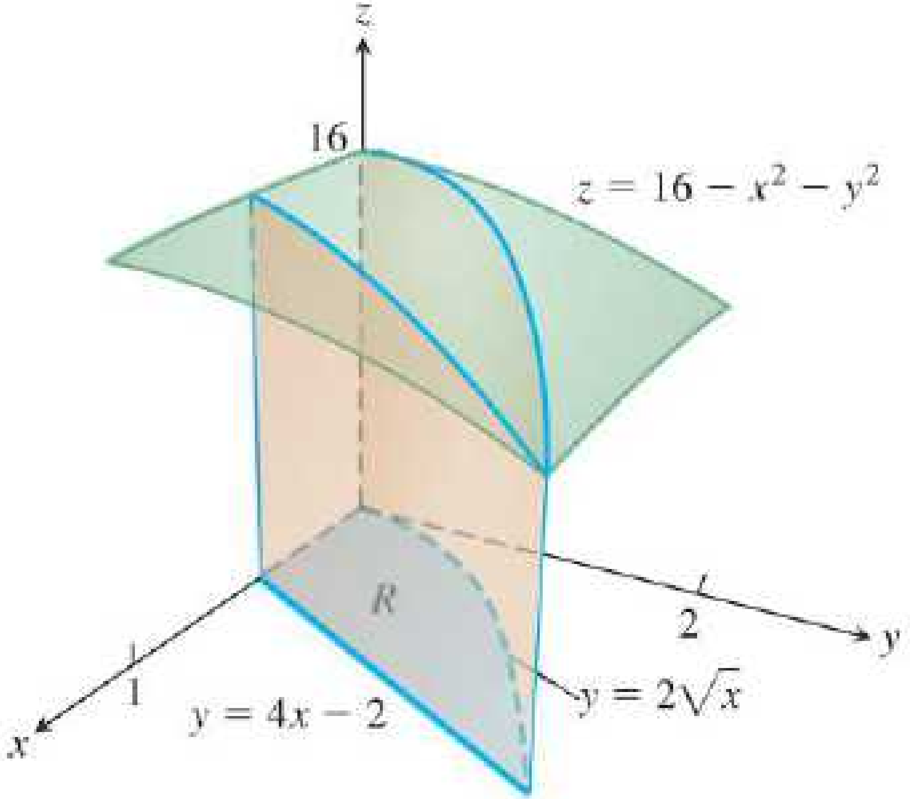
$$\iint_R f(x,y)\,dxdy = f(c_1, c_2)\,\text{Area}(R)$$

EXAMPLE 4

Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

Solution

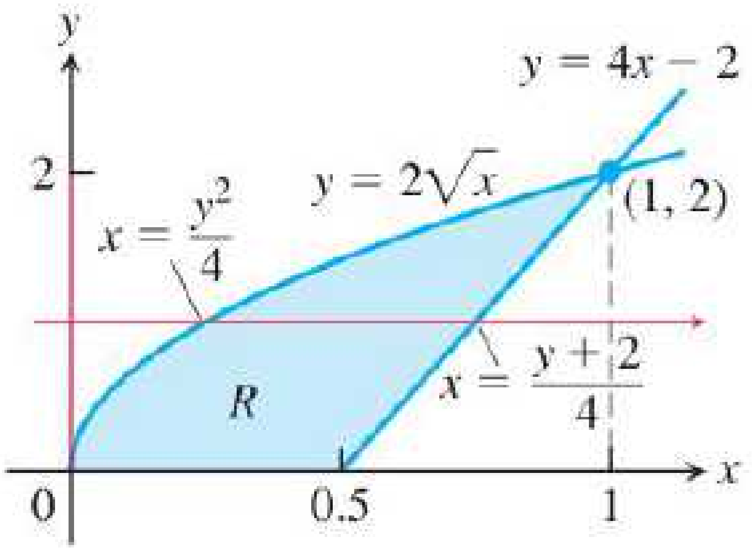
$$\iint_R (16 - x^2 - y^2)\, dA$$



$$\iint_R (16 - x^2 - y^2)\, dA = \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\, dx\, dy$$

$$= \int_0^2 \left[4(y + 2) - \frac{(y + 2)^3}{3 \cdot 64} - \frac{(y + 2)y^2}{4} - 4y^2 + \frac{y^6}{3 \cdot 64} + \frac{y^4}{4} \right] dy$$

$$= \frac{20803}{1680} \approx 12.4$$



15.3

Area by Double Integration

Areas of Bounded Regions in the Plane

DEFINITION The **area** of a closed, bounded plane region R is

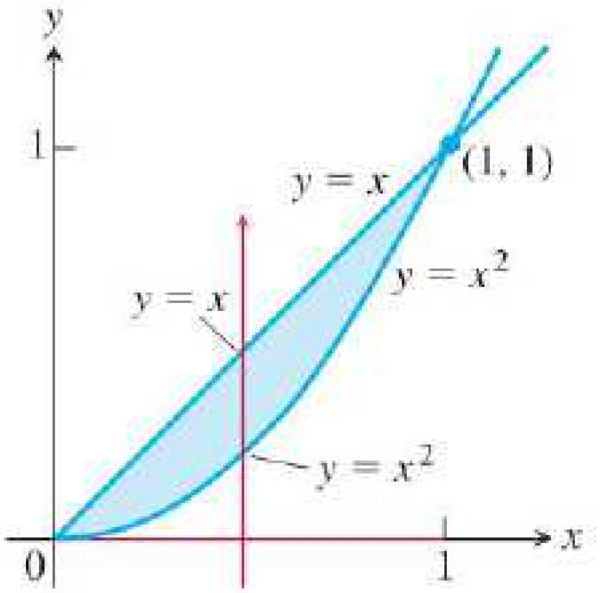
$$A = \iint_R dA.$$

If we take $f(x, y) = 1$

$$\lim_{||P||\to 0} \sum_{k=1}^{n} \Delta A_k = \iint_R dA.$$

EXAMPLE 1

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.



Solution $\iint\limits_R 1 dA$

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \Big[\, y \,\Big]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

EXAMPLE 2

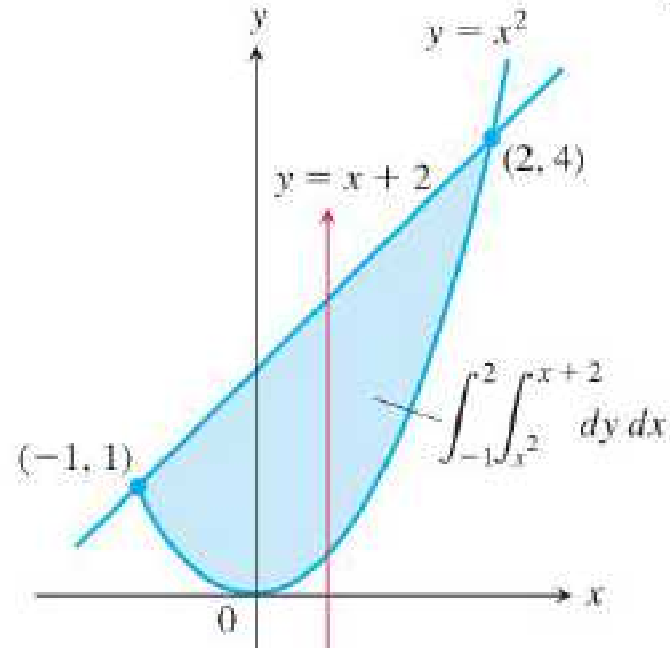
Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution

$$A = \int_{-1}^{2}\int_{x^2}^{x+2} dy\, dx = \int_{-1}^{2}(x + 2 - x^2)\, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}.$$

$$= \int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_{1}^{4}\int_{y-2}^{\sqrt{y}} dx\, dy.$$



EXAMPLE 3 Find the area of the playing field described by $R: -2 \le x \le 2,\ -1-\sqrt{4-x^2} \le y \le 1+\sqrt{4-x^2}$, using

(a) Fubini's Theorem (b) Simple geometry.

Solution (a)

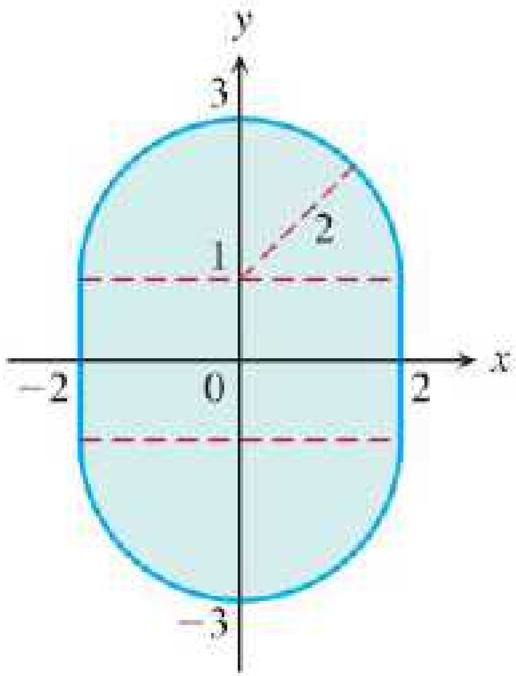
$$A = \iint_R dA = 4\int_0^2 \int_0^{1+\sqrt{4-x^2}} dy\,dx$$

$$= 4\int_0^2 \left(1+\sqrt{4-x^2}\right) dx$$

$$= 4\left[x + \frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 = 8 + 4\pi.$$

(b)

$$A = 8 + \pi 2^2 = 8 + 4\pi.$$



Average Value

$$\textbf{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f \, dA.$$

EXAMPLE 4

Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R: 0 \le x \le \pi,\ 0 \le y \le 1.$

Solution

$$\int_0^{\pi} \int_0^1 x \cos xy \, dy \, dx = \int_0^{\pi} \left[\sin xy \right]_{y=0}^{y=1} dx$$

$$= \int_0^{\pi} (\sin x - 0) \, dx = -\cos x \Big]_0^{\pi} = 1 + 1 = 2.$$

The area of R is π. The average value of f over R is $2/\pi$.

15.4

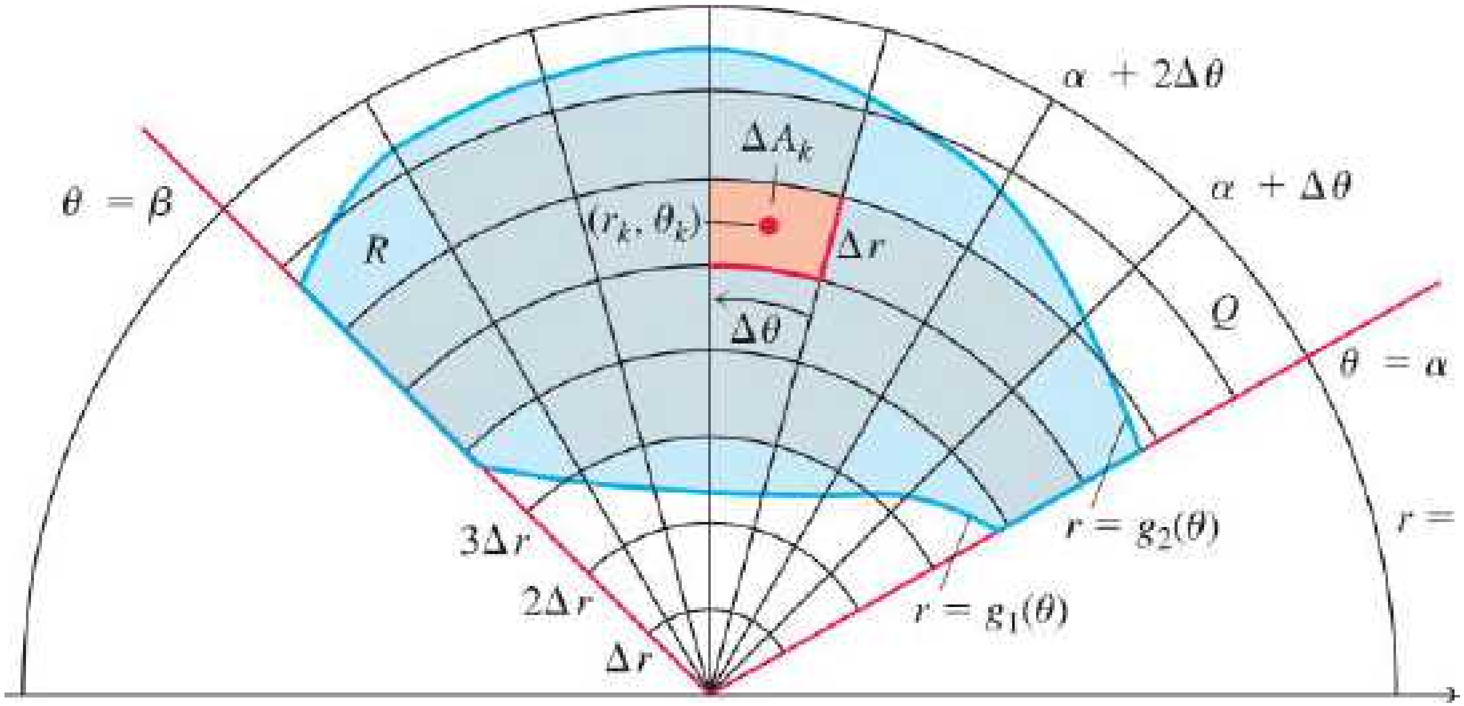
Double Integrals in Polar Form

极坐标形式的二重积分

Integrals in Polar Coordinates

Suppose that a function $f(r, \theta)$ is defined over a region R

The region R: $g_1(\theta) \le r \le g_2(\theta), \alpha \le \theta \le \beta,$



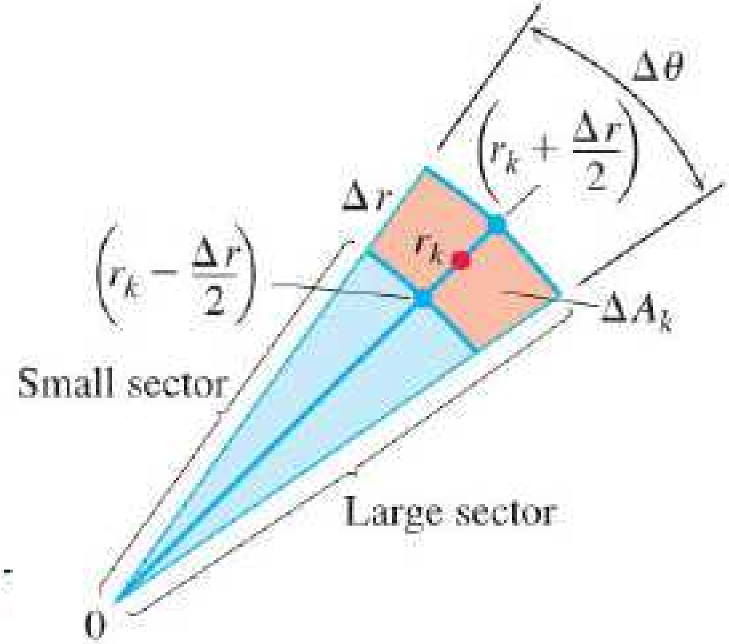
$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\,\Delta A_k. \qquad \lim_{n\to\infty} S_n = \iint_R f(r, \theta)\, dA.$$

$$\Delta A_k = \begin{pmatrix}\text{area of}\\ \text{large sector}\end{pmatrix} - \begin{pmatrix}\text{area of}\\ \text{small sector}\end{pmatrix}$$

leads to the formula $\Delta A_k = r_k\,\Delta r\,\Delta\theta$.

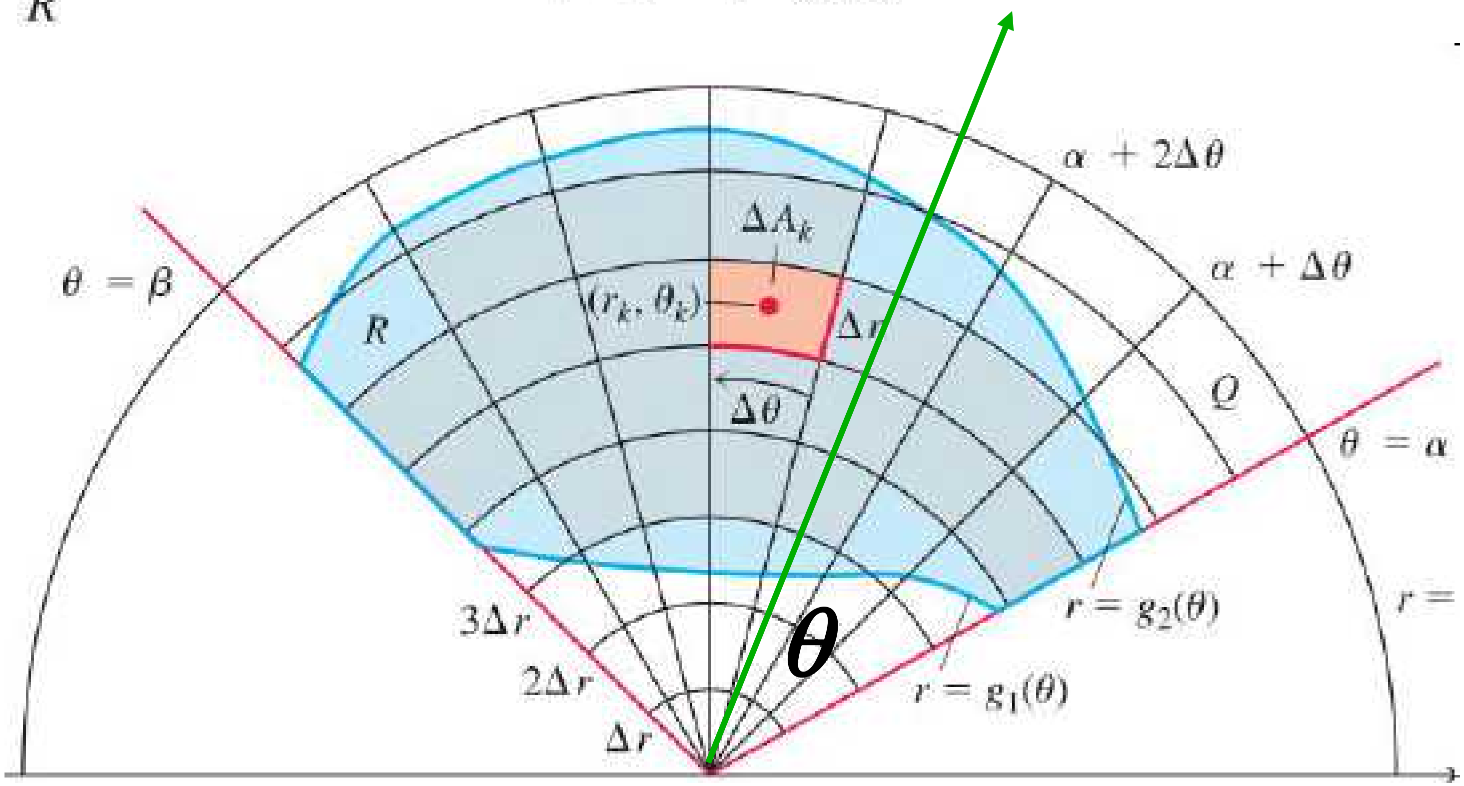
$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k\,\Delta r\,\Delta\theta.$$

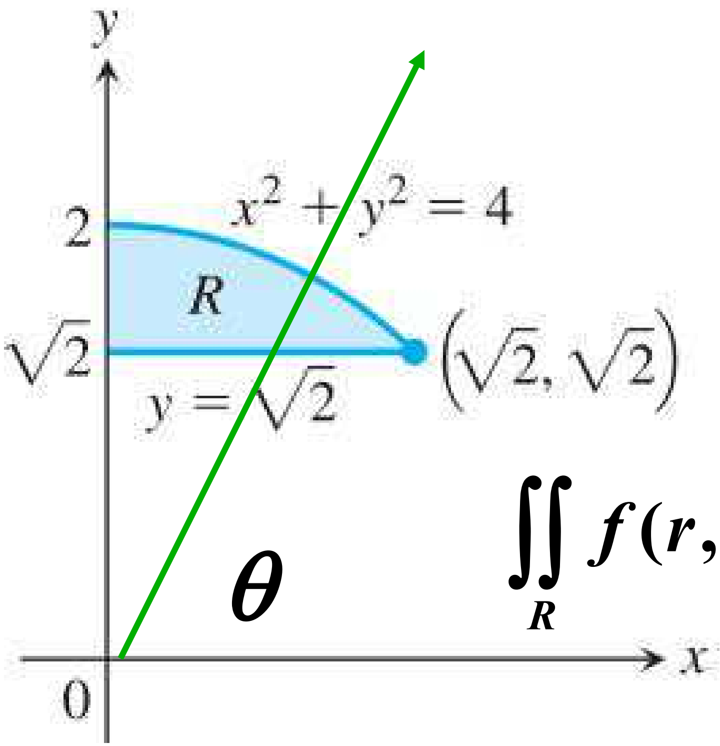
$$\lim_{n\to\infty} S_n = \iint_R f(r, \theta)\, r\, dr\, d\theta.$$



The region R: $g_1(\theta) \leq r \leq g_2(\theta)$, $\alpha \leq \theta \leq \beta$,

$$\iint_R f(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$





$$y = \sqrt{2} \Rightarrow r = \frac{\sqrt{2}}{\sin\theta}$$

$$x^2 + y^2 = 4 \Rightarrow r = 2$$

$$\iint_R f(r,\theta) r\,dr\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\int_{\frac{\sqrt{2}}{\sin\theta}}^{2} f(r,\theta) r\,dr \right) d\theta$$

$$R: x^2 + y^2 \le 2x$$

$$\iint_R f(r,\theta) r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{2\cos\theta} f(r,\theta) r\,dr \right) d\theta$$

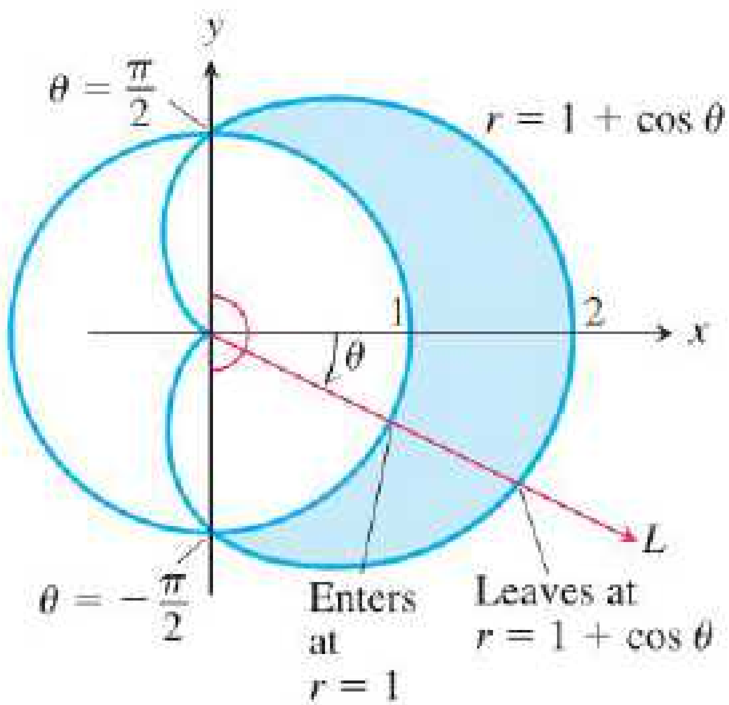
EXAMPLE 1

Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

$$\iint_R f(r,\theta)\,dA$$

$$= \iint_R f(r,\theta)\,r\,dr\,d\theta =$$

$$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta.$$



Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

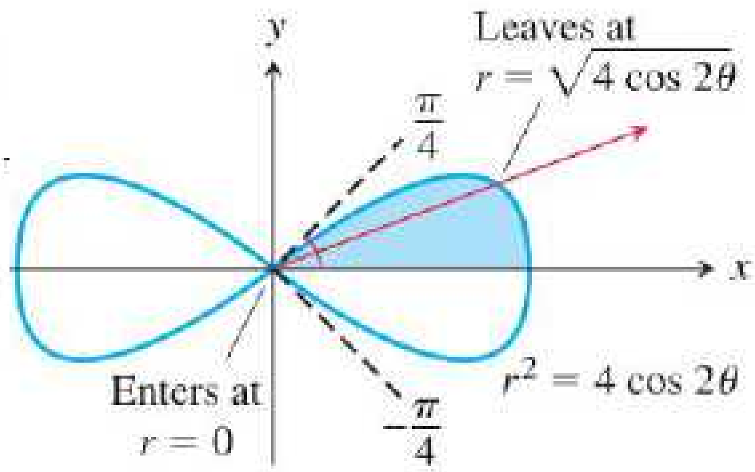
$$A = \iint_R r\,dr\,d\theta.$$

EXAMPLE 2 Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Solution

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$



Changing Cartesian Integrals into Polar Integrals

$$\iint_R f(x, y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta,$$

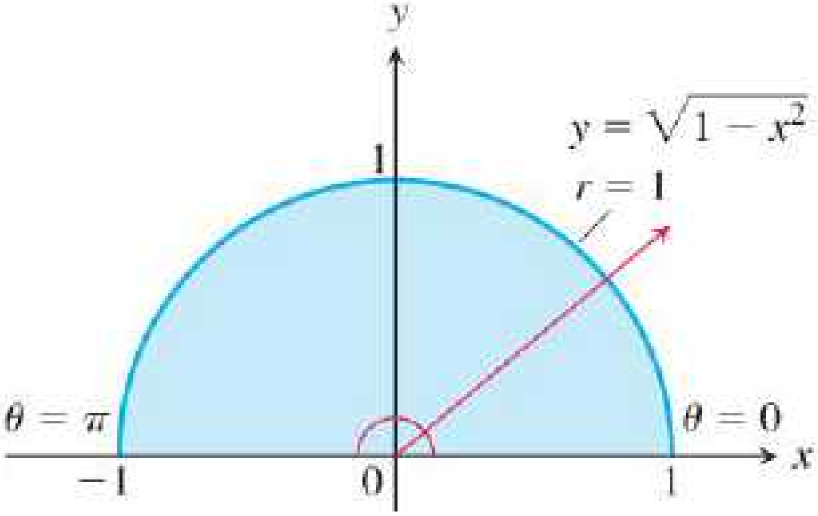
EXAMPLE 3 Evaluate $\displaystyle\iint_R e^{x^2+y^2}\,dy\,dx,$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$

Solution

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{1}{2}(e - 1)\,d\theta = \frac{\pi}{2}(e - 1).$$



EXAMPLE 4 Evaluate the integral $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$

Solution

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.$$

EXAMPLE 5

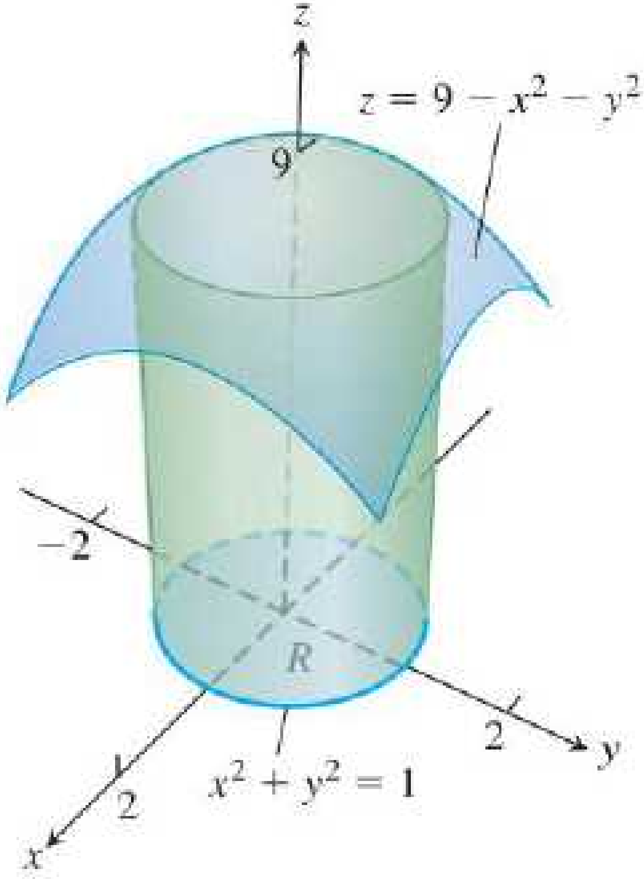
Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.

Solution

$$\iint_R (9 - x^2 - y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (9 - r^2)\, r\, dr\, d\theta$$

$$= \frac{17}{4} \int_0^{2\pi} d\theta = \frac{17\pi}{2}.$$

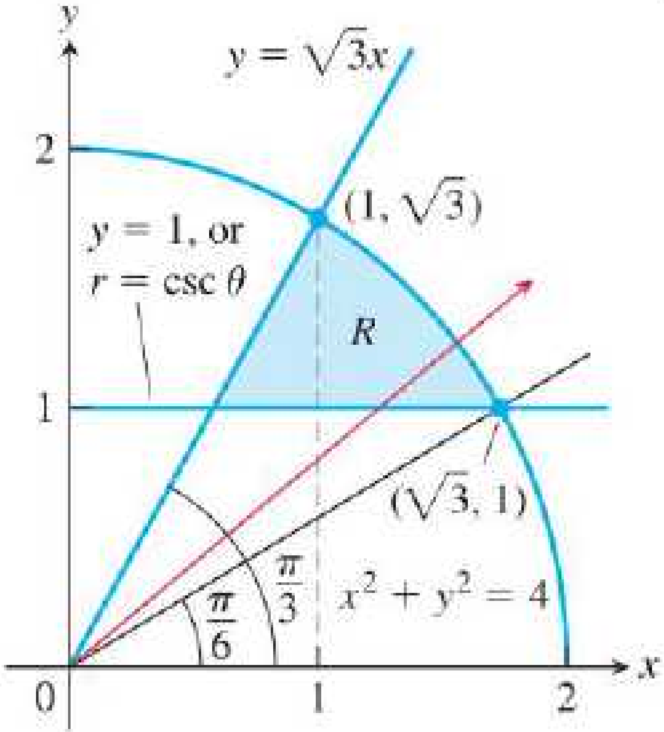


EXAMPLE 6

Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

Solution

$$\begin{cases} r = \dfrac{1}{\sin\theta} \\ r = 2 \end{cases} \Rightarrow \theta = \frac{\pi}{6}$$



$$\iint_R dA = \int_{\pi/6}^{\pi/3} \int_{\csc\theta}^{2} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{2}\left[4 - \csc^2\theta\right] d\theta$$

$$= \frac{1}{2}\left[4\theta + \cot\theta\right]_{\pi/6}^{\pi/3} = \frac{\pi - \sqrt{3}}{3}.$$

15.5

Triple Integrals in Rectangular Coordinates
直角坐标中的三重积分

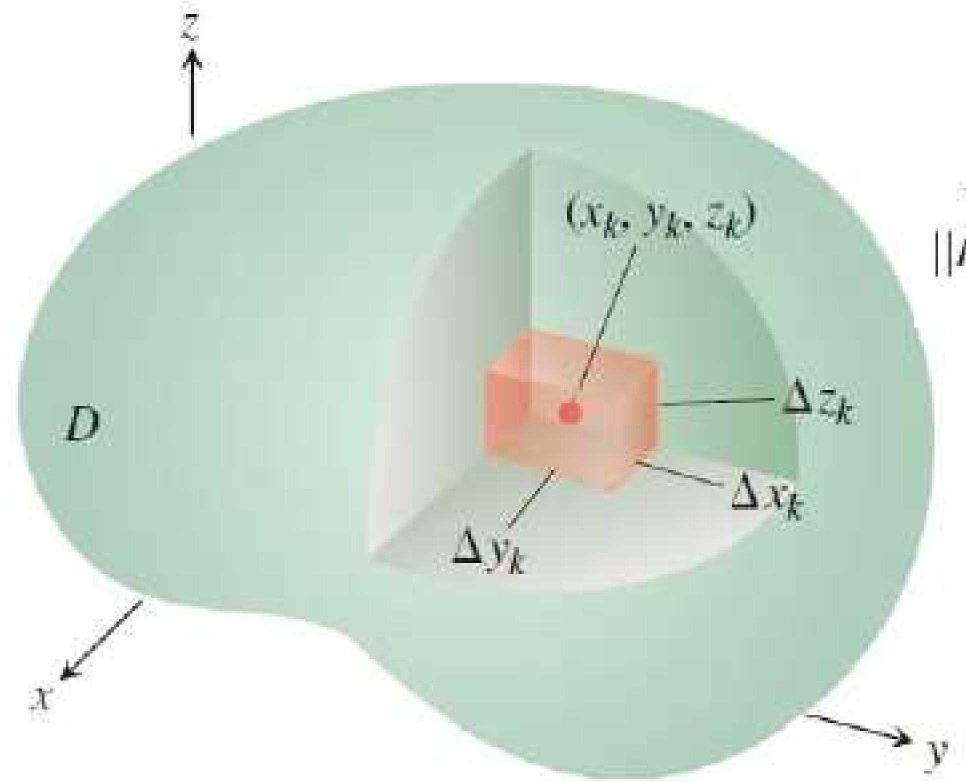


FIGURE 15.30 Partitioning a solid with rectangular cells of volume ΔV_k.

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\,\Delta V_k.$$

$$\lim_{\|P\|\to 0} S_n = \iiint_D F(x, y, z)\,dx\,dy\,dz.$$

$F(x, y, z)$: **density at** (x,y,z)

$$M = \iiint_D F(x, y, z)dxdydz$$

--mass on D

$F(x, y, z) = 1 \qquad V = \iiint_D 1dxdydz$

--volume on D

Volume of a Region in Space

DEFINITION The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

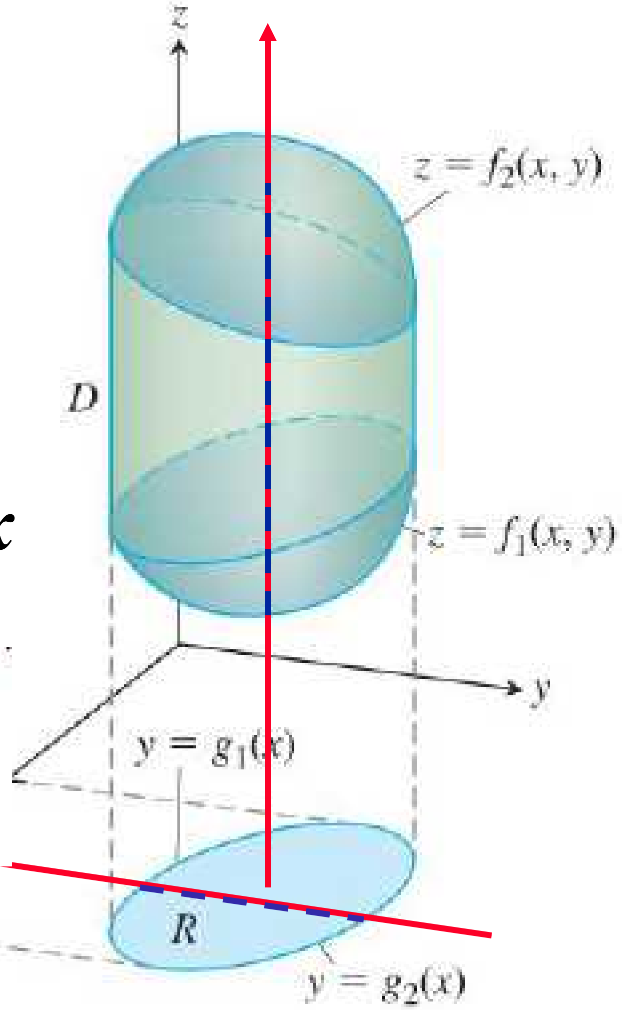
Finding Limits of Integration in the Order *dz dy dx*

To evaluate $\iiint_D F(x, y, z)\, dV$

$$= \iint_R \left[\int_{z_1(x,y)}^{z_2(x,y)} F(x,y,z)dz\right]dxdy$$

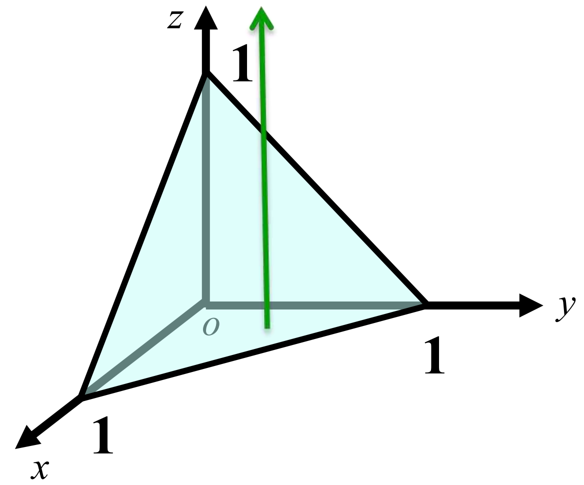
$$= \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \left[\int_{z_1(x,y)}^{z_2(x,y)} F(x,y,z)dz\right]dy\right]dx$$

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$



例 计算三重积分$\iiint\limits_{\Omega} z\,dx\,dy\,dz$，其中$\Omega$为三个坐标面及平面$x+y+z=1$所围成的闭区域.

解



$$\iiint\limits_{\Omega} z\,dx\,dy\,dz = \iint\limits_{R}\left[\int_0^{1-y-x} z\,dz\right]dA$$

$$= \int_0^1 dx\int_0^{1-x} dy\int_0^{1-y-x} z\,dz$$

$$= \int_0^1 dx\int_0^{1-x} \frac{1}{2}(1-x-y)^2\,dy$$

$$= \int_0^1 \frac{1}{6}(1-x)^3\,dx = \frac{1}{24}.$$

EXAMPLE 1

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution The volume is

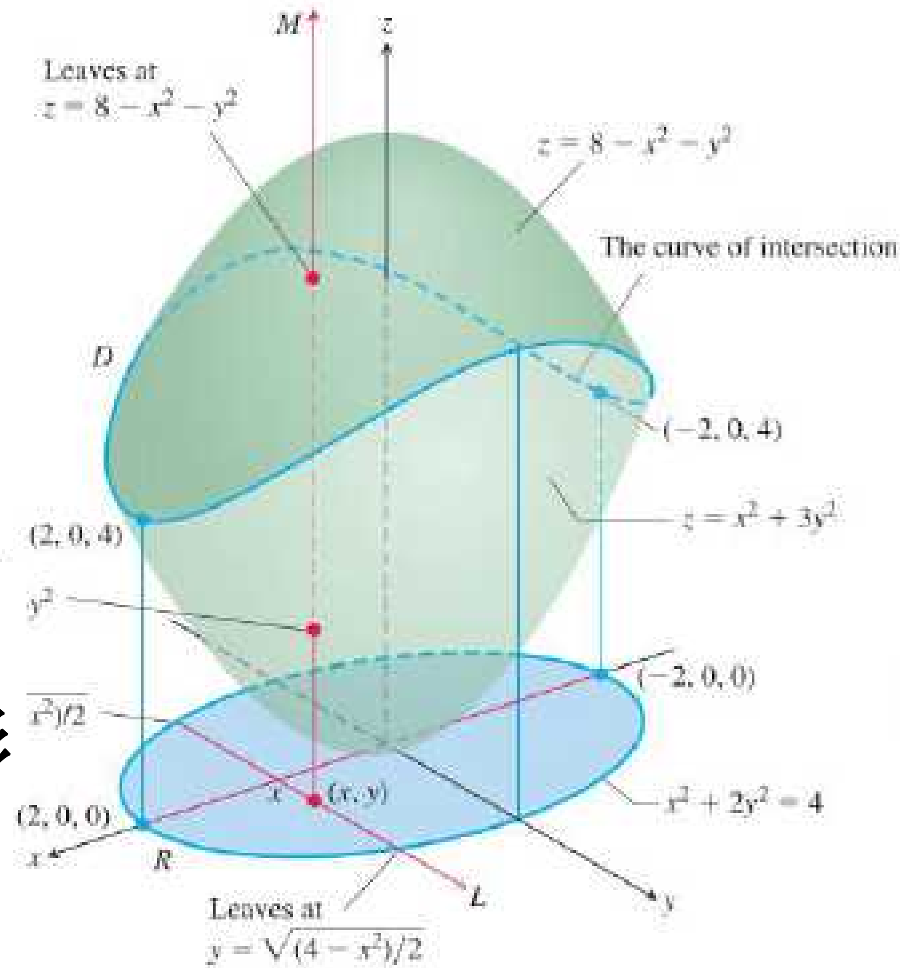
$$V = \iiint\limits_D dz\, dy\, dx,$$

交线$z = 8 - x^2 - y^2, z = x^2 + 3y^3$

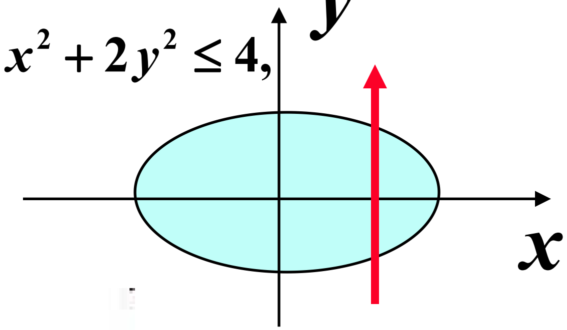
消去z变量

$x^2 + 2y^2 = 4$,在xy面上的投影

R: $x^2 + 2y^2 \le 4$,



$$V = \iiint_D dz\,dy\,dx = \iint_R \left[\int_{x^2+3y^2}^{8-x^2-y^2} dz\right] dA$$



$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\,dy\,dx$$

$$= \int_{-2}^{2} \left[(8 - 2x^2)y - \frac{4}{3}y^3 \right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx$$

$$= \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2}\,dx = 8\pi\sqrt{2}.$$

EXAMPLE 2

Set up the limits of integration for evaluating the triple integral

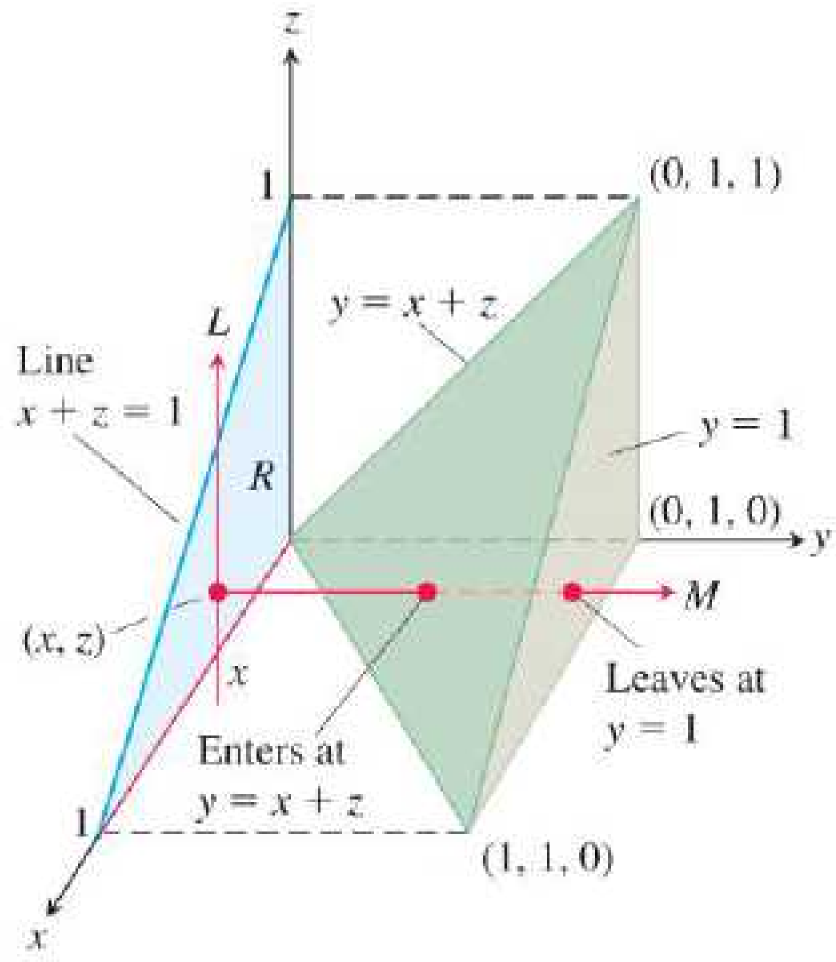
$$\iiint_D F(x, y, z)\, dV$$

the tetrahedron D with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$. Use the order of integration $dy\, dz\, dx$.

Solution

$$\iint_R \left[\int_{x+z}^{1} F(x,y,z)\,dz\right] dA$$

$$\int_0^1 \int_0^{1-x} \int_{x+z}^{1} F(x, y, z)\, dy\, dz\, dx.$$



EXAMPLE 3

Integrate $F(x, y, z) = 1$ over the tetrahedron D in Example 2 in the order $dz\,dy\,dx$, and then integrate in the order $dy\,dz\,dx$.

Solution

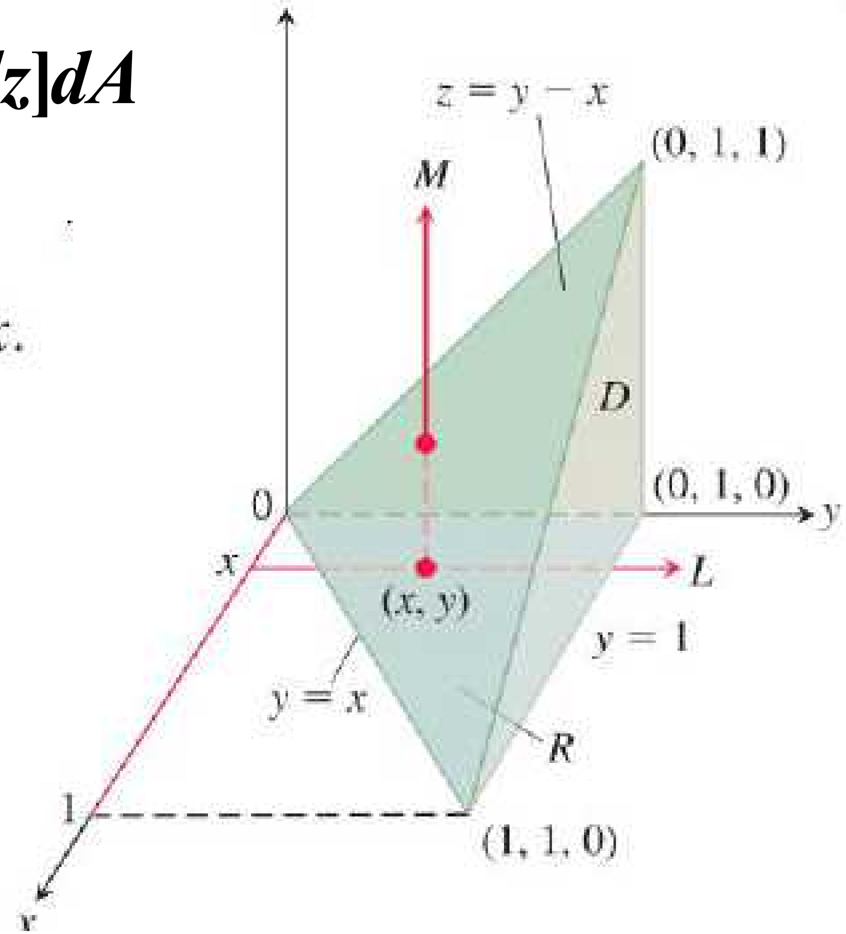
$$\iiint_D 1\,dV = \iint_R \left[\int_0^{y-x} dz\right] dA$$

$$\int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z)\,dz\,dy\,dx.$$

$$= \int_0^1 \int_x^1 (y - x)\,dy\,dx$$

$$= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2}x^2\right) dx$$

$$= \frac{1}{6}.$$



$$\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x, y, z)\, dy\, dz\, dx.$$

$$= \int_0^1 \int_0^{1-x} (1 - x - z) dz dx$$

$$= \int_0^1 [(1 - x)^2 - \frac{(1 - x)^2}{2}] dx$$

$$= \int_0^1 \frac{(1 - x)^2}{2} dx = \frac{1}{6}$$

Average Value of a Function in Space

$$\textbf{Average value} \text{ of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F \, dV.$$

EXAMPLE 4

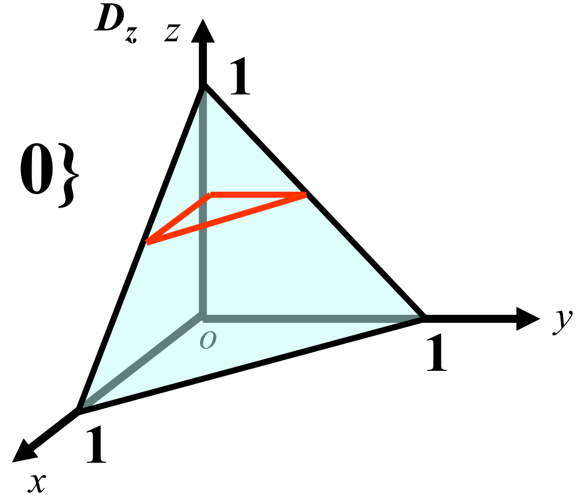
Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

Solution

$$\int_0^2 \int_0^2 \int_0^2 xyz \, dx \, dy \, dz = \int_0^2 \int_0^2 2yz \, dy \, dz = \int_0^2 4z \, dz = \Big[2z^2 \Big]_0^2 = 8.$$

$$\begin{array}{c}\text{Average value of} \\ xyz \text{ over the cube}\end{array} = \frac{1}{\text{volume}} \iiint_{\text{cube}} xyz \, dV = 1.$$

$$\iiint_{\Omega} z\,dxdydz = \int_0^1 \left(\iint_{D_z} z\,dxdy\right)dz = \int_0^1 z\,dz \iint_{D_z} dxdy,$$



$$D_z = \{(x, y) \mid x + y \le 1 - z, x \ge 0, y \ge 0\}$$

$$\iint_{D_z} dxdy = \frac{1}{2}(1 - z)(1 - z)$$

$$原式 = \int_0^1 z \cdot \frac{1}{2}(1 - z)^2 dz = \frac{1}{24}.$$

Properties of Triple Integrals

the same algebraic properties as double and single integrals.

if f is continuous on the D, then

there is $(c_1, c_2, c_2) \in D$ such that

$$\iiint_D f(x,y,z)\,dxdydz = f(c_1, c_2, c_3)\text{Volume}(D)$$

15.6

Moments and Centers of Mass

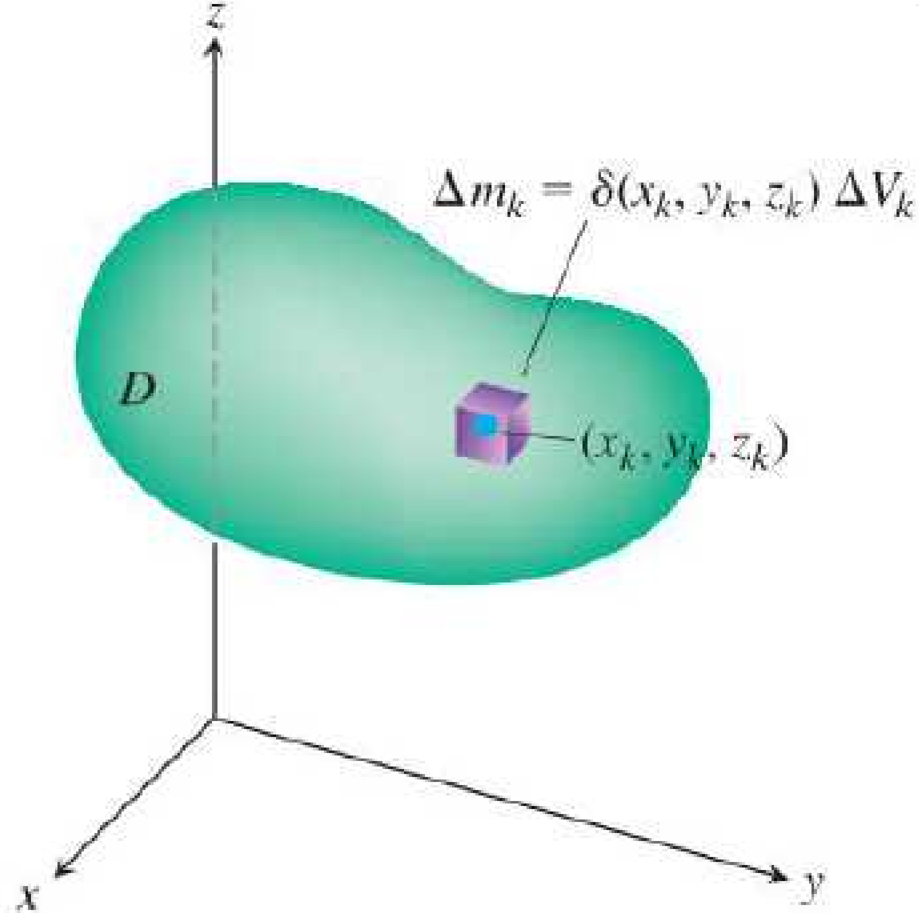
矩和质心

Masses and First Moments

$$M = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta m_k = \lim_{n\to\infty} \sum_{k=1}^{n} \delta(x_k, y_k, z_k)\, \Delta V_k = \iiint_D \delta(x, y, z)\, dV.$$

$$M_{yz} = \iiint_D x\delta(x, y, z)\, dV.$$

$$\bar{x} = M_{yz}/M.$$



THREE-DIMENSIONAL SOLID

Mass: $$M = \iiint\limits_D \delta \, dV$$ $\delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint\limits_D x \, \delta \, dV, \qquad M_{xz} = \iiint\limits_D y \, \delta \, dV, \qquad M_{xy} = \iiint\limits_D z \, \delta \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $$M = \iint_R \delta \, dA$$

$\delta = \delta(x, y)$ is the density at (x, y).

First moments: $$M_y = \iint_R x \, \delta \, dA, \qquad M_x = \iint_R y \, \delta \, dA$$

Center of mass: $$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

EXAMPLE 1

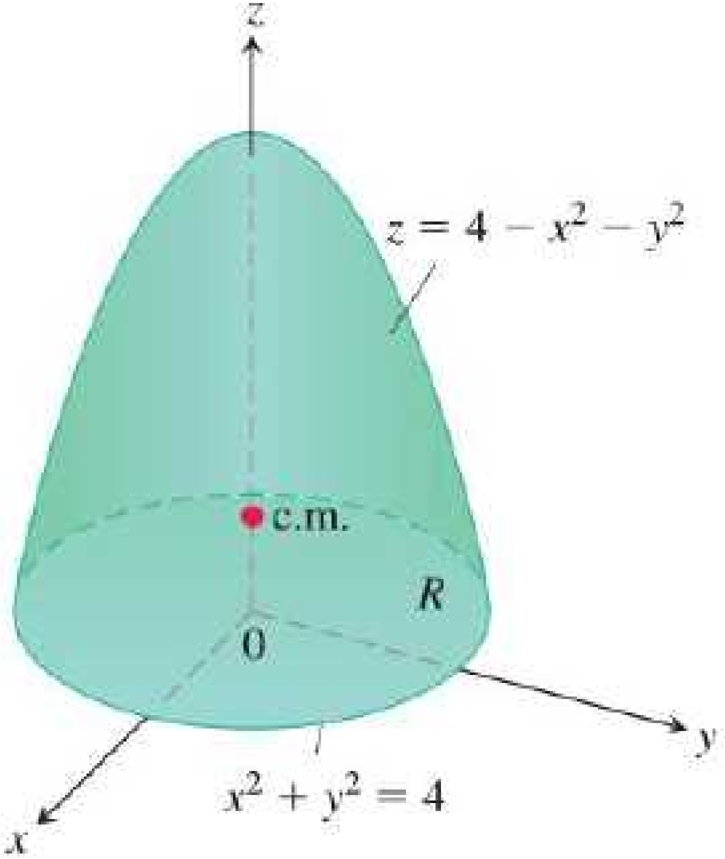
Find the center of mass of a solid of constant density δ bounded below by the disk $R: x^2 + y^2 \le 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$

Solution By symmetry $\bar{x} = \bar{y} = 0$.

$$M_{xy} = \iint_R \int_{z=0}^{z=4-x^2-y^2} z\,\delta\,dz\,dy\,dx$$

$$= \frac{\delta}{2} \iint_R (4 - x^2 - y^2)^2\,dy\,dx$$

$$= \frac{\delta}{2} \int_0^{2\pi} \int_0^2 (4 - r^2)^2 r\,dr\,d\theta$$



$$= \frac{\delta}{2}\int_0^{2\pi}\left[-\frac{1}{6}(4 - r^2)^3\right]_{r=0}^{r=2} d\theta = \frac{16\delta}{3}\int_0^{2\pi} d\theta = \frac{32\pi\delta}{3}.$$

$$M = \iint\limits_R \int_0^{4-x^2-y^2} \delta\, dz\, dy\, dx = 8\pi\delta.$$

$$\bar{z} = (M_{xy}/M) = 4/3$$

the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 4/3)$.

EXAMPLE 2

Find the centroid of the region in the first quadrant that is bounded above by the line $y = x$ and below by the parabola $y = x^2$.

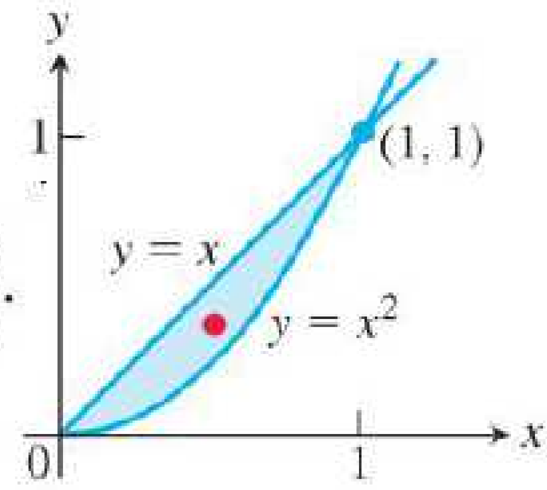
Solution

$$M = \int_0^1 \int_{x^2}^{x} 1\, dy\, dx = \int_0^1 (x - x^2)\, dx = \frac{1}{6}$$

$$M_x = \int_0^1 \int_{x^2}^{x} y\, dy\, dx = \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2}\right) dx = \frac{1}{15}$$

$$M_y = \int_0^1 \int_{x^2}^{x} x\, dy\, dx = \int_0^1 (x^2 - x^3)\, dx = \frac{1}{12}.$$

$$\bar{x} = \frac{M_y}{M} = \frac{1/12}{1/6} = \frac{1}{2} \qquad \bar{y} = \frac{M_x}{M} = \frac{1/15}{1/6} = \frac{2}{5}.$$



The centroid is the point $(1/2, 2/5)$.

Moments of Inertia

the moment of inertia for a solid in space.

If $r(x, y, z)$ is the distance from the point (x, y, z) in D to a line L,

$$\Delta m_k = \delta(x_k, y_k, z_k)\Delta V_k \qquad \Delta I_k = r^2(x_k, y_k, z_k)\Delta m_k.$$

$$I_L = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta I_k = \lim_{n\to\infty} \sum_{k=1}^{n} r^2(x_k, y_k, z_k)\,\delta(x_k, y_k, z_k)\,\Delta V_k$$

$$= \iiint_D r^2\delta\, dV.$$

THREE-DIMENSIONAL SOLID

$$I_x = \iiint (y^2 + z^2)\,\delta\,dV \qquad I_y = \iiint (x^2 + z^2)\,\delta\,dV$$

$$I_z = \iiint (x^2 + y^2)\,\delta\,dV \qquad I_L = \iiint r^2(x, y, z)\,\delta\,dV$$

TWO-DIMENSIONAL PLATE

$$I_x = \iint y^2\,\delta\,dA \qquad I_y = \iint x^2\,\delta\,dA$$

$$I_0 = \iint (x^2 + y^2)\,\delta\,dA = I_x + I_y \qquad I_L = \iint r^2(x, y)\,\delta\,dA$$

EXAMPLE 3

Find I_x, I_y, I_z for the rectangular solid of constant density δ shown in Figure

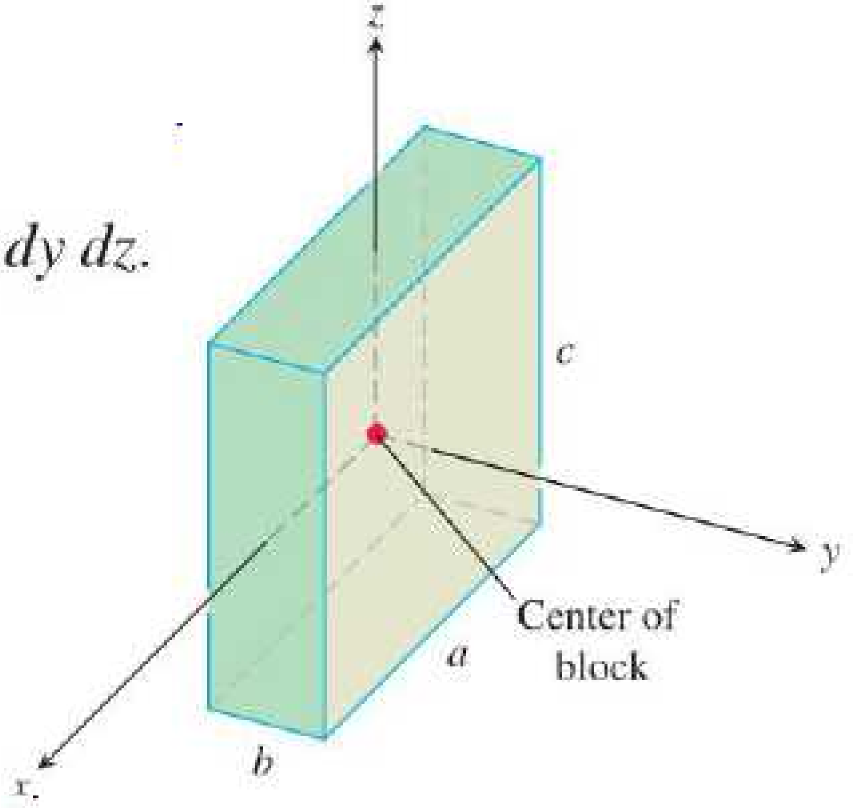
Solution

$$I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} (y^2 + z^2)\,\delta\,dx\,dy\,dz.$$

$$= 4a\delta \int_0^{c/2}\int_0^{b/2} (y^2 + z^2)\,dy\,dz$$

$$= 4a\delta \int_0^{c/2} \left(\frac{b^3}{24} + \frac{z^2 b}{2}\right) dz$$

$$= \frac{abc\delta}{12}(b^2 + c^2) = \frac{M}{12}(b^2 + c^2).$$



$$I_y = \frac{M}{12}(a^2 + c^2) \qquad \text{and} \qquad I_z = \frac{M}{12}(a^2 + b^2).$$

EXAMPLE 4

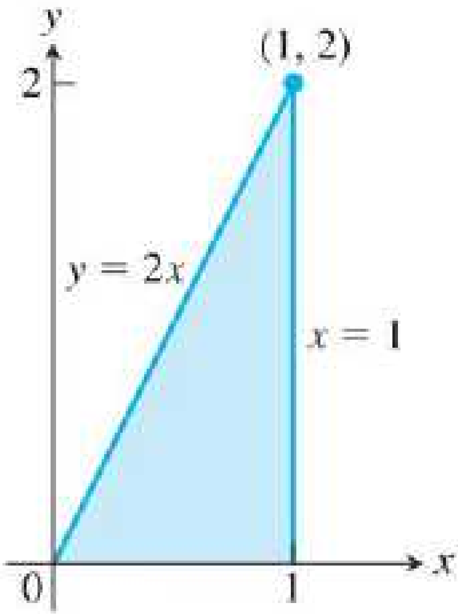
A thin plate covers the triangular region bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point (x, y) is $\delta(x, y) = 6x + 6y + 6$. Find the plate's moments of inertia about the coordinate axes and the origin.

Solution

$$I_x = \int_0^1 \int_0^{2x} y^2 \delta(x, y)\, dy\, dx$$

$$= \int_0^1 \int_0^{2x} (6xy^2 + 6y^3 + 6y^2)\, dy\, dx$$

$$= \int_0^1 (40x^4 + 16x^3)\, dx = 12.$$



$$I_y = \int_0^1 \int_0^{2x} x^2(6x+6y+6)\,dy\,dx$$

$$= 6\int_0^1 \int_0^{2x} (x^3 + x^2 y + x^2)\,dy\,dx$$

$$= 6\int_0^1 (2x^4 + 2x^4 + 2x^3)\,dx$$

$$= 12\int_0^1 (2x^4 + x^3)\,dx = 12\left(\frac{2}{5} + \frac{1}{4}\right) = \frac{39}{5}$$

$$I_0 = 12 + \frac{39}{5} = \frac{60 + 39}{5} = \frac{99}{5}.$$

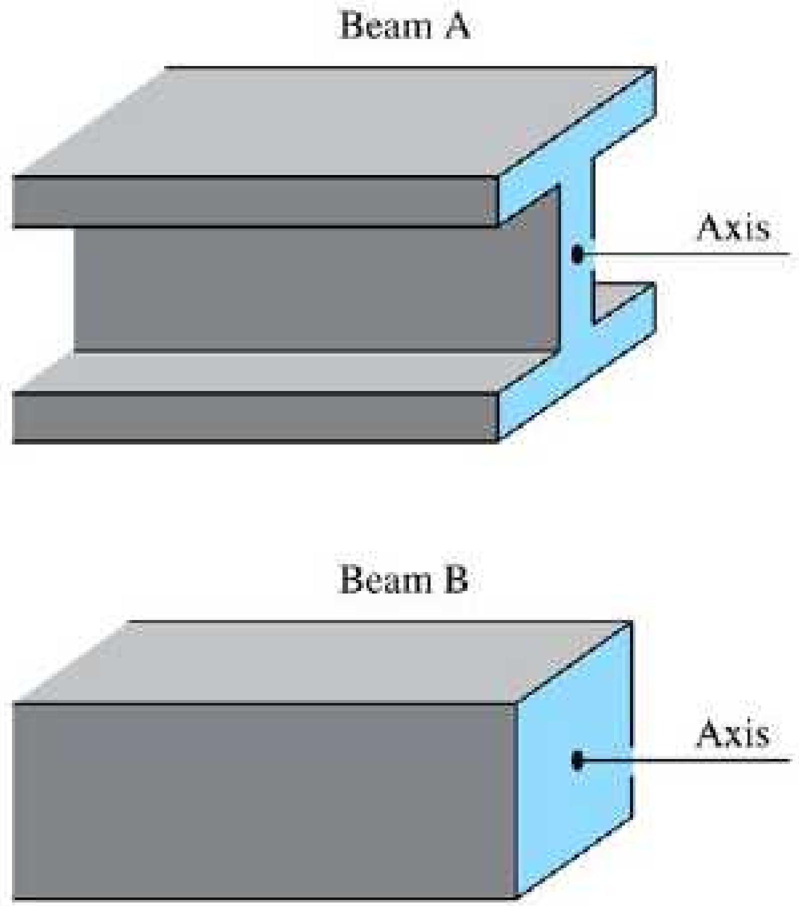


FIGURE 15.42 The greater the polar moment of inertia of the cross-section of a beam about the beam's longitudinal axis, the stiffer the beam. Beams A and B have the same cross-sectional area, but A is stiffer.

15.7

Triple Integrals in Cylindrical and Spherical Coordinates

柱面坐标和球坐标下的三重积分

Integration in Cylindrical Coordinates

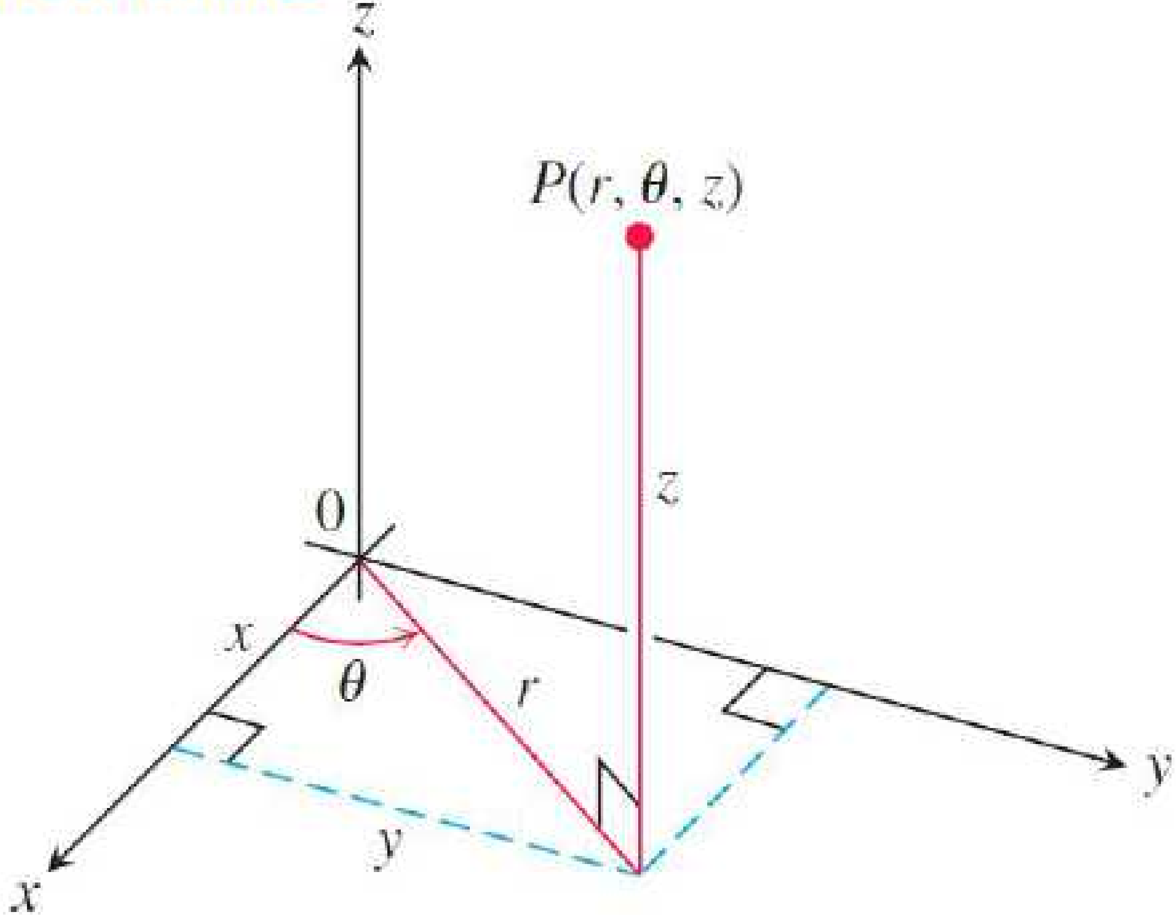
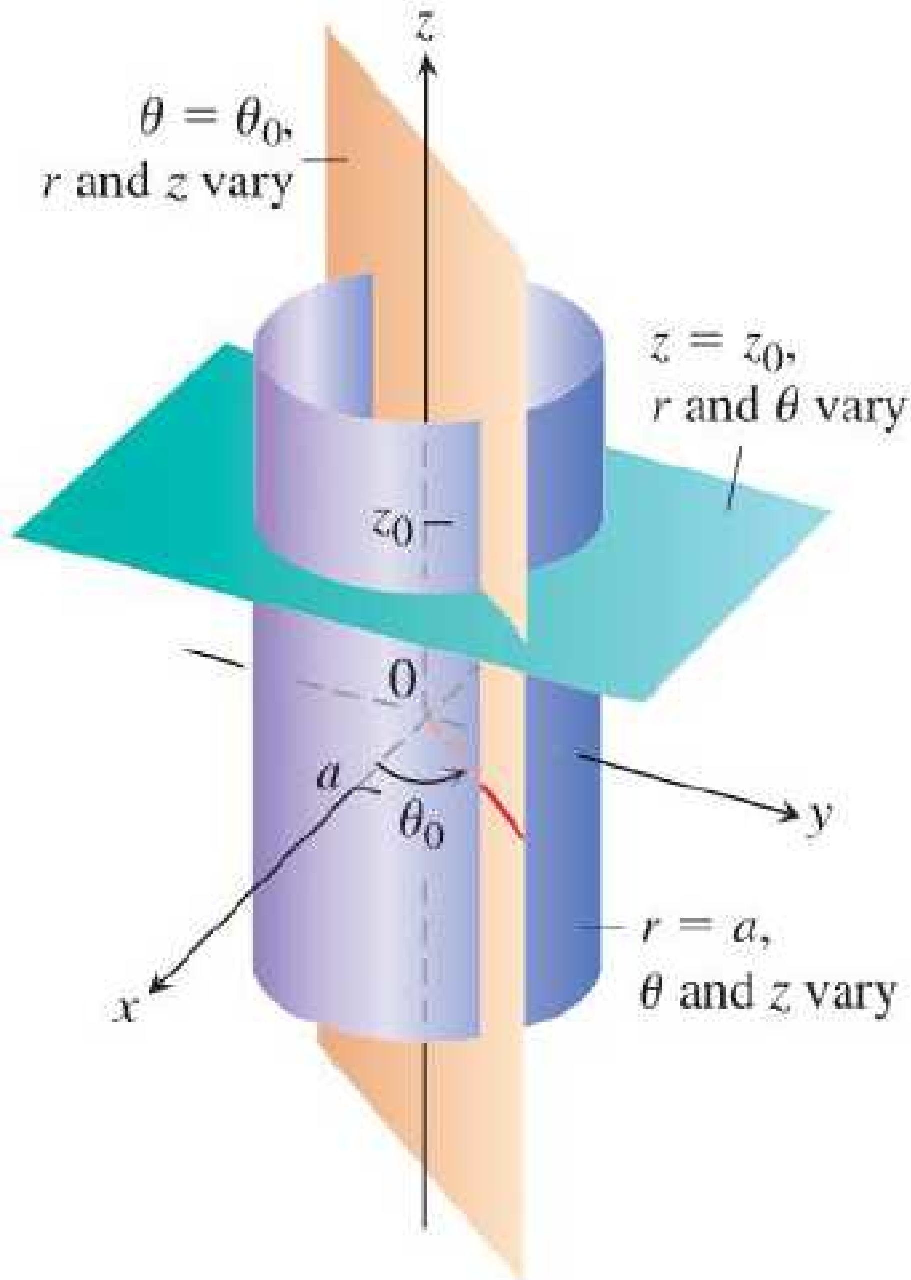


FIGURE 15.43 The cylindrical coordinates of a point in space are r, θ, and z.

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which $r \geq 0$,

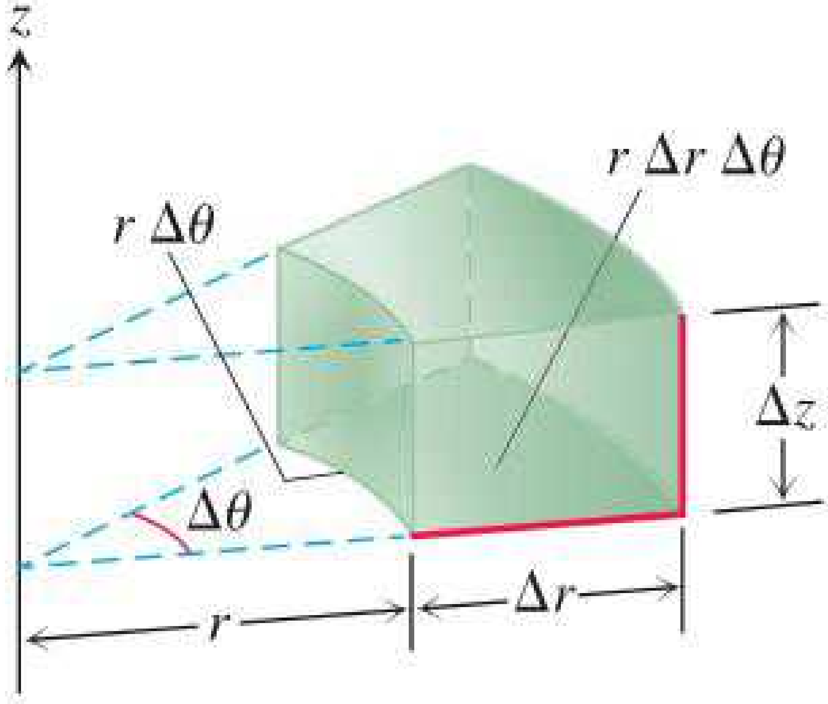
1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$


z
$\theta = \theta_0$,
r and z vary
$z = z_0$,
r and θ vary
z_0
O
a
θ_0
y
x
$r = a$,
θ and z vary

$$\Delta V_k = \Delta z_k \, r_k \, \Delta r_k \, \Delta\theta_k \qquad S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k) \, \Delta z_k \, r_k \, \Delta r_k \, \Delta\theta_k .$$

$$\lim_{n\to\infty} S_n = \iiint_D f \, dV = \iiint_D f \, dz \, r \, dr \, d\theta .$$

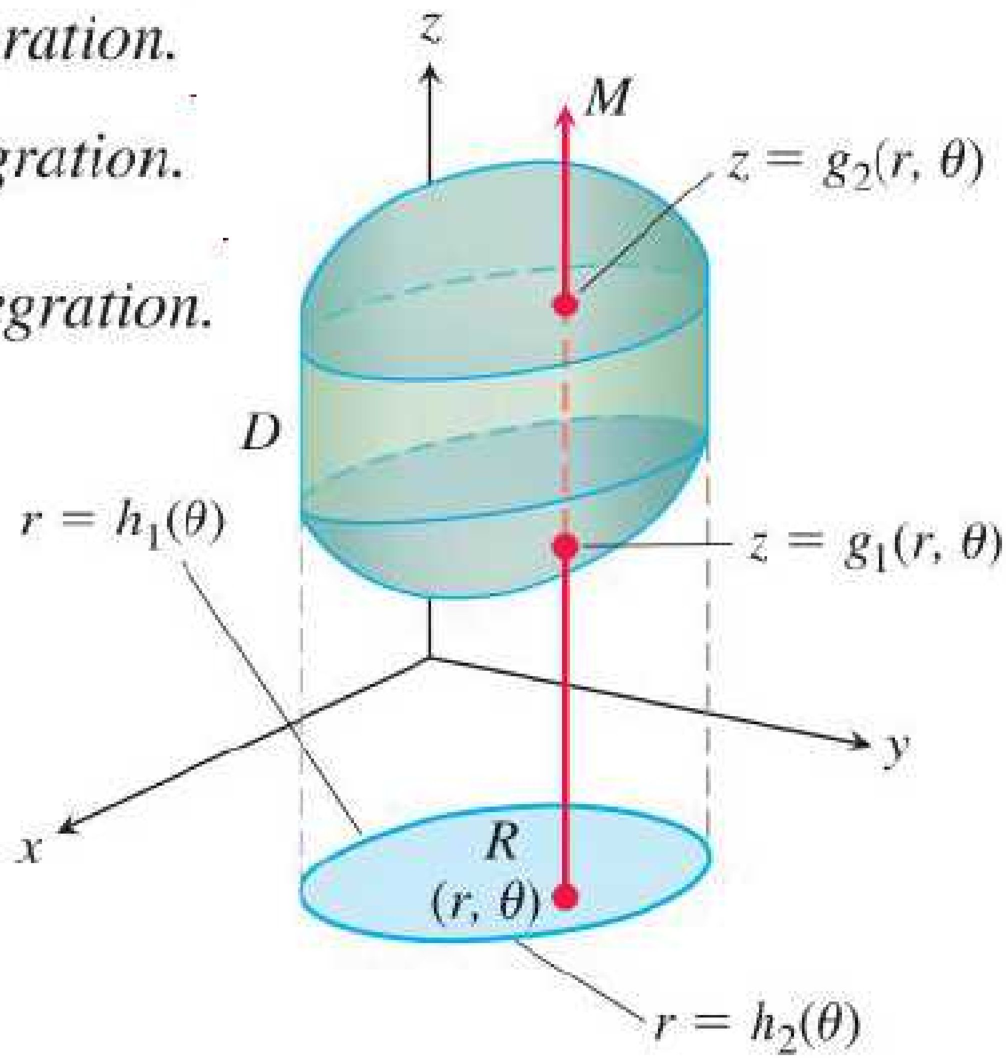


1. *Sketch*. Sketch the region D its projection R on the xy-plane.

2. *Find the z-limits of integration.*

3. *Find the r-limits of integration.*

4. *Find the θ-limits of integration.*



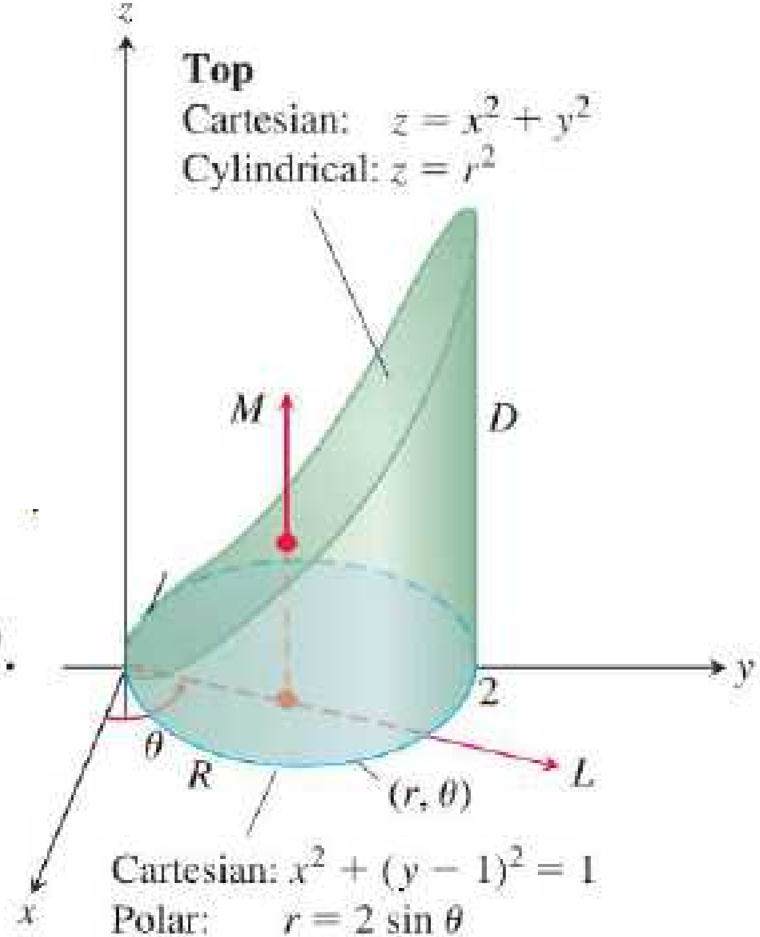
EXAMPLE 1

Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y - 1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

Solution

$$\iiint_D f(r, \theta, z)\, dV$$

$$= \int_0^{\pi} \int_0^{2 \sin \theta} \int_0^{r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$



EXAMPLE 2

Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.

Solution

$$M_{xy} = \iiint_D z\,dV = \int_0^{2\pi}\int_0^2\int_0^{r^2} z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \frac{r^5}{2}\,dr\,d\theta$$

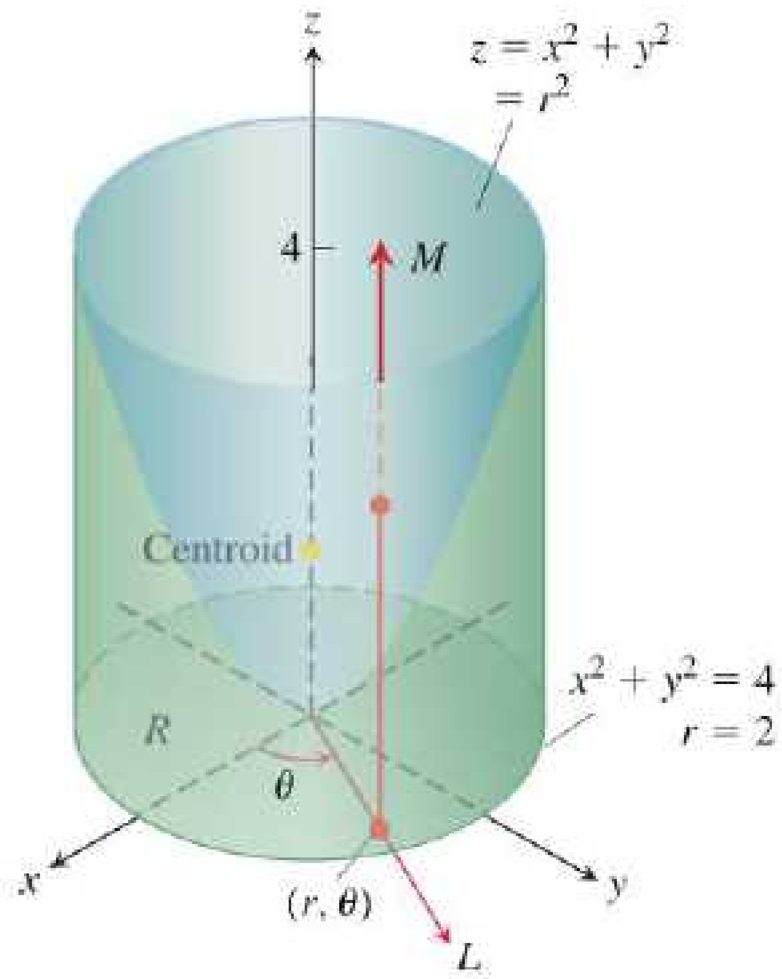
$$= \int_0^{2\pi} \frac{16}{3}\,d\theta = \frac{32\pi}{3}.$$

$$M = \int_0^{2\pi}\int_0^2\int_0^{r^2} dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = 8\pi.$$

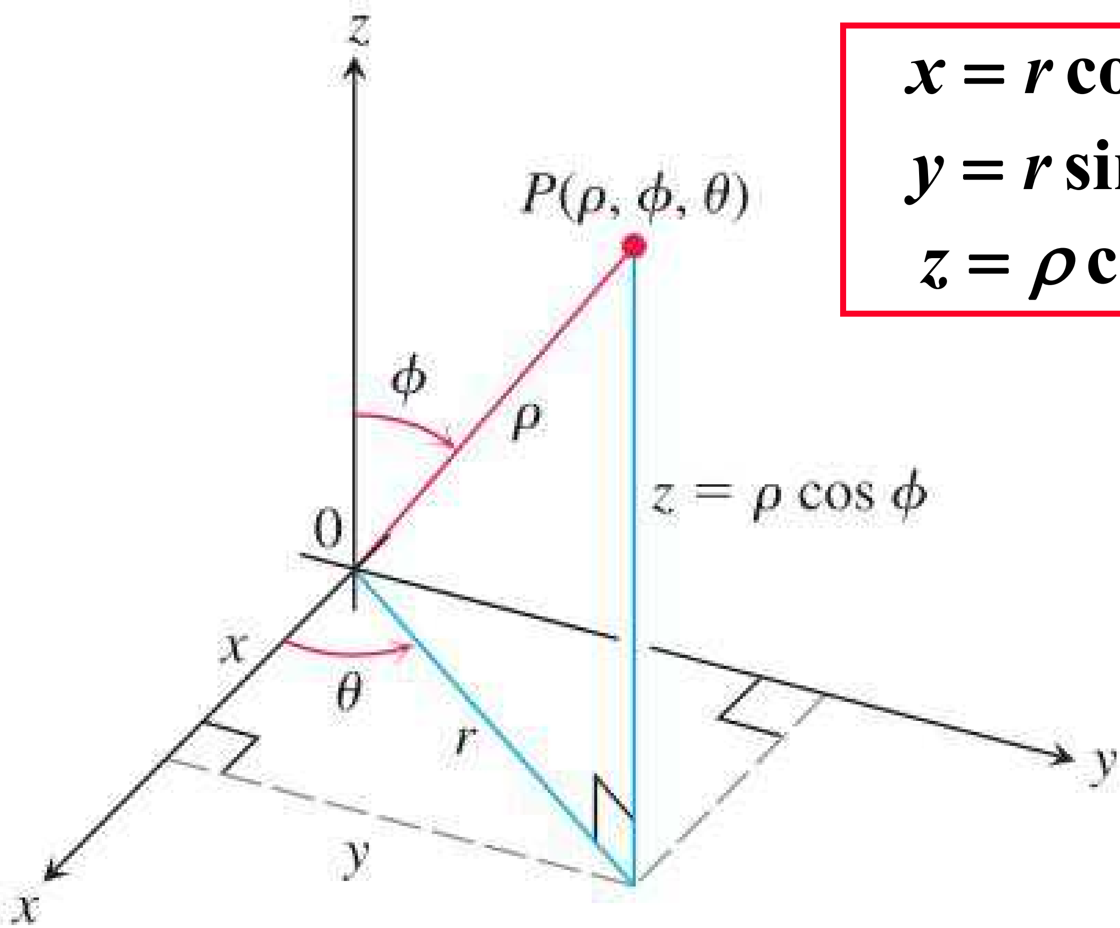
$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3}\frac{1}{8\pi} = \frac{4}{3},$$

the centroid is $(0, 0, 4/3)$.



z
$z = x^2 + y^2$
$= r^2$
4
M
Centroid
$x^2 + y^2 = 4$
$r = 2$
R
θ
x
(r, θ)
y
L

Spherical Coordinates and Integration



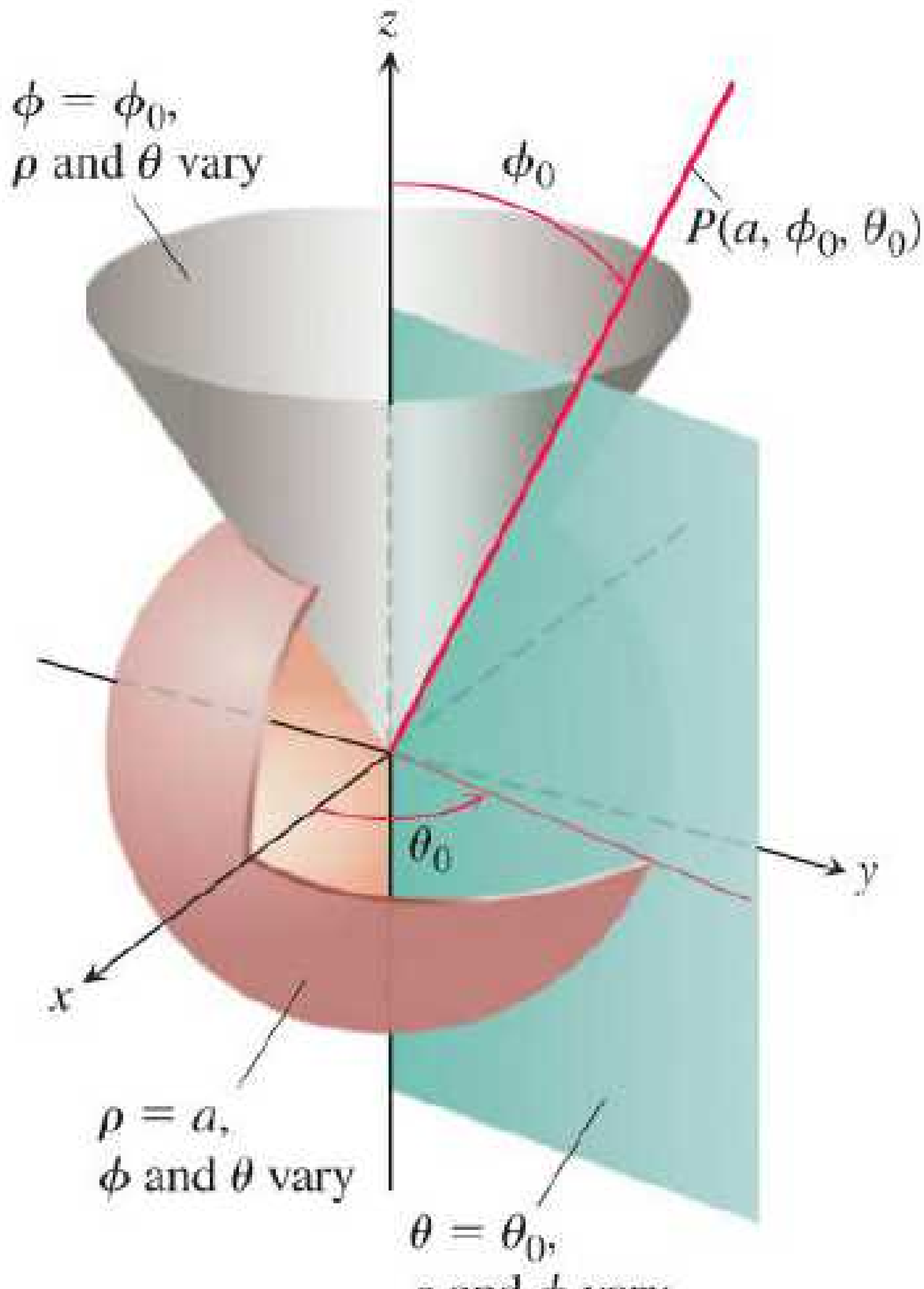
$$x = r\cos\theta \;= \rho\sin\varphi\cos\theta$$
$$y = r\sin\theta \;= \rho\sin\varphi\sin\theta$$
$$z = \rho\cos\varphi$$

DEFINITION **Spherical coordinates** represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin ($\rho \geq 0$).
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis ($0 \leq \phi \leq \pi$).
3. θ is the angle from cylindrical coordinates.

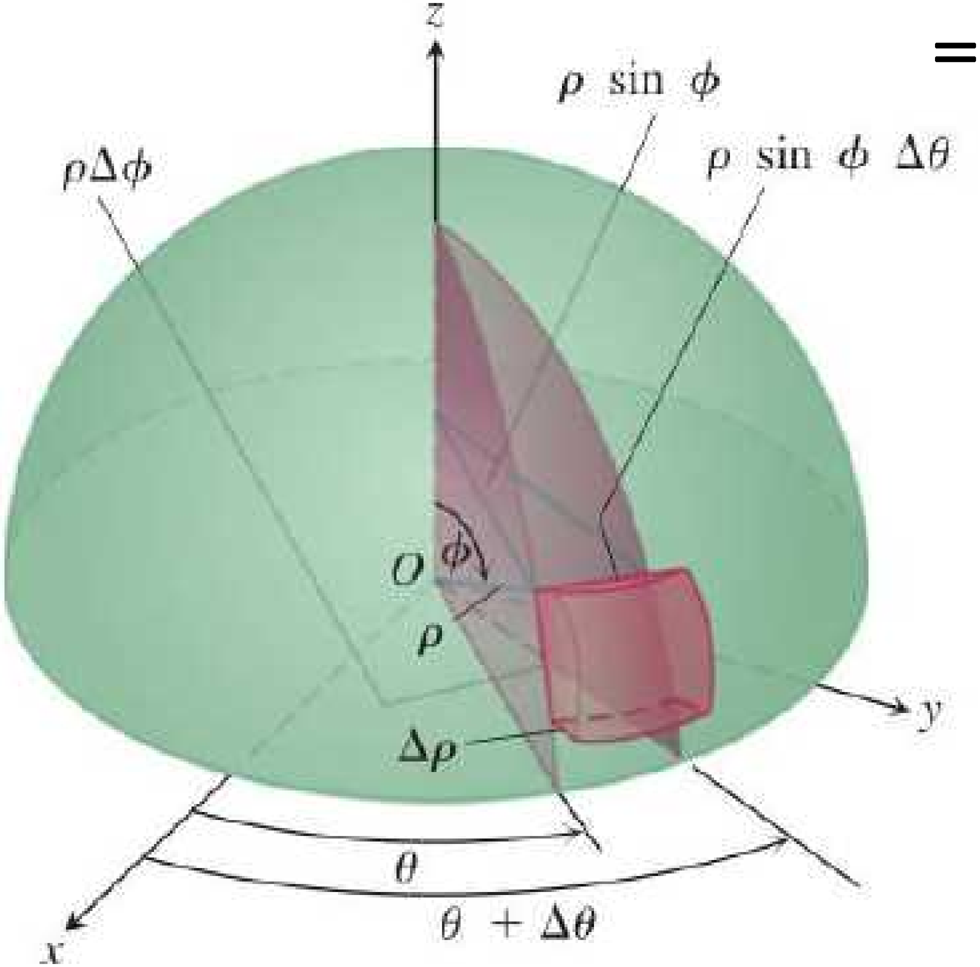
$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$

$$x^2 + y^2 + z^2 = \rho^2$$


$\phi = \phi_0$,
ρ and θ vary
z
ϕ_0
$P(a, \phi_0, \theta_0)$
θ_0
y
x
$\rho = a$,
ϕ and θ vary
$\theta = \theta_0$,
ρ and ϕ vary

$$\Delta V_k = \rho_k^{\,2} \sin \phi_k \, \Delta\rho_k \, \Delta\phi_k \, \Delta\theta_k$$

$$\Delta V = (\rho \sin\varphi \Delta\theta)(\rho \Delta\varphi)\Delta\rho$$
$$= \rho^2 \sin\varphi \Delta\rho \Delta\varphi \Delta\theta$$



$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$S_n = \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k)\, \rho_k^2 \sin \phi_k\, \Delta\rho_k\, \Delta\phi_k\, \Delta\theta_k.$$

$$\lim_{n\to\infty} S_n = \iiint_D f(\rho, \phi, \theta)\, dV = \iiint_D f(\rho, \phi, \theta)\, \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta.$$

$$dV = \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta$$

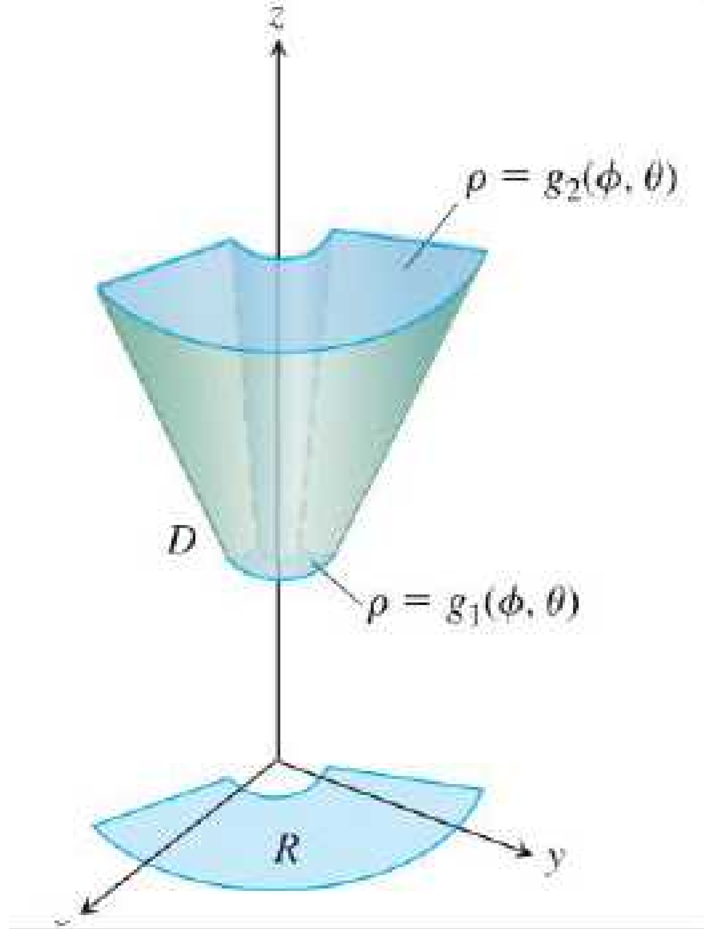
$$x = \rho \sin\varphi \cos\theta$$

$$y = \rho \sin\varphi \sin\theta$$

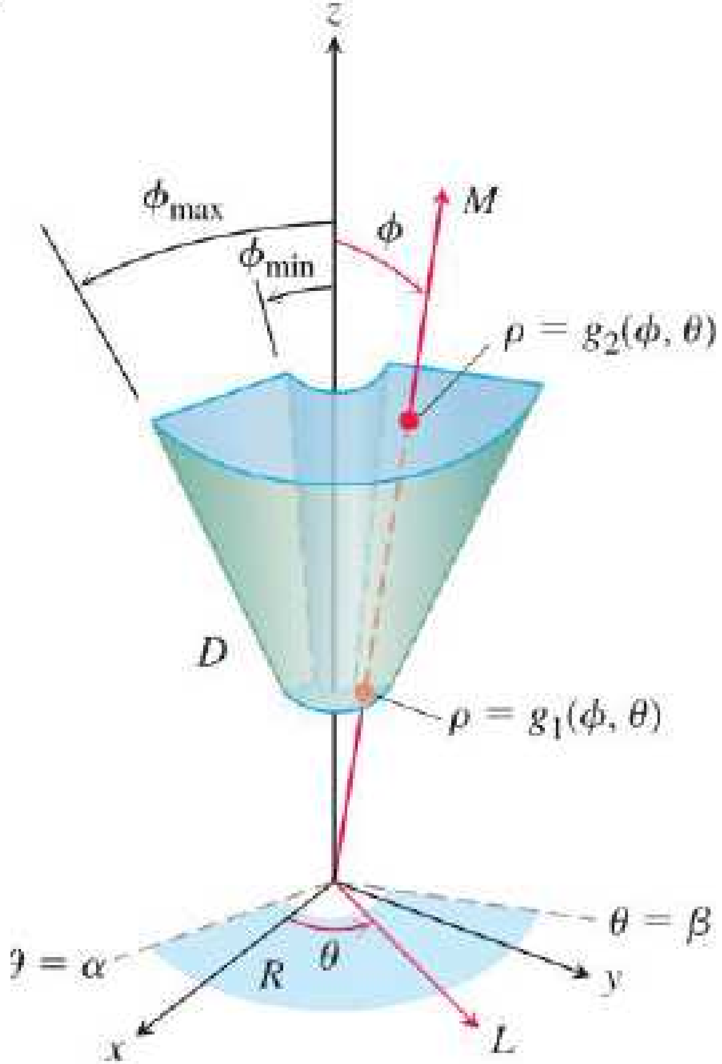
$$z = \rho \cos\varphi$$

How to Integrate in Spherical Coordinates

1. *Sketch*. Sketch the region D along with its projection R



2. *Find the ρ-limits of integration.* Draw a ray M from the origin through D,



3. *Find the ϕ-limits of integration.*

4. *Find the θ-limits of integration.* The ray L sweeps over R

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$$

EXAMPLE 5

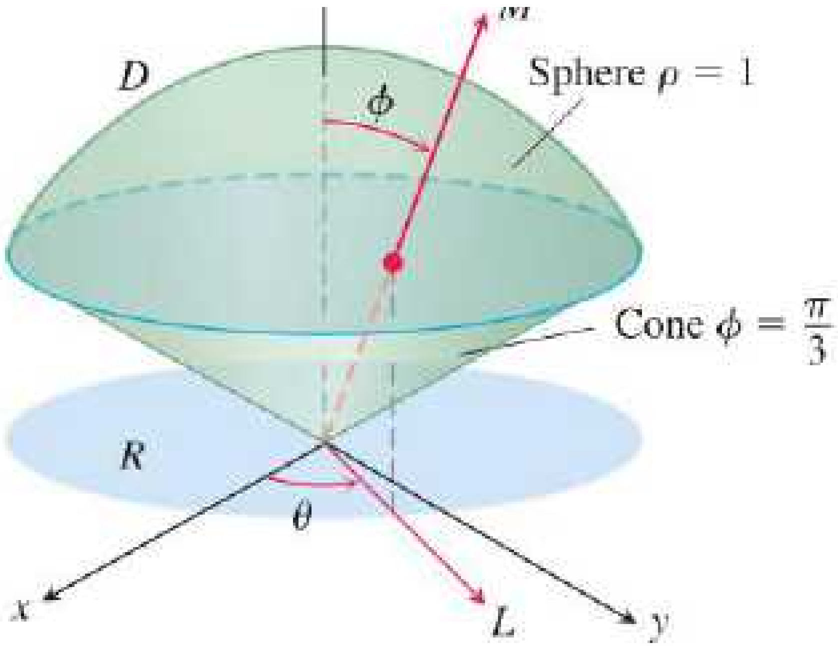
Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \pi/3$.

Solution The volume is $V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$,

$$V = \iiint \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left(-\frac{1}{6} + \frac{1}{3}\right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}.$$



EXAMPLE 6

A solid of constant density $\delta = 1$ occupies the region D in Example 5.

Find the solid's moment of inertia about the z-axis.

Solution

$$I_z = \iiint_D (x^2 + y^2)\, dV = \iiint_D \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta.$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{5} \int_0^{2\pi} \int_0^{\pi/3} (1 - \cos^2\phi) \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{5} \int_0^{2\pi} \frac{5}{24}\, d\theta = \frac{1}{24}(2\pi) = \frac{\pi}{12}.$$

$$x = \rho \sin\varphi \cos\theta$$

$$y = \rho \sin\varphi \sin\theta$$

$$z = \rho \cos\varphi$$

Find the integral $\iiint_D \sqrt{x^2+y^2+z^2}\,dV$,

D is bounded by the sphere $x^2+y^2+z^2=2z$.

solution

$$\iiint_D \sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\cos\varphi} \rho^3 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= 2\pi\int_0^{\frac{\pi}{2}}\int_0^{2\cos\varphi} \rho^3 \sin\varphi\, d\rho\, d\varphi \quad = 8\pi\int_0^{\frac{\pi}{2}} \cos^4\varphi \sin\varphi\, d\varphi$$

$$= \frac{8}{5}\pi$$

例　计算$I=\iiint\limits_{\Omega} z\,dx\,dy\,dz$，其中$\Omega$是球面$x^2+y^2+z^2=4$与抛物面$x^2+y^2=3z$所围的立体$(z\geq 0)$.

解

$$\begin{cases} x^2+y^2+z^2=4 \\ x^2+y^2=3z \end{cases} \qquad \begin{cases} x^2+y^2=3 \\ z=1 \end{cases}$$

把Ω投影到xoy面上，D：$x^2+y^2\leq 3$

$$I=\int_0^{2\pi}d\theta\int_0^{\sqrt{3}}dr\int_{\frac{r^2}{3}}^{\sqrt{4-r^2}} r\cdot z\,dz \;\boxed{=\frac{13}{4}\pi.}$$

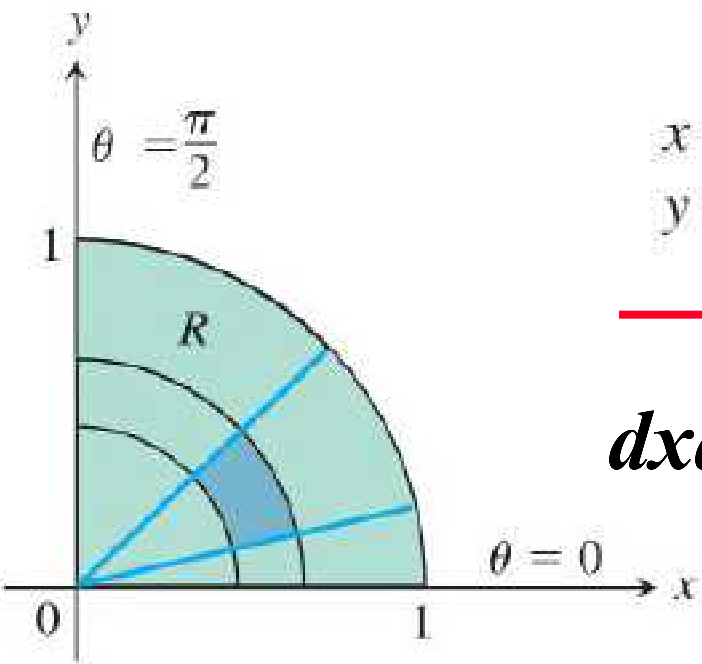
15.8

Substitutions in Multiple Integrals

重积分的变量替换（换元法）

Substitutions in Double Integrals

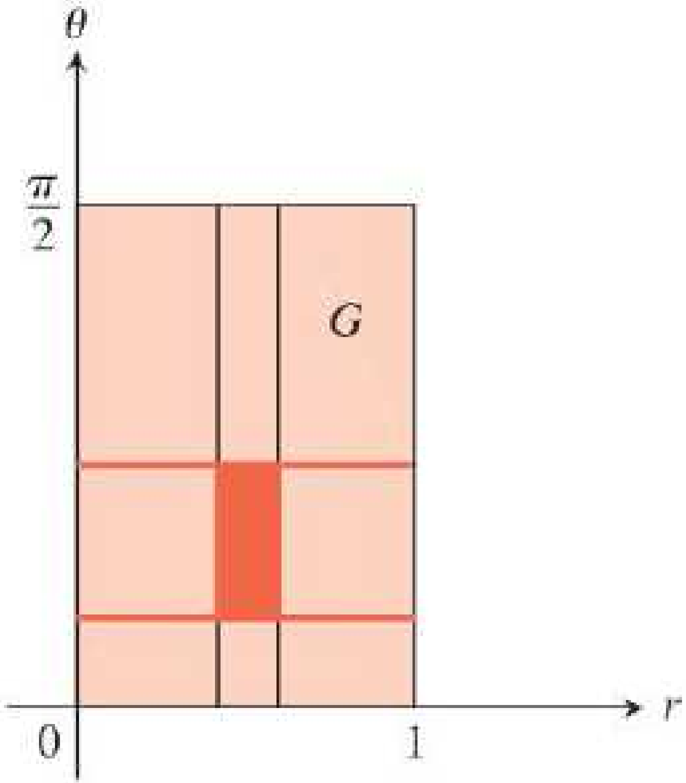
$$\iint\limits_{\substack{x^2+y^2\le 1\\ x,y\ge 0}} f(x,y)\,dxdy = \iint\limits_{\substack{0\le r\le 1\\ 0\le\theta\le\pi/2}} f(r\cos\theta, r\sin\theta)\,rdrd\theta$$



Cartesian xy-plane

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$dxdy = rdrd\theta$$



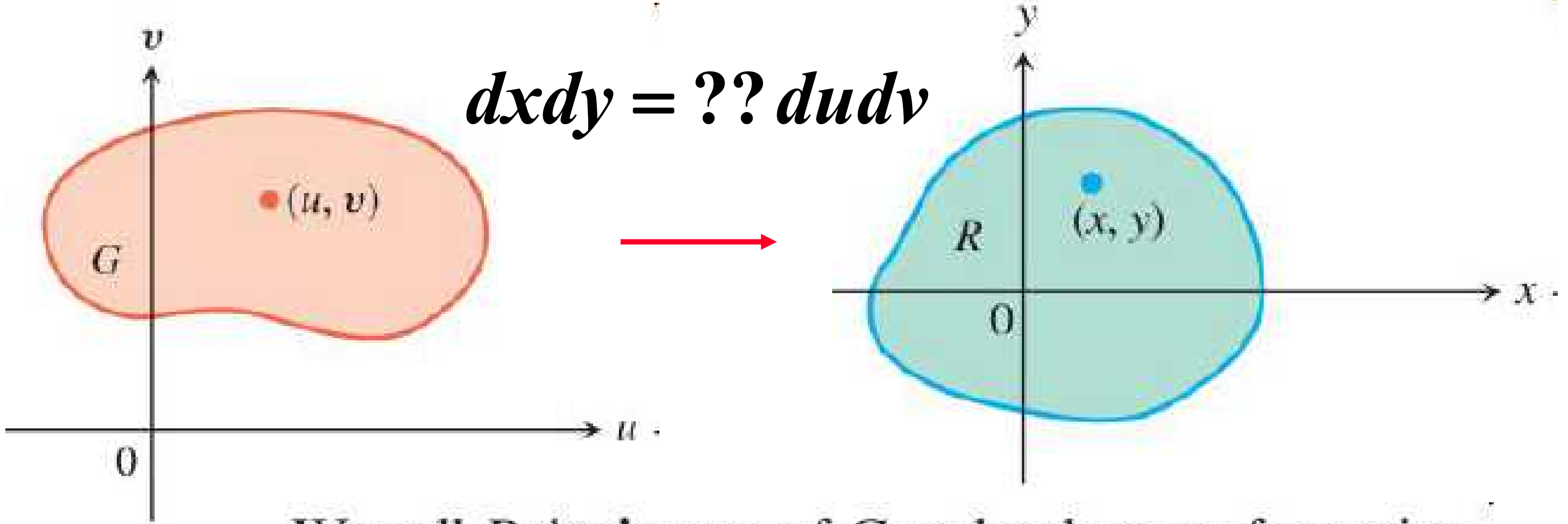
Cartesian $r\theta$-plane

Suppose that a region G in the uv-plane is transformed into the region R in the xy-plane by equations of the form

$$x = g(u, v), \qquad y = h(u, v),$$

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(g(u, v), h(u, v))\ ?\ ?\ du\, dv.$$

$$dxdy = ??\, dudv$$



We call R the **image** of G under the transformation,

DEFINITION The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$$

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv$$

THEOREM 3—Substitution for Double Integrals Suppose that $f(x, y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = g(u, v)$, $y = h(u, v)$, assumed to be one-to-one on the interior of G. If the functions g and h have continuous first partial derivatives within the interior of G, then

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv.$$

EXAMPLE 1 Find the Jacobian for the polar coordinate transformation use Equation (2) to write the Cartesian integral $\iint_R f(x, y)\, dx\, dy$ as a polar

Solution $x = r\cos\theta,\ y = r\sin\theta$ transform the $G: 0 \le r \le 1,\ 0 \le \theta \le \pi/2$, into the quarter circle R bounded by $x^2 + y^2 = 1$

$$J(r, \theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

EXAMPLE 2 Evaluate

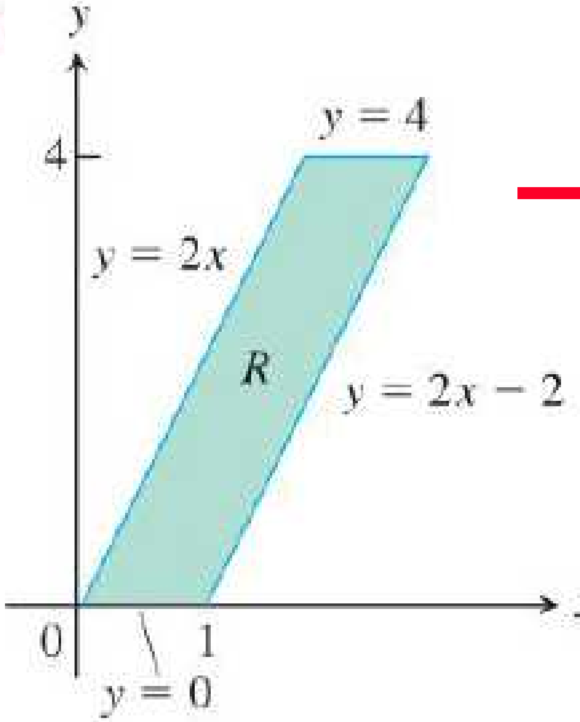
$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\,dx\,dy$$

by applying the transformation

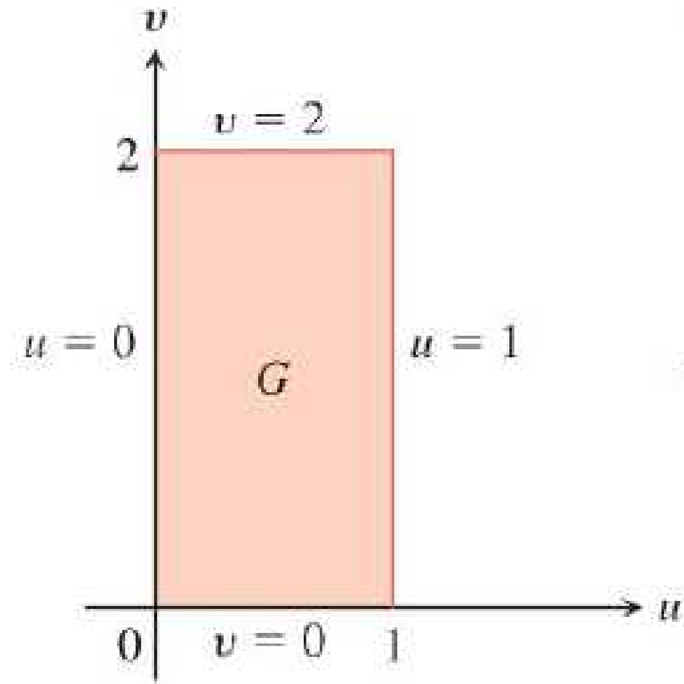
$$u = \frac{2x-y}{2}, \qquad v = \frac{y}{2}$$

and integrating over an appropriate region in the uv-plane.

Solution



$$x = u + v$$
$$y = 2v$$



xy-equations for the boundary of ***R***	Corresponding ***uv***-equations for the boundary of ***G***	Simplified ***uv***-equations
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial u}(u + v) & \dfrac{\partial}{\partial v}(u + v) \\ \dfrac{\partial}{\partial u}(2v) & \dfrac{\partial}{\partial v}(2v) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$

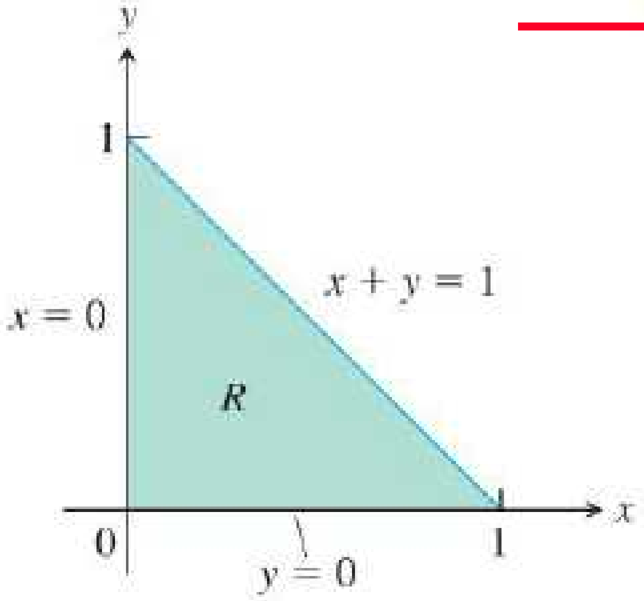
$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2}\, dx\, dy = \int_{v=0}^{v=2} \int_{u=0}^{u=1} u \left| J(u, v) \right| du\, dv$$

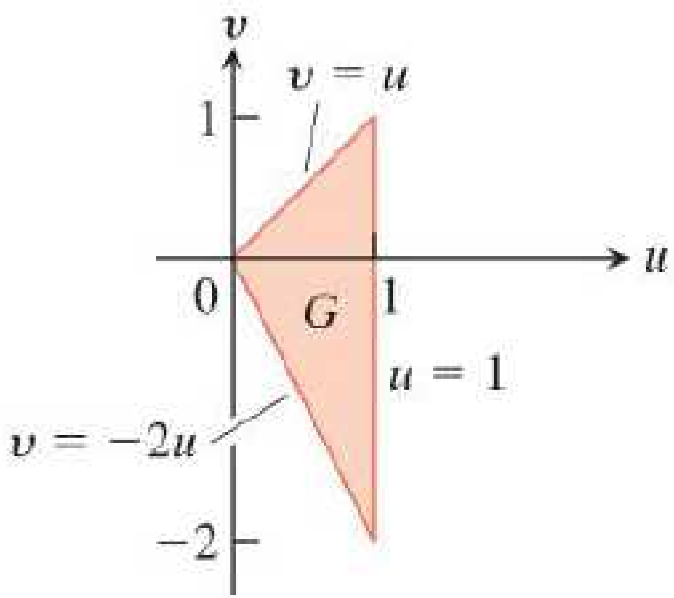
$$= \int_0^2 \int_0^1 (u)(2)\, du\, dv = \int_0^2 dv = 2.$$

EXAMPLE 3 Evaluate $$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx.$$

Solution the transformation $u = x + y$ and $v = y - 2x$.

$$x = \frac{u}{3} - \frac{v}{3}, \qquad y = \frac{2u}{3} + \frac{v}{3}.$$



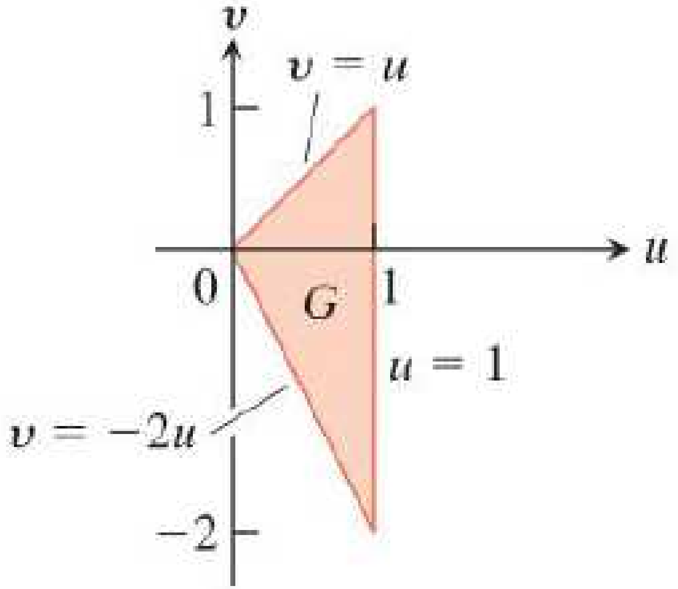


$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}.$$

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\, dy\, dx = \iint_G \sqrt{u}\, v^2\, \frac{1}{3}\, du dv$$

$$= \int_0^1 \int_{-2u}^{u} u^{1/2} v^2 \left(\frac{1}{3}\right) dv\, du$$

$$= \int_0^1 u^{7/2}\, du = \frac{2}{9} u^{9/2} \Big]_0^1 = \frac{2}{9}.$$

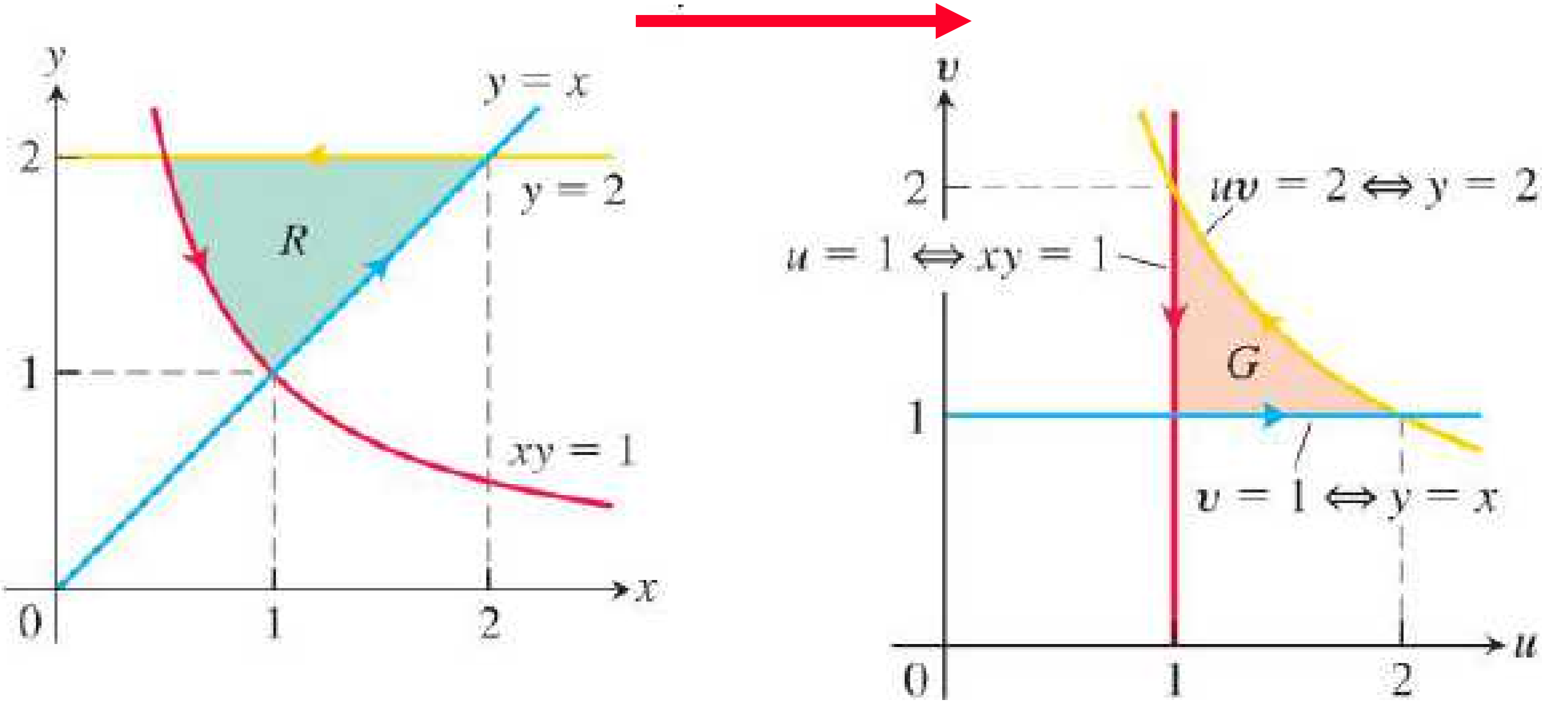


EXAMPLE 4 Evaluate the integral $\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy.$

Solution

$$u = \sqrt{xy} \text{ and } v = \sqrt{y/x}.$$

$$x = \frac{u}{v} \qquad \text{and} \qquad y = uv,$$



$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{v} & \dfrac{-u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}.$$

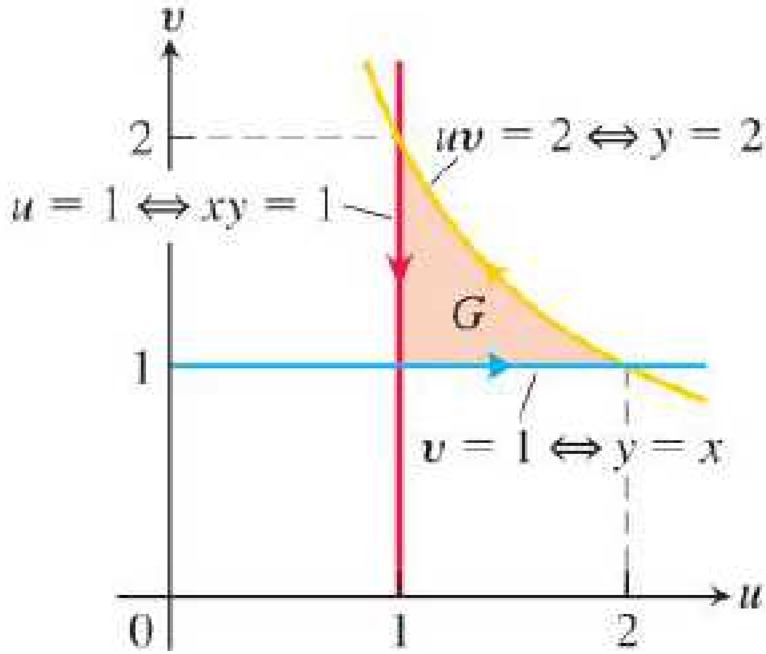
$$\iint_D \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \iint_G v e^u \frac{2u}{v}\, du\, dv = \iint_G 2ue^u\, du\, dv.$$

$$= \int_1^2 \int_1^{2/u} 2ue^u\, dv\, du.$$

$$= 2\int_1^2 (2e^u - ue^u)\, du$$

$$= 2\Big[(2-u)e^u + e^u\Big]_{u=1}^{u=2}$$

$$= 2e(e-2).$$



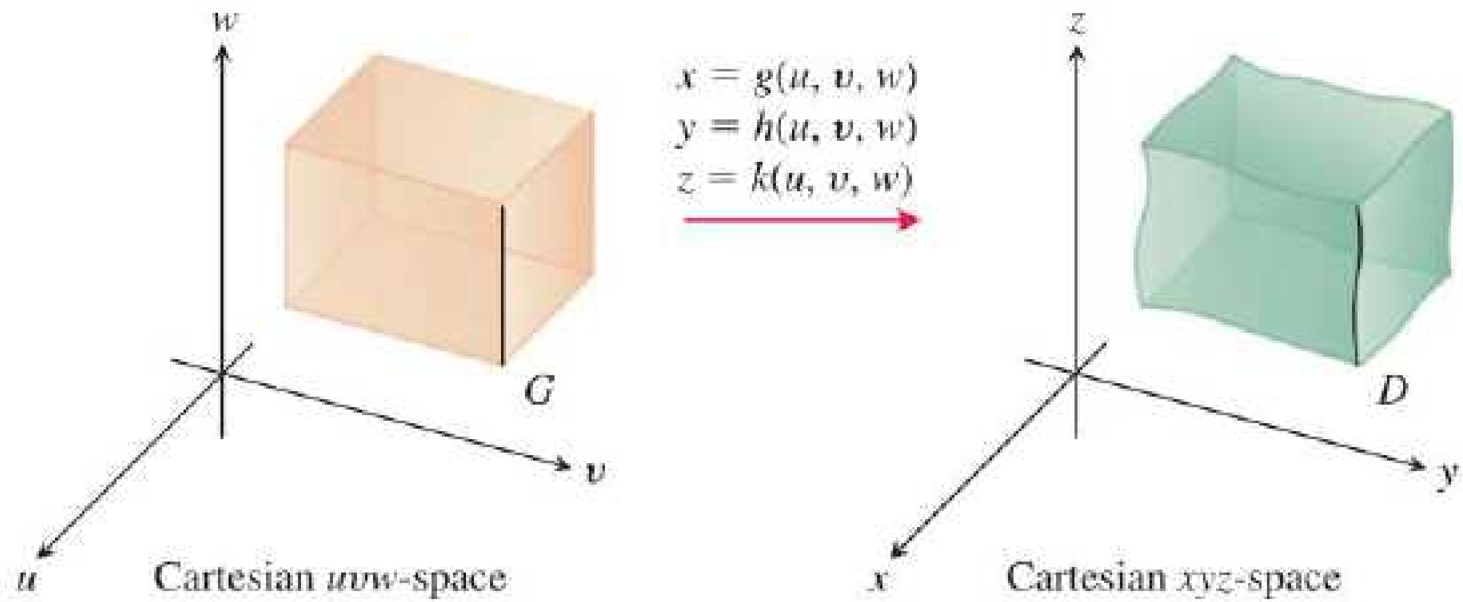
Substitutions in Triple Integrals

a region G in uvw-space is transformed one-to-one into the region D in xyz-space by differentiable equations of the form

$$x = g(u, v, w), \qquad y = h(u, v, w), \qquad z = k(u, v, w),$$

Then $$F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w)$$

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_G H(u, v, w)\left|J(u, v, w)\right|\, du\, dv\, dw.$$



$$J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

$$J(r, \theta, z) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_G H(r, \theta, z)|r|\, dr\, d\theta\, dz.$$

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad z = \rho \cos \phi$$

$$J(\rho, \phi, \theta) = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2 \sin \phi.$$

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_G H(\rho, \phi, \theta)\, |\rho^2 \sin \phi|\, d\rho\, d\phi\, d\theta.$$

EXAMPLE 5 Evaluate

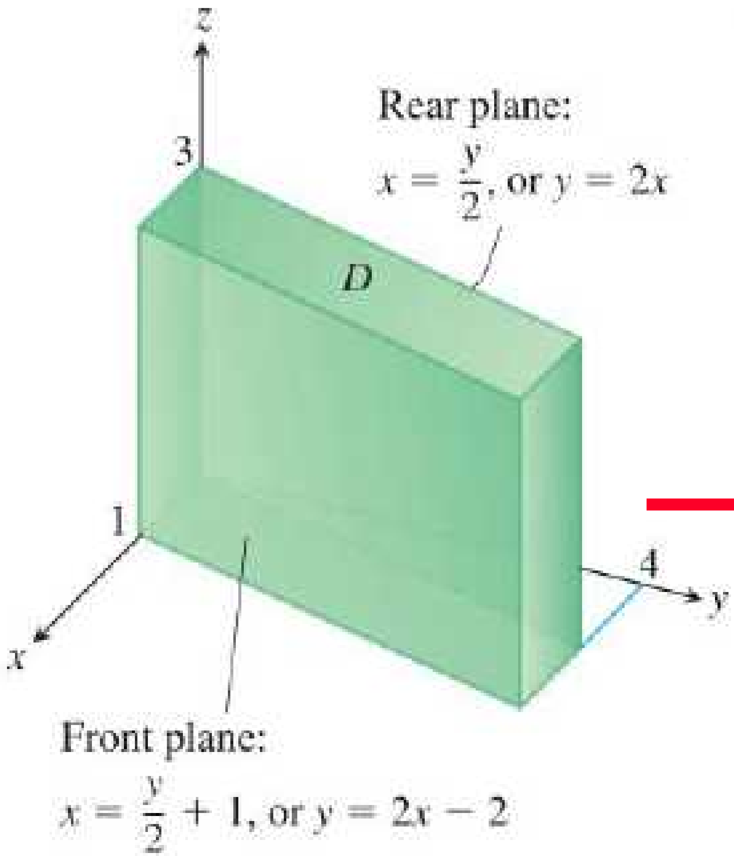
$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\, dy\, dz$$

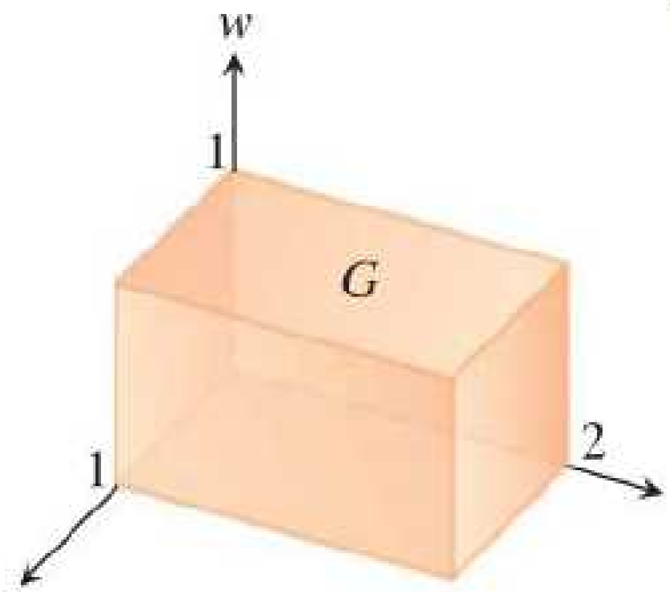
by applying the transformation

$$u = (2x - y)/2, \qquad v = y/2, \qquad w = z/3$$

Solution $x = u + v, \qquad y = 2v, \qquad z = 3w.$

xyz*-equations for the boundary of *D		**Simplified *uvw*-equations**
$x = y/2$		$u = 0$
$x = (y/2) + 1$		$u = 1$
$y = 0$	$\rightarrow$	$v = 0$
$y = 4$		$v = 2$
$z = 0$		$w = 0$
$z = 3$		$w = 1$





$$J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x - y}{2} + \frac{z}{3} \right) dx\, dy\, dz$$

$$= \int_0^1 \int_0^2 \int_0^1 (u + w)(6)\, du\, dv\, dw = 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dv\, dw$$

$$= 6 \int_0^1 (1 + 2w)\, dw = 12.$$

EXAMPLE 6 Evaluate $\iiint |xyz|\, dx\, dy\, dz$ over the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1.$$

Solution

$$x = a\rho \sin\varphi \cos\theta,\ y = b\rho \sin\varphi \sin\theta,\ z = c\rho \cos\varphi$$

$$J = abc\rho^2 \sin\varphi$$

$$\iiint_D |xyz|\, dxdydz \quad = 8\iiint_D xyz dxdydz$$

$$= 8\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 abc\rho^3 \sin^2\varphi \cos\varphi \cos\theta \sin\theta \cdot acb\rho^2 \sin\varphi d\rho d\varphi d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 (abc)^2 \rho^5 \sin^3\varphi \cos\varphi \cos\theta \sin\theta d\rho d\varphi d\theta \quad = \frac{1}{6}(abc)^2.$$