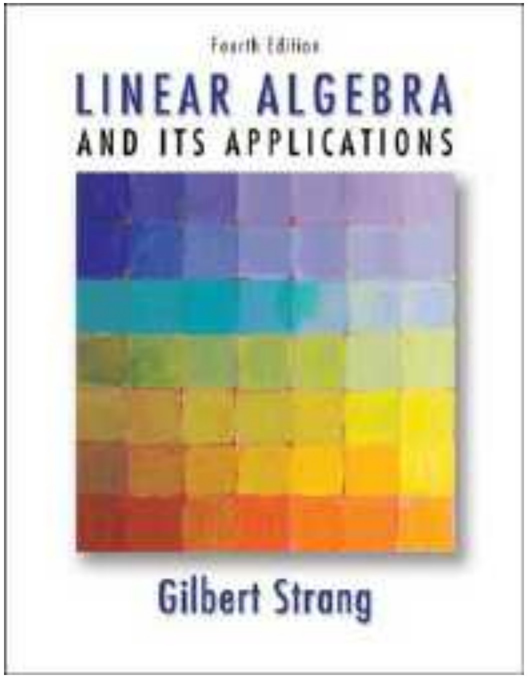
Linear Algebra



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1 Matrices and Gaussian Elimination

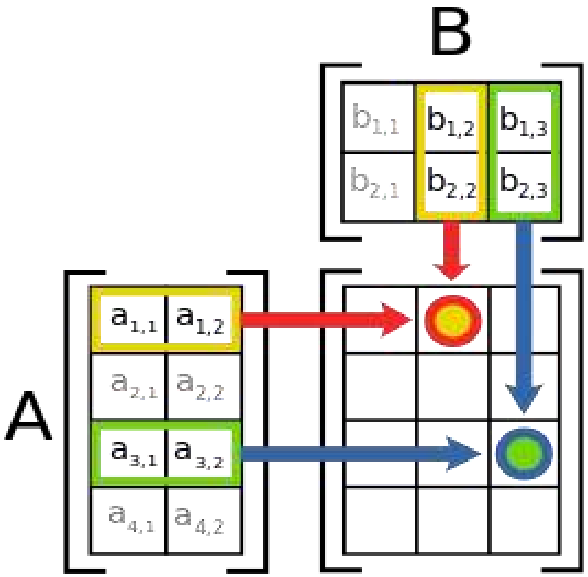
1.6

PARTITIONED MATRICES
（分块矩阵）

Introduction

Definition

Operations



- **Textbook: Problem Set1.4, 1.6**

I. Introduction

引例：数学中的矩阵分块

如：

Let $$\boldsymbol{A} = \left[\begin{array}{cc|c|cc} 1 & 2 & 0 & 0 & 0 \\ 3 & 7 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 9 & 5 \\ 0 & 0 & 0 & 7 & 4 \end{array}\right],$$

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}, \quad \boldsymbol{A}_2 = [1],$$

$$\boldsymbol{A}_3 = \begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix}.$$

Find A^k

Is A invertible?

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & & \\ & \boldsymbol{A}_2 & \\ & & \boldsymbol{A}_3 \end{bmatrix}$$

II. Definition

对于行数和列数较高的矩阵 A，经常采用**分块法**，使大矩阵的运算化成小矩阵的运算，这是矩阵运算中的一个重要技巧.

具体做法是：将矩阵 A 用若干条纵线和横线分成许多个小矩阵，每一个小矩阵称为 A 的**子块(blocks, submatrices)**，以子块为元素的形式上的矩阵称为**分块矩阵(partitioned matrices)**.

例1 已知5阶方阵

$$A=\left[\begin{array}{cc|ccc} -2 & 1 & 1 & 0 & -1 \\ 1 & 2 & 2 & -4 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

在 A 的第 2 行与第 3 行之间、第 2 列与第 3 列之间各加一条水平虚线和垂直虚线, 则 A 划分为 4 块.

记 $\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}=A_1$, $\begin{bmatrix} 1 & 0 & -1 \\ 2 & -4 & 0 \end{bmatrix}=A_2$,

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}=A_3$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=A_4$,

此时 A 可表示为

$$A=\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

如果把小矩阵 A_1, A_2, A_3, A_4 视为 4 个元素, 则 A可视为**形式上的 2 阶方阵**.

常用的分块方法

1. 按行分块

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}=\begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{bmatrix}$$

2. 按列分块

$$B=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}=\begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{bmatrix}$$

3. 按对角分块

当 n 阶矩阵 $\boldsymbol{C}$ 中非零元素都集中在主对角线附近，如果可以分块成如下分块矩阵

$$\boldsymbol{C}=\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}=\begin{bmatrix} \boldsymbol{C}_1 & & & \\ & \boldsymbol{C}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{C}_m \end{bmatrix},$$

其中 $\boldsymbol{C}_i$ 是 r_i 阶**方阵**，$i=1,2,\ldots,m$，$\sum_{i=1}^{m} r_i = n$，

则称 $\boldsymbol{C}$ 为**块对角矩阵(block diagonal matrix)** 或**准对角矩阵**.

Example 2 The matrix

$$C=\left[\begin{array}{cc:ccc:c} 0 & \cos\theta & 0 & 0 & 0 & 0 \\ \sin\theta & 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 4 \end{array}\right]=\begin{bmatrix} C_1 & & \\ & C_2 & \\ & & C_3 \end{bmatrix},$$

where

$$C_1=\begin{bmatrix} 0 & \cos\theta \\ \sin\theta & 1 \end{bmatrix},\quad C_2=\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix},\quad C_3=[4].$$

For $\left[\begin{array}{cc:cc} 1 & 2 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 3 \end{array}\right]=\begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}.$ Is it a block diagonal matrix?

NO

III. Operations

1. 分块矩阵的加法(Addition)

设矩阵 $\boldsymbol{A}$, $\boldsymbol{B}$ 为同型矩阵，将 $\boldsymbol{A}$, $\boldsymbol{B}$ 分块为

$$\boldsymbol{A}=\begin{bmatrix} \boxed{\boldsymbol{A}_{11}} & \cdots & \boxed{\boldsymbol{A}_{1t}} \\ \vdots & & \vdots \\ \boldsymbol{A}_{s1} & \cdots & \boldsymbol{A}_{st} \end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix} \boxed{\boldsymbol{B}_{11}} & \cdots & \boxed{\boldsymbol{B}_{1t}} \\ \vdots & & \vdots \\ \boldsymbol{B}_{s1} & \cdots & \boldsymbol{B}_{st} \end{bmatrix}$$

其中 $\boldsymbol{A}_{ij}$, $\boldsymbol{B}_{ij}$ $(i=1,2,\ldots,s;\ j=1,2,\ldots,t)$ 为**同型**矩阵, 则

$$\boldsymbol{A}\pm\boldsymbol{B}=\begin{bmatrix} \boxed{\boldsymbol{A}_{11}\pm\boldsymbol{B}_{11}} & \cdots & \boxed{\boldsymbol{A}_{1t}\pm\boldsymbol{B}_{1t}} \\ \vdots & & \vdots \\ \boldsymbol{A}_{s1}\pm\boldsymbol{B}_{s1} & \cdots & \boldsymbol{A}_{st}\pm\boldsymbol{B}_{st} \end{bmatrix}.$$

2. 分块矩阵的数量乘法(Scalar multiplication)

设分块矩阵 $\boldsymbol{A}=\begin{bmatrix} \boldsymbol{A}_{11} & \cdots & \boldsymbol{A}_{1r} \\ \vdots & & \vdots \\ \boldsymbol{A}_{s1} & \cdots & \boldsymbol{A}_{sr} \end{bmatrix}$，$k$ 为常数，则

$$k\boldsymbol{A}=\begin{bmatrix} k\boldsymbol{A}_{11} & \cdots & k\boldsymbol{A}_{1r} \\ \vdots & & \vdots \\ k\boldsymbol{A}_{s1} & \cdots & k\boldsymbol{A}_{sr} \end{bmatrix}.$$

- If matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same size and are partitioned in exactly the same way, then it is natural to make the same partition of the ordinary matrix sum $\boldsymbol{A}+\boldsymbol{B}$.
- In this case, each block of $\boldsymbol{A}+\boldsymbol{B}$ is the (matrix) sum of the corresponding blocks of $\boldsymbol{A}$ and $\boldsymbol{B}$.
- Multiplication of a partitioned matrix by a scalar is also computed block by block.

3. 分块矩阵的乘法
(Multiplication of partitioned matrices)

设 $\boldsymbol{A}$ 是 $m\times n$ 矩阵, $\boldsymbol{B}$ 是 $n\times p$ 矩阵. 将 $\boldsymbol{A}, \boldsymbol{B}$ 分块成

$$\boldsymbol{A}=\begin{bmatrix} \boldsymbol{A}_{11} & \cdots & \boldsymbol{A}_{1t} \\ \vdots & & \vdots \\ \boldsymbol{A}_{s1} & \cdots & \boldsymbol{A}_{st} \end{bmatrix}, \quad \boldsymbol{B}=\begin{bmatrix} \boldsymbol{B}_{11} & \cdots & \boldsymbol{B}_{1r} \\ \vdots & & \vdots \\ \boldsymbol{B}_{t1} & \cdots & \boldsymbol{B}_{tr} \end{bmatrix},$$

其中 **$\boldsymbol{A}$ 的列的分块法**和 **$\boldsymbol{B}$ 的行的分块法**完全相同, 则

$$\boldsymbol{AB}=\begin{bmatrix} \boldsymbol{C}_{11} & \cdots & \boldsymbol{C}_{1r} \\ \vdots & & \vdots \\ \boldsymbol{C}_{s1} & \cdots & \boldsymbol{C}_{sr} \end{bmatrix},$$

这里 $$\boldsymbol{C}_{ij}=\sum_{k=1}^{t}\boldsymbol{A}_{ik}\boldsymbol{B}_{kj}$$
$(i=1,2,\ldots,s;\ j=1,2,\ldots,r)$.

Partitioned matrices can be multiplied by the usual ***row—column rule*** as if the block entries were scalars, provided that for a product $\boldsymbol{AB}$, the *column* partition of $\boldsymbol{A}$ matches the *row* partition of $\boldsymbol{B}$.

Example 3 Let the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ be

$$\boldsymbol{A}=\left[\begin{array}{cc:ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hdashline 1 & 2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}\right], \quad \boldsymbol{B}=\left[\begin{array}{ccc:cc} 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \\ \hdashline -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right],$$

Find $\boldsymbol{AB}$.

Solution. $\boldsymbol{A}=\begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{0}_{2\times3} \\ \boldsymbol{A}_3 & \boldsymbol{I}_3 \end{bmatrix}$, where $\boldsymbol{A}_3=\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 0 \end{bmatrix}$;

(*B* 的行与 *A* 的列的分块法要一致, 但 *B* 的列可任意分.)

$$\boldsymbol{B}=\begin{bmatrix}\boldsymbol{B}_1 & \boldsymbol{I}_2\\ -\boldsymbol{I}_3 & \boldsymbol{0}_{3\times 2}\end{bmatrix},\quad \text{where } \boldsymbol{B}_1=\begin{bmatrix}2&3&0\\1&2&0\end{bmatrix}.$$

Thus

$$\boldsymbol{AB}=\begin{bmatrix}\boldsymbol{I}_2 & \boldsymbol{0}\\ \boldsymbol{A}_3 & \boldsymbol{I}_3\end{bmatrix}\begin{bmatrix}\boldsymbol{B}_1 & \boldsymbol{I}_2\\ -\boldsymbol{I}_3 & \boldsymbol{0}\end{bmatrix}=\begin{bmatrix}\boldsymbol{B}_1 & \boldsymbol{I}_2\\ \boldsymbol{A}_3\boldsymbol{B}_1-\boldsymbol{I}_3 & \boldsymbol{A}_3\end{bmatrix}$$

$$=\left[\begin{array}{ccc:cc}2&3&0&1&0\\1&2&0&0&1\\ \hdashline 3&7&0&1&2\\-1&-2&0&-1&1\\4&6&-1&2&0\end{array}\right].$$

(可以验算, $\boldsymbol{AB}$ 直接乘与分块相乘所得的结果一致.)

Example 4 Compute $\boldsymbol{A}^2$, where

$$\boldsymbol{A} = \left[\begin{array}{cc|c|cc} 1 & 2 & 0 & 0 & 0 \\ 3 & 7 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 9 & 5 \\ 0 & 0 & 0 & 7 & 4 \end{array}\right] = \begin{bmatrix} \boldsymbol{A}_1 & & \\ & \boldsymbol{A}_2 & \\ & & \boldsymbol{A}_3 \end{bmatrix}$$

$$\boldsymbol{A}^2 = \begin{bmatrix} \boldsymbol{A}_1{}^2 & & \\ & \boldsymbol{A}_2{}^2 & \\ & & \boldsymbol{A}_3{}^2 \end{bmatrix}$$

若 n 阶矩阵 $\boldsymbol{C}$ 和 $\boldsymbol{D}$ 分块成**同型**对角块矩阵, 即

$$\boldsymbol{C} = \mathrm{diag}(\boldsymbol{C}_1, \boldsymbol{C}_2, \cdots, \boldsymbol{C}_s), \quad \boldsymbol{D} = \mathrm{diag}(\boldsymbol{D}_1, \boldsymbol{D}_2, \cdots, \boldsymbol{D}_s)$$

其中 $\boldsymbol{C}_i$ 和 $\boldsymbol{D}_i$ 是同阶方阵 $(i = 1, 2, \cdots, s)$. 则

$$\boldsymbol{CD} = \mathrm{diag}(\boldsymbol{C}_1\boldsymbol{D}_1, \boldsymbol{C}_2\boldsymbol{D}_2, \cdots, \boldsymbol{C}_s\boldsymbol{D}_s)$$

4. 分块矩阵的转置(Transpose)

Let

$$\boldsymbol{A}=\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \cdots & \boldsymbol{A}_{1t} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \cdots & \boldsymbol{A}_{2t} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{A}_{s1} & \boldsymbol{A}_{s2} & \cdots & \boldsymbol{A}_{st} \end{bmatrix},$$

then

$$\boldsymbol{A}^{\mathrm{T}}=\begin{bmatrix} \boldsymbol{A}_{11}^{\mathrm{T}} & \boldsymbol{A}_{21}^{\mathrm{T}} & \cdots & \boldsymbol{A}_{s1}^{\mathrm{T}} \\ \boldsymbol{A}_{12}^{\mathrm{T}} & \boldsymbol{A}_{22}^{\mathrm{T}} & \cdots & \boldsymbol{A}_{s2}^{\mathrm{T}} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{A}_{1t}^{\mathrm{T}} & \boldsymbol{A}_{2t}^{\mathrm{T}} & \cdots & \boldsymbol{A}_{st}^{\mathrm{T}} \end{bmatrix}.$$

For $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} \end{bmatrix}$, we have $\boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{A}_{11}^{\mathrm{T}} & \boldsymbol{A}_{21}^{\mathrm{T}} \\ \boldsymbol{A}_{12}^{\mathrm{T}} & \boldsymbol{A}_{22}^{\mathrm{T}} \\ \boldsymbol{A}_{13}^{\mathrm{T}} & \boldsymbol{A}_{23}^{\mathrm{T}} \end{bmatrix}$.

Example: Let $\boldsymbol{A}=[\boldsymbol{\alpha}_1 \ \boldsymbol{\alpha}_2]$, $\boldsymbol{\alpha}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \boldsymbol{\alpha}_2 = \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix}$, then $\boldsymbol{A}^{\mathrm{T}}$=? where

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 6 & 7 \end{bmatrix} \quad \text{i.e.,} \quad \begin{bmatrix} \boldsymbol{\alpha}_1^{\mathrm{T}} \\ \boldsymbol{\alpha}_2^{\mathrm{T}} \end{bmatrix}$$

And if $\boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_m \end{bmatrix}$, then $\boldsymbol{B}^{\mathrm{T}} = \left[\boldsymbol{b}_1^{\mathrm{T}}, \boldsymbol{b}_2^{\mathrm{T}}, \cdots, \boldsymbol{b}_m^{\mathrm{T}} \right]$.

5. 可逆分块矩阵的逆矩阵(Inverse)

$$A=\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{bmatrix},\quad A^{-1}=\begin{bmatrix} A_1^{-1} & & & \\ & A_2^{-1} & & \\ & & \ddots & \\ & & & A_m^{-1} \end{bmatrix}.$$

块对角矩阵 A 可逆当且仅当 A_i $(i=1,2,\ldots,m)$都可逆.

(A block diagonal matrix is invertible if and only if each block on the diagonal is invertible.)

$$B=\begin{bmatrix} & & & B_1 \\ & & B_2 & \\ & \iddots & & \\ B_m & & & \end{bmatrix},\quad B^{-1}=\begin{bmatrix} & & & B_m^{-1} \\ & & \iddots & \\ & B_2^{-1} & & \\ B_1^{-1} & & & \end{bmatrix}.$$

Example 5 Find the inverses of the following matrices.

$$(1)\boldsymbol{A}=\begin{bmatrix}1 & 2 & 0 & 0 & 0\\ 3 & 7 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 9 & 5\\ 0 & 0 & 0 & 7 & 4\end{bmatrix}. \quad (2)\boldsymbol{A}=\begin{bmatrix}0 & a_1 & 0 & \cdots & 0\\ 0 & 0 & a_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & a_{n-1}\\ a_n & 0 & 0 & \cdots & 0\end{bmatrix}.$$

$$(1)\boldsymbol{A}^{-1}=\begin{bmatrix}7 & -2 & 0 & 0 & 0\\ -3 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 4 & -5\\ 0 & 0 & 0 & -7 & 9\end{bmatrix}\cdot(2)\boldsymbol{A}^{-1}=\begin{bmatrix}0 & 0 & 0 & \cdots & 1/a_n\\ 1/a_1 & 0 & 0 & \cdots & 0\\ & 1/a_2 & & \vdots & \vdots\\ 0 & 0 & \ddots & \cdots & 0\\ 0 & 0 & 0 & 1/a_{n-1} & 0\end{bmatrix}.$$

Example 6 Let $T=\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$, where A, D are $m\times m$ and $n\times n$ invertible matrices. Please find T^{-1}.

Solution Since $\begin{bmatrix} I_m & 0 \\ -CA^{-1} & I_n \end{bmatrix}\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}=\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$,

and $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}^{-1}=\begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}$,

we have $T^{-1}=\left[\begin{pmatrix} I_m & 0 \\ -CA^{-1} & I_n \end{pmatrix}^{-1}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right]^{-1}$

$$=\begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}\begin{bmatrix} I_m & 0 \\ -CA^{-1} & I_n \end{bmatrix}=\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}.$$

Summary

分块矩阵的概念与运算 (**Definition and Operations**)

分块矩阵之间与一般矩阵之间的运算性质类似:

(1) 加法(Addition) 同型矩阵, 采用相同的分块法.

(2) 数乘(Scalar Multiplication) 数 k 乘矩阵, 需数 k 乘矩阵的每个子块.

(3) 乘法(Multiplication of partitioned matrices) 要求 $\boldsymbol{A}$ 的列分块法和 $\boldsymbol{B}$ 的行分块法一致.

(4) 转置(Transpose) 行变为相应的列, 每个子块都转置.

(5) 分块矩阵的逆阵(Inverse).

Homework

See Blackboard announcement