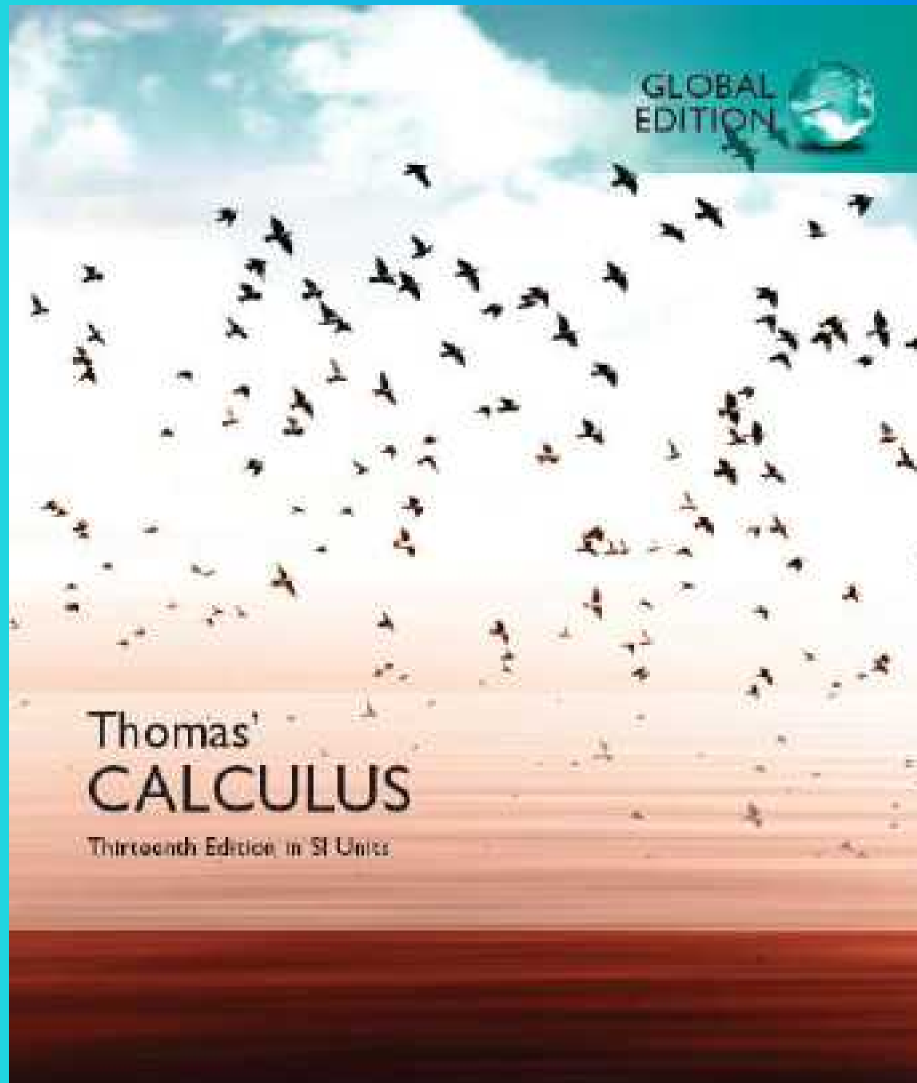
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Chapter 4

Applications of Derivatives

4.1

Extreme Values of Functions

函数的极值

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \le f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \ge f(c) \qquad \text{for all } x \text{ in } D.$$

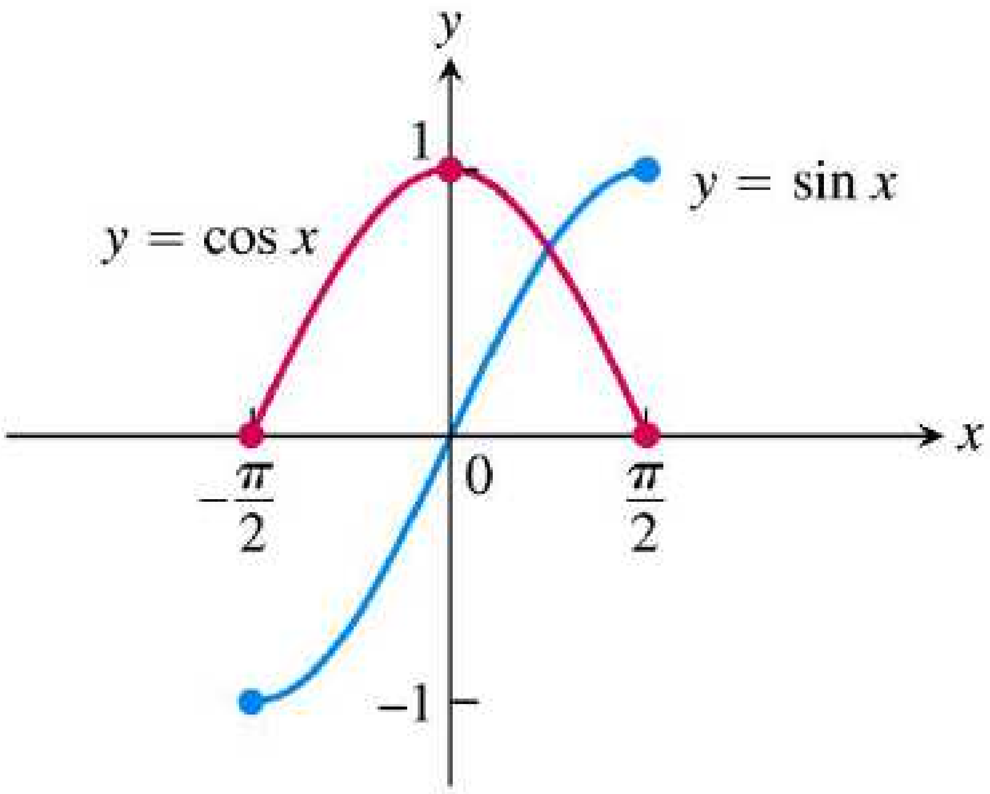


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

Ex.1

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d) $y = x^2$	$(0, 2)$	No absolute extrema.

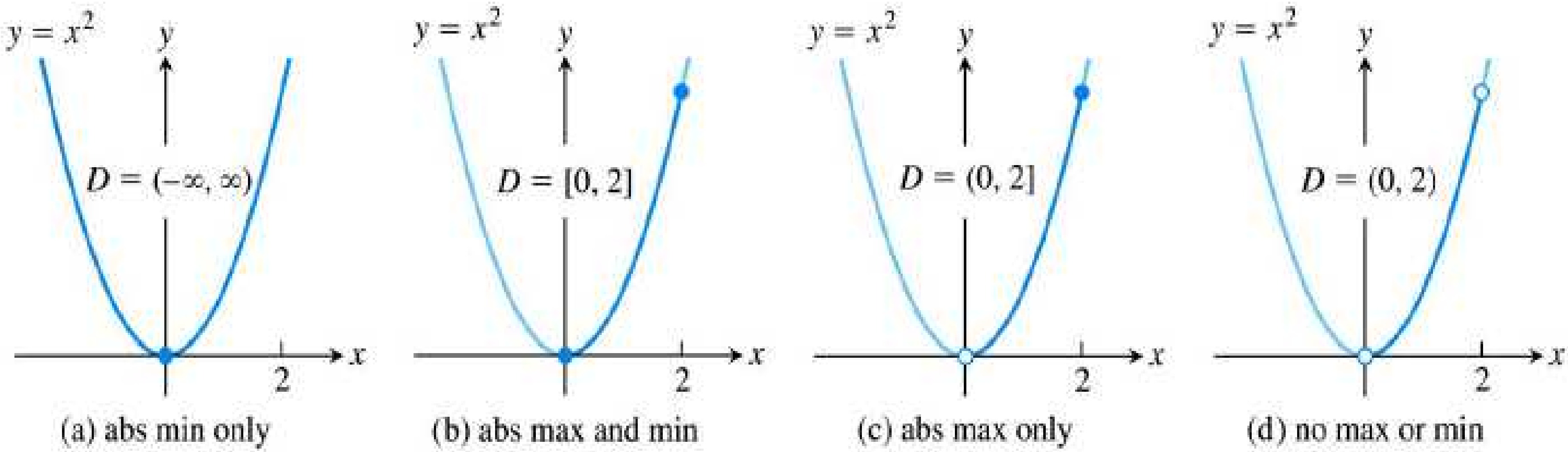


FIGURE 4.2 Graphs for Example 1.

什么函数一定在一个区间上有最值呢？

THEOREM 1—The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$.

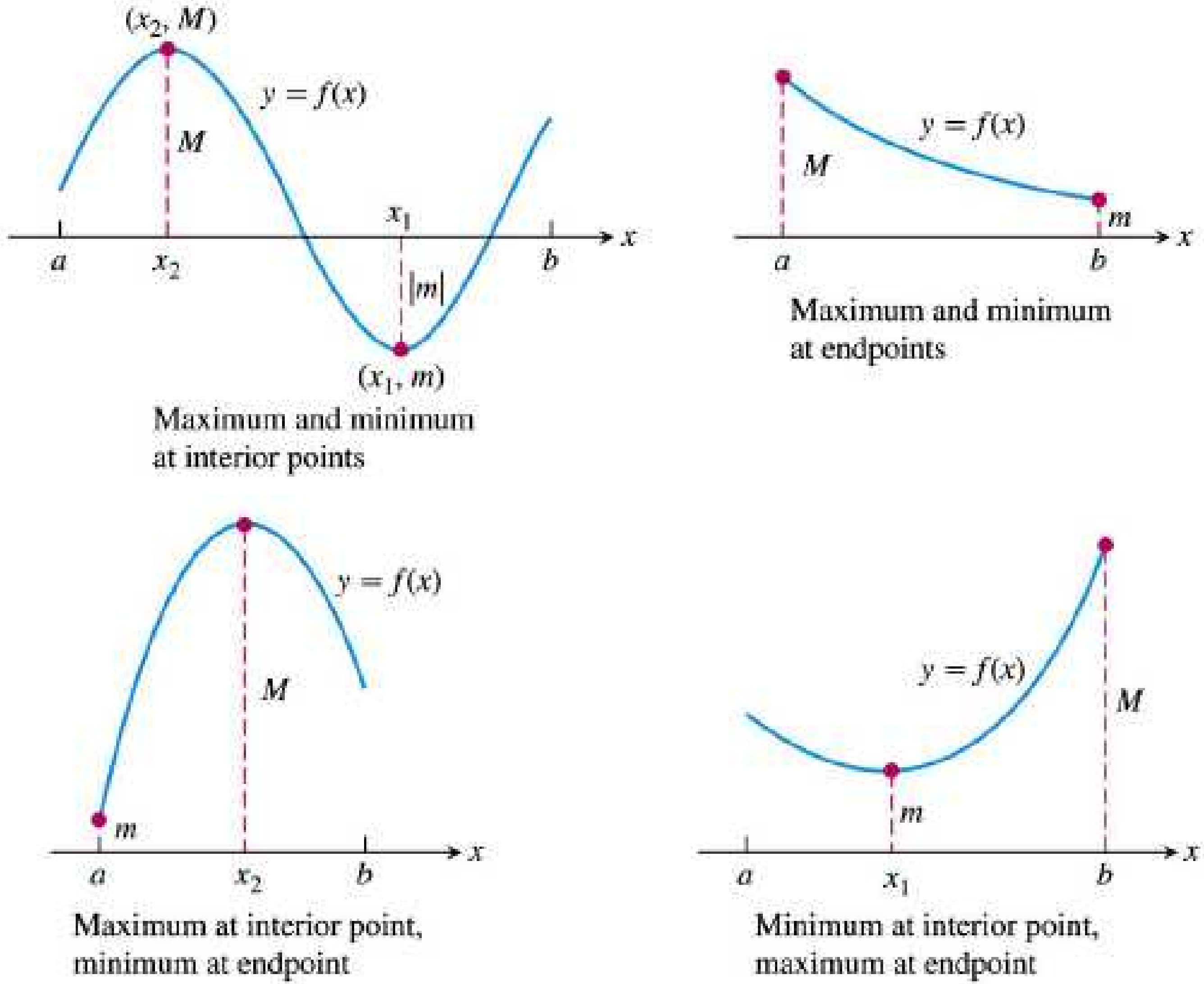


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

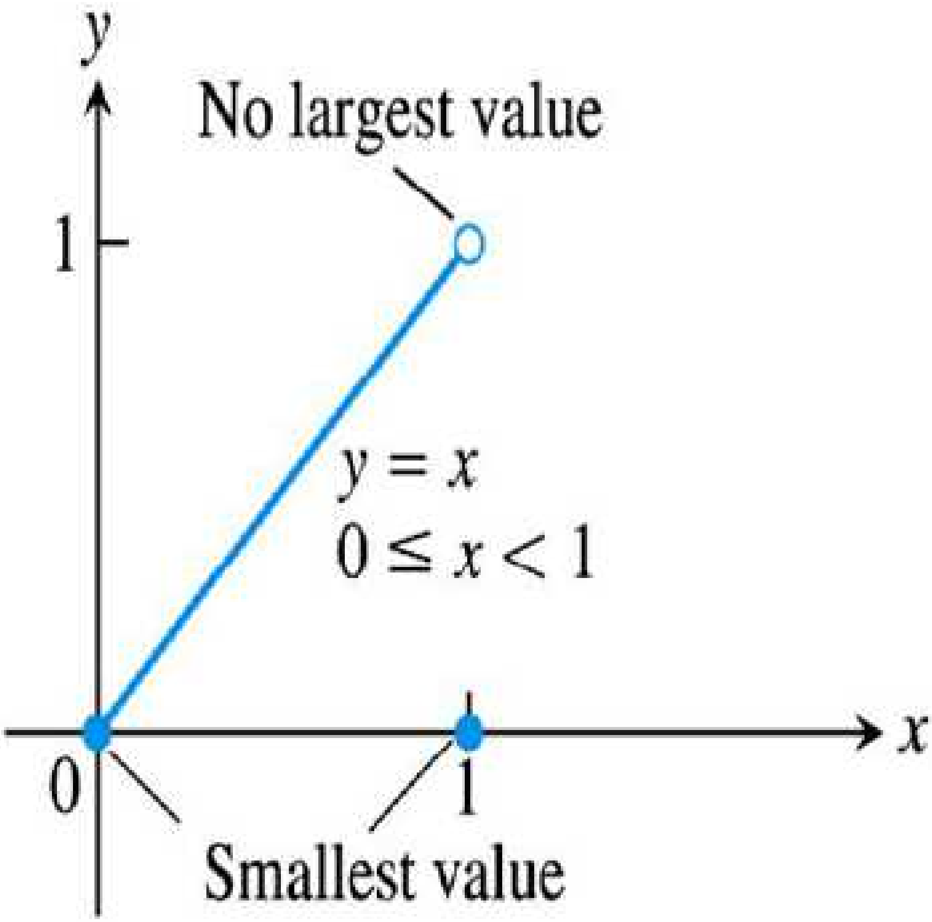


FIGURE 4.4 Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of [0, 1] except $x = 1$, yet its graph over [0, 1] does not have a highest point.

局部最大值最小值

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \le f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \ge f(c)$ for all $x \in D$ lying in some open interval containing c.

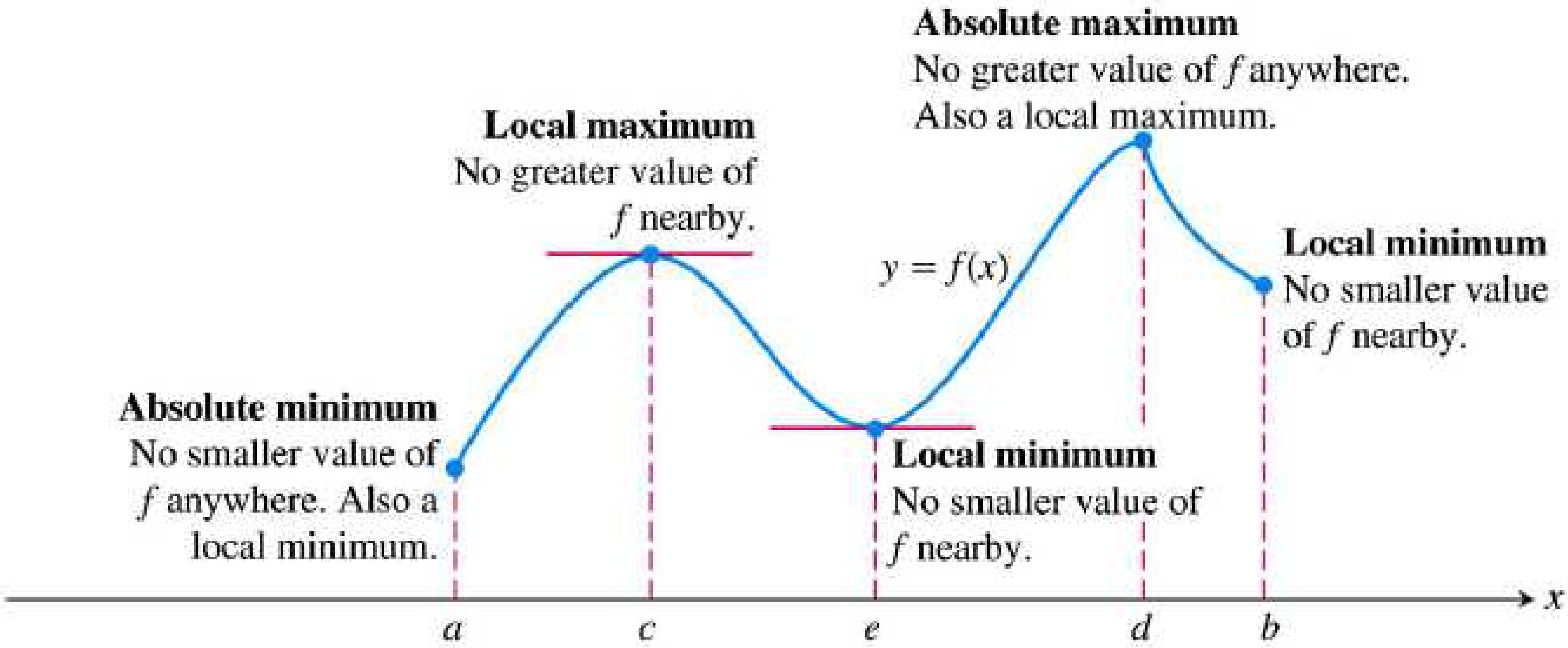


FIGURE 4.5 How to identify types of maxima and minima for a function with domain $a \le x \le b$.

可能取得极值的点：

导数为零的点，导数不存在的点，区间端点

可导函数在某点取得极值的必要条件

THEOREM 2—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

驻点

证 不妨设$f(c)$为极大值，则有 $f(x)-f(c)\le 0, x\in U(c)$

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}\le 0$$

$$=\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}\ge 0$$

$\therefore$ 只有 $f'(c)=0.$

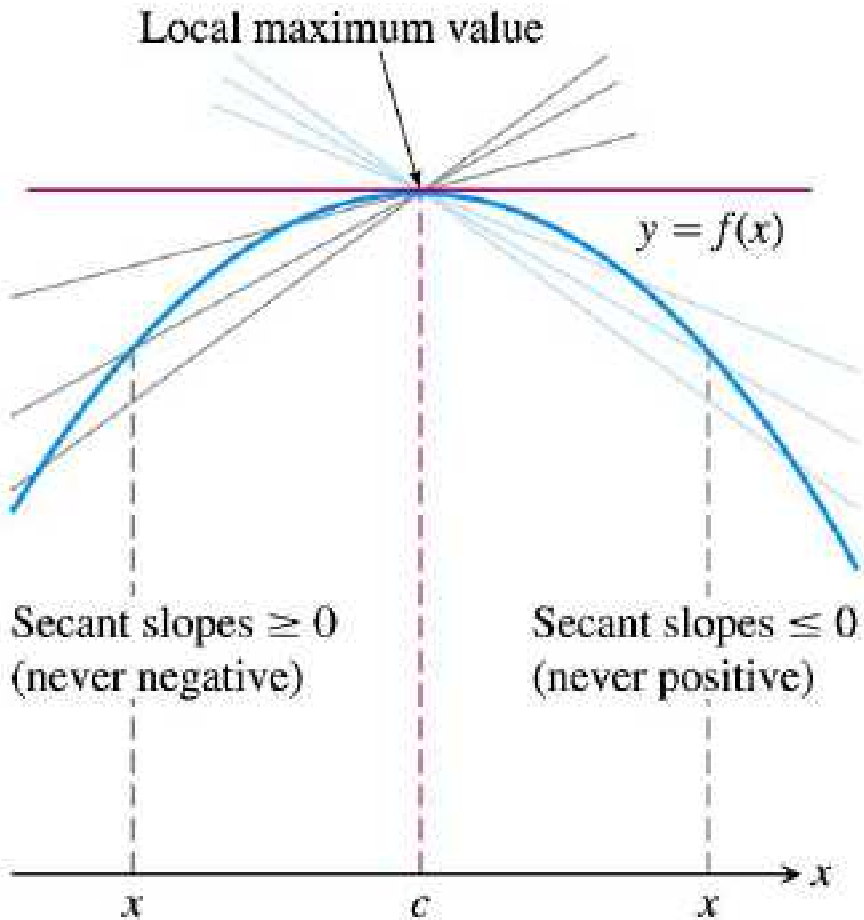
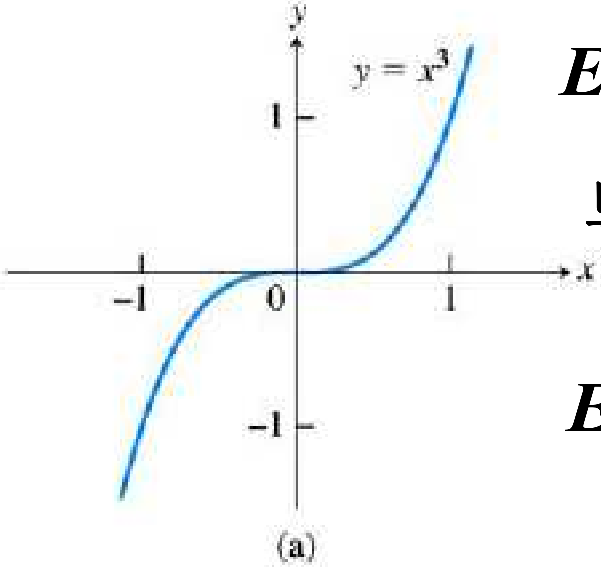


FIGURE 4.6 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.



$$Ex.1 \quad y = x^3 \qquad y'\big|_{x=0} = 0$$

导数为零的点不一定都是极值点

$$Ex.2 \quad y = x^{1/3} \qquad y'\big|_{x=0}$$ 不存在

导数不存在的点不一定都是极值点

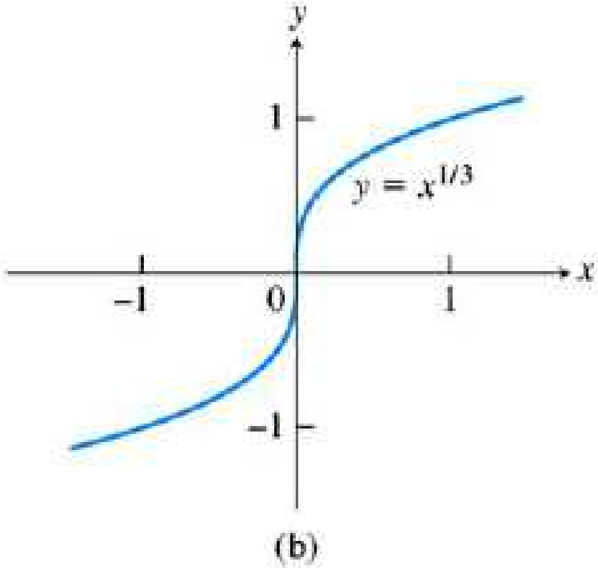


FIGURE 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

1.求出区间内所有的关键点、端点上的函数值

2.比大小得到最大最小值

$Ex.3$ **Find the absolute maxmum and minimum values of** $g(t) = 8t - t^4$ **on the interval** $[-2,1]$.

Solution $\because g'(t) = 8 - 4t^3$,

解方程 $g'(t) = 8 - 4t^3 = 0$,

得驻点 $t = \sqrt[3]{2} > 1$, 不在区间内!

$g(-2) = -32, g(1) = 7.$

min **max**

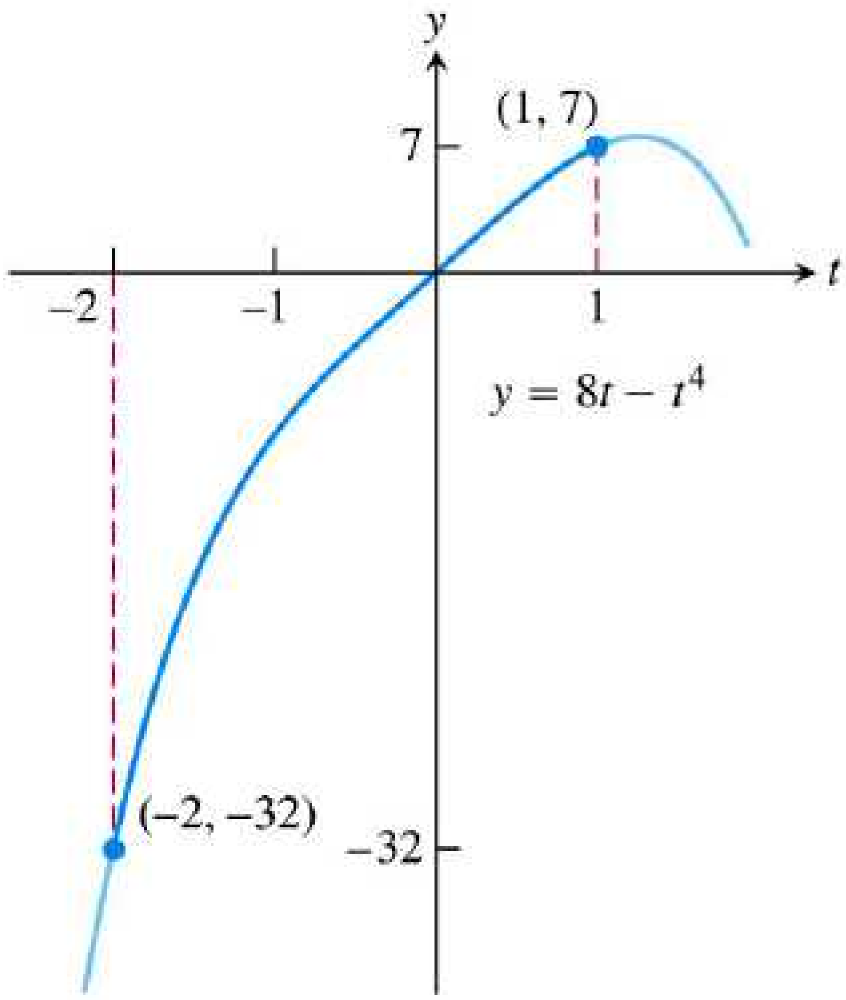


FIGURE 4.8 The extreme values of $g(t) = 8t - t^4$ on $[-2, 1]$ (Example 3).

Ex.4

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the $[-2,3]$

Solution $\because f'(x) = \frac{2}{3}x^{-1/3}$,

has no zeros but is undefined at the interior point $x = 0$.

$f(0) = 0$ $\quad f(-2) = \sqrt[3]{4}$ $\quad f(3) = \sqrt[3]{9}$

min **max**

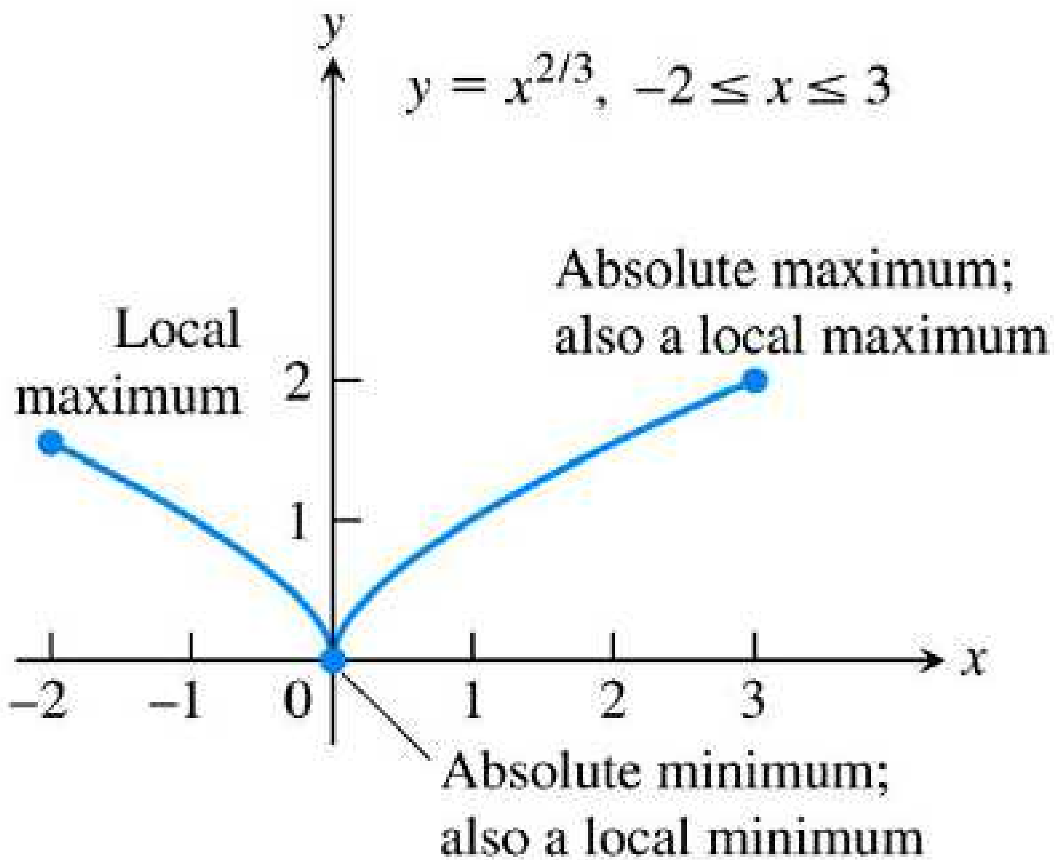


FIGURE 4.9 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

4.2

The Mean Value Theorem
中值定理

罗尔定理

THEOREM 3—Rolle's Theorem Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

证 由定理1可知函数在该区间上取得最大值最小值.

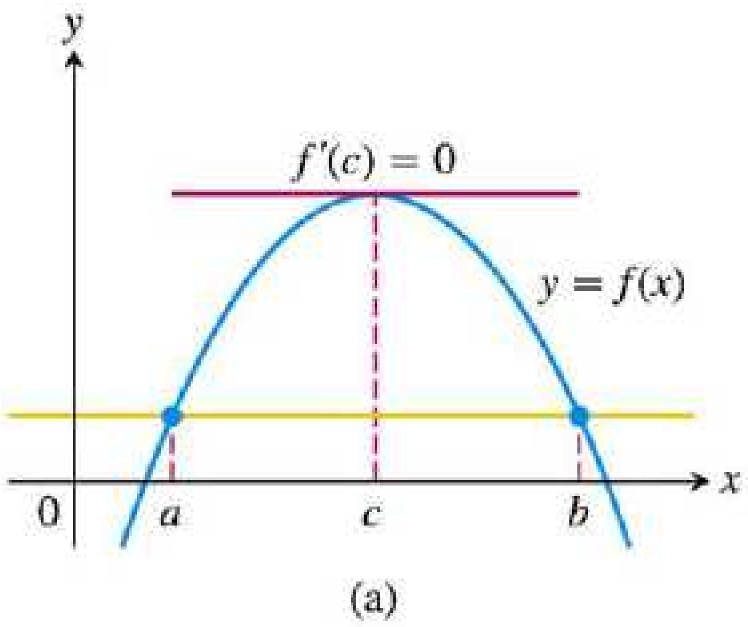
若最值都在端点取得，则

$f(x) = C$，$f'(x) = 0$，结论成立;

若最值至少有一个在区间内取得，

不妨设在 $c \in (a,b)$处，函数取得一个最值，

由定理2知，$f'(c) = 0$,结论成立.



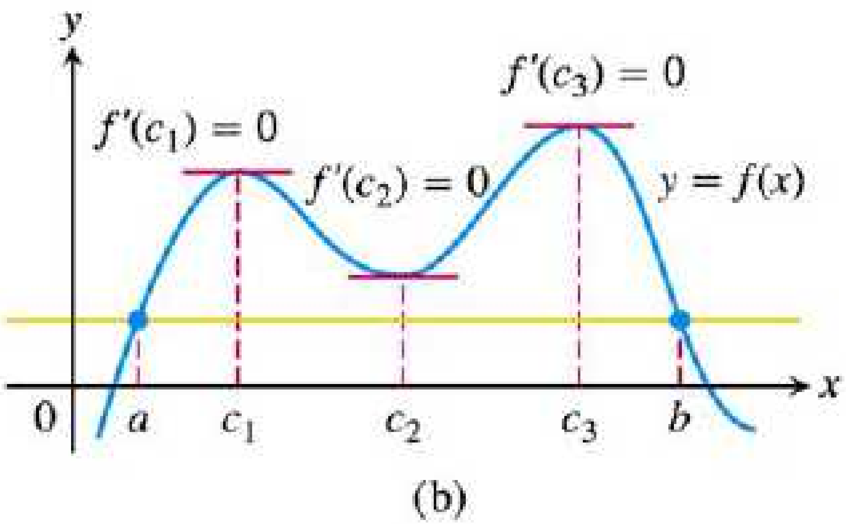
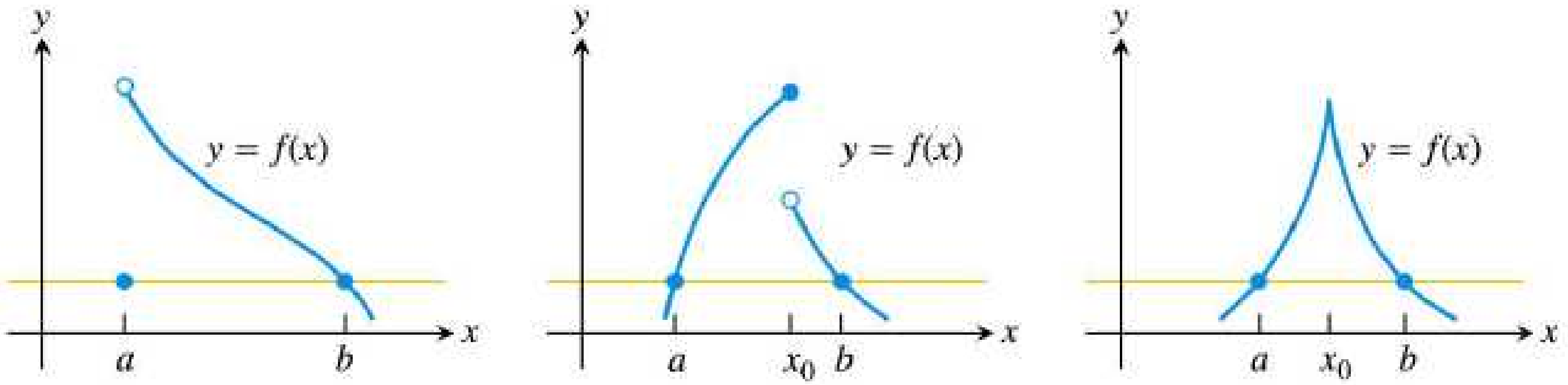


FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

FIGURE 4.11 There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

罗尔定理的条件是缺一不可的。

EXAMPLE 1 Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Solution We define the continuous function

$$f(x) = x^3 + 3x + 1.$$

Since $f(-1) = -3$ and $f(0) = 1$, f crosses the x-axis somewhere in the open interval $(-1, 0)$.

$$\exists\, c \in (-1,0), \text{s.t } f(c) = 0.$$

if there were even two points $x = a$ and $x = b$ where $f(x)$ was zero, Rolle's Theorem would guarantee the existence of a point in between them where f' was zero.

$f'(x) = 3x^2 + 3$ is never zero

Therefore, f has no more than one zero.

Ex. 1 证明方程 $x^3+3x+1=0$ 有且仅有一个实根.

证 设 $f(x)=x^3+3x+1$, 则 $f(x)$连续,

且 $f(-1)=-3, f(0)=1,$ 由介值定理

$\exists\, c\in(-1,0)$, 使 $f(c)=0$. 即为方程实根.

设另有 $x_1\in R, x_1\neq c,$ 使 $f(x_1)=0$.

$f(x)$ 在 $[x_1,c]$($[c,x_1]$) 满足罗尔定理的条件 ,

则至少存在一个 ξ (在 c,x_1 之间), 使得 $f'(\xi)=0$.

但 $f'(x)=3(x^2+1)>0$, $f'(x)$不可能有零点!

$\therefore$ 为唯一实根.

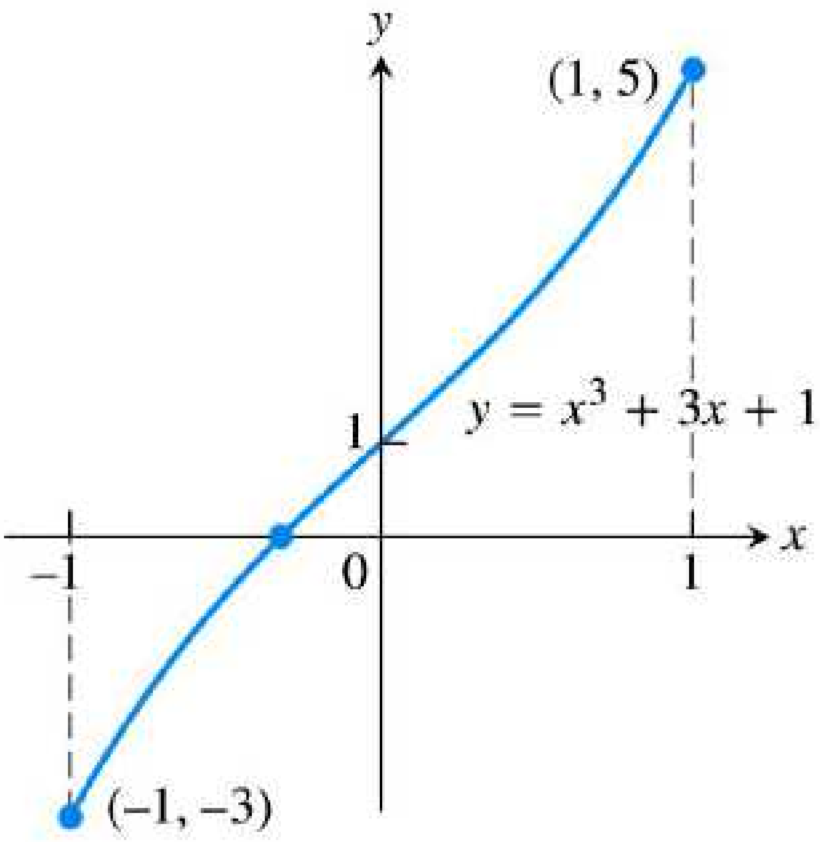


FIGURE 4.12 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the x-axis between -1 and 0 (Example 1).

例 $f(x)=x(x-1)(x-2)(x-3)(x-4)$，证明方程$f'(x)=0$在区间$(0,4)$内恰有4个实根。

证 $f(x)$在$[0,4]$连续,可导.

且 $f(0)=f(1)=f(2)=f(3)=f(4)=0.$

分别在$[0,1],[1,2],[2,3],[3,4]$上用罗尔定理，有

$\exists\, x_1\in(0,1)$,使 $f'(x_1)=0.$

$\exists\, x_2\in(1,2)$,使 $f'(x_2)=0.$

$\exists\, x_3\in(2,3)$,使 $f'(x_3)=0.$

$\exists\, x_4\in(3,4)$,使 $f'(x_4)=0.$

又$f'(x)=0$是4次代数方程，至多有4个实根，

所以，方程恰有4个实根。

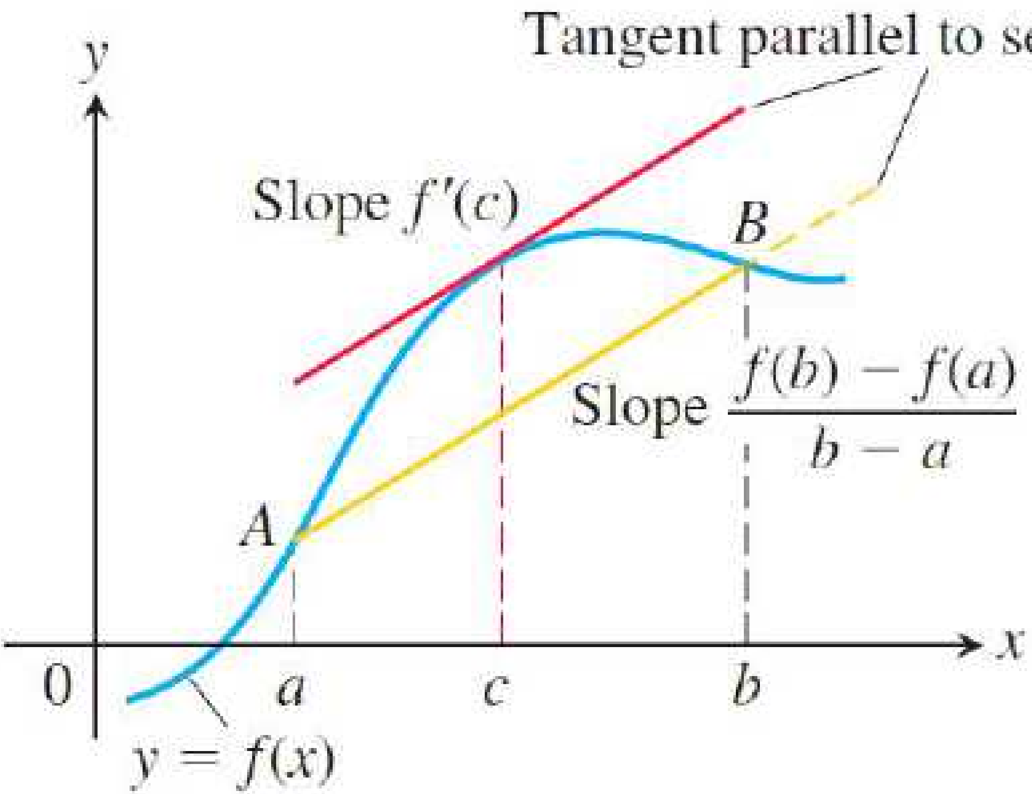


FIGURE 4.13 Geometrically, the Mean Value Theorem says that somewhere between a and b the curve has at least one tangent parallel to the secant joining A and B.

拉格朗日中值定理

THEOREM 4—The Mean Value Theorem Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

定理的物理解释:

$s = f(t)$是变速直线运动的位移 函数，它在$[a,b]$时段内连续、可导，则存在某 时刻的速度和该时段上 的平均速度相等.

证　分析: 条件中与罗尔定理相差 $f(a)=f(b)$.

化为罗尔定理的结论形式

欲证存在$c\in(a,b)$使得

$$\left(f(x)-\frac{f(b)-f(a)}{b-a}x\right)'\bigg|_{x=c}=0$$

作辅助函数　$F(x)=f(x)-\dfrac{f(b)-f(a)}{b-a}x.$

$F(x)$ 满足罗尔定理的条件,

则在(a,b)内至少存在一点c,使得 $F'(c)=0.$

即 $f'(c)-\dfrac{f(b)-f(a)}{b-a}=0$

或 $f(b)-f(a)=f'(c)(b-a).$

拉格朗日中值公式

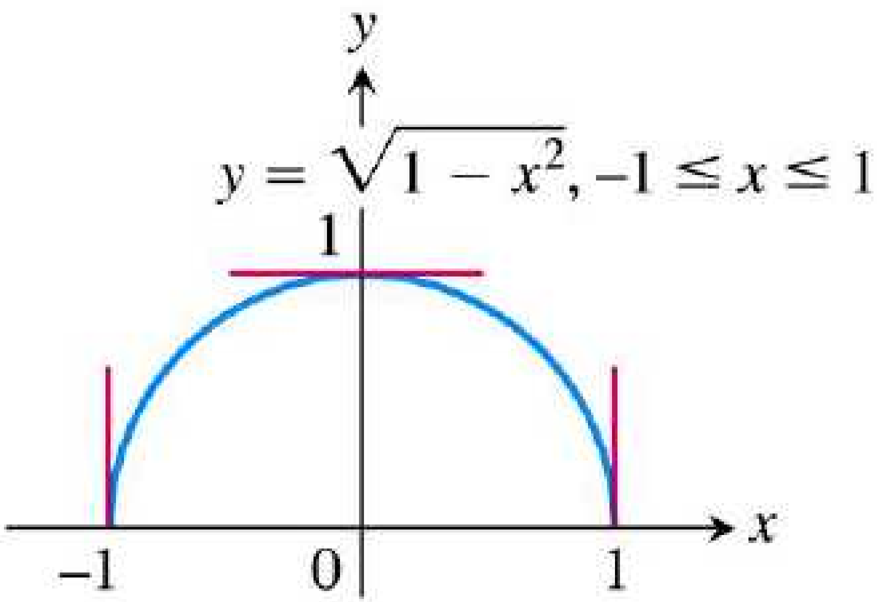


FIGURE 4.16 The function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1.

Ex. 2

验证 $f(x)=x^2$ 在区间[0,2]上Lagrange定理的结论成立.

解 $\dfrac{f(2)-f(0)}{2}=f'(c)$ $\quad 2=2c$

$c=1\in(0,2)$

Ex. 3

一辆汽车用时 $8s$ 从起步到开了 $176m$.解释汽车在这段时间内一定有某 时刻的速度是 $22m/s$.

解 平均速度 $\dfrac{176}{8}=22m/s$

由拉格朗日中值的物理 意义知结论成立.

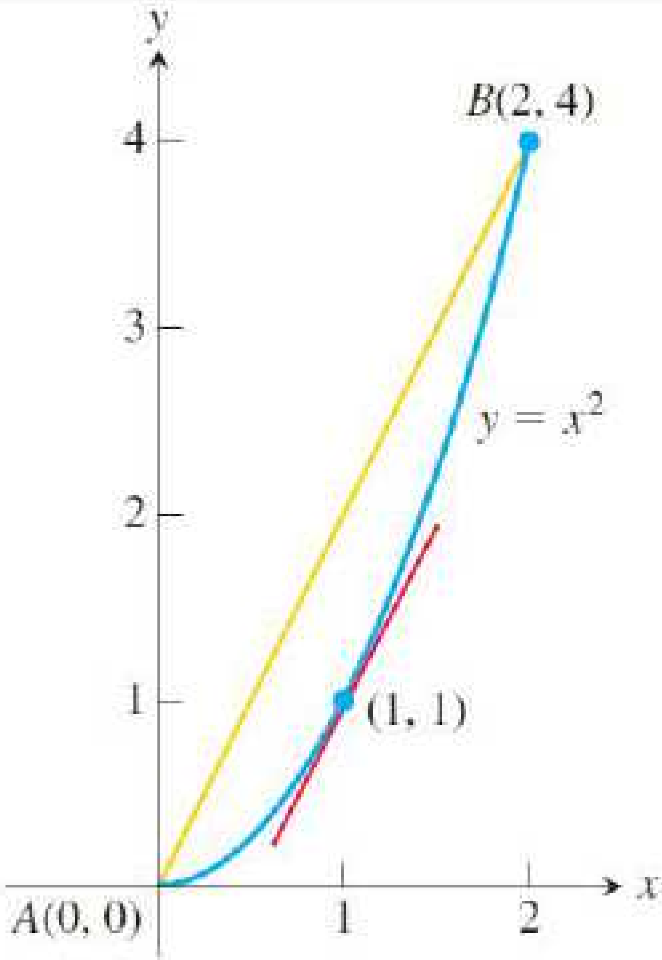


FIGURE 4.17 As we find in Example 2, $c = 1$ is where the tangent is parallel to the secant line.

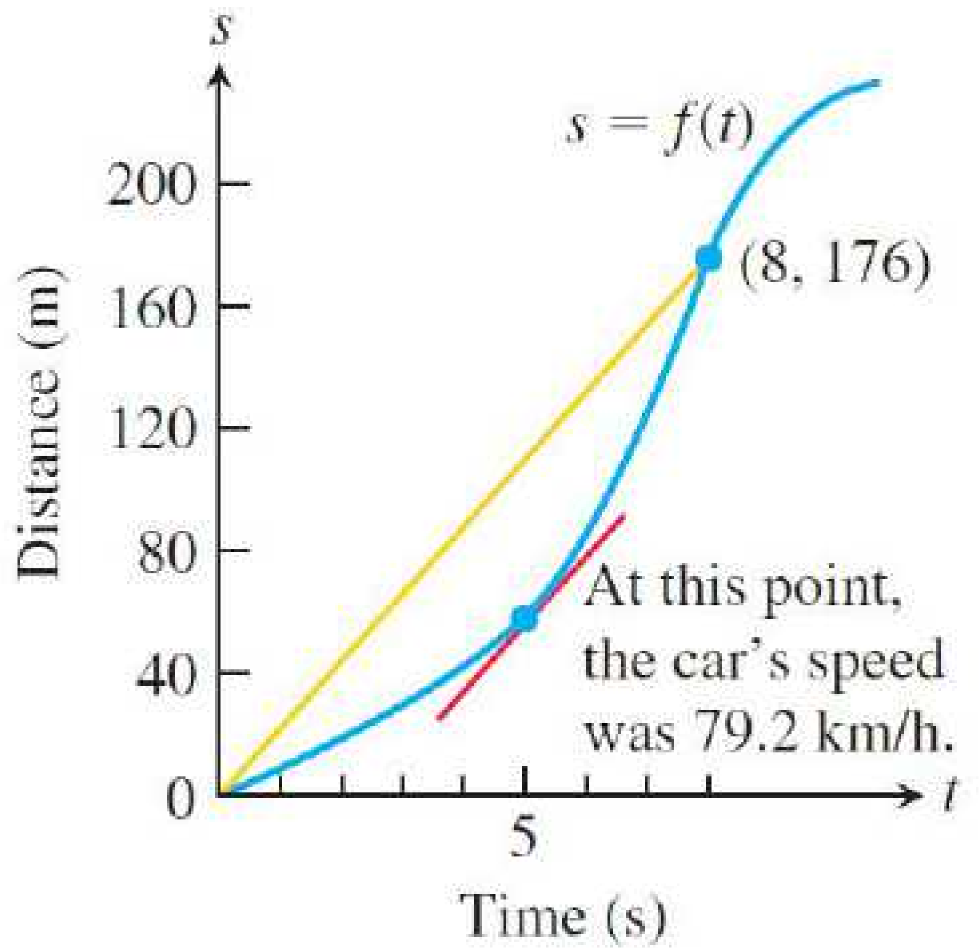


FIGURE 4.18 Distance versus elapsed time for the car in Example 3.

COROLLARY 1 If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

COROLLARY 2 If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

证明Corollary1: $\forall x_1, x_2 \in (a,b)$,

$f(x)$在$[x_1, x_2]$满足拉格朗日定理的条件

存在$c \in (x_1, x_2)$ 使得$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0$.

故$f(x_2) = f(x_1)$.

证明Corollary 2: 设$F(x) = f(x) - g(x)$

$F(x)$在(a,b)内恒有$F'(x) = 0$,

由Corollary1, 在(a,b)内$F(x) = C$,

故$f(x) = g(x) + C$.

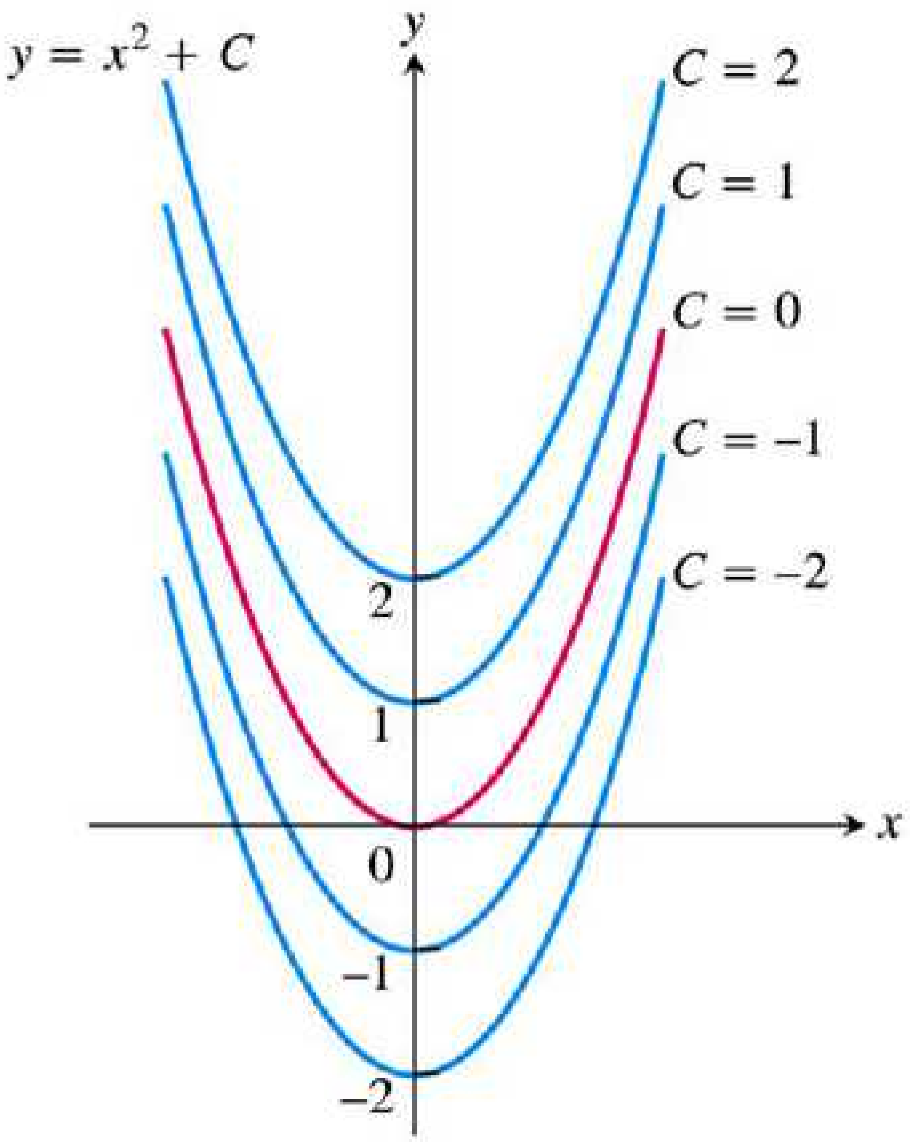


FIGURE 4.19 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives on an interval can differ only by a vertical shift there. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for selected values of C.

Ex. 4 已知$f'(x)=\sin x$,又知其图像过点 $(0,2)$,求$f(x)$.

解:
$$f'(x)=(-\cos x)',$$
$$f(x)=-\cos x+c,$$
$$f(0)=2,\quad 2=-1+c,\quad c=3,$$
$$f(x)=-\cos x+3.$$

Ex. 5 设某物体从静止自由落 体下来的加速度是 $9.8m/s^2$，设在 t 秒时求其路程是 $s(t)$，求 $s(t)$.

解：

$$s(0)=0, s'(0)=0.$$

$$s''(t)=9.8, \quad s'(t)=9.8t+c_1, \quad c_1=0,$$

$$s'(t)=9.8t,$$

$$s(t)=4.9t^2+c_2, \quad c_2=0,$$

$$s(t)=4.9t^2.$$

例　证明当$0 \le x_1 < x_2 < x_3 \le \pi$时，

$$\frac{\sin x_2 - \sin x_1}{x_2 - x_1} > \frac{\sin x_3 - \sin x_2}{x_3 - x_2}.$$

证　$\sin x$在$[x_1, x_2], [x_2, x_3]$上满足拉氏定理的条件，

$$\frac{\sin x_2 - \sin x_1}{x_2 - x_1} = \cos c_1, x_1 < c_1 < x_2$$

$$\frac{\sin x_3 - \sin x_2}{x_3 - x_2} = \cos c_2, x_2 < c_2 < x_3.$$

又$\cos x$在$[0, \pi]$上单调减少，且$c_1 < c_2$

$$\therefore \cos c_1 > \cos c_2$$

原式得证。

4.3

Monotonic Functions and the First Derivative Test

单调函数及一阶导数判别法

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

证 $\forall x_1, x_2 \in [a,b]$, 且 $x_1 < x_2$,应用拉氏定理,得

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \qquad (x_1 < \xi < x_2)$$

$\because x_2 - x_1 > 0,$

若在(a,b)内, $f'(x) > 0,$ 则 $f'(\xi) > 0,$

$\therefore f(x_2) > f(x_1).$ $\therefore y = f(x)$在$[a,b]$上单调增加.

若在(a,b)内, $f'(x) < 0,$ 则 $f'(\xi) < 0,$

$\therefore f(x_2) < f(x_1).$ $\therefore y = f(x)$在$[a,b]$上单调减少.

EXAMPLE 1

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

Solution $f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$

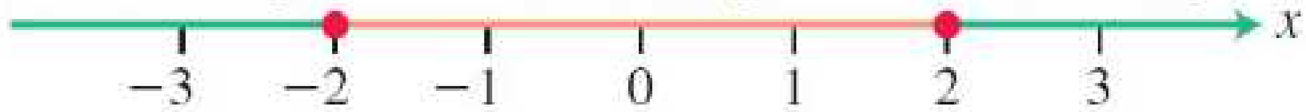
解方程$f'(x) = 0$,得critical point $x = \pm 2$.

在$(-\infty,-2)$内, $y' > 0$, is increasing on $-\infty < x \le -2$,

在$(-2,2)$内, $y' < 0$, decreasing on $-2 \le x \le 2$,

在$(2,+\infty)$内, $y' > 0$, increasing on $2 \le x < \infty$.

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing



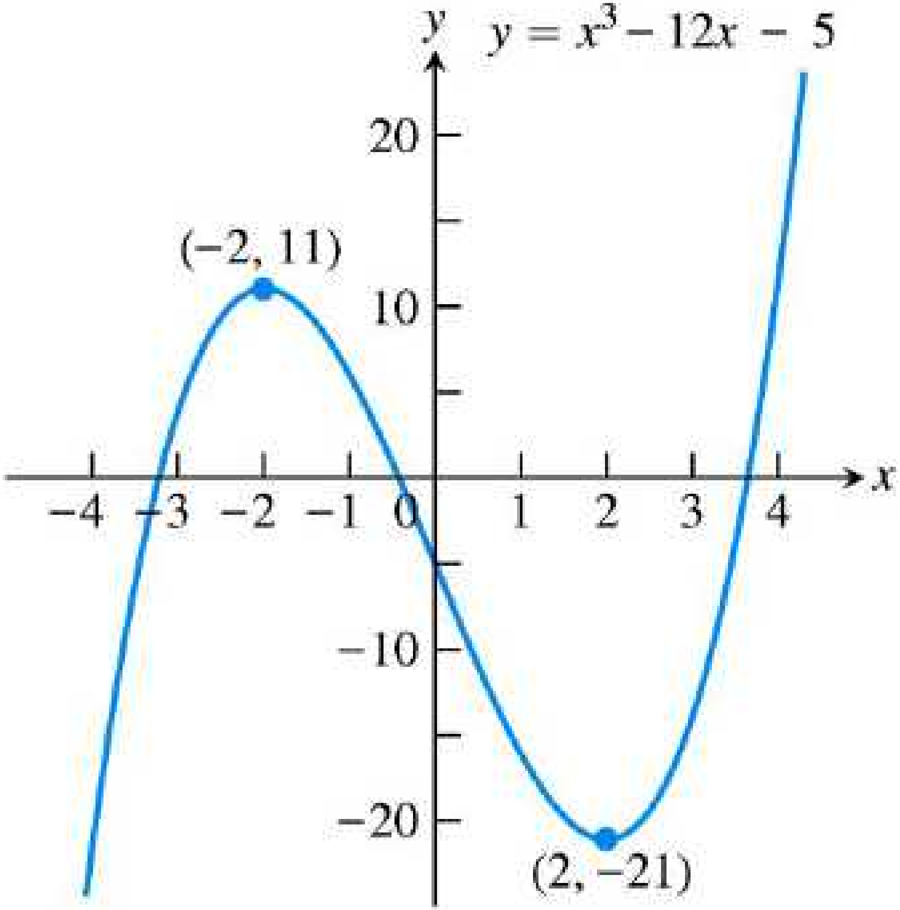


FIGURE 4.20 The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

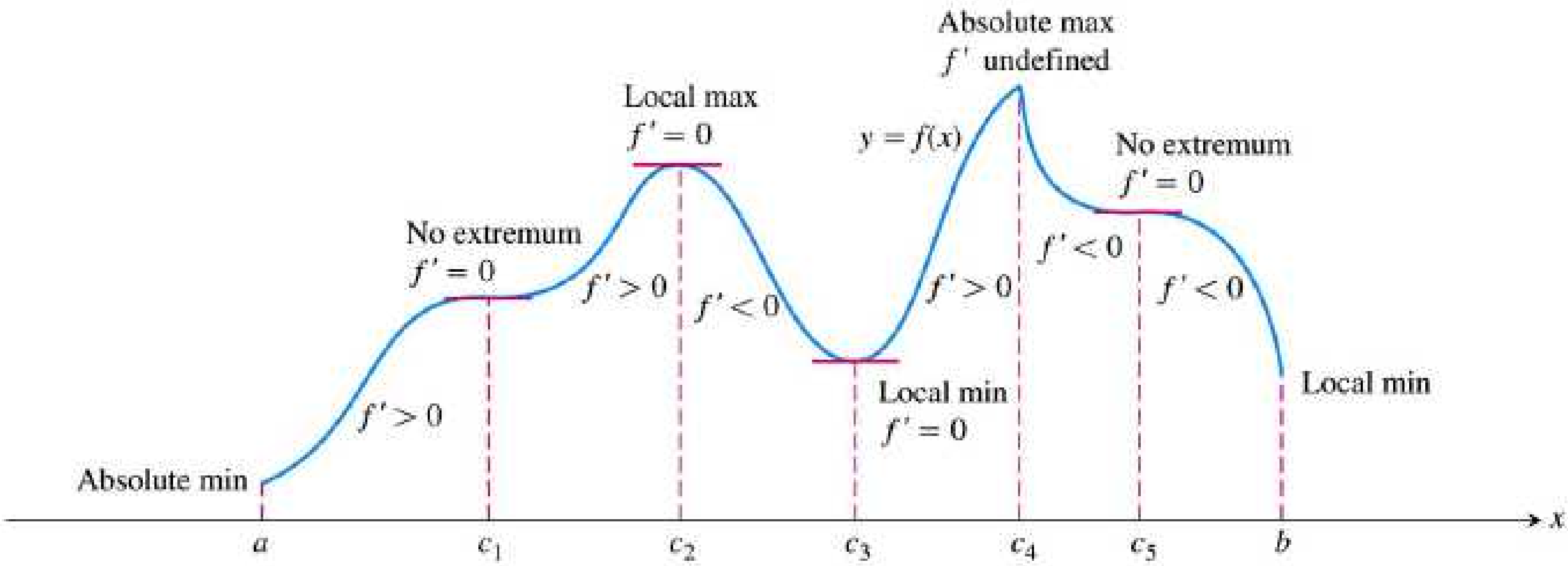


FIGURE 4.21 The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

Ex. 2 Find the critical points of

$$f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}.$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

解 $$f'(x) = \frac{4}{3}x^{-2/3}(x-1)$$

critical points : $x_1 = 1,\ x = 0,$

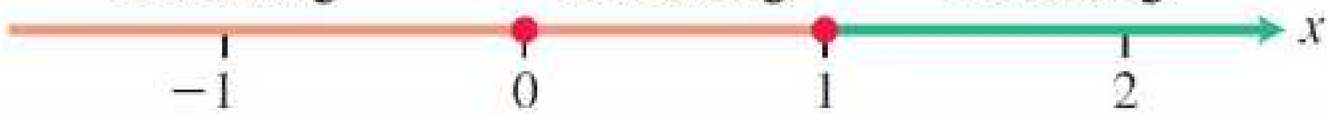
在$(-\infty,0)$内， $y' < 0,$ $\therefore$ 函数单调减少；

在$(0,1)$内， $y' < 0,$ $\therefore$ 函数单调减少；

在$(1,+\infty)$内， $y' > 0,$ $\therefore$ 函数单调增加。

the local minimum is $f(1) = 1^{1/3}(1-4) = -3.$

Interval	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing



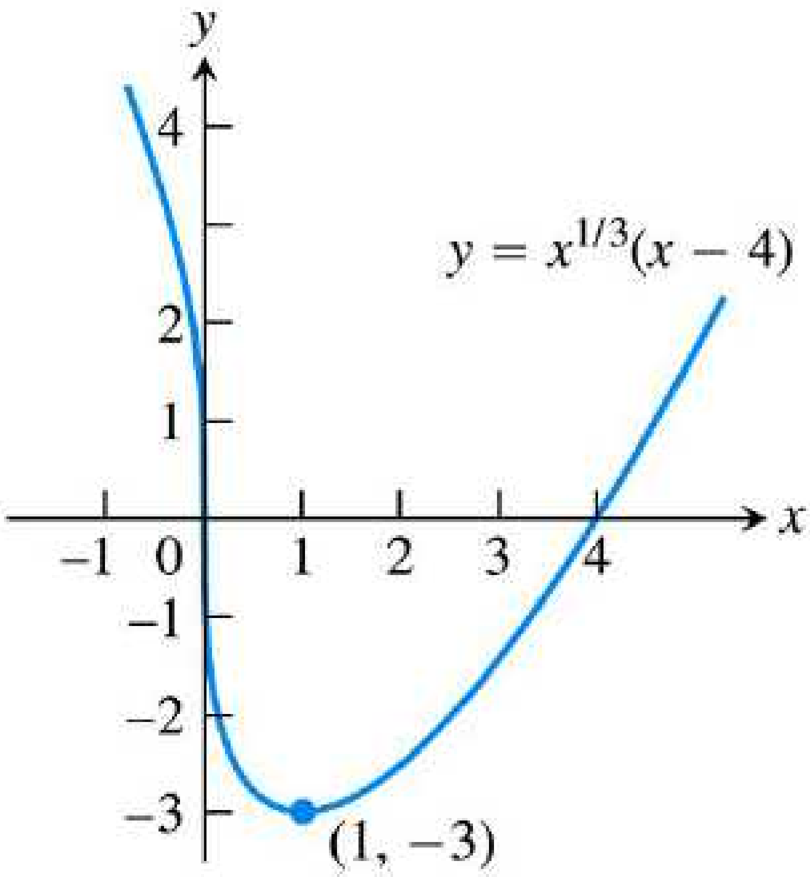


FIGURE 4.22 The function $f(x) = x^{1/3}(x - 4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

Ex. 3 求函数 $y=\sin^2 x-\sin x-1(0\le x\le 2\pi)$的关键点、单调区间以及局部极值和最值.

解 $f'(x)=2\sin x\cos x-\cos x=\cos x(2\sin x-1)$

解得关键点$x=\dfrac{\pi}{6},\dfrac{\pi}{2},\dfrac{5\pi}{6},\dfrac{3\pi}{2}$.

在$(0,\dfrac{\pi}{6})$内, $y'<0$, $\therefore$ 函数单调减少;

在$(\dfrac{\pi}{6},\dfrac{\pi}{2})$内, $y'>0$, $\therefore$ 函数单调增加。

在$(\dfrac{\pi}{2},\dfrac{5\pi}{6})$内, $y'<0$, $\therefore$ 函数单调减少;

在$(\dfrac{5\pi}{6},\dfrac{3\pi}{2})$内, $y'>0$, $\therefore$ 函数单调增加。

在$(\dfrac{3\pi}{2},2\pi)$内, $y'<0$, $\therefore$ 函数单调减少.

$f(\dfrac{\pi}{6})=-\dfrac{5}{4}$

$f(\dfrac{\pi}{2})=-1$

$f(\dfrac{5\pi}{6})=-\dfrac{5}{4}$

$f(\dfrac{3\pi}{2})=1$

局部极小值$f(\frac{\pi}{6}) = -\frac{5}{4}$ $\quad f(\frac{5\pi}{6}) = -\frac{5}{4}$ $\quad f(0) = -1$

局部极大值$f(\frac{\pi}{2}) = -1$ $\quad f(\frac{3\pi}{2}) = 1$ $\quad f(2\pi) = -1$

全局最大值$f(\frac{3\pi}{2}) = 1$ $\quad$ 全局最小值$f(\frac{5\pi}{6}) = -\frac{5}{4}$ $\quad f(\frac{\pi}{6}) = -\frac{5}{4}$

Interval	$\left(0, \frac{\pi}{6}\right)$	$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$	$\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
Sign of f'	$-$	$+$	$-$	$+$	$-$
Behavior of f	dec	inc	dec	increasing	decreasing

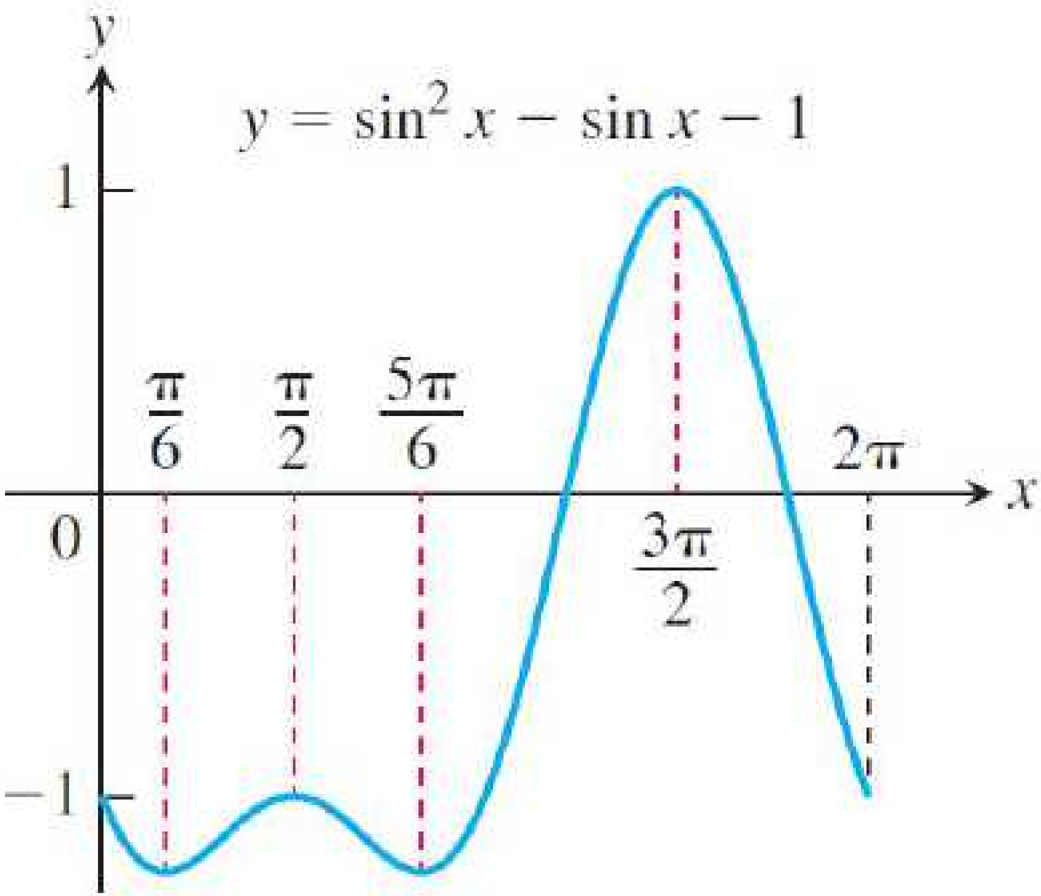


FIGURE 4.23 The graph of the function in Example 3.

例.设$0 < a < \dfrac{1}{2}$,确定方程$x^a - ax = 1 - 2a$在$(0,+\infty)$内的实根个数.

解.设$f(x) = x^a - ax - 1 + 2a$

$f(x)$在$[0,+\infty)$连续，在$(0,+\infty)$可导.

$$f'(x) = ax^{a-1} - a = a(x^{a-1} - 1)$$

当$x < 1$, $f'(x) > 0$, $f(x)$在$(0,1)$内递增,

当$x > 1$, $f'(x) < 0$, $f(x)$在$(1,+\infty)$内递减.

$f(1) = a > 0$, $f(0) = 2a - 1 < 0$, $f(x)$在$(0,1)$内有唯一实根,

又$f(1) = a > 0$, $\lim\limits_{x \to +\infty} f(x) = -\infty$, $f(x)$在$(1, +\infty)$内有唯一实根.

所以$f(x)$在$(0, +\infty)$内恰有2个实根.

例.证明：当 $x \in (0,\pi), \sin\frac{x}{2} > \frac{x}{\pi}$.

证明 设 $f(x) = \sin\frac{x}{2} - \frac{x}{\pi}$， $f(x)$在$[0,\pi]$连续，在$(0,\pi)$可导.

$$f'(x) = \frac{1}{2}\cos\frac{x}{2} - \frac{1}{\pi}, \qquad f''(x) = -\frac{1}{4}\sin\frac{x}{2} < 0$$

$f'(x)$在$[0,\pi]$内递减， $f'(0) = \frac{1}{2} - \frac{1}{\pi} > 0$， $f'(\pi) = -\frac{1}{\pi} < 0$，

$\exists x_0 \in (0,\pi)$，使 $f'(x_0) = 0$.

从而当 $0 < x < x_0, f'(x) > 0, \Rightarrow f(x)$在$(0,x_0)$增，

当 $x_0 < x < \pi, f'(x) < 0, \Rightarrow f(x)$在$(x_0,\pi)$减，

$f(x)$在$[0,\pi]$的最小值在端点取得： 而 $f(0) = 0, f(\pi) = 0$，

所以，当 $x \in (0,\pi), f(x) > 0$，即 $\sin\frac{x}{2} > \frac{x}{\pi}$.

4.4

Concavity and Curve Sketching
凹性和曲线作图

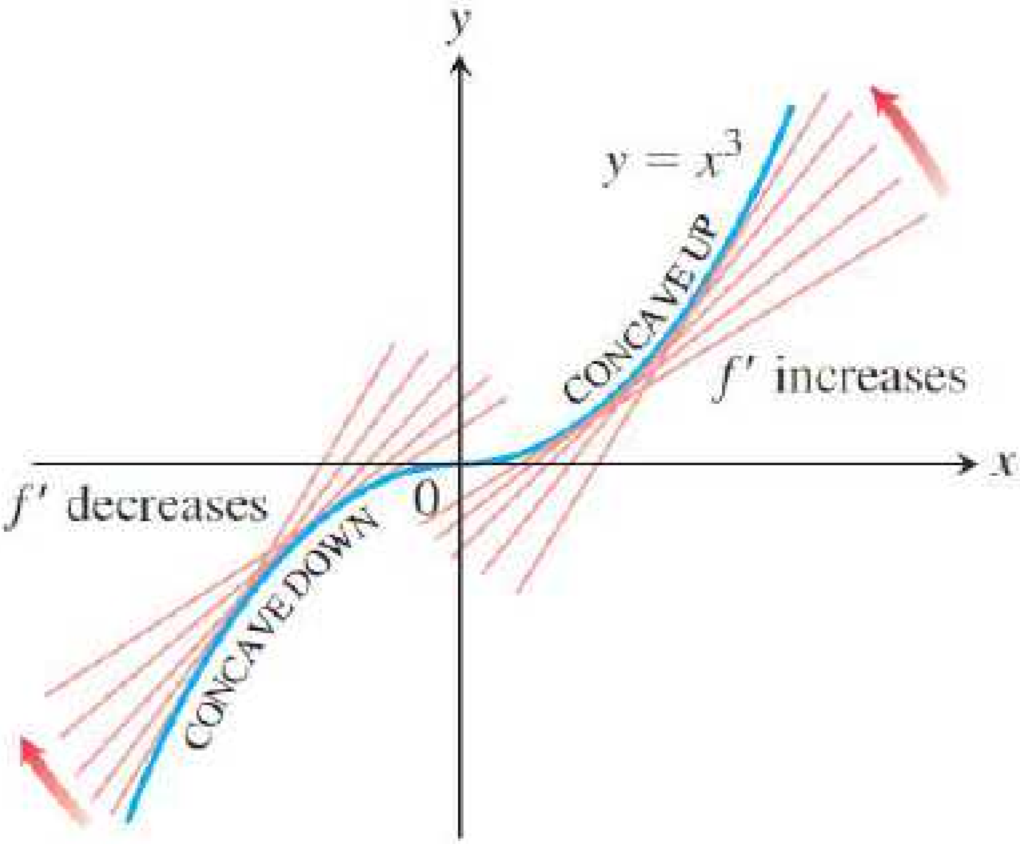


FIGURE 4.24 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I; 凹的

(b) **concave down** on an open interval I if f' is decreasing on I. 凸的

设$f(x)$在$[a,b]$上连续, 如果对$[a,b]$内任意两点$x_1 \neq x_2$, 恒有

$$f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2},$$

那末称 $f(x)$在$[a,b]$内的图形是凹的;

若恒有 $$f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2},$$

那末称 $f(x)$在$[a,b]$内的图形是凸的;

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Ex. 1 **Ditermine the concavity of the graph of**

$$(a)\, y = x^3 \quad \text{and} \quad (b)\, y = x^2.$$

解 $(a)\, y'' = 6x, \quad x > 0, y'' > 0,$ **concave up;**

$x < 0, y'' < 0,$ **concave down.**

$(b)\, y'' = 2, \quad \forall x \in R, y'' > 0,$ **concave up.**

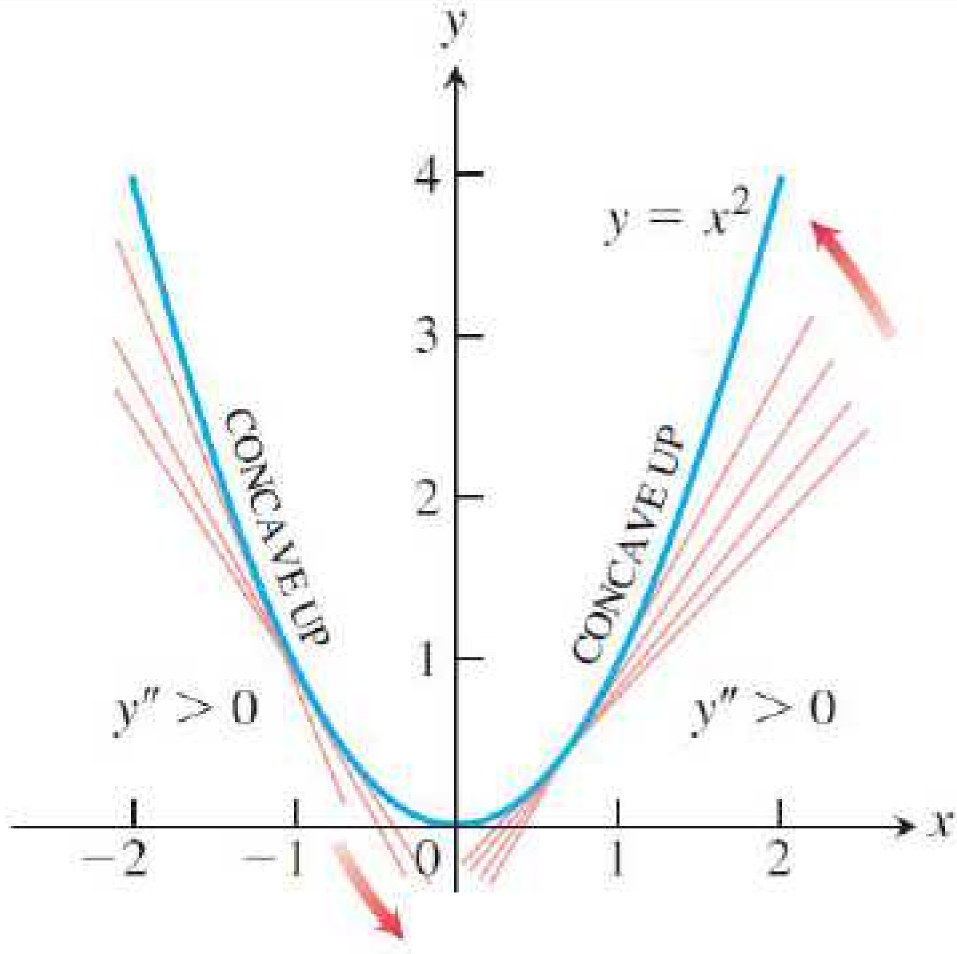


FIGURE 4.25 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

Ex. 2 Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution $y'' = -\sin x,$

$0 < x < \pi, y'' < 0,$ concave down ,

$\pi < x < 2\pi, y'' > 0,$ concave up.

$y''(\pi) = 0.$

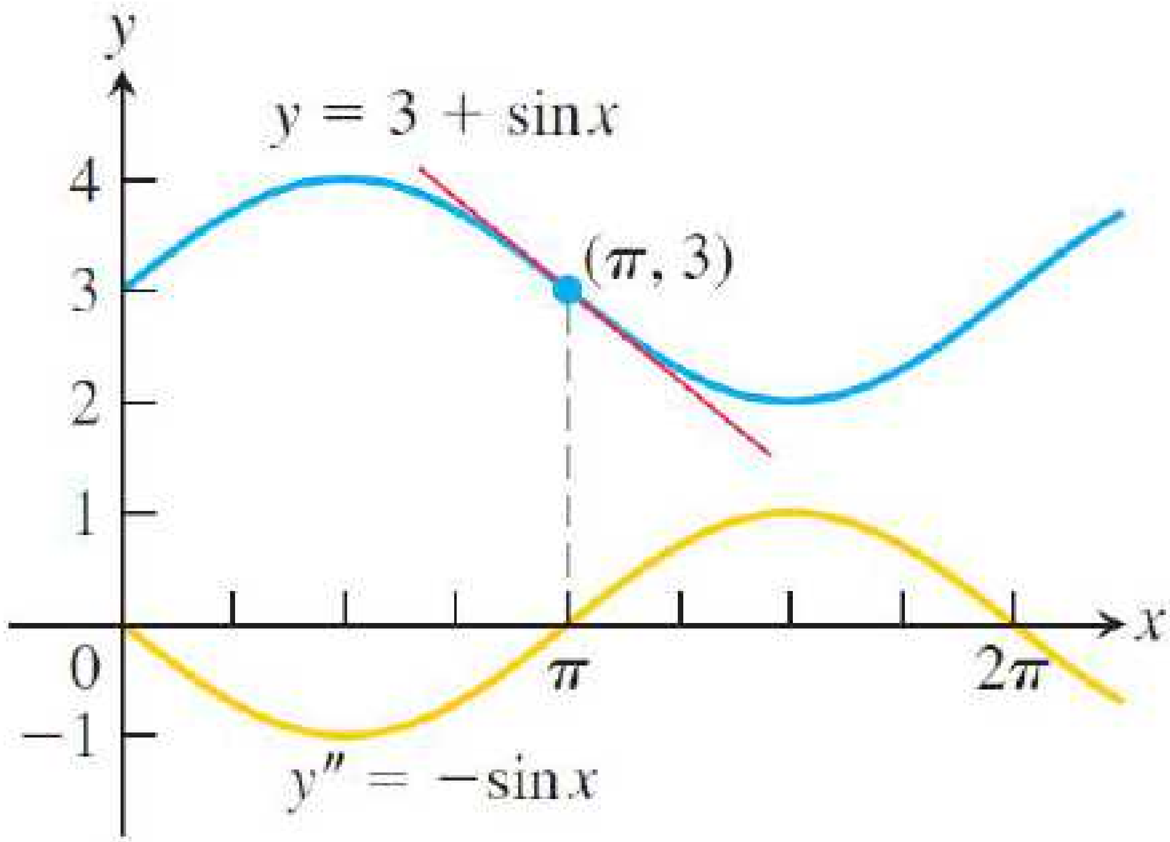


FIGURE 4.26 Using the sign of y'' to determine the concavity of y (Example 2).

DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection.** 拐点

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

Ex. 3 **Determine the concavity of $y = x^{5/3}$ on $[0,2\pi]$ and its points of inflection .**

Solution $$y'' = \frac{10}{9\sqrt[3]{x}},$$

$$-\infty < x < 0, y'' < 0, \textit{concave down},$$

$$0 < x < \infty, y'' > 0, \textit{concave up}.$$

the point of inflection is (0,0)

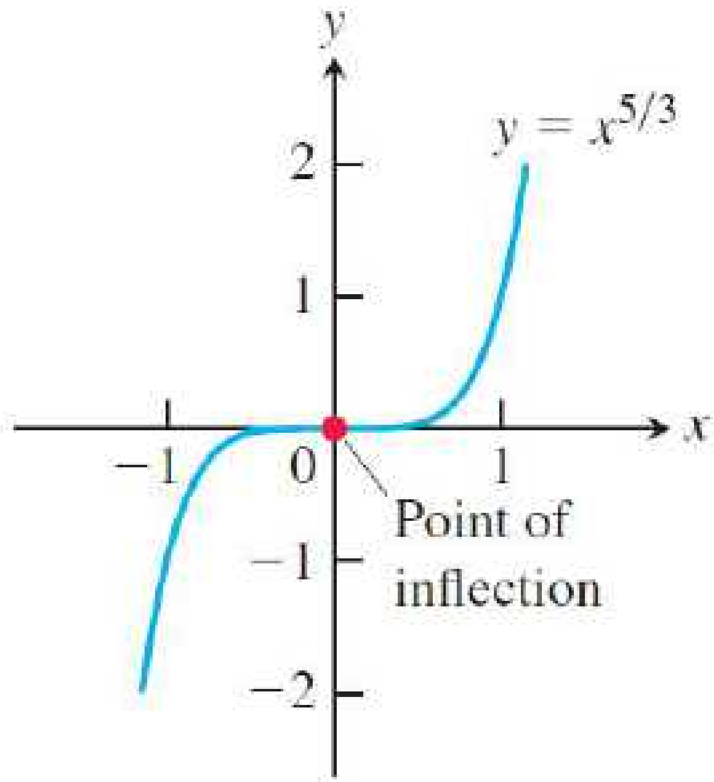


FIGURE 4.27 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$ (Example 3).

二阶导数为零的点和二阶导数不存在的点一定是拐点的横坐标吗?

Ex. 4 Determine the concavity of $y = x^4$ and its points of inflection .

Solution $y'' = 12x^2, \quad y''(0) = 0$

$-\infty < x < 0, y'' > 0, \textit{concave up},$

$0 < x < \infty, y'' > 0, \textit{concave up}.$

$(0,0)$ is not a point of inflection.

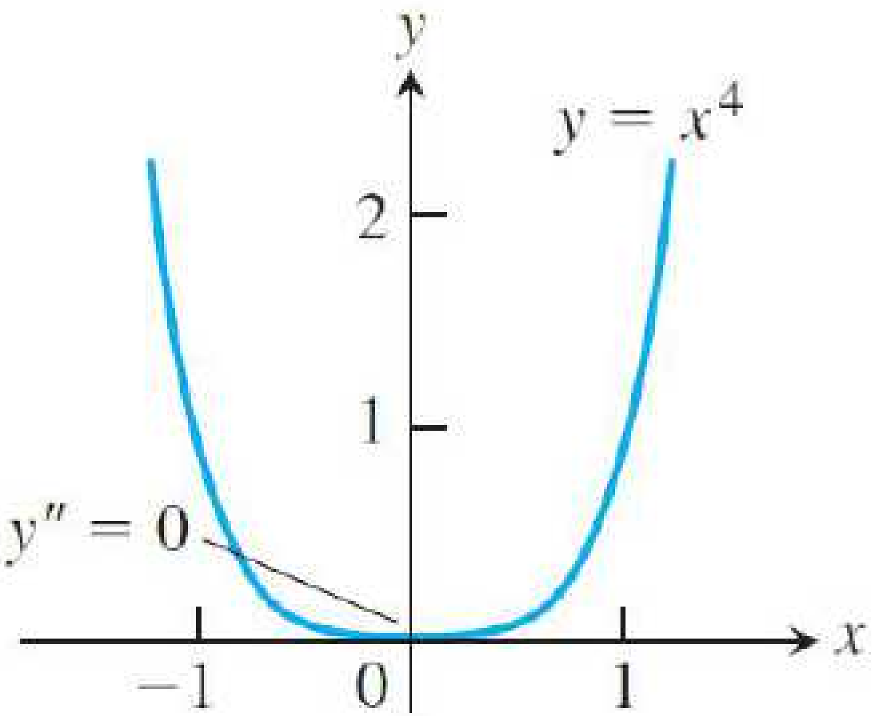


FIGURE 4.28 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 4).

Ex. 5 Find the point of inflection of $y=\sqrt[3]{x}$.

解 $y'=\frac{1}{3}x^{-\frac{2}{3}}$, $y''=-\frac{4}{9}x^{-\frac{5}{3}}$, $x\neq 0,$

$x\in(-\infty,0), y''>0,$ the graph on $(-\infty,0]$ in is concave up.

$x\in(0,+\infty), y''<0,$ the graph on $[0,+\infty)$ is concave down.

$\therefore (0,0)$ is the point of inflection for $y=\sqrt[3]{x}$.

Ex. 曲线 $y=\sqrt[3]{x^2}$ 有无拐点?

解 当$x\neq 0$时, $y'=\frac{2}{3}x^{-\frac{1}{3}}$, $y''=-\frac{2}{9}x^{-\frac{4}{3}}$,

在$(-\infty,0)$内, $y''<0,$ 曲线在 $[0,+\infty)$ concave down.

在$(0,+\infty)$内, $y''<0,$ 曲线在 $[0,+\infty)$ concave down.

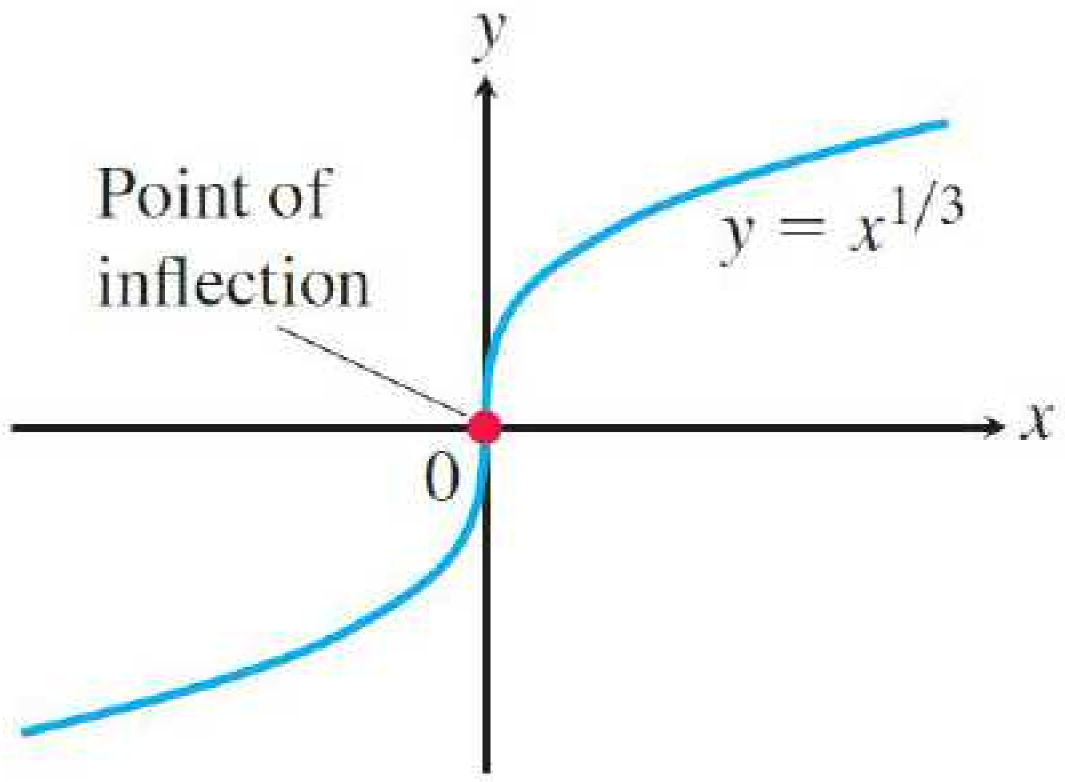


FIGURE 4.29 A point of inflection where y' and y'' fail to exist (Example 5).

Ex. 6 设一个粒子沿着数轴做 水平运动，数轴的正向 向右. 已知该粒子在 t时刻的位置是 $s(t)=2t^3-14t^2+22t-5$, $t\geq 0$. 求速度和加速度，并描 述粒子的运动 .

解

$$v(t)=s'(t)=6t^2-28t+22=2(t-1)(3t-11)$$

$$a(t)=v'(t)=s''(t)=12t-28=4(3t-7)$$

Interval	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	$+$	$-$	$+$
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

Interval	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	$-$	$+$
Graph of s	concave down	concave up

THEOREM 5—Second Derivative Test for Local Extrema Suppose f'' is continuous on an open interval that contains $x = c$.

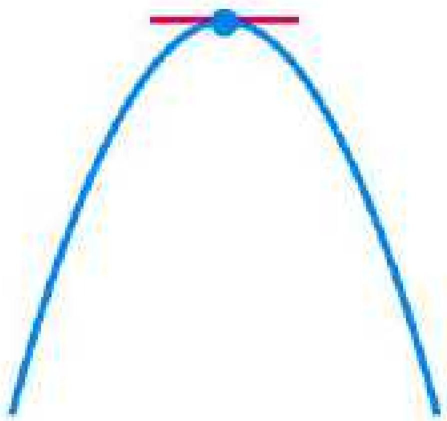
1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

证明1. $f''(c) = \lim_{x \to c} \dfrac{f'(x) - f'(c)}{x - c} = \lim_{x \to c} \dfrac{f'(x)}{x - c} < 0,$

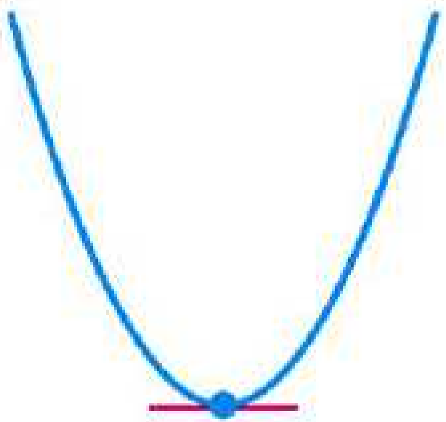
当$x < c, f'(x) > 0,$当$x > c, f'(x) < 0, \therefore f(c)$是局部极大值．

$$f(x) = x^3, f(x) = x^4, f(x) = -x^4$$

$$x = 0.$$



$f' = 0, f'' < 0$
$\Rightarrow$ local max



$f' = 0, f'' > 0$
$\Rightarrow$ local min

Ex. 7 Sketch the graph of $y = x^4 - 4x^3 + 10.$

(a)Find where the curve is monotonic.

(b)Find where the curve is concave up, or concave down.

(c)Find the local extremaand the points of inflection.

Solution $D:(-\infty,+\infty)$ $\quad y' = 4x^2(x-3),$

$$y'' = 12x(x-2).$$

$$x_1 = 0,\ x_2 = 3,\quad x_3 = 2.$$

Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

Interval	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

$x < 0$		$0 < x < 2$		$2 < x < 3$		$3 < x$
decreasing	拐点	decreasing	拐点	decreasing	极值点	increasing
concave up		concave down		concave up		concave up

$$f(0) = 10,\ f(2) = -6,\ f(3) = -17$$

$$f(0) = 10,$$
$$f(3) = -17,$$
$$f(2) = -6.$$

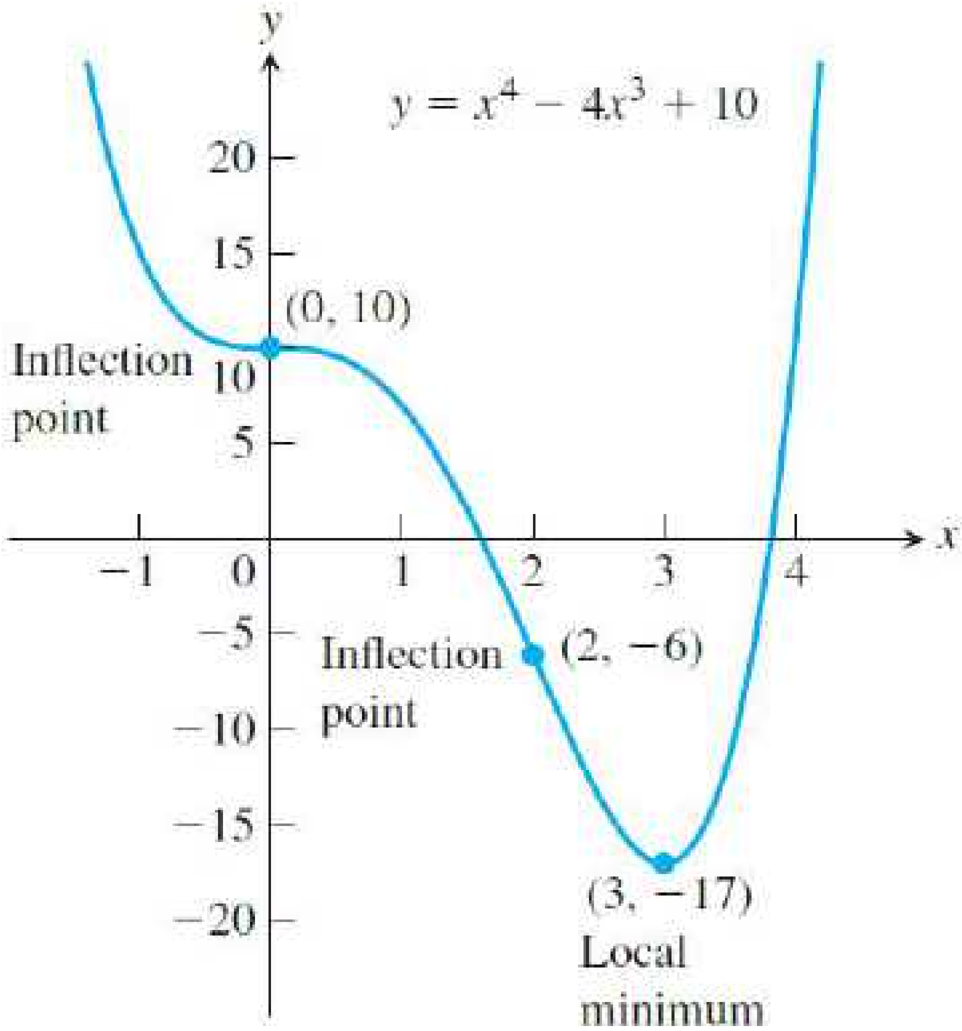


FIGURE 4.30 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 7).

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

Ex. 8 **Sketch the graph of** $f(x)=\dfrac{(x+1)^2}{1+x^2}$.

Solution: $D: R$, **No symmetries.**

$$f'(x)=\frac{2(1-x^2)}{(1+x^2)^2}, \qquad f''(x)=\frac{4x(x^2-3)}{(1+x^2)^3},$$

Let $f'(x)=0$, **we have** $x=\pm 1$,

Let $f''(x)=0$, **we have** $x=0, \pm\sqrt{3}$.

列表 $f'(x)=\dfrac{2(1-x^2)}{(1+x^2)^2}$, $f''(x)=\dfrac{4x(x^2-3)}{(1+x^2)^3}$,

x	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},-1)$	$(-1,0)$	$(0,1)$	$(1,\sqrt{3})$	$(\sqrt{3},+\infty)$
$f'(x)$	$-$		$-$	$+$	$+$	$-$	$-$
$f''(x)$	$-$		$+$	$+$	$-$	$-$	$+$
$f(x)$							

$f(x)=\dfrac{(x+1)^2}{1+x^2}$

拐点 $(-\sqrt{3},1-\frac{\sqrt{3}}{2})$

极小值 $f(-1)=0$

拐点 $(0,1)$

极大值 $f(1)=2$

拐点 $(\sqrt{3},1+\frac{\sqrt{3}}{2})$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(x+1)^2}{1+x^2} = 1$$

得水平渐近线 $y=1$

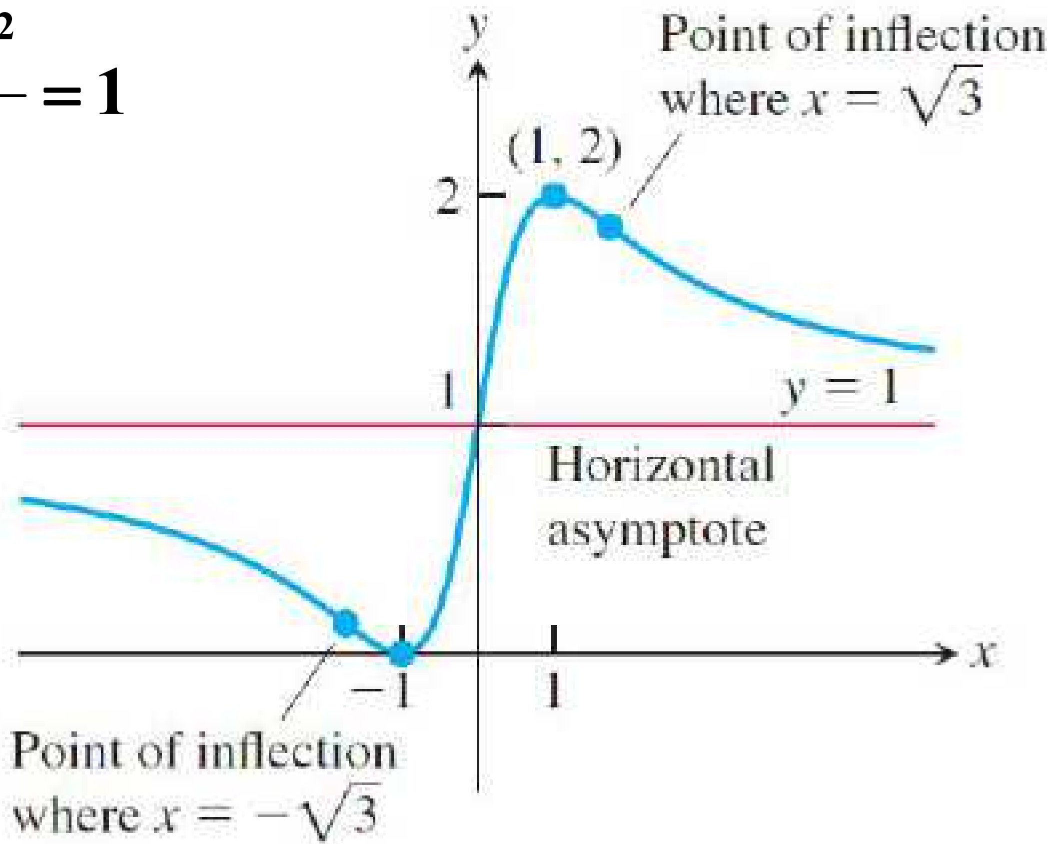


FIGURE 4.31 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 8).

Ex. 9 **Sketch the graph of** $f(x)=\dfrac{x^2+4}{2x}$.

Solution: $D: x\neq 0,$ **symmetric about the origin.**

$f'(x)=\dfrac{x^2-4}{2x^2},\qquad f''(x)=\dfrac{4}{x^3},$

令 $f'(x)=0,$ 得驻点 $x=\pm 2,$ $\qquad f(x)=\dfrac{x}{2}+\dfrac{2}{x}$,

斜渐近线 $y=\dfrac{x}{2}$. $\lim\limits_{x\to 0^{\pm}} f(x)=\lim\limits_{x\to 0^{\pm}}(\dfrac{x}{2}+\dfrac{2}{x})=\pm\infty$,

竖直渐近线 $x=0$.

设斜渐近线 $y=ax+b$ ，则 $a=\lim\limits_{x\to\infty}\dfrac{f(x)}{x}\quad b=\lim\limits_{x\to\infty}(f(x)-ax)$

$a=\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=\dfrac{1}{2}\qquad b=\lim\limits_{x\to\infty}(\dfrac{x}{2}+\dfrac{2}{x}-\dfrac{x}{2})=0$

$$f(x)=\frac{x}{2}+\frac{2}{x}\ ,\qquad f'(x)=\frac{x^2-4}{2x^2},\qquad f''(x)=\frac{4}{x^3},$$

x	$(0,2)$	2	$(2,\infty)$
$f'(x)$	$-$		$+$
$f''(x)$	$+$	极小值	$+$
$f(x)$		$f(2)=2$	

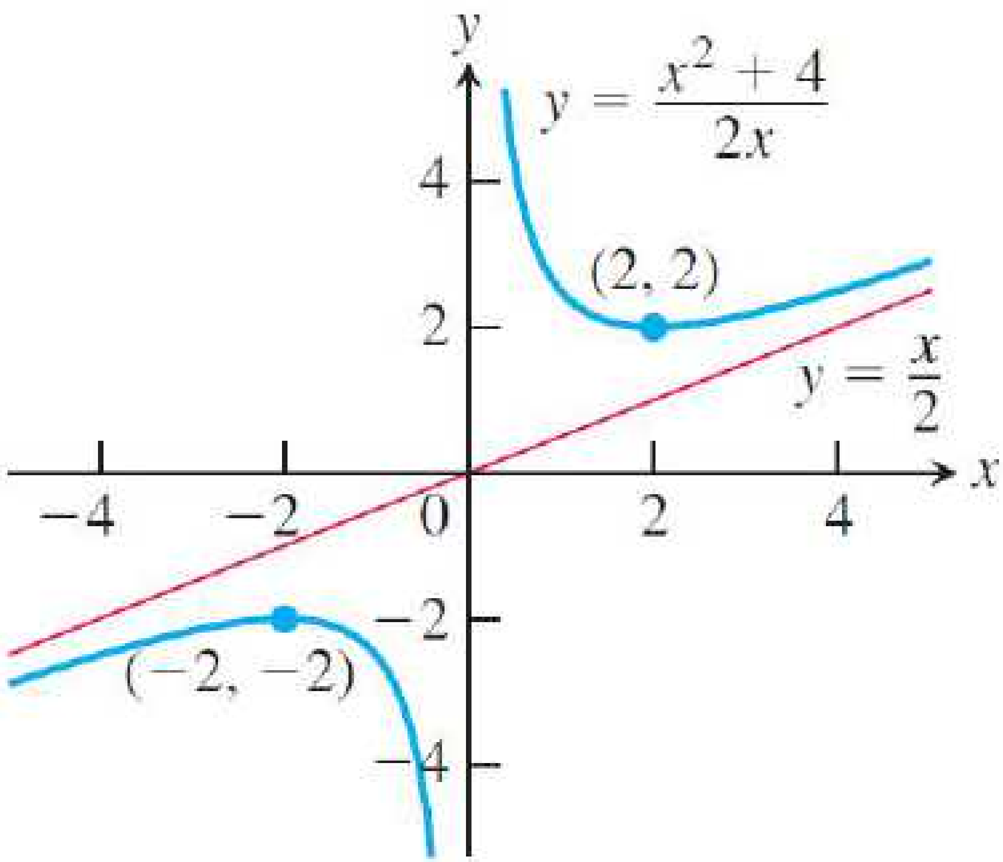


FIGURE 4.32 The graph of $y = \dfrac{x^2 + 4}{2x}$ (Example 9).

Ex. 10 Sketch the graph of $f(x)=\cos x-\frac{\sqrt{2}}{2}x$

$(0\le x\le 2\pi)$.

Solution: $D:[0,2\pi]$ No symmetries.

$f'(x)=-\sin x-\frac{\sqrt{2}}{2}$, $f''(x)=-\cos x$,

令 $f'(x)=0$, 得驻点 $x=\frac{5\pi}{4},\frac{7\pi}{4}$,

令$f''(x)=0$, 得点 $x=\frac{\pi}{2},\frac{3\pi}{2}$,

$$f(x)=\cos x-\frac{\sqrt{2}}{2}x \quad f'(x)=-\sin x-\frac{\sqrt{2}}{2}, \quad f''(x)=-\cos x,$$

x	$(0,\frac{\pi}{2})$	$(\frac{\pi}{2},\frac{5\pi}{4})$	$(\frac{5\pi}{4},\frac{3\pi}{2})$	$(\frac{3\pi}{2},\frac{7\pi}{4})$	$(\frac{7\pi}{4},2\pi)$
$f'(x)$	$-$	$-$	$+$	$+$	$-$
$f''(x)$	$-$	$+$	$+$	$-$	$-$
$f(x)$					

极小值 $f(\frac{5\pi}{4})=-3.48$ 极大值 $f(\frac{7\pi}{4})=-3.18$ 拐点 $(\pi/2,-1.11)$

$(3\pi/2,-3.33)$

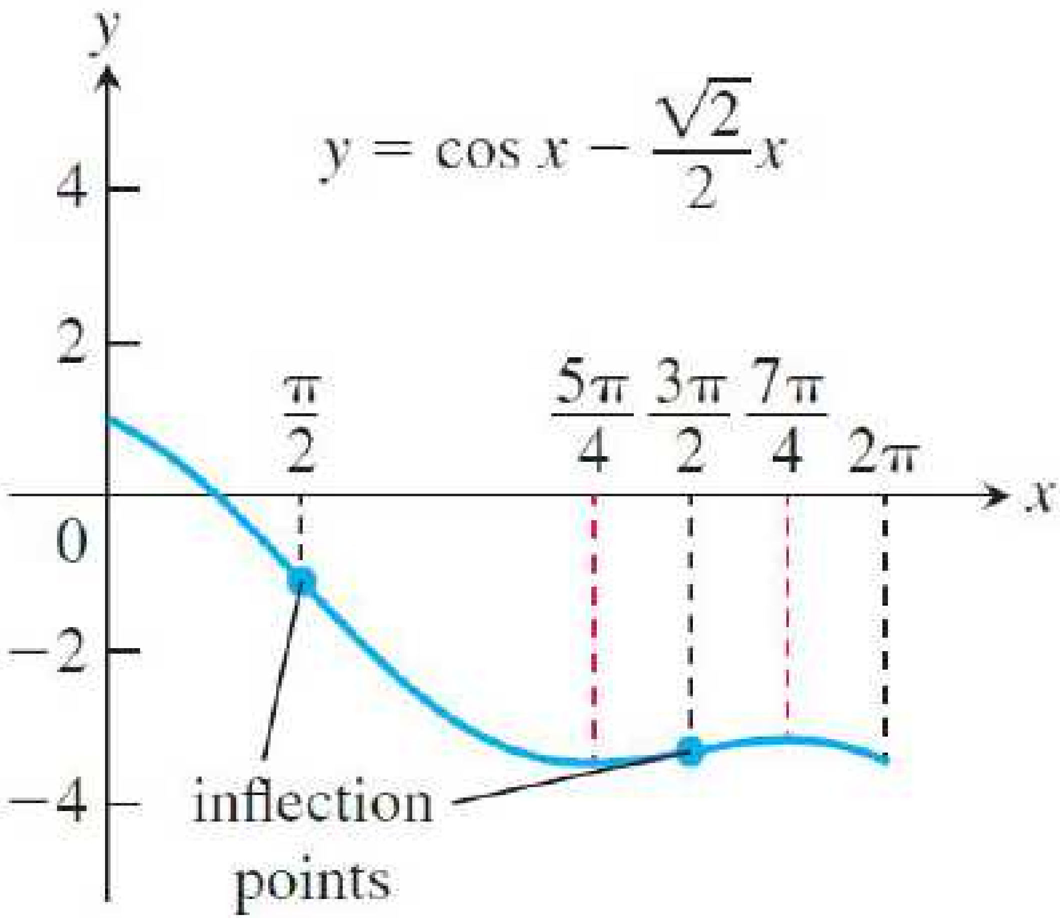
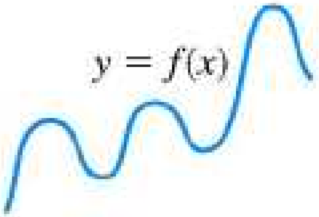
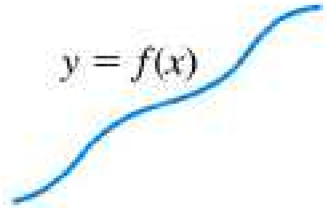


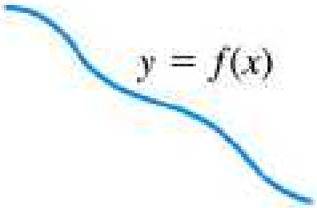
FIGURE 4.33 The graph of the function in Example 10.



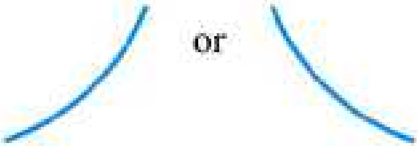
Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall



$y' > 0 \Rightarrow$ rises from left to right; may be wavy

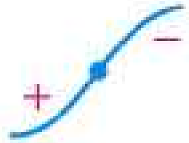


$y' < 0 \Rightarrow$ falls from left to right; may be wavy



$y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall

$y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall



y'' changes sign at an inflection point

y' changes sign $\Rightarrow$ graph has local maximum or local minimum

$y' = 0$ and $y'' < 0$ at a point; graph has local maximum

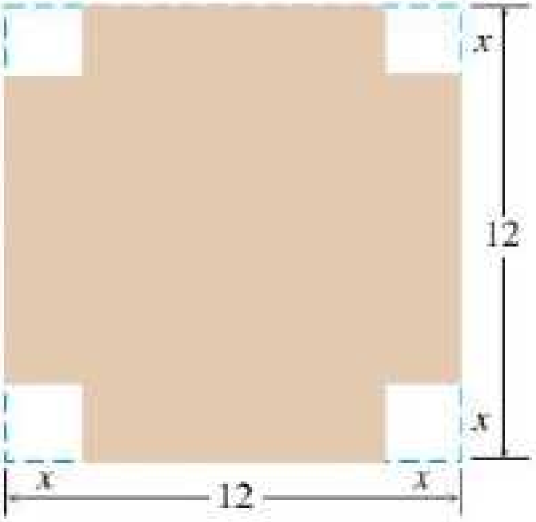
$y' = 0$ and $y'' > 0$ at a point; graph has local minimum

4.5

Applied Optimization
应用优化问题

EXAMPLE 1

An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?



Solution the corner squares are x in.

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3.$$

$$\frac{dV}{dx} = 12(2 - x)(6 - x).$$

only $x = 2$ lies in the interior of the function's

Critical point value: $V(2) = 128$

Endpoint values: $V(0) = 0, \qquad V(6) = 0.$

The maximum volume is 128 in^3. The cutout squares should be 2 in. on a side.

Ex. 1

将边长为$12cm$的正方形纸板的四个角 剪掉一个小正方形后再折起称为一个 无盖的纸箱，问剪掉的 小正方形的边长为多少时， 可以使得纸箱的容积最 大?

解: 设剪掉的小正方形边长 为x,

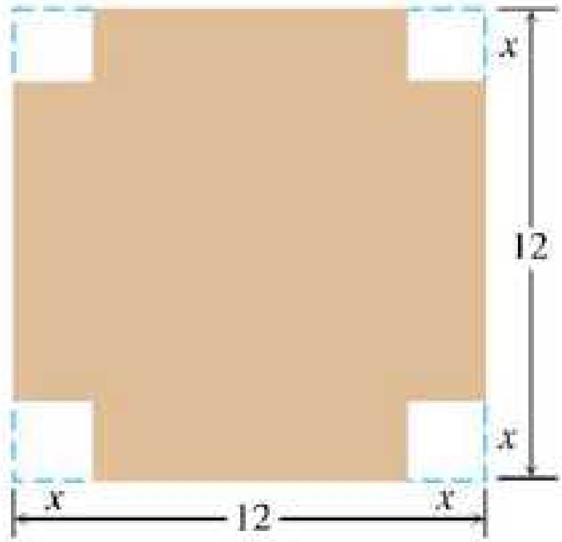
纸箱的体积为V,则 $V = x(12-2x)^2, 0 \le x \le 6$

$$V' = 12(2-x)(6-x)$$

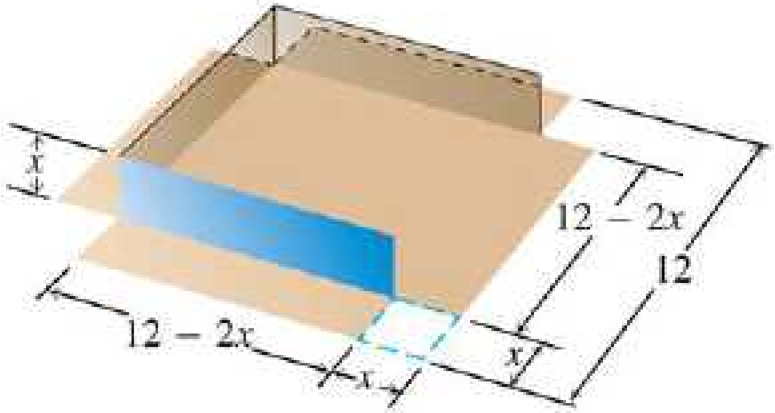
令$V' = 12(2-x)(6-x) = 0$，解得驻点 $x = 2$

$$V(2) = 128 \qquad V(0) = 0 \qquad V(6) = 0$$

$$\therefore V_{\max} = 128(cm^3)$$



(a)



(b)

FIGURE 4.34 An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

Solving Applied Optimization Problems

1. *Read the problem*. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture*. Label any part that may be important to the problem.
3. *Introduce variables*. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity*. If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown*. Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

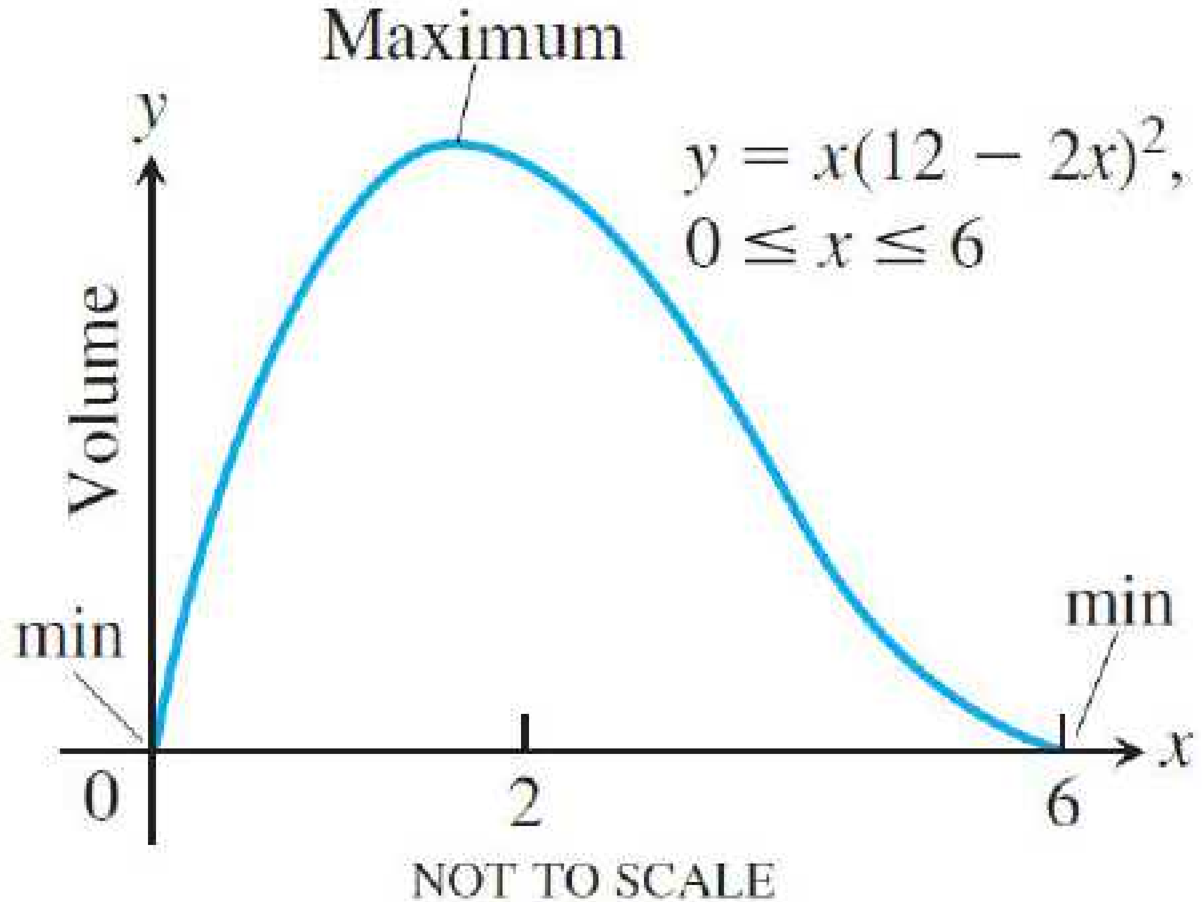
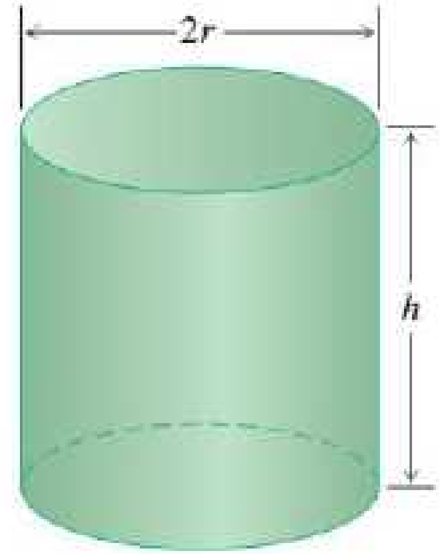


FIGURE 4.35 The volume of the box in Figure 4.34 graphed as a function of x.

EXAMPLE 2

You have been asked to design a one-liter can shaped like a right circular cylinder What dimensions will use the least material?

Solution $\pi r^2 h = 1000.$



Surface area of can: $A = 2\pi r^2 + 2\pi r h$

$$= 2\pi r^2 + \frac{2000}{r}.$$

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2} \qquad 0 = 4\pi r - \frac{2000}{r^2}$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \qquad A'' = 4\pi + \frac{4000}{r^3} > 0$$

A at $r = \sqrt[3]{500/\pi}$ is an absolute minimum. $h = \dfrac{1000}{\pi r^2} = 2\sqrt[3]{\dfrac{500}{\pi}} = 2r.$

Ex. 2 欲做一个1升容量的圆柱形水罐， 问如何设计罐的尺寸可以使得用料 最省？

解: 设圆柱形的底半径为 r， 高为 h,则其表面积

$$A = 2\pi rh + 2\pi r^2, \qquad \pi r^2 h = 1000,$$

$$A = \frac{2000}{r} + 2\pi r^2, r > 0. \qquad A' = 4\pi r - \frac{2000}{r^2}$$

令 $V' = 0$, 解得驻点 $r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42.$

$$A'' = 4\pi + \frac{4000}{r^3} > 0$$

$\therefore$ 当 $r = \sqrt[3]{\frac{500}{\pi}}, h = 2\sqrt[3]{\frac{500}{\pi}}$ 时用料最省.

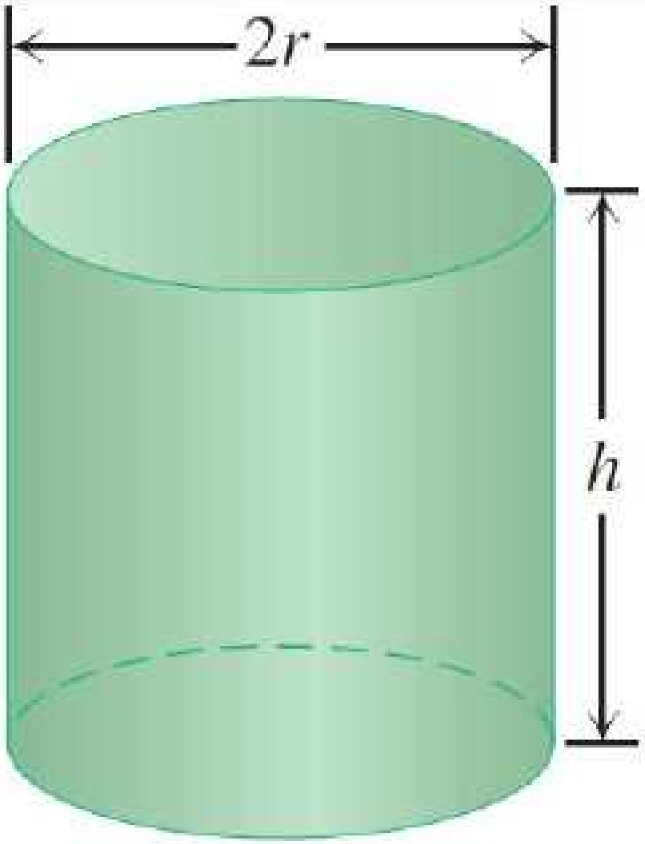


FIGURE 4.36 This one-liter can uses the least material when $h = 2r$ (Example 2).

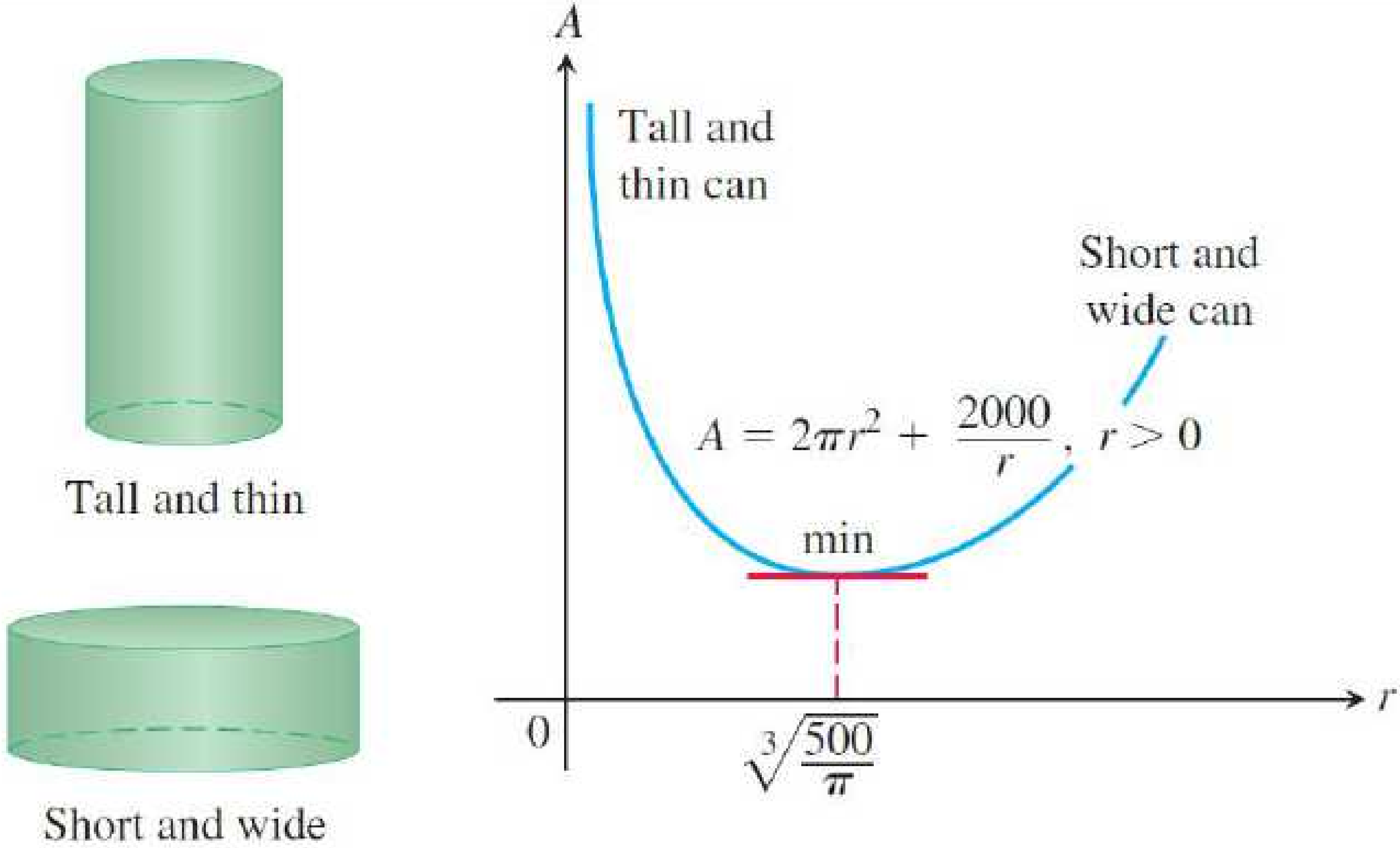


FIGURE 4.37 The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

EXAMPLE 3

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

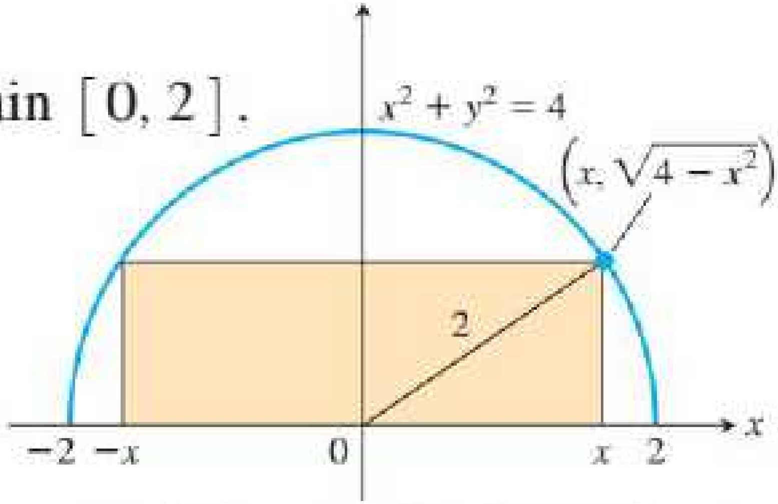
Solutior Let $\left(x, \sqrt{4-x^2}\right)$ be the coordinates of the corner of the rectangle

$A(x) = 2x\sqrt{4-x^2}$ on the domain $[0, 2]$.

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2}$$

$$\frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2} = 0$$

$$x = \pm\sqrt{2}.$$



Critical point value: $A\left(\sqrt{2}\right) = 2\sqrt{2}\sqrt{4-2} = 4$

Endpoint values: $A(0) = 0, \quad A(2) = 0.$

The area has a maximum value of 4 when the rectangle is $\sqrt{4-x^2} = \sqrt{2}$ ı

Ex. 3

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

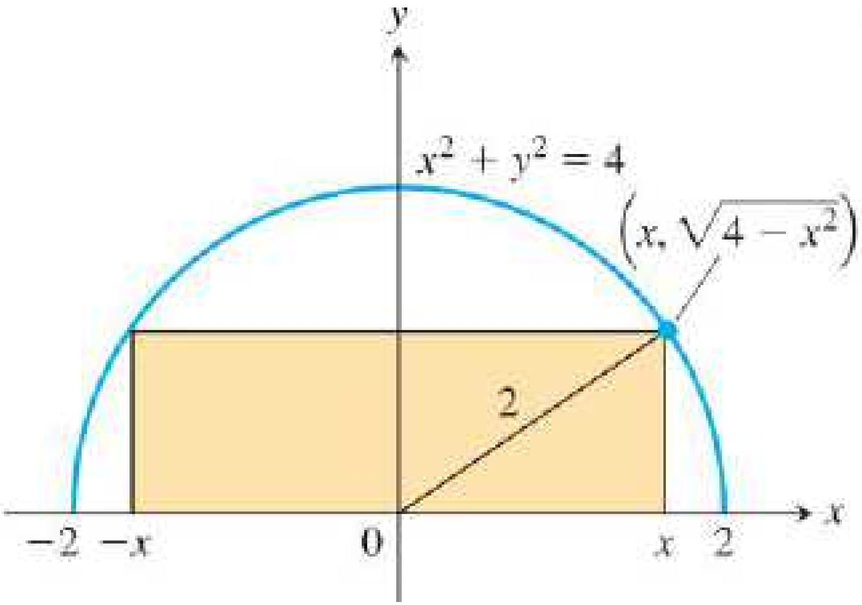
解： 如图建立坐标系，设 $(x, \sqrt{4-x^2})$在圆周上，则

$$A = 2x\sqrt{4-x^2}, 0 \le x \le 2$$

$$A' = \frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2}$$

令$V' = 0$，解得驻点 $x = \sqrt{2}$.

$$A(0) = 0, A(2) = 0, A(\sqrt{2}) = 4$$

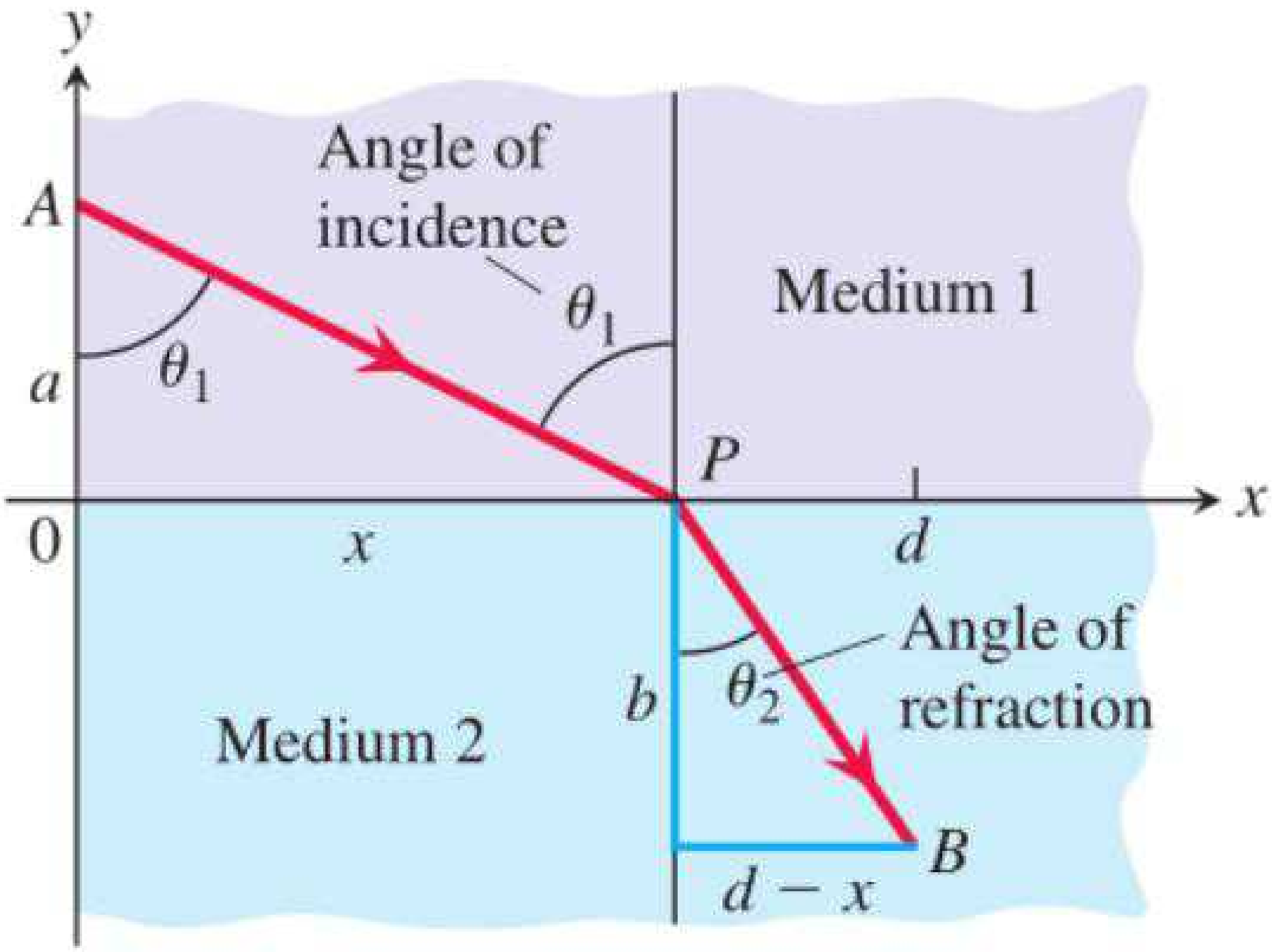


$\therefore$ 面积最大为 4，此时的长 $2\sqrt{2}$，宽为$\sqrt{2}$.

The speed of light depends on the medium through which it travels, and is generally slower in denser media.

Fermat's principle in optics states that light travels from one point to another along a path for which the time of travel is a minimum.

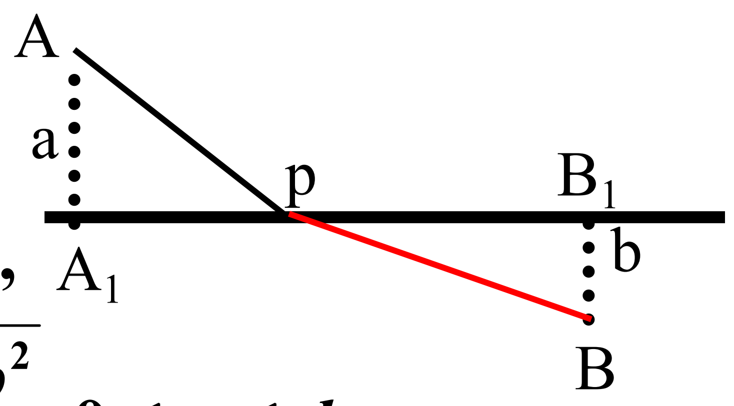
Describe the path that a ray of light will follow in going from a point A in a medium where the speed of light is c_1 to a point B in a second medium where its speed is c_2.


y
Angle of incidence
A
θ_1
Medium 1
a
θ_1
P
0
x
d
x
Angle of refraction
θ_2
b
Medium 2
B
d − x

Ex.4 由“光行最速原理”推　出“光的折射原理”.

设光在介质 A 中的速度是 c_1，在介质 B 中的速度是 c_2.

解：设光从 A 照射到 B 经过点 P,并设



$A_1P = x$,则 $B_1P = d - x$,

则 $AP = \sqrt{x^2 + a^2}, PB = \sqrt{(d-x)^2 + b^2}$,

$$\text{时间}\, T(x) = \frac{\sqrt{x^2 + a^2}}{c_1} + \frac{\sqrt{(d-x)^2 + b^2}}{c_2}, 0 \le x \le d,$$

$$T'(x) = \frac{x}{c_1\sqrt{x^2 + a^2}} - \frac{d-x}{c_2\sqrt{(d-x)^2 + b^2}}, 0 \le x \le d,$$

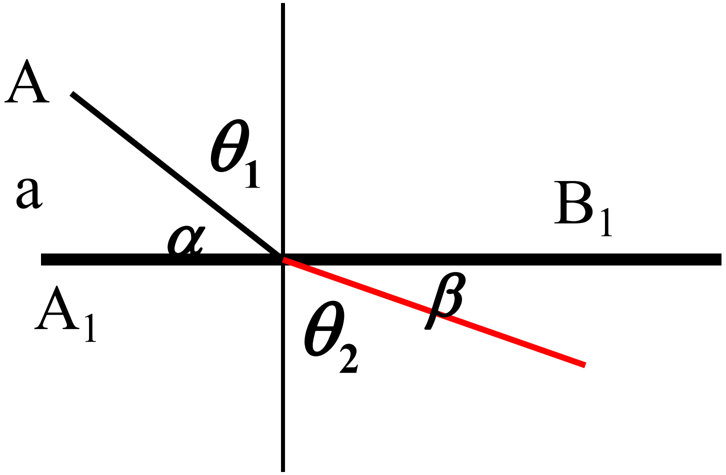
$T'(0) < 0$,　$T'(d) > 0$,　$\exists x_0 \in (0,d)$, 使 $T'(x_0) = 0$,

$$T''(x) = \frac{a^2}{c_1(\sqrt{x^2 + a^2})^3} + \frac{b^2}{c_2(\sqrt{(d-x)^2 + b^2})^2} > 0,$$

x_0 是最小值点.

$$\therefore \frac{x_0}{c_1\sqrt{x_0^{\ 2}+a^2}} = \frac{d-x_0}{c_2\sqrt{(d-x_0)^2+b^2}},$$

即 $\dfrac{\cos\alpha}{c_1} = \dfrac{\cos\beta}{c_2}$,



即 $\dfrac{\sin\theta_1}{c_1} = \dfrac{\sin\theta_2}{c_2}$.

Snell's Law

导数在经济学中的应用

成本函数 (cost) $c(x)$,

收益函数 (revenue) $r(x)$,

利润函数 (profit) $p(x) = r(x) - c(x)$.

边际成本 (marginal cost) $c'(x)$,

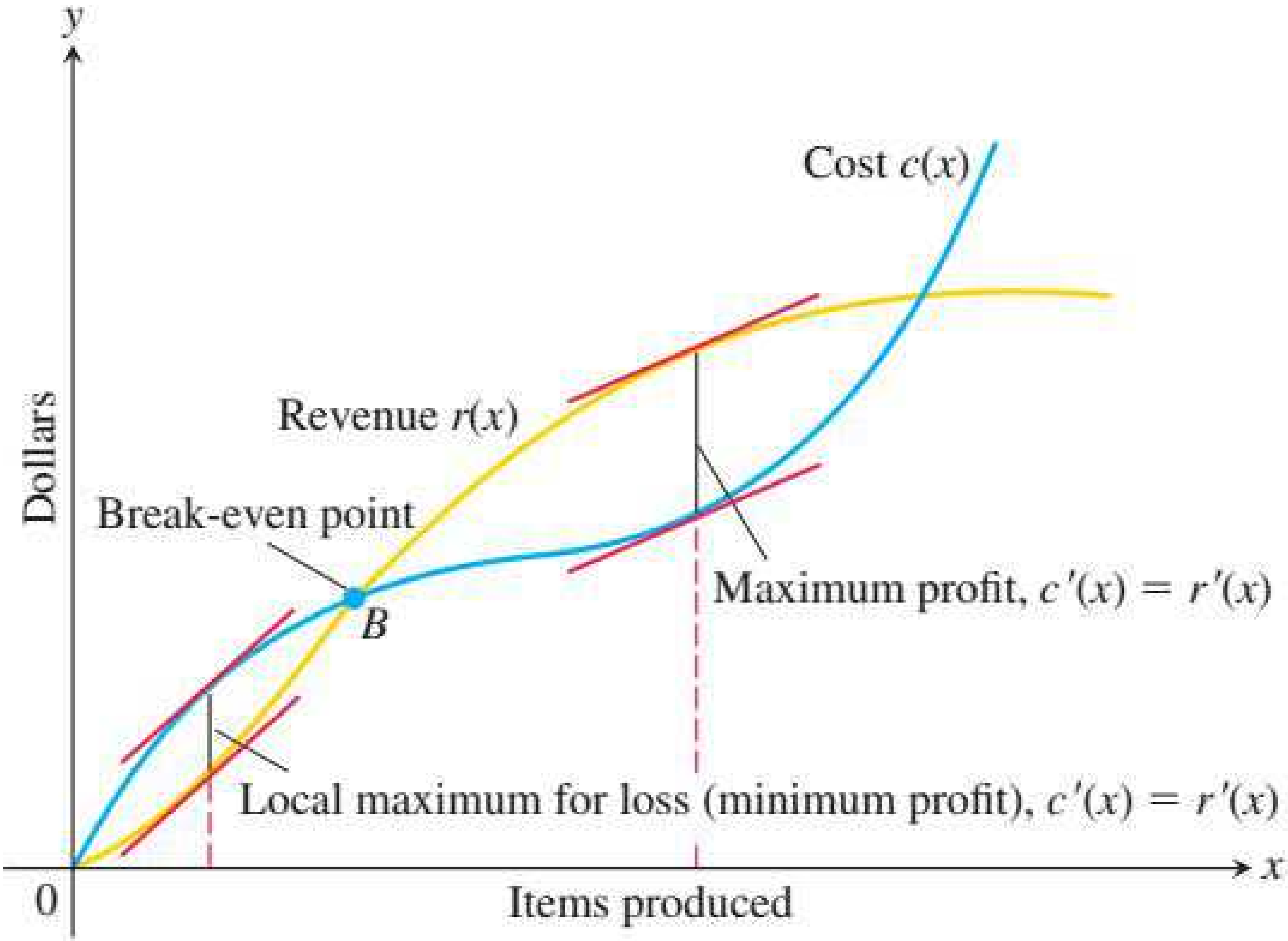
边际收益 (marginal revenue) $r'(x)$,

边际利润 (marginal profit) $p'(x) = r'(x) - c'(x)$.

若利润在 $x = x_0$时达到最大，则

$p'(x_0) = r'(x_0) - c'(x_0) = 0$, 即 $r'(x_0) = c'(x_0)$.

At a production level yielding maximum profit, marginal revenue equals marginal cost (Figure 4.41).


y
Cost $c(x)$
Revenue $r(x)$
Dollars
Break-even point
B
Maximum profit, $c'(x) = r'(x)$
Local maximum for loss (minimum profit), $c'(x) = r'(x)$
0
x
Items produced

EXAMPLE 5

Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where x represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

Solution Marginal cost $c'(x) = 3x^2 - 12x + 15$,

Marginal revenue $r'(x) = 9$,

the production level that maximizes profit:

$$3x^2 - 12x + 15 = 9, \qquad x_1 = 2 - \sqrt{2}, \quad x_2 = 2 + \sqrt{2}$$

$$x^2 - 4x + 2 = 0, \qquad x_1 \approx 0.568, \quad x_2 = 3.414$$

$$p''(x) = -(6x - 12) = 6(2 - x),$$

$$p''(x_1) = 6(2 - 0.568) > 0, \; p''(x_2) = 6(2 - 3.414) < 0,$$

the production level for maximum profit is

$x \approx 3.42$(单位)

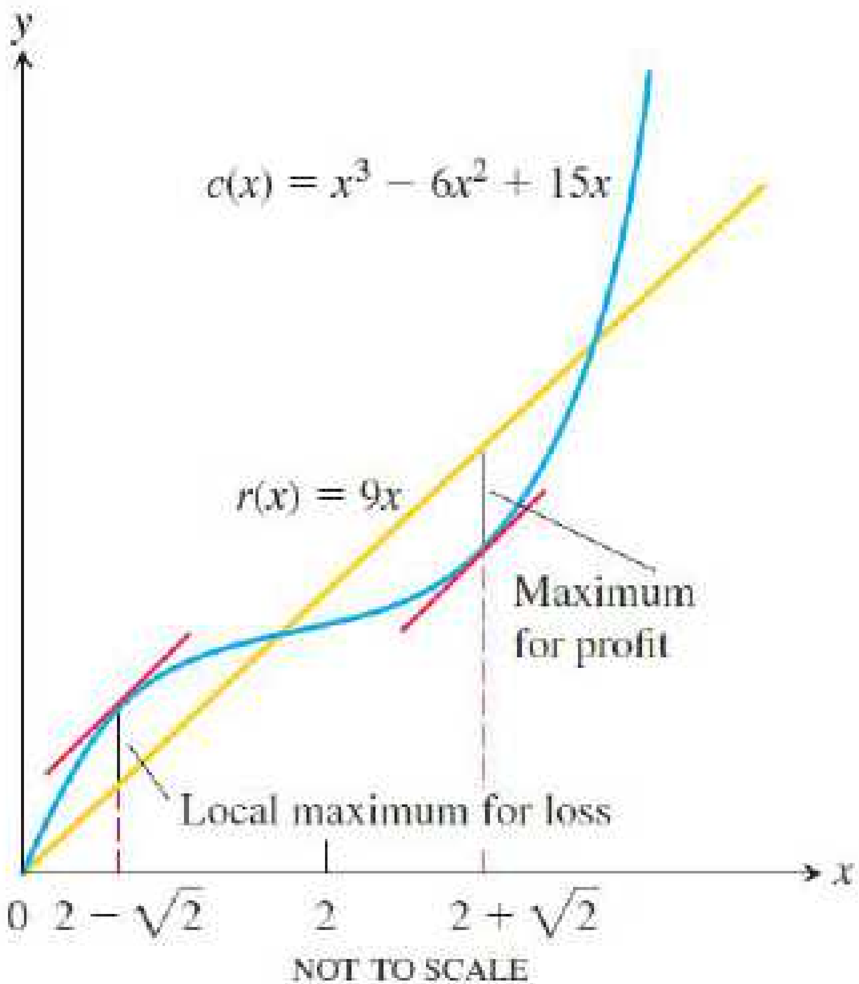


FIGURE 4.42 The cost and revenue curves for Example 5.

Ex.6 设家具生产厂每天用红 木生产 5个桌子.1集装箱的红木是 5000元，而一个单元 (制作一个桌子所用木材)的储存费是 10元 / 天.为了使每天的平均成本最小，厂家应该一 次订购多少木材？多长时间进一次木材？ 14天 70单位

解：设工厂 x天进一次木材，则必须 订购$5x$(单位).

木材的总储存费用为 $\dfrac{50(x-1)x}{2}$,

故在一个生产周期中的 成本是 $5000+\dfrac{50x(x-1)}{2}$,

平均成本 $c(x)=\dfrac{5000}{x}+25(x-1),\quad x>0$

令$c'(x)=-\dfrac{5000}{x^2}+25=0$ $\quad x=\sqrt{200}\approx 14.14$

$c''(x)=\dfrac{10000}{x^3}>0$ $\quad c(\sqrt{200})$是全局极小值 .

4.6

Newton’s Method

牛顿切线法

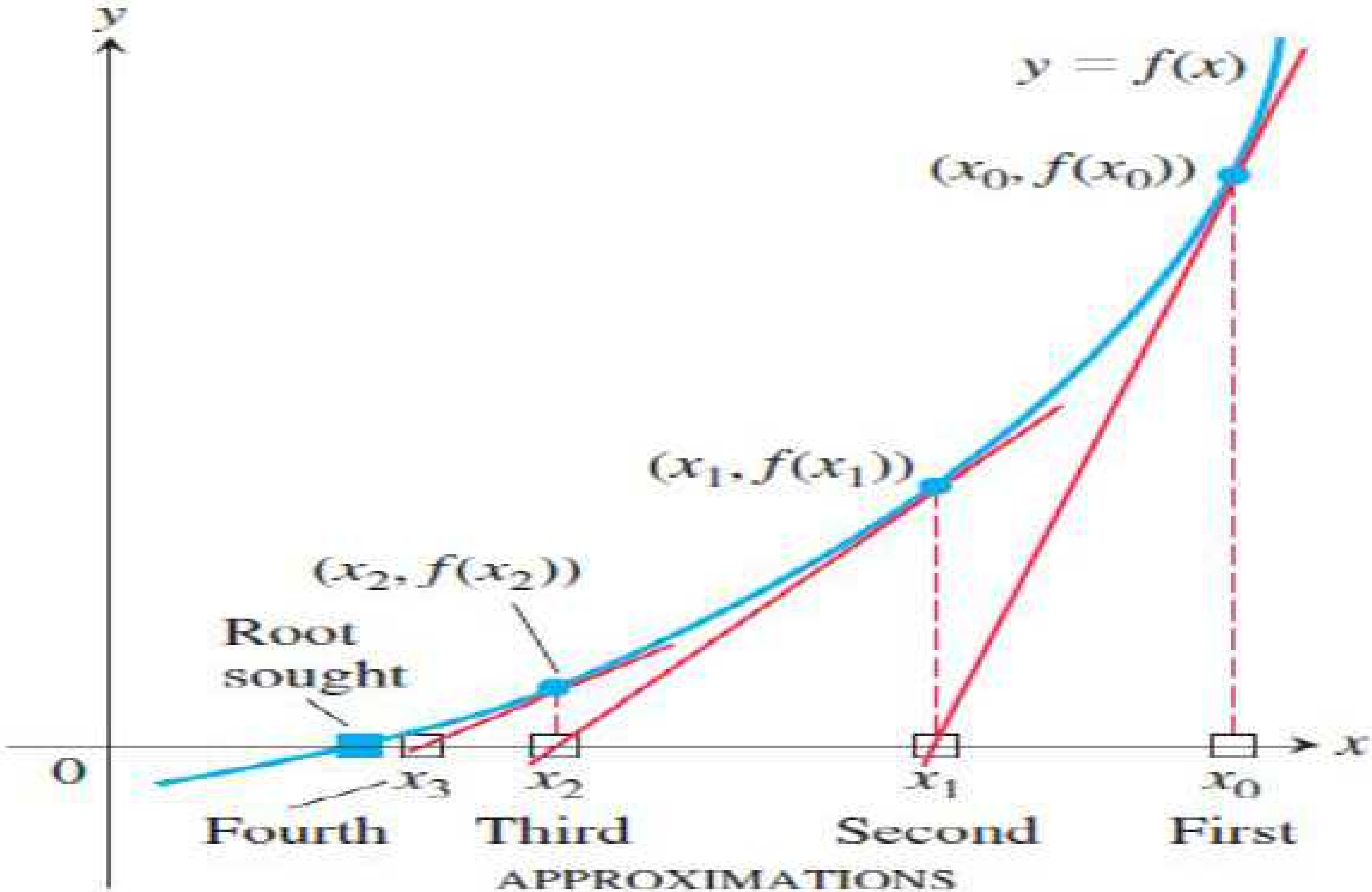


FIGURE 4.44 Newton's method starts with an initial guess x_0 and (under favorable circumstances) improves the guess one step at a time.

$y = f(x)$在$(x_0, f(x_0))$的切线 $y - f(x_0) = f'(x_0)(x - x_0)$

切线与x轴的交点的横坐标 $0 - f(x_0) = f'(x_0)(x - x_0)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{if } f'(x_n) \neq 0. \tag{1}$$

Find the root of the equition

$$x^3 - x - 1 = 0.$$

Let $f(x) = x^3 - x - 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_0 = 1, f(1) = -1, f'(1) = 2,$$

$$x_1 = 1 + \frac{1}{2} = 1.5,$$

$$f(1.5) = \frac{7}{8}, f'(1.5) = \frac{23}{4},$$

$$x_2 = 1.5 - \frac{7}{46} = 1.3479,$$

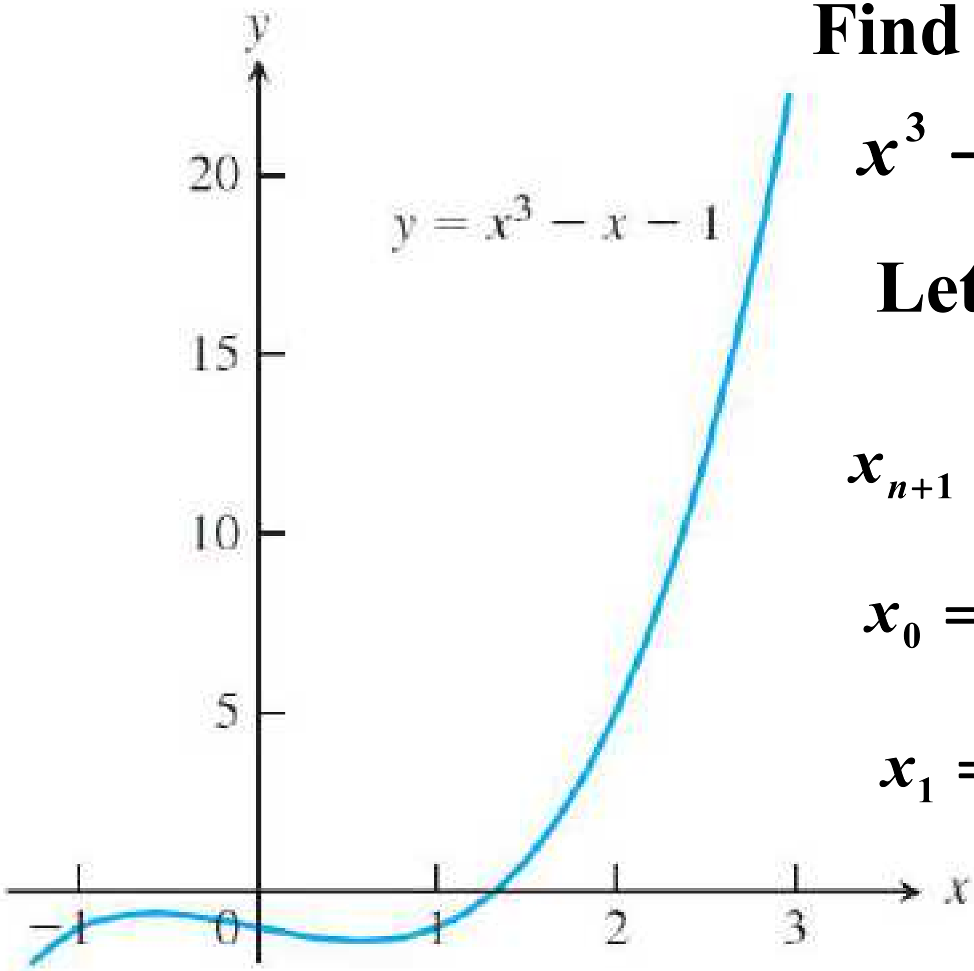


FIGURE 4.46 The graph of $f(x) = x^3 - x - 1$ crosses the x-axis once; this is the root we want to find (Example 2).

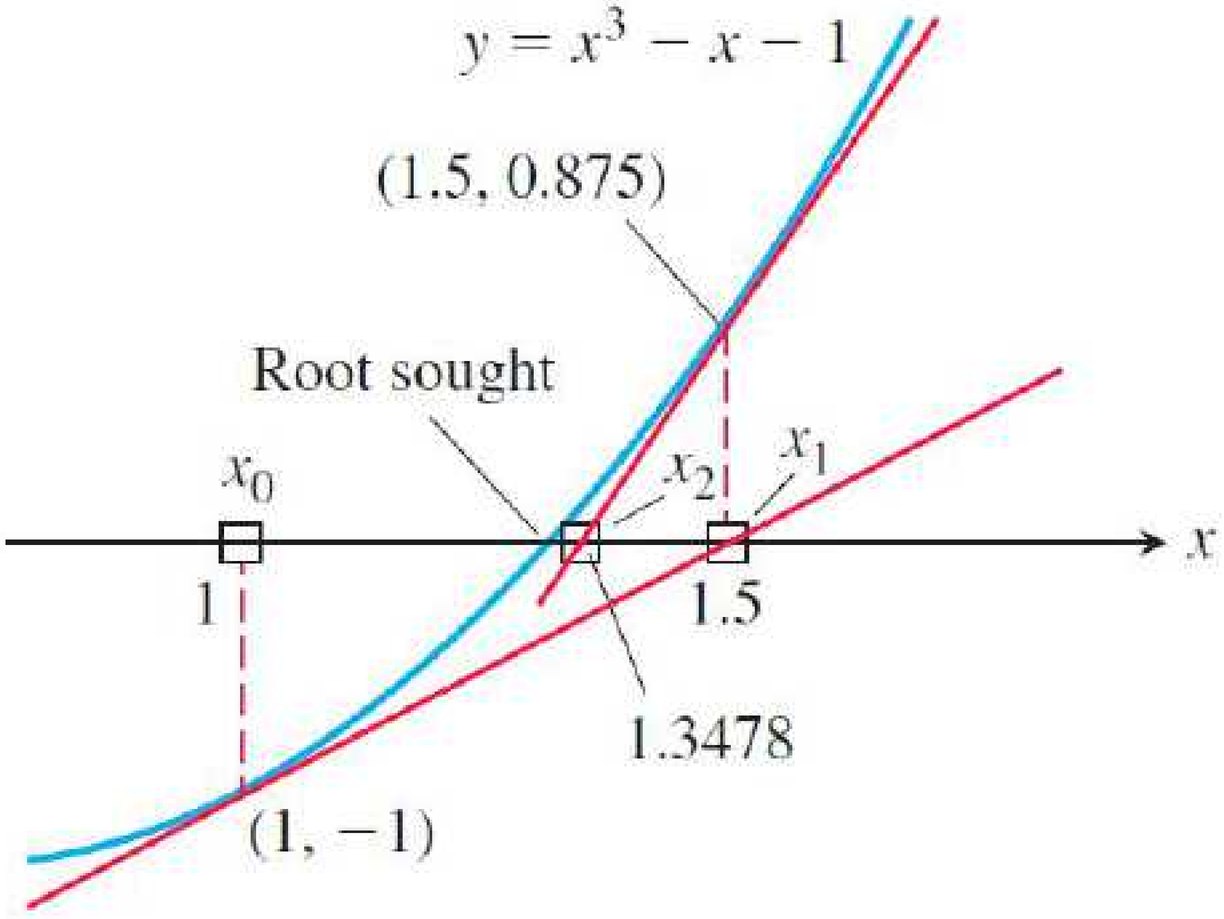


FIGURE 4.47 The first three x-values in Table 4.1 (four decimal places).

TABLE 4.1 The result of applying Newton's method to $f(x) = x^3 - x - 1$ with $x_0 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
0	1	−1	2	1.5
1	1.5	0.875	5.75	1.3478 26087
2	1.3478 26087	0.1006 82173	4.4499 05482	1.3252 00399
3	1.3252 00399	0.0020 58362	4.2684 68292	1.3247 18174
4	1.3247 18174	0.0000 00924	4.2646 34722	1.3247 17957
5	1.3247 17957	−1.8672E-13	4.2646 32999	1.3247 17957

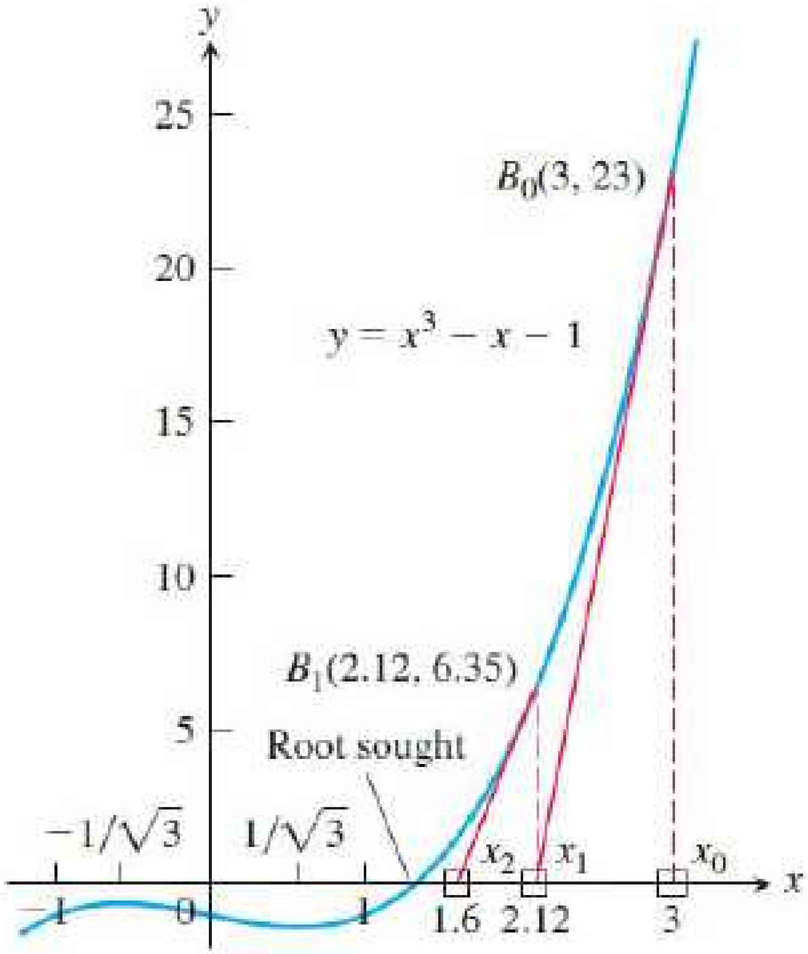


FIGURE 4.48 Any starting value x_0 to the right of $x = 1/\sqrt{3}$ will lead to the root in Example 2.

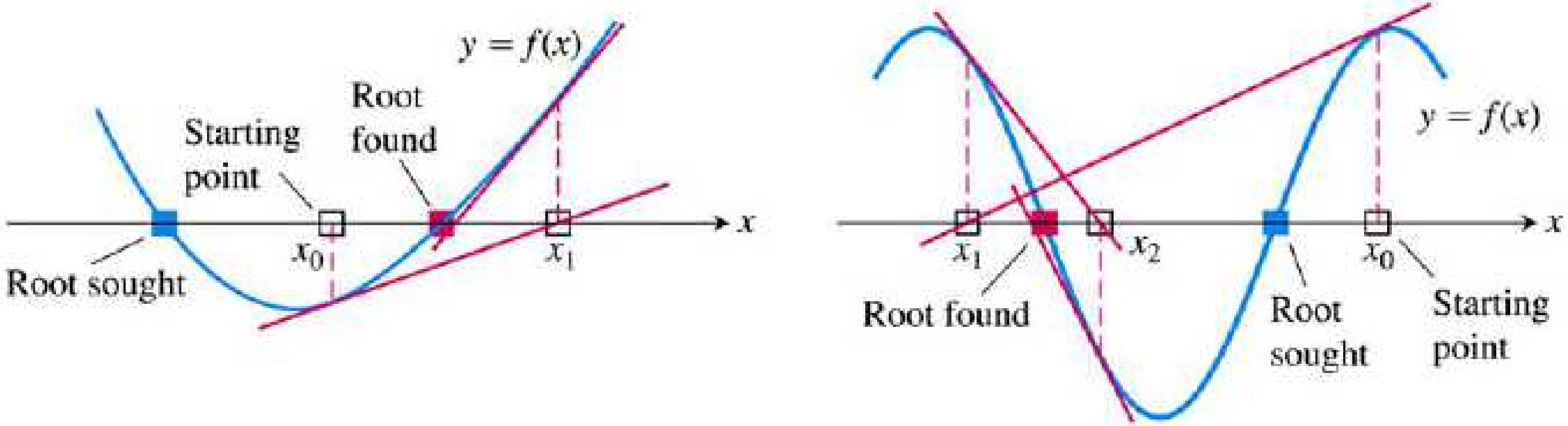


FIGURE 4.50 If you start too far away, Newton's method may miss the root you want.

4.7

Antiderivatives

原函数

DEFINITION A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

原函数

Ex.1 Find the antiderivatives of

$$(a)\, f(x) = 2x, \quad (b)\, g(x) = \cos x,$$

$$(c)\, h(x) = \sec^2 x + \frac{1}{2\sqrt{x}}.$$

Solution $(a)\ \left(x^2\right)' = 2x \qquad F(x) = x^2.$

$(b)\ (\sin x)' = \cos x, \qquad G(x) = \sin x$

$(c)\ H(x) = \tan x + \sqrt{x}.$

THEOREM 8 If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

若 $G(x)$ 也是 $f(x)$ 的原函数，

则 $$[G(x) - F(x)]' = G'(x) - F'(x) = f(x) - f(x) = 0$$

$\therefore \quad G(x) = F(x) + C$ （C为任意常数）

Ex.2

Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Solution $F(x) = x^3 + C,$

$$F(1) = -1, \Rightarrow C = -2,$$

$$\therefore F(x) = x^3 - 2.$$

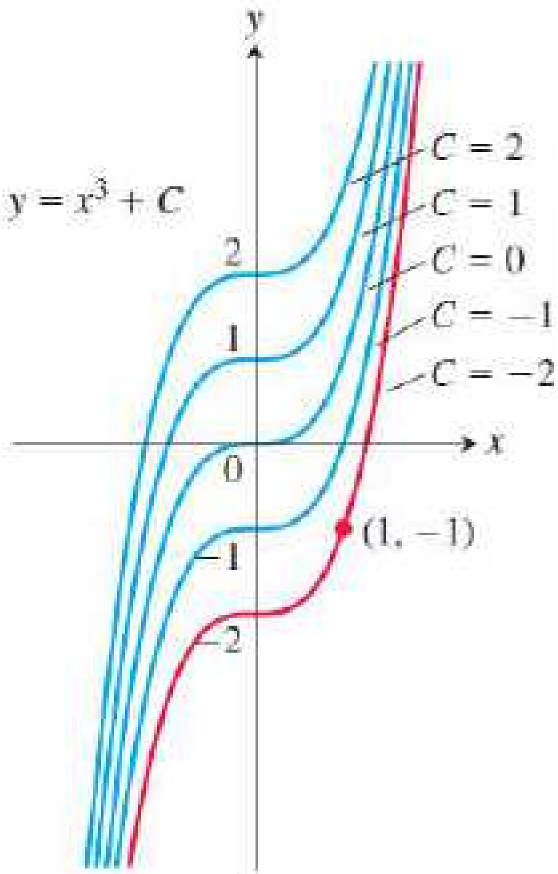


FIGURE 4.51 The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 2, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

TABLE 4.2 Antiderivative formulas, k a nonzero constant

Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$
3. $\cos kx$	$\frac{1}{k}\sin kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$

TABLE 4.3 Antiderivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule:*	$kf(x)$	$kF(x) + C, \quad k$ a constant
2.	*Negative Rule:*	$-f(x)$	$-F(x) + C,$
3.	*Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

Ex.3 Find an antiderivative of

$$(a)\ f(x) = x^5, \qquad (b)\ g(x) = \sqrt{x},$$

$$(c)\ h(x) = \sin 2x, \qquad (d)\ i(x) = \cos\frac{x}{2}.$$

Ex.4

Find the general antiderivative of $f(x) = 3\sqrt{x} + \sin 2x.$

Initial Value Problems and Differential Equations

Differential Equations 微分方程:

凡含有未知函数的导数或微分的方程叫微分方程.

$$y' = f(x, y)$$

初始条件: $y\big|_{x=x_0} = y_0$

初值问题: 求微分方程满足初始条件的解的问题.

$$\begin{cases} y' = f(x, y) \\ y\big|_{x=x_0} = y_0 \end{cases}$$

Ex. 5 一个热气球以 $3.6m/s$的速度上升，在距离地面$24.5m$时一个包裹脱落下降， 这个包裹需要多久可以到达地面 ？

解： 设下落后t时刻包裹距离地面为$s(t)$,

$$s(0)=24.5, s'(0)=3.6.$$

$$s''(t)=-9.8,\ s'(t)=-9.8t+c_1, \qquad c_1=3.6,$$

$$s'(t)=-9.8t+3.6,$$

$$s(t)=-4.9t^2+3.6t+c_2, \qquad c_2=24.5,$$

$$s(t)=-4.9t^2+3.6t+24.5,$$

$-4.9t^2+3.6t+24.5=0,$解得 $t=2.63s$

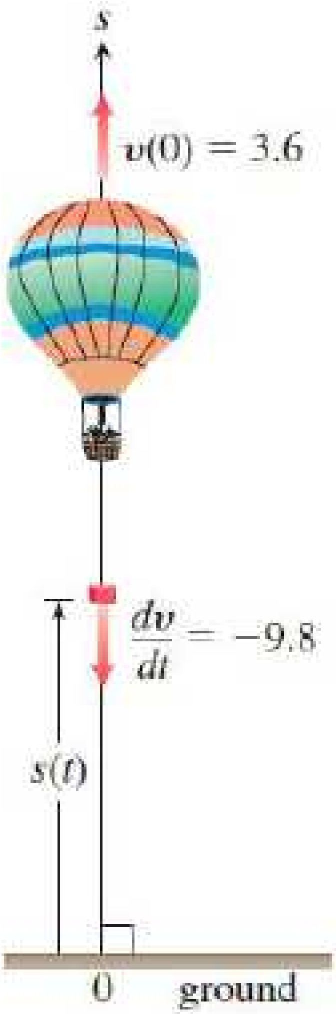


FIGURE 4.52 A package dropped from a rising hot-air balloon (Example 5).

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

$$\boxed{\int}\,\boxed{f(x)}\,dx = F(x) + \boxed{C}$$

积分号

被积函数

积分变量

任意常数

Ex. 6 Evaluate $\int (x^2 - 2x + 5\sin x)dx$

Solution $\int (x^2 - 2x + 5\sin x)dx$

$$= \frac{x^3}{3} - x^2 - 5\cos x + C.$$