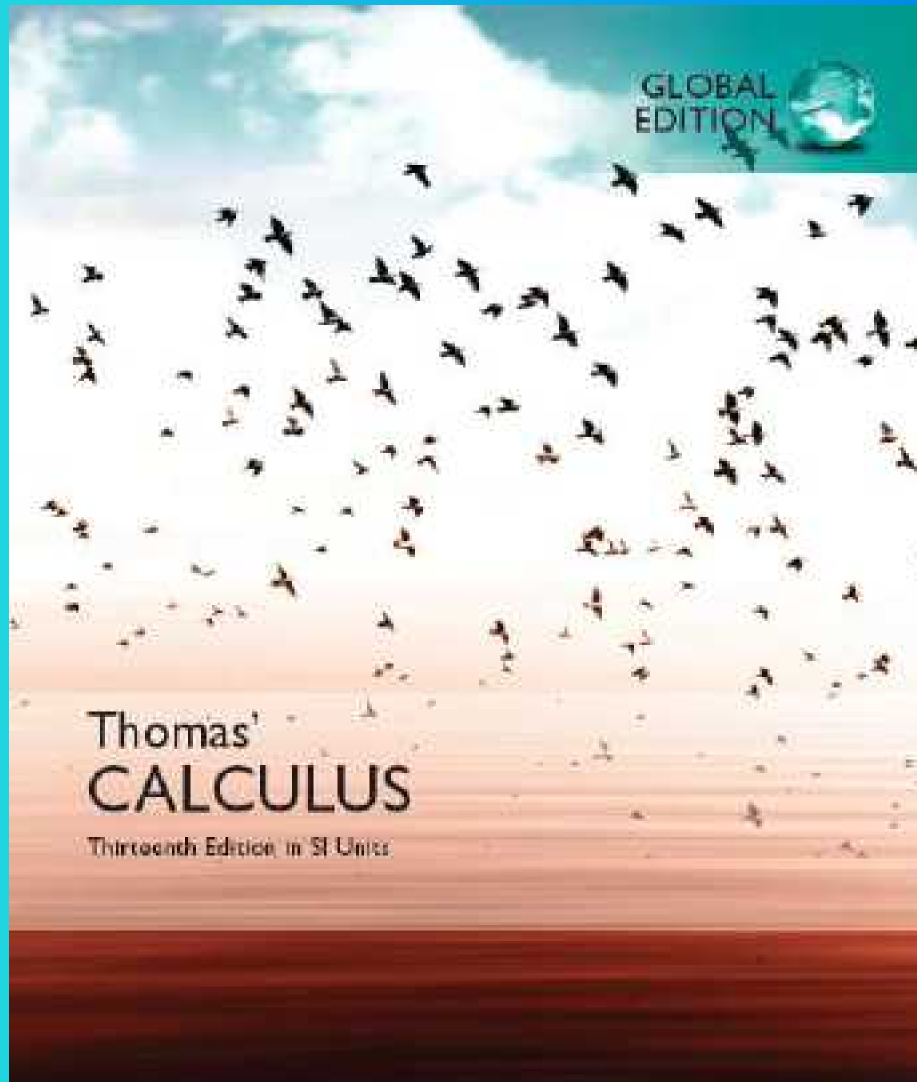

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 11

Parametric Equations and Polar Coordinates

参数方程和极坐标

11.1

Parametrizations of Plane Curves
平面曲线的参数方程

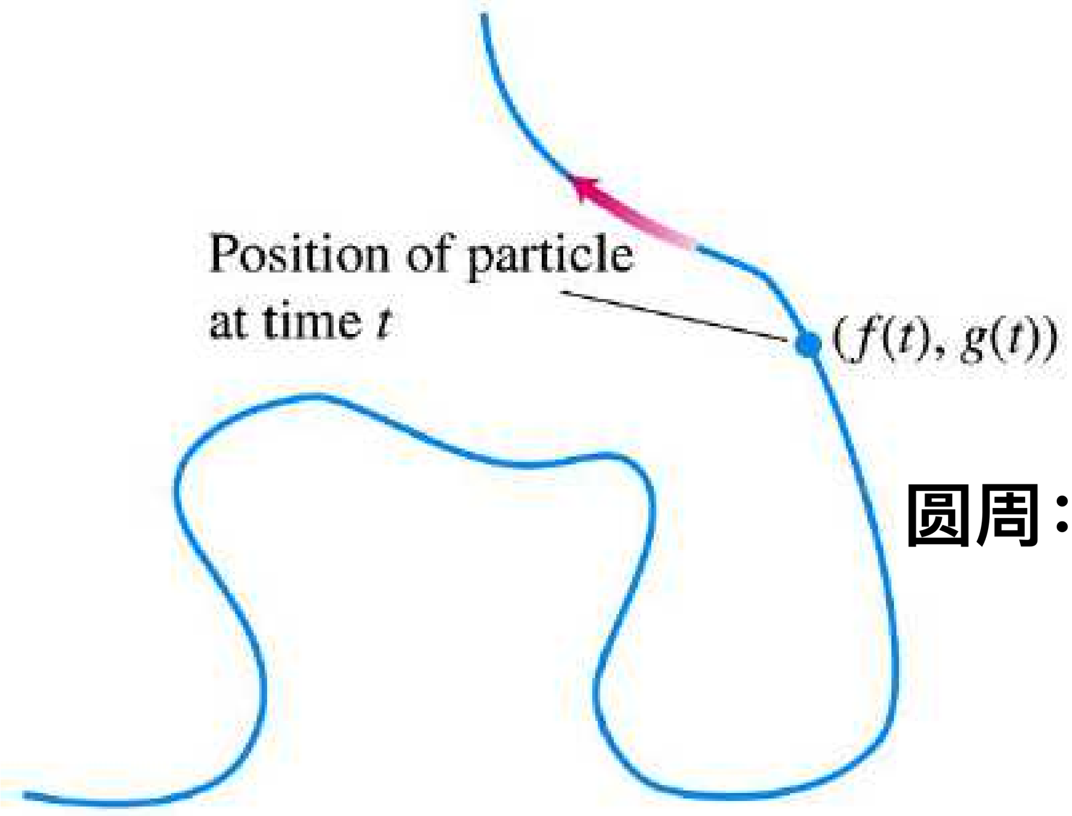


FIGURE 11.1 The curve or path traced by a particle moving in the xy-plane is not always the graph of a function or single equation.

圆周 : $\begin{cases} x = a\cos t \\ y = a\sin t, \end{cases} \quad t \in [0, 2\pi]$

DEFINITION If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve.** The equations are **parametric equations** for the curve.

parameter interval. I is a closed interval, $a \le t \le b$,

$(f(a), g(a))$ is the **initial point**

$(f(b), g(b))$ is the **terminal point**.

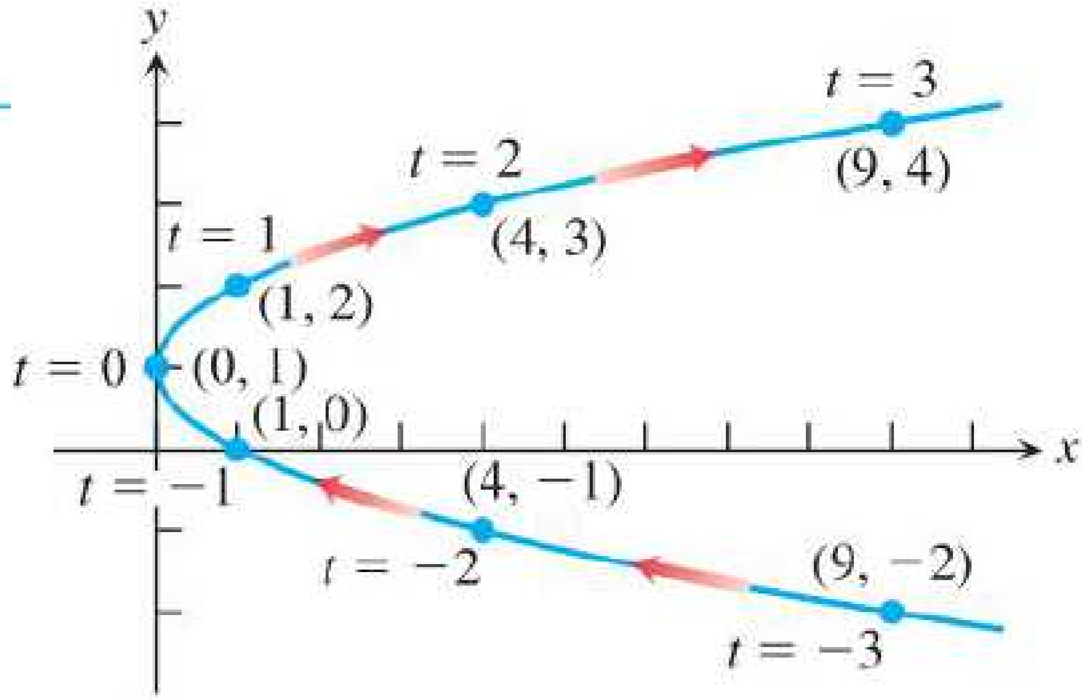
parametrized the curve.

EXAMPLE 1 Sketch the curve defined by the parametric equations

$$x = t^2, \qquad y = t + 1, \qquad -\infty < t < \infty.$$

Solution We make a brief table of values (Table 11.1),

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4



EXAMPLE 2 obtaining an algebraic equation in x and y.

$$x = t^2, \qquad y = t + 1,$$

Solution by eliminating the parameter t

$$x = t^2 = (y - 1)^2 = y^2 - 2y + 1.$$

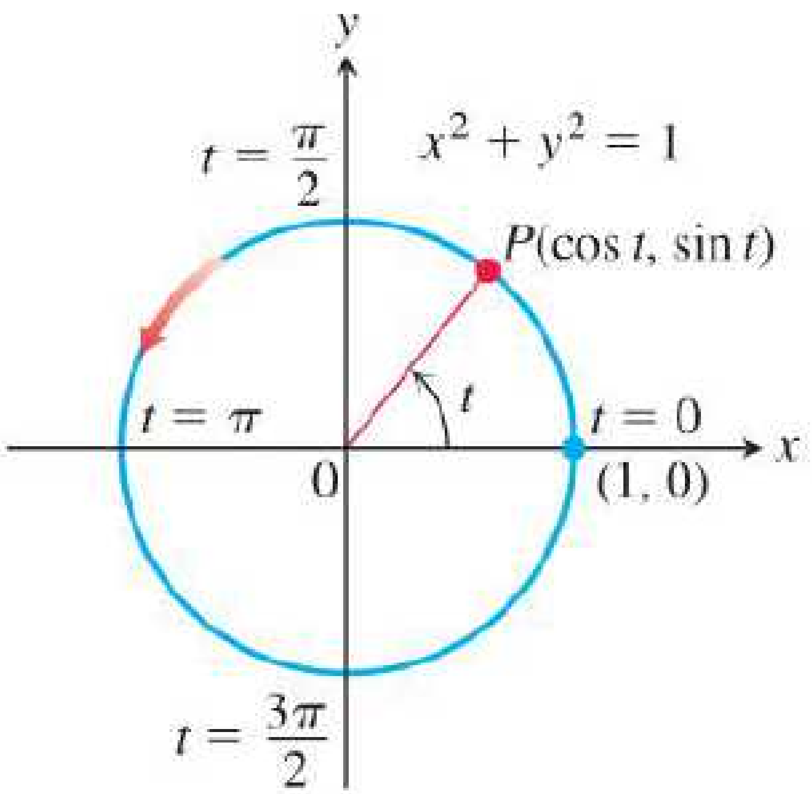
represents a parabola,

EXAMPLE 3 Graph the parametric curves

(a) $x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le 2\pi.$

(b) $x = a\cos t, \qquad y = a\sin t, \qquad 0 \le t \le 2\pi.$

Solution **(a)** Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the unit circle



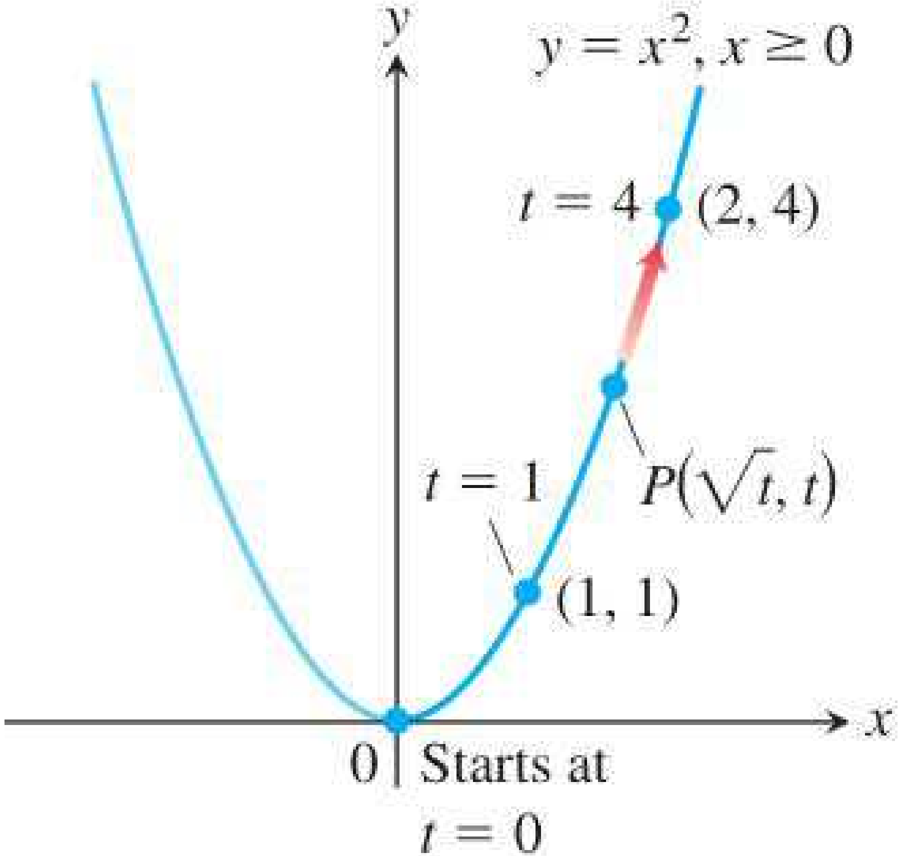
(b) the circle $x^2 + y^2 = a^2$

EXAMPLE 4 **Graph the curve**

$$x = \sqrt{t}, \qquad y = t, \qquad t \geq 0.$$

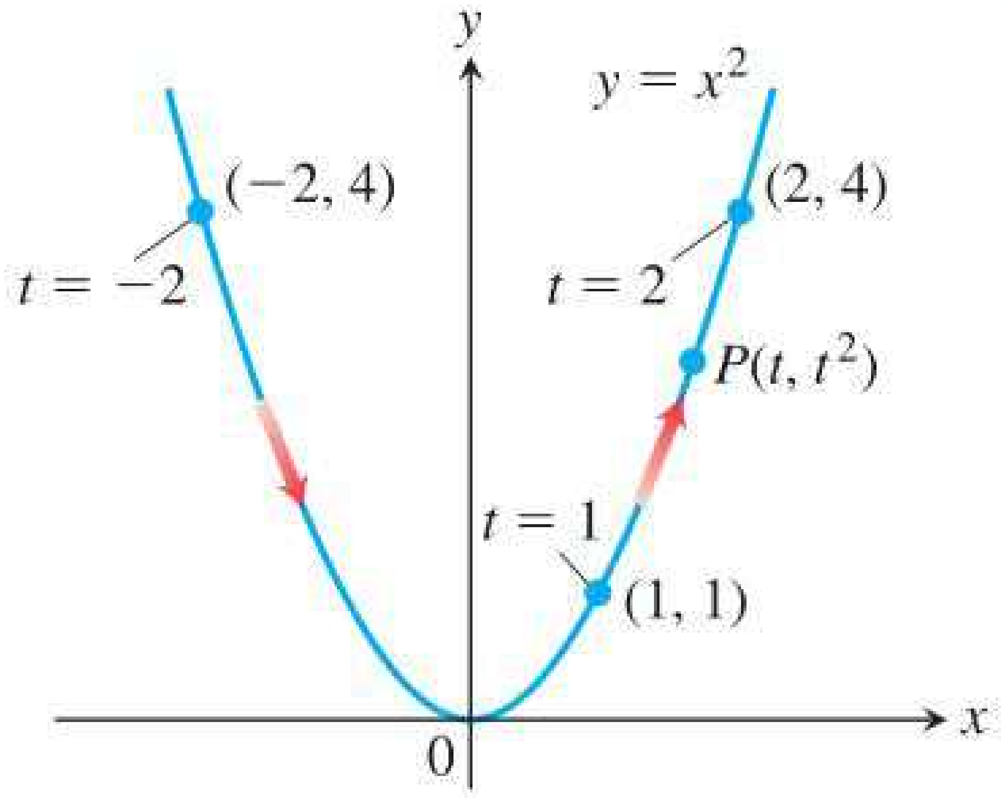
Solution

$$y = t = \left(\sqrt{t}\right)^2 = x^2.$$



EXAMPLE 5 A parametrization of the function $f(x) = x^2$ i

Solution $x = t, \quad y = f(t) = t^2, \quad -\infty < t < \infty.$



EXAMPLE 6

Find a parametrization for the line through the point (a, b) having slope m.

Solution $y - b = m(x - a)$. we set $t = x - a$,

$$x = a + t, \qquad y = b + mt, \qquad -\infty < t < \infty$$

EXAMPLE 7

Sketch and identify the path traced by the point $P(x, y)$ if

$$x = t + \frac{1}{t}, \qquad y = t - \frac{1}{t}, \qquad t > 0.$$

Solution

$$x - y = \left(t + \frac{1}{t}\right) - \left(t - \frac{1}{t}\right) = \frac{2}{t}. \qquad (x - y)(x + y) = 4,$$

$$x + y = \left(t + \frac{1}{t}\right) + \left(t - \frac{1}{t}\right) = 2t.$$

TABLE 11.2 Values of $x = t + (1/t)$ and $y = t - (1/t)$ for selected values of t.

t	$1/t$	x	y
0.1	10.0	10.1	−9.9
0.2	5.0	5.2	−4.8
0.4	2.5	2.9	−2.1
1.0	1.0	2.0	0.0
2.0	0.5	2.5	1.5
5.0	0.2	5.2	4.8
10.0	0.1	10.1	9.9

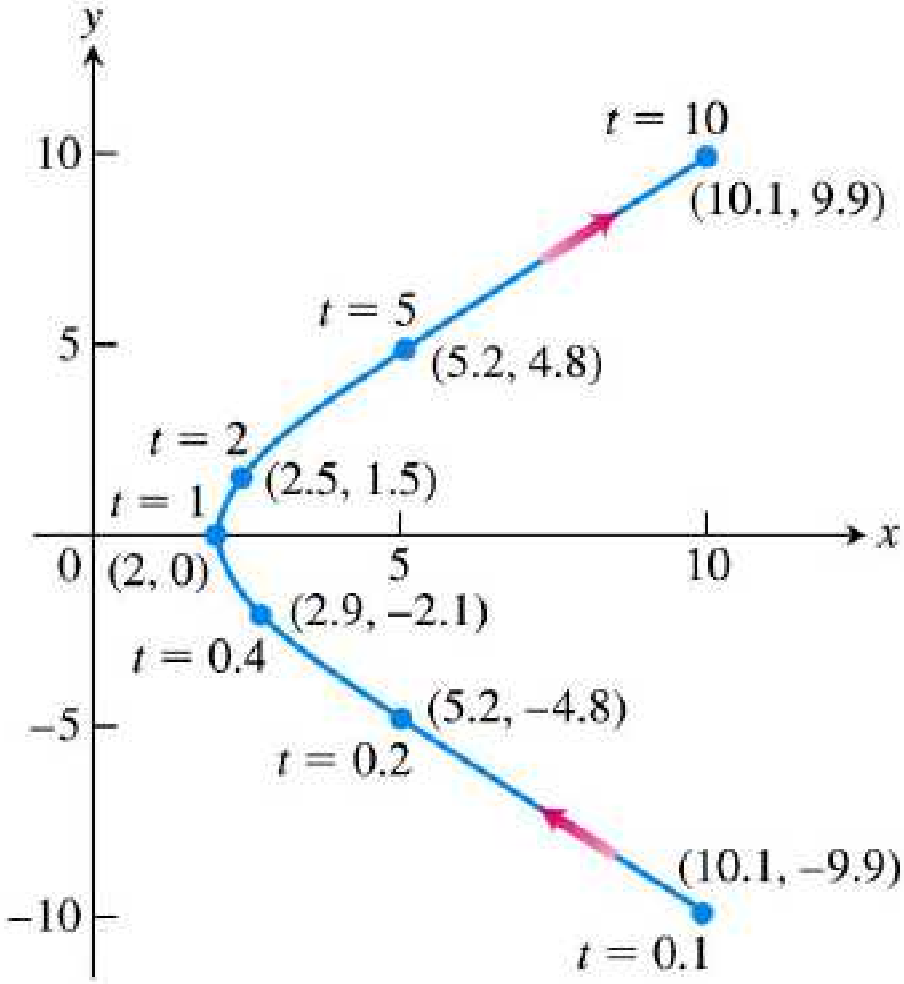


FIGURE 11.6 The curve for $x = t + (1/t)$, $y = t - (1/t)$, $t > 0$ in Example 7. (The part shown is for $0.1 \leq t \leq 10$.)

$$(x - y)(x + y) = 4,$$

$$x = \sqrt{4 + t^2}, \qquad y = t, \qquad -\infty < t < \infty,$$

$$x = 2 \sec t, \qquad y = 2 \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

The cycloid curve

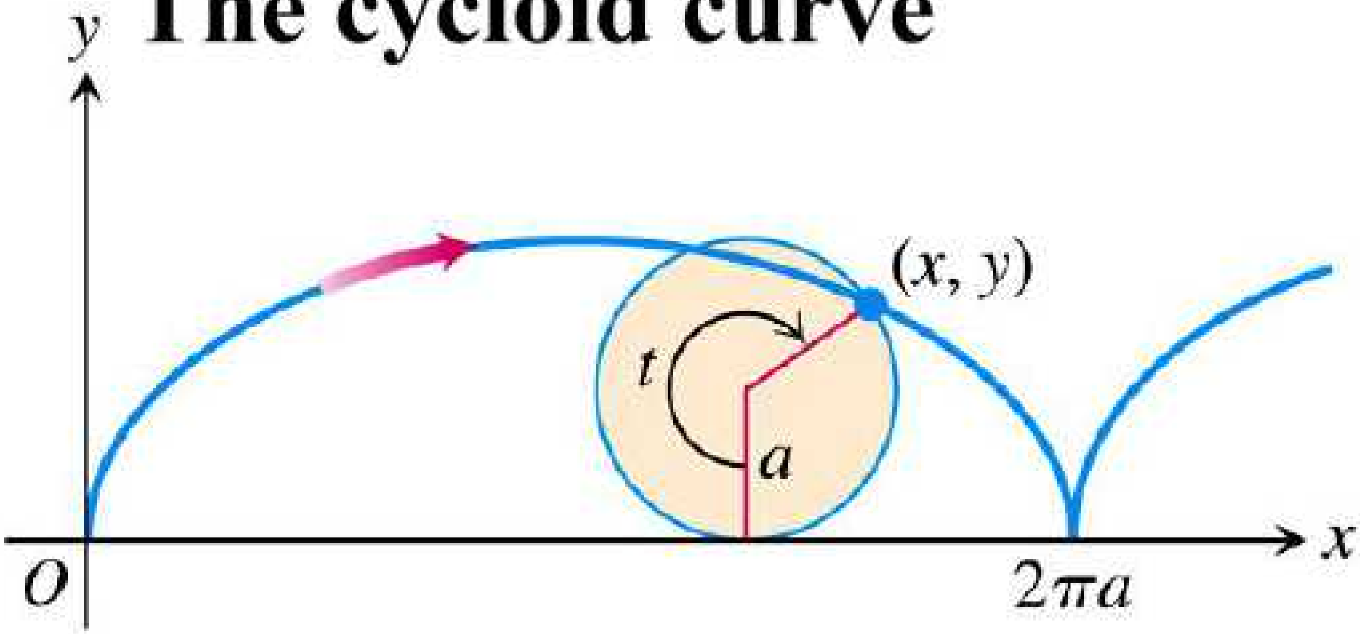


FIGURE 11.9 The cycloid curve $x = a(t - \sin t), y = a(1 - \cos t)$, for $t \geq 0$.

frequency is independent of the amplitude.

EXAMPLE 8 A wheel of radius a rolls along a horizontal straight line. Find paramet-ric equations for the path traced by a point P on the wheel's circumference. The path is called a **cycloid**.

Solution We take the line to be the x-axis, mark a point P on the wheel, start the wheel with P at the origin, and roll the wheel to the right. we use the angle t through which the wheel turns, in radians. the coordinates of P are

$$x = at + a\cos\theta, \qquad y = a + a\sin\theta.$$

To express θ in terms of t, we observe that $t + \theta = 3\pi/2$

$$x = at - a\sin t, \qquad y = a - a\cos t.$$

$$x = at - a \sin t, \qquad y = a - a \cos t.$$

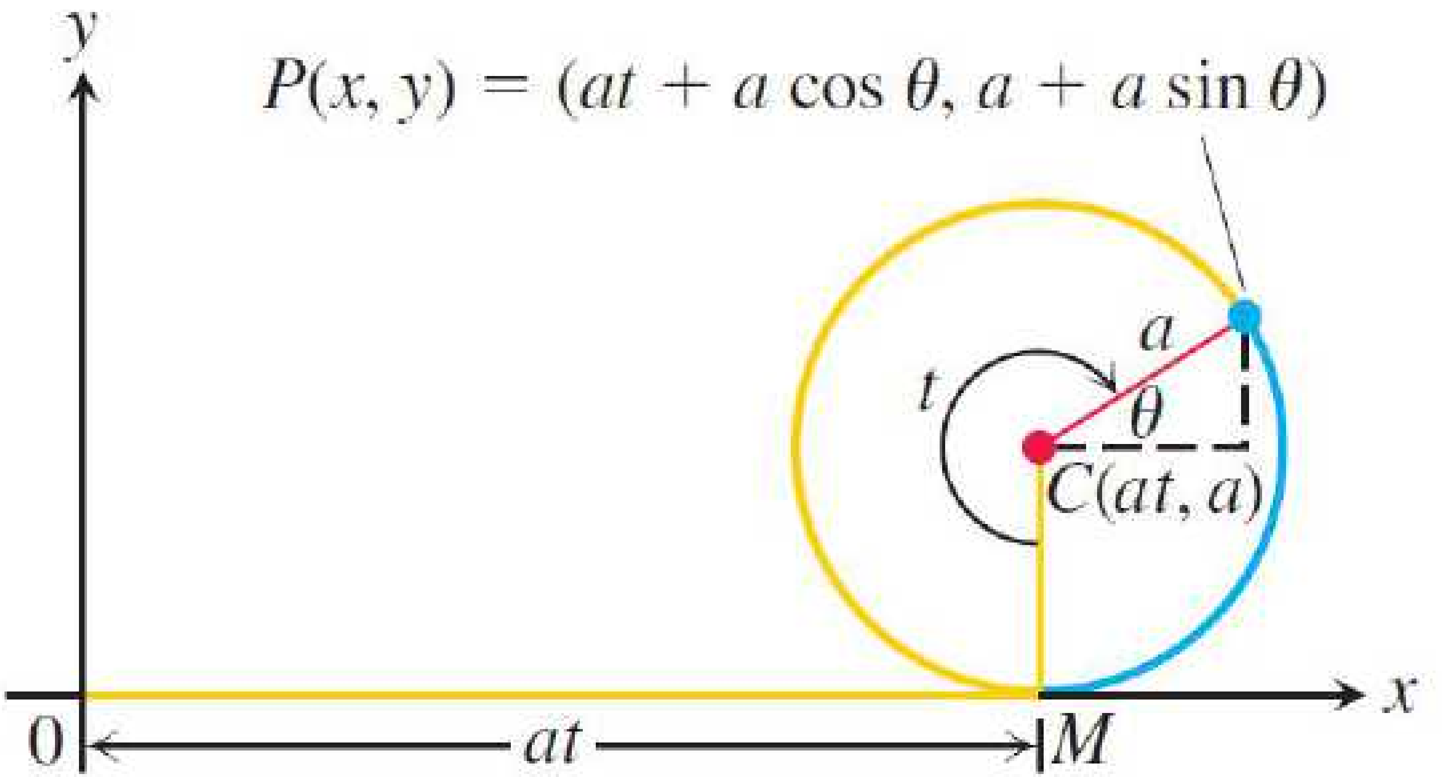


FIGURE 11.8 The position of $P(x, y)$ on the rolling wheel at angle t (Example 8).

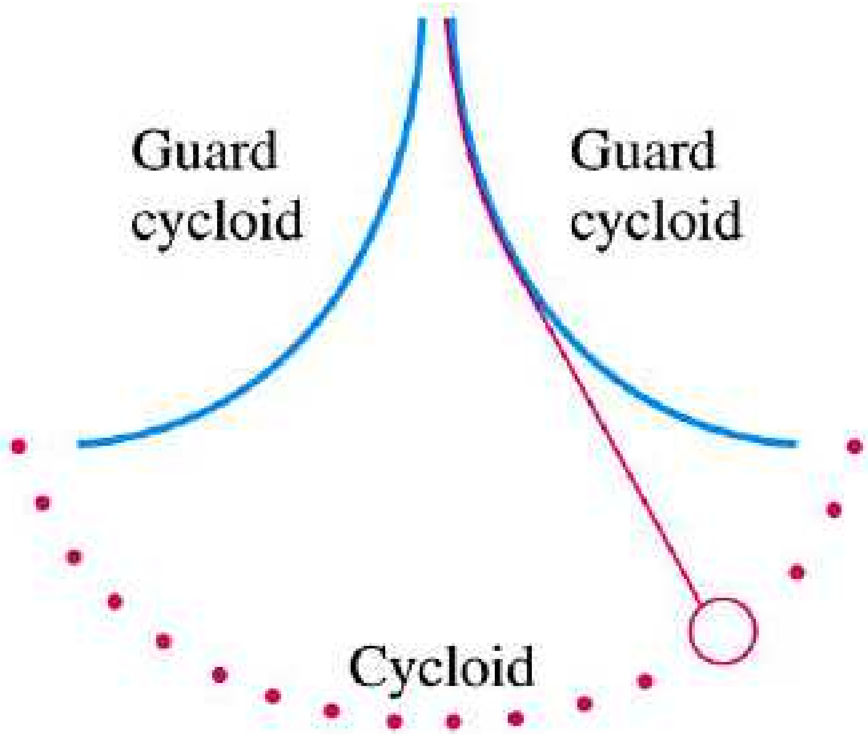
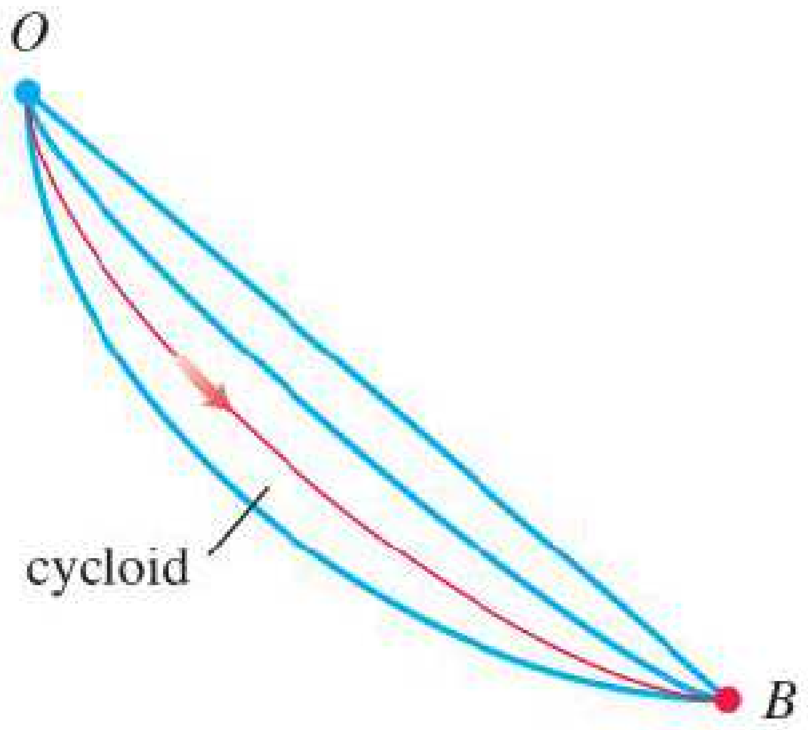
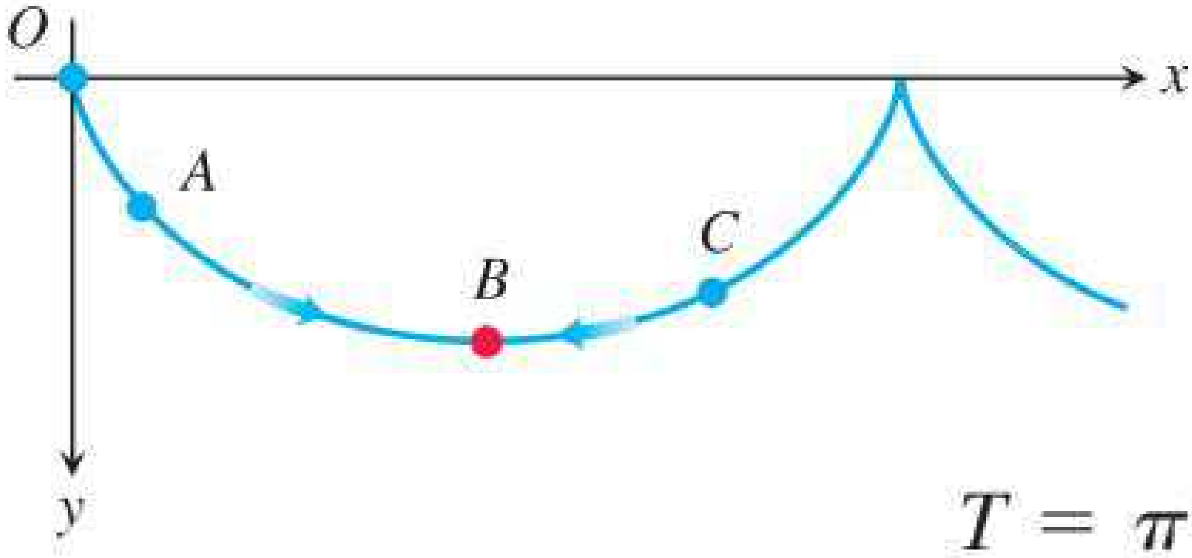


FIGURE 11.7 In Huygens' pendulum clock, the bob swings in a cycloid, so the frequency is independent of the amplitude.

$$T_f = \int_{x=0}^{x=a\pi} \sqrt{\frac{1 + (dy/dx)^2}{2gy}}\, dx.$$

minimize the value of this integral





$$T = \pi\sqrt{a/g}.$$

11.2

Calculus with Parametric Curves
参数曲线的微积分

Tangents and Areas

A parametrized curve $x = f(t)$ and $y = g(t)$

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

by the Chain Rule: $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}.$$

EXAMPLE 1 Find the tangent to the curve

$$x = \sec t, \qquad y = \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $(\sqrt{2}, 1)$, where $t = \pi/4$ (Figure 11.13).

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t} = \frac{1}{\sin t}$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \sqrt{2}.$$

The tangent line is $y - 1 = \sqrt{2}\left(x - \sqrt{2}\right)$

$$y = \sqrt{2}\,x - 1.$$

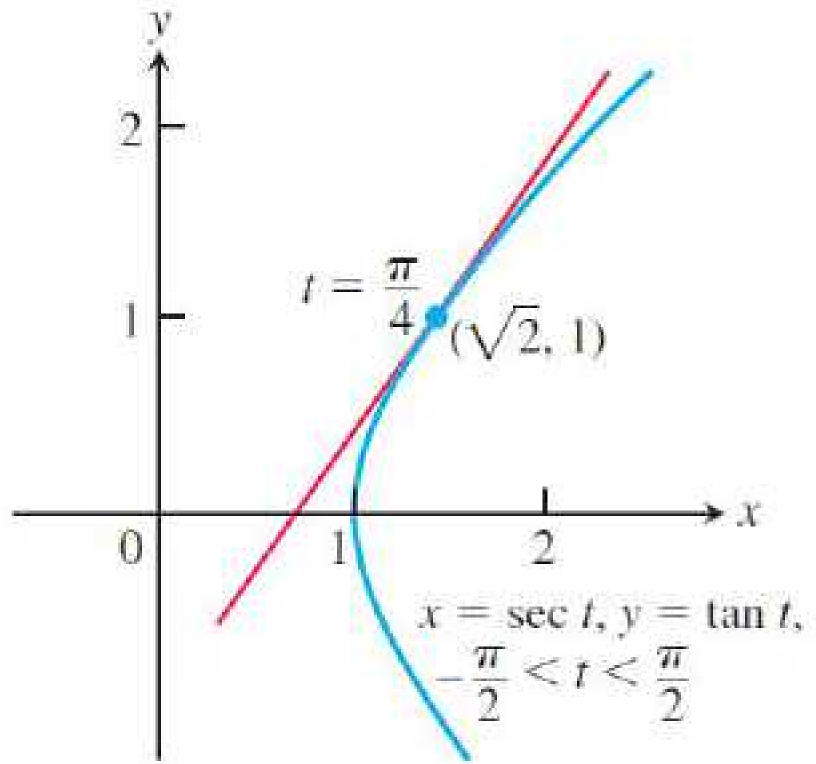


FIGURE 11.13 The curve in Example 1 is the right-hand branch of the hyperbola $x^2 - y^2 = 1$.

EXAMPLE 2

Find d^2y/dx^2 as a function of t if $x = t - t^2$ and $y = t - t^3$.

Solution

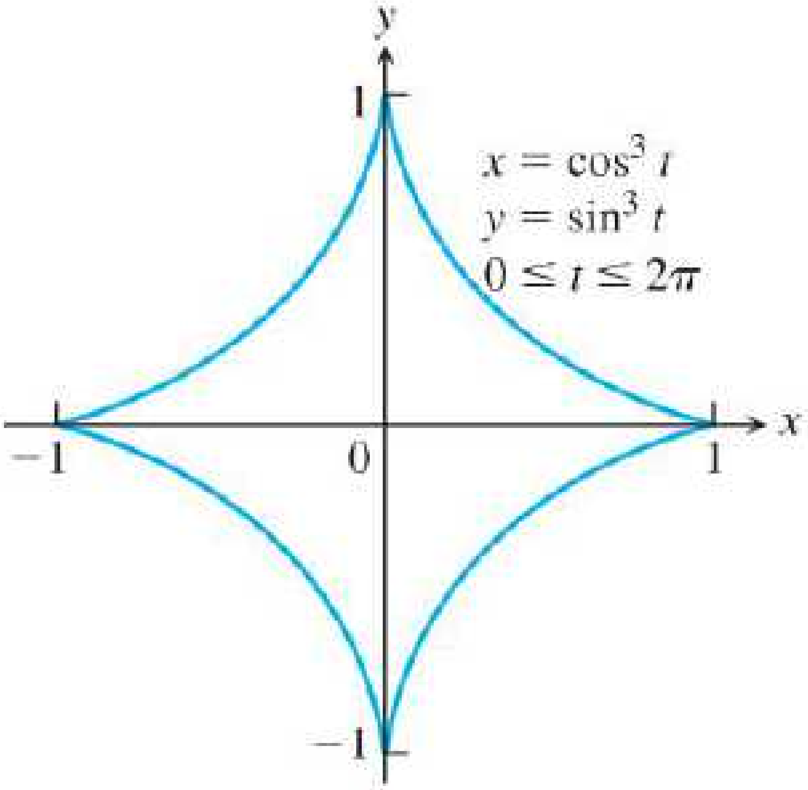
$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{1 - 2t}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1 - 3t^2}{1 - 2t}\right) = \frac{2 - 6t + 6t^2}{(1 - 2t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(2 - 6t + 6t^2)/(1 - 2t)^2}{1 - 2t} = \frac{2 - 6t + 6t^2}{(1 - 2t)^3}$$

EXAMPLE 3 Find the area enclosed by the astroid (Figure 11.14)

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$



Solution By symmetry,

$$A = 4\int_0^1 y\,dx = 4\int_0^{\pi/2} \sin^3 t \cdot 3\cos^2 t \sin t\,dt$$

$$= 12\int_0^{\pi/2} \left(\frac{1 - \cos 2t}{2}\right)^2 \left(\frac{1 + \cos 2t}{2}\right) dt$$

$$= \frac{3}{2}\int_0^{\pi/2} (1 - 2\cos 2t + \cos^2 2t)(1 + \cos 2t)\,dt$$

$$= \frac{3}{2}\int_0^{\pi/2} (1 - \cos 2t - \cos^2 2t + \cos^3 2t)\,dt$$

$$= \frac{3\pi}{8}.$$

Length of a Parametrically Defined Curve

Let C be a curve given parametrically by the equations

$$x = f(t) \qquad \text{and} \qquad y = g(t), \qquad a \le t \le b.$$

f and g are **continuously differentiable**

$f'(t)$ and $g'(t)$ are not simultaneously zero,

$$L = \int_{f(a)}^{f(b)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

DEFINITION If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

EXAMPLE 5 Find the length of the astroid

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

Solution

$$\left(\frac{dx}{dt}\right)^2 = \left[3\cos^2 t(-\sin t)\right]^2 = 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = \left[3\sin^2 t(\cos t)\right]^2 = 9\sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3\left|\cos t \sin t\right| = 3\cos t \sin t.$$

Because of the curve's symmetry

$$\text{Length of first-quadrant portion} = \int_0^{\pi/2} 3\cos t \sin t \, dt = \frac{3}{2}.$$

The length of the astroid is four times this: $4(3/2) = 6$.

EXAMPLE 6 Find the perimeter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution $x = a \sin t$

$y = b \cos t,\ a > b$ and $0 \le t \le 2\pi$.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2 \cos^2 t + b^2 \sin^2 t = a^2 - (a^2 - b^2)\sin^2 t$$

$$= a^2[1 - e^2 \sin^2 t]$$

$e = 1 - \frac{b^2}{a^2},$

$$P = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t}\, dt.$$

elliptic integral

$|e \sin t| \le e < 1$

$$\sqrt{1 - e^2 \sin^2 t} = 1 - \frac{1}{2} e^2 \sin^2 t - \frac{1}{2 \cdot 4} e^4 \sin^4 t - \cdots,$$

$$P = 2\pi a\left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{e^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{e^6}{5} - \cdots\right].$$

The Arc Length Differential

curve $x = f(t)$ and $y = g(t)$, $a \le t \le b$,

$$s(t) = \int_a^t \sqrt{[f'(z)]^2 + [g'(z)]^2}\, dz.$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

$$ds = \sqrt{1 + f'(x)^2}\, dx$$

$$ds = \sqrt{dx^2 + dy^2}.$$

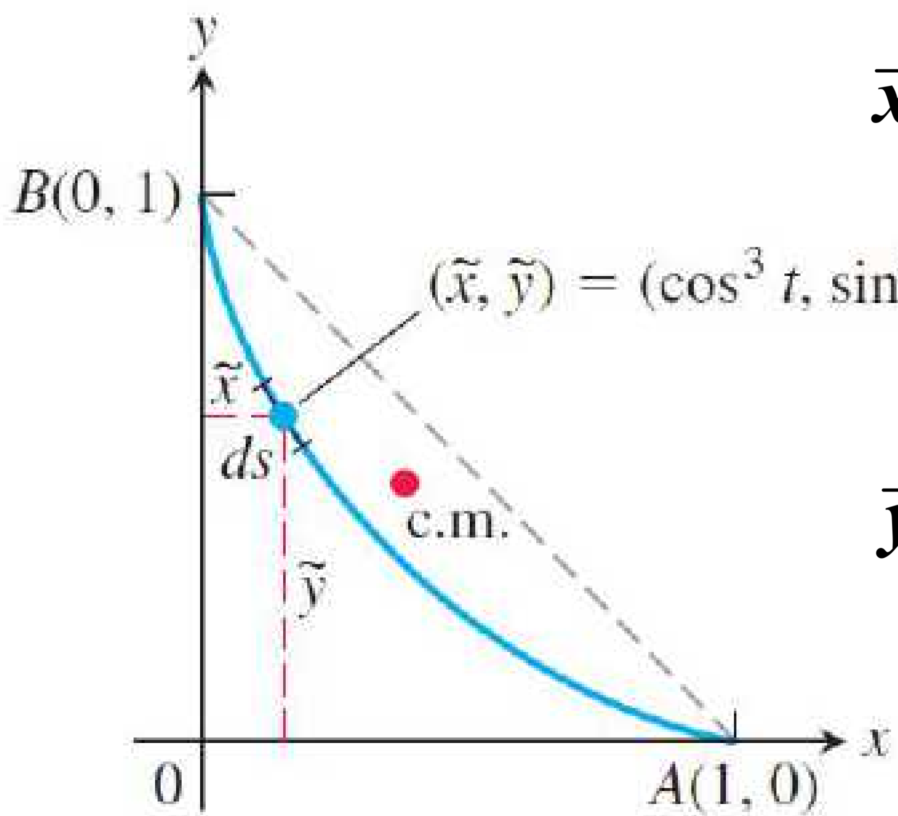


FIGURE 11.17 The centroid (c.m.) of the astroid arc in Example 7.

$$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} = \frac{\int \tilde{x}\,\delta\,ds}{\int \delta\,ds}$$

$$\bar{y} = \frac{\int \tilde{y}\,dm}{\int dm} = \frac{\int \tilde{y}\,\delta\,ds}{\int \delta\,ds}$$

EXAMPLE 7 Find the centroid of the first-quadrant arc of the astroid

Solution take the curve's density to be $\delta = 1$

mass is symmetric about the line $y = x$, so $\bar{x} = \bar{y}$.

$$dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 3 \cos t \sin t \, dt.$$

$$M = \int_0^{\pi/2} dm = \int_0^{\pi/2} 3 \cos t \sin t \, dt = \frac{3}{2}.$$

$$M_x = \int \tilde{y} \; dm = \int_0^{\pi/2} \sin^3 t \cdot 3 \cos t \sin t \, dt = 3 \cdot \frac{\sin^5 t}{5}\Bigg]_0^{\pi/2} = \frac{3}{5}.$$

$$\bar{y} = \frac{M_x}{M} = \frac{3/5}{3/2} = \frac{2}{5}.$$

The centroid is the point $(2/5, 2/5)$.

Areas of Surfaces of Revolution

$S = \int 2\pi y \, ds$ for revolution about the x-axis,

$x = f(t)$ and $y = g(t), a \leq t \leq b,$

1. Revolution about the x-axis ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

2. Revolution about the y-axis ($x \geq 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

EXAMPLE 9

The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the xy-plane is

$$x = \cos t, \qquad y = 1 + \sin t, \qquad 0 \le t \le 2\pi.$$

find the area of the surface swept out by revolving the circle about the x-axis

Solution

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt = 4\pi^2.$$

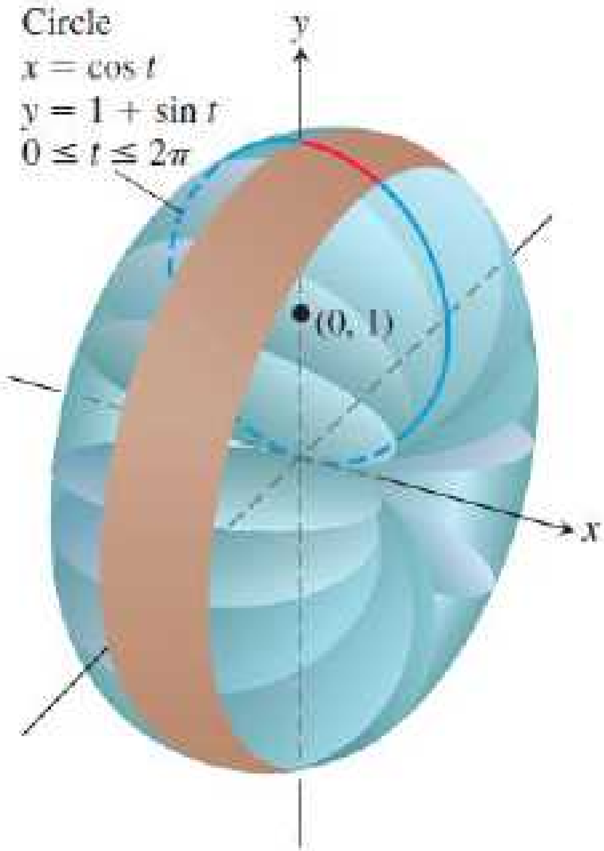
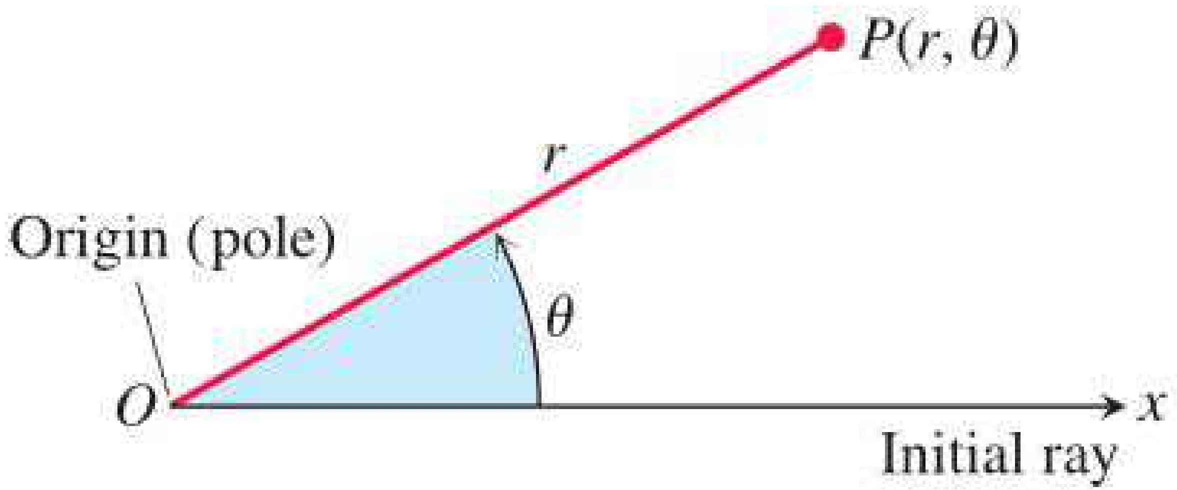


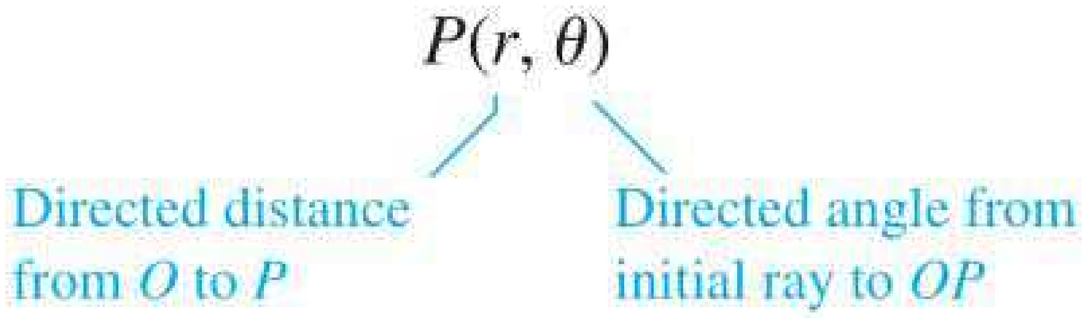
FIGURE 11.18 In Example 9 we calculate the area of the surface of revolution swept out by this parametrized curve.

11.3

Polar Coordinates
极坐标



$P(r, \theta)$
r
Origin (pole)
θ
O
x
Initial ray



$P(r, \theta)$
Directed distance from O to P
Directed angle from initial ray to OP

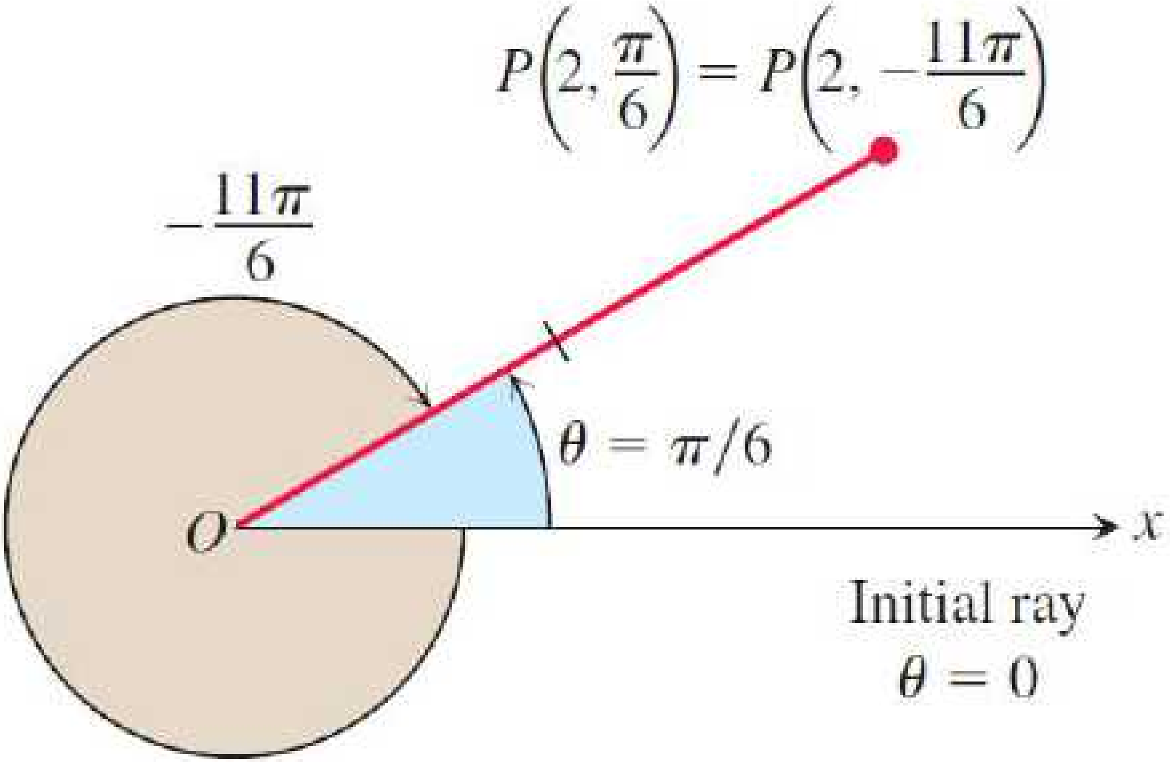


FIGURE 11.20 Polar coordinates are not unique.

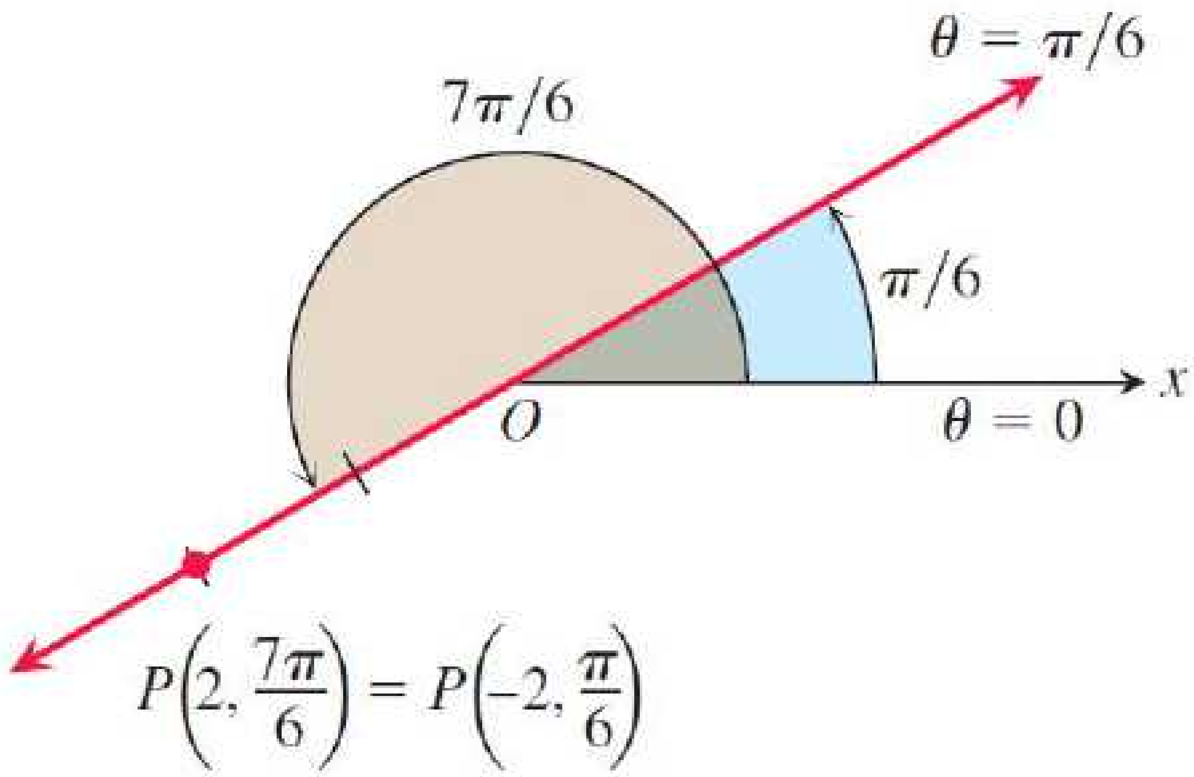


FIGURE 11.21 Polar coordinates can have negative r-values.

EXAMPLE 1 Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \ldots .$$

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \ldots .$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots .$$

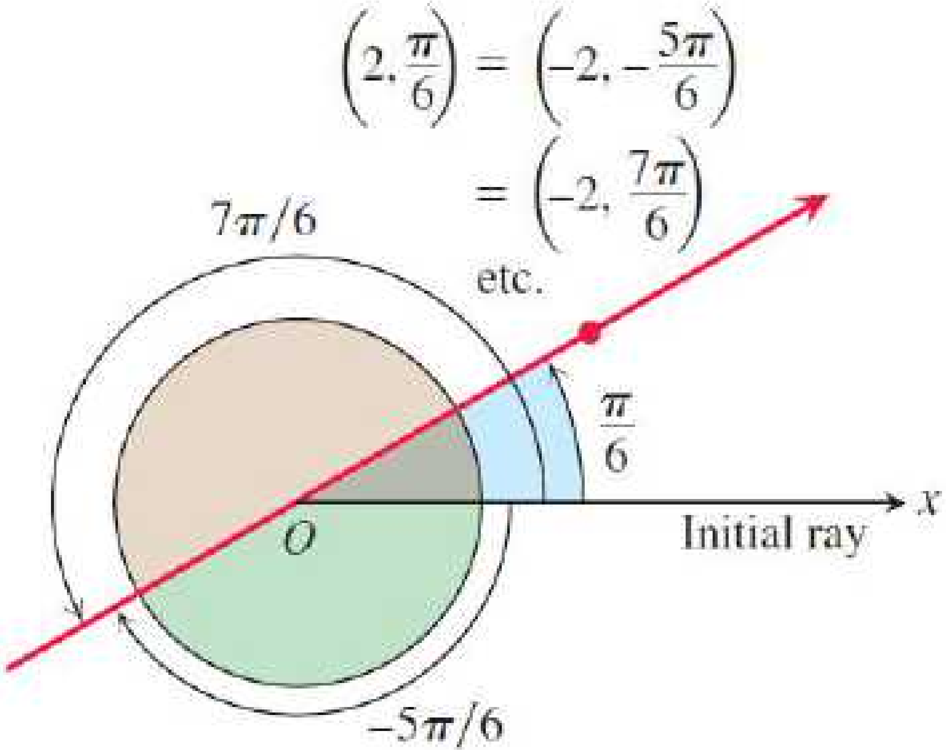
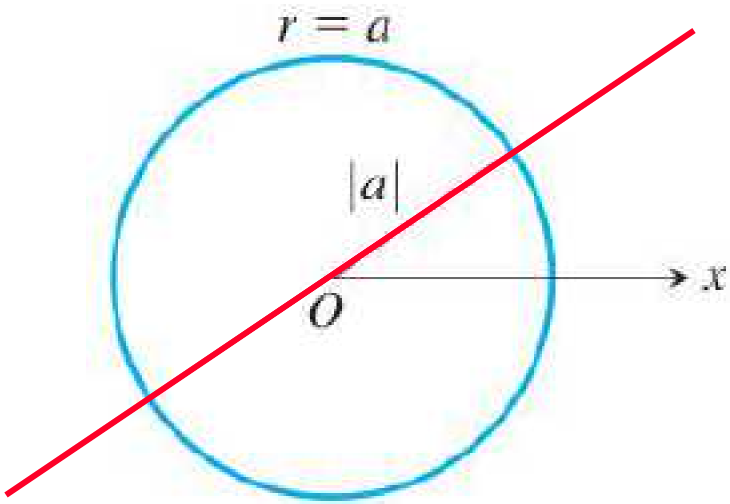


FIGURE 11.22 The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs (Example 1).

Polar Equations and Graphs

$\theta = \theta_0$ and let r vary between $-\infty$ and ∞,



$r = a \neq 0$, θ varies over any interval of length 2π,

EXAMPLE 2 A circle or line can have more than one polar equation.

(a) $r = 1$ and $r = -1$ are equations for the circle of radius 1 centered at O.

(b) $\theta = \pi/6, \theta = 7\pi/6$, and $\theta = -5\pi/6$ are equations for the line in Figure 11.22.

EXAMPLE 3 Graph the sets of points whose polar coordinates satisfy

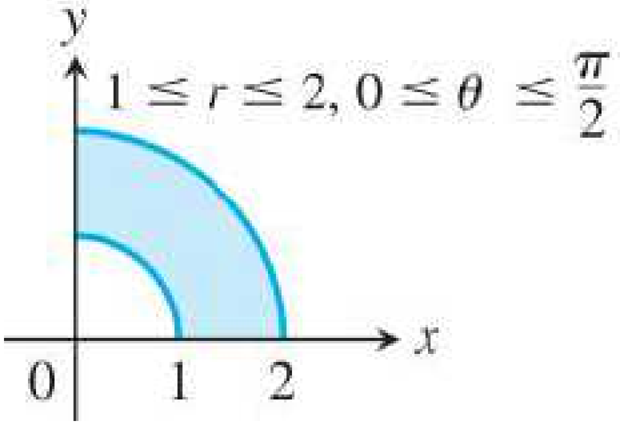
(a) $1 \le r \le 2 \qquad \text{and} \qquad 0 \le \theta \le \dfrac{\pi}{2}$

(b) $-3 \le r \le 2 \qquad \text{and} \qquad \theta = \dfrac{\pi}{4}$

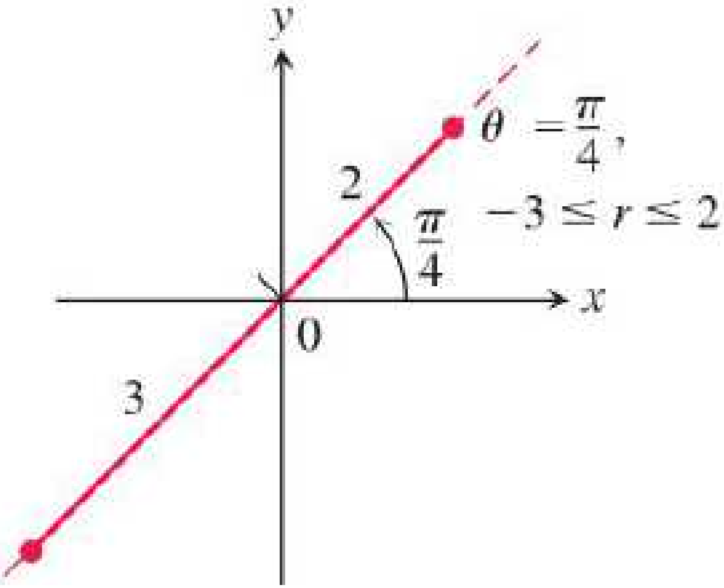
(c) $\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6} \qquad$ (no restriction on r)

Solution

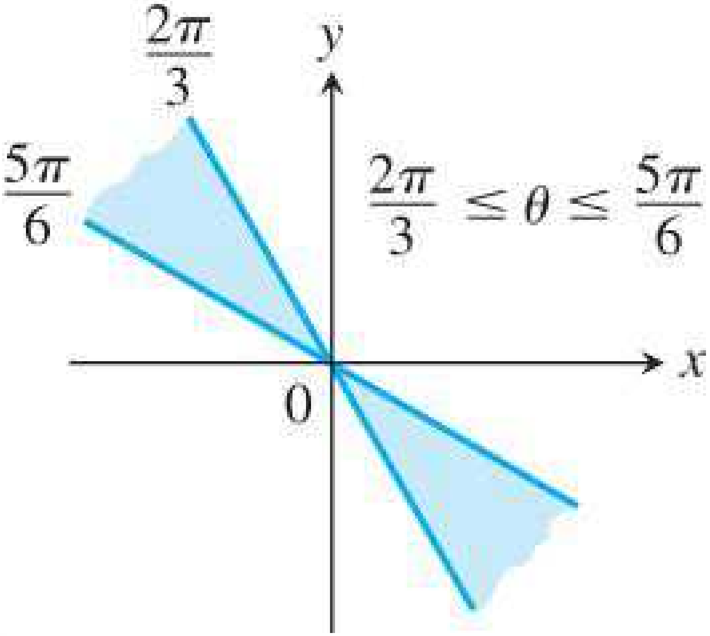
(a)



(b)



(c)

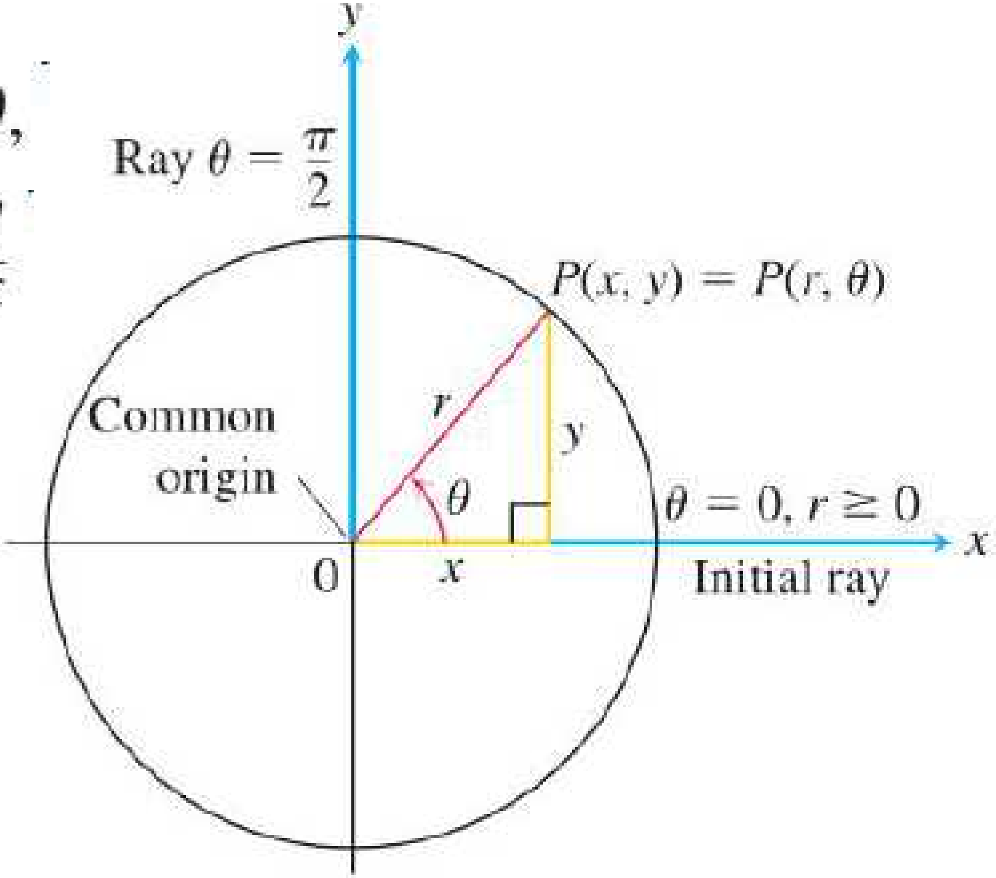


Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x-axis.

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$



there is a unique $\theta \in [0, 2\pi)$

$$\mathbf{0} < \boldsymbol{r} < \infty$$

EXAMPLE 4 Here are some plane curves expressed in terms of both

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

EXAMPLE 5 Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$

Solution

$$x^2 + y^2 - 6y = 0$$

$$r^2 - 6r\sin\theta = 0$$

$$r = 0 \quad \text{or} \quad r - 6\sin\theta = 0$$

$$r = 6\sin\theta$$

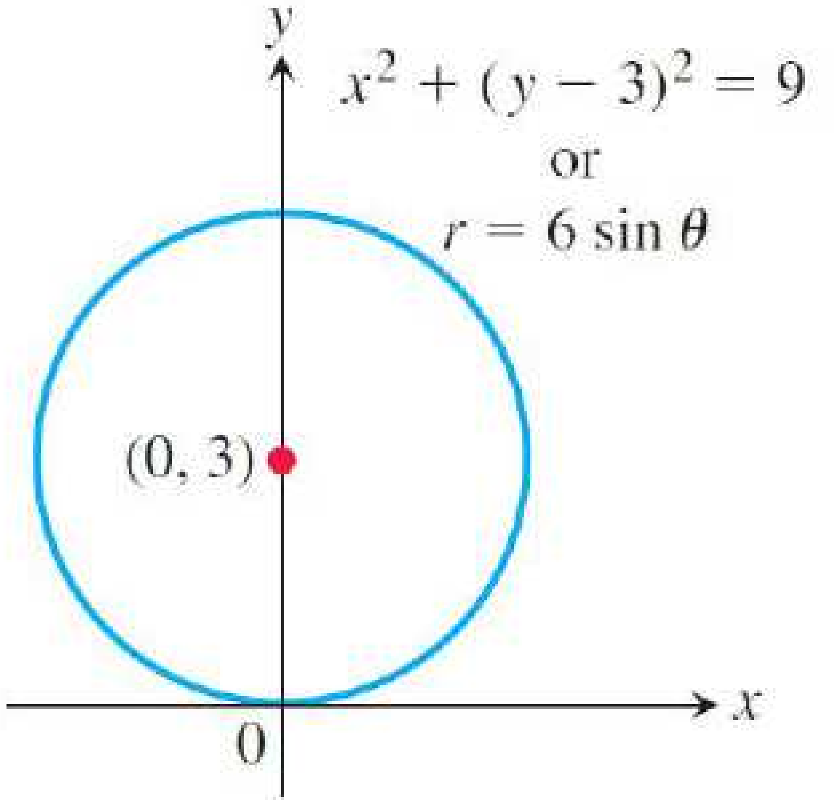


FIGURE 11.26 The circle in Example 5.

EXAMPLE 6

Replace the following polar equations by equivalent Cartesian equations and identify their graphs.

(a) $r \cos \theta = -4$ **(b)** $r^2 = 4r \cos \theta$ **(c)** $r = \dfrac{4}{2 \cos \theta - \sin \theta}$

Solution $r \cos \theta = x$, $r \sin \theta = y$, and $r^2 = x^2 + y^2$.

(a) $x = -4$ **(b)** $x^2 + y^2 = 4x$

(c) $r(2 \cos \theta - \sin \theta) = 4$ $2x - y = 4$

11.4

Graphing Polar Coordinate Equations

极坐标方程的做图

Slope

The slope of a polar curve $r = f(\theta)$ in the xy-plane is still given by dy/dx,

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\dfrac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\dfrac{df}{d\theta}\cos\theta - f(\theta)\sin\theta}$$

Slope of the Curve $r = f(\theta)$ in the Cartesian xy-Plane

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

provided $dx/d\theta \neq 0$ at (r, θ).

例. 求螺线 $r=\theta$ 在 $\theta=\pi$ 处的斜率.

解. $x=r\cos\theta,\ y=r\sin\theta,$

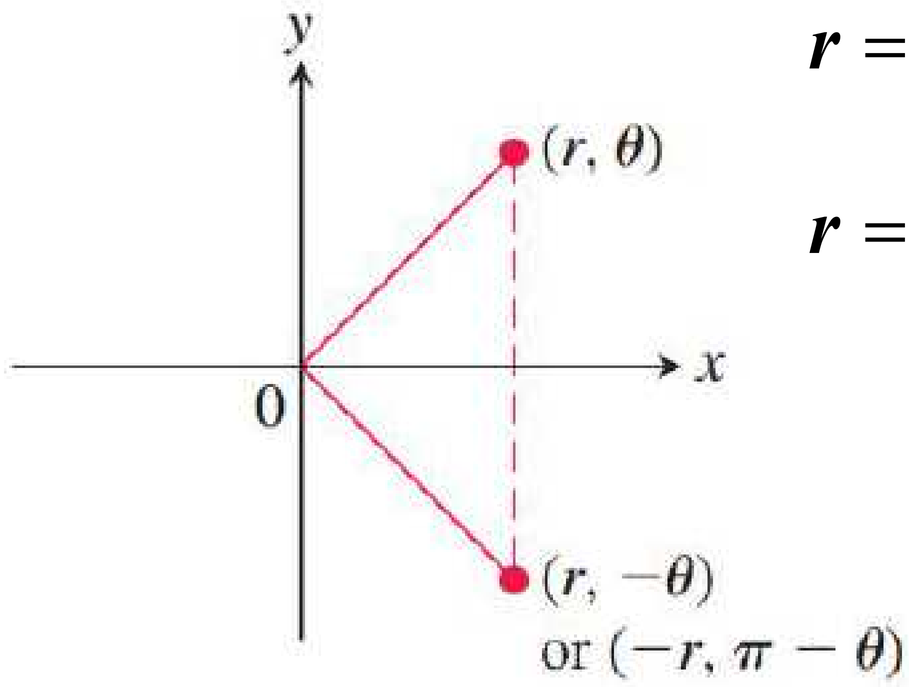
$x=\theta\cos\theta,\ y=\theta\sin\theta,$

$$\frac{dy}{dx}=\frac{\sin\theta+\theta\cos\theta}{\cos\theta-\theta\sin\theta},$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi}=\frac{-\pi}{-1}=\pi.$$

Symmetry

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 11.27a).

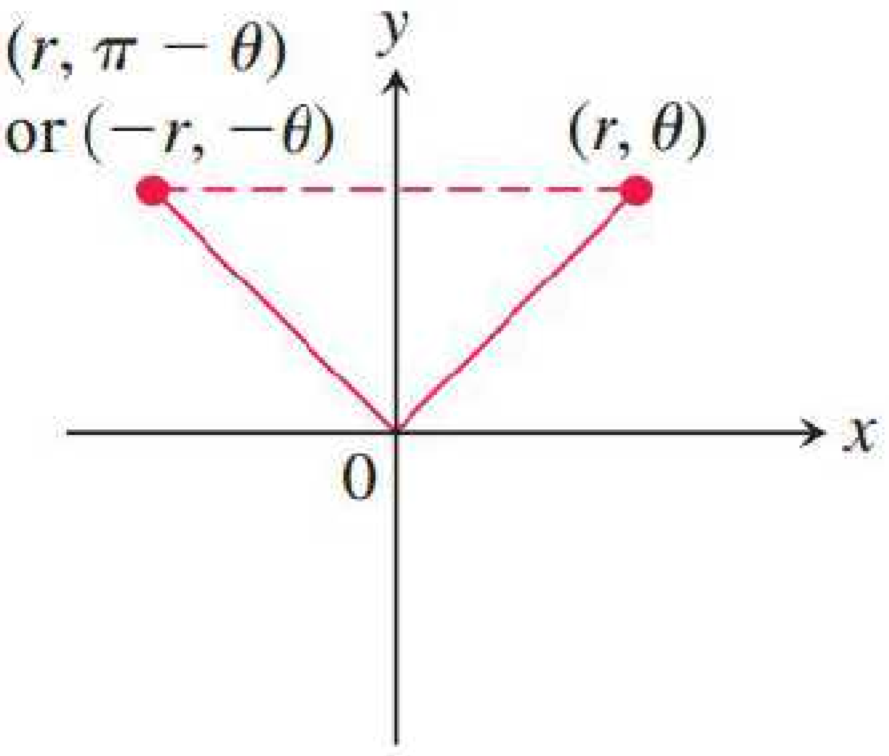


(a) About the x-axis

$$r = \cos\theta$$

$$r = \cos 2\theta$$

2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 11.27b).

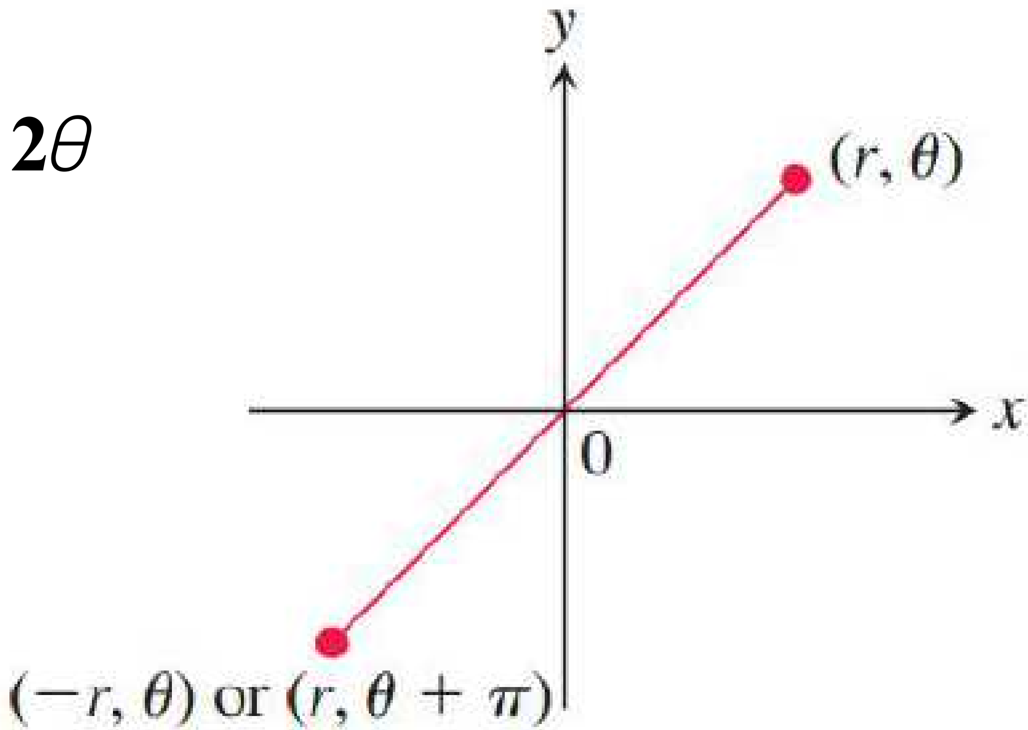


(b) About the y-axis

$$r = \cos 2\theta$$

3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 11.27c).

$$r = \cos 2\theta$$



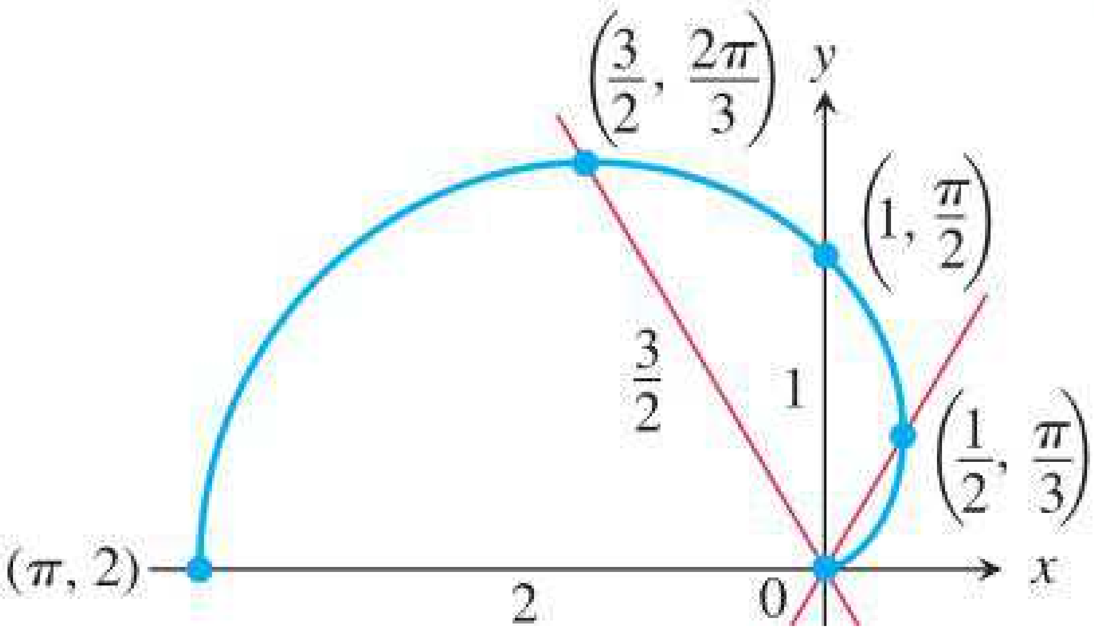
(c) About the origin

EXAMPLE 1 Graph the curve $r = 1 - \cos\theta$ in the Cartesian xy-plane.

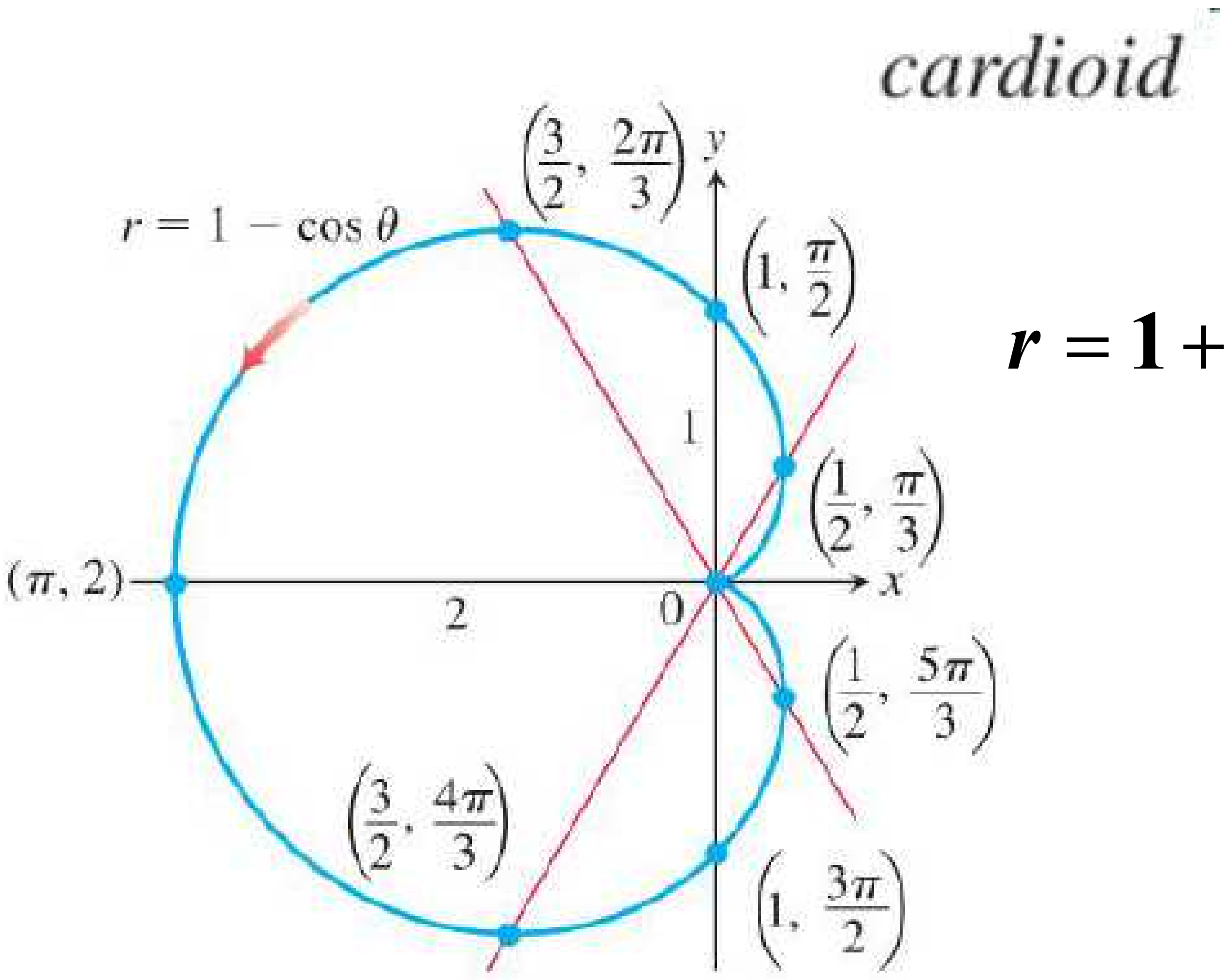
Solution The curve is symmetric about the x-axis

$$(r, \theta) \text{ on the graph } \Rightarrow (r, -\theta) \text{ on the graph.}$$

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2



$$r = 1 + \cos\theta$$



cardioid
$r = 1 - \cos\theta$
$\left(\frac{3}{2}, \frac{2\pi}{3}\right)$
y
$\left(1, \frac{\pi}{2}\right)$
1
$\left(\frac{1}{2}, \frac{\pi}{3}\right)$
$(\pi, 2)$
2
0
x
$\left(\frac{1}{2}, \frac{5\pi}{3}\right)$
$\left(\frac{3}{2}, \frac{4\pi}{3}\right)$
$\left(1, \frac{3\pi}{2}\right)$

EXAMPLE 2 Graph the curve $r^2 = 4\cos\theta$ in the Cartesian xy-plane.

Solution The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$,

θ from $-\pi/2$ to $\pi/2$.

The curve is symmetric about the x-axis

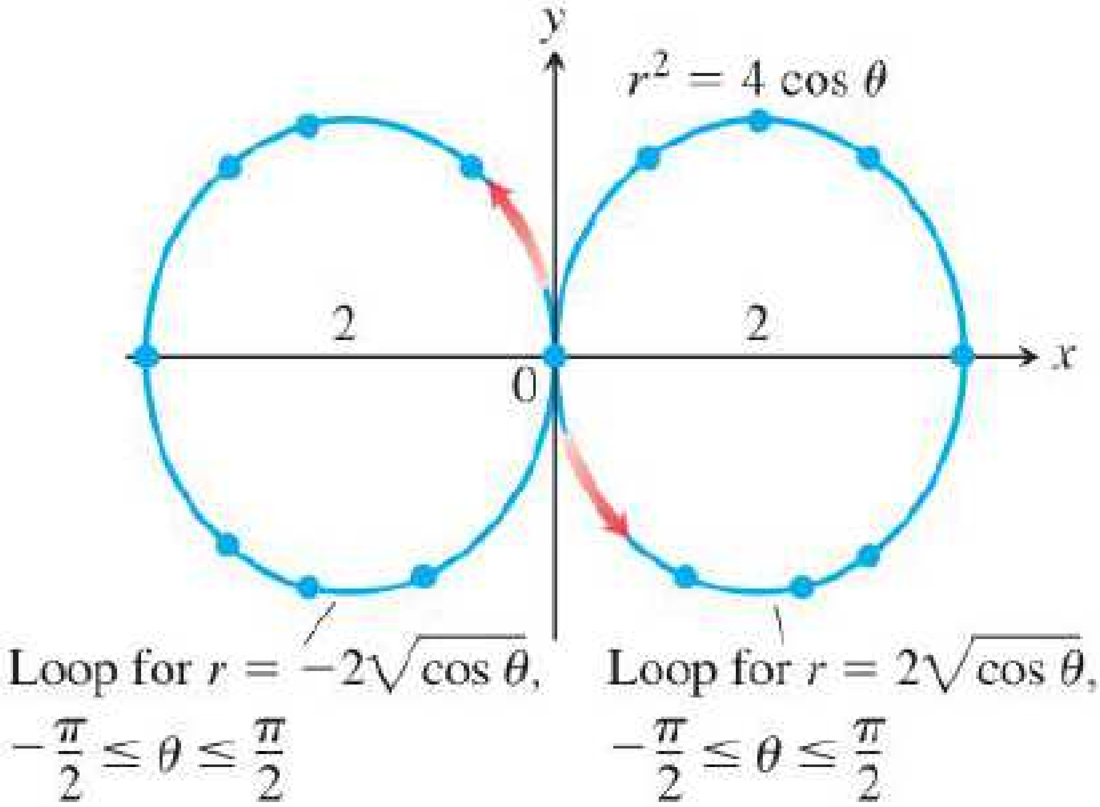
$$(r, \theta) \text{ on the graph} \Rightarrow (r, -\theta) \text{ on the graph.}$$

The curve is also symmetric about the origin

$$(r, \theta) \text{ on the graph} \Rightarrow (-r, \theta) \text{ on the graph.}$$

$$r = \pm 2\sqrt{\cos\theta}.$$

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0



11.5

Areas and Lengths in Polar Coordinates

极坐标下计算面积和长度

Area in the Plane

The region is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. n nonoverlapping fan-shaped circular sectors based on a partition P of angle TOS.

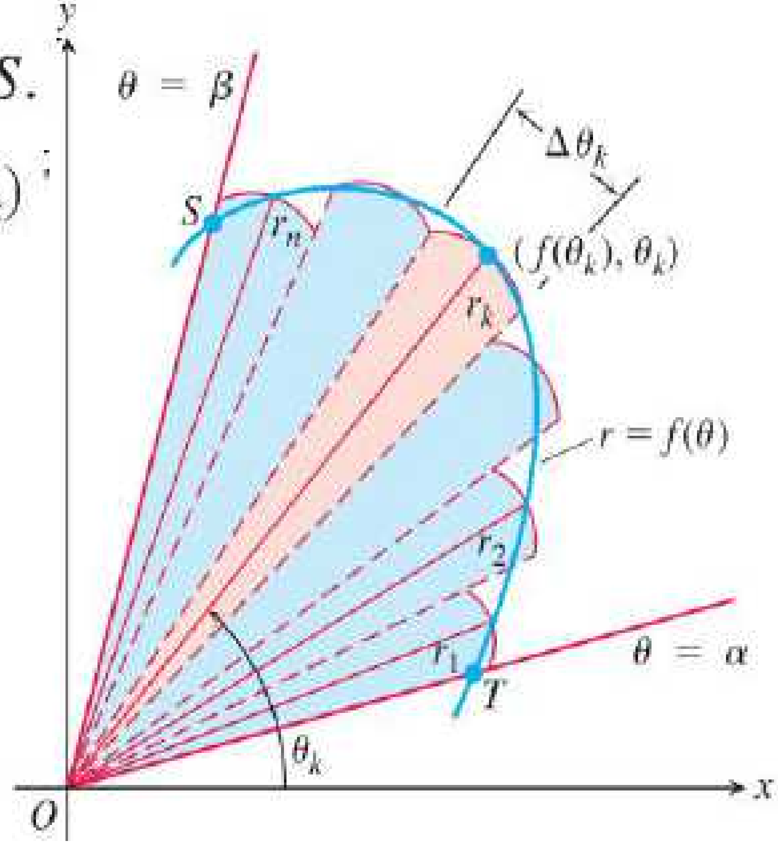
The typical sector has radius $r_k = f(\theta_k)$

$$A_k = \frac{1}{2} r_k^{\,2} \, \Delta\theta_k = \frac{1}{2} \left(f(\theta_k) \right)^2 \Delta\theta_k.$$

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2} \left(f(\theta_k) \right)^2 \Delta\theta_k.$$

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \frac{1}{2} \left(f(\theta_k) \right)^2 \Delta\theta_k$$

$$= \int_{\alpha}^{\beta} \frac{1}{2} \left(f(\theta) \right)^2 d\theta.$$



Area of the Fan-Shaped Region Between the Origin and the Curve

$r = f(\theta), \quad \alpha \leq \theta \leq \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

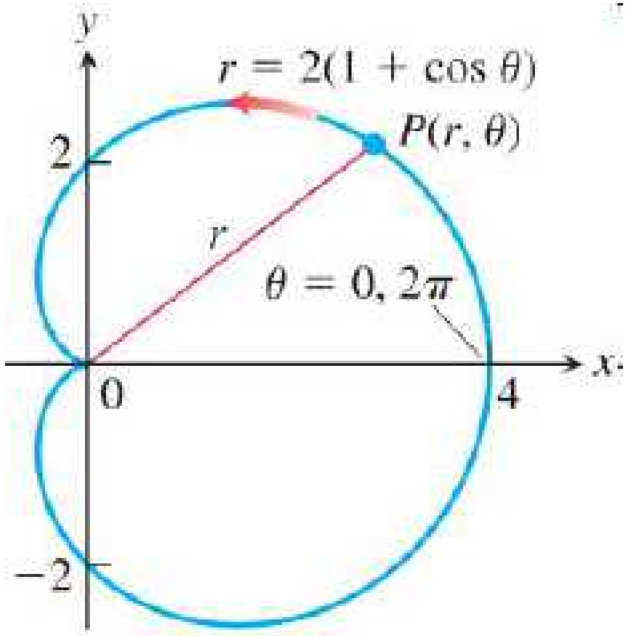
the **area differential** $\quad dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \left(f(\theta) \right)^2 d\theta.$

EXAMPLE 1

Find the area of the region in the xy-plane enclosed by the cardioid

$r = 2(1 + \cos \theta).$

Solution

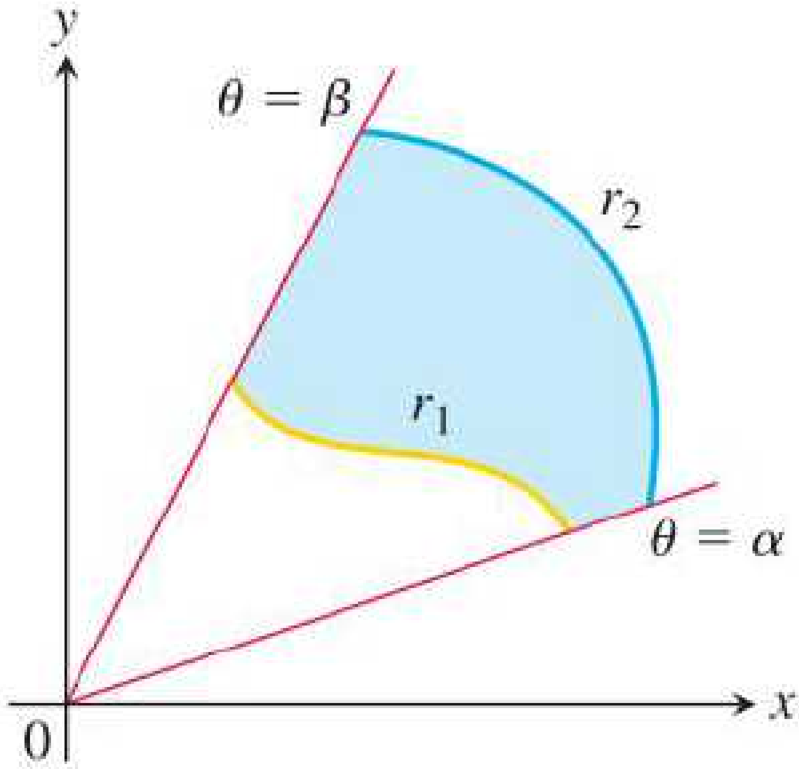


$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta$$

$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta = 6\pi.$$

Area of the Region $0 \leq r_1(\theta) \leq r \leq r_2(\theta), \alpha \leq \theta \leq \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} \left(r_2^2 - r_1^2 \right) d\theta$$



EXAMPLE 2

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

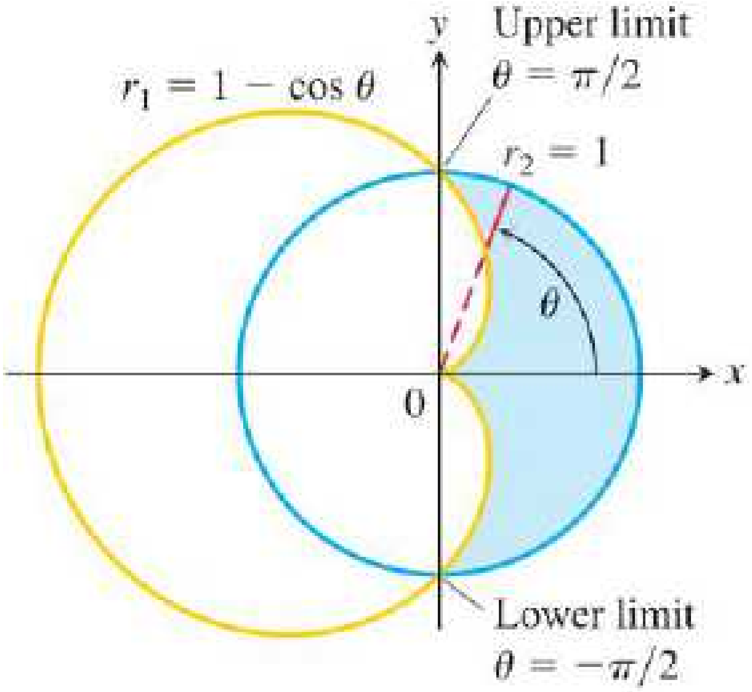
Solution

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^{\,2} - r_1^{\,2}\right) d\theta$$

$$= 2\int_{0}^{\pi/2} \frac{1}{2}\left(r_2^{\,2} - r_1^{\,2}\right) d\theta$$

$$= \int_{0}^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta$$

$$= 2 - \frac{\pi}{4}.$$



Length of a Polar Curve

$r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$

$P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β,

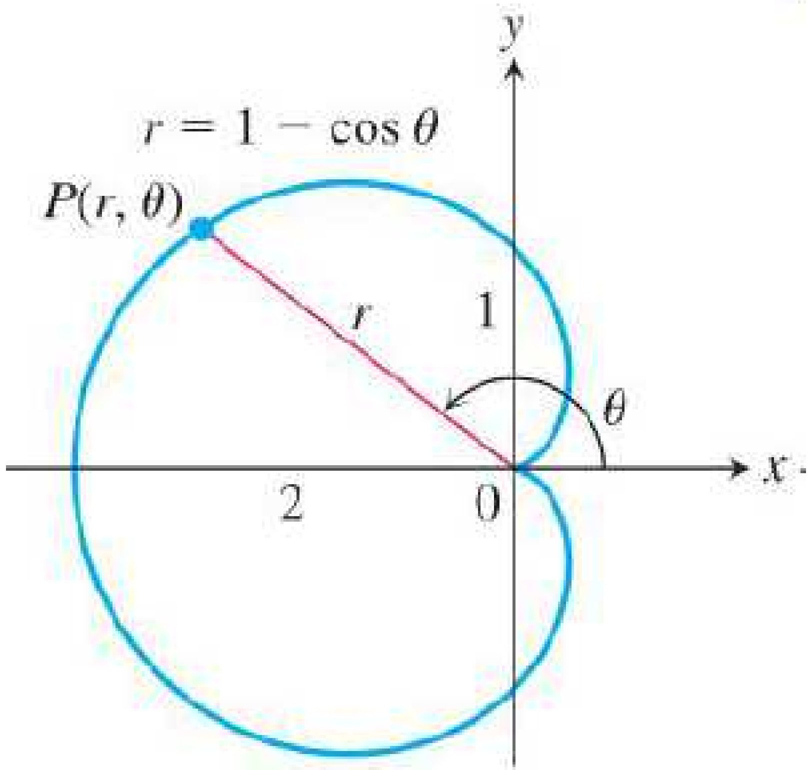
$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

EXAMPLE 3 Find the length of the cardioid $r = 1 - \cos\theta$.

Solution

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \int_{0}^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta = 8.$$



y
r = 1 − cos θ
P(r, θ)
r
1
θ
x
2
0

11.6

Conic Sections

圆锥曲线

11.7

Conics in Polar Coordinates