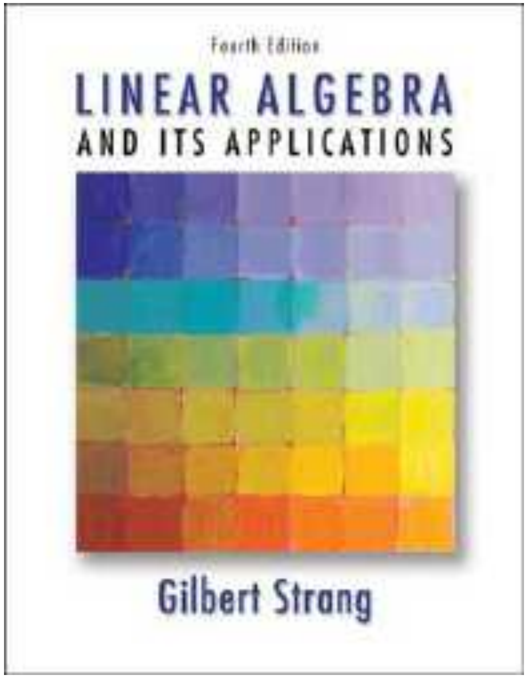
Linear Algebra



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1 Matrices and Gaussian Elimination

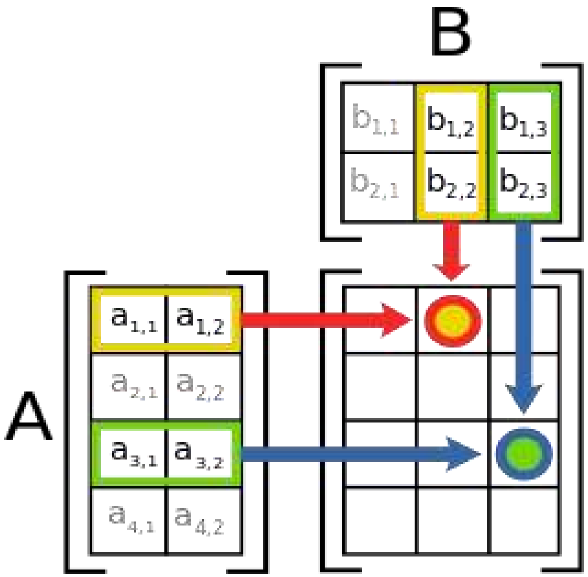
1.7

TRIANGULAR FACTORS AND ROW EXCHANGES

（矩阵的三角分解和换行）

LU Factorization

Row Exchanges



*** Textbook: Section 1.5 + Section 1.6 (part)**

I. Triangular Factors (矩阵的LU分解)

Example 1 将矩阵 $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

分解成为主对角元为1的下三角矩阵L和上三角矩阵U的乘积，即 $A=LU$（称为矩阵的LU分解 or *Triangular factorization* $A = LU$）.

解 利用倍加初等变换(Replacement)把 A 变为上三角矩阵:

$$E_{12}(-\frac{1}{2})A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad E_{23}(-\frac{2}{3})E_{12}(-\frac{1}{2})A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

$$E_{34}(-\frac{3}{4})E_{23}(-\frac{2}{3})E_{12}(-\frac{1}{2})A=\boxed{\begin{bmatrix}2&1&0&0\\0&\frac{3}{2}&1&0\\0&0&\frac{4}{3}&1\\0&0&0&\frac{5}{4}\end{bmatrix}}=U$$

$$A=E_{12}^{-1}(-\frac{1}{2})E_{23}^{-1}(-\frac{2}{3})E_{34}^{-1}(-\frac{3}{4})U=E_{12}(\frac{1}{2})E_{23}(\frac{2}{3})E_{34}(\frac{3}{4})U=LU$$

其中 $L=E_{12}(\frac{1}{2})E_{23}(\frac{2}{3})E_{34}(\frac{3}{4}).$

$$L=\begin{bmatrix}1&0&0&0\\\frac{1}{2}&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&\frac{2}{3}&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&\frac{3}{4}&1\end{bmatrix}=\boxed{\begin{bmatrix}1&0&0&0\\\frac{1}{2}&1&0&0\\0&\frac{2}{3}&1&0\\0&0&\frac{3}{4}&1\end{bmatrix}}$$

Remark: In Example 1, $\boldsymbol{A}$ ($n \times n$ matrix) is written in the form $\boldsymbol{A} = \boldsymbol{LU}$, where $\boldsymbol{L}$ is an $n \times n$ lower triangular matrix with 1's on the diagonal and $\boldsymbol{U}$ is an $n \times n$ upper triangular matrix.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & 1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$$

Divide out of U a diagonal pivot matrix D

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix}}_{\boldsymbol{L}} \underbrace{\begin{bmatrix} \frac{2}{1} & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}}_{\boldsymbol{D}} \underbrace{\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{V}}$$

$\boldsymbol{A}$ is symmetric:

$\boldsymbol{A}=\boldsymbol{LDL}^{\mathrm{T}}$.

习惯上仍记为$\boldsymbol{U}$

(此时为单位上三角矩阵)

The triangular factorization can be written $\boldsymbol{A} = \boldsymbol{LDU}$*, where* $\boldsymbol{L}$ *and* $\boldsymbol{U}$ *have* 1*'s on the diagonal and* $\boldsymbol{D}$ *is the diagonal matrix of pivots.*

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Remarks: 1. *The LDU factorization is* ***uniquely*** *determined by* $\boldsymbol{A}$ *if* $\boldsymbol{A}$ *is invertible.*
(Proof: P53, Problem Set1.6, #17)

2. Some matrices *cannot* be factored into $\boldsymbol{A} = \boldsymbol{LU}$ ***or*** $\boldsymbol{LDU}$.

For instance, $\boldsymbol{A} = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$.

练习 将矩阵

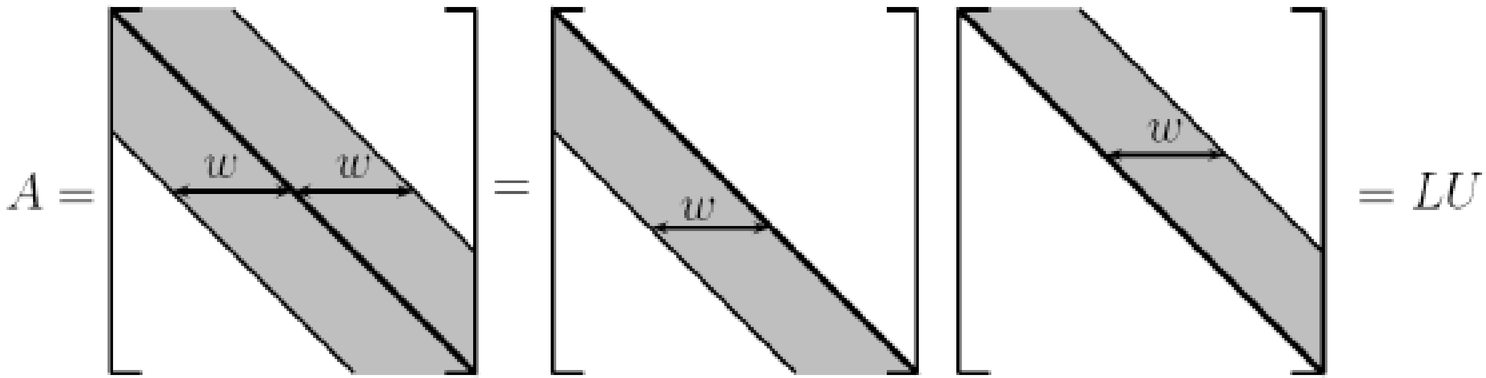
$$A=\begin{bmatrix}1&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}$$

分解成为主对角元为1的下三角矩阵$\boldsymbol{L}$ (invertible, unit lower triangular matrix)和上三角矩阵$\boldsymbol{U}$ (upper triangular matrix)的乘积, 即 $\boldsymbol{A}=\boldsymbol{L}\boldsymbol{U}$.

解

$$A=\begin{bmatrix}1&-1&&\\-1&2&-1&\\&-1&2&-1\\&&-1&2\end{bmatrix}=\begin{bmatrix}1&&&\\-1&1&&\\&-1&1&\\&&-1&1\end{bmatrix}\begin{bmatrix}1&-1&&\\&1&-1&\\&&1&-1\\&&&1\end{bmatrix}.$$

Remark: band matrix (带状矩阵) (P61, Figure 1.8)



A band matrix ***A*** and its factors ***L*** and ***U***.

A band matrix ***A*** has $a_{ij} = 0$ except in the band $|i - j| < w$.

w : "half bandwidth"

$w = 1$: a diagonal matrix,

$w = 2$: a tridiagonal matrix (三对角矩阵),

$w = n$: a full matrix.

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Example 2 Solve $\boldsymbol{Ax} = \boldsymbol{b}$

$$\begin{aligned} x_1 - x_2 &= 1 \\ -x_1 + 2x_2 - x_3 &= 1 \\ - x_2 + 2x_3 - x_4 &= 1 \\ - x_3 + 2x_4 &= 1 \end{aligned}$$

This is the previous matrix $\boldsymbol{A}$ with a right-hand side

$\boldsymbol{b} = (1,1,1,1)^{T}$.

$$(\boldsymbol{LU})\boldsymbol{x} = \boldsymbol{b} \Rightarrow$$
$$\boldsymbol{Ux} = \boldsymbol{c} \;\&\; \boldsymbol{Lc} = \boldsymbol{b}$$

$\boldsymbol{Ax} = \boldsymbol{b}$ splits into $\boldsymbol{Lc} = \boldsymbol{b}$ and $\boldsymbol{Ux} = \boldsymbol{c}$

$$Lc = b \qquad \begin{aligned} c_1 &= 1 \\ -c_1 + c_2 &= 1 \\ - c_2 + c_3 &= 1 \\ - c_3 + c_4 &= 1 \end{aligned} \quad \text{gives} \quad \boldsymbol{c} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{bmatrix}.$$

solved *forward*

$$Ux = c \qquad \begin{aligned} x_1 - x_2 &= 1 \\ x_2 - x_3 &= 2 \\ x_3 - x_4 &= 3 \\ x_4 &= 4 \end{aligned} \quad \text{gives} \quad \boldsymbol{x} = \begin{bmatrix} \mathbf{10} \\ \mathbf{9} \\ \mathbf{7} \\ \mathbf{4} \end{bmatrix}.$$

solved *backward*

One Linear System = Two Triangular Systems

Splitting of $Ax = b$

First $Lc = b$ *and then* $Ux = c$.

1. ***Factor*** (from ***A*** find its factors ***L*** and ***U***).
2. ***Solve*** (from ***L*** and ***U*** and ***b*** find the solution ***x***).

$$\boldsymbol{A} = \begin{bmatrix} 1 & & & \\ * & 1 & & \\ * & * & 1 & \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * \\ & \blacksquare & * & * \\ & & \blacksquare & * \\ & & & \blacksquare \end{bmatrix}$$

$n \times n$ $\quad$ $n \times n$ $\quad$ $n \times n$

What if – ***A*** is an $m \times n$ matrix ?

LU factorization

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\qquad\qquad L \qquad\qquad\qquad\qquad U$$

Notes: Assume that $\boldsymbol{A}$ is an $m \times n$ matrix that can be row reduced to echelon form, *without row interchanges*.

Then $\boldsymbol{A}$ can be written in the form $\boldsymbol{A} = \boldsymbol{LU}$, where $\boldsymbol{L}$ is an $m \times m$ lower triangular matrix with 1's on the diagonal and $\boldsymbol{U}$ is an $m \times n$ echelon form of $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix},$$ which needs a row exchange,

cannot be factored into $\boldsymbol{A} = \boldsymbol{LU}$.

II. Row Exchanges and Permutation Matrices

$$\boldsymbol{A}=\begin{bmatrix}0 & 2\\ 3 & 4\end{bmatrix},\quad \text{cannot be factored into } \boldsymbol{A}=\boldsymbol{LU}\ .$$

Remedy: *Exchange the two rows*

$$\boldsymbol{P}_{12}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\qquad \boldsymbol{P}_{12}\boldsymbol{A}=\begin{bmatrix}3 & 4\\ 0 & 2\end{bmatrix}.$$

Permutation matrix (置换矩阵)

A permutation matrix has the same rows as the identity matrix but in some order.

There is a single “1” in every row and every column.

How many permutation matrices do we have for $n = 2$? $n = 3$? …

$n = 2$

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{P}_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$n = 3$

$$\boldsymbol{I} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \qquad \boldsymbol{P}_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \qquad \boldsymbol{P}_{32}\boldsymbol{P}_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix},$$

$$\boldsymbol{P}_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \qquad \boldsymbol{P}_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}, \qquad \boldsymbol{P}_{21}\boldsymbol{P}_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

There are $n! = n(n-1)\ldots(1)$ permutations of size n.

A zero in the pivot location raises two possibilities:
The trouble may be easy to fix, or it may be serious.

$$\boldsymbol{A} = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{bmatrix},$$

$$\boldsymbol{P}_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{P}_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\boxed{\boldsymbol{P}_{23}\boldsymbol{P}_{13}}\boldsymbol{A} = \begin{bmatrix} d & e & f \\ 0 & a & b \\ 0 & 0 & c \end{bmatrix}$$

$$\begin{aligned} d = 0 &\implies \text{no first pivot} \\ a = 0 &\implies \text{no second pivot} \\ c = 0 &\implies \text{no third pivot.} \end{aligned}$$

$$\boldsymbol{P} = \boldsymbol{P}_{23}\boldsymbol{P}_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

With the rows in the right order $\boldsymbol{PA}$, any nonsingular matrix is ready for elimination.

Elimination in a Nutshell: $PA = LU$

In the *nonsingular* case, there is a permutation matrix $\boldsymbol{P}$ that reorders the rows of $\boldsymbol{A}$ to avoid zeros in the pivot positions. Then $\boldsymbol{Ax} = \boldsymbol{b}$ has a *unique solution*.

With the rows reordered in advance, $\boldsymbol{PA}$ can be factored into $\boldsymbol{LU}$.

In the *singular* case, no $\boldsymbol{P}$ can produce a full set of pivots: elimination fails.

Remark:

In practice, we also consider a row exchange when the original pivot is *near* zero — even if it is not exactly zero. Choosing a larger pivot reduces the roundoff error. (***partial pivoting***)

Homework

- See Blackboard announcement
- ***Hardcover* textbook + Supplementary problems**

Deadline (DDL):

- Next tutorial class