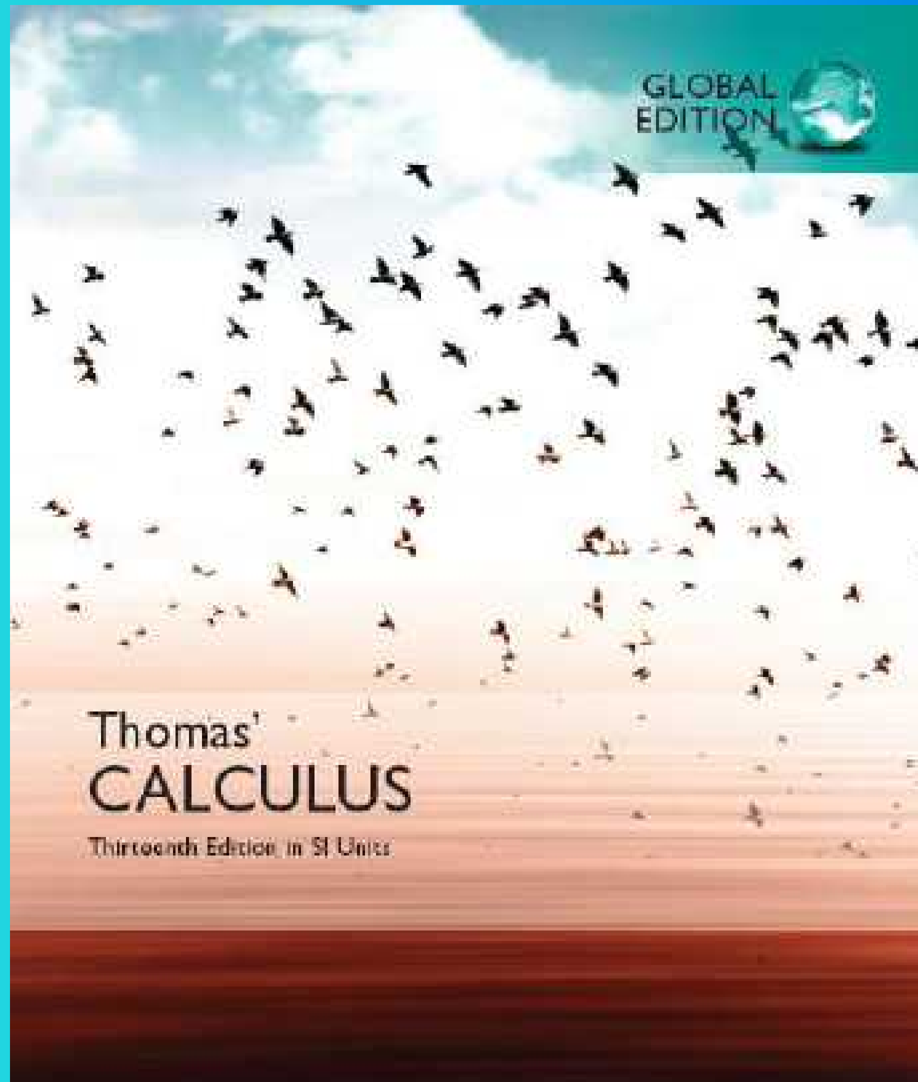

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 7

Transcendental Functions

超越函数

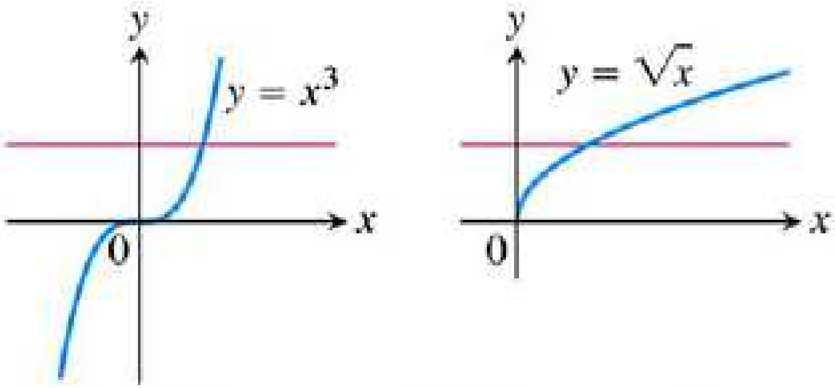
7.1

Inverse Functions and Their Derivatives

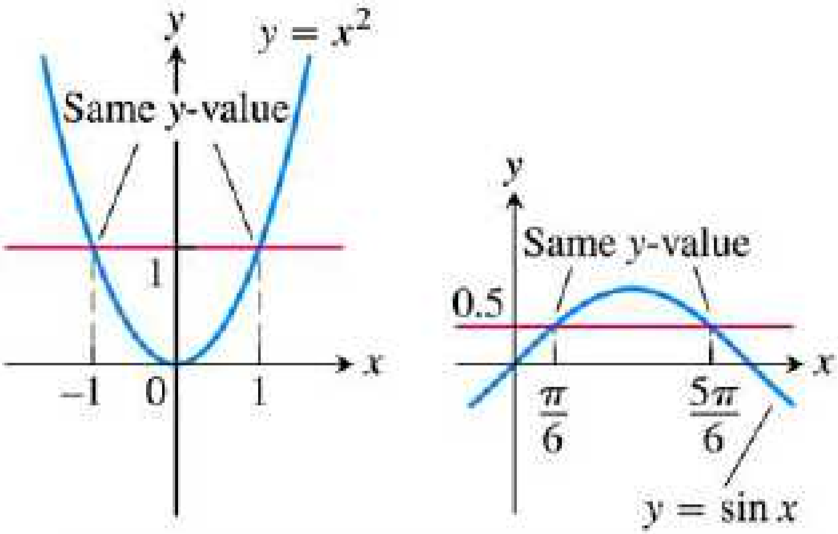
反函数及其导数

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Ex.1



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

FIGURE 7.1 (a) $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$. (b) $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

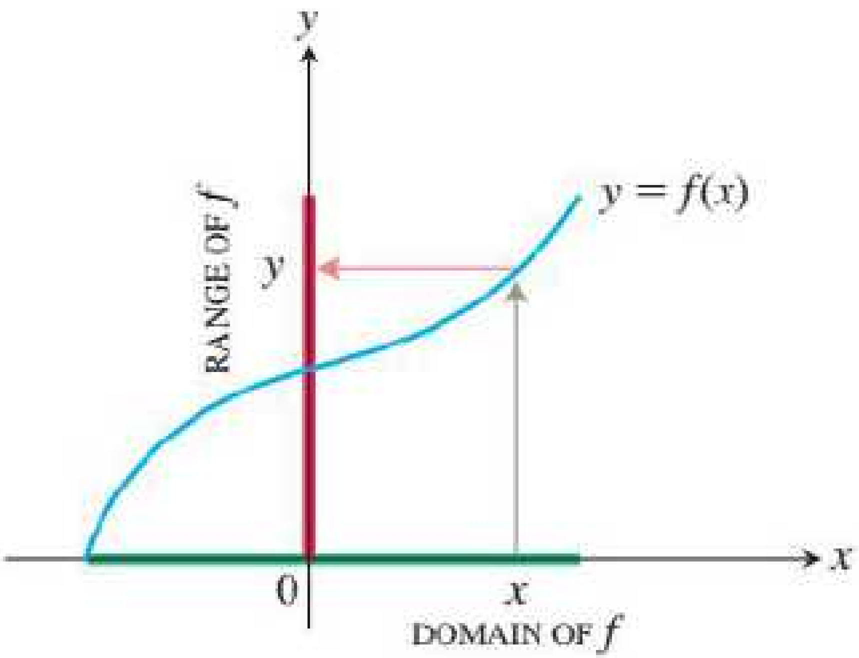
$$f^{-1}(b) = a \text{ if } f(a) = b.$$

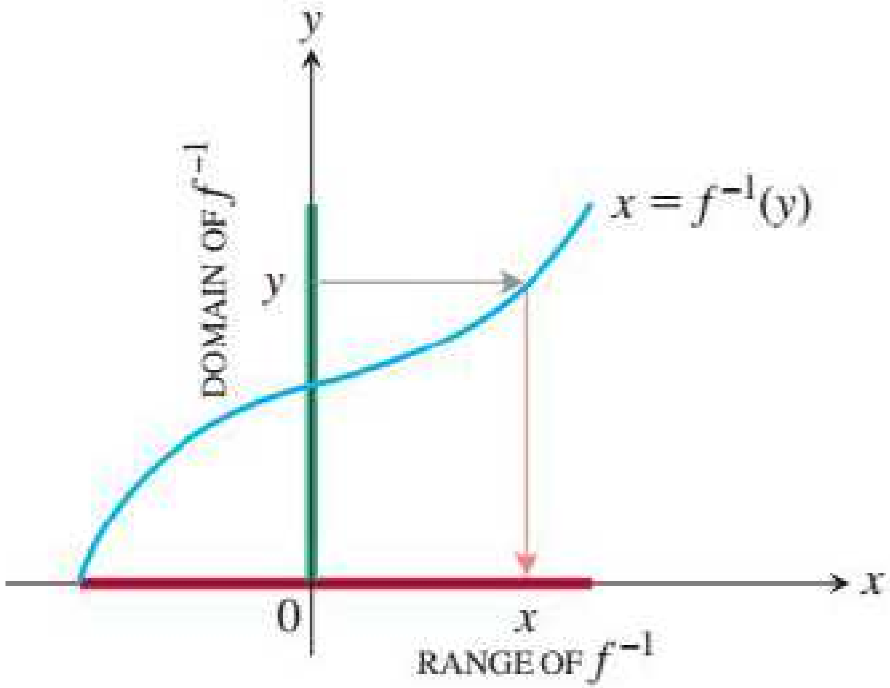
The domain of f^{-1} is R and the range of f^{-1} is D.

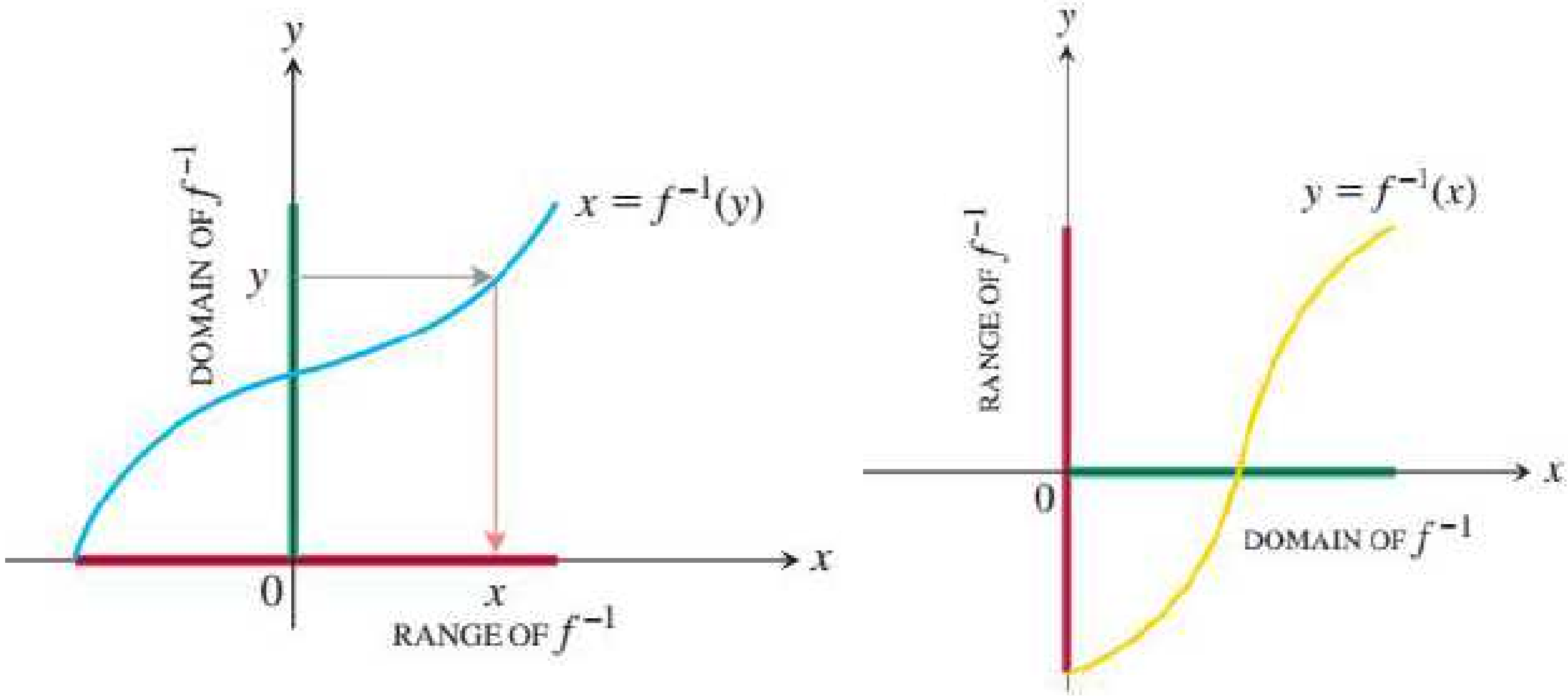
$$R = \{b \mid b = f(a), a \in D\}$$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x, \forall x \in D,$$

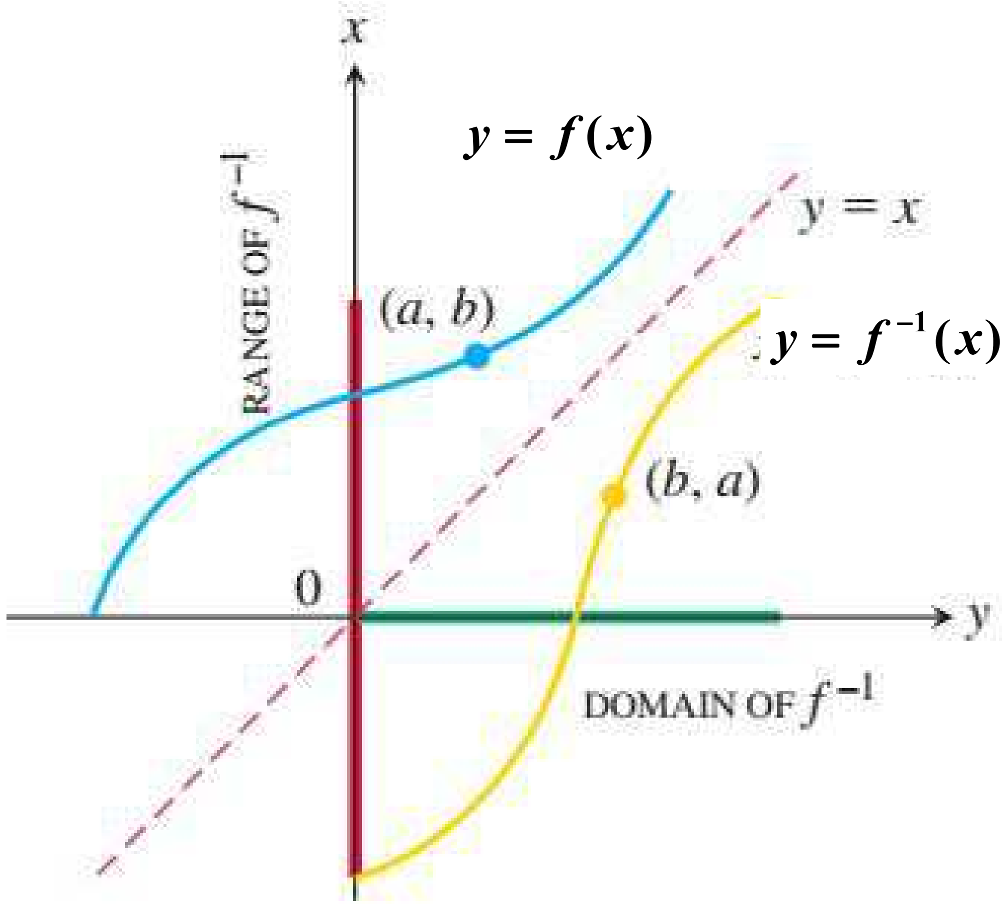
$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = y, \forall y \in R.$$


y
RANGE OF f
y
$y = f(x)$
0
x
x
DOMAIN OF f


y
DOMAIN OF f^{-1}
y
$x = f^{-1}(y)$
0
x
x
RANGE OF f^{-1}



y
DOMAIN OF f^{-1}
y
$x = f^{-1}(y)$
0
x
x
RANGE OF f^{-1}
y
RANGE OF f^{-1}
$y = f^{-1}(x)$
0
x
DOMAIN OF f^{-1}



x
y = f(x)
y = x
RANGE OF f^{-1}
(a, b)
y = f^{-1}(x)
(b, a)
0
y
DOMAIN OF f^{-1}

$$y = ax + b \quad (a \neq 0)$$

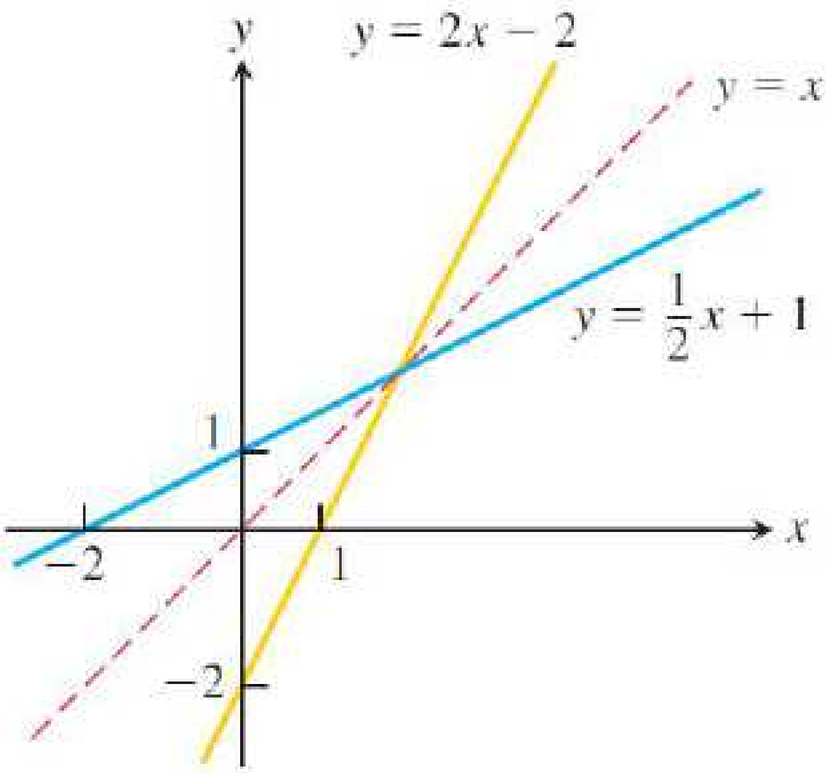


FIGURE 7.3 Graphing the functions $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$ (Example 3).

$$\downarrow \quad x = \frac{y}{a} - \frac{b}{a}$$

$$y = \frac{x}{a} - \frac{b}{a}$$

任何关于$y=x$对称的直线的斜率互为倒数

reciprocal

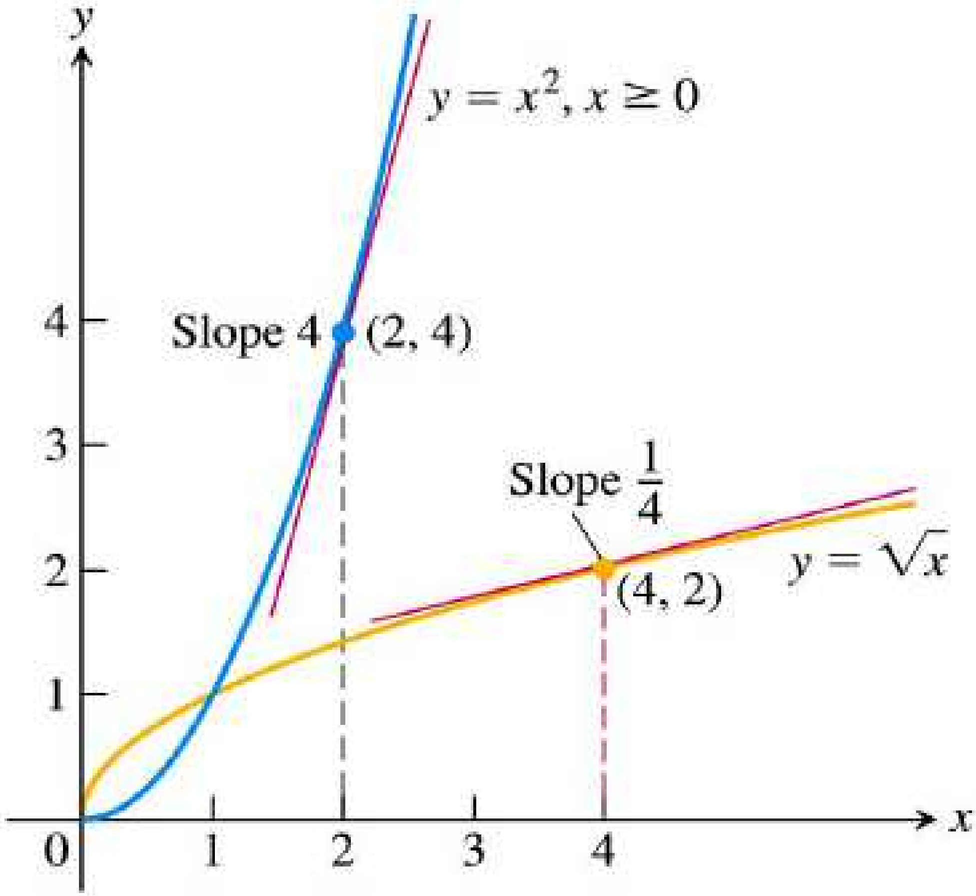


FIGURE 7.6 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 5).

反函数和直接函数在任何对应点上的切线的斜率互为倒数

反函数和直接函数在任何对应点上的导数互为倒数

THEOREM 1—The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \tag{1}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

$$(f^{-1})'(b) = \frac{1}{f'(a)}, f(a) = b$$

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = y, \forall y \in R.$$

$$f'(f^{-1}(y))[f^{-1}(y)]' = 1, \forall y \in R.$$

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f'(x)}, \forall y \in R,$$

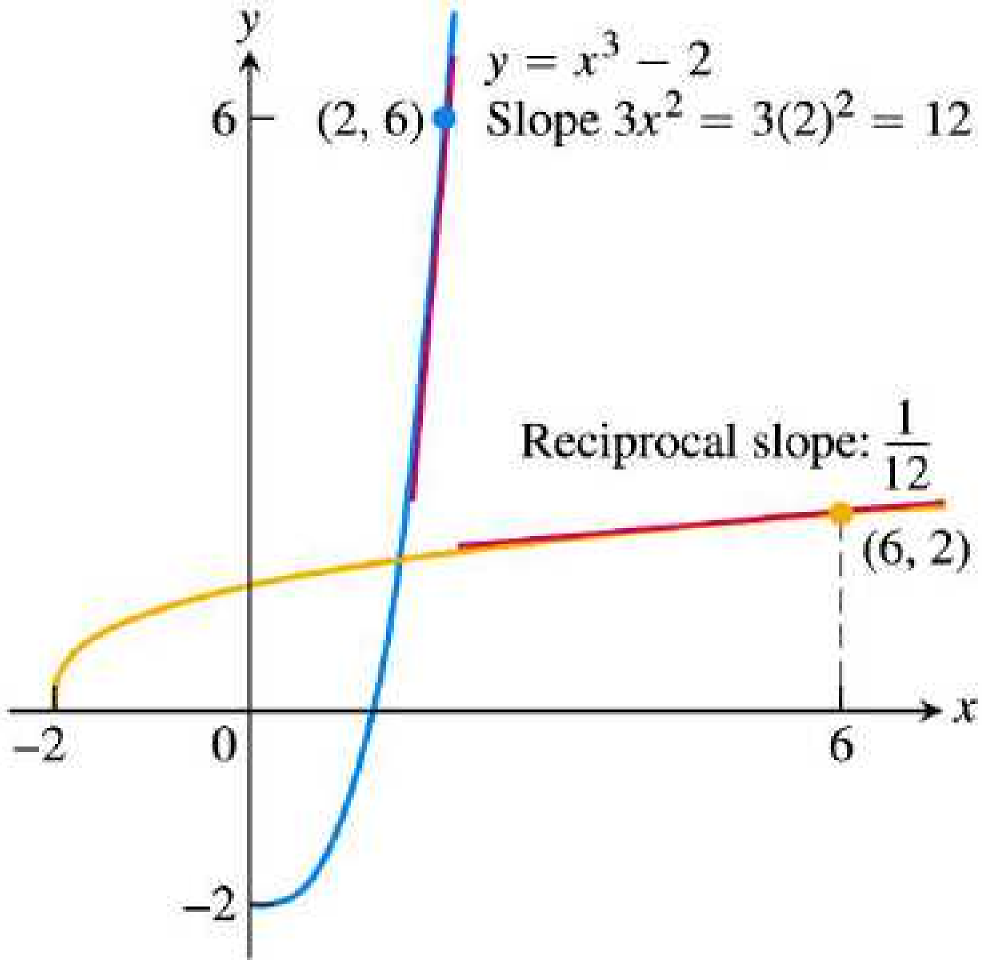


FIGURE 7.7 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 6).

7.2

Natural Logarithms
自然对数

DEFINITION The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0. \tag{1}$$

$$\ln 1 = \int_1^1 \frac{1}{x}\,dx = 0 \qquad (\ln x)' = \left(\int_1^x \frac{1}{t}\,dt\right)' = \frac{1}{x}$$

连续、可导、严格递增

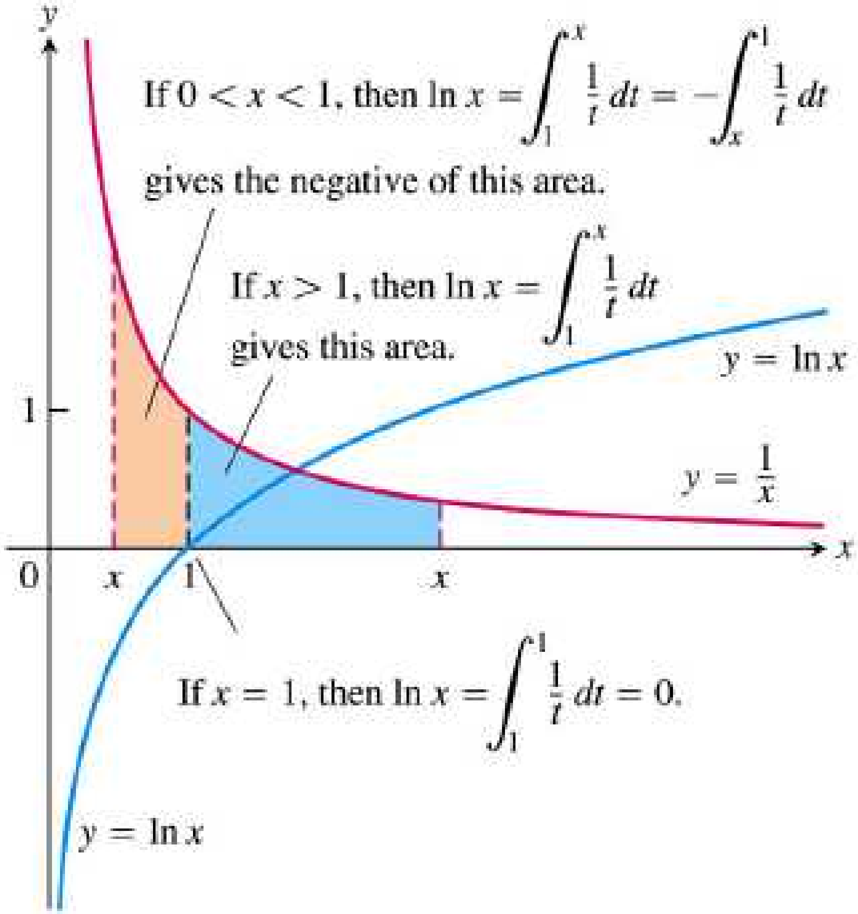


FIGURE 7.8 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the x-axis as x moves from 1 to the left.

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0,$$

if $0 < x < 1$,

$$\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$$

if $x > 1$,

$$\ln x = \int_1^x \frac{1}{t}\,dt > 0$$

TABLE 7.1 Typical 2-place values of ln x

x	$\ln x$
0	undefined
0.05	-3.00
0.5	-0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

利用面积可以近似计算函数值

$\ln 2 < 1, \ln 3 > 1,$

存在某数 $e, 1 < e < 3,$

使 $\ln e = 1.$

DEFINITION The **number e** is that number in the domain of the natural logarithm satisfying

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1.$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad x \neq 0 \tag{4}$$

$$x > 0, (\ln|x|)' = \left(\int_1^x \frac{1}{t}\,dt\right)' = \frac{1}{x}$$

$$x < 0, (\ln|x|)' = \left(\int_1^{-x} \frac{1}{t}\,dt\right)' = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Ex.1 **Find derivatives of the functions.**

(a) $y = \ln 2x.$

(b) $y = \ln(x^2 + 3).$

(c) $y = \sqrt{\ln(1 + x^2)}.$

Solution (a) $y' = \dfrac{2}{2x} = \dfrac{1}{x}, x > 0.$

(b) $y' = \dfrac{2x}{x^2 + 3}.$

(c) $y' = \dfrac{1}{2\sqrt{\ln(1 + x^2)}} \cdot \dfrac{2x}{1 + x^2}.$

对数函数的性质

THEOREM 2—Algebraic Properties of the Natural Logarithm For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln bx = \ln b + \ln x$
2. *Quotient Rule*: $\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ For r rational

$$(\ln bx)' = \frac{b}{bx} = \frac{1}{x} \qquad (\ln b + \ln x)' = \frac{1}{x} \qquad \ln bx = \ln b + \ln x + C$$

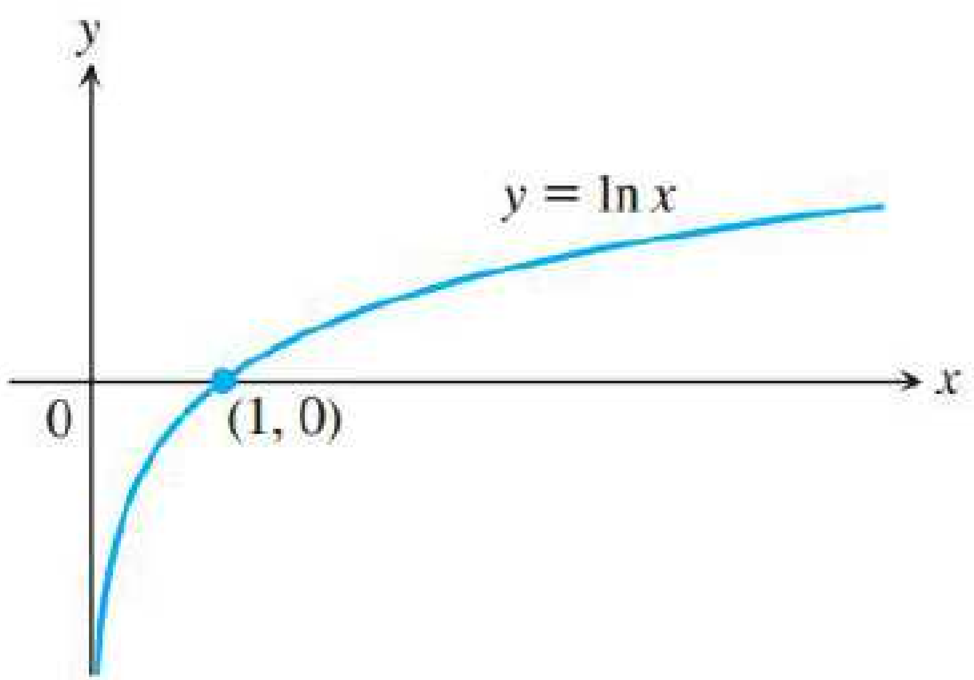
令$x = 1$,则$C = 0$ $\qquad \ln bx = \ln b + \ln x$

$\ln x$的图像当 $x\to\infty$时是递增的，

但上有界吗？下有界吗 ？

$$\ln 2^n = n\ln 2 \quad \to \infty \qquad \lim_{x\to\infty}\ln x = \infty.$$

$$\lim_{x\to 0^+}\ln x = \lim_{t\to\infty}\ln\frac{1}{t} = -\lim_{t\to\infty}\ln t = -\infty.$$



If u is a differentiable function that is never zero,

$$\int \frac{1}{u}\,du = \ln|u| + C. \qquad (3)$$

$$\int \frac{1}{f(x)}\,df(x) = \ln|f(x)| + C.$$

Ex.3 **Evaluate an integral of** $\int_{-\pi/2}^{\pi/2} \frac{4\cos x}{3+2\sin x}\,dx.$

Solution $\int_{-\pi/2}^{\pi/2} \frac{4\cos x}{3+2\sin x}\,dx = 2\int_{-\pi/2}^{\pi/2} \frac{1}{3+2\sin x}\,d(3+2\sin x)$

$$= 2\ln|3+2\sin x|\Big|_{-\pi/2}^{\pi/2} = 2\ln 5 - 2\ln 1 = 2\ln 5.$$

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan u\,du = \ln|\sec u| + C \qquad \int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{\cos x} d\cos x = -\ln|\cos x| + C$$

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{1}{\sin x} d\sin x = \ln|\sin x| + C$$

$$\int \sec x dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{1}{\sec x + \tan x} d(\sec x + \tan x) = \ln|\sec x + \tan x| + C$$

Ex.4 Evaluate integral $\int_0^{\pi/6} \tan 2x dx$.

Solution $\int_0^{\pi/6} \tan 2x dx = \frac{1}{2}\int_0^{\pi/6} \tan 2x d2x$

$$= -\frac{1}{2}\ln|\cos 2x|\Big|_0^{\pi/6} = -\frac{1}{2}(\ln\frac{1}{2} - \ln 1) = \frac{1}{2}\ln 2$$

Ex.5 Find the derivative of $y = \dfrac{(x^2+1)\sqrt{x+3}}{x-1}, \quad x > 1.$

Solution $\ln y = \ln(x^2+1) + \dfrac{1}{2}\ln(x+3) - \ln(x-1),$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}$$

$$\frac{dy}{dx} = y\left[\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}\right].$$

Find $\dfrac{dy}{dx}$, $y = \dfrac{(x^2+1)\sqrt{x+3}}{x-1}, -3 < x < 1.$

$$y = -\frac{(x^2+1)\sqrt{x+3}}{1-x}, -3 < x < 1,$$

$$-y = \frac{(x^2+1)\sqrt{x+3}}{1-x}, -3 < x < 1,$$

$$\ln(-y) = \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(1-x),$$

$$\frac{(-1)}{(-y)}\frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{-1}{(1-x)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}$$

$$\frac{dy}{dx} = y[\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}].$$

Ex.6 Find $\frac{dy}{dx}$, $y = x^{\sin x}$, $x > 0$.

Solution $\ln y = \sin x \ln x$, $x > 0$.

$$\frac{1}{y} y' = \cos x \ln x + \frac{\sin x}{x}, \quad x > 0.$$

$$y' = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right), \quad x > 0.$$

Ex.7 Show that the function $y=(1+x)^{\frac{1}{x}}\,(x>0)$ is decreasing.

Proof $\ln y=\dfrac{\ln(1+x)}{x},\quad x>0.$

$$\frac{1}{y}y'=\frac{\dfrac{x}{1+x}-\ln(1+x)}{x^2}=\frac{x-(1+x)\ln(1+x)}{x^2(1+x)},$$

Let $g(x)=x-(1+x)\ln(1+x),\qquad g'(x)=-\ln(1+x)<0,$

$\Rightarrow g(x)$ is decreasing, if $x>0$, then $g(x)<g(0)=0.$

so $y'=y\dfrac{x-(1+x)\ln(1+x)}{x^2(1+x)}<0,$

所以当$x>0$时，$y(x)$单调递减.

7.3

Exponential Functions

$y = \ln x.$

定义域$(0,\infty)$，值域$(-\infty,\infty)$，

严格递增，连续可导， 故有反函数.

反函数记为 $y = \exp(x)$

定义域$(-\infty,\infty)$，值域$(0,\infty)$，

严格递增，连续可导.

$\because 1 = \ln e, \therefore \exp(1) = e$

$\because 0 = \ln 1, \therefore \exp(0) = 1.$

对于任何有理数 r, $e^r > 0$,

$\because \ln e^r = r, \quad \therefore \exp(r) = e^r.$

$$e^x = \exp(x).$$

$$\lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to-\infty} e^x = 0.$$

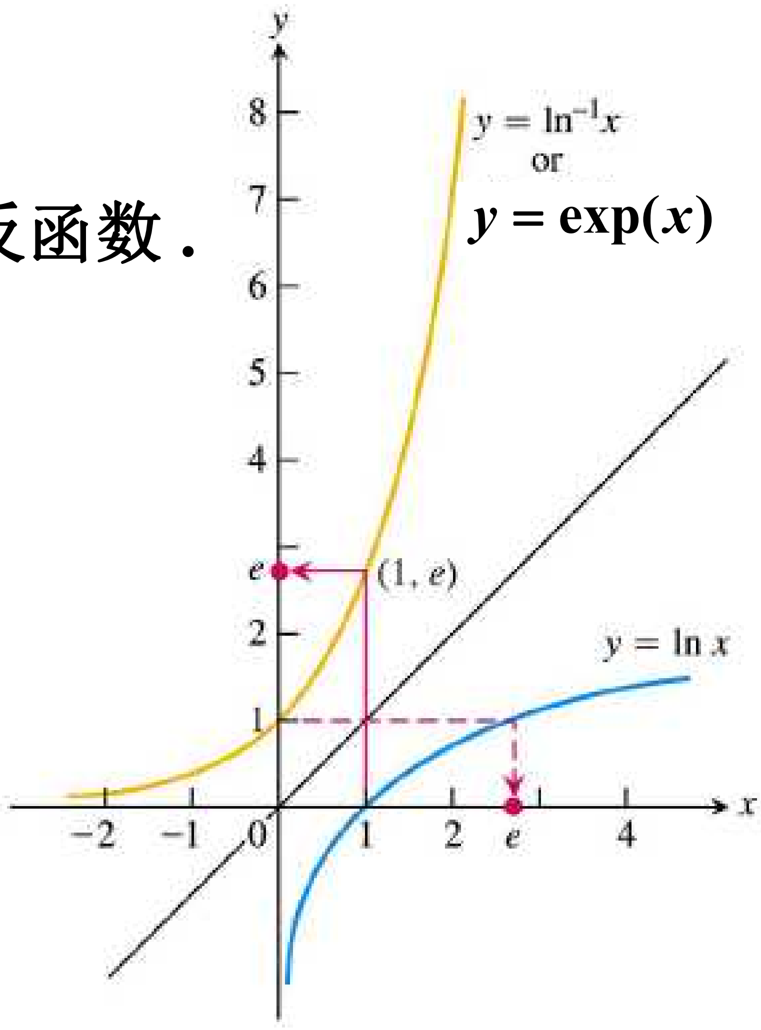


FIGURE 7.10 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

DEFINITION For every real number x, we define the **natural exponential function** to be $e^x = \exp x$.

$$e^0 = \exp(0) = 1$$

$$e^1 = \exp(1) = e$$

$$e^{r_1 + r_2} = \exp(r_1 + r_2) = e^{r_1} \cdot e^{r_2} \ ?$$

THEOREM 3 For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^{x_1})^r = e^{r x_1}$, if r is rational

Let $y_1 = e^{x_1}, y_2 = e^{x_2}$, then $\ln y_1 = x_1, \ln y_2 = x_2$,

$\ln y_1 + \ln y_2 = x_2 + x_1$, $\ln(y_1 \cdot y_2) = x_2 + x_1$, $y_1 \cdot y_2 = e^{x_2+x_1}$,

$\ln y_1 - \ln y_2 = x_1 - x_2$, $\ln(\dfrac{y_1}{y_2}) = x_1 - x_2$, $\dfrac{y_1}{y_2} = e^{x_1-x_2}$,

Let $y = e^x$, then $\ln y = x$,

$\ln\dfrac{1}{y} = -x$, $\dfrac{1}{y} = e^{-x}$,

$\ln y = x$, $r\ln y = rx$, $\ln y^r = rx$, $y^r = e^{rx}$, $(e^x)^r = e^{rx}$,

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0)$$

$$\ln(e^x) = x \qquad (\text{all } x)$$

$$f^{-1}(f(x)) = x, \forall x \in D,$$

Let $f(x) = \ln x,$

$$f(f^{-1}(x)) = x, \forall x \in \mathrm{R}^+.$$

Let $f^{-1}(x) = e^x.$

Ex.1 Solve the equation $e^{2x-6}=4$ for x.

Solution $\ln e^{2x-6}=\ln 4$

$2x-6=2\ln 2$ $\quad$ $x=3+\ln 2.$

Ex.2 A line with slope m pass through the origin and is tangent to the graph of $y=\ln x$. Find the value of m.

Solution Let the point of tengency $(a,\ln a)$, then

$$\begin{cases} ma=\ln a, \\ m=\dfrac{1}{a} \end{cases} \qquad \begin{cases} a=e, \\ m=\dfrac{1}{e}. \end{cases}$$

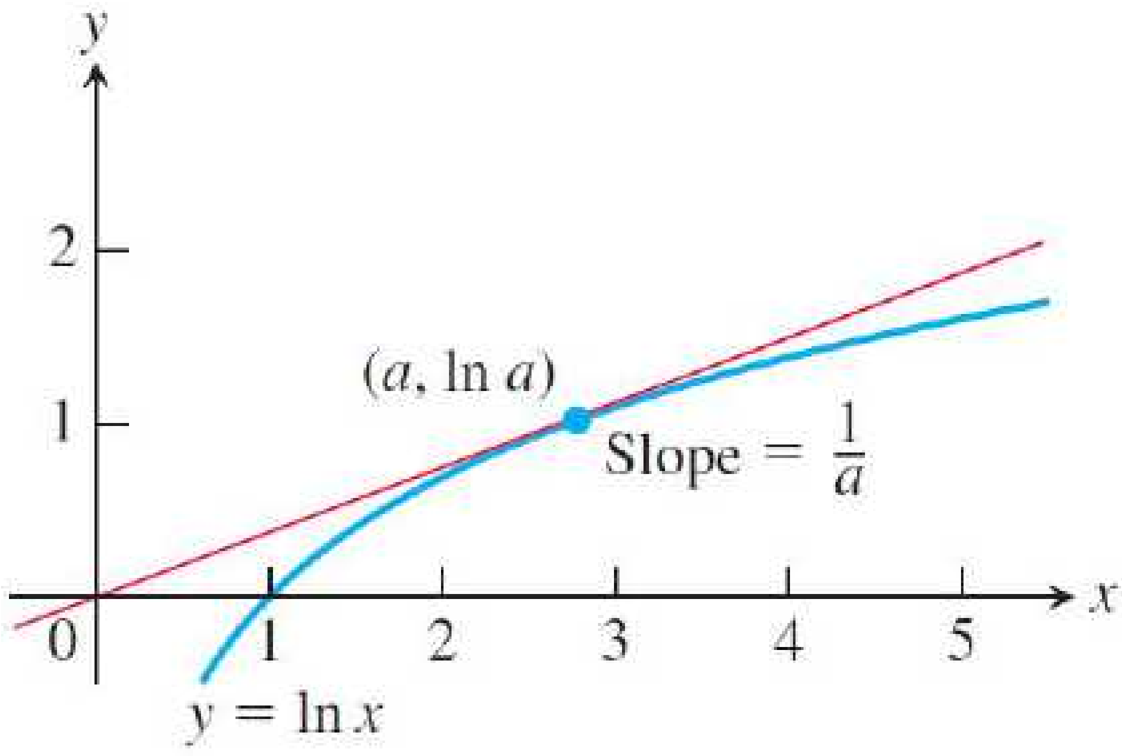


FIGURE 7.11 The tangent line intersects the curve at some point $(a, \ln a)$, where the slope of the curve is $1/a$ (Example 2).

$\because \ln e^{x} = x \qquad \therefore \dfrac{(e^{x})'}{e^{x}} = 1 \qquad (e^{x})' = e^{x}$

Ex.3 Find derivatives of the functions.

(a) $y = 5e^{2x}$. (b) $y = e^{-x}$.

(c) $y = e^{\sin x}$. (d) $y = e^{\sqrt{3x+1}}$

Solution (a) $y' = 10e^{2x}$. (b) $y' = -e^{-x}$.

(c) $y' = e^{\sin x}\cos x$. (d) $y' = e^{\sqrt{3x+1}}\dfrac{3}{2\sqrt{3x+1}}$.

The general antiderivative of the exponential function

$$\int e^u\,du = e^u + C \qquad \int e^{f(x)}df(x) = e^{f(x)} + C.$$

Ex.4 Evaluate

$$\text{(a)}\int_0^{\ln 2} e^{3x}dx. \qquad \text{(b)}\int_0^{\pi/2} e^{\sin x}\cos x\,dx.$$

Solution (a) $$\int_0^{\ln 2} e^{3x}dx = \frac{1}{3}\int_0^{\ln 2} e^{3x}d3x = \frac{1}{3}e^{3x}\Big|_0^{\ln 2} = \frac{7}{3}$$

$$\text{(b)}\int_0^{\pi/2} e^{\sin x}\cos x\,dx = \int_0^{\pi/2} e^{\sin x}d\sin x = e^{\sin x}\Big|_0^{\pi/2} = e-1.$$

$\because x = e^{\ln x} (x > 0)$ $\therefore$ $a = e^{\ln a}$ for any positive number a,

then for any rational r, $a^r = e^{r \ln a}$,

for an irrational x, a^x hasn't meaning yet.

DEFINITION For any numbers $a > 0$ and x, the **exponential function with base a** is

$$a^x = e^{x \ln a}.$$

与前类似地可以推出一般指数函数的性质.

DEFINITION For any $x > 0$ and for any real number n,

$$x^n = e^{n \ln x}.$$

General Power Rule for Derivatives

For $x > 0$ and any real number n,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $x \le 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} all exist.

$$x > 0,\ y = x^n = e^{n\ln x} \qquad y' = e^{n\ln x}\frac{n}{x} = x^n\frac{n}{x} = nx^{n-1}.$$

$$x < 0,\ y = x^n = (-1)^n(-x)^n = (-1)^n e^{n\ln(-x)}$$

$$y' = (-1)^n e^{n\ln(-x)}\frac{n}{x} = (-1)^n(-x)^n\frac{n}{x} = x^n\frac{n}{x} = nx^{n-1}.$$

$$y'\big|_{x=0} = \lim_{x\to 0}\frac{x^n - 0}{x} = \lim_{x\to 0}x^{n-1} = \begin{cases} 0. & n > 1, \\ 1, & n = 1. \end{cases}$$

Ex.5 **Differentiate** $f(x) = x^x \, (x > 0)$.

Solution $f(x) = e^{x \ln x} \, (x > 0)$.

$$f'(x) = e^{x \ln x}(\ln x + 1), (x > 0).$$

THEOREM 4—The Number e as a Limit The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

证明 已知 $\ln x\big|_{x=1} = 0, (\ln x)'\big|_{x=1} = 1$

$$1 = (\ln x)'\big|_{x=1} = \lim_{h \to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h \to 0} \frac{\ln(1+h)}{h}$$

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1 \qquad \lim_{x \to 0} \ln(1+x)^{\frac{1}{x}} = 1 \qquad \ln[\lim_{x \to 0}(1+x)^{\frac{1}{x}}] = 1$$

$$\lim_{x \to 0}(1+x)^{\frac{1}{x}} = e \qquad \lim_{x \to \infty}(1+\frac{1}{x})^{x} = e \qquad \lim_{x \to -\infty}(1+\frac{1}{x})^{x} = e$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1 \qquad \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e \qquad \lim_{x\to 0}\frac{e^{x}-1}{x}=1$$

$$\lim_{x\to 0}\frac{e^{x}-1}{x}=\lim_{u\to 0}\frac{u}{\ln(u+1)}=1$$

$$\lim(\cos x)^{\frac{1}{x^{2}}}=\lim[(1+(\cos x-1))^{\frac{1}{\cos x-1}}]^{\frac{\cos x-1}{x^{2}}}=e^{-\frac{1}{2}}$$

$$\lim_{x\to 0}\frac{\cos x-1}{x^{2}}=-\lim_{x\to 0}\frac{2\sin^{2}\frac{x}{2}}{x^{2}}=-\lim_{x\to 0}\frac{\sin^{2}\frac{x}{2}}{(\frac{x}{2})^{2}}\frac{1}{2}=-\frac{1}{2}$$

$$\lim_{x\to 0}\frac{e^{\tan x}-e^{\sin x}}{x^{3}}=\lim_{x\to 0}e^{\sin x}\frac{e^{\tan x-\sin x}-1}{\tan x-\sin x}\frac{\tan x-\sin x}{x^{3}}$$

$$=\lim_{x\to 0}\frac{\tan x\,(1-\cos x)}{x^{3}}=\frac{1}{2}$$

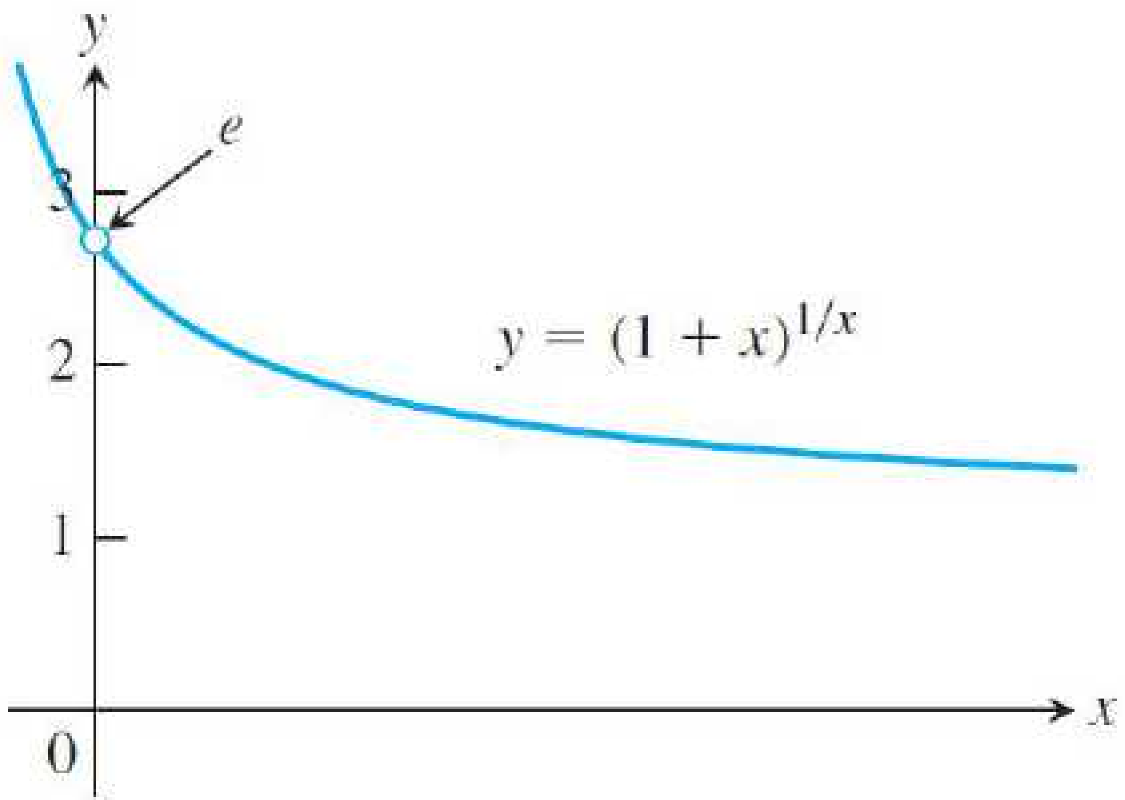


FIGURE 7.12 The number e is the limit of the function graphed here as $x \to 0$.

$$a^x = e^{x \ln a} \quad (a > 0).$$

$$(a^x)' = e^{x \ln a} \ln a = a^x \ln a.$$

$$(a^x)'' = a^x (\ln a)^2.$$

$$\boxed{\frac{\mathrm{d}(a^x)}{dx} = a^x \ln a.}$$

$$\boxed{\int a^x dx = \frac{a^x}{\ln a} + C.}$$

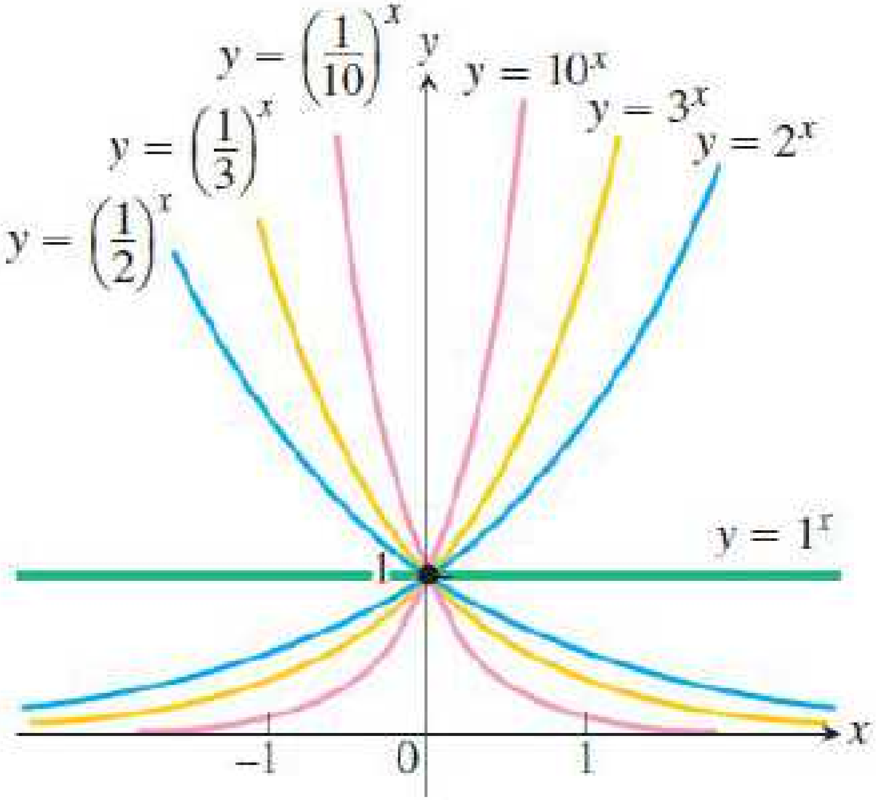


FIGURE 7.13 Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and $a^x \to \infty$ if $a > 1$. As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and $a^x \to 0$ if $a > 1$.

Ex.6 **Find derivatives and integral.**

(a) $(3^x + 2^{-x})'$. (b) $(3^{x+\sin x})'$.

(c) $\int 3^{-x}\,dx$. (d) $\int 2^{\cos x}\sin x\,dx$.

Solution (a) $(3^x + 2^{-x})' = 3^x \ln 3 - 2^{-x}\ln 2.$

(b) $(3^{x+\sin x})' = 3^{x+\sin x}(1+\cos x)\ln 3.$

(c) $\int 3^{-x}\,dx = -\int 3^{-x}\,d(-x) = -\dfrac{3^{-x}}{\ln 3} + C.$

(d) $\int 2^{\cos x}\sin x\,dx = -\int 2^{\cos x}\,d\cos x = -\dfrac{2^{\cos x}}{\ln 2} + C.$

$(2^x)' = 2^x \ln 2 > 0$ 函数具有递增的反函数

记为$y = \log_2 x$

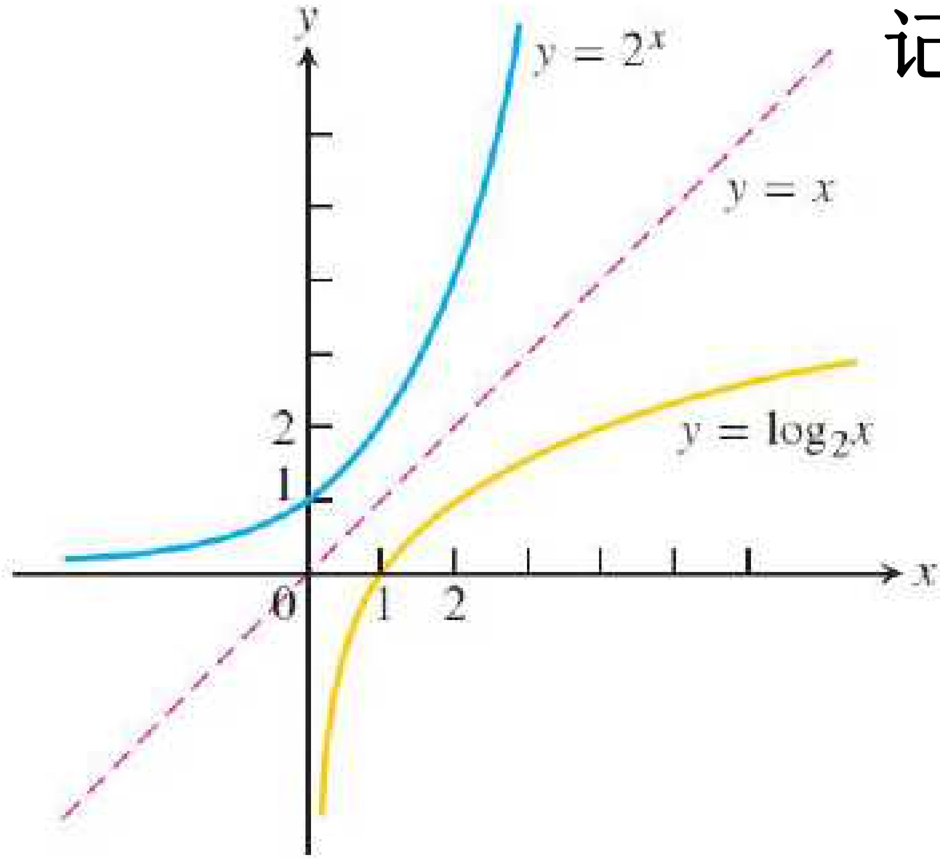


FIGURE 7.14 The graph of 2^x and its inverse, $\log_2 x$.

DEFINITION For any positive number $a \neq 1$,

$\log_a x$ is the inverse function of a^x.

对数函数和指数函数的定义过程：

定义 $\ln x = \int_1^x \frac{1}{t} dt$,

将$y = \ln x$的反函数定义为 $y = e^x$

定义一般指数函数 $y = a^x = e^{x \ln a}$

将$y = a^x$的反函数定义为一般对 数函数$y = \log_a x$

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0)$$
$$\log_a (a^x) = x \qquad (\text{all } x)$$

$$y = \log_a x (a > 0), \quad x = a^y (a > 0), \quad \ln x = y \ln a, \quad y = \frac{\ln x}{\ln a}.$$

$$\log_a x = \frac{\ln x}{\ln a}. \tag{5}$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \frac{du}{dx}$$

$$\log_a x = \frac{\ln x}{\ln a}.$$

TABLE 7.2 Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:*

   $$\log_a xy = \log_a x + \log_a y$$

2. *Quotient Rule:*

   $$\log_a \frac{x}{y} = \log_a x - \log_a y$$

3. *Reciprocal Rule:*

   $$\log_a \frac{1}{y} = -\log_a y$$

4. *Power Rule:*

   $$\log_a x^y = y \log_a x$$

Ex.7 Find derivatives and integrals.

(a) $(\log_{10}(3x+1))'$. (b) $\int \frac{\log_2(2x+1)}{2x+1}dx$.

Solution (a) $(\log_{10}(3x+1))' = (\frac{\ln(3x+1)}{\ln 10})' = \frac{3}{(3x+1)\ln 10}$.

(b) $\int \frac{\log_2(2x+1)}{2x+1}dx = \frac{1}{\ln 2}\int \frac{\ln(2x+1)}{2x+1}dx$.

$= \frac{1}{2\ln 2}\int \ln(2x+1)d\ln(2x+1) = \frac{\ln^2(2x+1)}{4\ln 2} + C$.

7.4

Exponential Change and Separable Differential Equations
指数变化与可分离变量的微分方程

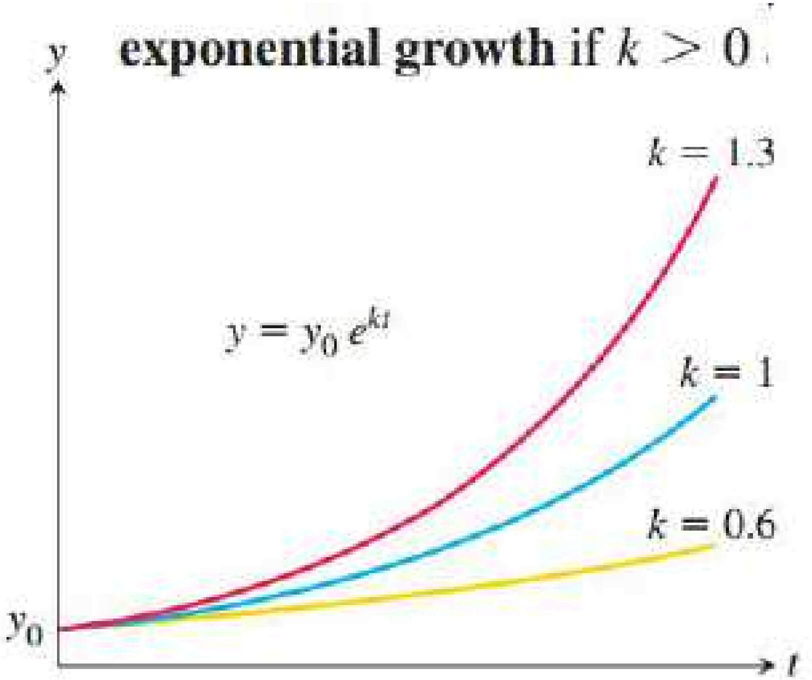
社会经济现象中一些量的变化律有类似的规律，如人口增长、放射性物质的衰变等等。

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0 \qquad\qquad y = y_0 e^{kt}$$

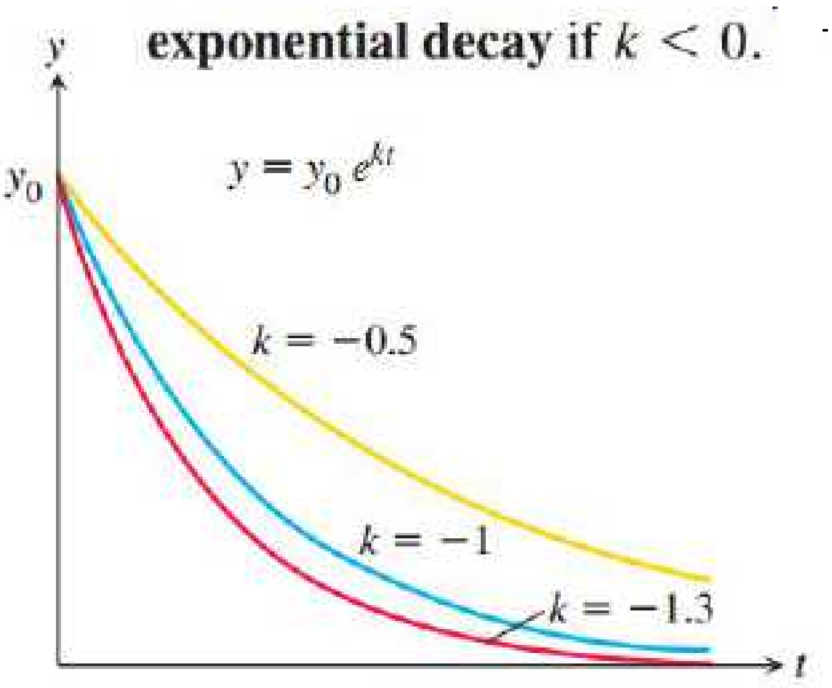
$$\frac{dy}{y} = kdt \qquad\qquad \int \frac{dy}{y} = \int kdt$$

$$\ln|y| = kt + C \qquad\qquad |y| = e^{kt+C}$$

$$y = \pm e^C e^{kt} = Ae^{kt} \qquad y_0 = y(0) = A \qquad y = y_0 e^{kt}$$


exponential growth if $k > 0$
y
$k = 1.3$
$y = y_0 e^{kt}$
$k = 1$
$k = 0.6$
y_0
t

(a)


exponential decay if $k < 0$.
y
$y = y_0 e^{kt}$
y_0
$k = -0.5$
$k = -1$
$k = -1.3$
t

(b)

Separable Differential Equations

More generally, $\frac{dy}{dx} = f(x, y),$

A solution a differentiable function $y = y(x)$

such that $\frac{d}{dx} y(x) = f(x, y(x))$

The **general solution** it always contains an arbitrary constant.

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}, \qquad h(y)\, dy = g(x)\, dx.$$

$$\int h(y) dy = \int h(y) \frac{dy}{dx} dx = \int h(y) \frac{g(x)}{h(y)} dx = \int g(x) dx$$

$$\int h(y)\, dy = \int g(x)\, dx. \qquad H(y) = G(x) + C$$

Ex.1 **Solve the differential equation**

$$\frac{dy}{dx} = (1+y)e^x, \quad y > -1.$$

Solution

$$\frac{dy}{(1+y)} = e^x dx,$$

$$\int \frac{dy}{(1+y)} = \int e^x dx,$$

$$\ln|y+1| = e^x + C.$$

$$y+1 = e^{e^x + C} = Ae^{e^x}.$$

Ex.2 **Solve the differential equation**

$$y(x+1)\frac{dy}{dx} = x(1+y^2).$$

Solution $$\frac{ydy}{1+y^2} = \frac{x}{1+x}dx,$$

$$\int \frac{ydy}{1+y^2} = \int \frac{x}{1+x}dx,$$

$$\frac{1}{2}\ln(1+y^2) = x - \ln|x+1| + C.$$

实际中有很多问题可以抽象成微分方程的

初值问题： $\dfrac{dy}{dt} = ky,\qquad y(0) = y_0\qquad y = y_0 e^{kt}$

无限制的种群增长模型

Ex.3 The biomass of a yeast cultured in an experiment is initially 29 grams. After 30 minutes the mass is 37 grams. Assuming that the equation for unlimit-ed population growth gives a good model for the growth of the yeast when the mass is below 100 grams, how long will it take for the mass to double from its initial value?

Solution $\dfrac{dy}{dt} = ky,\quad y(0) = 29\quad y = 29e^{kt}$

$$37 = 29e^{k\,30}\qquad 30k = \ln\frac{37}{29}\qquad k = \frac{1}{30}\ln\frac{37}{29} \approx 0.008118.$$

$$58 = 29e^{0.008118t}\qquad 0.008118t = \ln 2\qquad t = \frac{\ln 2}{0.008118} \approx 85.38$$

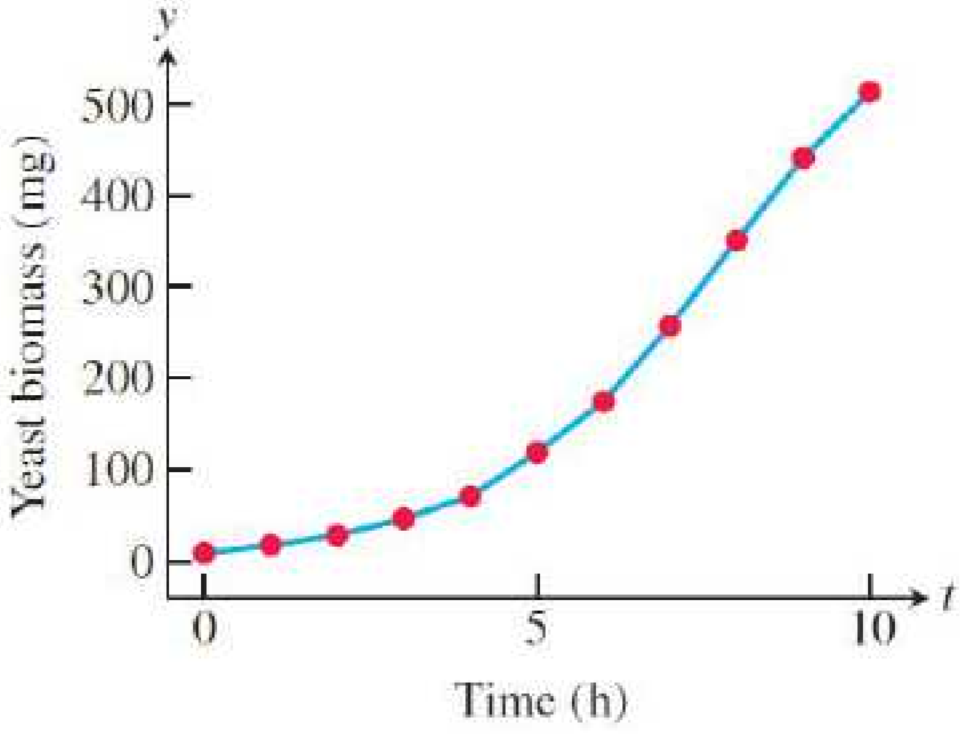


FIGURE 7.16 Graph of the growth of a yeast population over a 10-hour period, based on the data in Example 3.

Ex.4 One model for the way diseases die out when properly treated assumes that the rate dy/dt at which the number of infected people changes is proportional to the number y. The number of the people cured is proportional to the number y that are infected with the disease. Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there are 10000 cases today, how many years will it take to reduce the number to 1000?

Solution $\frac{dy}{dt} = ky, \quad y(0) = y_0 \quad y = y_0 e^{kt}$

$y_0 = 10000 \quad y = 10000 e^{kt}$

当$t=1$时，$y=10000(1-20\%)=8000$

由$y=10000e^{kt} \quad \Rightarrow 8000=10000e^{k} \Rightarrow k=\ln 0.8$

$$\therefore\ y=10000e^{t\ln 0.8}$$

$$1000=10000e^{t\ln 0.8} \qquad e^{t\ln 0.8}=0.1 \qquad t=\frac{\ln 0.1}{\ln 0.8}\approx 10.32$$

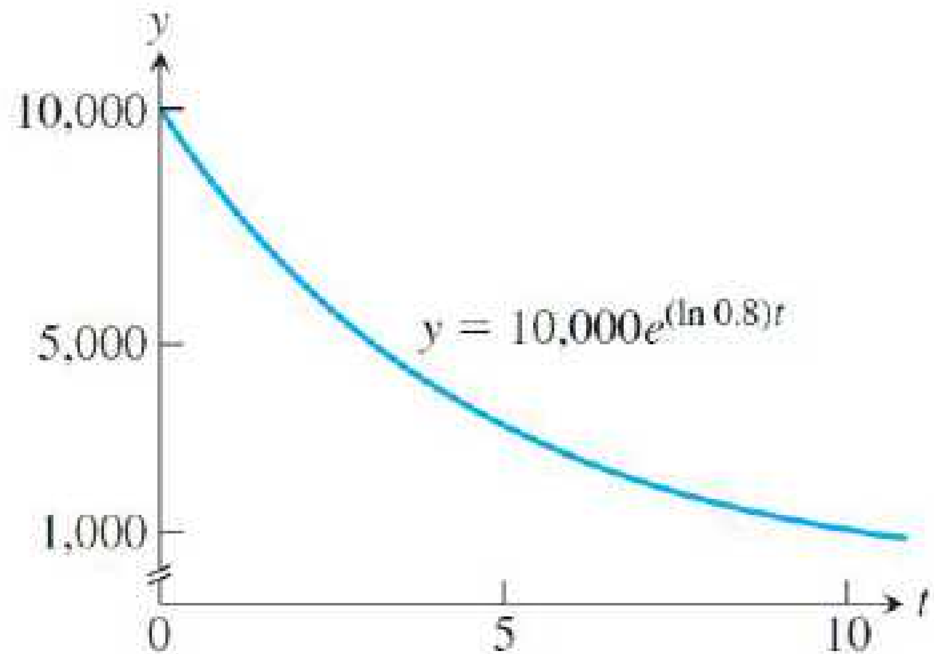


FIGURE 7.17 A graph of the number of people infected by a disease exhibits exponential decay (Example 4).

$$\frac{dy}{dt} = -ky, \quad y(0) = y_0 \quad y = y_0 e^{-kt} \, (k > 0)$$

放射性物质的衰减模型

放射性物质的半衰期----half-life

$$y_0 e^{-kt} = \frac{y_0}{2} \qquad e^{-kt} = \frac{1}{2} \qquad t = \frac{\ln 2}{k}.$$

$$\boxed{\text{Half-life} = \frac{\ln 2}{k} \qquad (7)}$$

钋 - 210: $y = y_0 e^{-5\times10^{-3} t}$ $\quad t = \dfrac{\ln 2}{5\times10^{-3}} \approx 139$天.

氡 - 222: $y = y_0 e^{-0.18t}$ $\quad t = \dfrac{\ln 2}{0.18} \approx 3.9$天.

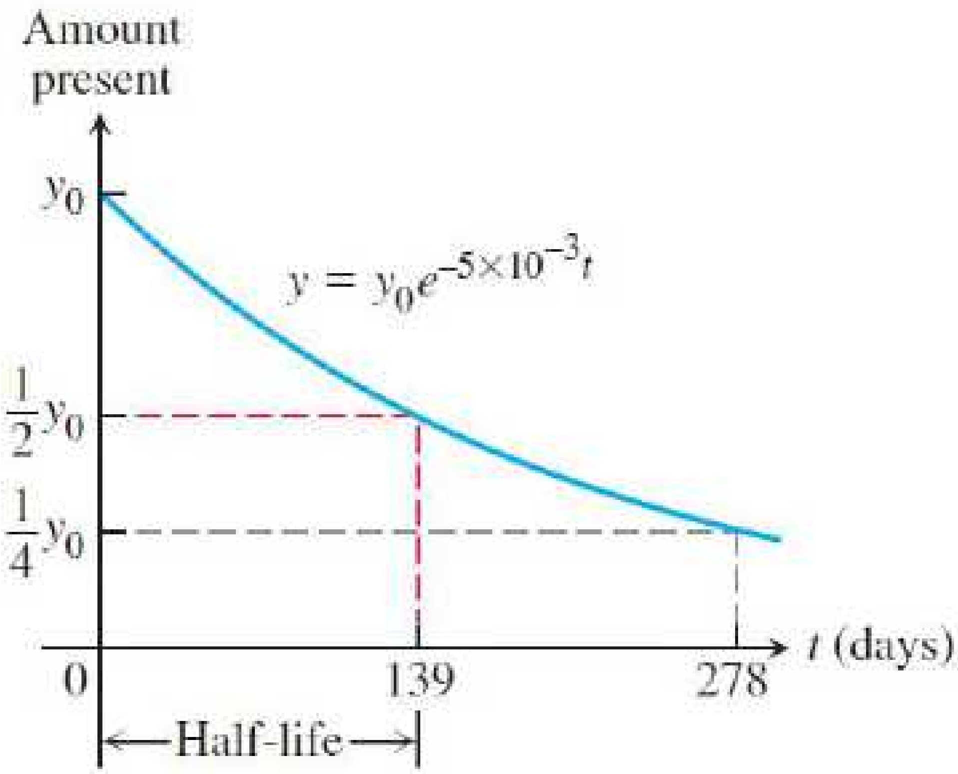


FIGURE 7.18 Amount of polonium-210 present at time t, where y_0 represents the number of radioactive atoms initially present.

Carbon-14的half-life是5730年！ $5730 = \dfrac{\ln 2}{k}.$

Ex.5 人们发现，当生物活着 时，它器官中的碳 14与普通碳的比是一个常数，但 是当生物死去后这个量 会衰减，其规律符合 $y = y_0 e^{-kt}$.现在有一个生物样品其 中的碳14只有现在活着的同种生 物的90%,问这个样品至今多久了？

Solution $y = y_0 e^{-kt}$ $\quad 0.9 y_0 = y_0 e^{-kt}$ $\quad 0.9 = e^{-kt}$

$$5730 = \frac{\ln 2}{k}, \qquad \therefore k = \frac{\ln 2}{5730},$$

$$kt = -\ln 0.9 \qquad t = -\frac{\ln 0.9}{k} = -\frac{5730 \ln 0.9}{\ln 2} \approx 871年$$

荷兰野战军保卫部门 → 戈林家中 → 维梅尔油画

梅赫伦 → 戈林家中

梅赫伦 → 梅赫伦被捕

赝品！

梅赫伦被捕 → 涉事作品为仿制品

叛国罪？ 伪造罪？

“赝品鉴定问题”

《耶稣和他的门徒》

伪作？

《织花边的女工》

仿品？

1967年，卡内基 - 梅伦大学的科研团队利 用铅元素的衰减规律证明了梅赫伦 所卖油画只有几十年的 历史，而不是将近300年.

设 t 时刻每克白铅中铅 -210的数量：$N(t)$,

$t=0$时刻每克白铅中铅 -210 的数量：N_0

每克白铅中镭 -226每分钟衰变为铅 -210的数量：R

铅 -210 含量应满足

$$\begin{cases} \dfrac{dN}{dt} = -\lambda N + R \\ N(t)\big|_{t=0} = N_0 \end{cases}$$

$$N(t)=\frac{R}{\lambda}[1-e^{-\lambda t}]+N_0e^{-\lambda t} \qquad t=\frac{1}{\lambda}\ln\frac{\lambda N_0-R}{\lambda N-R}$$

由此式可知，若 $t=300$ 年，则 λN 不超过 30000

λ 已知，$\lambda N(t)$，R 可测得

<table>
<tr><th>油画名称</th><th>测算 λN</th><th>论断</th></tr>
<tr><td>1. 耶稣和他的门徒</td><td>95053>30000</td><td>赝品</td></tr>
<tr><td>2. 濯足</td><td>157138>30000</td><td>赝品</td></tr>
<tr><td>3. 看乐谱的女人</td><td>127340>30000</td><td>赝品</td></tr>
<tr><td>4. 织花边的女工</td><td>1274</td><td rowspan="2">不是几十年内的仿品</td></tr>
<tr><td>5. 笑女</td><td>-10181</td></tr>
</table>

热传导问题：牛顿冷却定理

Ex.6 人们发现，热物体温度 为H，它在凉环境 (温度H_S)中温度降低的速度与温度 差$(H-H_S)$成比例．现在，有一个温度为 98^0 是鸡蛋放到温度为 18^0的河水中，5分钟后鸡蛋的温度是 38^0．还需多少时间可以让鸡 蛋的温度降到20^0？

Solution $\dfrac{dH}{dt}=-k(H-H_S)$ 设$y=H-H_S$, $\dfrac{dy}{dt}=-ky$,

$y=y_0e^{-kt}$, $H-H_S=(H_0-H_S)e^{-kt}$,

$38-18=(98-18)e^{-5k}$, $k=0.2\ln 4$ $H-H_S=(H_0-H_S)e^{-(0.2\ln 4)t}$,

$2=80e^{-(0.2\ln 4)t}$, $t=\dfrac{\ln 40}{0.2\ln 4}\approx 13$分

it will take about 8 min more to reach 20°C.

7.5

Indeterminate Forms and L′Hopital's Rule

未定式和罗比达法则

未定式 $\frac{0}{0}$ 型的极限

THEOREM 5—L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

等价无穷小的性质

若$\lim\limits_{x\to c} f(x)=0$,则称当$x\to c$时$f(x)$是无穷小（量）.

1.有限个无穷小的和是无穷小.

2.有限个无穷小的积是无穷小.

3.无穷小与有界变量的乘积是无穷小.

4.若$\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=1,\lim\limits_{x\to c}\dfrac{h(x)}{w(x)}=1, w(x), g(x)\neq 0$,则

$$\lim_{x\to c}\frac{f(x)}{h(x)}=\lim_{x\to c}\frac{g(x)}{w(x)}.$$

常用等价无穷小

当$x \to 0$时

$\sin x \sim x$　　$\tan x \sim x$　　$1-\cos x \sim \dfrac{x^2}{2}$

$\ln(1+x) \sim x$　　$e^x - 1 \sim x$

$(1+x)^\alpha - 1 \sim \alpha x$

例.$\displaystyle\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{\tan x(1-\cos x)}{x^3}=\lim_{x\to 0}\frac{x\frac{x^2}{2}}{x^3}=\frac{1}{2}.$

例.$\displaystyle\lim_{x\to 1}\frac{x^x-1}{x-1}=\lim_{x\to 1}\frac{e^{x\ln x}-1}{x-1}=\lim_{x\to 1}\frac{x\ln x}{x-1}=\lim_{x\to 1}\frac{x\ln(1+x-1)}{x-1}=1$

例1　求 $\lim_{x\to 0}\frac{\tan x - x}{x^3}$.　　$(\frac{0}{0})$

解 原式 $=\lim_{x\to 0}\frac{(\tan x - x)'}{(x^3)'}=\lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}=\lim_{x\to 0}\frac{\tan^2 x}{3x^2}$

$=\frac{1}{3}.$

例2　求 $\lim_{x\to 1}\frac{x^3 - 3x + 2}{x^3 - x^2 - x + 1}$.　　$(\frac{0}{0})$

解 原式 $=\lim_{x\to 1}\frac{3x^2 - 3}{3x^2 - 2x - 1}=\lim_{x\to 1}\frac{6x}{6x - 2}=\frac{3}{2}.$

例3 求 $\lim\limits_{x\to0}\dfrac{\sqrt{1+x}-1-x/2}{x^2}$. $\left(\dfrac{0}{0}\right)$

解 原式 $=\lim\limits_{x\to0}\dfrac{(1/2)(1+x)^{-1/2}-1/2}{2x}$

$$=\lim_{x\to0}\frac{-(1/2)^2(1+x)^{-3/2}}{2}=-\frac{1}{8}.$$

例4 求 $\lim\limits_{x\to0}\dfrac{x-\sin x}{x^3}$. $\left(\dfrac{0}{0}\right)$

解 原式 $=\lim\limits_{x\to0}\dfrac{1-\cos x}{3x^2}$

$$=\lim_{x\to1}\frac{\sin x}{6x}=\frac{1}{6}.$$

例5 求 $(a)\lim\limits_{x\to0^+}\dfrac{\sin x}{x^2}$, $(b)\lim\limits_{x\to0^-}\dfrac{\sin x}{x^2}$. $(c)\lim\limits_{x\to0}\dfrac{(1+x)^\alpha-1}{\alpha x}$

$(d)\lim\limits_{x\to0}\dfrac{\sqrt[m]{1+2x^2}-\sqrt[n]{1+3x^2}}{x^2}$

解 (a)原式 $=\lim\limits_{x\to0^+}\dfrac{\cos x}{2x}=\infty.$

(b)原式 $=\lim\limits_{x\to0^-}\dfrac{\cos x}{2x}=-\infty.$

(c)原式 $=\lim\limits_{x\to0}\dfrac{\alpha(x+1)^{\alpha-1}}{\alpha}=1$ 当 $x\to0,(1+x)^\alpha-1\sim\alpha x$

(d)原式 $=\lim\limits_{x\to0}\dfrac{(\sqrt[m]{1+2x^2}-1)-(\sqrt[n]{1+3x^2}-1)}{x^2}=\dfrac{2}{m}-\dfrac{3}{n}.$

未定式 $\frac{\infty}{\infty}$ 型的极限

设 (1) 当 $x \to a$ 时，函数 $f(x)$ 及 $g(x)$ 都趋于 $\pm\infty$;

(2) 在 a 点的某领域内 (点 a 本身可以除外), $f'(x)$ 及 $g'(x)$ 都存在且 $g'(x) \neq 0$;

(3) $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ 存在 (或为无穷大);

那末 $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}.$

例6 求 $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\tan x}{\tan 3x}$. $\left(\dfrac{\infty}{\infty}\right)$

解 原式 $=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sec^2 x}{3\sec^2 3x}=\dfrac{1}{3}\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos^2 3x}{\cos^2 x}$

$$=\frac{1}{3}\lim_{x\to\frac{\pi}{2}}\frac{-6\cos 3x\sin 3x}{-2\cos x\sin x}=\lim_{x\to\frac{\pi}{2}}\frac{\sin 6x}{\sin 2x}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{6\cos 6x}{2\cos 2x}=3.$$

例7 求 $\lim\limits_{x\to+\infty}\dfrac{\ln^{\alpha}x}{x^{n}}(\alpha>0,\quad n=1,2,3,\cdots).$

解 原式 $=\lim\limits_{x\to+\infty}\dfrac{\alpha\ln^{\alpha-1}x\frac{1}{x}}{nx^{n-1}}$

$x\to+\infty$时，$\ln^{\alpha}x \ll x^{n}$

$$=\lim_{x\to+\infty}\frac{\alpha\ln^{\alpha-1}x}{nx^{n}}$$

$$=\lim_{x\to+\infty}\frac{\alpha(\alpha-1)\ln^{\alpha-2}x}{n^{2}x^{n}}=\cdots\quad=0.$$

例8 求 $\lim\limits_{x\to+\infty}\dfrac{a^x}{x^{100}}(a>1)$.

解 原式 $=\lim\limits_{x\to+\infty}\dfrac{a^x\ln a}{100x^{99}}$

$x\to+\infty$时，$x^n << a^x$

$$=\lim_{x\to+\infty}\frac{a^x(\ln a)^2}{100\cdot 99x^{98}}$$

$$=\cdots=\lim_{x\to+\infty}\frac{a^x(\ln a)^{100}}{100!}=\infty.$$

$0\cdot\infty,\infty-\infty,0^0,1^\infty,\infty^0$型未定式解法

$0\cdot\infty$ 型

例9 求 $\lim\limits_{x\to 0^+} x^2 \ln x.$

解 原式 $= \lim\limits_{x\to 0^+} \dfrac{\ln x}{\dfrac{1}{x^2}} = \lim\limits_{x\to +0} \dfrac{\dfrac{1}{x}}{-\dfrac{2}{x^3}} = \lim\limits_{x\to +0} \dfrac{x^2}{-2} = 0.$

一般地，$\lim\limits_{x\to 0^+} x^\alpha \ln x = 0$（$\alpha > 0$）.

$\infty-\infty$ 型

例10 求 $\lim\limits_{x\to0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$.

解 原式 $=\lim\limits_{x\to0}\frac{x-\sin x}{x\cdot\sin x}\quad=\lim\limits_{x\to0}\frac{x-\sin x}{x^2}$

$$=\lim_{x\to0}\frac{1-\cos x}{2x}=0.$$

$0^0, 1^\infty, \infty^0$ 型

If $\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

例11 求 $\lim\limits_{x\to 0^+} x^x$. (0^0)

解 原式 $= \lim\limits_{x\to 0^+} e^{x\ln x} = e^{\lim\limits_{x\to 0^+} x\ln x} = e^{\lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}}$

$$= e^{\lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}} = e^0 = 1.$$

例12 求 $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$. (1^∞)

解 原式 $=\lim\limits_{x\to 0} e^{\frac{\ln(\cos x)}{x^2}} = e^{\lim\limits_{x\to 0}\frac{-\sin x}{2x\cos x}} = e^{-\frac{1}{2}}$

例13 求 $\lim\limits_{x\to 0^+}(\cot x)^{\frac{1}{\ln x}}$. (∞^0)

解 $(\cot x)^{\frac{1}{\ln x}} = e^{\frac{1}{\ln x}\cdot\ln(\cot x)}$,

$$\because \lim_{x\to 0^+}\frac{1}{\ln x}\cdot\ln(\cot x) = \lim_{x\to 0^+}\frac{-\dfrac{1}{\cot x}\cdot\dfrac{1}{\sin^2 x}}{\dfrac{1}{x}}$$

$$=\lim_{x\to 0^+}\frac{-x}{\cos x\cdot\sin x} = -1,\quad \therefore \text{原式} = e^{-1}.$$

THEOREM 5—L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

THEOREM 6—Cauchy's Mean Value Theorem Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

证 $g(a)\neq g(b)$, 否则$\exists c\in(a,b)$, 使$g'(c)=0$.

设函数$F(x)=f(x)-\dfrac{f(b)-f(a)}{g(b)-g(a)}g(x)$

在$[a,b]$满足罗尔定理的条件,

则在(a,b)内至少存在一点c, 使得 $F'(c)=0$.

即 $f'(c)-\dfrac{f(b)-f(a)}{g(b)-g(a)}\cdot g'(c)=0$, $\therefore \dfrac{f(b)-f(a)}{g(b)-g(a)}=\dfrac{f'(c)}{g'(c)}$.

证 $\dfrac{f(x)}{g(x)}=\dfrac{f(x)-f(a)}{g(x)-g(a)}=\dfrac{f'(c)}{g'(c)}$

当$x\to a$时,$c\to a$, $\boxed{\therefore \lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{c\to a}\dfrac{f'(c)}{g'(c)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}.}$

例14 求 $\lim\limits_{x\to\infty}\dfrac{x+\cos x}{x}$.

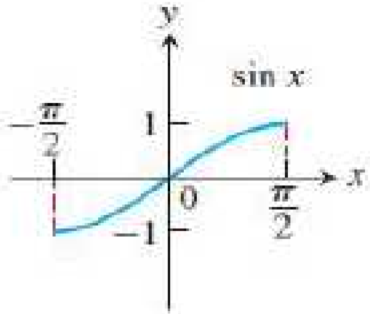
解 原式 $=\lim\limits_{x\to\infty}\dfrac{1-\sin x}{1}=\lim\limits_{x\to\infty}(1-\sin x).$

洛必达法则失效。

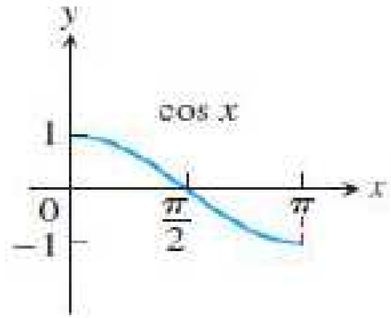
原式 $=\lim\limits_{x\to\infty}(1+\dfrac{1}{x}\cos x)=1.$

7.6

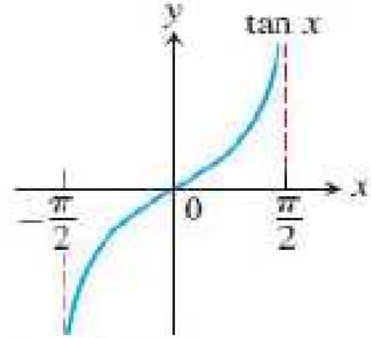
Inverse Trigonometric Functions
反三角函数



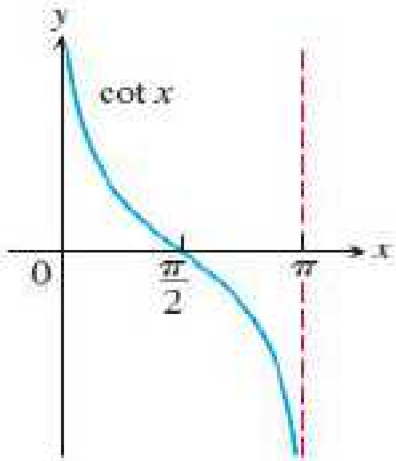
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



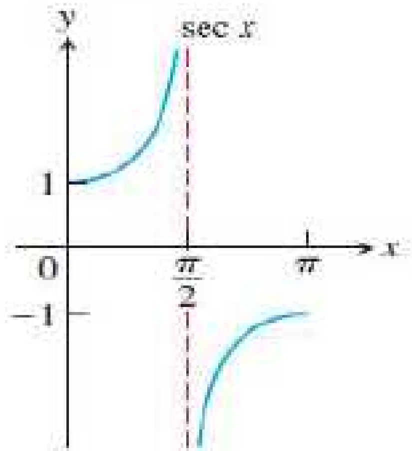
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



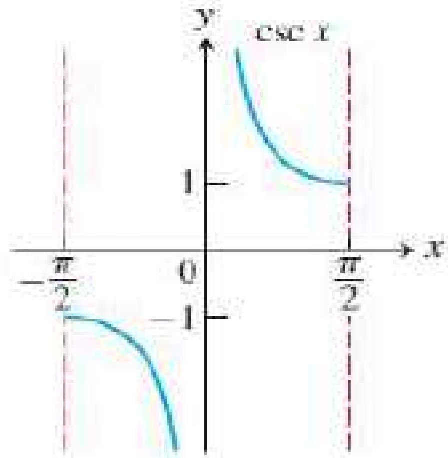
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$

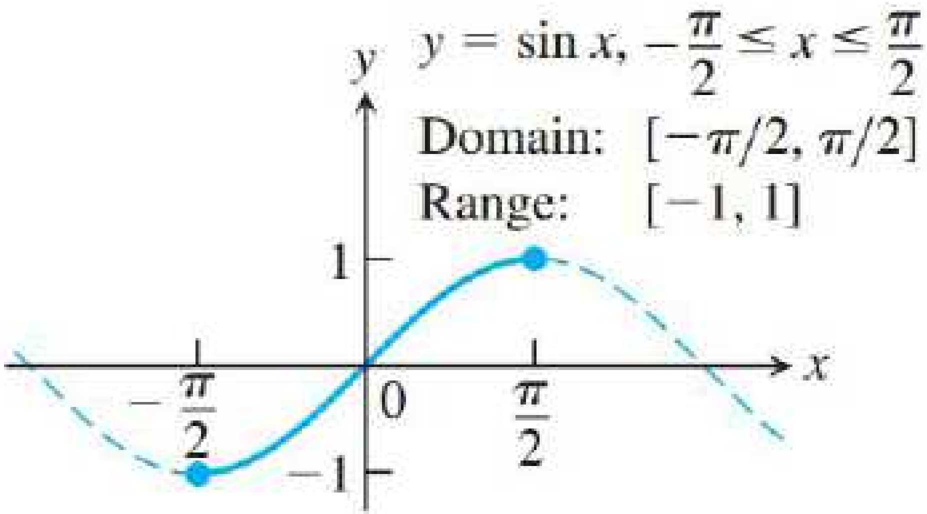


$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$

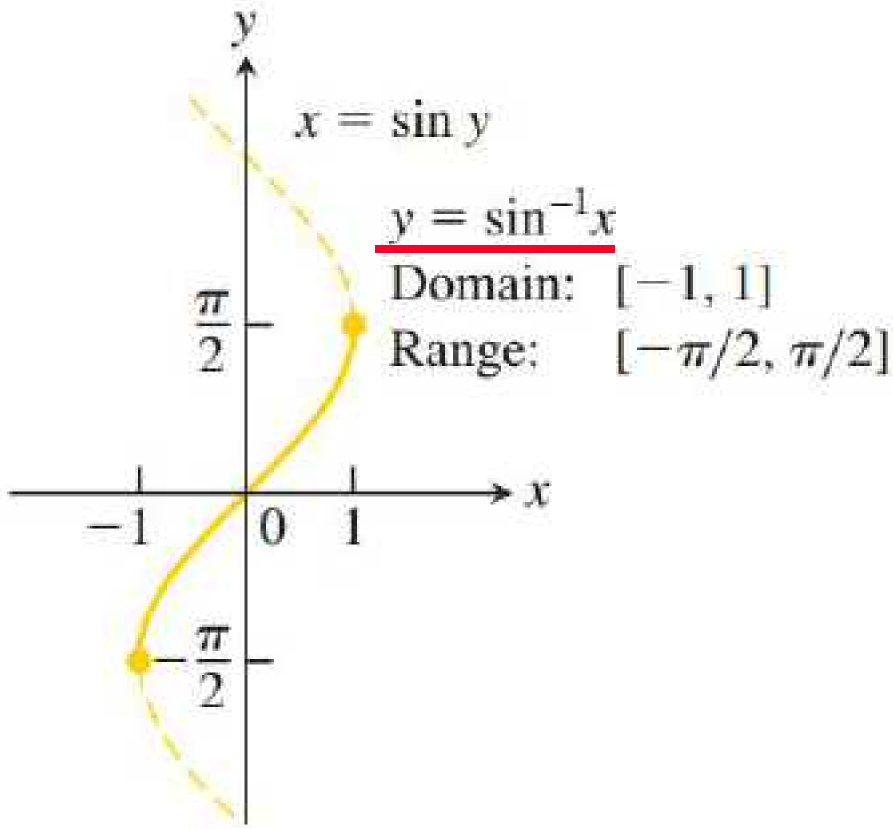


$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

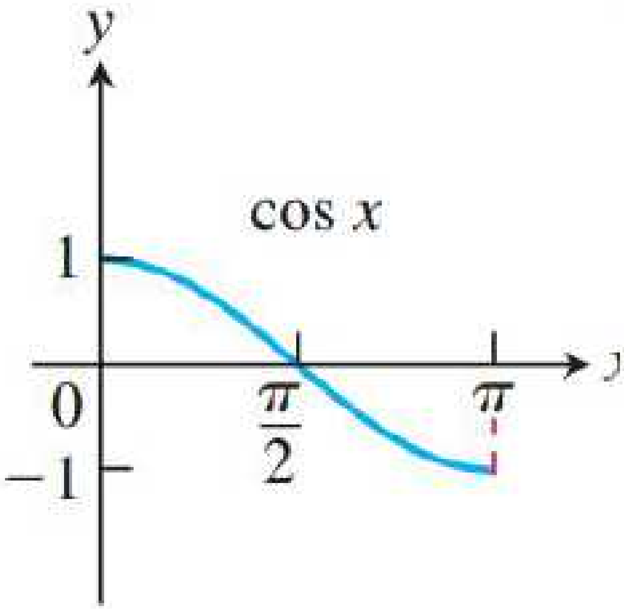
$$\sin^{-1}(-x) = -\sin^{-1} x$$



$y = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$
y
x
1
−1
0
$-\frac{\pi}{2}$
$\frac{\pi}{2}$



$x = \sin y$
$y = \sin^{-1}x$
Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$
y
x
−1
0
1
$\frac{\pi}{2}$
$-\frac{\pi}{2}$



$y = \cos x$

Domain: $[0, \pi]$

Range: $[-1, 1]$

$$y = \cos^{-1} x$$

$$x = \cos y$$

$$-x = -\cos y$$

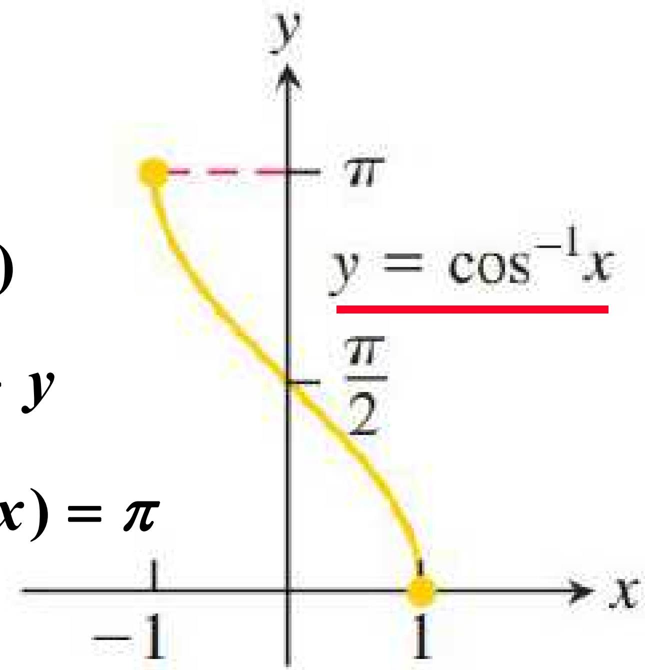
$$-x = \cos(\pi - y)$$

$$\cos^{-1}(-x) = \pi - y$$

$$\cos^{-1}(-x) + \cos^{-1}(x) = \pi$$

Domain: $-1 \le x \le 1$

Range: $0 \le y \le \pi$



$$\cos^{-1}(-x) + \cos^{-1} x = \pi$$

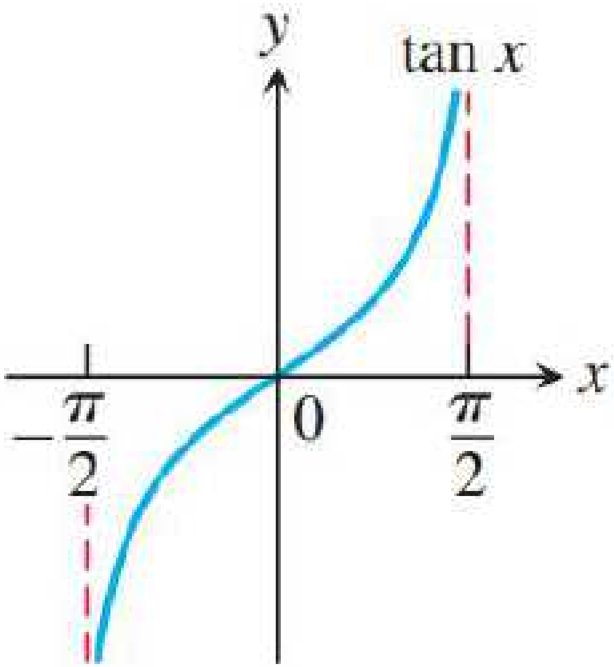
DEFINITION

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

$$\sin^{-1}(-x) = -\sin^{-1} x$$

$$\cos^{-1}(-x) + \cos^{-1} x = \pi$$



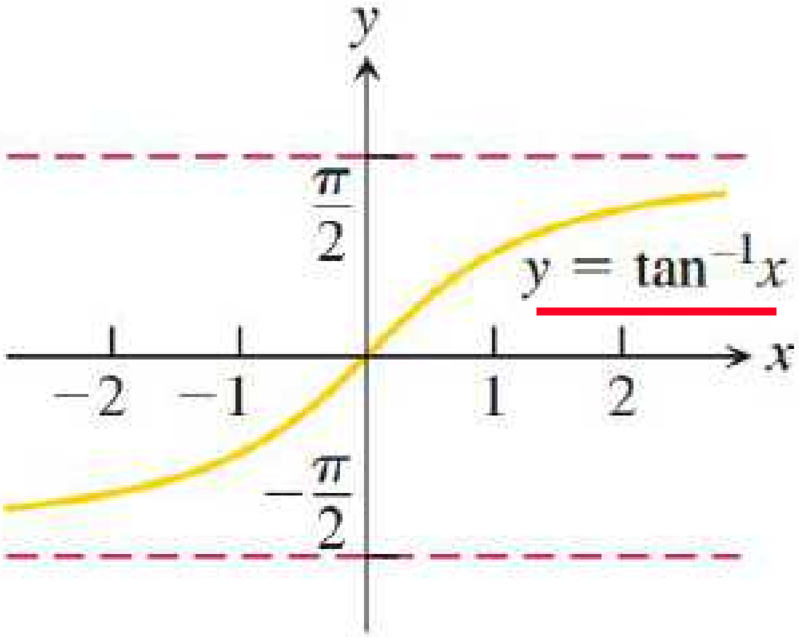
$y = \tan x$

Domain: $(-\pi/2, \pi/2)$

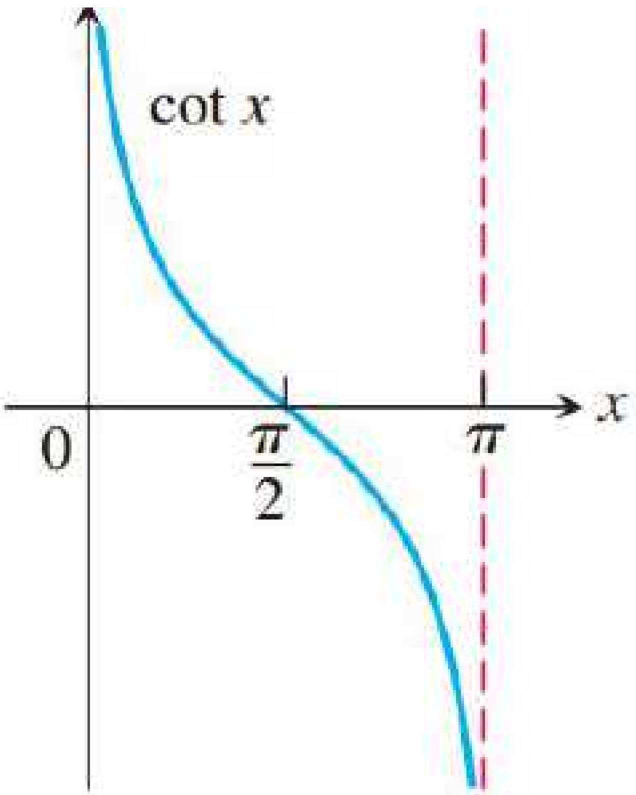
Range: $(-\infty, \infty)$

Domain: $-\infty < x < \infty$

Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

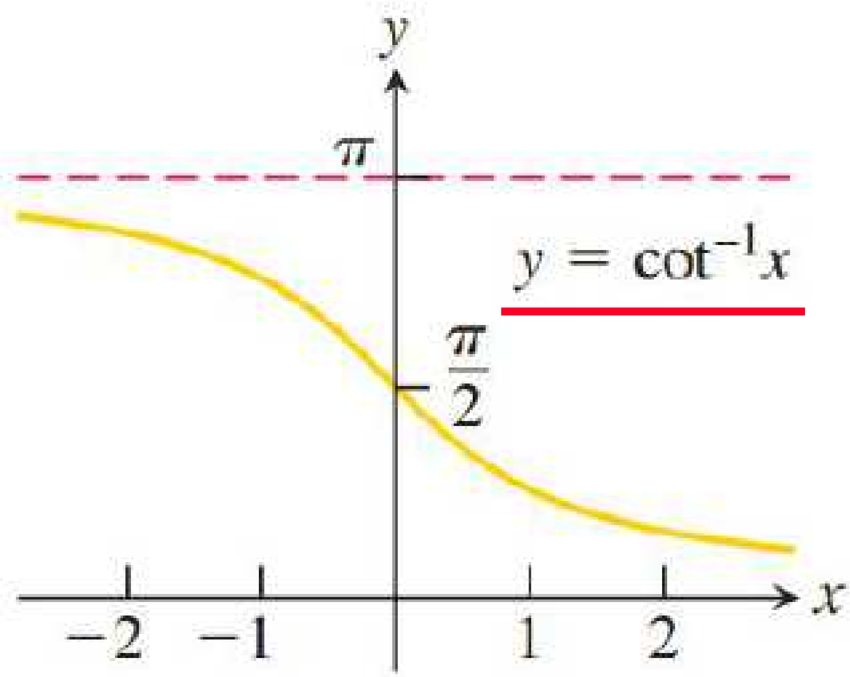


$$\tan^{-1}(-x) = -\tan^{-1} x$$

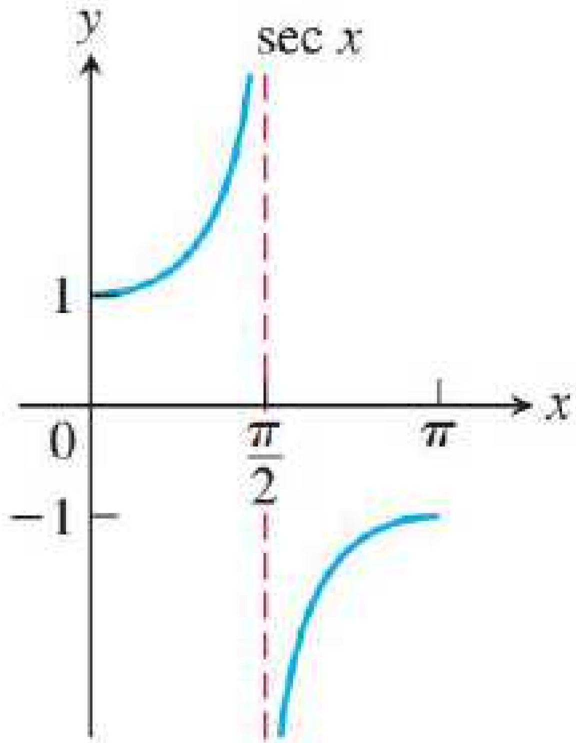


$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$

Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$



$$\cot^{-1}(-x) + \cot^{-1} x = \pi$$



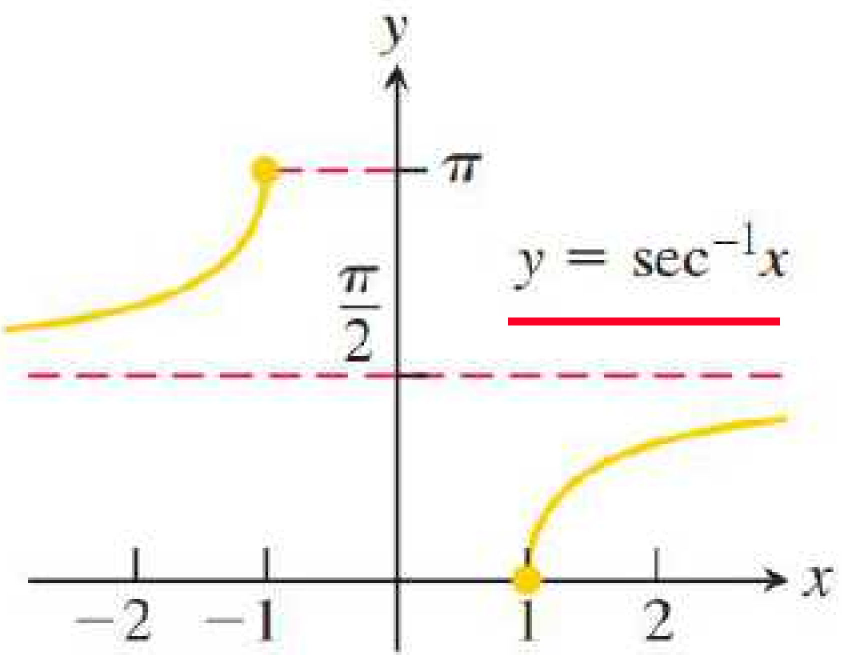
$y = \sec x$

Domain: $[0, \pi/2) \cup (\pi/2, \pi]$

Range: $(-\infty, -1] \cup [1, \infty)$

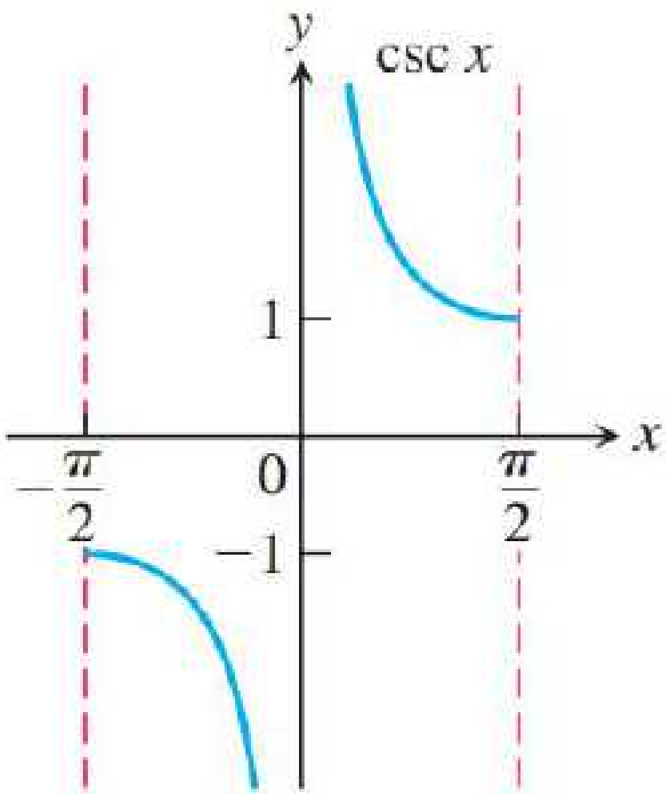
Domain: $x \le -1$ or $x \ge 1$

Range: $0 \le y \le \pi, y \ne \frac{\pi}{2}$



$$\sec^{-1} x + \sec^{-1}(-x) = \pi$$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$



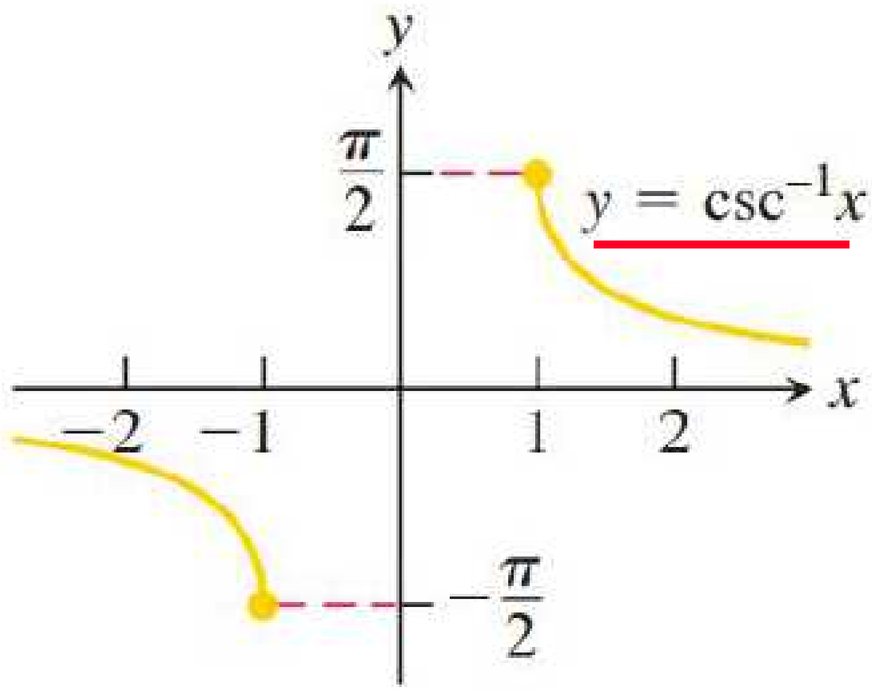
$y = \csc x$

Domain: $[-\pi/2, 0) \cup (0, \pi/$

Range: $(-\infty, -1] \cup [1, \infty)$

Domain: $x \le -1$ or $x \ge 1$

Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$



$$\csc^{-1}(-x) = -\csc^{-1}(x)$$

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$

DEFINITIONS

$y = \tan^{-1} x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$.

$y = \sec^{-1} x$ is the number in $[0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$.

$y = \csc^{-1} x$ is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$.

$$\tan^{-1}(-x) = -\tan^{-1} x \qquad \sec^{-1} x + \sec^{-1}(-x) = \pi$$

$$\cot^{-1}(-x) + \cot^{-1} x = \pi \qquad \csc^{-1}(-x) = -\csc^{-1}(x)$$

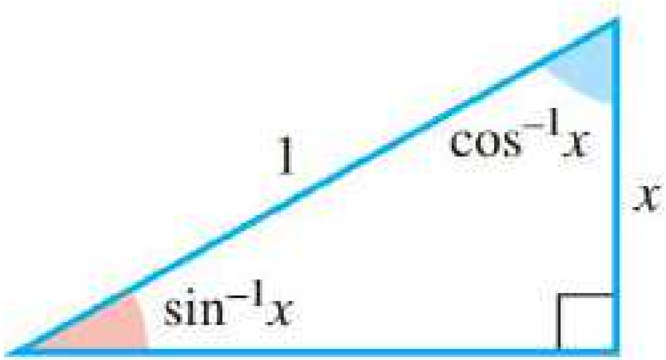
$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) \qquad \csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$

Ex.1 Evaluate (a) $\sin^{-1}(\frac{\sqrt{3}}{2})$, (b) $\cos^{-1}(-\frac{1}{2})$.

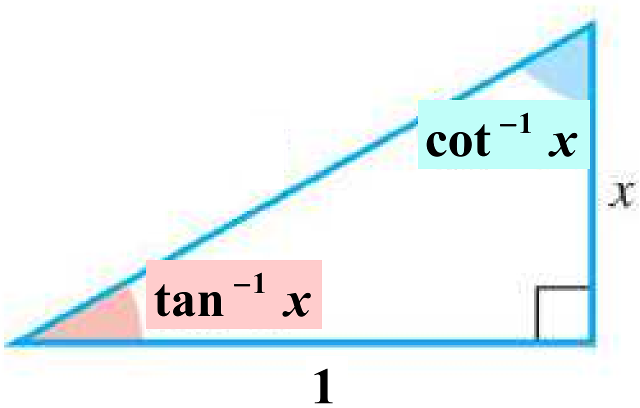
Solution (a) $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \therefore \sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$.

(b) $\cos\frac{2\pi}{3} = -\frac{1}{2}, \therefore \cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$.

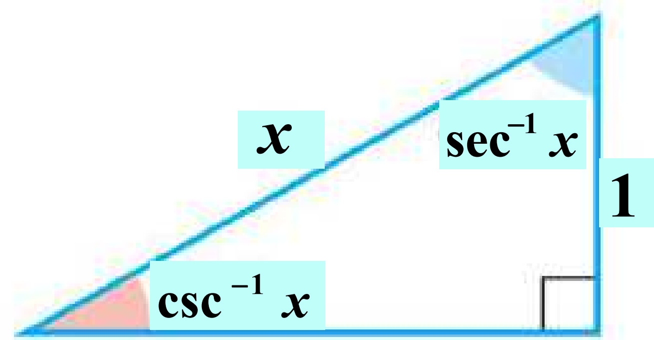
x	$\sin^{-1}x$	$\cos^{-1}x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$
	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[0, \pi]$



$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$



$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$



$$\sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}$$

Ex.3

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

$$[-\frac{\pi}{2}, \frac{\pi}{2}]$$

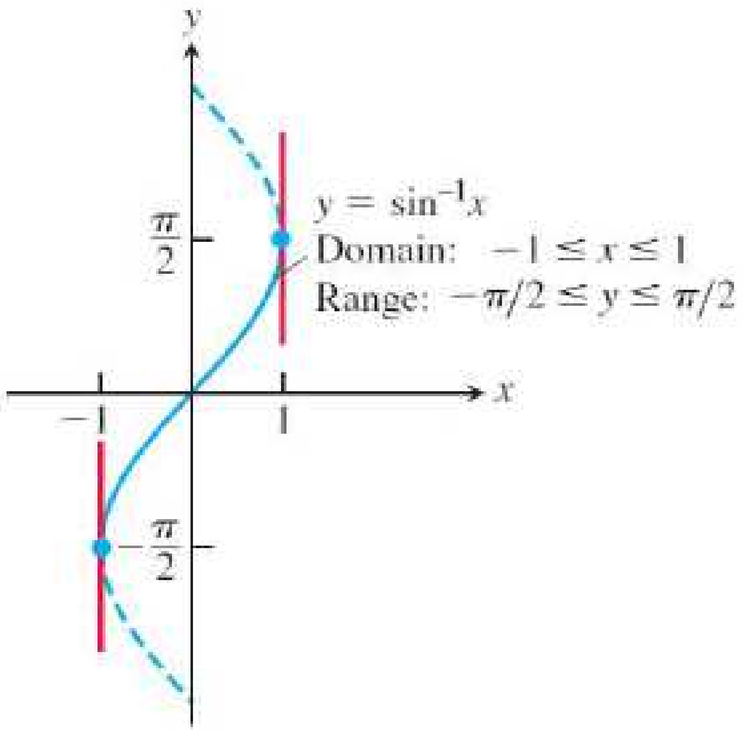


FIGURE 7.30 The graph of $y = \sin^{-1} x$ has vertical tangents at $x = -1$ and $x = 1$.

$$y = \sin^{-1} x, 求 \frac{dy}{dx}$$

$$x = \sin y,$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1 - x^2}}$$

$$(\tan^{-1} x)' = \frac{1}{(\tan y)'} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1.$$

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}.$$

Ex.3 Calculate the derivatives of

$$(a)\, y = \sin^{-1}(x^2), (b)\, y = \tan^{-1}(10x + \sin^2 x).$$

Solution $$(a)\, y' = \frac{2x}{\sqrt{1-x^4}}.$$

$$(b)\, y' = \frac{10 + \sin 2x}{1 + (10x + \sin^2 x)^2}.$$

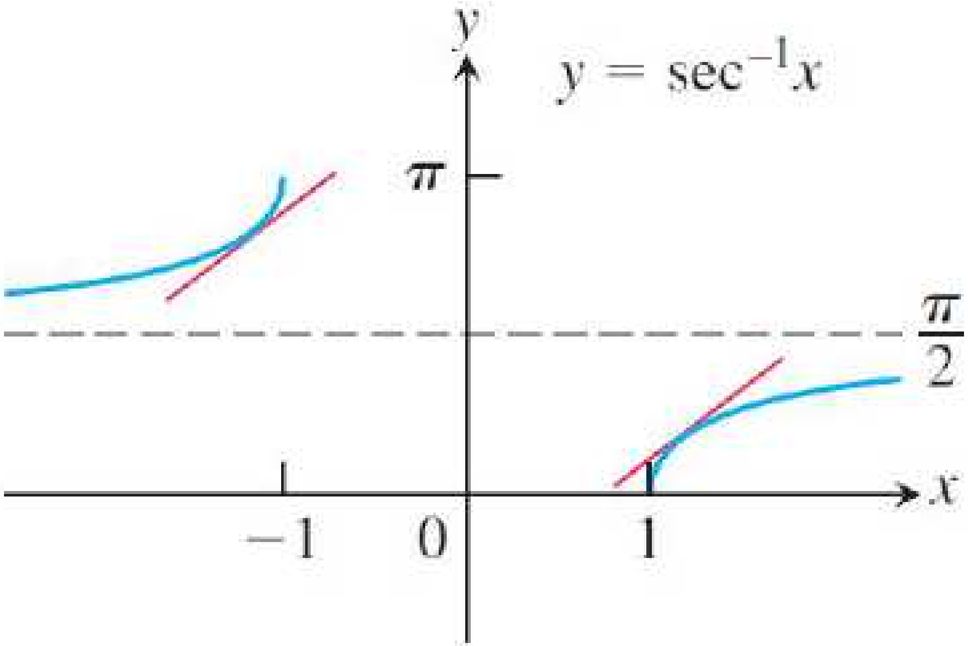


FIGURE 7.31 The slope of the curve $y = \sec^{-1}x$ is positive for both $x < -1$ and $x > 1$.

$$y=\sec^{-1}x, |x|>1,$$

$$\sec y = x,$$

两边对 x 求导， $\sec y \tan y \dfrac{dy}{dx}=1,$

$$\frac{dy}{dx}=\frac{1}{\sec y \tan y}=\pm\frac{1}{\sec y\sqrt{\sec^2 y-1}}=\pm\frac{1}{x\sqrt{x^2-1}}$$

$$(\sec^{-1}x)'=\begin{cases}\dfrac{1}{x\sqrt{x^2-1}}, & x>1\\ -\dfrac{1}{x\sqrt{x^2-1}}, & x<-1\end{cases}=\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad |u| > 1.$$

Ex.3 Calculate the derivative s of

$$(a)\, y = \sec^{-1}(5x^4), (b)\, y = \sec^{-1}\sqrt{1 + x^2}.$$

Solution $$(a)\, y' = \frac{20x^3}{5x^4\sqrt{25x^8 - 1}} = \frac{4}{x\sqrt{25x^8 - 1}}.$$

$$(b)\, y' = \frac{1}{\sqrt{1 + x^2}\sqrt{x^2}} \cdot \frac{x}{\sqrt{1 + x^2}} = \frac{x}{|x|(1 + x^2)}.$$

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

$$(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\cot^{-1} x)' = -\frac{1}{1+x^2}$$

$$(\csc^{-1} x)' = -\frac{1}{|x|\sqrt{x^2-1}}$$

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1} x + C \qquad \int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}|x| + C$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a^2}\int \frac{1}{1+\frac{x^2}{a^2}}dx = \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}d\left(\frac{x}{a}\right) = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}d\left(\frac{x}{a}\right) = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\int \frac{1}{\frac{x}{a}\sqrt{\left(\frac{x}{a}\right)^2-1}}d\left(\frac{x}{a}\right) = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \qquad \text{(Valid for } u^2 < a^2\text{)}$$

2. $$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \qquad \text{(Valid for all } u\text{)}$$

3. $$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad \text{(Valid for } |u| > a > 0\text{)}$$

例6 计算

$$(a)\int_{\sqrt{2}/2}^{\sqrt{3}/2}\frac{1}{\sqrt{1-x^2}}dx = \arcsin x\Big|_{\sqrt{2}/2}^{\sqrt{3}/2} = \frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}$$

$$(b)\int\frac{1}{\sqrt{3-4x^2}}dx=\frac{1}{2}\int\frac{1}{\sqrt{(\sqrt{3})^2-(2x)^2}}d(2x)$$

$$=\frac{1}{2}\arcsin(\frac{2x}{\sqrt{3}})+C.$$

$$(c)\int\frac{1}{\sqrt{e^{2x}-6}}dx=\int\frac{e^x}{e^x\sqrt{(e^x)^2-(\sqrt{6})^2}}dx$$

$$=\int\frac{1}{e^x\sqrt{(e^x)^2-(\sqrt{6})^2}}d(e^x) = \frac{1}{\sqrt{6}}\sec^{-1}(\frac{e^x}{\sqrt{6}})+C.$$

例7　计算

$$(a)\int \frac{1}{\sqrt{4x-x^2}}dx = \int \frac{1}{\sqrt{2^2-(x-2)^2}}d(x-2)$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right)+C.$$

$$(b)\int \frac{1}{4x^2+4x+2}dx = \frac{1}{4}\int \frac{1}{x^2+x+1/2}dx$$

$$= \frac{1}{4}\int \frac{1}{(x+1/2)^2+(1/2)^2}dx = \frac{1}{2}\tan^{-1}(2x+1)+C.$$

7.7

Hyperbolic Functions

双曲函数

Definition

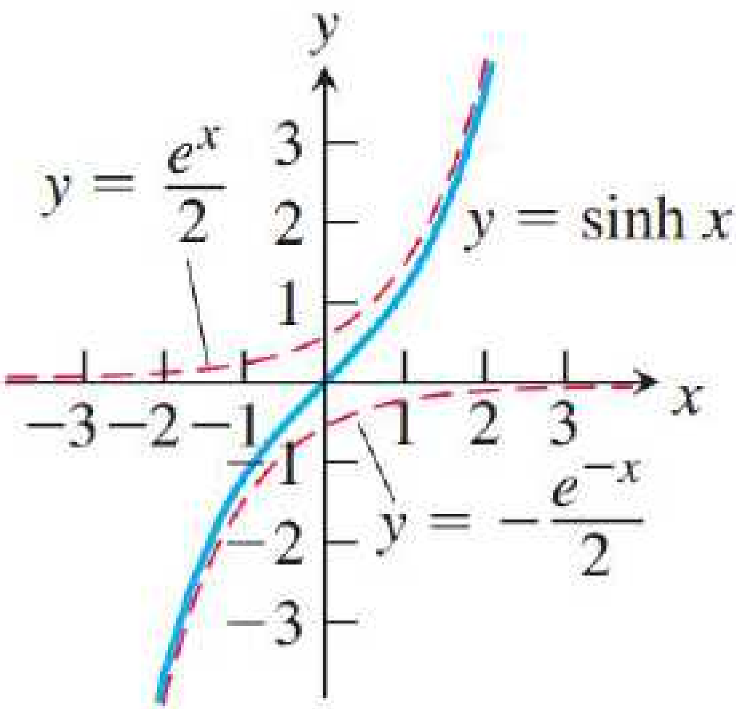
双曲正弦 $\sinh x = \dfrac{e^x - e^{-x}}{2}$　　双曲余弦 $\cosh x = \dfrac{e^x + e^{-x}}{2}$

双曲正切 $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

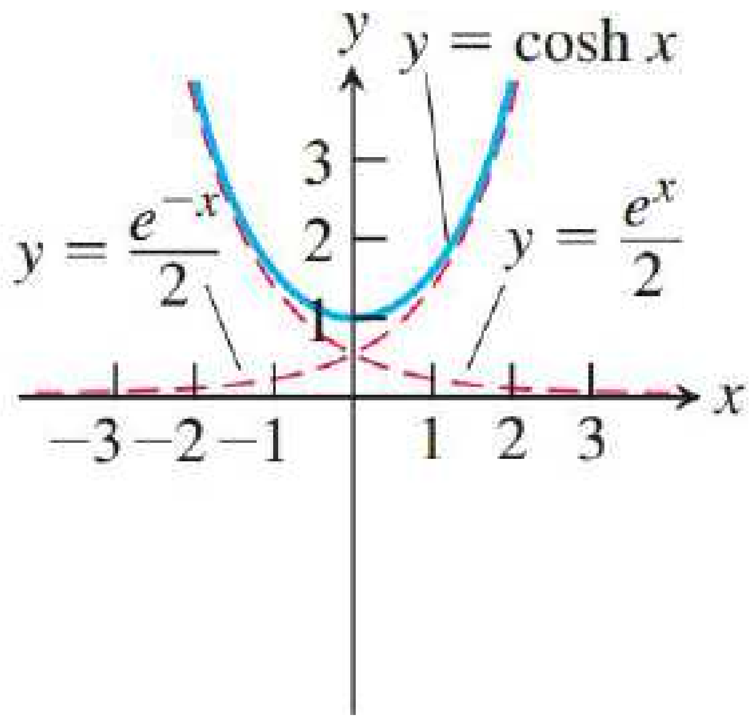
双曲余切 $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

双曲正割 $\mathrm{sech}\, x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

双曲余割 $\mathrm{csch}\, x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$



(a)



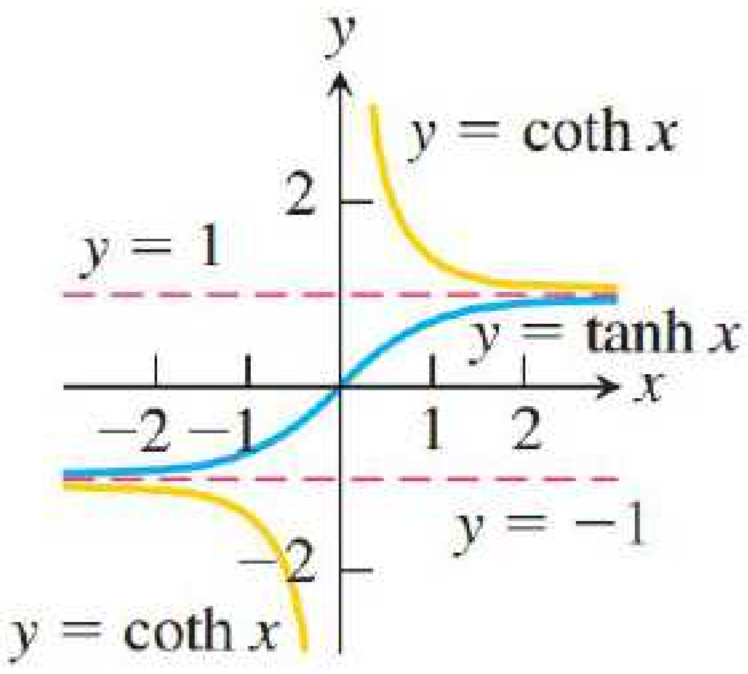
(b)

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



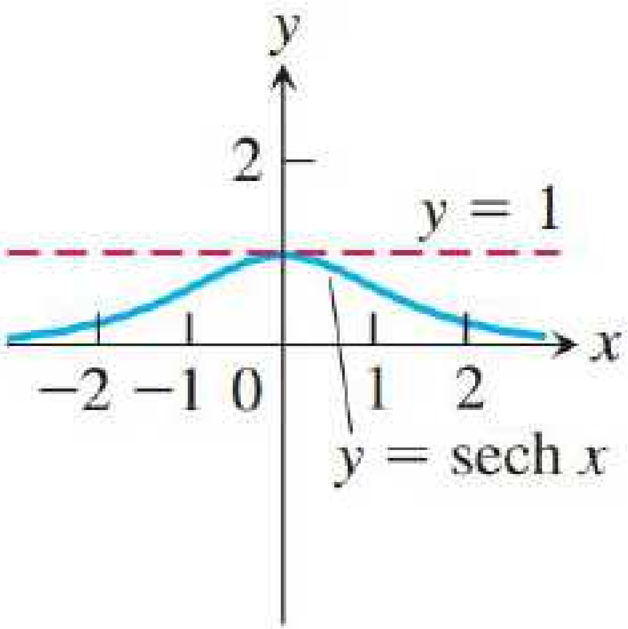
(c)

Hyperbolic tangent:

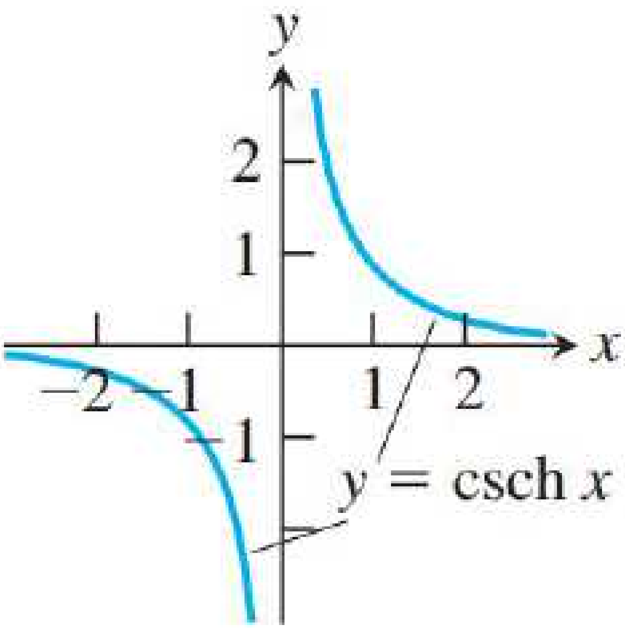
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)



(e)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

基本恒等式

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

求导公式和积分公式

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u \, du = \cosh u + C$$

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

例1 计算导数和积分

$$(a)(\tanh\sqrt{1+t^2})' = \mathrm{sech}^2\sqrt{1+t^2} \cdot \frac{t}{\sqrt{1+t^2}}$$

$$(b)\int \coth 5x dx = \frac{1}{5}\int \frac{\cosh 5x}{\sinh 5x} d(5x) = \frac{1}{5}\int \frac{1}{\sinh 5x} d(\sinh 5x)$$

$$= \frac{1}{5}\ln|\sinh 5x| + C.$$

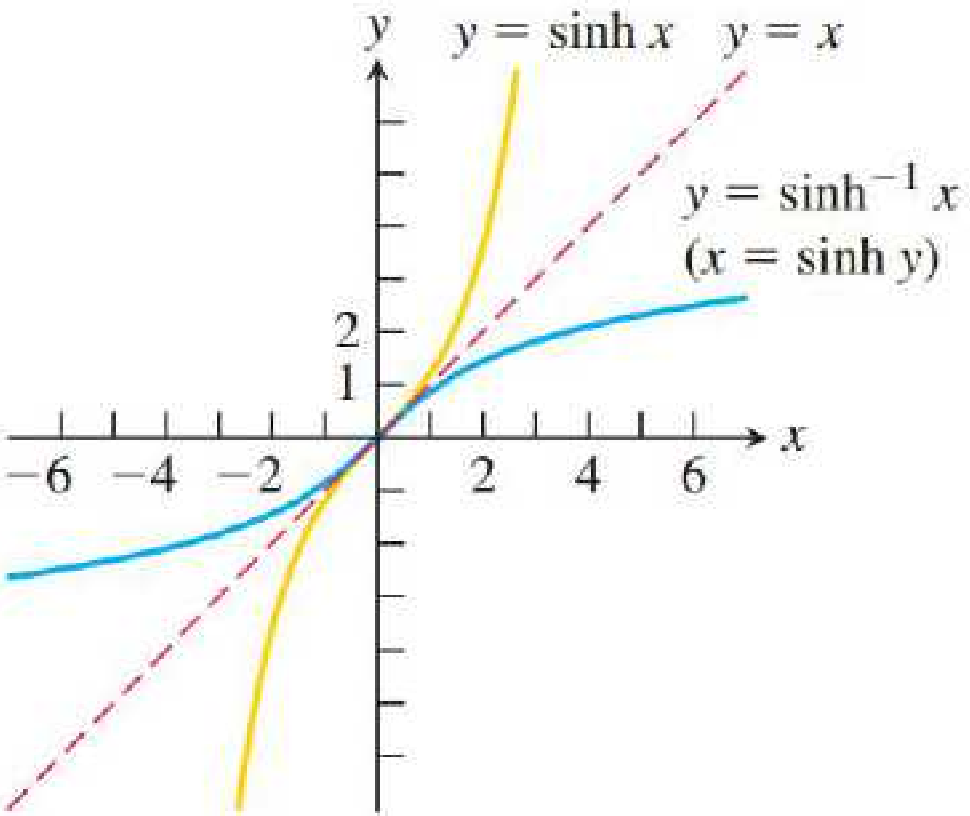
$$(c)\int_0^1 \sinh^2 x dx = \int_0^1 \frac{\cosh 2x - 1}{2} dx$$

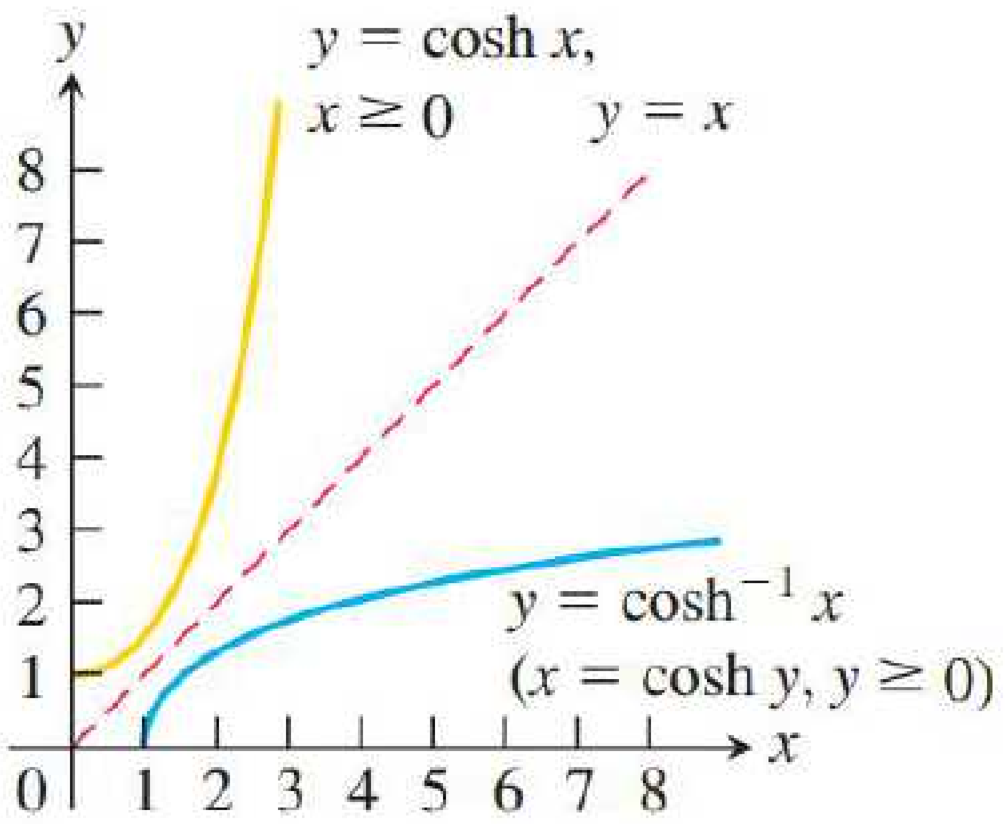
$$= \frac{1}{4}\int_0^1 \cosh 2x d2x - \int_0^1 \frac{1}{2} dx = \frac{\sinh 2x}{4}\Big|_0^1 - \frac{1}{2} = \frac{\sinh 2}{4} - \frac{1}{2}$$

$$(d)\int_0^{\ln 2} 4e^x \sinh x dx = \int_0^{\ln 2} (2e^{2x} - 2) dx$$

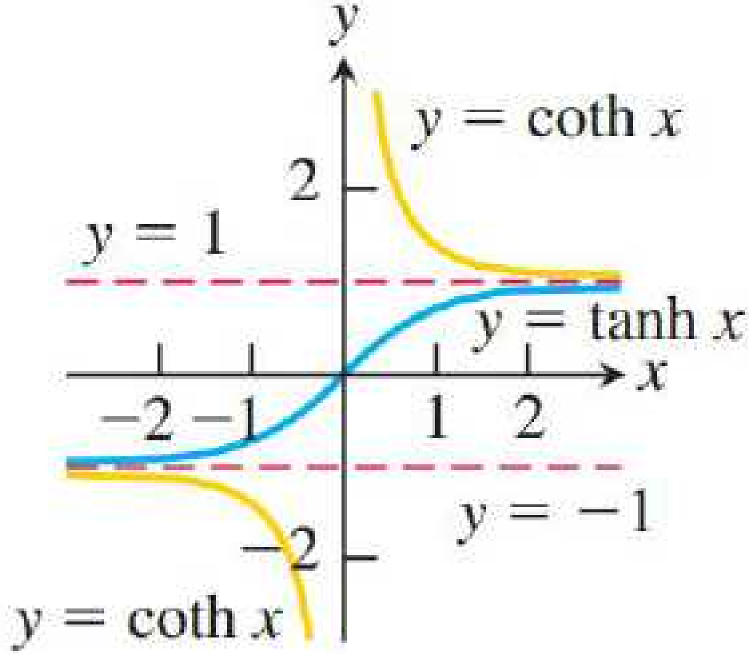
$$= e^{2x}\Big|_0^{\ln 2} - 2\ln 2 = 3 - 2\ln 2$$

反双曲函数 Inverse Hyperbolic Functions

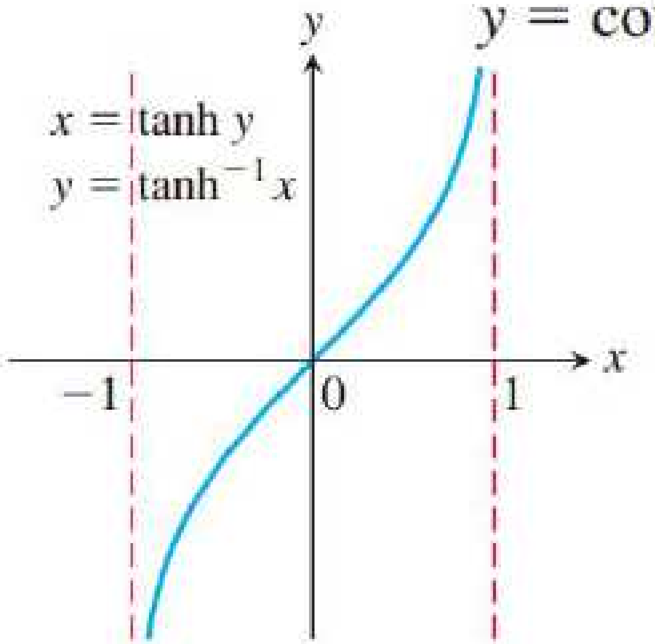




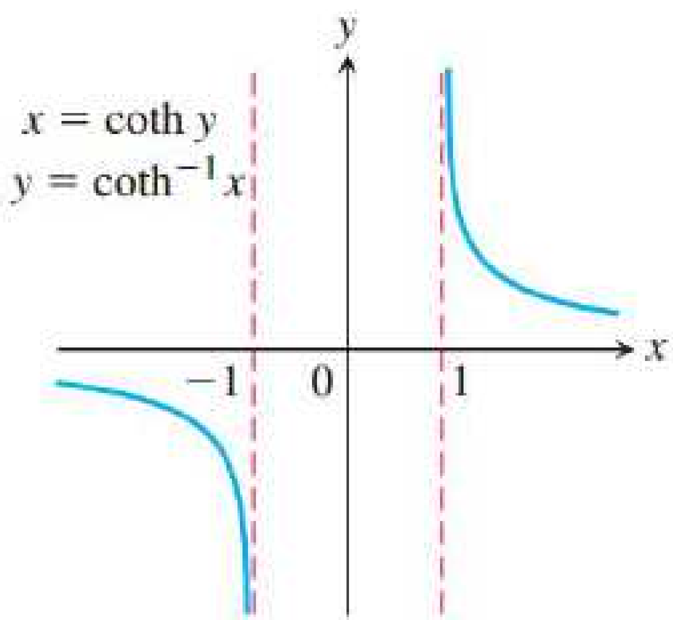
y
$y = \cosh x,$
$x \geq 0$
$y = x$
$y = \cosh^{-1} x$
$(x = \cosh y, y \geq 0)$
x
0
1
2
3
4
5
6
7
8



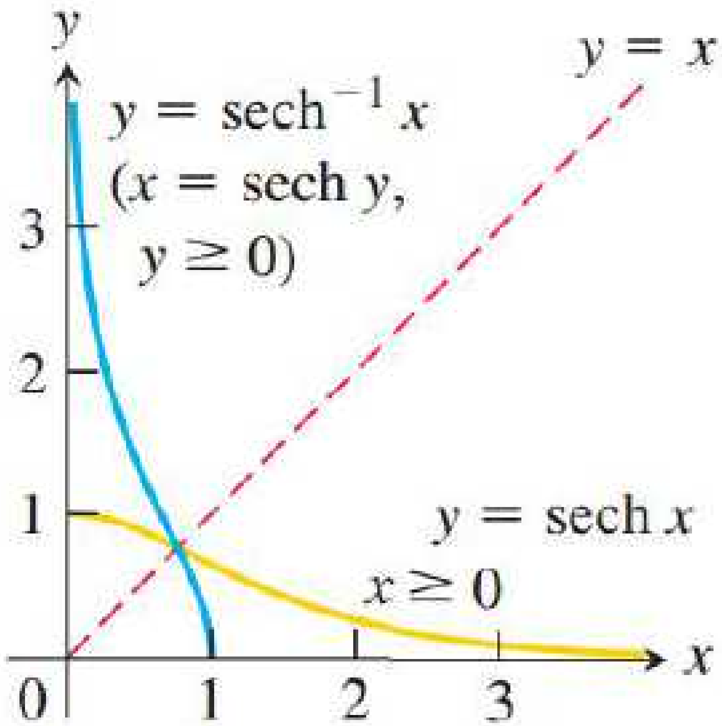
y
y = coth x
2
y = 1
y = tanh x
x
−2 −1
1 2
y = −1
−2
y = coth x

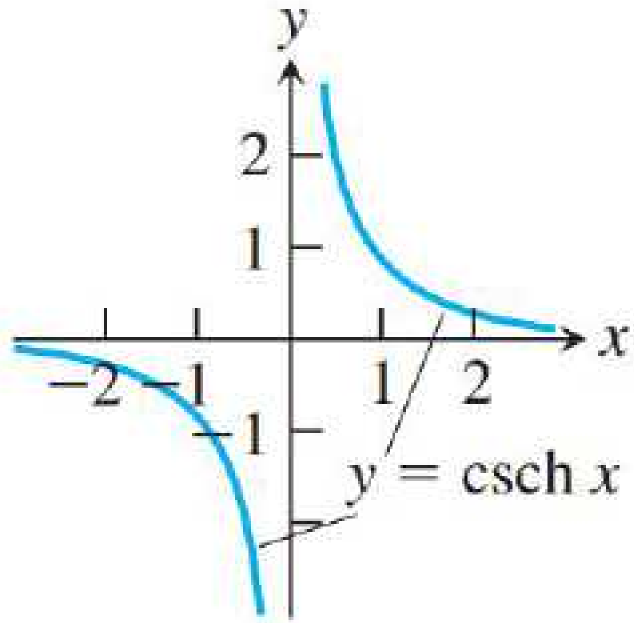


y
x = tanh y
y = tanh⁻¹ x
x
−1
0
1



y
x = coth y
y = coth⁻¹ x
x
−1
0
1


y
y = x
y = sech^{-1} x
(x = sech y,
y ≥ 0)
3
2
1
y = sech x
x ≥ 0
0
1
2
3
x


y
2
1
x
−2
−1
1
2
−1
y = csch x

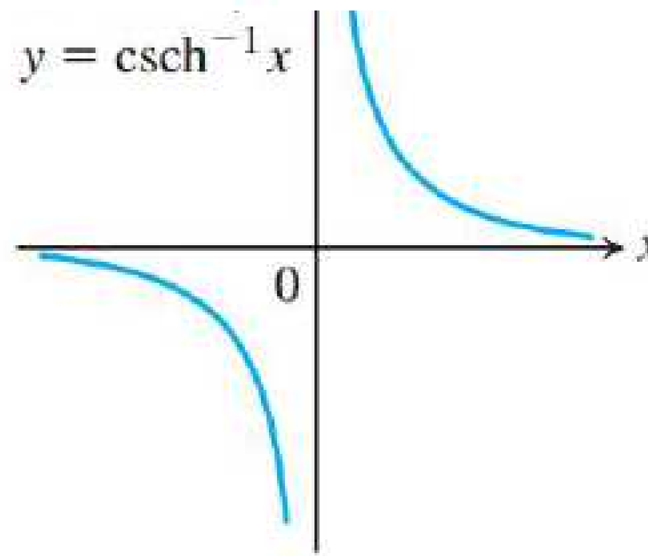

y = csch^{-1} x
x
0

TABLE 7.9 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

TABLE 7.10 Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1 - u^2} \frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1 - u^2} \frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = -\frac{1}{u\sqrt{1 - u^2}} \frac{du}{dx}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{1 + u^2}} \frac{du}{dx}, \qquad u \neq 0$$

EXAMPLE 2 Show that if u is a differentiable function of x whose values are greater than 1, then

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx}.$$

Solution $f(x) = \cosh x$ and $f^{-1}(x) = \cosh^{-1} x$.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sinh(\cosh^{-1} x)}$$

$$= \frac{1}{\sqrt{\cosh^2(\cosh^{-1} x) - 1}} \qquad \cosh^2 u - \sinh^2 u = 1,$$

$$= \frac{1}{\sqrt{x^2 - 1}}$$

TABLE 7.11 Integrals leading to inverse hyperbolic functions

1. $$\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

2. $$\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

3. $$\int \frac{du}{a^2 - u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C, & u^2 < a^2 \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C, & u^2 > a^2 \end{cases}$$

4. $$\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

5. $$\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$$

EXAMPLE 3 Evaluate $\displaystyle\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}$.

Solution

$u = 2x, \quad du = 2\,dx, \quad a = \sqrt{3}$

$$\int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}}$$

$$= \sinh^{-1}\left(\frac{u}{a}\right) + C = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg]_0^1$$

$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0) = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) \approx 0.98665.$$

7.8

Relative Rates of Growth

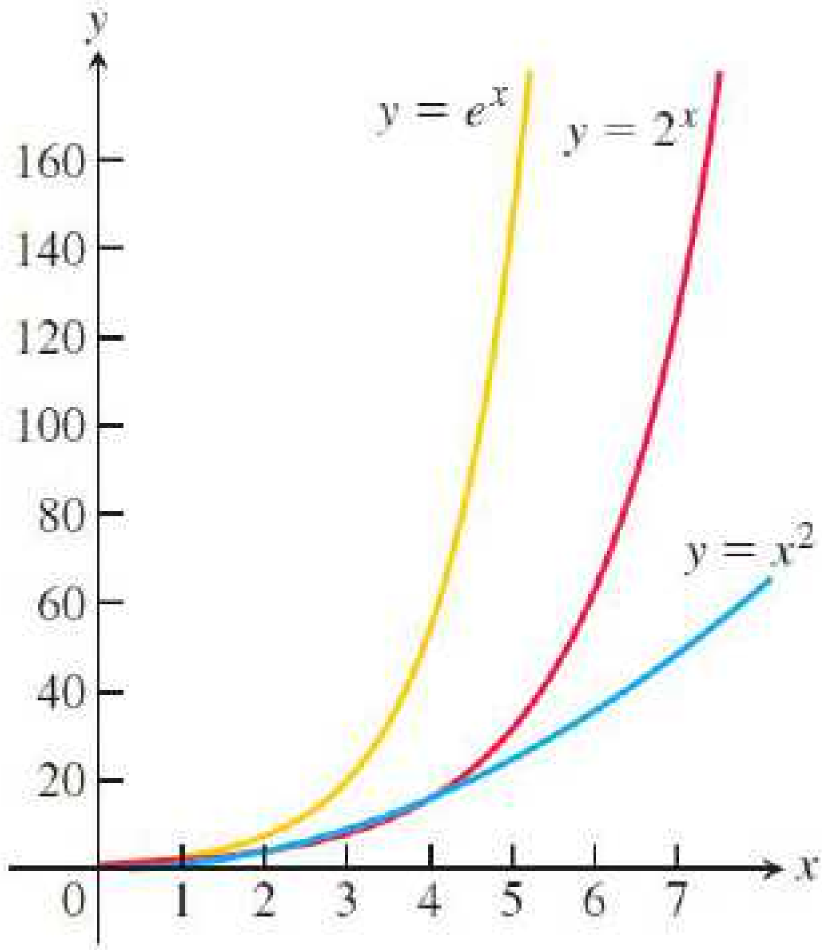


FIGURE 7.34 The graphs of e^x, 2^x, and x^2.

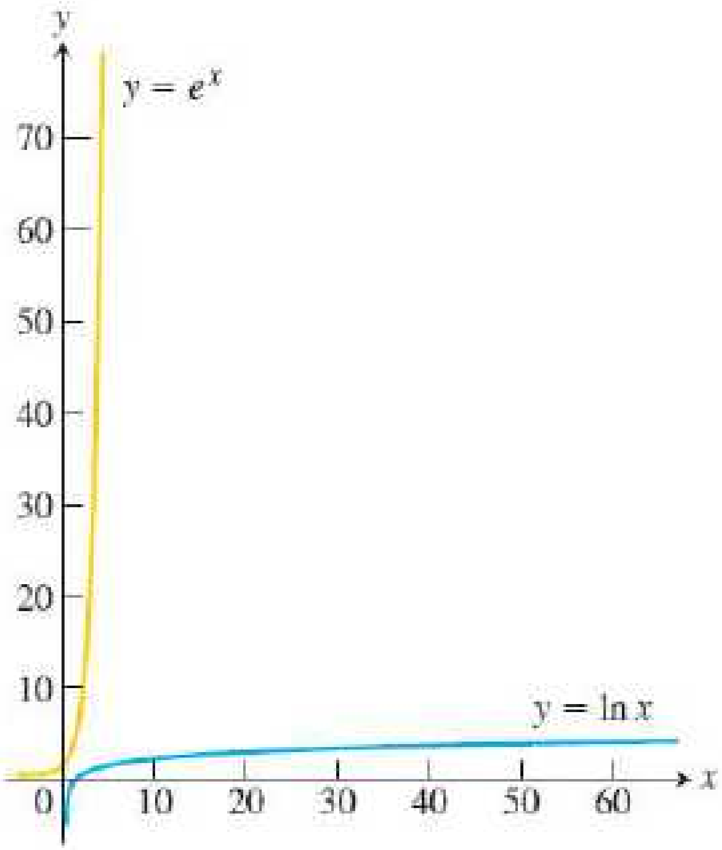


FIGURE 7.35 Scale drawings of the graphs of e^x and $\ln x$.

DEFINITION Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f **grows faster than** g as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$$

or, equivalently, if

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0.$$

We also say that g **grows slower than** f as $x \to \infty$.

2. f and g **grow at the same rate** as $x \to \infty$ if $\displaystyle\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$

where L is finite and positive.

EXAMPLE 1

Let's compare the growth rates of several common functions.

(a) e^x grows faster than x^2 as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

(b) 3^x grows faster than 2^x as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{3^x}{2^x} = \lim_{x\to\infty} \left(\frac{3}{2}\right)^x = \infty.$$

(c) x^2 grows faster than $\ln x$ as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{x^2}{\ln x} = \lim_{x\to\infty} \frac{2x}{1/x} = \lim_{x\to\infty} 2x^2 = \infty.$$

(d) $\ln x$ grows slower than $x^{1/n}$ as $x \to \infty$ for any positive integer n

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/n}} = \lim_{x\to\infty} \frac{1/x}{(1/n)x^{(1/n)-1}} = \lim_{x\to\infty} \frac{n}{x^{1/n}} = 0.$$

(e) If $a > b > 0$, then a^x grows faster than b^x. Since $(a/b) > 1$,

$$\lim_{x\to\infty} \frac{a^x}{b^x} = \lim_{x\to\infty} \left(\frac{a}{b}\right)^x = \infty.$$

(f) with different bases $a > 1$ and $b > 1$

always grow at the same rate as $x \to \infty$:

$$\lim_{x\to\infty} \frac{\log_a x}{\log_b x} = \lim_{x\to\infty} \frac{\ln x/\ln a}{\ln x/\ln b} = \frac{\ln b}{\ln a}.$$

If f grows at the same rate as g as $x \to \infty$,

and g grows at the same rate as h as $x \to \infty$,

then f grows at the same rate as h as $x \to \infty$.

$$\lim_{x \to \infty} \frac{f}{g} = L_1 \qquad \text{and} \qquad \lim_{x \to \infty} \frac{g}{h} = L_2$$

$$\lim_{x \to \infty} \frac{f}{h} = \lim_{x \to \infty} \frac{f}{g} \cdot \frac{g}{h} = L_1 L_2.$$

If L_1 and L_2 are finite and nonzero, then so is $L_1 L_2$.

EXAMPLE 2

Show that $\sqrt{x^2+5}$ and $(2\sqrt{x}-1)^2$ grow at the same rate as $x \to \infty$.

Solution

$$\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{x} = \lim_{x\to\infty} \sqrt{1+\frac{5}{x^2}} = 1,$$

$$\lim_{x\to\infty} \frac{(2\sqrt{x}-1)^2}{x} = \lim_{x\to\infty} \left(\frac{2\sqrt{x}-1}{\sqrt{x}}\right)^2 = \lim_{x\to\infty} \left(2-\frac{1}{\sqrt{x}}\right)^2 = 4.$$

DEFINITION A function f is **of smaller order than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

We indicate this by writing $f = o(g)$ ("f is little-oh of g").

EXAMPLE 3 Here we use little-oh notation.

(a) $\ln x = o(x)$ as $x \to \infty$ because $\lim_{x\to\infty} \frac{\ln x}{x} = 0$

(b) $x^2 = o(x^3 + 1)$ as $x \to \infty$ because $\lim_{x\to\infty} \frac{x^2}{x^3+1} = 0$

设$f(x), g(x) > 0$.

$$\begin{cases} 若\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 0, & 称f(x)比g(x)较低阶; \\ 若\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = L(正数), & 称f(x)与g(x)同阶. \end{cases}$$

共同点：当$x \to \infty$时，$\dfrac{f(x)}{g(x)} \le M$.

称$f(x)$至多与$g(x)$同阶。

DEFINITION

Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \le M, \quad \text{for } x \text{ sufficiently large.}$$

We indicate this by writing $\boldsymbol{f = O(g)}$ ("f is big-oh of g").

EXAMPLE 4 Here we use big-oh notation.

(a) $x + \sin x = O(x)$ as $x \to \infty$ because

$$\frac{x + \sin x}{x} \leq 2 \text{ for } x \text{ sufficiently large.}$$

(b) $e^x + x^2 = O(e^x)$ as $x \to \infty$ because

$$\frac{e^x + x^2}{e^x} \to 1 \text{ as } x \to \infty.$$

(c) $x = O(e^x)$ as $x \to \infty$ because $\dfrac{x}{e^x} \to 0$ as $x \to \infty$.