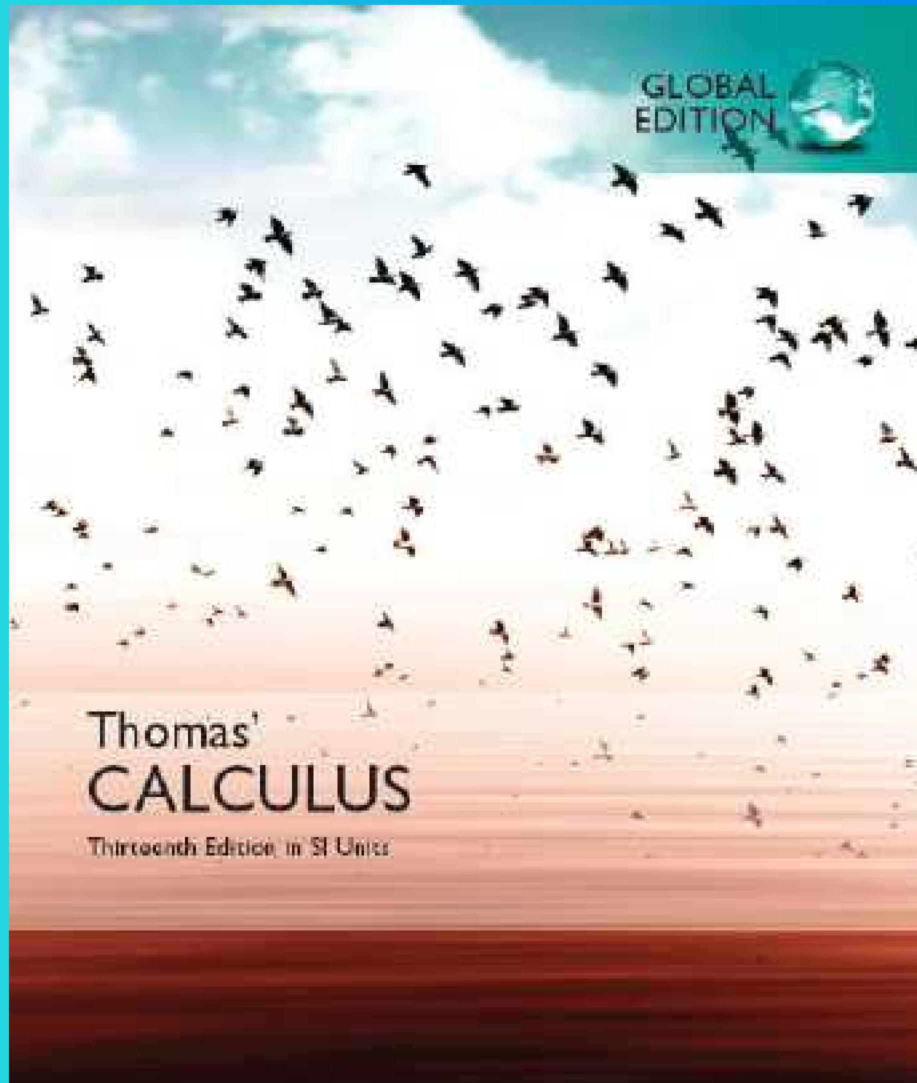

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 3

Differentiation

微分法

3.1

Tangents and the Derivative at a Point

在一点的切线与导数

曲线的切线：

割线的极限位置

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

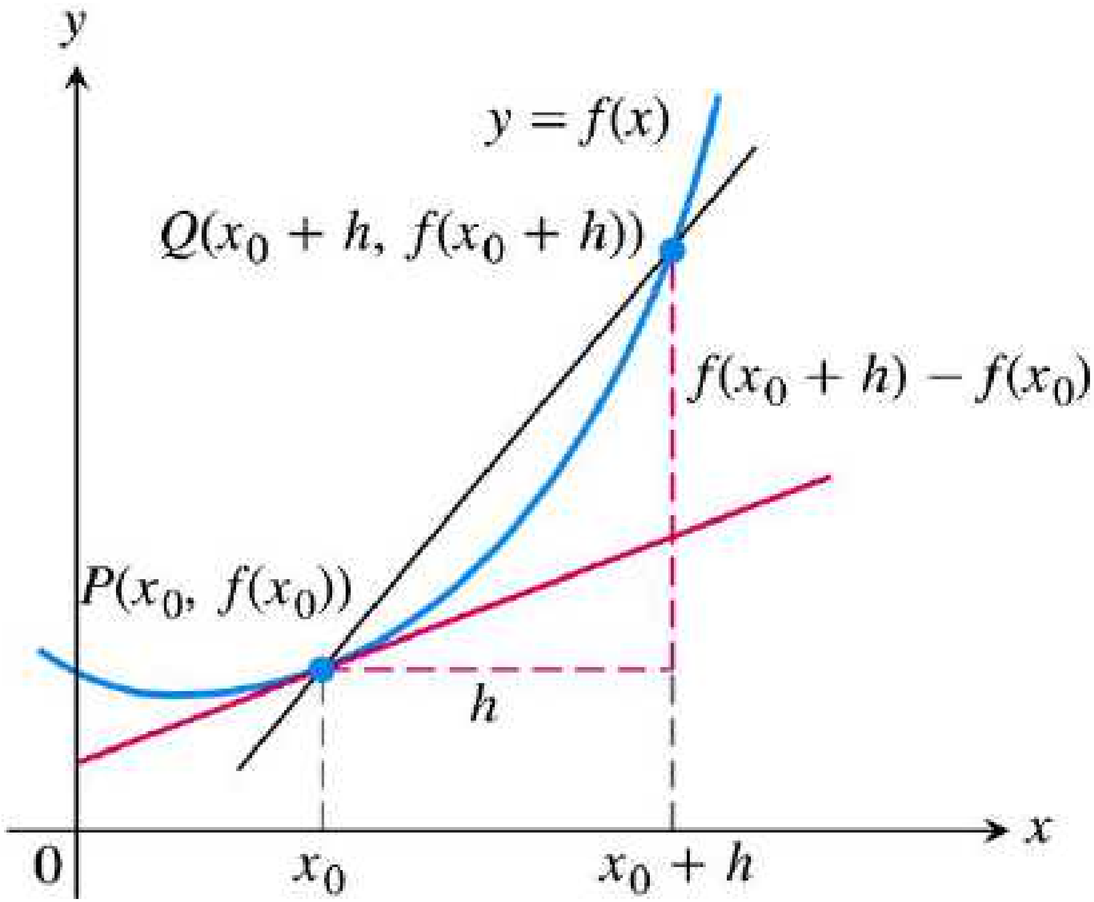


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

EXAMPLE 1

(a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the s
point $x = -1$? slope at the

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes

解(a): $$k = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h \to 0} \frac{-1}{a(a+h)} = \frac{-1}{a^2}$$

$$k\Big|_{x=-1} = -1.$$

解(b): $k=-\frac{1}{a^2}=-\frac{1}{4}$, 得$a=\pm 2$,

得切点 $(2,\frac{1}{2}),(-2,-\frac{1}{2})$.

解(c): $k=-\frac{1}{a^2}$,

$\lim_{a\to 0^+}(-\frac{1}{a^2})=-\infty$, $\lim_{a\to 0^-}(-\frac{1}{a^2})=-\infty$,

$\lim_{a\to \pm\infty}(-\frac{1}{a^2})=0$.

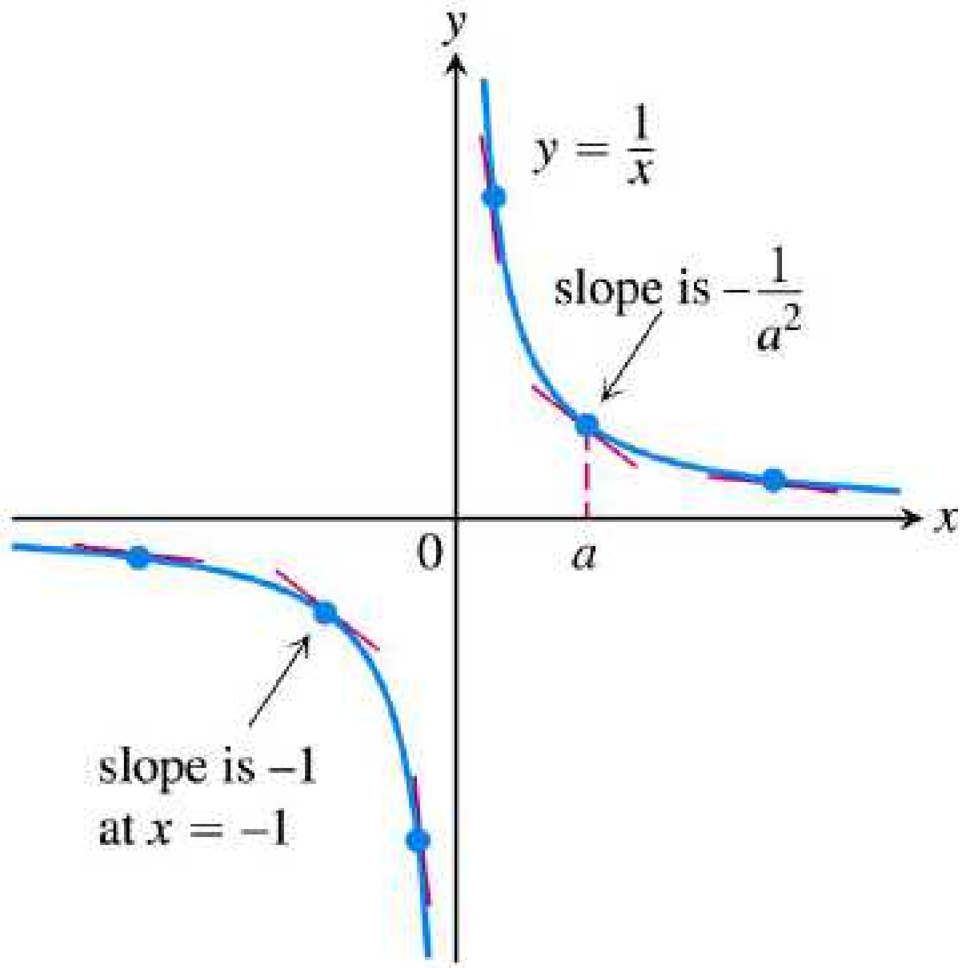


FIGURE 3.2 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away (Example 1).

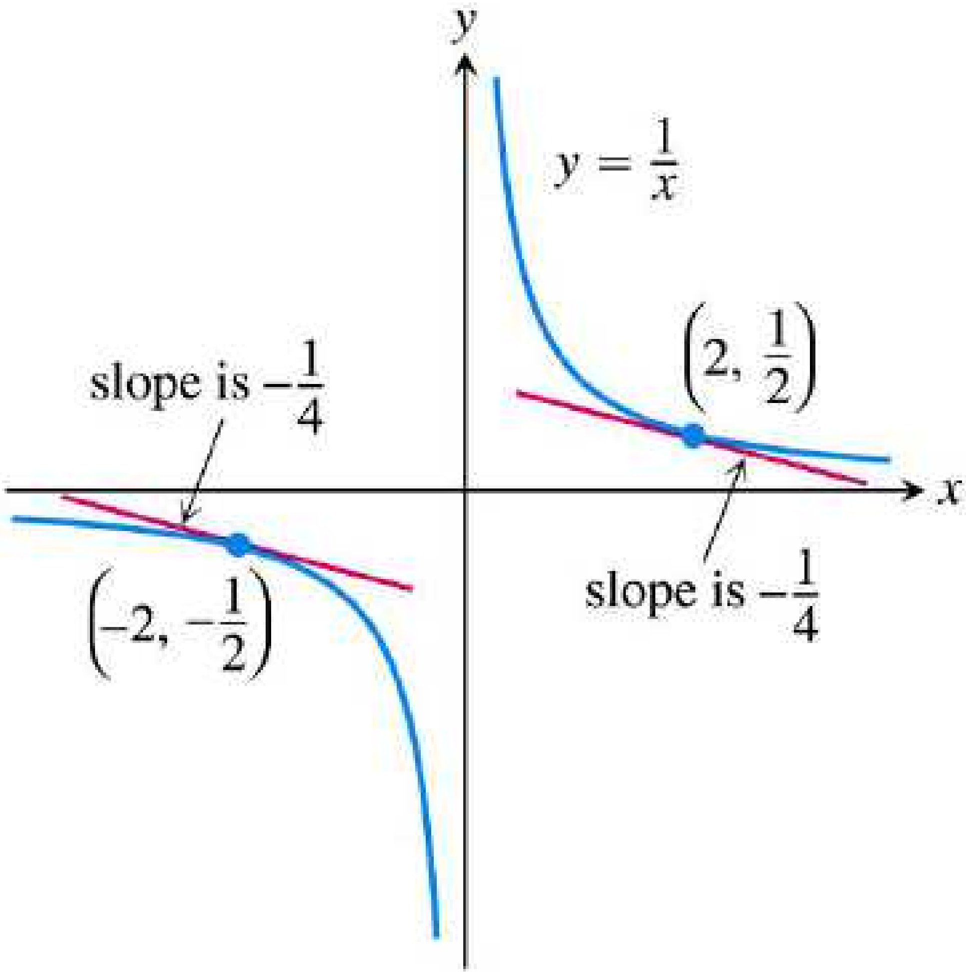


FIGURE 3.3 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 1).

difference quotient of f at x_0 with increment h.

$$\frac{f(x_0+h)-f(x_0)}{h}, \quad h \neq 0$$

函数在一点的变化率：平均变化率的极限

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

导数的另外形式

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

3.2

The Derivative as a Function

导函数

导函数

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

EXAMPLE 1 Differentiate $f(x) = \dfrac{x}{x-1}$.

use the definition of derivative.

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}$$

EXAMPLE 2

(a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution

$$(a)\ f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x}$$

$$= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$(b)\ f'(x) = \frac{1}{2\sqrt{x}},\quad f'(4) = \frac{1}{4}$$

切线： $y - 2 = \frac{1}{4}(x - 4)$

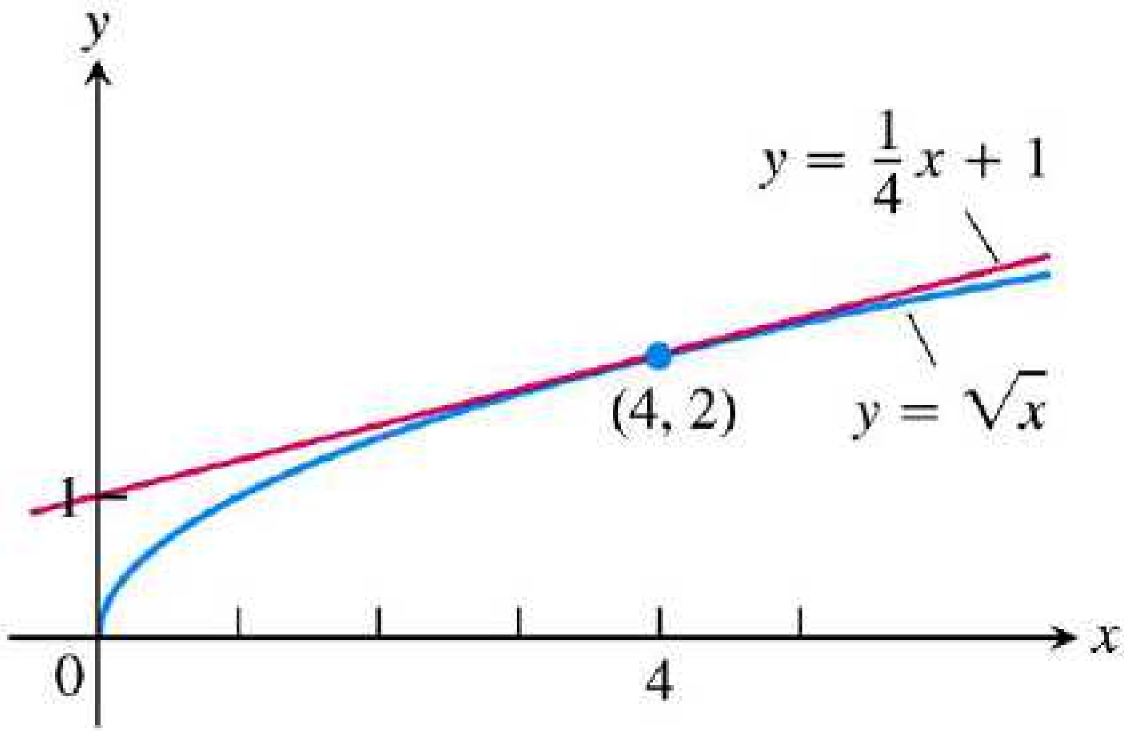
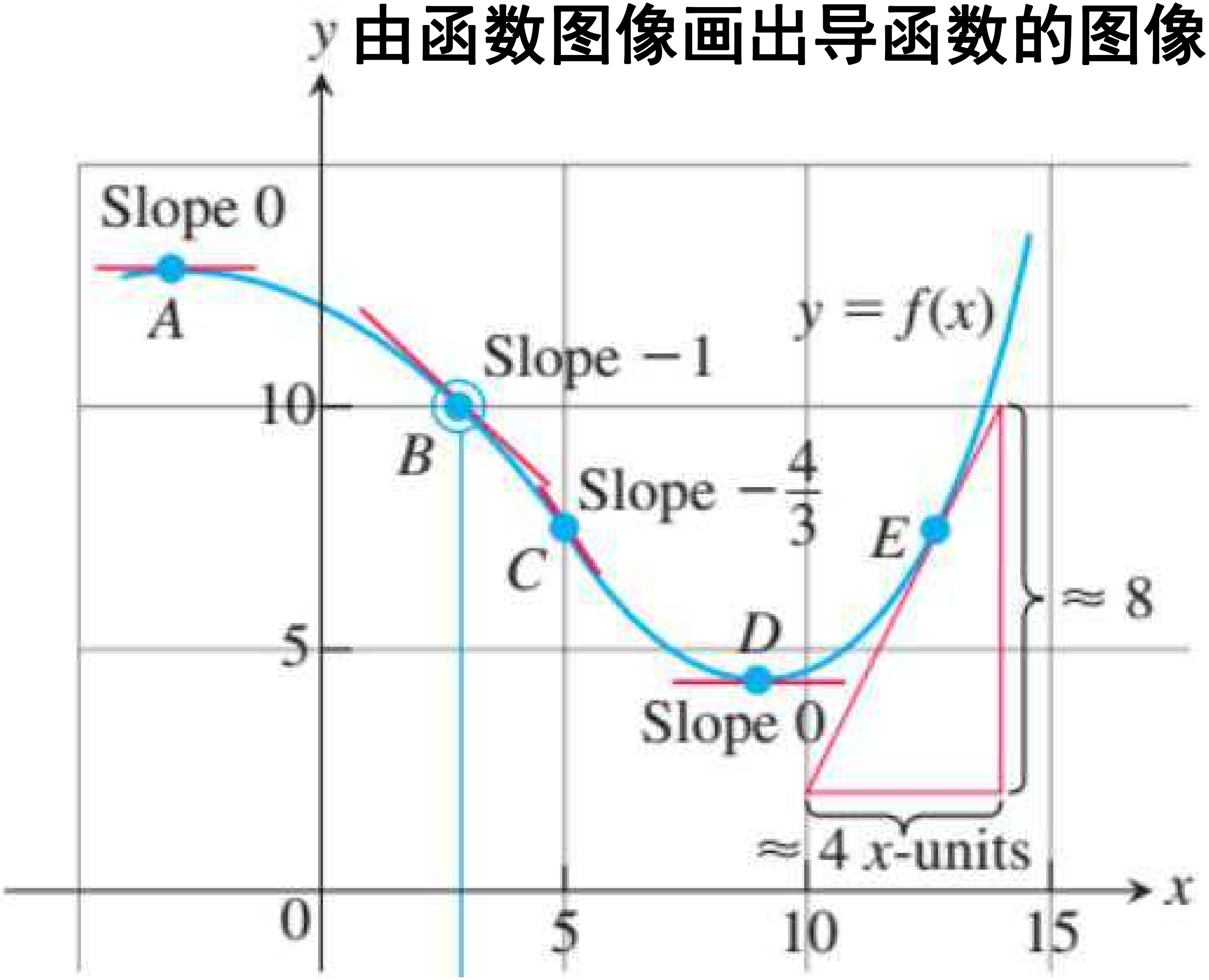
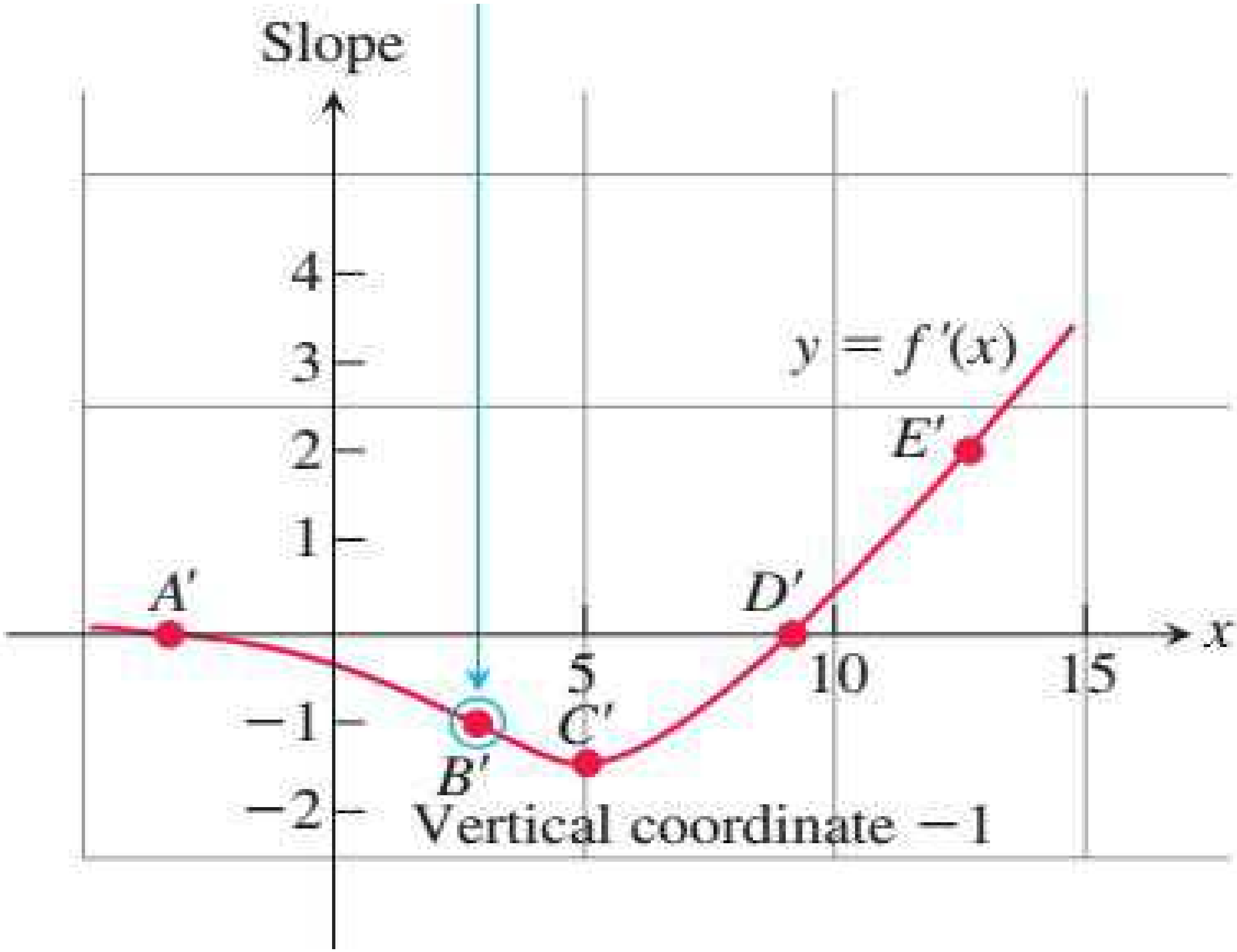


FIGURE 3.5 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

EXAMPLE 3 Graph the derivative of the function $y = f(x)$

由函数图像画出导函数的图像





关注：

1)导数为零的点.

2)导数为正的区间.

3)导数为负的区间.

4)导数的最大最小点.

One-Sided Derivatives

1.左导数: Left-hand derivative:

$$f'_-(x_0)=\lim_{x\to x_0-0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{\Delta x\to -0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x};$$

2.右导数: Right-hand derivative

$$f'_+(x_0)=\lim_{x\to x_0+0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{\Delta x\to +0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x};$$

$f'(x_0)$存在 $\Leftrightarrow$ $f'_+(x_0)$与$f'_-(x_0)$都存在且相等.

Differentiable on an Interval;

It is **differentiable on a closed interval** $[a, b]$ if it

is differentiable on the interior (a, b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } \boldsymbol{a}$$

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } \boldsymbol{b}$$

exist at the endpoints (Figure 3.7).

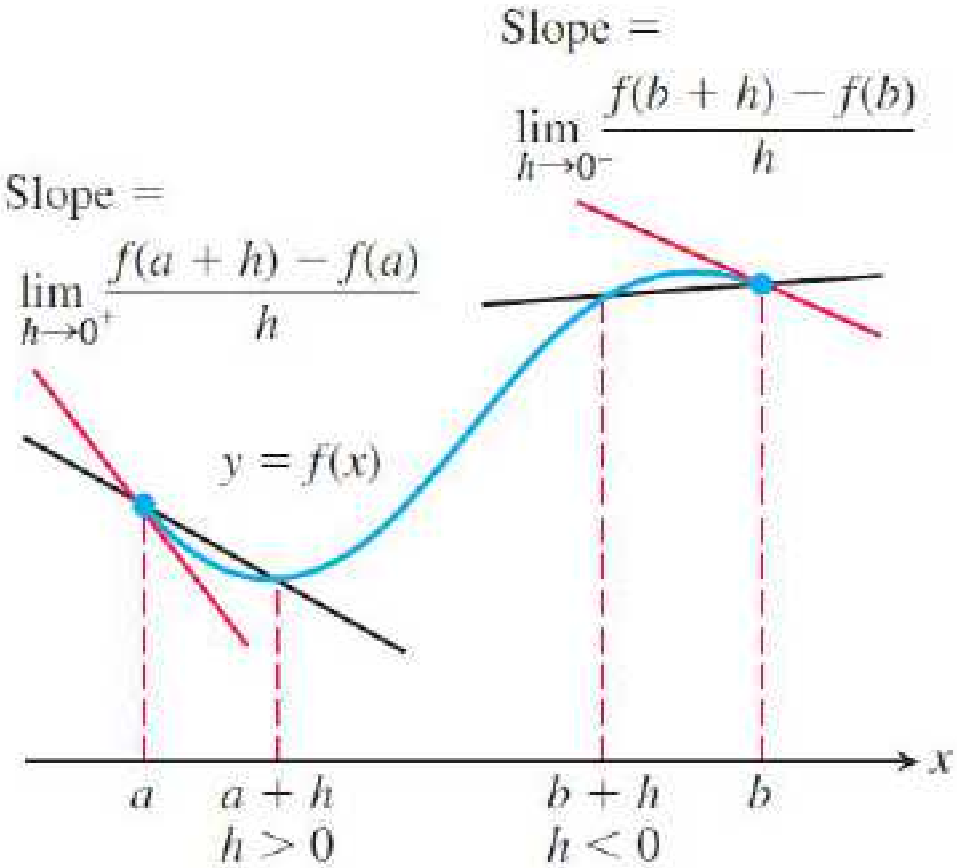


FIGURE 3.7 Derivatives at endpoints of a closed interval are one-sided limits.

Ex.4 讨论函数 $f(x)=|x|$ 的可导性.

解 $\because \dfrac{f(0+h)-f(0)}{h}=\dfrac{|h|}{h}$,

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{h}{h}=1,$$

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{-h}{h}=-1.$$

$\therefore$ 函数$y=f(x)$在$x=0$点不可导.

当$x>0$时,$(|x|)'=(x)'=1$

当$x<0$时,$(|x|)'=(-x)'=-1.$

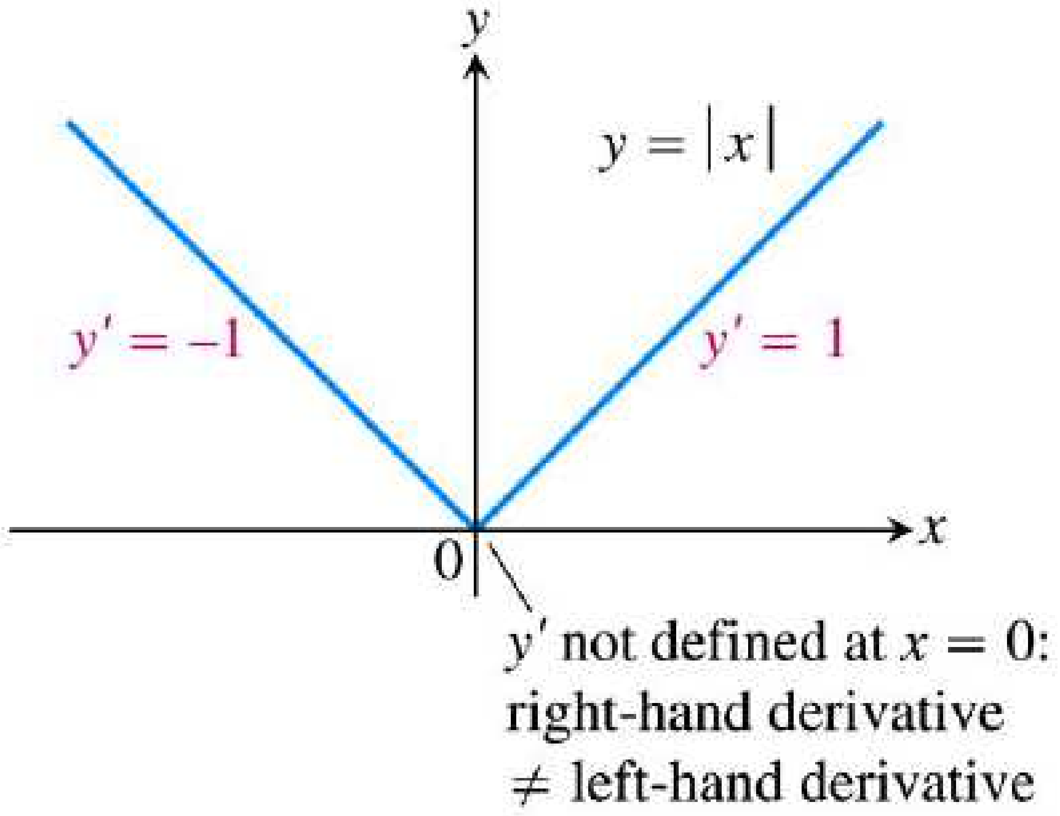


FIGURE 3.8 The function $y = |x|$ is not differentiable at the origin where the graph has a "corner" (Example 4).

Ex.5 证明函数 $f(x)=\sqrt{x}$ 在$(0,+\infty)$内可导，

但在$x=0$处不可导.

解：已知 $\dfrac{d}{dx}(\sqrt{x})=\dfrac{1}{2\sqrt{x}}\ (x>0)$. 但是

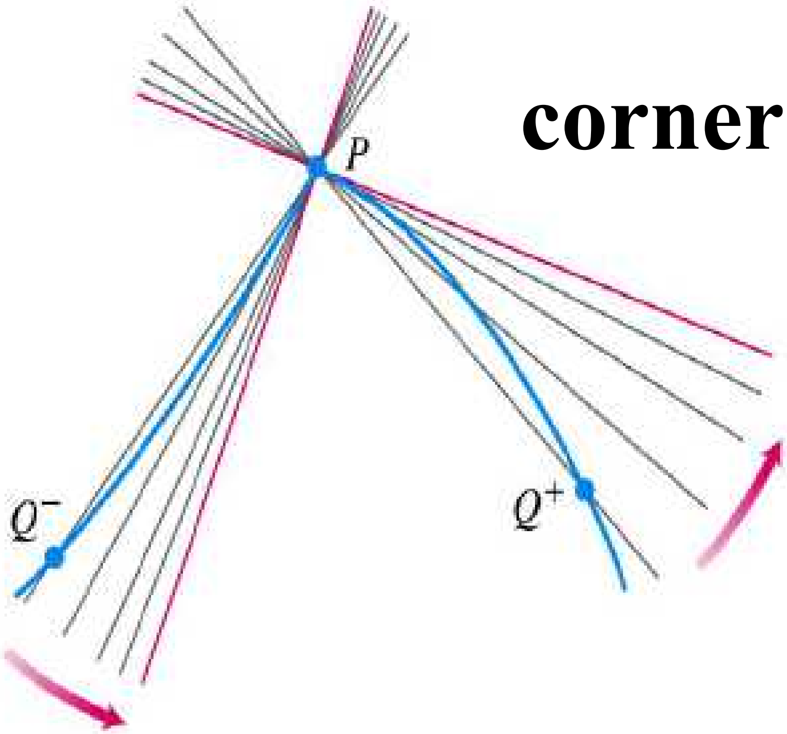
$$\lim_{h\to 0^+}\frac{\sqrt{0+h}-\sqrt{0}}{h}=\lim_{h\to 0^+}\frac{1}{\sqrt{h}}=+\infty.$$

导数不存在。

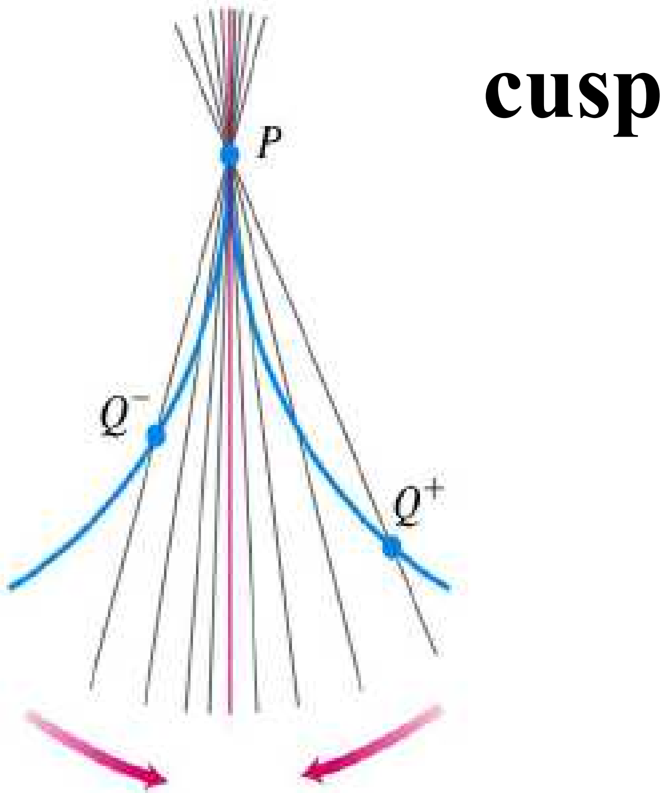
注意，曲线在这点的右切线存在。

可导 $\underset{?}{\rightleftarrows}$ 曲线的切线存在。

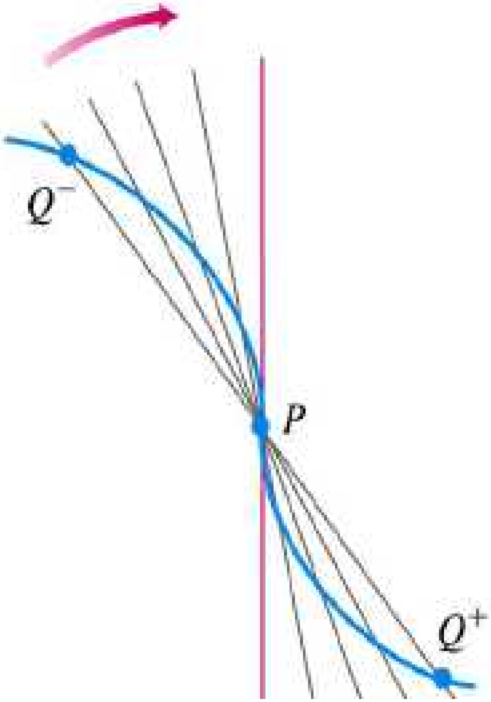
从函数的图像上如何看出不可导的点？有竖的切线？

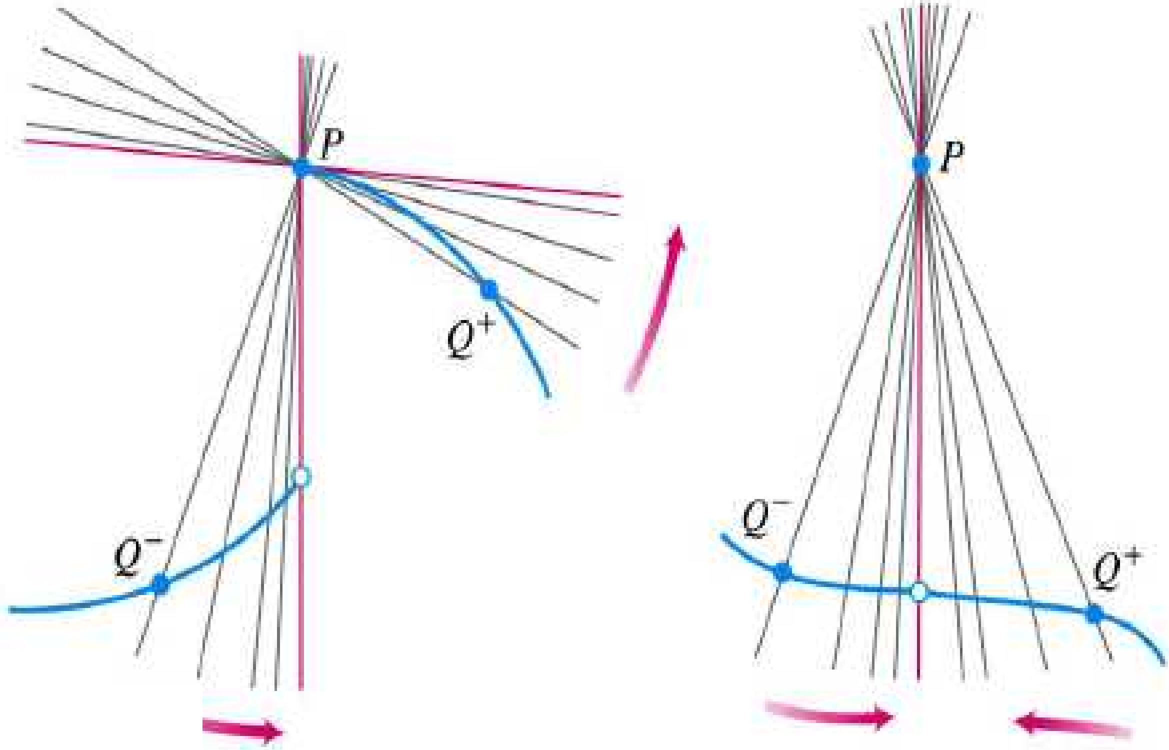


1. a *corner*, where the one-sided derivatives differ.



2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.





Vertical tangent

3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).

4. a *discontinuity* (two examples shown).

可导与连续的关系

THEOREM 1—Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

证明： $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

当$x \neq c$时， $f(x) = f(c) + \dfrac{f(x) - f(c)}{x - c} \cdot (x - c)$

取极限 $x \to c$， 得 $\lim\limits_{x \to c} f(x) = f(c).$

所有的间断点处皆不可导！

3.3

Differentiation Rules

求导法则

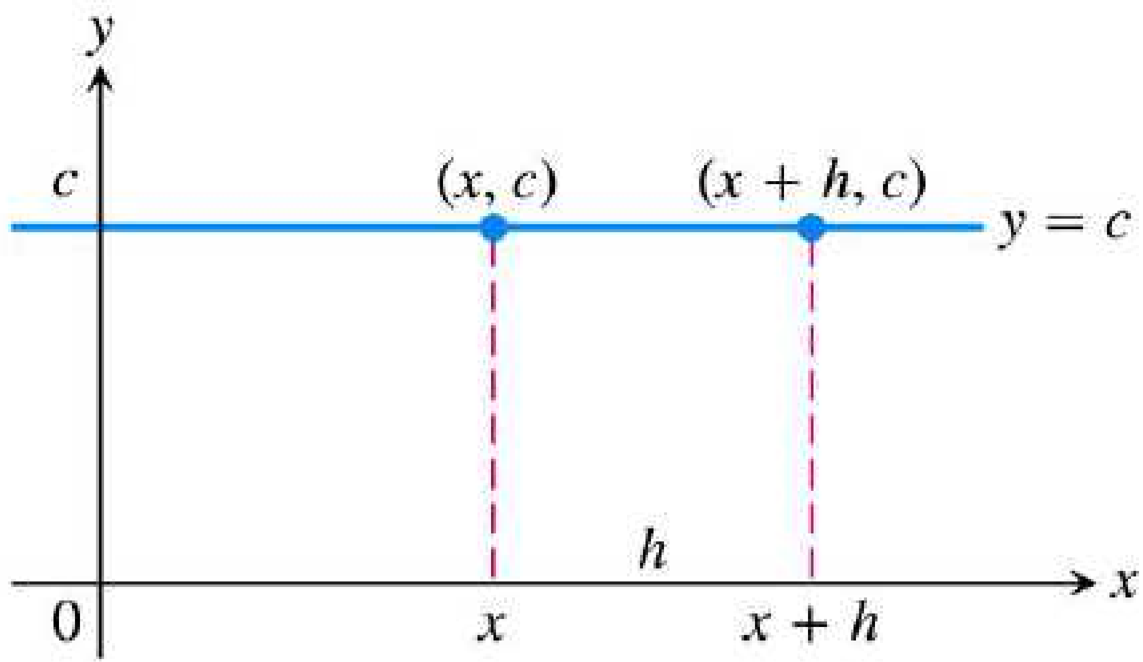


FIGURE 3.9 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0.$$

Derivative of a Positive Integer Power

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{z^n - x^n}{z - x}$$

$$= \lim_{z \to x} (z^{n-1} + z^{n-2}x + z^{n-3}x^2 \cdots + x^{n-1}) \quad = nx^{n-1}.$$

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2} \qquad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

Ex.1 求下列函数的导数：

$$x^3, \quad x^{\frac{2}{3}}, \quad x^{\sqrt{2}}, \quad x^{-4}, \quad x^{-\frac{4}{3}}, \quad \sqrt{x^{2+\pi}}$$

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

$$\frac{d(cu(x))}{dx} = \lim_{h\to 0}\frac{cu(x+h)-cu(x)}{h}$$

$$= c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

$$= c\frac{d(u(x))}{dx}$$

Ex.2 Find the slope of

$$y = 3x^2$$

at the point (1,3).

Solution $y' = 6x$

slope $= y'|_{x=1} = 6x|_{x=1} = 6.$

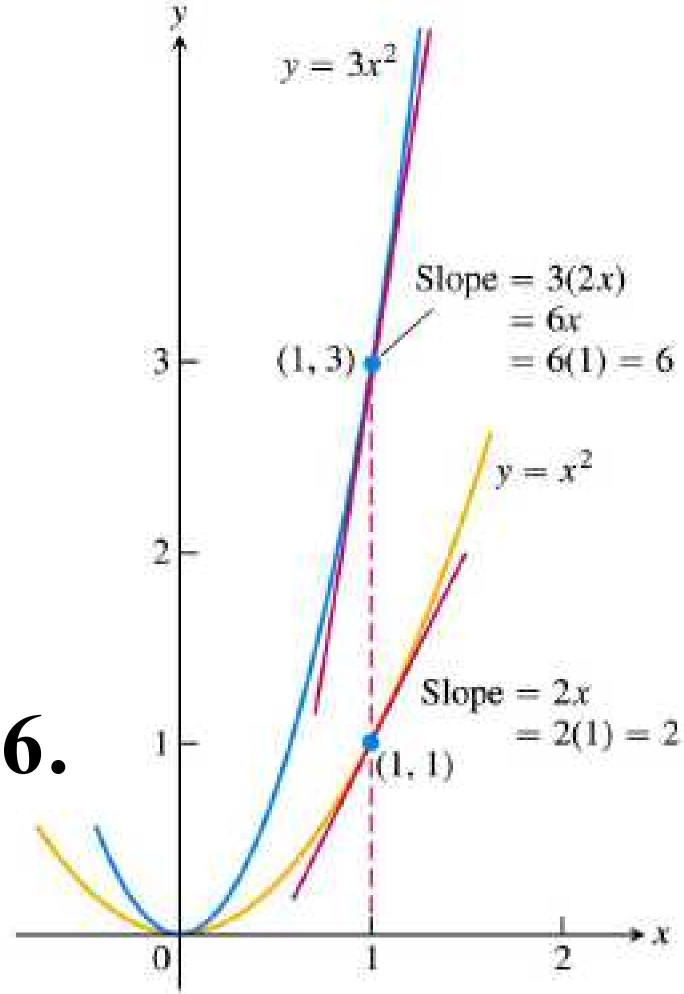


FIGURE 3.10 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinate triples the slope (Example 2).

正弦余弦的求导公式

$$\frac{d(\sin x)}{dx} = \cos x \qquad \frac{d(\cos x)}{dx} = -\sin x$$

证明：$(\sin x)' = \lim_{h\to 0}\dfrac{\sin(x+h)-\sin x}{h}$

$$= \lim_{h\to 0} 2\cos\left(x+\frac{h}{2}\right)\cdot\frac{\sin\frac{h}{2}}{h} = \cos x.$$

$$(\sin x)'\Big|_{x=\frac{\pi}{4}} = \cos x\Big|_{x=\frac{\pi}{4}} = \frac{\sqrt{2}}{2}.$$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

$$\frac{d(u(x)+v(x))}{dx} \qquad \frac{d(u-v)}{dx} = \frac{d(u)}{dx} - \frac{d(v)}{dx}$$

$$= \lim_{h\to 0} \frac{[u(x+h)+v(x+h)]-[u(x)+v(x)]}{h}$$

$$= \lim_{h\to 0} [\frac{u(x+h)-u(x)}{h} + \frac{v(x+h)-v(x)}{h}$$

$$= \frac{d(u(x))}{dx} + \frac{d(v(x))}{dx}$$

例3

Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

解 $$y' = 3x^2 + \frac{8}{3}x - 5$$

例4

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents?

解 $y' = 4x^3 - 4x = 0$, 得$x = -1,0,1$

有水平切线的切点为 $(-1,1),(0,2),(1,1)$.

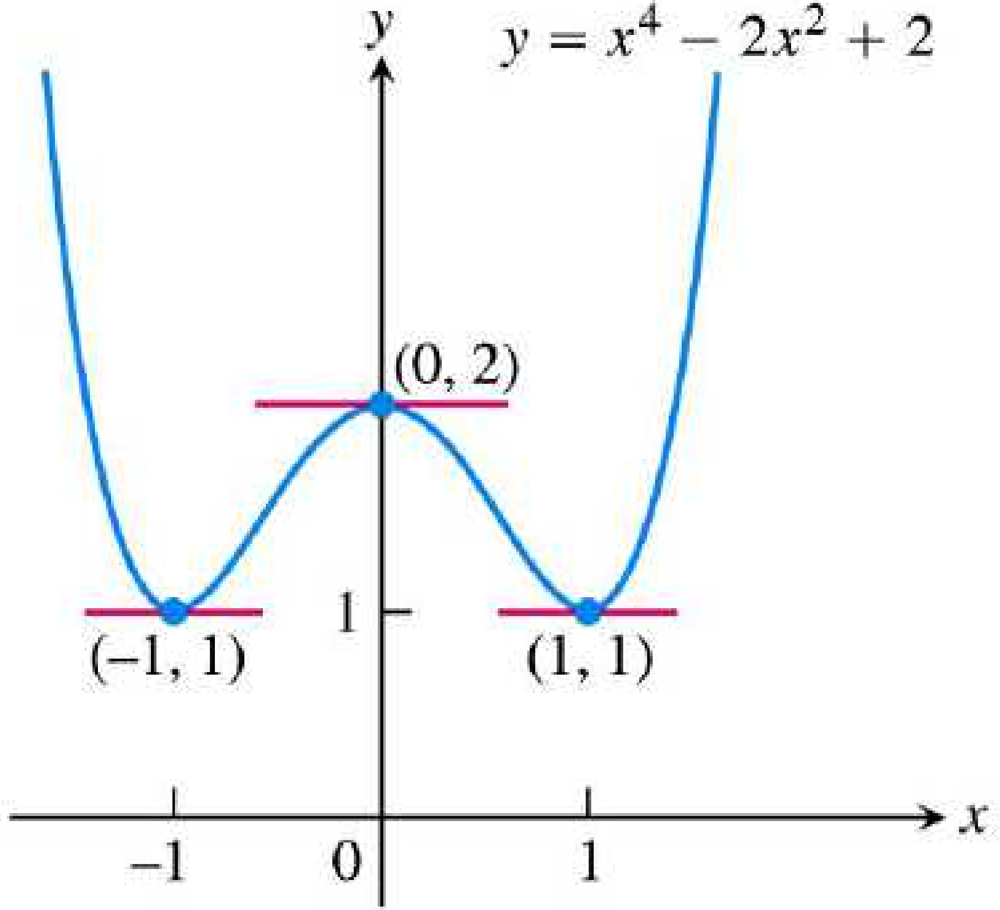


FIGURE 3.11 The curve in Example 4 and its horizontal tangents.

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

设 $f(x)=u(x)v(x)$, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x)}{h} \quad = u(x)v'(x)+u'(x)v(x)$$

$$=\lim_{h\to 0}\frac{u(x+h)(v(x+h)-v(x))+v(x)(u(x+h)-u(x))}{h}$$

$$=\lim_{h\to 0}\left[\frac{u(x+h)(v(x+h)-v(x))}{h}+\frac{v(x)(u(x+h)-u(x))}{h}\right]$$

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

设 $f(x)=\dfrac{u(x)}{v(x)}, (v(x)\neq 0),\quad f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{\dfrac{u(x+h)}{v(x+h)}-\dfrac{u(x)}{v(x)}}{h}$$

$$=\lim_{h\to 0}\frac{u(x+h)v(x)-u(x)v(x+h)}{v(x+h)v(x)h}$$

$$= \lim_{h \to 0} \frac{[u(x+h)-u(x)]v(x)-u(x)[v(x+h)-v(x)]}{v(x+h)v(x)h}$$

$$= \lim_{h \to 0} \frac{\dfrac{u(x+h)-u(x)}{h} \cdot v(x) - u(x) \cdot \dfrac{v(x+h)-v(x)}{h}}{v(x+h)v(x)}$$

$$= \frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}$$

$\therefore f(x)$在x处可导.

例 Find the derivative for $y = \tan x$.

解 $y' = (\tan x)' = \left(\dfrac{\sin x}{\cos x}\right)'$ $\qquad (\tan x)' = \sec^2 x.$

$$= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

同理可得 $\qquad (\cot x)' = -\csc^2 x.$

例 Find the derivative for $y = x\tan x$.

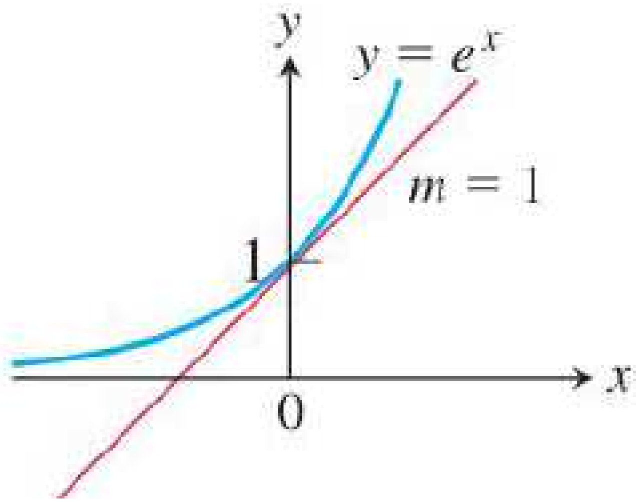
解

$$y' = (x\tan x)' = \tan x + x\sec^2 x$$

Derivatives of Exponential Functions

$$\frac{d}{dx}(e^x) = e^x$$

$$(e^x)' = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0}\frac{e^h - e^0}{h} = e^x$$



EXAMPLE 5

Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

Solution $y = mx,$

it is tangent to the graph at the point (a, e^a),

$m = (e^a - 0)/(a - 0).$

$e^a = e^a/a.$ $a = 1$ $m = e,$

the tangent line is $y = ex.$

EXAMPLE 6

Find the derivative of **(a)** $y = \frac{1}{x}\left(x^2 + e^x\right)$, **(b)** $y = e^{2x}$.

Solution **(a)**

$$\frac{d}{dx}\left[\frac{1}{x}(x^2 + e^x)\right] = \frac{1}{x}(2x + e^x) + (x^2 + e^x)\left(-\frac{1}{x^2}\right)$$

$$= 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2} = 1 + (x - 1)\frac{e^x}{x^2}.$$

(b) $$\frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x \cdot e^x) = e^x \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(e^x)$$

$$= 2e^x \cdot e^x = 2e^{2x}$$

EXAMPLE 7 Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution

$$\frac{d}{dx}\left[(x^2+1)(x^3+3)\right] = (x^2+1)(3x^2) + (x^3+3)(2x)$$
$$= 5x^4 + 3x^2 + 6x.$$

EXAMPLE 8

Find the derivative of **(a)** $y = \dfrac{t^2 - 1}{t^3 + 1}$, **(b)** $y = e^{-x}$.

Solution

(a) $$\frac{dy}{dt} = \frac{(t^3+1)\cdot 2t - (t^2-1)\cdot 3t^2}{(t^3+1)^2}$$
$$= \frac{-t^4 + 3t^2 + 2t}{(t^3+1)^2}.$$

(b) $$\frac{d}{dx}(e^{-x}) = \frac{d}{dx}\left(\frac{1}{e^x}\right) = \frac{e^x \cdot 0 - 1 \cdot e^x}{(e^x)^2} = \frac{-1}{e^x} = -e^{-x}$$

(b) $$\frac{d}{dx}(e^{-x}) = \frac{d}{dx}\left(\frac{1}{e^x}\right) = \frac{e^x \cdot 0 - 1 \cdot e^x}{(e^x)^2} = \frac{-1}{e^x} = -e^{-x}$$

Ex. Find the derivative of

$$y = \frac{(x-1)(x^2-2x)}{x^4}.$$

Solution

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4}$$

$$= x^{-1} - 3x^{-2} + 2x^{-3}$$

$$\frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4}$$

$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$

Second- and Higher-Order Derivatives

高阶导数

问题: 变速直线运动的加速度.

设 $s = f(t)$，则瞬时速度为 $v(t) = f'(t)$

$\because$ 加速度 a 是速度 v 对时间 t 的变化率

$\therefore a(t) = v'(t) = [f'(t)]'.$

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x{}^2 f(x).$$

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

二阶和二阶以上的导数统称为**高阶导数**.

Ex. 10 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

Solution

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

The general polynomial of degree n has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where $a_n \neq 0$. **Find $P^{(n)}(x)$ and $P^{(n+1)}(x)$**

$$P^{(n)}(x) = a_n n! \qquad P^{(n+1)}(x) = 0$$

3.4

The Derivative as a Rate of Change

导数的实际意义：变化率

DEFINITION The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Ex. 1 The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4} D^2.$$

when the diameter is 10 m?

How fast does the area change with respect to the diameter

解 $$\frac{dA}{dD} = \frac{\pi}{2} D \qquad \left.\frac{dA}{dD}\right|_{D=10} = 5\pi = 15.71 m^2 / m.$$

DEFINITION **Velocity (instantaneous velocity)** is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

DEFINITION **Speed** is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

例2　一个粒子在数轴上做往返运动，其速度为右图。根据图像说明粒子运动状态。

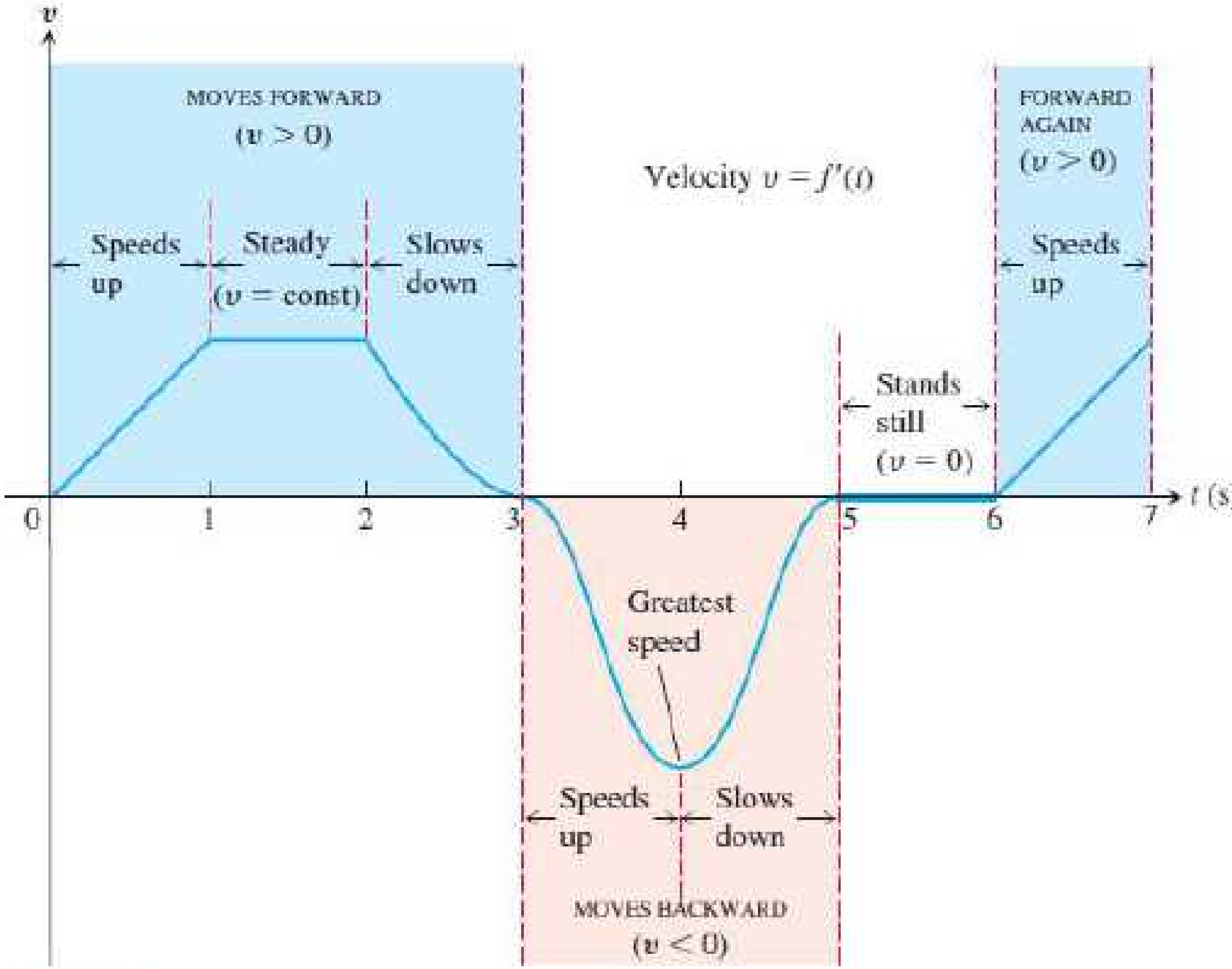


FIGURE 3.14 The velocity graph of a particle moving along a horizontal line, discussed in Example 2.

DEFINITIONS **Acceleration** is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

EXAMPLE 4

A dynamite blast blows a heavy rock straight up with a launch velocity of **49m/s,** It reaches a height of $s = 49t - 4.9t^2$ after t sec.

(a) How high does the rock go?

(b) What are the velocity and speed of the rock when it is **78.4m** above the ground on the way up? On the way down?

(c) What is the acceleration of the rock at any time t during its flight

(d) When does the rock hit the ground again?

Solution $(a)\ v = s' = 49 - 9.8t = 0$得$t = 5$, $s(5) = 122.5m$.

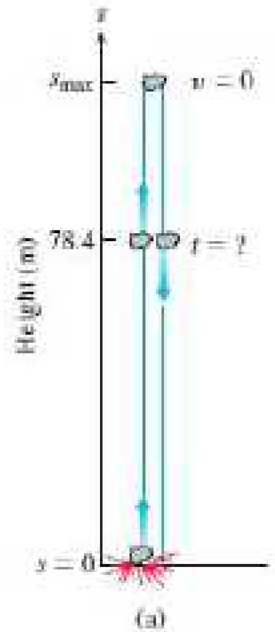
$(b)\ s = 49t - 4.9t^2 = 78.4,$得$t = 2, 8.$

$s'(2) = 49 - 9.8 \times 2 = 29.4m/s,$

$s'(8) = 49 - 9.8 \times 8 = -29.4m/s.$

$(c)\ a = s'' = -9.8m/s^2.$

$(d)\ s = 49t - 4.9t^2 = 0,$得$t = 0, 10.\quad t = 10s.$



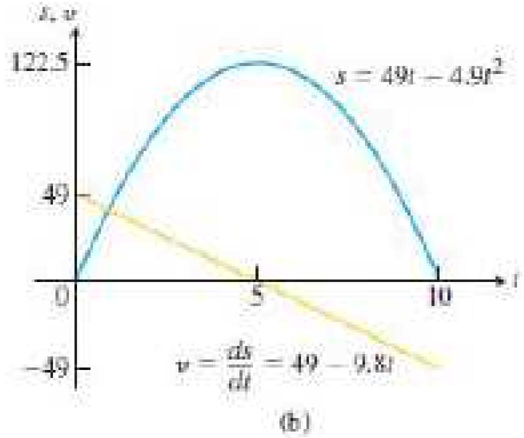


FIGURE 3.16 (a) The rock in Example 4. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

经济学中的导数应用

The marginal cost of production i

设成本函数 $C=C(x)$，x是产量.

则称 $C'(x_0)$为当产量为 x_0时的边际成本 .

边际成本$C'(x_0)$的经济学意义是什么呢？

$$C'(x_0)=\lim_{h\to 0}\frac{C(x_0+h)-C(x_0)}{h}$$

实际中h的最小数是1件， $C'(x_0)\approx C(x_0+1)-C(x_0)$

边际成本$C'(x_0)$的经济学意义是：

生产第x_0+1件(One More)产品的成本的估计 .

设收益函数（Revenue） $r = r(x)$，x是销量.

则称 $r'(x_0)$为当销量为 x_0时的边际收益

(Marginal Revenue).

$$r'(x_0) \approx r(x_0 + 1) - r(x_0)$$

边际收益 $r'(x_0)$的经济学意义是：

从销售第 $x_0 + 1$件(One More)产品获得的收益估计．

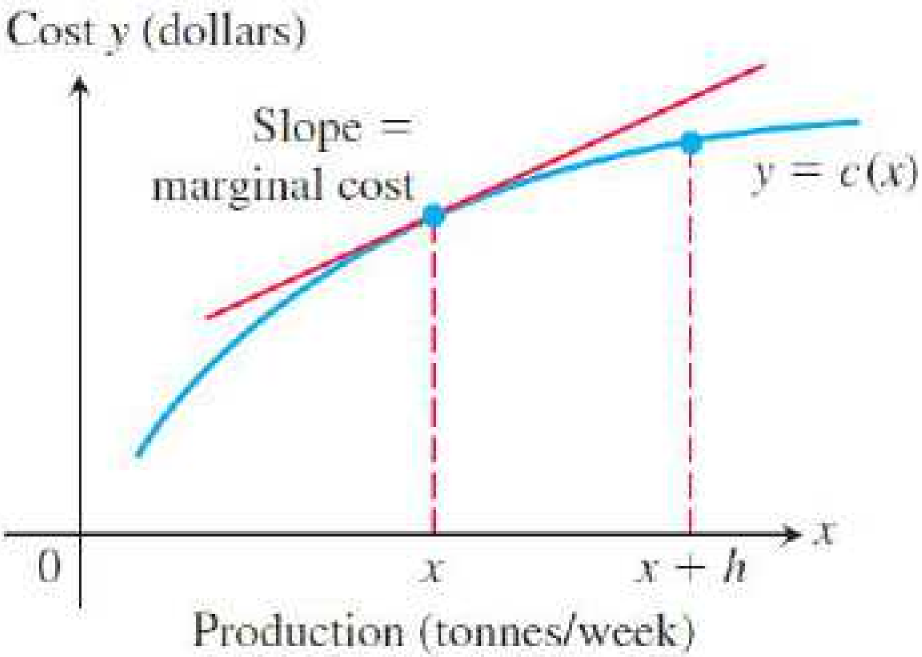


FIGURE 3.17 Weekly steel production: $c(x)$ is the cost of producing x tonnes per week. The cost of producing an additional h tonnes is $c(x + h) - c(x)$.

EXAMPLE 5 Suppose that it costs $c(x) = x^3 - 6x^2 + 15x$ dollars to produce x radiators when 8 to 30 radiators are produced and that

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x radiators. Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day, and what is your estimated increase in revenue for selling 11 radiators a day?

Solution

$$C'(x) = 3x^2 - 12x + 15,$$

$$C'(10) = 3\times 10^2 - 12\times 10 + 15 = 195,$$

$$r'(x) = 3x^2 - 6x + 12,$$

$$r'(10) = 3\times 10^2 - 9\times 10 + 12 = 252.$$

3.5

Derivatives of Trigonometric Functions
三角函数的导数

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$(\cos x)' = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

$$= \lim_{h\to 0}\left(\frac{\cos x(\cos h - 1)}{h} - \frac{\sin x\sin h}{h}\right) = -\sin x.$$

Ex.1 We find derivatives

$$(a)\, y = x^2 - \sin x; \quad (b)\, y = x^2 \sin x; \quad (c)\, y = \frac{\sin x}{x}$$

Ex.2 求下列函数的导数：

$$(a)\, y = 5x + \cos x; (b)\, y = \sin x \cos x; (c)\, y = \frac{\cos x}{1 - \sin x}$$

Ex.3 设简谐振动的位移函数 ： $y = 5\cos t$,

求运动的速度和加速度 .

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Ex.4 $$\lim_{x\to 0}\frac{\sqrt{2+\sec x}}{\cos(\pi-\tan x)}$$

$$=\frac{\sqrt{2+\sec 0}}{\cos(\pi-\tan 0)}=-\sqrt{3}$$

Ex.5 已知$f'(0)=3$,且$f(0)=0$,求$\lim\limits_{x\to 0}\dfrac{(e^{2\tan x}-1)f(x)}{x^2}$

原式$=2\lim\limits_{x\to 0}\dfrac{e^{2\tan x}-1}{2\tan x}\dfrac{\tan x}{x}\dfrac{f(x)-f(0)}{x}=6$

3.6

The Chain Rule

链式法则

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

证 由$y=f(u)$在点u可导，

$$\therefore \lim_{z\to u}\frac{f(z)-f(u)}{z-u}=f'(u) \qquad y=f(g(x)),$$

$$\therefore \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(g(x+\Delta x))-f(g(x))}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{f(g(x+\Delta x))-f(g(x))}{g(x+\Delta x)-g(x)}\cdot\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

$$=\lim_{z\to u}\frac{f(z)-f(u)}{z-u}\cdot\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

$$=f'(u)g'(x)$$

$$g(x+\Delta x)-g(x)\neq 0$$

例1 求函数 $y=(x^2+1)^{10}$ 的导数．

解 $$\frac{dy}{dx}=10(x^2+1)^9\cdot(x^2+1)'$$

$$=10(x^2+1)^9\cdot 2x=20x(x^2+1)^9.$$

例2 求位移函数 $x(t)=\cos(t^2+1)$ 的速度．

解 $$x'(t)=-\sin(t^2+1)\cdot(2t)$$

例3 $y=\tan(5-\sin 2t)$，求y'.

$$y'=\sec^2(5-\sin 2t)\cdot(-\cos 2t)\cdot 2$$

例4 $(a)\,y=(5x^3-x^4)^7;(b)\,y=\dfrac{1}{3x-2};(c)\,y=\sin^5 x.$求$y'$.

例5 $y=\sqrt{x+\sqrt{x+\sqrt{x}}}$

例6 $y = f(x^2)$,其中f 2阶导数存在，求 $\dfrac{d^2y}{dx}$.

解 $\dfrac{dy}{dx} = f'(x^2)2x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x^2)4x^2 + 2f'(x^2)$$

例7 $y = \sin(x^o)$,求y'.

解 $y = \sin\dfrac{\pi}{180}x$ $\qquad y' = \dfrac{\pi}{180}\cos\dfrac{\pi}{180}x$

3.7

Implicit Differentiation
隐函数的微分法

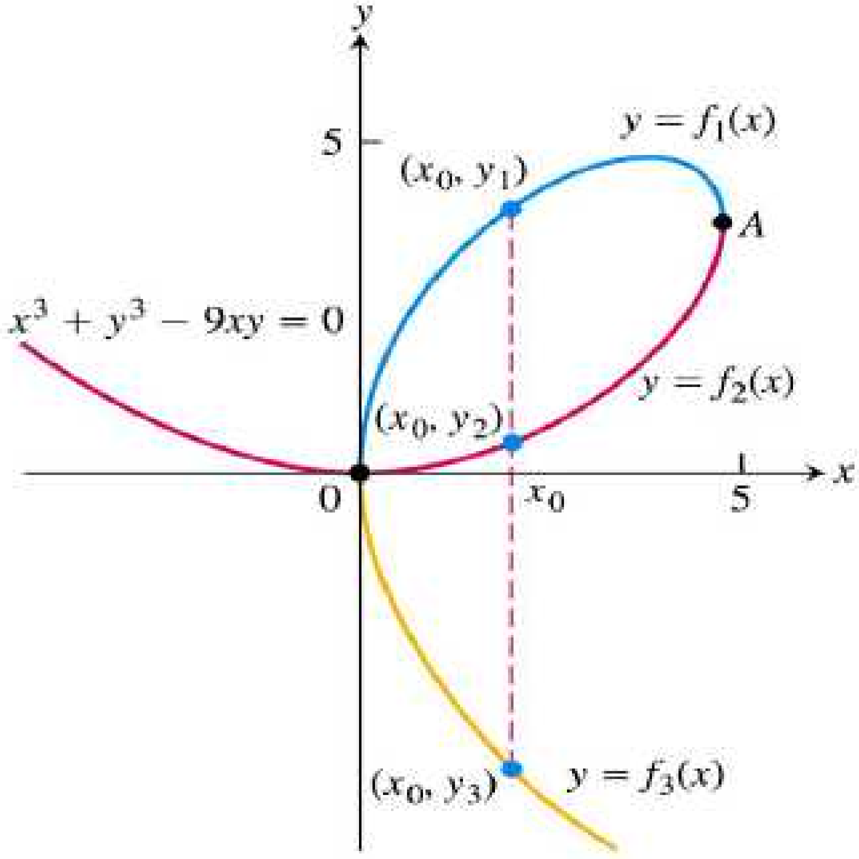


FIGURE 3.26 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x. The curve can, however, be divided into separate arcs that *are* the graphs of functions of x. This particular curve, called a *folium*, dates to Descartes in 1638.

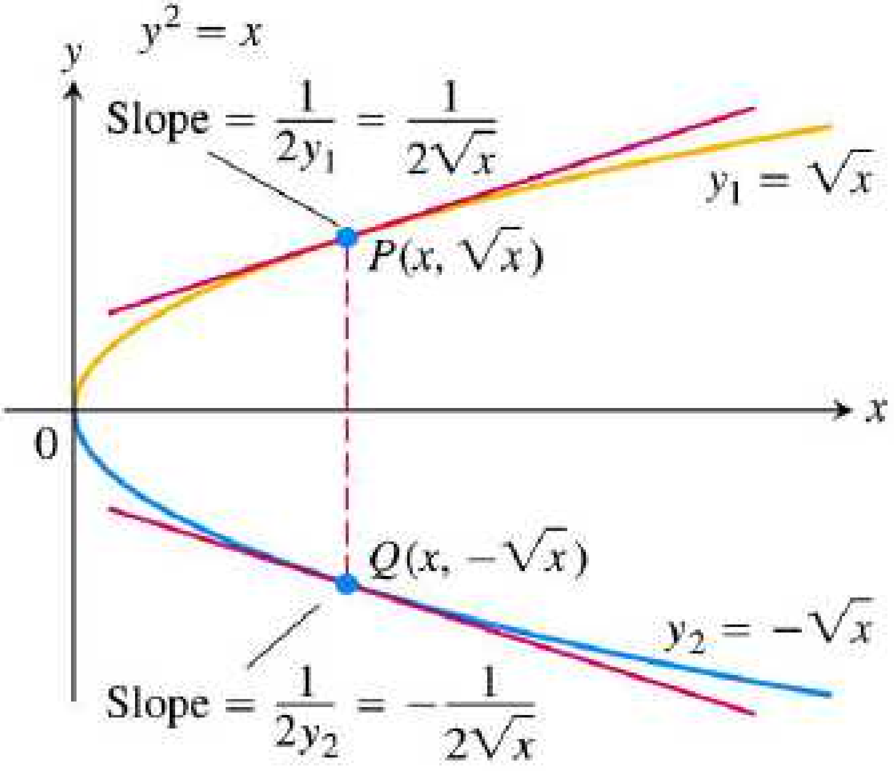


FIGURE 3.27 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x > 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y.

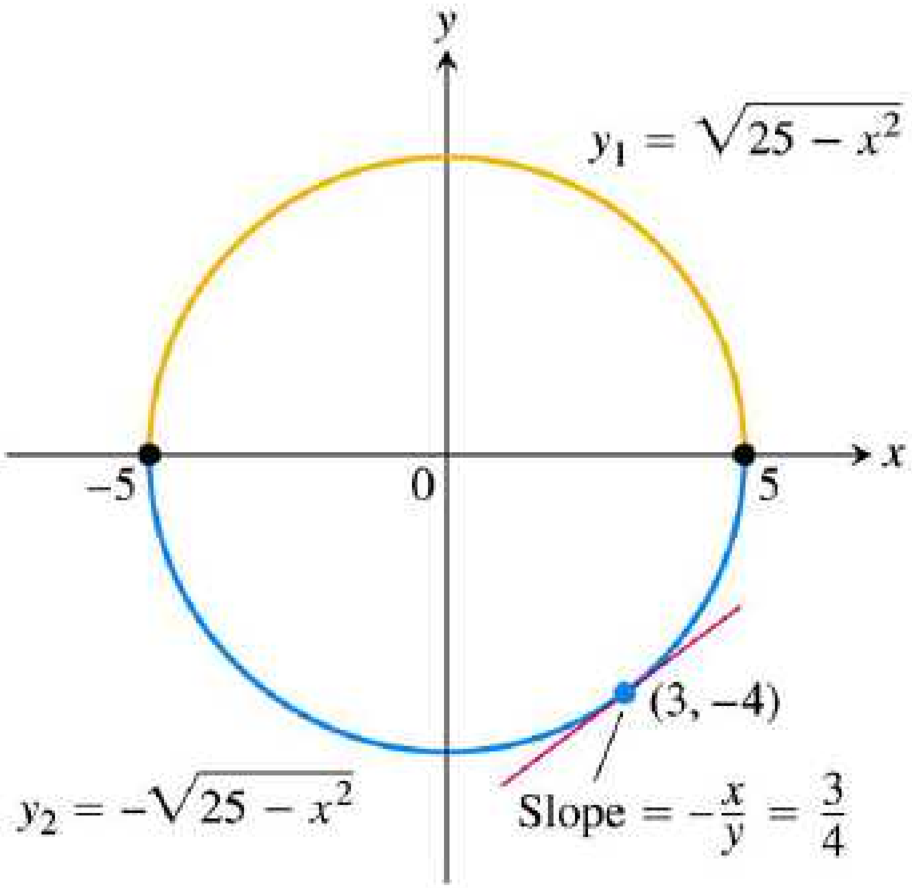


FIGURE 3.28 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$.

如何不用解出显函数直接求隐函数的导数？

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

例1 $y^2 = x$，求y'.

解 $2yy' = 1 \qquad y' = \dfrac{1}{2y}$

例2 求曲线$x^2 + y^2 = 25$在$(3,-4)$处的斜率.

解 $2x + 2yy' = 0 \qquad y' = -\dfrac{x}{y} \qquad y'\big|_{(3,-4)} = \dfrac{3}{4}$

例3 $y^2 = x^2 + \sin(xy)$, 求y'.

解 $2yy' = 2x + \cos(xy)(y + xy')$

$$y' = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}$$

例4 求曲线$x^3 + y^3 - 9xy = 0$在(2,4)处的切线与法线.

解 $3x^2 + 3y^2y' - 9(y + xy') = 0$ $\quad y' = \dfrac{3y - x^2}{y^2 - 3x}$ $\quad y'\big|_{(2,4)} = \dfrac{4}{5}$

切线 $y - 4 = \dfrac{4}{5}(x - 2)$ 法线 $y - 4 = -\dfrac{5}{4}(x - 2)$

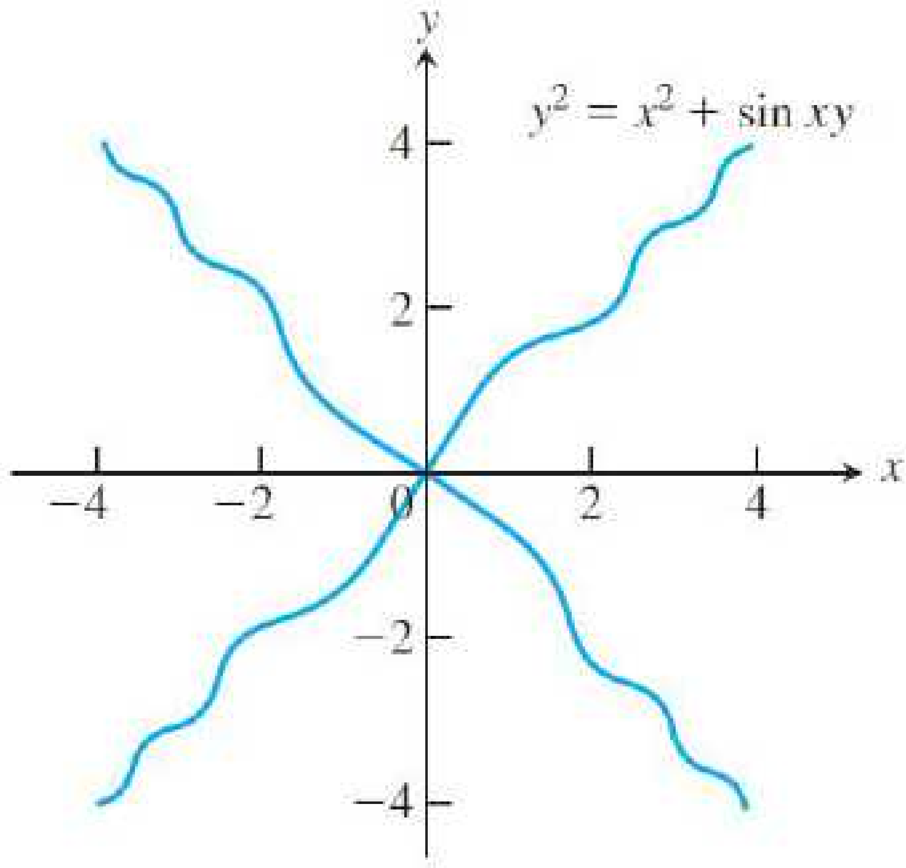


FIGURE 3.29 The graph of the equation in Example 3.

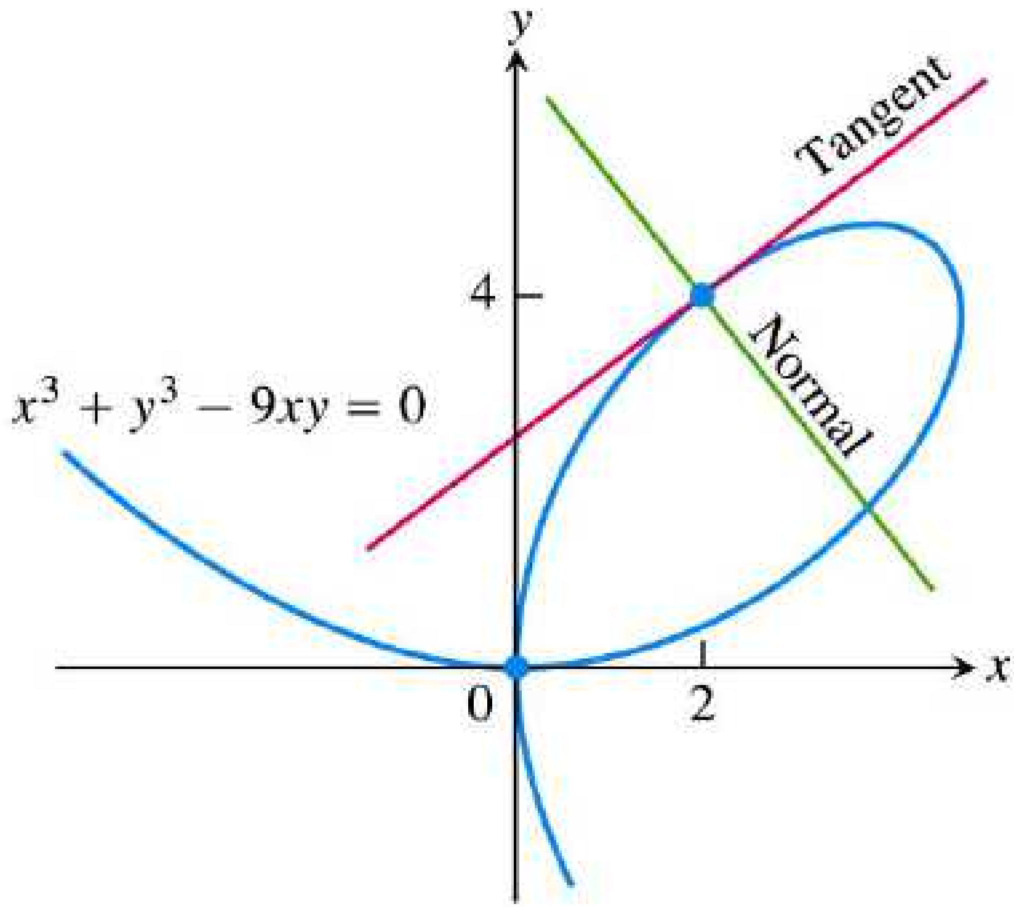


FIGURE 3.31 Example 5 shows how to find equations for the tangent and normal to the folium of Descartes at (2, 4).

隐函数如何求高阶导数？

例5 $2x^3 - 3y^2 = 8$，求y''.

解 $6x^2 - 6yy' = 0$ $\qquad y' = \dfrac{x^2}{y}$

$$y'' = \left(\frac{x^2}{y}\right)' = \frac{2xy - x^2 y'}{y^2} = \frac{2x}{y} - \frac{x^4}{y^3} \quad (y \neq 0)$$

3.8

Related Rates
相关变化率

EXAMPLE 1 Water runs into a conical tank at the rate of 9 ft^3/min.
has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

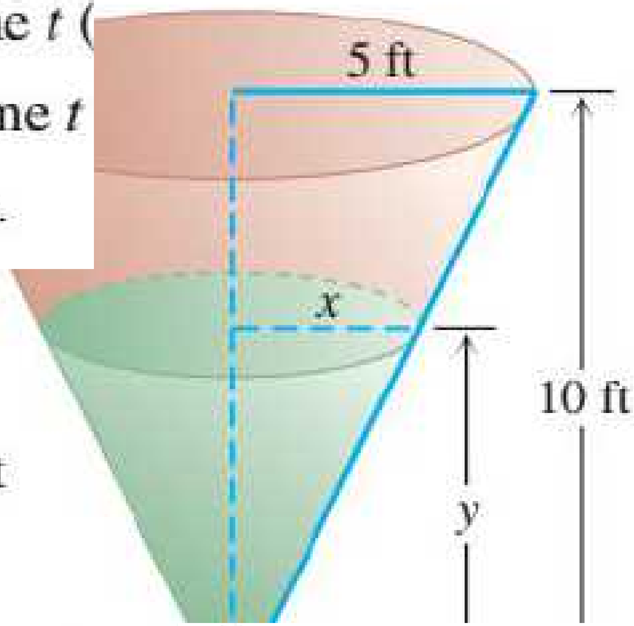
Solution

$$\frac{dV}{dt} = 9 \text{ ft}^3/\text{min}$$

V = volume (ft^3) of the water in the tank at time t (

x = radius (ft) of the surface of the water at time t

y = depth (ft) of the water in the tank at time t.



$$V = \frac{1}{3}\pi x^2 y.$$

$$\frac{dy}{dt} = ?$$
when $y = 6$ ft

$$\frac{x}{y} = \frac{5}{10} \quad \text{or} \quad x = \frac{y}{2}.$$

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3 \qquad \frac{dV}{dt} = \frac{\pi}{12}\cdot 3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}. \qquad \frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

例1 向圆锥形水罐 (如图)注水：$0.25 m^3/\min$.

求当水面高度是 $1.8m$时水面上升的速度 .

解 设经过 t min 后，水面的高度为 y,

水面的半径为 x,罐子内的水体积为 V,

$$\text{则} V = \frac{1}{3}\pi x^2 y \quad \text{而} x = \frac{y}{2},$$

$$\text{则} V = \frac{1}{12}\pi y^3 \qquad \frac{dV}{dt} = 0.25, \quad y = 1.8$$

$$\text{求导得} \quad \frac{dV}{dt} = \frac{1}{4}\pi y^2 \frac{dy}{dt} \qquad 0.25 = \frac{\pi}{4} \cdot 1.8^2 \frac{dy}{dt}$$

$$\frac{dy}{dt} = 0.098 m^3/\min$$

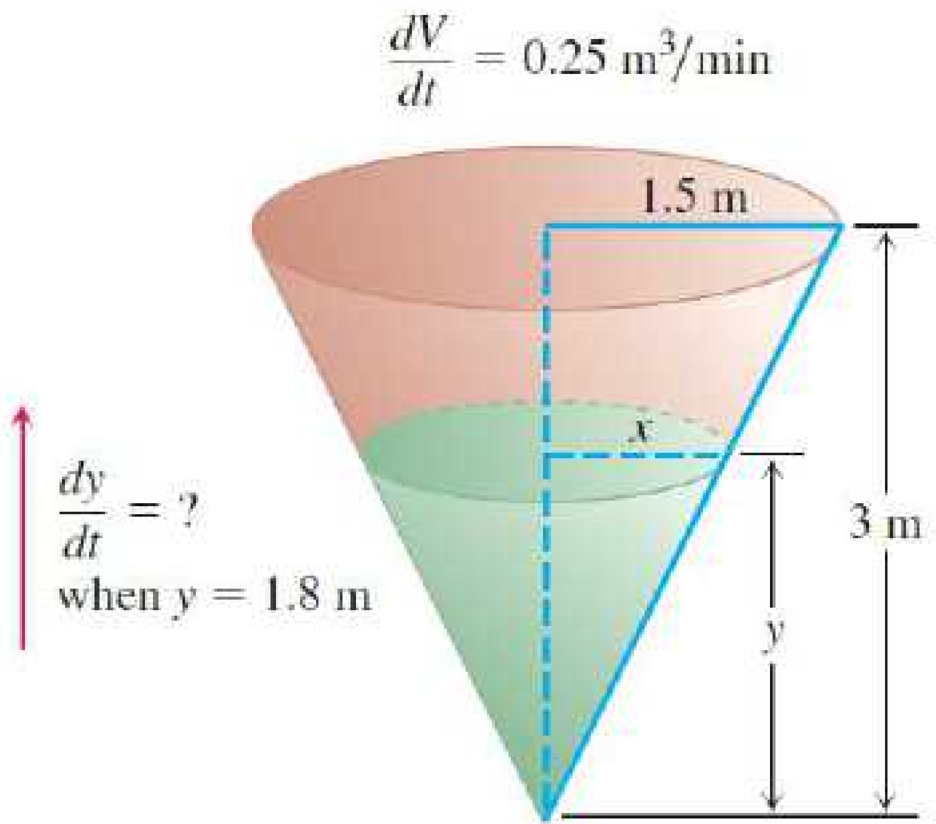
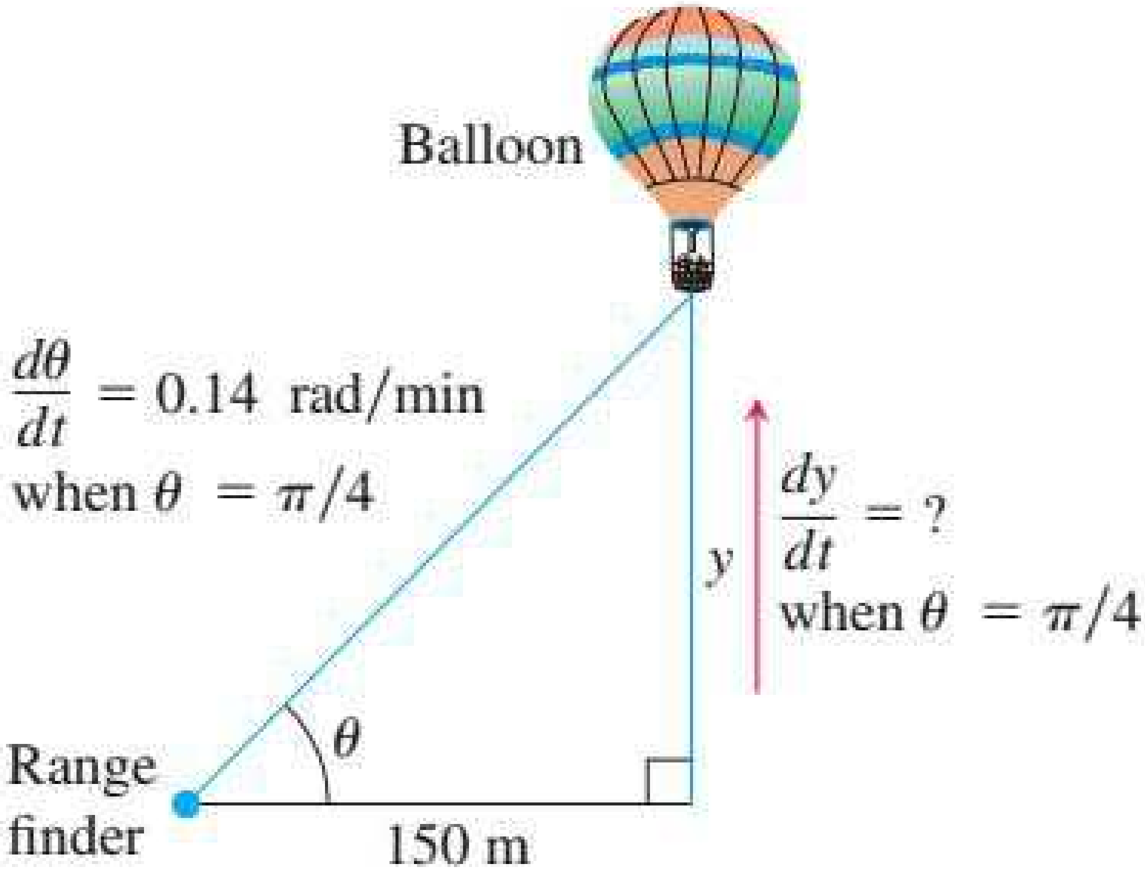


FIGURE 3.32 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants*. Use t for time. Assume that all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables*. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t*. Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate*. Use known values to find the unknown rate.

EXAMPLE 2 A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?



Solution

$\theta =$ the angle in radians the range finder makes with the ground

$y =$ the height in meters of the balloon above the ground.

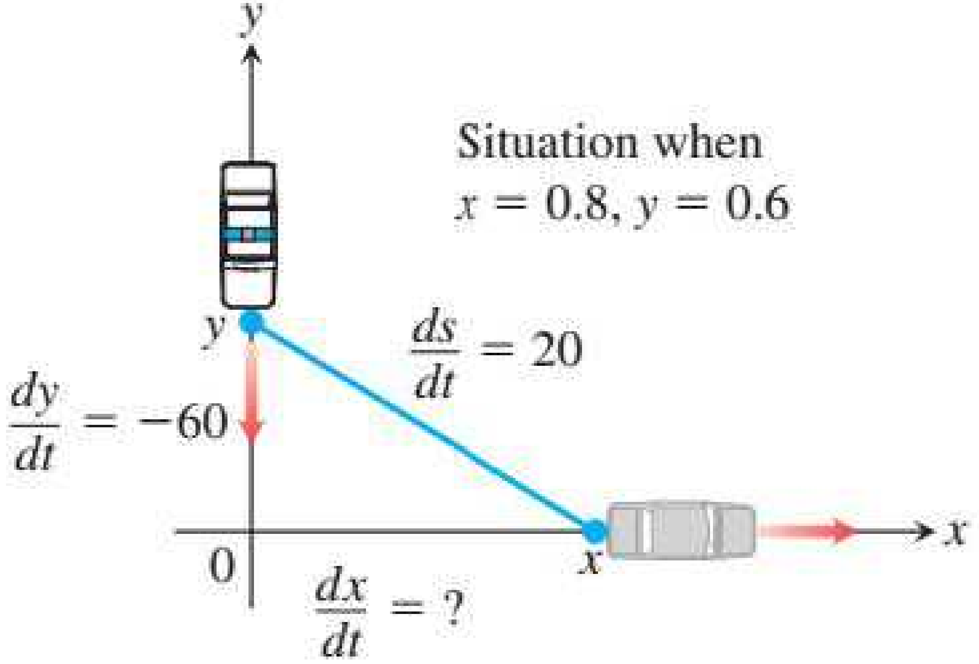
$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \qquad \text{when} \qquad \theta = \frac{\pi}{4}$$

$$\tan\theta = \frac{y}{150} \qquad \sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{150} \cdot \frac{dy}{dt}$$

$$\because \frac{d\theta}{dt} = 0.14/\text{min}, \qquad \text{当}\ \theta = 45^o,\ \sec^2\theta = 2$$

$$\therefore \frac{dy}{dt} = 150 \times 2 \times 0.14 = 42m/\text{min}$$

EXAMPLE 3 A police cruiser, approaching a right-angled intersection from the north is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?



Solution x = position of car at time t

y = position of cruiser at time t

s = distance between car and cruiser at time t.

$$x = 0.8 \text{ mi}, \qquad y = 0.6 \text{ mi}, \qquad \frac{dy}{dt} = -60 \text{ mph}, \qquad \frac{ds}{dt} = 20 \text{ mph}.$$

$$s^2 = x^2 + y^2 \qquad\qquad 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$x\frac{dx}{dt} = s\frac{ds}{dt} - y\frac{dy}{dt}$$

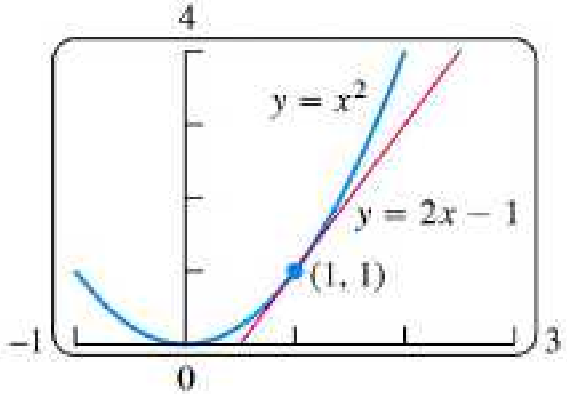
$$\frac{dx}{dt} = \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph.

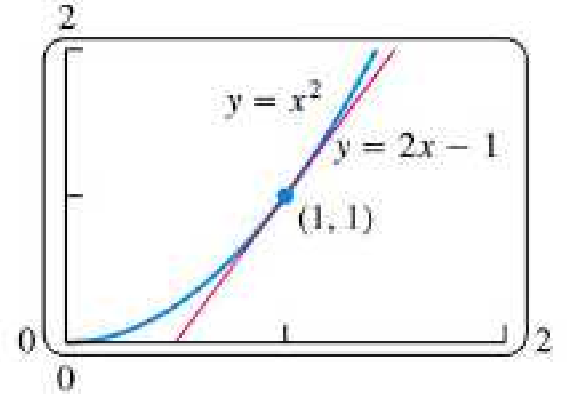
3.9

Linearization and Differentials

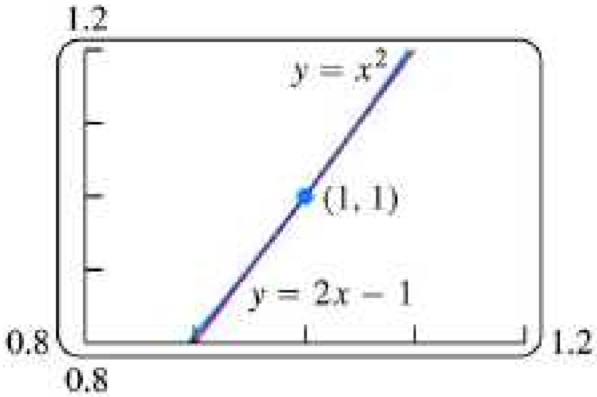
线性化和微分



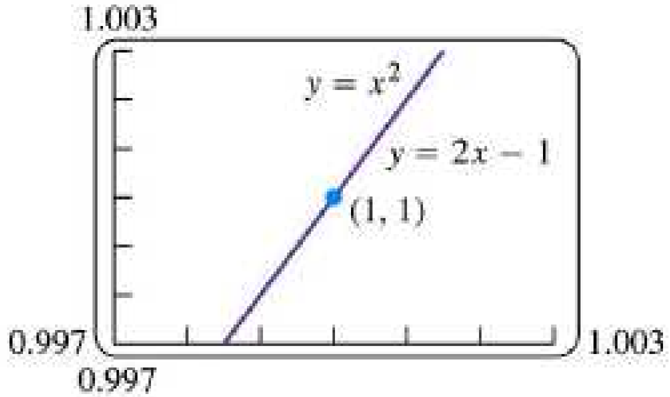
$y = x^2$ and its tangent $y = 2x - 1$ at (1, 1).



Tangent and curve very close near (1, 1).



Tangent and curve very close throughout entire x-interval shown.



Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this x-interval.

FIGURE 3.38 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

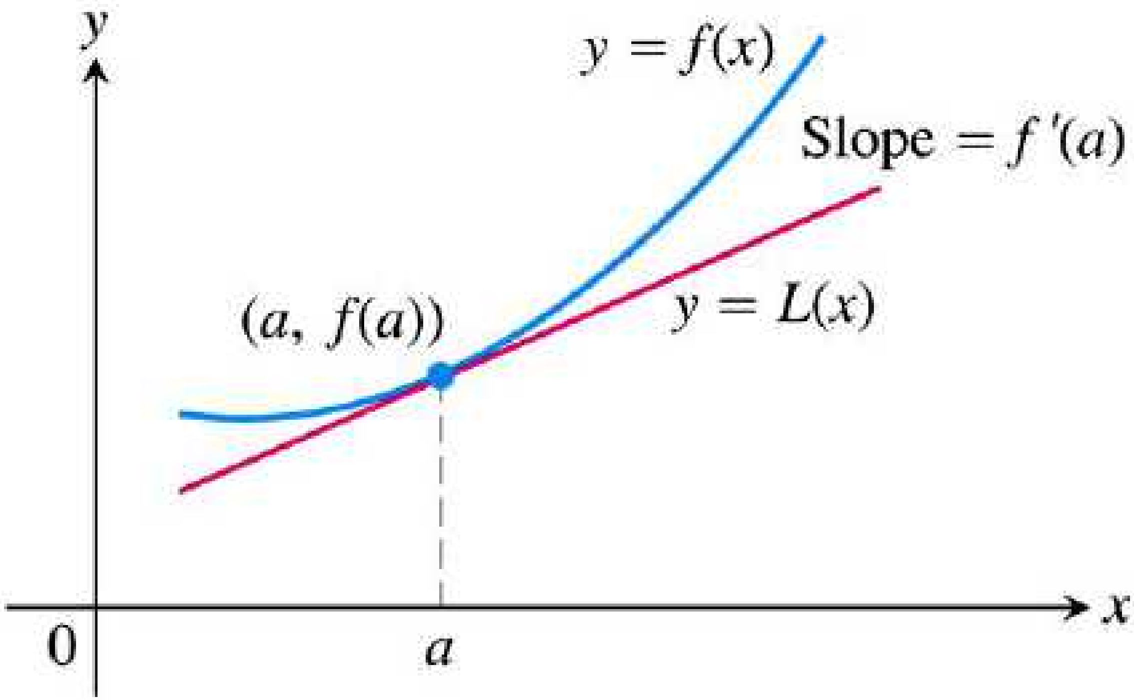


FIGURE 3.39 The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

$$y = f(x) \qquad\qquad y = f(a) + f'(a)(x - a)$$

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

EXAMPLE 1 Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$

Solution $$f'(x) = \frac{1}{2\sqrt{1+x}} \qquad f(0) = 1,\ f'(0) = \frac{1}{2}$$

$$L(x) = 1 + \frac{x}{2}.$$

EXAMPLE 2 Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 3$.

Solution $$f'(x) = \frac{1}{2\sqrt{1+x}} \qquad f(3) = 2,\ f'(3) = \frac{1}{4}$$

$$L(x) = 2 + \frac{x-3}{4} = \frac{5}{4} + \frac{x}{4}.$$

$$f(3.2) \approx L(3.2) = \frac{5}{4} + \frac{3.2}{4} = 2.050.$$

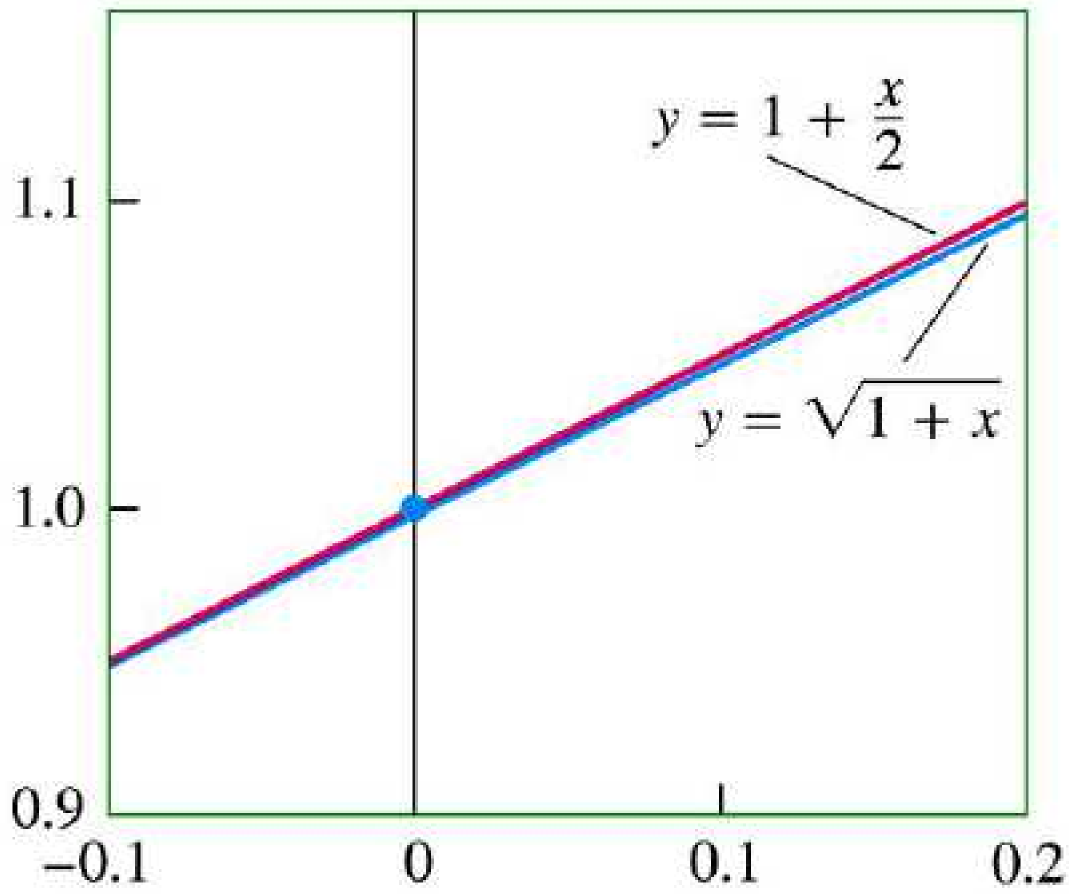


FIGURE 3.41 Magnified view of the window in Figure 3.40.

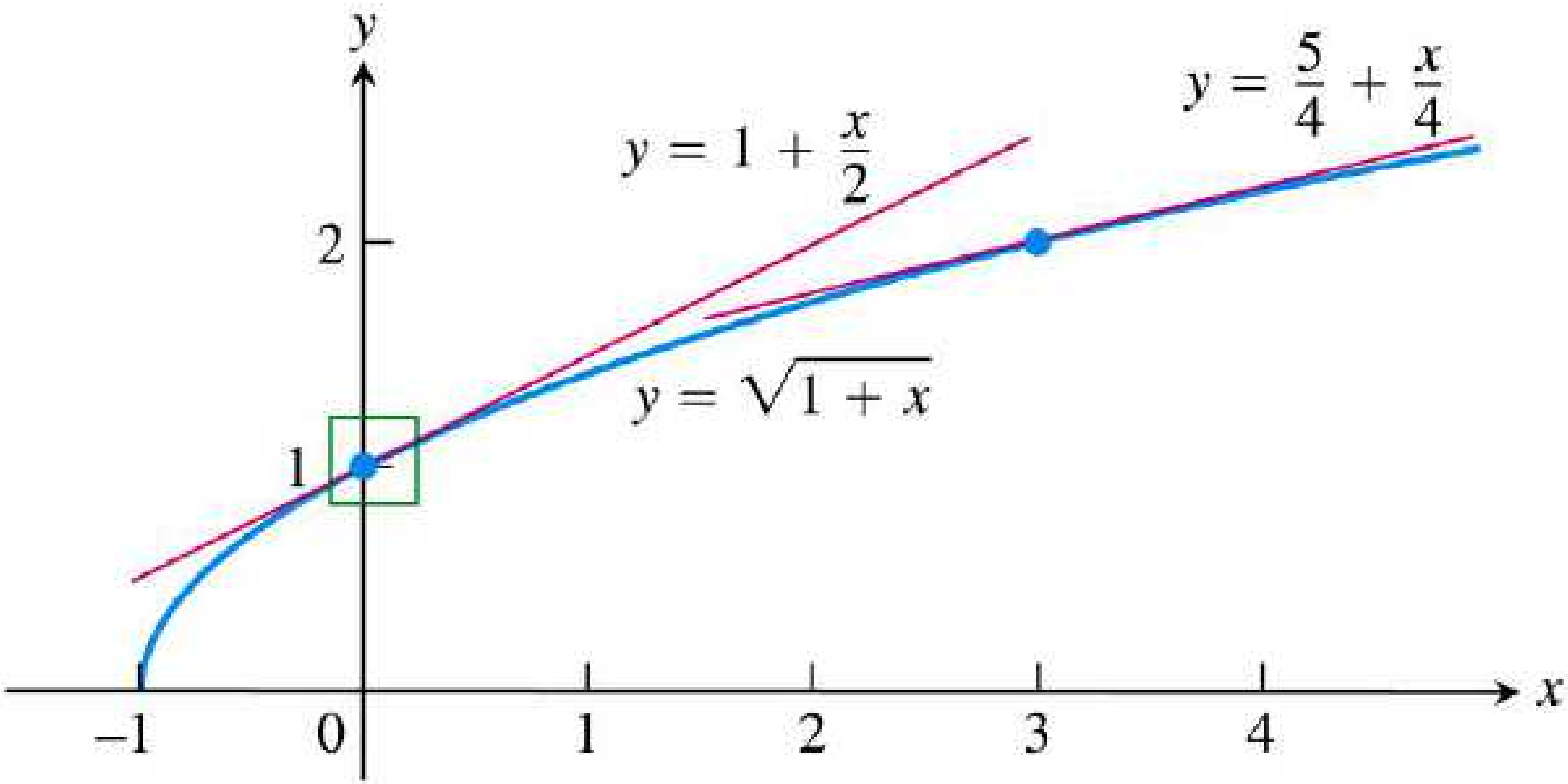


FIGURE 3.40 The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.41 shows a magnified view of the small window about 1 on the y-axis.

Approximation	True value	$\lvert$True value − approximation$\rvert$
$\sqrt{1.2} \approx 1 + \dfrac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$
$\sqrt{1.05} \approx 1 + \dfrac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$\sqrt{1.005} \approx 1 + \dfrac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$

EXAMPLE 3 Find the linearization of $f(x) = \cos x$ at $x = \pi/2$

Solution $f'(x) = -\sin x \qquad f(\frac{\pi}{2}) = 0, f'(\frac{\pi}{2}) = -1,$

$$L(x) = \frac{\pi}{2} - x.$$

Ex. Find the linearization of $f(x) = (1+x)^k$ at $x = 0$,

and show that $\dfrac{1}{\sqrt{1-x^2}} \approx 1 + \dfrac{x^2}{2}.$

Solution $f'(x) = k(1+x)^{k-1} \qquad f(0) = 1, f'(0) = k,$

$L(x) = 1 + kx. \qquad (1+x)^k \approx 1 + kx,$

Let $k = -\dfrac{1}{2}, x = -u^2$, we have $\dfrac{1}{\sqrt{1-u^2}} \approx 1 + \dfrac{u^2}{2}.$

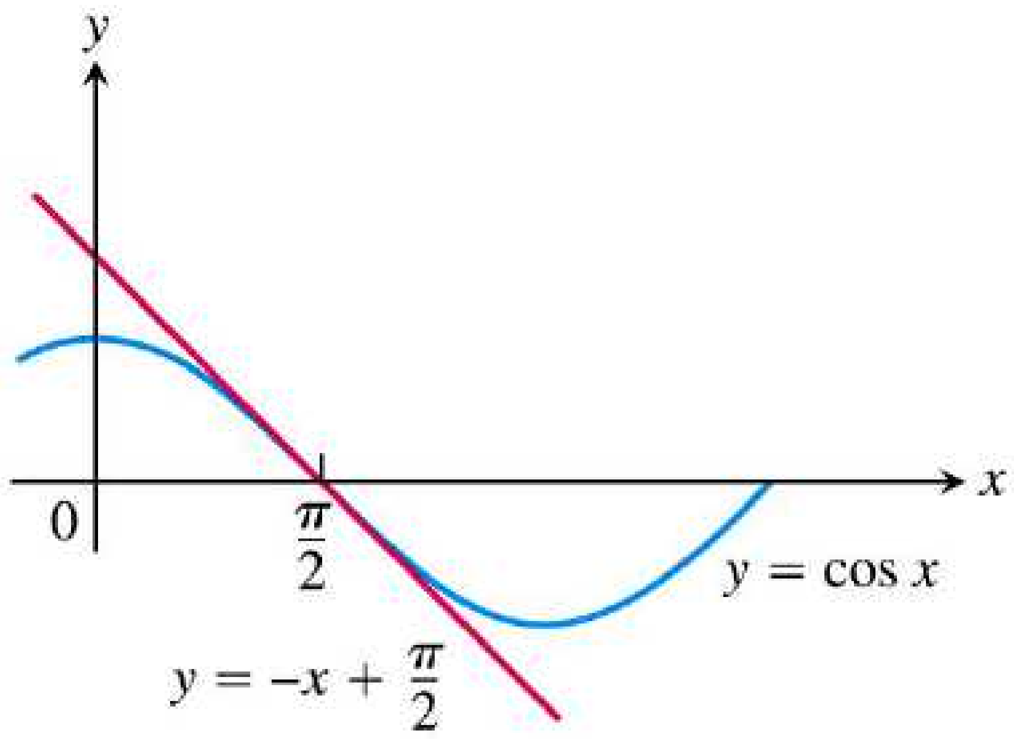


FIGURE 3.42 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$ (Example 3).

微分的定义

$$f(x_0+dx)-f(x_0)\approx f'(x_0)dx$$

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is

$$dy = f'(x)\,dx.$$

$$f'(x)=\frac{dy}{dx}$$

EXAMPLE 4

(a) Find dy if $y = x^5 + 37x$.

(b) Find the value of dy when $x = 1$ and $dx = 0.2$.

Solution

$$(a)\,dy=(5x^4+37)dx$$

$$(b)\,dy\big|_{x=1,dx=0.2}=42\times 0.2=8.4$$

$$\approx(1.2)^5+37\times 1.2-38$$

$$=8.88832$$

微分的几何意义

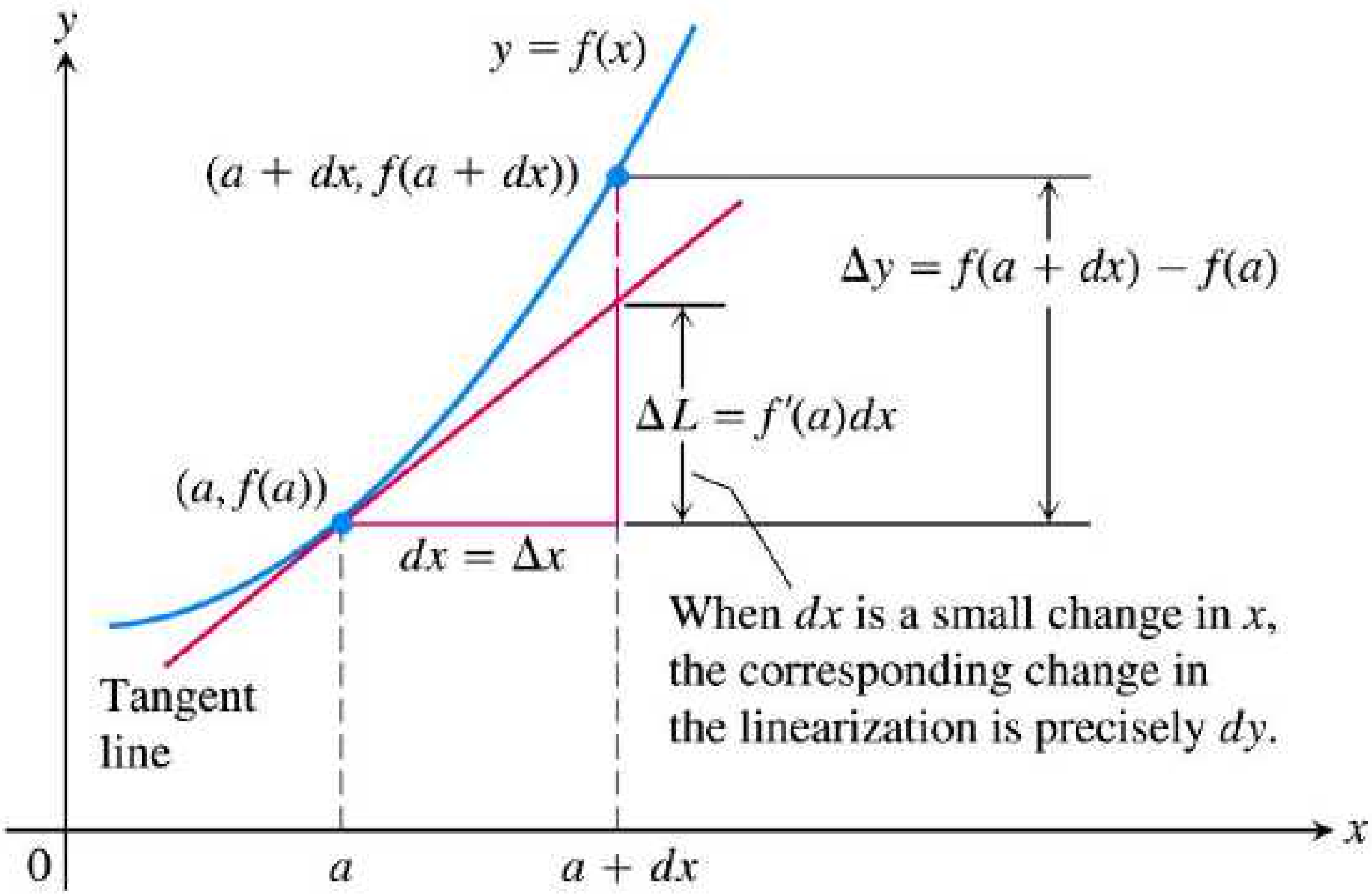


FIGURE 3.43 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

常用的微分公式和微分法则

$$d(C)=0 \qquad d(x^{\mu})=\mu x^{\mu-1}dx$$

$$d(\sin x)=\cos xdx \qquad d(\cos x)=-\sin xdx$$

$$d(\tan x)=\sec^2 xdx \qquad d(\cot x)=-\csc^2 xdx$$

$$d(\sec x)=\sec x\tan xdx \qquad d(\csc x)=-\csc x\cot xdx$$

$$d(u\pm v)=du\pm dv \qquad d(Cu)=Cdu$$

$$d(uv)=vdu+udv \qquad d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}$$

微分形式不变性

公式$dy=f'(u)du$当$y=f(u),u=g(x)$形式也正确！

因为$dy=f'(u)g'(x)dx=f'(u)du$

EXAMPLE 5 (a) $d(\tan 2x) =$

(b) $d\left(\dfrac{x}{x+1}\right)$

$$(a)\,d(\tan 2x) = \sec^2 2x\,d(2x) = 2\sec^2 2x\,dx$$

$$(b)\,d\left(\frac{x}{x+1}\right) = \frac{(x+1)dx - x\,d(x+1)}{(x+1)^2}$$

$$= \frac{xdx + dx - xdx}{(x+1)^2} = \frac{dx}{(x+1)^2}$$

设曲线由方程$\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases}$则 $\dfrac{dy}{dx}=\dfrac{\psi'(t)dt}{\varphi'(t)dt}=\dfrac{\psi'(t)}{\varphi'(t)}$

例. 设$\begin{cases} x=\cos^3 t, \\ y=\sin^3 t, \end{cases}$ 求$\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$.

解. $\dfrac{dy}{dx}=\dfrac{3\sin^2 t\cos t}{-3\cos^2 t\sin t}=-\tan t.$

$$\frac{d^2y}{dx^2}=\frac{d(-\tan t)}{dx}=\frac{-\sec^2 t dt}{-3\cos^2 t\sin t dt}$$

$$=\frac{1}{3\cos^4 t\sin t}.$$

$$dy = f'(x)dx \approx f(x+dx) - f(x)$$

we can estimate $f(x+dx) - f(x)$ with dy.

EXAMPLE 6

The radius r of a circle increases from $a = 10$ m to 10.1 m

Use dA to estimate the increase in the circle's area A.

compare your estimate to the true area found by direct calculation.

Solution Since $A = \pi r^2$,

$$\Delta A \approx dA\Big|_{a=10,da=0.1} = 2\pi a da\Big|_{a=10,da=0.1} = 2\pi(m^2).$$

$$\Delta A = \pi(a+da)^2 - \pi a^2 = 102.01\pi - 100\pi = 2.01\pi$$

$$|\Delta A - dA| = 0.01\pi(m^2)$$

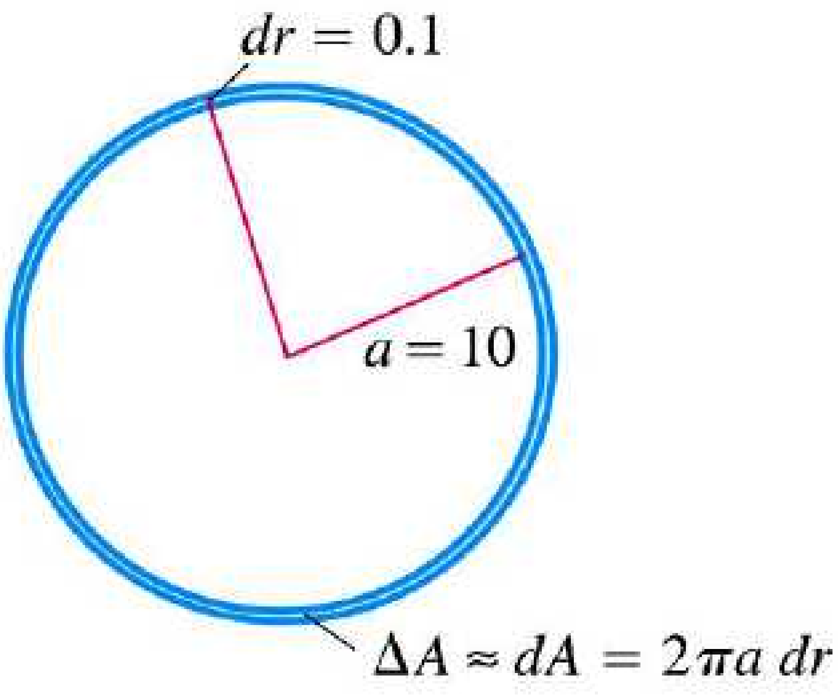


FIGURE 3.44 When dr is small compared with a, the differential dA gives the estimate $A(a + dr) = \pi a^2 + dA$ (Example 6).

$$f(x+dx) \approx f(x) + f'(x)dx$$

EXAMPLE 7 Use differentials to estimate

(a) $7.97^{1/3}$

(b) $\sin(\pi/6 + 0.01)$.

Solution

$$(a)\ \sqrt[3]{7.97} = \sqrt[3]{8+(-0.03)} = f(x+dx)$$

$$\text{Let}\quad f(x) = \sqrt[3]{x},\ \ x = 8, dx = -0.03$$

$$f(x+dx) \approx f(x) + f'(x)dx$$

$$= \sqrt[3]{8} + \frac{1}{3} \times \frac{1}{(\sqrt[3]{8})^2}(-0.03) = 2 + \frac{-0.03}{12} = 1.9975$$

$$(b) \sin\left(\frac{\pi}{6} + 0.01\right) \approx \sin\frac{\pi}{6} + \cos\frac{\pi}{6} \times 0.01 = \frac{1}{2} + \frac{\sqrt{3}}{2} \times 0.01$$

$$= 0.5087.$$

Error in Differential Approximation

$$f'(x)dx = f(x+dx) - f(x) + ?$$ 误差多少？

$$f(x+\Delta x) - f(x) - f'(x)\Delta x$$

$$= \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} - f'(x)\right)\Delta x = \varepsilon \Delta x \qquad \lim_{\Delta x \to 0} \varepsilon = 0$$

$$f(x+\Delta x) - f(x) = f'(x)\Delta x + \varepsilon \Delta x$$

误差是$\varepsilon \Delta x$，$\lim_{\Delta x \to 0} \varepsilon = 0$

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$$\Delta y = f'(a)\,\Delta x + \epsilon\,\Delta x \qquad (1)$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

微分的等价定义

Proof of the Chain Rule

$$y = f(u), \quad u = g(x), \qquad \left.\frac{dy}{dx}\right|_{x=x_0} = f'(g(x_0)) \cdot g'(x_0).$$

$$\Delta u = g'(x_0)\Delta x + \epsilon_1 \Delta x = (g'(x_0) + \epsilon_1)\Delta x,$$

$$\Delta y = f'(u_0)\Delta u + \epsilon_2 \Delta u = (f'(u_0) + \epsilon_2)\Delta u,$$

$$= (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1)\Delta x,$$

$$\frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) + \epsilon_2 g'(x_0) + f'(u_0)\epsilon_1 + \epsilon_2\epsilon_1.$$

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) = f'(g(x_0)) \cdot g'(x_0).$$

Sensitivity to Change

$$\Delta f = f(x+dx) - f(x) \qquad df = f'(x)dx$$

	True	Estimated
Absolute change	$\Delta f = f(a+dx) - f(a)$	$df = f'(a)\,dx$
Relative change	$\dfrac{\Delta f}{f(a)}$	$\dfrac{df}{f(a)}$
Percentage change	$\dfrac{\Delta f}{f(a)} \times 100\ \%$	$\dfrac{df}{f(a)} \times 100\ \%$

例8 某人向水井里投一个石头，记录该石头落水的时间，用公式$s = 4.9t^2$来测其深度．若记录时间的误差是0.1s,那么这样估计出的水井深度的误差是多少？

EXAMPLE 8

$$s = 4.9t^2$$

You want to calculate the depth of a well from the equation by timing how long it takes a heavy stone you drop to splash into the water below. How sensi tive will your calculations be to a 0.1-sec error in measuring the time?

Solution 当$dt = 0.1$时，$\Delta s \approx ds = 9.8t\,dt = 9.8 \times 0.1t$

若$t = 2\text{s}$，$\Delta s \approx ds = 0.98 \times 2 = 1.96\text{m}$ $\dfrac{\Delta s}{s} \approx \dfrac{ds}{s} = \dfrac{1.96}{19.6} = 0.1$

若$t = 5\text{s}$，$\Delta s \approx ds = 0.98 \times 5 = 4.9\text{m}$ $\dfrac{\Delta s}{s} \approx \dfrac{ds}{s} = \dfrac{4.9}{122.5} = 0.04$