

# *Linear Algebra*



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## 5

# Eigenvalues and Eigenvectors (特征值与特征向量)

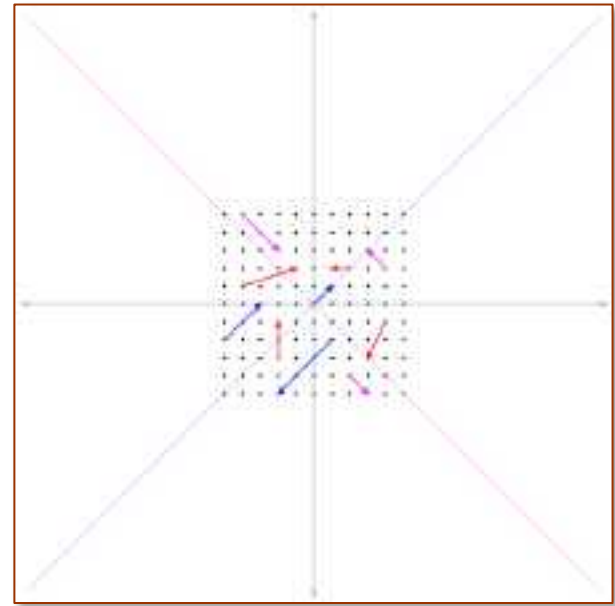
## 5.2

## DIAGONALIZATION OF A MATRIX (矩阵的对角化)

Conditions

Examples

Powers and Products



# I. Diagonalization – Conditions

A matrix  $A$  is called **diagonalizable** (可对角化) if there exists an invertible matrix  $S$  such that  $S^{-1}AS$  is a diagonal matrix. The matrix  $S$  is sometimes called a *diagonalizing matrix* for the matrix  $A$ .

**Example 1** Let  $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ .

Then the eigenvalues of  $A$  are **2** and **-1**, and two eigenvectors are  $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , respectively.

Let  $S = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$ , then  $S^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}$ .

And

$$S^{-1}AS = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

**Theorem 1** Let  $\mathbf{A}$  be a matrix of degree  $n$ , and have  $n$  **linearly independent** eigenvectors. Let  $\mathbf{S}$  be a matrix with columns being the eigenvectors. Then  $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$  is a diagonal matrix  $\mathbf{\Lambda}$ . The eigenvalues of  $\mathbf{A}$  are on the diagonal of  $\mathbf{\Lambda}$ :

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n).$$

**Proof.** Let  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  be the  $n$  **linearly independent** eigenvectors of  $\mathbf{A}$ , corresponding to the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ , respectively.

Then  $\mathbf{S} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_n]$ , and  $\mathbf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i$  with  $1 \leq i \leq n$ , so

$$\mathbf{A}\mathbf{S} = \mathbf{A}[\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_n] = [\lambda_1\mathbf{x}_1 \ \lambda_2\mathbf{x}_2 \ \dots \ \lambda_n\mathbf{x}_n]$$

$$= [\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \mathbf{S}\mathbf{\Lambda}.$$

*Crucial to keep these matrices in the right order!*

$\mathbf{S}$  is invertible, because its columns (the eigenvectors) were assumed to be independent.

Therefore,  $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ .

**Remarks:**

(1)

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n).$$

Equivalently,  $\mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{\Lambda}$  ;  $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$ .

We also call  $\mathbf{S}$  the “*eigenvector matrix*” (“*diagonalizing matrix*”) and  $\mathbf{\Lambda}$  the “*eigenvalue matrix*”.

(2) The diagonalizing matrix  $\mathbf{S}$  *is not unique*, since an eigenvector  $\mathbf{x}$  can be multiplied by a constant, and remains an eigenvector.

For the trivial example  $\mathbf{A} = \mathbf{I}$ , any invertible  $\mathbf{S}$  will do:  $\mathbf{S}^{-1}\mathbf{I}\mathbf{S}$  is always diagonal ( $\mathbf{\Lambda}$  is just  $\mathbf{I}$ ). All vectors are eigenvectors of the identity matrix.

**Remarks:**

(3) Not all matrices possess  $n$  linearly independent eigenvectors, so *not all matrices are diagonalizable*. (并非所有方阵都可以对角化)

An example: “defective matrix”  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

Its eigenvalues are  $\lambda_1 = \lambda_2 = 0$ , since it is triangular with zeros on the diagonal.

All eigenvectors of this  $A$  are multiples of the vector  $(1,0)^T$ .

$\lambda = 0$  is a **double** eigenvalue. But there is **only one** independent eigenvector. We can't construct  $S$ .

(A more direct proof: Since  $\lambda_1 = \lambda_2 = 0$ ,  $A$  would have to be the zero matrix. But if  $S^{-1}AS = A = \mathbf{0}$ , then  $A = \mathbf{0}$ , which is not true.)

For  $\lambda = 0$ :

The *algebraic multiplicity* is 2. But the *geometric multiplicity* is 1.

(Explain in the next few slides.)

**Lemma 1** If a matrix  $A$  has **no repeated eigenvalues**, i.e.,  $\lambda_1, \lambda_2, \dots, \lambda_n$  are **distinct**, then its  $n$  eigenvectors are **linearly independent**, and  $A$  is diagonalizable.

(In short, a matrix with  $n$  distinct eigenvalues can be diagonalized.  
具有 $n$ 个互不相同特征值的 $n$ 阶方阵, 一定可以对角化.)

**Proof** Suppose first that  $k = 2$ , and that some combination of  $\mathbf{x}_1$  and  $\mathbf{x}_2$  produces zero:  $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{0}$ . (\*)

Multiplying (\*) by  $A$ , we find  $c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 = \mathbf{0}$ .

Multiplying (\*) by  $\lambda_2$ , we find  $c_1\lambda_2\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 = \mathbf{0}$ .

Subtraction makes the vector  $\mathbf{x}_2$  disappear:  $c_1(\lambda_1 - \lambda_2)\mathbf{x}_1 = \mathbf{0}$ .

Since  $\lambda_1 \neq \lambda_2$  and  $\mathbf{x}_1 \neq \mathbf{0}$ , we are forced into  $c_1 = 0$ .

Similarly  $c_2 = 0$ , and the two vectors are independent.

*By mathematical induction*, eigenvectors that come from **distinct** eigenvalues are automatically **independent**.

**Example 2** Determine if the following matrix is diagonalizable.

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}.$$

**Solution** Yes. The matrix is triangular, and its eigenvalues are obviously 5, 0, and  $-2$ .

Since  $A$  is a  $3 \times 3$  matrix with three distinct eigenvalues,  $A$  is diagonalizable.

**Remark:**

*Diagonalization can fail only if there are repeated eigenvalues.*

(只有当矩阵存在重复特征值时, 才有可能不能对角化)

Even then, it does not always fail.

Example:  $A = I$  has repeated eigenvalues  $1, 1, \dots, 1$ , but it is already diagonal! There is no shortage of eigenvectors in that case.

*What if -- there are repeated eigenvalues?*



*What if -- there are repeated eigenvalues?*

$$A \xrightarrow{\quad} |A - \lambda I| = 0 \xrightarrow{\quad} (A - \lambda_i I)x = 0$$

求特征值  $\lambda_i$

求特征向量

The set of *all* solutions of  $(A - \lambda_i I)x = 0$  is just the nullspace of the matrix  $A - \lambda_i I$ .

So this set is a *subspace* of  $\mathbf{R}^n$  and is called the **eigenspace** (特征子空间) of  $A$  corresponding to  $\lambda_i$ .

The eigenspace consists of the zero vector and all the eigenvectors corresponding to  $\lambda_i$ .

algebraic multiplicity vs. geometric multiplicity of an eigenvalue  $\lambda_i$

- **algebraic multiplicity** (代数重数) : multiplicity of  $\lambda_i$  as a root of the characteristic polynomial
- **geometric multiplicity** (几何重数) : dimension of the eigenspace for  $\lambda_i$ .

For  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , the failure of diagonalization was *not* a result of  $\lambda = 0$ . It came from  $\lambda_1 = \lambda_2$ :

$$A_1 = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

**Eigenvalues:**  $3, 3$   $1, 1$

**Eigenvectors:**  $(1,0)^T$   $(1,1)^T$

For  $\lambda = 3$  of  $A_1$  and  $\lambda = 1$  of  $A_2$  :

The algebraic multiplicity is 2. But the geometric multiplicity is 1.

$A_1$  and  $A_2$  are not singular!

The problem is the shortage of independent eigenvectors — which are needed for constructing  $S$ .

*Attention :*

- *Diagonalizability of  $A$  depends on enough independent eigenvectors.*
- *Invertibility of  $A$  depends on nonzero eigenvalues.*

**Example 3** Diagonalize the following matrix, if possible:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

**Solution** The characteristic equation of  $\mathbf{A}$ :

$$0 = |\mathbf{A} - \lambda \mathbf{I}| = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2.$$

The eigenvalues are  $\lambda_1 = 1$  and  $\lambda_2 = -2$  (algebraic multiplicity = 2).

However, it is easy to verify that each eigenspace is only one-dimensional:

Basis for the eigenspace of  $\lambda_1 = 1$ :  $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$

Basis for the eigenspace of  $\lambda_2 = -2$ :  $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$

(geometric multiplicity = 1)

There are no other eigenvalues, and every eigenvector of  $\mathbf{A}$  is a multiple of either  $\mathbf{x}_1$  or  $\mathbf{x}_2$ . Thus  $\mathbf{A}$  is *not* diagonalizable.

## Theorem 2

Let  $A$  be an  $n \times n$  matrix whose distinct eigenvalues are  $\lambda_1, \lambda_2, \dots, \lambda_p$ .

- (1) For  $1 \leq i \leq p$ , the dimension of the eigenspace for  $\lambda_i$  is less than or equal to the multiplicity of the eigenvalue  $\lambda_i$  as a root of characteristic polynomial.
- (2) The matrix  $A$  is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals  $n$ , and this happens if and only if: the dimension of the eigenspace for each  $\lambda_i$  equals the multiplicity of  $\lambda_i$ .

*algebraic multiplicity  $\geq$  geometric multiplicity*

(几何重数总是不超过代数重数)

WHY?

The matrix  $A$  is diagonalizable if and only if *algebraic multiplicity  $\equiv$  geometric multiplicity for each eigenvalue  $\lambda_i$ .*

(矩阵  $A$  可以对角化 当且仅当 对于每一个特征值  $\lambda_i$  都有: 其代数重数与几何重数相等)

**Example 4** Diagonalize the following matrix, if possible:

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}.$$

**Solution** The eigenvalues of  $\mathbf{A}$  are 5 and  $-3$ , each with multiplicity 2.

$$\text{For } \lambda_1 = 5: \mathbf{x}_1 = \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{x}_2 = \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}.$$

$$\text{For } \lambda_2 = -3: \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So the matrix  $\mathbf{S} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3 \ \mathbf{x}_4]$  is invertible, and  $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$ , where  $\mathbf{\Lambda} = \text{diag}(5, 5, -3, -3)$ .

The next theorem shows that *diagonalizing matrix*  $\mathbf{S}$  **must** be formed by eigenvectors.

**Theorem 3** Let  $\mathbf{A}$  be a matrix of degree  $n$ , and assume that  $\mathbf{S}$  is an invertible matrix such that

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \text{diag}(d_1, d_2, \dots, d_n).$$

Then  $d_1, d_2, \dots, d_n$  are the eigenvalues of  $\mathbf{A}$ , and column  $j$  of  $\mathbf{S}$  is an eigenvector of  $\mathbf{A}$  corresponding to  $d_j$ .

**Proof.** Let  $\mathbf{S} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n]$ , i.e.,  $\mathbf{v}_j$  is the  $j$ -th column of  $\mathbf{S}$ , and let  $\mathbf{D} = \text{diag}(d_1, d_2, \dots, d_n)$ . Then  $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$ , and so  $\mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{D}$ . Thus

$$\begin{aligned} [\mathbf{A}\mathbf{v}_1 \ \mathbf{A}\mathbf{v}_2 \ \dots \ \mathbf{A}\mathbf{v}_n] &= \mathbf{A}[\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n] = \mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{D} \\ &= [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n] \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} = [d_1\mathbf{v}_1 \ d_2\mathbf{v}_2 \ \dots \ d_n\mathbf{v}_n]. \end{aligned}$$

Therefore,  $\mathbf{A}\mathbf{v}_j = d_j\mathbf{v}_j$  for  $1 \leq j \leq n$ , i.e.,  $d_j$  is an eigenvalue of  $\mathbf{A}$  and  $\mathbf{v}_j$  is an eigenvector of  $\mathbf{A}$  corresponding to  $d_j$ .

algebraic multiplicity  $\geq$  geometric multiplicity

**定理** 设 $\lambda_0$ 是 $n$ 阶矩阵 $A$ 的 $k$ 重特征值, 属于 $\lambda_0$ 的线性无关的特征向量的最大个数为 $l$ , 则  $k \geq l$ .

(代数重数  $\geq$  几何重数)

**证** 由  $Ax_i = \lambda_0 x_i, \quad x_i \neq 0, \quad i=1, \dots, l$  (1)

将 $\{x_1, x_2, \dots, x_l\}$ 扩充为 $\mathbf{R}^n$ 的基 $\{x_1, \dots, x_l, x_{l+1}, \dots, x_n\}$ ,  
 $x_{l+1}, \dots, x_n$ 一般不是特征向量, 但  $Ax_j \in \mathbf{R}^n$  ( $j = l+1, \dots, n$ ),  
 可用 $\mathbf{R}^n$ 的这组基表示:

$$Ax_j = b_{1j}x_1 + \dots + b_{lj}x_l + b_{l+1,j}x_{l+1} + \dots + b_{nj}x_n, \\ j = l+1, \dots, n \quad (2)$$

将(1)、(2)式中的 $n$ 个等式写成一个矩阵等式:

$$A[\mathbf{x}_1, \cdots \mathbf{x}_l, \mathbf{x}_{l+1}, \cdots, \mathbf{x}_n]$$

$$= [\mathbf{x}_1, \cdots \mathbf{x}_l, \mathbf{x}_{l+1}, \cdots, \mathbf{x}_n] \begin{bmatrix} \lambda_0 & \cdots & 0 & b_{1,l+1} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \lambda_0 & b_{l,l+1} & \cdots & b_{ln} \\ 0 & \cdots & 0 & b_{l+1,l+1} & \cdots & b_{l+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & b_{n,l+1} & \cdots & b_{nn} \end{bmatrix} \quad (3)$$

其中 $\lambda_0$ 有 $l$ 个.

记 $P=[\mathbf{x}_1, \cdots, \mathbf{x}_l, \mathbf{x}_{l+1}, \cdots, \mathbf{x}_n]$ , (3)式为:

$$P^{-1}AP = \begin{bmatrix} \lambda_0 I_l & B_1 \\ \mathbf{0} & B_2 \end{bmatrix}$$



$$P^{-1}AP = \begin{bmatrix} \lambda_0 I_l & B_1 \\ \mathbf{0} & B_2 \end{bmatrix}$$

因为

$$\begin{aligned} |A - \lambda I| &= |P^{-1}| \cdot |A - \lambda I| \cdot |P| = |P^{-1}(A - \lambda I)P| \\ &= |P^{-1}AP - \lambda I| \\ &= \begin{vmatrix} (\lambda_0 - \lambda)I_l & -B_1 \\ \mathbf{0} & B_2 - \lambda I_{n-l} \end{vmatrix} = (\lambda_0 - \lambda)^l |B_2 - \lambda I_{n-l}|. \end{aligned}$$

由于  $|B_2 - \lambda I_{n-l}|$  是  $\lambda$  的  $n-l$  次多项式,

所以,  $\lambda_0$  是  $A$  的 **大于或等于  $l$  重** 的特征值,

因此  $k \geq l$ .

已知:  $\lambda_0$  是  $n$  阶矩阵  $A$  的  $k$  重特征值

**推论:**  $n$  阶方阵的线性无关的特征向量的个数不会超过  $n$ .

## II. Diagonalization – Examples

### Example 5 (Projection matrix)

Let  $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ .

Then the eigenvalue matrix of  $A$  is  $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ .

The eigenvectors go into the columns of  $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ .

And

$$AS = S\Lambda = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Therefore  $S^{-1}AS = \Lambda$ .

**Example 6 (Rotation matrix)**

Let  $\mathbf{K} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ . (90° rotation)

Then the characteristic polynomial is  $|\mathbf{K} - \lambda \mathbf{I}| = \lambda^2 + 1$ .

It has two roots—but those roots are *not real*.

The eigenvalues of  $\mathbf{K}$  are *imaginary numbers*,  $\lambda_1 = i$  and  $\lambda_2 = -i$ .

The eigenvectors are also not real.

$$(\mathbf{K} - \lambda_1 \mathbf{I})\mathbf{x}_1 = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{and } \mathbf{x}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

$$(\mathbf{K} - \lambda_2 \mathbf{I})\mathbf{x}_2 = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{and } \mathbf{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

The eigenvalues are *distinct*, even if imaginary, and the eigenvectors are *independent*. They go into the columns of  $\mathbf{S}$ :

$$\mathbf{S} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \quad \text{and} \quad \mathbf{S}^{-1} \mathbf{K} \mathbf{S} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

**Remark:** *complex numbers may be needed even for real matrices.*

### III. Diagonalization – Powers and Products

The eigenvalue of  $\mathbf{A}^2$  are exactly  $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$ , and every eigenvector of  $\mathbf{A}$  is also an eigenvector of  $\mathbf{A}^2$ .

**Corollary 1** If  $\mathbf{A}$  is diagonalizable, then  $\mathbf{A}^k$  is diagonalizable, and has same diagonalizing matrix.

This is true because when  $\mathbf{S}$  diagonalizes  $\mathbf{A}$ , it also diagonalizes  $\mathbf{A}^k$ .

$$\mathbf{A}^k = (\mathbf{S}^{-1}\mathbf{A}\mathbf{S})(\mathbf{S}^{-1}\mathbf{A}\mathbf{S}) \dots (\mathbf{S}^{-1}\mathbf{A}\mathbf{S}) = \mathbf{S}^{-1}\mathbf{A}^k\mathbf{S}.$$

Each  $\mathbf{S}^{-1}$  cancels an  $\mathbf{S}$ , except for the first  $\mathbf{S}^{-1}$  and the last  $\mathbf{S}$ .

If  $\mathbf{A}$  is invertible this rule also applies to its inverse (the power  $k = -1$ ).

**Example 7** If  $\mathbf{K}$  is rotation through  $90^\circ$ , then  $\mathbf{K}^2$  is rotation through  $180^\circ$  (which means  $-\mathbf{I}$ ) and  $\mathbf{K}^{-1}$  is rotation through  $-90^\circ$  :

$$\mathbf{K} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{K}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } \mathbf{K}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of  $\mathbf{K}$  are  $i$  and  $-i$ ; their squares are  $-1$  and  $-1$ ; their reciprocals are  $\frac{1}{i} = -i$  and  $\frac{1}{-i} = i$ .

Then  $\mathbf{K}^4$  is a complete rotation through  $360^\circ$  :

$$\mathbf{K}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and also  $\mathbf{A}^4 = \begin{bmatrix} i^4 & 0 \\ 0 & (-i)^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

**Question:** If  $\lambda$  is an eigenvalue of  $\mathbf{A}$  and  $\mu$  is an eigenvalue of  $\mathbf{B}$ , then  $\mathbf{AB}$  has the eigenvalue  $\lambda\mu$ ??

$\mathbf{A} + \mathbf{B}$  has the eigenvalue  $\lambda + \mu$ ??

*Usually not.*

**False proof**  $\mathbf{AB}\mathbf{x} = \mathbf{A}\mu\mathbf{x} = \mu\mathbf{A}\mathbf{x} = \mu\lambda\mathbf{x}$ .

The **mistake** lies in assuming that  $\mathbf{A}$  and  $\mathbf{B}$  share the *same* eigenvector  $\mathbf{x}$ .

**Counterexample (反例):**

We could have two matrices with zero eigenvalues, while  $\mathbf{AB}$  has  $\lambda = 1$ :

$$\mathbf{AB} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvectors of this  $\mathbf{A}$  and  $\mathbf{B}$  are completely different, which is typical.

For the same reason, the eigenvalues of  $\mathbf{A} + \mathbf{B}$  generally have nothing to do with  $\lambda + \mu$ .

If the eigenvector is the same for  $\mathbf{A}$  and  $\mathbf{B}$ , then  $\mathbf{AB}$  has the eigenvalue  $\lambda\mu$ .

And finally, we have a nice result for product of matrices.

**Theorem 4** Let  $\mathbf{A}, \mathbf{B}$  be two diagonalizable matrices of degree  $n$ . Then they have same eigenvectors if and only if  $\mathbf{AB} = \mathbf{BA}$ .

**Proof.** Suppose first that a matrix  $\mathbf{S}$  diagonalizes both  $\mathbf{A}, \mathbf{B}$ , i.e.,  $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{D}_1$  and  $\mathbf{S}^{-1}\mathbf{BS} = \mathbf{D}_2$  are two diagonal matrices. Then

$$\mathbf{AB} = (\mathbf{SD}_1\mathbf{S}^{-1})(\mathbf{SD}_2\mathbf{S}^{-1}) = \mathbf{SD}_1\mathbf{D}_2\mathbf{S}^{-1},$$

$$\mathbf{BA} = (\mathbf{SD}_2\mathbf{S}^{-1})(\mathbf{SD}_1\mathbf{S}^{-1}) = \mathbf{SD}_2\mathbf{D}_1\mathbf{S}^{-1}.$$

Since  $\mathbf{D}_1\mathbf{D}_2 = \mathbf{D}_2\mathbf{D}_1$  (diagonal matrices always commute), we have  $\mathbf{AB} = \mathbf{BA}$ . *(see next slide)*

**Theorem 4** Let  $\mathbf{A}$ ,  $\mathbf{B}$  be two diagonalizable matrices of degree  $n$ . Then they have same eigenvectors if and only if  $\mathbf{AB} = \mathbf{BA}$ .

**Proof.** (*continued*)

Conversely, assume that  $\mathbf{AB} = \mathbf{BA}$ . Suppose that  $\mathbf{Ax} = \lambda\mathbf{x}$ . Then

$$\mathbf{ABx} = \mathbf{BAx} = \mathbf{B}\lambda\mathbf{x} = \lambda\mathbf{Bx}.$$

Thus  $\mathbf{Bx}$  is also an eigenvector of  $\mathbf{A}$  corresponding to the same eigenvalue  $\lambda$ .

We only complete the proof for the simpler case where all eigenvalues of  $\mathbf{A}$  are distinct.

Then the eigenspaces are all of dimension 1, so  $\mathbf{Bx}$  must be a multiple of  $\mathbf{x}$ , i.e.,  $\mathbf{Bx} = \mu\mathbf{x}$ , and  $\mathbf{x}$  is an eigenvector of  $\mathbf{B}$ , as claimed.

(The proof with repeated eigenvalues is a little longer.)



**例** 设三阶方阵  $A$  的特征值为  $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 4$ ,  
对应的特征向量是  $\xi_1 = (1, 1, 0)^T, \xi_2 = (-1, 0, 1)^T$ ,  
 $\xi_3 = (1, 1, 2)^T$ , 求  $A, A^{-1}$ .

**解** 作矩阵

$$P = [\xi_1 \ \xi_2 \ \xi_3] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

因为  $P$  为可逆矩阵, 且

$$P^{-1} = \begin{bmatrix} -1/2 & 3/2 & -1/2 \\ -1 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}.$$

所以

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 4 \end{bmatrix},$$

$$\mathbf{A} = \mathbf{P} \begin{bmatrix} 1 & & \\ & 3 & \\ & & 4 \end{bmatrix} \mathbf{P}^{-1} = \frac{1}{2} \begin{bmatrix} 9 & -7 & 3 \\ 3 & -1 & 3 \\ 2 & -2 & 8 \end{bmatrix},$$

$$\mathbf{A}^{-1} = \mathbf{P} \begin{bmatrix} 1 & & \\ & 1/3 & \\ & & 1/4 \end{bmatrix} \mathbf{P}^{-1} = \frac{1}{24} \begin{bmatrix} -1 & 25 & -9 \\ -9 & 33 & -9 \\ -2 & 2 & 6 \end{bmatrix}.$$

## Key words:

*Conditions*

*Examples*

*Powers and Products*

## Homework

**See Blackboard**

