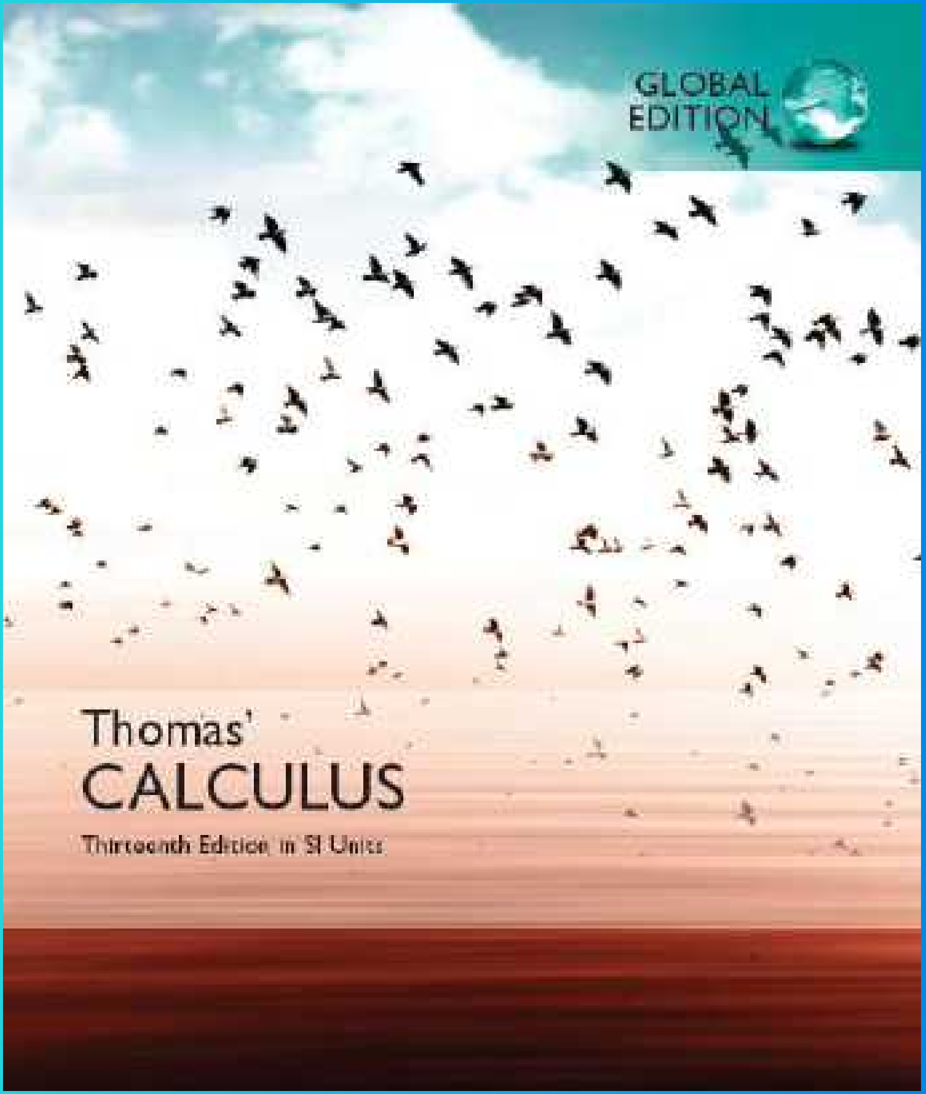

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 5

Integrals

5.1

Area and Estimating with Finite Sums

面积及其估计

平面图形面积的估算

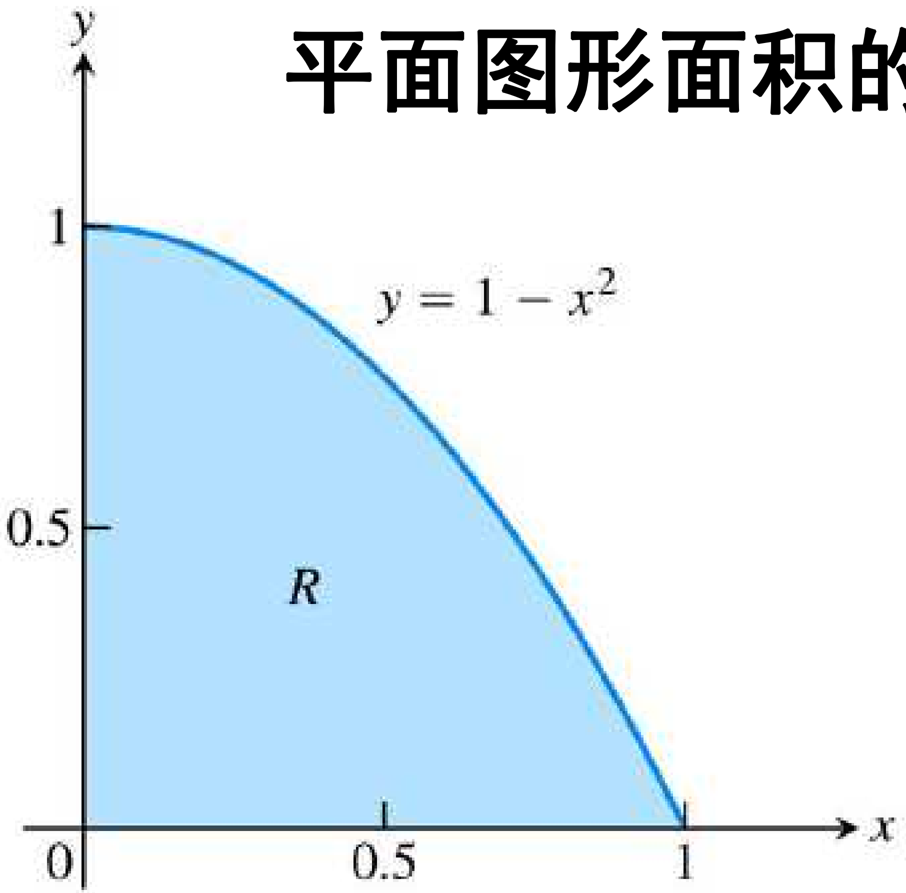


FIGURE 5.1 The area of the region R cannot be found by a simple formula.

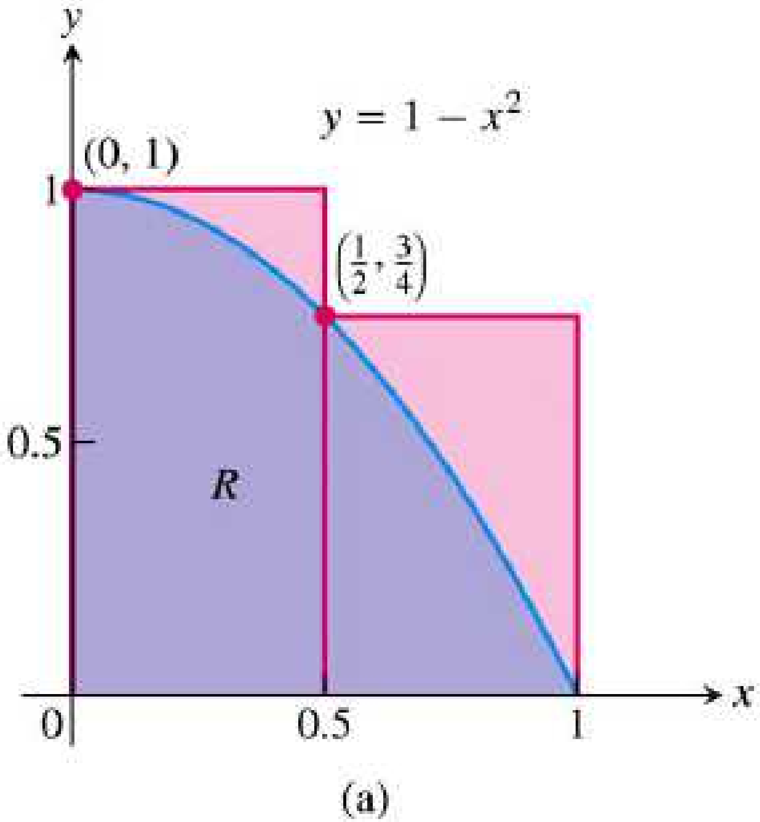


FIGURE 5.2 (a) We get an upper estimate of the area of R by using two rectangles containing R. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area by the amount shaded in light red.

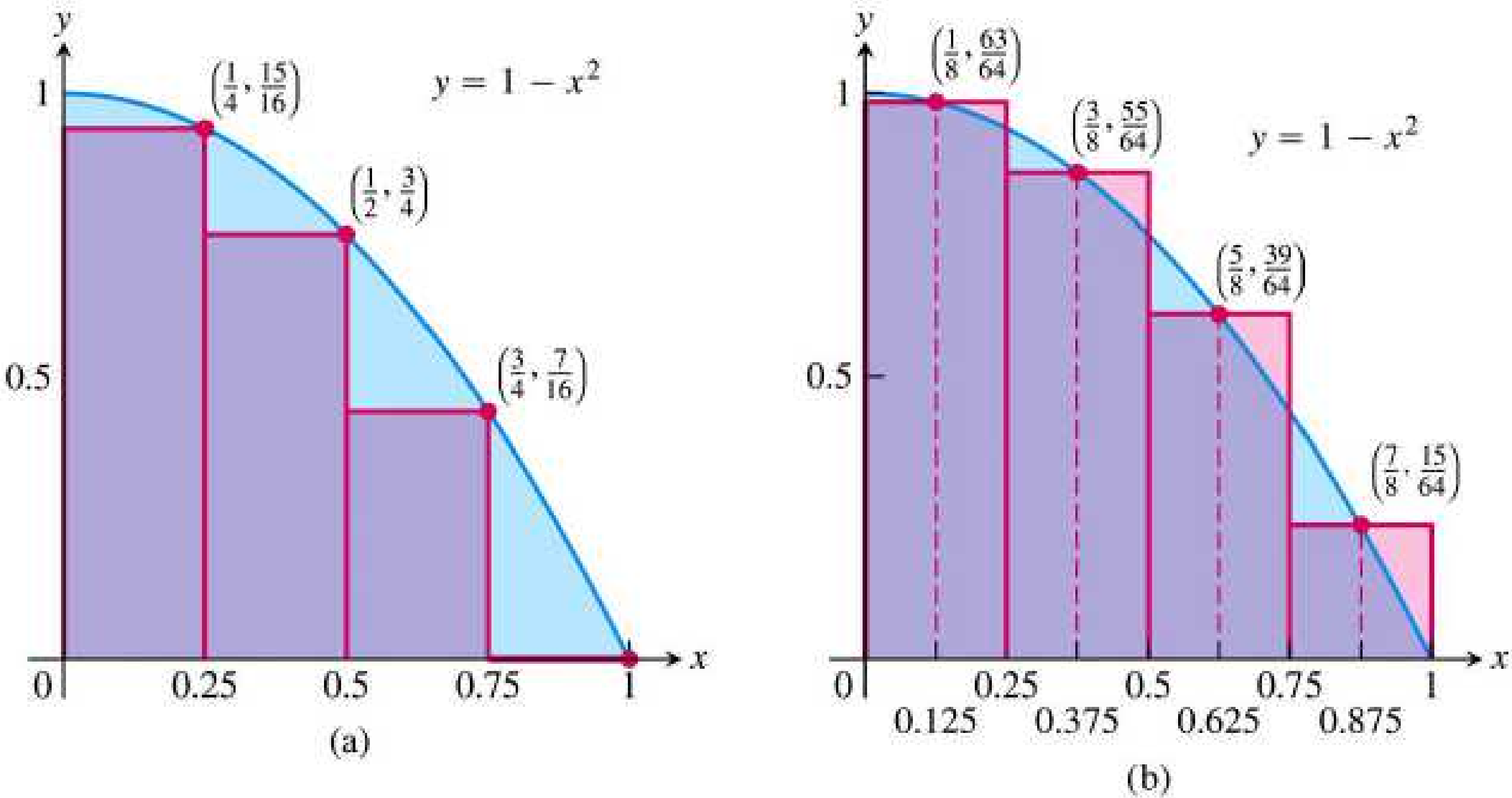
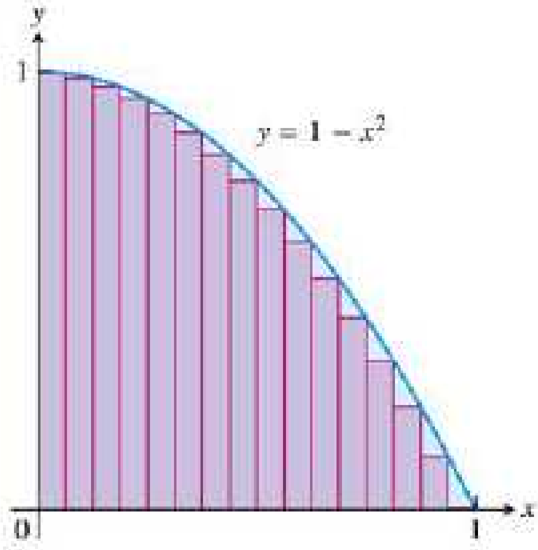
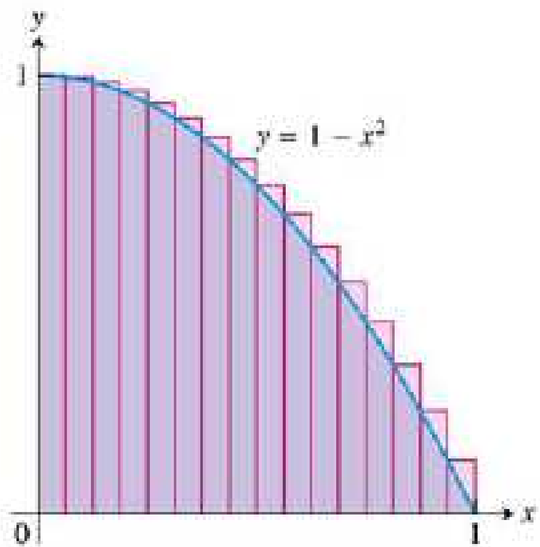


FIGURE 5.3 (a) Rectangles contained in R give an estimate for the area that undershoots the true value by the amount shaded in light blue. (b) The midpoint rule uses rectangles whose height is the value of $y = f(x)$ at the midpoints of their bases. The estimate appears closer to the true value of the area because the light red overshoot areas roughly balance the light blue undershoot areas.



(a)



(b)

FIGURE 5.4 (a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$. (b) An upper sum using 16 rectangles.

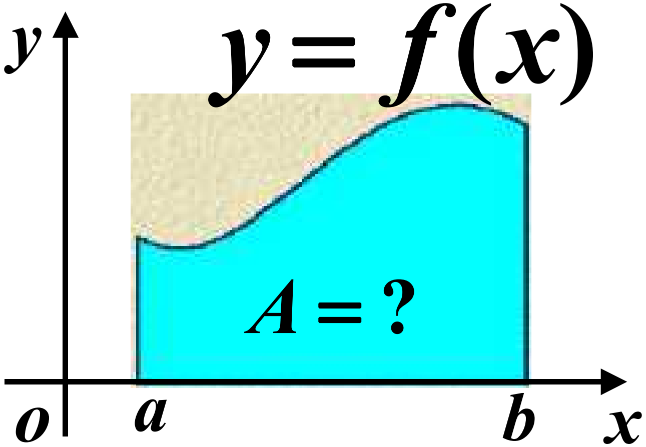
实例1 （求曲边梯形的面积）

曲边梯形由连续曲线

$y = f(x)(f(x) \geq 0)$、

x轴与两条直线$x = a$、

$x = b$所围成.

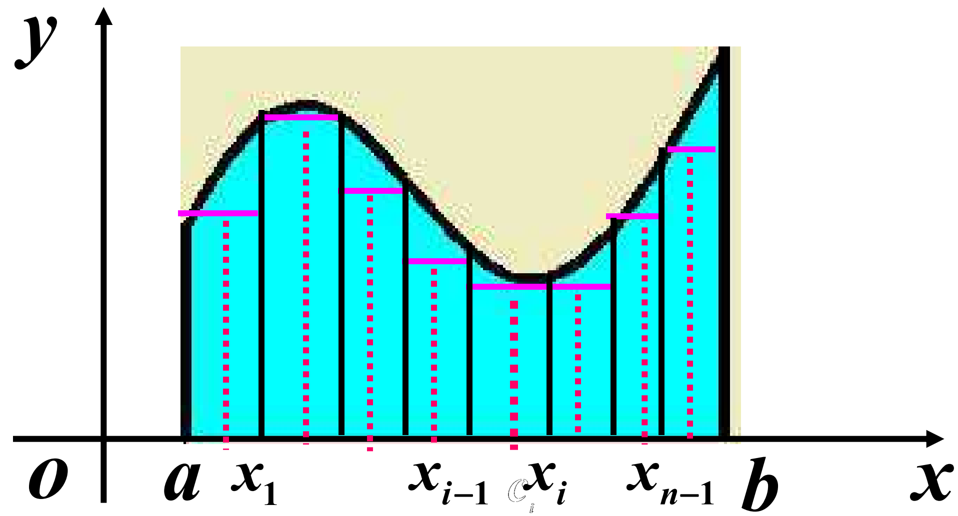


曲边梯形如图所示． 在区间 $[a,b]$ 内插入若干个分点，$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$, 分割

把区间 $[a,b]$ 分成 n 个小区间 $[x_{i-1}, x_i]$,

长度为 $\Delta x_i = x_i - x_{i-1}$;

在每个小区间 $[x_{i-1}, x_i]$ 上任取一点 c_i,



以 $[x_{i-1}, x_i]$为底，$f(\xi_i)$ 为高的小矩形面积为

$$A_i = f(c_i)\Delta x_i$$ 近似

曲边梯形面积的近似值为

$$A \approx \sum_{i=1}^{n} f(c_i)\Delta x_i$$

求和

当分割无限加细,即小区间的最大长度

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \cdots \Delta x_n\}$$

趋近于零 $(\|P\| \to 0)$ 时,

曲边梯形面积为 $$A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

取极限

实例2 （求变速直线运动的路程）

设某物体作直线运动，已知速度$v=v(t)$是时间间隔$[T_1,T_2]$上t的一个连续函数，且$v(t)\geq 0$，求物体在这段时间内所经过的路程.

(1) 分割 $T_1=t_0<t_1<t_2<\cdots<t_{n-1}<t_n=T_2$

(2) 近似 $\Delta t_i=t_i-t_{i-1}$ $\Delta s_i\approx v(c_i)\Delta t_i$

部分路程值

某时刻的速度

(3) 求和 $s\approx\sum_{i=1}^{n}v(c_i)\Delta t_i$

(4) 取极限 $\|P\|=\max\{\Delta t_1,\Delta t_2,\cdots,\Delta t_n\}$

路程的精确值 $s=\lim_{\|P\|\to 0}\sum_{i=1}^{n}v(c_i)\Delta t_i$

实例3 求闭区间上函数的平均值

设函数 $f(x)$ 在闭区间 $[a,b]$ 上连续，求函数在其上的平均值.

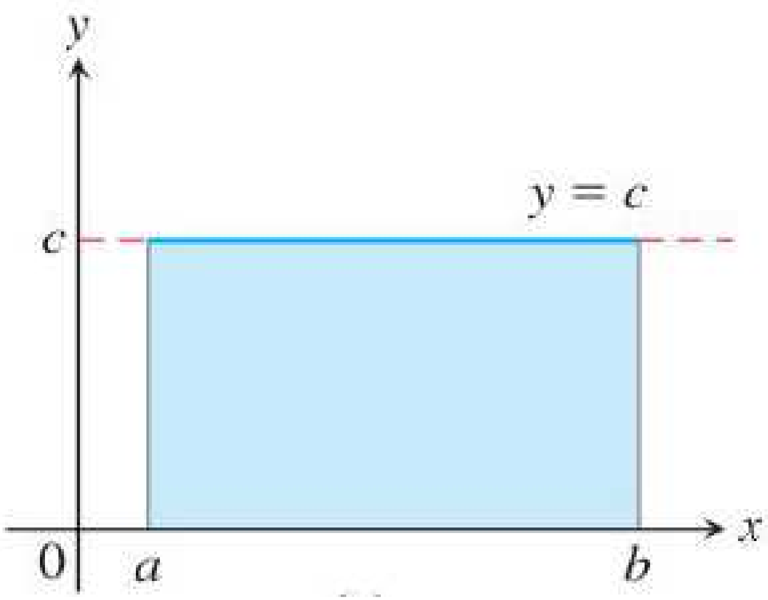
（1）分割 $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

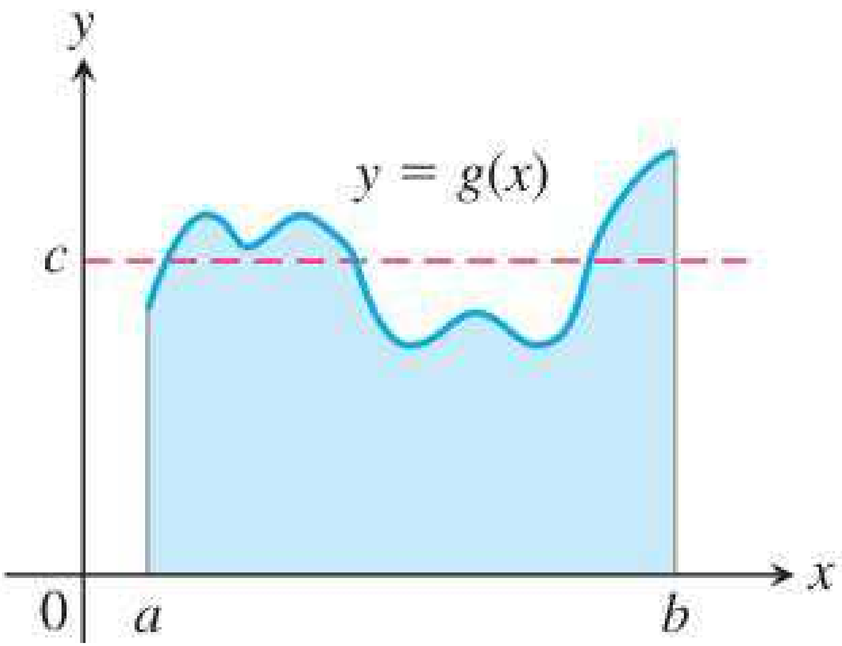
（2）近似 $\Delta x_i = x_i - x_{i-1}$ $f(c_i)\dfrac{\Delta x_1}{b-a}$， $i = 1,2,\cdots,n$

（3）求和 $\bar{f} \approx \sum_{i=1}^{n} f(c_i)\dfrac{\Delta x_i}{b-a}$

（4）取极限 $$\boxed{\bar{f} = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\frac{\Delta x_i}{b-a}}$$

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \cdots \Delta x_n\}$$


y
y = c
c
0
a
b
x


y
y = g(x)
c
0
a
b
x

5.2

Sigma Notation and Limits of Finite Sums

$$A = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$s = \lim_{\|P\|\to 0} \sum_{i=1}^{n} v(c_i)\Delta t_i$$

$$\bar{f} = \frac{\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i}{b-a}$$

函数f在区间$[a,b]$上的黎曼和

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

Riemann Sum

Algebra Rules for Finite Sums

1. *Sum Rule:* $$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

2. *Difference Rule:* $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

3. *Constant Multiple Rule:* $$\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$$ (Any number c)

4. *Constant Value Rule:* $$\sum_{k=1}^{n} c = n \cdot c$$ (c is any constant value.)

The first n squares: $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

The first n cubes: $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

数学归纳法易证

构造$f(x)$在区间$[a,b]$上的黎曼和：

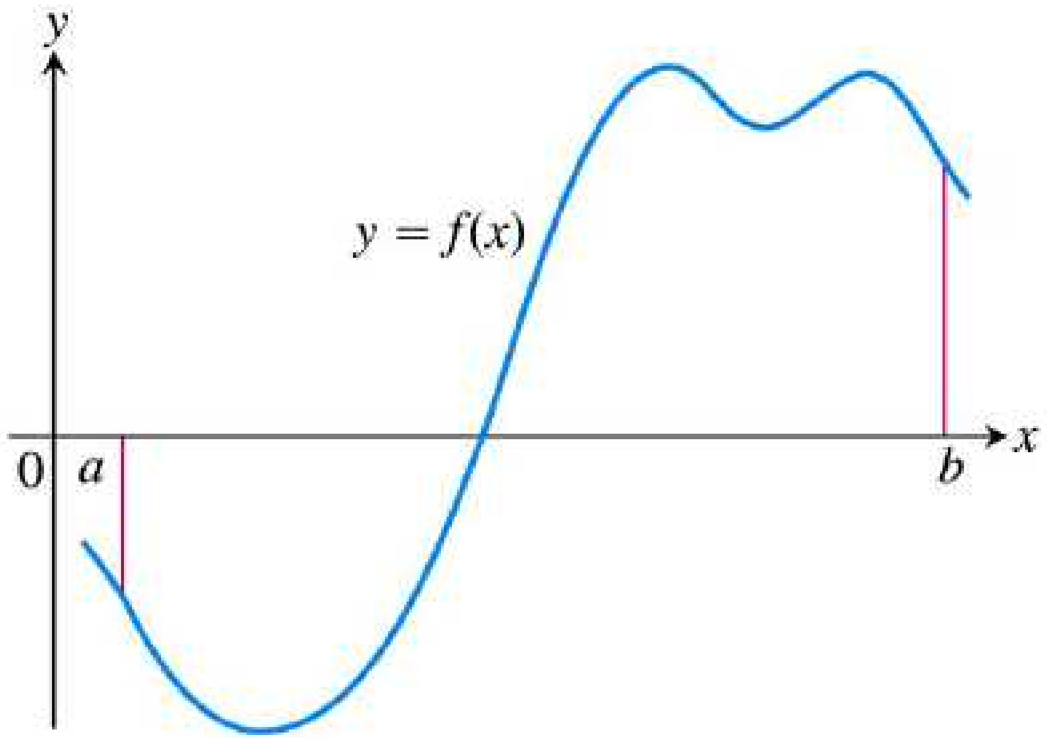


FIGURE 5.8 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

an arbitrary bounded function f defined on a closed interval $[a, b]$.

choose $n - 1$ points $\{x_1, x_2, x_3, \ldots, x_{n-1}\}$ between a and b satisfying

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$ is called a **partition** of $[a, b]$.

The partition P divides $[a, b]$ into n closed subintervals

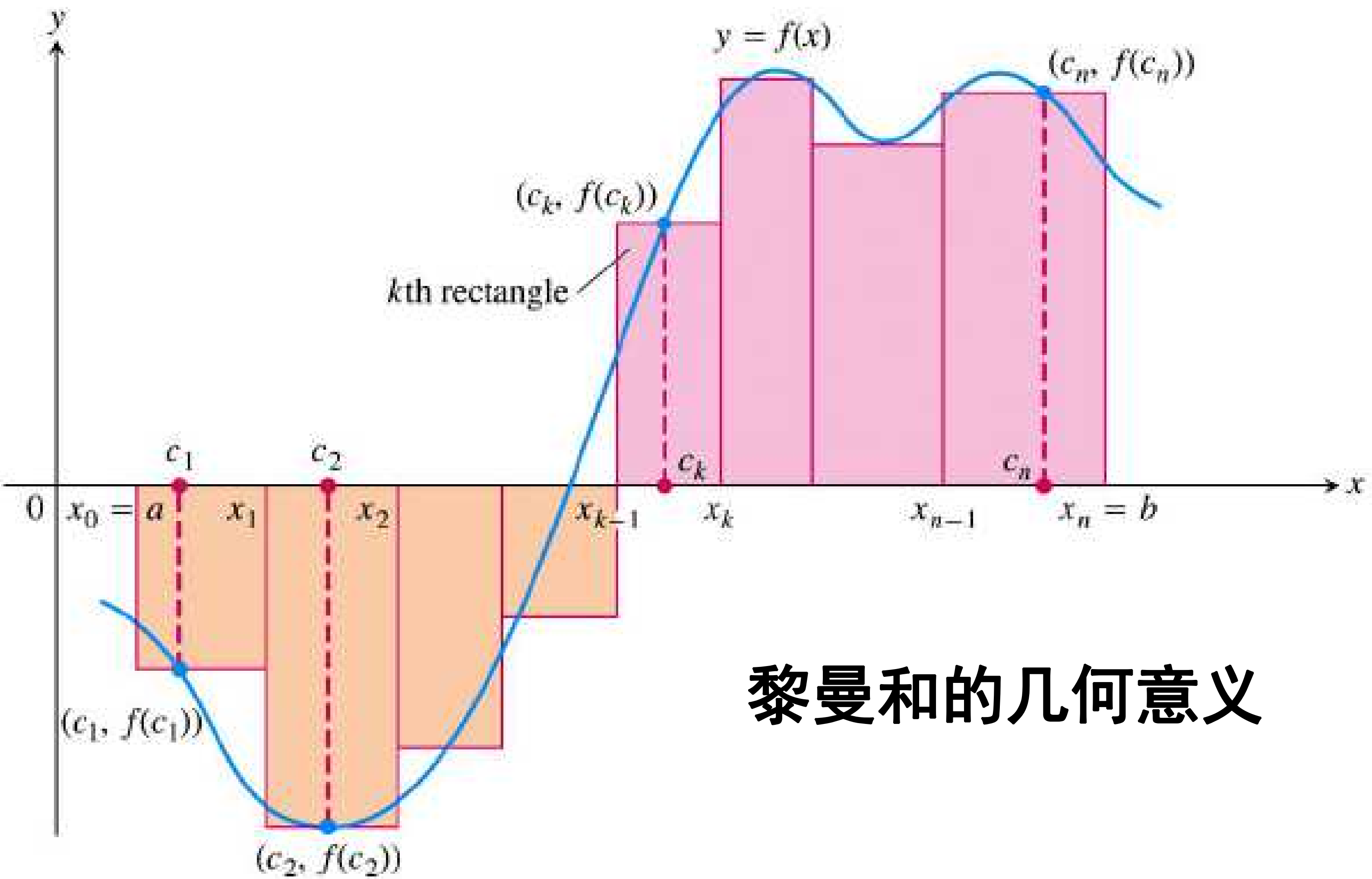
$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]. \qquad \Delta x_k = x_k - x_{k-1}.$$

In each subinterval we select some point. $c_k \in [x_{k-1}, x_k], k = 1,2,3,\cdots,n.$

$$f(c_k) \cdot \Delta x_k. \qquad k = 1,2,3,\cdots,n.$$

$$S_P = \sum_{k=1}^{n} f(c_k)\, \Delta x_k.$$

Riemann sum for f on the interval $[a, b]$.



黎曼和的几何意义

FIGURE 5.9 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis. Figure 5.8 has been enlarged to enhance the partition of $[a, b]$ and selection of points c_k that produce the rectangles.

黎曼和的几何意义

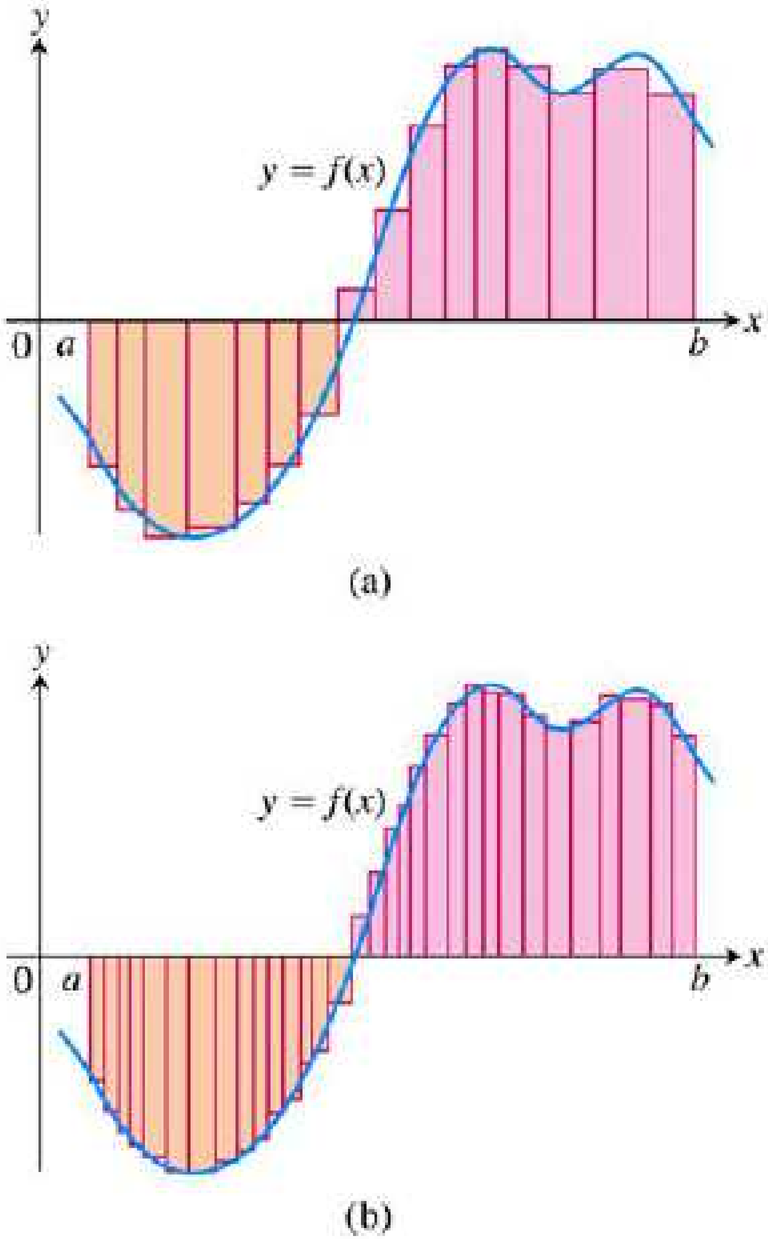


FIGURE 5.10 The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x-axis with increasing accuracy.

5.3

The Definite Integral
定积分

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\|P\| = \max_{1\le k\le n}(\Delta x_k)$$

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \epsilon .$$

The Definite Integral

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

The function is the integrand.

Upper limit of integration

x is the variable of integration.

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

$$A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx.$$

$$s = \lim_{\|P\| \to 0} \sum_{i=1}^{n} v(c_i)\Delta t_i = \int_a^b v(t)dt$$

$$\bar{f} = \frac{\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i}{b-a} = \frac{\int_a^b f(x)dx}{b-a}$$

注意：

（1）积分值仅与被积函数及积分区间有关，

而与积分变量的字母无关． dummy variable

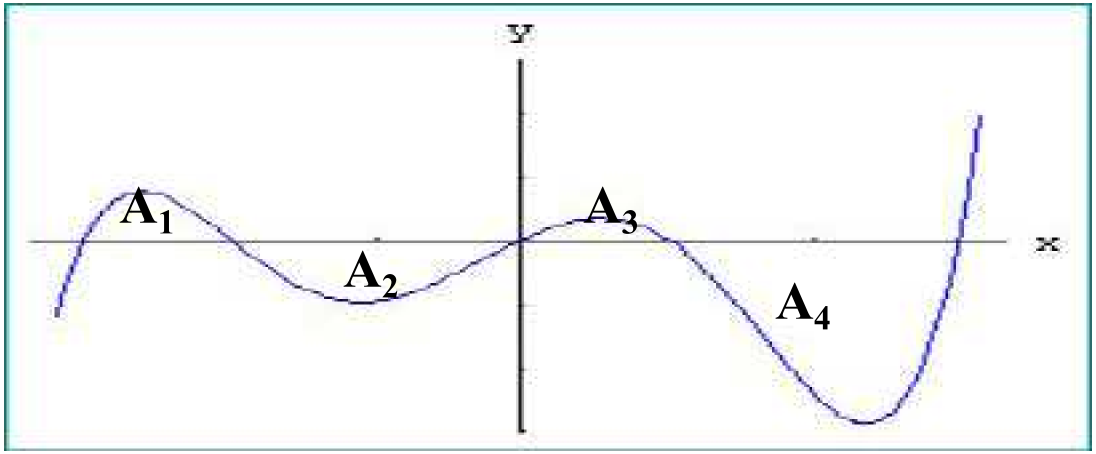
$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du$$

（2）定义中区间的分法和c_i的取法是任意的．

定积分的几何意义

$f(x) > 0,\quad \int_a^b f(x)dx = A$ 曲边梯形的面积

$f(x) < 0,\quad \int_a^b f(x)dx = -A$ 曲边梯形的面积的负值



$$\int_a^b f(x)dx = A_1 - A_2 + A_3 - A_4$$

THEOREM 1—Integrability of Continuous Functions If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$.

可积函数：闭区间上连续函数或至多有有限个第一类间断点的函数

Ex. 1 **Using the definition of definite integral to proof** $\int_a^b 1dx = b - a.$

Proof

$$\int_a^b 1dx = \lim_{\|P\|\to 0}\sum_{i=1}^{n} \Delta x_i = b - a.$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right)\left(\frac{b-a}{n}\right)$$

Ex. 2 compute $\int_0^b x\,dx \qquad (b>0)$

Solution: Subdivide the $[0,b]$ by $x_i = \frac{ib}{n}, \quad i = 1,2,\cdots,n.$

the width of $[x_{i-1}, x_i]$ is $\Delta x_i = \frac{b}{n}, \quad i = 1,2,\cdots,n.$

choose $c_i = x_i = \frac{ib}{n}, \qquad i = 1,2,\cdots,n.$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} c_i \Delta x_i = \sum_{i=1}^{n} \frac{ib}{n} \cdot \frac{b}{n},$$

$$= \frac{b^2}{n^2}\sum_{i=1}^{n} i = \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2}$$

$$\int_0^b x\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} c_i \Delta x_i = \lim_{n\to\infty} \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2} = \frac{b^2}{2}.$$

Ex. 3 Show that the integral on the interval [0,1] does not exist for the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

Solution

$$\int_0^1 f(x)dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$= \begin{cases} \lim_{\|P\|\to 0} \sum_{i=1}^{n} 1\Delta x_i = 1, & c_i \text{ chosen rational}, \\ \lim_{\|P\|\to 0} \sum_{i=1}^{n} 0\Delta x_i = 0, & c_i \text{ chosen irrational}, \end{cases}$$

Since the limit depends on tchoices of c_k, the function f is not integrable.

Properties of Definite Integrals

定积分的性质

THEOREM 2 When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules in Table 5.6.

1. *Order of Integration:* $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

2. *Zero Width Interval:* $\displaystyle\int_a^a f(x)\,dx = 0$

3. *Constant Multiple:* $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

4. *Sum and Difference:* $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Proof 3,4:

$$\int_a^b kf(x)dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} kf(c_i)\Delta x_i$$

$$= k \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i \quad = k\int_a^b f(x)dx$$

$$\int_a^b [f(x) \pm g(x)]dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} [f(c_i) \pm g(c_i)]\Delta x_i$$

$$= \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i \pm \lim_{\|P\|\to 0} \sum_{i=1}^{n} g(c_i)\Delta x_i$$

$$= \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

Proof 5: 若$a<b<c$,

$$\int_a^c f(x)dx = \lim_{\|P\|\to 0}\sum_{i=1}^{n} f(c_i)\Delta x_i$$ 取b为一个分点

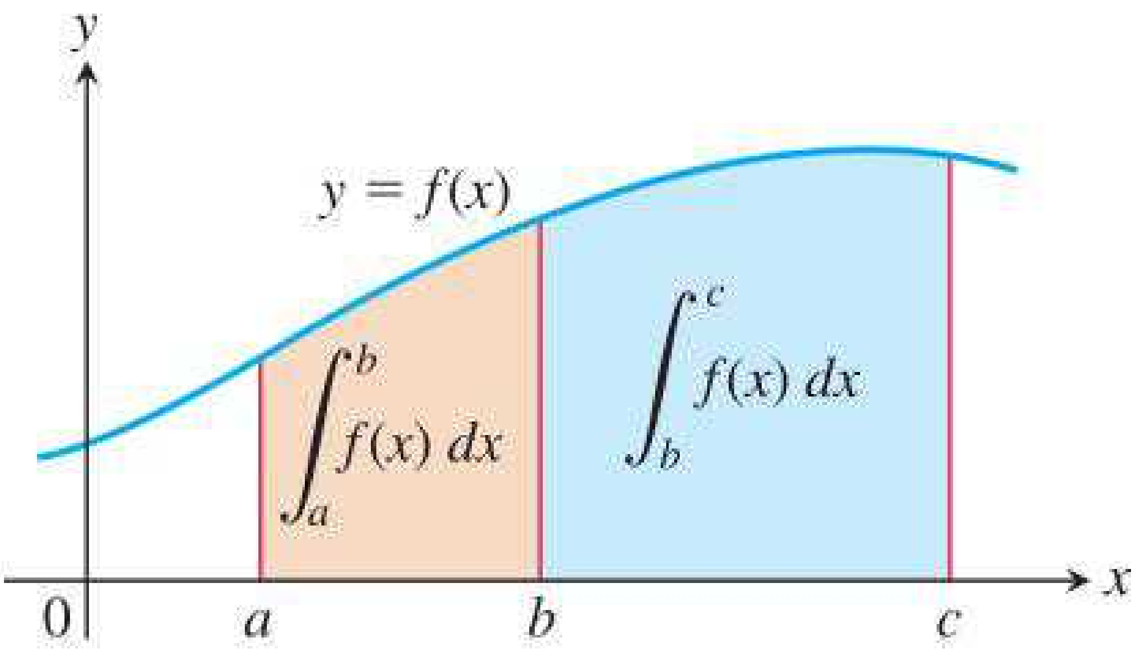
$$= \lim_{\|P\|\to 0}\sum_{[x_{i-1},x_i]_i\subset[a,b]} f(c_i)\Delta x_i + \lim_{\|P\|\to 0}\sum_{[x_{i-1},x_i]_i\subset[b,c]} f(c_i)\Delta x_i$$

$$= \int_a^b f(x)dx + \int_b^c f(x)dx$$

若$a<c<b$,则$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

$$\int_a^c f(x)dx = \int_a^b f(x)dx - \int_c^b f(x)dx$$

$$= \int_a^b f(x)dx + \int_b^c f(x)dx$$



(d) *Additivity for Definite Integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

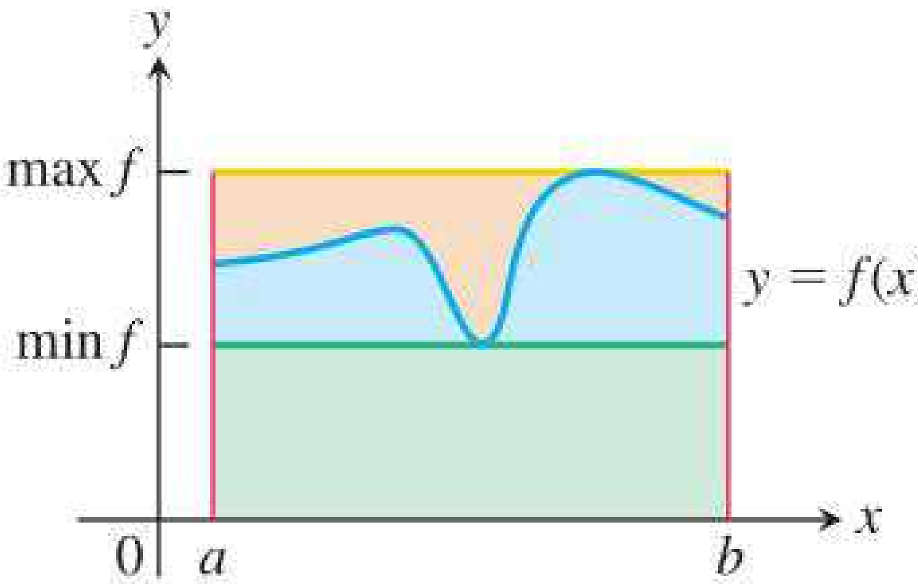
6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:*

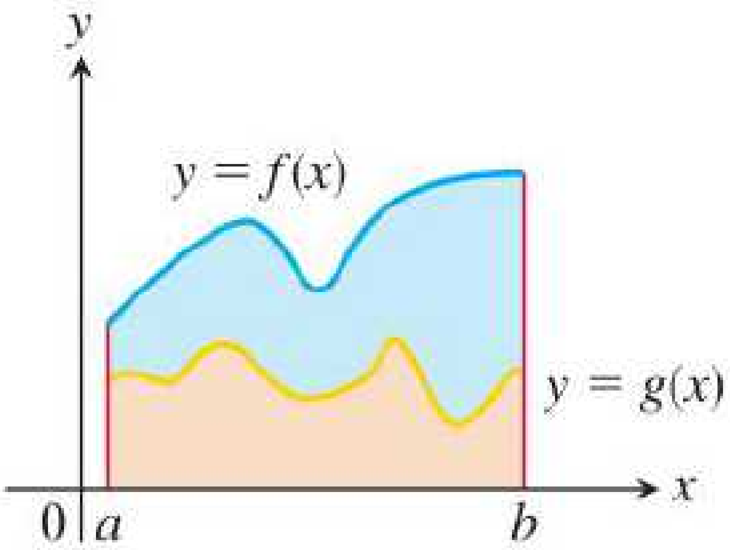
$$f(x) \ge g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

$$f(x) \ge 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$$ (Special case)



(e) *Max-Min Inequality:*

$$\min f \cdot (b-a) \leq \int_a^b f(x)\,dx$$
$$\leq \max f \cdot (b-a)$$



(f) *Domination:*

$f(x) \geq g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Proof 7:

If $f(x) \geq 0$ on $[a,b]$, then $\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i \geq 0$

$$\therefore \int_a^b f(x)dx \geq 0 \quad (a < b)$$

If $f(x) \geq g(x)$ on $[a,b]$, then $f(x) - g(x) \geq 0$, on $[a,b]$,

$$\therefore \int_a^b [f(x) - g(x)]dx \geq 0, \qquad \therefore \int_a^b f(x)dx - \int_a^b g(x)dx \geq 0.$$

Proof 6: $\because m \leq f(x) \leq M$,

$$\therefore \int_a^b m\,dx \leq \int_a^b f(x)dx \leq \int_a^b M\,dx,$$

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Ex. 4

Show that the value of $\int_0^1 \sqrt{1+\cos x}\,dx$ is less than or equal to $\sqrt{2}$.

Solution $$\int_0^1 \sqrt{1+\cos x}\,dx \le \max f \cdot (b-a)$$
$$= \sqrt{2}\cdot(1-0) = \sqrt{2}.$$

例.设 $M = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin x}{1+x^2}\cos^4 x dx$， $N = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^3 x + \cos^4 x)dx$，

$P = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^2 \sin^3 x - \cos^4 x)dx$，　　比较 M, N, P 的大小。

解：$P < M < N$。

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

Ex. 5 compute $\int_0^b x\,dx \qquad (b > 0)$

$$\int_0^b x\,dx = \frac{b^2}{2}.$$

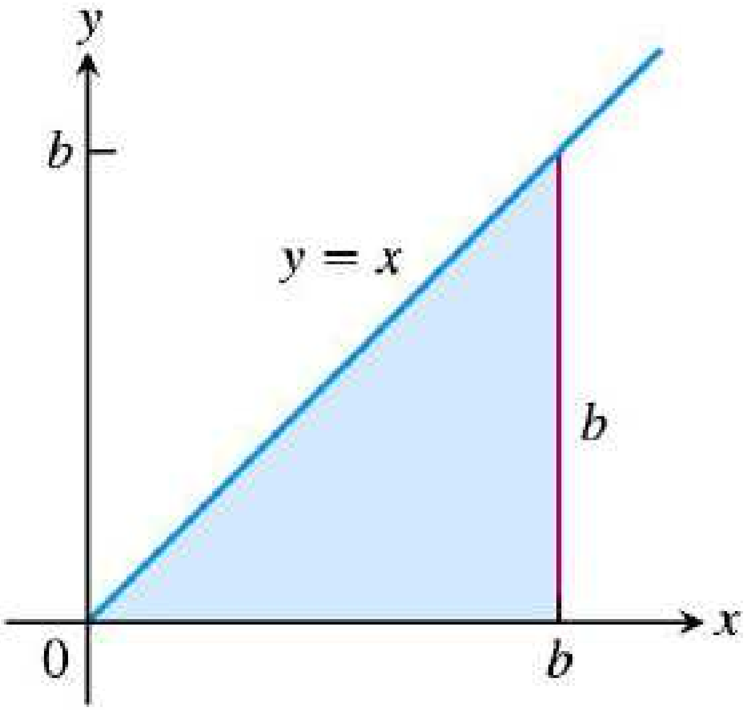
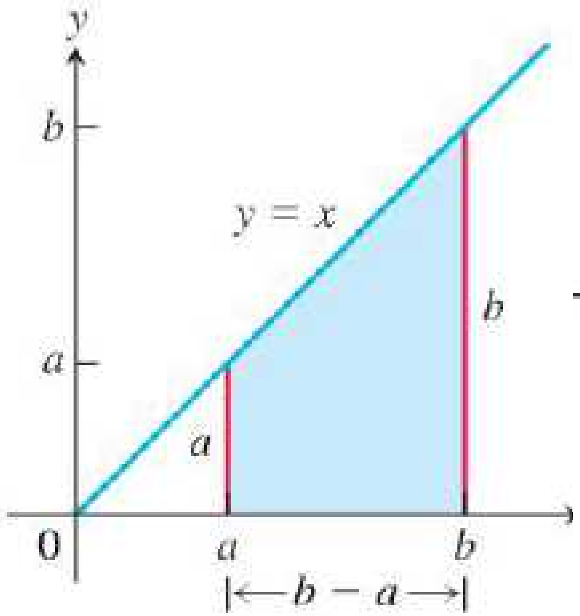
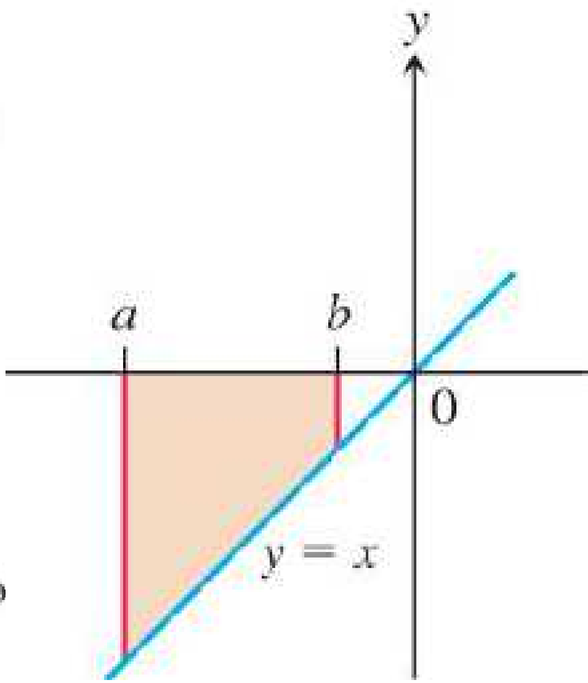
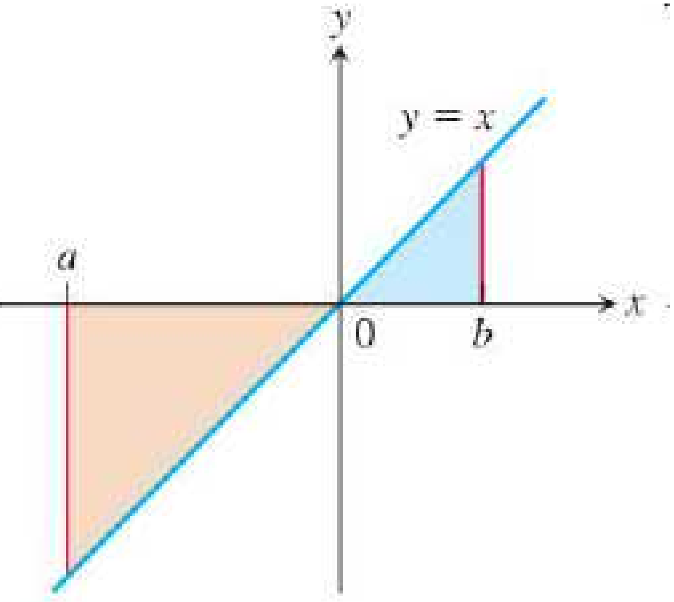


FIGURE 5.12 The region in Example 4 is a triangle.







$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}.$$

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}, \qquad a < b \tag{1}$$

$$\int_a^b c\,dx = c(b - a), \qquad c \text{ any constant} \tag{2}$$

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}, \qquad a < b \tag{3}$$

Ex.6 Using the definition of integral to evaluate $\int_0^1 x^2 dx$.

Solution Subdivide the interval $[0,1]$ by $x = \frac{i}{n}, i = 1,2,\cdots,n$

The width of $[x_{i-1}, x_i]$ is $\Delta x_i = \frac{1}{n}$, $(i = 1,2,\cdots,n)$

choose $c_i = \frac{i}{n}$, $i = 1,2,\cdots,n$, Riemann Sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$\int_0^1 x^2 dx = \lim_{\|P\|\to 0}\sum_{i=1}^{n} c_i^{\,2}\Delta x_i = \lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{1}{3}.$$

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is

$$\operatorname{av}(f) = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

Ex.7.Find the average of $f(x)=\sqrt{4-x^2}$ on [–2,2].

Solution

$$av(f)=\frac{\int_{-2}^{2}\sqrt{4-x^2}\,dx}{4}=\frac{2\pi}{4}=\frac{\pi}{2}$$

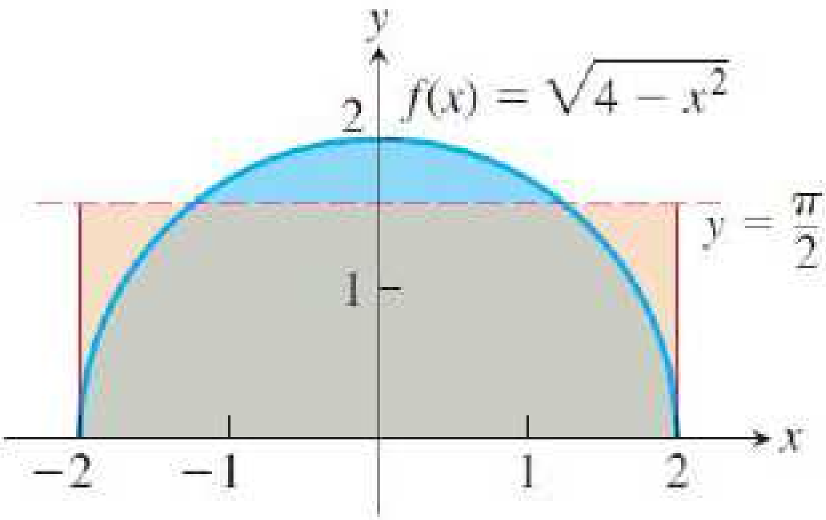


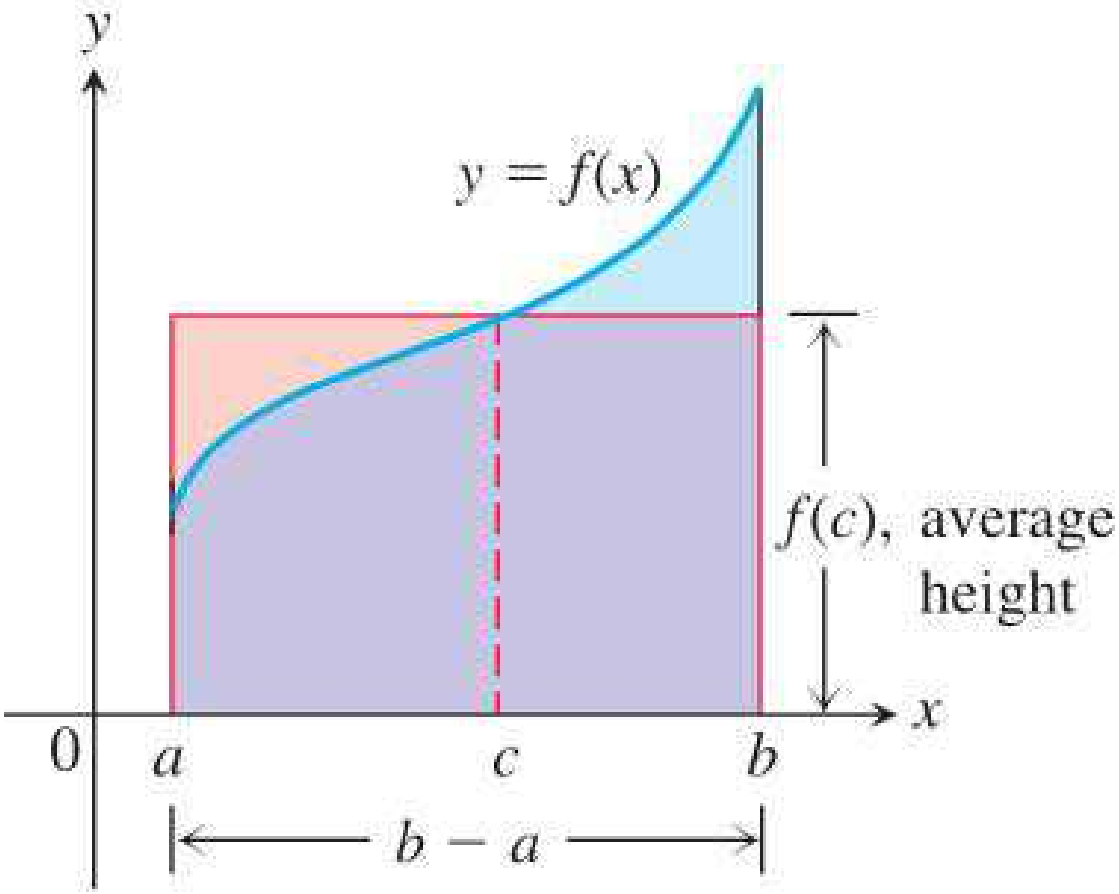
FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5). The area of the rectangle shown here is $4 \cdot (\pi/2) = 2\pi$, which is also the area of the semicircle.

5.4

The Fundamental Theorem of Calculus

微积分基本定理

$$\frac{\int_a^b f(x)dx}{b-a} = f(c)$$



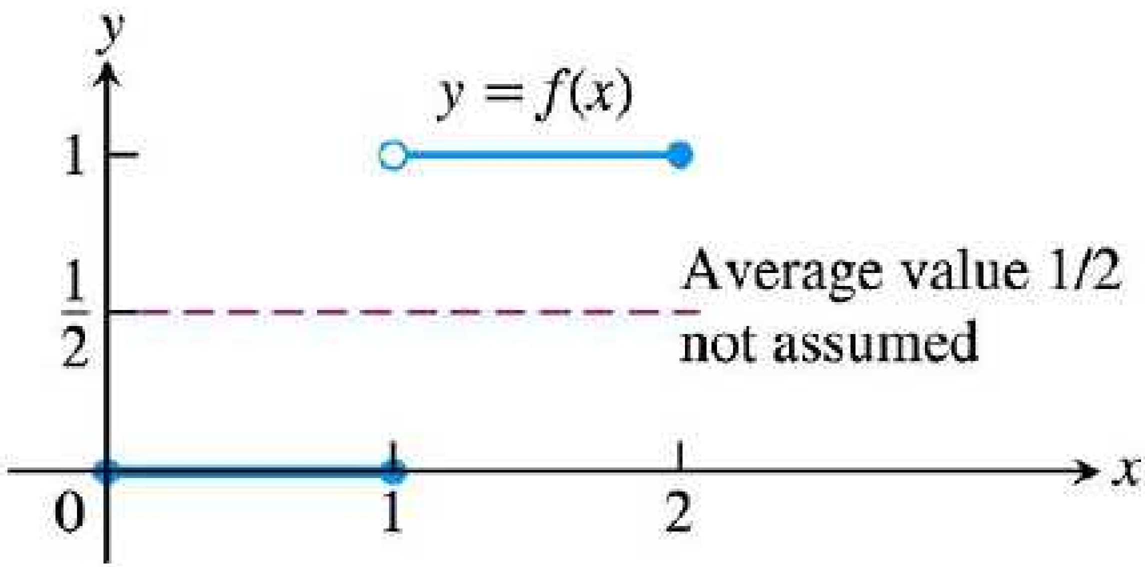


FIGURE 5.17 A discontinuous function need not assume its average value.

THEOREM 3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$$(b-a)\min f \le \int_a^b f(x)dx \le (b-a)\max f,$$

$$\min f \le \frac{\int_a^b f(x)dx}{b-a} \le \max f.$$

by the Intermediate Value theorem, there exist

$c \in [a,b]$ **such that** $$f(c) = \frac{\int_a^b f(x)dx}{b-a}.$$

Ex.5 Show that if f is continuous on $[a,b], a \neq b$, and if $\int_a^b f(x)dx = 0$, then $f(x) = 0$ at least once in $[a,b]$.

Solution $$av(f) = \frac{\int_a^b f(x)dx}{b-a} = 0$$

By the Mean Value Theorem, f assumes this value at some point $c \in [a,b]$.

设某物体作直线运动，$v(t)$是时间段$[T_1,T_2]$上t时刻的速度，是连续函数，且$v(t)\geq 0$，则在这段时间内物体所走过的路程为

$$\int_{T_1}^{T_2} v(t)\,dt$$

另一方面设s(t)是t时刻物体所走过的路程, 则在这段时间内物体所走过的路程为

$$s(T_2)-s(T_1)$$

$$\therefore \int_{T_1}^{T_2} v(t)\,dt = s(T_2)-s(T_1).$$ 其中 $s'(t)=v(t)$.

$$\boxed{\int_a^b f(x)\,dx \stackrel{?}{=} F(b)-F(a).}$$

$$s(x) - s(a) = \int_a^x v(t)\,dt$$

$$\left(\int_a^x v(t)\,dt\right)' = s'(x) = v(x)$$

$$F(x) = \int_a^x f(t)\,dt$$

$$F'(x) \stackrel{?}{=} f(x)$$

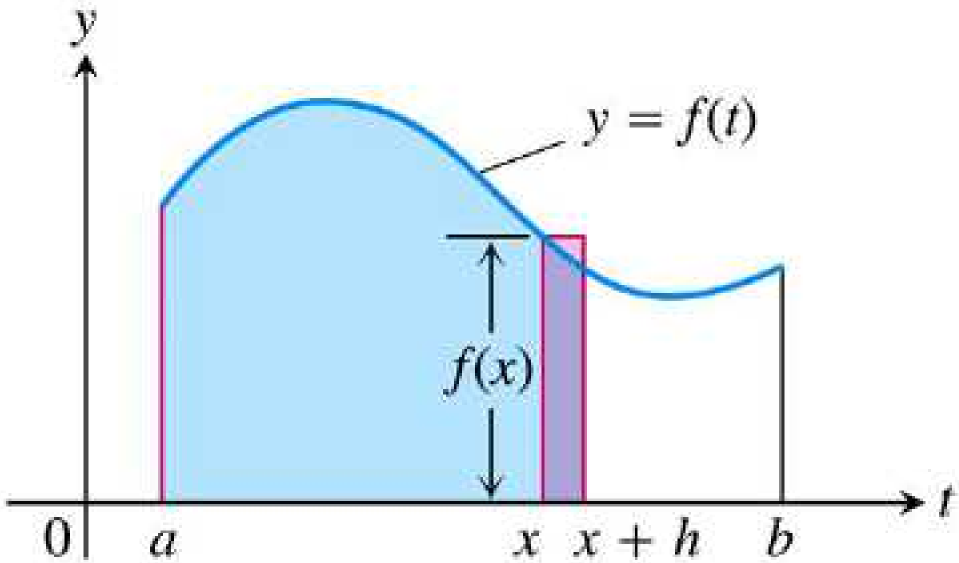


FIGURE 5.19 In Equation (1), $F(x)$ is the area to the left of x. Also, $F(x+h)$ is the area to the left of $x+h$. The difference quotient $[F(x+h)-F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

$$\forall x \in (a, b)$$

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x},$$

$$= \lim_{\Delta x \to 0} \frac{f(c)\Delta x}{\Delta x},$$

$$= \lim_{c \to x} f(c)$$

$$= f(x).$$

THEOREM 4—The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

***Ex*.2** **Find** $\frac{dy}{dx}$ **if**

(a) $y = \int_a^x (t^3+1)dt$

(b) $y = \int_x^5 3t\sin t dt$

(c) $y = \int_1^{x^2} \cos t dt$

(d) $y = \int_{1+3x^2}^4 \frac{1}{2+t} dt$

(e) $y = \int_{e^x}^{x^2} \sin(t^2)\mathrm{d}t$

$$y' = \left(\int_{e^x}^0 \sin(t^2)\mathrm{d}t + \int_0^{x^2} \sin(t^2)\mathrm{d}t\right)' = \sin(x^4)2x - \sin(e^{2x})e^x$$

$Ex.$ 设$y=\int_0^x (x-t)f(t)dt$，其中f连续,$\int_0^1 f(x)dx=1$,求$\frac{dy}{dx}\Big|_{x=1}$.

$$y=x\int_0^x f(t)dt-\int_0^x tf(t)dt, \qquad y'=\int_0^x f(t)dt, \qquad y'\big|_{x=1}=\int_0^1 f(t)dt=1.$$

$Ex.$ 设$y=y(x)$是由方程$x-\int_1^{x+y} e^{-t^2}dt=0$所确定的隐函数，求$\frac{dy}{dx}\Big|_{x=0}$.

$$1-e^{-(x+y)^2}(1+y')=0,$$

在方程$x-\int_1^{x+y} e^{-t^2}dt=0$中令$x=0$,得$\int_1^y e^{-t^2}dt=0$, 显然$y(0)=1$,

将$x=0$带入方程 $1-e^{-(x+y)^2}(1+y')=0$,得$y'(0)=e-1$.

$Ex.$求$\lim\limits_{x\to 0}\dfrac{\int_0^x e^{-t^2}dt}{x}=\lim\limits_{x\to 0}\dfrac{e^{-c^2}x}{x}=\lim\limits_{x\to 0}e^{-c^2}=1.$

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2
If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

证 $\because$ 已知$F(x)$是$f(x)$的一个原函数，

又$\because$ $\Phi(x)=\int_a^x f(t)dt$也是$f(x)$的一个原函数，

$\therefore F(x)-\Phi(x)=C \quad x\in[a,b]$

$x=a \quad \Rightarrow \quad F(a)-\Phi(a)=C, \quad \Phi(a)=0 \Rightarrow C=F(a)$

$\therefore \int_a^x f(t)dt = F(x)-F(a),$

$x=b \Rightarrow \quad \int_a^b f(x)dx = F(b)-F(a).$

牛顿—莱布尼茨公式

$$\int_a^b f(x)dx = F(b) - F(a).$$

The evaluation theorem

注意 当$a > b$时，$\int_a^b f(x)dx = F(b) - F(a)$仍成立.

***Ex*.3 Calculate several definite integrals using the Evaluation Theorem.**

(a) $\int_0^{\pi} \cos x dx$ $= 0$

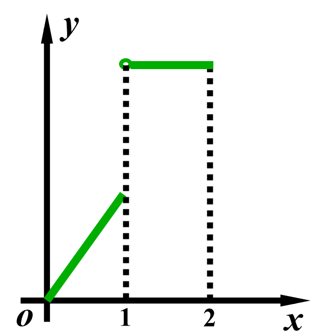
(b) $\int_{-\pi/4}^{0} \sec x \tan x dx$ $= 1 - \sqrt{2}$

(c) $\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)dx$ $= 4$

(d) $\int_0^2 f(x)dx \quad f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 5 & 1 < x \le 2 \end{cases}$

$$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$$

$$= \int_0^1 2x dx + \int_1^2 5dx = 6.$$



THEOREM 5—The Net Change Theorem The net change in a differentiable function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\,dx. \tag{6}$$

$Ex.4$ Interpretation of the Net Change Theorem.

$$\int_{t_1}^{t_2} s'(t)dt = s(t_2) - s(t_1).$$

Ex.5 一次爆炸将一块岩石以初速度$49m/s$上抛，t秒时其速度是$v(t)=49-9.8t(m/s)$.

(a)求在时段$[0,8]$内岩石的位移量(displacement)?

(b)求在时段$[0,8]$内岩石运动的总距离(disdance)?

解(a) $$\int_0^8 (49-9.8t)dt=(49t-4.9t^2)\Big|_0^8=78.4m.$$

(b) $$\int_0^8 |49-9.8t|dt=\int_0^5 |49-9.8t|dt+\int_5^8 |49-9.8t|dt$$

$$=\int_0^5 (49-9.8t)dt-\int_5^8 (49-9.8t)dt$$

$$=(49t-4.9t^2)\Big|_0^5-(49t-4.9t^2)\Big|_5^8$$

$$=122.5-(-44.1)=166.6$$

Ex.6 $f(x) = \sin x$ Compute

(a) the definite integral of $f(x)$ over $[0, 2\pi]$.

(b) the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.

Solution

$$\int_0^{2\pi} \sin x\,dx = -\cos x\Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

$$\int_0^{\pi} \sin x\,dx = -\cos x\Big]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2$$

$$\int_{\pi}^{2\pi} \sin x\,dx = -\cos x\Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2$$

$$\text{Area} = |2| + |-2| = 4.$$

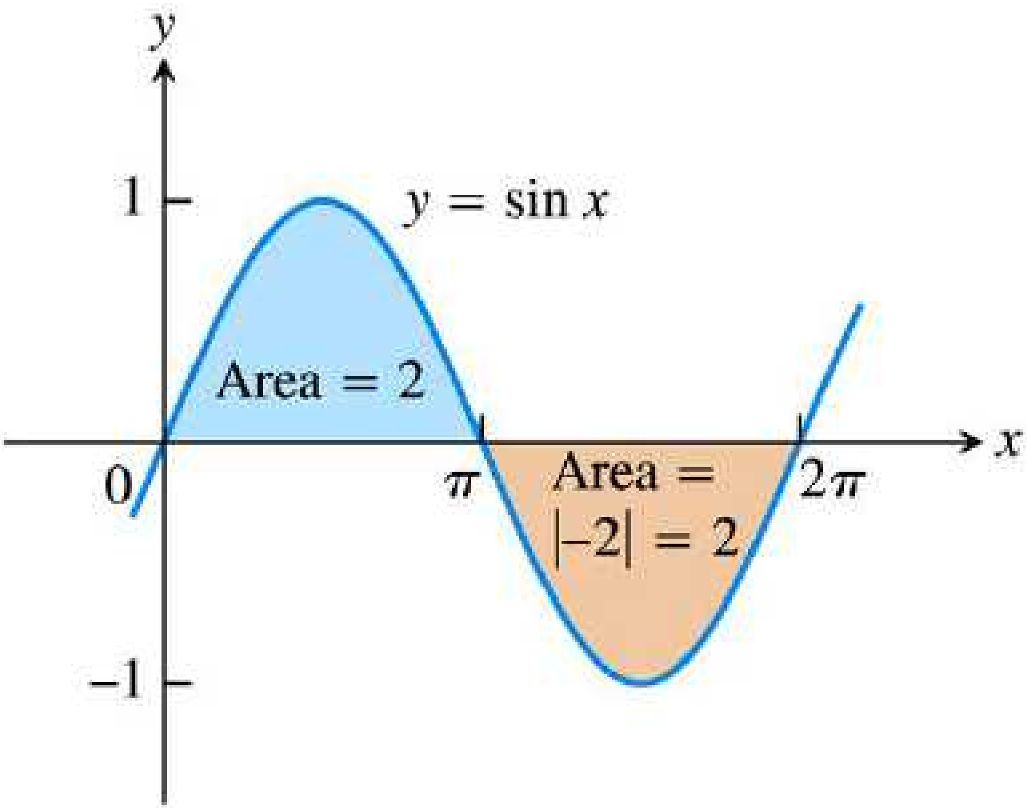


FIGURE 5.21 The total area between $y = \sin x$ and the x-axis for $0 \le x \le 2\pi$ is the sum of the absolute values of two integrals (Example 7).

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Ex.7

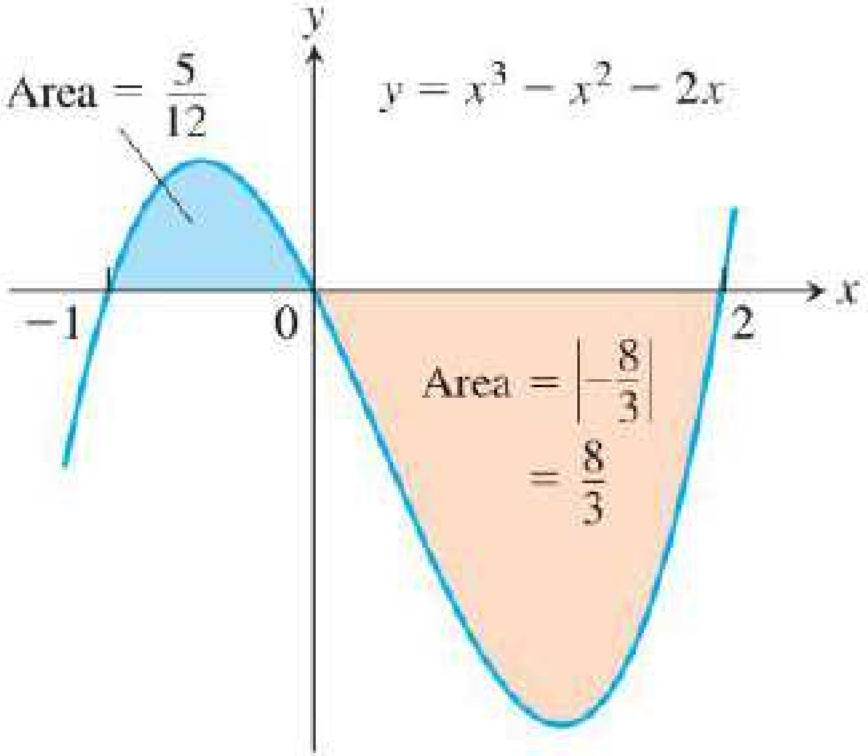
Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x,\ -1 \le x \le 2.$

Solution the zeros are $x = 0, -1$, and 2

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x + 1)(x - 2),$$

$$A = \int_{-1}^{0}(x^3 - x^2 - 2x)\,dx - \int_{0}^{2}(x^3 - x^2 - 2x)\,dx$$

$$= \frac{5}{12} + \frac{8}{3} = \frac{37}{12}.$$



Ex.8 Evaluate $\int_0^{\frac{\pi}{2}} \left| \frac{1}{2} - \sin x \right| dx$

Solute $\int_0^{\frac{\pi}{2}} \left| \frac{1}{2} - \sin x \right| dx = \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \sin x\right) dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\sin x - \frac{1}{2}\right) dx$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \quad - \frac{\pi}{6} + \frac{\sqrt{3}}{2} \qquad = \sqrt{3} - 1 - \frac{\pi}{12}.$$

Ex.9 Evaluate $\lim_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}} \ (p \neq -1)$

Solute $\lim_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}} = \lim_{n\to\infty} \left(\left(\frac{1}{n}\right)^p + \left(\frac{2}{n}\right)^p + \cdots + \left(\frac{n}{n}\right)^p \right) \frac{1}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^p \frac{1}{n} \quad = \int_0^1 x^p dx = \frac{1}{p+1}$$

5.5

Indefinite Integrals and the Substitution Method
不定积分和换元法

$$\int f(x)dx = F(x) + C$$

$$\int_a^b f(x)dx = F(b) - F(a) \quad = \left(\int f(x)dx\right)\Big|_a^b$$

(1) $\int kdx = kx + C$ （k是常数）;

(2) $\int x^k dx = \dfrac{x^{k+1}}{k+1} + C$ （k常数）

(3) $\int \cos xdx = \sin x + C$;

(4) $\int \sin xdx = -\cos x + C$;

(5) $\int \dfrac{dx}{\cos^2 x} = \int \sec^2 xdx = \tan x + C$;

(6) $\int \dfrac{dx}{\sin^2 x} = \int \csc^2 xdx = -\cot x + C$;

(7) $\int \sec x \tan xdx = \sec x + C$;

(8) $\int \csc x \cot xdx = -\csc x + C$;

THEOREM 6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du. \qquad u = g(x)$$

Proof: suppose $F'(u) = f(u)$,

Then $\int f(u)du = F(u) + C$ 连续函数的原函数存在

$\because \quad (F[g(x)])' = f[g(x)]g'(x)$

$$\therefore \quad \int f[g(x)]g'(x)dx = F[g(x)] + C = (F(u) + C)_{u=g(x)}$$

$$= [\int f(u)du]_{u=\varphi(x)}$$

The Substitution Method to evaluate $\int f(g(x))g'(x)\,dx$

1. Substitute $u = g(x)$ and $du = (du/dx)\,dx = g'(x)\,dx$ to obtain $\int f(u)\,du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

***Ex*.1** **Find the integral** $\int (x^3 + x)^5 (3x^2 + 1)dx$.

solution $\int (x^3 + x)^5 (3x^2 + 1)dx$

$$= \int (x^3 + x)^5 d(x^3 + x)$$

$$= \int u^5 du (u = x^3 + x) = \frac{u^6}{6} + C = \frac{(x^3 + x)^6}{6} + C.$$

Ex. 2 Find $\int \sin 2x dx$.

Solution 1. $\int \sin 2x dx = \frac{1}{2}\int \sin 2x d(2x) = -\frac{1}{2}\cos 2x + C$;

2. $\int \sin 2x dx = 2\int \sin x \cos x dx = 2\int \sin x d(\sin x)$

$= (\sin x)^2 + C$;

3. $\int \sin 2x dx = 2\int \sin x \cos x dx = -2\int \cos x d(\cos x)$

$= -(\cos x)^2 + C$.

Ex. 3 求 $\int \sec^2(5x+1)dx$.

Solution $\int \sec^2(5x+1)dx = \frac{1}{5}\int \sec^2(5x+1)d(5x+1)$

$$= \frac{1}{5}\int \sec^2 u du = \frac{1}{5}\tan u + C = \frac{1}{5}\tan(5x+1) + C$$

Ex. 4 Find $\int x^2 \cos x^3 dx$.

Solution $\int x^2 \cos x^3 dx = \frac{1}{3}\int \cos x^3 d(x^3) = \frac{1}{3}\sin x^3 + C$.

Ex. 5 Find (a) $\int \cos^2 x dx$. (b) $\int \sin^2 x dx$. (c) $\int (1-2\sin^2 x)\sin 2x dx$.

Solution

(a) $\int \cos^2 x dx = \frac{1}{2}\int (1+\cos 2x)dx = \frac{x}{2} + \frac{1}{2}\cdot\frac{\sin 2x}{2} + C$.

(b) $\int \sin^2 x dx = \frac{1}{2}\int (1-\cos 2x)dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

(c) $\int (1-2\sin^2 x)\sin 2x dx = \int \cos 2x \sin 2x dx$

$= \frac{1}{2}\int \sin 4x dx = -\frac{\cos 4x}{8} + C$.

Ex. 6 Find $\displaystyle\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz.$

Solution $\displaystyle\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz = \int \frac{1}{\sqrt[3]{z^2+1}}\,d(z^2+1) = \int u^{-1/3}\,du$

$$= \frac{3}{2}u^{2/3} + C = \frac{3}{2}(z^2+1)^{2/3} + C$$

Solution $\sqrt[3]{z^2+1} = u, du = \dfrac{1}{3}(z^2+1)^{-2/3}2zdz,\ du = \dfrac{1}{3}\dfrac{1}{u^2}2zdz$

$$\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz = \int \frac{2z}{u}\,dz = 3\int u\,du = \frac{3}{2}u^2 + C \qquad 2zdz = 3u^2du$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$

Ex. 7 **Find** $\int x\sqrt{2x+1}\,dx.$

Solution $u=\sqrt{2x+1}, du=\dfrac{1}{\sqrt{2x+1}}dx=\dfrac{dx}{u}, x=\dfrac{u^2-1}{2},$

$$dx = udu$$

$$\int x\sqrt{2x+1}\,dx = \int \frac{(u^2-1)u^2}{2}du = \frac{1}{2}\int (u^4-u^2)du$$

$$= \frac{u^5}{10}-\frac{u^3}{6}+C = \frac{\sqrt{2x+1}^{\,5}}{10}-\frac{\sqrt{2x+1}^{\,3}}{6}+C.$$

Find $\int \sin^4 x dx$.

Find $\int \sin^3 x dx$.

Find $\int \tan^3 x \cdot \sec^5 x dx$.

solution

$$\int \tan^3 x \cdot \sec^5 x dx = \int \tan^2 x \cdot \sec^4 x d \sec x$$

$$= \int (\sec^2 x - 1) \cdot \sec^4 x d \sec x$$

$$= \int (\sec^6 x - \sec^4 x) d \sec x$$

$$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$$

Find $\int \cos 3x \cos 2x dx.$ $\int \sin 3x \sin 2x dx.$

solution $\int \sin 3x \cos 2x dx.$

$$\cos 3x \cos 2x = \frac{1}{2}(\cos x + \cos 5x),$$

$$\int \cos 3x \cos 2x dx = \frac{1}{2}\int (\cos x + \cos 5x) dx$$

$$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C.$$

5.6

Definite Integral Substitutions and the Area Between Curves

定积分换元法和曲线间的面积

THEOREM 7—Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

证明：设 $F(u)$ 是 $f(u)$ 的一个原函数，

$$\int_{g(a)}^{g(b)} f(u)du = F(g(b)) - F(g(a)),$$

$$\left(F(g(x))\right)' = f[g(x)]g'(x),$$

$\therefore F(g(x))$ 是 $f[g(x)]g'(x)$ 的一个原函数.

$$\int_a^b f[g(x)]g'(x)dx = F[g(b)] - F[g(a)]$$

$$\int_a^b f[g(x)]g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Ex. 1 Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx.$

解 $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_{-1}^{1}\sqrt{x^3+1}\,d(x^3+1)$

$= \int_{0}^{2}\sqrt{u}\,du\,(u = x^3+1) = \frac{2}{3}u^{3/2}\Big|_0^2 = \frac{4\sqrt{2}}{3}.$

Ex. 2 Evaluate (a) $\int_{\pi/4}^{\pi/2}\cot x\csc^2 x\,dx.$ (b) $\int_{-\pi/4}^{\pi/4}\tan x\,dx.$

解

(a) $\int_{\pi/4}^{\pi/2}\cot x\csc^2 x\,dx = -\int_{\pi/4}^{\pi/2}\cot x\,d\cot x$

$= -\int_{1}^{0} u\,du = \frac{1}{2}.$

(b) $\int_{-\pi/4}^{\pi/4}\tan x\,dx = 0.$

Ex. 3 **Evaluate** $\int_0^1 x\sqrt{1-x^2}\,dx.$

解 $\int_0^1 x\sqrt{1-x^2}\,dx. = -\frac{1}{2}\int_0^1 \sqrt{1-x^2}\,d(1-x^2)$

$= -\frac{1}{2}\int_1^0 \sqrt{u}\,du (u = 1 - x^2)$

$= \frac{1}{3}u^{3/2}\Big|_0^1 = \frac{1}{3}.$

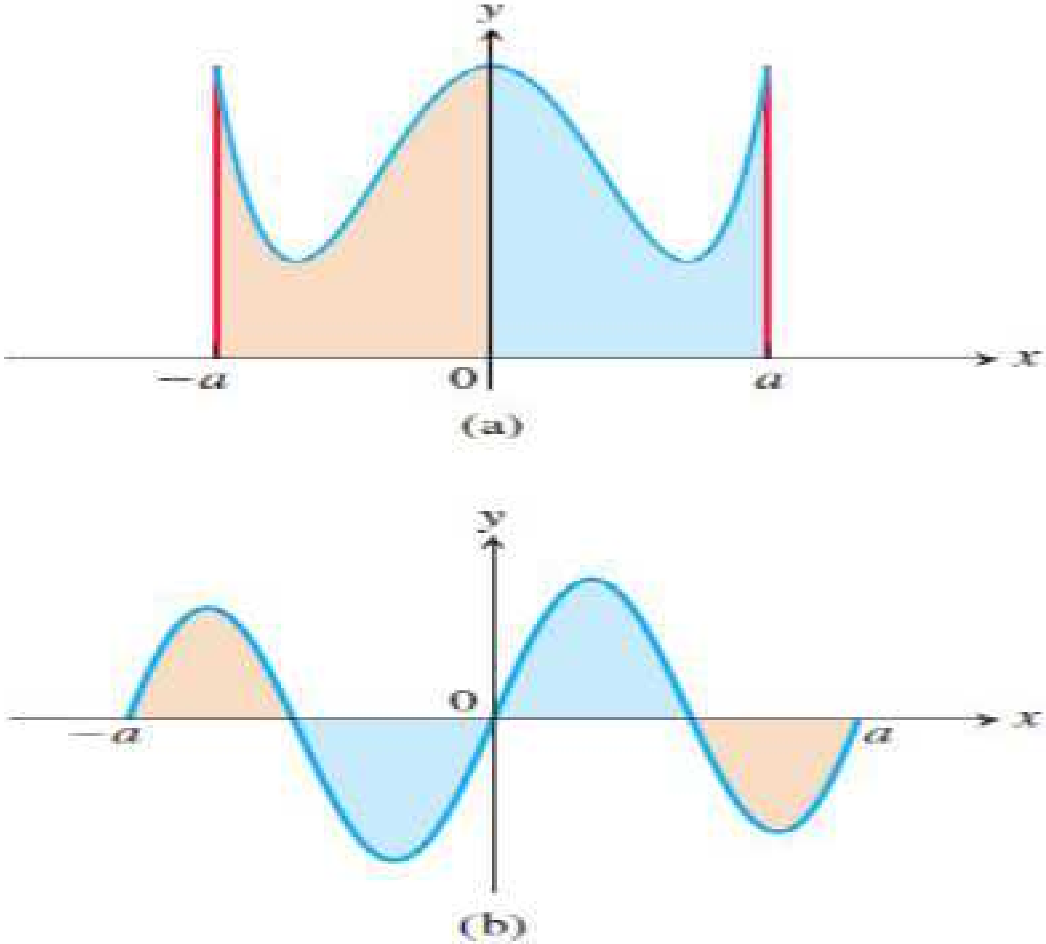


FIGURE 5.23 (a) For f an even function, the integral from $-a$ to a is twice the integral from 0 to a. (b) For f an odd function, the integral from $-a$ to a equals 0.

THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

证 $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx,$

在$\int_{-a}^{0} f(x)dx$中令$x = -t,$

$$\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-t)dt = \int_{0}^{a} f(-t)dt = \int_{0}^{a} f(t)dt = \int_{0}^{a} f(x)dx$$

Ex.4 Evaluate $\int_{-1}^{1} \frac{2x^2 + x\cos x}{1+\sqrt{1-x^2}}\,dx.$

Solution $\int_{-1}^{1} \frac{2x^2 + x\cos x}{1+\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \frac{2x^2}{1+\sqrt{1-x^2}}\,dx + \int_{-1}^{1} \frac{x\cos x}{1+\sqrt{1-x^2}}\,dx$

$$= 4\int_{0}^{1} \frac{x^2}{1+\sqrt{1-x^2}}\,dx = 4\int_{0}^{1} \frac{x^2(1-\sqrt{1-x^2})}{1-(1-x^2)}\,dx$$

$$= 4\int_{0}^{1} (1-\sqrt{1-x^2})\,dx = 4 - 4\int_{0}^{1} \sqrt{1-x^2}\,dx = 4-\pi.$$

Ex. Proof $\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx.$

Proof Let $x = \frac{\pi}{2} - u$, then

$$\int_0^{\pi/2} \sin^n x dx = -\int_{\pi/2}^{0} \cos^n u du = \int_0^{\pi/2} \cos^n x dx$$

计算$\int_0^{\frac{\pi}{2}} \frac{\cos^{10} x}{\sin^{10} x + \cos^{10} x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^{10} x}{\sin^{10} x + \cos^{10} x} dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^{10} x}{\sin^{10} x + \cos^{10} x} dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^{10} x}{\sin^{10} x + \cos^{10} x} dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 dx = \pi / 4.$$

例 若 $f(x)$ 是在 $(-\infty,+\infty)$ 内以 T 为周期的连续函数，证明：对于任何的实数 a，有

$$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx.$$

证： $\int_0^T f(x)dx = \int_0^a f(x)dx + \int_a^{a+T} f(x)dx + \int_{a+T}^T f(x)dx$

令 $u = x - T$， $\int_{a+T}^T f(x)dx = \int_a^0 f(u+T)du = -\int_0^a f(u)du$

所以 $\int_0^T f(x)dx = \int_a^{a+T} f(x)dx.$

计算 $\int_2^{2+100\pi} |\sin x|\,dx = \int_0^{100\pi} |\sin x|\,dx$

$$= \int_0^{\pi} |\sin x|\,dx + \int_{\pi}^{2\pi} |\sin x|\,dx + \cdots + \int_{99\pi}^{100\pi} |\sin x|\,dx$$

$$= 100\int_0^{\pi} |\sin x|\,dx = 100\int_0^{\pi} \sin x dx = 200.$$

例 计算 $\dfrac{d(\int_0^1 f(x-t)dt)}{dx}$,其中$f(x)$连续。

解. 令$x-t=u$, $\int_0^1 f(x-t)dt = -\int_x^{x-1} f(u)du$

$$\therefore \frac{d(\int_0^1 f(x-t)dt)}{dx} = \frac{d\int_{x-1}^{x} f(u)du}{dx} = f(x)-f(x-1)$$

设$F'(x)=f(x)$,

$$\int_0^1 f(x-t)dt = -\int_0^1 f(x-t)d(x-t)$$

$$= -\int_x^{x-1} f(u)du = -F(u)\Big|_x^{x-1} = F(x)-F(x-1)$$

$$\therefore \frac{d(\int_0^1 f(x-t)dt)}{dx} = f(x)-f(x-1)$$

曲线之间的面积

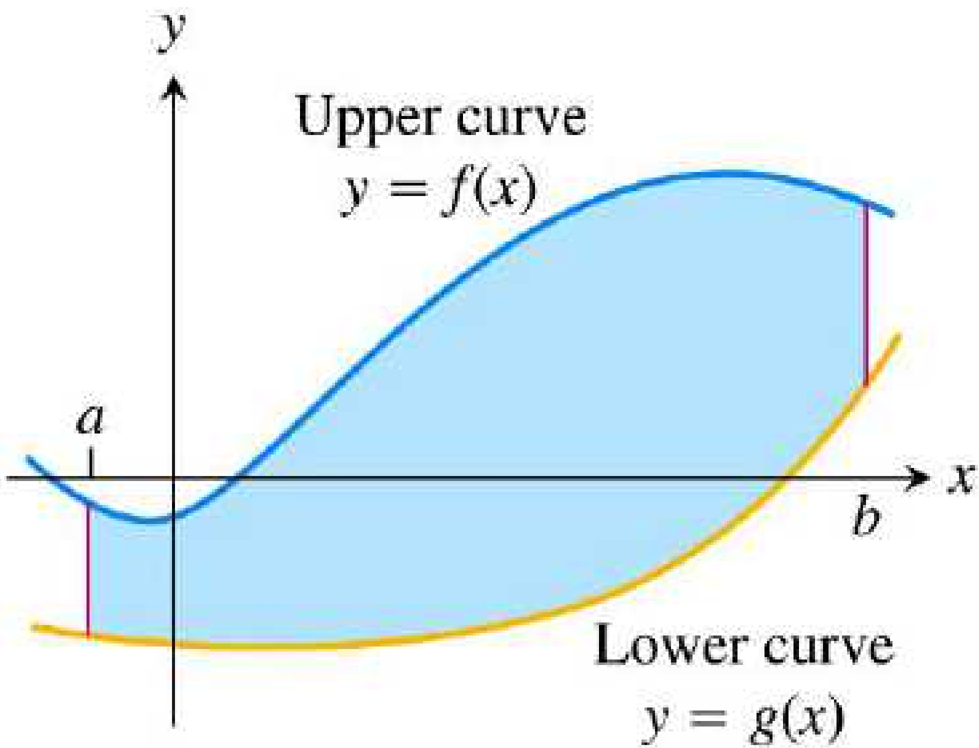


FIGURE 5.25 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

分割 $$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

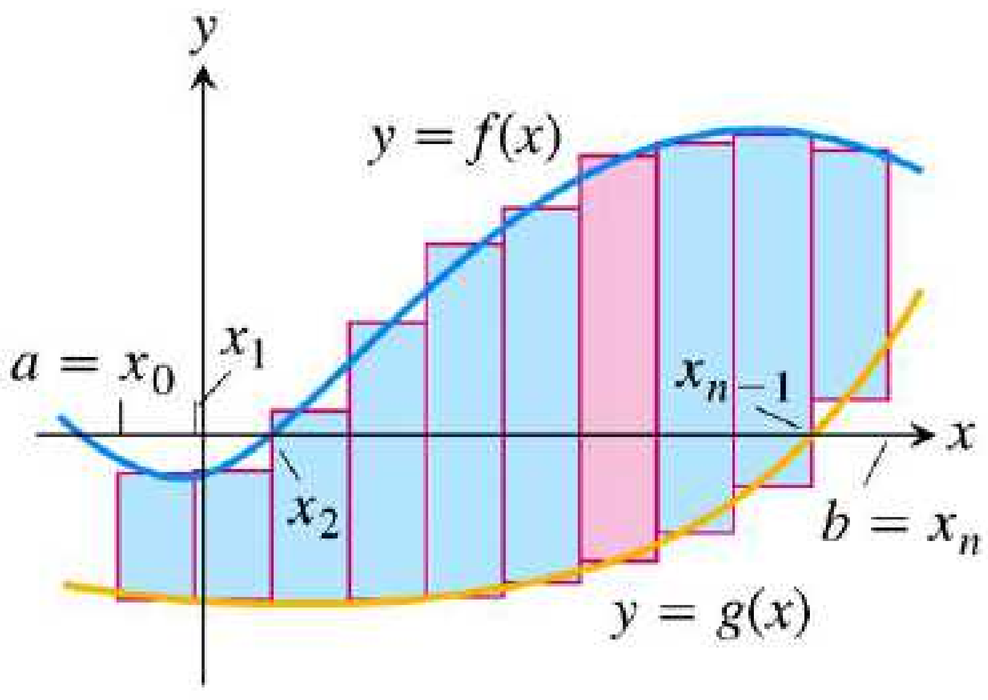


FIGURE 5.26 We approximate the region with rectangles perpendicular to the x-axis.

近似

$$\Delta A_k \approx (f(c_k) - g(c_k))\Delta x_k$$

求和

$$A \approx \sum_{k=1}^{n} (f(c_k) - g(c_k))\Delta x_k$$

取极限

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} (f(c_k) - g(c_k))\Delta x_k$$

$$A = \int_a^b [f(x) - g(x)]dx$$

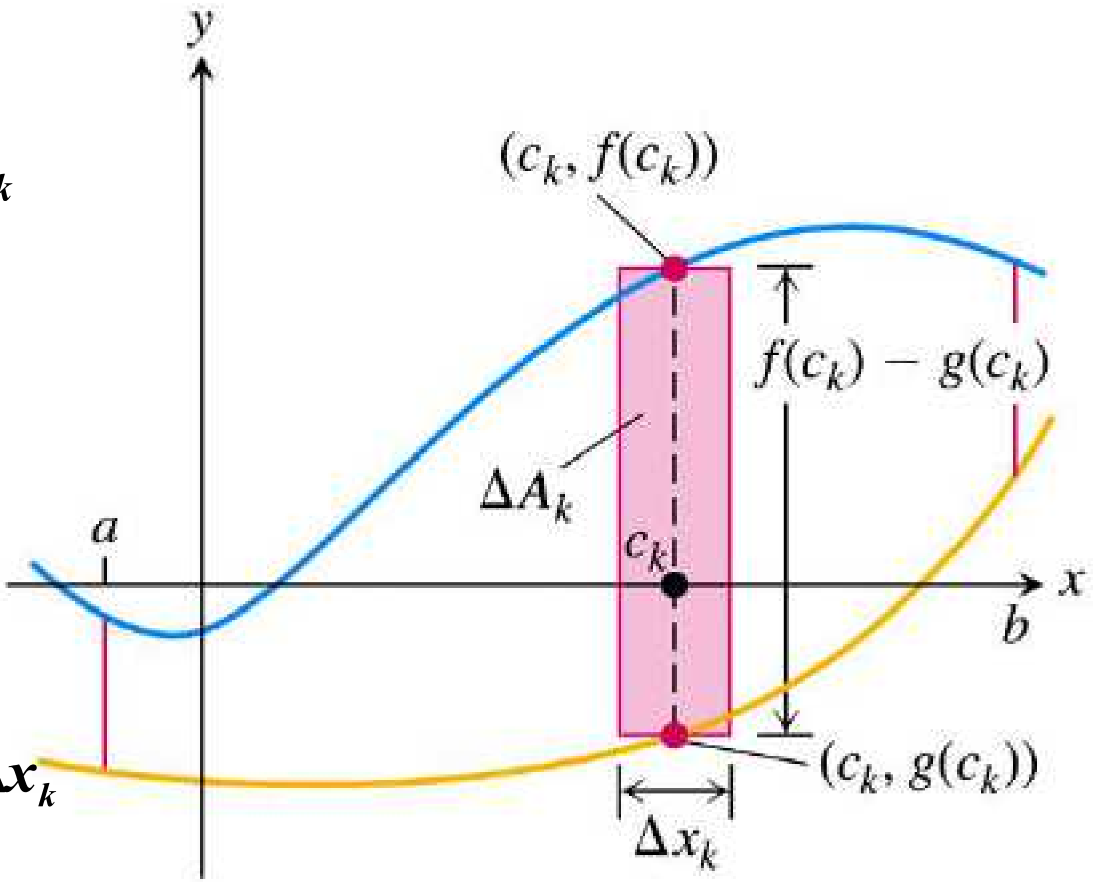


FIGURE 5.27 The area ΔA_k of the kth rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k.

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

Ex.5 Find the area of the region enclosed by $y = 2 - x^2$ and $y = -x$.

Solution $A = \int_{-1}^{2} (2 - x^2 + x)\,dx$

$$= \frac{9}{2}.$$

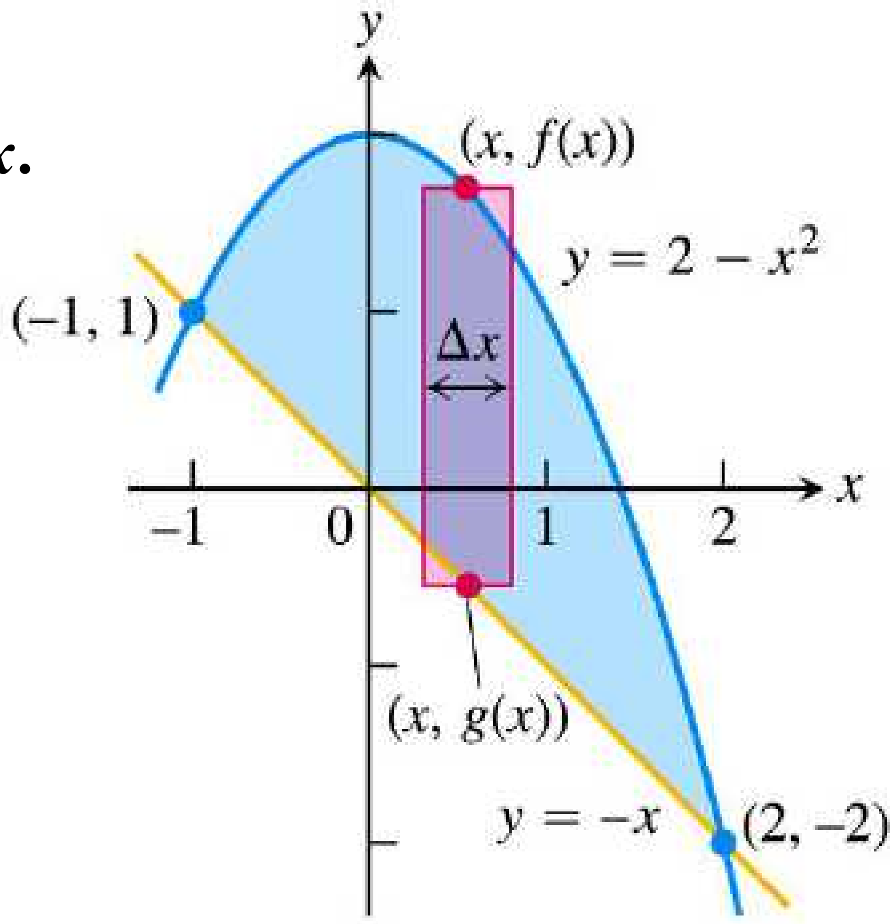


FIGURE 5.28 The region in Example 4 with a typical approximating rectangle.

Ex.6 **Find the area of the region in the first quadrant enclosed by $y=\sqrt{x}$, x - axis and $y = x-2$.**

Solution $A=\int_0^2 \sqrt{x}\,dx$

$$+\int_2^4 [\sqrt{x}-(x-2)]dx$$

$$=\int_0^4 \sqrt{x}\,dx-\int_2^4 (x-2)dx$$

$$=\frac{10}{3}.$$

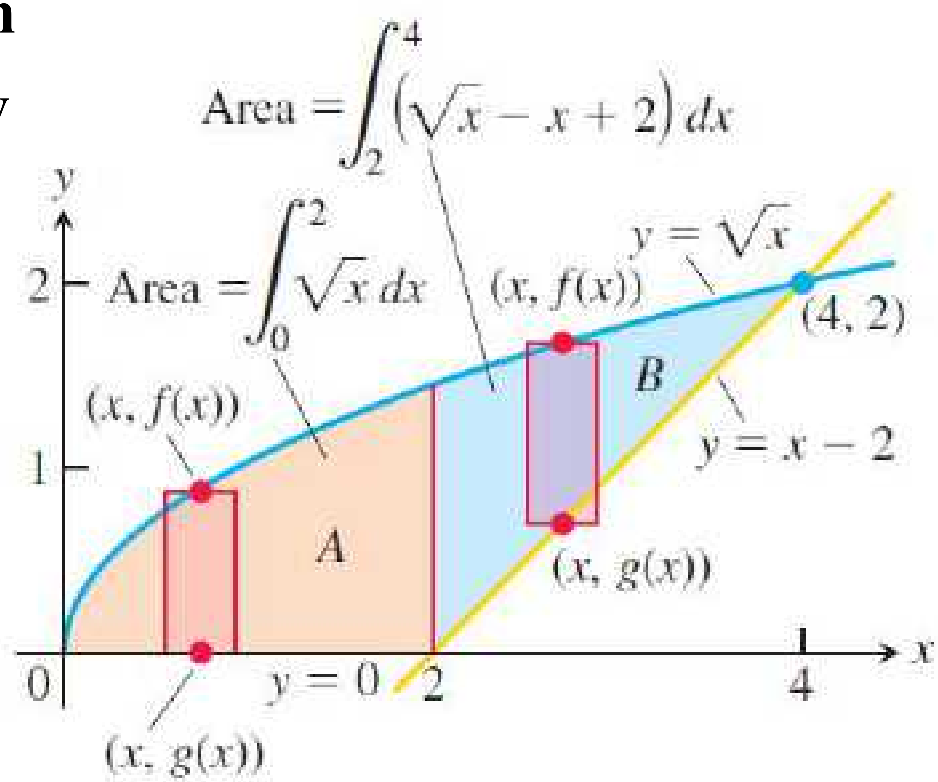
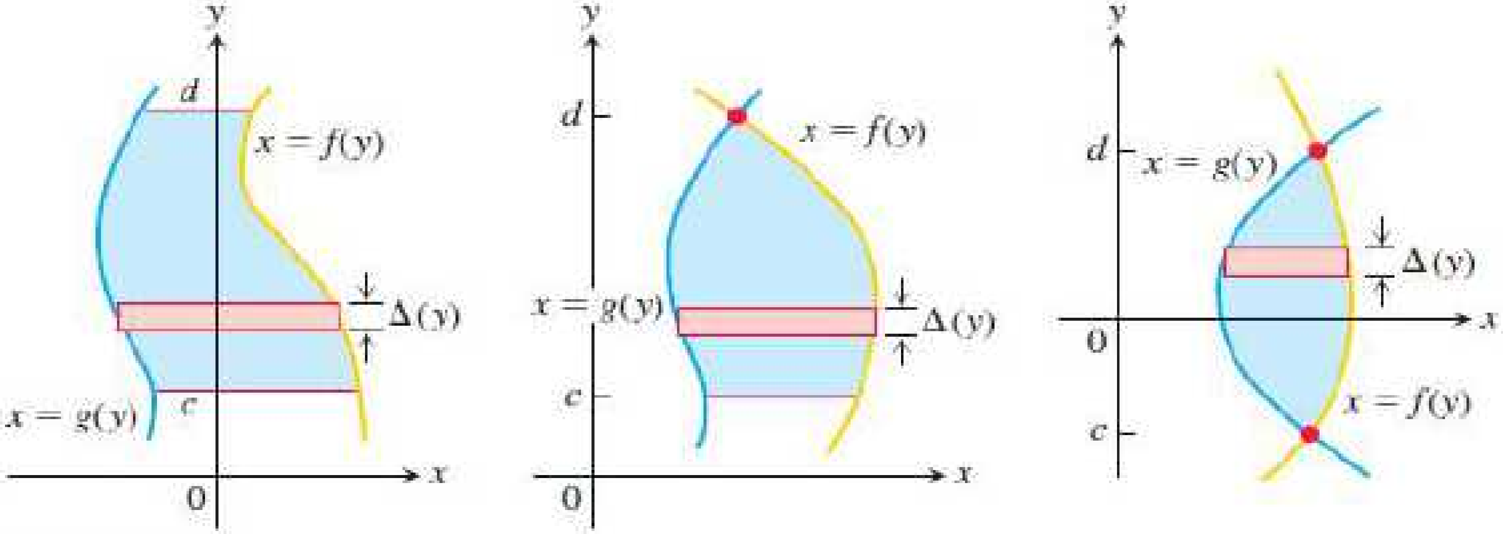


FIGURE 5.28 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 6.

Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these:



use the formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

Ex.7 Find the area of the region in the first quadrant enclosed by $y=\sqrt{x}$, x-axis and $y=x-2$.

Solution $A=\int_0^2 [y+2-y^2]dy$

$$=\frac{10}{3}.$$

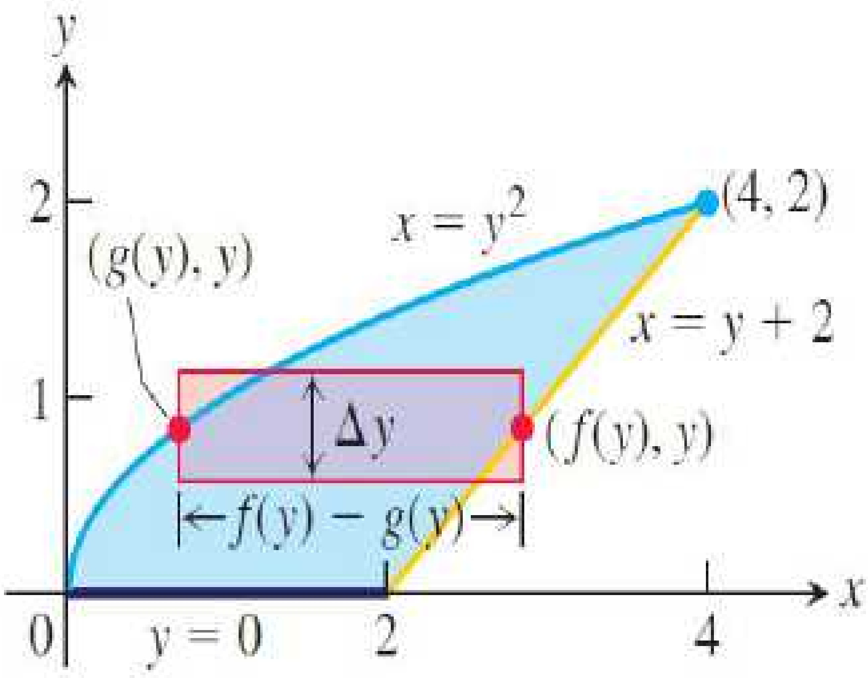


FIGURE 5.29 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 6).

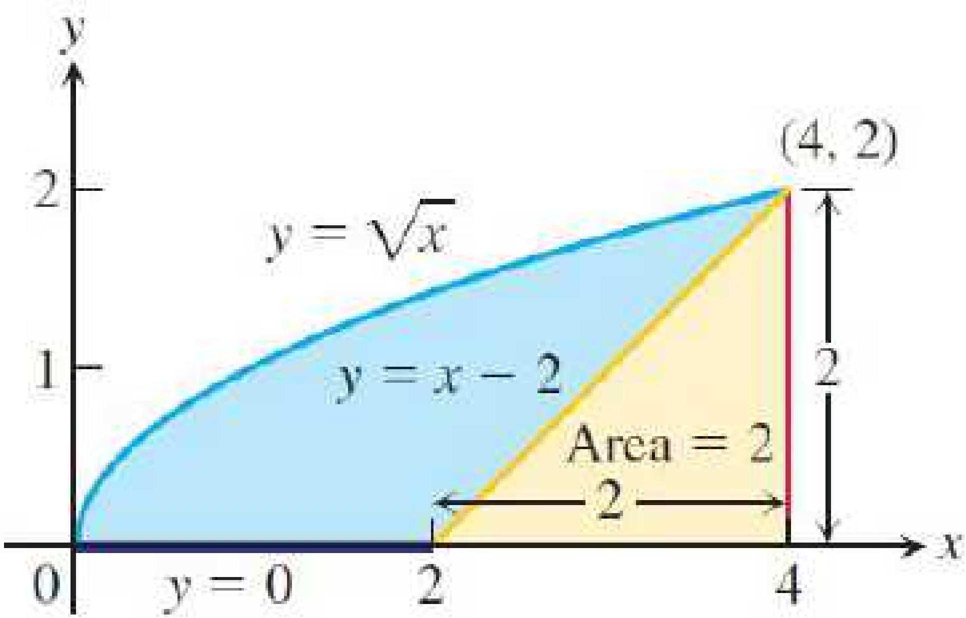


FIGURE 5.30 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

Ex.8 **Find the area of the region bounded below by $y = 2 - x$, and above by $y = \sqrt{2x - x^2}$.**

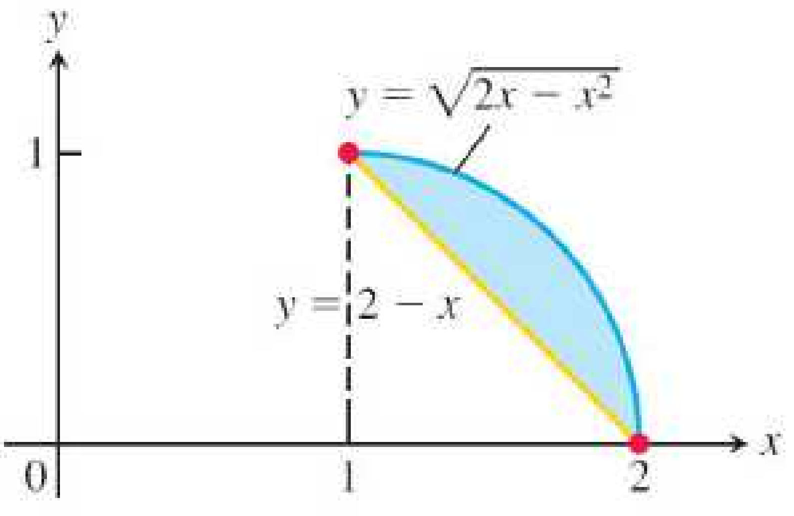


FIGURE 5.31 The region described by the curves in Example 7.

Solution $A = \int_1^2 [\sqrt{2x - x^2} - (2 - x)]dx$

$$= \int_0^1 [(1 + \sqrt{1 - y^2}) - (2 - y)]dy = \int_0^1 [\sqrt{1 - y^2} + y - 1]dy$$

$$= \frac{\pi}{4} + \frac{1}{2} - 1 = \frac{\pi}{4} - \frac{1}{2}.$$