



Chapter 4: Expected Values (期望值)

- The Expected Value of a Random Variable(随机变量的期望)
- Variance and Standard Deviation(方差和标准差)
- Covariance and Correlation Coefficient(协方差和相关系数)
- Conditional Expectation (条件期望)



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Review

The properties of conditional **frequency** function

$$\textcircled{1} P\{X = x_i | Y = y_j\} \geq 0 \quad (i = 1, 2, \dots)$$

$$\textcircled{2} \sum_{i=1}^{\infty} P\{X = x_i | Y = y_j\} = 1$$

The properties of conditional **density** function

$$\textcircled{1} f_{X|Y}(x | y) \geq 0$$


$$\textcircled{2} \int_{-\infty}^{\infty} f_{X|Y}(u | y) du = 1$$

It will be used to define conditional expectation.



When $X = x$, the conditional expectation definition of Y is

$$E(Y | X = x) = \sum_y y \underline{p_{Y|X}(y | x)} \quad (\text{Discrete})$$


$$E(Y | X = x) = \int y \underline{f_{Y|X}(y | x)} dy \quad (\text{Continuous})$$

Generally, the conditional expectation of **function $h(Y)$** is

$$E[\underline{h(Y)} | X = x] = \sum_y \underline{h(y)} p_{Y|X}(y | x) \quad (\text{Discrete})$$

$$E[\underline{h(Y)} | X = x] = \int \underline{h(y)} f_{Y|X}(y | x) dy \quad (\text{Continuous})$$



Understand **Conditional expectation** $E(Y | X)$

If x is any value within the range of X , $E(Y | (X = x))$ always exists

This is the **function of X** , then
it is **r.v.**, noted as **$E(Y | X)$**

In the same way we can define the expectation and variance for conditional expectation $E(Y | X)$ as follows:

$$E[E(Y | X)]$$

$$D[E(Y | X)]$$

Theorem $E(Y) = E[E(Y | X)]$

$$D(Y) = D[E(Y | X)] + E[D(Y | X)]$$



Theorem $E(Y) = E[E(Y | X)]$ (Law of Total Expectation)

Prove: In discrete case (continuous case is similar)

$$\begin{aligned} E[E(Y | X)] &= \sum_x E(Y | X = x) p_X(x) \\ &= \sum_x [\sum_y y p_{Y|X}(y | x)] p_X(x) \\ &= \sum_y y \sum_x p_{Y|X}(y | x) p_X(x) \\ &= \sum_y y p_Y(y) \\ &= E[Y]. \end{aligned}$$

Law of total probability

$$P(A) = \sum_{i=1}^n P(A | B_i) P(B_i)$$

Note: The expectation of a random variable Y can be calculated by first based on the condition of X to find $E(Y | X)$, and then averaging this quantity with respect to X .
i.e. weighting conditional expectations appropriately and summing or integrating.



Theorem $D(Y) = D[E(Y | X)] + E[D(Y | X)]$

Prove: In discrete case (continuous case is similar)

$$D(X) = E(X - E(X))^2 = E(X^2) - [E(X)]^2$$

$$D(Y) = E(Y^2) - [E(Y)]^2$$

By the Law of Total Expectation $E(Y) = E[E(Y | X)]$

$$\begin{aligned} D(Y) &= E(E(Y^2 | X)) - \{E[E(Y | X)]\}^2 \\ &= E(E(Y^2 | X)) - E\{[E(Y | X)]^2\} + E\{[E(Y | X)]^2\} - \{E[E(Y | X)]\}^2 \end{aligned}$$

$$D[E(Y | X)] = E\{[E(Y | X)]^2\} - \{E[E(Y | X)]\}^2$$

$$E[D(Y | X)] = E\{E(Y^2 | X) - [E(Y | X)]^2\} = E[E(Y^2 | X)] - E\{[E(Y | X)]^2\}$$

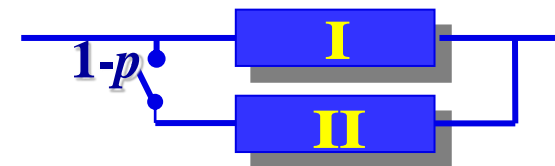
$$D(Y) = D[E(Y | X)] + E[D(Y | X)]$$



§ 4 条件期望

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Example: Assume in a system shown in the diagram, the average life span of the component and its backup part all is μ . If the component breaks, the backup part will replace it automatically in the system. However, the probability of an error occurring during the replacement process is p . What is the average life span of the system?



$$E(Y) = E[E(Y | X)] = \sum_x E(Y | X = x) p_X(x)$$

Solve: Let T denote the life span of the system. $E(T)$

$$X = \begin{cases} 1, & \text{The replacement of backup part is successful} \\ 0, & \text{The replacement of backup part is failed} \end{cases}$$

Then $E(T | X = 1) = 2\mu, \quad E(T | X = 0) = \mu,$

Therefore

$$\begin{aligned} E(T) &= E(T | X = 1)P\{X = 1\} + E(T | X = 0)P\{X = 0\} \\ &= 2\mu(1 - p) + \mu p = \mu(2 - p) \end{aligned}$$



Random Sum (随机和)

$$T = \sum_{i=1}^N X_i$$

What is the expectation and the variance of T ?

N is r.v. and has limited values with expectation and variance;
 $X_i (i = 1, \dots, N)$ has the same mean value $E(X)$ and variance $D(X)$;
 X_i and N are independent.



Insurance companies received N claims at a certain time. Using r.v. X_i to denote the cost of each claim.

Real cases



There are N customers in a shopping mall. The consumption amount of i_{th} customer is r. v. X_i



The amount of tasks of a service line is N . The time that is needed for i_{th} service is r. v. X_i .



$X_i (i = 1, \dots, N)$ has the same mean value $E(X)$ and variance $D(X)$; X_i and N are independent.

$$T = \sum_{i=1}^N X_i$$

Solve:

$$\text{For } E(T | N = n) = nE(X), \quad E(T | N) = NE(X),$$

$$D(T | N) = ND(X),$$

Then

$$E(T) = E[E(T | N)] = E[NE(X)] = E(N)E(X).$$

$$\boxed{E(T) = E(N)E(X)}$$

- Thus **the average cost for N claims** is the average value of the random claim number N multiplying the average cost for one claim.
- Thus **the average consumption amount for N customers** is the average value of the random customer number N multiplying the average amount for one customer.
- Thus **the average time for N services** is the average value of the random number N multiplying the average time for one service.



$X_i (i = 1, \dots, N)$ has the same mean value $E(X)$ and variance $D(X)$; X_i and N are independent.

Solve: In discrete case (continuous case is similar)

If X and Y are independent $D(X + Y) = D(X) + D(Y)$

$$E(T | N) = NE(X), D(T | N) = ND(X),$$

$$D(Y) = D[E(Y | X)] + E[D(Y | X)]$$

So $D[E(T | N)] = D[NE(X)] = [E(X)]^2 D(N),$

$$E[D(T | N)] = E(N)D(X),$$

Therefore $D(T) = D[E(T | N)] + E[D(T | N)]$

$$D(T) = [E(X)]^2 D(N) + E(N)D(X).$$

$$\text{And } D(T | N = n) = D\left(\sum_{i=1}^n X_i\right) = nD(X),$$

Thus the uncertainty of total time T is from the randomness of N and X .
[If $N=n$ is fixed, then $D(T) = nD(X)$]



Exercise: The number of insurance claims N in a certain time period has the expected value being equal to 900 and its standard deviation is 30. As would be the case if the number N were a Poisson random variable with expected value 900. Suppose that the average claim value X is \$1000 and the standard deviation is \$500. What is the expected value of the total claim T ? What is the variance of T ? $E(T) = ?$ $D(T) = ?$

$$E(N) = ? \quad D(N) = ? \quad E(X) = ? \quad D(X) = ?$$

$$E(T) = E[E(T | N)] = E[NE(X)] = E(N)E(X).$$

$$D(T) = [E(X)]^2 D(N) + E(N)D(X).$$

The **Expected value** of the total claims T :

$$E(T) = E(X) \times E(N) = 1000 \times 900 = 900,000$$

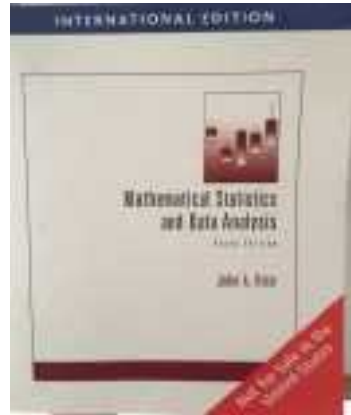
The **Variance** of the total claims T :

$$D(T) = [E(X)]^2 D(N) + E(N)D(X).$$

$$= 1000^2 \times 30^2 + 900 \times 500^2 = 1.125 \times 10^9$$



Homework



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Supplementary Questions

1. Suppose X and Y are independent, prove that $E(X|Y = y) = E(X)$.

2. Suppose the joint density function of (X, Y) is

$$f(x, y) = \begin{cases} ke^{-(x+y)}, & 0 \leq y \leq x \\ 0, & \text{otherwise} \end{cases}$$

(1). Compute $Cov(X, Y)$ and ρ_{XY}

(2). Compute $E(X|Y = y)$, $E(Y|X = x)$

(3). Find the density function of $E(X|Y)$ and $E(Y|X)$.