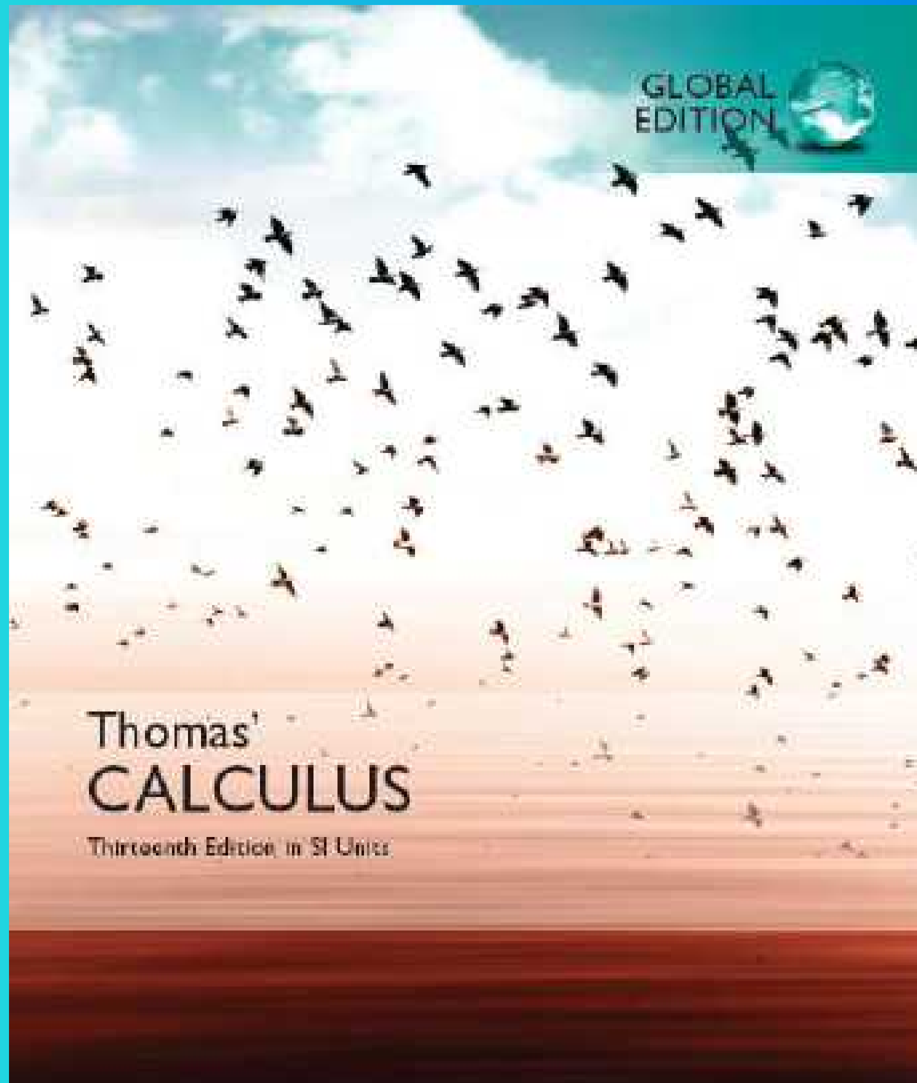

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 13

Vector-Valued Functions and Motion in Space

向量值函数和运动

13.1

Curves in Space and Their Tangents

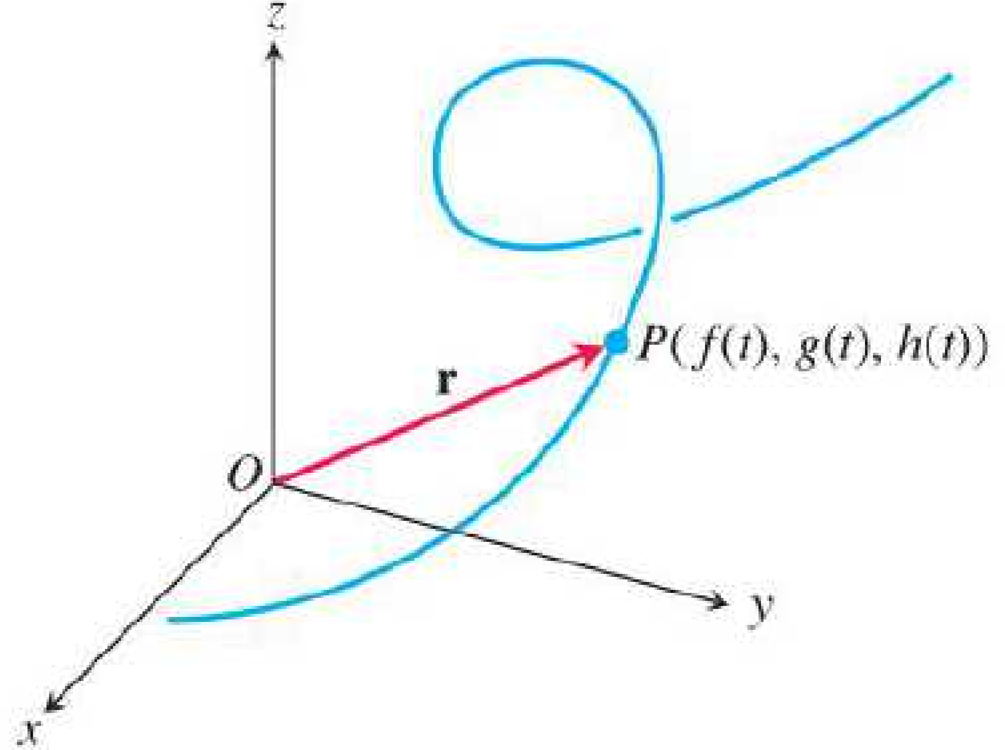
空间曲线和切线

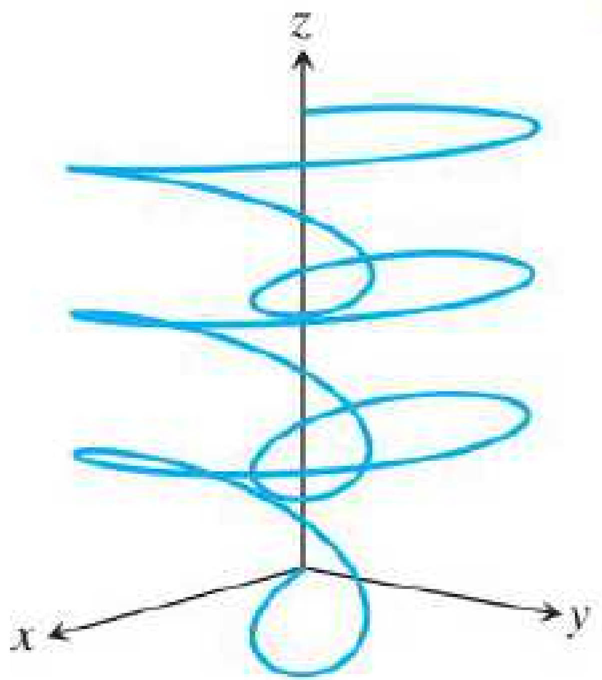
$$x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad t \in I.$$

$$\mathbf{r}(t) = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

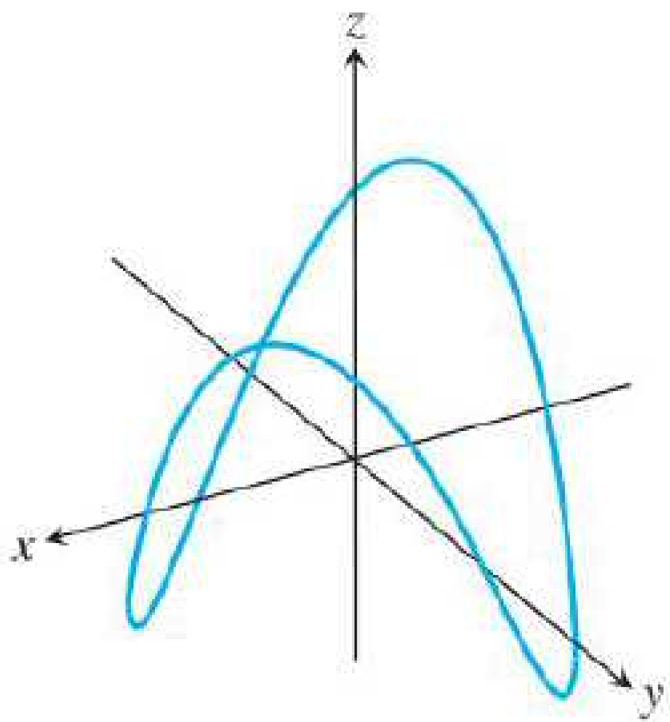
f, g, and h are the **component functions**

a **vector-valued function** or **vector function** on a domain set D

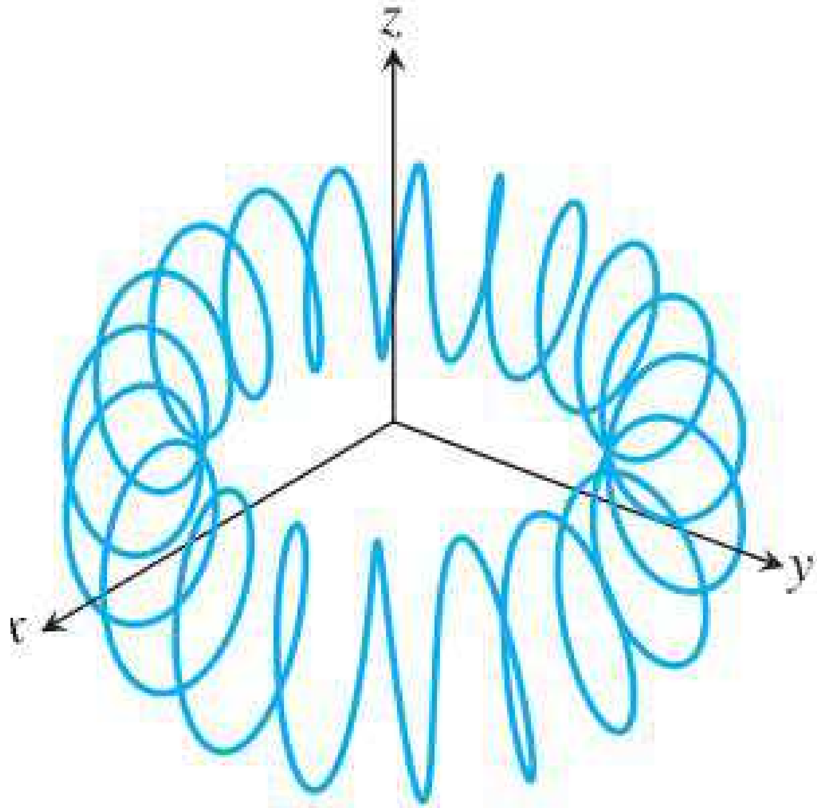




$$\mathbf{r}(t) = (\sin 3t)(\cos t)\mathbf{i} + (\sin 3t)(\sin t)\mathbf{j} + t\mathbf{k}$$



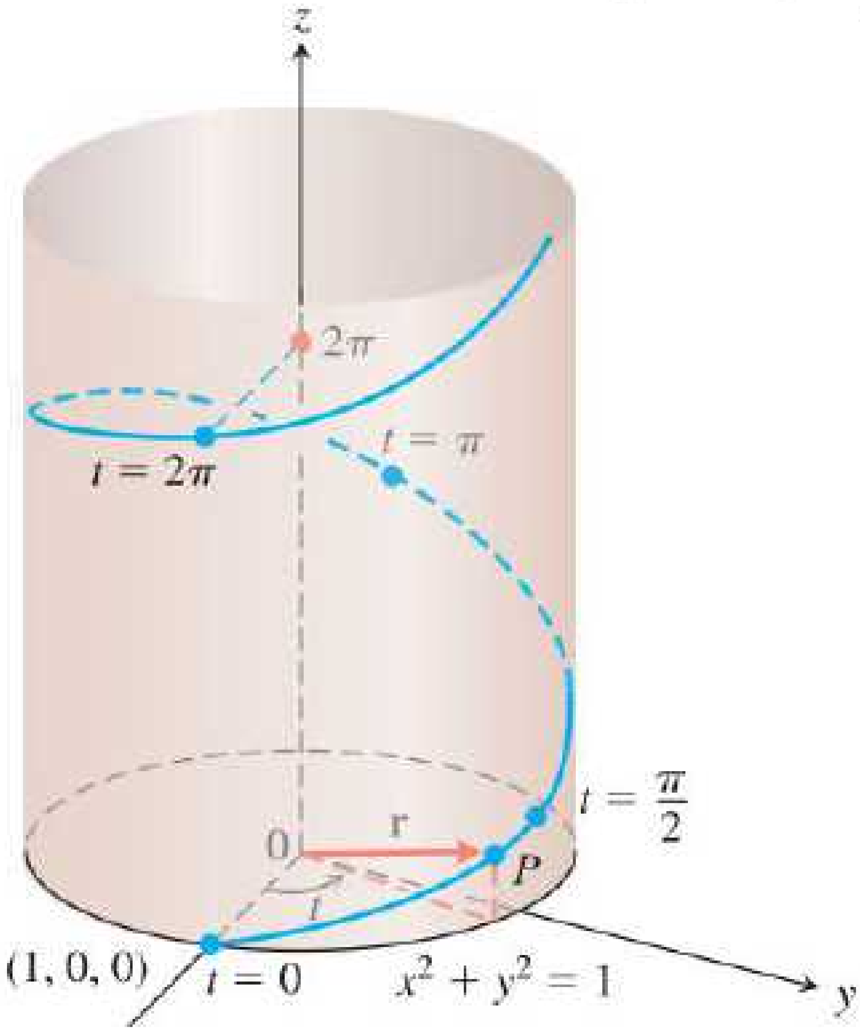
$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}$$

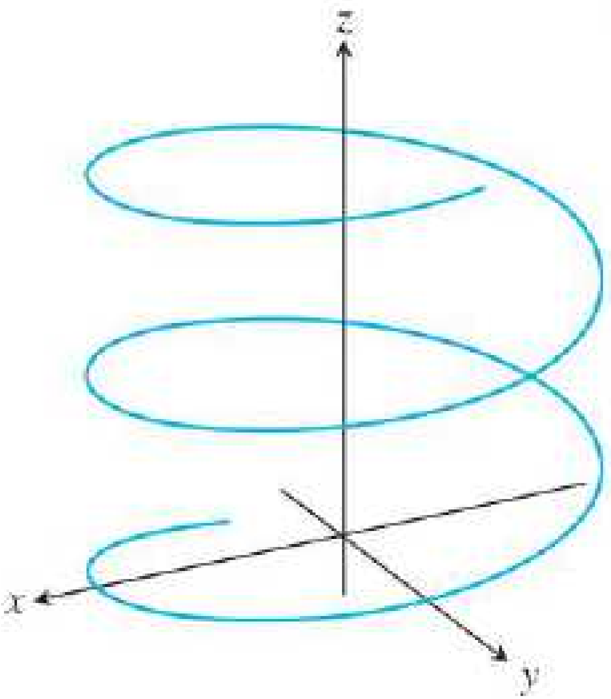


$$\mathbf{r}(t) = (4 + \sin 20t)(\cos t)\mathbf{i} + (4 + \sin 20t)(\sin t)\mathbf{j} + (\cos 20t)\mathbf{k}$$

EXAMPLE 1 Graph the vector function

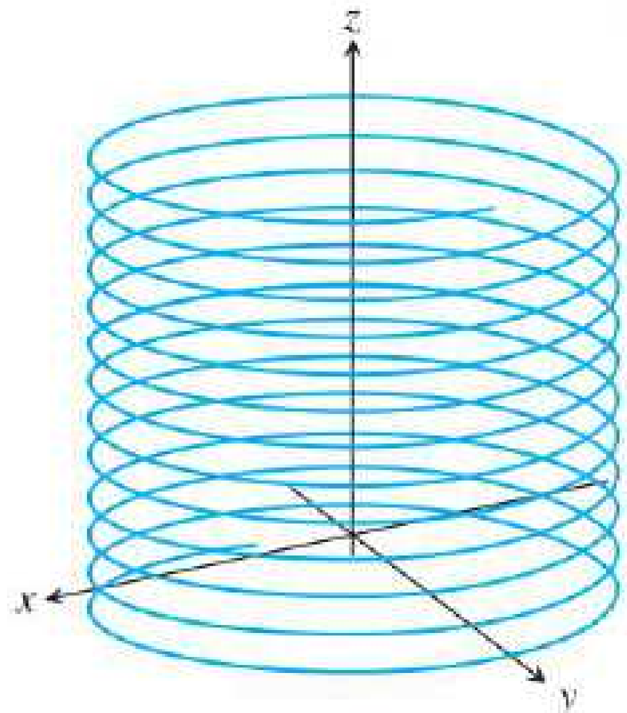
$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}.$$





$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0.3t\mathbf{k}$$



Limits and Continuity

DEFINITION Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function with domain D, and $\mathbf{L}$ a vector. We say that $\mathbf{r}$ has **limit L** as t approaches t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$|\mathbf{r}(t) - \mathbf{L}| < \epsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

$$\mathbf{r}(t) = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

If $\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$, then it can be shown that $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$

$$|r(t) - L| < \varepsilon$$

$$\sqrt{(f(t) - L_1)^2 + (g(t) - L_2)^2 + (h(t) - L_3)^2} < \varepsilon$$

precisely when

$$\lim_{t \to t_0} f(t) = L_1, \qquad \lim_{t \to t_0} g(t) = L_2, \qquad \text{and} \qquad \lim_{t \to t_0} h(t) = L_3.$$

$$\lim_{t \to t_0} \mathbf{r}(t) = \left(\lim_{t \to t_0} f(t)\right)\mathbf{i} + \left(\lim_{t \to t_0} g(t)\right)\mathbf{j} + \left(\lim_{t \to t_0} h(t)\right)\mathbf{k}$$

a practical way to calculate limits of vector functions.

EXAMPLE 2 If $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, then

$$\lim_{t\to\pi/4} \mathbf{r}(t) = \left(\lim_{t\to\pi/4} \cos t\right)\mathbf{i} + \left(\lim_{t\to\pi/4} \sin t\right)\mathbf{j} + \left(\lim_{t\to\pi/4} t\right)\mathbf{k}$$

$$= \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}.$$

DEFINITION A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous over its interval domain.

$$\lim_{t \to t_0} r(t) = r(t_0)$$

$$\lim_{t \to t_0} f(t)\vec{\mathrm{i}} + \lim_{t \to t_0} g(t)\vec{\mathrm{j}} + \lim_{t \to t_0} h(t)\vec{\mathrm{k}} = f(t_0)\vec{\mathrm{i}} + g(t_0)\vec{\mathrm{j}} + h(t_0)\vec{\mathrm{k}}$$

$$\lim_{t \to t_0} f(t) = f(t_0), \quad \lim_{t \to t_0} g(t) = g(t_0), \quad \lim_{t \to t_0} h(t) = h(t_0)$$

$\mathbf{r}(t)$ is continuous at $t = t_0$ if and only if each component function is continuous there

EXAMPLE 3

$$\mathbf{r}(t) = (\sin 3t)(\cos t)\mathbf{i} + (\sin 3t)(\sin t)\mathbf{j} + t\mathbf{k}$$

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}$$

$$\mathbf{r}(t) = (4 + \sin 20t)(\cos t)\mathbf{i} + (4 + \sin 20t)(\sin t)\mathbf{j} + (\cos 20t)\mathbf{k}$$

$$\mathbf{r}(t) = (\cos 5t)\mathbf{i} + (\sin 5t)\mathbf{j} + t\mathbf{k}$$

(a) All the space curves are continuous at every value of t in $(-\infty, \infty)$.

(b) The function $\mathbf{g}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \lfloor t \rfloor\mathbf{k}$ is discontinuous at every integer,

Derivatives and Motion

a curve in space $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

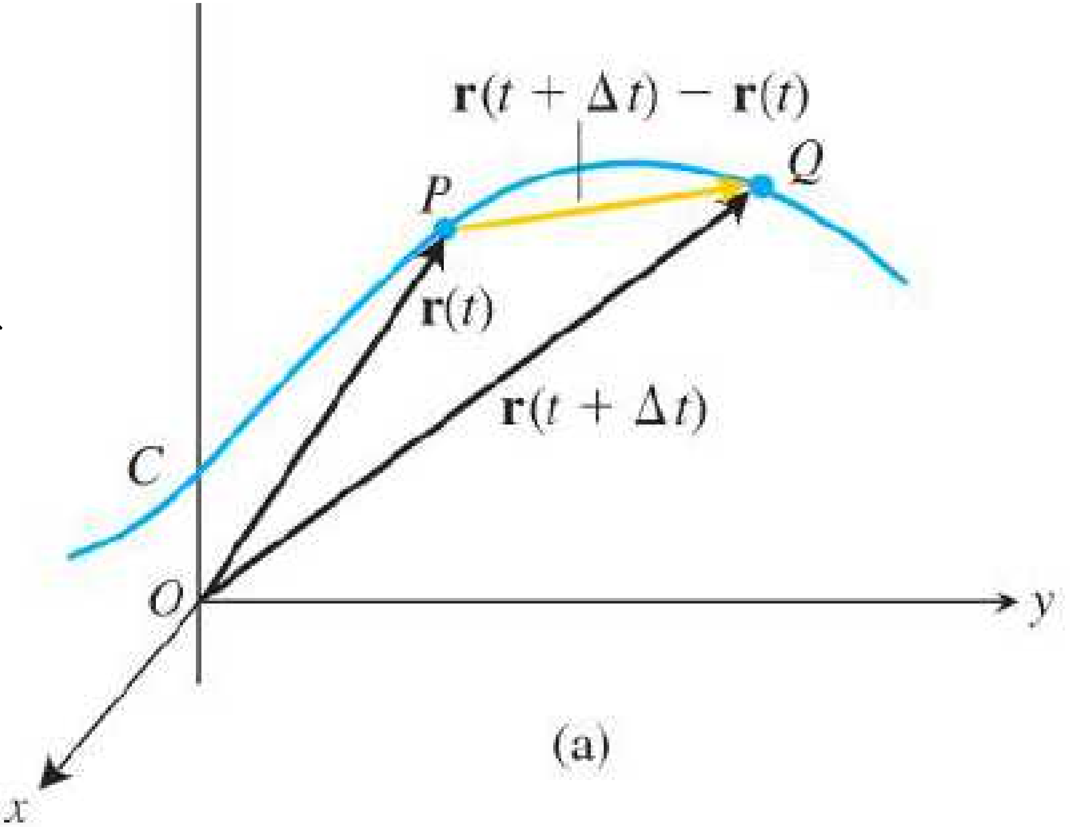
f, g, and h are differentiable functions of t.

Find the vector **tangent** to the curve at P.

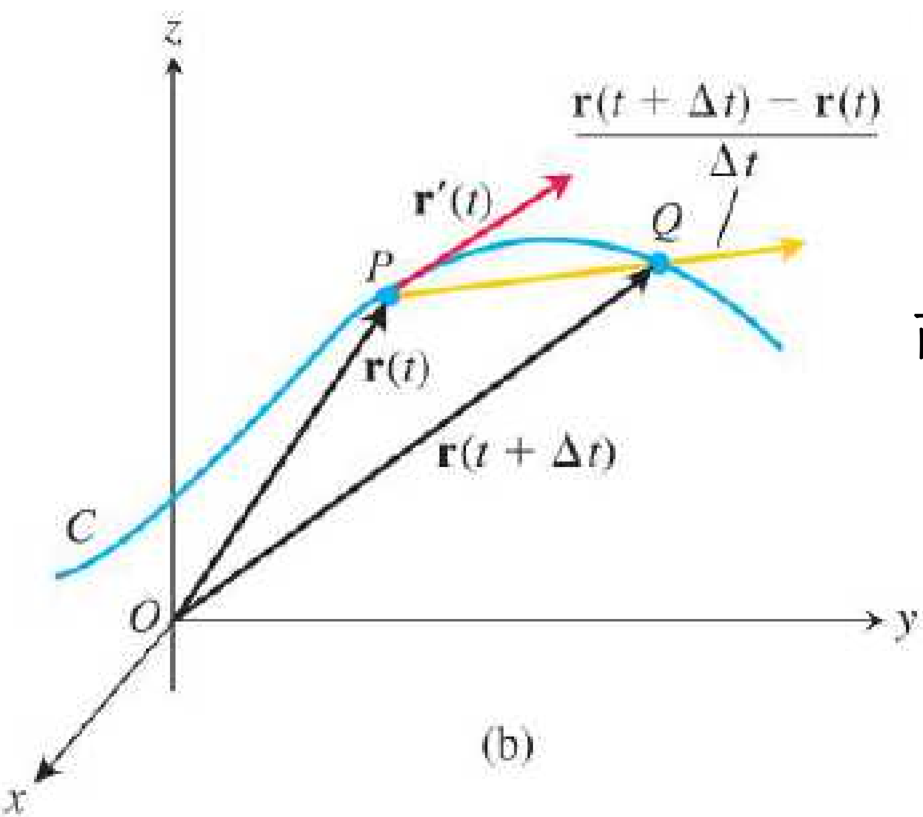
$$\frac{r(t+\Delta t)-r(t)}{\Delta t}$$

物理上：表示粒子位置变化的速度.

几何上：表示曲线割线向量.



(a)



(b)

若$\Delta t > 0$，则$\mathbf{r}(t+\Delta t)-\mathbf{r}(t)$指向参数$t$增加的方向，

故$\dfrac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}$指向增加方向；

若$\Delta t < 0$，则$\mathbf{r}(t+\Delta t)-\mathbf{r}(t)$指向参数$t$减少的方向，

故$\dfrac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}$指向增加方向.

几何上：表示曲线割线 向量，且指向参数增加的方向.

物理上：表示位置变化速度，且指向运动的方向.

$$
\begin{aligned}
\Delta \mathbf{r} &= \mathbf{r}(t+\Delta t) - \mathbf{r}(t) \\
&= \left[f(t+\Delta t)\mathbf{i} + g(t+\Delta t)\mathbf{j} + h(t+\Delta t)\mathbf{k} \right] \\
&\quad - \left[f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \right] \\
&= \left[f(t+\Delta t) - f(t) \right]\mathbf{i} + \left[g(t+\Delta t) - g(t) \right]\mathbf{j} + \left[h(t+\Delta t) - h(t) \right]\mathbf{k}.
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} &= \left[\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \right]\mathbf{i} + \left[\lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} \right]\mathbf{j} \\
&\quad + \left[\lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \right]\mathbf{k} \\
&= \left[\frac{df}{dt} \right]\mathbf{i} + \left[\frac{dg}{dt} \right]\mathbf{j} + \left[\frac{dh}{dt} \right]\mathbf{k}.
\end{aligned}
$$

DEFINITION The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative (is differentiable) at t** if f, g, and h have derivatives at t. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

物理上：表示位置变化瞬时速度，且指向运动的方向.

几何上：表示曲线切线向量，且指向参数增加的方向.

The curve traced by $\mathbf{r}$ is **smooth** if $d\mathbf{r}/dt$ is continuous and never $\mathbf{0}$, On a smooth curve, there are no sharp corners or cusps.

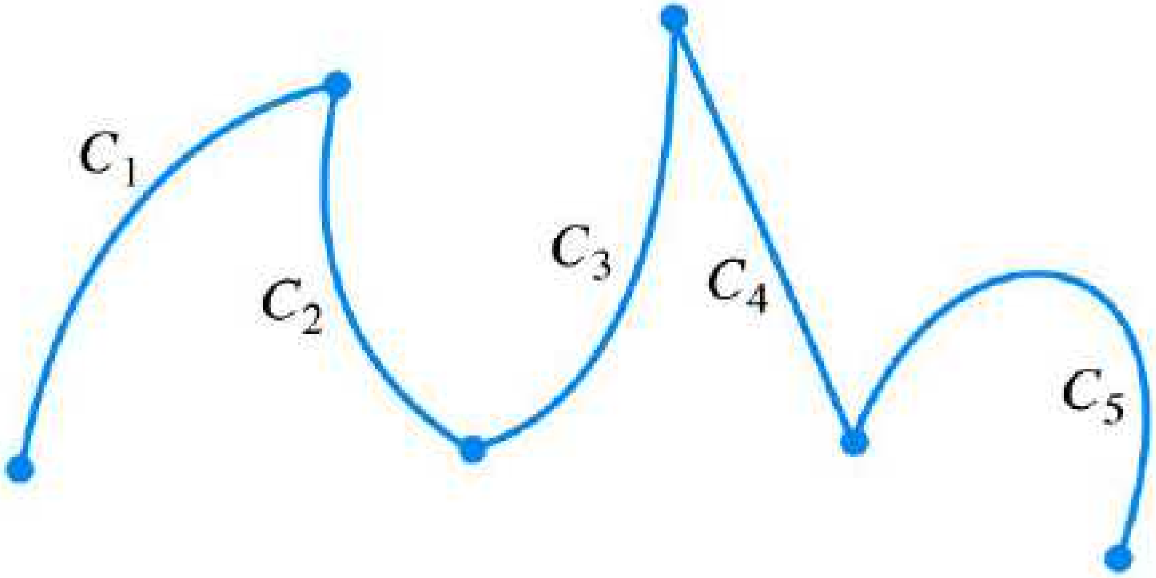


FIGURE 13.6 A piecewise smooth curve made up of five smooth curves connected end to end in a continuous fashion. The curve here is not smooth at the points joining the five smooth curves.

DEFINITIONS If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then

1. Velocity is the derivative of position: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.

   is the **direction of motion**,

2. Speed is the magnitude of velocity: $\text{Speed} = |\mathbf{v}|$.

3. Acceleration is the derivative of velocity: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$.

4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t.

EXAMPLE 4

Find the velocity, speed, and acceleration of a particle whose motion is given by the position vector $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 5\cos^2 t\,\mathbf{k}$.

Solution

$$\mathbf{v}(t) = \mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 10\cos t\sin t\,\mathbf{k}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} - 10\cos 2t\,\mathbf{k},$$

$$|\mathbf{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2} = \sqrt{4 + 25\sin^2 2t}.$$

Differentiation Rules for Vector Functions

Let **u** and **v** be differentiable vector functions of t, **C** a constant vector,

$$\frac{d}{dt}\mathbf{C} = \mathbf{0} \qquad \frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$$

Proof of the Chain Rule

differentiable

Suppose that $\mathbf{u}(s) = a(s)\mathbf{i} + b(s)\mathbf{j} + c(s)\mathbf{k}$ $\quad s = f(t)$

$$\begin{aligned}
\frac{d}{dt}[\mathbf{u}(s)] &= \frac{da}{dt}\mathbf{i} + \frac{db}{dt}\mathbf{j} + \frac{dc}{dt}\mathbf{k} \\
&= \frac{da}{ds}\frac{ds}{dt}\mathbf{i} + \frac{db}{ds}\frac{ds}{dt}\mathbf{j} + \frac{dc}{ds}\frac{ds}{dt}\mathbf{k} \\
&= \frac{ds}{dt}\left(\frac{da}{ds}\mathbf{i} + \frac{db}{ds}\mathbf{j} + \frac{dc}{ds}\mathbf{k}\right) \\
&= \frac{ds}{dt}\frac{d\mathbf{u}}{ds} = f'(t)\mathbf{u}'(f(t)).
\end{aligned}$$

Vector Functions of Constant Length

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = c^2$$

$$\frac{d}{dt}\left[\mathbf{r}(t) \cdot \mathbf{r}(t)\right] = 0$$

$$\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

$$2\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0.$$

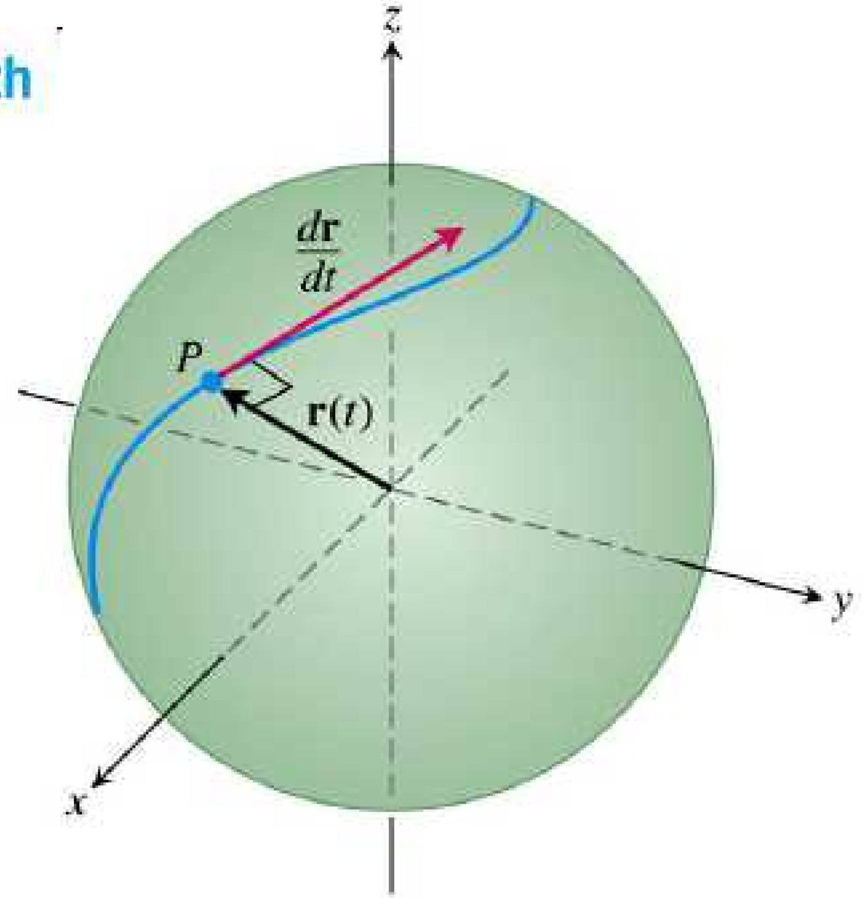


FIGURE 13.8 If a particle moves on a sphere in such a way that its position $\mathbf{r}$ is a differentiable function of time, then $\mathbf{r} \cdot (d\mathbf{r}/dt) = 0$.

If $\mathbf{r}$ is a differentiable vector function of t of constant length, then

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0.$$

13.2

Integrals of Vector Functions; Projectile Motion

向量值函数的积分 抛物运动

DEFINITION The **indefinite integral** of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}. \qquad d\mathbf{R}/dt = \mathbf{r}$$

EXAMPLE 1

$$\int ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\,dt = \left(\int \cos t\,dt\right)\mathbf{i} + \left(\int dt\right)\mathbf{j} - \left(\int 2t\,dt\right)\mathbf{k}$$

$$= (\sin t + C_1)\mathbf{i} + (t + C_2)\mathbf{j} - (t^2 + C_3)\mathbf{k}$$

$$= (\sin t)\mathbf{i} + t\mathbf{j} - t^2\mathbf{k} + \mathbf{C} \qquad C = C_1\mathbf{i} + C_2\mathbf{j} - C_3\mathbf{k}$$

DEFINITION If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is $\mathbf{r}$, and the **definite integral** of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

EXAMPLE 2

$$\int_0^{\pi} ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\,dt = \left(\int_0^{\pi} \cos t\,dt\right)\mathbf{i} + \left(\int_0^{\pi} dt\right)\mathbf{j} - \left(\int_0^{\pi} 2t\,dt\right)\mathbf{k}$$

$$= \left[\sin t\right]_0^{\pi}\mathbf{i} + \left[t\right]_0^{\pi}\mathbf{j} - \left[t^2\right]_0^{\pi}\mathbf{k} = \pi\mathbf{j} - \pi^2\mathbf{k}$$

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

EXAMPLE 3 a hang glider, $\mathbf{a}(t) = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}$.

the glider departed from the point (4, 0, 0) with velocity $\mathbf{v}(0) = 3\mathbf{j}$.

Find the glider's position as a function of t.

Solution $$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{v}(t) = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\mathbf{k} + \mathbf{C}_1.$$

$\mathbf{v}(0) = 3\mathbf{j}$ and $\mathbf{r}(0) = 4\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$. $\mathbf{C}_1 = \mathbf{0}$.

$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k} + \mathbf{C}_2$. $\mathbf{r}(0) = 4\mathbf{i}$

$\mathbf{C}_2 = \mathbf{i}$. $\mathbf{r}(t) = (1 + 3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$.

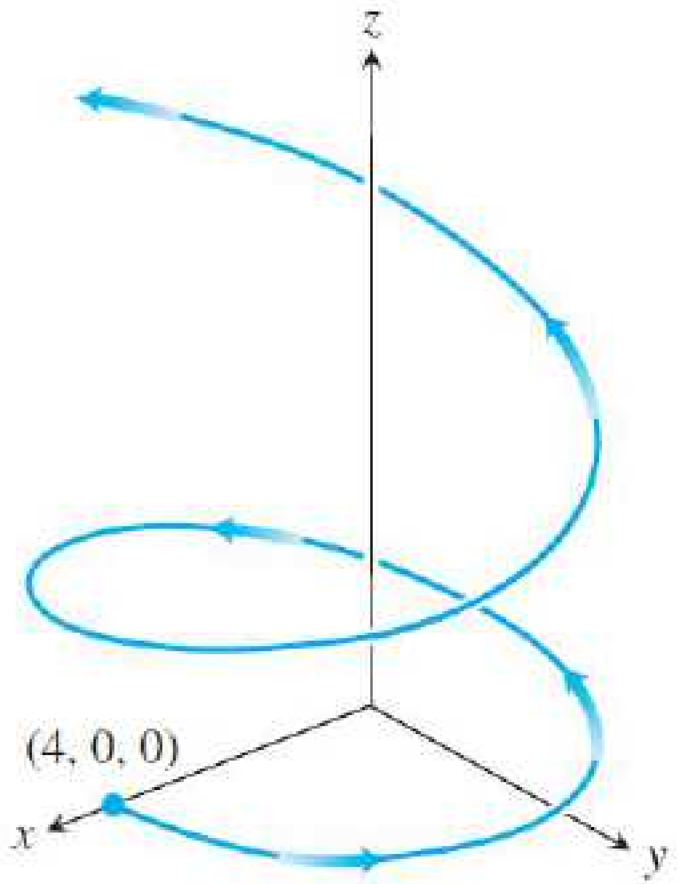
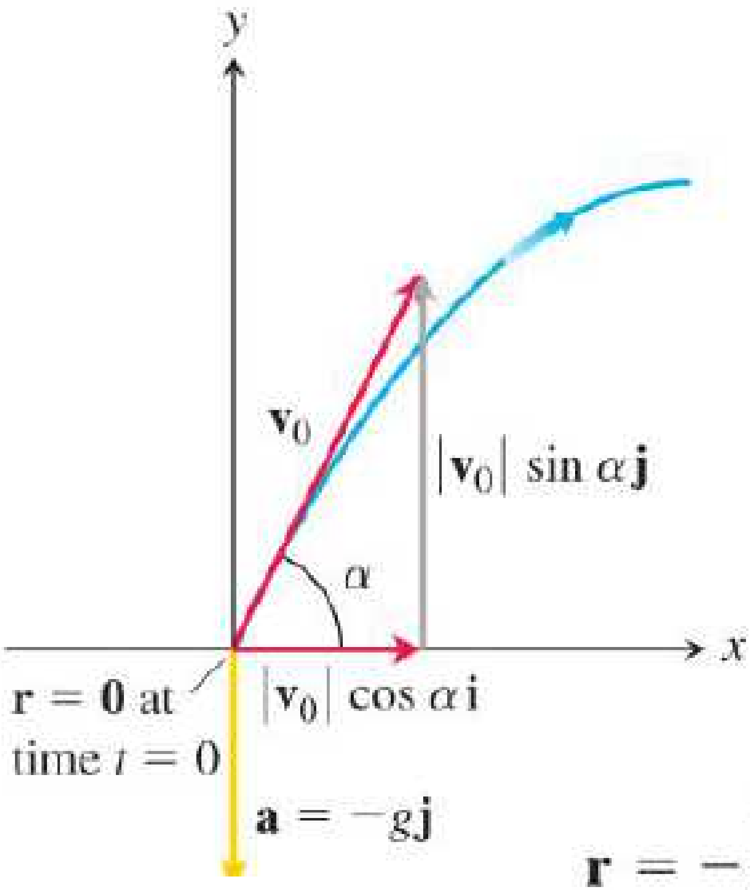


FIGURE 13.9 The path of the hang glider in Example 3. Although the path spirals around the z-axis, it is not a helix.

The Vector and Parametric Equations for Ideal Projectile Motion



$$\mathbf{v}_0 = (|\mathbf{v}_0| \cos \alpha)\mathbf{i} + (|\mathbf{v}_0| \sin \alpha)\mathbf{j}.$$

$$\mathbf{v}_0 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j}.$$

$$\mathbf{r}_0 = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}.$$

$$m\frac{d^2\mathbf{r}}{dt^2} = -mg\mathbf{j} \qquad \frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{j},$$

$$\frac{d\mathbf{r}}{dt} = -(gt)\mathbf{j} + \mathbf{v}_0.$$

$$\mathbf{r} = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0.$$

$$\mathbf{r} = -\frac{1}{2}gt^2\mathbf{j} + \underbrace{(v_0 \cos \alpha)t\mathbf{i} + (v_0 \sin \alpha)t\mathbf{j}}_{\mathbf{v}_0 t} + \mathbf{0}.$$

Ideal Projectile Motion Equation

$$\mathbf{r} = (v_0 \cos \alpha)t\mathbf{i} + \left((v_0 \sin \alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}.$$

$$x = (v_0 \cos \alpha)t \qquad \text{and} \qquad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2,$$

EXAMPLE 4

A projectile is fired from the origin over horizontal ground at an initial speed of 500 m/sec and a launch angle of 60°. Where 10 sec later?

Solution

$$\mathbf{r} = (500)\left(\frac{1}{2}\right)(10)\mathbf{i} + \left((500)\left(\frac{\sqrt{3}}{2}\right)10 - \left(\frac{1}{2}\right)(9.8)(100)\right)\mathbf{j}$$

$$\approx 2500\mathbf{i} + 3840\mathbf{j}$$

$$x = (v_0 \cos \alpha)t \quad \text{and} \quad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2,$$

$$y = -\left(\frac{g}{2{v_0}^2 \cos^2 \alpha}\right)x^2 + (\tan \alpha)x.$$

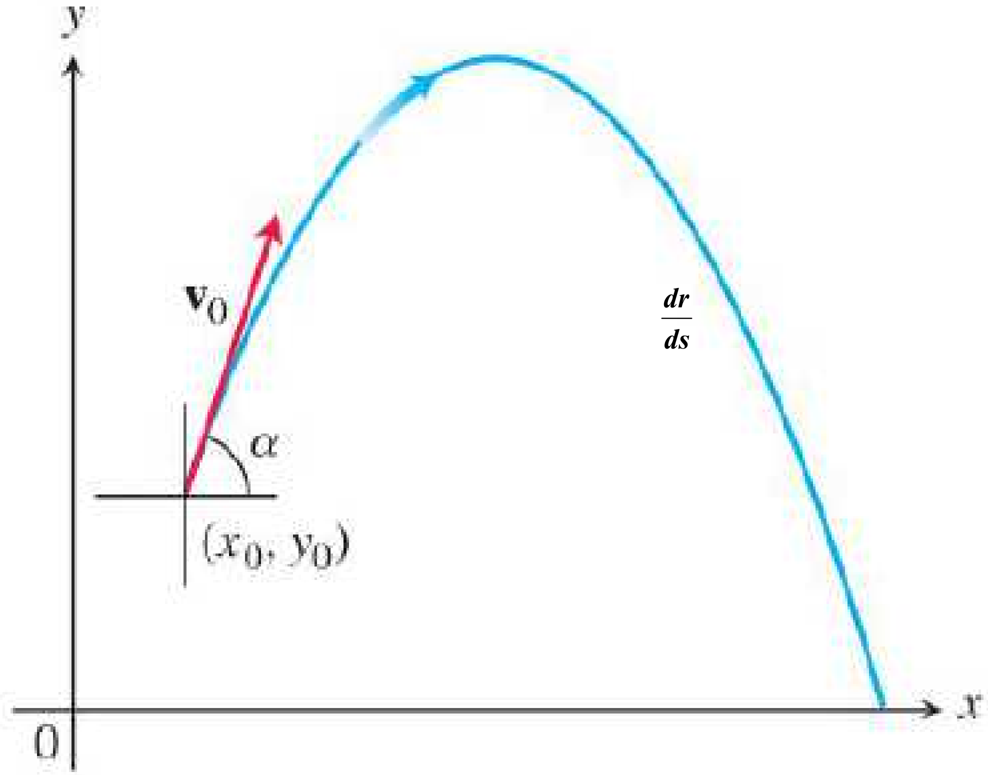
$$\mathbf{r} = (v_0 \cos \alpha)t\mathbf{i} + \left((v_0 \sin \alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}.$$

Maximum height: $$y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g}$$

Flight time: $$t = \frac{2v_0 \sin \alpha}{g}$$

Range: $$R = \frac{{v_0}^2}{g} \sin 2\alpha.$$

$$\mathbf{r} = (x_0 + (v_0 \cos \alpha)t)\mathbf{i} + \left(y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2\right)\mathbf{j},$$



$$m\frac{d^2\mathbf{r}}{dt^2} = -mg\mathbf{j}$$

EXAMPLE 5

A baseball is hit when it is 3 ft above the ground. It leaves the bat with initial speed of 152 ft/sec, making an angle of 20° with the horizontal. an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component to the ball's initial velocity (8.8 ft/sec = 6 mph).

(a) Find a vector equation (position vector) for the path of the baseball.

(b) How high does the baseball go, and when does it reach maximum

(c) Assuming that the ball is not caught, find its range and flight time.

Solution (a) $$\mathbf{r} = (x_0 + (v_0 \cos \alpha)t)\mathbf{i} + \left(y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2\right)\mathbf{j},$$

$$\mathbf{r}_0 = 0\mathbf{i} + 3\mathbf{j}.$$ 初速度的x分量加了$-8.8\mathbf{i}$

$$\mathbf{r} = (152\cos 20^o - 8.8)\,t\,\vec{\mathbf{i}} + (3 + (152\sin 20^o)\,t - 16t^2)\,\vec{\mathbf{j}}$$

$$\mathbf{r} = (152\cos 20^o - 8.8)\,t\,\vec{\mathbf{i}} + (3 + (152\sin 20^o)\,t - 16t^2)\,\vec{\mathbf{j}}$$

(b) $\frac{dy}{dt} = 152\sin 20^\circ - 32t = 0.$ $t = \frac{152\sin 20^\circ}{32} \approx 1.62$ sec.

$$y_{\max} = 3 + (152\sin 20^\circ)(1.62) - 16(1.62)^2 \quad \approx 45.2\text{ft}$$

(c) $3 + (152\sin 20^\circ)t - 16t^2 = 0$

$3 + (51.99)t - 16t^2 = 0.$ $t = 3.3$ sec

$$R = (152\cos 20^\circ - 8.8)(3.3)$$
$$\approx 442\text{ ft.}$$

13.3

Arc Length in Space

空间中的弧长

DEFINITION The **length** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt. \qquad (1)$$

Arc Length Formula

$$L = \int_a^b |\mathbf{v}|\, dt$$

EXAMPLE 1

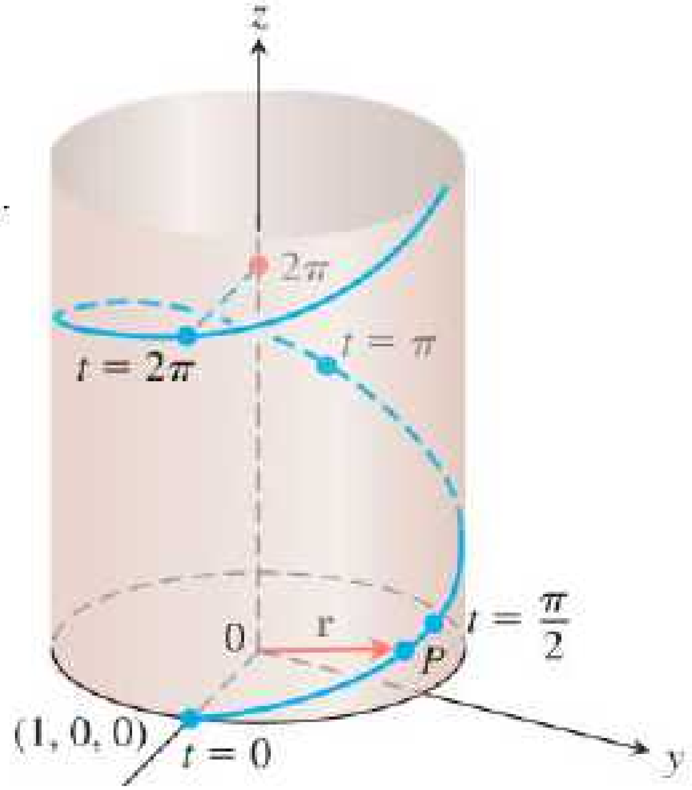
A glider is soaring upward along the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$. How long is the glider's path from $t = 0$ to $t = 2\pi$?

Solution

$$L = \int_a^b |\mathbf{v}|\, dt$$

$$= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2} \text{ units of length.}$$



Arc Length Parameter with Base Point $P(t_0)$

Unit Tangent Vector

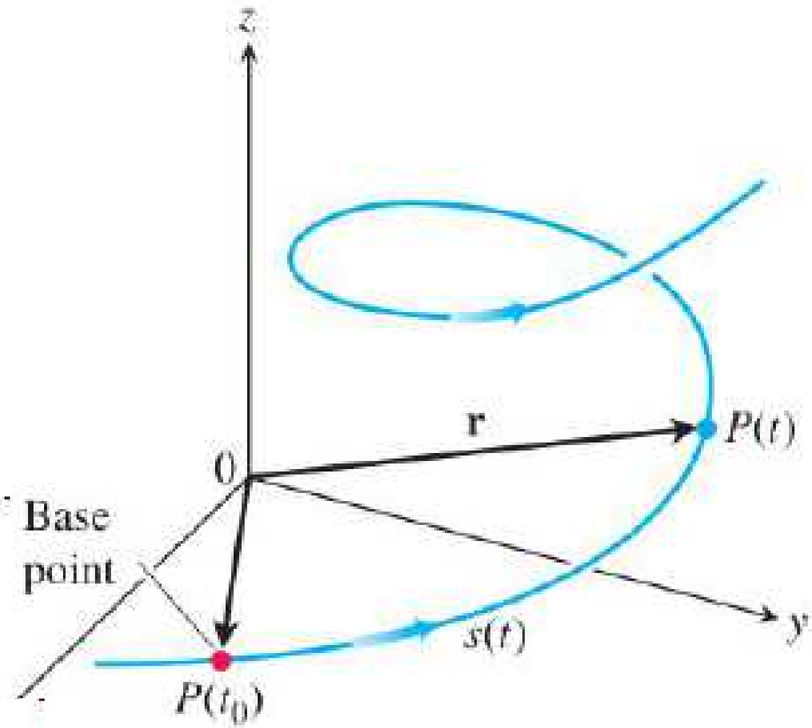
$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau \qquad (3)$$

$$\frac{ds}{dt} = |v(t)| > 0, \quad s(t) \text{ increasing.}$$

$$t = t(s).$$

$$\mathbf{r} = \mathbf{r}(t(s)).$$

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \mathbf{v}\frac{1}{|\mathbf{v}|} = \frac{\mathbf{v}}{|\mathbf{v}|} = \mathbf{T}.$$



unit tangent vector $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

EXAMPLE 2 we can actually find the arc length param etrization of a curve. If $t_0 = 0$, the arc length parameter along the helix

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

Solution from t_0 to t is

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau = \int_0^t \sqrt{2}\, d\tau = \sqrt{2}\, t.$$

$t = s/\sqrt{2}.$

$$\mathbf{r}(t(s)) = \left(\cos\frac{s}{\sqrt{2}}\right)\mathbf{i} + \left(\sin\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}.$$

$$\frac{\mathbf{dr}}{\mathbf{ds}} = \left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \qquad \left|\frac{dr}{ds}\right| = 1$$

$\mathbf{v} = d\mathbf{r}/dt$ is tangent to the curve $\mathbf{r}(t)$

unit tangent vector $\quad \mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

EXAMPLE 3 Find the unit tangent vector of the curve

$$\mathbf{r}(t) = (1 + 3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$$

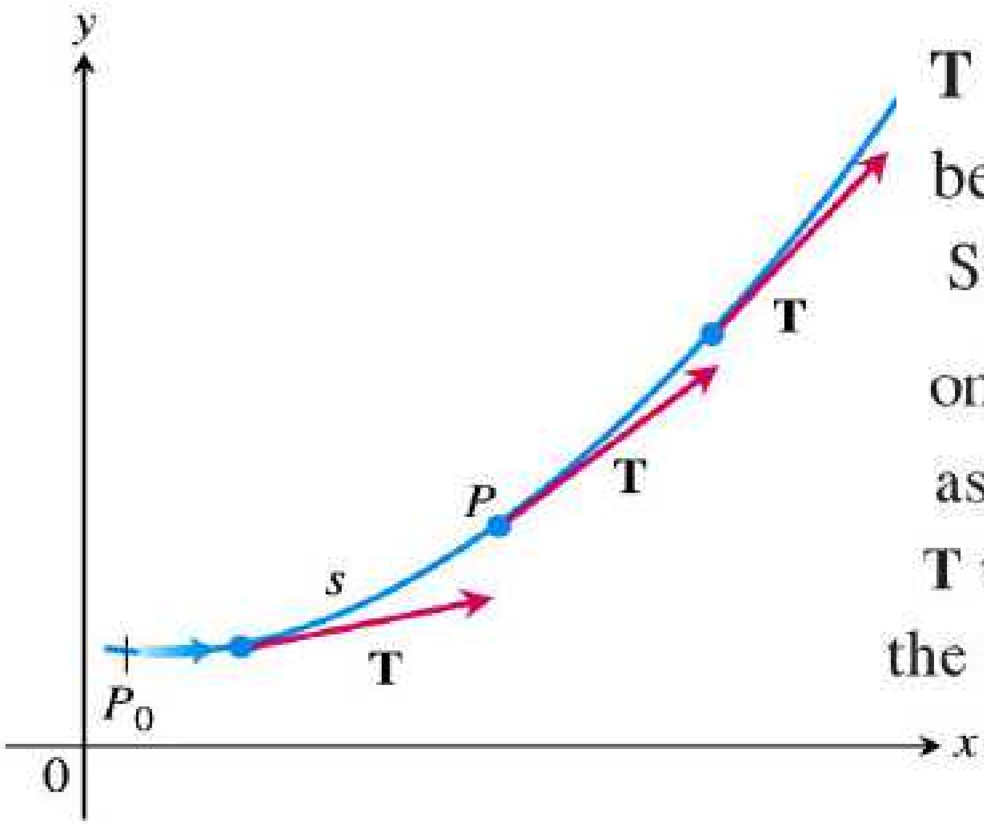
Solution $\quad \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\mathbf{k} \quad |\mathbf{v}| = \sqrt{9 + 4t^2}.$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{3\sin t}{\sqrt{9 + 4t^2}}\mathbf{i} + \frac{3\cos t}{\sqrt{9 + 4t^2}}\mathbf{j} + \frac{2t}{\sqrt{9 + 4t^2}}\mathbf{k}.$$

13.4

Curvature and the Normal Vector of a Curve

曲线的曲率和法向量



$\mathbf{T} = d\mathbf{r}/ds$ turns as the curve bends.

Since $\mathbf{T}$ is a unit vector, only its direction changes as the particle moves

$\mathbf{T}$ turns per unit of length along the curve is called the *curvature*

$$\left|\frac{\Delta \mathbf{T}}{\Delta s}\right| \text{---平均曲率}$$

$$\lim_{\Delta s \to 0}\left|\frac{\Delta \mathbf{T}}{\Delta s}\right| = \left|\frac{d\mathbf{T}}{ds}\right| \text{---曲率}$$

FIGURE 13.17 As P moves along the curve in the direction of increasing arc length, the unit tangent vector turns. The value of $|d\mathbf{T}/ds|$ at P is called the *curvature* of the curve at P.

DEFINITION If $\mathbf{T}$ is the unit vector of a smooth curve, the **curvature** function of the curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

If $|d\mathbf{T}/ds|$ is large, $\mathbf{T}$ turns sharply as the particle passes through P,

If $|d\mathbf{T}/ds|$ is close to zero, $\mathbf{T}$ turns more slowly

若$r = r(s)$, $T = \frac{dr}{ds}$, $\kappa = \left|\frac{d^2 r}{ds^2}\right|$

若$r = r(t)$,

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| = \frac{1}{|ds/dt|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|.$$

Formula for Calculating Curvature

If $\mathbf{r}(t)$ is a smooth curve, then the curvature is the scalar function

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|,$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

EXAMPLE 1 A straight line is parametrized by $\mathbf{r}(t) = \mathbf{C} + t\mathbf{v}$ constant vectors $\mathbf{C}$ and $\mathbf{v}$. **Find the curvature of the line.**

Solution Thus, $\mathbf{r}'(t) = \mathbf{v}$, $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{|\mathbf{v}|}|\mathbf{0}| = 0.$$

EXAMPLE 2

Here we find the curvature of a circle. We begin with the parametrization

$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$ of a circle of radius a. Then,

Solution

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{(-a \sin t)^2 + (a \cos t)^2} = \sqrt{a^2} = |a| = a.$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} \qquad \frac{d\mathbf{T}}{dt} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1. \qquad \kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{a}(1) = \frac{1}{a}$$

$\mathbf{T}$ has constant length

$d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$

$$\left|\frac{dT}{ds}\right| = \kappa,$$

$\dfrac{1}{\kappa}\dfrac{d\mathbf{T}}{ds}$, a *unit* vector orthogonal to $\mathbf{T}$

DEFINITION At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

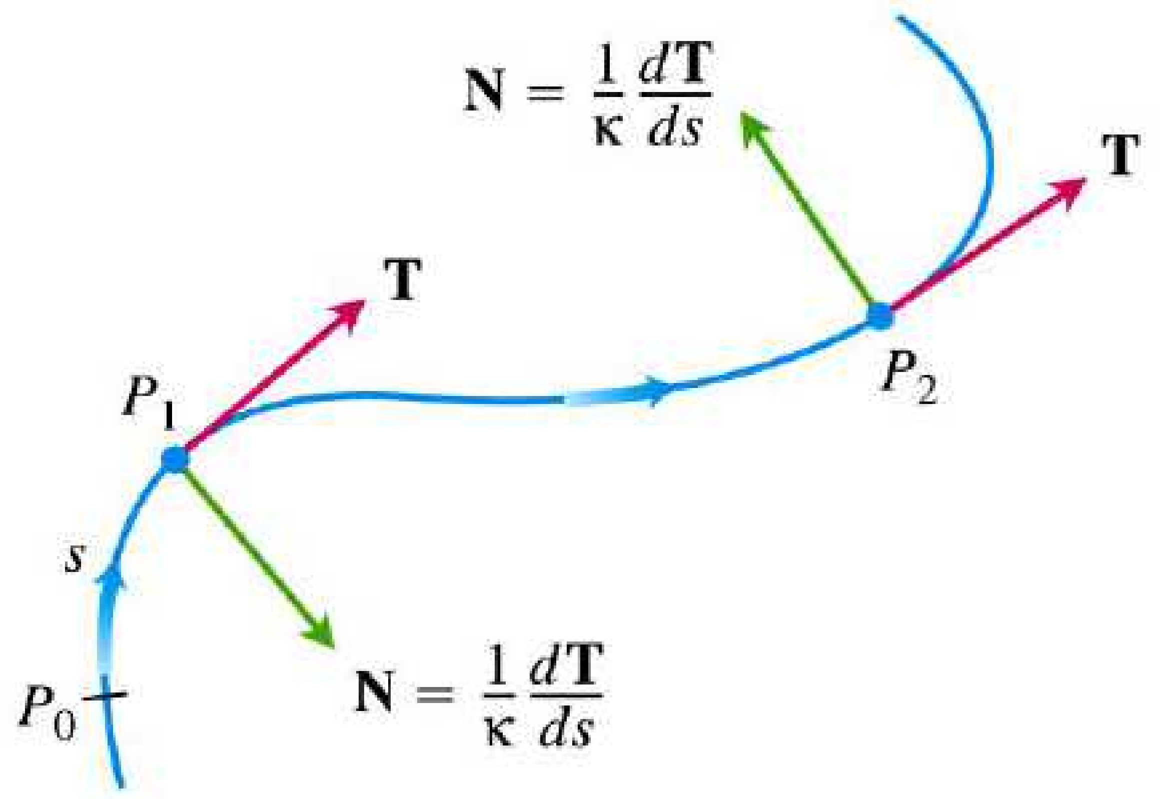


FIGURE 13.19 The vector $d\mathbf{T}/ds$, normal to the curve, always points in the direction in which $\mathbf{T}$ is turning. The unit normal vector $\mathbf{N}$ is the direction of $d\mathbf{T}/ds$.

If a smooth curve $\mathbf{r}(t)$ is already given

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{(d\mathbf{T}/dt)(dt/ds)}{|d\mathbf{T}/dt||dt/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}.$$

Formula for Calculating N

If $\mathbf{r}(t)$ is a smooth curve, then the principal unit normal is

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

EXAMPLE 3 Find **T** and **N** for the circular motion

$$\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j}.$$

Solution We first find **T**:

$$\mathbf{v} = -(2\sin 2t)\mathbf{i} + (2\cos 2t)\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j}.$$

$$\frac{d\mathbf{T}}{dt} = -(2\cos 2t)\mathbf{i} - (2\sin 2t)\mathbf{j}$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 2t} = 2$$

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = -(\cos 2t)\mathbf{i} - (\sin 2t)\mathbf{j}.$$

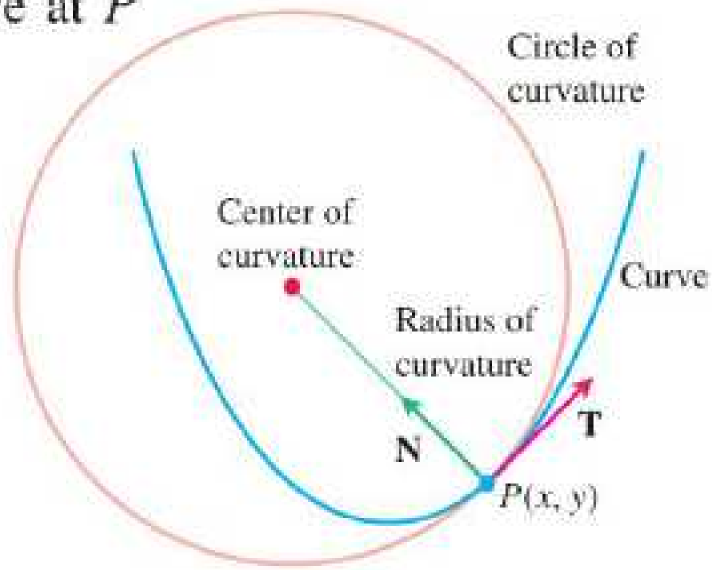
Circle of Curvature for Plane Curves

The **circle of curvature** or **osculating circle** at a point P on a plane curve

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P
3. has center that lies toward the concave or inner side of the curve

The **radius of curvature** of the curve at P

$$\text{Radius of curvature} = \rho = \frac{1}{\kappa}.$$



EXAMPLE 4

Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

Solution $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j}$ $\quad |\mathbf{v}| = \sqrt{1 + 4t^2}$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (1 + 4t^2)^{-1/2}\mathbf{i} + 2t(1 + 4t^2)^{-1/2}\mathbf{j}.$$

$$\frac{d\mathbf{T}}{dt} = -4t(1 + 4t^2)^{-3/2}\mathbf{i} + \left[2(1 + 4t^2)^{-1/2} - 8t^2(1 + 4t^2)^{-3/2}\right]\mathbf{j}.$$

$$\kappa(0) = \frac{1}{|\mathbf{v}(0)|}\left|\frac{d\mathbf{T}}{dt}(0)\right| = \frac{1}{\sqrt{1}}|0\mathbf{i} + 2\mathbf{j}| = 2.$$

the radius of curvature is $1/\kappa = 1/2$.

$$(x - 0)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$

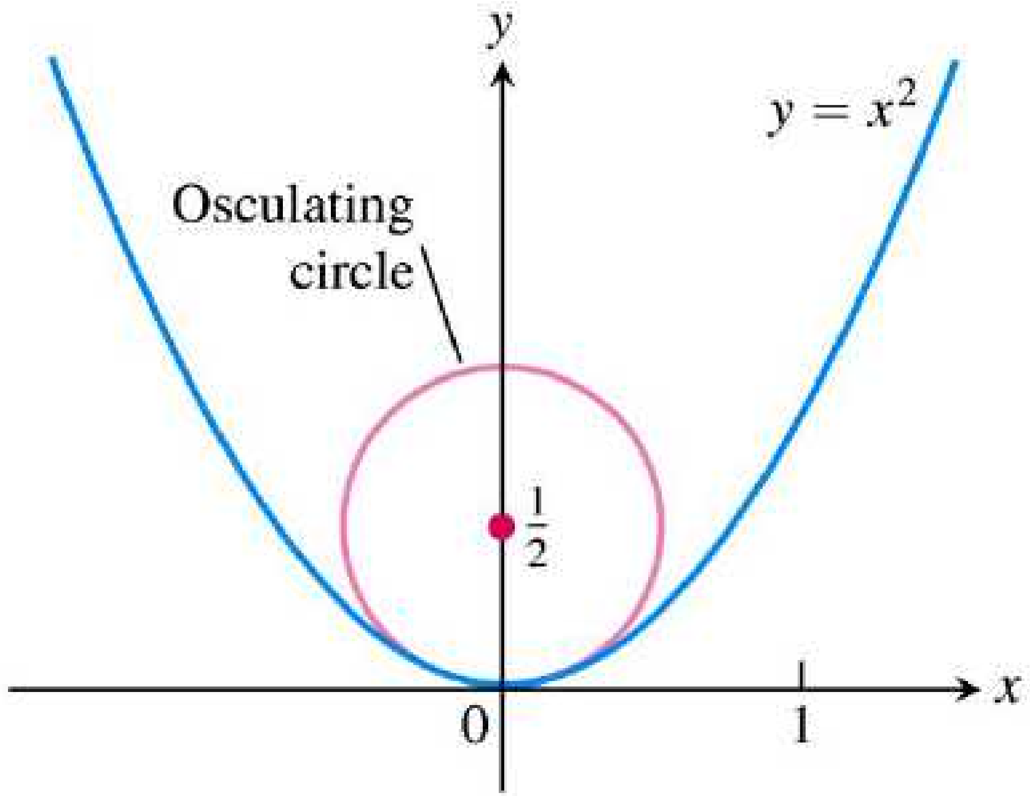


FIGURE 13.21 The osculating circle for the parabola $y = x^2$ at the origin (Example 4).

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$$

EXAMPLE 5 Find the curvature for the helix

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}, \qquad a, b \geq 0, \qquad a^2 + b^2 \neq 0.$$

Solution

$$\mathbf{v} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}$$

$$|\mathbf{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{a^2 + b^2}}\left[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}\right].$$

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{a^2 + b^2}}\left|\frac{1}{\sqrt{a^2 + b^2}}\left[-(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}\right]\right| = \frac{a}{a^2 + b^2}.$$

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}.$$

the **principal unit normal**

EXAMPLE 6

Find **N** for the helix in Example 5 and describe how the vector is pointing.

Solution

$$\frac{d\mathbf{T}}{dt} = -\frac{1}{\sqrt{a^2+b^2}}\left[(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}\right]$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2\cos^2 t + a^2\sin^2 t} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = -\frac{\sqrt{a^2+b^2}}{a}\cdot\frac{1}{\sqrt{a^2+b^2}}\left[(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}\right]$$

$$= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}.$$

Thus, **N** is parallel to the xy-plane and always points toward the z-axis.

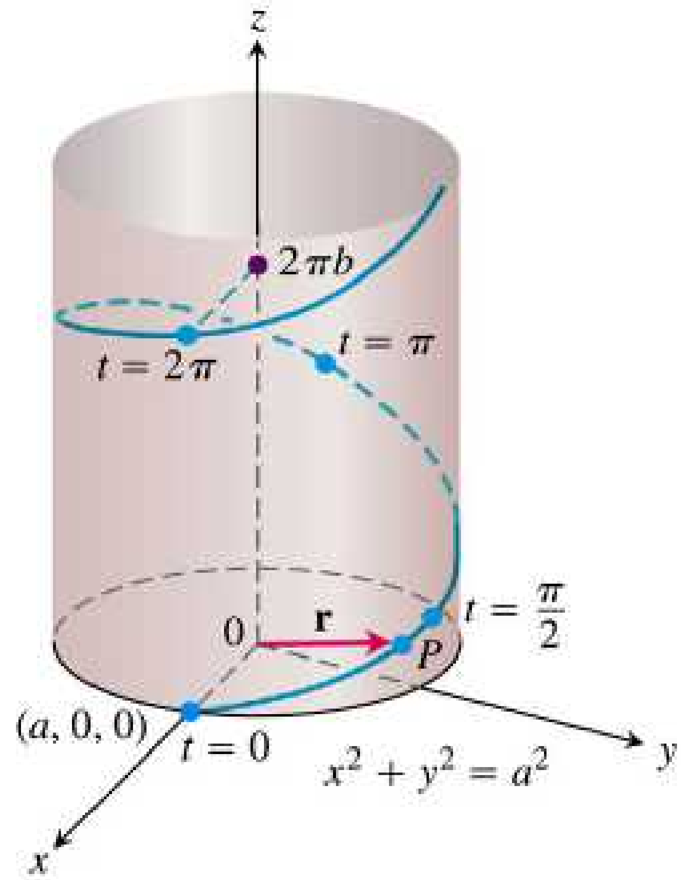


FIGURE 13.22 The helix

$$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k},$$

drawn with a and b positive and $t \geq 0$ (Example 5).

13.5

Tangential and Normal Components of Acceleration

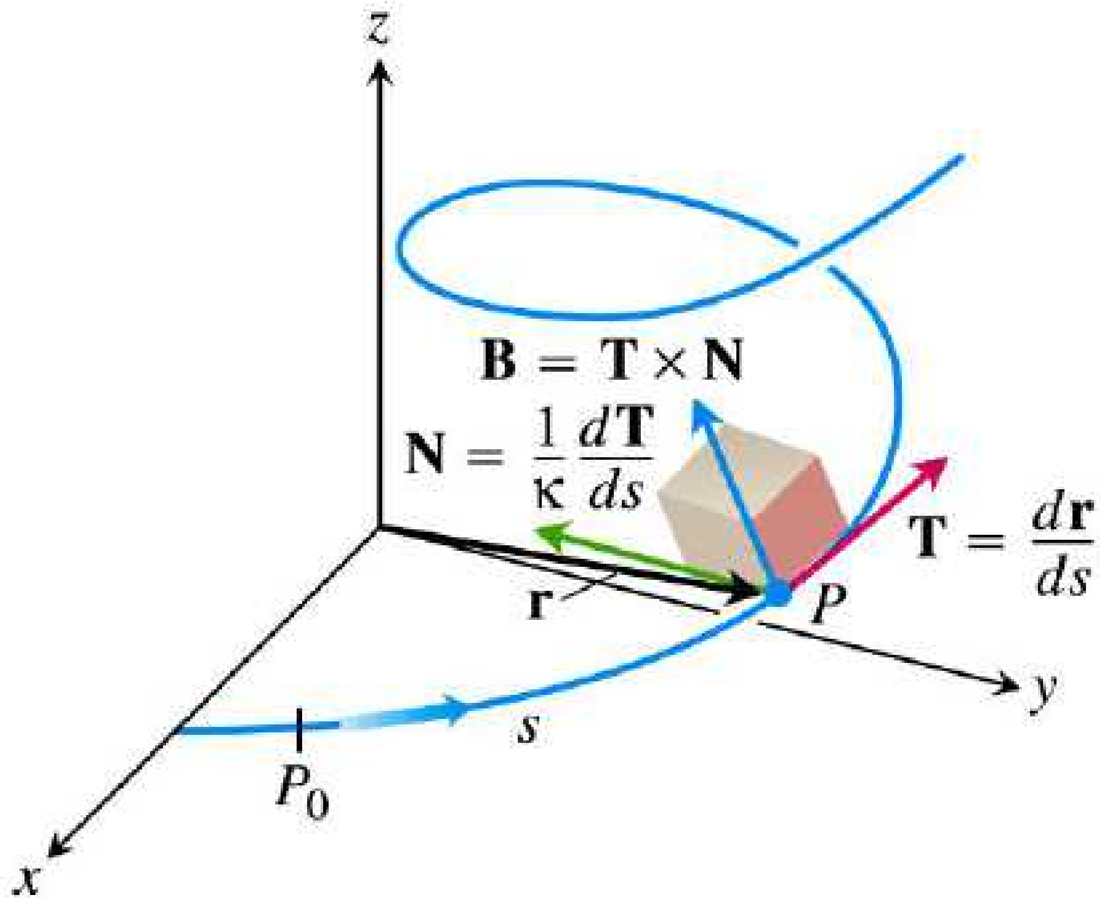


FIGURE 13.23 The **TNB** frame of mutually orthogonal unit vectors traveling along a curve in space.

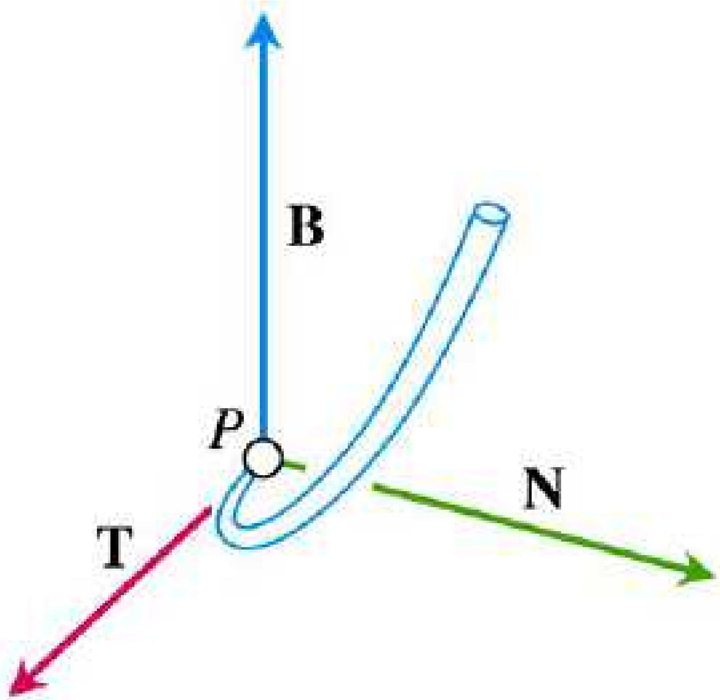


FIGURE 13.24 The vectors **T**, **N**, and **B** (in that order) make a right-handed frame of mutually orthogonal unit vectors in space.

DEFINITION If the acceleration vector is written as

$$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}, \tag{1}$$

then

$$a_{\mathrm{T}} = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}| \quad \text{and} \quad a_{\mathrm{N}} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2 \tag{2}$$

are the **tangential** and **normal** scalar components of acceleration.

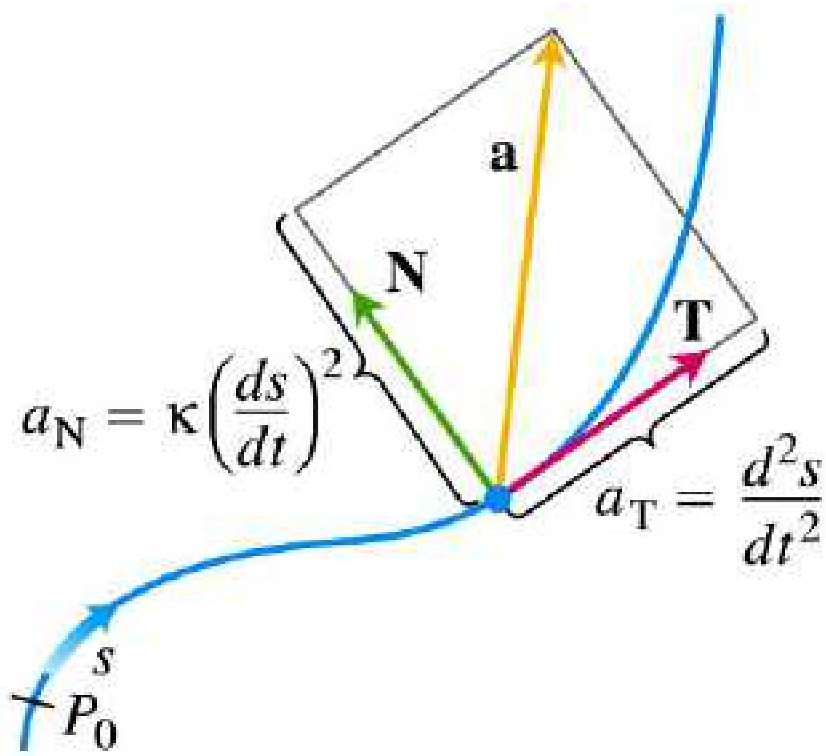


FIGURE 13.25 The tangential and normal components of acceleration. The acceleration **a** always lies in the plane of **T** and **N**, orthogonal to **B**.

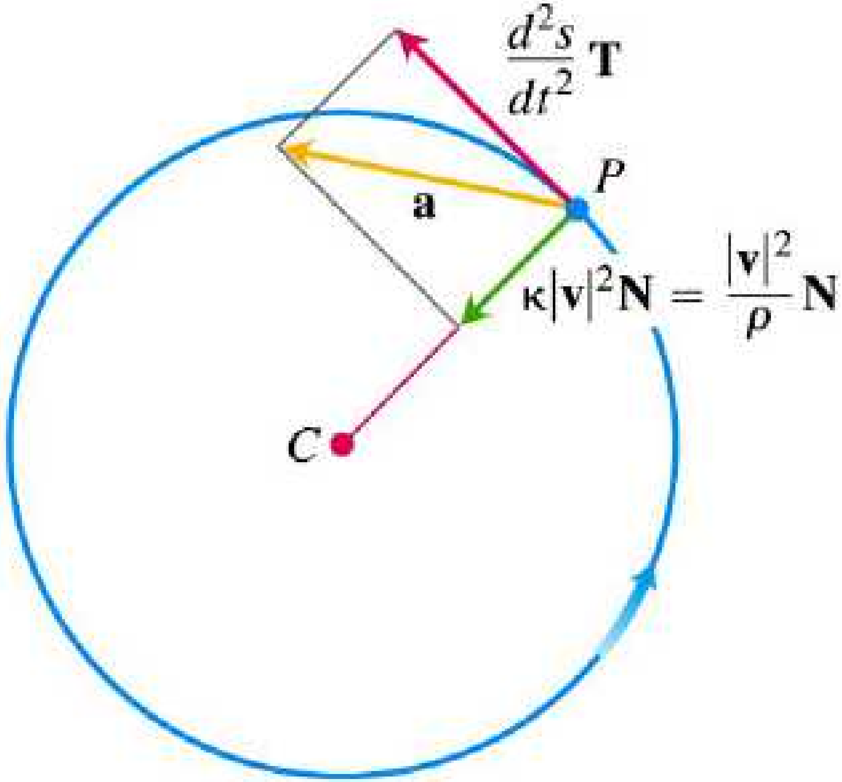


FIGURE 13.26 The tangential and normal components of the acceleration of an object that is speeding up as it moves counterclockwise around a circle of radius ρ.

Formula for Calculating the Normal Component of Acceleration

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} \tag{3}$$

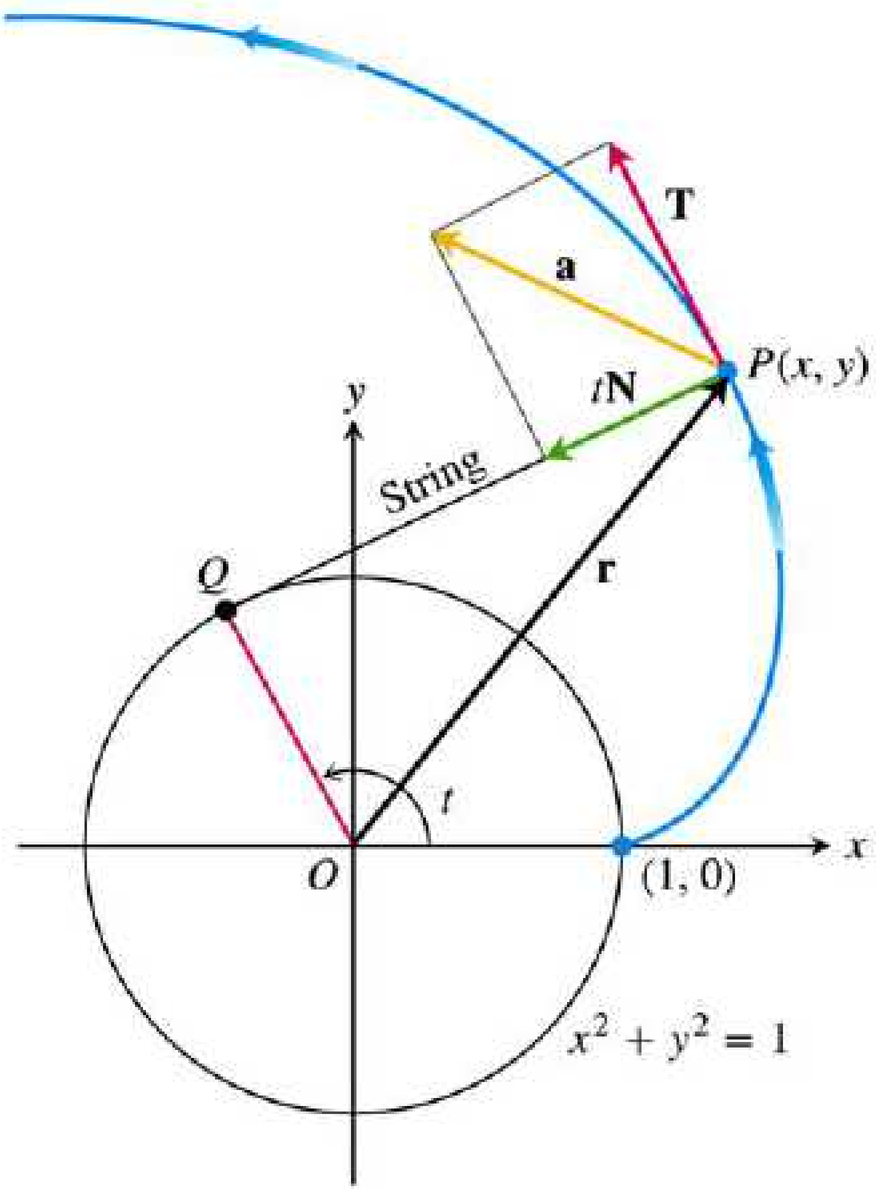


FIGURE 13.27 The tangential and normal components of the acceleration of the motion $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}$, for $t > 0$. If a string wound around a fixed circle is unwound while held taut in the plane of the circle, its end P traces an involute of the circle (Example 1).

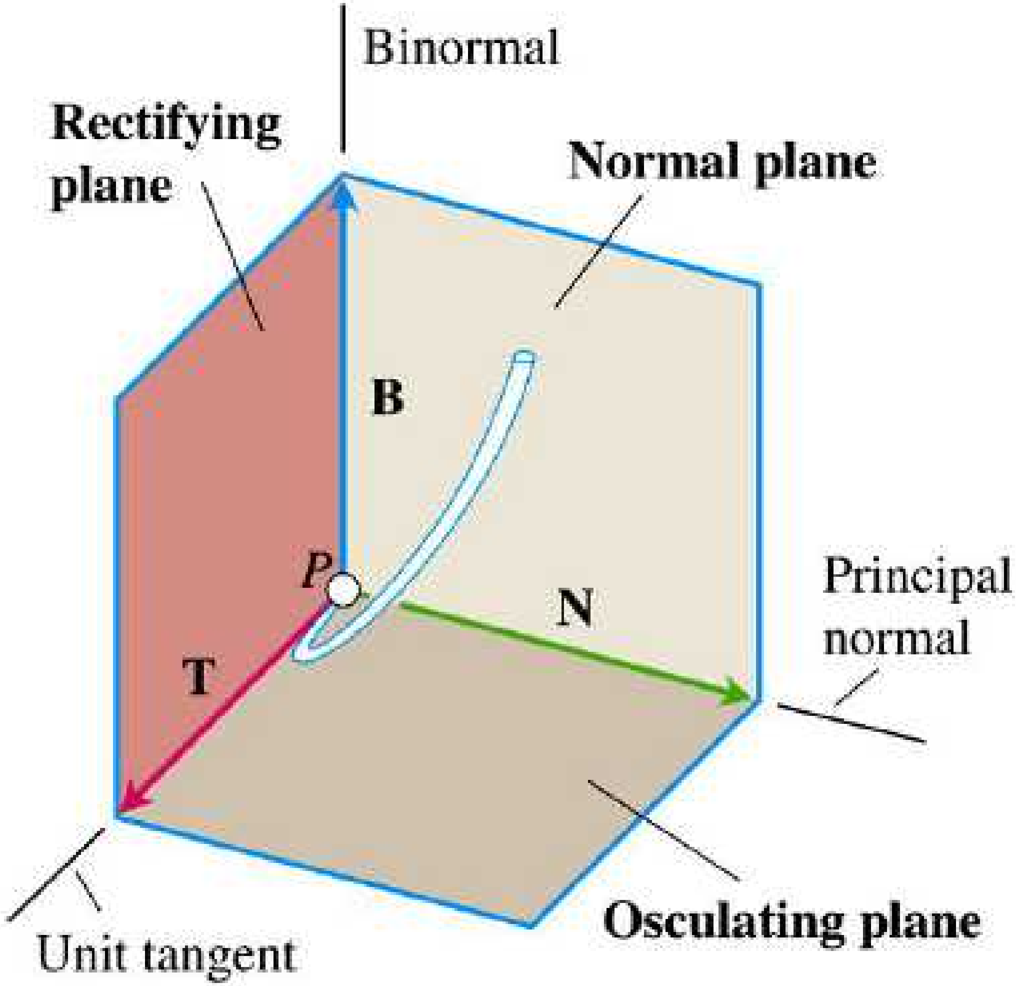


FIGURE 13.28 The names of the three planes determined by **T**, **N**, and **B**.

DEFINITION Let $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The **torsion** function of a smooth curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}. \tag{4}$$

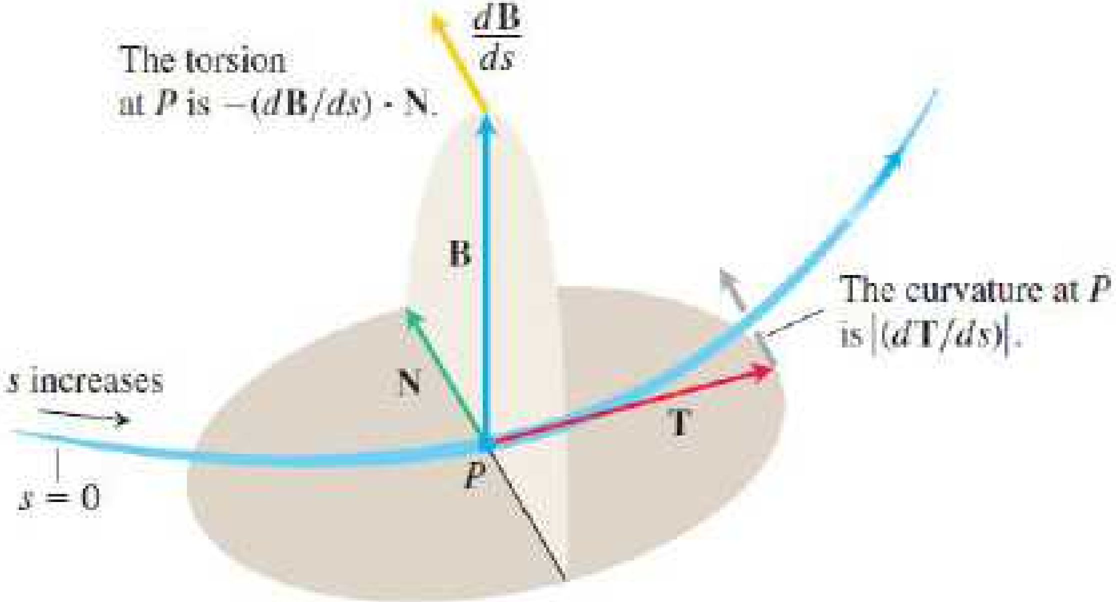


FIGURE 13.29 Every moving body travels with a **TNB** frame that characterizes the geometry of its path of motion.

Vector Formula for Curvature

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \qquad (5)$$

Formula for Torsion

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} \qquad (\text{if } \mathbf{v} \times \mathbf{a} \neq \mathbf{0}) \qquad (6)$$

Computation Formulas for Curves in Space

Unit tangent vector: $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

Binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

Torsion: $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$

Tangential and normal scalar components of acceleration: $\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$

$$a_{\mathrm{T}} = \frac{d}{dt}|\mathbf{v}|$$

$$a_{\mathrm{N}} = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_{\mathrm{T}}^2}$$

13.6

Velocity and Acceleration in Polar Coordinates

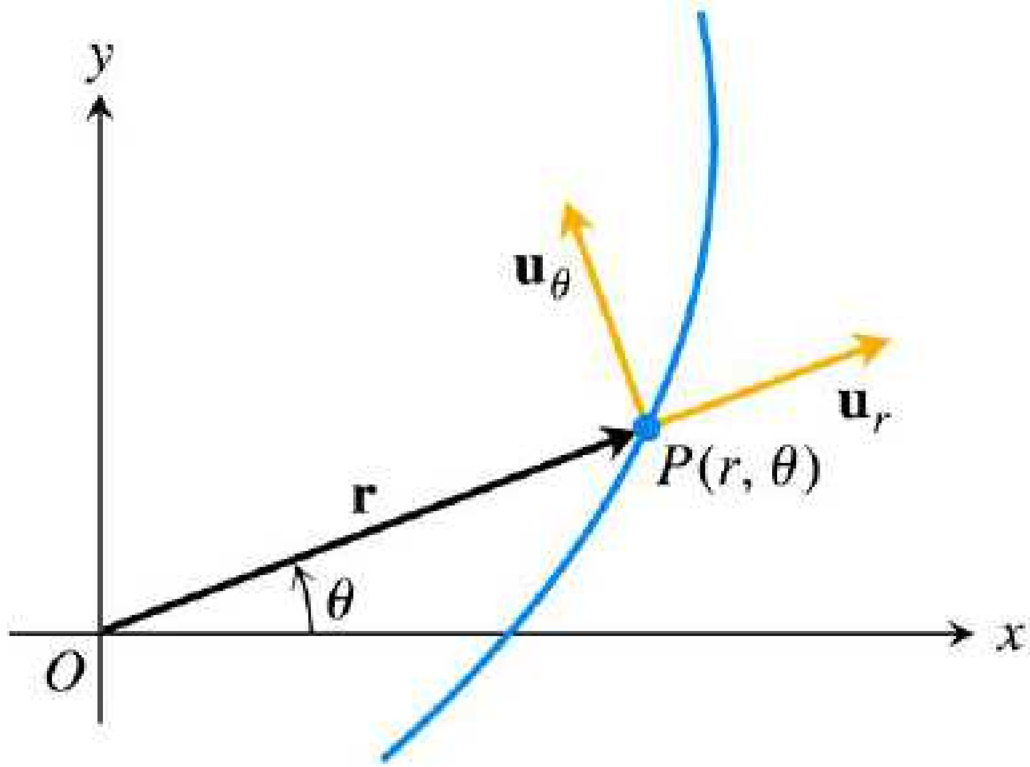


FIGURE 13.30 The length of $\mathbf{r}$ is the positive polar coordinate r of the point P. Thus, $\mathbf{u}_r$, which is $\mathbf{r}/|\mathbf{r}|$, is also $\mathbf{r}/r$. Equations (1) express $\mathbf{u}_r$ and $\mathbf{u}_\theta$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

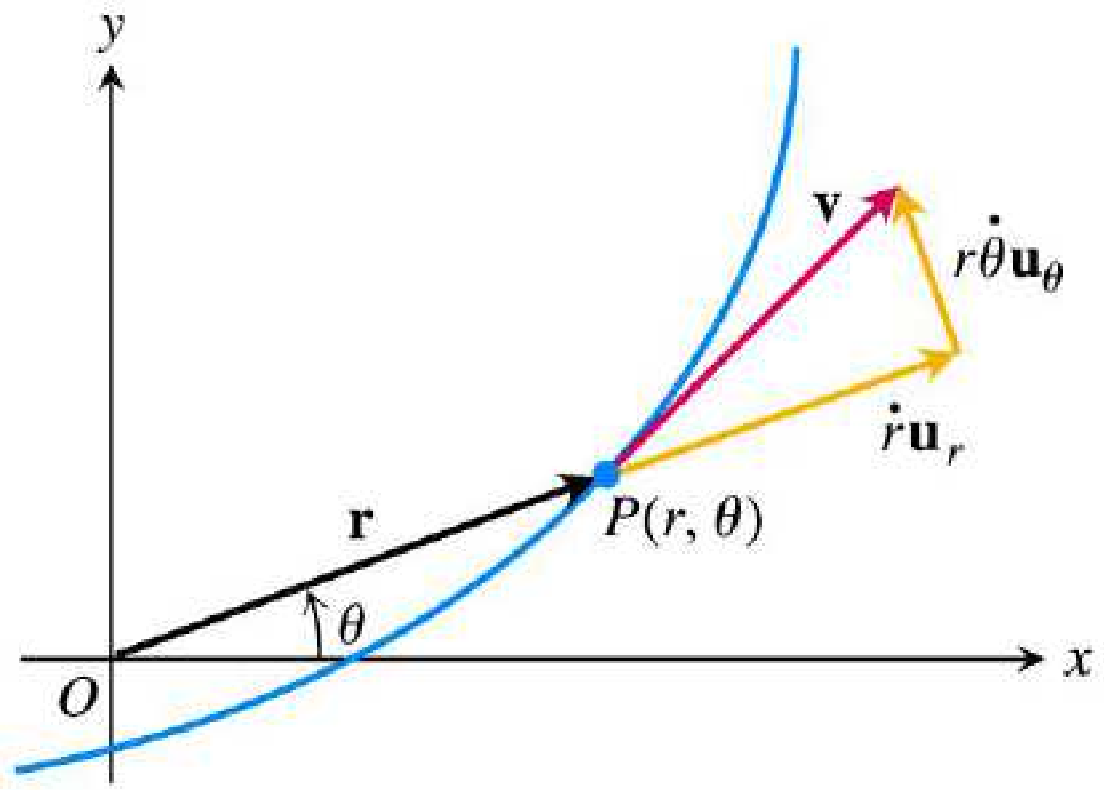


FIGURE 13.31 In polar coordinates, the velocity vector is

$$\mathbf{v} = \dot{r}\,\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta.$$

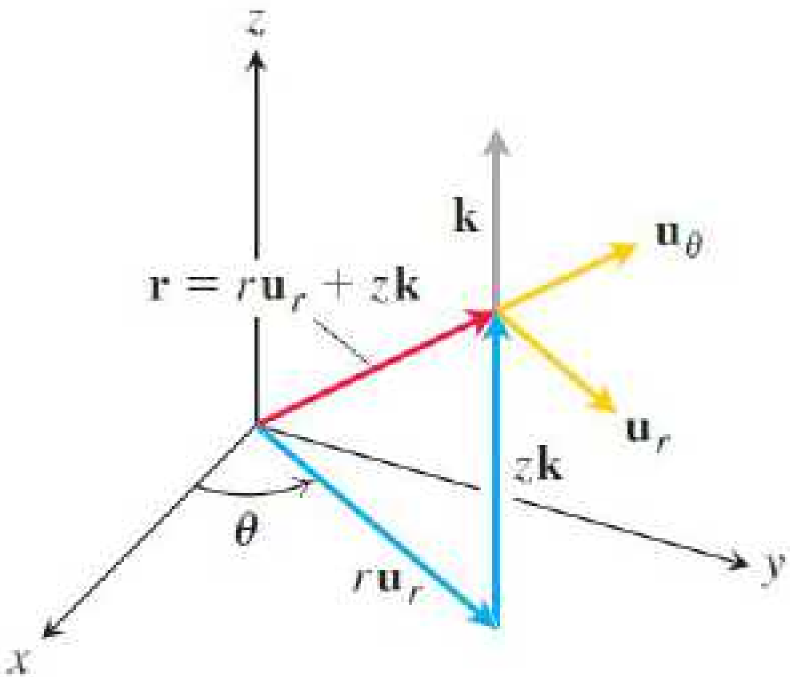


FIGURE 13.32 Position vector and basic unit vectors in cylindrical coordinates. Notice that $|\mathbf{r}| \neq r$ if $z \neq 0$ since $|\mathbf{r}| = \sqrt{r^2 + z^2}$.

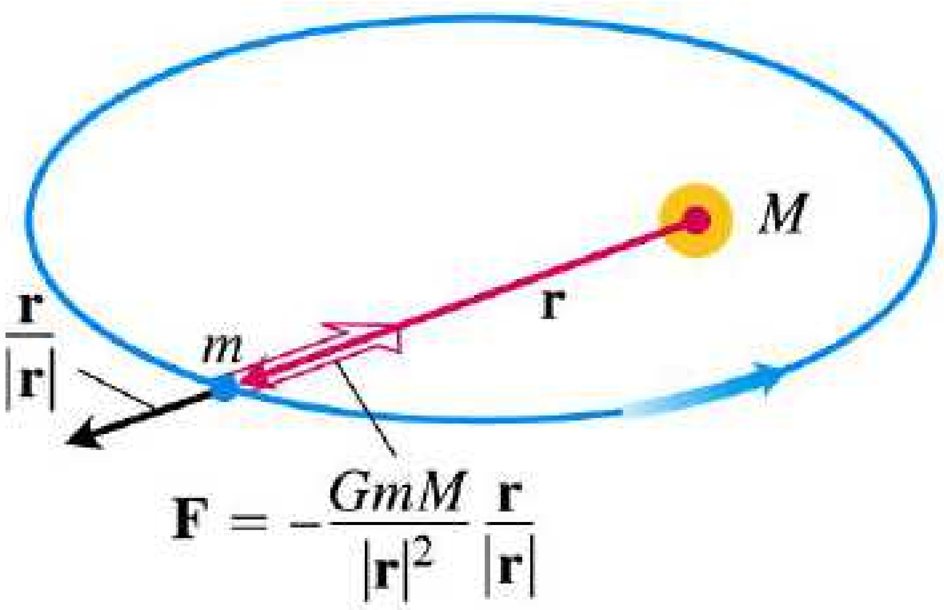


FIGURE 13.33 The force of gravity is directed along the line joining the centers of mass.

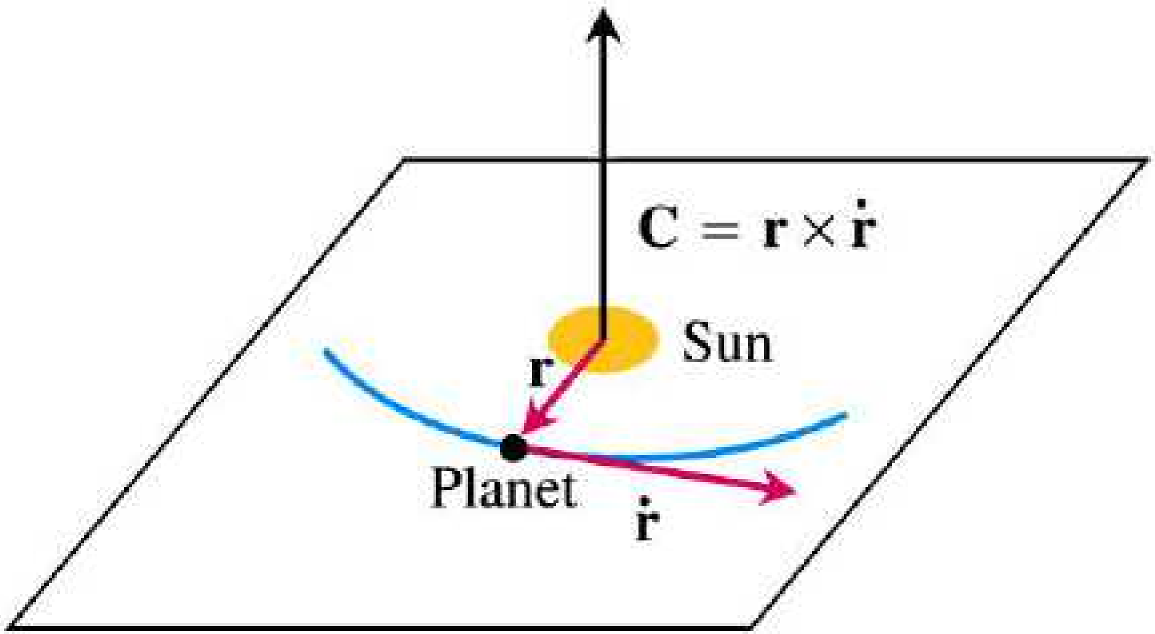


FIGURE 13.34 A planet that obeys Newton's laws of gravitation and motion travels in the plane through the sun's center of mass perpendicular to $\mathbf{C} = \mathbf{r} \times \dot{\mathbf{r}}$.

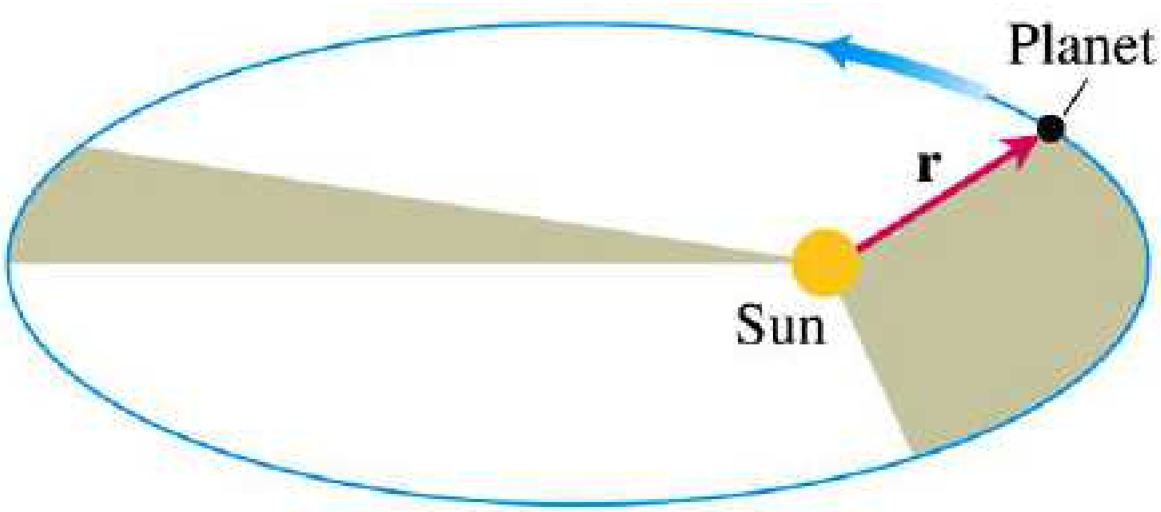


FIGURE 13.35 The line joining a planet to its sun sweeps over equal areas in equal times.

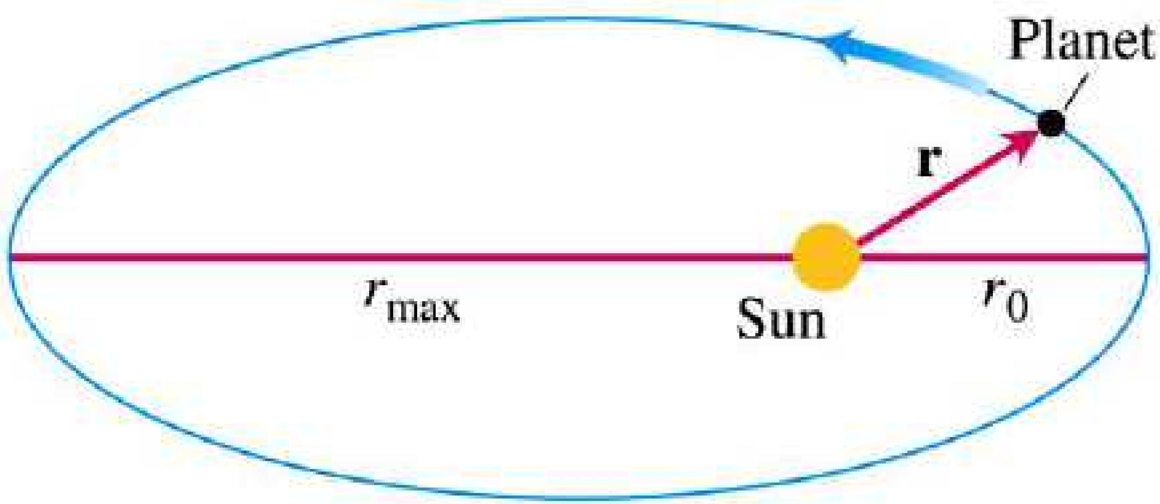


FIGURE 13.36 The length of the major axis of the ellipse is $2a = r_0 + r_{max}$.