Probability and Statistics
Tutorial 10

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Outline

Review

1. Covariance
   - (DEF) $Cov(X, Y) = E[(X - E(X))(Y - E(Y))]$.
   - (Property) $Cov(X, Y) = E(XY) - (EX)(EY)$.
   - (Property) $Cov(X, Y) = Cov(Y, X)$
   - (Property) $Var(X) = Cov(X, X)$.
   - (Property) $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$
   - (Property) $Var(X_1 + X_2 + ... + X_n) = \sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j) = \sum_{i=1}^{n} Var(X_i) + 2 \sum_{i<j} Cov(X_i, X_j)$.
   - (Property) $Cov(aX + bY, Z) = aCov(X, Z) + bCov(Y, Z)$.
   - (Property) $Cov(aX, bY) = abCov(X, Y)$.
   - (Property) If X and Y are independent, then $Cov(X, Y) = 0$.

Review

2. Correlation ρ_{XY}
   - (DEF) $\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{D(X)}\sqrt{D(Y)}}$.
   - (Property) $|\rho_{XY}| \leq 1$, that is ,$|Cov(X, Y)| \leq \sqrt{D(X)}\sqrt{D(Y)}$.
   - (Property) $|\rho_{XY}| = 1$ iff $Y = a + bX$ a.e.
   - (Property) $min_{a,b}E[(Y - (a + bX))^2] = D(Y)(1 - \rho_{XY}^2)$, where $b_{min} = \frac{Cov(X,Y)}{D(X)}$, $a_{min} = E(Y) - b_{min}E(X)$.

Review

3. Conditional Expectation
   - Discrete Case:
     - (DEF) $E(X|Y=y) = \sum_{i=1}^{\infty} n_i P(X = n_i | Y = y)$.
     - (Property) $E(h(X)|Y=y) = \sum_{i=1}^{\infty} h(n_i) P(X = n_i | Y = y)$.
   - Continuous Case:
     - (DEF) $E(X|Y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$.
     - (Property) $Eh(X) = \int_{-\infty}^{\infty} h(x) f_{X|Y}(x|y) dx$.
   - Properties
     - $E[E(X|Y)] = E[X]$.
     - $D(Y) = D(E(Y|X)) + E(D(Y|X))$.
     - The above means $D(Y) \leq D(E(Y|X))$ (Variance reduction!)
     - If X and Y are independent, then $E(X|Y) = E(X)$.
     - If $X = f(Y)$, then $E(X|Y) = f(Y)$.

Homework

54. 令 X, Y 和 Z 为不相关的随机变量，方差分别为 σ_X^2, σ_Y^2 和 σ_Z^2. 令

$$U = Z + X$$
$$V = Z + Y$$

计算 $\mathrm{Cov}(U, V)$ 和 ρ_{UV}.

Solution

54. Solution.

$$Cov(U,V) = Cov(Z+X, Z+Y) = Cov(Z,Z) = \sigma_Z^2$$

$$Var(U) = Var(Z) + Var(X) = \sigma_Z^2 + \sigma_X^2$$

$$Var(V) = Var(Z) + Var(Y) = \sigma_Z^2 + \sigma_Y^2$$

$$\rho_{UV} = \frac{\sigma_Z^2}{\sqrt{\sigma_Z^2 + \sigma_X^2} \cdot \sqrt{\sigma_Z^2 + \sigma_Y^2}} . \quad \square$$

Homework

60. 令 Y 的密度函数关于原点对称，令 $X = SY$，其中 S 是另一个独立的随机变量，以概率 $\frac{1}{2}$ 分别取值 $+1$ 和 -1。证明 $\mathrm{Cov}(X, Y) = 0$，但 X 和 Y 不是独立的。

Solution

60. Proof. $\mathbb{E}Y = \int_{-\infty}^{+\infty} y f_Y(y)\,dy = 0.$

$$\mathbb{E}[X] = \mathbb{E}[SY] = \mathbb{E}S\,\mathbb{E}Y = 0.$$

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] = \mathbb{E}[SY^2]$$

$$= \mathbb{E}[S]\,\mathbb{E}[Y^2] = 0$$

$$\begin{cases} f_Y(y) = \frac{1}{2} f_X(x) + \frac{1}{2} f_X(-x) \\ \mathbb{P}(Y = (-1)^i x \mid X = x) = \frac{1}{2}, \quad i = 1, 2. \end{cases}$$

Hence, X and Y not independent.

Homework

1. 设随机变量(X,Y)的概率密度为

$$f(x,y)=\begin{cases}\frac{1}{8}(x+y), & 0\leqslant x\leqslant 2,\ 0\leqslant y\leqslant 2,\\ 0, & \text{其他},\end{cases}$$

求 $E(X)$, $E(Y)$, $Cov(X,Y)$, ρ_{XY}, $D(X+Y)$.

Solution

1 Solution $f_X(x)=\begin{cases}\int_0^2 \frac{1}{8}(x+y)\,dy=\frac{1}{4}(x+1), & x\in[0,2]\\ 0, & \text{other}\end{cases}$

$f_Y(y)=\begin{cases}\frac{1}{4}(y+1), & y\in[0,2]\\ 0, & \text{other}\end{cases}$

$\mathbb{E}(X)=\int_0^2 x\cdot\frac{1}{4}(x+1)\,dx=\frac{7}{6}$

$\mathbb{E}(Y)=\frac{7}{6}$

$\mathbb{E}[XY]=\int_0^2\int_0^2 xy\,\frac{1}{8}(x+y)\,dx\,dy=\frac{4}{3}$

$Cov(X,Y)=\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]=-\frac{1}{36}$

$\mathbb{E}(X^2)=\int_0^2 x^2\,\frac{1}{4}(x+1)\,dx=\frac{5}{3}$

$D(X)=\mathbb{E}(X^2)-(\mathbb{E}X)^2=\frac{11}{36}$

$D(Y)=\frac{11}{36}$

$D(X+Y)=D(X)+D(Y)+2Cov(X,Y)=\frac{5}{9}$

Homework

2. 设随机变量X和Y独立同分布于$N(\mu,\sigma^2)$，令$Z=\alpha X+\beta Y$，$W=\alpha X-\beta Y$，求$Cov(Z,W)$，ρ_{ZW}.

Solution

2. Solution:

$$\mathrm{Cov}(Z,W) = \mathrm{Cov}(\alpha X, \alpha X) - \mathrm{Cov}(\beta Y, \beta Y)$$
$$= (\alpha^2 - \beta^2)\sigma^2$$

$$\mathrm{Var}(Z) = \mathrm{Var}(W) = (\alpha^2+\beta^2)\sigma^2.$$

$$\rho_{Z,W} = \frac{(\alpha^2-\beta^2)\sigma^2}{(\alpha^2+\beta^2)\sigma^2} = \frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}$$

Homework

设随机变量X的概率密度为$f(x)=\frac{1}{2}e^{-|x|}, -\infty < x < +\infty$,

(1)求出$E(X), D(X)$.

(2)X与$|X|$是否独立? 说明理由.

(3)X与$|X|$是否相关? 说明理由.

Solution

3. Solution.

(1) $\mathbb{E}(X) = \int_{-\infty}^{+\infty} x \cdot \frac{1}{2} e^{-|x|} dx = 0$

$D(X) = \mathbb{E}(X^2) = \int_{-\infty}^{+\infty} x^2 \cdot \frac{1}{2} e^{-|x|} dx$

$= \int_{0}^{+\infty} x^2 e^{-x} dx = 2$

(2) $\mathbb{P}(X \in (0,1), |X| \in (1,2)) = 0$

$\mathbb{P}(X \in (0,1)) \neq 0$, $\mathbb{P}(|X| \in (1,2)) \neq 0$.

Hence, X and $|X|$ are not independent.

(3) $|X| \sim Exp(1)$. $\mathbb{E}[|X|] = 1$

$\mathbb{E}[X|X|] = \mathbb{E}[X^2 1_{\{X>0\}}] + \mathbb{E}[-X^2 1_{\{X<0\}}]$

$= \int_{0}^{+\infty} x^2 \cdot \frac{1}{2} e^{-x} dx + \int_{-\infty}^{0} -x^2 \frac{1}{2} e^{x} dx$

$= 0$

Thus, $Cov(X, |X|) = \mathbb{E}[X|X|] - \mathbb{E}[X]\mathbb{E}[|X|]$

Homework

67. 随机矩形构造如下：底 X 选自 $[0,1]$ 上的均匀随机变量，生成完底部之后，取宽为 $[0,X]$ 上的均匀随机变量. 利用 4.4.1 节定理 4.4.1.1 的全期望公式计算矩形的期望周长和期望面积.

Solution

67. Solution:

$X \sim U(0,1)$ $\quad$ $Y \mid X = x \sim U(0,x)$.

$L = 2(X+Y)$ $\quad$ $S = XY$.

$\mathbb{E}[L] = 2\mathbb{E}[X] + 2\mathbb{E}[Y]$

$= 1 + 2\mathbb{E}[\mathbb{E}[Y \mid X]] = 1 + 2\mathbb{E}[\frac{X}{2}] = \frac{3}{2}$.

$\mathbb{E}[S] = \mathbb{E}[\mathbb{E}[XY \mid X]] = \mathbb{E}[X\,\mathbb{E}[Y \mid X]]$

$= \mathbb{E}[X \cdot \frac{X}{2}] = \frac{1}{6}$. $\square$

Homework

77. 令 X 和 Y 具有联合密度函数

$$f(x,y)=e^{-y},\quad 0\leqslant x\leqslant y$$

a. 计算 $\mathrm{Cov}(X,Y)$，以及 X 与 Y 的相关系数.

b. 计算 $E(X|Y=y)$ 和 $E(Y|X=x)$.

c. 推导出随机变量 $E(X|Y)$ 和 $E(Y|X)$ 的密度函数.

Solution

77. Solution.

a. $f_X(x) = \begin{cases} \int_x^{+\infty} e^{-y} dy = e^{-x}, & x > 0 \\ 0, & \text{other.} \end{cases}$

$f_Y(y) = \begin{cases} \int_0^y e^{-y} dx = y e^{-y}, & y > 0 \\ 0, & \text{other} \end{cases}$

$\mathbb{E}(X) = 1, \ \mathbb{E}(X^2) = 2, \ D(X) = 1$

$\mathbb{E}(Y) = 2, \ \mathbb{E}(Y^2) = 6, \ D(Y) = 2$

$\mathbb{E}(XY) = \int_0^{\infty} \int_0^y xy e^{-y} dx dy$

$= \int_0^{\infty} \frac{1}{2} y^3 e^{-y} dy = 3.$

$\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 1.$

Solution

6. For $y>0$,

$$f_{X|Y}(x|y)=\begin{cases}\frac{1}{y}, & x\in(0,y)\\ 0, & \text{other}\end{cases}$$

$$\mathbb{E}[X|Y=y]=\frac{y}{2},\quad \text{for } y>0.$$

For $x>0$,

$$f_{Y|X}(y|x)=\begin{cases}e^{-(y-x)}, & y\in(x,+\infty)\\ 0, & \text{other.}\end{cases}$$

$$\mathbb{E}[Y|X=x]=1+x,\quad \text{for } x>0.$$

Solution

c. Let $U = \mathbb{E}[X|Y]$, $V = \mathbb{E}[Y|X]$.

Then, $U = \frac{1}{2}Y$, $V = 1 + X$.

Hence, $f_U(u) = \begin{cases} 4u \cdot e^{-2u}, & u > 0 \\ 0, & \text{other} \end{cases}$

$$f_V(v) = \begin{cases} e^{-(v-1)}, & v > 1 \\ 0, & \text{other.} \end{cases}$$

Homework

1. 如果X和Y是两独立的随机变量，证明：$E(X|Y=y)=E(X)$.

Homework

Solution

4. Prof. ① X, Y discrete.

$$\mathbb{E}[X \mid Y=y] = \sum_{k=1}^{\infty} k\,\mathbb{P}(X=k \mid Y=y)$$

$$= \sum_{k=1}^{\infty} k\,\mathbb{P}(X=k) = \mathbb{E}[X].$$

② X, Y continuous,

$$\mathbb{E}[X \mid Y=y] = \int_{\mathbb{R}} x f_{X|Y}(x|y)\,dx$$

$$= \int_{\mathbb{R}} x f_X(x)\,dx = \mathbb{E}[X].$$

2. 设随机变量(X,Y)的概率密度为

$$f(x,y)=\begin{cases} ke^{-(x+y)}, & 0\leqslant y\leqslant x, \\ 0, & \text{其他.} \end{cases}$$

(1) 计算$Cov(X,Y)$, ρ_{XY};

(2) 计算$E(X|Y=y)$ 和$E(Y|X=x)$;

(3) 推导随机变量$E(X|Y)$和$E(Y|X)$的概率密度.

Solution

5 Solution

$$1 = k\int_0^{+\infty}\int_0^{x} e^{-(x+y)}\,dy\,dx = \frac{k}{2}, \quad k=2$$

$$(1)\ f_X(x) = \begin{cases} \int_0^{x} 2e^{-(x+y)}\,dy = 2e^{-x}-2e^{-2x}, & x>0 \\ 0, & \text{other} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_y^{+\infty} 2e^{-(x+y)}\,dx = 2e^{-2y}, & y>0 \\ 0, & \text{other} \end{cases}$$

$$\mathbb{E}(X) = 2-\frac{1}{2} = \frac{3}{2} \qquad \mathbb{E}(X^2) = 4-\frac{1}{2} = \frac{7}{2}$$

$$D(X) = \frac{5}{4}.$$

$$\mathbb{E}(Y) = \frac{1}{2}, \quad \mathbb{E}(Y^2) = \frac{1}{2}, \quad D(Y) = \frac{1}{4}$$

$$\mathbb{E}(XY) = \int_0^{+\infty}\int_0^{x} (xy)\cdot\left(2e^{-(x+y)}\right)dy\,dx$$

$$= \int_0^{+\infty} 2xe^{-x}\left(1-(1+x)e^{-x}\right)dx$$

$$= 2-\frac{1}{2}-\frac{1}{2} = 1$$

$$Cov(X,Y) = \mathbb{E}(XY)-\mathbb{E}(X)\mathbb{E}(Y) = \frac{1}{4}.$$

$$\rho_{XY} = \frac{\sqrt{5}}{5}.$$

Solution

(1) For $y>0$,

$$f_{X|Y}(x|y)=\begin{cases} e^{-(x-y)}, & x\in(y,+\infty) \\ 0, & \text{other} \end{cases}$$

$$\mathbb{E}[X|Y=y]=1+y.$$

For $x>0$,

$$f_{Y|X}(y|x)=\begin{cases} \frac{1}{1-e^{-x}}e^{-y}, & y\in(0,x) \\ 0, & \text{other}. \end{cases}$$

$$\mathbb{E}[Y|X=x]=\frac{1}{1-e^{-x}}\int_0^x ye^{-y}dy$$

$$=\frac{1-(x+1)e^{-x}}{1-e^{-x}}=1-\frac{xe^{-x}}{1-e^{-x}}$$

Thank you!