Probability and Statistics
Tutorial 3

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Outline

Review

1. Definition of Conditional Probability $P(A|B)$
   - Given two events A, B where $P(B) > 0$, define $P(A|B) = \frac{P(AB)}{P(B)}$.
2. Properties of Conditional Probability $P(A|B)$
   - $P(A|B) \in [0, 1]$
   - $P(\Omega|B) = 1$
   - $P(\bigcup_{i=1}^{\infty} A_i|B) = \sum_{i=1}^{\infty} P(A_i|B)$, where $A_i \cap A_j = \emptyset$, for any $i \neq j$. (Countable Additivity)
   - $P(AB) = P(A|B)P(B)$
   - $P(A_1A_2...A_n) = P(A_n|A_1A_2..A_{n-1})P(A_1A_2...A_{n-1}) = P(A_n|A_1A_2..A_{n-1})P(A_{n-1}|A_1A_2...A_{n-2})P(A_1A_2...A_{n-2}) = ... = P(A_n|A_1A_2..A_{n-1})P(A_{n-1}|A_1A_2...A_{n-2})...P(A_2|A_1)P(A_1)$.
   - $P(A_1A_2A_3) = P(A_1|A_2A_3)P(A_2A_3) = P(A_1|A_2A_3)P(A_2|A_3)P(A_3)$.

Review

3. Law of Total Probability
   - Suppose $\Omega = \bigcup_{i=1}^{n} B_i$, where $B_i \cap B_j = \emptyset$ for any $i \neq j$. For any event A, we have $P(A) = \sum_{i=1}^{n} P(AB_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$.
4. Bayes Formula
   - Suppose $\Omega = \bigcup_{i=1}^{n} B_i$, where $B_i \cap B_j = \emptyset$ for any $i \neq j$ and $P(A) > 0, P(B_k) > 0$ for any k. Then, we have $P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{n} P(A|B_j)P(B_j)}$.

Review

5. Independence
   - Given two events A and B, we say they are independent if $P(AB) = P(A)P(B)$.
   - Property: If A and B are independent, then so does $\overline{A}$ and $\overline{B}$.
6. Pairwise Independence and Mutual Independence
   - For event $A_1, A_2, .., A_n$, we say they are pairwisely independent if $P(A_i A_j) = P(A_i A_j)$ for any $i \neq j$.
   - For event $A_1, A_2, .., A_n$, we say they are mutually independent if $P(A_{i_1} A_{i_2} ... A_{i_k}) = P(A_{i_1})P(A_{i_2})...P(A_{i_k})$ for any $1 \leq i_1 < i_2 < ... < i_k \leq n$.
   - Mutual Independence implies Pairwise Independence; the converse is not ture.

Homework

46. A 盒中有 3 个红球和 2 个白球，B 盒中有 2 个红球和 5 个白球. 抛掷一枚质地均匀硬币. 如果硬币正面朝上，就从 A 盒中抽取一球，否则从 B 盒中抽取.

a. 抽到红球的概率是多少？

b. 如果抽到红球，那么硬币正面朝上的概率是多少？

Solution

Let $R = \{Draw\ a\ red\ ball\}$, $W = \{Draw\ a\ white\ ball\}$, $A = \{Select\ box\ A\}$ *and* $B = \{Select\ box\ B\}$. *Then, we have* $P(A) = P(B) = \frac{1}{2}$, $P(R|A) = \frac{3}{5}$ *and* $P(R|B) = \frac{2}{7}$.

a. $P(R) = P(RA) + P(RB) = P(R|A)P(A) + P(R|B)P(B) = \frac{31}{70}$

b. $P(A|R) = \frac{P(AR)}{P(R)} = \frac{P(R|A)P(A)}{P(R)} = \frac{21}{31}$.

Homework

53. 火险公司有高、中和低三种类型的风险客户，他们的年度索赔概率分别是 0.02、0.01、0.0025，三类客户的市场份额分别是 0.10、0.20、0.70。每一年来自高风险客户索赔的概率是多少？

Homework

Solution

Let $H = \{$*High risk*$\}$, $M = \{$*Medium risk*$\}$, $L = \{$*Low risk*$\}$ $A = \{$*Claim filed*$\}$. *Then, we have* $P(H) = 0.1$, $P(M) = 0.2$, $P(L) = 0.7$ *and* $P(A|H) = 0.02$, $P(A|M) = 0.01$, $P(A|L) = 0.025$.

$P(A) = P(A|H)P(H) + P(A|M)P(M) + P(A|L)P(L)$

$P(H|A) = \frac{P(HA)}{P(A)} = \frac{P(A|H)P(H)}{P(A|H)P(H)+P(A|M)P(M)+P(A|L)P(L)} = \frac{8}{23}$.

Homework

54. 该习题介绍一个简单的气象模型，更复杂的版本参见气象学文献. 考虑连续几天的天气，令 R_i 表示 i 天下雨这个事件. 假设 $P(R_i|R_{i-1}) = \alpha$ 和 $P(R_i^c|R_{i-1}^c) = \beta$. 进一步假设只有今天的天气才与明天的天气预报有关，即 $P(R_i|R_{i-1} \cap R_{i-2} \cap \cdots \cap R_0) = P(R_i|R_{i-1})$.

a. 如果今天下雨的概率是 p，那么明天下雨的概率是多少?

b. 后天下雨的概率是多少?

c. n 天之后下雨的概率是多少？当 n 趋于无穷时又会怎样？

Homework

Solution

Let today be day 1.

$a.P(R_2) = P(R_2|R_1)P(R_1) + P(R_2|R_1^c)P(R_1^c) = \alpha p + (1-\beta)(1-p) = (\alpha+\beta-1)p + 1 - \beta.$

$b.P(R_3) = P(R_3|R_2)P(R_2) + P(R_3|R_2^c)P(R_2^c) = \alpha[(\alpha+\beta-1)p+1-\beta] + (1-\beta)(1-((\alpha+\beta-1)p+1-\beta)) = ((\alpha-1)^2+(\beta-1)^2+2\alpha\beta-1)p + (\alpha+\beta-\alpha\beta-\beta^2).$

$c.$*Let* $P_n = P(R_n)$. *We have* $P_{n+1} = \alpha P_n + (1-\beta)(1-P_n) = (\alpha+\beta-1)P_n + (1-\beta) = (\alpha+\beta-1)^n p + (1-\beta)\sum_{i=1}^{n}(\alpha+\beta-1)^{i-1}.$

$\lim_{n\to\infty} P_n = \frac{1-\beta}{2-\alpha-\beta}.$

Homework

63. 假设人活到 70 岁的概率是 0.6，活到 80 岁的概率是 0.2. 如果一个人已经活到 70 岁，他将庆祝第 80 个生日的概率是多少?

Homework

Solution

Let $S = \{Age \geq 70\}$ *and* $E = \{Age \geq 80\}$.
$P(E|S) = \frac{P(ES)}{P(S)} = \frac{P(E)}{P(S)} = \frac{1}{3}$.

Homework

68. 如果 A 和 B 独立，B 和 C 独立，那么 A 和 C 亦独立. 若该陈述为假，给出反例，否则证明之.

Homework

Solution

False.
$\Omega = \{1,2\}, A = C = \{1\}, B = \emptyset, P(\{1\}) = P(\{2\}) = \frac{1}{2}$.
Then, $P(AB) = 0 = P(A)P(B)$, $P(BC) = 0 = P(B)P(C)$. *But*
$P(AC) = P(\{1\}) = \frac{1}{2} \neq P(A)P(C)$.

Homework

71．证明：如果 A，B 和 C 相互独立，那么 $A \cap B$ 和 C 是独立的，$A \cup B$ 和 C 是独立的．

Homework

Solution

$P(ABC) = P(A)P(B)P(C) = P(AB)P(C)$.
$P((A \cup B) \cap C) = P((AC) \cup (BC)) = P(AC) + P(BC) - P(ABC) = P(A)P(C) + P(B)P(C) - P(AB)P(C) = [P(A) + P(B) - P(AB)]P(C) = P(A \cup B)P(C)$.

Homework

74. 如果每个单元独立地工作，且失效的概率是 p，那么下面的系统正常工作的概率是多少（见图 1.5）？

Homework

Solution

$P_u = 2p - p^2, P_m = p, P_l = 2p - p^2.$
$Then,\ P = 1 - (2p - p^2)^2 p = 1 - 4p^3 + 4p^4 - p^5.$

Homework

77. 玩家向目标投飞镖，每次试验都独立地进行，他命中靶心的概率是 0.05. 他要多少次才能使命中靶心至少一次的概率为 0.5?

Homework

Solution

$P_n = 1 - (1 - 0.05)^n = 1 - 0.95^n.$
$N = min_{n \in \mathbf{N}} \{n : P_n \geq 0.5\} = 14.$

Homework

79. 很多人类疾病是遗传的（例如，血友病或泰萨二氏病）。这里是此类疾病的一个简单模型，基因型 aa 是有病的，在交配之前死亡。基因型 Aa 是一个携带者，但是没有病。基因型 AA 不是携带者，也没有病。

   a. 如果两个携带者交配，他们的后代是这三种基因型之一的概率分别是多少？

   b. 如果两个携带者的男性后代没有疾病，他是疾病携带者的概率是多少？

   c. 假设 b 项的无病后代与没有家族病史的个体交配，并设其配偶是病毒携带者的概率是 p（p 是一个非常小的数），那么他们的第一代具有基因型 AA，Aa 和 aa 的概率是多少？

   d. 假设 c 项的第一代没有疾病，那么基于此证据，其父亲是病毒携带者的概率是多少？

Homework

Solution

a. $P(\{AA\}) = P(\{aa\}) = \frac{1}{4}, P(\{Aa\}) = \frac{1}{2}$.
b. $P(M \cap \{Aa\} | M \cap \{Aa, AA\}) = \frac{2}{3}$.
c. *Let the gene of the first offspring be* O. *Then,*
$P(O = AA) = P(O = AA | H = AA, W = AA) P(H = AA, W = AA) + P(O = AA | H = AA, W = Aa) P(H = AA, W = Aa) + P(O = AA | H = Aa, W = AA) P(H = Aa, W = AA) + P(O = AA | H = Aa, W = Aa) P(H = Aa, W = Aa) = \frac{1-p}{3} + \frac{1}{2} * \frac{p}{3} + \frac{1}{2} * \frac{2(1-p)}{3} + \frac{1}{4} * \frac{2p}{3} = \frac{2}{3} - \frac{1}{3}p$.
$P(O = Aa) = P(O = Aa | H = AA, W = AA) P(H = AA, W = AA) + P(O = Aa | H = AA, W = Aa) P(H = AA, W = Aa) + P(O = Aa | H = Aa, W = AA) P(H = Aa, W = AA) + P(O = Aa | H = Aa, W = Aa) P(H = Aa, W = Aa) = \frac{1}{2} * \frac{p}{3} + \frac{1}{2} * \frac{2(1-p)}{3} + \frac{1}{2} * \frac{2p}{3} = \frac{1}{3} + \frac{1}{6}p$.

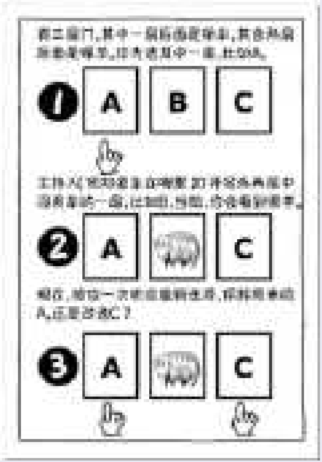
Homework

Solution

$P(O = aa) = P(O = aa|H = AA, W = AA)P(H = AA, W = AA) + P(O = aa|H = AA, W = Aa)P(H = AA, W = Aa) + P(O = aa|H = Aa, W = AA)P(H = Aa, W = AA) + P(O = aa|H = Aa, W = Aa)P(H = Aa, W = Aa) = \frac{1}{4} * \frac{2p}{3} = \frac{1}{6}p.$

d. $P(H = Aa \text{ or } W = Aa|O = AA \text{ or } Aa) = \frac{P(\{H=Aa \text{ or } W=Aa\}\cap\{O=AA \text{ or } Aa\})}{P(O=AA \text{ or } Aa)} = \frac{\frac{2}{3}-\frac{1}{6}p}{1-\frac{1}{6}p} = \frac{4-p}{6-p} \approx \frac{2}{3}.$

Exercise 1



Supplement Exercises

Solution

Let $M_i = \{Car\ in\ door\ i\}$, $N_i = \{Host\ opens\ door\ i\}$, $L_i = \{You\ choose\ door\ i\}$, $i = A, B, C$. *We want to compare* $P(M_A|N_B L_A)$ *and* $P(M_C|N_B L_A)$.

$P(N_B L_A) = P(N_B L_A M_A) + P(N_B L_A M_B) + P(N_B L_A M_C)$.

$P(N_B L_A M_A) = P(N_B|M_A L_A)P(M_A L_A) = P(N_B|M_A L_A)P(M_A)P(L_A) = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{18}$.

$P(N_B L_A M_B) = 0$.

$P(N_B L_A M_C) = P(N_B|M_C L_A)P(M_C L_A) = P(N_B|M_C L_A)P(M_C)P(L_A) = 1 \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

$P(N_B L_A) = \frac{1}{18} + \frac{1}{9} = \frac{1}{6}$.

$P(M_A|N_B L_A) = \frac{P(M_A N_B L_A)}{P(N_B L_A)} = \frac{1}{3}$.

$P(M_C|N_B L_A) = \frac{P(M_C N_B L_A)}{P(N_B L_A)} = \frac{2}{3}$.

Hence, You should change your choice.

Exercise 2

2. 据以往资料表明,某一3口之家,患某种传染病的概率有以下规律:

$P\{孩子得病\}=0.6$,

$P\{母亲得病|孩子得病\}=0.5$,

$P\{父亲得病|母亲及孩子得病\}=0.4$,

求母亲及孩子得病但父亲未得病的概率.

Solution

$Let\ F = \{Father\ is\ sick.\}, M = \{Mother\ is\ sick.\} and C = \{Child\ is\ sick.\}.$
$P(MC) = 0.6 \cdot 0.5 = 0.3,\ P(MCF) = 0.4 \cdot 0.3 = 0.12,$
$P(MC\overline{F}) = 0.3 - 0.12 = 0.18.$

Supplement Exercises

Exercise 3

5. 对以往数据分析结果表明，当机器调整得良好时，产品的合格率为0.98；而当机器发生某种故障时，产品的合格率为0.55。每天早上机器开动时，机器调整良好的概率为0.95。试求已知某日早上的第一件产品是合格品时，机器调整得良好的概率。

Supplement Exercises

Solution

$$P = \frac{0.98 \cdot 0.95}{0.95 \cdot 0.98 + 0.05 \cdot 0.55} = \frac{0.931}{0.9585} = \frac{1862}{1917}.$$

Supplement Exercises

Exercise 4

1. 设两个独立事件 A 和 B 都不发生的概率为 1/9，A 发生 B 不发生的概率与 B 发生 A 不发生的概率相同。求事件 A 发生的概率。

Solution

Let $p = P(A), q = P(B)$. *We have* $(1-p)(1-q) = \frac{1}{9}$ *and* $p(1-q) = q(1-p)$. *Then,* $p = \frac{2}{3}$.

Exercise 5

2. 设两两相互独立的三事件 A,B,C 满足条件：$ABC=\varnothing$, $P(A)=P(B)=P(C)$,且已知

$P(A\cup B\cup C)=9/16$,求 $P(A)$.

Supplement Exercises

Solution

Let $p = P(A) = P(B) = P(C)$. *Then, we have* $\frac{9}{16} = P(A \cup B \cup C) = 3p - 3p^2$. *Hence,* $p = \frac{3}{4}$ *or* $p = \frac{1}{4}$. *However, since* $p = P(A) \leq P(A \cup B \cup C) = \frac{9}{16}$, *then* $P(A) = p = \frac{1}{4}$.

Thank you!