Probability and Statistics
Tutorial 1

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Outline

General Information

Quiz Arrangement

- 4th, 7th, 11th, 15th week
- Quiz length: 50 minutes

Homework

- Deadline: Every Mon 18:30
- All homework last week

Midterm

- Time: Nov 8th 10:00-12:00 a.m.

Review

1. Probability Theory
   - A branch of Mathematics.
   - Take it as a mathematical model of chance phenomenon. (Motivation)
2. Probability Space
   - A probability space is a triple $(\Omega, \mathcal{F}, P)$.
   - Ω: Sample space. (The **set** of all possible outcomes.)
   - $\mathcal{F}$: Event space. (A **collection** of some **subsets** of Ω.) ('σ-field').
   - P: Probability measure. (Defined on $\mathcal{F}$).

3. Set/Event Operation

- Communicative Law: $A \cap B = B \cap A$, $A \cup B = B \cup A$.
- Associative Law: $A \cap (B \cap C) = (A \cap B) \cap C$,
  $A \cup (B \cup C) = (A \cup B) \cup C$.
  (Hence, $A \cap B \cap C$ and $A \cup B \cup C$ are well-defined.)
- Distributive Law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- De Morgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$, $\overline{A \cap B} = \overline{A} \cup \overline{B}$. (Then, we have $\overline{\bigcup\limits_{i=1}^{n} A_i} = \bigcap\limits_{i=1}^{n} \overline{A_i}$ and $\overline{\bigcap\limits_{i=1}^{n} A_i} = \bigcup\limits_{i=1}^{n} \overline{A_i}$.)
- $A = B \iff A \subset B, B \subset A$.
- $A \subset B \iff$ For any $x \in A$, we have $x \in B$.
- $A - B = A \cap \overline{B}$ and $(A - B) \cup B = A \cup B$.

Review

4. Definition of Probability Measure P in $(\Omega, \mathcal{F}, P)$
    - $P : \mathcal{F} \to [0, 1]$.
    - $P(\Omega) = 1$.
    - $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$, where $A_i \cap A_j = \emptyset$, for any $i \neq j$. (Countable Additivity)
5. Countable Additivity and Finite Additivity.
    - $P(\bigcup_{i=1}^{N} A_i) = \sum_{i=1}^{N} P(A_i)$, where $A_i \cap A_j = \emptyset$, for any $i \neq j$. (Finite Additivity)
    - Countable Additivity implies Finite Additivity.

Homework

P20, 5

Let A and B be arbitrary events. Let C be the event that either A occurs or B occurs, but not both. Express C in terms of A and B using any of the basic operations of union, intersection, and complement.

Solution

We have $\{A \text{ or } B \text{ occurs}\} = A \cup B$ *and* $\{\text{Both } A \text{ and } B \text{ occur}\} = A \cap B$. *Then,* $C = (A \cup B) \cap \overline{(A \cap B)}$.

Homework

P20, 6

Two six-sided dice are thrown sequentially, and the face values that come up are recorded.

a. List the sample space.

b. List the elements that make up the following events: (1) $A =$ the sum of the two values is at least 5, (2) $B =$ the value of the first die is higher than the value of the second, (3) $C =$ the first value is 4.

c. List the elements of the following events: (1) $A \cap C$, (2) $B \cup C$, (3) $A \cap (B \cup C)$.

Homework

Solution

a. $\Omega = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),$ *(2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)*$\}$.

b. (1) $A = \{(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,2),(3,3),$ *(3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)*$\}$.

(2) $B = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),$ *(5,4), (6,1), (6,2), (6,3), (6,4), (6,5)*$\}$.

(3) $C = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$.

c. (1) $A \cap C = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$.

(2) $B \cup C = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3),(4,4),(4,5),$ *(4,6), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5)*$\}$.

(3) $A \cap (B \cup C) = \{(3,2),(4,1),(4,2),(4,3),(4,4),(4,5),$ *(4,6), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5)* $\}$.

Exercise 1

1. 写出下列随机试验的样本空间：

(1) 抛三枚硬币；

(2) 抛三颗骰子；

(3) 连续抛一枚硬币，直至出现正面为止；

(4) 口袋中有黑、白、红球各一个，从中任取两个球；先从中取出一个，放回后再取出一个；

(5) 口袋中有黑、白、红球各一个，从中任取两个球；先从中取出一个，不放回后再取出一个。

Supplement Exercises

Solution

解 (1) $\Omega=\{(0,0,0),(0,0,1),(0,1,0),(1,0,0),(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}$，共含有 $2^3=8$ 个样本点，其中 0 表示反面，1 表示正面. (3) 中的 0 与 1 也是此意.

(2) $\Omega=\{(x,y,z):x,y,z=1,2,3,4,5,6\}$，共含有 $6^3=216$ 个样本点.

(3) $\Omega=\{(1),(0,1),(0,0,1),(0,0,0,1),\cdots\}$，共含有可列个样本点.

(4) $\Omega=\{$黑黑，黑白，黑红，白黑，白白，白红，红黑，红白，红红$\}$.

(5) $\Omega=\{$黑白，黑红，白黑，白红，红黑，红白$\}$.

Exercise 2

3. 设 A,B,C 为三事件，试表示下列事件：

(1) A,B,C 都发生或都不发生；

(2) A,B,C 中不多于一个发生；

(3) A,B,C 中不多于两个发生；

(4) A,B,C 中至少有两个发生.

Supplement Exercises

Solution

解 (1) $ABC \cup \overline{A}\,\overline{B}\,\overline{C}$.

(2) $\overline{A}\,\overline{B}\,\overline{C} \cup A\overline{B}\,\overline{C} \cup \overline{A}B\overline{C} \cup \overline{A}\,\overline{B}C$.

(3) $\Omega - ABC = \overline{ABC} = \overline{A} \cup \overline{B} \cup \overline{C}$.

(4) $AB \cup AC \cup BC$.

Exercise 3

1. 设随机事件A,B满足条件$AB=\bar{A}\bar{B}$. 试求$A\cup B$.↵

Supplement Exercises

Solution

Since $A \cap B = \overline{A} \cap \overline{B}$, *then we have* $\overline{A} \cap \overline{B} = A \cap B \cap \overline{B} = \emptyset$. *Hence,* $A \cup B = \overline{(\overline{A} \cap \overline{B})} = \overline{\emptyset} = \Omega$.

Supplement Exercises

Exercise 4

2. 试把事件 $A_1 \cup A_2 \cup \Lambda \cup A_n$ 表示成 n 个两两互不相容事件之并。

Supplement Exercises

Solution

Let $B_1 = A_1$ *and* $B_k = A_k - \bigcup_{i=1}^{k-1} A_i$ *for* $k = 2, .., n$. *Then, we will prove the following by induction:* $\bigcup_{i=1}^{k} B_i = \bigcup_{i=1}^{k} A_i$, *for* $k = 1, .., n$.

First, we have $B_1 = A_1$.

Then, assume we have $\bigcup_{i=1}^{k} B_i = \bigcup_{i=1}^{k} A_i$, *then we consider* $\bigcup_{i=1}^{k+1} B_i$. *We have*

$$\bigcup_{i=1}^{k+1} B_i = B_{k+1} \cup (\bigcup_{i=1}^{k} B_i) = (A_{k+1} - \bigcup_{i=1}^{k} A_i) \cup (\bigcup_{i=1}^{k} A_i) = \bigcup_{i=1}^{k+1} A_i.$$

Hence, by induction, we have $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$.

Now we need to prove $B_i \cap B_j = \emptyset$, *for* $i \neq j$.

Assume $i < j$, *then we have*

$$B_i \cap B_j = B_i \cap (A_j - \bigcup_{l=1}^{j-1} A_l) = B_i \cap (A_j - \bigcup_{l=1}^{j-1} B_l) = \emptyset.$$

Supplement Exercises

Exercise 5

4. 指出下列事件等式成立的条件.

(1) $A \cup B = A$;

(2) $AB = A$.

Supplement Exercises

Solution

解 (1) $A \supset B$;

(2) $A \subset B$.

Supplement Exercises

Exercise 6

5. 设 X 为随机变量，其样本空间为 $\Omega=\{0\leqslant X\leqslant 2\}$，记事件 $A=\{0.5<X\leqslant 1\}$，$B=\{0.25\leqslant X<1.5\}$，写出下列各事件：

(1) $\bar{A}B$;　(2) $\bar{A}\cup B$;　(3) $\overline{AB}$;　(4) $\overline{A\cup B}$.

Supplement Exercises

Solution

(1) $\overline{A}B = (\{0 \le X \le 0.5\} \cup \{1 < X \le 2\}) \cap \{0.25 \le X < 1.5\}$

$= \{0.25 \le X \le 0.5\} \cup \{1 < X < 1.5\}$.

(2) $\overline{A} \cup B = \{0 \le x \le 2\} = \Omega$.

(3) 由于 $A \subset B$,所以 $AB = A$,故

$$\overline{AB} = \overline{A} = \{0 \le X \le 0.5\} \cup \{1 < X \le 2\}.$$

(4) 由于 $A \subset B$,所以 $A \cup B = B$,故

$$\overline{A \cup B} = \overline{B} = \{0 \le X < 0.25\} \cup \{1.5 \le X \le 2\}.$$

Further Reading

1. Countability
   - (Definition) For a set A, if there exists a bijection $f : A \to \mathbb{N}$, then we say A is **countable/countable infinite**.
   - Finite set: $\{1\}$, $\{1, 2, 3\}$.
   - Countable set: $\mathbb{N}, \{-2, -1, 0, 1, 2, ...\}, \mathbb{Z}, \mathbb{Q}, \mathbb{N}^2$.
   - Uncountable infinte set: $\mathbb{R}, [0, 1], \mathbb{R}\backslash\mathbb{Q}$.

Further Reading

2. Self-study Resources and Methods
    - Stack Exchange: Math Stack Exchange, Mathoverflow, Stat Stack Exchange, Quant Stack Exchange, ...
    - If you want to learn some topic by yourself, you can google 'topic+lecture notes' to find some notes from top university.
    - Wikipedia
    - Self-study is essential in university.
3. Some Interesting Math Channel.
    - 3Blue1Brown
    - Numberphile

Thank you!