CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

Dr. QI WANG

Department of Computer Science and Engineering
Office: Room413, CoE South Tower
Email: wangqi@sustech.edu.cn

Euler's Formula

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- **Corollary 1** If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

  Corollary 2 If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.

  Corollary 3 In a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

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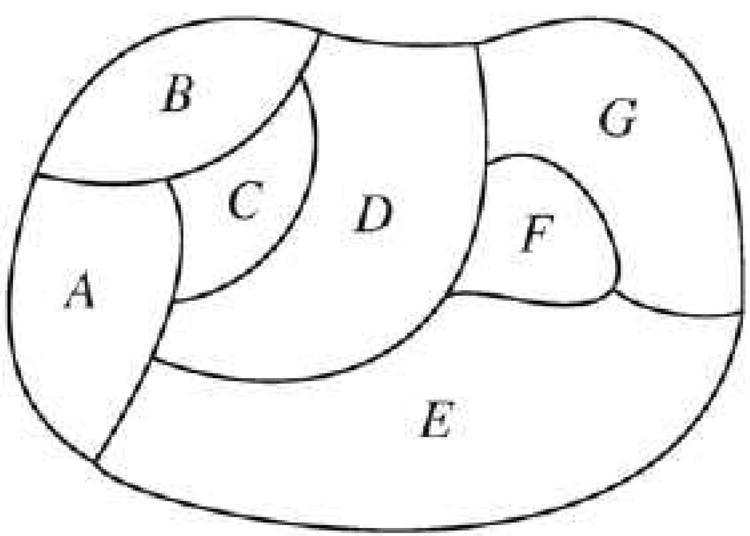

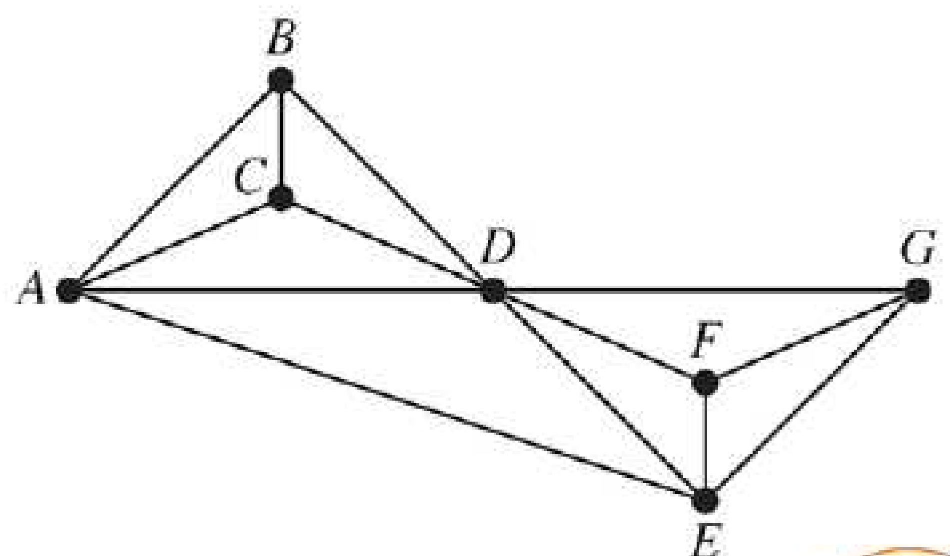

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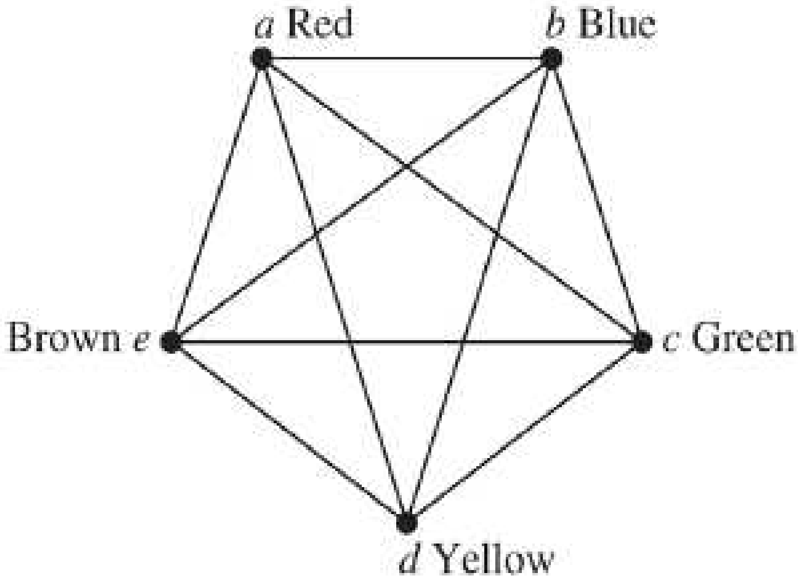

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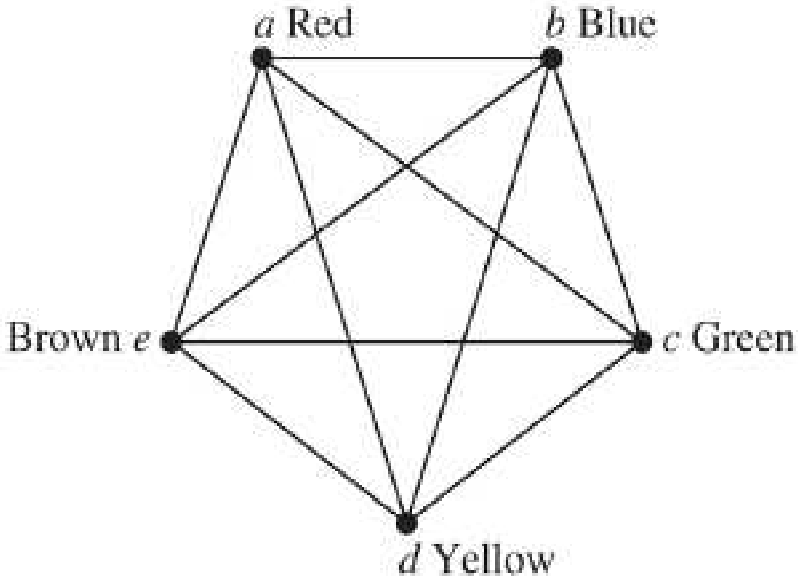

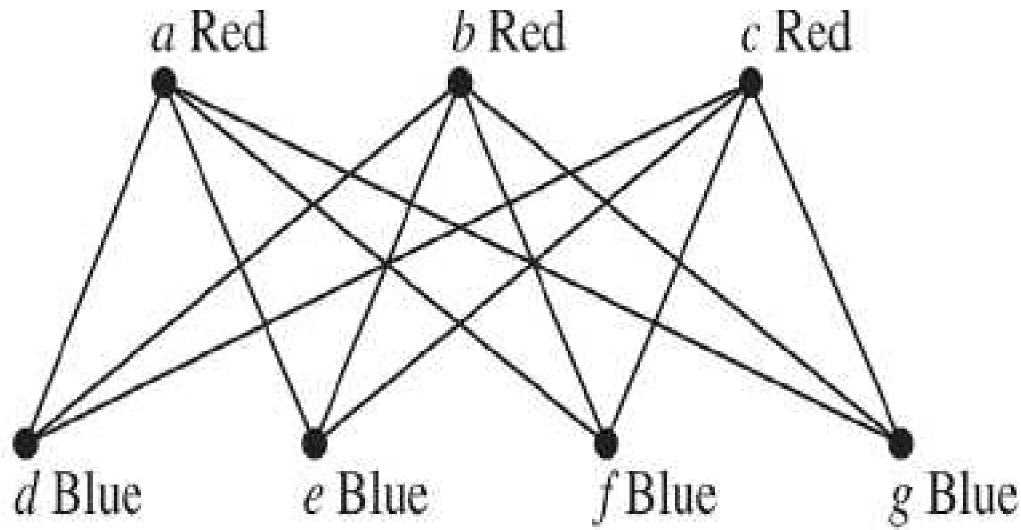

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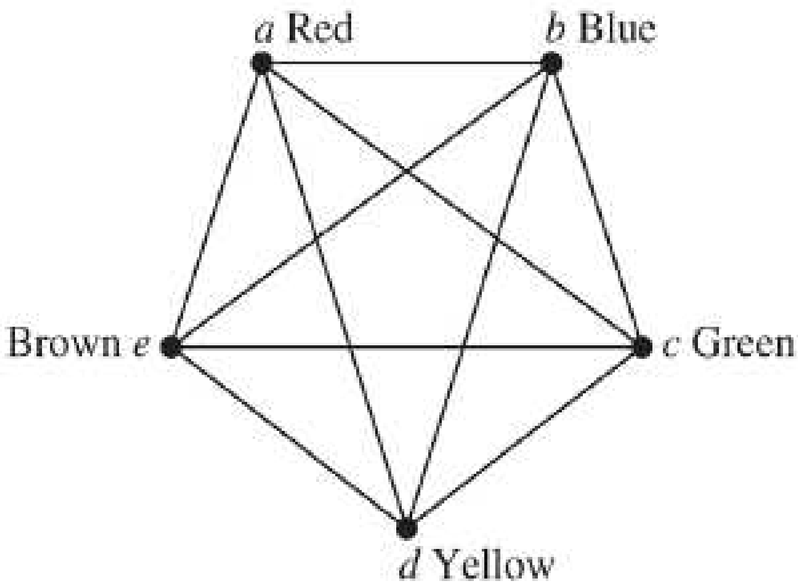

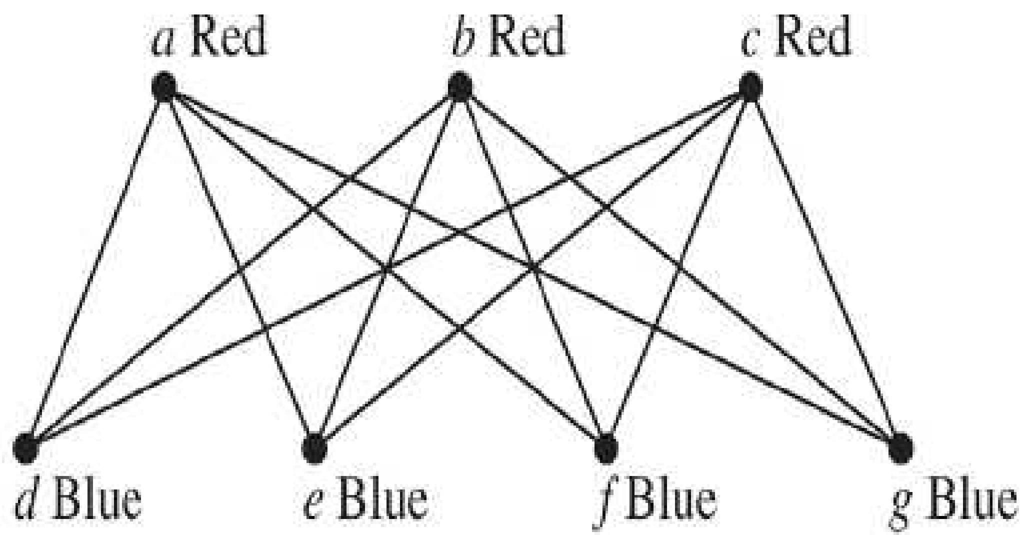

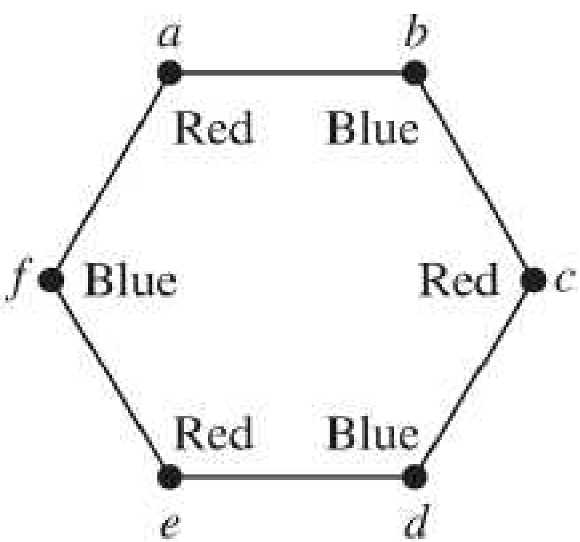

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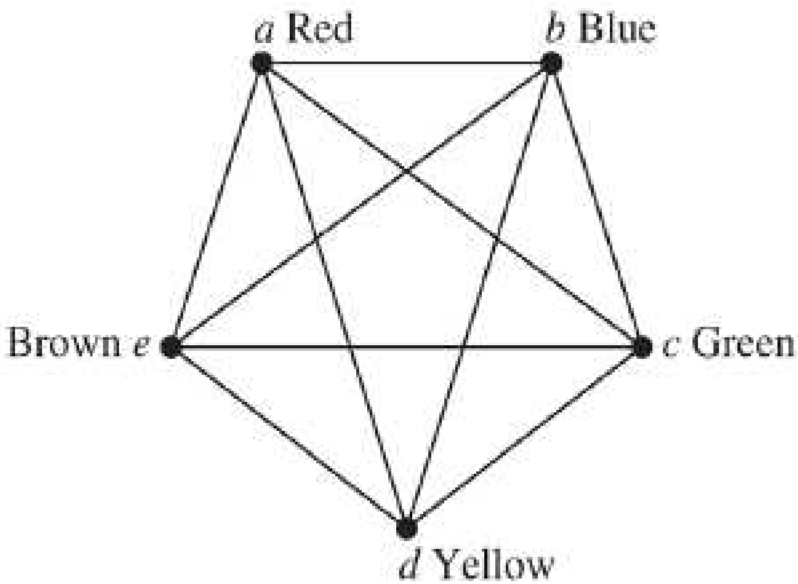
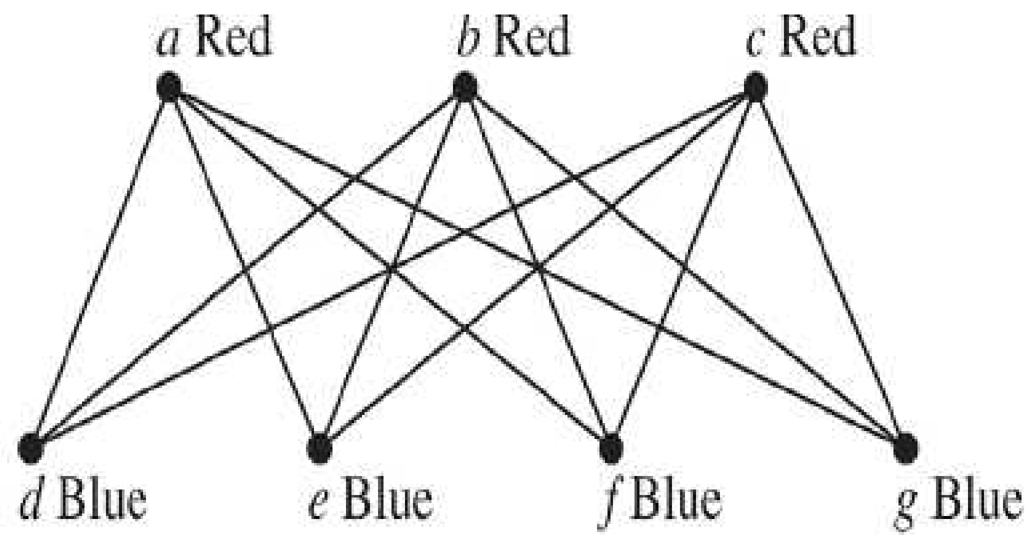
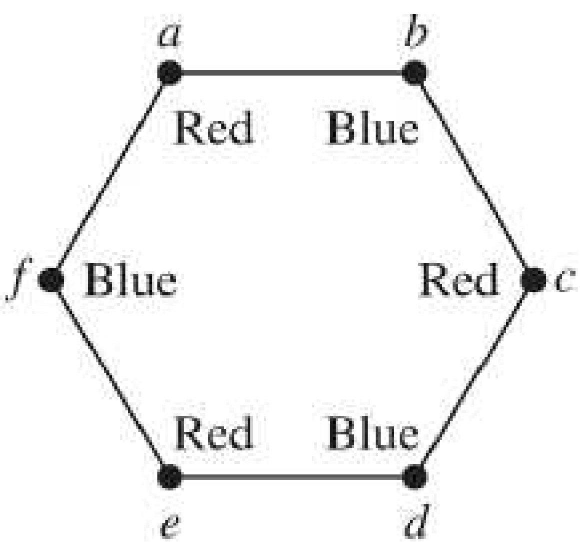
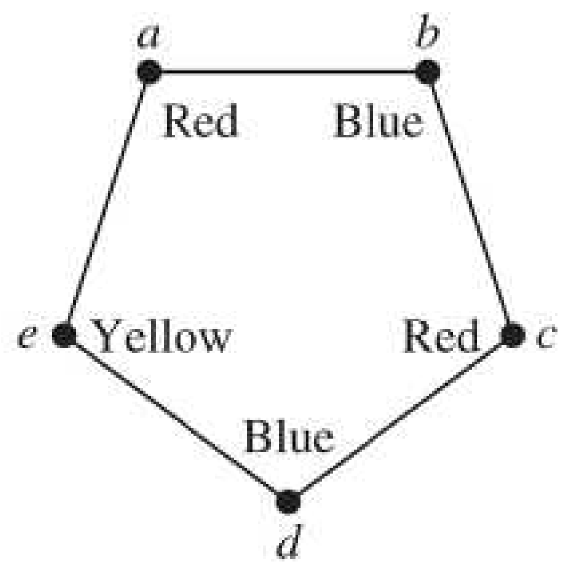




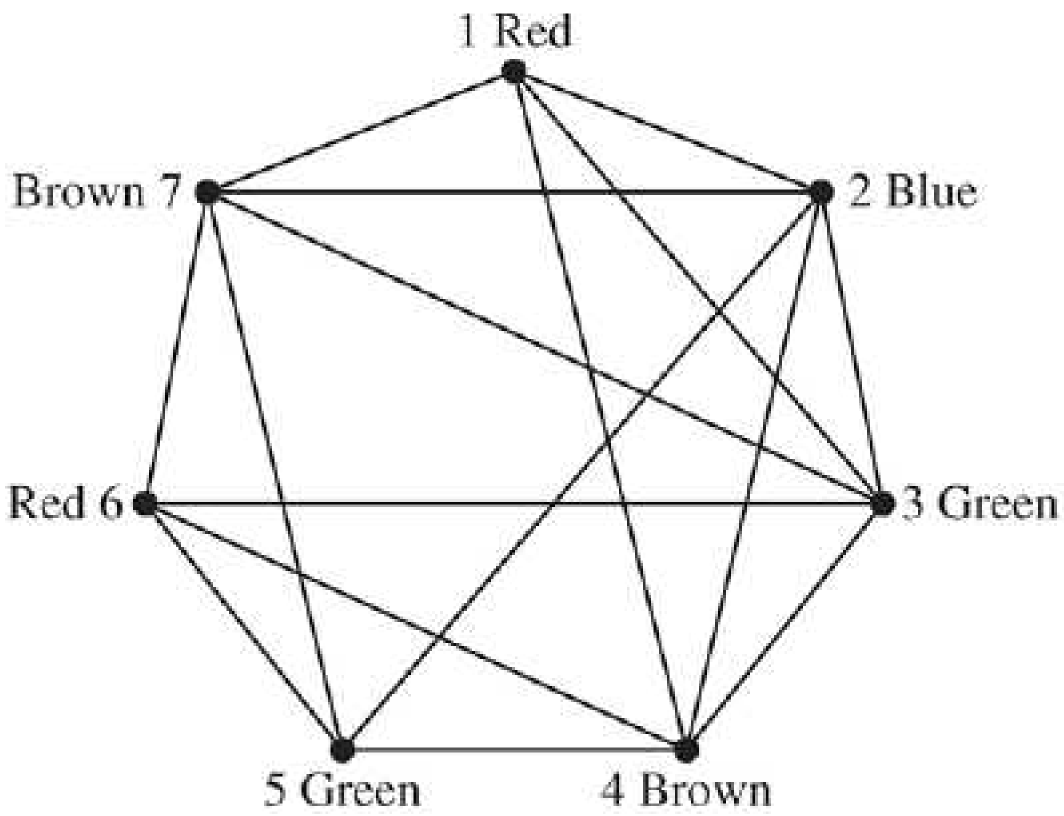




Applications of Graph Coloring

- Scheduling Final Exams

Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

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- Channel Assignments

  Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?

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Graph Coloring $\in$ NPC

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  In a *Zero Knowledge Proof*, Alice will prove to Bob that a statement P is true. Bob will be completely convinced that P is true, but will not learn anything as a result of this process. That is, Bob will gain zero knowledge.

Applications of ZKPs

- *Protocol design*. A *protocol* is an algorithm for interactive parties to achieve a certain goal. However, in crypto, we often want to design protocols that should achieve security even when one of the parties is "cheating". Alice can prove in *zero knowledge* that she followed the instructions.

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Proofs that Yield Nothing But their Validity
and a Methodology of Cryptographic Protocol Design

(Extended Abstract)

Oded Goldreich
Dept. of Computer Sc.
Technion
Haifa, Israel

Silvio Micali
Lab. for Computer Sc.
MIT
Cambridge, MA 02139

Avi Wigderson
Inst. of Math. and CS
Hebrew University
Jerusalem, Israel

Applications of ZKPs

- *Identification scheme*. How should Alice prove to Bob that she is who she claimed to be? For example, how to design a control access system to the CSE dept.?

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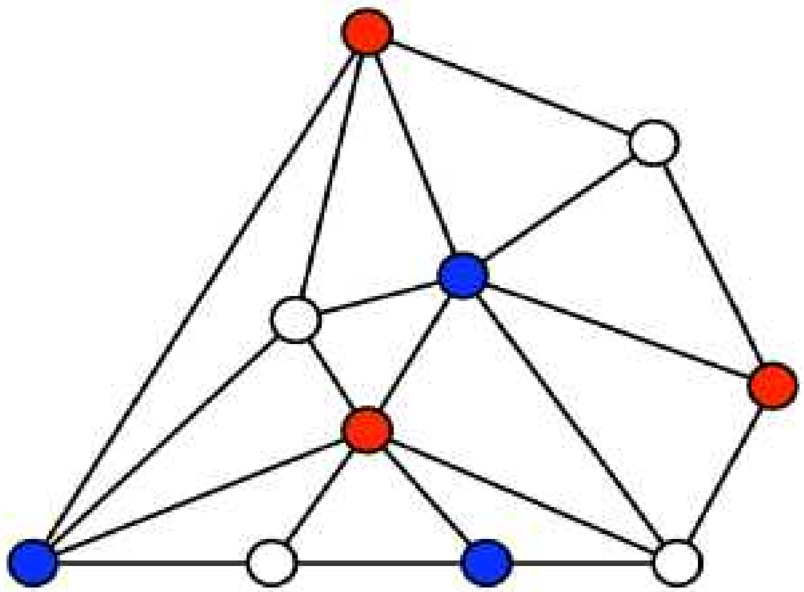
  Ideas using ZKPs:
  - Let the box contain an *instance* of a hard problem.
  - Give the authorized people the *solution* to the instance.
  - The authorized people will *prove* to the box that they know the solution in zero knowledge.

An Example



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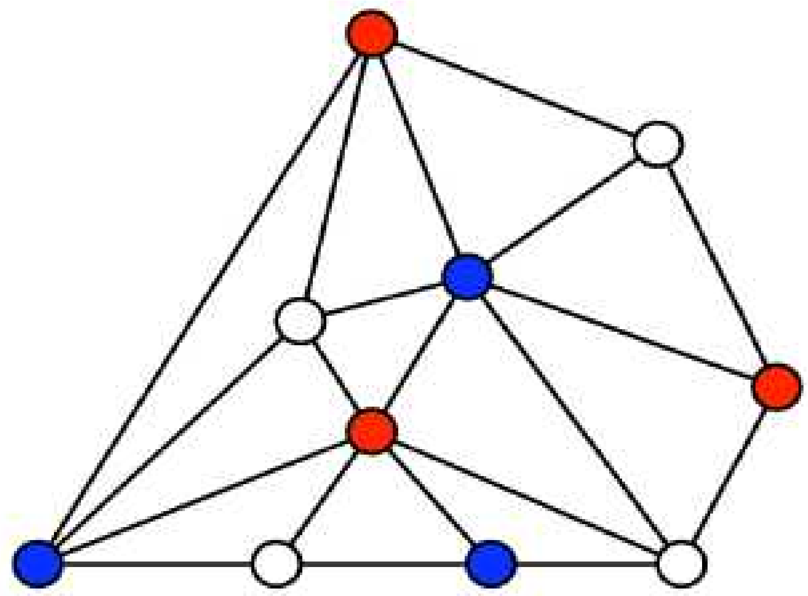

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  - can impress your friends
  - useful for identification

An Example

- How can Alice convince Bob that she can 3-color the graph without
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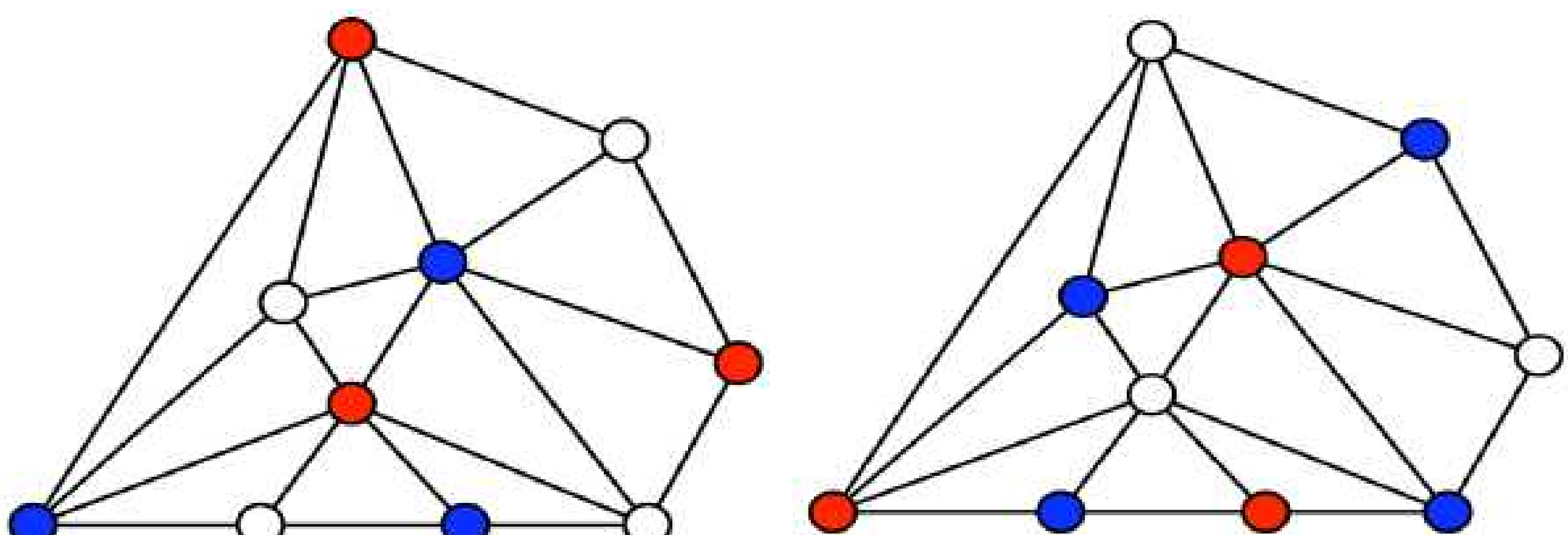
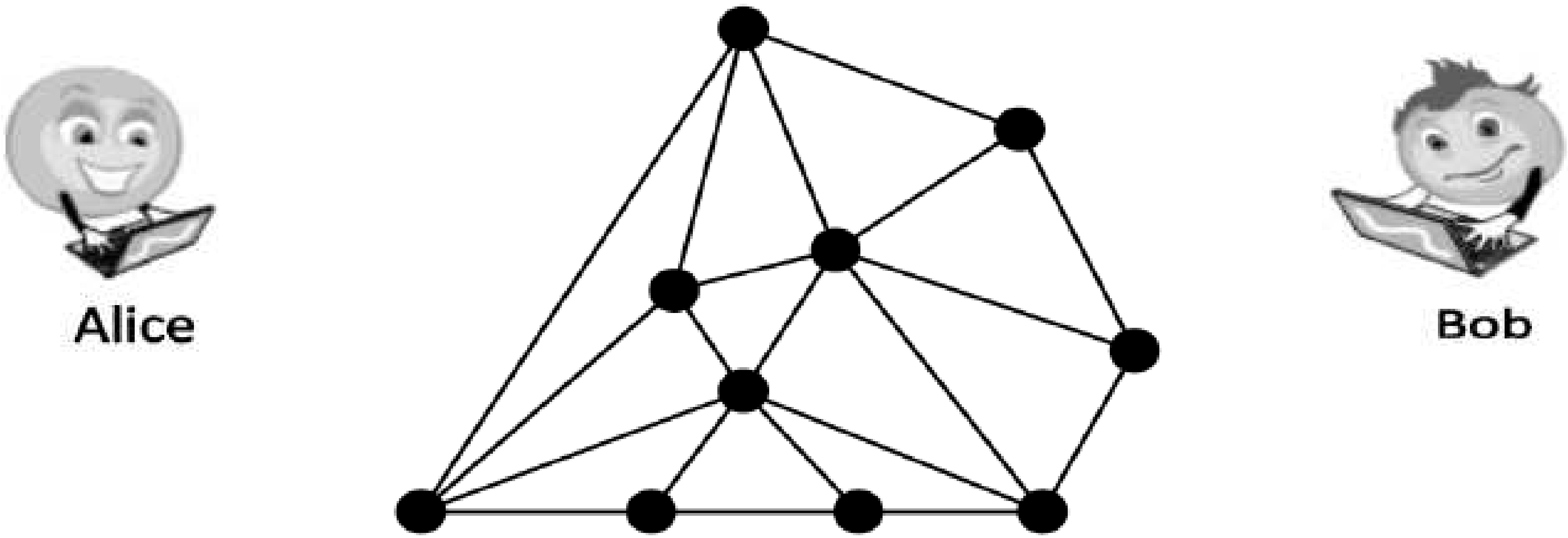
Alice may permute the vertex colors.



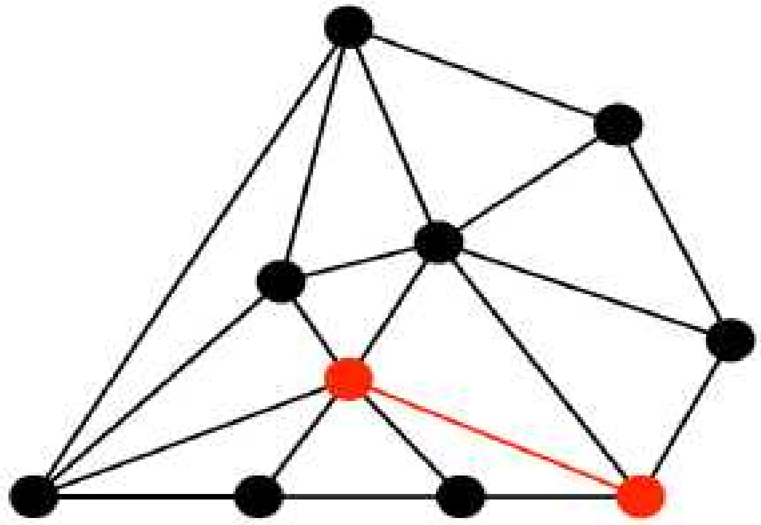
An Example

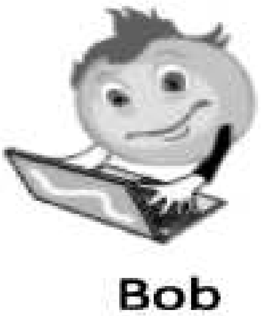
- Alice then encrypts all vertex colors (one key per vertex), and sends the graph to Bob.



An Example

- Bob picks an edge at random.





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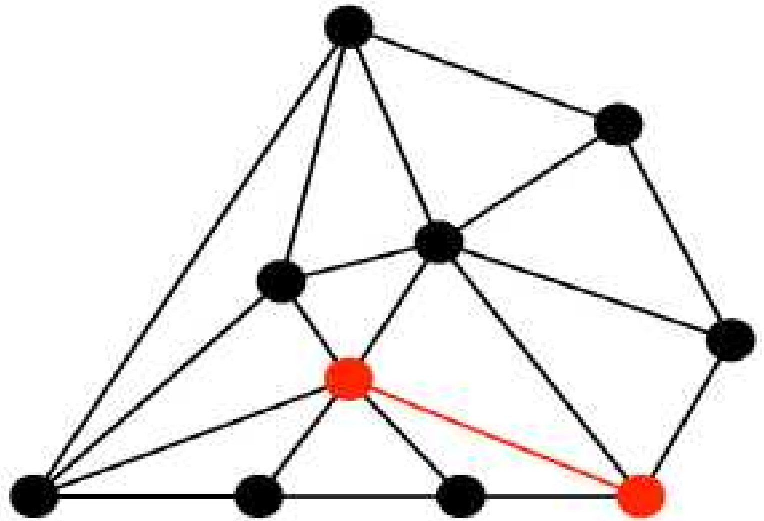
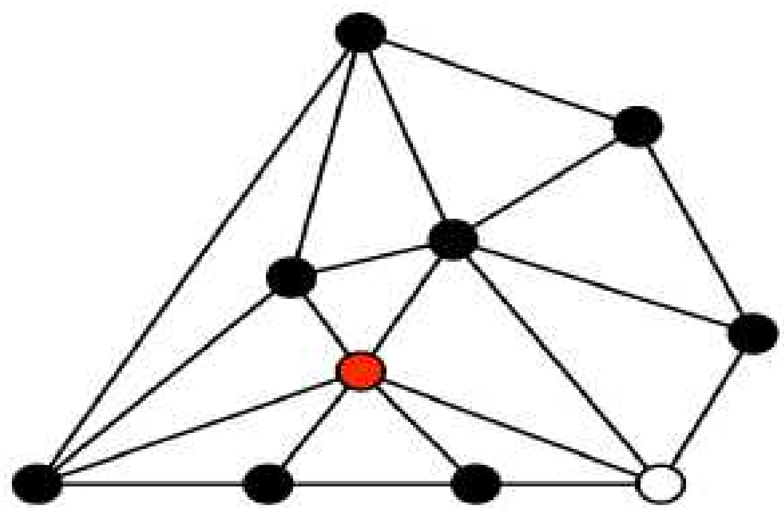


Alice reveals colors of those two keys.



An Example

- Repeat as much as needed:
  - Alice permutes graph coloring
  - Alice encrypts all vertices with distinct keys
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After k repetitions, the probability she fools Bob is $(1 - \frac{1}{|E|})^k$.

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- What does Bob see?
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  Because Bob could have generated those keys and colors by himself, he learns nothing from the graph coloring.

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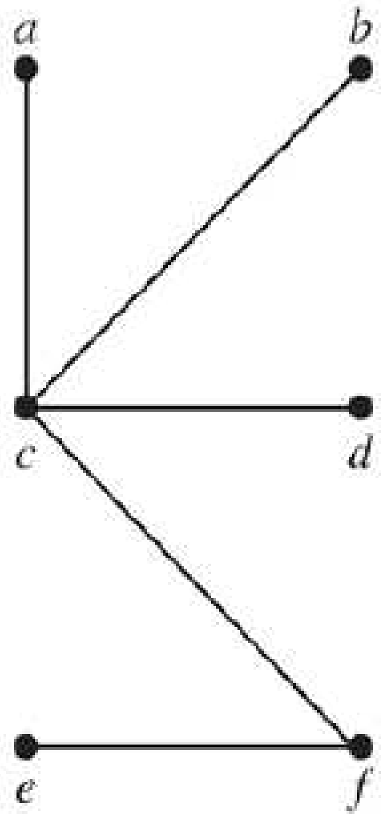

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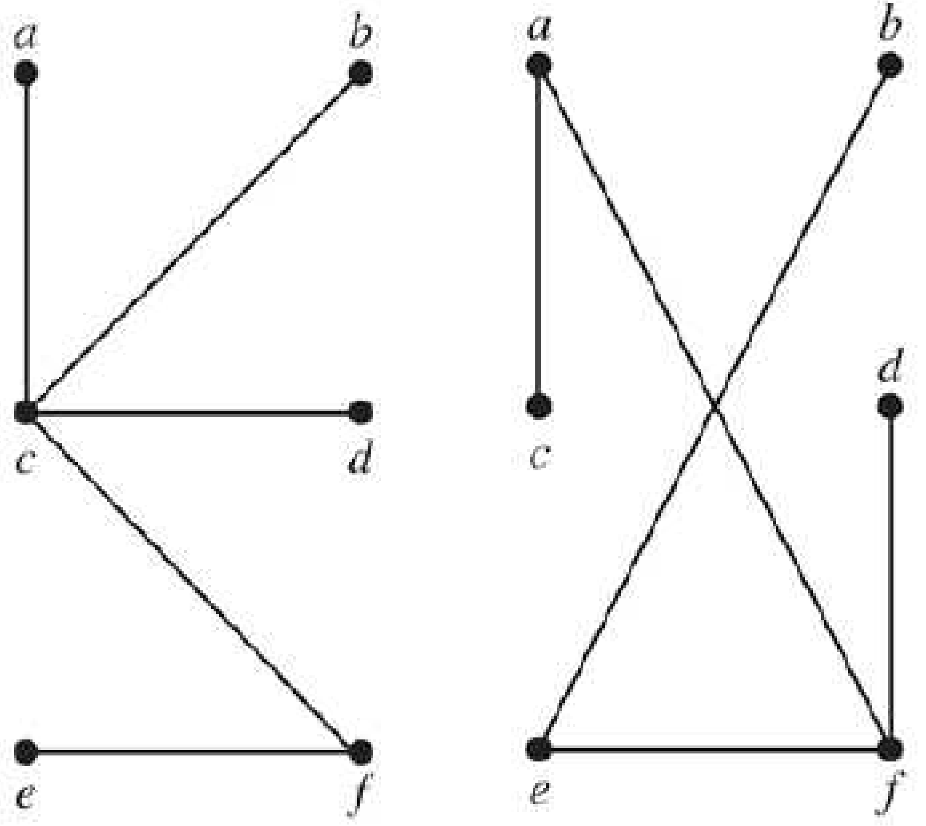

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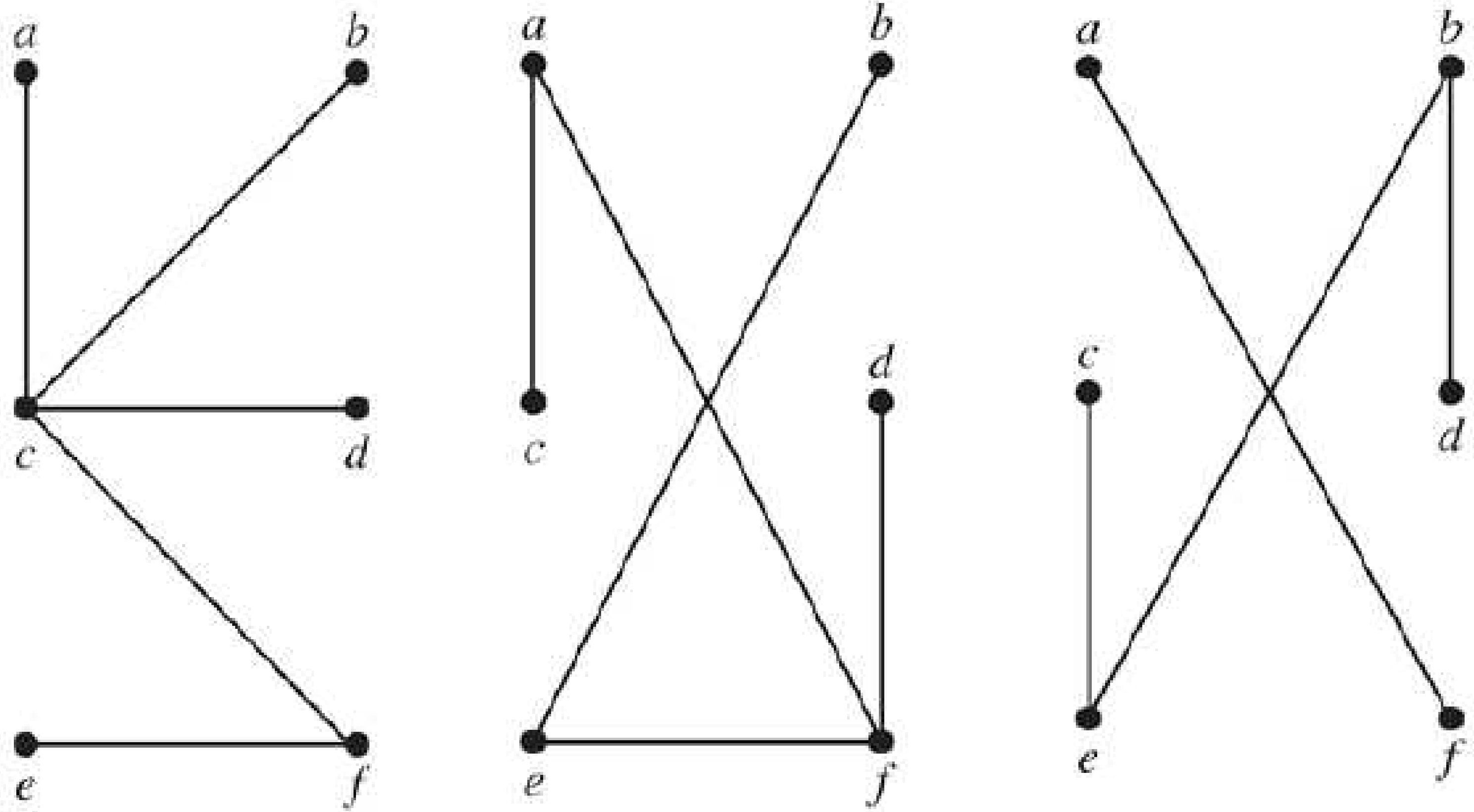

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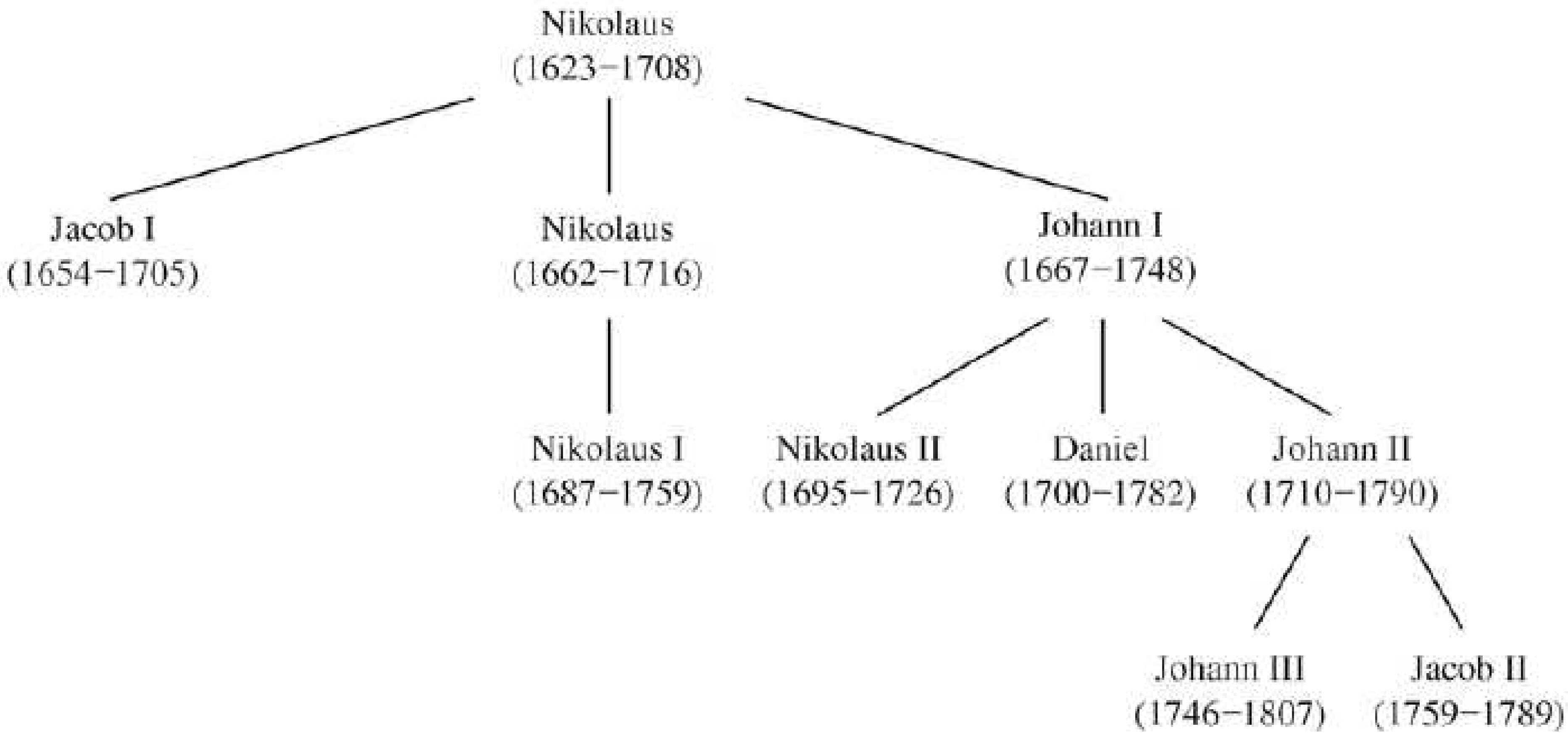

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Two properties of tree: connected, no circuit

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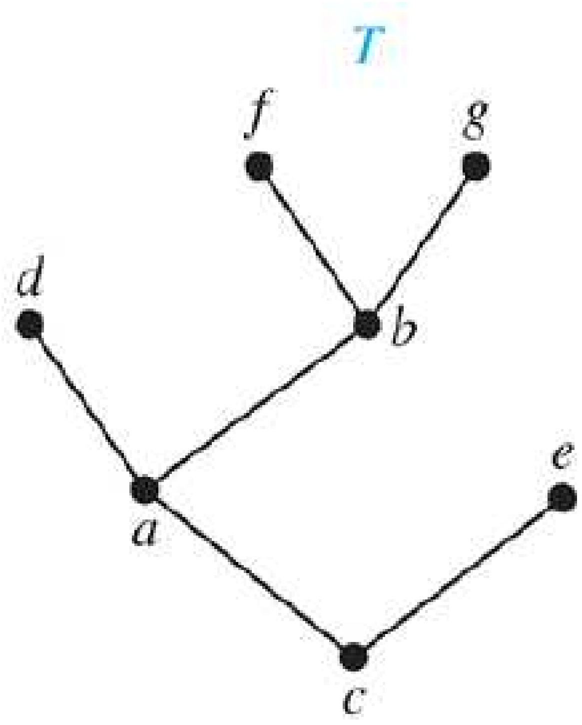

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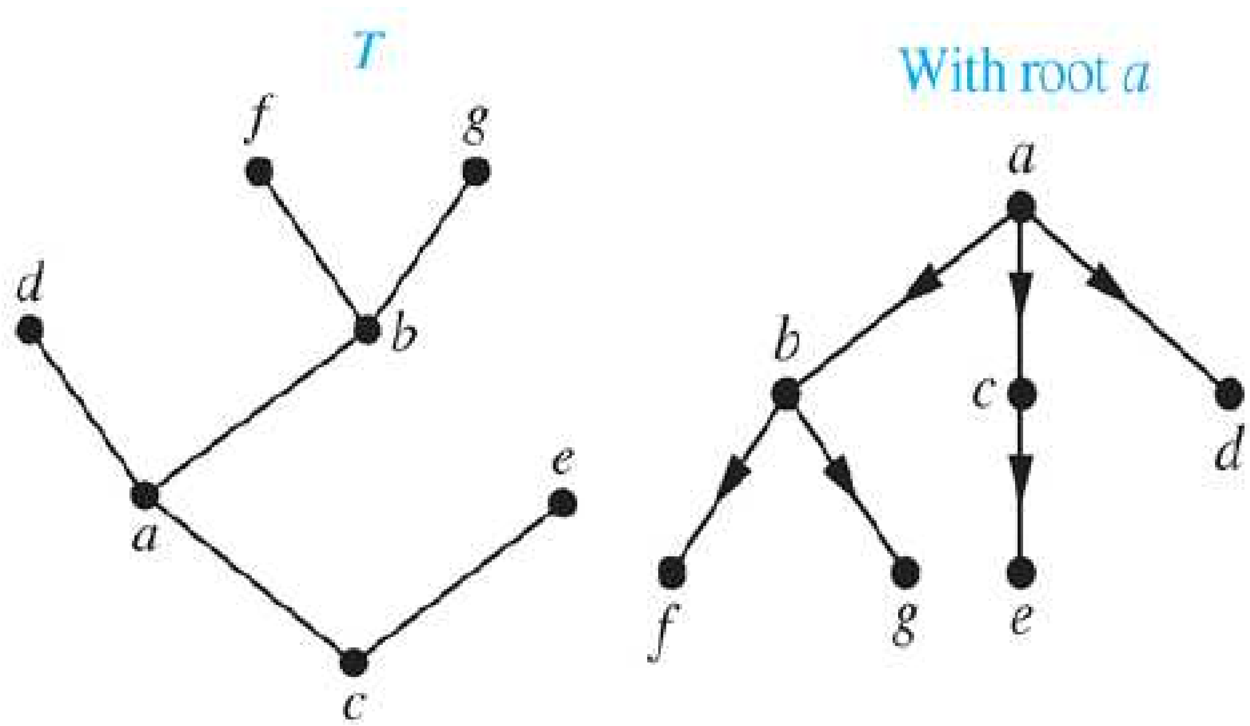

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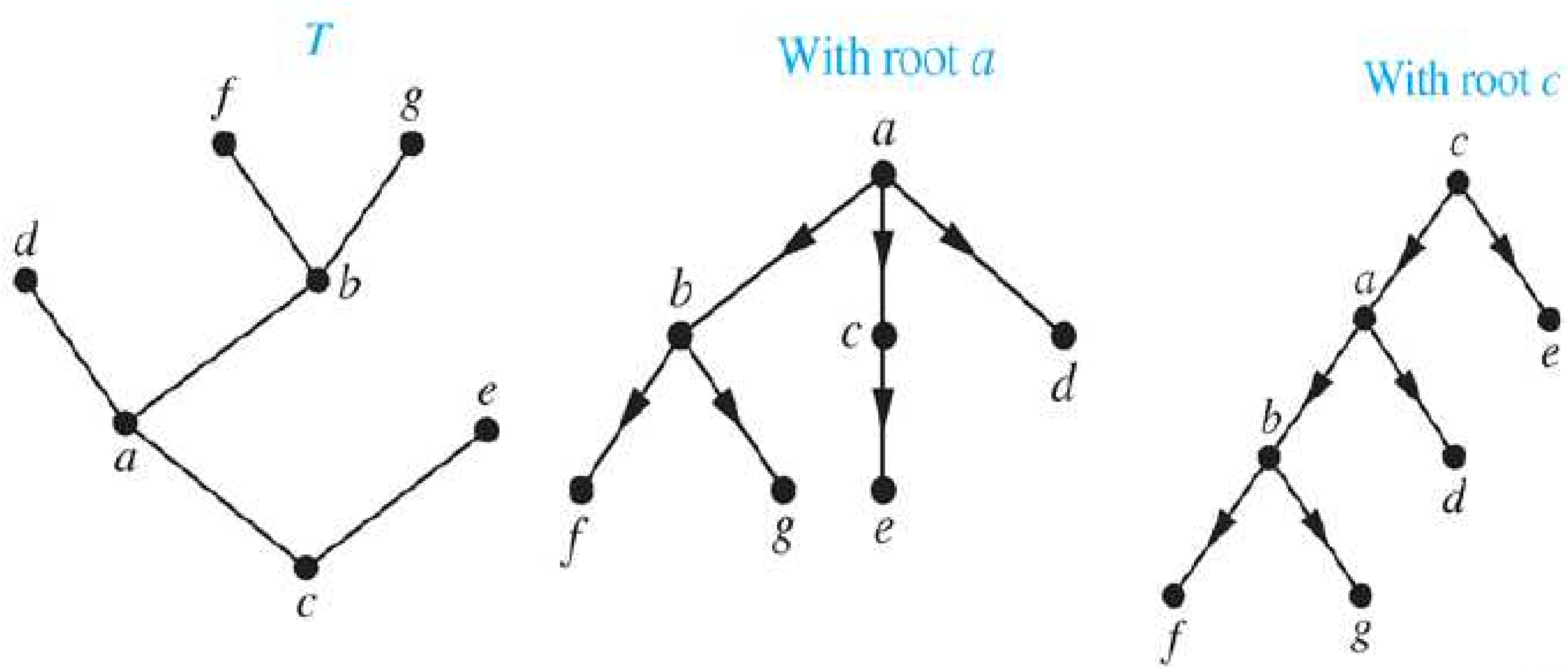

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  subtree with a as its root: consists of a and its descendants and all edges incident to these descendants

m-Ary Trees

- **Definition** A rooted tree is called an *m-ary tree* if every internal vertex has no more than m children. The tree is called a *full m-ary tree* if every internal vertex has exactly m children. In particular, an m-ary tree with $m = 2$ is called a *binary tree*.

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  left subtree, *right subtree*

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  using $n = mi + 1$ and $n = i + \ell$

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  Definition A rooted m-ary tree of height h is *balanced* if all leaves are at levels h or $h-1$. (differ no greater than 1)

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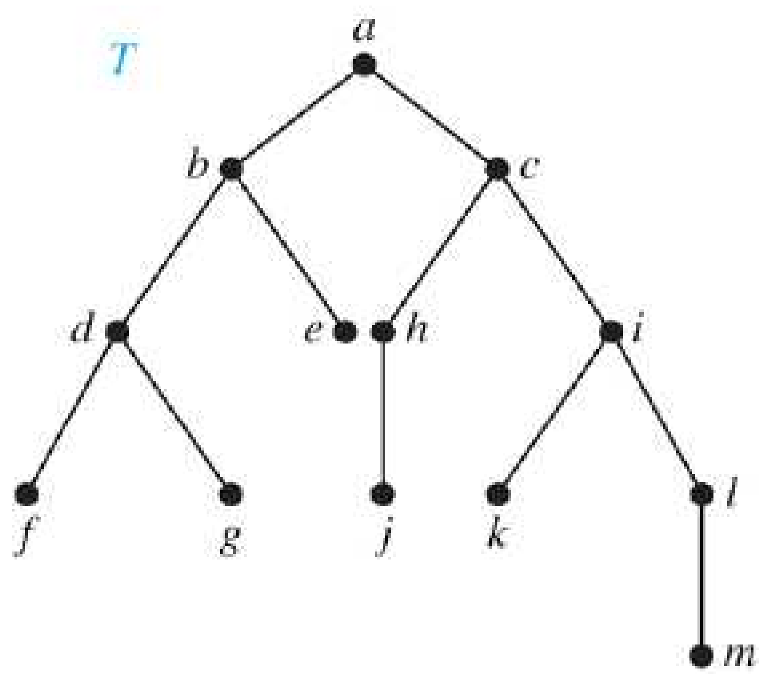

Tree Traversal

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  The three most commonly used traversals are *preorder traversal*, *inorder traversal*, *postorder traversal*.

Preorder Traversal

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *preorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *preorder traversal* begins by visiting r, and continues by traversing T_1 in preorder, then T_2 in preorder, and so on, until T_n is traversed in preorder.

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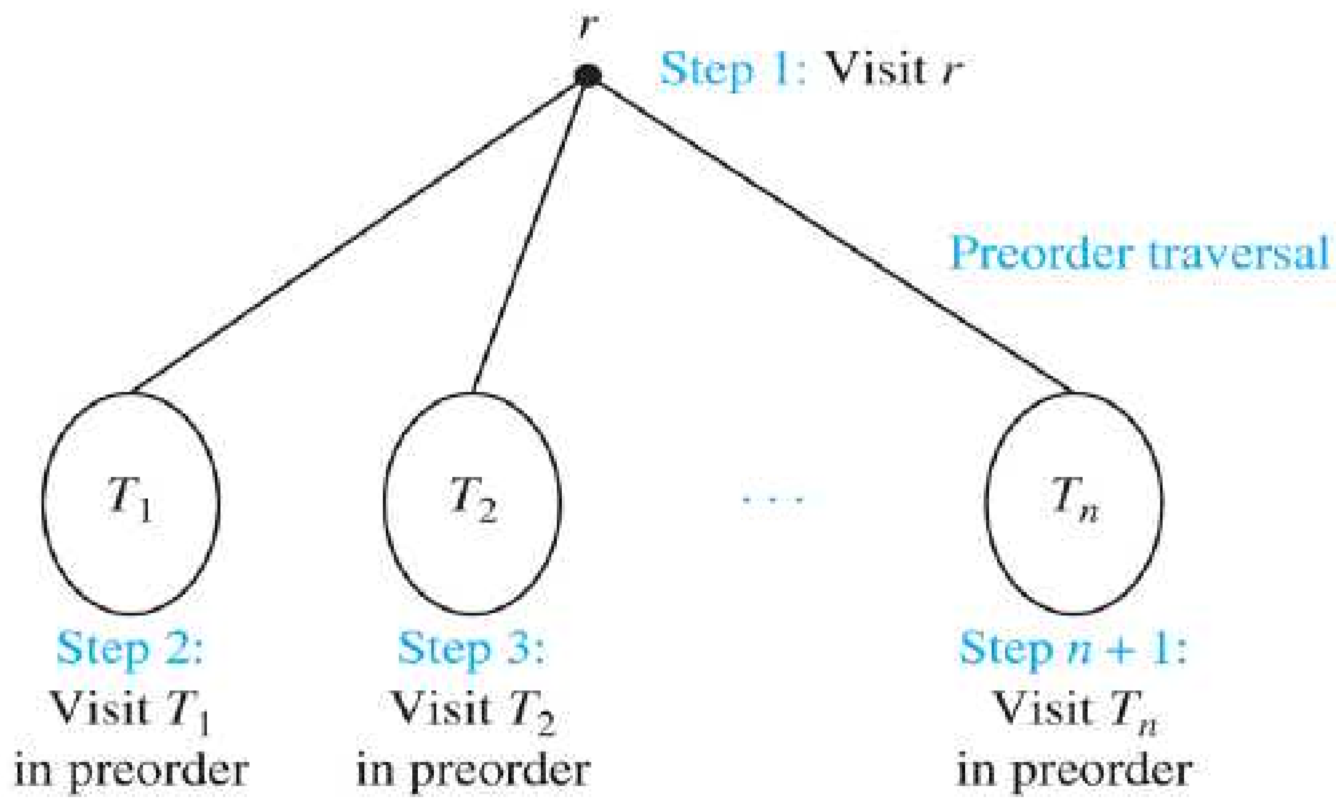
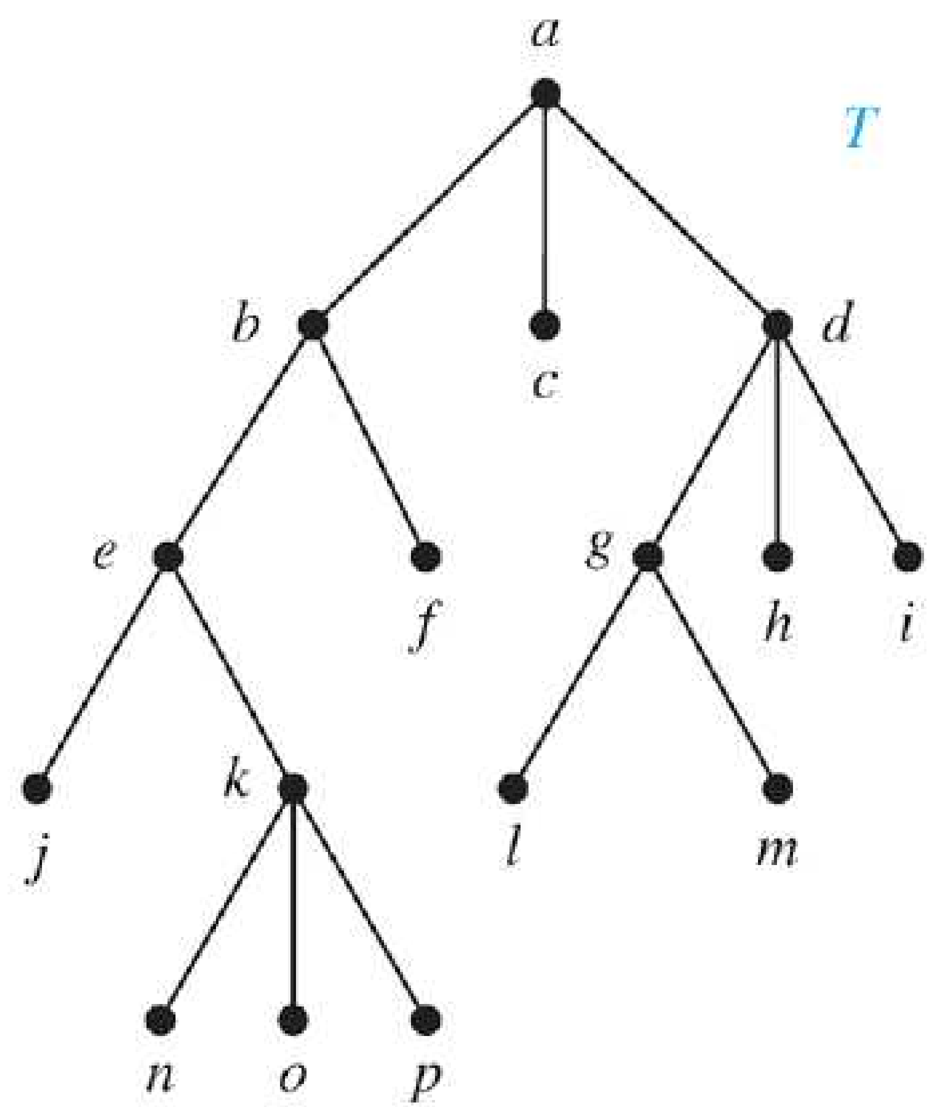


- **Example**



Preorder Traversal

procedure *preorder* (T: ordered rooted tree)
$r :=$ root of T
list r
for each child c of r from left to right
    $T(c) :=$ subtree with c as root
    $preorder(T(c))$

Inorder Traversal

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *inorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *inorder traversal* begins by traversing T_1 in inorder, then visiting r, and continues by traversing T_2 in inorder, and so on, until T_n is traversed in inorder.

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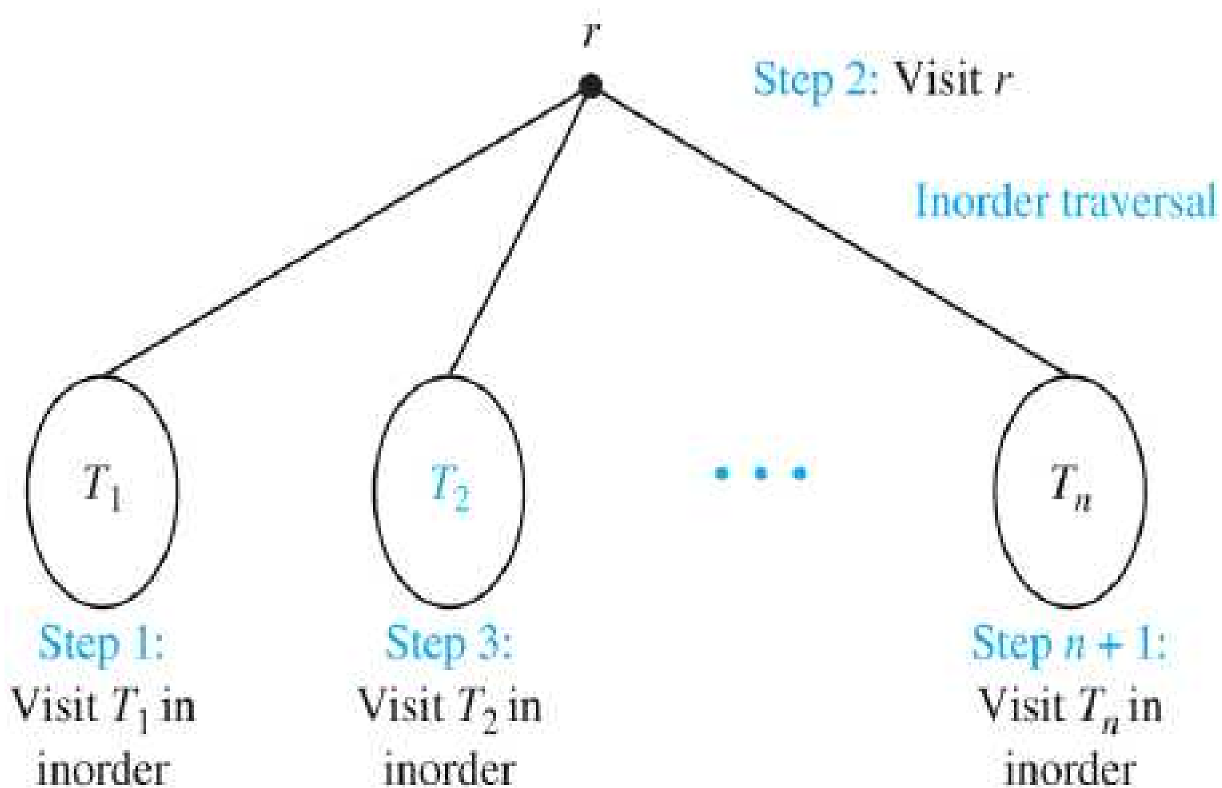
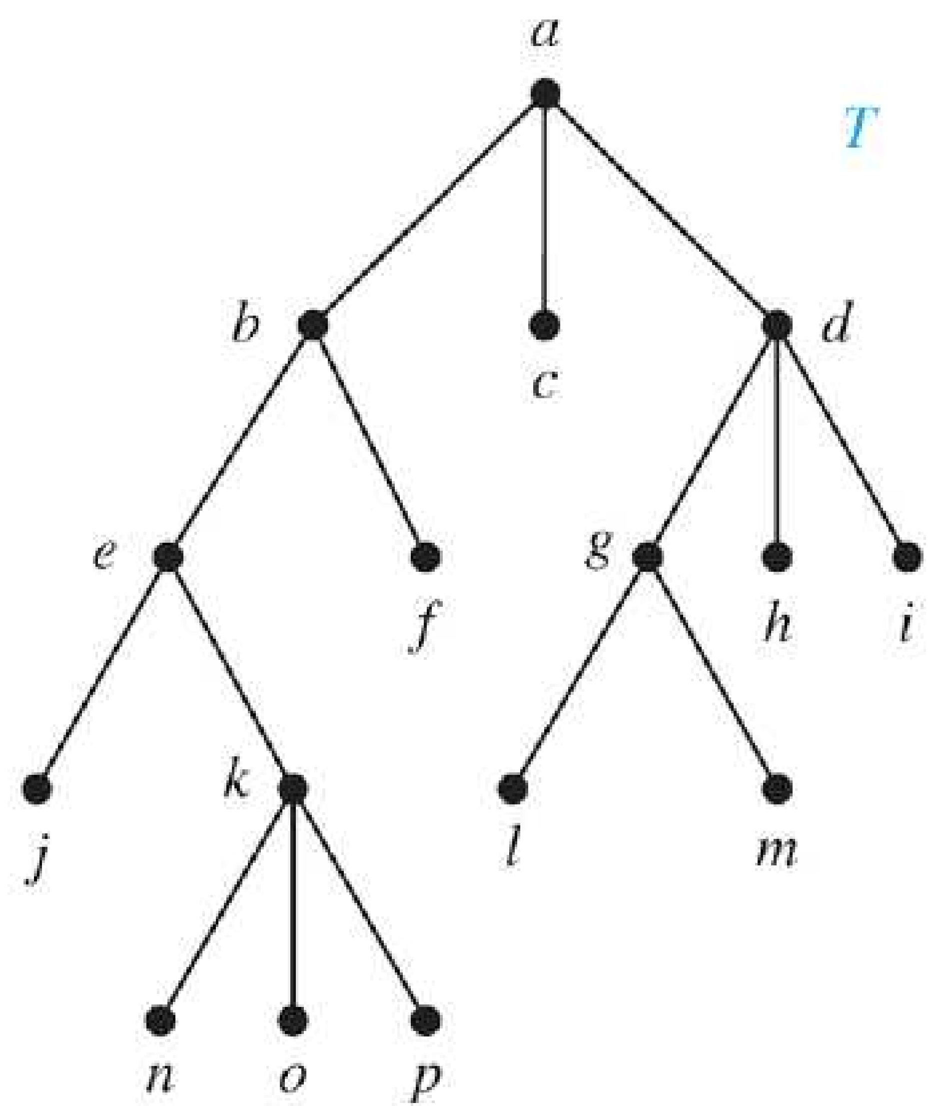


- **Example**



Inorder Traversal

procedure *inorder* (T: ordered rooted tree)
r := root of T
if r is a leaf **then** list r
else
    l := first child of r from left to right
    $T(l)$:= subtree with l as its root
    inorder($T(l)$)
    list(r)
    for each child c of r from left to right
        $T(c)$:= subtree with c as root
        inorder($T(c)$)

Postorder Traversal

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *postorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *postorder traversal* begins by traversing T_1 in postorder, then T_2 in postorder, and so on, after T_n is traversed in postorder, r is visited.

Postorder Traversal

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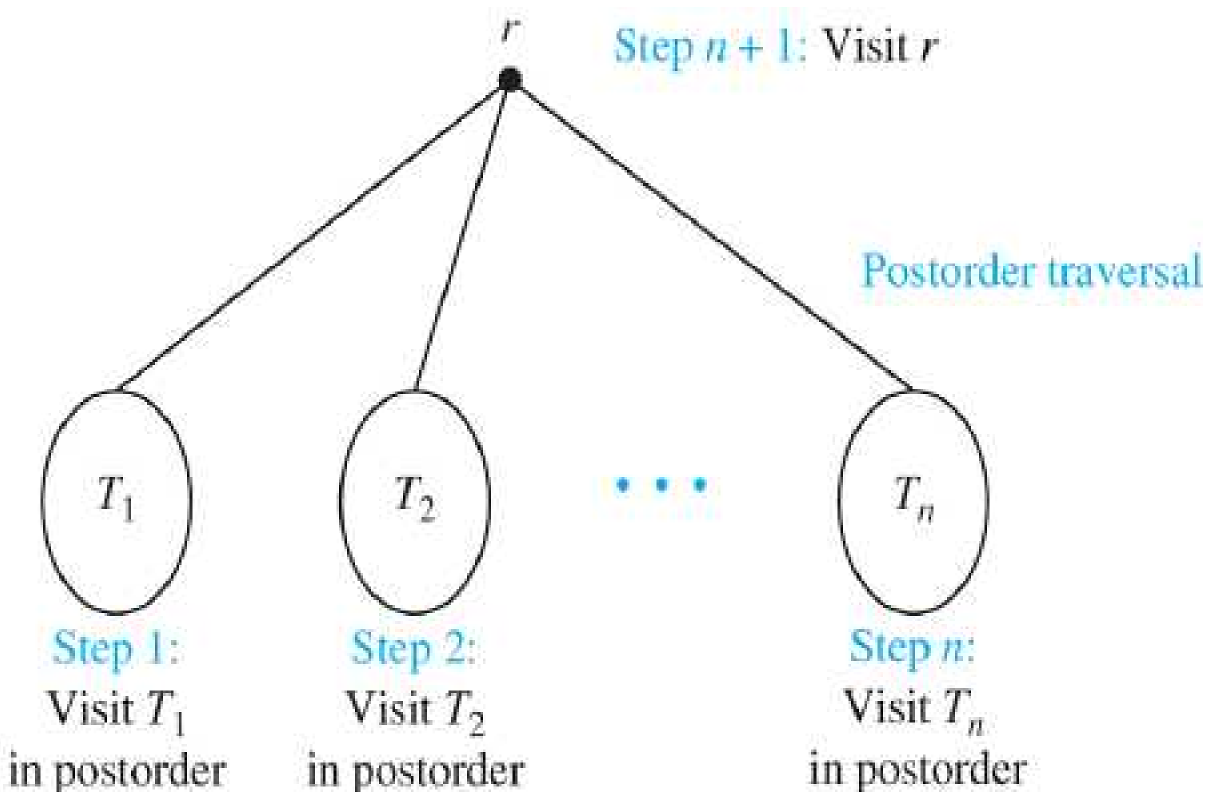

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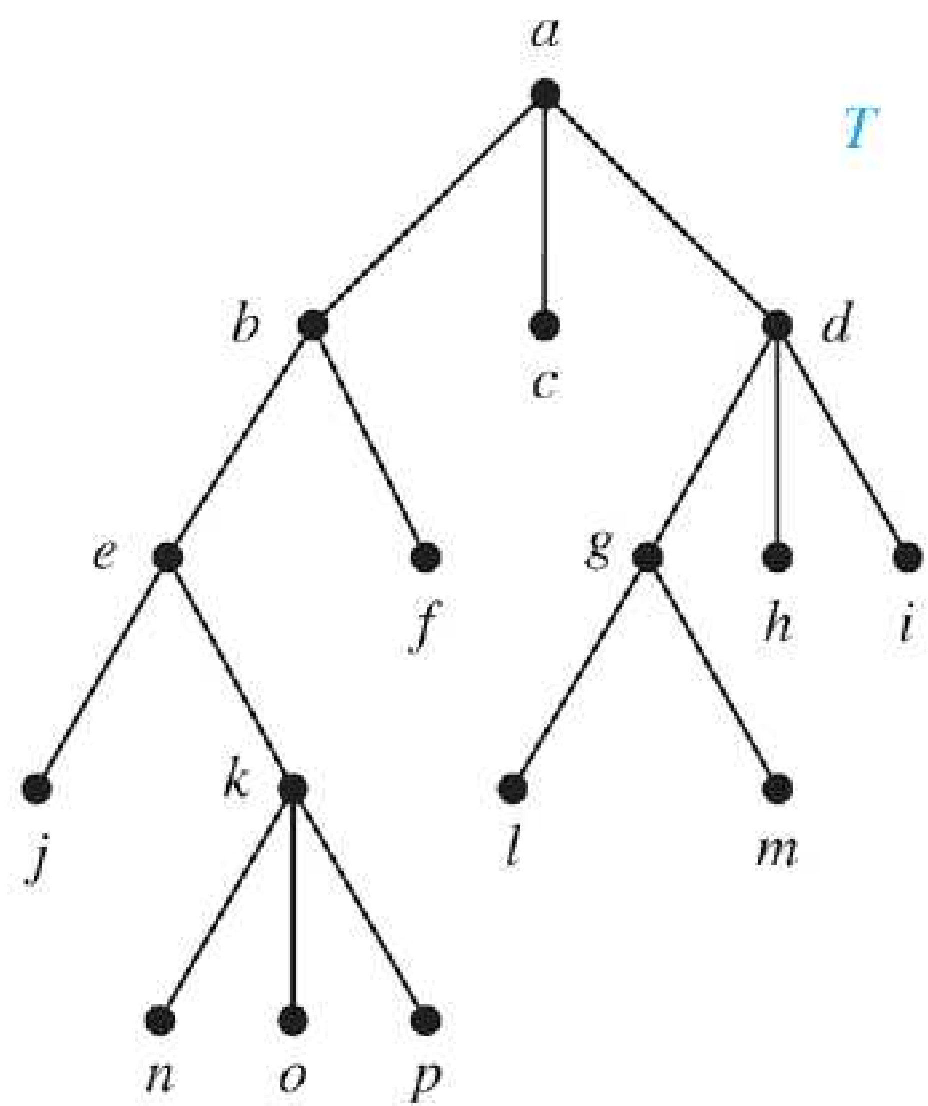
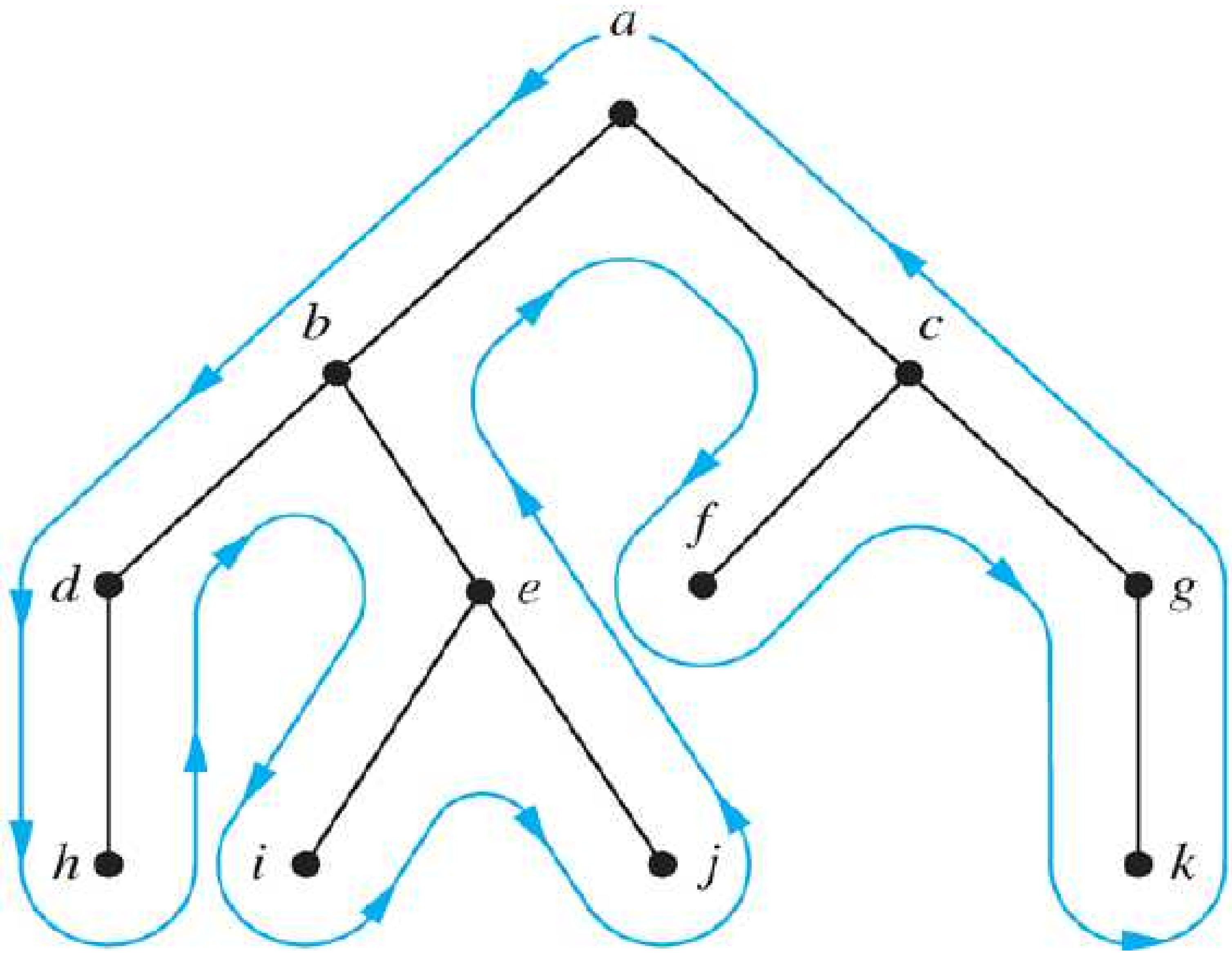


Postorder Traversal

procedure *postordered* (T: ordered rooted tree)
r := root of T
for each child c of r from left to right
    $T(c)$:= subtree with c as root
    postorder($T(c)$)
list r

Preorder, Inorder, Postorder Traversal



Expression Trees

- Complex expressions can be represented using ordered rooted trees

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Example

consider the expression $((x + y) \uparrow 2) + ((x - 4)/3)$

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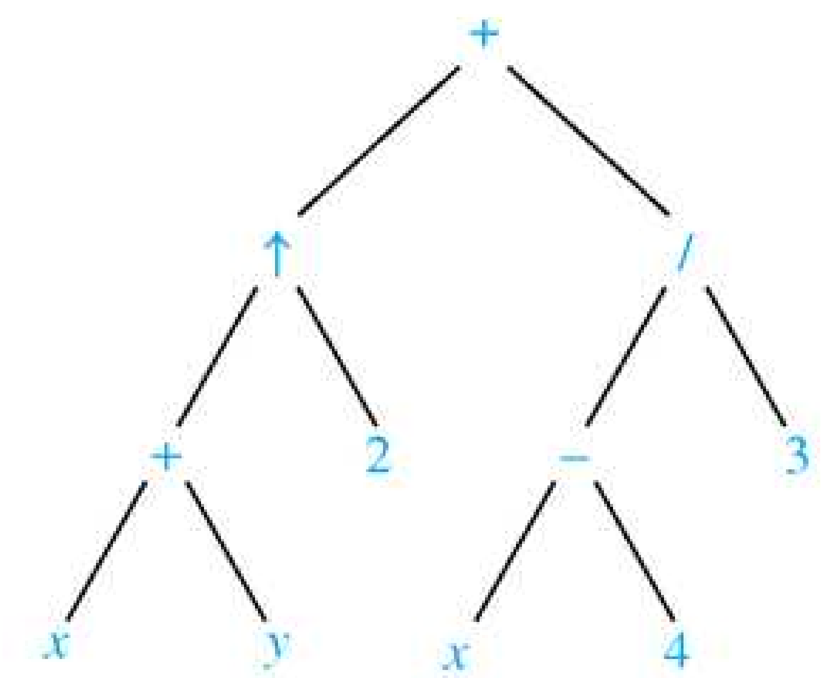

Infix Notation

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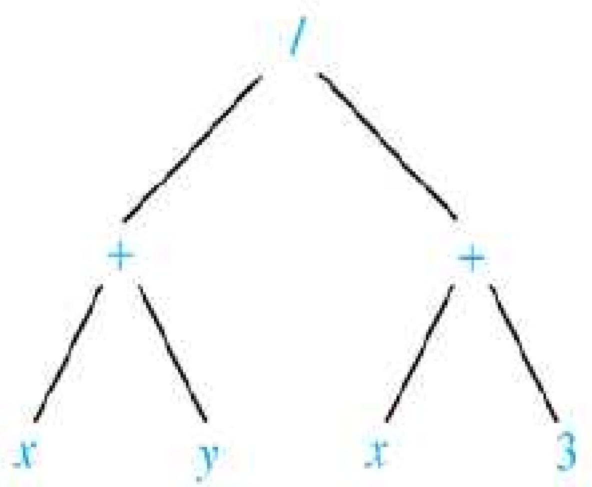

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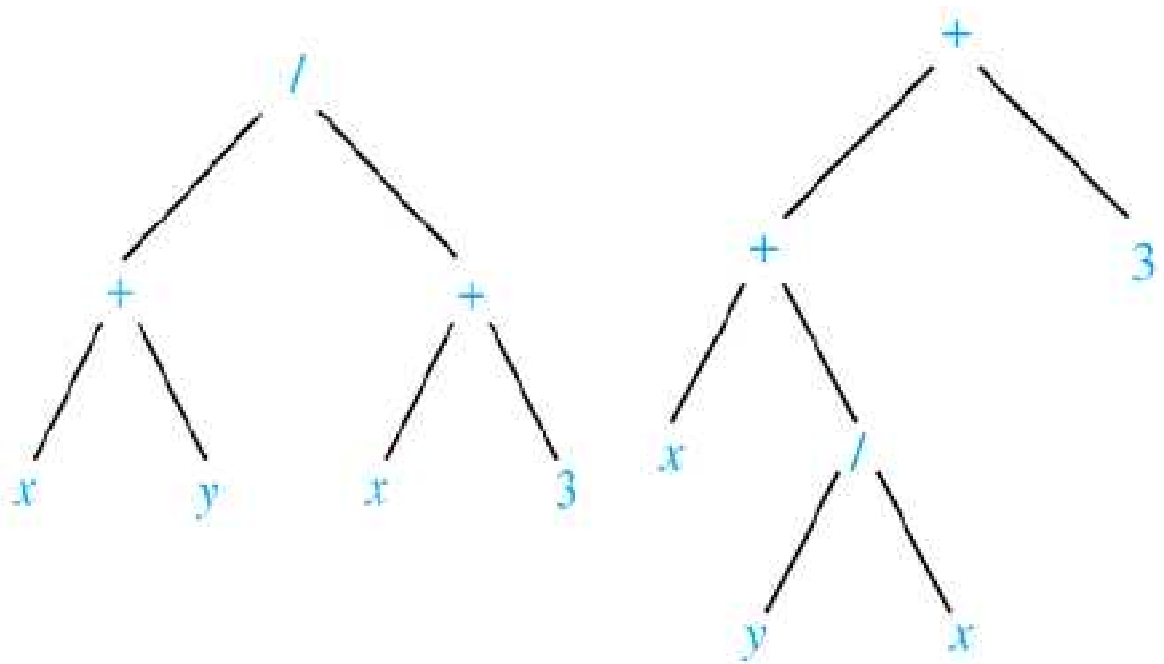

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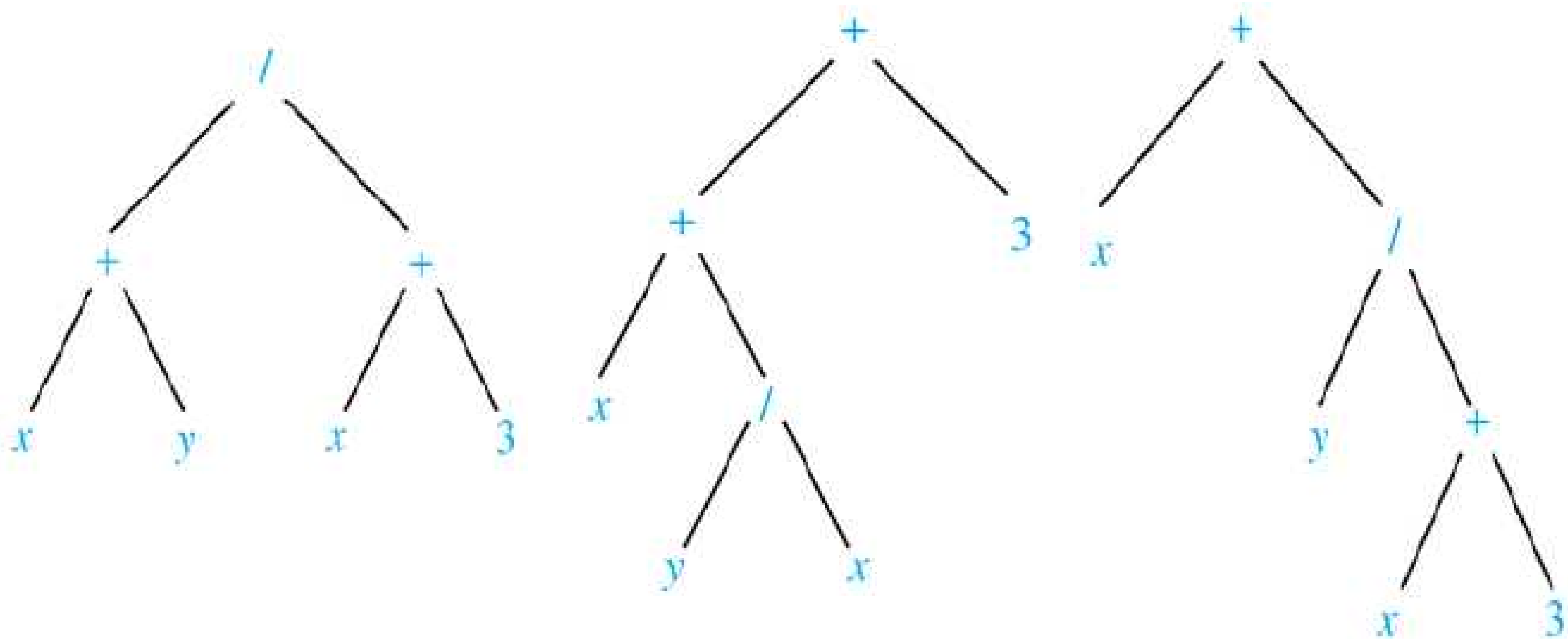

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  Prefix expressions are evaluated by working from right to left. When we encounter an operator, we perform the operation with the two operands to the right.

Prefix Notation

- **Example**

  $+ \ - \ * \, 2 \ 3 \ 5 \ / \ \uparrow \ 2 \ 3 \ 4$

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  $+ \; - \; * \; 2 \; 3 \; 5 \; / \; \uparrow \; 2 \; 3 \; 4$

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad / \quad \underbrace{\uparrow \quad 2 \quad 3}_{2 \uparrow 3 = 8} \quad 4$$

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad \underbrace{/ \quad 8 \quad 4}_{8 / 4 = 2}$$

$$+ \quad - \quad \underbrace{* \quad 2 \quad 3}_{2 * 3 = 6} \quad 5 \quad 2$$

$$+ \quad \underbrace{- \quad 6 \quad 5}_{6 - 5 = 1} \quad 2$$

$$\underbrace{+ \quad 1 \quad 2}_{1 + 2 = 3}$$

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Postfix Notation

- **Example**

  $7\ 2\ 3\ *\ -\ 4\ \uparrow\ 9\ 3\ /\ +$

Postfix Notation

- **Example**

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$$7\ \underbrace{2\ \ 3\ \ *}_{2*3=6}\ \ -\ \ 4\ \ \uparrow\ \ 9\ \ 3\ \ /\ \ +$$

$$\underbrace{7\ \ 6\ \ -}_{7-6=1}\ \ 4\ \ \uparrow\ \ 9\ \ 3\ \ /\ \ +$$

$$\underbrace{1\ \ 4\ \ \uparrow}_{1^4=1}\ \ 9\ \ 3\ \ /\ \ +$$

$$1\ \ \underbrace{9\ \ 3\ \ /}_{9/3=3}\ \ +$$

$$\underbrace{1\ \ 3\ \ +}_{1+3=4}$$

Spanning Trees

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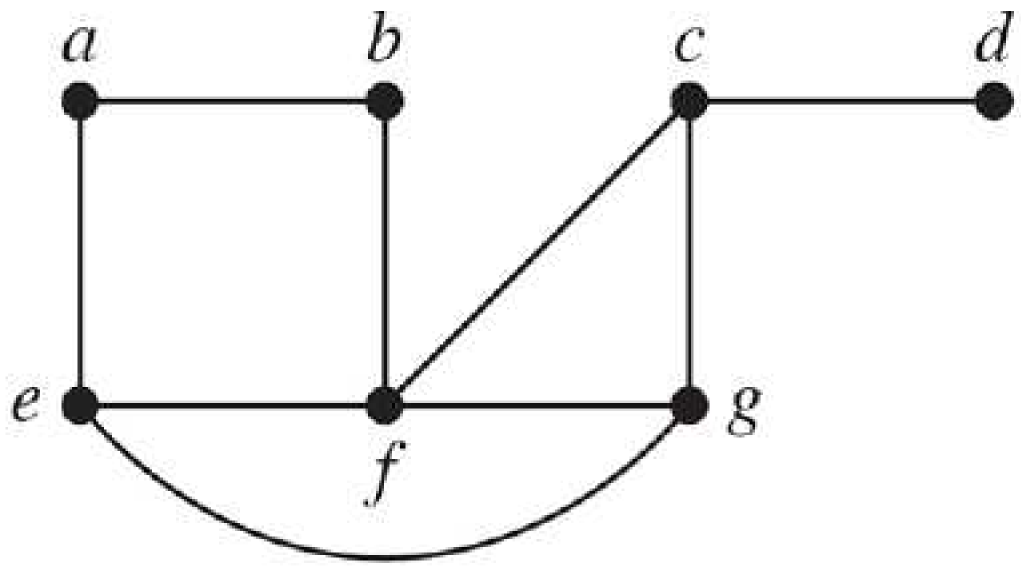

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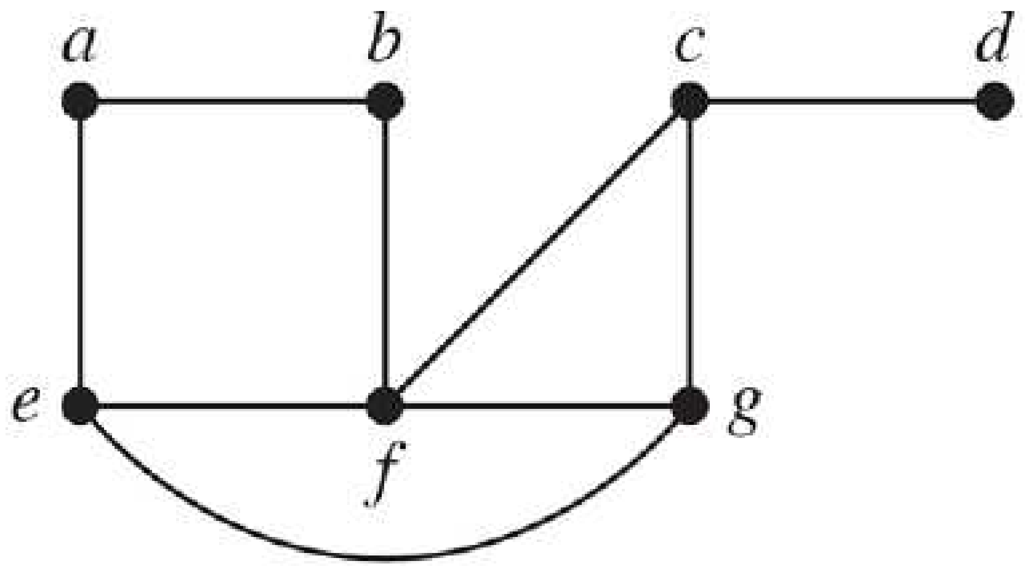

remove edges to avoid circuits

Spanning Trees

- **Theorem** A simple graph is connected if and only if it has a spanning tree.

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easy

Depth-First Search

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  Continue adding to this path as long as possible.

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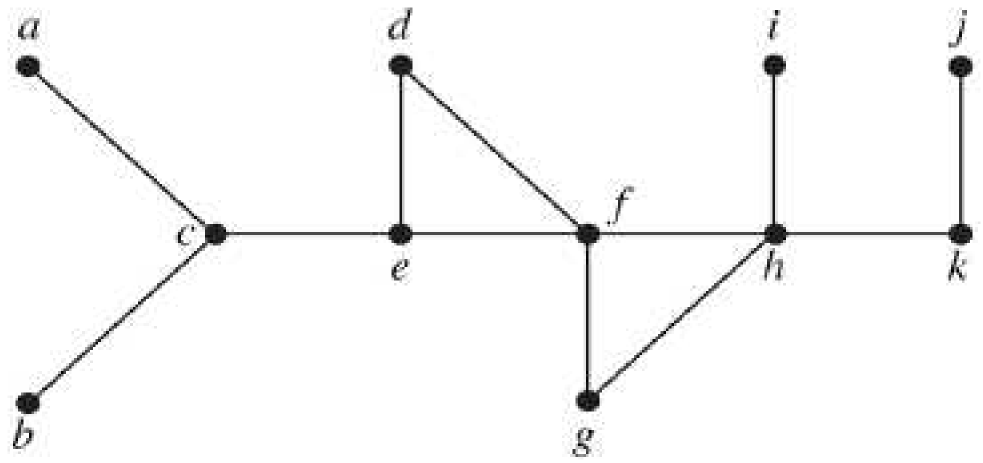
  ◇ Otherwise, move back to some vertex to repeat this procedure (*backtracking*)

- **Example**



Depth-First Search

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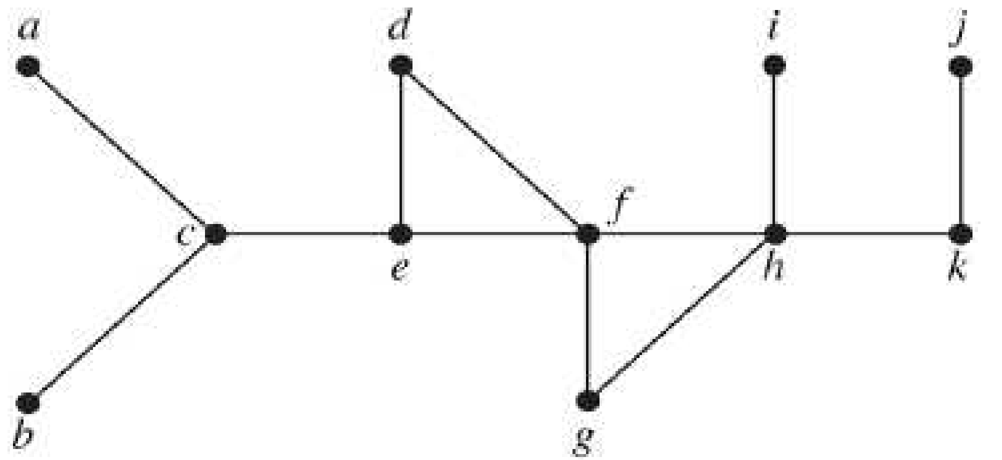

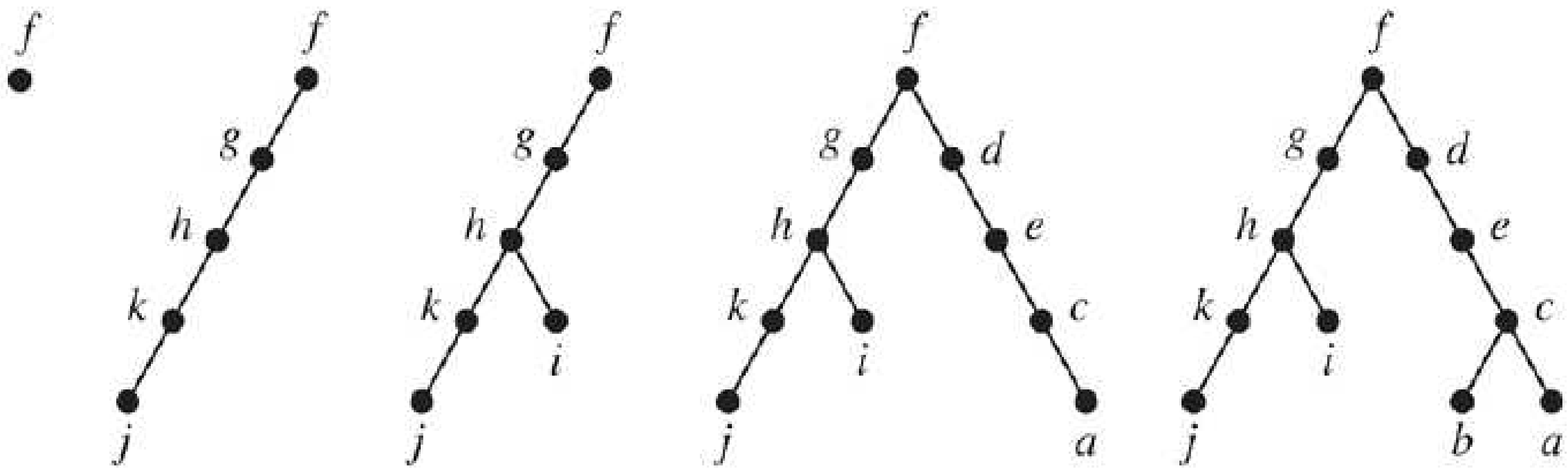

Depth-First Search Algorithm

procedure *DFS*(G: connected graph with vertices $v_1, v_2, \ldots, v_n$)
T := tree consisting only of the vertex v_1
visit(v_1)

procedure *visit*(v: vertex of G)
for each vertex w adjacent to v and not yet in T
  add vertex w and edge $\{v, w\}$ to T
  visit(w)

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time complexity: $O(e)$

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  ◇ First arbitrarily choose a vertex of the graph as the root.

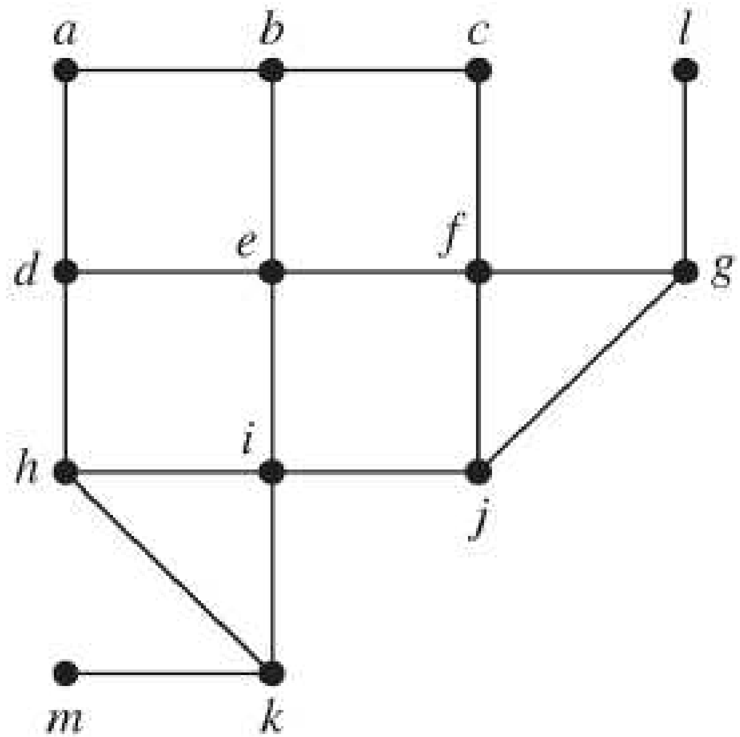
  ◇ Form a path by adding all edges incident to this vertex and the other endpoint of each of these edges

  ◇ For each vertex added at the previous level, add edge incident to this vertex, as long as it does not produce a simple circuit.

  ◇ Continue in this manner until all vertices have been added.

Breadth-First Search

- **Example**



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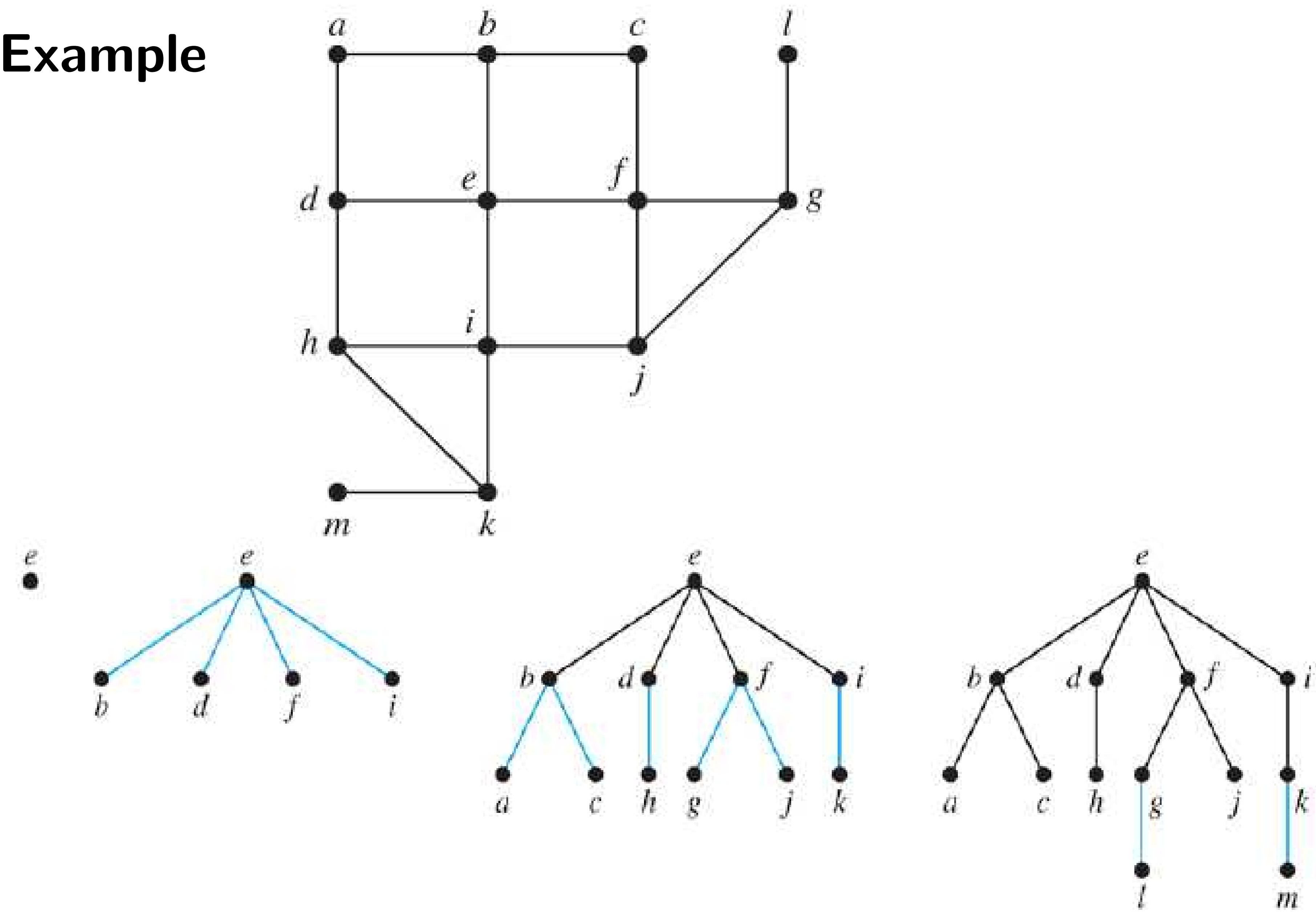

Breadth-First Search

procedure *BFS*(G: connected graph with vertices $v_1, v_2, \ldots, v_n$)
T := tree consisting only of the vertex v_1
L := empty list *visit*(v_1)
put v_1 in the list L of unprocessed vertices
while L is not empty
    remove the first vertex, v, from L
    for each neighbor w of v
        if w is not in L and not in T **then**
            add w to the end of the list L
            add w and edge $\{v,w\}$ to T

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Applications of DFS, BFS

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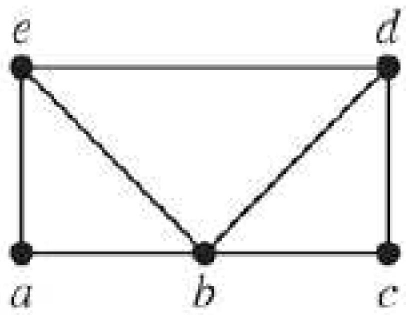
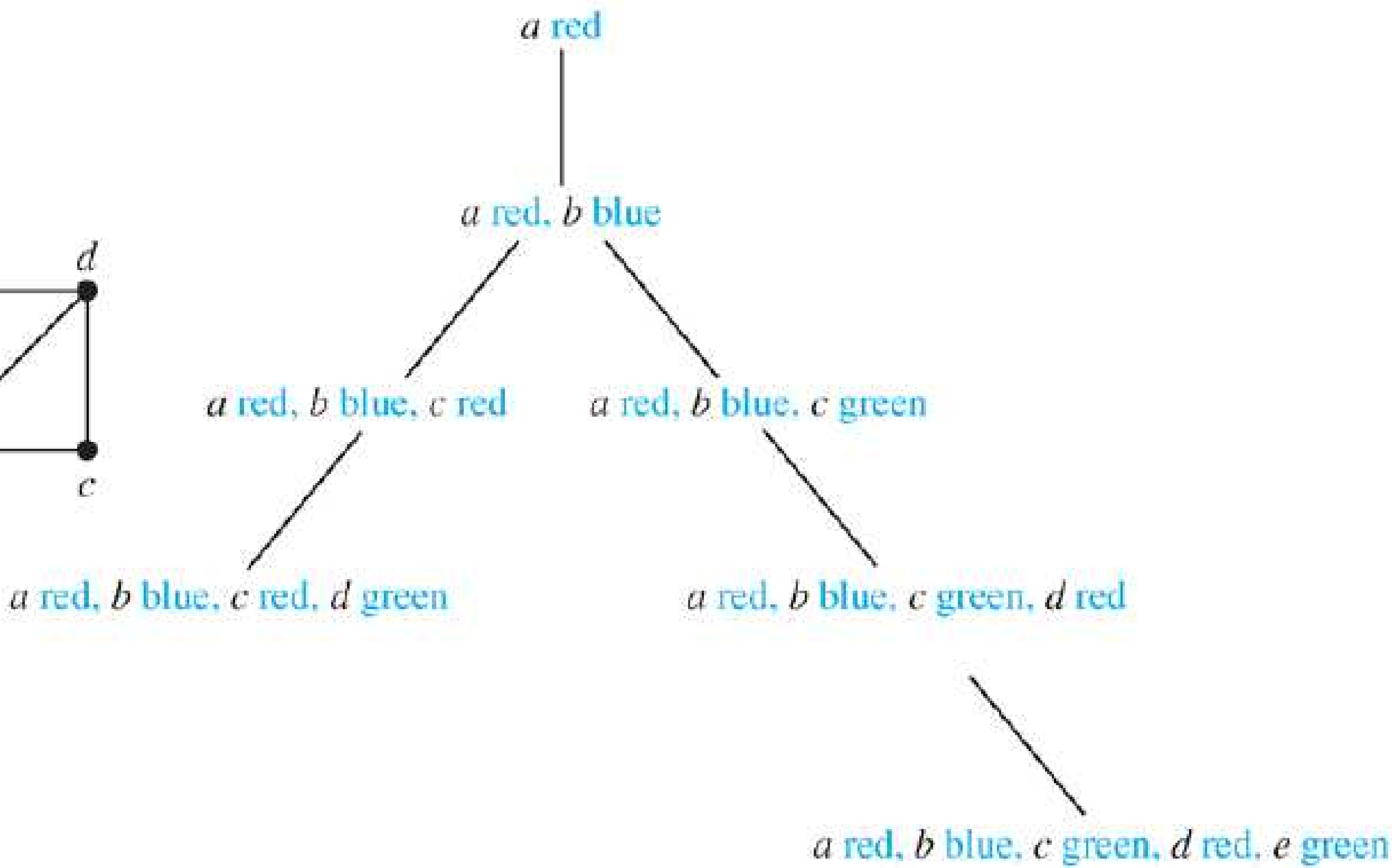
  graph coloring, sums of subsets, ...

Applications of DFS, BFS

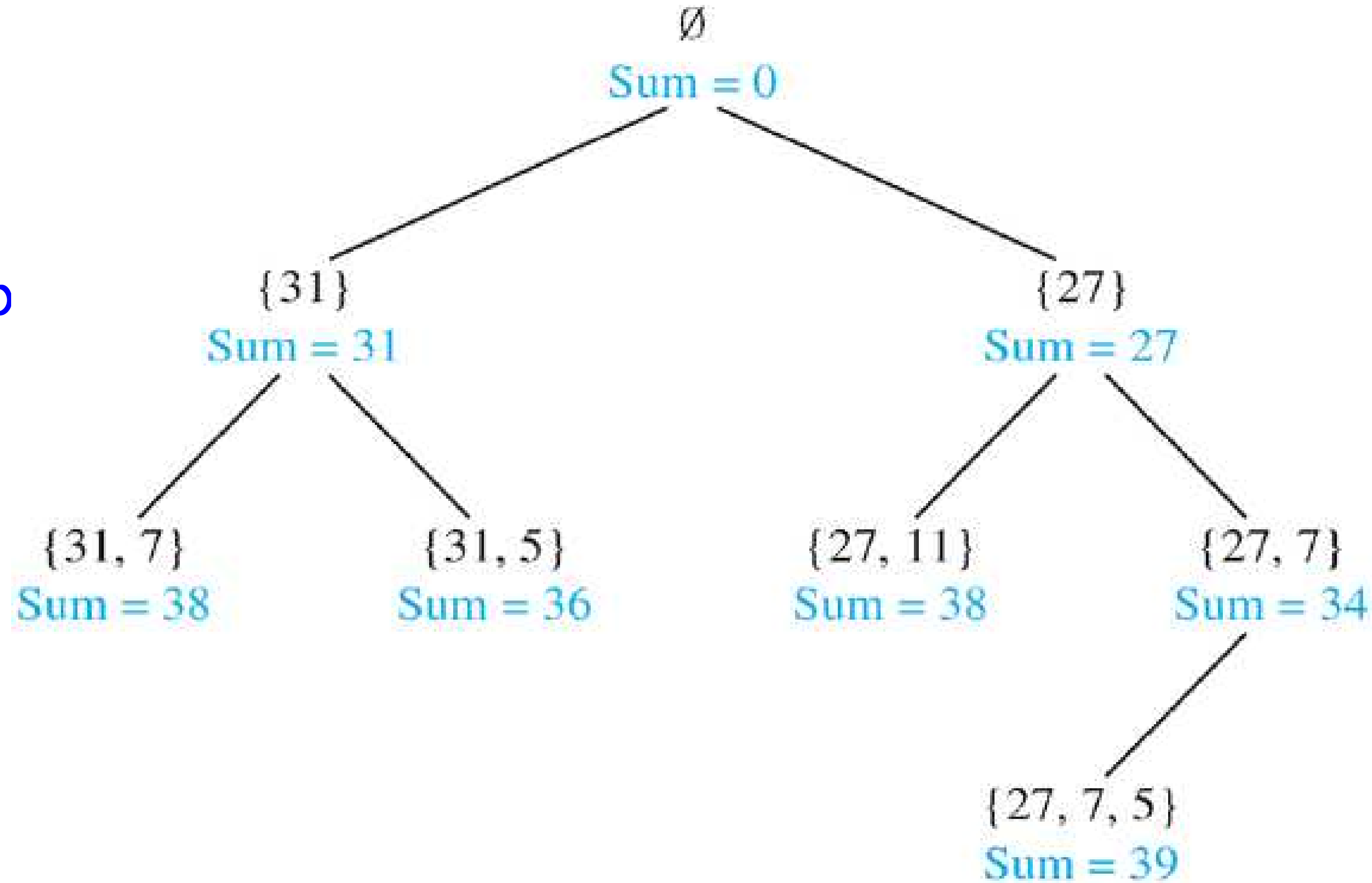
- 



- find ces, ...

  find

  grap



find a subset of $\{31, 27, 15, 11, 7, 5\}$ with the sum 39

Minimum Spanning Trees

- **Definition** A *minimum spanning tree* in a connected weighted graph is a spanning tree that has the smallest possible sum of weights of its edges.

Minimum Spanning Trees

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two greedy algorithms: Prim's Algorithm, Kruscal's Algorithm

Prim's Algorithm

ALGORITHM 1 Prim's Algorithm.

procedure *Prim*(G: weighted connected undirected graph with n vertices)
T := a minimum-weight edge
for $i := 1$ **to** $n - 2$
    e := an edge of minimum weight incident to a vertex in T and not forming a simple circuit in T if added to T
    $T := T$ with e added
return T {T is a minimum spanning tree of G}

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time complexity: $e \log v$

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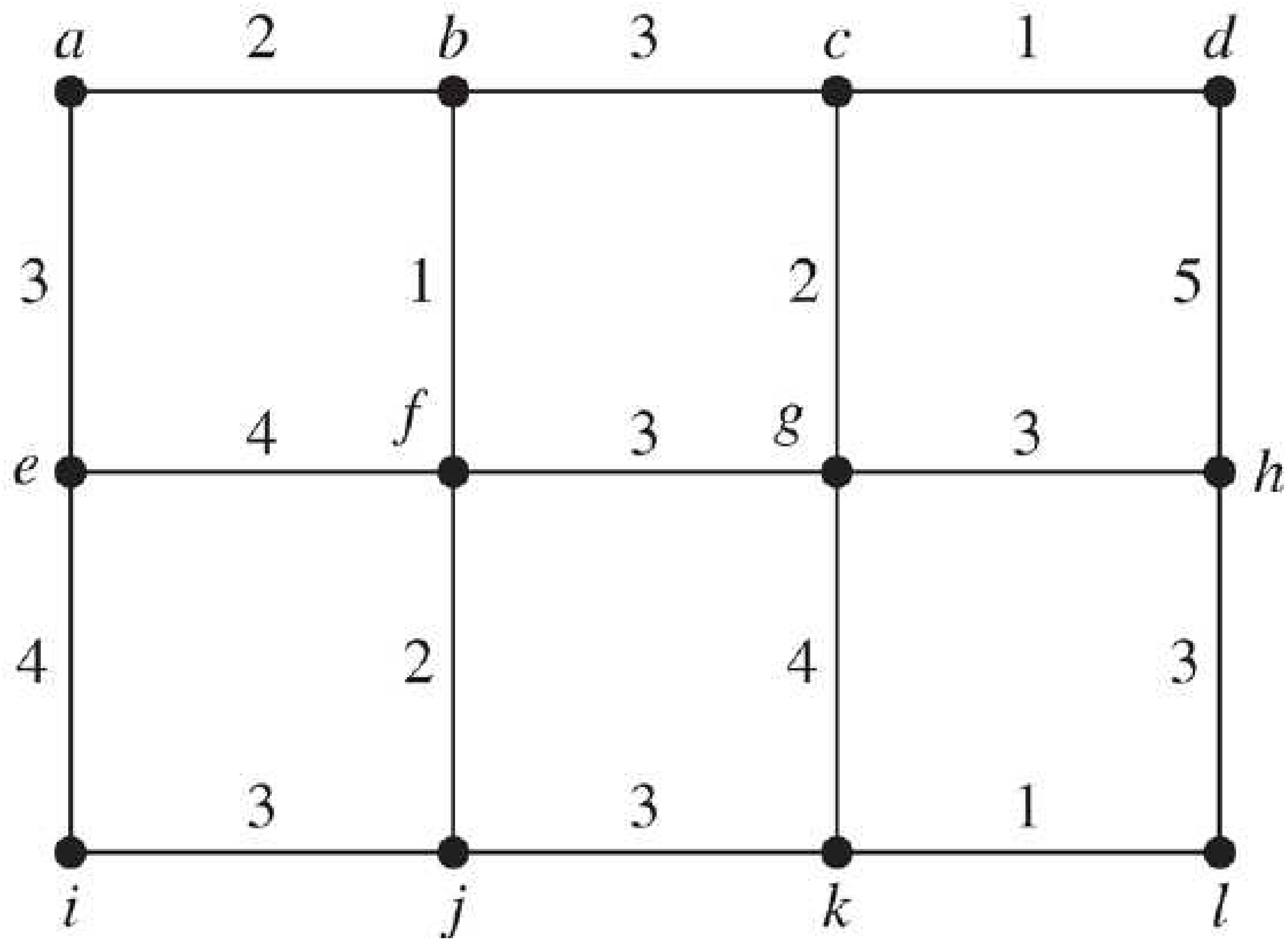

Kruscal's Algorithm

ALGORITHM 2 Kruskal's Algorithm.

procedure *Kruskal*(G: weighted connected undirected graph with n vertices)
T := empty graph
for $i := 1$ **to** $n - 1$
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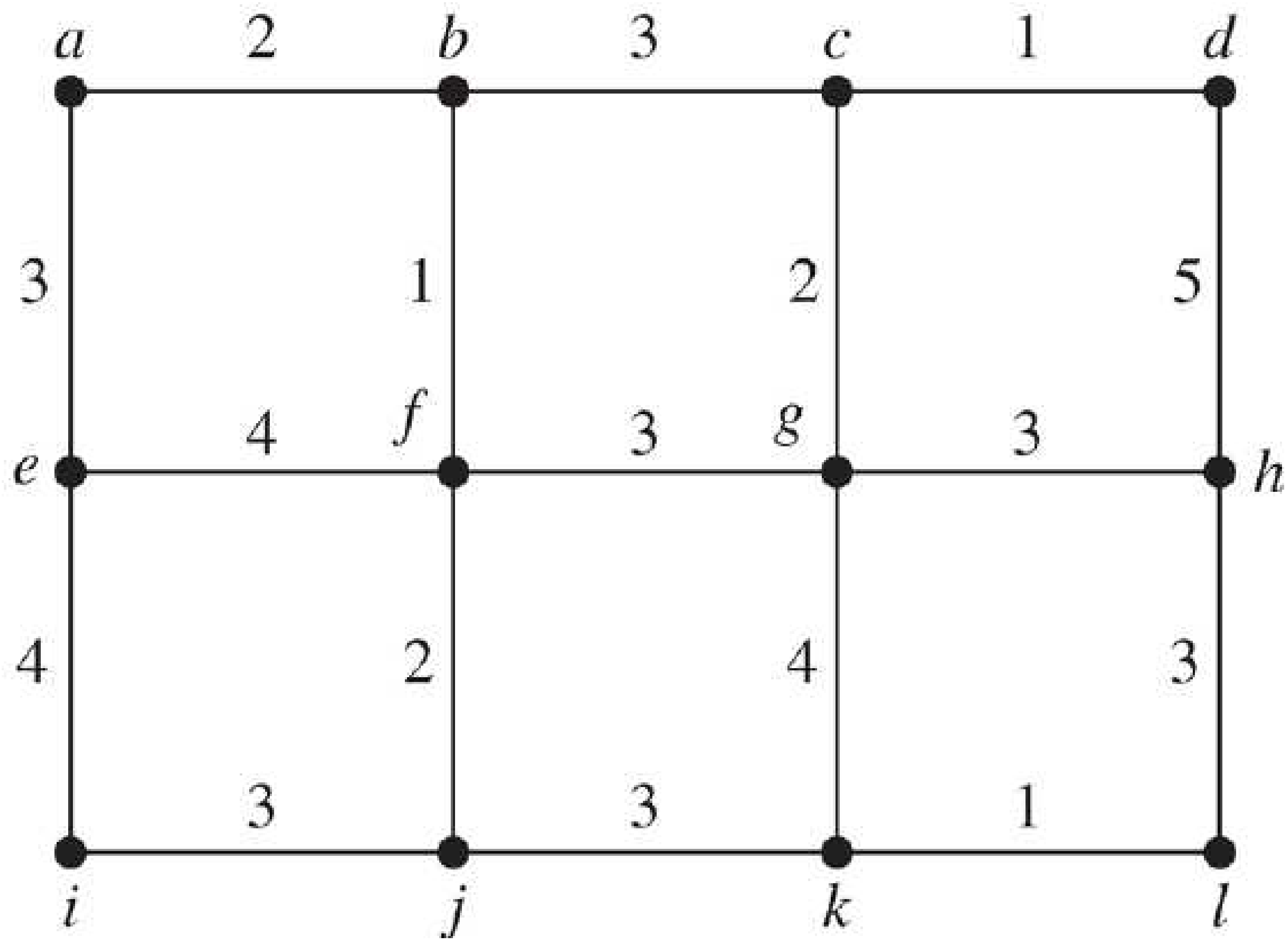
see *CLRS* / *Algorithm Design*, J. Kleinberg, E. Tardos

Kruscal's Algorithm

- **Example**



Next Lecture

- course review ...