Character Tables for Representations of Finite Groups

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Motivation for Representation Theory

Groups arise naturally as sets of symmetries of some object which are closed under composition and taking inverses. For example,

1. The **symmetric group** of degree n, S_n, is the group of all symmetries of the set $\{1, \ldots, n\}$.
2. The **dihedral group** of order $2n$, D_n, is the group of all symmetries of the regular n-gon in the plane.

In these two examples, S_n acts on the set $\{1, \ldots, n\}$ and D_n acts on the regular n-gon in a natural manner. One may wonder more generally: Given an abstract group G, which objects X does G act on? This is the basic question of representation theory, which attempts to classify all such X up to isomorphism.

The Definition of a Representation

Definition

Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is any group homomorphism

$$\rho\colon G \to GL(V).$$

Definition

The **dimension** of a representation $\rho\colon G \to GL(V)$ is the dimension of the vector space V.

Examples of Representations

Example

Let V be an n-dimensional vector space. The map $\rho\colon G \to GL(V)$ defined by $\rho(g) = \mathrm{Id}_V$ for all $g \in G$ is a representation of G called the **trival representation** of dimension n.

Example

If G is a finite group that acts on a finite set X, and F is any field, then there is an associated **permutation representation** on the vector space V over F with basis $\{e_x : x \in X\}$. We let G act on the basis elements by the permutation $g \cdot e_x = e_{gx}$ for all $x \in X$ and $g \in G$. This representation has dimension $|X|$.

The Regular Representation

Example

A special case of a permutation representation is that when a finite group acts on itself by left multiplication. Consider the vector space V_{reg} which has a basis given by the formal symbols $\{e_g | g \in G\}$, and let $h \in G$ act by

$$\rho_{\text{reg}}(h)(e_g) = e_{hg}.$$

This representation is called the **regular representation** of G, has dimension $|G|$.

Example

Let $G = C_2 \times C_2 = \langle \sigma, \tau | \sigma^2 = \tau^2 = e, \sigma\tau = \tau\sigma \rangle$ be the Klein four-group. Let V be the vector space with basis $\{b_e, b_\sigma, b_\tau, b_{\sigma\tau}\}$. Left multiplication by σ gives a permutation

$$\begin{aligned} b_e &\mapsto b_\sigma \\ b_\sigma &\mapsto b_e \\ b_\tau &\mapsto b_{\sigma\tau} \\ b_{\sigma\tau} &\mapsto b_\tau. \end{aligned}$$

We can similarly compute $\rho_{\text{reg}}(\tau)$. Thus, in our basis, the regular representation $\rho_{\text{reg}} \colon G \to GL(V)$ is given by the matrices

$$\rho_{\text{reg}}(\sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \rho_{\text{reg}}(\tau) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The Alternating Representation

Example

For any symmetric group S_n, the **alternating representation** on $\mathbb{C}$ is given by the map

$$\begin{aligned} \rho\colon S_n &\to GL(\mathbb{C}) = \mathbb{C}^\times \\ \sigma &\mapsto \operatorname{sgn}(\sigma). \end{aligned}$$

G-linear maps

Definition

A **homomorphism** between two representations $\rho_1 \colon G \to GL(V)$ and $\rho_2 \colon G \to GL(W)$ is a linear map $\psi \colon V \to W$ that interwines with the action of G, i.e. such that

$$\psi \circ \rho_1(g) = \rho_2(g) \circ \psi \quad \forall g \in G.$$

In this case, we also refer to ψ as a **G-linear map**.

Definition

An **isomorphism** of representations is a G-linear map that is also invertible.

Representations as matrices

Example

Given any representation (ρ, V), where V is a vector space of dimension n over the field K, we can fix a basis for V to obtain an isomorphism of vector spaces $\psi\colon V \to K^n$. This yields a representation ϕ of G on K^n by defining

$$\phi(g) = \psi \circ \rho(g) \circ \psi^{-1}$$

for all $g \in G$. This representation is isomorphic to our original representation (ρ, V). In particular, we can always choose to view complex n-dimensional representations of G as representations on $\mathbb{C}^n$, where each $\phi(g)$ is given by an invertible $n \times n$ matrix with entries in $\mathbb{C}$.

Example (2-dim rep of D_4.)

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$.

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$$\rho(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\rho(\sigma) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \rho(\sigma\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\rho(\sigma^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \rho(\sigma^2\tau) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho(\sigma^3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \rho(\sigma^3\tau) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

The direct sum of representations

Definition

Let V and W be representations of G. Then $V \oplus W$ admits a natural representation of G, called the **direct sum representation** of V and W, which we define by

$$\rho_{V \oplus W} \colon G \to GL(V \oplus W)$$
$$\rho_{V \oplus W}(g) \colon (x, y) \mapsto (\rho_V(g)(x), \rho_W(g)(y)).$$

Irreducible representations and complete reducibility

Definition

A **subrepresentation** of V is a G-invariant subspace $W \leq V$; that is, a subspace $W \leq V$ with the property that $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$. Note that W itself is a representation of G under the action of the restriction of $\rho(g)$ to W.

Definition

A representation V is said to be **irreducible** if it has no subrepresentations other than the trivial subrepresentations $0 \leq V$ and $V \leq V$. A representation is called **completely reducible** if it decomposes into a direct sum of irreducible representations. We sometimes write **irrep** as shorthand for irreducible representation.

Example (A 2-dimensional irrep)

Let $G = D_3 = \langle \sigma, \tau | \sigma^3 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$. (Note that $D_3 \cong S_3$). Consider the regular triangle centered at the origin with vertices

$$(1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2}).$$

We can let σ act as rotation by $\frac{2\pi}{3}$ and let τ act as reflection over the x-axis to obtain an action of G on $\mathbb{C}^2$ given (under the standard basis) by the matrices

$$\rho(\sigma) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$
$$\rho(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Example (A 2-dimensional irrep cont.)

Suppose ρ has a non-trivial subrepresentation W. We must have dim $W = 1$. Since W is invariant under the action of both $\rho(\sigma)$ and $\rho(\tau)$, there must be some mutual eigenvector for $\rho(\sigma)$ and $\rho(\tau)$ that spans W. The eigenvectors of $\rho(\sigma)$ are

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} \quad (\lambda_1 = e^{\frac{2\pi i}{3}}) \quad \text{and} \quad \begin{bmatrix} 1 \\ i \end{bmatrix} \quad (\lambda_2 = e^{-\frac{2\pi i}{3}}).$$

The eigenvectors of $\rho(\tau)$ are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\lambda_1 = 1) \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\lambda_2 = -1).$$

Thus we see that there is no such W, and our representation is irreducible.

Representations of finite abelian groups

Theorem

If $A_1, A_2, \ldots, A_r$ are linear operators on V and each A_i is diagonalizable, then $\{A_i\}$ are simultaneously diagonalizable if and only if they commute.

Representations of finite abelian groups

Theorem

Every complex representation of a finite abelian group is completely reducible into irreducible representations of dimension 1.

Proof.

Take an arbitrary element $g \in G$. Since G is finite, we can find an integer n such that $g^n = 1$ and $\rho(g)^n = Id$. The minimal polynomial of $\rho(g)$ divides $x^n - 1$, which has n distinct roots over $\mathbb{C}$, so it factors into distinct linear factors over $\mathbb{C}$, i.e. $\rho(g)$ is diagonalizable. Now, given any two elements $g_1, g_2 \in G$ we have $\rho(g_1)\rho(g_2) = \rho(g_2)\rho(g_1)$. Since the matrices $\{\rho(g)\}$ commute, $\{\rho(g)\}$ are simultaneously diagonalizable, say with respect to basis $\{e_1, ..., e_k\}$. Then we have $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \ldots \oplus \mathbb{C}e_k$, with each subspace $\mathbb{C}e_i$ invariant under the action of G since e_i is an eigenvector for every $\rho(g)$. □

Question

Does every irreducible representation of a finite abelian group still have dimension 1 when the field is not algebraically closed?

Answer

No. Consider the representation of the cyclic group of order 4, $C_4 = \langle g \rangle$, on $\mathbb{R}^2$ given by

$$\rho(g) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $\rho(g)$ is not diagonalizable over $\mathbb{R}$, since the characteristic polynomial of $\rho(g)$ is $x^2 + 1$. Thus, we cannot decompose ρ into a direct sum of 1 dimensional representations over $\mathbb{R}$.

Question:

Is every finite dimensional representation of a group completely reducible?

Answer:

No, in general. The full answer to this question is given by Maschke's Theorem.

Maschke's Theorem

Theorem (Maschke's Theorem)

Let G be a finite group and let F be a field such that char$(F) \nmid |G|$. If V is any finite dimensional representation of G over F, and $W \leq V$ is a subrepresentation of V, then there exists a complementary subrepresentation $U \leq V$ to W, i.e. there is a G-invariant subspace $U \leq V$ such that

$$V = W \oplus U.$$

Example (Maschke's Theorem fails in the modular case)

Let F be a field whose characteristic divides $|G|$. Suppose that Maschke's Theorem holds in this case. Then FG is Artinian and semisimple. Recall that a ring is Artinian and semisimple iff it has no nonzero nilpotent ideals. We will obtain a contradiction by exhibiting a nonzero nilpotent ideal of FG. Consider the element

$$x = \sum_{g \in G} g \in FG.$$

Then $gx = x$ for every $g \in G$, and the ideal (x) generated by x is precisely the F-vector space ${}_F\langle x \rangle$ spanned by x. Moreover

$$x^2 = |G|x = 0.$$

It follows that the nonzero ideal ${}_F\langle x \rangle$ is nilpotent, so the group algebra FG is not semisimple.

Example (Maschke's Theorem fails when the group is infinite)

Consider the additive group $G = (F, +)$, which we can view as a subgroup of $GL_2(F)$ by identifying $t \in F$ with the matrix

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Then consider the linear action of G on $V = K^2$ given by $t \cdot (x, y) = (x + ty, y)$. Any one-dimensional subspace spanned by a vector $(x_1, y_1) \in V$ is G-invariant precisely when for all $t \in F$ there exist $\lambda_t \in F$ such that

$$t \cdot (x_1, y_1) = \lambda_t (x_1, y_1).$$

But this requires $y_1 = 0$, so the only one-dimensional G-subrepresentation of V is spanned by $(1, 0)$. Therefore this subrepresentation has no G-invariant direct complement.

Definition

Let W be a subspace of V. A **linear projection** V onto W is a linear map $f: V \to W$ such that $f\restriction_W = \mathrm{Id}_W$. If W is a subrepresentation of V and the projection f is G-invariant, then we say that f is a **G-linear projection**.

Lemma

Let V be a G-representation, and $W \leq V$ be a G-subrepresentation. Suppose we have a G-linear projection

$$f: V \to W.$$

Then Ker(f) is a complementary subrepresentation to W, i.e. Ker(f) is a G-invariant subspace of V such that

$$V = \mathit{Ker}(f) \oplus W$$

Maschke's Theorem

Proof.

It will suffice to find a G-linear projection from V onto W. Fix a basis $\{b_1, \ldots, b_m\}$ for W and extend it to a basis $\{b_1, \ldots, b_m, b_{m+1}, \ldots, b_n\}$ for V. Let U be the complementary subspace to W given by $U = \langle b_{m+1}, \ldots, b_n \rangle$. Then we have a natural projection $f\colon W \oplus U \to W$. There is no reason to think that f should be G-linear, but we can fix this by averaging over G. Define $\widetilde{f}\colon V \to V$ by

$$\widetilde{f}(x) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}))(x).$$

We claim that $\widetilde{f}$ is a G-linear projection from V onto W.

Maschke's Theorem

Proof.

First we check that $\mathrm{Im}(\tilde{f}) \leq W$. If $x \in V$ and $g \in G$, then

$$f(\rho(g^{-1})(x)) \in W$$

and so

$$\rho(g)(f(\rho(g^{-1})(x))) \in W$$

since W is G-invariant. Thus

$$\widetilde{f}(x) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}))(x) \in W.$$

Maschke's Theorem

Proof.

Next we check that $\widetilde{f}\restriction_W = \mathrm{Id}_W$. Let $y \in W$. For any $g \in G$, we know that $\rho(g^{-1})(y)$ is also in W, so $f(\rho(g^{-1})(y)) = \rho(g^{-1})(y)$. Then

$$\begin{aligned}\widetilde{f}(y) &= \frac{1}{|G|}\sum_{g\in G}\rho(g)(f(\rho(g^{-1})(y))) \\ &= \frac{1}{|G|}\sum_{g\in G}\rho(g)(\rho(g^{-1})(y)) \\ &= \frac{1}{|G|}\sum_{g\in G}(y) = \frac{|G|y}{|G|} = y\end{aligned}$$

so indeed $\widetilde{f}$ is a linear projection of V onto W.

Maschke's Theorem

Proof.

Finally, we check that $\widetilde{f}$ is G-linear. If $x \in V$ and $h \in G$, then

$$\begin{aligned}
(\widetilde{f} \circ \rho(h))(x) &= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}) \circ \rho(h))(x) \\
&= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}h))(x) \\
&= \frac{1}{|G|} \sum_{g \in G} (\rho(hg) \circ f \circ \rho(g^{-1}))(x) \quad (g \mapsto hg) \\
&= (\rho(h) \circ \widetilde{f})(x).
\end{aligned}$$

$\square$

Consequences of Machke's Theorem

Corollary

Let G be a finite group and let F be a field such that $\mathrm{char}(F) \nmid |G|$. Then any finite-dimensional representation of G over F is completely reducible.

Proof.

Let V be a representation of G over F of dimension n. If V is irreducible, then V is, in particular, completely reducible. If not, then V contains a proper subrepresentation $W \leq V$. From Maschke's Theorem, we know there exists a subrepresentation $U \leq V$ such that

$$V = W \oplus U. \tag{1}$$

Both W and U have dimension less than n, so by induction we know that W and U are completely reducible. We deduce that V is completely reducible. □

Proposition

Suppose we have representations $\rho_V : G \to GL(V)$ *and* $\rho_W : G \to GL(W)$ *of* G*. Then there is a natural representation of* G *on the vector space* $\mathit{Hom}(V, W)$ *given for all* $g \in G$ *by*

$$\begin{aligned} \rho_{\mathit{Hom}(V,W)}(g) \colon \mathit{Hom}(V, W) &\to \mathit{Hom}(V, W) \\ f &\mapsto \rho_W(g) \circ f \circ \rho_V(g^{-1}). \end{aligned}$$

Definition

Let V and W be two representations of G. The set of G-linear maps from V to W, which we denote by $\mathbf{Hom_G(V, W)}$, forms a subspace of $\mathrm{Hom}(V, W)$. In other words, $\mathrm{Hom}_G(V, W)$ is the vector space consisting of all *homomorphisms of representations* between V and W.

Definition

Let $\rho\colon G \to GL(V)$ be a representation. We define the **invariant subrepresentation** $V^G \leq V$ to be the set

$$\{v \in V \mid \rho(g)(v) = v, \quad \forall g \in G\}.$$

Remark

$\mathrm{Hom}_G(V, W) = (\mathrm{Hom}(V, W))^G$.

Theorem (Schur's Lemma over $\mathbb{C}$.)

If V is a complex irreducible representation of G, then $\mathrm{End}_G(V) = \{\lambda \mathrm{Id}_v | \lambda \in \mathbb{C}\}$.

Proof.

Let $\phi \colon V \to V$ be a G-linear endomorphism of V, and let λ be an eigenvalue of ϕ. We claim that the eigenspace E_λ is G-invariant. If $v \in E_\lambda$, then $\phi(v) = \lambda v$. This implies that $\phi(gv) = g\phi(v) = g(\lambda v) = \lambda(gv)$, i.e. $gv \in E_\lambda$. Since g was arbitrary, E_λ is indeed G-invariant. Now $E_\lambda \neq 0$, so since V is irreducible, $E_\lambda = V$. Thus $\phi = \lambda \mathrm{Id}$. □

Corollary

Let V and W be irreducible representations. If V and W are isomorphic, the space $\mathrm{Hom}_G(V, W)$ is 1-dimensional, and in this case any non-zero G-linear map from V to W is an isomorphism. Otherwise, $\mathrm{Hom}_G(V, W) = \{0\}$.

Proof.

Suppose $\mathrm{Hom}_G(V, W) \neq \{0\}$ and let $\phi \in \mathrm{Hom}_G(V, W)$ be a nonzero G-linear map. Since $\ker(\phi)$ and $\mathrm{im}(\phi)$ are both G-invariant, irreducibility yields ($\ker(\phi) = 0$ or V) and ($\mathrm{im}(\phi) = 0$ or W) as the only possibilities. Since $\phi \neq 0$, then $\ker(\phi) = 0$, $\mathrm{im}(\phi) = W$, and ϕ is an isomorphism. Let ψ be another nonzero G-linear map from V to W. Then $\phi^{-1} \circ \psi \in \mathrm{End}_G(V)$. We can apply Schur's Lemma over $\mathbb{C}$ to see that $\phi^{-1} \circ \psi = \lambda \mathrm{Id}$, hence $\psi = \lambda \phi$. So ϕ spans $\mathrm{Hom}_G(V, W)$. $\square$

Proposition

Let V and W be irreducible representations of G. Then

$$\text{dim Hom}_G(V,W) = \begin{cases} 1 & \text{if } V \text{ and } W \text{ are isomorphic} \\ 0 & \text{if } V \text{ and } W \text{ are not isomorphic} \end{cases}$$

Proposition

Let $\rho\colon G \to GL(V)$ *be a representation, let*

$$V = U_1 \oplus \ldots \oplus U_s$$

be a decomposition of V *into irreps, and let* W *be any irrep of* G*. Then the number of irreps in the set* $\{U_1, \ldots, U_s\}$ *which are isomorphic to* W *equals the dimension of* $\mathrm{Hom}_G(V, W)$*.*

Proof.

Have:

$$\mathrm{Hom}_G(V, W) = \bigoplus_{i=1}^{s} \mathrm{Hom}_G(U_i, W),$$

so taking the dimension of both sides yields

$$\dim \mathrm{Hom}_G(V, W) = \sum_{i=1}^{s} \dim \mathrm{Hom}_G(U_i, W).$$

By previous Proposition, this sum is exactly the $\#$ of irreps in $\{U_1, \dots, U_s\}$ which are isomorphic to W. $\square$

Theorem (Uniqueness of decomposition into irreducibles.)

Let $\rho\colon G \to GL(V)$ be a representation, and let

$$V = U_1 \oplus \ldots \oplus U_s$$
$$V = \widetilde{U_1} \oplus \ldots \oplus \widetilde{U_r}$$

be two decompositions of V into irreducible subrepresentations. Then $s = r$, and (after reordering if necessary) U_i and $\widetilde{U}_i$ are isomorphic for every $i \in \{1, \ldots, s\}$.

Proof.

The number of irreps in either decomposition that are isomorphic to any irrep W is equal to dim $\mathrm{Hom}_G(V, W)$. So the two decompositions contain the same number of factors isomorphic to W for any irrep W of G. $\square$

The definition of a Character

Definition

The **character** of a representation $\rho \colon G \to GL(V)$ is the function

$$\chi_V \colon G \to \mathbb{C}$$

defined by

$$\chi_V(g) = \mathrm{Tr}(\rho(g)).$$

Note

The character of a representation is not a homomorphism in general, since $\mathrm{Tr}(MN) \neq \mathrm{Tr}(M)\mathrm{Tr}(N)$ in general.

Basic properties of Characters

Proposition

Let V be a representation of G.

- χ_V *is conjugation invariant:* $\chi_V(hgh^{-1}) = \chi_V(g) \quad \forall g, h \in G$.
- $\chi_V(e) =$ *dim* V.
- $\chi_V(g^{-1}) = \overline{\chi_V(g)} \quad \forall g \in G$.
- $\chi_{V^*}(g) = \overline{\chi_V(g)} \quad \forall g \in G$.

Proposition

Let V and W be representations of G.

- $\chi_{V \oplus W} = \chi_V + \chi_W$.
- $\chi_{V \otimes W} = \chi_V \cdot \chi_W$.

Proposition

Isomorphic representations have the same character.

Proof.

Isomorphic representations can be described by the same set of matrices with the right choice of bases. Thus each $\rho(g)$ has the same trace as $\widetilde{\rho(g)}$ for any representation $\widetilde{\rho}$ isomorphic to ρ. $\square$

Definition

Let $\mathbb{C}^G$ denote the vector space of all functions from G to $\mathbb{C}$. A basis for $\mathbb{C}^G$ is given by the set of functions

$$\{\delta_g | g \in G\}$$

defined by

$$\delta_g \colon h \mapsto \begin{cases} 1 & \text{if } h = g \\ 0 & \text{if } h \neq g. \end{cases}$$

Definition

Let $\varphi, \psi \in \mathbb{C}^G$. We define a **Hermetian inner product** on $\mathbb{C}^G$ by

$$\langle \varphi | \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}.$$

Inner product of Characters

Theorem

Let $\rho_V : G \to GL(V)$ *and* $\rho_W : G \to GL(W)$ *be representations of* G*, and let* χ_V, χ_W *be their characters. Then*

$$\langle \chi_W | \chi_V \rangle = \mathit{dim}\ \mathit{Hom}_G(V, W).$$

Corollary

Let $\chi_1, \ldots, \chi_r$ be characters of pairwise non-isomorphic irreducible representations of G. Then

$$\langle \chi_i | \chi_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Proof.

Let χ_i and χ_j be the characters of the irreducible representations U_i, U_j. Then

$$\langle \chi_i | \chi_j \rangle = \dim \operatorname{Hom}_G(U_j, U_i) = \begin{cases} 1 & \text{if } U_i, U_j \text{ are isomorphic} \\ 0 & \text{if } U_i, U_j \text{ are not isomorphic.} \end{cases}$$

□

Corollary

Let χ be any character of G. Then χ is irreducible if and only if

$$\langle \chi | \chi \rangle = 1$$

Proof.

Write χ as a linear combination of irreducible characters

$$\chi = m_1 \chi_1 + \ldots + m_k \chi_k$$

where each m_i is a non-negative integer. Then

$$\begin{aligned} \langle \chi | \chi \rangle &= \sum_{i,j \in [1,k]} m_i m_j \langle \chi_i | \chi_j \rangle \\ &= m_1^2 + \ldots + m_k^2. \end{aligned}$$

So $\langle \chi | \chi \rangle = 1$ if and only if exactly one of the $m_i = 1$ and the rest are 0. □

Example

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$. Recall the two dimensional representation W of D_4 given earlier. We compute the character of this representation by taking the trace of the matrices from that example:

$$
\begin{aligned}
\chi_W(e) &= 2 & \chi_W(\tau) &= 0 \\
\chi_W(\sigma) &= 0 & \chi_W(\sigma\tau) &= 0 \\
\chi_W(\sigma^2) &= -2 & \chi_W(\sigma^2\tau) &= 0 \\
\chi_W(\sigma^3) &= 0 & \chi_W(\sigma^3\tau) &= 0.
\end{aligned}
$$

Then

$$\langle \chi_W | \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_W(g)\overline{\chi_W(g)} = \frac{1}{8}(4+4) = 1$$

so we conclude that W is irreducible.

Corollary

Let V and W be representations of G. Then V and W are isomorphic if and only if $\chi_V = \chi_W$.

Proof.

Suppose $\chi_V = \chi_W$. We can find non-negative integers m_i and l_j such that

$$V = U_1^{m_1} \oplus \ldots \oplus U_r^{m_r} \quad \text{and} \quad W = U_1^{l_1} \oplus \ldots \oplus U_r^{l_r}$$

where $U_1, \ldots, U_r$ are distinct irreps of G. Then

$$\chi_V = m_1\chi_1 + \ldots + m_r\chi_r \quad \text{and} \quad \chi_W = l_1\chi_1 + \ldots + l_r\chi_r.$$

It follows that

$$m_i = \langle \chi_V | \chi_i \rangle = \langle \chi_W | \chi_i \rangle = l_i$$

for all $i \in \{1, \ldots, r\}$ since $\chi_V = \chi_W$. $\square$

Lemma

$$\chi_{\text{reg}}(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$$

Proposition

The multiplicity of any irreducible representation in the regular representation equals its dimension.

Proof.

Let V be an irreducible representation of G. Then

$$\begin{aligned} \langle \chi_{\text{reg}}, \chi_V \rangle &= \frac{1}{|G|} \chi_{\text{reg}}(e) \overline{\chi_V(e)} \\ &= \frac{1}{|G|} |G| (\dim V) = \dim V. \end{aligned}$$

□

Corollary

There are finitely many irreducible representations of G, up to isomorphism.

Corollary

Let $U_1, \ldots, U_r$ be the irreducible representations of G with degrees $d_1, \ldots, d_r$. Then

$$|G| = \sum_{i=1}^{n} d_i^2$$

Definition

We define **the character table of** G to be the table of complex numbers whose:

- rows are index by the isomorphism classes of irreducible representations of G,
- columns are indexed by the conjugacy classes of G,
- i, j entry is given by value of the character corresponding to row i evaluated at the conjugacy class corresponding to column j.

Note

To find the inner product of χ_V and χ_W, we only need to calculate χ_V and χ_W once on each conjugacy class, i.e.

$$\langle \chi_V | \chi_W \rangle = \frac{1}{|G|} \sum_{[g]} |[g]| \chi_V(g) \overline{\chi_W(g)}.$$

Character table of D_3

Example

Consider $G = D_3 = \langle \sigma, \tau | \sigma^3 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$. We have seen three irreducible representations of D_3, namely the 1-dimensional trivial representation, the 1-dimensional alternating representation, and the 2-dimensional irreducible representation W constructed geometrically. Observe that

$$|D_3| = 6 = 1^2 + 1^2 + 2^2$$

so these are all of the irreducible representations of D_3 up to isomorphism.

Character table of D_3

Example

The conjugacy classes of D_3 are $\{e\}$, $\{\sigma, \sigma^2\}$, and $\{\tau, \tau\sigma, \tau\sigma^2\}$. Thus, the character table of D_3 is given by

Character table of D_3			
Conjugacy class representative $[g]$	$[e]$	$[\tau]$	$[\sigma]$
χ_1 (1-d trivial reprn)	1	1	1
χ_{sgn} (1-d sign reprn)	1	-1	1
χ_W (2-d reprn obtained geometrically)	2	0	-1

Character Table of D_4

Example

Let $G = D_4$. Let $U_1, \ldots, U_r$ be the irreducible representations of D_4, with dimensions $d_1, \ldots, d_r$ respectively, and let U_1 be the 1-dimensional trivial representation. Then

$$d_2^2 + \ldots + d_r^2 = |G| - d_1^2 = 8 - 1 = 7.$$

There are two possibilities:
1. $r = 8$, and $d_i = 1$ for all $1 \leq i \leq 8$.
2. or $r = 5$, and $d_2 = d_3 = d_4 = 1$, $d_5 = 2$.

We saw earlier that G has a two-dimensional irreducible representation, so in fact (2) holds.

Character Table of D_4

Example

The remaining 1-dimensional representations are easy to find, since they must satisfy the relations $\rho(\sigma)^2 = 1$ and $\rho(\tau)^2 = 1$. Thus the character table for D_4 is as follows:

Character table of D_4					
Conjugacy class	$\{1\}$	$\{\sigma, \sigma^3\}$	$\{\sigma^2\}$	$\{\tau, \sigma^2\tau\}$	$\{\sigma\tau, \sigma^3\tau\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	1	1	-1
χ_4	1	-1	1	-1	1
χ_W (2-d reprn)	2	0	-2	0	0