CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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RSA Variant

$\mathcal{Q}$: Consider the RSA system. Let (e, d) be a key pair for the RSA. Define

$$\lambda(n) = \mathrm{lcm}(p-1, q-1)$$

and compute $d' = e^{-1} \bmod \lambda(n)$. Will decryption using d' instead of d still work? (prove $C^{d'} \bmod n = M$)

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Case I: $\gcd(M, n) = 1$

$$\begin{aligned} C^{d'} \bmod n &= M^{ed'} \bmod n = M^{k\lambda(n)+1} \bmod n \\ &= (M^{k\lambda(n)} \bmod n) M \bmod n \\ &= \left(M^{(p-1)(q-1)/\gcd(p-1,q-1)} \bmod n\right)^k M \bmod n \end{aligned}$$

By Fermat's theorem, $M^{(p-1)(q-1)/\gcd(p-1,q-1)} \bmod p = \left(M^{(q-1)/\gcd(p-1,q-1)}\right)^{p-1} \bmod p = 1$ and $M^{(p-1)(q-1)/\gcd(p-1,q-1)} \bmod q = 1$. Then by Chinese Remainder Theorem, we have $C^{d'} \bmod n = M$.

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Case II: $\gcd(M, n) = p$

$M = tp$ for some integer $0 < t < q$. We have $\gcd(M, q) = 1$ and $ed' = k\lambda(n) + 1$ for some integer k. By Fermat's theorem, we have

$$(M^{k\lambda(n)} - 1) \bmod q = (M^{k(p-1)(q-1)/\gcd(p-1,q-1)} - 1) \bmod q$$

Then

$$\begin{aligned}(M^{ed'} - M) \bmod n &= M(M^{ed'-1} - 1) \bmod n \\ &= tp(M^{k\lambda(n)} - 1) \bmod pq \\ &= 0\end{aligned}$$

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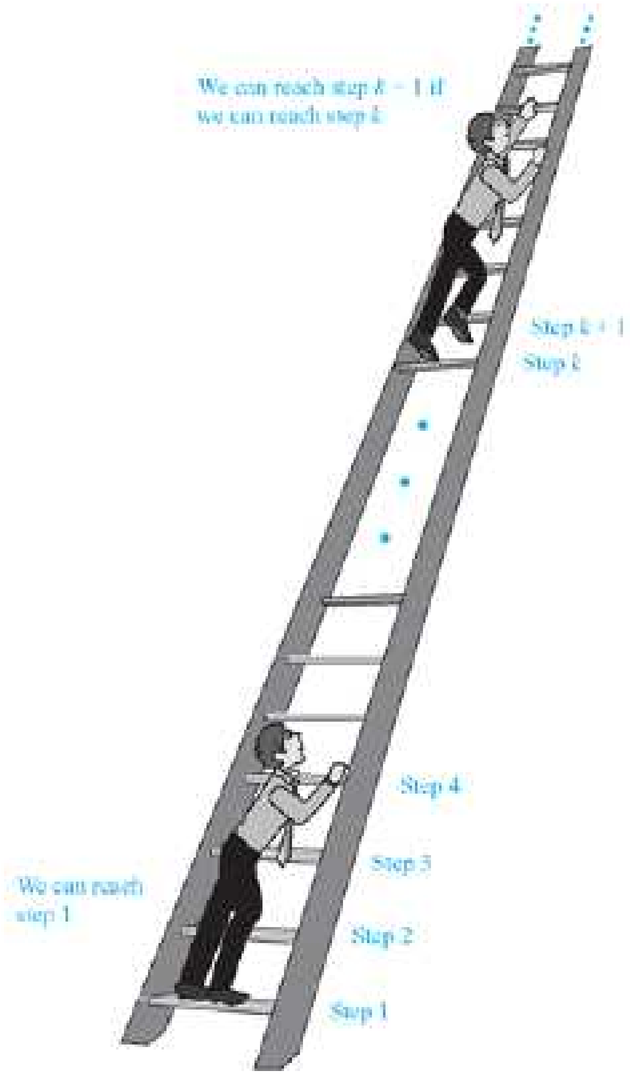
Case III: $\gcd(M, n) = q$

Similar to Case II.

Case IV: $\gcd(M, n) = pq$

Trivial.

Mathematical Induction



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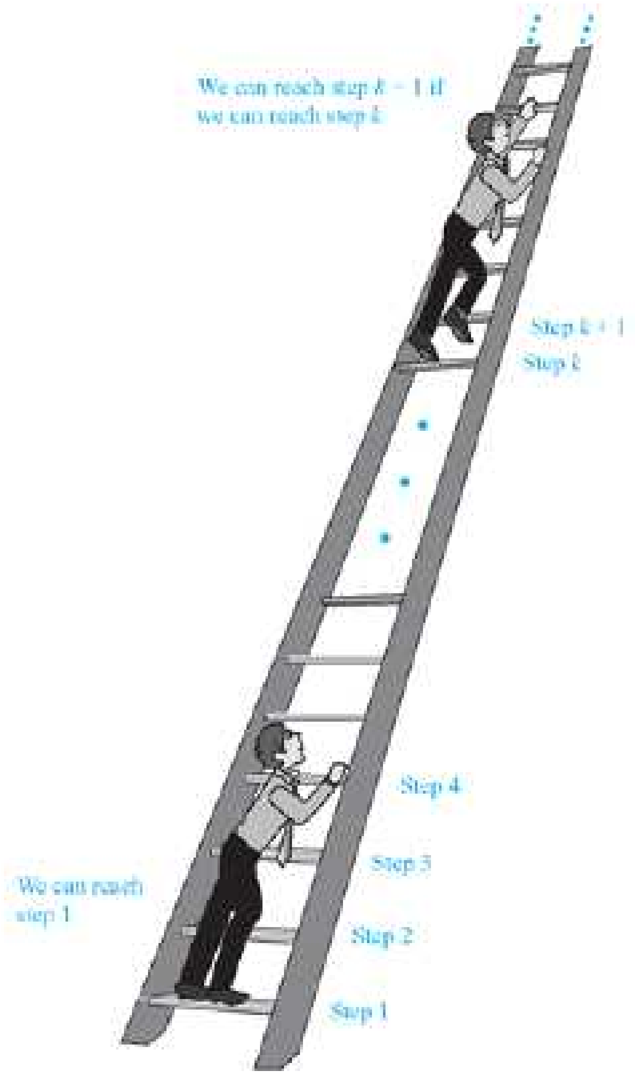

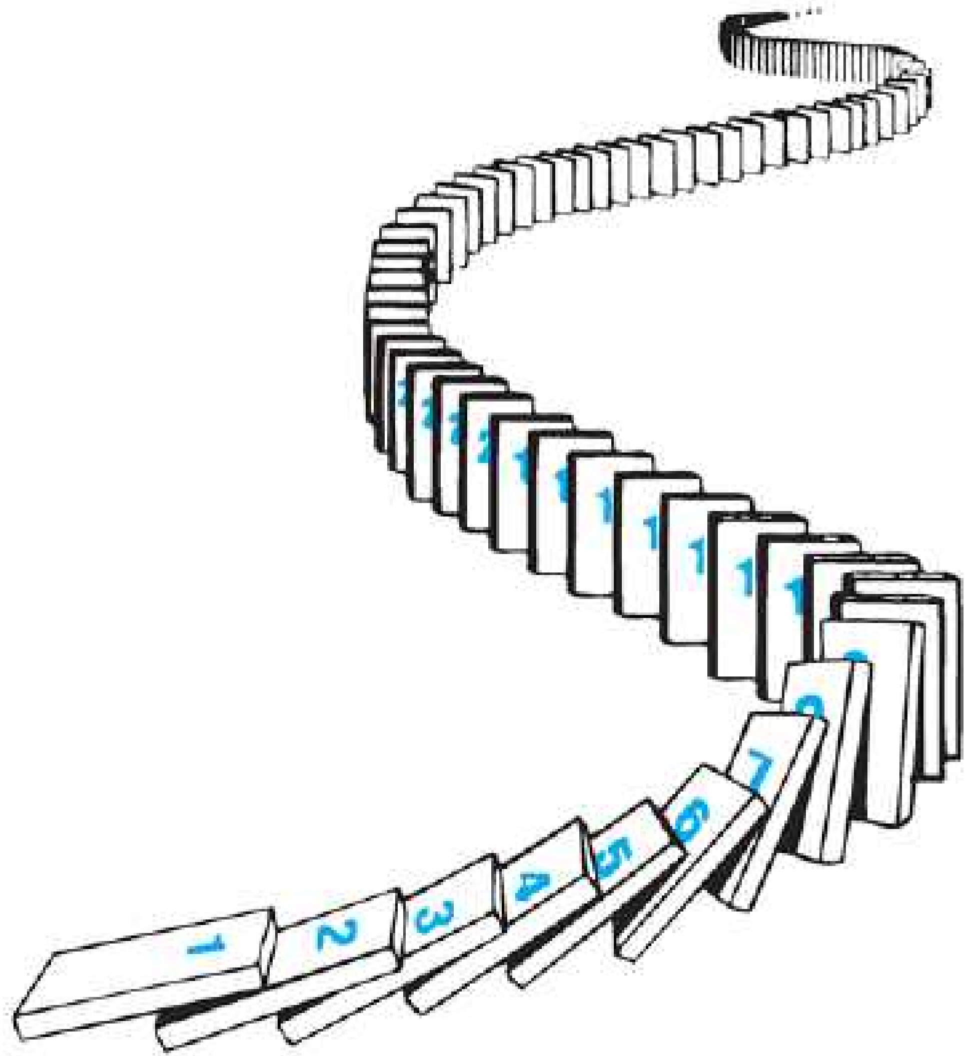

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- We conclude by distinguishing between the *weak principle* of mathematical induction and the *strong principle* of mathematical induction.

  The *strong principle* can actually be derived from the *weak principle*.

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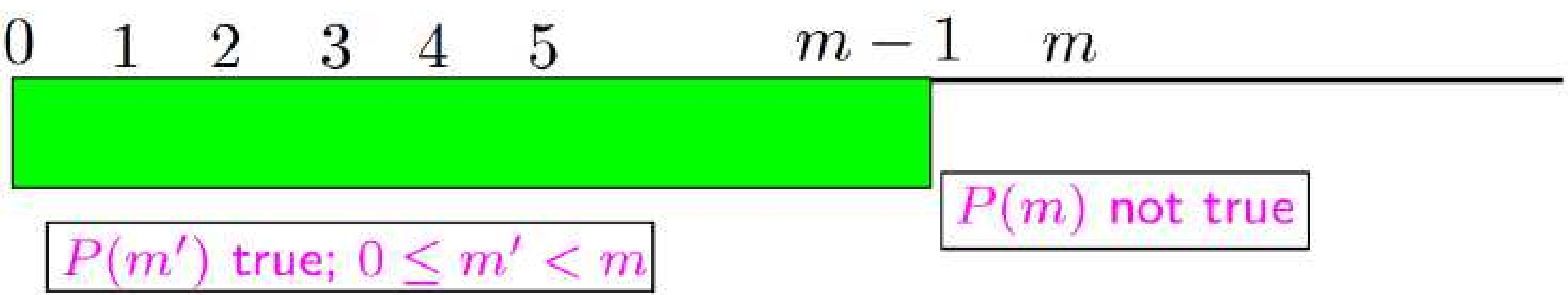

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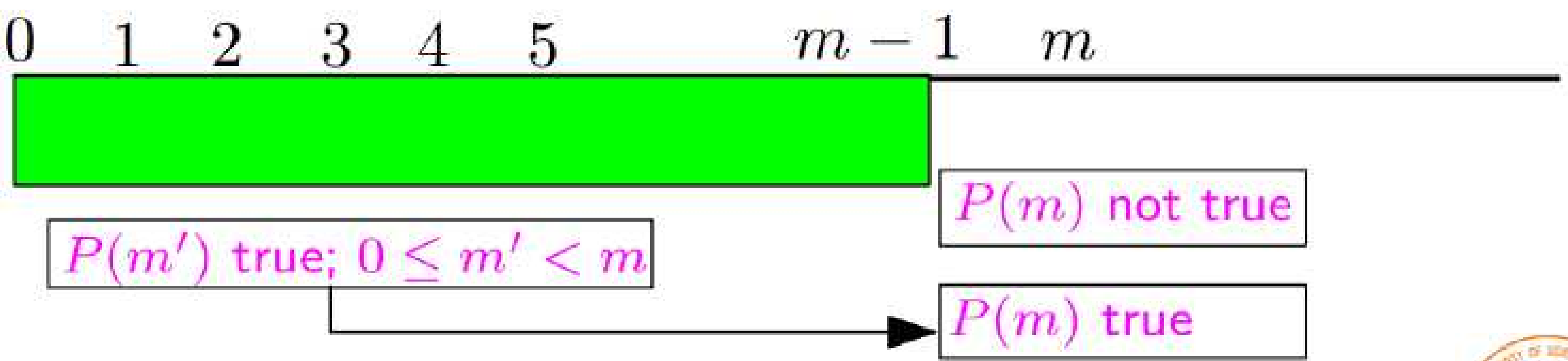

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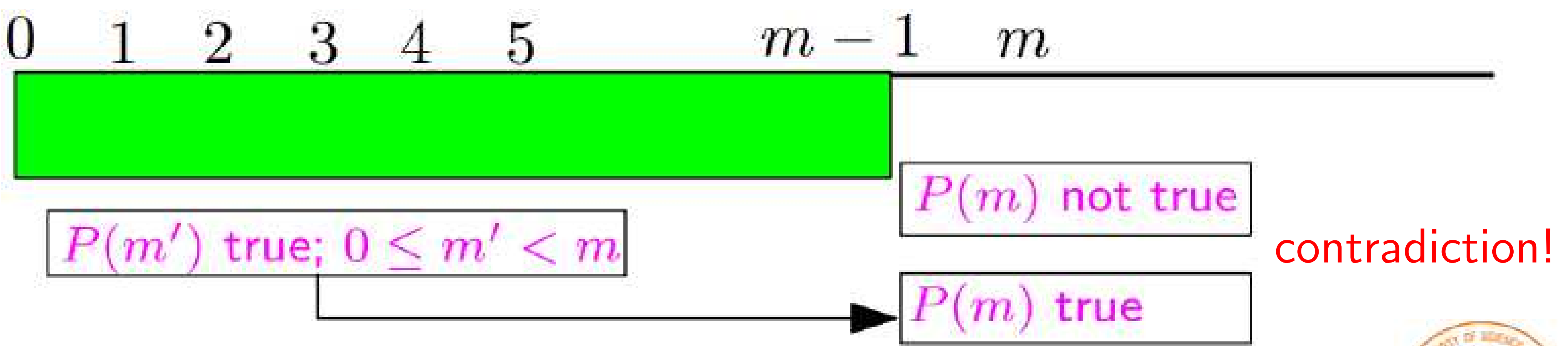

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  ◇ The smallest counterexample n is larger than 0

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  ◇ Therefore, $(*)$ holds for all positive integers n.

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  The key step was proving that

  $$P(n-1) \to P(n)$$

  where $P(n)$ is the statement

  $$1+2+\cdots+n=\frac{n(n+1)}{2}$$

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When a for all quantifier is false, there must be some n for which it is false. Let n be the smallest nonnegative integer for which $2^{n+1} \not\geq n^2 + 2$.

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  Then setting $i = n - 1$ gives
  $$2^{(n-1)+1} \geq (n-1)^2 + 2.$$

  or
  $$(*) \quad 2^n \geq n^2 - 2n + 1 + 2 = n^2 - 2n + 3$$

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  Thus, we write
  $$\begin{aligned} 2^{n+1} &\geq 2n^2 - 4n + 6 \\ &= (n^2 + 2) + (n^2 - 4n + 4) \\ &= n^2 + 2 + (n - 2)^2 \\ &\geq n^2 + 2. \end{aligned}$$

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  (b) if $n > 0$, then $P(n-1) \rightarrow P(n)$

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  ◇ Thus, $P(n)$ is true for all $n \in N$.

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Since $P(n-1) \rightarrow P(n)$, we see that

$P(0)$ implies $P(1)$, $P(1)$ implies $P(2)$, ...

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  (a) – *Basic Step* *Inductive Hypothesis*

  (b) – *Inductive Step* *Inductive Conclusion*

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By mathematical induction, $\forall n > 0,\ 2^{n+1} \geq n^2 + 2$.

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Inductive Hypothesis

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Inductive Step

Hence, we've just prove that for $n > 2$, $P(n-1) \to P(n)$.

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Inductive Conclusion

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  - ◇ Iterating gives us a proof of $P(n)$ for all n

Strong Induction

- **Principle** (*Strong Principle of Mathematical Induction*)

  (a) If the statement $P(b)$ is true

  (b) for all $n > b$, the statement
  $P(b) \wedge P(b+1) \wedge \cdots \wedge P(n-1) \rightarrow P(n)$ is true.

  then $P(n)$ is true for all integers $n \geq b$.

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- Prove that every positive integer is a power of a prime or the product of powers of primes.

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  ◇ Thus, by the strong principle of mathematical induction, every positive integer is a power of a prime or a product of powers of primes.

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- In reality, they are equivalent to each other in that the weak form is a special case of the strong form, and the strong form can be derived from the weak form.

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- A *typical* proof by mathematical induction, showing that a statement $P(n)$ is true for all integers $n \geq b$ consists of three steps:

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  3. We conclude on the basis of the principle of mathematical induction that $P(n)$ is true for all $n \geq b$.

Recursion

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- A classical example of *recursion* is the **Towers of Hanoi** Problem.

Towers of Hanoi

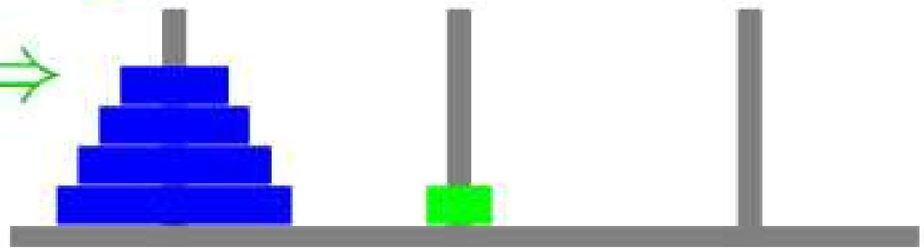
Towers of Hanoi

- 3 pegs; n disks of different sizes
- A *legal move* takes a disk from one peg and moves it onto another peg so that it is not on top of a smaller disk
- **Problem**: Find a (efficient) way to move all of the disks from one peg to another

Towers of Hanoi

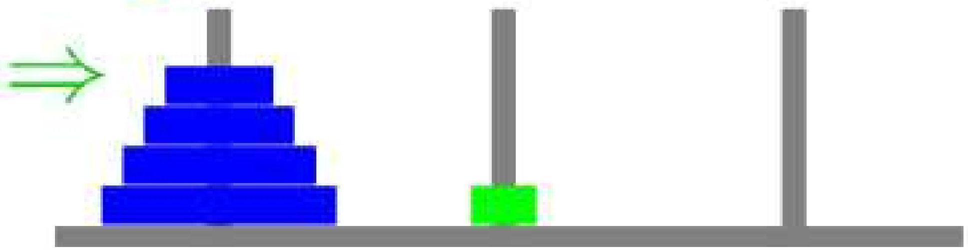
Towers of Hanoi

legal move

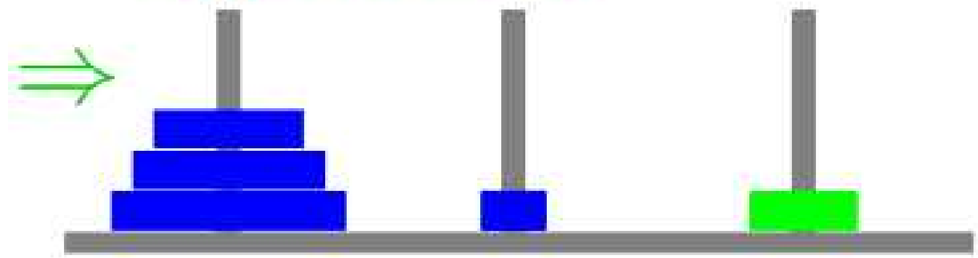
$\Rightarrow$ 

Towers of Hanoi

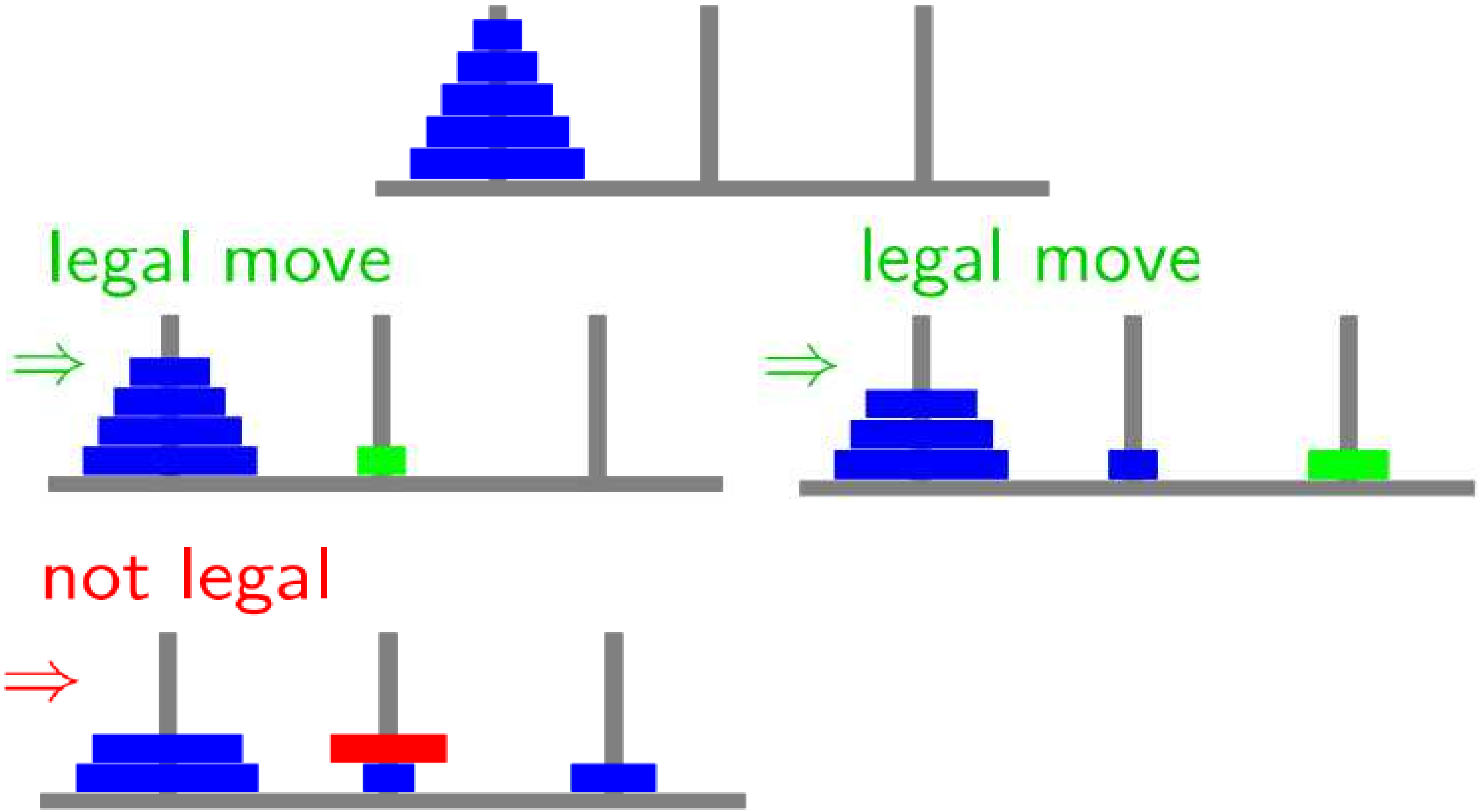
legal move



legal move



Towers of Hanoi



Towers of Hanoi

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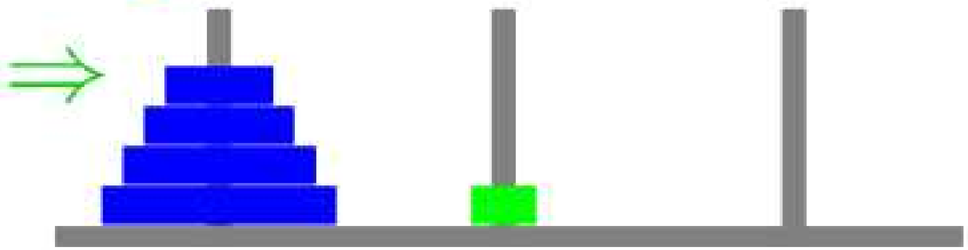

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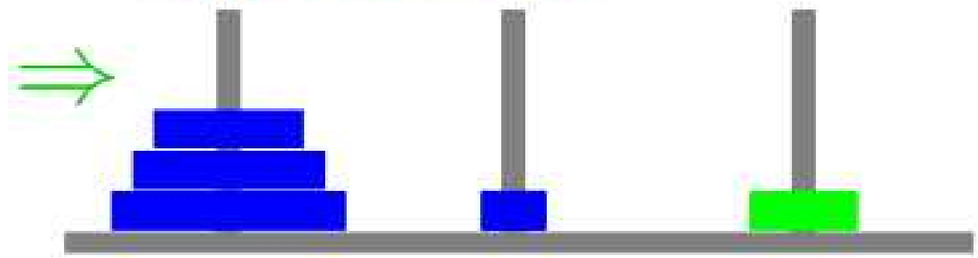
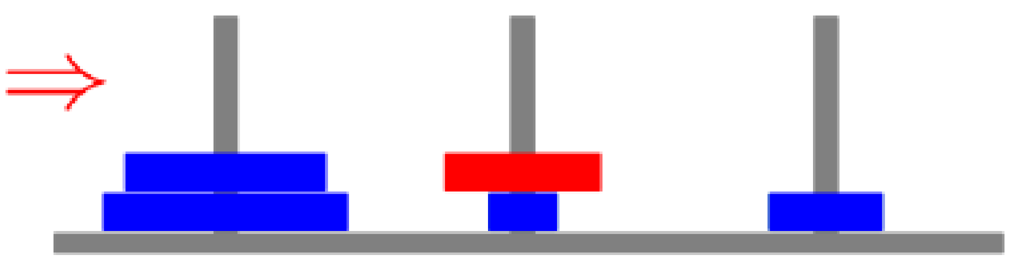


not legal



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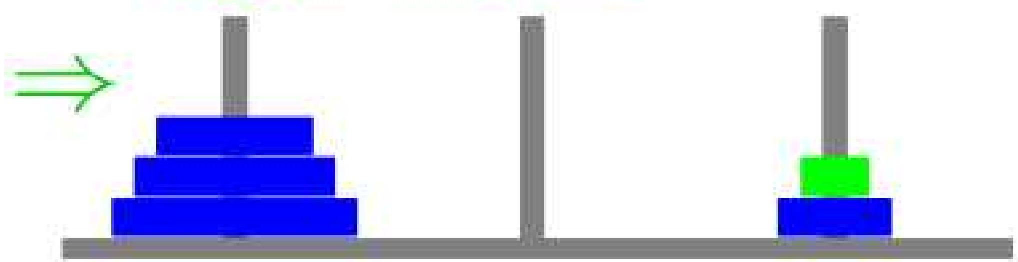

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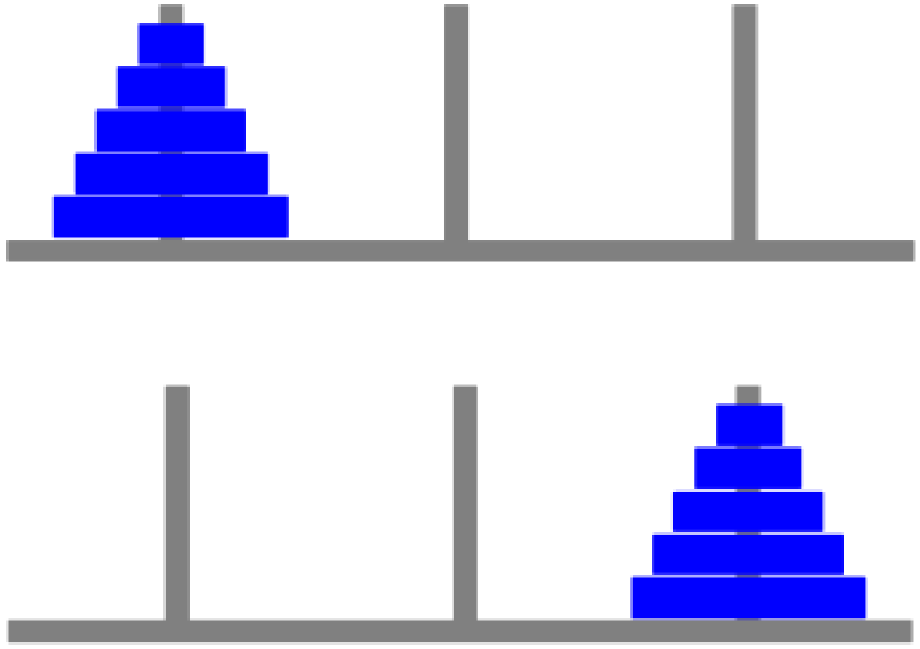

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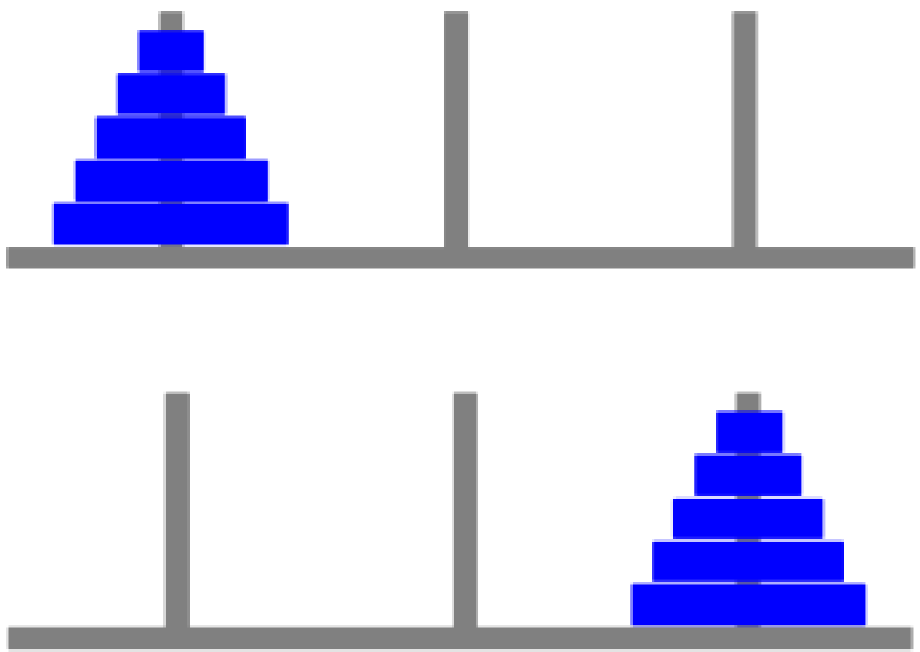

Given $i, j \in \{1,2,3\}$, let
$\overline{\{i,j\}} = \{1,2,3\} - \{i\} - \{j\}$,
i.e., $\overline{\{1,2\}} = \{3\}$, $\overline{\{1,3\}} = \{2\}$,
$\overline{\{2,3\}} = \{1\}$.

Towers of Hanoi

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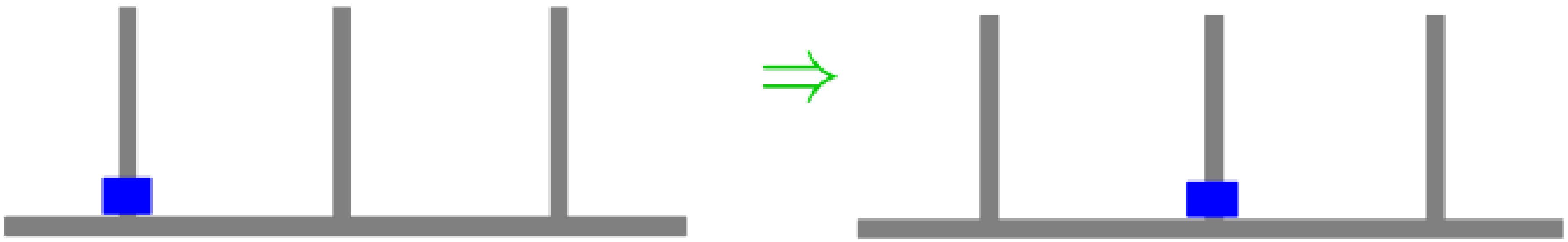

Towers of Hanoi

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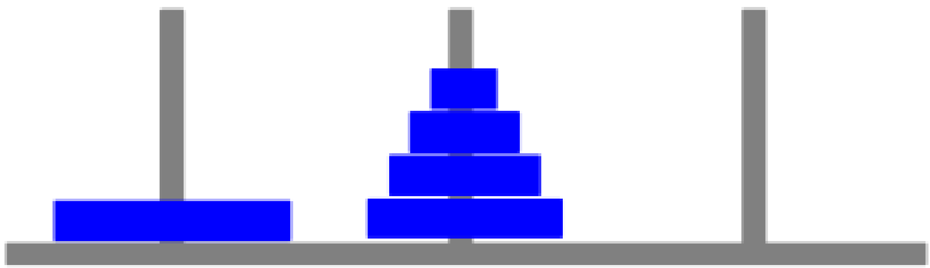

move top $n-1$ disks from i to $\overline{\{i,j\}}$

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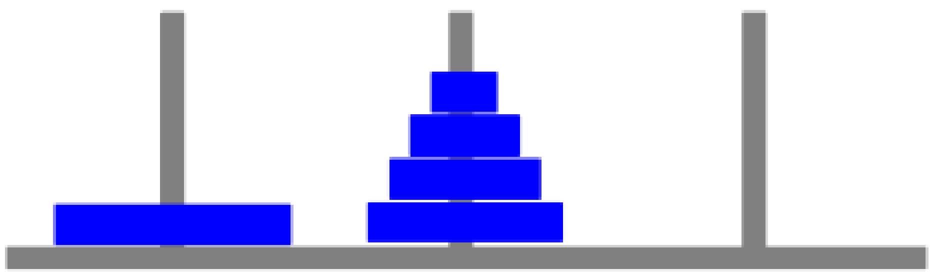
1) 

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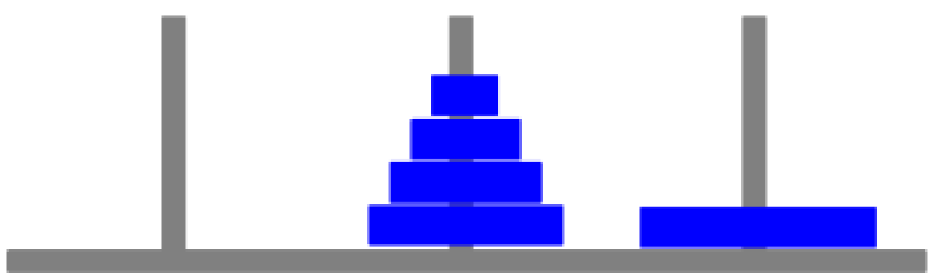
2) 

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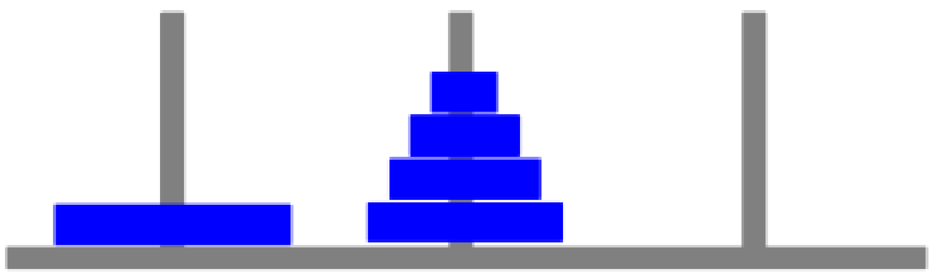
1) 

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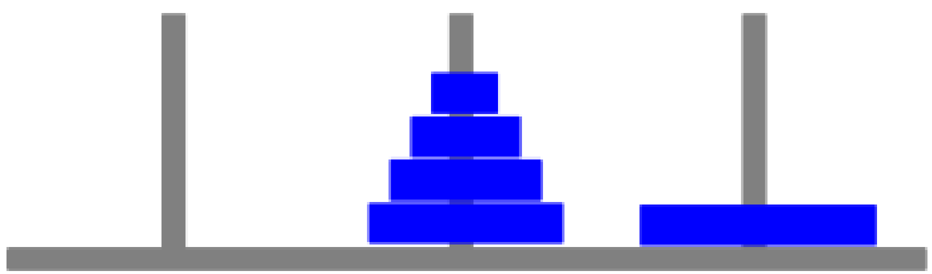
2) 

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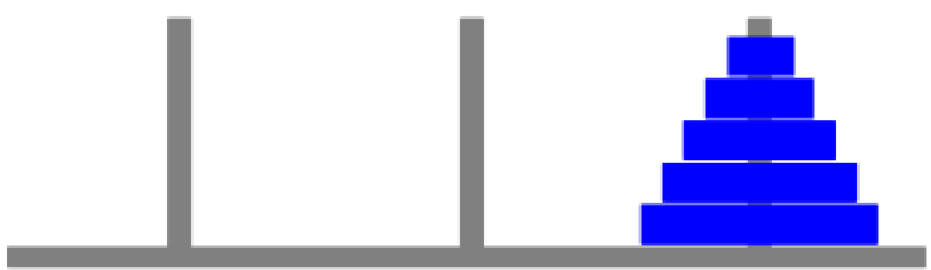
3) 

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Towers of Hanoi

```java
public class Hanoi
{

    public void move(int n, char a, char b, char c)
    {
        if (n == 1)
            System.out.println("plate " + n + " from " + a + " to " + c);
        else
        {
            move(n-1,a,c,b);
            System.out.println("plate " + n + " from " + a + " to " + c);
            move(n-1,b,a,c);
        }

    }

```

Towers of Hanoi

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i) move top $n-1$ disks from i to $\overline{\{i,j\}}$

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Towers of Hanoi

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- $p(n-1) \to p(n)$ is *recursion* statement that

  if our algorithm works for $n-1$ disks, then we can build a correct solution for n disks

Towers of Hanoi

- Running time

  $M(n)$ is number of disk moves needed for n disks

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$M(1) = 1$

if $n > 1$, then $M(n) = 2M(n-1) + 1$

Towers of Hanoi

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  Later, we'll also see how to solve without guessing

Towers of Hanoi

- Formally, given

$$M(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2M(n-1) + 1 & \text{otherwise} \end{cases}$$

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Towers of Hanoi

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Proof. (by induction)

The base case $n = 1$ is true, since $2^1 - 1 = 1$.

For the inductive step, assume that $M(n-1) = 2^{n-1} - 1$ for $n > 1$.

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Then $M(n) = 2M(n-1) + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1$

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Towers of Hanoi

Fibonacci Sequence

$$F(n) = \begin{cases} 1 & \text{if } n = \\ F(n-1)+F(n-2) & \text{other} \end{cases}$$

- **Example 2**: Let $S(n)$ be the number of subsets of a set of size n. What is the formula for $S(n)$?

  The empty set, of size $n = 0$ has only one subset (itself), so $S(0) = 1$.

  It is not difficult to see that

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  The empty set, of size $n = 0$ has only one subset (itself), so $S(0) = 1$.

  It is not difficult to see that

  $S(1) = 2$, $S(2) = 4$, $S(3) = 8$

  We "guess" that $S(n) = 2^n$. But, in order to prove formula, we'll need to think recursively.

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  This suggests that the recurrence for the number of subsets of an n-element set $\{1, 2, \ldots, n\}$ is

$$S(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2S(n-1) & \text{if } n \geq 1 \end{cases}$$

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  Proof by induction is easy.

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  Can we generalize this to find a closed-form solution?

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  Guess $T(n) = r^nT(0) + a\sum_{i=0}^{n-1} r^i$

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  T(0) &= b \\
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  This would lead to the same guess

  $$T(n) = r^n b + a \sum_{i=0}^{n-1} r^i.$$

Formula of Recurrences

- **Theorem** If $T(n) = rT(n-1) + a$, $T(0) = b$, and $r \neq 1$, then

$$T(n) = r^n b + a\frac{1 - r^n}{1 - r}$$

for all nonnegative integers n.

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Proof by induction

The base case:

$$T(0) = r^0 b + a\frac{1-r^0}{1-r} = b.$$

So the formula is true when $n = 0$.

Now assume that $n > 0$ and

$$T(n-1) = r^{n-1} b + a\frac{1-r^{n-1}}{1-r}.$$

- **Proof by induction**

$$
\begin{aligned}
T(n) &= rT(n-1) + a \\
&= r\left(r^{n-1}b + a\frac{1-r^{n-1}}{1-r}\right) + a \\
&= r^n b + \frac{ar - ar^n}{1-r} + a \\
&= r^n b + \frac{ar - ar^n + a - ar}{1-r} \\
&= r^n b + a\frac{1-r^n}{1-r}.
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$T(n) = 3T(n-1) + 2$ with $T(0) = 5$

Plugging $r = 3, a = 2, b = 5$ in the formula, gives
$$T(n) = 3^n \cdot 5 + 2\frac{1-3^n}{1-3} = 3^n \cdot 6 - 1$$

First-Order Linear Recurrences

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  ◇ Linear because $T(n-1)$ only appears to the first power.

  Something like $T(n) = (T(n-1))^2 + 3$ would be a non-linear first-order recurrence relation.

Next Lecture

- recurrence ...