

# *Linear Algebra*



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## 1

## Matrices and Gaussian Elimination

## 1.2

## GEOMETRY OF LINEAR EQUATIONS

(线性方程组的几何解释)



GAUSS

Divide line 1 by 3

$$\begin{array}{ccc|c} 1) & 3 & -5 & 9 & -1.6 \\ 2) & -1 & 7 & 0 & 8 \\ 3) & 3 & 3 & 4 & -2 \end{array}$$

line 2 - (Line 1 \* -1)

$$\begin{array}{ccc|c} 1) & 1 & -1.667 & 3 & -0.533 \\ 2) & -1 & 7 & 0 & 8 \\ 3) & 3 & 3 & 4 & -2 \end{array}$$

line 3 - (Line 1 \* 3)

$$\begin{array}{ccc|c} 1) & 1 & -1.667 & 3 & -0.533 \\ 2) & 0 & 5.333 & 3 & 7.467 \\ 3) & 3 & 3 & 4 & -2 \end{array}$$

$$\begin{array}{ccc|c} 1) & 1 & -1.667 & 3 & -0.533 \\ 2) & 0 & 1 & 0.563 & 1.4 \\ 3) & 0 & 0 & -9.5 & -11.6 \end{array}$$

$$\begin{array}{ccc|c} 1) & 1 & -1.667 & 3 & -0.533 \\ 2) & 0 & 1 & 0.563 & 1.4 \\ 3) & 0 & 0 & 1 & 1.221 \end{array}$$

$$\begin{array}{ccc|c} 1) & 1 & 0 & 0 & -3.0081 \\ 2) & 0 & 1 & 0 & 0.7126 \\ 3) & 0 & 0 & 1 & 1.221 \end{array}$$

$$X_1 = -3.0081$$

$$X_2 = 0.7126$$

$$X_3 = 1.221$$



- A system of linear equations has

1. *exactly one solution*, or **1&3: consistent(相容)**

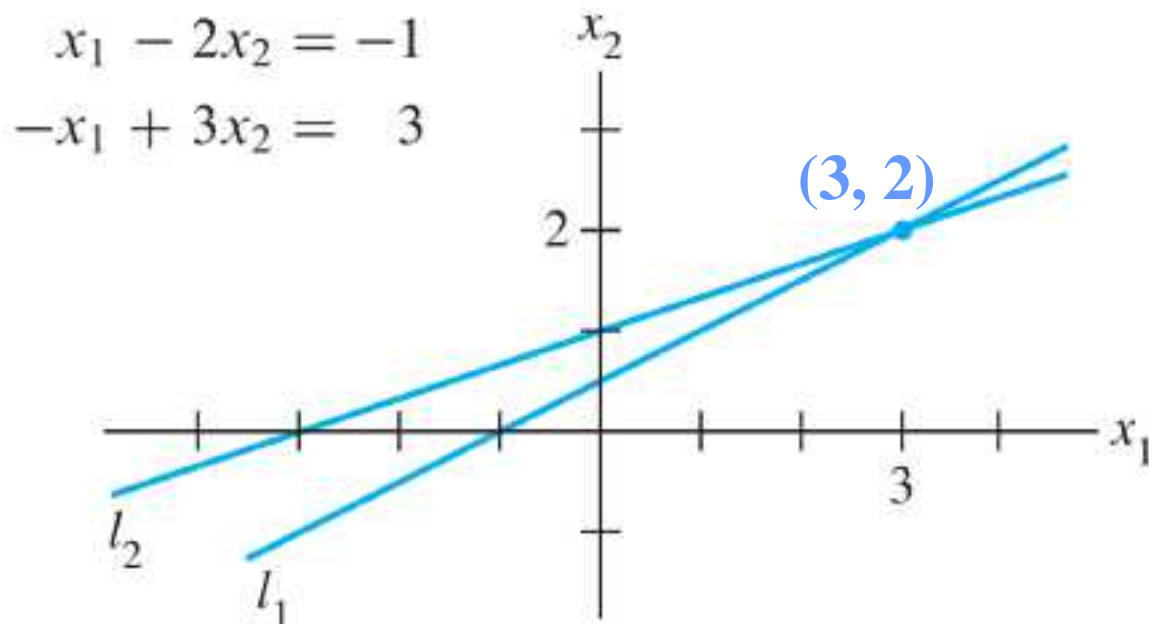
2. *no solution*, or

3. *infinitely many solutions*.

**2&3: Singular(奇异) cases:**

*none or too many solutions*

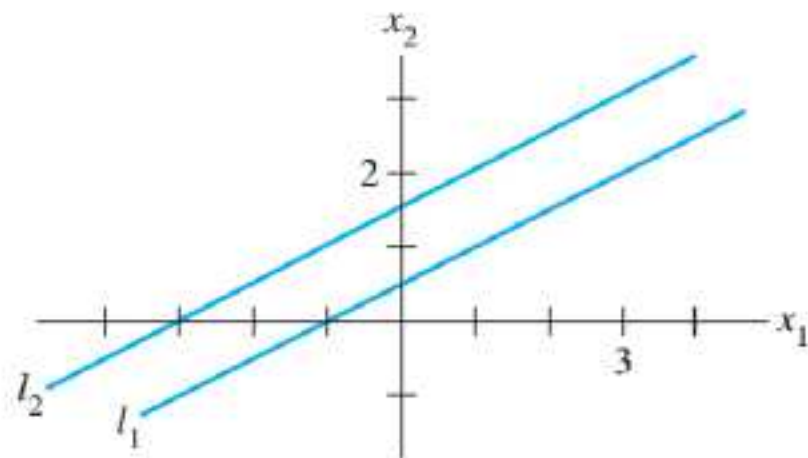
**Why?**



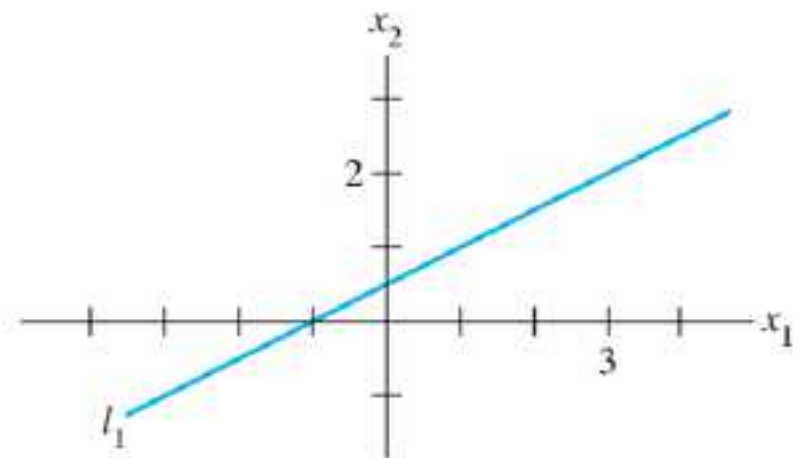
**FIGURE 1** Exactly one solution.

(a)  $x_1 - 2x_2 = -1$   
 $-x_1 + 2x_2 = 3$

(b)  $x_1 - 2x_2 = -1$   
 $-x_1 + 2x_2 = 1$



(a)



(b)

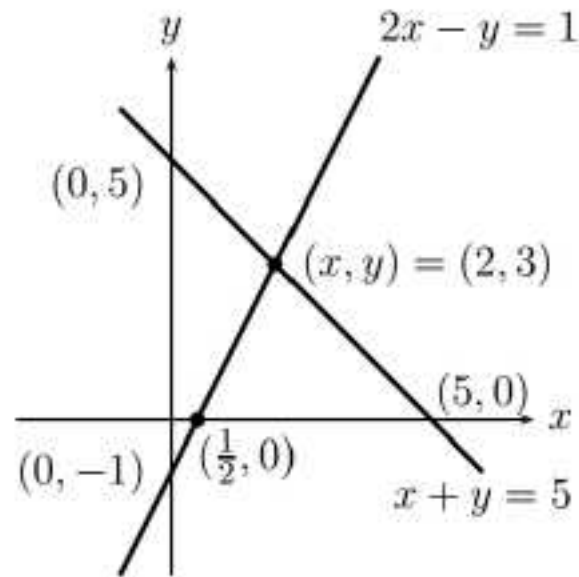
**FIGURE 2** (a) No solution. (b) Infinitely many solutions.

$$n = 2:$$

***Row Picture***

$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$$

We can look at that system by rows **and** by columns.



(a) Lines meet at  $x = 2, y = 3$

**Row picture**  
**(two lines)**

理解Column Picture之前，回顾：What is a vector (向量)?

## $n$ 维向量

**定义1**  $n$  个有序的数  $a_1, a_2, \dots, a_n$  所组成的数组称为  $n$  维向量, 记为

$$\alpha = (a_1, a_2, \dots, a_n),$$

其中  $a_i$  称为向量  $\alpha$  的第  $i$  个分量(或坐标).

分量全为零的向量称为零向量, 记为  $\mathbf{0} = (0, 0, \dots, 0)$ .


分量全为实数的向量称为实向量(real vector),

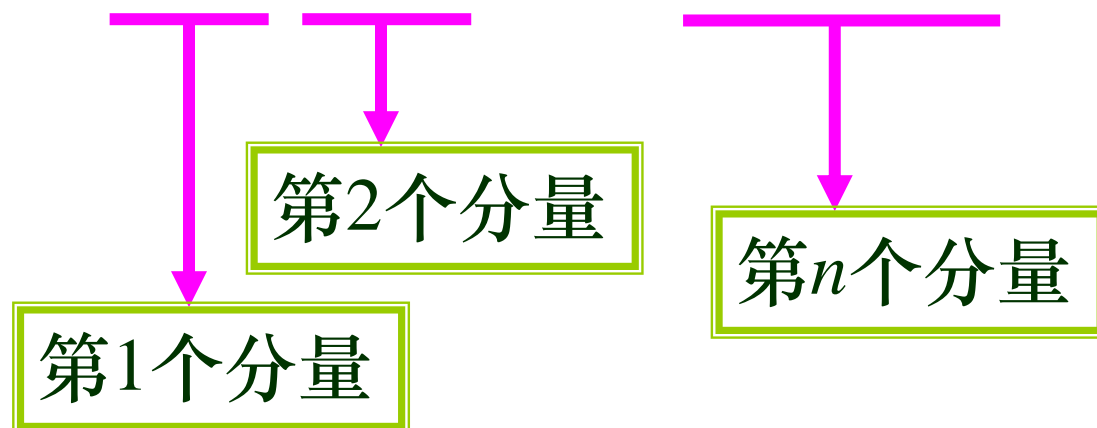
分量为复数的向量称为复向量(complex vector).

全体  $n$  维实向量的集合记为  $\mathbf{R}^n$ .

例如

$(1, 2, 3, \dots, n)$    $n$  维实向量

$(1 + 2i, 2 + 3i, \dots, n + (n + 1)i)$    $n$  维复向量



$(1, 2i, 3+4i)$   3维复向量

$n$  维向量写成一行, 称为**行向量(row vector)**, 如

$$\alpha = (a_1, a_2, \dots, a_n);$$

$n$  维向量写成一列, 称为**列向量(column vector)**, 如

$$\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = (a_1, a_2, \dots, a_n)^T.$$

1. 在做运算时, 行向量和列向量总被看作是两个不同的向量.
2. 当未说明是行向量还是列向量时, 都当作列向量.



**定义2** 设  $\alpha = (a_1, a_2, \dots, a_n) \in P^n$ ,  $\beta = (b_1, b_2, \dots, b_n) \in P^n$ ,  $\lambda \in P$ ,

$P$ 为数域

(1)  $\alpha = \beta$  当且仅当  $a_i = b_i$ ,  $i=1, 2, \dots, n$

(2) **向量加法** ( $\alpha$  与  $\beta$  之和) : *addition of vectors*

$$\alpha + \beta = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

(3) **向量数乘** (数量乘法, 数  $\lambda$  与  $\alpha$  之乘积) :

$$\lambda \alpha = (\lambda a_1, \lambda a_2, \dots, \lambda a_n) \quad \textit{multiplication by a scalar}$$

向量的加法与数量乘法统称为向量的线性运算

$\lambda = -1$  时,  $-\alpha = (-a_1, -a_2, \dots, -a_n)$  **负向量**

$$\beta - \alpha = \beta + (-\alpha)$$

## 线性方程组的向量表示

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$
  

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = b$$

Diagram illustrating the vector representation of a system of linear equations. The first part shows a system of  $m$  equations in  $n$  variables. The second part shows the compact vector form, where the coefficients are grouped into vectors  $\alpha_i$  and the right-hand side is  $b$ . Arrows indicate the mapping from the individual coefficients in the first part to the vectors in the second part.

## 系数矩阵

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$= [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n]$$

## 增广矩阵

$$(A, b) = [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n \ b]$$

线性方程组与增广矩阵的列向量组之间一一对应。

## 线性组合(linear combination)

— *one of the central ideas of linear algebra*

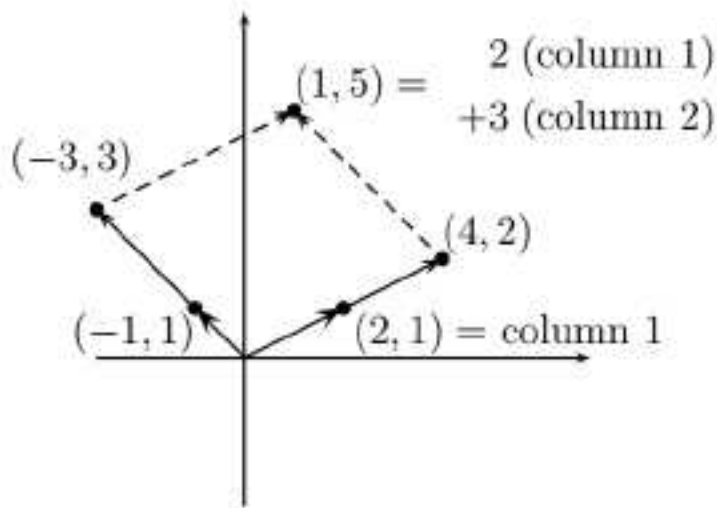
**定义3** 给定向量组  $\beta, \alpha_1, \alpha_2, \dots, \alpha_m$ , 若存在一组数  $k_1, k_2, \dots, k_m$ , 使得

$$\beta = k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_m \alpha_m = \sum_{i=1}^m k_i \alpha_i,$$

则称  $\beta$  为向量组  $\alpha_1, \alpha_2, \dots, \alpha_m$  的**线性组合**, 或称向量  $\beta$  可由向量组  $\alpha_1, \alpha_2, \dots, \alpha_m$  **线性表示 (线性表出)** .

It uses *both* of the basic operations; vectors are *multiplied by numbers and then added*. The result is called a ***linear combination***.

$n = 2$ :  
**Column Picture**



(b) Columns combine with 2 and 3

Column picture  
(combine columns)

$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$$

Column form

$$x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

*vector  
equation*

Linear combination  
(线性组合)

The problem is *to find the combination of the column vectors on the left side that produces the vector on the right side.*

$$n = 3$$

$$\begin{cases} 2u + v + w = 5 \\ 4u - 6v = -2 \\ -2u + 7v + 2w = 9 \end{cases}$$

$$u=1, v=1, w=2$$

*Linear algebra can operate with **any number** of equations.*

The first equation produces an  $(n-1)$ -dimensional plane in  $n$  dimensions, ... **Assuming all goes well**, every new plane (every new equation) reduces the dimension by one. At the end, when all  $n$  planes are accounted for, the intersection has dimension zero. It is a *point*, it lies on all the planes, and its coordinates satisfy all  $n$  equations. **It is the solution!**

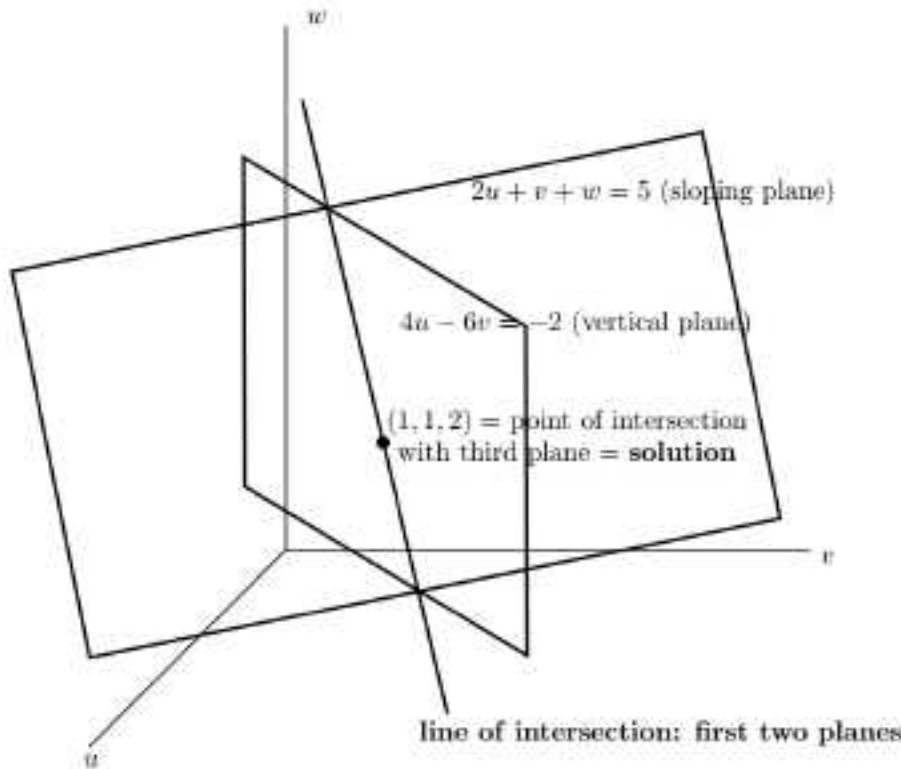
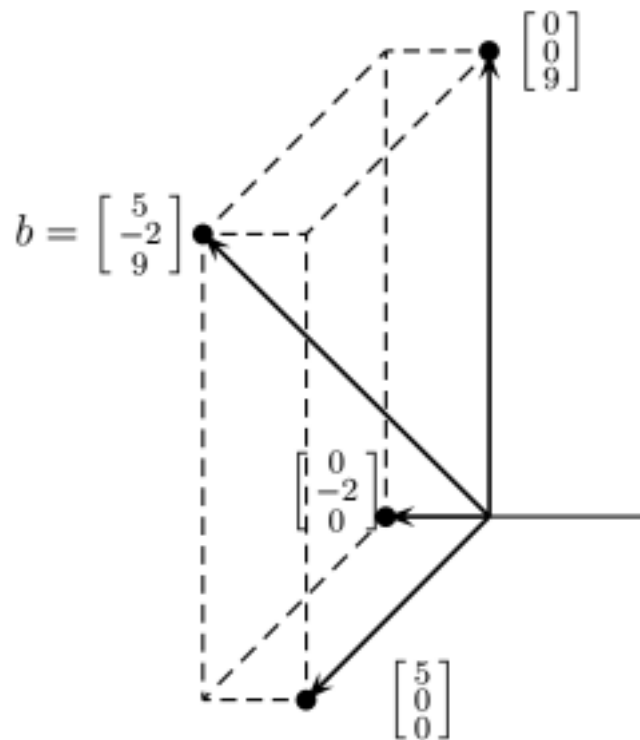


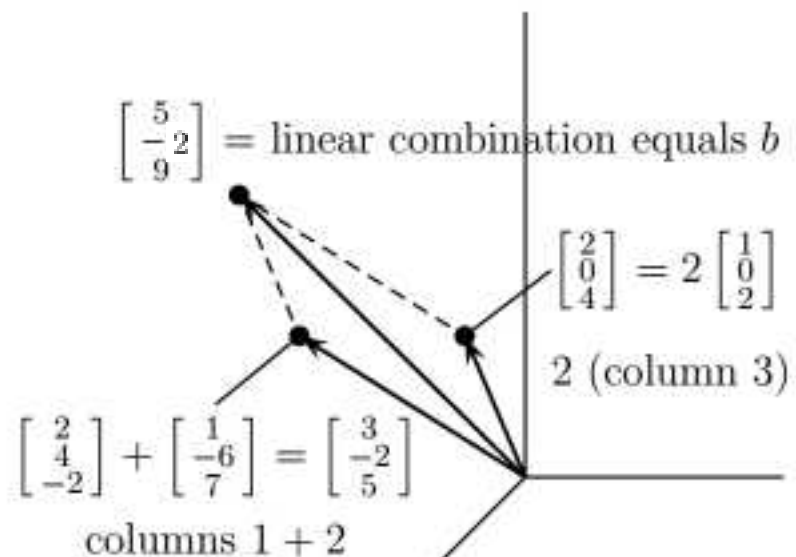
Figure 1.3: The row picture: three intersecting planes from three linear equations.

Column form

$$1 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \mathbf{b}.$$



(a) Add vectors along axes



(b) Add columns 1 + 2 + (3 + 3)

**Figure 1.4:** The column picture: linear combination of columns equals  $\mathbf{b}$ .

Linear combination  
(线性组合)

$$1 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}.$$

$n$  dimensions

- With  $n$  equations in  $n$  unknowns, there are  $n$  planes in the row picture.
- There are  $n$  vectors in the column picture, plus a vector  $\mathbf{b}$  on the right side. The equations ask for a *linear combination of the  $n$  columns that equals  $\mathbf{b}$* .

*(For certain equations that will be impossible.)*

- **Row picture:** Intersection of planes (“平面”的交点)
- **Column picture:** Combination of columns (列的组合)

$$n = 3:$$

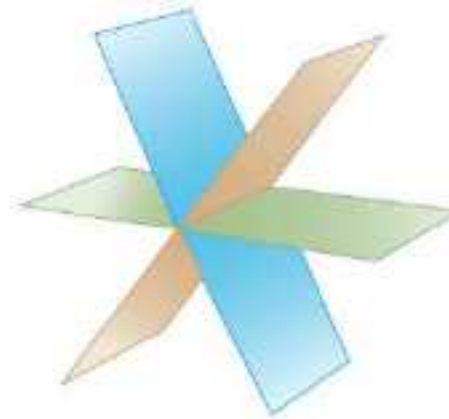
## *Row Picture*

The solutions  $(x,y,z)$  of a single linear equation

$$ax + by + cz = d$$

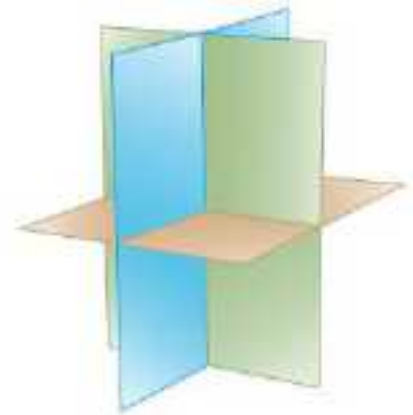
form a plane in  $\mathbf{R}^3$  when  $a$ ,  $b$  and  $c$  are not all zero.

Construct sets of three linear equations whose graphs  
(a) intersect in a single line,  
(b) intersect in a single point,  
and  
(c) have no points in common.



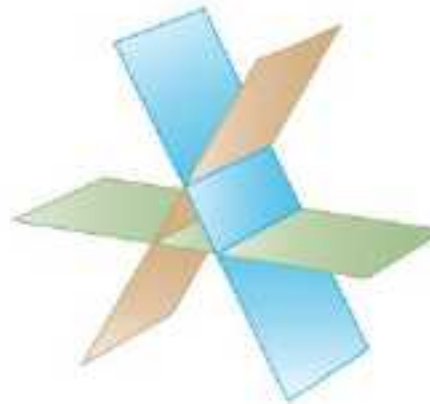
Three planes intersecting in a line

(a)



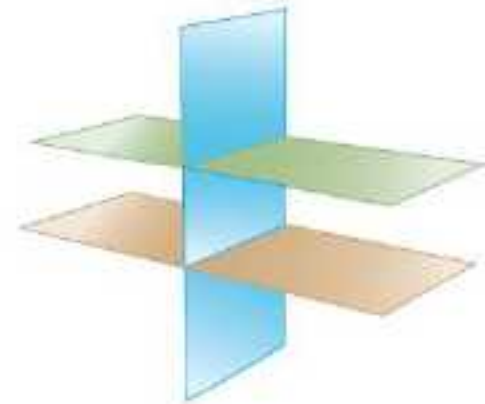
Three planes intersecting in a point

(b)



Three planes with no intersection

(c)



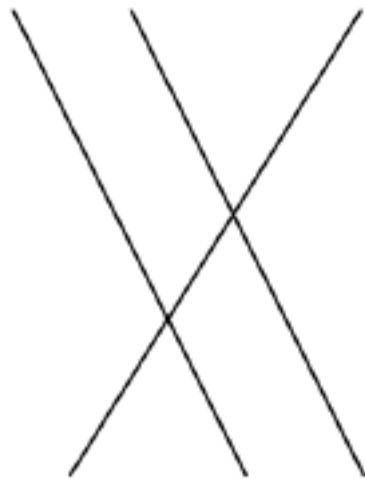
Three planes with no intersection

(c')



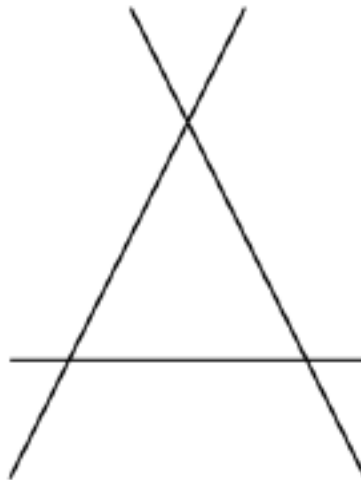
***Singular Cases 1 (  $n = 3$  ): Row picture -- no solution***

$$\begin{cases} 2u + v + w = 5 \\ 4u + 2v + 2w = 11 \\ -2u + 7v + 2w = 9 \end{cases}$$



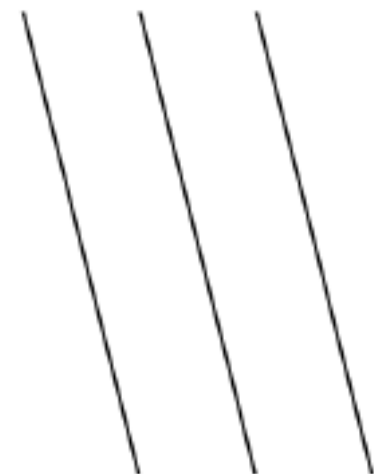
two parallel planes

$$\begin{cases} u + v + w = 2 \\ 2u + 3w = 5 \\ 3u + v + 4w = 6 \end{cases}$$



no intersection

$$\begin{cases} 2u + v + w = 5 \\ 4u + 2v + 2w = 11 \\ 6u + 3v + 3w = 20 \end{cases}$$



all planes parallel

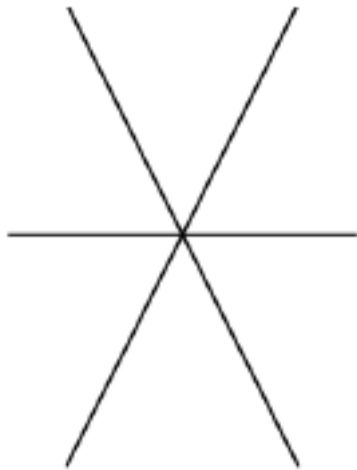
no solution (end view)

*Singular Cases 2 (  $n = 3$  ):*  
***Row picture*** -- infinitely many solutions

$$\begin{cases} u + v + w = 2 \\ 2u + 3w = 5 \\ 3u + v + 4w = 7 \end{cases}$$

$$\begin{cases} 2u + v + w = 2 \\ 4u + 2v + 2w = 4 \\ 6u + 3v + 3w = 6 \end{cases}$$

$$\begin{cases} 2u + v + w = 0 \\ 4u + 2v + 2w = 0 \\ 6u + 3v + 3w = 0 \end{cases}$$



line of intersection



move over to the same plane

an infinity of solutions (end view)

# Singular Cases ( $n = 3$ ): the column picture

**Why?**

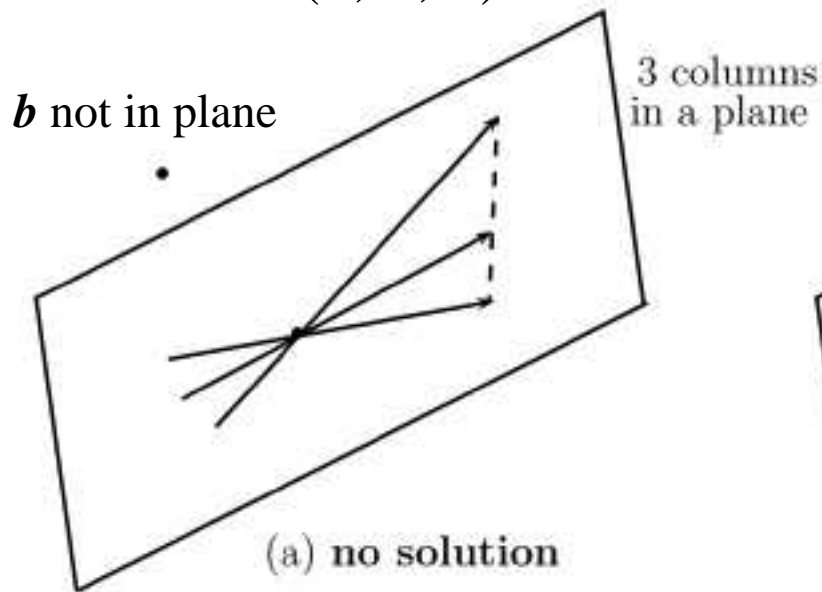
$$u \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + v \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \mathbf{b}$$

**Three columns in the same plane;**

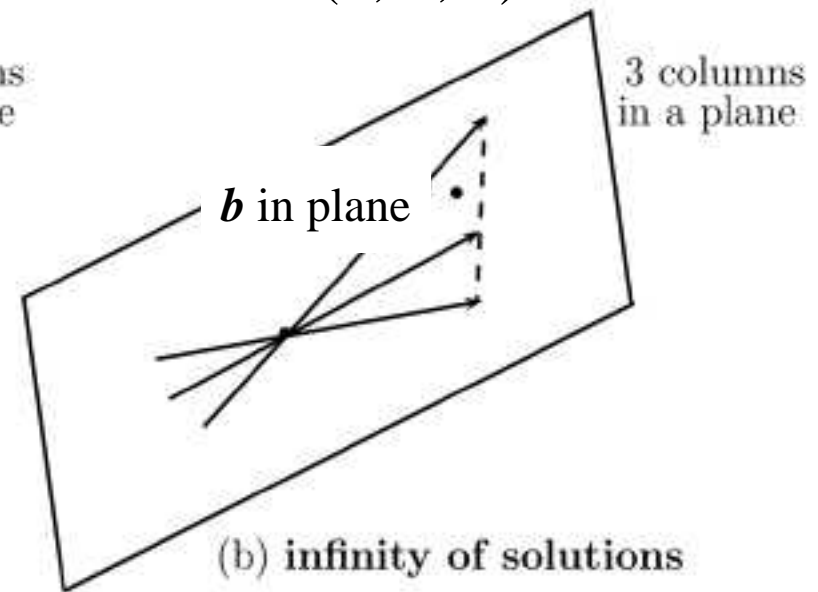
**Solvable only for  $\mathbf{b}$  in that plane**

<https://www.geogebra.org/m/cgtdgg3x>

$$\mathbf{b} = (2, 5, 6)^T$$



$$\mathbf{b} = (2, 5, 7)^T$$



**Figure 1.6:** Singular cases:  $\mathbf{b}$  outside or inside the plane with all three columns.

## *Three columns lie in the same plane?*

Find a combination of the columns that adds to zero.

$$k^* \quad 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad \text{Only two columns are } \textit{independent}.$$

The vector  $\mathbf{b} = (2, 5, 7)^T$  is in that plane of the columns:

$$+ \quad 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}, \quad \text{so } (1, 0, 1)^T \text{ is a solution.}$$

We can add a *multiple* of the combination  $(3, -1, -2)^T$  that gives  $\mathbf{b} = (0, 0, 0)^T$ .

So there is a whole line of solutions—as we know from the row picture.

The truth is that we *knew* the columns would combine to give zero, because the rows did.

*That is a fact of mathematics, not of computation*—and it remains true in dimension  $n$ .

*If the  $n$  planes have no point in common, or infinitely many points, then the  $n$  columns lie in the same plane.*

If the row picture breaks down, so does the column picture.

With a decent notation (*matrix notation*) and a decent algorithm (*elimination*), this becomes clear.

And it will be even more interesting after we learn the concept of *the rank of a matrix* (矩阵的秩).

This is one of the most important theorems in linear algebra.

*“row rank = column rank.”*

# Homework



- See Blackboard announcement
- ***Hardcover* textbook + Supplementary problems**

## Deadline (DDL):

- Next tutorial class

