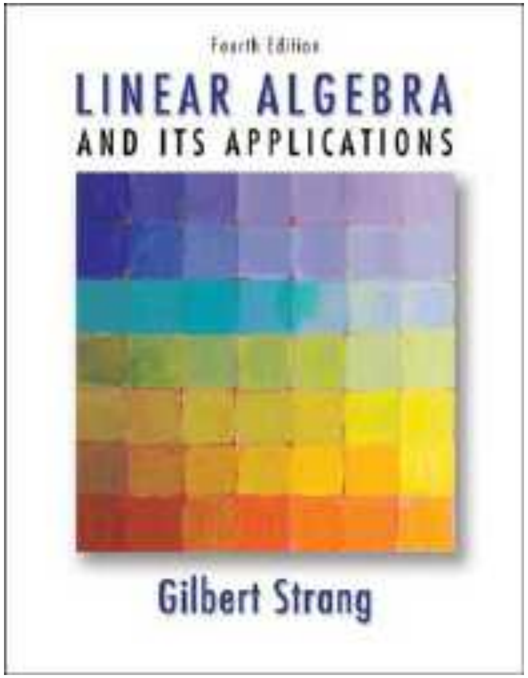
Linear Algebra



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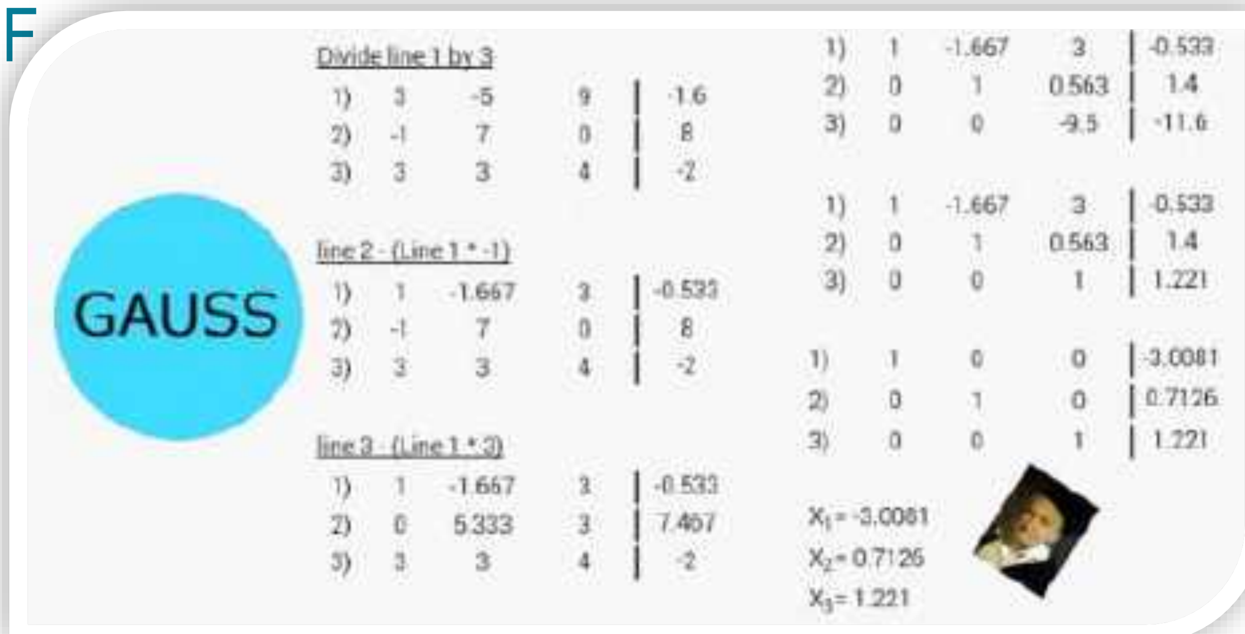
1 Matrices and Gaussian Elimination

1.3

EXAMPLES AND DISCUSSIONS OF GAUSSIAN ELIMINATION



- ***Two fundamental questions*** about a linear system are as follows:
  1. Is the system consistent; that is, does at least one solution *exist*?
  2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

- A system of linear equations has
  1. *no solution*, or
  2. *exactly one solution*, or
  3. *infinitely many solutions*.

I. Row Reduction and Echelon Forms

（行化简与阶梯形式）(P78-79)

II. Solutions of Linear Systems;

Existence and Uniqueness Theorem

III. The Breakdown of Elimination (P13)

I. Row Reduction and Echelon Forms

Elementary Row Operations (ERO): Notations

Recall Three kinds of **elementary row operations (初等行变换)：**

1) 对换变换 (*Interchange*)： interchange the *i-th row* and the *j-th row*, denoted as $r_i \leftrightarrow r_j$,

2) 倍乘变换 (*Scaling*)： multiply all entries in the *i-th row* by a nonzero constant *k*, denoted as kr_i,

3) 倍加变换(*Replacement*)： Replace the *i-th row* by the sum of itself and a multiple of the *j-th row* (将第 *j* 行的所有元素的 *k* 倍加到第 *i* 行对应元素上，记为 $r_i + kr_j$).

相应地，可以定义三种**初等列(column)变换:**

$c_i \leftrightarrow c_j$, kc_j, $c_i + kc_j$. (暂时不用)

Example 1 Apply elementary row operations (ERO) to the following matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & -1 & 0 & 3 & -1 \\ 2 & -2 & -1 & 2 & 4 & -2 \\ 3 & -3 & -1 & 4 & 5 & -3 \\ 1 & -1 & 1 & 1 & 8 & 2 \end{bmatrix} \xrightarrow[r_4 - r_1]{\substack{r_2 - 2r_1 \\ r_3 - 3r_1}} \begin{bmatrix} 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 2 & 4 & -4 & 0 \\ 0 & 0 & 2 & 1 & 5 & 3 \end{bmatrix}$$

$$\xrightarrow[r_4 - 2r_2]{r_3 - 2r_2} \begin{bmatrix} 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 9 & 3 \end{bmatrix} \xrightarrow{r_3 \leftrightarrow r_4} \begin{bmatrix} 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & -3 & 9 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \boldsymbol{B}.$$

ECHELON FORM
行阶梯形

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

$$B = \begin{bmatrix} 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & -3 & 9 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. All *nonzero* rows are above any rows of all zeros.
2. Each *leading entry* of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

A ***nonzero* row or column (非零行/列)** in a matrix means a row or column that contains at least one nonzero entry;
A **leading entry (非零首元)** of a row refers to the leftmost nonzero entry (in a nonzero row).

Continue elementary row operations (ERO) on the matrix $\boldsymbol{B}$:

$$\boldsymbol{B} = \begin{bmatrix} 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & -3 & 9 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{(-\frac{1}{3})r_3} \begin{bmatrix} 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{r_2 - 2r_3} \begin{bmatrix} 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 + r_2} \begin{bmatrix} 1 & -1 & 0 & 0 & 7 & 1 \\ 0 & 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \boldsymbol{C}.$$

REDUCED ECHELON FORM
最简行阶梯形

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 & 7 & 1 \\ 0 & 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

  4. The leading entry in each nonzero row is 1.

  5. Each leading 1 is the only nonzero entry in its column.

- A (**reduced**) **echelon matrix** is one that is in (reduced) echelon form.

The following matrices are in ***echelon form***. The leading entries (■) may have any nonzero value; the starred entries (*) may have any values (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

The following matrices are in ***reduced echelon form***, because the leading entries are 1's, and there are 0's below *and above* each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Gaussian Elimination - Continued

(主元)Pivot &

Pivot columns

Pivot

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 & 7 & 1 \\ 0 & 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

Pivot

$$B = \begin{bmatrix} 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & -3 & 9 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

A **pivot position** in a matrix ***C*** is a location in ***C*** that corresponds to a leading 1 in the reduced echelon form of ***C***. A **pivot column** is a column of ***C*** that contains a pivot position.

思考:

是否所有的矩阵都可以化为最简行阶梯形?

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

- Any *nonzero* matrix may be **row reduced** (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is *unique*.

THEOREM 1（定理）

Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to *one and only one* reduced echelon matrix. (每个矩阵行等价于唯一的最简阶梯形矩阵.)

- **Example 2**: Apply elementary row operations to transform the following matrix first into *echelon form* and then into *reduced echelon form*:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Pivot column

- **Solution:**
- **STEP 1**: Begin with the leftmost nonzero column.
- **STEP 2**: Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Pivot

$$\xrightarrow{r_1 \longleftrightarrow r_3} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

- **STEP 3**: Use row replacement operations to create zeros in all positions below the pivot.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

- **STEP 4**: Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

echelon form

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{r_3 - (3/2) r_2} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

New pivot column

Pivot

- **STEP 1 ~4**: ***forward phase***
- **STEP 5**: Beginning with the rightmost pivot and working upward and to the left, *create zeros above each pivot*. If a pivot is not 1, *make it 1 by a scaling operation*. ***backward phase***

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow[r_2+(-2)r_3]{r_1+(-6)r_3} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{(1/2)r_2} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{r_1+9r_2} \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{(1/3)r_1} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

reduced echelon form

II. Solutions of Linear Systems

- The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.
- **Example 3**: Suppose that the augmented matrix of a linear system has been changed into the equivalent *reduced* echelon form.

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- The associated system of equations is

$$\begin{cases} x_1 \quad -5x_3 = 1 \\ \quad x_2 + x_3 = 4 \\ \qquad\quad 0 = 0 \end{cases}$$

- The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables** or **pivot variables (基本变量/主元变量)**.
- The other variable, x_3, is called a **free variable (自由变量)**.

$$\begin{cases} x_1 \quad -5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases} \quad (*)$$

- x_1, x_2 : **basic variables (基本变量/主元变量)**.
- x_3: **free variable (自由变量)**.

- The statement "x_3 is free" means that you are free to choose any value for x_3. Once that is done, the formulas in (*) determine the values for x_1 and x_2.
- For instance, when x_3=0, the solution is (1,4,0); when x_3=1, the solution is (6,3,1).
- *Each different choice of x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_3.*

Existence and Uniqueness Theorem

THEOREM 2

Existence and Uniqueness Theorem

A linear system is consistent if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \ \dots \ 0 \quad b] \text{ with } b \text{ nonzero.}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

存在与唯一性定理

THEOREM 2

（存在与唯一性定理）

线性方程组相容的充要条件是增广矩阵的阶梯形没有形如

$$[0\ \ldots\ 0\ \ b],\ \ b\neq 0$$

的行.

若线性方程组相容, 它的解集可能有两种情形:

(i) 当没有自由变量时, 有唯一解;

(ii) 当至少有一个自由变量, 有无穷多解.

Gaussian Elimination - Continued

- ***Recall:*** A system of linear equations has
  1. *no solution*, or
  2. *exactly one solution*, or
  3. *infinitely many solutions*.

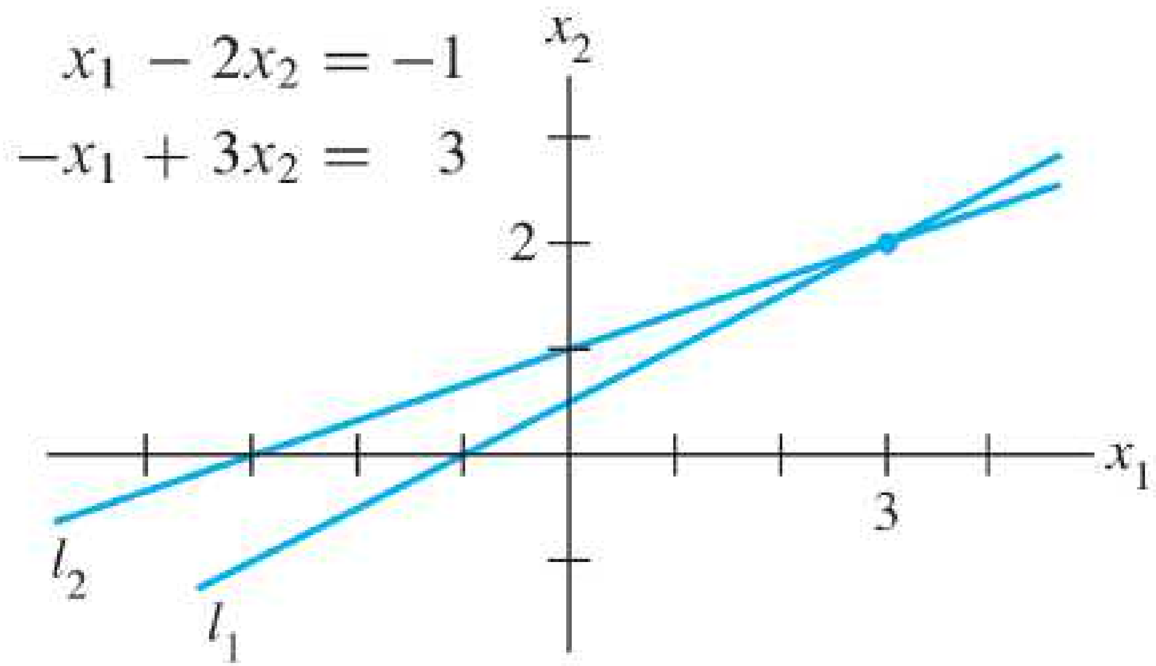


FIGURE 1 Exactly one solution.

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

(a) $\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$

(b) $\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$

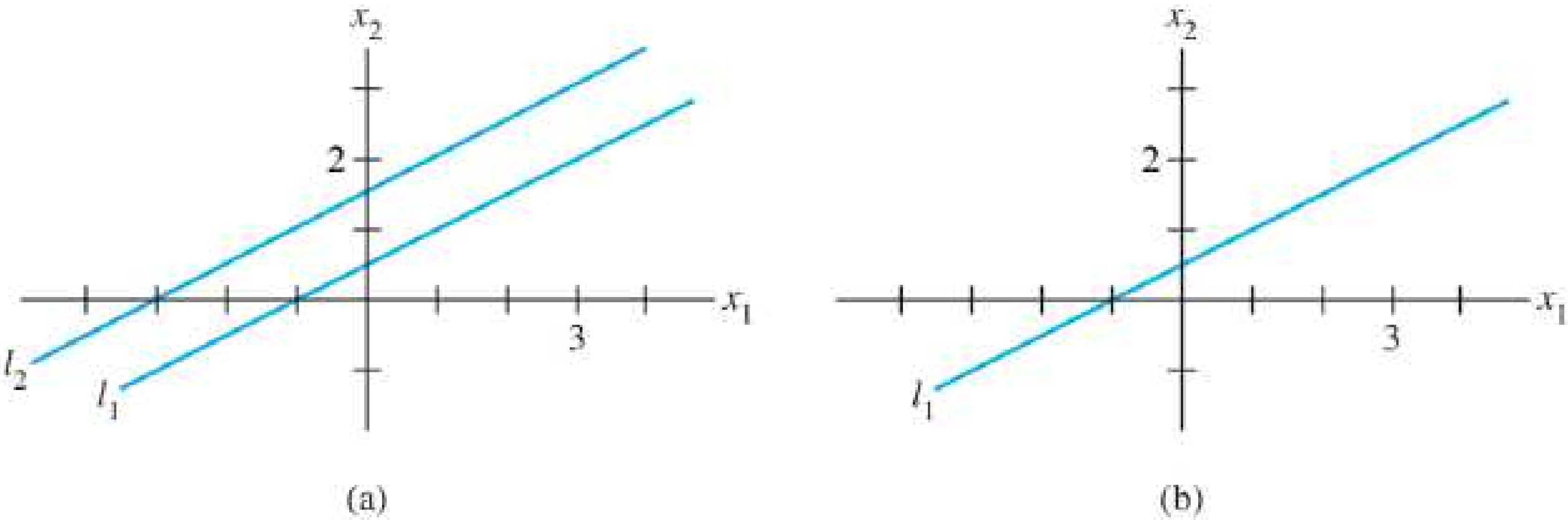


FIGURE 2 (a) No solution. (b) Infinitely many solutions.

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 2 & 3 \end{bmatrix} \downarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 2 & 1 \end{bmatrix} \downarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

应用行化简算法解线性方程组

1. 写出方程组的增广矩阵.
2. 应用行化简算法把增广矩阵化为阶梯形. 确定方程组是否有解, 如果没有解则停止; 否则进行下一步.
3. 继续行化简算法得到它的简化阶梯形.
4. 写出由第3步所得矩阵所对应的方程组.
5. 把第4步所得的每个方程改写为用自由变量表示基本变量的形式.

III. The breakdown of elimination

Under what circumstances could the process break down?

- Something ***must*** go wrong in the singular case, *and*
- Something ***might*** go wrong in the nonsingular case.

  - With *a full set of n pivots* (for *n* equations in *n* unknowns), there is only one solution.
  - The system is nonsingular, and it is solved by forward elimination and back-substitution.
  - (See next page)

Recall :

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Forward elimination

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

triangular form

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- But if a zero appears in a pivot position, elimination has to stop -- *either temporarily or permanently*.
- The system might or might not be singular.

Nonsingular

(cured by exchanging equations 2 and 3→ a full set of pivots)

$$\begin{array}{rcrcrcl} u & + & v & + & w & = & \_\_ \\ 2u & + & 2v & + & 5w & = & \_\_ \\ 4u & + & 6v & + & 8w & = & \_\_ \end{array} \rightarrow \begin{array}{rcrcrcl} u & + & v & + & w & = & \_\_ \\ & & & & 3w & = & \_\_ \\ & & 2v & + & 4w & = & \_\_ \end{array} \rightarrow \begin{array}{rcrcrcl} u & + & v & + & w & = & \_\_ \\ & & 2v & + & 4w & = & \_\_ \\ & & & & 3w & = & \_\_ \end{array}$$

Singular

(incurable: no 3rd pivot)

$$\begin{array}{rcrcrcl} u & + & v & + & w & = & \_\_\_\_ \\ 2u & + & 2v & + & 5w & = & \_\_\_\_ \\ 4u & + & 4v & + & 8w & = & \_\_\_\_ \end{array} \longrightarrow \begin{array}{rcrcrcl} u & + & v & + & w & = & \_\_\_\_ \\ & & & & 3w & = & \_\_\_\_ \\ & & & & 4w & = & \_\_\_\_ \end{array}$$

Homework

- See Blackboard announcement
- ***Hardcover* textbook + Supplementary problems**

Deadline (DDL):

- Next tutorial class