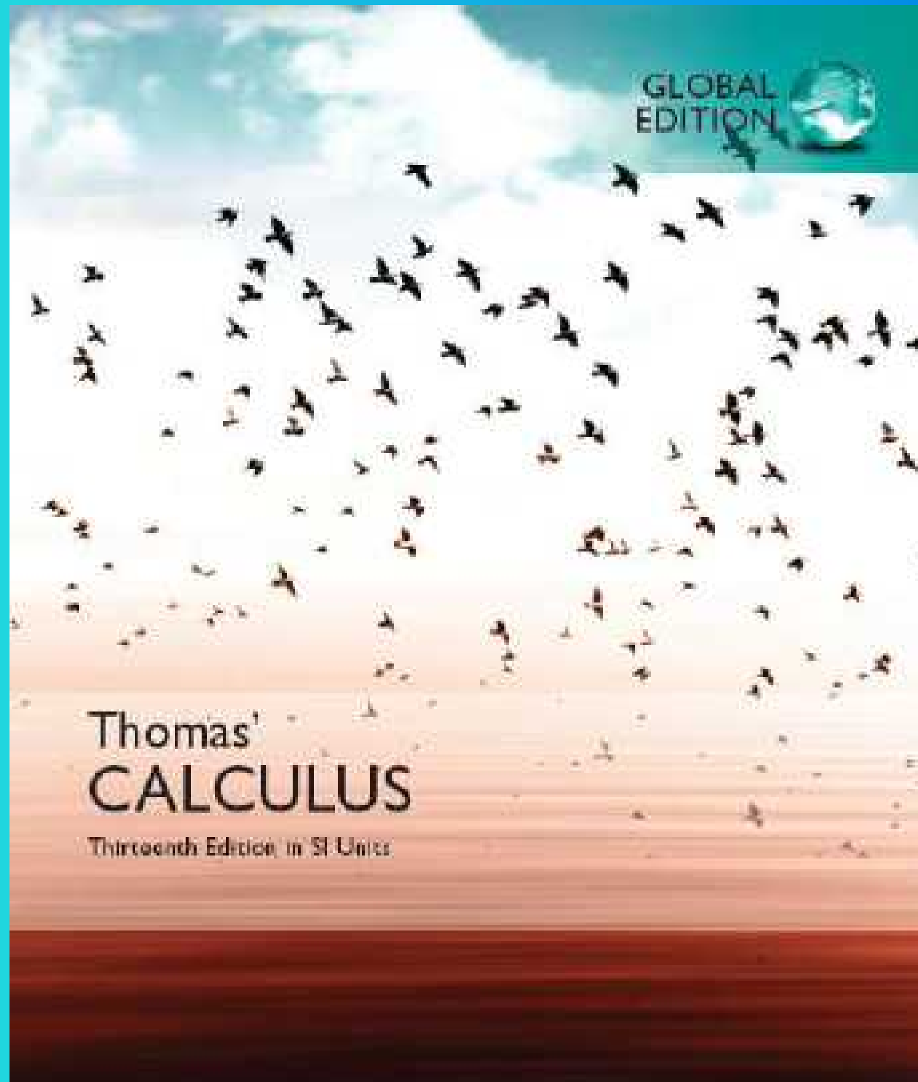

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 8

Techniques of Integration

积分技术

8.1

Using Basic Integration Formulas
用积分积分公式计算积分

1. $\int k\,dx = kx + C \qquad \text{(any number } k\text{)}$

2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$

3. $\int \dfrac{dx}{x} = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \qquad (a > 0,$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \cos x\,dx = \sin x + C$

8. $\int \sec^2 x\,dx = \tan x + C$

9. $\int \csc^2 x\,dx = -\cot x + C$

10. $\int \sec x \tan x\,dx = \sec x + C$

11. $\int \csc x \cot x\,dx = -\csc x + C$

12. $\int \tan x \, dx = \ln |\sec x| + C$

13. $\int \cot x \, dx = \ln |\sin x| + C$

14. $\int \sec x \, dx = \ln |\sec x + \tan x| + C$

15. $\int \csc x \, dx = -\ln |\csc x + \cot x| + C$

16. $\int \sinh x \, dx = \cosh x + C$

17. $\int \cosh x \, dx = \sinh x + C$

17. $\displaystyle\int \cosh x\, dx = \sinh x + C$

18. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

19. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

20. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

21. $\displaystyle\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad (a > 0)$

22. $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x > a > 0)$

Ex. 1 **Evaluate** $\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}dx.$

Solution $\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}dx$

$$= \int_3^5 \frac{1}{\sqrt{x^2-3x+1}}d(x^2-3x+1).$$

$$= 2\sqrt{x^2-3x+1}\Big|_3^5 = 2(\sqrt{11}-1)$$

Ex. 2 Complete the square to evaluate $\int \frac{dx}{\sqrt{8x-x^2}}$.

Solution $\int \frac{dx}{\sqrt{8x-x^2}} = \int \frac{dx}{\sqrt{16-(x-4)^2}}$

$$= \int \frac{d(x-4)}{\sqrt{4^2-(x-4)^2}} = \sin^{-1}\left(\frac{x-4}{4}\right) + C.$$

Ex. 3 Evaluate the integral $\int \cos x \sin 2x dx$.

Solution

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$

$$\int \cos x \sin 2x dx = \frac{1}{2}\int (\sin 3x + \sin x)dx = -\frac{\cos 3x}{6} - \frac{\cos x}{2} + C.$$

Ex. 4 Find $\int_0^{\pi/4} \frac{1}{1-\sin x} dx.$

Solution $\int_0^{\pi/4} \frac{1}{1-\sin x} dx = \int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x} dx$

$$= \int_0^{\pi/4} (\sec^2 x + \tan x \sec x) dx = (\tan x + \sec x)\Big|_0^{\pi/4} = \sqrt{2}$$

Ex. 5 Evaluate the integral $\int \frac{3x^2 - 7x}{3x+2} dx.$

Solution $\int \frac{3x^2 - 7x}{3x+2} dx = \int (x - 3 + \frac{6}{3x+2}) dx$

$$= \int (x-3) dx + 2\int \frac{1}{3x+2} d(3x+2)$$

$$= \frac{(x-3)^2}{2} + 2\ln|3x+2| + C.$$

Ex. 6 Evaluate the integral $\int \frac{x+1}{x^2+x+1}dx.$

Solution $\int \frac{x+1}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+2}{x^2+x+1}dx$

$$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx + \frac{1}{2}\int \frac{1}{x^2+x+1}dx$$

$$= \frac{1}{2}\int \frac{d(x^2+x+1)}{x^2+x+1} + \frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}d(x+\frac{1}{2})$$

$$= \frac{1}{2}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + C.$$

Ex. 7 **Evaluate the integral** $\displaystyle\int \frac{1}{(1+\sqrt{x})^3}\,dx.$

Solution $u=\sqrt{x},\ x=u^2,\quad dx=2u\,du$

$$\int \frac{1}{(1+\sqrt{x})^3}\,dx = 2\int \frac{u}{(1+u)^3}\,du$$

$$= 2\int \frac{u+1-1}{(1+u)^3}\,du = 2\int \frac{1}{(1+u)^2}\,du - 2\int \frac{1}{(1+u)^3}\,du$$

$$= -\frac{2}{1+u} + \frac{1}{(1+u)^2} + C = -\frac{2}{1+\sqrt{x}} + \frac{1}{(1+\sqrt{x})^2} + C$$

EXAMPLE 7 Evaluate

$$\int \frac{dx}{(1+\sqrt{x})^3}.$$

Solution

$$u = 1 + \sqrt{x},\; du = \frac{1}{2\sqrt{x}}\,dx;$$

$$dx = 2\sqrt{x}\,du = 2(u-1)\,du$$

$$\int \frac{dx}{(1+\sqrt{x})^3} = \int \frac{2(u-1)\,du}{u^3} = \int \left(\frac{2}{u^2} - \frac{2}{u^3}\right) du$$

$$= -\frac{2}{u} + \frac{1}{u^2} + C = C - \frac{1+2\sqrt{x}}{(1+\sqrt{x})^2}.$$

EXAMPLE 8 Evaluate $\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx$.

Solution

$$\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx = 0.$$

例9 求 $\int \frac{1}{x^2-a^2}dx.$

解 $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\int (\frac{1}{x-a} - \frac{1}{x+a})dx$

$$= \frac{1}{2a}\int \frac{1}{x-a}d(x-a) - \frac{1}{2a}\int \frac{1}{x+a}d(x-a)$$

$$= \frac{1}{2a}(\ln|x-a| - \ln|x+a|) + C.$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C.$$

例10 求 $\int \frac{1}{1+e^x}dx.$

解 $$\int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx$$

$$= \int \left(1-\frac{e^x}{1+e^x}\right)dx = \int dx - \int \frac{e^x}{1+e^x}dx$$

$$= \int dx - \int \frac{1}{1+e^x}d(1+e^x)$$

$$= x - \ln(1+e^x) + C.$$

例11 设 $f'(\sin^2 x) = \cos^2 x$，求 $f(x)$.

解 令 $u = \sin^2 x \quad \Rightarrow \quad \cos^2 x = 1 - u$,

$$f'(u) = 1 - u,$$

$$f(u) = \int (1-u)du = u - \frac{1}{2}u^2 + C,$$

$$f(x) = x - \frac{1}{2}x^2 + C.$$

例12　求 $\int \frac{1}{\sqrt{1-x^2}\arcsin x}dx.$

解 $$\int \frac{1}{\sqrt{1-x^2}\arcsin x}dx$$

$$=\int \frac{1}{\arcsin x}d(\arcsin x) \quad = \ln|\arcsin x| + C.$$

例13 求 $\int \frac{\arctan^2 x}{1+x^2}dx.$

解 $\int \frac{\arctan^2 x}{1+x^2}dx$

$$=\int \arctan^2 x d(\arctan x) = \frac{\arctan^3 x}{3}+C.$$

例 求 $\int \frac{x}{\sqrt{1-x^4}}dx$.

例 求 $\int \frac{x^2}{1+x^6}dx$.

例 求 $\int \frac{1}{x(3+2\ln x)}dx$.

例 求 $\int \frac{1}{x}\sqrt{1+\ln x^3}\,dx$.

8.2

Integration by Parts

分部积分法

问题 $\int xe^{-x}dx = ?$

两个函数乘积的求导法则:

设函数$u = u(x)$和$v = v(x)$具有连续导数,

$$(uv)' = u'v + uv', \qquad uv' = (uv)' - u'v,$$

$$\int uv'dx = uv - \int u'vdx, \qquad \boxed{\int udv = uv - \int vdu.}$$

$$\int_a^b uv'dx = \int_a^b (uv)'dx - \int_a^b u'vdx,$$

$$\boxed{\int_a^b udv = uv\Big|_a^b - \int_a^b vdu,}$$

分部积分公式

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du.$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du,$$

例1 求积分 $\int x\cos x dx$.

解 $$\int x\cos x dx = \int \cos x d\left(\frac{x^2}{2}\right)$$

$$= \frac{x^2}{2}\cos x + \int \frac{x^2}{2}\sin x dx$$

显然，u, v' 选择不当，积分更难进行.

解 $$\int x\cos x dx = \int x d\sin x$$

$$= x\sin x - \int \sin x dx$$

$$= x\sin x + \cos x + C.$$

例2　求积分 $\int \ln x dx.$　$\int x^2 \ln x dx.$

解 $\int \ln x dx = x\ln x - \int x d\ln x = x\ln x - x + C.$

例3　求积分 $\int x^2 e^x dx.$

解 $\int x^2 e^x dx = \int x^2 d(e^x)$

$= x^2 e^x - 2\int x e^x dx = x^2 e^x - 2\int x de^x$

（再次使用分部积分法）

$= x^2 e^x - 2(xe^x - e^x) + C.$

例4　求积分$\int e^x \sin x dx$.　　　　$\int e^{ax}\sin bx dx$.

解　$\boxed{\int e^x \sin x dx} = \int \sin x de^x$

$= e^x \sin x - \int e^x d(\sin x)$

$= e^x \sin x - \int e^x \cos x dx = e^x \sin x - \int \cos x de^x$

$= e^x \sin x - (e^x \cos x - \int e^x d\cos x)$

$= e^x(\sin x - \cos x) - \boxed{\int e^x \sin x dx}$ 注意循环形式

$\therefore \int e^x \sin x dx = \dfrac{e^x}{2}(\sin x - \cos x) + C$.

例5 求积分的 $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$ 递推公式

解 $\int \sin^n x dx = -\int \sin^{n-1} x d\cos x$

$= -\cos x \sin^{n-1} x + \int \cos x d\sin^{n-1} x$

$= -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x dx$

$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx$

$nI_n = -\left[\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + (n-1)I_{n-2} = (n-1)I_{n-2}$

$I_n = \frac{(n-1)}{n} I_{n-2}$ $I_{10} = \int_0^{\frac{\pi}{2}} \sin^{10} x dx$ $I_{10} = \frac{9}{10}\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$

Ex.6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from x=0 to x=4.

Solution $A = \int_0^4 xe^{-x}dx \quad = -\int_0^4 xde^{-x}$

$$= -xe^{-x}\Big|_0^4 + \int_0^4 e^{-x}dx$$

$$= -4e^{-4} - e^{-x}\Big|_0^4 \quad = 1 - 5e^{-4} \approx 0.91$$

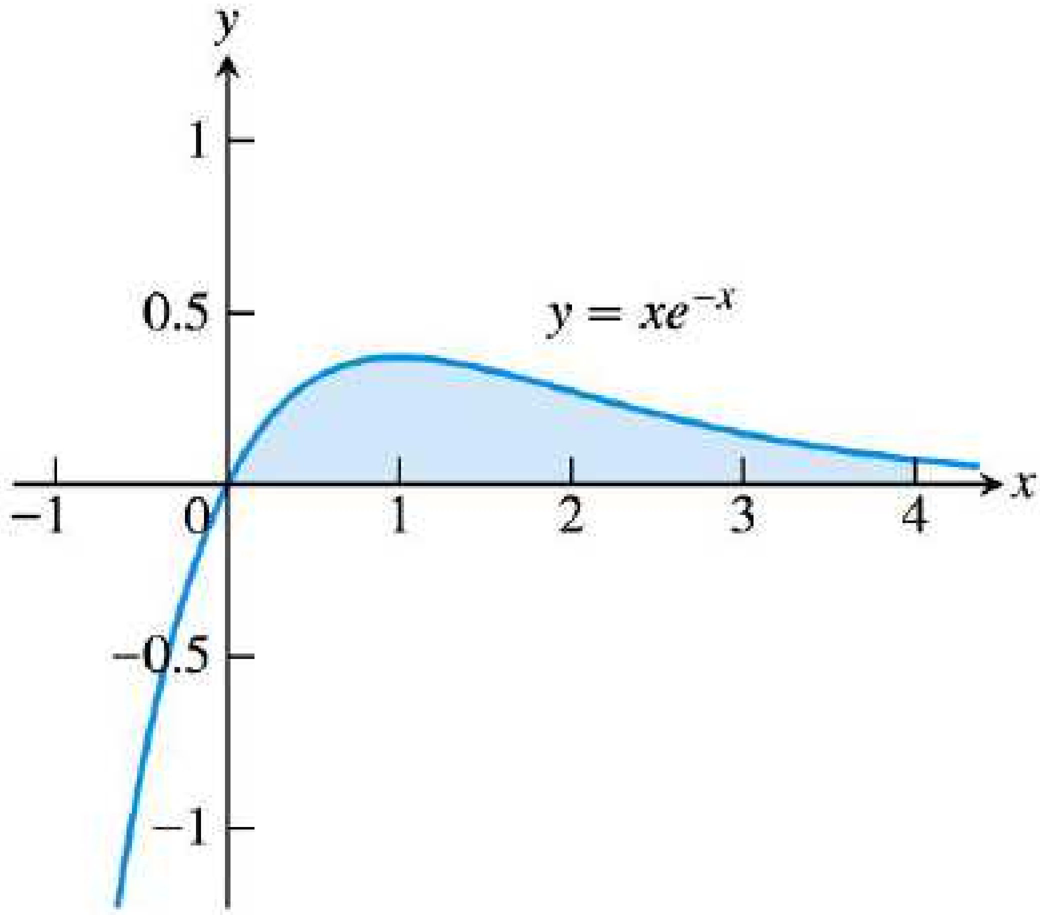


FIGURE 8.1 The region in Example 6.

例7　求积分 $\int x\arctan x dx$.

解
$$\int x\arctan x dx = \int \arctan x d\frac{x^2}{2}$$
$$= \frac{x^2}{2}\arctan x - \int \frac{x^2}{2} d(\arctan x)$$
$$= \frac{x^2}{2}\arctan x - \int \frac{x^2}{2}\cdot\frac{1}{1+x^2}dx$$
$$= \frac{x^2}{2}\arctan x - \int \frac{1}{2}\cdot\left(1-\frac{1}{1+x^2}\right)dx$$
$$= \frac{x^2}{2}\arctan x - \frac{1}{2}(x-\arctan x)+C.$$

Tabular integration by part

$$\int f(x)g(x)dx = \int f(x)dg_1(x)$$

$$= f(x)g_1(x) - \int f'(x)g_1(x)dx$$

$$= f(x)g_1(x) - \int f'(x)dg_{11}(x)$$

$$= f(x)g_1(x) - f'(x)g_{11}(x) + \int f''(x)g_{11}(x)dx$$

$$= f(x)g_1(x) - f'(x)g_{11}(x) + \int f''(x)dg_{111}(x)$$

$$= f(x)g_1(x) - f'(x)g_{11}(x) + f''(x)g_{111}(x) - \int f'''(x)g_{111}(x)dx$$

$$= f(x)g_1(x) - f'(x)g_{11}(x) + f''(x)g_{111}(x) - f'''(x)g_{1111}(x)$$

$$+ \int f''''(x)g_{1111}(x)dx$$

if $f^{(k)}(x) = 0$, then $\int f(x)g(x)dx$

$$= f(x)g_1(x) - f'(x)g_{11}(x) + f''(x)g_{111}(x) - \cdots + (-1)^k f^{(k-1)}(x)g_{(k)}(x)$$

EXAMPLE 7 Evaluate $\displaystyle\int x^2 e^x\,dx$.

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + C.$$

EXAMPLE 8 Find the integral $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

for $f(x) = 1$ on $[-\pi, 0)$ and $f(x) = x^3$ on $[0, \pi]$, where n is a positive integer.

Solution

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x^3\cos nx\,dx$$

$$= \frac{1}{n\pi}\sin nx\Big]_{-\pi}^{0} + \frac{1}{\pi}\int_{0}^{\pi} x^3\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x^3\cos nx\,dx.$$

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\cos nx$
$3x^2$	$(-)$	$\frac{1}{n}\sin nx$
$6x$	$(+)$	$-\frac{1}{n^2}\cos nx$
6	$(-)$	$-\frac{1}{n^3}\sin nx$
0		$\frac{1}{n^4}\cos nx$

$$\frac{1}{\pi}\int_0^{\pi} x^3 \cos nx\,dx \qquad = \frac{1}{\pi}\left(\frac{3\pi^2}{n^2}(-1)^n - \frac{6((-1)^n - 1)}{n^4}\right)$$

$$= \frac{1}{\pi}\left(\frac{x^3}{n}\sin nx + \frac{3x^2}{n^2}\cos nx - \frac{6x}{n^3}\sin nx - \frac{6}{n^4}\cos nx\right)_0^{\pi}$$

8.3

Trigonometric Integrals
三角函数积分法

Products of Powers of Sines and Cosines

$$\int \sin^m x \cos^n x \, dx,$$

Case 1 If m is odd, $\quad \sin x dx = -d(\cos x)$

Case 2 If m is even and n is odd $\quad \cos x dx = d(\sin x)$

Case 3 If both m and n are even

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Ex. 1 **Evaluate** $\int \sin^3 x \cos^2 x dx.$

Solution $\int \sin^3 x \cos^2 x dx = -\int \sin^2 x \cos^2 x d\cos x$

$$= -\int (\cos^2 x - \cos^4 x) d\cos x$$

$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.$$

Ex. 2 **Evaluate** $\int \cos^5 x dx$.

Solution $\int \cos^5 x dx = \int \cos^4 x d \sin x$

$$= \int (1 - \sin^2 x)^2 d \sin x$$

$$= \int (1 - 2\sin^2 x + \sin^4 x) d \sin x$$

$$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C.$$

Ex. 3 Evaluate $\int \sin^2 x \cos^4 x dx$.

Solution $\int \sin^2 x \cos^4 x dx = \frac{1}{4}\int \sin^2 2x \cos^2 x dx$

$$= \frac{1}{4}\int \sin^2 2x \frac{1+\cos 2x}{2} dx$$

$$= \frac{1}{8}\int \sin^2 2x dx + \frac{1}{8}\int \sin^2 2x \cos 2x dx$$

$$= \frac{1}{16}\int (1-\cos 4x) dx + \frac{1}{16}\int \sin^2 2x d(\sin 2x)$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C.$$

Ex. 4 Find $\int_0^{\pi/2} \sqrt{1+\cos 4x}\,dx.$

Solution $\int_0^{\pi/2} \sqrt{1+\cos 4x}\,dx = \int_0^{\pi/2} \sqrt{2\cos^2 2x}\,dx$

$$= \sqrt{2}\int_0^{\pi/2} |\cos 2x|\,dx$$

$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx - \sqrt{2}\int_{\pi/4}^{\pi/2} \cos 2x\,dx$$

$$= \left[\frac{\sqrt{2}\sin 2x}{2}\right]_0^{\pi/4} - \left[\frac{\sqrt{2}\sin 2x}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \sqrt{2}.$$

Integrals of Powers of tan x and sec x

Ex. 5 Evaluate $\int \tan^4 x dx$.

Solution $\int \tan^4 x dx \quad = \int \tan^2 x(\sec^2 x - 1)dx$

$$= \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx$$

$$= \int \tan^2 x d \tan x - \int (\sec^2 x - 1)dx$$

$$= \frac{\tan^3 x}{3} - \tan x + x + C.$$

Ex. 6 Evaluate $\int \sec^3 x\,dx$.

Solution $\int \sec^3 x\,dx = \int \sec x\,d\tan x$

$$= \sec x \tan x - \int \tan^2 x \sec x\,dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x)\,dx$$

$$\int \sec^3 x\,dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C.$$

Ex. 7 Evaluate $\int \tan^4 x \sec^4 x dx$.

Solution

$$\int \tan^4 x \sec^4 x dx$$

$$= \int \tan^4 x \sec^2 x d \tan x$$

$$= \int \tan^4 x (1 + \tan^2 x) d \tan x$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C.$$

8.4

Trigonometric Substitutions
积分的三角代换

例1 求 $\int \frac{1}{\sqrt{x^2+a^2}}dx \quad (a>0).$

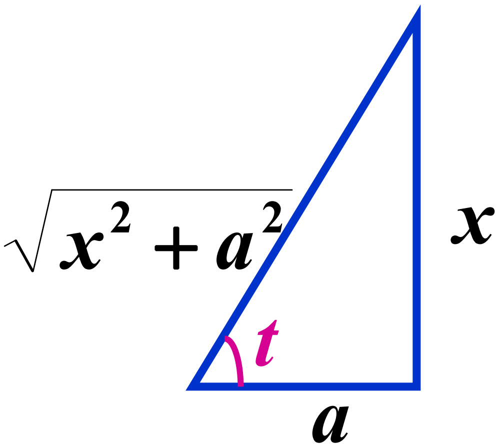
解 令 $x = a\tan t \Rightarrow dx = a\sec^2 t dt \qquad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{1}{a\sec t}\cdot a\sec^2 t dt$$

$$= \int \sec t dt = \ln|\sec t + \tan t| + C$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C.$$

$$= \ln\left(x + \sqrt{x^2+a^2}\right) + C.$$



例2 求 $\int x^3\sqrt{4-x^2}dx.$

解 令 $x=2\sin t \quad dx=2\cos t dt \quad t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

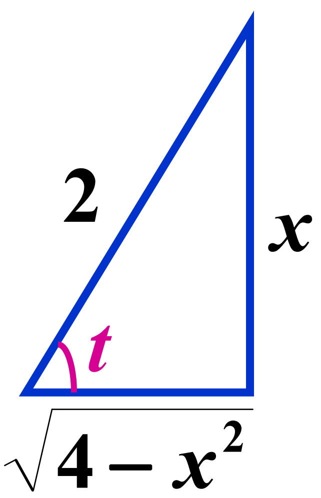
$$\int x^3\sqrt{4-x^2}dx=\int(2\sin t)^3\sqrt{4-4\sin^2 t}\cdot 2\cos t dt$$

$$=32\int\sin^3 t\cos^2 t dt=32\int\sin t(1-\cos^2 t)\cos^2 t dt$$

$$=-32\int(\cos^2 t-\cos^4 t)d\cos t$$

$$=-32\left(\frac{1}{3}\cos^3 t-\frac{1}{5}\cos^5 t\right)+C$$

$$=-\frac{4}{3}\left(\sqrt{4-x^2}\right)^3+\frac{1}{5}\left(\sqrt{4-x^2}\right)^5+C.$$



例3 求 $\int \frac{1}{\sqrt{x^2-a^2}}dx \quad (a>0)$.

$x > a$

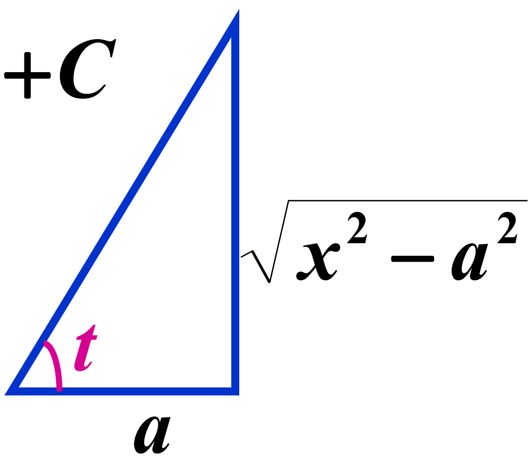
解 令$x = a\sec t \quad dx = a\sec t\tan t\,dt \quad t \in \left(0, \frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{a\sec t \cdot \tan t}{a\tan t}dt$$

$$= \int \sec t\,dt = \ln|\sec t + \tan t| + C$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C.$$

$$= \ln|x + \sqrt{x^2-a^2}| + C.$$



当 $x < -a$

$$x = -a\sec t \qquad dx = -a\sec t\tan t dt \qquad t \in \left(0, \frac{\pi}{2}\right)$$

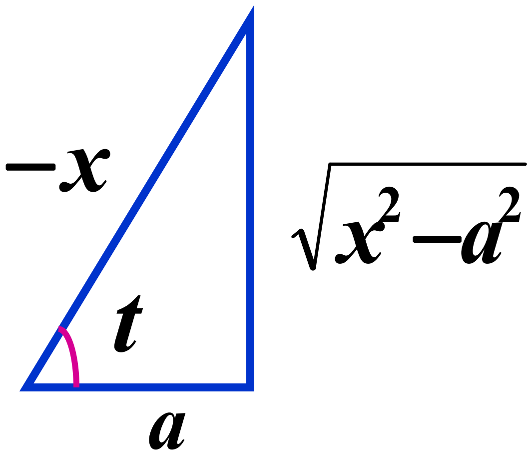
$$\int \frac{1}{\sqrt{x^2 - a^2}}dx = -\int \frac{a\sec t \cdot \tan t}{a\tan t}dt$$

$$= -\int \sec t dt \quad = -\ln|\sec t + \tan t| + C$$

$$= -\ln\left|\frac{-x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C.$$

$$= -\ln|-x + \sqrt{x^2 - a^2}| + C.$$

$$= \ln|x + \sqrt{x^2 - a^2}| + C.$$



EXAMPLE 4 Evaluate
$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \qquad x > \frac{2}{5}.$$

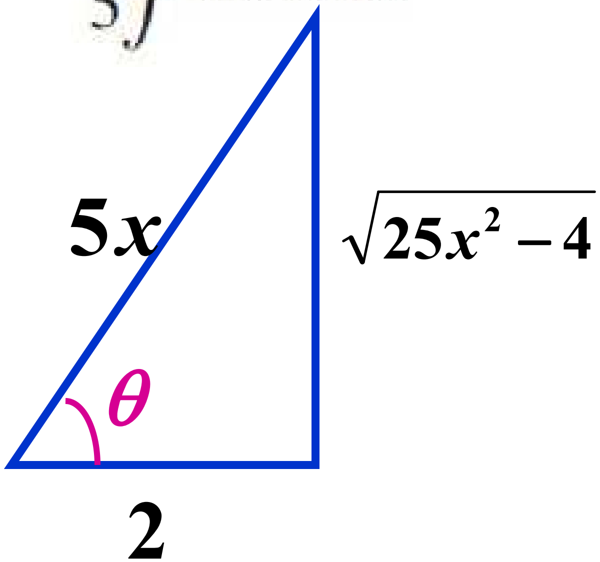
Solution We then substitute $x = \frac{2}{5}\sec\theta, \quad 0 < \theta < \frac{\pi}{2}$

$$dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta,$$

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5\cdot(2/5)\tan\theta} = \frac{1}{5}\int \sec\theta\, d\theta$$

$$= \frac{1}{5}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C.$$



例8 求积分 $\int \frac{\sin x}{1+\sin x+\cos x}dx.$

解 $\int \frac{\sin x}{1+\sin x+\cos x}dx = \int \frac{\sin x(1-\sin x-\cos x)}{1-(\sin x+\cos x)^2}dx$

$$= -\int \frac{\sin x(1-\sin x-\cos x)}{2\sin x\cos x}dx$$

$$= -\frac{1}{2}\int(\sec x-\tan x-1)dx$$

$$= -\frac{1}{2}(\ln|\sec x+\tan x|+\ln|\cos x|-x)+C.$$

例8 求积分 $\int \frac{\sin x}{1+\sin x+\cos x}dx.$

解法2　由万能代换公式　$\tan\frac{x}{2}=u,$

$$\sin x=\frac{2u}{1+u^2},\quad \cos x=\frac{1-u^2}{1+u^2}\quad dx=\frac{2}{1+u^2}du,$$

$$\int \frac{\sin x}{1+\sin x+\cos x}dx=\int \frac{2u}{(1+u)(1+u^2)}du$$

$$=\int \frac{2u+1+u^2-1-u^2}{(1+u)(1+u^2)}du$$

$$= \int \frac{1+u}{1+u^2} du - \int \frac{1}{1+u} du$$

$$= \arctan u + \frac{1}{2} \ln(1+u^2) - \ln|1+u| + C$$

$$\because \quad u = \tan \frac{x}{2}$$

$$= \frac{x}{2} + \ln\left|\sec \frac{x}{2}\right| - \ln\left|1 + \tan \frac{x}{2}\right| + C.$$

8.5

Integration of Rational Functions by Partial Fractions

有理函数的定义：

两个多项式的商表示的函数称之.

$$\frac{P(x)}{Q(x)}=\frac{a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_{m-1}x+b_m}$$

其中m、n都是非负整数；$a_0,a_1,\cdots,a_n$及

$b_0,b_1,\cdots,b_m$都是实数，并且$a_0\neq 0$，$b_0\neq 0$.

假定分子与分母之间没有公因式

(1) $n < m$，这有理函数是**真分式**；

(2) $n \ge m$，这有理函数是**假分式**；

利用多项式除法，假分式可以化成一个多项式和一个真分式之和.

例 $$\frac{x^3+x+1}{x^2+1} = x + \frac{1}{x^2+1}.$$

下面只考虑真分式的积分

$$\frac{1}{x^2-5x+6}=\frac{1}{(x-2)(x-3)}=\frac{1}{x-3}-\frac{1}{x-2}$$

$$\frac{x+3}{x^2-5x+6}=\frac{x+3}{x-3}-\frac{x+3}{x-2}=\frac{-5}{x-2}+\frac{6}{x-3}.$$

$$\frac{1}{(x^2+1)x}=\frac{1+x^2-x^2}{(x^2+1)x}=\frac{1}{x}-\frac{x}{x^2+1}$$

$$\frac{x+1}{(x^2+1)x^2}=\frac{1+x^2-x^2+x}{(x^2+1)x^2}=\frac{1}{x^2}-\frac{1}{x^2+1}+\frac{1}{x}-\frac{x}{x^2+1}$$

$$\frac{1}{(x^2+1)^2x}=\frac{1+x^2-x^2}{(x^2+1)^2x}=\frac{1}{(1+x^2)x}-\frac{x}{(x^2+1)^2}$$

$$=\frac{1}{x}-\frac{x}{x^2+1}-\frac{x}{(x^2+1)^2}$$

有理函数化为部分分式之和的一般规律：

（1）分母中若有因式 $(x-r)^k$ ，则拆项后有

$$\frac{A_1}{(x-r)^k}+\frac{A_2}{(x-r)^{k-1}}+\cdots+\frac{A_k}{x-r},$$

其中$A_1, A_2, \cdots, A_k$都是常数.

（2）分母中若有因式 $(x^2+px+q)^k$，其中 $p^2-4q<0$ 则拆项后有

$$\frac{B_1x+C_1}{(x^2+px+q)^k}+\frac{B_2x+C_2}{(x^2+px+q)^{k-1}}+\cdots+\frac{B_kx+C_k}{x^2+px+q}$$

其中 B_i, C_i 都是常数$(i=1,2,\cdots,k)$.

Ex. 1 Evaluate $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx.$

Solution $\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$

$$= \frac{A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)}{(x-1)(x+1)(x+3)}$$

$$A+B+C=1, \quad 4A+2B=4, \quad 3A-3B-C=1,$$

$$A=3/4, \quad B=1/2, \quad C=-1/4.$$

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx \quad = \int \left(\frac{3/4}{x-1} + \frac{1/2}{x+1} + \frac{-1/4}{x+3}\right)dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C.$$

Ex. 2 Evaluate $\int \frac{1}{x(x-1)^2}dx.$

Solution $\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)^2} + \frac{C}{x-1},$

$$1 = A(x-1)^2 + Bx + Cx(x-1)$$

取 $x = 0, \Rightarrow A = 1$ 取 $x = 1, \Rightarrow B = 1$

取 $x = 2,$ 并将 A, B 值代入 $\Rightarrow C = -1$

$$\int \frac{1}{x(x-1)^2}dx = \int \left[\frac{1}{x} + \frac{1}{(x-1)^2} - \frac{1}{x-1}\right]dx$$

$$= \ln|x| - \frac{1}{x-1} - \ln|x-1| + C.$$

Ex. 3 Evaluate $\displaystyle\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$

Solution $\displaystyle\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3} = \frac{2}{x+1} + \frac{3}{x-3}$$

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx = \int \left(2x + \frac{2}{x+1} + \frac{3}{x-3}\right)dx$$

$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C.$$

Ex. 4 Evaluate $\displaystyle\int \frac{-2x+4}{(x-1)^2(1+x^2)}dx.$

Solution $$\frac{-2x+4}{(x-1)^2(1+x^2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{1+x^2}$$

$$A(x-1)(1+x^2) + B(1+x^2) + (x-1)^2(Cx+D) = -2x+4$$

let $x=1$, find $B=1$, let $x=i$, find $C=2$, $D=1$.

let $x=0$, find $A=-2$.

$$\int \frac{-2x+4}{(x-1)^2(1+x^2)}dx = \int \left(\frac{-2}{x-1} + \frac{1}{(x-1)^2} + \frac{2x+1}{1+x^2}\right)dx$$

$$= -2\ln|x-1| - \frac{1}{x-1} + \ln(1+x^2) + \tan^{-1}x + C.$$

Ex. 5 Evaluate $\int \frac{1}{x(1+x^2)^2}dx.$

Solution $\frac{1}{x(1+x^2)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

$$1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

let $x = 0$, find $A = 1$, let $x = i$, find $E = 0$, $D = -1$.

equate the coefficient of x^4, find $B = -1$,

equate the coefficient of x^3, find $C = 0$.

$$\int \frac{1}{x(1+x^2)^2}dx = \int \left(\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right)dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C.$$

例6 求积分 $\int \frac{1}{1+e^{\frac{x}{2}}+e^{\frac{x}{3}}+e^{\frac{x}{6}}}dx.$

解 令 $t=e^{\frac{x}{6}} \Rightarrow x=6\ln t$, $dx=\frac{6}{t}dt$,

$$\int \frac{1}{1+e^{\frac{x}{2}}+e^{\frac{x}{3}}+e^{\frac{x}{6}}}dx = \int \frac{1}{1+t^3+t^2+t}\cdot\frac{6}{t}dt$$

$$=6\int \frac{1}{t(1+t)(1+t^2)}dt=\int\left(\frac{6}{t}-\frac{3}{1+t}-\frac{3t+3}{1+t^2}\right)dt$$

$$= \int \left(\frac{6}{t} - \frac{3}{1+t} - \frac{3t+3}{1+t^2} \right) dt$$

$$= 6\ln t - 3\ln(1+t) - \frac{3}{2}\int \frac{d(1+t^2)}{1+t^2} - 3\int \frac{1}{1+t^2}\,dt$$

$$= 6\ln t - 3\ln(1+t) - \frac{3}{2}\ln(1+t^2) - 3\arctan t + C$$

$$= x - 3\ln(1+e^{\frac{x}{6}}) - \frac{3}{2}\ln(1+e^{\frac{x}{3}}) - 3\arctan(e^{\frac{x}{6}}) + C.$$

8.7

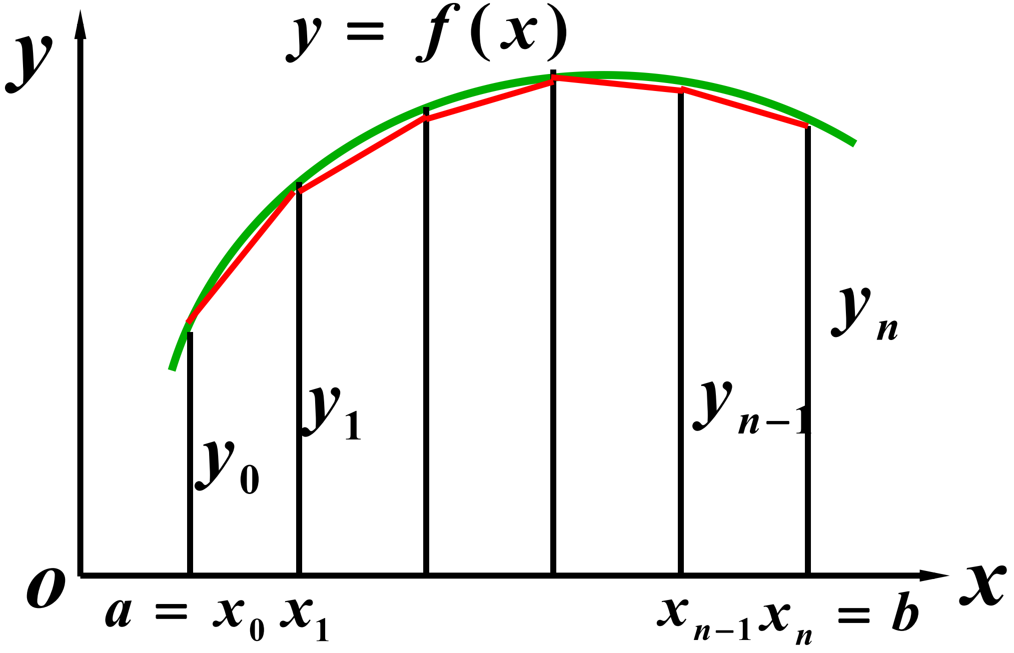
Numerical Integration

数值积分

like $\sin(x^2)$, $1/\ln x$, and $\sqrt{1+x^4}$,

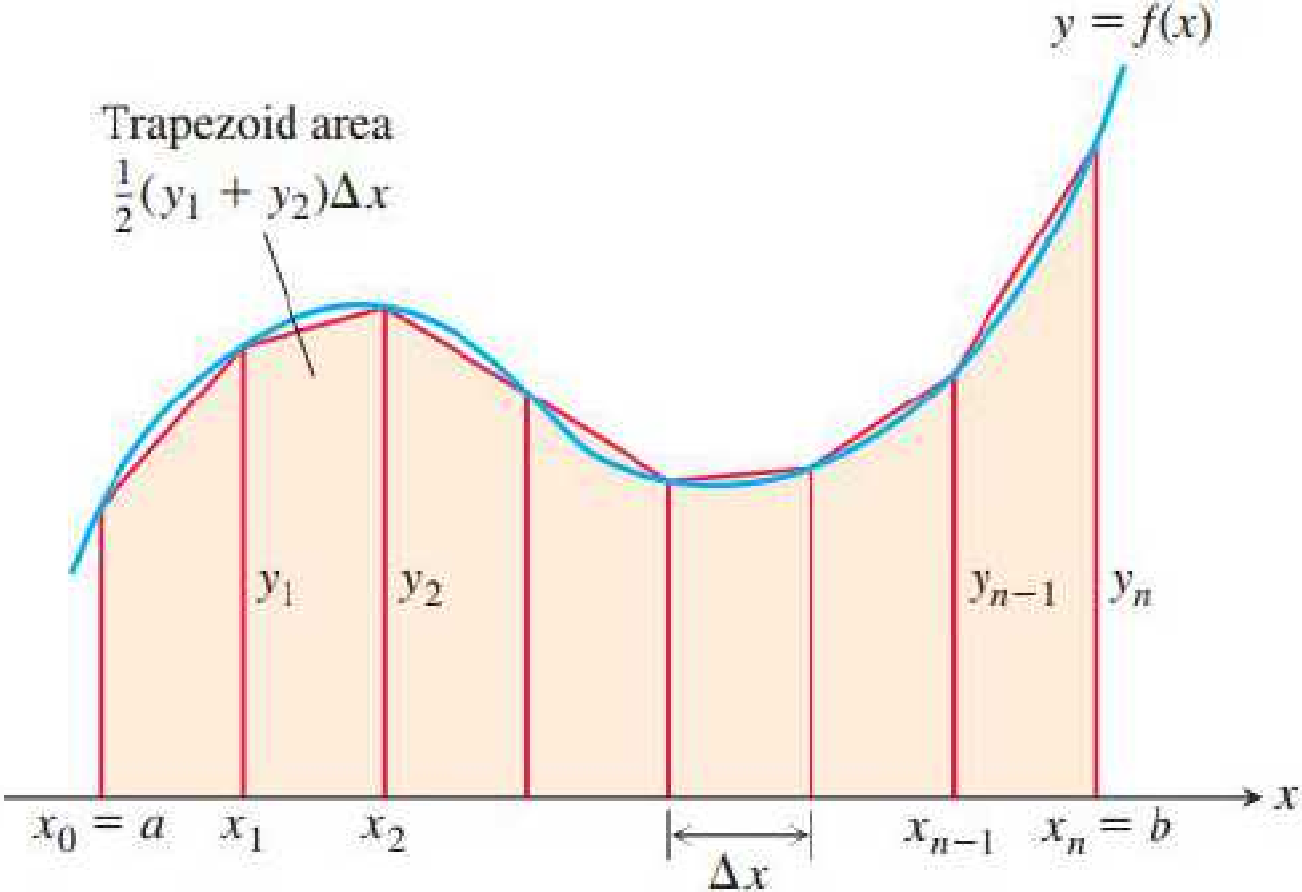
we cannot find a workable antiderivative for a function f

To approximate $\int_a^b f(x)\,dx$,



Trapezoidal Approximations

$$\Delta x = \frac{b-a}{n}.$$



The Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x$$

$$+ \cdots + \frac{1}{2}(y_{n-1} + y_n)\Delta x$$

$$= \frac{b-a}{n}[\frac{1}{2}(y_0 + y_n) + y_1 + y_2 + \cdots + y_{n-1}]$$

$$= \frac{b-a}{2n}[(y_0 + y_n) + 2y_1 + 2y_2 + \cdots + 2y_{n-1}]$$

$y_0 = f(a), \quad y_1 = f(x_1), \quad \ldots, \quad y_{n-1} = f(x_{n-1}), \quad y_n = f(b).$

EXAMPLE 1

Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$.

Solution Partition $[1, 2]$ into four subintervals of equal length

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right)$$

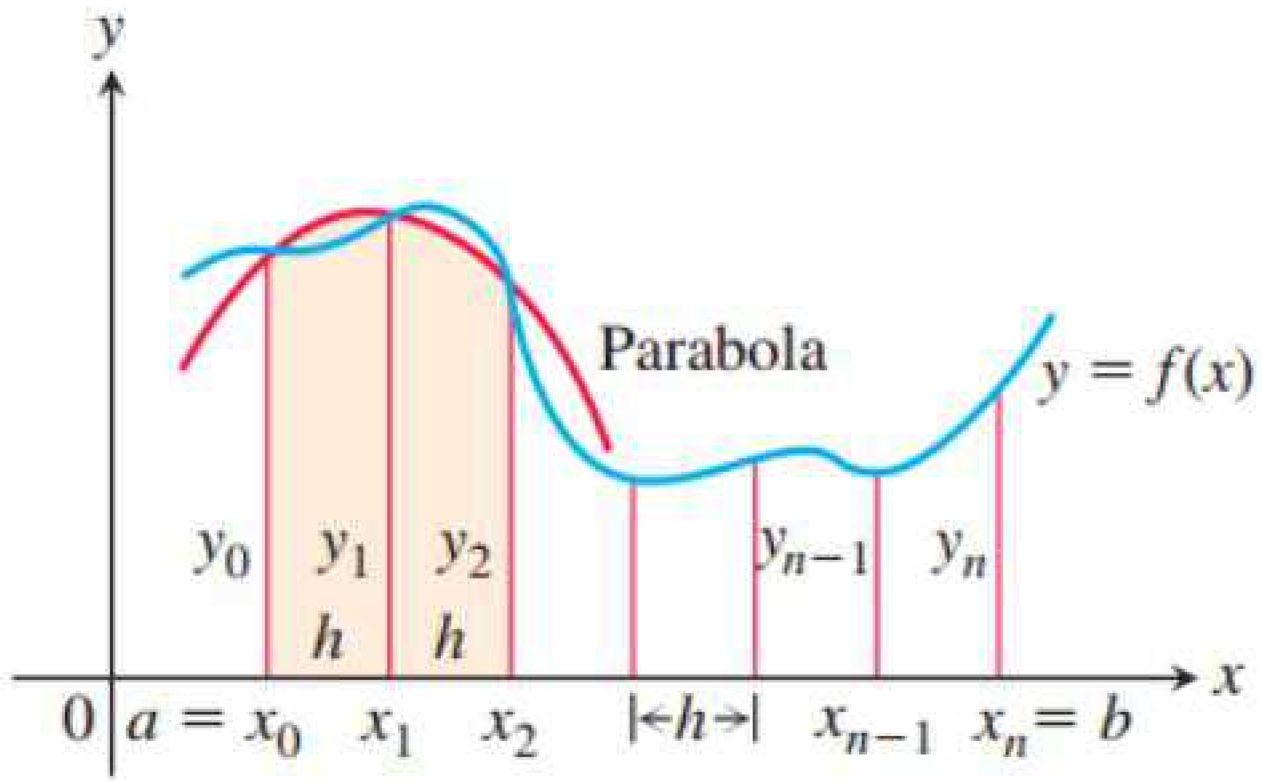
$$= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right)$$

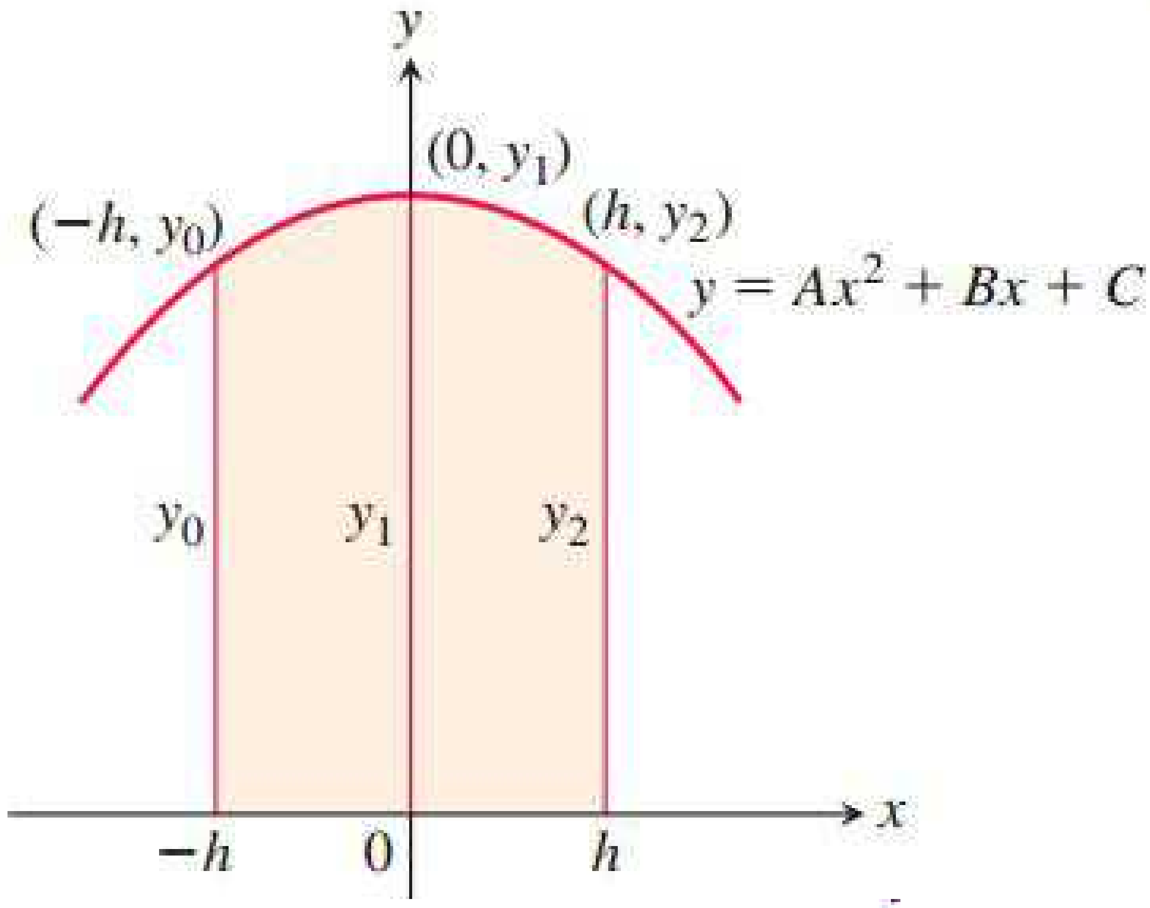
$$= \frac{75}{32} = 2.34375.$$

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\bigg]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

$$(2.34375 - 7/3)/(7/3) \approx 0.00446, \text{ or } 0.446\%.$$

Simpson's Rule: Approximations Using Parabolas





$$A_p = \int_{-h}^{h} (Ax^2 + Bx + C)\, dx$$

$$A_p = \int_{-h}^{h} (Ax^2 + Bx + C)\, dx = \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \right]_{-h}^{h}$$

$$= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C).$$

Since the curve passes through the three points $(-h, y_0)$, $(0, y_1)$, and (h, y_2),

$$y_0 = Ah^2 - Bh + C, \qquad y_1 = C, \qquad y_2 = Ah^2 + Bh + C,$$

from which we obtain

$$C = y_1,$$

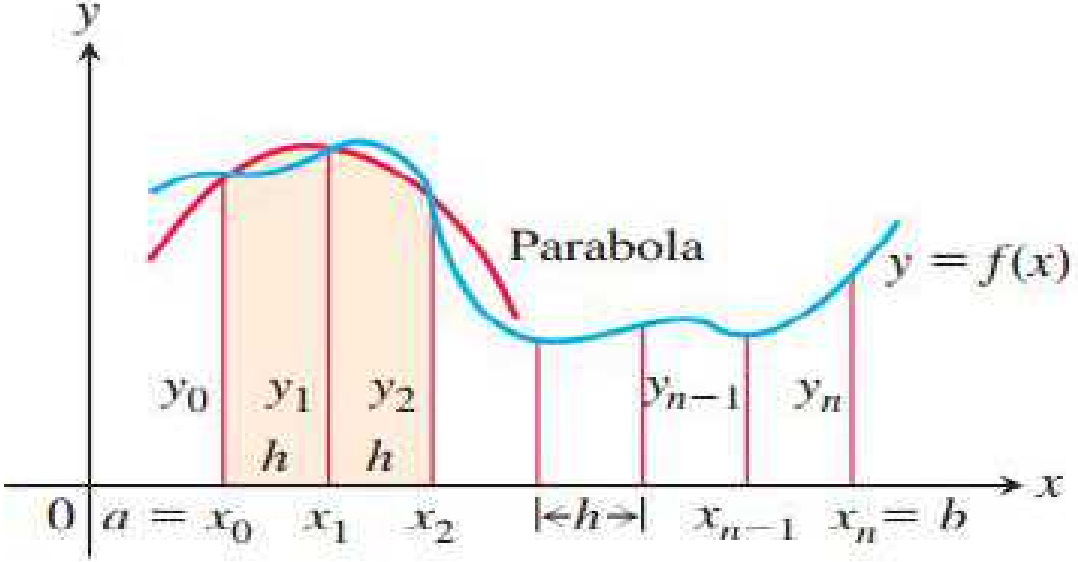
$$Ah^2 - Bh = y_0 - y_1,$$

$$Ah^2 + Bh = y_2 - y_1,$$

$$2Ah^2 = y_0 + y_2 - 2y_1.$$

$$A_p = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}((y_0 + y_2 - 2y_1) + 6y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Simpson's Rule



$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots$$

$$+ \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{b-a}{3n}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

EXAMPLE 2

Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

Solution

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$$= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right)$$

$$= 32\frac{1}{12}.$$

绝对误差$=\dfrac{1}{12}$。

This estimate differs from the exact value (32) by only 1/12,

Error Analysis

THEOREM 1—Error Estimates in the Trapezoidal and Simpson's Rules
If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \le \frac{M(b-a)^3}{12n^2}. \qquad \text{Trapezoidal Rule}$$

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_S| \le \frac{M(b-a)^5}{180n^4}. \qquad \text{Simpson's Rule}$$

EXAMPLE 3

Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's Rule with $n = 4$ (Example 2).

Solution $$|E_S| \le \frac{M(b-a)^5}{180n^4}.$$

$f^{(4)}(x) = 120$, we take $M = 120$. With $b - a = 2$ and $n = 4$,

$$|E_S| \le \frac{M(b-a)^5}{180n^4} = \frac{120(2)^5}{180 \cdot 4^4} = \frac{1}{12}.$$

EXAMPLE 4

Estimate the minimum number of subintervals needed to approximate the integral in Example 3 using Simpson's Rule with an error of magnitude less than 10^{-4}.

Solution

$$\frac{M(b-a)^5}{180n^4} < 10^{-4},$$

we have $M = 120$ and $b - a = 2$, so we want n to

$$\frac{120(2)^5}{180n^4} < \frac{1}{10^4} \qquad n^4 > \frac{64 \cdot 10^4}{3}.$$

$$n > 10\left(\frac{64}{3}\right)^{1/4} \approx 21.5. \qquad n = 22.$$

EXAMPLE 5 the value of ln 2 can be calculated from the integral

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx.$$

n	T_n	|Error| less than . . .	S_n	|Error| less than . . .
10	0.6937714032	0.0006242227	0.6931502307	0.0000030502
20	0.6933033818	0.0001562013	0.6931473747	0.0000001942
30	0.6932166154	0.0000694349	0.6931472190	0.0000000385
40	0.6931862400	0.0000390595	0.6931471927	0.0000000122
50	0.6931721793	0.0000249988	0.6931471856	0.0000000050
100	0.6931534305	0.0000062500	0.6931471809	0.0000000004

EXAMPLE 6

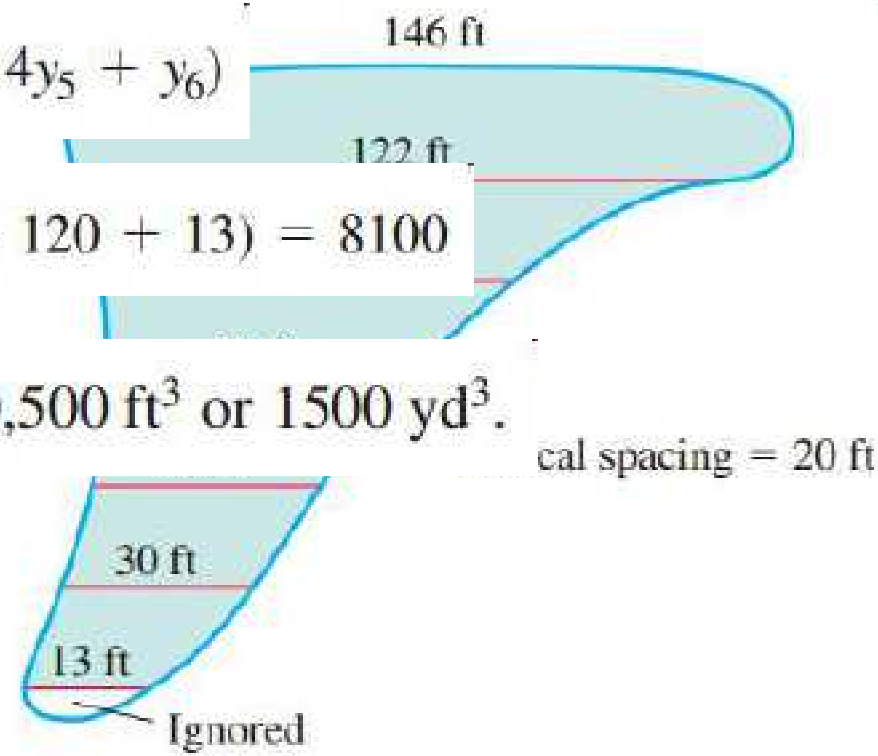
A town wants to drain and fill a small polluted swamp. About how many cubic yards of dirt will it take to fill the area after the swamp is drained? The swamp averages 5 ft deep.

Solution

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$$

$$= \frac{20}{3}(146 + 488 + 152 + 216 + 80 + 120 + 13) = 8100$$

The volume is about $(8100)(5) = 40{,}500\ \text{ft}^3$ or $1500\ \text{yd}^3$.

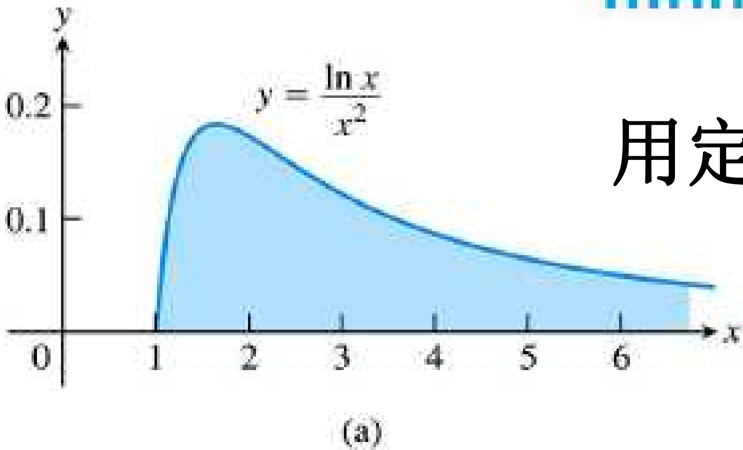


8.8

Improper Integrals

反常积分（广义积分）

Infinite Limits of Integration



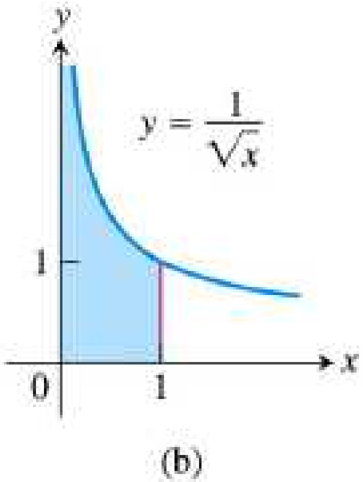


FIGURE 8.12 Are the areas under these infinite curves finite? We will see that the answer is yes for both curves.

用定积分表示蓝色部分的面积

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx. \qquad \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^2}\, dx.$$

$$= \lim_{b\to\infty} \left(1 - \frac{1+\ln b}{b}\right) = 1$$

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx.$$

$$\lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a\to 0^+} (2 - 2\sqrt{a}) = 2.$$

$$\int_0^1 \frac{1}{x}\, dx.$$

$$\lim_{a\to 0^+} \int_a^1 \frac{1}{x}\, dx = \lim_{a\to 0^+} (-\ln a) = \infty.$$

DEFINITION

Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

if the limit is finite we say that the improper integral **converges**

If the limit fails to exist, the improper integral **diverges**.

Ex. 1 Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is the value?

Solution

$$\int_1^\infty \frac{\ln x}{x^2}\,dx.$$

$$= \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx.$$

$$= \lim_{b\to\infty}\int_1^b \ln x\, d\left(-\frac{1}{x}\right)$$

$$= \lim_{b\to\infty}\left(-\frac{\ln x}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx\right)$$

$$= \lim_{b\to\infty}\left(-\frac{\ln b}{b} + 1 - \frac{1}{b}\right) = 1$$

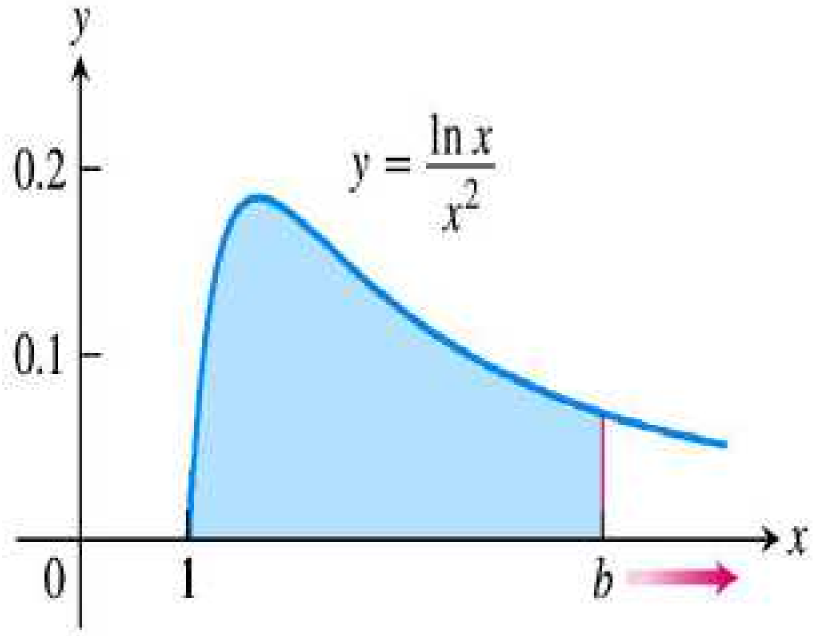


FIGURE 8.14 The area under this curve is an improper integral (Example 1).

Ex. 2 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$ $= \pi$

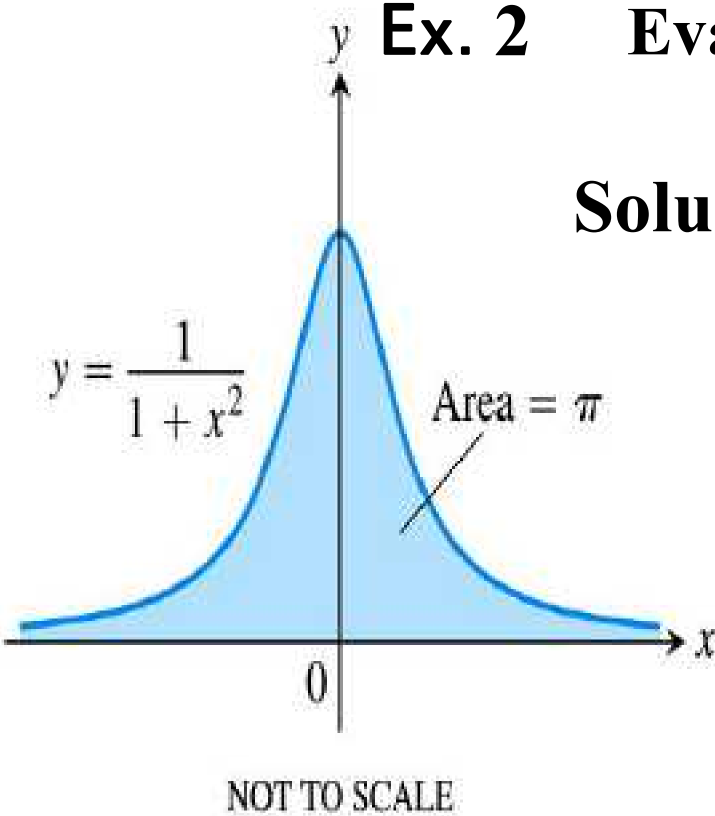


FIGURE 8.15 The area under this curve is finite (Example 2).

Solution $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} (-\tan^{-1} a) = \pi/2$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to-\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} (\tan^{-1} b) \;\; = \pi/2$$

Ex. 3 Investigate the convergence of $\int_1^{+\infty} \frac{1}{x^p} dx$.

Solution if $p = 1$,

$$\int_1^{+\infty} \frac{1}{x^p} dx = \int_1^{+\infty} \frac{1}{x} dx = [\ln x]_1^{+\infty} = \lim_{x\to\infty} \ln x = +\infty,$$

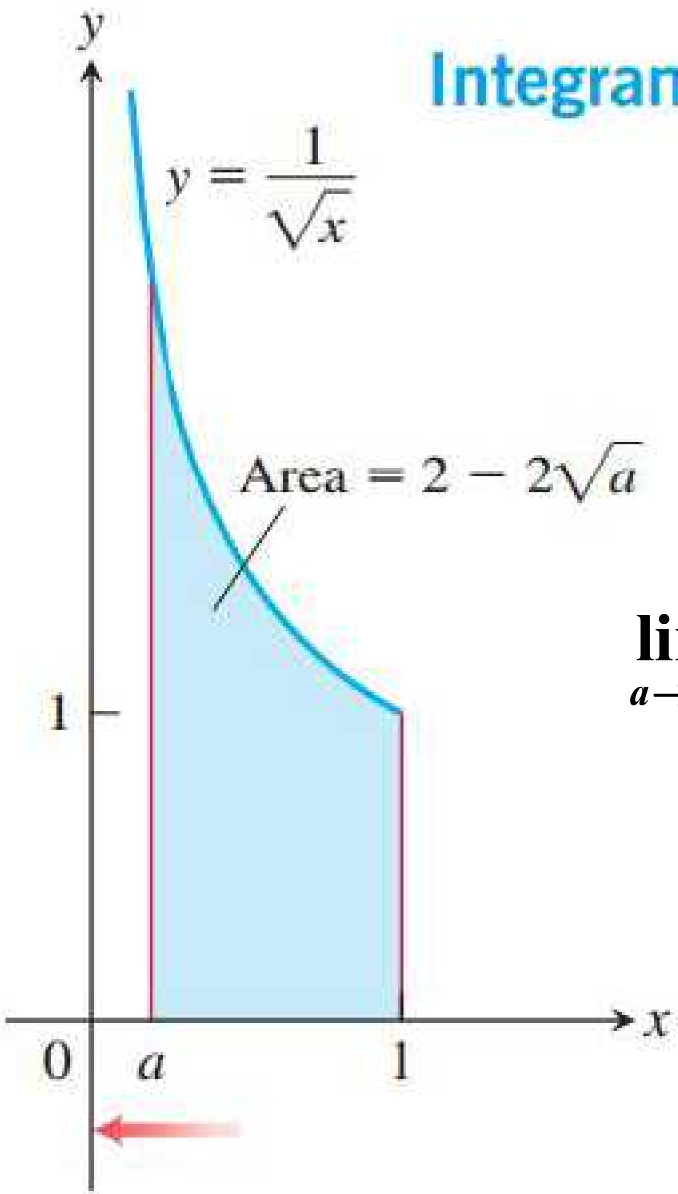
if $p \neq 1$,

$$\int_1^{+\infty} \frac{1}{x^p} dx = \left[\frac{x^{1-p}}{1-p}\right]_1^{+\infty} = \begin{cases} +\infty, & p < 1 \\ \frac{1}{p-1}, & p > 1 \end{cases}$$

Therefore, the integral converges to $\frac{1}{p-1}$ if $p > 1$

and it diverges if $p \leq 1$.

Integrands with Vertical Asymptotes



$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left(2-2\sqrt{a}\right) = 2.$$

DEFINITION

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and $\lim\limits_{x \to a^+} f(x) = \pm\infty$ then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and $\lim\limits_{x \to b^-} f(x) = \pm\infty$ then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is $\lim\limits_{x \to c} f(x) = \pm\infty$ where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

converges
diverges.

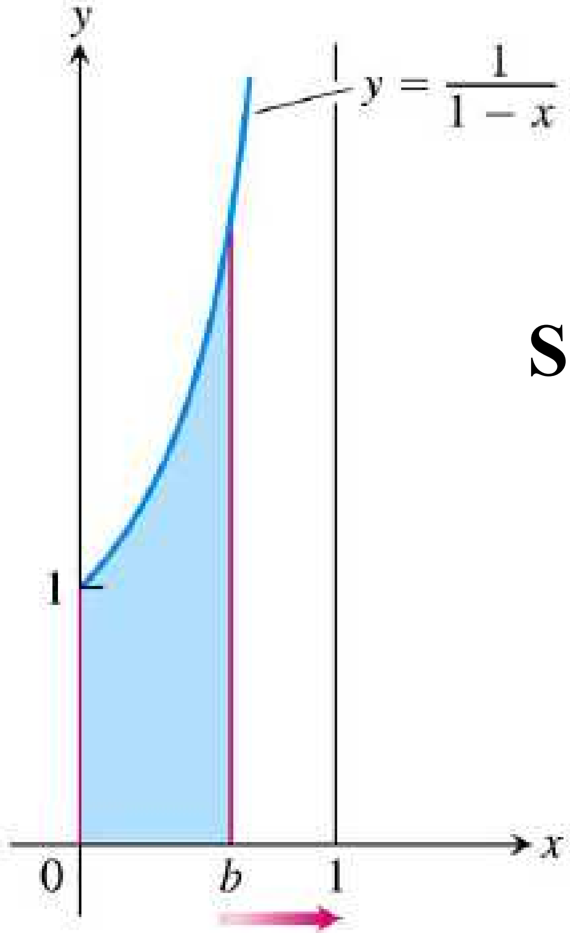


FIGURE 8.17 The area beneath the curve and above the x-axis for $[0, 1)$ is not a real number (Example 4).

Ex. 4 Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

Solution

$$\int_0^1 \frac{1}{1-x}\,dx = \lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx$$

$$= \lim_{b\to 1^-}\left(-\ln(1-x)\Big|_0^b\right)$$

$$= \lim_{b\to 1^-}\left(-\ln(1-b)\right)$$

$$= +\infty,$$

So the integral diverges.

Ex. 5 Evaluate $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx.$

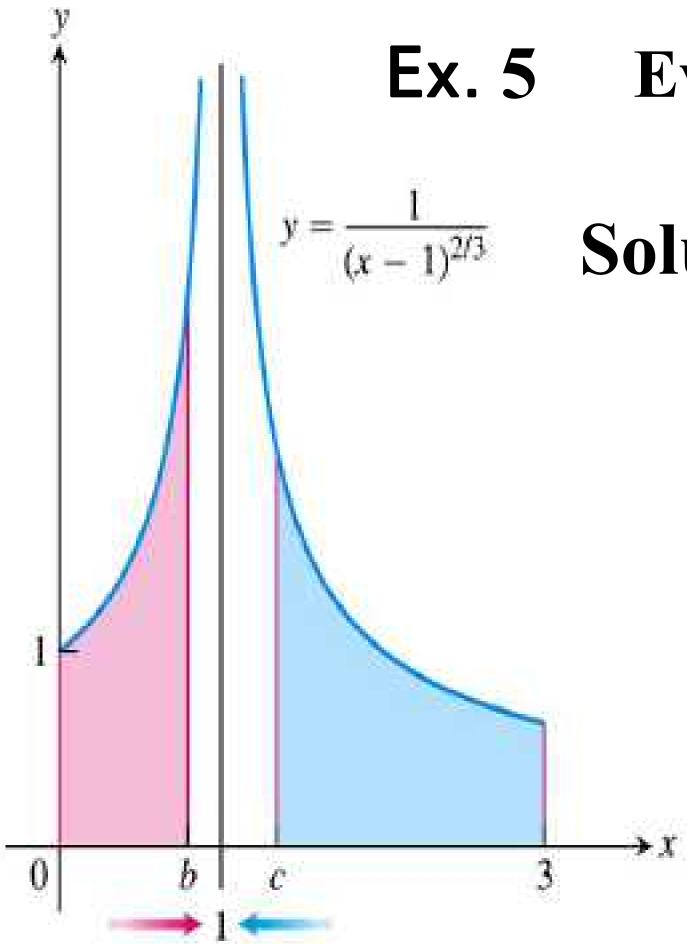


FIGURE 8.18 Example 5 shows that the area under the curve exists (so it is a real number).

Solution $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx =$

$$\int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\,dx$$

$$\int_0^1 \frac{1}{(x-1)^{2/3}}\,dx = \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^{2/3}}\,dx$$

$$= \lim_{b\to 1^-} 3\sqrt[3]{x-1}\,\Big|_0^b = \lim_{b\to 1^-} 3(\sqrt[3]{b-1}+1) = 3$$

$$\int_1^3 \frac{1}{(x-1)^{2/3}}\,dx = 3\sqrt[3]{x-1}\,\Big|_1^3 = 3\sqrt[3]{2}$$

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3 + 3\sqrt[3]{2}$$

Ex. **Investigate the convergence of** $\int_0^1 \frac{1}{x^q}dx \quad \int_a^b \frac{1}{(x-a)^q}dx$

Solution

(1) $q=1$, $\int_0^1 \frac{1}{x^q}dx = \int_0^1 \frac{1}{x}dx = [\ln x]_0^1 = +\infty$,

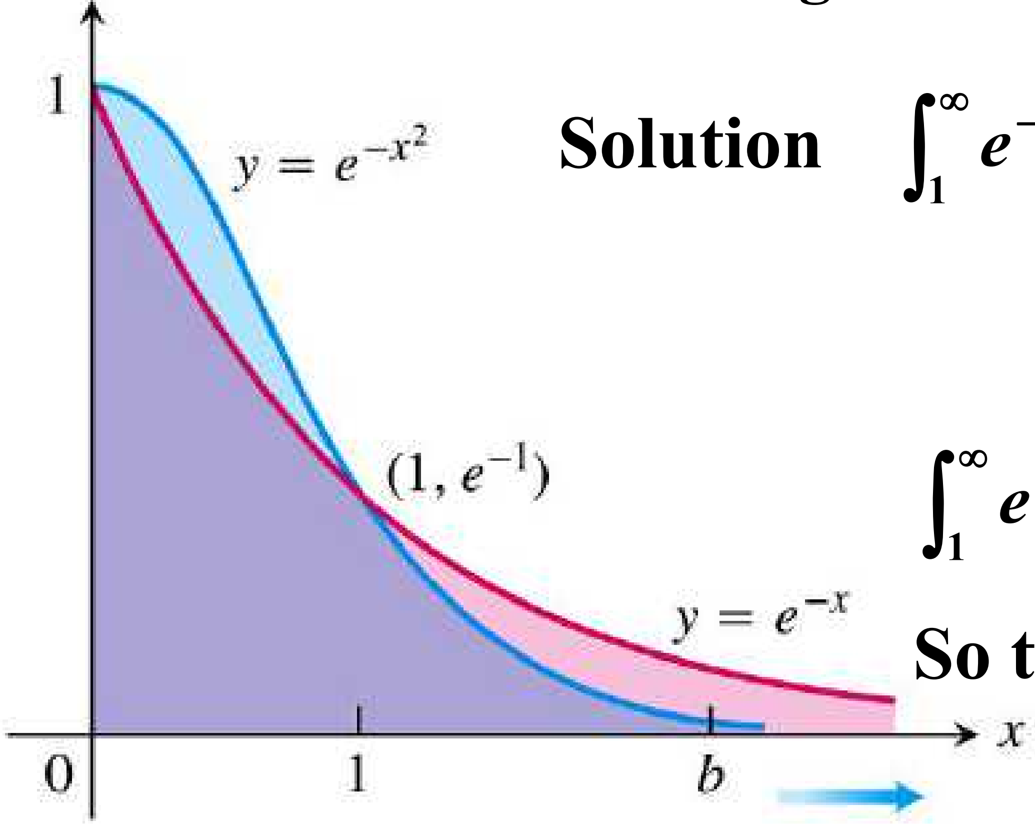
(2) $q \neq 1$, $\int_0^1 \frac{1}{x^q}dx = \left[\frac{x^{1-q}}{1-q}\right]_0^1 = \begin{cases} +\infty, & q>1 \\ \frac{1}{1-q}, & q<1 \end{cases}$

因此当$q<1$时反常积分收敛，其值为

$\frac{1}{1-q}$； 当$q \geq 1$时反常积分发散.

Tests for Convergence and Divergence

Ex. 6 Does the integral $\int_1^{\infty} e^{-x^2}\,dx$ converge?



Solution $\int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_1^{b} e^{-x}\,dx$

$$= \lim_{b\to\infty} (e^{-1} - e^{-b}) = e^{-1}$$

$$\int_1^{\infty} e^{-x^2}\,dx \le \int_1^{\infty} e^{-x}\,dx$$

So the integral converges.

FIGURE 8.19 The graph of e^{-x^2} lies below the graph of e^{-x} for $x > 1$ (Example 6).

比较检验法

THEOREM 2—Direct Comparison Test Let f and g be continuous on $[a,b)$ with $0 \le f(x) \le g(x)$ for all $[a,b)$ Then

1. $\int_a^b f(x)dx$ converges if $\int_a^b g(x)dx$ converges.

2. $\int_a^b g(x)dx$ diverges if $\int_a^b f(x)dx$ diverges.

$$\int_a^b f(x)dx \le \int_a^b g(x)dx \qquad \lim_{b\to\infty}\int_a^b f(x)dx \le \lim_{b\to\infty}\int_a^b g(x)dx$$

Ex. Test the convergence for $\int_1^{\infty} \frac{1}{x\sqrt{2x+1}} dx$

Solution $\int_1^{\infty} \frac{1}{x\sqrt{2x+1}} dx = \int_{\sqrt{3}}^{\infty} \frac{2}{u^2-1} du$ $\quad \sqrt{2x+1} = u,$

$$= \int_{\sqrt{3}}^{\infty} \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du = \ln\left|\frac{u-1}{u+1}\right|\Bigg|_{\sqrt{3}}^{+\infty} = \ln\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right|$$

$$\frac{1}{x\sqrt{2x+1}} < \frac{1}{x\sqrt{2x}} = \frac{1}{\sqrt{2}} \frac{1}{x^{3/2}}$$

converges

Ex. Test the convergence for $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}} dx$

Solution $\frac{\cos x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$

converges

$$\int_1^{\infty} \frac{1}{x\sqrt{2x-1}} dx?$$

比较检验法的极限形式

THEOREM 3—Limit Comparison Test If the positive functions f and g are continuous on $\boxed{[a,b)}$ and if

$$\boxed{\lim_{x\to b^-}\frac{f(x)}{g(x)}=L} \quad 0<L<\infty,$$

then

$$\boxed{\int_a^b f(x)dx} \quad \text{and} \quad \boxed{\int_a^b g(x)dx}$$

both converge or both diverge.

$$\frac{L}{2}<\frac{f(x)}{g(x)}\le\frac{3L}{2}(x>M>a) \quad \frac{L}{2}g(x)<f(x)\le\frac{3L}{2}g(x)(x>M>a)$$

Ex. 7 Test the convergence for the next integrals

$$(a)\int_2^{\infty}\frac{\cos\frac{1}{x}}{\sqrt{x(x-1)(x+1)}}\,dx$$

converges

$$(b)\int_1^{\infty}\frac{1}{\sqrt{x^2-0.1}}\,dx$$

diverges

$$(c)\int_0^{\pi/2}\frac{\cos x}{\sqrt{x}}\,dx$$

converges

Ex. 8 Shou that $\int_1^{\infty} \frac{1}{x\sqrt{1+x^2}} dx$ converges by comparison test.

Solution $\lim_{x\to\infty} \frac{1}{x\sqrt{1+x^2}} / \frac{1}{x^2} = 1$

$\int_1^{\infty} \frac{1}{x\sqrt{1+x^2}} dx$ converges because $\int_1^{\infty} \frac{1}{x^2} dx$ converges.

Ex. 9 Investigate the converges of $\int_1^{\infty} \frac{1-e^{-x}}{x} dx$

Solution $\lim_{x\to\infty} \frac{1-e^{-x}}{x} / \frac{1}{x} = 1$

$\int_1^{\infty} \frac{1-e^{-x}}{x} dx$ diverges because $\int_1^{\infty} \frac{1}{x} dx$ diverges.

例 判别下列反常积分的敛 散性：

$$(1)\int_0^1 \frac{e^x dx}{\sqrt{1-x}}, \qquad (2)\int_0^1 \frac{\ln x dx}{\sqrt{x}}.$$

解（1） $\because$ 被积函数在点 $x=1$ 的左邻域内无界.

$$\lim_{x\to 1-0} \frac{\dfrac{e^x}{\sqrt{1-x}}}{\dfrac{1}{\sqrt{1-x}}} = e,$$ 所给反常积分（1）收敛.

解（2） $$\int_0^1 \frac{\ln x dx}{\sqrt{x}} = \int_0^1 \ln x d2\sqrt{x}$$

$$= 2\sqrt{x}\ln x\Big|_0^1 - 2\int_0^1 \frac{1}{\sqrt{x}} dx = -4$$

所给反常积分（2）收敛.

Testing for Convergence

$$\int_{-1}^{1} \ln|x|\, dx$$

$$\int_{0}^{1} \frac{dt}{t - \sin t}$$

$$\int_{1}^{\infty} \frac{\sqrt{x+1}}{x^2}\, dx$$

$$\int_{0}^{\pi} \frac{dt}{\sqrt{t} + \sin t}$$

$$\int_{\pi}^{\infty} \frac{1 + \sin x}{x^2}\, dx$$

$$\int_{e^e}^{\infty} \ln(\ln x)\, dx$$

For what value or values of a does

$$\int_1^{\infty} \left(\frac{ax}{x^2 + 1} - \frac{1}{2x} \right) dx$$

converge? Evaluate the corresponding integral(s).

$$\boldsymbol{a = \frac{1}{2}} \qquad \boldsymbol{-\frac{1}{4}\ln 2}$$