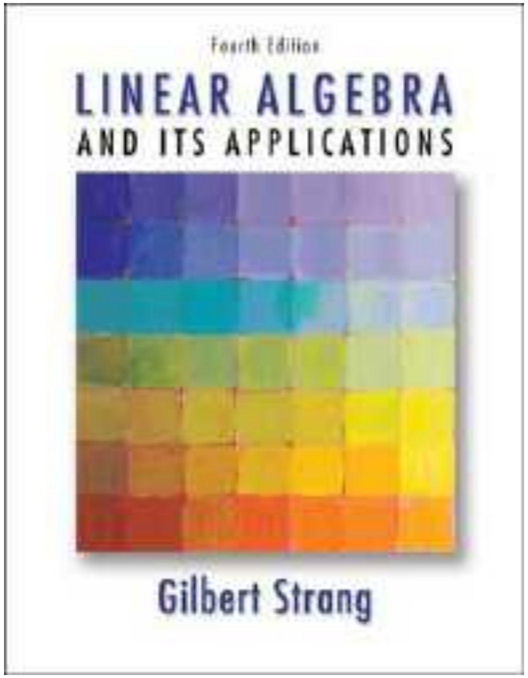
Linear Algebra



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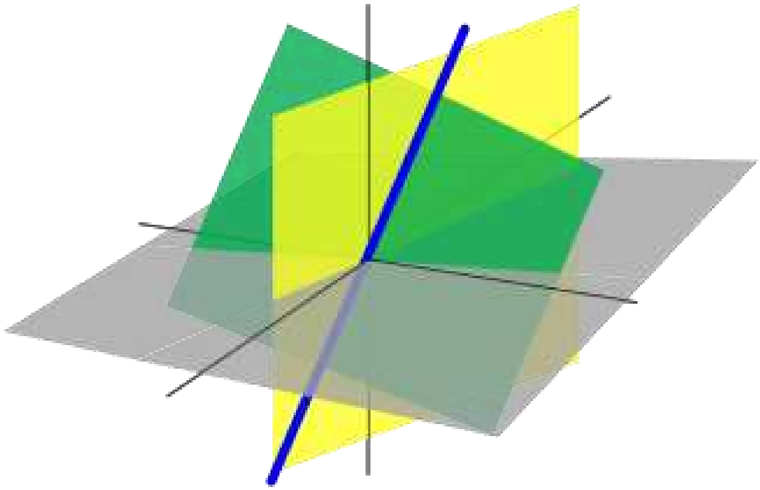
2 Vector Spaces (向量空间)

2.3

LINEAR INDEPENDENCE, BASIS AND DIMENSION （线性无关性、基和维数）

- Linear Independence
- Basis
- Coordinates (坐标)
- Dimension



I. Introduction

在三维几何向量空间 $\mathbf{R}^3$ 中,

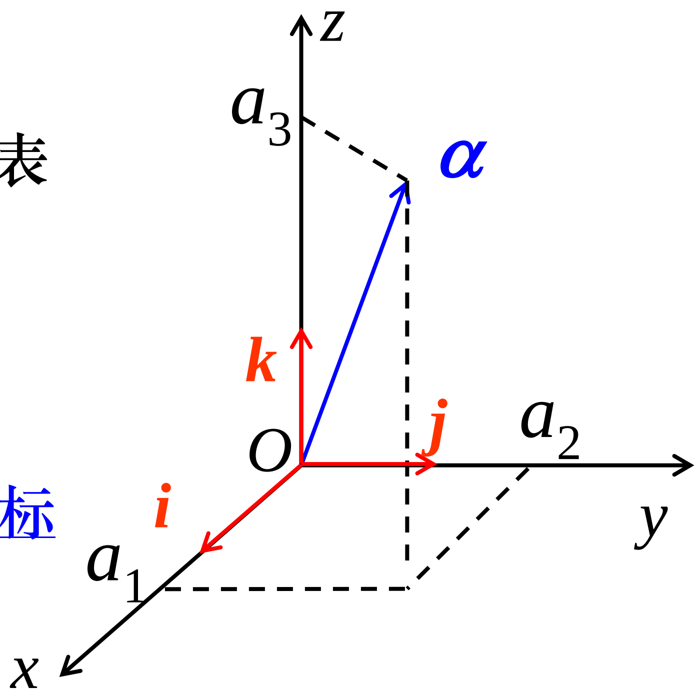
$$\boldsymbol{i} = (1,0,0)^{\mathrm{T}},\ \boldsymbol{j} = (0,1,0)^{\mathrm{T}},\ \boldsymbol{k} = (0,0,1)^{\mathrm{T}}.$$

$$\boldsymbol{\alpha} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}.$$

向量$(a_1, a_2, a_3)^{\mathrm{T}}$是 $\boldsymbol{\alpha}$ 关于一组基 $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ 的坐标.

- 向量组$\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$线性无关
- 向量组$\{\boldsymbol{\alpha}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$线性相关
- $\mathbf{R}^3$ 中的任何一个向量可以由$\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$线性表示, 但不可以仅由它的子集表示:

  $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$是$\mathbf{R}^3$ 的一组基
- $\mathbf{R}^3$的维数是3 (基含有的向量个数)
- 系数$(a_1, a_2, a_3)^{\mathrm{T}}$是向量$\boldsymbol{\alpha}$在这组基下的坐标



Use your geometric experience with $\mathbf{R}^2$ and $\mathbf{R}^3$ to visualize general concepts

In $\mathbf{R}^n$, let

$$\boldsymbol{e}_1 = (1,0,0,\ \ldots,\ 0,0)^{\mathrm{T}},$$
$$\boldsymbol{e}_2 = (0,1,0,\ \ldots,\ 0,0)^{\mathrm{T}},$$
$$\ldots$$
$$\boldsymbol{e}_n = (0,0,0,\ \ldots,\ 0,1)^{\mathrm{T}}.$$

Then $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is a spanning set of $\mathbf{R}^n$, since each vector $\boldsymbol{v} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is a linear combination of them:

$$\boldsymbol{v} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \cdots + x_n\boldsymbol{e}_n.$$

It is obvious that no *proper subset* (真子集) of $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is a spanning set of $\mathbf{R}^n$.

Let V be a subspace of $\mathbf{R}^n$.

A set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ is called a ***basis* (一组基)** if it is a *spanning set* of V but any proper subset is not.

Bases are important in the study of vector spaces.

Recall: **Spanning Sets**

Recall that, if V is a subspace of $\mathbf{R}^n$ and A is a subset of $\mathbf{R}^n$ such that

$$V = \mathrm{span}(A)$$

then V is the **span** of A, and A is a **spanning set** for V.

Example 1 Let $A = \{\boldsymbol{u}, \boldsymbol{v}\}$, defined below.

(1) If $\boldsymbol{u} = (1,1)^T$ and $\boldsymbol{v} = (2,2)^T$, then

$$\mathrm{span}(A) = \{(\alpha, \alpha)^T \mid \alpha \in \mathbf{R}\}.$$

(2) For $\boldsymbol{u} = (1,1)^T$ and $\boldsymbol{v} = (1,2)^T$, then

$$\mathrm{span}(A) = \{(\alpha, \beta)^T \mid \alpha, \beta \in \mathbf{R}\} = \mathbf{R}^2.$$

Example 2 Let $A = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$, defined below.

(1) If $\boldsymbol{u} = (1,0,0)^T$, $\boldsymbol{v} = (0,1,0)^T$, and $\boldsymbol{w} = (2,3,0)^T$, then

$$\mathrm{span}(A) = \{(\alpha, \beta, 0)^T \mid \alpha, \beta \in \mathbf{R}\}.$$

(2) Let $\boldsymbol{u} = (1,0,0)^T$, $\boldsymbol{v} = (0,1,0)^T$ and $\boldsymbol{w} = (0,0,1)^T$, then

$$\mathrm{span}(A) = \mathbf{R}^3.$$

A natural question is how to determine whether a given set of vectors is a spanning set of a vector space?

Example 3 Let $A = \{(1,1,1)^T, (1,3,5)^T, (1,2,3)^T\}$. Is A a spanning set for $\mathbf{R}^3$?

We need to see if *any* vector $\boldsymbol{v} = (a, b, c)^T \in \mathbf{R}^3$ is a linear combination of A. Let

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

for some scalars x_1, x_2, x_3. This forms a system of linear equations.

If this system has solutions for *any* $(a, b, c)^T$, then A is a spanning set; otherwise, A is not a spanning set .

Answer?

What if -- Let $A = \{(1,1,1)^T, (1,3,5)^T, (1,2,4)^T\}$?

II. Linear Independence

实例

某调料公司用6种成分制造了6 种调味品

每包调味品所需各成分的量

成分 \ 调味品	A	B	C	D	E	F
红辣椒	3	1.5	4.5	7.5	9	4.5
姜黄	2	4	0	8	1	6
胡椒	1	2	0	4	2	3
大蒜粉	0.5	1	0	2	2	1.5
盐	0.5	1	0	2	2	1.5
丁香油	0.25	0.5	0	2	1	0.75

- 顾客是否可只买其中部分调味品，并配出其余几种?
- 最少要购买几种调味品？哪几种?
- 能否配制出下列新调味品?

红辣椒:18; 姜黄:18; 胡椒:9; 大蒜粉:4.8; 盐:4.5; 丁香油: 3.25.

II. Linear Independence

Definition 1 (*Linear dependence*). Let $A = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ ($k \geq 2$) be a set of vectors of $\mathbf{R}^n$. Then

(i) A is called **linearly dependent (线性相关)** if *one of* the vectors can be expressed as a linear combination of the others,

i.e., there exists $\boldsymbol{v}_i \in A$ such that

$$\boldsymbol{v}_i = \sum_{j \neq i} \lambda_j \boldsymbol{v}_j$$

where λ_j's are numbers;

(ii) A is called **linearly independent (线性无关)** if *no* vector in A is a linear combination of the others.

A set containing a single vector $\boldsymbol{v}$ ($k=1$) is linearly independent if and only if $\boldsymbol{v} \neq \boldsymbol{0}$. (仅有一个向量$\boldsymbol{v}$的集合线性无关当且仅当$\boldsymbol{v} \neq \boldsymbol{0}$)

For example,

- $\{(1,1)^T, (2, 2)^T\}$ is linearly dependent.
- $\{(1, 1)^T, (1, 2)^T\}$ is linearly independent.
- $\{(1, 0, 0)^T, (0, 1, 0)^T, (2, 3, 0)^T\}$ is linearly dependent.
- $\{(1, 0, 0)^T, (0, 1, 0)^T, (2, 3, 1)^T\}$ is linearly independent.

The next theorem provides us with a method for deciding whether a set of vectors is linearly independent.

Theorem 1 *Let* $A = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ *be a set of vectors of* $\mathbf{R}^n$. *Then*

(1) *A is* linearly independent *if and only if*

$$\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 + \cdots + \lambda_k\boldsymbol{v}_k = \boldsymbol{0}$$

holds ***only for*** $\boldsymbol{\lambda_1 = \lambda_2 = \cdots = \lambda_k} = 0$*; equivalently*

(2) *A is* linearly dependent *if and only if*

$$\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2 + \cdots + \lambda_k\boldsymbol{v}_k = \boldsymbol{0}$$

holds ***for some scalars*** $\boldsymbol{\lambda_1, \lambda_2, \ldots, \lambda_k}$ ***which are not all zeros***.

Note: A set of vectors containing the zero vector must be linearly dependent. (含零向量的向量组必定线性相关.)

***A Method*:** Given $A = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\} \subset \mathbf{R}^n$, the following process decides if A is linearly independent.

Step 1. Set up a vector equation

$$\lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2 + \cdots + \lambda_k \boldsymbol{v}_k = \boldsymbol{0}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are unknowns.

Step 2. Write this vector equation into a system of linear equations with unknowns $\lambda_1, \lambda_2, \ldots, \lambda_k$ (关于$\lambda_1, \lambda_2, \ldots, \lambda_k$的线性方程组).

Step 3. Solve the system of linear equations.

Step 4. Discussion: If $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$ is the only solution (该线性方程组只有零解, trivial), then A is linearly independent (线性无关).

If there is a solution such that some $\lambda_i \neq 0$ (该线性方程组有非零解, 即某些λ_i的取值非零, nontrivial), then A is linearly dependent (线性相关).

Example 4 Let

$$\boldsymbol{u} = (1, 0, -1, 0)^T, \boldsymbol{v} = (1, 1, 0, 2)^T, \boldsymbol{w} = (0, 3, 1, -2)^T.$$

Is $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ linearly independent?

Solution The key to answer this question is to solve the equation

$$\lambda_1\boldsymbol{u} + \lambda_2\boldsymbol{v} + \lambda_3\boldsymbol{w} = \boldsymbol{0}.$$

In terms of components, we have a system of linear equations

$$\lambda_1\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}+\lambda_2\begin{bmatrix}1\\1\\0\\2\end{bmatrix}+\lambda_3\begin{bmatrix}0\\3\\1\\-2\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

To solve this system of equations, we reduce the augmented matrix to row echelon form

$$\left[\begin{array}{ccc|c}1 & 1 & 0 & 0\\0 & 1 & 3 & 0\\-1 & 0 & 1 & 0\\0 & 2 & -2 & 0\end{array}\right]\to\left[\begin{array}{ccc|c}1 & 1 & 0 & 0\\0 & 1 & 3 & 0\\0 & 0 & -2 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

This system only has solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and so $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly independent.

Example 5 Let

$$\boldsymbol{u} = (1, 0, -1, 0)^T, \boldsymbol{v} = (1, 1, 0, 2)^T, \boldsymbol{w} = (1, 3, 2, 6)^T.$$

Is $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ linearly independent?

Solution The key to answer this question is to solve the equation

$$\lambda_1\boldsymbol{u} + \lambda_2\boldsymbol{v} + \lambda_3\boldsymbol{w} = \boldsymbol{0}.$$

In terms of components, we have a system of linear equations. To solve this system of equations, we reduce the augmented matrix to row echelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 2 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now, the system of equations has non-trivial solutions, so $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly dependent.

Two independent rows;
Two independent columns

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

For instance, $(\lambda_1, \lambda_2, \lambda_3) = (2, -3, 1)$ is a solution, and hence

$$2\boldsymbol{u} - 3\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{0}.$$

Example 6 The columns of the following triangular matrix are *linearly independent*. It has **no zeros on the diagonal**.

$$\boldsymbol{A}=\begin{bmatrix}3 & 4 & 2\\ 0 & 1 & 5\\ 0 & 0 & 2\end{bmatrix}$$

Look for a combination of the columns that makes zero:

Solve $\boldsymbol{Ac} = \boldsymbol{0}$:

$$c_1\begin{bmatrix}3\\0\\0\end{bmatrix}+c_2\begin{bmatrix}4\\1\\0\end{bmatrix}+c_3\begin{bmatrix}2\\5\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The only combination to produce the zero vector is the trivial combination ($c_1 = c_2 = c_3 = 0$).

***The nullspace of A contains only the zero vector*.**

***The columns of A are independent exactly when $N(A) = \{\boldsymbol{0}\}$*.**

III. Basis

Definition 2 (*Basis*) Let V be a subspace of $\mathbf{R}^n$, and let A be a set of vectors of V. Then A is called a **basis (基)** for V if

- A is a spanning set for V, i.e., $V = \text{span}(A)$, and ***(not too few vectors)***
- A is linearly independent. ***(not too many vectors)***

A basis is: **最小的生成集**和**最大的线性无关组**

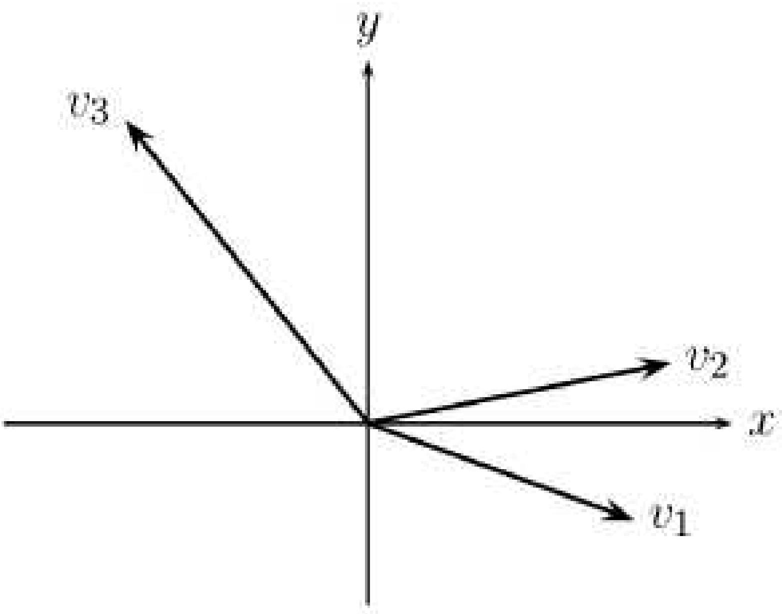
a **minimal** spanning set *(It cannot be made smaller and still span the space)*, and a **maximal** linearly independent set *(It cannot be made larger without losing independence)*.

For example,

- $\{(1,0)^T, (0,1)^T\}$ is a basis for $\mathbf{R}^2$, and so is $\{(1,0)^T, (1,1)^T\}$.
- $\{(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T\}$ is a basis for $\mathbf{R}^3$, and so is $\{(1, 0, 0)^T, (0, 2, 0)^T, (1, 1, 1)^T\}$.
- In $\mathbf{R}^n$, the set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is a basis for $\mathbf{R}^n$, called the **standard basis (标准基, 或自然基)** of $\mathbf{R}^n$.

The *x*-*y* plane in the figure is just $\mathbf{R}^2$. The vector $\boldsymbol{v}_1$ by itself is linearly independent, but it fails to span $\mathbf{R}^2$. The three vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ certainly span $\mathbf{R}^2$, but are not independent. *Any two* of these vectors, say $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, have both properties—they span, and they are independent. So they form a basis.

Notice again that *a vector space does not have a unique basis*. (向量空间的基不唯一)



A spanning set: $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$.
Bases: $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_1$, $\boldsymbol{v}_3$ and $\boldsymbol{v}_2$, $\boldsymbol{v}_3$.

A Method for deciding if A is a basis:

Show that

(1) each vector in A lies in V,

(2) A is linearly independent, i.e., none of the vectors in A is a linear combination of the others,

(3) A is a spanning set for V, i.e., every vector of V is a linear combination of A.

Example 7 Let $V = \{(x, y, z)^{\mathrm{T}} | x + y + z = 0\}$. Show that V is a subspace of $\mathbf{R}^3$, and find a basis for V.

(1) Show that V is a subspace of $\mathbf{R}^3$.

1) $(0,0,0)^{\mathrm{T}} \in V$ since $0 + 0 + 0 = 0$.

2) $\forall \boldsymbol{u} = (x_1, y_1, z_1)^{\mathrm{T}}, \boldsymbol{v} = (x_2, y_2, z_2)^{\mathrm{T}} \in V$, then $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$, and therefore

$$\boldsymbol{u} + \boldsymbol{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)^{\mathrm{T}} \in V$$

since

$$\begin{aligned}&(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2)\\&= (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0 + 0 = 0\end{aligned}$$

3) $\forall k \in \mathbf{R}, \forall \boldsymbol{u} = (x, y, z)^{\mathrm{T}} \in V$,

$$k\boldsymbol{u} = (kx, ky, kz)^{\mathrm{T}} \in V$$

since $kx + ky + kz = k(x + y + z) = 0$.

So V is a subspace of $\mathbf{R}^3$.

Example 7 Let $V = \{(x,y,z)^{\mathrm{T}} | x+y+z=0\}$. Show that V is a subspace of $\mathbf{R}^3$, and find a basis for V.

(2) Find a basis for V.

$\forall \boldsymbol{u} = (x,y,z)^{\mathrm{T}} \in V$, then $x+y+z=0$ and

$$\boldsymbol{u} = (-y-z,y,z)^{\mathrm{T}} = y(-1,1,0)^{\mathrm{T}} + z(-1,0,1)^{\mathrm{T}}.$$

<u>We claim that $\{(-1,1,0)^{\mathrm{T}}, (-1,0,1)^{\mathrm{T}}\}$ is a basis for V</u> since

1) Clearly $(-1,1,0)^{\mathrm{T}}, (-1,0,1)^{\mathrm{T}} \in V$.

2) $(-1,1,0)^{\mathrm{T}}, (-1,0,1)^{\mathrm{T}}$ are linearly independent since

$$c_1(-1,1,0)^{\mathrm{T}} + c_2(-1,0,1)^{\mathrm{T}} = (0,0,0)^{\mathrm{T}}$$

$\Longrightarrow (-c_1-c_2, c_1, c_2)^{\mathrm{T}} = (0,0,0)^{\mathrm{T}} \Longrightarrow c_1 = c_2 = 0.$

3) $\forall \boldsymbol{u} = (x,y,z)^{\mathrm{T}} \in V$,

$$\boldsymbol{u} = (-y-z,y,z)^{\mathrm{T}} = y(-1,1,0)^{\mathrm{T}} + z(-1,0,1)^{\mathrm{T}}$$

Every vector of V is a linear combination of $(-1,1,0)^{\mathrm{T}}$ and $(-1,0,1)^{\mathrm{T}}$.

IV. Coordinates

An important property of a basis is that it provides a unique representation for each vector.

Theorem 2 *Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for a space V. Then for each vector $\mathbf{w} \in V$, there is a* ***unique*** *choice of scalars $a_1, a_2, \ldots, a_k$ such that*

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k .$$

We call the scalars $a_1, a_2, \ldots, a_k$ the **coordinates (坐标)** of $\mathbf{w}$ in the basis B, denoted by $[\mathbf{w}]_B$.

Why -- unique?

$$\left.\begin{aligned} \mathbf{w} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k \\ \mathbf{w} &= b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k \end{aligned}\right\} \Longrightarrow$$

$$\mathbf{0} = (a_1-b_1)\mathbf{v}_1 + (a_2-b_2)\mathbf{v}_2 + \cdots + (a_k-b_k)\mathbf{v}_k$$

Example 8

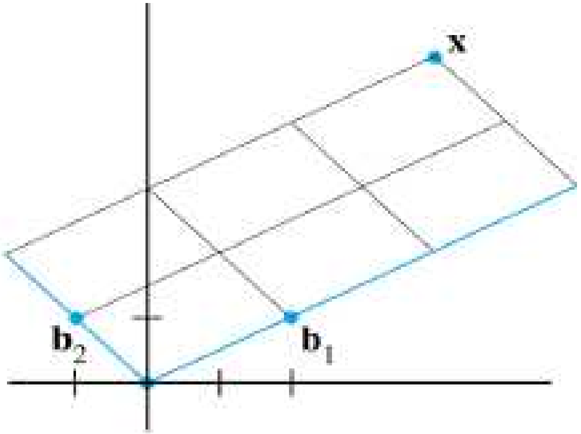
- $A = \{(1, 0)^T, (0, 1)^T\}$ is a basis for $\mathbf{R}^2$, and so is $B = \{(2, 1)^T, (-1, 1)^T\}$.

The vector $\boldsymbol{x} = (4, 5)^T$ has coordinates

$$[\boldsymbol{x}]_A = (4, 5)^T,$$
$$[\boldsymbol{x}]_B = (3, 2)^T.$$

$$c_1\begin{bmatrix}2\\1\end{bmatrix}+c_2\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}4\\5\end{bmatrix}\text{ i.e., }\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}4\\5\end{bmatrix}$$

This equation can be solved by row operations on an augmented matrix or by using the inverse of the coefficient matrix on the left.



The $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is $(3, 2)^T$.

Example 9 The standard basis $B_1=\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is a basis for $\mathbf{R}^n$, and so is $B_2=\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n\}$,

where $\boldsymbol{\beta}_1=(1,-1,0,\cdots,0)^{\mathrm{T}}$, $\boldsymbol{\beta}_2=(0,1,-1,0,\cdots,0)^{\mathrm{T}}$, …,

$\boldsymbol{\beta}_{n-1}=(0,\cdots,0,1,-1)^{\mathrm{T}}$, $\boldsymbol{\beta}_n=(0,\cdots,0,1)^{\mathrm{T}}$.

Determine the coordinates of the vector $\boldsymbol{\alpha}=(a_1,a_2,\cdots,a_n)^{\mathrm{T}}$ in B_1 and B_2.

Solution It is easy to see that

$$\boldsymbol{\alpha}=a_1\boldsymbol{e}_1+a_2\boldsymbol{e}_2+\ldots+a_n\boldsymbol{e}_n,$$

so

$$[\boldsymbol{\alpha}]_{B_1}=(a_1,a_2,\ldots,a_n)^{\mathrm{T}}.$$

For the basis $B_2=\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n\}$, we suppose that

$$\boldsymbol{\alpha}=x_1\boldsymbol{\beta}_1+x_2\boldsymbol{\beta}_2+\cdots+x_n\boldsymbol{\beta}_n$$

$$\boldsymbol{\beta}_1 = (1,-1,0,\cdots,0)^{\mathrm{T}}, \quad \boldsymbol{\beta}_2 = (0,1,-1,0,\cdots,0)^{\mathrm{T}}, \ \ldots,$$
$$\boldsymbol{\beta}_{n-1} = (0,\cdots,0,1,-1)^{\mathrm{T}}, \quad \boldsymbol{\beta}_n = (0,\cdots,0,1)^{\mathrm{T}},$$

$$\boldsymbol{\alpha} = x_1\boldsymbol{\beta}_1 + x_2\boldsymbol{\beta}_2 + \cdots + x_n\boldsymbol{\beta}_n = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

Place the vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n$ into column forms of the system, then

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}.$$

By solving the system, we can get

$$[\boldsymbol{\alpha}]_{B_2} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$$
$$= \begin{pmatrix} a_1 \\ a_1 + a_2 \\ \vdots \\ a_1 + a_2 + \ldots + a_{n-1} + a_n \end{pmatrix}.$$

V. Dimension

The most fundamental parameter for a vector space would be its dimension.

Theorem 3 *Every basis for a subspace V of $\mathbf{R}^n$ contains the same number of vectors.*

(*Proof: see next slide or G. Strang: LA and its applications, p97*)

Definition 3 (*Dimension*) The number of vectors in a basis for a vector space V is called the **dimension (维数)** of V .

For example,

- $B = \{(1, 1)^T, (1, 2)^T\}$ is a basis for $\mathbf{R}^2$, and so is $C = \{(1, 0)^T, (1, 1)^T\}$. $\mathbf{R}^2$ is of dimension 2.
- $B = \{(2, 0, 0)^T, (1, 1, 0)^T, (1, 1, 1)^T\}$ is a basis for $\mathbf{R}^3$, and so is $C = \{(1, 0, 0)^T, (0, 2, 0)^T, (1, 1, 1)^T\}$. $\mathbf{R}^3$ is of dimension 3.

Proof of Theorem 3 (If $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ and $\boldsymbol{w}_1, \dots, \boldsymbol{w}_n$ are both bases for the same vector space, then $m = n$.)

Proof. Suppose there are more $\boldsymbol{w}$'s than $\boldsymbol{v}$'s ($n > m$).
Since the $\boldsymbol{v}$'s form a basis, they must span the space.
Every $\boldsymbol{w}_j$ can be written as a combination of the $\boldsymbol{v}$'s:

$$\boldsymbol{w}_1 = a_{11}\boldsymbol{v}_1 + \cdots + a_{m1}\boldsymbol{v}_m, \ \ldots, \boldsymbol{w}_n = a_{1n}\boldsymbol{v}_1 + \cdots + a_{mn}\boldsymbol{v}_m.$$

So we have

$$[\boldsymbol{w}_1 \quad \boldsymbol{w}_2 \quad \cdots \quad \boldsymbol{w}_n] = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \cdots \quad \boldsymbol{v}_m]\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

which can be rewritten as

$$\boldsymbol{W} = \boldsymbol{V}\boldsymbol{A}. \quad (\boldsymbol{A} \text{ is } m \text{ by } n)$$

There is a nonzero solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ (since $n > m$),

$$\boxed{r \le m < n}$$

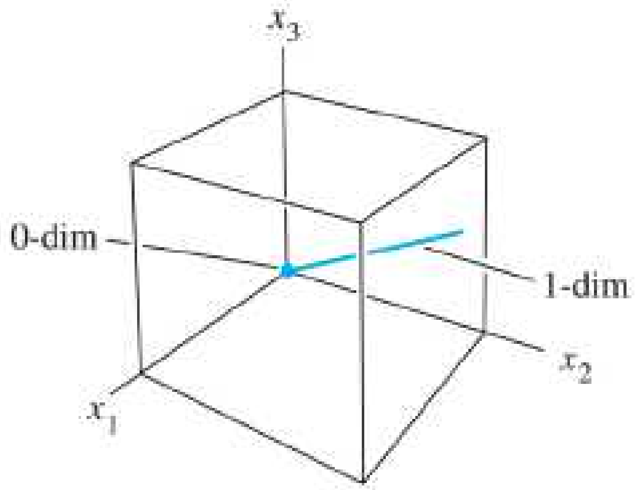
then $\boldsymbol{V}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$,
which is $\boldsymbol{W}\boldsymbol{x} = \boldsymbol{0}$. (*A combination of the $\boldsymbol{w}$'s gives zero!*)

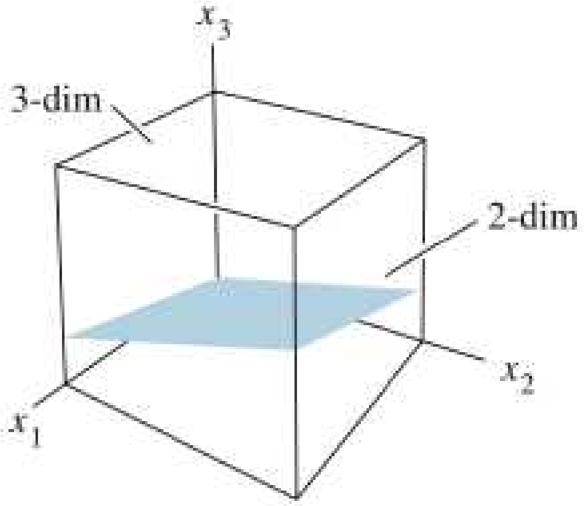
$$\boxed{n - r > 0}$$

The $\boldsymbol{w}$'s could not be a basis. So we cannot have $n > m$.
Similarly, we cannot have $m > n$.
The only way to avoid a contradiction is to have $m = n$.

Corollary (推论): *Let V be a space of dimension k. Then*
(a) *any set of more than k vectors is linearly dependent,*
(b) *any set of less than k vectors is not a spanning set,*
(c) *a proper subspace of* $\mathbf{R}^n$ *has dimension less than n.*

For example,
- Any set of 4 vectors of $\mathbf{R}^3$ is linearly dependent.
- Any set of 2 vectors of $\mathbf{R}^3$ is not a spanning set.
- A proper subspace of $\mathbf{R}^3$ has dimension at most 2.





The results in the following theorem are important and very useful.

Theorem 4 *Let V be a space. Then*

(1) *any linearly independent set of V can be extended to be a basis;*

(2) *any spanning set for V contains a basis.*

Example 10 (*Basis from a spanning set*) Let

$$A = \{(1, 1, -2)^T, (-2, -2, 4)^T, (-1, -2, 3)^T, (1, -1, 0)^T\},$$

and let $V = \text{span}(A)$. Find a basis contained in A.

Solution

$$\begin{array}{cccc} & & & \\ \left[\begin{array}{cccc} 1 & -2 & -1 & 1 \\ 1 & -2 & -2 & -1 \\ -2 & 4 & 3 & 0 \end{array}\right] \\ \uparrow\ \ \ \uparrow\ \ \ \uparrow\ \ \ \uparrow \\ \boldsymbol{b}_1\ \ \boldsymbol{b}_2\ \ \boldsymbol{b}_3\ \ \boldsymbol{b}_4 \end{array} \rightarrow \begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{c} \left[\begin{array}{cccc} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \uparrow\ \ \ \uparrow\ \ \ \uparrow\ \ \ \uparrow \\ \boldsymbol{c}_1\ \ \boldsymbol{c}_2\ \ \boldsymbol{c}_3\ \ \boldsymbol{c}_4 \end{array}$$

$\boldsymbol{b}_2 = -2\boldsymbol{b}_1$ $\quad$ $\boldsymbol{b}_4 = 3\boldsymbol{b}_1 + 2\boldsymbol{b}_3$ $\qquad$ $\boldsymbol{c}_2 = -2\boldsymbol{c}_1$ $\quad$ $\boldsymbol{c}_4 = 3\boldsymbol{c}_1 + 2\boldsymbol{c}_3$

初等行变换不改变列向量之间的线性相关性！ Why?

Basis: $\{\boldsymbol{b}_1, \boldsymbol{b}_3\}$, or $\{\boldsymbol{b}_2, \boldsymbol{b}_3\}$, or $\{\boldsymbol{b}_1, \boldsymbol{b}_4\}$, or $\{\boldsymbol{b}_2, \boldsymbol{b}_4\}$, or $\{\boldsymbol{b}_3, \boldsymbol{b}_4\}$.

Example 11 Find the dimension and a basis for the vector space $\mathbf{R}^{2\times 2}$.

Solution. Let

$$\boldsymbol{K}_{11}=\begin{bmatrix}1&0\\0&0\end{bmatrix},\ \boldsymbol{K}_{12}=\begin{bmatrix}0&1\\0&0\end{bmatrix},\ \boldsymbol{K}_{21}=\begin{bmatrix}0&0\\1&0\end{bmatrix},\ \boldsymbol{K}_{22}=\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

Then $\boldsymbol{K}_{11},\boldsymbol{K}_{12},\boldsymbol{K}_{21},\boldsymbol{K}_{22}$ are linearly independent.

In fact, by $a\boldsymbol{K}_{11}+b\boldsymbol{K}_{12}+c\boldsymbol{K}_{21}+d\boldsymbol{K}_{22}=\boldsymbol{0}$, i.e., $\begin{bmatrix}a&b\\c&d\end{bmatrix}=\boldsymbol{0}$, we have $a=b=c=d=0$.

And $\forall \boldsymbol{A}=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\in\mathbf{R}^{2\times 2}$, it can be expressed as

$$\boldsymbol{A}=a_{11}\boldsymbol{K}_{11}+a_{12}\boldsymbol{K}_{12}+a_{21}\boldsymbol{K}_{21}+a_{22}\boldsymbol{K}_{22},$$

so $\boldsymbol{K}_{11},\boldsymbol{K}_{12},\boldsymbol{K}_{21},\boldsymbol{K}_{22}$ is a basis for $\mathbf{R}^{2\times 2}$, and the dimension of space $\mathbf{R}^{2\times 2}$ is 4.

Remark. The coordinates of the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in this coordinate system are $(a_{11}, a_{12}, a_{21}, a_{22})$.

In general, the vector space $\mathbf{R}^{m \times n}$ is of dimension $m \times n$,

$$K_{ij} = \begin{array}{c} \begin{array}{cc} \begin{bmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & 1 & \\ & & & \ddots & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & 0 \end{bmatrix} & \begin{array}{c} \\ \\ \text{Row } i \\ \\ \\ \\ \\ \end{array} \end{array} \\ \text{column } j \end{array} \qquad \begin{array}{l} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, n \end{array}$$

is a basis for $\mathbf{R}^{m \times n}$.

$$\forall A = [a_{ij}] \in \mathbf{R}^{m \times n}, \qquad A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} K_{ij} .$$

Key words:

Linearly dependent, linearly independent, spanning set, basis, dimension, coordinate

Homework

See Blackboard

Note: 20(c): If $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = 0$, then $\boldsymbol{u}$, $\boldsymbol{v}$ are called **perpendicular (垂直)**.