



CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Some Thoughts on Algorithm Design

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Some Thoughts on Algorithm Design

- **Algorithm Design**, is mainly about designing algorithms that have **small Big- O running time**.
- Being able to do good algorithm design lets you identify the **hard parts** of your problem and deal with them **effectively**.
- Too often, programmers try to solve problems using **brute force techniques** and end up with **slow complicated code**!
- A few hours of abstract thought devoted to algorithm design could have **speeded up the solution substantially and simplified it**!



Dealing with Hard Problems

- What happens if you **can't** find an efficient algorithm for a given problem?



Dealing with Hard Problems

- What happens if you **can't** find an efficient algorithm for a given problem?

Blame yourself.



I couldn't find a polynomial-time algorithm.
I guess I am too dumb.

Dealing with Hard Problems

- What happens if you **can't** find an efficient algorithm for a given problem?

Show that **no**-efficient algorithm exists.



I couldn't find a polynomial-time algorithm,
because **no** such algorithm exists.



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We will now learn about **NP-Complete** problems, which provide us with a way to approach this question.



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- Researchers have spent innumerable man-years trying to find efficient solutions to these problems but **failed**.
- So, **NP-Complete** problems are very likely to be **hard**.
- What do you do: prove that **your problem is NP-Complete**.



Introduction

What do you actually do:



I couldn't find a polynomial-time algorithm,
but neither could all these other smart people!

Encoding the Inputs of Problems

- **Complexity** of a problem is measure w.r.t **the size of input**.



Encoding the Inputs of Problems

- **Complexity** of a problem is measure w.r.t **the size of input**.
- In order to formally discuss how hard a problem is, we need to be **much more** formal than before about the **input size** of a problem.



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Definition The **input size** of a problem is the **minimum number** of bits ($\{0,1\}$) needed to **encode** the input of the problem.

- The **exact** input size s , determined by an **optimal** encoding method, is **hard** to compute in most cases.

However, we do **not** need to determine s **exactly**.

For most problems, it is sufficient to choose some **natural**, and (usually) **simple**, encoding and use the size s of this encoding.



Input Size Example: Composite

■ Example:

Given a positive integer n , are there integers $j, k > 1$ such that $n = jk$? (i.e., **is n a composite number?**)



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Any integer $n > 0$ can be represented in the **binary number system** as a string $a_0a_1 \cdots a_k$ of length $\lceil \log_2(n+1) \rceil$.

Thus, a natural measure of input size is $\lceil \log_2(n+1) \rceil$ (or just **$\log_2 n$**)



Input Size Example: Sorting

- **Example:**

Sort n integers a_1, \dots, a_n



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Using fixed length encoding, we write a_i as a binary string of length $m = \lceil \log_2 \max(|a_i| + 1) \rceil$.

This coding gives an input size nm .



Complexity in terms of Input Size

- **Example:** (Composite)

The naive algorithm for determining whether n is composite compares n with the first $n - 1$ numbers to see if **any of them divides n**



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The naive algorithm for determining whether n is composite compares n with the first $n - 1$ numbers to see if **any of them divides n**

This makes $\Theta(n)$ comparisons, so it might seem **linear** and very **efficient**.

But, note that the input size of this problem is $size(n) = \log_2 n$, so the number of comparisons performed is actually $\Theta(n) = \Theta(2^{size(n)})$, which is **exponential**.



Input Size of Problems

- **Definition** Two positive functions $f(n)$ and $g(n)$ are of the same type if

$$c_1 g(n^{a_1})^{b_1} \leq f(n) \leq c_2 g(n^{a_2})^{b_2}$$

for all large n , where $a_1, b_1, c_1, a_2, b_2, c_2$ are some positive constants.



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Example:

All polynomials are of the **same type**, but *polynomials* and *exponentials* are of **different types**.



Input Size Example: Integer Multiplication

- **Example:** (Integer Multiplication problem)

Compute $a \times b$.



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The minimum input size is

$$s = \lceil \log_2(a + 1) \rceil + \lceil \log_2(b + 1) \rceil.$$

A natural choice is to use $t = \log_2 \max(a, b)$ since $\frac{s}{2} \leq t \leq s$.



Decision Problems

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If L is the problem, and x is the input, we will often write $x \in L$ to denote a *yes* answer and $x \notin L$ to denote a *no* answer.



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Examples:

Knapsack vs. *Decision Knapsack* (DKnapsack)



Knapsack vs. DKnapsack

- We have a knapsack of capacity W (a positive integer) and n objects with weights w_1, \dots, w_n and values v_1, \dots, v_n , where v_i and w_i are positive integers.



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- We have a knapsack of capacity W (a positive integer) and n objects with weights w_1, \dots, w_n and values v_1, \dots, v_n , where v_i and w_i are positive integers.

Optimization problem: (Knapsack)

Find the **largest value** $\sum_{i \in T} v_i$ of any subset T that fits in the knapsack, i.e., $\sum_{i \in T} w_i \leq W$.

Decision problem: (DKnapsack)

Given k , **is there a subset** of the objects that fits in the knapsack and has total value **at least k** ?



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First solve the **optimization problem**, then check the **decision problem**. If it does, answer **yes**, otherwise **no**.

Thus, if we prove that a given **decision problem** is hard to solve efficiently, then it is **obvious** that the **optimization problem** must be (at least as) **hard**.



Complexity Classes

- The **Theory of Complexity** deals with
 - ◇ the classification of certain “**decision problems**” into several classes:
 - ◇ the class of “easy” problems
 - ◇ the class of “hard” problems
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Question:

How to classify decision problems?

A. Use **polynomial-time algorithms**.



Polynomial-Time Algorithms

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Example:

The standard multiplication algorithm has time $O(m_1 m_2)$, where m_1, m_2 denote the number of digits in the two integers, respectively.



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Let's return to the *Composite* problem.

- ◇ it checks, in time $\Theta((\log N)^2)$, whether K divides N for each K with $2 \leq K \leq N - 1$.
- ◇ The complete algorithm therefore uses $\Theta(N(\log N)^2)$ time.



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In terms of the *input size*, the complexity is $\Theta(2^n n^2)$.



Polynomial- vs. Nonpolynomial-Time

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In reality, an $O(n^{20})$ algorithm is **not** really practical.



Polynomial-Time Solvable Problems

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Definition (The **Class P**) The class P consists of all **decision problems** that are solvable in **polynomial time**. That is, there exists an algorithm that will decide in **polynomial time** if any given input is a **yes-input** or a **no-input**.



The Class P

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How to prove that a decision problem is in P?



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The Class P

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Question:

How to prove that a decision problem is **not** in P?

A. You need to prove that there is **no** polynomial-time algorithm for this problem. (much much **harder**)



Certificates and Verifying Certificates

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Verifying a certificate: Given a presumed yes-input and its corresponding certificate, by making use of the given certificate, we verify that the input is actually a *yes-input*.



The Class NP

- **Definition** The class **NP** consists of all decision problems such that, for each **yes-input**, there exists a *certificate* which allows one to verify in **polynomial time** that the input is indeed a **yes-input**.



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NP – “nondeterministic polynomial-time”



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A. An integer a ($1 < a < n$) with the property that $a|n$.



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- ◇ This can be done in $O((\log n)^2)$.
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DKnapsack \in NP



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Just being able to verify a certificate in **polynomial time** does **not** necessarily mean we can tell **whether an input is a yes-input or a no-input in polynomial time**.



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- Observe that $P \subseteq NP$.
- Intuitively, $NP \subseteq P$ is **doubtful**.

Just being able to verify a certificate in **polynomial time** does **not** necessarily mean we can tell **whether an input is a yes-input or a no-input in polynomial time**.

However, we are still **no** closer to solving it and do not know the answer. The search for a solution, though, has provided us with deep insights into **what distinguishes an “easy” problem from a “hard” one**.



Application of Number Theory

- G. H. Hardy (1877 - 1947)

In his 1940 autobiography *A Mathematician's Apology*, Hardy wrote “The great modern achievements of applied mathematics have been in relativity and quantum mechanics, and these subjects are, at present, **almost as ‘useless’ as the theory of numbers.**”



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If he could see the world now, Hardy would be spinning in his grave.



Number Theory

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- *Number theory* is a branch of mathematics that explores integers and their properties, is the basis of **cryptography**, **coding theory**, **computer security**, **e-commerce**, etc.
- At one point, the largest employer of mathematicians in the United States, and probably the world, was the **National Security Agency** (NSA). The NSA is the largest spy agency in the US (bigger than CIA, Central Intelligence Agency), and has the responsibility for code design and breaking.



Division

- If a and b are integers with $a \neq 0$, we say that a divides b if there is an integer c such that $b = ac$, or equivalently b/a is an integer. In this case, we say that a is a *factor* or *divisor* of b , and b is a *multiple* of a . (We use the notations $a|b$, $a \nmid b$)



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Example

◇ $4 \mid 24$

◇ $3 \nmid 7$



Divisibility

- **All integers divisible by $d > 0$** can be **enumerated** as:
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- **Question:** Let n and d be two positive integers. How many positive integers **not exceeding n** are divisible by d ?



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 $\dots, -kd, \dots, -2d, -d, 0, d, 2d, \dots, kd, \dots$
- **Question:** Let n and d be two positive integers. How many positive integers **not exceeding n** are divisible by d ?

Answer: Count the number of integers such that $0 < kd \leq n$. Therefore, there are $\lfloor n/d \rfloor$ such positive integers.



Divisibility

■ Properties

Let a, b, c be integers. Then the following hold:

- (i) if $a|b$ and $a|c$, then $a|(b + c)$
- (ii) if $a|b$ then $a|bc$ for all integers c
- iii) if $a|b$ and $b|c$, then $a|c$



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Proof.



Divisibility

- **Corollary** If a, b, c are integers, where $a \neq 0$, such that $a|b$ and $a|c$, then $a|(mb + nc)$ whenever m and n are integers.



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Proof. By part (ii) and part (i) of Properties.



The Division Algorithm

- If a is an integer and d a positive integer, then there are **unique** integers q and r , with $0 \leq r < d$, such that $a = dq + r$. In this case, d is called the *divisor*, a is called the *dividend*, q is called the *quotient*, and r is called the *remainder*.



The Division Algorithm

- If a is an integer and d a positive integer, then there are **unique** integers q and r , with $0 \leq r < d$, such that $a = dq + r$. In this case, d is called the *divisor*, a is called the *dividend*, q is called the *quotient*, and r is called the *remainder*.

In this case, we use the notations $q = a \text{ div } d$ and $r = a \bmod d$.



Congruence Relation

- If a and b are integers and m is a positive integer, then a is *congruent to b modulo m if m divides $a - b$* , denoted by $a \equiv b \pmod{m}$. This is called *congruence* and m is its *modulus*.



Congruence Relation

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Example

- ◇ $15 \equiv 3 \pmod{6}$
- ◇ $-1 \equiv 11 \pmod{6}$



More on Congruences

- Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$.



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Proof.

“only if” part

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- $a \equiv b \pmod{m}$ and $a \bmod m = b$ are different.
 - ◇ $a \equiv b \pmod{m}$ is a **relation** on the set of integers
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Then $a \equiv b \bmod m$ if and only if $a \bmod m = b \bmod m$



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Congruences of Sums and Products

- Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$



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- If $a \equiv b \pmod{m}$, then
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 - $a/c \equiv b/c \pmod{m}$?



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$$14 \equiv 8 \pmod{6} \text{ but } 7 \not\equiv 4 \pmod{6}$$



Computing the mod Function

- **Corollary** Let m be a positive integer and let a and b be integers. Then

$$(a + b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$$

$$ab \bmod m = ((a \bmod m)(b \bmod m)) \bmod m$$



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Example

$$\diamond 7 +_{11} 9 = ?$$

$$\diamond 7 \cdot_{11} 9 = ?$$



Arithmetic Modulo m

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- **Distributivity**: if $a, b, c \in \mathbf{Z}_m$, then
 $a \cdot_m (b +_m c) = (a \cdot_m b) +_m (a \cdot_m c)$ and
 $(a +_m b) \cdot_m c = (a \cdot_m c) +_m (b \cdot_m c)$



Representations of Integers

- We may use *decimal* (*base 10*) or *binary* or *octal* or *hexadecimal* or other notations to represent integers.



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- We may use *decimal* (*base 10*) or *binary* or *octal* or *hexadecimal* or other notations to represent integers.
- Let $b > 1$ be an integer. Then if n is a positive integer, it can be expressed **uniquely in the form**
$$n = a_k b^k + a_{k-1} b^{k-1} + \dots + a_1 b + a_0,$$
 where k is nonnegative, a_i 's are nonnegative integers less than b . The representation of n is called *the base- b expansion of n* and is denoted by $(a_k a_{k-1} \dots a_1 a_0)_b$.



Base- b Expansions

- To get the decimal expansion is easy.



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Example

- ◇ $(10101111)_2 = 2^8 + 2^6 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 351$
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- Conversions between binary, octal, hexadecimal expansions are easy.

Example

- ◇ $(101011111)_2 = (\underline{101}\overline{011}\underline{111}) = (537)_8$
- ◇ $(7016)_8 = (\underline{111}\overline{000}\underline{001}\overline{110})_2$
 $= (\underline{111}\overline{000}\underline{001}\overline{110})_2 = (E0E)_{16}$



Base- b Expansions

$$\begin{aligned}n &= a_k b^k + a_{k-1} b^{k-1} + a_{k-2} b^{k-2} + \cdots + a_2 b^2 + a_1 b + a_0 \\&= b(a_k b^{k-1} + a_{k-1} b^{k-2} + a_{k-2} b^{k-3} + \cdots + a_2 b + a_1) + \textcolor{red}{a_0} \\&= b(b(a_k b^{k-2} + a_{k-1} b^{k-3} + a_{k-2} b^{k-4} + \cdots + a_2) + \textcolor{red}{a_1}) + \textcolor{blue}{a_0} \\&= \cdots\end{aligned}$$



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To construct the base- b expansion of an integer n ,

- Divide n by b to obtain $\textcolor{blue}{n = bq_0 + a_0}$, with $0 \leq a_0 < b$
- The remainder a_0 is the rightmost digit in the base- b expansion of n . Then divide q_0 by b to get $\textcolor{blue}{q_0 = bq_1 + a_1}$ with $0 \leq a_1 < b$
- a_1 is the second digit from the right. Continue by successively dividing the quotients by b until **the quotient is 0**



Algorithm: Constructing Base- b Expansions

```
procedure base b expansion( $n, b$ : positive integers with  $b > 1$ )  
   $q := n$   
   $k := 0$   
  while ( $q \neq 0$ )  
     $a_k := q \bmod b$   
     $q := q \operatorname{div} b$   
     $k := k + 1$   
  return( $a_{k-1}, \dots, a_1, a_0$ ) {  $(a_{k-1} \dots a_1 a_0)_b$  is base  $b$  expansion of  $n$  }
```



Example

- $(12345)_{10} = (30071)_8$



Example

■ $(12345)_{10} = (30071)_8$

$$12345 = 8 \cdot 1543 + 1$$

$$1543 = 8 \cdot 192 + 7$$

$$192 = 8 \cdot 24 + 0$$

$$24 = 8 \cdot 3 + 0$$

$$3 = 8 \cdot 0 + 3$$



Binary Addition of Integers

$$a = (a_{n-1}a_{n-2} \dots a_1a_0), \quad b = (b_{n-1}b_{n-2} \dots b_1b_0)$$

```
procedure add(a, b: positive integers)
{the binary expansions of a and b are  $(a_{n-1}, a_{n-2}, \dots, a_0)_2$  and  $(b_{n-1}, b_{n-2}, \dots, b_0)_2$ , respectively}
c := 0
for j := 0 to n - 1
    d :=  $\lfloor (a_j + b_j + c) / 2 \rfloor$ 
    sj :=  $a_j + b_j + c - 2d$ 
    c := d
sn := c
return(s0, s1, ..., sn) {the binary expansion of the sum is  $(s_n, s_{n-1}, \dots, s_0)_2$ }
```



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$O(n)$ bit additions



Algorithm: Binary Multiplication of Integers

$$a = (a_{n-1}a_{n-2} \dots a_1a_0)_2, \quad b = (b_{n-1}b_{n-2} \dots b_1b_0)_2$$

$$\begin{aligned} ab &= a(b_02^0 + b_12^1 + \dots + b_{n-1}2^{n-1}) \\ &= a(b_02^0) + a(b_12^1) + \dots + a(b_{n-1}2^{n-1}) \end{aligned}$$

```
procedure multiply(a, b: positive integers)
{the binary expansions of a and b are  $(a_{n-1}, a_{n-2}, \dots, a_0)_2$  and  $(b_{n-1}, b_{n-2}, \dots, b_0)_2$ , respectively}
for j := 0 to n - 1
    if  $b_j = 1$  then  $c_j = a$  shifted j places
    else  $c_j := 0$ 
{ $c_0, c_1, \dots, c_{n-1}$  are the partial products}
p := 0
for j := 0 to n - 1
    p := p +  $c_j$ 
return p {p is the value of ab}
```



Algorithm: Binary Multiplication of Integers

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```

$O(n^2)$ shifts and $O(n^2)$ bit additions



Algorithm: Computing div and mod

```
procedure division algorithm (a: integer, d: positive integer)
  q := 0
  r := |a|
  while r ≥ d
    r := r - d
    q := q + 1
  if a < 0 and r > 0 then
    r := d - r
    q := -(q+1)
  return (q, r) {q = a div d is the quotient, r = a mod d is the
  remainder }
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$O(q \log a)$ bit operations. But there exist more efficient algorithms with complexity $O(n^2)$, where $n = \max(\log a, \log d)$



Algorithm: Binary Modular Exponentiation

$$b^n = b^{a_{k-1} \cdot 2^{k-1} + \dots + a_1 \cdot 2 + a_0} = b^{a_{k-1} \cdot 2^{k-1}} \dots b^{a_1 \cdot 2} \cdot b^{a_0}$$

Successively finds $b \bmod m$, $b^2 \bmod m$, $b^4 \bmod m$, \dots , $b^{2^{k-1}} \bmod m$, and multiplies together the terms $b^{2^j} \bmod m$ where $a_j = 1$.

```
procedure modular_exponentiation(b: integer,  $n = (a_{k-1}a_{k-2}\dots a_1a_0)_2$ , m: positive integers)
  x := 1
  power := b mod m
  for i := 0 to k - 1
    if  $a_i = 1$  then x := (x · power) mod m
    power := (power · power) mod m
  return x {x equals  $b^n \bmod m$ }
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     $power := (power \cdot power) \bmod m$ 
  return  $x$  { $x$  equals  $b^n \bmod m$ }
```

$O((\log m)^2 \log n)$ bit operations



Next Lecture

- number theory, cryptography ...

