# CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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# Properties of Relations

- **Reflexive Relation**: A relation $R$ on a set $A$ is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  **Irreflexive Relation**: A relation $R$ on a set $A$ is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

  **Symmetric Relation**: A relation $R$ on a set $A$ is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

  **Antisymmetric Relation**: A relation $R$ on a set $A$ is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

  **Transitive Relation**: A relation $R$ on a set $A$ is called *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

# Connectivity

- **Lemma**: Let $A$ be a set with $n$ elements, and $R$ a relation on $A$. If there is a path from $a$ to $b$ with $a \neq b$, then there exists a path of length $\leq n-1$.

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**Theorem**: The transitive closure of a relation $R$ equals the connectivity relation $R^*$.

**Recall** Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

# Equivalence Relation

- **Definition** A relation $R$ on a set $A$ is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

- **Definition** Let $R$ be an equivalence relation on a set $A$. The set of all elements that are related to an element $a$ of $A$ is called the *equivalence class* of $a$, denoted by $[a]_R$. When only one relation is considered, we use the notation $[a]$.

$$[a]_R = \{b : (a, b) \in R\}$$

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**Theorem** Let $\{A_1, A_2, \ldots, A_i, \ldots\}$ be a partition of $S$. Then there is an equivalence relation $R$ on $S$, that has the sets $A_i$ as its equivalence classes.

# Equivalence Relation

Don Zagier

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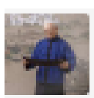

他四十年前来中国，在合肥一周就写了四百中国汉字，说是在日本学的中日通用。他在波恩讲演之前，先问听众赞成用英语，德语，。。分别举手，他用多数人赞成的语言讲。张贤科给他起中文名：查吉尔，或查杰尔。他选后者。。。。

# Partial Ordering

- **Definition** A relation $R$ on a set $S$ is called a *partial ordering*, or *partial order*, if it is reflexive, antisymmetric, and transitive. A set $S$ together with a partial ordering $R$ is called a *partially ordered set*, or *poset*, denoted by $(S, R)$. Members of $S$ are called *elements of the poset*.

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  $2, 4$ are comparable, $3, 5$ are incomparable.

# Total Ordering

- **Definition** If $(S, \preccurlyeq)$ is a poset and every two elements of $S$ are comparable, $S$ is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

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# Lexicographic Ordering

- **Definition** Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that $(a_1, a_2)$ is less than $(b_1, b_2)$, i.e., $(a_1, a_2) \preccurlyeq (b_1, b_2)$, either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ then $a_2 \preccurlyeq_2 b_2$.

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  - $\textit{discreet} \prec \textit{discrete}$
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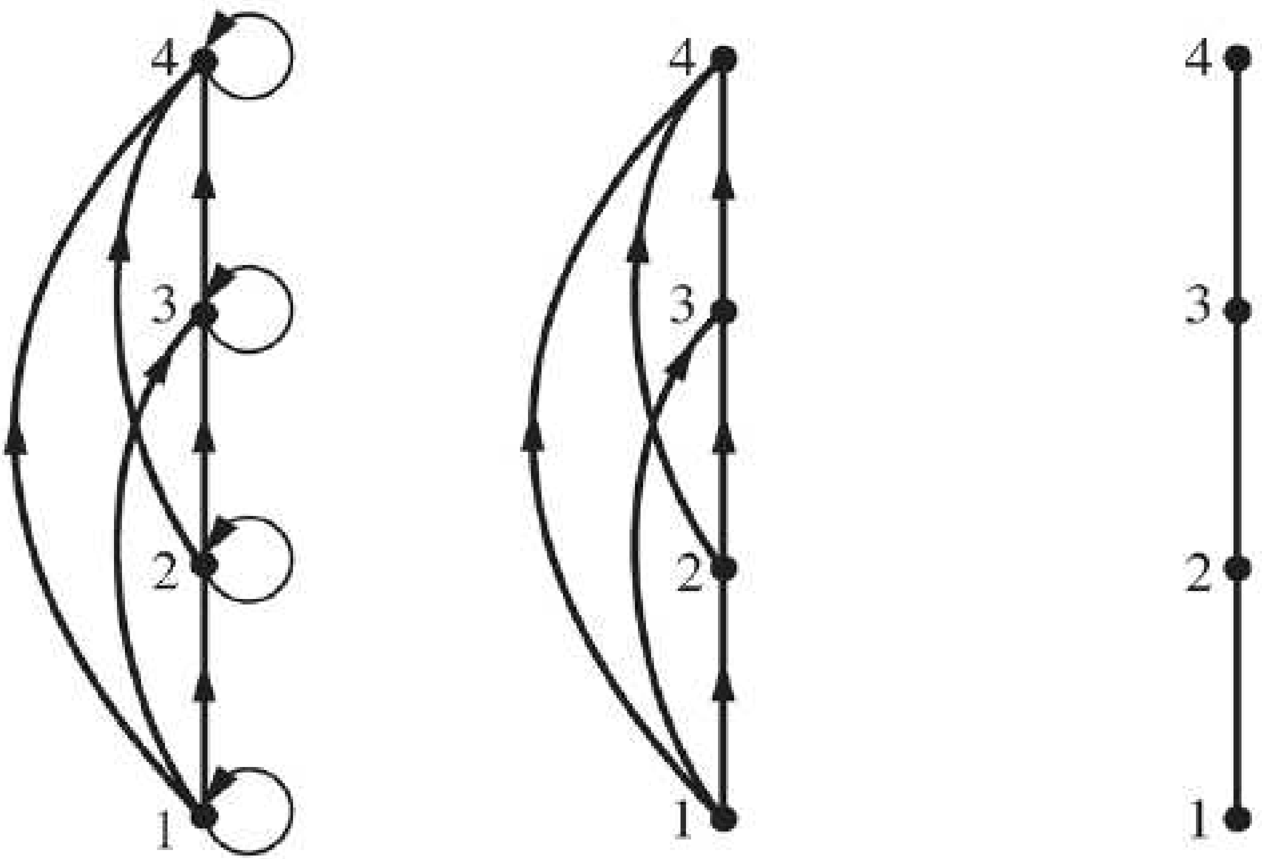
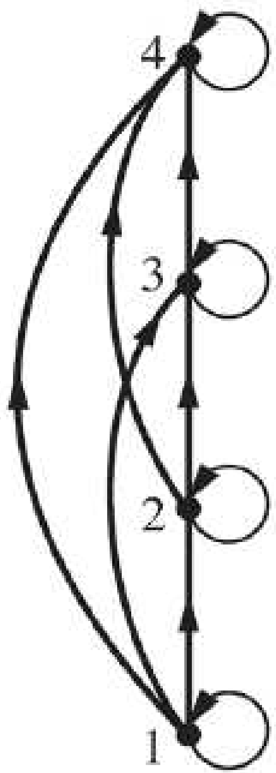


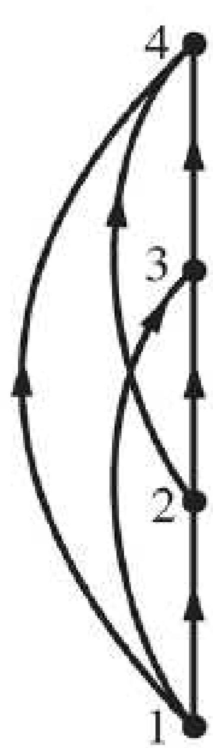
# Hasse Diagram

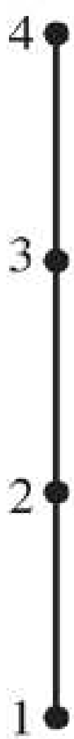
- (a) A partial ordering. The loops are due to the reflexive property

  (b) The edges that must be present due to the transitive property are deleted

  (c) The Hasse diagram for the partial ordering (a)



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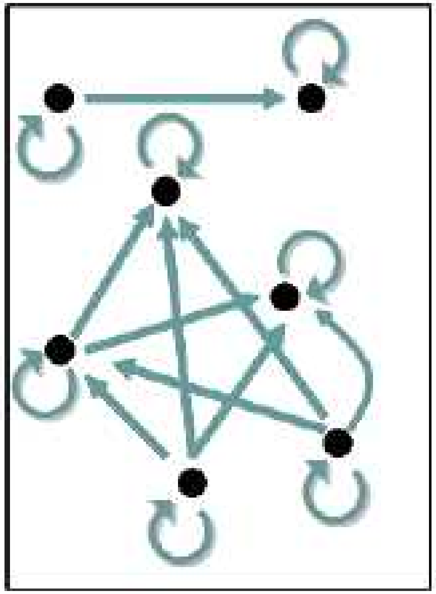
  ◇ Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

# Hasse Diagram Example



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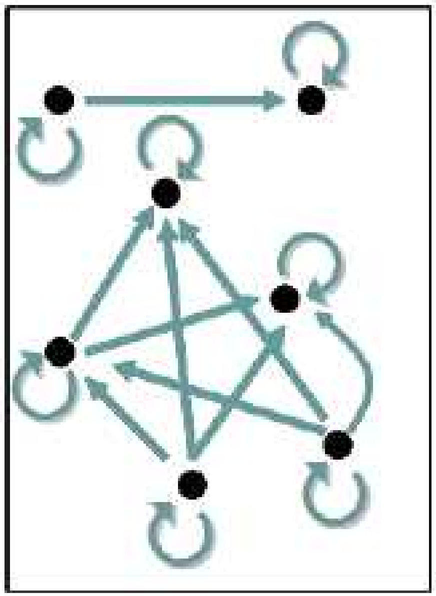

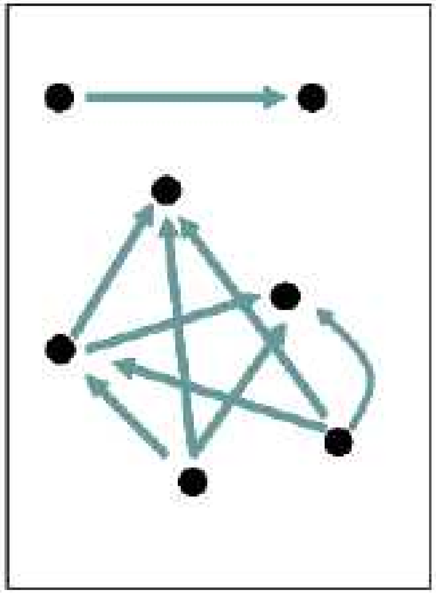

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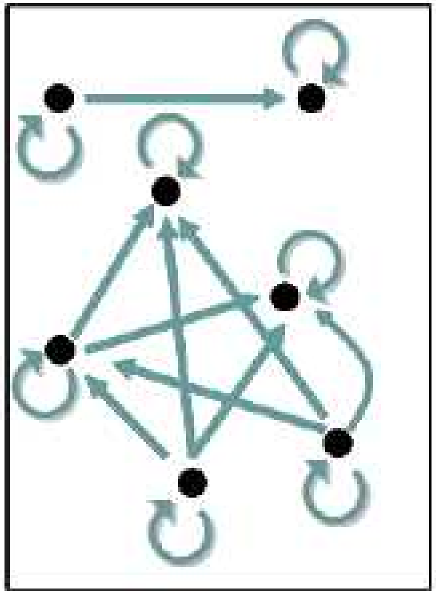

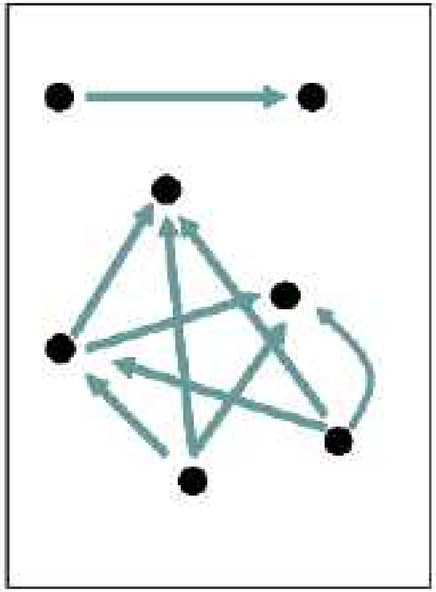

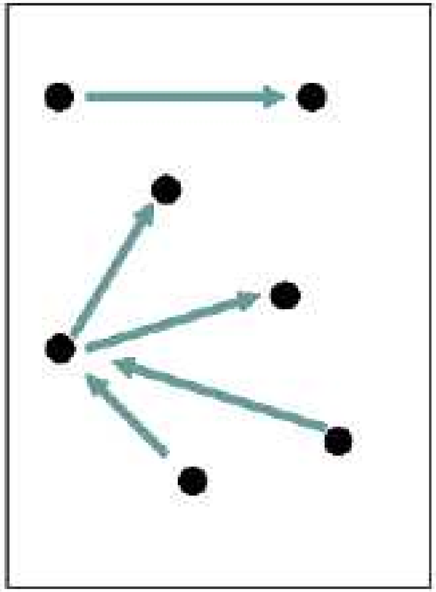

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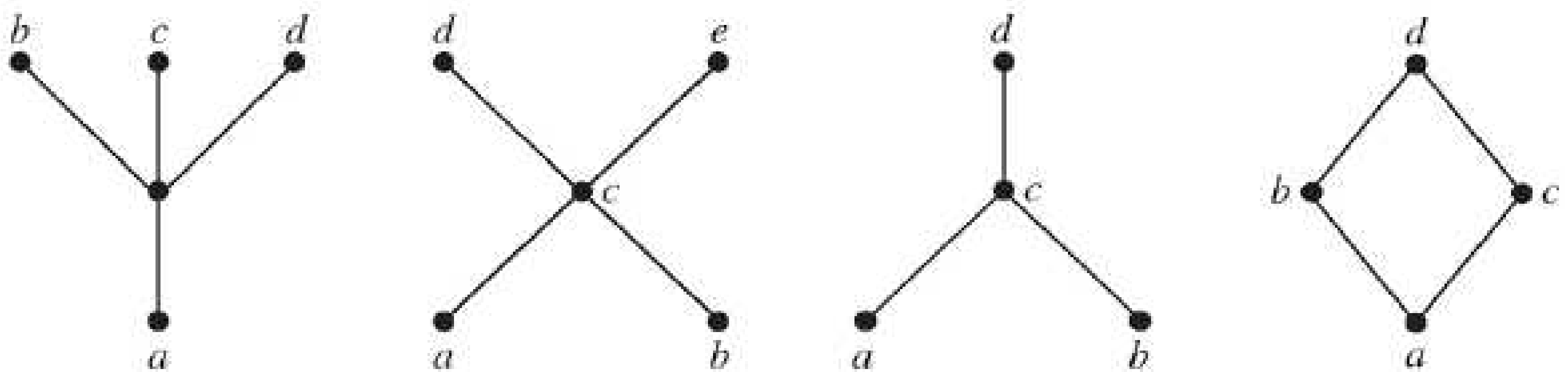


(a) (b) (c) (d)

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**Example** Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

# Well-Ordered Set

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**The Principle of Well-Ordered Induction** Suppose that $S$ is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if

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p. 620, Theorem 1

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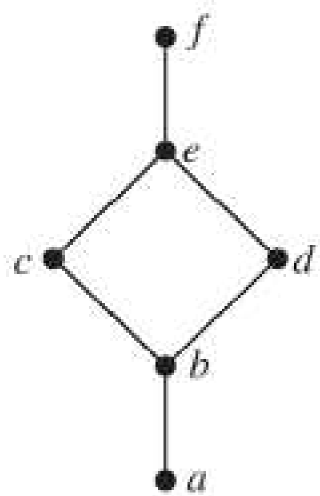


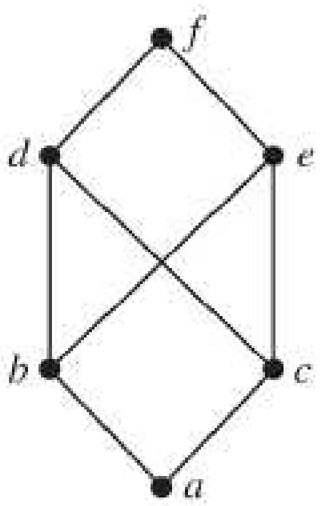

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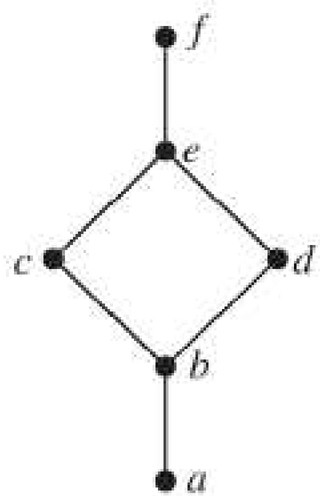


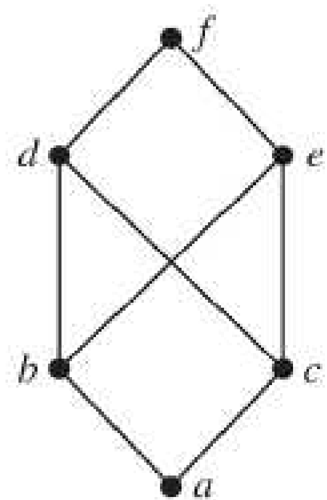


**Example** Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

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- Motivation: A project is made up of 20 different tasks. Some tasks can be completed only after others have been finished. How can an order be found for these tasks?

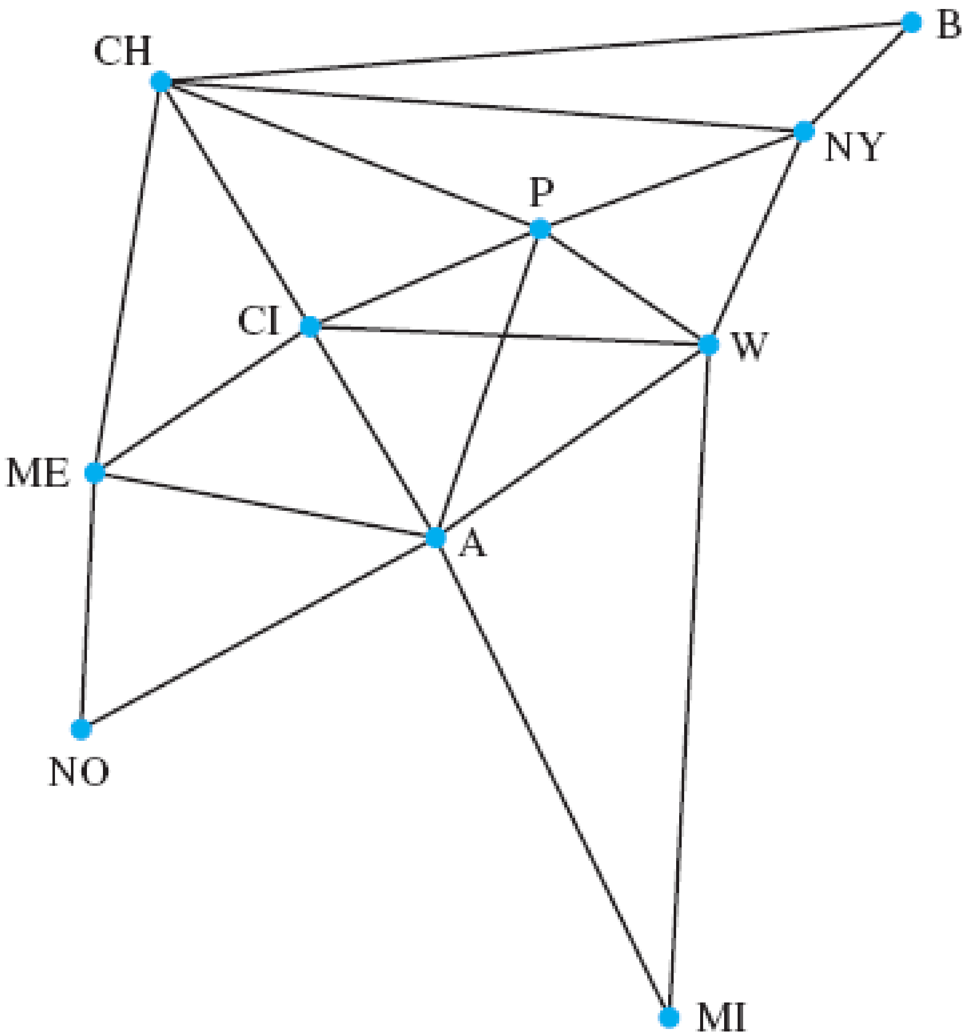
*Topological sorting*: Given a partial ordering $R$, find a total ordering $\preccurlyeq$ such that $a \preccurlyeq b$ whenever $a\ R\ b$. $\preccurlyeq$ is said *compatible with $R$*.

# Topological Sorting for Finite Posets

**procedure** topological_sort ($S$: finite poset)

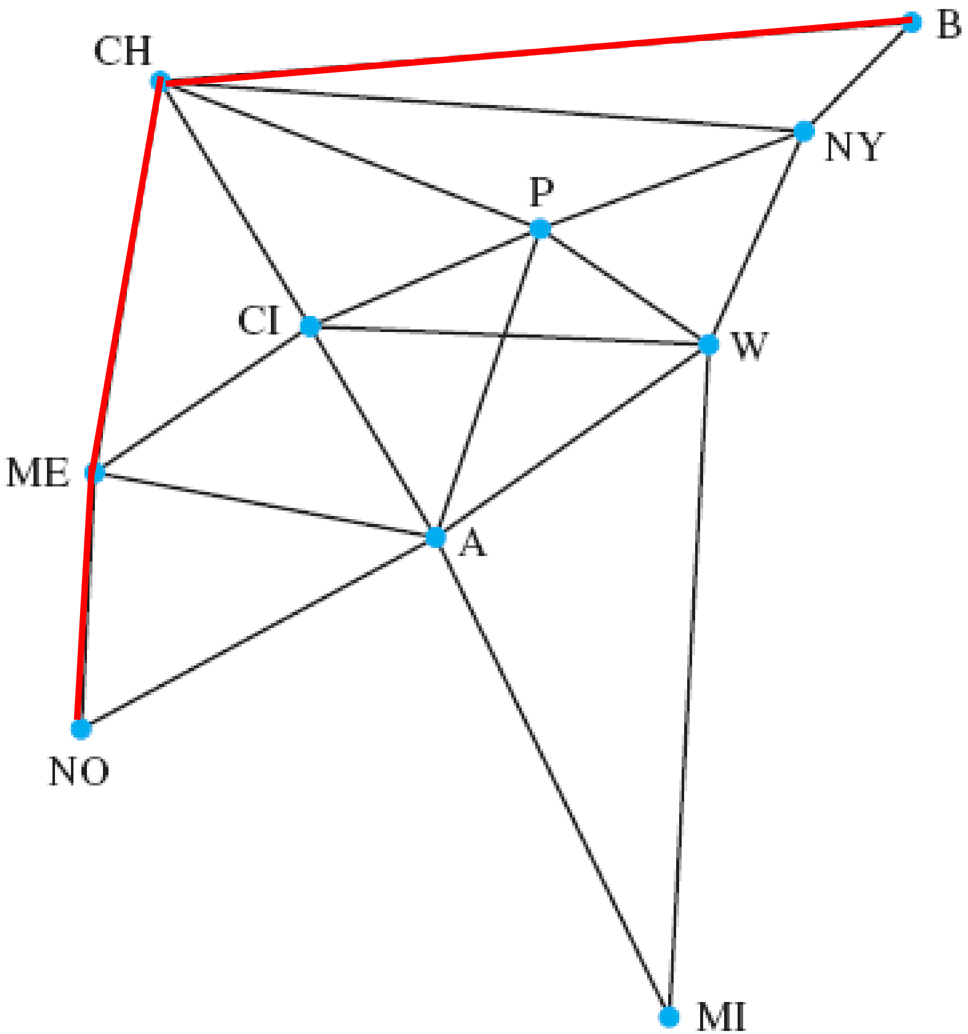
$k := 1$;
**while** $S \neq \emptyset$
  $a_k :=$ a minimal element of $S$
  $S := S \setminus \{a_k\}$
  $k := k + 1$
**end while**
// $\{a_1, a_2, \ldots, a_n\}$ is a compatible total ordering of $S$

# Example




What is the minimum number of links to send a message from B to NO?

# Example

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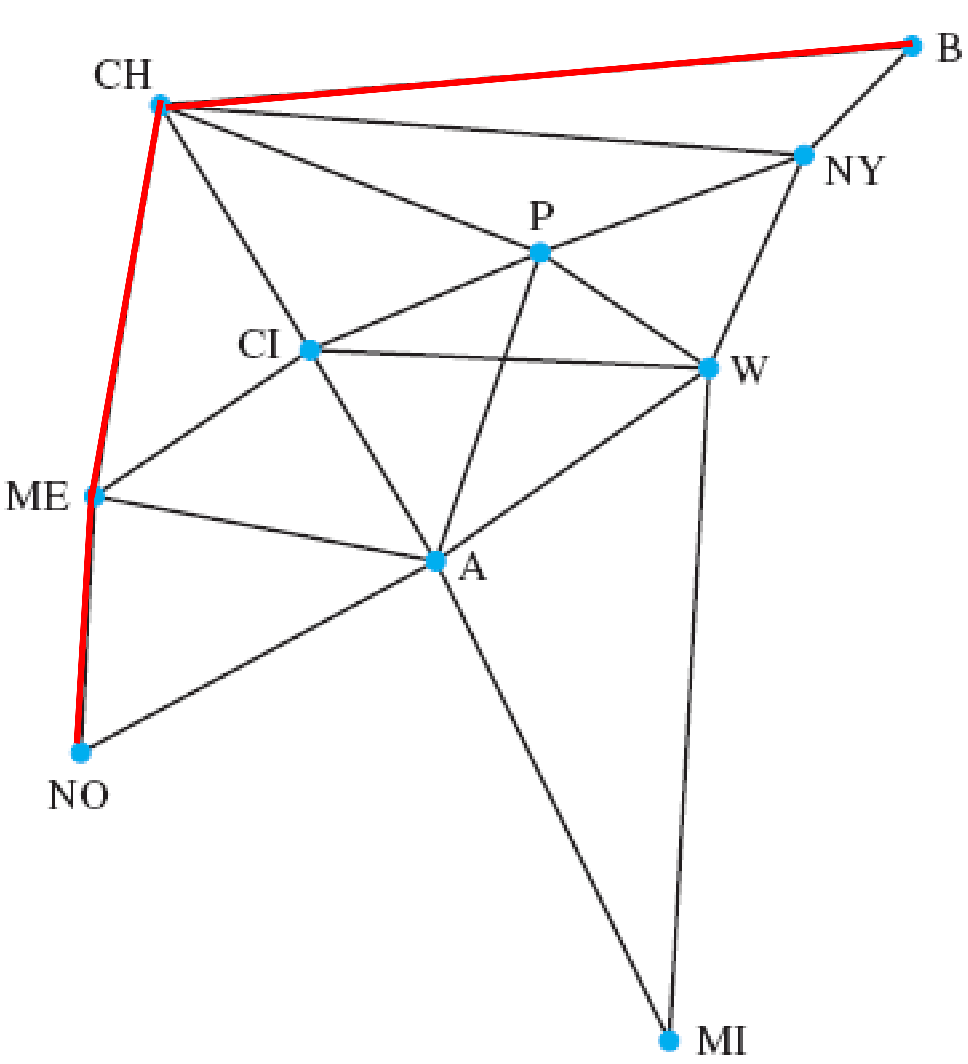


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3: B - CH - ME - NO

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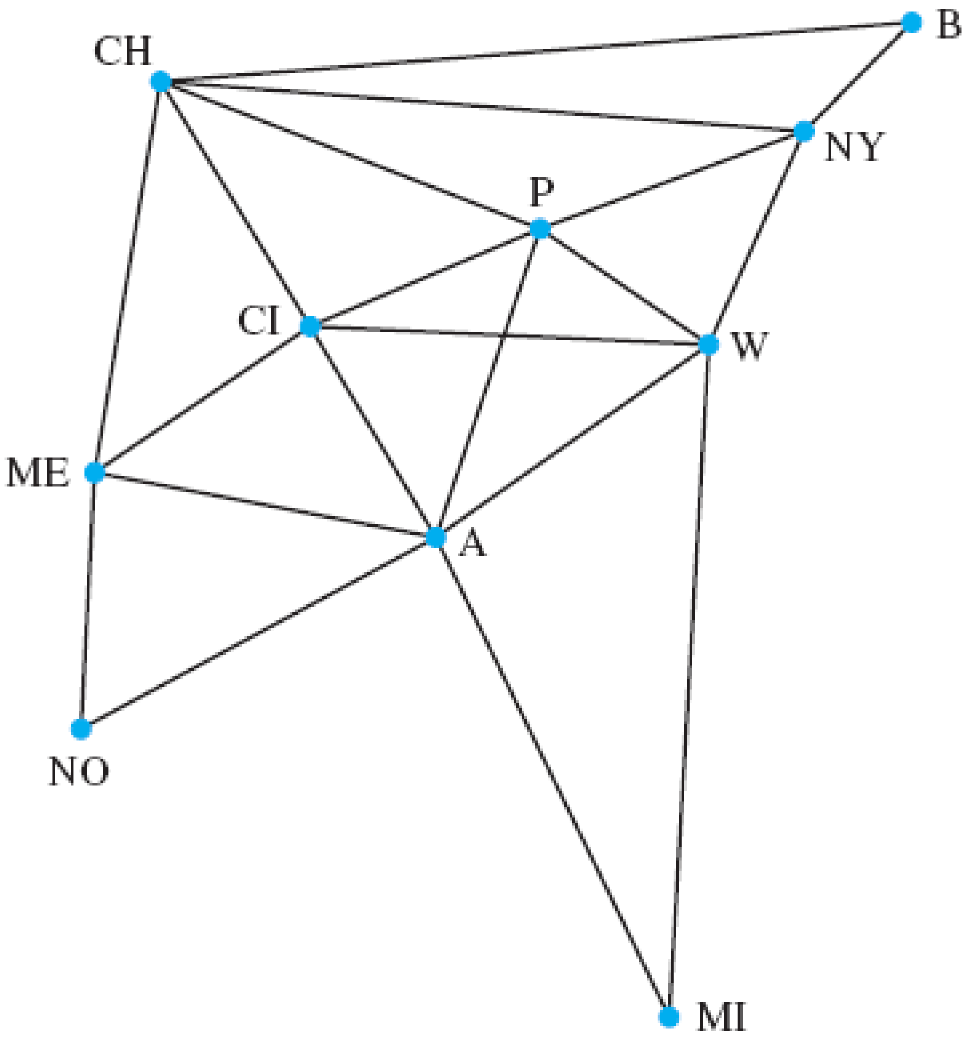


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Which city/cities has/have the most communication links emanating from it/them?

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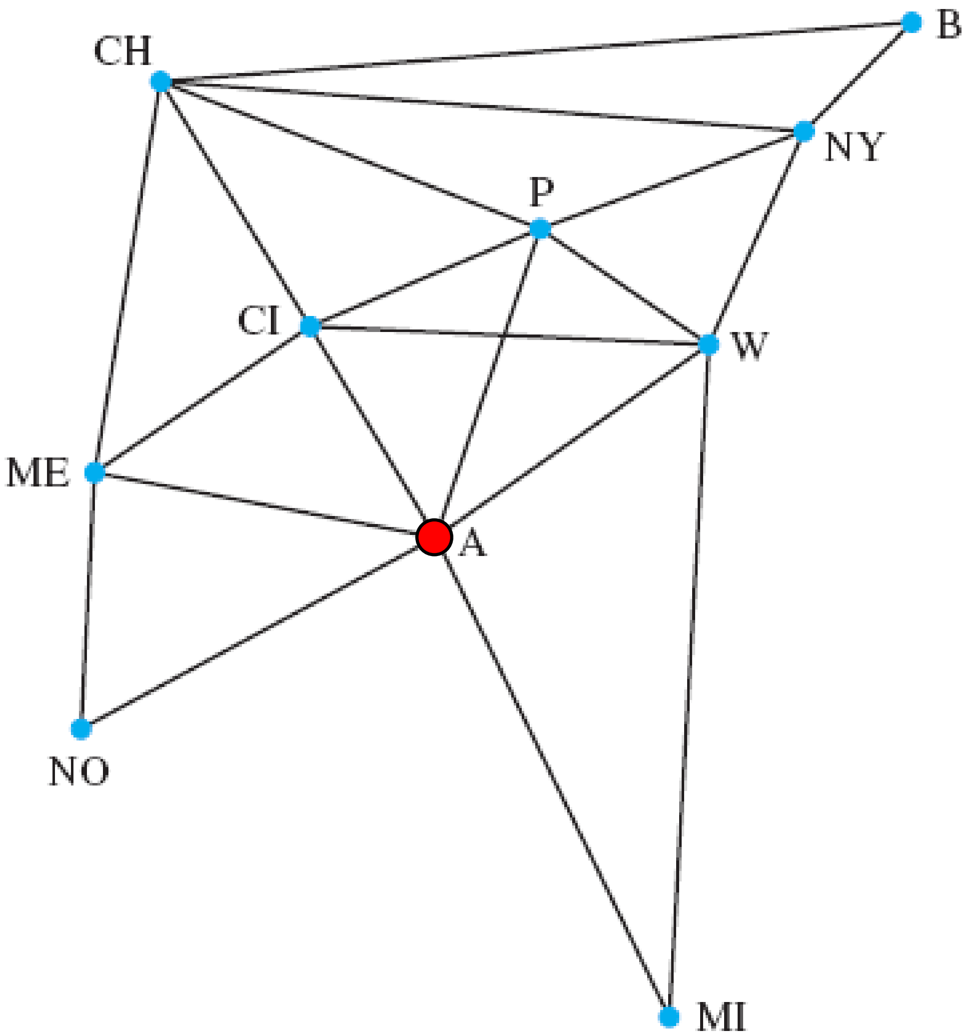


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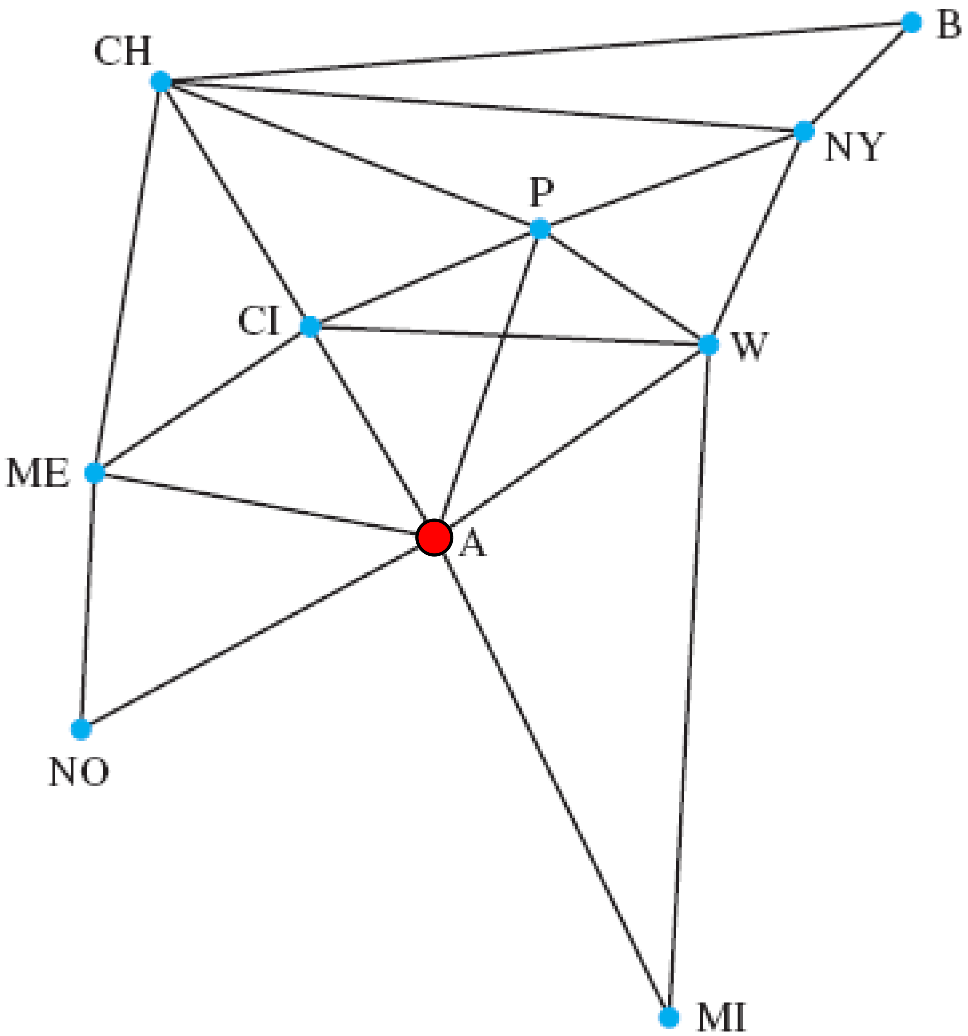


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*A*: 6 links

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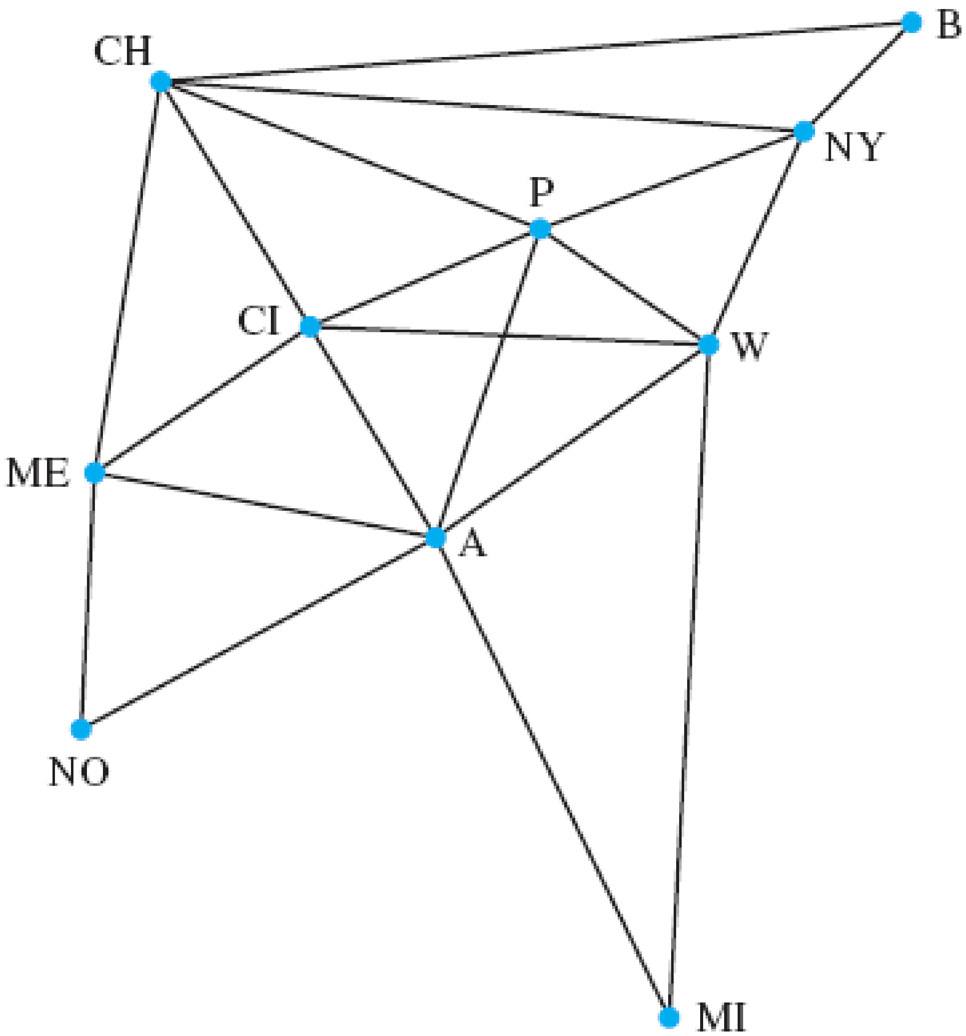


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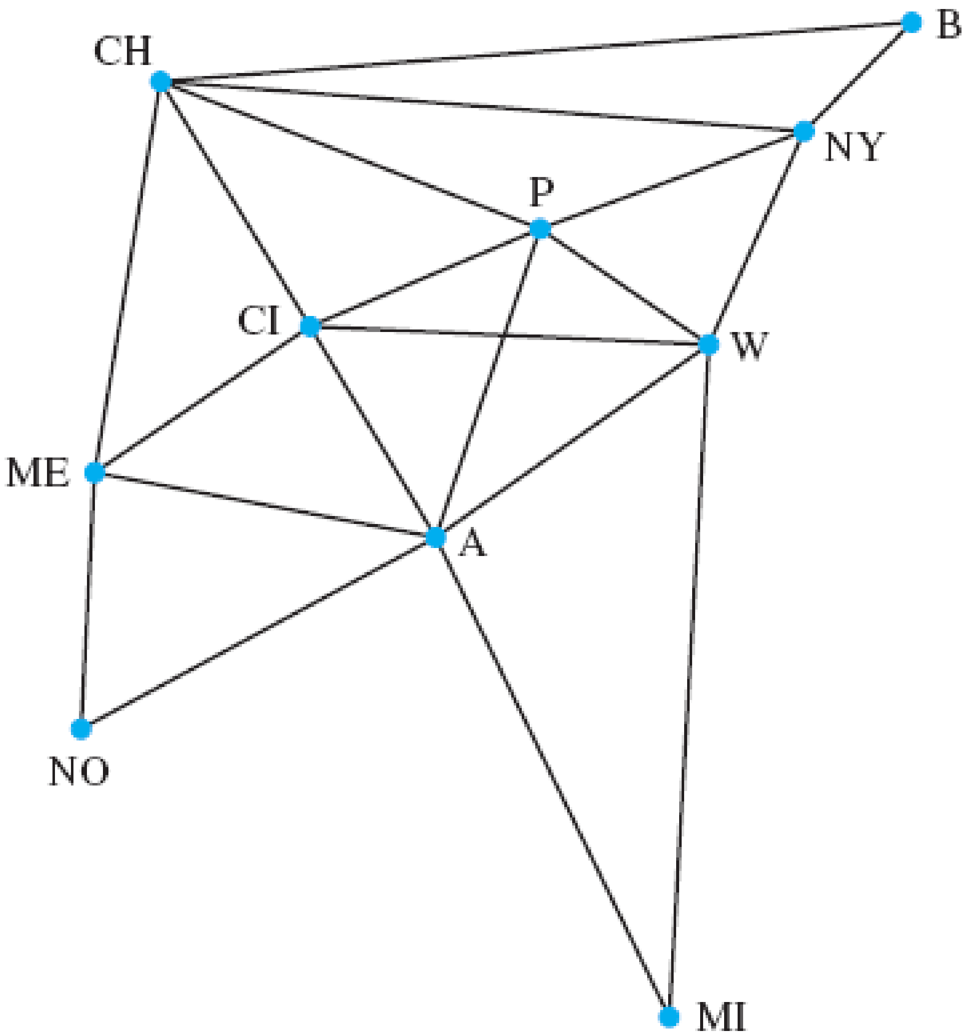


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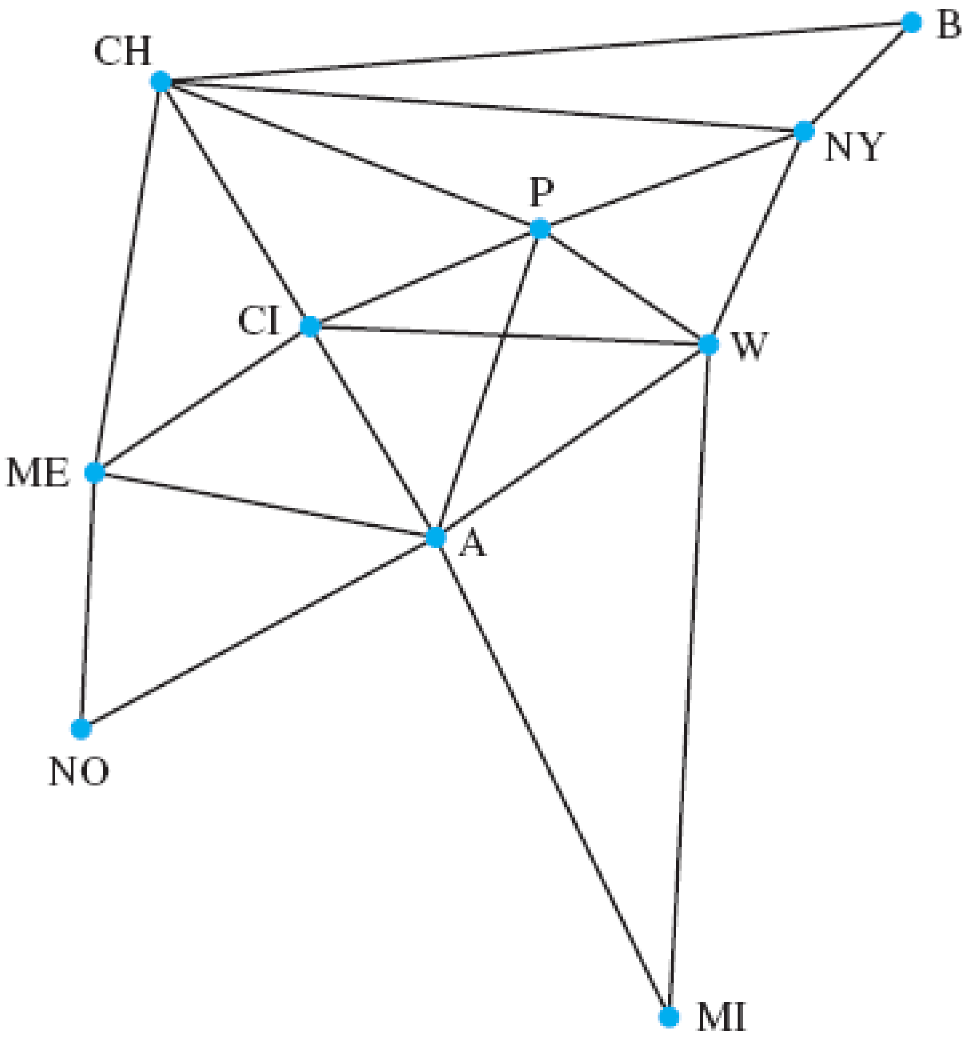
# Graph $G$



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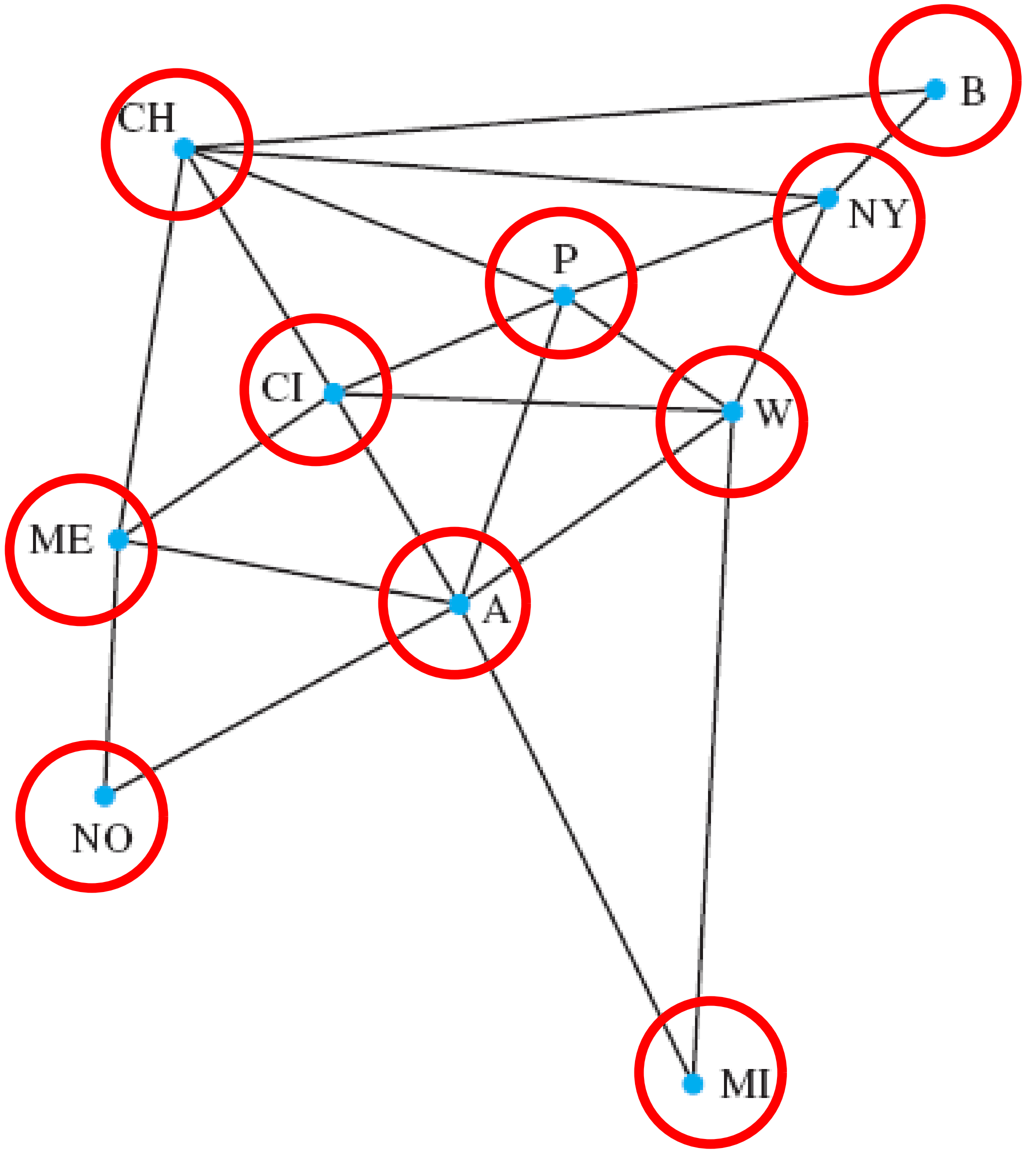


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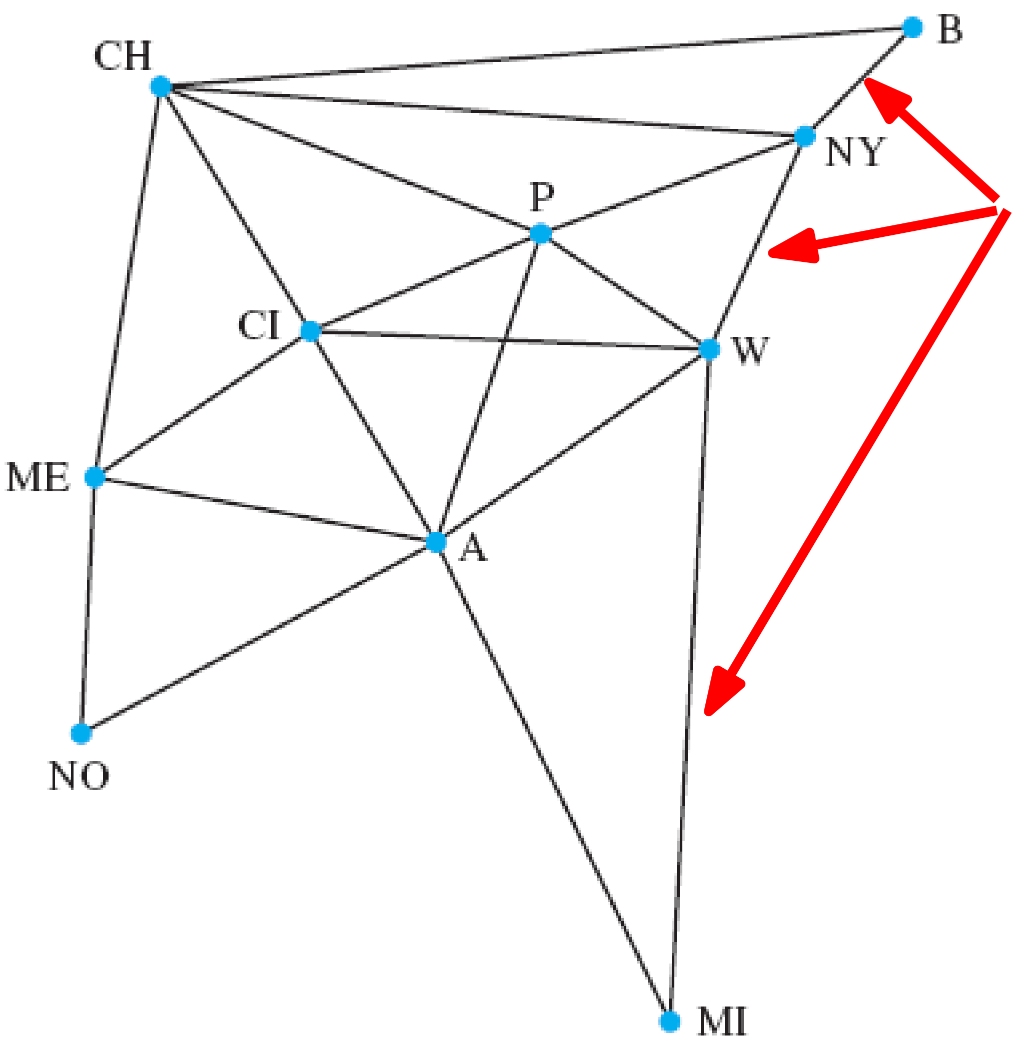


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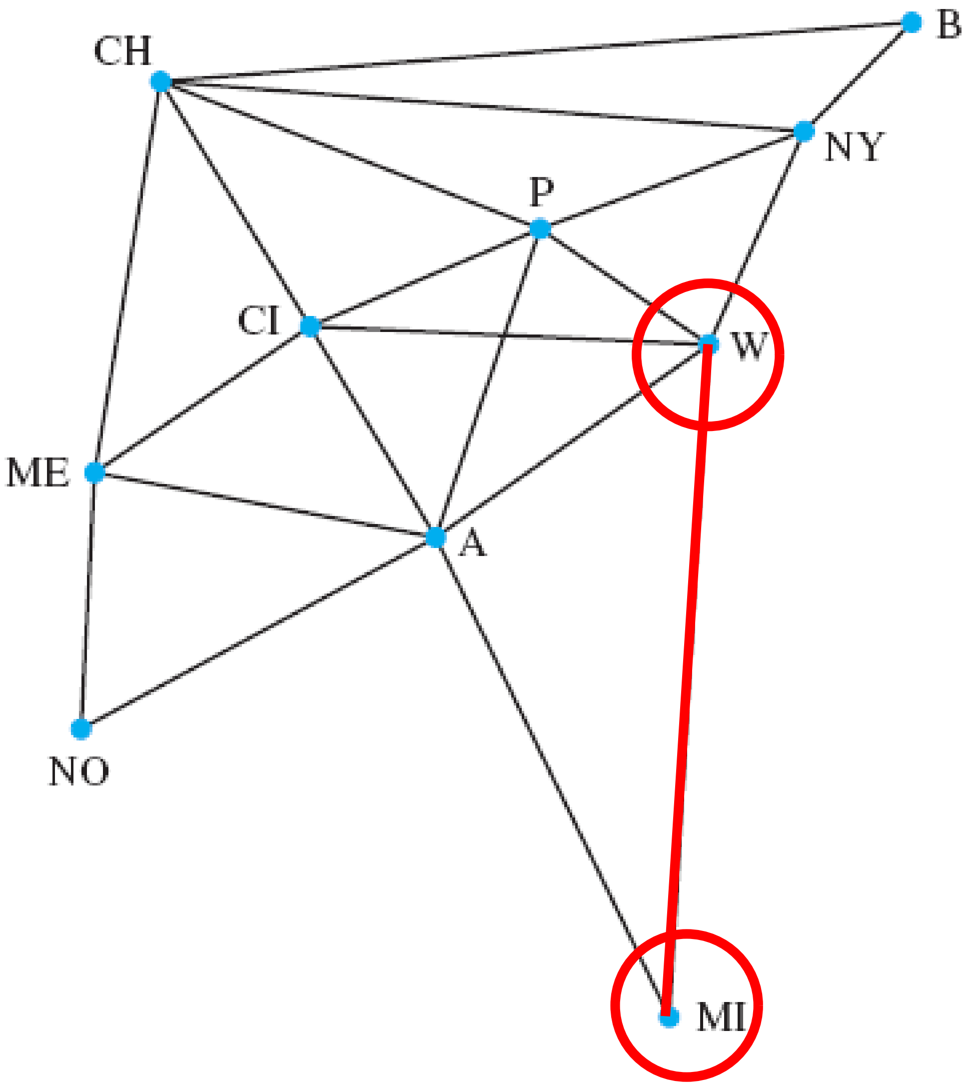


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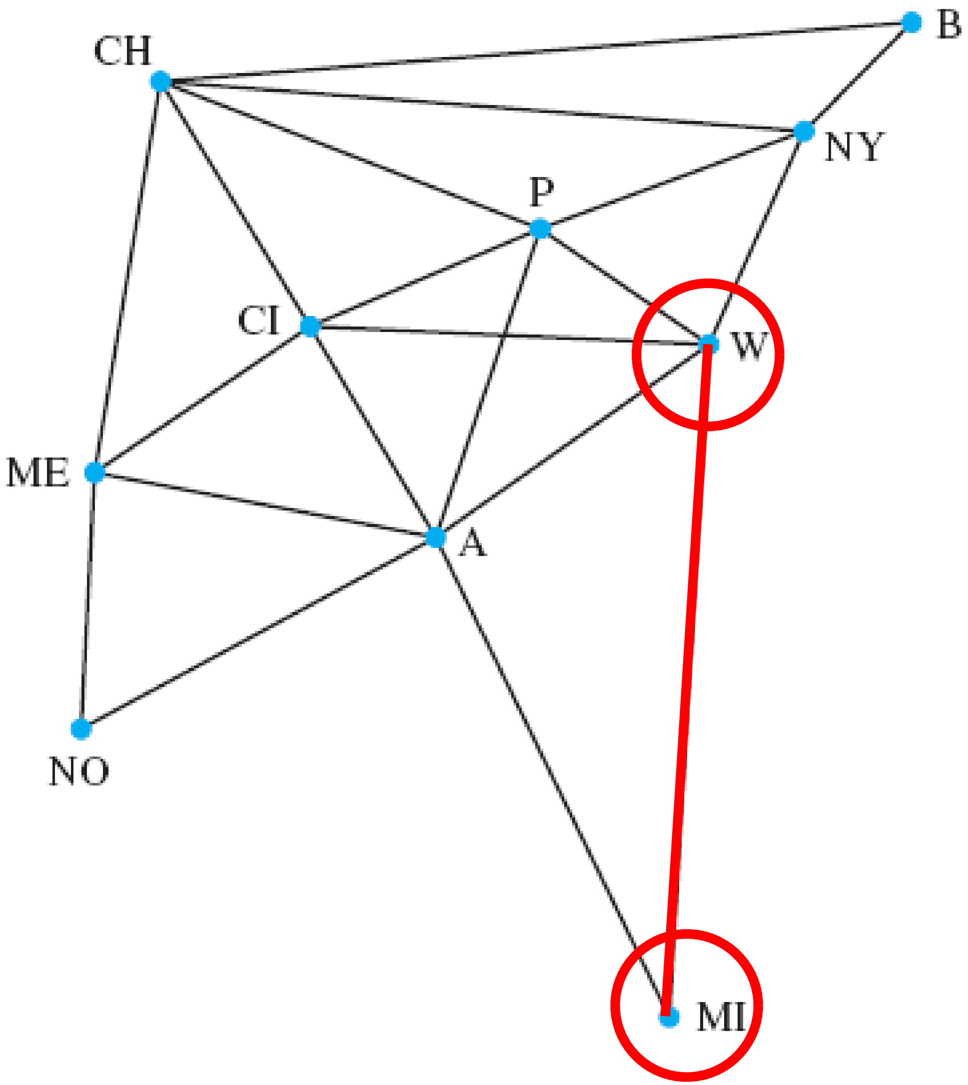


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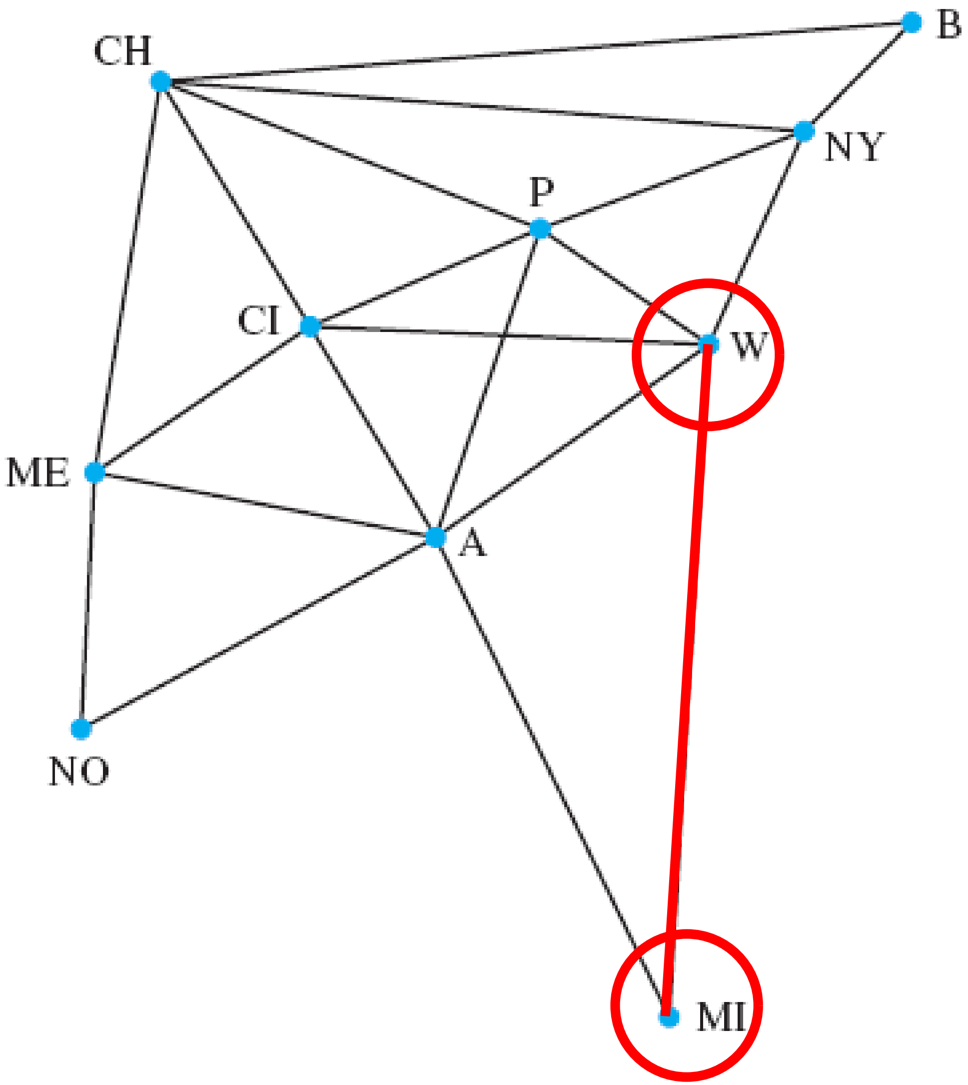


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When a vertex is an endpoint of an edge, we say that the edge and the vertex are incident to each other

# Definition of a Graph

- **Definition**. A *graph* $G = (V, E)$ consists of a nonempty set $V$ of *vertices* (or *nodes*) and a set $E$ of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to be *incident to* (or *connect* its endpoints.

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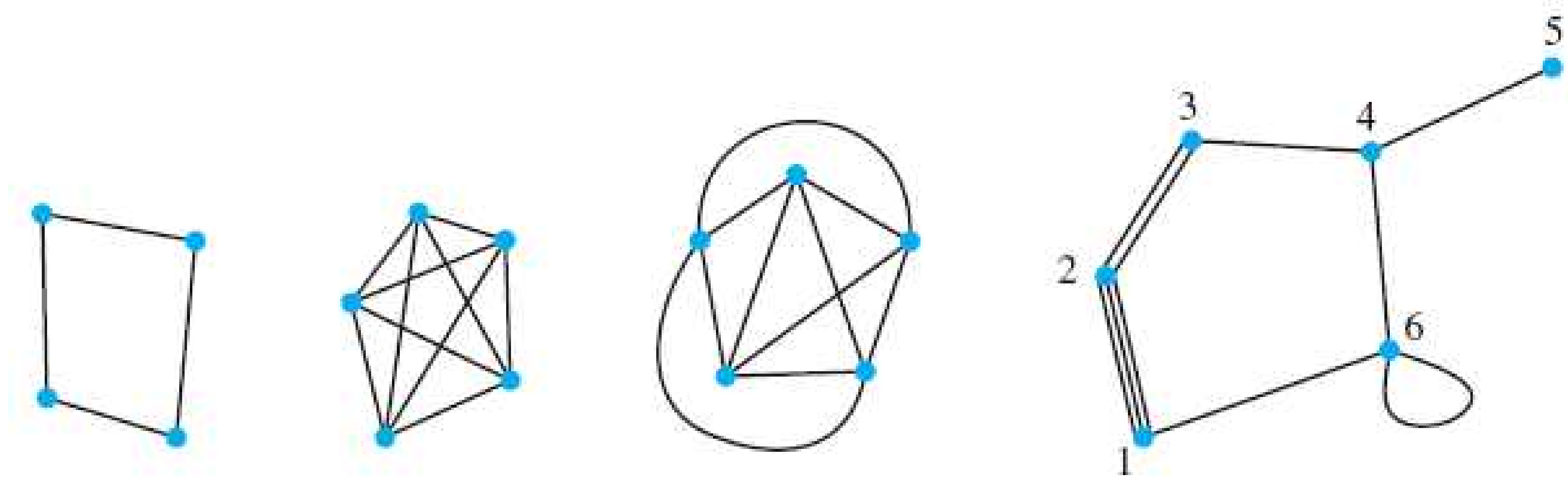


a b c d

# More Definitions

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  *Complete graph* $K_n$

  A graph with $n$ vertices that has an edge between each pair of vertices

# Graphs

- Graphs and graph theory can be used to model:
  - ◇ Computer networks
  - ◇ Social networks
  - ◇ Communication networks
  - ◇ Information networks
  - ◇ Software design
  - ◇ Transportation networks
  - ◇ Biological networks

# Graph Models

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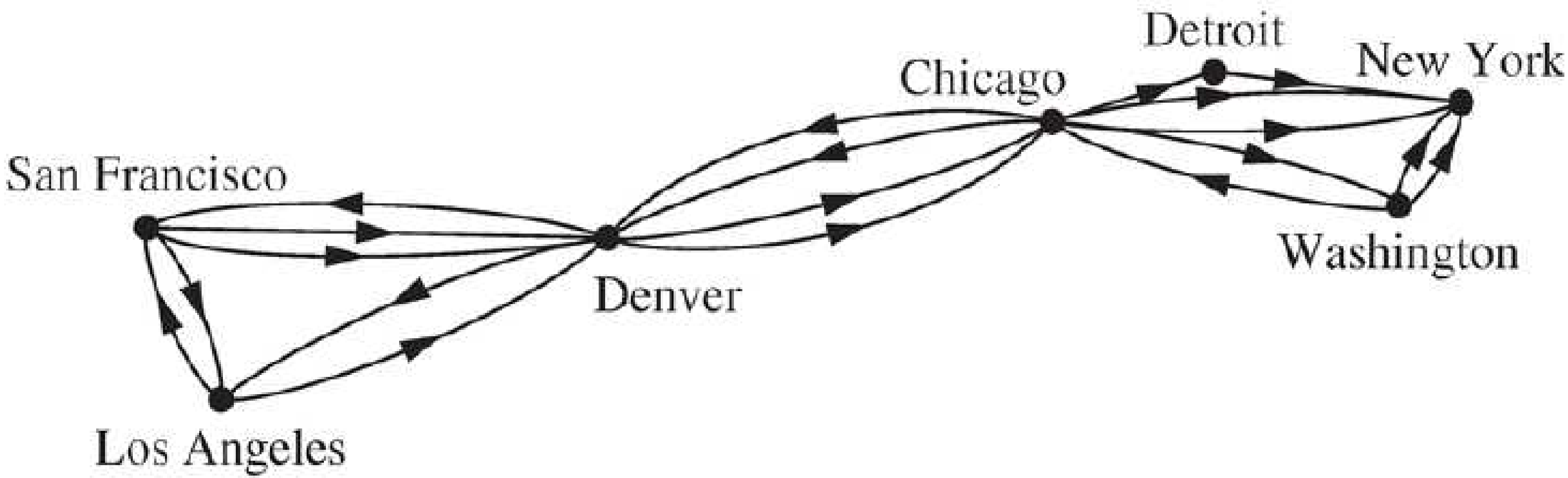

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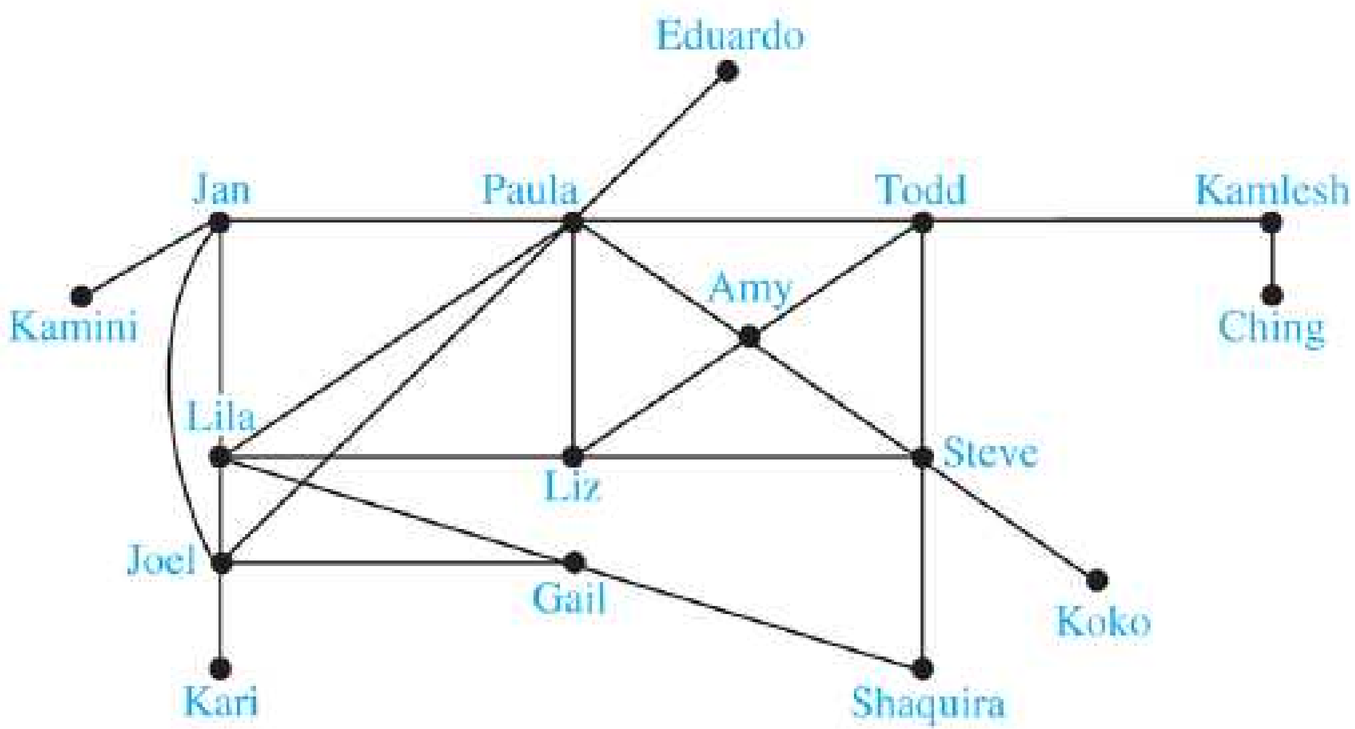

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  **Example**

  the Hollywood graph

  the Erdős number

# The Erdős Number

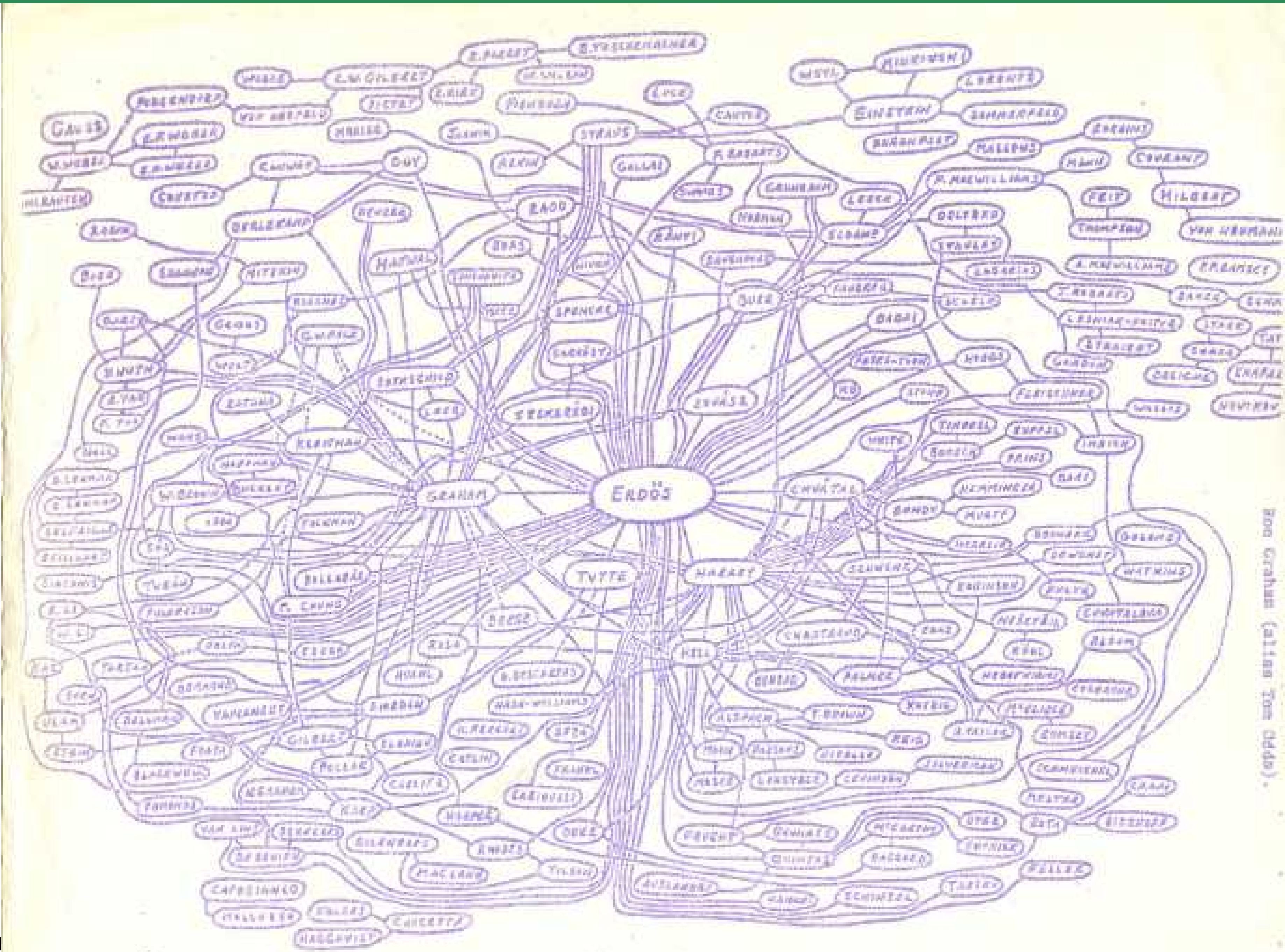


Figure 1
To appear in Topics in Graph Theory (F. Harary, ed.) New York Academy of Sciences (1979).

# The Erdős Number

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```
Erdös number  0  ---      1 person
Erdös number  1  ---    504 people
Erdös number  2  ---   6593 people
Erdös number  3  ---  33605 people
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**Statistics on Mathematical Collaboration, 1903-2016**

| | #Laureates | #Erdős | %Erdős | Min | Max | Average | Median |
|---|---|---|---|---|---|---|---|
| Fields Medal | 56 | 56 | 100.0% | 2 | 6 | 3.36 | 3 |
| Nobel Economics | 76 | 47 | 61.84% | 2 | 8 | 4.11 | 4 |
| Nobel Chemistry | 172 | 42 | 24.42% | 3 | 10 | 5.48 | 5 |
| Nobel Medicine | 210 | 58 | 27.62% | 3 | 12 | 5.50 | 5 |
| Nobel Physics | 200 | 159 | 79.50% | 2 | 12 | 5.63 | 5 |

- **Definition** Two vertices $u, v$ in an undirected graph $G$ are called *adjacent* (or *neighbors*) in $G$ if there is an edge $e$ between $u$ and $v$. Such an edge $e$ is called *incident* with the vertices $u$ and $v$ and $e$ is said to connect $u$ and $v$.

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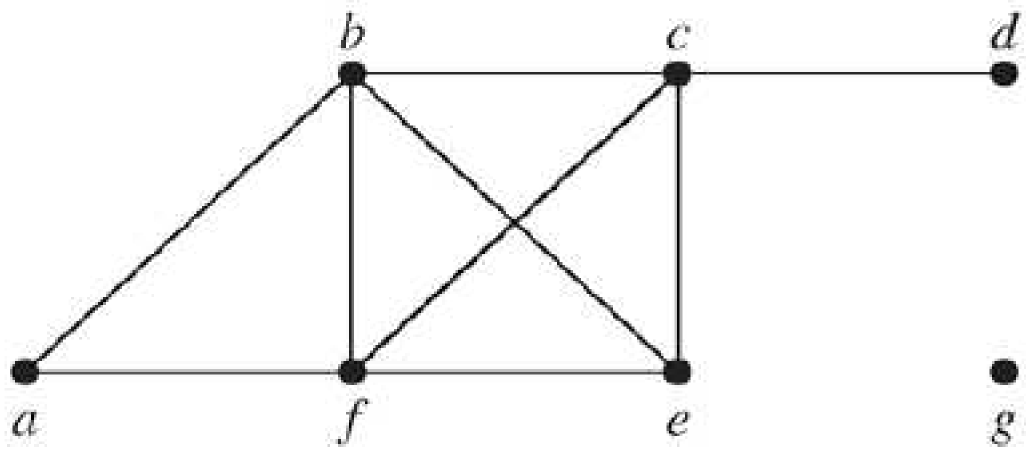
  **Definition** The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex $v$ is denoted by $\deg(v)$.

- **Example**: What are the degrees and neighborhoods of the vertices in the graph $G$?



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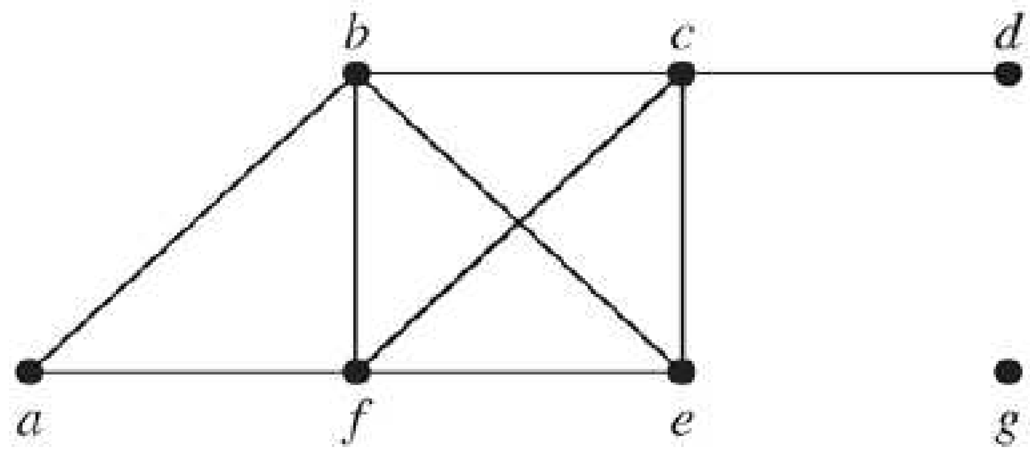


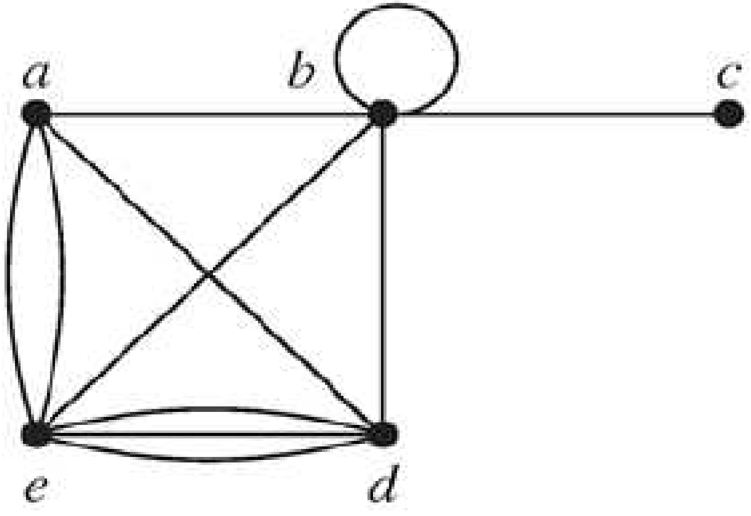

- **Theorem 1** (Handshaking Theorem) If $G = (V, E)$ is an undirected graph with $m$ edges, then

$$2m = \sum_{v \in V} \deg(v)$$

**Proof**

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  **Definition** Let $(u, v)$ be an edge in $G$. Then $u$ is the *initial vertex* of the edge and is *adjacent to* $v$ and $v$ is the *terminal vertex* of this edge and is *adjacent from* $u$. The initial and terminal vertices of a loop are the same.

- **Definition** The *in-degree* of a vertex $v$, denoted by $\deg^-(v)$, is the number of edges which terminate at $v$. The *out-degree* of $v$, denoted by $\deg^+(v)$, is the number of edges with $v$ as their initial vertex. Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

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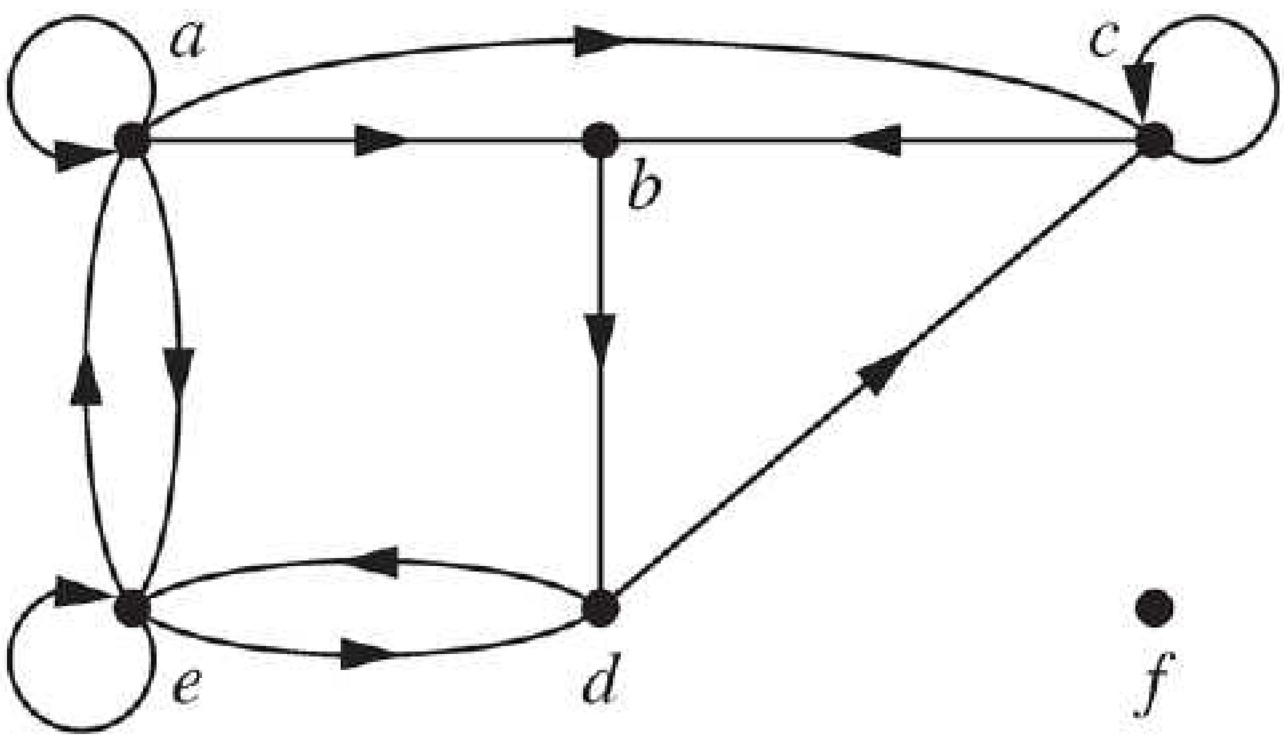

# Directed Graphs

- **Theorem 3** Let $G = (V, E)$ be a graph with directed edges. Then

$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)$$

**Proof**

# Complete Graphs

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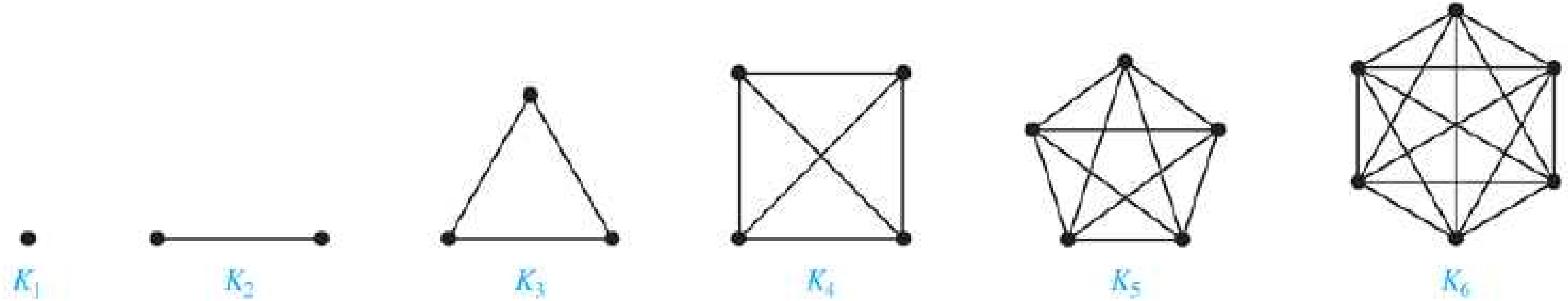

## Cycles

- A *cycle* $C_n$ for $n \geq 3$ consists of $n$ vertices $v_1, v_2, \ldots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

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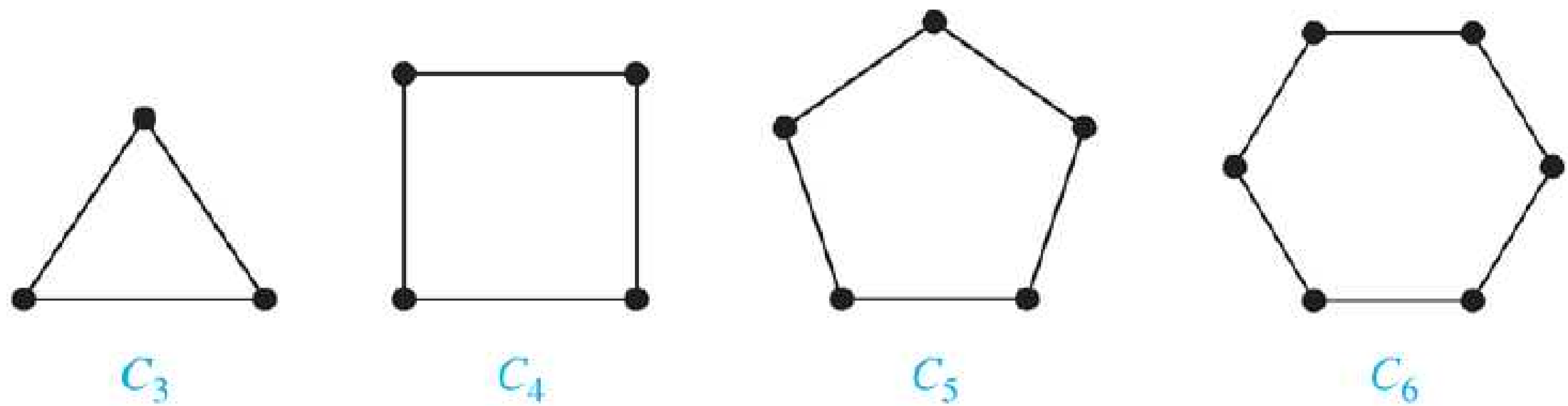

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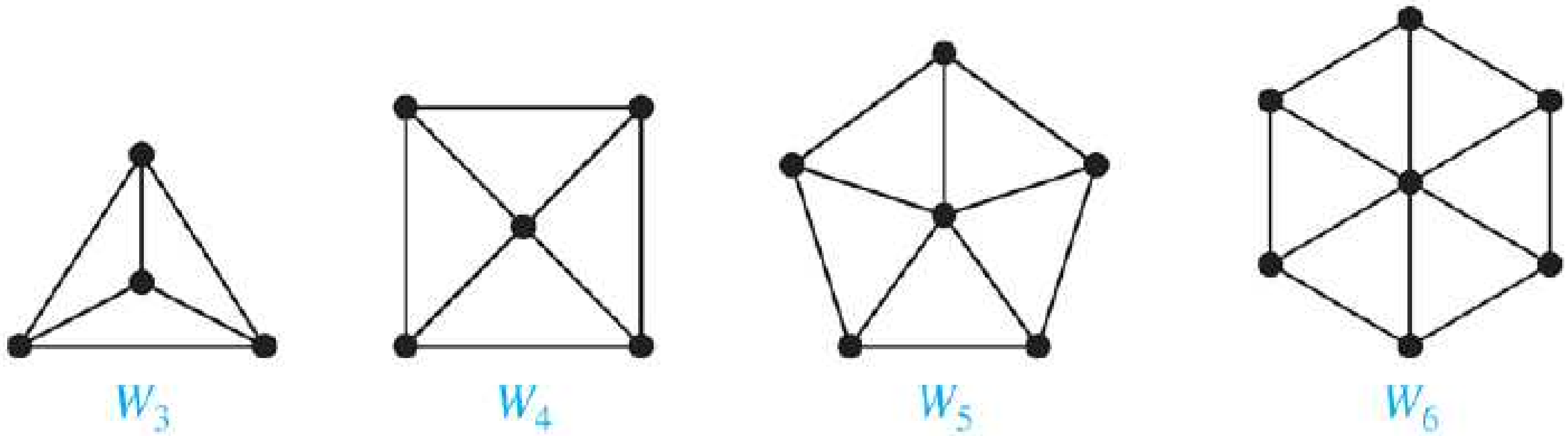

# $N$-dimensional Hypercube

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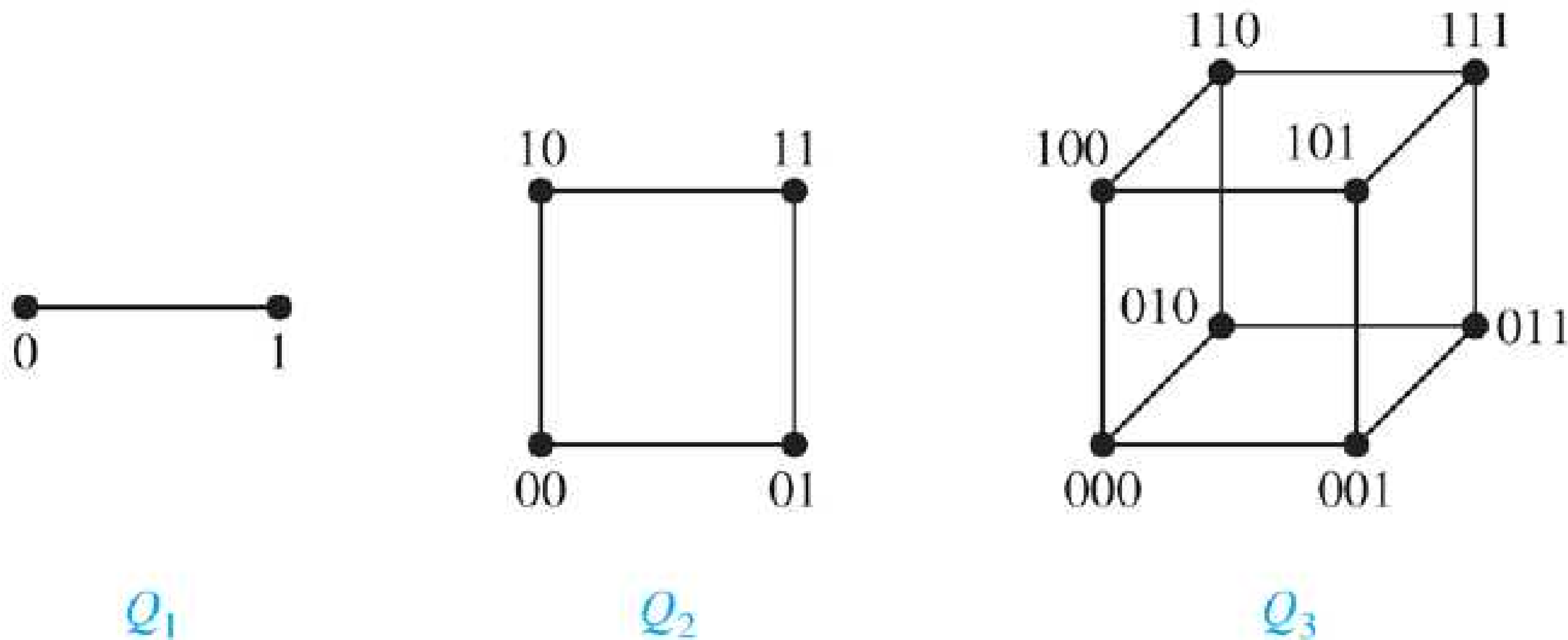

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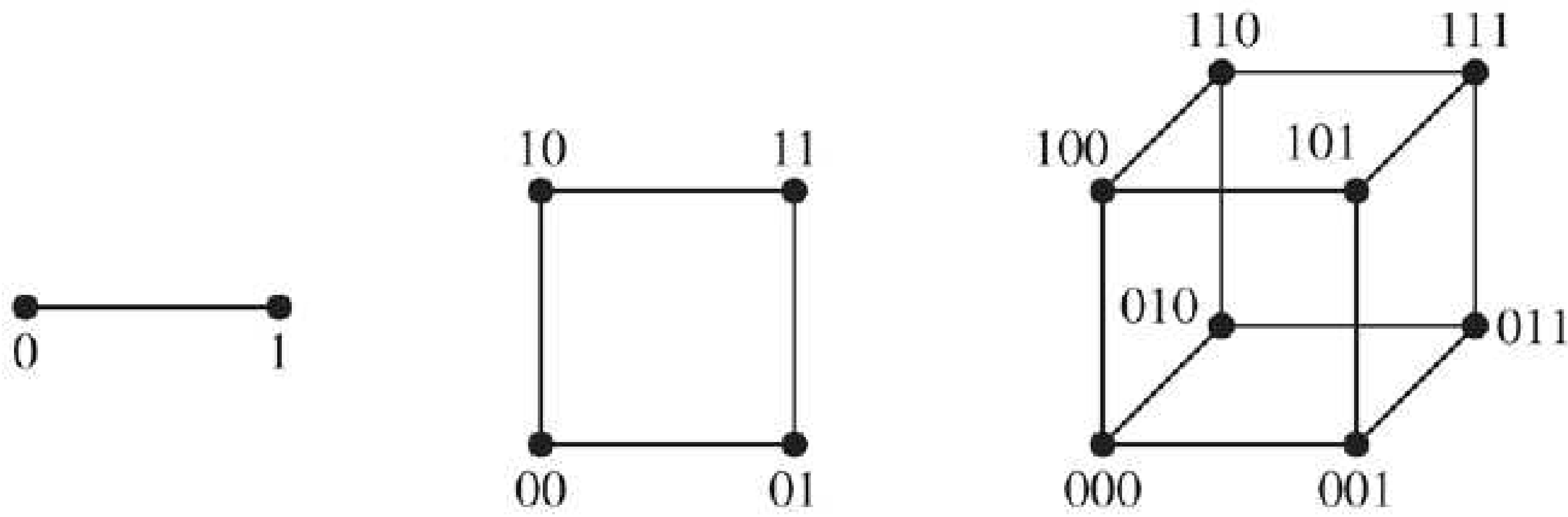


$Q_1$ $Q_2$ $Q_3$

How many vertices? How many edges?

# Bipartite Graphs

- **Definition** A simple graph $G$ is *bipartite* if $V$ can be partitioned into two disjoint subsets $V_1$ and $V_2$ such that every edge connects a vertex in $V_1$ and a vertex in $V_2$.

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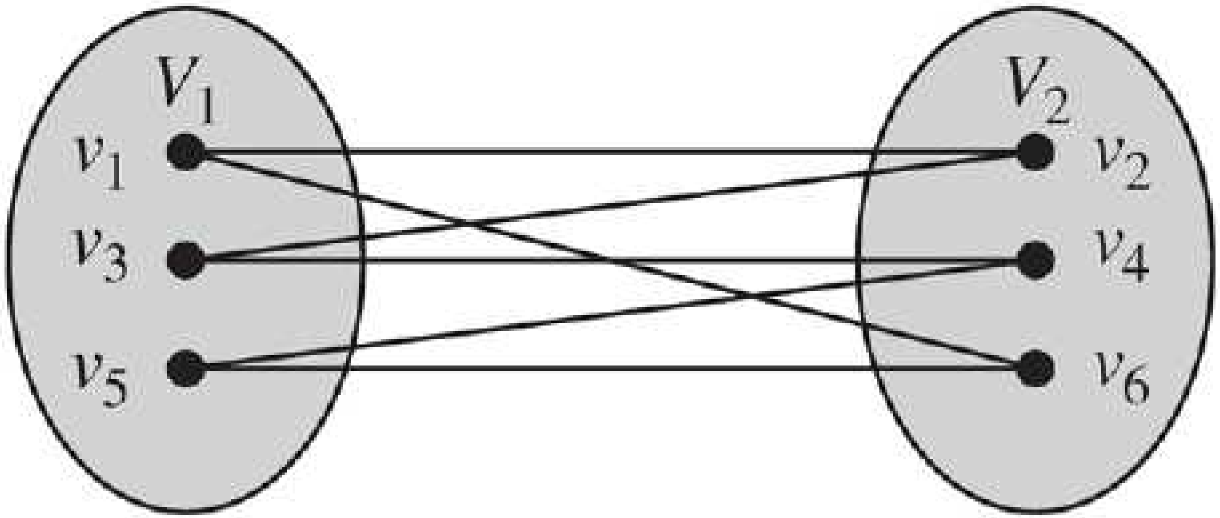

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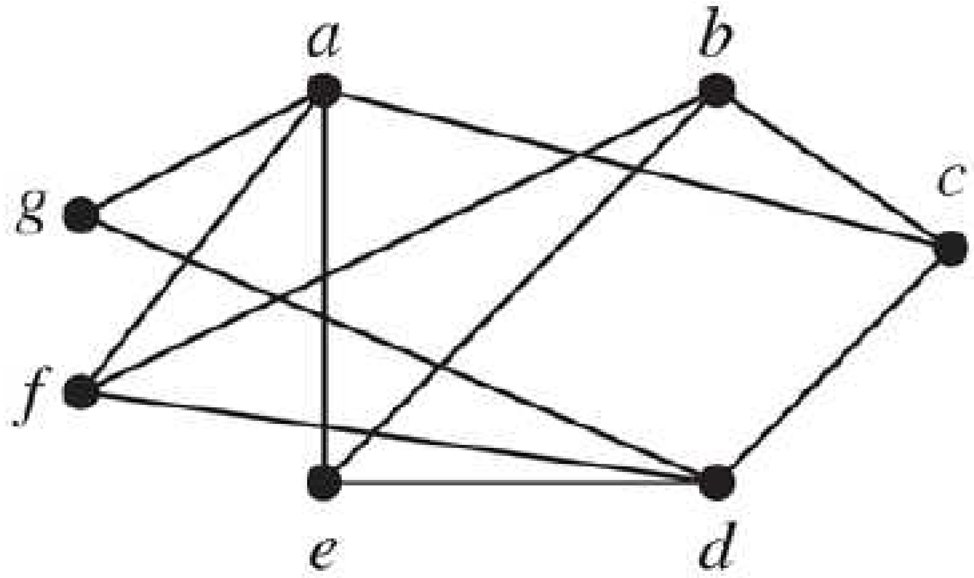

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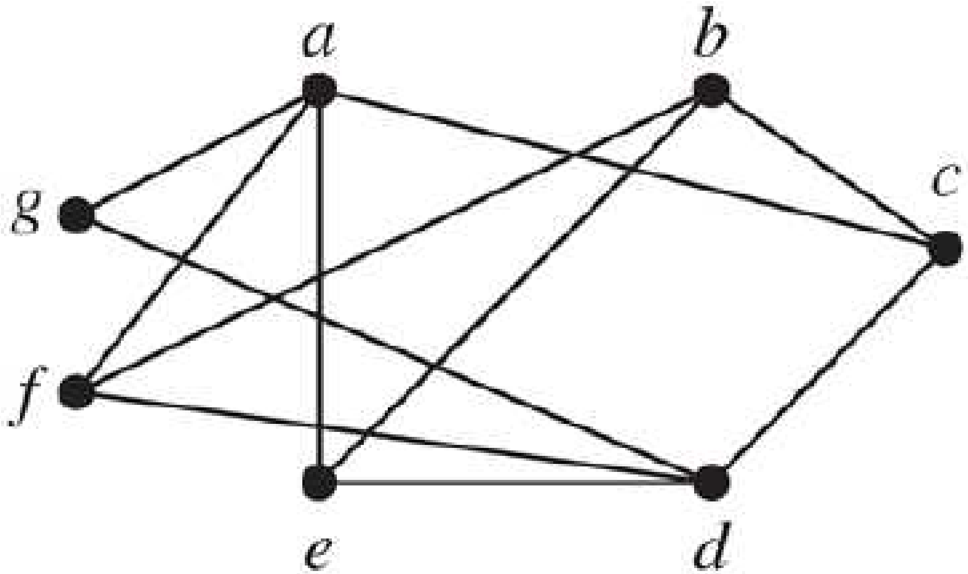


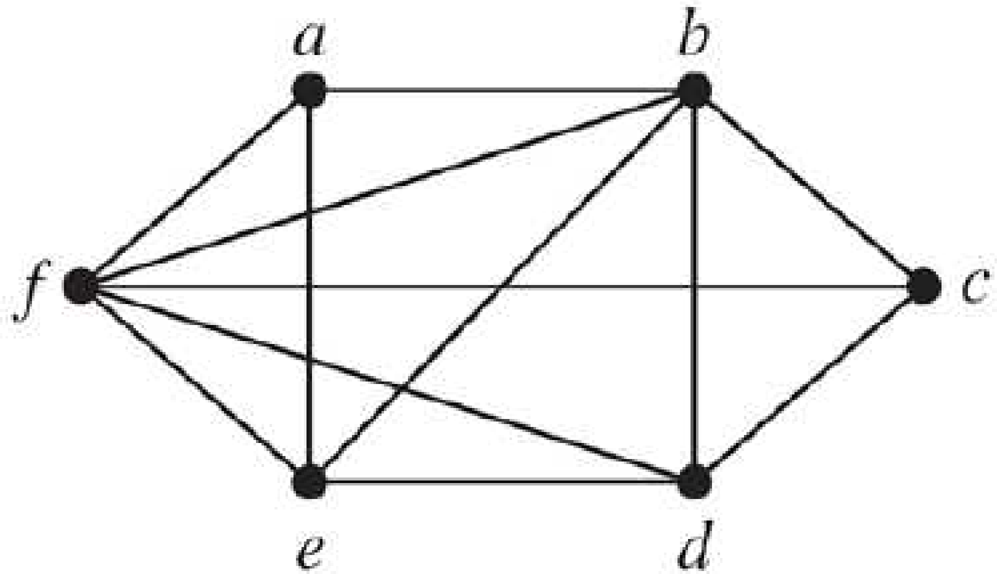

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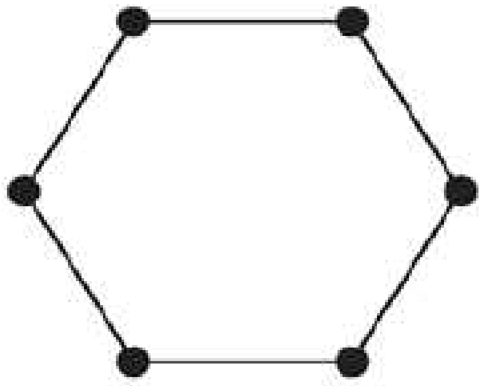

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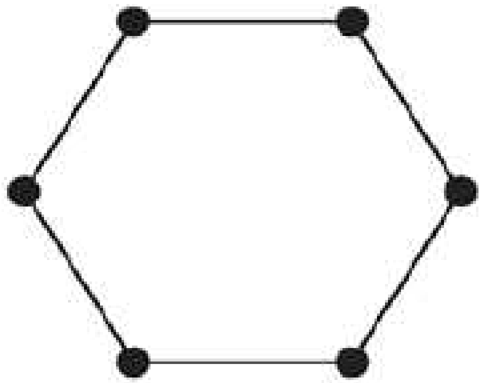
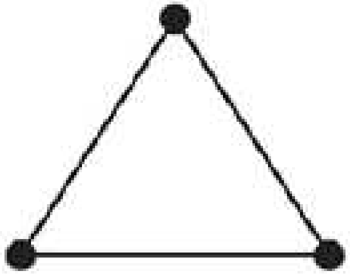


$C_6$

**Example** Show that $C_3$ is not bipartite.



$C_3$

## Complete Bipartite Graphs

- **Definition** A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets $V_1$ of size $m$ and $V_2$ of size $n$ such that there is an edge from every vertex in $V_1$ to every vertex in $V_2$.

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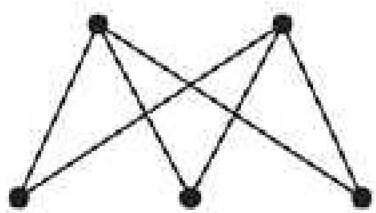
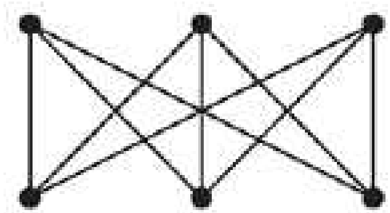
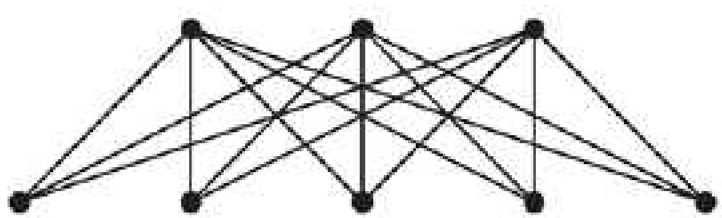


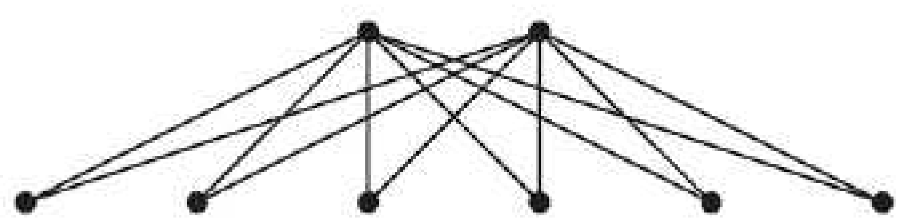
$K_{2,3}$



$K_{3,3}$



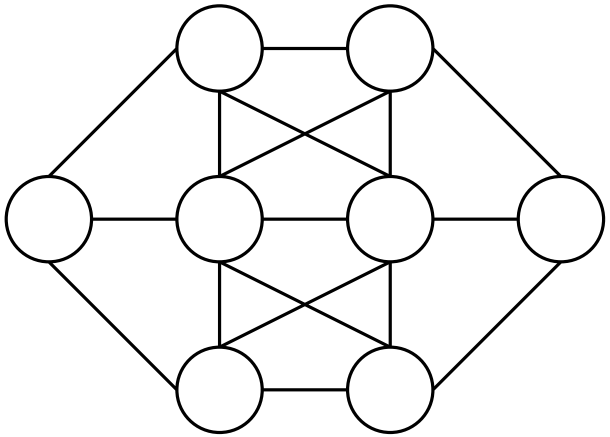
$K_{3,5}$



$K_{2,6}$

# Puzzles using Graphs

- **The eight-circles problem** Place the letters A, B, C, D, E, F, G, H into the eight circles in the figure, in such a way that no letter is adjacent to a letter that is next to it in the alphabet.



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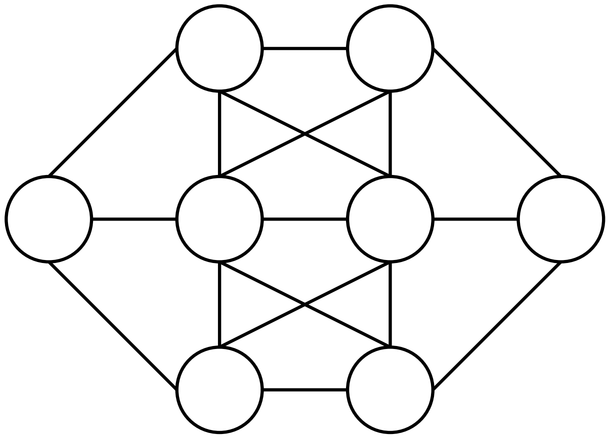

- **Six people at a party** Show that, in any gathering of six people, there are either three people who all know each other, or three people none of which knows either of the other two.

# Next Lecture

- graph theory ...