UNIVERSITY OF MISSOURI

MASTER'S PROJECT

A Survey on Character Tables for Representations of Finite Groups

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Chapter 1

Basic Notions of Representation Theory

1.1 Group Actions

Definition 1.1. A ***(left)* group action** of a group G on a set X is a map $\rho\colon G \times X \to X$ (written as $g \cdot a$, for all $g \in G$ and $a \in A$) that satisfies the following two axioms:

$$1 \cdot x = x \qquad \forall x \in X \tag{1.1.1}$$

$$(gh) \cdot x = g \cdot (h \cdot x) \qquad \forall g, h \in G, x \in X \tag{1.1.2}$$

Note. We could likewise define the concept of a *right* group action, where the set elements would be multiplied by group elements on the right instead of on the left. Throughout we shall use the term *group action* to mean a *left* group action.

Proposition 1.2. *Let G act on the set X. For any fixed $g \in G$, the map σ_g from X into X defined by $\sigma_g(x) = g \cdot x$ is a permutation of the set X. That is, $\sigma_g \in S_X$.*

Proof. We show that σ_g is a permutation of X by finding a two-sided inverse map, namely $\sigma_{g^{-1}}$. Observe that for any $x \in X$, we have

$$\begin{aligned}
(\sigma_{g^{-1}} \circ \sigma_g)(x) &= \sigma_{g^{-1}}(\sigma_g(x)) \\
&= g^{-1} \cdot (g \cdot x) && \text{(by definition of } \sigma_g \text{ and } \sigma_{g^{-1}}) \\
&= (g^{-1}g) \cdot x && \text{(by axiom 1.1.1 of an action)} \\
&= 1 \cdot x \\
&= x && \text{(by axiom 1.1.2 of an action).}
\end{aligned}$$

Thus $\sigma_{g^{-1}} \circ \sigma_g$ is the identity map on X. We can reverse the roles of g and g^{-1} to see that $\sigma_g \circ \sigma_{g^{-1}}$ is also the identity map on X. Having a two-sided inverse, we conclude that σ_g is a permutation of X. $\square$

Proposition 1.3. *Let G act on the set X. The map from G into the symmetric group S_X defined by $g \mapsto \sigma_g(x) = g \cdot x$ is a group homomorphism.*

Proof. Define the map $\rho\colon G \to S_X$ by $\rho(g) = \sigma_g$. We have seen from Proposition 1.2 that σ_g is indeed an element of S_X. It remains to show that $\rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2)$ for any $g_1, g_2 \in G$. Observe that

$$\begin{aligned}
\rho(g_1g_2)(x) &= \sigma_{g_1g_2}(x) && \text{(by definition of } \rho\text{)}\\
&= (g_1g_2)\cdot x && \text{(by definition of } \sigma_{g_1g_2}\text{)}\\
&= g_1\cdot(g_2\cdot x) && \text{(by axiom 1.1.1 of an action)}\\
&= \sigma_{g_1}(\sigma_{g_2}(x)) && \text{(by definition of } \sigma_{g_1} \text{ and } \sigma g_2\text{)}\\
&= \rho(g_1)(\rho(g_2)(x)) && \text{(by definition of } \rho\text{)}\\
&= (\rho(g_1)\circ\rho(g_2))(x) && \text{(by definition of function composition).}
\end{aligned}$$

Since the values of $\rho(g_1g_2)$ and $\rho(g_1)\circ\rho(g_2)$ agree on every element $x \in X$, these two permutations are equal. We conclude that ρ is a homomorphism, since g_1 and g_2 were arbitrary elements of G. □

Proposition 1.4. *Any homomorphism ψ from the group G into the symmetric group S_X on a set X gives rise to an action of G on X, defined by taking $g \cdot x = \psi(g)(x)$.*

Proof. Suppose that we have a homomorphism ψ from G into S_X. We can define a map from $G \times X$ to X by $g \cdot x = \psi(g)(x)$. We verify that this map satisfies the definition of a group action of G on X:
(axiom 1.1.1) $1 \cdot x = \psi(1)(x) = id_X(x) = x$
(axiom 1.1.2) $(gh)\cdot x = \psi(gh)(x) = (\psi(g)\psi(h))(x) = \psi(g)(\psi(h)(x)) = g\cdot(h\cdot x)$ □

Corollary 1.5. *The actions of G on the set X are in bijective correspondence with the homomorphisms from G into the symmetric group S_X.*

Proof. By Proposition 1.3, any action of G on X yields a homomorphism from G into S_X. Conversely, any homomorphism from G into S_X establishes an action of G on X by Proposition 1.4. □

1.2 The Definition of a Representation

Definition 1.6. Let G be a group. A **representation** of G is a homomorphism $\rho\colon G \to GL_n(\mathbb{C})$ for some positive integer n.

Definition 1.7. Two representations $\rho_1\colon G \to GL_n(\mathbb{C})$ and $\rho_2\colon G \to GL_n(\mathbb{C})$ of G are **equivalent** if there exists $P \in GL_n(\mathbb{C})$ such that $\rho_2 = P^{-1}\rho_1 P$.

Equivalent representations are fundamentally "the same" in some sense, but to make this precise we need to shift our thinking to linear maps instead of matrices.

Definition 1.8. Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is any group homomorphism $\rho\colon G \to GL(V)$. If we fix a basis for V, we get a representation in the previous sense.

Definition 1.9 (Alternative definition)**.** Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is an action of G on V which preserves the linear structure of V, i.e. an action of G on V such that

$$g\cdot(v_1+v_2) = g\cdot v_1 + g\cdot v_2 \qquad \forall g \in G, v_1, v_2 \in V \tag{1.9.1}$$
$$g\cdot(kv) = k(g\cdot v) \qquad \forall g \in G, v \in V, k \in F \tag{1.9.2}$$

Note. Unless otherwise specificed, we use *representation* to mean *finite-dimensional complex representation*.

Proposition 1.10. *The definitions of a linear representation given in 1.8 and 1.9 above are equivalent.*

Proof. ($\rightarrow$) Suppose that we have a homomorphism $\rho\colon G \to GL(V)$. Note that $GL(V)$ is a subgroup of the symmetric group S_V on V, so we can apply Proposition 1.4 to obtain an action of G on V by $g \cdot v = \rho(g)(v)$. We check that this action preserves the linear structure of V.

1.9.1 For any $g \in G$, $v_1, v_2 \in V$ we have $g \cdot (v_1 + v_2) = \rho(g)(v_1 + v_2) = \rho(g)(v_1) + \rho(g)(v_2) = g \cdot v_1 + g \cdot v_2$.

1.9.2 For any $g \in G, v \in V, k \in F$ we have $g \cdot (kv) = \rho(g)(kv) = k(\rho(g)(v)) = k(g \cdot v)$.

($\leftarrow$) Suppose that we have an action of G on V which preserves the linear structure of V in the sense of Definition 1.9. We can apply Proposition 1.3 to obtain a homorphism $\rho\colon G \to S_V$ given by $\rho(g) = \sigma_g$ where $\sigma_g(v) = g \cdot v$. It remains to show that the image $\rho(G)$ of G under ρ is actually contained in $GL(V)$, i.e. that for each $g \in G$ the map σ_g is linear. Fix an element $g \in G$. For any $k \in F$ and $v \in V$, we have

$$
\begin{aligned}
\sigma_g(kv) &= g \cdot (kv) && \text{(by definition of } \sigma_g\text{)} \\
&= k(g \cdot v) && \text{(by property 1.9.1)} \\
&= k(\sigma_g(v)) && \text{(by definition of } \sigma_g\text{).}
\end{aligned}
$$

Also, for any $v_1, v_2 \in V$ we have

$$
\begin{aligned}
\sigma_g(v_1 + v_2) &= g \cdot (v_1 + v_2) && \text{(by definition of } \sigma_g\text{)} \\
&= g \cdot v_1 + g \cdot v_2 && \text{(by property 1.9.2)} \\
&= \sigma_g(v_1) + \sigma_g(v_2) && \text{(by definition of } \sigma_g\text{).}
\end{aligned}
$$

Thus σ_g is linear, and $\rho(G) \subset GL(V)$ proves that we have a homomorphism $\rho\colon G \to GL(V)$.

□

Definition 1.11. Let G be a group, let F be a field, let V be a vector space over F, and let $\rho\colon G \to GL(V)$ be a representation of G. The **dimension** of the representation is the dimension of V over F.

Example 1.12.

1. Let V be a 1-dimensional vector space over the field F. The map $\rho\colon G \to GL(V)$ defined by $\rho(g) = 1$ for all $g \in G$ is a representation called the *trival representation* of G. The trivial representation has dimension 1.

2. If G is a finite group that acts on a finite set X, and F is any field, then there is an associated *permutation representation* on the vector space V over F with basis $\{e_x \colon x \in X\}$. We let G act on the basis elements by the permutation $g \cdot e_x = e_{gx}$ for all $x \in X$ and $g \in G$. This representation has dimension $|X|$.

3. A fundamental special case of a permutation representation that we shall return to later on is that when a finite group acts on itself by left multiplication. In this case, the elements of G form a basis for V, and each $g \in G$ permutes the basis

elements by $g \cdot g_i = gg_i$. This representation is called the *regular representation* of G and has dimension $|G|$. We shall see later that this representation encodes information about all other representations of G.

4. For any symmetric group S_n, the *alternating representation* on $V = \mathbb{C}$ is given by the map $\rho\colon S_n \to GL(\mathbb{C}) = \mathbb{C}^\times$ defined by $\rho(\sigma) = \mathrm{sgn}(\sigma)$. More generally, for any group G with a subgroup H of index 2, we can define an *alternating representation* $\rho\colon G \to GL(\mathbb{C})$ by letting $\rho(g) = 1$ if $g \in H$ and $\rho(g) = -1$ if $g \notin H$. (We recover our original example by taking $G = S_n$ and $H = A_n$.)

Definition 1.13. A **homomorphism** between two representations $\rho_1\colon G \to GL(V)$ and $\rho_2\colon G \to GL(W)$ is a linear map $\psi\colon V \to W$ that interwines with (respects) the G-action, i.e. a map ψ such that

$$\psi(\rho_1(g)(v)) = \rho_2(g)(\psi(v)) \quad \forall v \in V, g \in G$$

An **isomorphism** of representations is a homomorphism of representations that is also an invertible map.

Note. If we have representations (ρ_1, V) and (ρ_2, W) and an isomorphism of vector spaces $\psi\colon V \to W$ then we can rewrite the compatibility requirement above as $\rho_2(g) = \psi \circ \rho_1(g) \circ \psi^{-1}$ for all $g \in G$.

Given any representation (ρ, V) of a group G on a vector space V over a field F of dimension n, we can fix a basis for V to obtain an isomorphism of vector spaces $\psi\colon V \to F^n$. This yields a representation ϕ of G on F^n by defining $\phi(g) = \psi \circ \rho(g) \circ \psi^{-1}$ for all $g \in G$. Clearly, this representation is isomorphic to our original representation (ρ, V). In particular, this means we can always choose to view n-dimensional complex representations as representations on $\mathbb{C}^n$ where each $\phi(g)$ is given by an $n \times n$ matrix with entries in $\mathbb{C}$.

Suppose that we have two representations $\rho_1\colon G \to GL_n(F)$ and $\rho_2\colon G \to GL_m(F)$ given by $\rho_1(g) = X_g$ and $\rho_2(g) = Y_g$. A homomorphism between these representations is then an $m \times n$ matrix A such that $AX_g = Y_g A$ for all $g \in G$. An isomorphism is given precisely when such A is square and invertible. Thus, two representations $\rho_1\colon G \to GL_n(F)$ and $\rho_2\colon G \to GL_n(F)$ are isomorphic if and only if there exists $A \in GL_n(F)$ such that $\rho_1(g) = A\rho_2(g)A^{-1}$ for all $g \in G$. This establishes the following proposition:

Proposition 1.14. ?? *The isomorphism classes of n-dimensional representations of G on $\mathbb{C}$ are in bijection with the quotient $Hom(G; GL_n(\mathbb{C}))/GL_n(\mathbb{C})$ of group homomorphisms $G \to GL_n(\mathbb{C})$ modulo the conjugation action of $GL_n(\mathbb{C})$.*

1.3 Representations of Cyclic Groups

Example 1.15 (Representations of $\mathbb{Z}$)**.** We want to classify all representations of the group $\mathbb{Z}$ under addition. Consider an n-dimensional representation $\rho\colon \mathbb{Z} \to GL_n(\mathbb{C})$. For ρ to be a group homomorphism requires that $\rho(0) = \mathrm{Id}$. Observe that for any $0 \neq n \in \mathbb{Z}$, we have $\rho(n) = \rho(1 + \ldots + 1) = \rho(1)^n$. Thus ρ is completely determined by the matrix $\rho(1) \in GL_n(\mathbb{C})$, and any such matrix determines a representation of $\mathbb{Z}$. It follows that the n-dimensional isomorphism classes of representations of $\mathbb{Z}$ are in bijection with the conjugacy classes in $GL_n(\mathbb{C})$. These conjugacy classes can be parameterized by the *Jordan canonical form*.

Example 1.16 (Representations of the cyclic group of order n)**.** We shall classify all representations of the cyclic group $G = \{g, g^2, \ldots, g^{n-1}, g^n = 1\}$ of order n. Consider a representation $\rho\colon G \to GL(V)$. As in the previous example, we know that $\rho(1) = \mathrm{Id}$ and $\rho(g^k) = \rho(g)^k$. Thus our representation ρ is determined completely by the linear transformation $\rho(g)$. It will be helpful to fix a basis of V so that we may view $\rho(g)$ as a matrix. Recall from linear algebra that there exists a basis in which $\rho(g)$ takes the *Jordan canonical form*

$$\rho(g) = \begin{bmatrix} J_1 & 0 & \ldots & 0 \\ 0 & J_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & J_m \end{bmatrix}$$

where each *Jordan block* J_k is of the form

$$J_k = \begin{bmatrix} \lambda & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda & 1 & \ddots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda \end{bmatrix}.$$

Now $I = \rho(g)^n$ is a block-diagonal matrix with diagonal blocks J_k^n, so we must have that each block $J_k^n = \mathrm{Id}$. Observe that we can write each block J_k as $J_k = \lambda\mathrm{Id} + N$ where N is the Jordan block with $\lambda = 0$. Thus we have

$$\mathrm{Id} = J_k^n = (\lambda\mathrm{Id} + N)^n = \lambda^n\mathrm{Id} + \binom{n}{1}\lambda^{n-1}N + \binom{n}{2}\lambda^{n-2}N^2 + \ldots + \binom{n}{n-1}\lambda N^{n-1} + N^n$$

. The following lemma will show that in fact $N = 0$.

Lemma 1.17. *Let N be the Jordan block with $\lambda = 0$ of size $n \times n$. For any integer p with $1 \le p \le n-1$, then N^p is the matrix with ones in the positions (i, j) where $j = i + p$ and zeroes everywhere else. (The ones lie along a line parallel to the diagonal, p steps above it.)*

Proof. (By induction.)

Base case: This is simply the definition of N.

Inductive step: Suppose that the lemma holds for N^p. We compute the (i, j) entry of N^{p+1}:

$$(N^{p+1})_{i,j} = \sum_{k=1}^{n} (N^p)_{i,k} N_{k,j} = (N^p)_{i,i+p} N_{i+p,j} = N_{i+p,j} = \begin{cases} 1 & \text{if} j = i + (p+1) \\ 0 & \text{otherwise} \end{cases}$$

∎

Now, if $N \neq 0$ then each term $\binom{n}{k}\lambda^{n-k}N^k$ for $k > 0$ would yield some non-zero non-diagonal entries (in the positions (i, j) where $j = i + k$) and hence our sum could not equal the identity matrix. We must conclude that $N = 0$, $J_k = \lambda\mathrm{Id}$ is a 1×1 block, and $J_k^n = \lambda^n\mathrm{Id}$. Thus $\rho(g)$ is a diagonal matrix with nth roots of unity as diagonal entries.

To summarize, every m-dimensional representation ρ of the cyclic group $G = \langle g \rangle$ of order n can be seen to act (with the right choice of basis) as $m \times m$ diagonal matrices all with nth roots of unity along the diagonal. In particular, these representations are determined completely by the value of $\rho(g)$ and are classified up to isomorphism by unordered m-tuples of nth roots of unity.

1.4 Constructions from Linear Algebra

Definition 1.18. A **subrepresentation** of V is a G-invariant subspace $W \subseteq V$; that is, a subspace $W \subseteq V$ with the property that $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$. Note that W itself is a representation of G under the action $\rho(g)\restriction_W$.

From elementary linear algebra, we know that given a subspace $W \subseteq V$, we can form the **quotient space** V/W consisting of cosets $v + W$ in V. If W is a subrepresentation of V, we would like to define an action of G on V/W by the natural choice of $g(v + W) = \rho(g)(v) + W$. It remains to verify that this action is well defined. If we choose another $v' \in v + W$, then $v - v' \in W$, so that $\rho(g)(v - v') \in W$ since W is G-invariant. Thus, the cosets $\rho(g)(v) + W$ and $\rho(g)(v') + W$ agree and this action is indeed well defined. This justifies the following definition:

Definition 1.19. Let W be a G-subrepresentation of V. Then V/W forms a representation of G called the **quotient representation** of V under W with the action $g(v + W) = \rho(g)(v) + W$.

Recall also from linear algebra that given two vector spaces V_1 and V_2, we can form the **direct sum** $V_1 \oplus V_2$ consisting of ordered pairs (v_1, v_2) where $v_1 \in V_1, v_2 \in V_2$.

Definition 1.20. Let V_1 and V_2 be representations of G. Then $V_1 \oplus V_2$ forms a representation of G called the **direct sum representation** of V_1 and V_2 with the action $g(v_1, v_2) = (g \cdot v_1, g \cdot v_2)$.

1.5 Complete Reducibility and Unitarity

Definition 1.21. A representation is said to be **irreducible** if it contains no proper invariant subspaces. It is called **completely reducible** if it decomposes into a direct sum of irreducible subrepresentations.

Example 1.22. 1. Any irreducible representation is, in particular, completely reducible. 2. Any 1-dimensional representations has no proper subspaces, and is thus irreducible.

Theorem 1.23. *If $A_1, A_2, \ldots, A_r$ are linear operators on V and each A_i is diagonalizable, they are simultaneously diagonalizable if and only if they commute.*

Proof. See Conrad [2, Theorem 5.1]. □

Theorem 1.24. *Every complex representation of a finite abelian group is completely reducible into irreducible representations of dimension 1.*

Proof. Take an arbitrary element $g \in G$. Since G is finite, we can find an integer n such that $g^n = 1$ and $\rho(g)^n = Id$. Hence the minimal polynomial of $\rho(g)$ divides $x^n - 1$. Recall that $x^n - 1$ has n distinct roots over $\mathbb{C}$, which are generated by taking powers of $\xi = e^{\frac{2\pi i}{n}}$. This means that the minimal polynomial $\rho(g)$ factors into linear factors only over $\mathbb{C}$ so that $\rho(g)$ is diagonalizable. We conclude that each $\rho(g)$ is (separately) diagonalizable since $g \in G$ was arbitrary.

Now, given any two elements $g_1, g_2 \in G$ we have

$$\begin{aligned}
\rho(g_1)\rho(g_2) &= \rho(g_1 g_2) && \text{(since } \rho \text{ is a homomorphism)} \\
&= \rho(g_2 g_1) && \text{(since } G \text{ is abeilian)} \\
&= \rho(g_2)\rho(g_1) && \text{(since } \rho \text{ is a homomorphism).}
\end{aligned}$$

Thus the matrices $\{\rho(g)\}$ commute, so we may apply theorem 1.23 to conclude that $\{\rho(g)\}$ are simultaneously diagonalizable, say with basis $\{e_1, ..., e_k\}$. Then we have $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \ldots \oplus \mathbb{C}e_n$, with each subspace $\mathbb{C}e_1$ invariant under the action of G. □

Definition 1.25. Let W be a subspace of V. A **linear projection** V onto W is a linear map $f \colon V \to W$ such that $f\restriction_W = \mathrm{Id}_W$. If W is a subrepresentation of V and the map f is G-invariant, then we say that f is a **G-linear projection**.

Lemma 1.26. *Let $\rho \colon G \to GL(V)$ be a representation, and $W \subset V$ be a subrepresentation. Suppose we have a G-linear projection*

$$f \colon V \to W.$$

Then $Ker(f)$ is a complementary subrepresentation to W, i.e. $Ker(f)$ is a G-invariant subspace of V such that

$$V = Ker(f) \oplus W$$

Proof. First we note that $\mathrm{Ker}(f)$ is G-invariant, since if $x \in \mathrm{Ker}(f)$, then $0 = g0 = gf(x) = f(gx)$ for every $g \in G$. Now if $y \in \mathrm{Ker}(f) \cap W$ then $y = f(y) = 0$, so $\mathrm{Ker}(f) \cap W = 0$. Finally $\mathrm{Im}(f) = W$, so by the Rank-Nullity theorem

$$\dim \mathrm{Ker}(f) + \dim W = \dim V.$$

Thus $V = \mathrm{Ker}(f) \oplus W$. □

Theorem 1.27 (Maschke's Theorem). *Let G be a finite group and let F be a field such that $char(F) \nmid |G|$. If V is any finite-dimensional representation of G over F, and $W \subset V$ is a subrepresentation of V, then there exists a complementary subrepresentation $U \subset V$, i.e. there is a G-invariant subspace $U \subset V$ such that*

$$V = W \oplus U$$

.

Proof. By the previous Lemma 1.26 it will suffice to find a G-linear projection from V onto W. Fix a basis $\{b_1, \ldots, b_m\}$ for W and extend it to a basis $\{b_1, \ldots, b_m, b_{m+1}, \ldots, b_n\}$ for V. Let $U = \langle b_{m+1}, \ldots, b_n \rangle$. Then U is certainly a complementary subspace to W, and we have a natural projection $f \colon W \oplus U \to W$ of V onto W with kernel U. There is no reason to think that f should be G-linear, but we can fix this by averaging over G. Define $\widetilde{f} \colon V \to V$ by

$$\widetilde{f}(x) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}))(x).$$

We claim that $\widetilde{f}$ is a G-linear projection from V onto W. First we check that $\mathrm{Im}(f) \subset W$. If $x \in V$ and $g \in G$, then

$$f(\rho(g^{-1})(x)) \in W$$

and so

$$\rho(g)(f(\rho(g^{-1})(x))) \in W$$

since W is G-invariant. Thus $\widetilde{f}(x) \in W$. Next we check that $\widetilde{f}\restriction_W = \mathrm{Id}_W$. Let $y \in W$. For any $g \in G$, we know that $\rho(g^{-1})(y)$ is also in W, so

$$f(\rho(g^{-1})(y)) = \rho(g^{-1})(y).$$

Then

$$\begin{aligned}
\widetilde{f}(y) &= \frac{1}{|G|}\sum_{g\in G}\rho(g)(f(\rho(g^{-1})(y))) \\
&= \frac{1}{|G|}\sum_{g\in G}\rho(g)(\rho(g^{-1})(y)) \\
&= \frac{1}{|G|}\sum_{g\in G}\rho(gg^{-1})(y) \\
&= \frac{1}{|G|}\sum_{g\in G}(y) \\
&= \frac{|G|y}{|G|}
\end{aligned}$$

so indeed $\widetilde{f}$ is a linear projection of V onto W. Finally, we check that $\widetilde{f}$ is G-linear. If $x \in V$ and $h \in G$, then

$$\begin{aligned}
(\widetilde{f}\circ\rho(h))(x) &= \frac{1}{|G|}\sum_{g\in G}(\rho(g)\circ f\circ\rho(g^{-1})\circ\rho(h))(x) \\
&= \frac{1}{|G|}\sum_{g\in G}(\rho(g)\circ f\circ\rho(g^{-1}h))(x) \\
&= \frac{1}{|G|}\sum_{g\in G}(\rho(hg)\circ f\circ\rho(g^{-1}))(x) \quad \text{(relabelling } g \mapsto hg\text{)} \\
&= (\rho(h)\circ\widetilde{f})(x).
\end{aligned}$$

$\square$

Corollary 1.28. *Let G be a finite group and let F be a field such that char$(F) \nmid |G|$. then any finite-dimensional representation of G over F is completely reducible.*

Proof. Let V be a representation of G over F of dimension n. If V is irreducible, then V is, in particular, completely reducible. If not, then V contains a proper subrepresentation $W \subset V$. From Maschke's Theorem (1.27), we know there exists a subrepresentation $U \subset V$ such that

$$V = W \oplus U. \tag{1.28.1}$$

Both W and U have dimension less than n, so by induction we know that W and U are completely reducible. We deduce from 1.28.1 that V is completely reducible. $\square$

1.6 Vector Spaces of Linear Maps

Definition 1.29. Let V and W be vector spaces. Recall that the set $\mathbf{Hom}(\mathbf{V}, \mathbf{W})$ of linear maps from V to W is itself a vector space. If f_1, f_2 are two linear maps from V to W, then we define their sum by

$$\begin{aligned}(f_1 + f_2) \colon V &\to W \\ x &\mapsto f_1(x) + f_2(x)\end{aligned}$$

and we define scalar multiplication of $\lambda \in \mathbb{C}$ by

$$\begin{aligned}(\lambda f_1) \colon V &\to W \\ x &\mapsto \lambda f_1(x).\end{aligned}$$

Now suppose we have representations $\rho_V \colon G \to GL(V)$ and $\rho_W \colon G \to GL(W)$ of G. Then there is a natural representation of G on the vector space $\mathrm{Hom}(V, W)$ given by

$$\begin{aligned}\rho_{\mathrm{Hom}(V,W)}(g) \colon \mathrm{Hom}(V, W) &\to \mathrm{Hom}(V, W) \\ f &\mapsto \rho_W(g) \circ f \circ \rho_V(g^{-1})\end{aligned}$$

for all $g \in G$. Note that $\rho_{\mathrm{Hom}(V,W)}(g)(f)$ is certainly a linear map from V to W since the composition of linear maps is linear.

Proposition 1.30. *$\rho_{Hom(V,W)}$ is a representation of G. That is, the map*

$$\begin{aligned}\rho_{Hom(V,W)} \colon G &\to GL(Hom(V, W)) \\ g &\mapsto \rho_{Hom(V,W)}(g).\end{aligned}$$

is a homomorphism.

Proof. We must check two things:

1. The map $g \mapsto \rho_{\mathrm{Hom}(V,W)}(g)$ is a homomorphism.
2. For every $g \in G$, $\rho_{\mathrm{Hom}(V,W)}(g)$ is invertible.

First, we check that

$$\begin{aligned}\rho_{\mathrm{Hom}(V,W)}(g) \circ \rho_{\mathrm{Hom}(V,W)}(h) \colon f &\mapsto \rho_{\mathrm{Hom}(V,W)}(g)(\rho_W(h) \circ f \circ \rho_V(h^{-1})) \\ &= \rho_W(g) \circ \rho_w(h) \circ f \circ \rho_V(h^{-1}) \circ \rho_V(g^{-1}) \\ &= \rho_W(gh) \circ f \circ \rho_V(g^{-1}h^{-1}) \\ &= \rho_{\mathrm{Hom}(V,W)}(gh)(f)\end{aligned}$$

so indeed $\rho_{\mathrm{Hom}(V,W)}$ is a homomorphism. We can use this fact to see that $\rho_{\mathrm{Hom}(V,W)}(g^{-1})$ is inverse to $\rho_{\mathrm{Hom}(V,W)}(g)$ as

$$\begin{aligned}\rho_{\mathrm{Hom}(V,W)}(g) \circ \rho_{\mathrm{Hom}(V,W)}(g^{-1}) &= \rho_{\mathrm{Hom}(V,W)}(e) \\ &= \mathrm{Id}_{\mathrm{Hom}(V,W)} \\ &= \rho_{\mathrm{Hom}(V,W)}(g^{-1}) \circ \rho_{\mathrm{Hom}(V,W)}(g).\end{aligned}$$

Thus $\rho_{\mathrm{Hom}(V,W)}(g)$ is invertible for every $g \in G$, and $\rho_{\mathrm{Hom}(V,W)}$ is a representation of G. $\square$

Definition 1.31. Let V and W be two representations of G. The set of G-linear maps from V to W forms a subspace of $\mathrm{Hom}(V, W)$, which we denote by $\mathbf{Hom_G(V, W)}$. In other words, $\mathrm{Hom}_G(V, W)$ is the vector space consisting of all *homomorphisms of representations* between V and W.

Definition 1.32. Let $\rho\colon G \to GL(V)$ be a representation. We define the **invariant subrepresentation** $V^G \subset V$ to be the set

$$\{v \in V \mid \rho(g)(v) = v, \quad \forall g \in G\}.$$

Note that V^G is a subspace of V, and is also clearly a subrepresentation. It is isomorphic to a trivial representation of some dimension.

Proposition 1.33. *Let $\rho_V\colon G \to GL(V)$ and $\rho_W\colon G \to GL(W)$ be representations of G. Then the subrepresentation*

$$Hom_G(V, W) \subset Hom(V, W)$$

is precisely the invariant subrepresentation $Hom(V, W)^G$ of $Hom(V, W)$.

Proof. Let $f \in \mathrm{Hom}(V, W)$. Then f is an element of the invariant subrepresentation $\mathrm{Hom}(V, W)^G$ iff we have

$$\begin{aligned} & f = \rho_{\mathrm{Hom}(V,W)}(g)(f) \quad \forall g \in G \\ \iff & f = \rho_W(g) \circ f \circ \rho_V(g^{-1}) \quad \forall g \in G \\ \iff & f \circ \rho_V(g) = \rho_W(g) \circ f \quad \forall g \in G \end{aligned}$$

which is exactly the condition that f is G-linear, i.e. that $f \in \mathrm{Hom}_G(V, W)$. □

Lemma 1.34. *Let A and B be two representations of G. Then*

$$(A \oplus B)^G = A^G \oplus B^G.$$

Proof. Observe that

$$\begin{aligned} (a, b) \in (A \oplus B)^G &\iff \rho_{A\oplus B}(g)(a, b) = (a, b) && \forall g \in G \\ &\iff (\rho_A(g)(a), \rho_B(g)(b)) = (a, b) && \forall g \in G \\ &\iff (a, b) \in A^G \oplus B^G. \end{aligned}$$

□

Lemma 1.35. *Let $\psi\colon A \to B$ be an isomorphism between representations of G. Then ψ induces an isomorphism between their invariant subrepresentations*

$$\psi\restriction_{A^G}\colon A^G \to B^G.$$

Proof. Clearly the restriction of ψ to $A^G \subset A$ induces an isomorphism to some subrepresentation of B, but we must check that the image of this restriction actually equals B^G. We verify that

$$\begin{aligned} a \in A^G &\iff \rho_A(g)(a) = a && \forall g \in G \\ &\iff \psi(\rho_A(g)(a)) = \psi(a) && \forall g \in G \\ &\iff \rho_B(g)\psi(a) = \psi(a) && \forall g \in G \\ &\iff \psi(a) \in B^G. \end{aligned}$$

□

1.7 Schur's Lemma

Theorem 1.36. *[Schur's Lemma over $\mathbb{C}$.] If V is an irreducible G-representation over $\mathbb{C}$, then evey linear operator $\phi\colon V \to V$ commuting with G is a scalar.*

Proof. Let λ be an eigenvalue of ϕ. Observe that the eigenspace E_λ is G-invariant: If $v \in E_\lambda$, then $\phi(v) = \lambda v$. This implies that $\phi(gv) = g\phi(v) = g(\lambda v) = \lambda(gv)$, i.e. $gv \in E_\lambda$. Since g was arbitrary, E_λ is indeed G-invariant. Now $E_\lambda \neq 0$, so by irreducibility $E_\lambda = V$. Thus $\phi = \lambda\mathrm{Id}$. □

Corollary 1.37. *If V and W are irreducible, the space $Hom_G(V, W)$ is 1-dimensional if the representations are isomorphic, and in this case any non-zero map is an isomorphism. Otherwise, $Hom_G G(V, W) = \{0\}$.*

Proof. Suppose V and W aren't isomorphic. Then by Schur's Lemma, the zero map is the only G-linear map from V to W, so

$$\mathrm{Hom}_G(V, W) = \{0\}.$$

On the other hand, suppose that $\phi\colon V \to W$ is an isomorphism. Let ψ be another interwining operator from V to W. Then $\phi^{-1} \circ \psi \in \mathrm{Hom}_G(V, V)$. We can apply Schur's Lemma over $\mathbb{C}$ to see that $\phi^{-1} \circ \psi = \lambda\mathrm{Id}$, hence $\psi = \lambda\phi$. So phi spans $\mathrm{Hom}_G(V, W)$. □

More definitions are required before we can state a more general Schur's Lemma (not restricted to just $\mathbb{C}$).

Definition 1.38. An **algebra** over a field K is a ring with unit, containing a distinguished copy of K that commutes with every algebra element, and with $1 \in K$ begin the algebra unit. A **division ring** is a ring where every non-zero element is invertible, and a **division algebra** is a division ring which is also a K-algebra.

Definition 1.39. Let V be a representation of G over K. The **endomorphism algebra** $\mathrm{End}^G(V)$ is the space of linear self-maps $\phi\colon V \to V$ which commute with the group action, that is, $\rho(g) \circ \phi = \phi \circ \rho(g) \quad \forall g \in G$. The addition is the usual addition of linear maps (pointwise), and the multiplication is function composition. The distinguished copy of K is given by $K\mathrm{Id}$.

Theorem 1.40. *[Schur's Lemma] If V is an irreducible finite-dimensional representation of G over K, then $End^G(V)$ is a finite-dimensional division algebra over K.*

1.8 Isotypical Decomposition

Lemma 1.41. *Let U, V, W be three vector spaces. Then we have natural isomorphisms*

1. *$Hom(V, U \oplus W) = Hom(V, U) \oplus Hom(V, W)$*
2. *$Hom(U \oplus W, V) = Hom(U, V) \oplus Hom(W, V)$.*

Additionally, if U, V, W carry representations of G, then (1) and (2) are isomorphisms of representations.

Proof. We have inclusion and projection maps

$$U \underset{\pi_U}{\overset{\iota_U}{\rightleftarrows}} U \oplus W \underset{\iota_W}{\overset{\pi_W}{\rightleftarrows}} W$$

given by

$$\begin{aligned} \iota_U &\colon x \mapsto (x, 0) \\ \pi_U &\colon (x, y) \mapsto x \end{aligned}$$

and similarly for ι_W and π_W. It is clear that

$$\mathrm{Id}_{U \oplus W} = \iota_U \circ \pi_U + \iota_W \circ \pi_W.$$

We also note that the four spaces under consideration all have dimension $(\dim V)(\dim W + \dim U)$.

(1) We define a map

$$\begin{aligned} \psi \colon \mathrm{Hom}(V, U \oplus W) &\to \mathrm{Hom}(V, U) \oplus \mathrm{Hom}(V, W) \\ f &\mapsto (\pi_U \circ f, \pi_W \circ f). \end{aligned}$$

We claim that this map has an inverse given by

$$\begin{aligned} \psi^{-1} \colon \mathrm{Hom}(V, U) \oplus \mathrm{Hom}(V, W) &\to \mathrm{Hom}(V, U \oplus W) \\ (f_U, f_W) &\mapsto \iota_U \circ f_U + \iota_W \circ f_W. \end{aligned}$$

Check that

$$\begin{aligned} \psi^{-1} \circ \psi \colon f &\mapsto \iota_U \circ \pi_U \circ f + \iota_W \circ \pi_W \circ f \\ &= (\iota_U \circ \pi_U + \iota_W \circ \pi_W) \circ f \\ &= \mathrm{Id}_{\mathrm{Hom}(V,W)} \circ f = f. \end{aligned}$$

Since both vector spaces have the same dimension, $\psi \circ \psi^{-1}$ must be the identity map as well, and ψ is an isomorphism of vector spaces. Now suppose we have representations ρ_V, ρ_W, ρ_U of G on V, W and U. Then we claim ψ is G-linear. Recall that by definition,

$$\rho_{\mathrm{Hom}(V, U \oplus W)}(g)(f) = \rho_{U \oplus W}(g) \circ f \circ \rho_V(g^{-1}).$$

Observe that for any $g \in G$ and $f \in \mathrm{Hom}(V, U \oplus W)$,

$$\begin{aligned} \pi_U \circ (\rho_{\mathrm{Hom}(V, U \oplus W)}(g)(f)) &= \pi_U \circ \rho_{U \oplus W}(g) \circ f \circ \rho_V(g^{-1}) \\ &= \rho_U(g) \circ \pi_U \circ f \circ \rho_V(g^{-1}) \quad (\text{since } \pi_U \text{ is } G\text{-linear}) \\ &= \rho_{\mathrm{Hom}(U,V)}(g)(f) \end{aligned}$$

and similarly for W, so that

$$\begin{aligned} \psi(\rho_{\mathrm{Hom}(V, U \oplus W)}(g)(f)) &= (\pi_U \circ \rho_{\mathrm{Hom}(V, U \oplus W)}(g)(f), \pi_W \circ \rho_{\mathrm{Hom}(V, U \oplus W)}(g)(f)) \\ &= (\rho_{\mathrm{Hom}(V,U)}(g)(\pi_U \circ f), \rho_{\mathrm{Hom}(V,W)}(g)(\pi_W \circ f)) \\ &= \rho_{\mathrm{Hom}(V,U) \oplus \mathrm{Hom}(V,W)}(g)(\pi_U \circ f, \pi_W \circ f). \end{aligned}$$

Thus ψ is G-linear, and we've proved (1).

(2) Define a map

$$\phi\colon \mathrm{Hom}(U \oplus W, V) \to \mathrm{Hom}(U, V) \oplus \mathrm{Hom}(W, V)$$
$$= (f \circ \iota_U, f \circ \iota_W).$$

We [finish me]. The book says proof is like (1) □

Corollary 1.42. *If U, V, W are representations of G, then there are natural isomorphisms*

1. *$Hom_G(V, U \oplus W) = Hom_G(V, U) \oplus Hom_G(V, W)$*
2. *$Hom_G(U \oplus W, V) = Hom_G(U, V) \oplus Hom_G(W, V)$*

Proof. (1). By Lemma (1.41), we have an isomorphism of representations

$$\psi\colon \mathrm{Hom}(V, U \oplus W) \to \mathrm{Hom}(V, U) \oplus \mathrm{Hom}(V, W).$$

We can apply Lemma (1.35) to obtain an isomorphism on the invariant subrepresentations

$$\mathrm{Hom}(V, U \oplus W)^G \cong (\mathrm{Hom}(V, U) \oplus \mathrm{Hom}(V, W))^G.$$

Then Lemma (1.34) implies that

$$\mathrm{Hom}(V, U \oplus W)^G \cong \mathrm{Hom}(V, U)^G \oplus \mathrm{Hom}(V, W)^G.$$

The statement now follows from Proposition (1.33).

(2). The argument is similar to the one above. □

Proposition 1.43. *Let V and W be irreducible representations of G. Then*

$$dim\ Hom_G(V, W) = \begin{cases} 1 & \textit{if } V \textit{ and } W \textit{ are isomorphic} \\ 0 & \textit{if } V \textit{ and } W \textit{ are not isomorphic} \end{cases}$$

Proof. Suppose V and W are not isomorphic. Then Schur's Lemma states that the only G-linear map from V to W is the zero map, hence $\mathrm{Hom}_G(V, W) = \{0\}$.

On the other hand, suppose that $f\colon V \to W$ is an isomorphism. Then for any $h \in \mathrm{Hom}_G(V, W)$, we have $f^{-1} \circ h \in \mathrm{Hom}_G(V, V)$. By Schur's Lemma, $f^{-1} \circ h = \lambda \mathrm{Id}_V$ for some $\lambda \in \mathbb{C}$, i.e. $h = \lambda f$. Thus f spans $\mathrm{Hom}_G(V, W)$. □

Proposition 1.44. *Let $\rho\colon G \to GL(V)$ be a representation, and let*

$$V = U_1 \oplus \ldots \oplus U_s$$

be a decomposition of V into irreducibles. Let W be any irreducible representation of G. Then the number of irreducible representations in the set $\{U_1, \ldots, U_s\}$ which are isomorphic to W is equal to the dimension of $Hom_G(V, W)$, and also equal to the dimension of $Hom_G(W, V)$.

Proof. We know from Proposition (1.43) that the number of irreducibles representations in the set $\{U_1, \ldots, U_s\}$ which are isomorphic to W is equal to

$$\sum_{i=1}^{s} \dim \mathrm{Hom}_G(U_i, W).$$

By Corollary (1.42),

$$\mathrm{Hom}_G(V, W) = \bigoplus_{i=1}^{s} \mathrm{Hom}_G(U_i, W)$$

so that

$$\dim \mathrm{Hom}_G(V, W) = \sum_{i=1}^{s} \dim \mathrm{Hom}_G(U_i, W).$$

The same argument works if we consider $\mathrm{Hom}_G(W, V)$ and $\mathrm{Hom}_G(W, U_i)$ in place of $\mathrm{Hom}_G(V, W)$ and $\mathrm{Hom}_G(U_i, W)$. □

Theorem 1.45. *Let $\rho\colon G \to GL(V)$ be a representation, and let*

$$V = U_1 \oplus \ldots \oplus U_s$$
$$V = \widetilde{U_1} \oplus \ldots \oplus \widetilde{U_r}$$

be two decompositions of V into irreducible subrepresentations. Then $s = r$, and (after reordering if necessary) U_i and $\widetilde{U_i}$ are isomorphic for every $i \in \{1, \ldots, s\}$.

Proof. Let W be any irreducible representation of G. By Proposition (1.44), the number of irreducible subprepresentations in the first decomposition that are isomorphic to W is equal to $\dim \mathrm{Hom}_G(V, W)$. On the other hand, the number of irreducible subrepresentations in the second decomposition that are isomorphic to W is also equal to $\dim \mathrm{Hom}_G(V, W)$. So for any irreducible representation W, the two decompositions contain the same number of factors isomorphic to W. □

1.9 Duals and Tensor Products

Definition 1.46. Let V be a vector space. Recall that we define the **dual vector space** to be

$$V^* = \mathrm{Hom}(V, \mathbb{C}).$$

This is a special case of $\mathrm{Hom}(V, W)$ where $W = \mathbb{C}$. We know that if $\{b_1, \ldots, b_n\}$ is a basis for V, then there is a **dual basis** $\{f_1, \ldots, f_n\}$ for V defined by

$$f_i(b_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\rho_V\colon G \to GL(V)$ be a representation of G, and let $\mathbb{C}$ be the 1-dimensional trival representation of G. Then we have seen that V^* carries a representation of G defined by

$$\rho_{\mathrm{Hom}(V,\mathbb{C})}(g)(f) = f \circ \rho_V(g^{-1})$$

We call this the **dual representation** to ρ_V and denote it by ρ_V^*.

Proposition 1.47. *If we fix a basis for V, then $\rho_V^*(g)$ is given by the matrix*

$$(\rho_V(g^{-1}))^T$$

with respect to the dual basis.

Proof. Fix a basis $\{a_1, \ldots, a_n\}$ for V. Let $\rho_V(g^{-1})$ be described by the matrix M, so that

$$\rho_V(g^{-1})(a_j) = \sum_{1 \le i \le n} M_{ij} a_i.$$

Let $\rho_V^*(g)$ be described by the matrix N with respect to the dual basis $\{\alpha_1, \ldots, \alpha_n\}$, so that

$$\rho_V^*(g)(\alpha_j) = \sum_{1 \le i \le n} N_{ij} \alpha_i.$$

Then

$$\begin{aligned}
N_{ji} &= \sum_{1 \le k \le n} N_{ki} \delta_{kj} \\
&= \sum_{1 \le k \le n} N_{ki} (\alpha_k a_j) \\
&= \left(\sum_{1 \le k \le n} N_{ki} \alpha_k \right) a_j \\
&= (\rho_V^*(g)(\alpha_i))(a_j) \\
&= (\alpha_i \circ \rho_V(g^{-1}))(a_j) \quad \text{(by definition of the dual representation)} \\
&= \alpha_i(\rho_V(g^{-1})(a_j)) \\
&= \alpha_i \left(\sum_{1 \le k \le n} M_{kj} a_k \right) \\
&= \sum_{1 \le k \le n} M_{kj} \alpha_i a_k \\
&= \sum_{1 \le k \le n} M_{kj} \delta_{ik} \\
&= M_{ij}.
\end{aligned}$$

That is, $\rho_V^*(g) = (\rho_V(g^{-1})^T$ □

Definition 1.48. Suppose V and W are two vector spaces over a field K. Then we define a new vector space called the **tensor product** of V and W, denoted by $V \otimes_K W$. This space is the quotient of the free vector space on $V \times W$ (with basis given by formal symbols $v \otimes w, v \in V, w \in W$), by the subspace D spanned by all elements of the form

$$\begin{aligned}
(v_1 + v_2, w) - (v_1, w) - (v_2, w) \\
(v, w_1 + w_2) - (v, w_1) - (v, w_2) \\
(k \cdot v, w) - (v, k \cdot w)
\end{aligned}$$

for $v, v_1, v_2 \in V, w, w_1, w_2 \in W$, and $k \in K$. When the ground field K is clear it can be omitted from the notation. The elements of $V \otimes W$ are called **tensors**, and the coset $v \otimes w$ of (v, w) in $V \otimes W$ is called a **simple tensor**. We have the relations

$$\begin{aligned}
(v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \\
v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2 \\
(k \cdot v) \otimes w &= v \otimes (k \cdot w) = k \cdot (v \otimes w).
\end{aligned}$$

Definition 1.49. Let V and W be vector spaces over K. A map $\phi\colon V \times W \to K$ is called **K-balanced** if

$$\begin{aligned} \phi(v_1 + v_2, w) &= \phi(v_1, w) + \phi(v_2, w) \\ \phi(v, w_1 + w_2) &= \phi(v, w_1) + \phi(v, w_2) \\ \phi(v, kw) &= \phi(kv, w) \end{aligned}$$

for all $v \in V, w \in W, k \in K$.

Example 1.50. Mapping $V \times W$ to the free K-vector space on $V \times W$, and then passing to the quotient defines a map $\iota\colon V \times W \to V \otimes W$ with $\iota(v, w) = v \otimes w$. From the relations satisfied by the tensor product, we see that the map ι is K-balanced.

Theorem 1.51. *[Universal property of the tensor product] Suppose V, W, and U are vector spaces over the field K. Let $\varphi\colon V \times W \to U$ be a K-balanced map, and let ι be the map above. Then there is a unique linear map $\varphi\colon V \otimes W \to U$ such that φ factors through ι, i.e., $\varphi = \varphi \circ \iota$.*

Proof. The map φ extends by linearity to a linear transformation $\widetilde{\varphi}$ from the free vector space on $V \times W$ to U such that $\widetilde{\varphi}(v, w) = \varphi(v, w)$ for all $v \in V, w \in W$. Since φ is K-balanced, $\widetilde{\varphi}$ maps each of the elements which span the subspace D from the definition of the tensor product to 0. For example,

$$\widetilde{\varphi}((kv, w) - (v, kw)) = \varphi(kv, w) - \varphi(v, kw) = 0.$$

Thus the kernel of $\widetilde{\varphi}$ contains D, and so $\widetilde{\varphi}$ induces a linear map $\varphi\colon V \otimes W \to U$. Then

$$\varphi(v \otimes w) = \widetilde{\varphi}(v, w) = \varphi(v, w)$$

i.e., $\varphi = \varphi \circ \iota$. Note that φ is completely determined by this equation since the elements $v \otimes w$ span $V \otimes W$. □

Proposition 1.52. *Let $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ be bases for V and W. Then $\{e_i \otimes f_j | i \in I, j \in J\}$ is a basis for $V \otimes W$.*

Proof. An elementary tensor in $V \otimes W$ has the form $v \otimes w$. Write $v = \sum_i a_i e_i$ and $w = \sum_j b_j f_j$, where all but finitely many of a_i and b_j are 0. Then

$$m \otimes n = \sum_i a_i e_i \otimes \sum_j b_j f_j = \sum_{i,j} a_i b_j e_i \otimes f_j$$

is a linear combination of the tensors $e_i \otimes f_j$. Since every tensor can be written as a sum of elementary tensors, the elements $e_i \otimes f_j$ span $V \otimes W$.

Now, we must show that this spanning set is linearly independent. Suppose that $\sum_{i,j} c_{ij} e_i \otimes f_j = 0$, where all but finitely many c_{ij} are 0. We want to show that $c_{ij} = 0$ for every $i \in I, j \in J$. Fix two elements $i_0 \in I$ and $j_0 \in J$. To show that $c_{i_0 j_0} = 0$, consider the K-balanced map

$$\begin{aligned} V \times W &\to K \\ (v, w) &\mapsto a_{i_0} b_{j_0} \end{aligned}$$

where $v = \sum_i a_i e_i$ and $w = \sum_j b_j f_j$. By the universal property of tensor products, there is a linear map $f_0\colon V \otimes W \to K$ such that $f_0(v \otimes w) = a_{i_0} b_{j_0}$ on any elementary tensor $v \otimes w$. In particular, $f_0(e_{i_0} \otimes f_{j_0}) = 1$ and $f_0(e_i \otimes f_j) = 0$ for $(i, j) \neq (i_0, j_0)$.

Applying f_0 to our assumption that $\sum_{i,j} c_{ij} e_i \otimes f_j = 0$ in $V \otimes W$ tells us that $c_{i_0 j_0} = 0$ in K. □

Proposition 1.53. *There are natural isomorphisms*

1. $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$
2. $(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$.

Proof. (1.) For each fixed $w \in W$, the mapping $(u, v) \mapsto u \otimes (v \otimes w)$ is K-balanced, so by Theorem 1.51 there is a unique linear map from $U \otimes V$ to $U \otimes (V \otimes W)$ with $u \otimes v \mapsto u \otimes (v \otimes w)$. This shows that the map from $(U \otimes V) \times W$ to $U \otimes (V \otimes W)$ given by $(u \otimes v, w) \mapsto u \otimes (v \otimes w)$ is well defined. This map is also K-balanced, and thus another application of Theorem 1.51 shows that it induces a linear map $(U \otimes V) \otimes W \to U \otimes (V \otimes W)$ such that $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$. In a similar manner, we can construct a map $U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ with $u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$ which is inverse to our first map. This proves the isomorphism.

(2.) The map $(U \oplus V) \times W \to (U \oplus W) \otimes (V \oplus W)$ defined by $((u, v), w) \mapsto (u \otimes w, v \otimes w)$ is clearly K-balanced. Thus it induces a linear map $f\colon (U \oplus V) \otimes W \to (U \otimes W) \oplus (V \otimes W)$ with

$$f((u, v) \otimes w) = (u \otimes w, v \otimes w).$$

In the other direction, we use the K-balanced maps $U \times W \to (U \oplus V) \otimes W$ and $V \times W \to (U \oplus V) \otimes W)$ given by $(u, w) \mapsto (u, 0) \otimes w$ and $(v, w) \mapsto (0, v) \otimes w$ to obtain linear maps from $U \otimes W$ and $V \otimes W$ to $(U \oplus V) \otimes W$. Together these maps give a linear transformation g from the direct sum $(U \otimes W) \oplus (V \otimes W)$ to $(U \oplus V) \otimes W)$ with

$$g(u \otimes w_1, v \otimes w_2) = (u, 0) \otimes w_1 + (0, v) \otimes w_2.$$

It is straightforward to see that f and g are inverse linear transformations, and the isomorphism holds. □

Now let V and W be two representations of G.

Definition 1.54. We can define a representation of G on $V \otimes W$ called the **tensor product representation**. We let

$$\rho_{V \otimes W}(g)\colon V \otimes W \to V \otimes W$$

be the linear map given by

$$\rho_{V \otimes W}(g)\colon a_i \otimes b_j \mapsto \rho_V(g)(a_i) \otimes \rho_W(g)(b_j).$$

Suppose $\rho_V(g)$ is described by the matrix M and $\rho_W(g)$ is described by the matrix N in the given bases $\{a_1, \ldots, a_n\}$ for V and $\{b_1, \ldots, b_m\}$ for W. Then

$$\begin{aligned}\rho_{V \otimes W}(g)\colon a_i \otimes b_t \mapsto & \left(\sum_{j=1}^{n} M_{ji} a_j\right) \otimes \left(\sum_{s=1}^{m} N_{st} b_s\right) \\ &= \sum_{\substack{j \in [1,n] \\ s \in [1,m]}} M_{ji} N_{st} a_j \otimes b_s.\end{aligned}$$

So $\rho_{V \otimes W}$ is described by the $nm \times nm$ matrix $M \otimes N$ whose entries are

$$[M \otimes N]_{js,it} = M_{ji} N_{st}.$$

This matrix has nm rows, and to specify a row we need a pair of numbers (j, s) where $j \in \{1, \dots, n\}$ and $s \in \{1, \dots, m\}$.

Proposition 1.55. *Let V and W be representations of G. Then $V \otimes W$ is isomorphic to $Hom(V^*, W)$.*

Proof. Let $\{a_1, \dots, a_n\}$ be a basis for V, let $\{\alpha_1, \dots, \alpha_n\}$ be the corresponding dual basis for V^*, and let $\{b_1, \dots, b_m\}$ be a basis for W. Then $\mathrm{Hom}(V^*, W)$ has a basis $\{f_{it} | 1 \leq i \leq n, 1 \leq t \leq m\}$ where

$$f_{it}(\alpha_j) = \begin{cases} b_t & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

We obtain an isomorphism of vector spaces between $\mathrm{Hom}(V^*, W)$ and $V \otimes W$ by the map

$$\psi(f_{it}) = a_i \otimes b_t$$

extended to all of $\mathrm{Hom}(V^*, W)$ by linearity. It remains to show that this isomorphism of vector spaces yields an isomorphism of representations, i.e. we need to check that

$$\psi \circ \rho_{\mathrm{Hom}(V^*,W)}(g) = \rho_{V \otimes W}(g) \circ \psi$$

for all $g \in G$. Fix $g \in G$, and let M and N denote the matrices which describe $\rho_V(g)$ and $\rho_W(g)$ in the given bases. By definition,

$$\rho_{\mathrm{Hom}(V^*,W)}(g)(f_{it}) = \rho_W(g) \circ f_{it} \circ \rho_{V^*}(g^{-1}).$$

Now $\rho_{V^*}(g^{-1})$ is given by the matrix M^T in the dual basis (CITE ME!!!), so

$$\rho_{V^*}(g^{-1})(\alpha_k) = \sum_{j=1}^{n} M_{kj}\alpha_j.$$

Then

$$f_{it} \circ \rho_{V^*}(g^{-1})(\alpha_k) = M_{ki} b_t$$

which means that

$$\rho_W(g) \circ f_{it} \circ \rho_{V^*}(g^{-1})(\alpha_k) = M_{ki} \left(\sum_{s=1}^{m} N_{st} b_s \right).$$

Thus, if we write $\rho_W(g) \circ f_{it} \circ \rho_{V^*}(g^{-1})$ in terms of the basis $\{f_{js}\}$, we have

$$\rho_W(g) \circ f_{it} \circ \rho_{V^*}(g^{-1}) = \sum_{\substack{j \in [1,n] \\ s \in [1,m]}} M_{ji} N_{st} f_{js}$$

(since both sides agree on every basis vector α_k). Therefore,

$$\begin{aligned}
\psi \circ \rho_{\mathrm{Hom}(V^*,W)}(g)(f_{it}) &= \psi\left(\rho_W(g) \circ f_{it} \circ \rho_{V^*}(g^{-1})\right) \\
&= \sum_{\substack{j \in [1,n] \\ s \in [1,m]}} M_{ji} N_{st} a_j \otimes b_s \\
&= \rho_{V \otimes W}(g)(a_i \otimes b_t) \quad \text{(by definition of the tensor product representation)} \\
&= \rho_{V \otimes W}(g) \circ \psi(f_{it})
\end{aligned}$$

□

1.10 Character Theory

Definition 1.56. The **character** of a representation $\rho \colon G \to GL(V)$ is the function $\chi_V : G \to \mathbb{C}$ defined by $\chi_V(g) = \mathrm{Tr}(\rho(g))$.

Note. The character is of a representation is not a homorphism in general, since $\mathrm{Tr}(MN) \neq \mathrm{Tr}(M)\mathrm{Tr}(N)$ in general.

Proposition 1.57. *(Basic Properties)*

1. *χ_V is conjugation invariant: $\chi_V(hgh^{-1}) = \chi_V(g)$ for all $g, h \in G$.*
2. *$\chi_V(1) = \dim V$.*
3. *$\chi_V(g^{-1}) = \overline{\chi_V(g)}$ for all $g \in G$.*

Proof. 1. $\chi_V(hgh^{-1} = \mathrm{Tr}(hgh^{-1}) = \mathrm{Tr}(ghh^{-1}) = \mathrm{Tr}(g) = \chi_V(g)$ for any $g, h \in G$.

2. $\chi_V(1) = \mathrm{Tr}(\mathrm{Id}_V) = \dim V$.
3. Since G is finite, we have seen that $\rho(g)$ is a diagonal matrix with roots of unity along the diagonal with the right choice of basis. The inverse of a root of unity is given by its complex conjugate, so under this same basis, $\rho(g)^{-1}$ is clearly given by $\overline{\rho(g)}$. Thus, $\chi_V(g^{-1}) = \mathrm{Tr}(\rho(g^{-1})) = \mathrm{Tr}(\rho(g)^{-1}) = \mathrm{Tr}(\overline{\rho(g)}) = \overline{\mathrm{Tr}(\rho(g))} = \overline{\chi_V(g)}$.

□

Definition 1.58. A **class function** on G is a function on G whose values are invariant by conjugation of elements in G. The value of a class function at an element $g \in G$ depends only on the conjugacy class of g. We may therefore view class functions as functions on the set of conjugacy classes of G.

Note. The character χ_V of a representation V of G is a class function on G.

Proposition 1.59. *Isomorphic representations have the same character.*

Proof. We have seen in Proposition **??** that isomorphic representations can be described by the same set of matrices with the right choice of bases. □

We will see later that the converse is true - if two representations have the same character, then they are isomorphic.

Proposition 1.60. *Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be representations of G with characters χ_V and χ_W.*

1. $\chi_{V\oplus W} = \chi_V + \chi_W$.
2. $\chi_{V\otimes W} = \chi_V \cdot \chi_W$.

Proof. 1. Pick bases for V and W, so that $\rho_V(g)$ and $\rho_W(g)$ are described by matrices M and N. Then $\rho_{V\oplus W}(g)$ is described by the block-diagonal matrix

$$\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

So we have $\mathrm{Tr}(\rho_{V\oplus W}(g)) = \mathrm{Tr}(M) + \mathrm{Tr}(N) = \mathrm{Tr}(\rho_V(g)) + \mathrm{Tr}(\rho_W(g))$.

2. $\rho_{V\otimes W}(g)$ is given by the matrix

$$[M \otimes N]_{js,it} = M_{ji}N_{st}$$

so

$$\begin{aligned} \mathrm{Tr}(M \otimes N) &= \sum_{i,t} [M \otimes N]_{js,it} \\ &= \sum_{i,t} (M_{ii}N_{tt}) \\ &= \mathrm{Tr}(M)\mathrm{Tr}(N). \end{aligned}$$

Thus $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$.

$\square$

Proposition 1.61. *1. Let $\{V_i\}$ be the irreducible representations of G, with d_i the dimension of V_i and χ_i the corresponding irreducible character. Then*

$$\chi_{reg} = d_1\chi_1 + \ldots + d_r\chi_r$$

2.

$$\chi_{reg}(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$$

Definition 1.62. Let $\mathbb{C}^G$ denote the set of all functions from G to $\mathbb{C}$. Then $\mathbb{C}^G$ is a vector space with the sum of two functions defined pointwise and with scalar multiplication defined for $f \in \mathbb{C}^G, \lambda \in \mathbb{C}$ by

$$\begin{aligned} \lambda f \colon G &\to \mathbb{C} \\ g &\mapsto \lambda f(g). \end{aligned}$$

A basis for $\mathbb{C}^G$ is clearly given by the set of functions

$$\{\delta_g | g \in G\}$$

defined by

$$\delta_g \colon h \mapsto \begin{cases} 1 & \text{if } h = g \\ 0 & \text{if } h \neq g \end{cases}$$

Definition 1.63. Let $\varphi, \psi \in \mathbb{C}^G$. We define an **inner product** on $\mathbb{C}^G$ by

$$\langle \varphi | \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}.$$

It is easy to see that $\langle \varphi | \psi \rangle$ is linear in the first variable, conjugate-linear in the second variable (i.e., $\langle \varphi | \lambda \psi \rangle = \overline{\lambda} \langle \varphi | \psi \rangle$) , and that $\langle \varphi | \psi \rangle = \overline{\langle \psi | \varphi \rangle}$. These three properties are the definition of a Hermitian inner product. Note that our basis elements δ_g are orthogonal with respect to this inner product, but not orthonormal since

$$\langle \delta_g | \delta_g \rangle = \begin{cases} \frac{1}{|G|} & \text{if } h = g \\ 0 & \text{if } h \neq g \end{cases}.$$

The characters of G are elements of $\mathbb{C}^G$, so we can evaluate this inner product on pairs of characters. The answer turns out to be very useful, but before we can begin the proof we require two quick lemmas:

Lemma 1.64. *Let V be a vector space with subspace $U \subset V$, and let $\pi \colon V \to V$ be a projection onto U. Then*

$$Tr(\pi) = dim\ U.$$

Proof. Recall that $V = U \oplus \text{Ker}\ (\pi)$. (FROM WHENCE IS THIS RECOLLECTION?? MASCHKE??) If we fix bases for U and Ker (π), which together give a basis for V, then π is given by the block-diagonal matrix

$$\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{0} \end{bmatrix}$$

where dim U is the size of the upper left block and dim Ker (π) is the size of the bottom right block. So $\text{Tr}(\pi) = \text{Tr}(\mathbf{1}_U) = \dim U$. □

Lemma 1.65. *Let $\rho \colon G \to GL(V)$ be any representation. Consider the linear map*

$$\begin{aligned} \Psi \colon V &\to V \\ x &\mapsto \frac{1}{|G|} \sum_{g \in G} \rho(g)(x). \end{aligned}$$

Then Ψ is a projection from V onto the invariant subspace V^G.

Proof. We need to check that $\Psi(x) \in V^G$ for all $x \in V$. For any $h \in G$,

$$\begin{aligned} \rho(h)(\Psi(x)) &= \frac{1}{|G|} \sum_{g \in G} \rho(h)\rho(g)(x) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho(hg)(x) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho(g)(x) \quad \text{(by relabelling } g \mapsto h^{-1}g) \\ &= \Psi(x). \end{aligned}$$

Thus Ψ is a linear map $V \to V^G$. Next we need to check that Ψ is a projection. Let $x \in V^G$. Then

$$\begin{aligned}\Psi(x) &= \frac{1}{|G|}\sum_{g\in G}\rho(g)(x)\\ &= \frac{1}{|G|}\sum_{g\in G}(x) = x.\end{aligned}$$

□

Theorem 1.66. *Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be representations of G, and let χ_V, χ_W be their characters. Then*

$$\langle \chi_W | \chi_V \rangle = \mathit{dim}\ \mathit{Hom}_G(V, W).$$

In particular, the inner product of two characters is always a non-negative integer. (Whereas in general, the inner product of two arbitrary functions can be any complex number.)

Proof. We have seen in Proposition 1.33 that

$$\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}(V, W)$$

as the invariant subrepresentation, and by the previous lemma we have a projection

$$\begin{aligned}\Psi\colon \mathrm{Hom}(V, W) &\to \mathrm{Hom}(V, W)\\ f &\mapsto \frac{1}{|G|}\sum_{g\in G}\rho_{\mathrm{Hom}(V,W)}(g)(f).\end{aligned}$$

We claim that

$$\mathrm{Tr}(\Psi) = \langle \chi_W | \chi_V \rangle.$$

Once this claim is established, then Lemma 1.64 will prove the theorem. We proceed by calculating $\mathrm{Tr}(\Psi)$. Fix bases $\{a_1, \ldots, a_n\}$ for V and $\{b_1, \ldots, b_m\}$ for W. Then $\mathrm{Hom}(V, W)$ has an associated basis

$$\{f_{ji} | 1 \le i \le n, 1 \le j \le m\}$$

where

$$f_{ji}(a_i) = \begin{cases} b_j & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

We may calculate $\mathrm{Tr}(\Psi)$ as follows: For each i, j, compute the expression of $\Psi(f_{ji})$ in this basis, and take the coefficient of the basis element f_{ji}. This is a diagonal entry in the matrix for Ψ. Summing these values over all i and j will gives us $\mathrm{Tr}(\Psi)$.

Let $\widetilde{\rho_V}, \widetilde{\rho_W}$ be the matrix representations obtained by writing ρ_V and ρ_W in the given bases. We know that

$$\mathrm{Hom}(V, W) = V^* \otimes W$$

so if we write $\rho_{\mathrm{Hom}(V,W)}$ in the basis $\{f_{ji}\}$ then we get the tensor product of $\widetilde{\rho_{V^*}}$ and $\widetilde{\rho_W}$. Thus

$$\begin{aligned}\rho_{\mathrm{Hom}(V,W)}(g)(f_{ji}) &= \rho_W(g)\circ f_{ji}\circ \rho_V(g^{-1})\\ &= \sum_{\substack{k\in[1,n]\\ t\in[1,m]}} \widetilde{\rho_V}(g^{-1})_{ik}\widetilde{\rho_W}(g)_{tj} f_{kt}. \quad \text{Show another step?}\end{aligned}$$

Then

$$\begin{aligned}\Psi(f_{ji}) &= \frac{1}{|G|}\sum_{g\in G}\rho_{\mathrm{Hom}(V,W)}(g)(f)\\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{k\in[1,n]\\ t\in[1,m]}} \widetilde{\rho_V}(g^{-1})_{ik}\widetilde{\rho_W}(g)_{tj} f_{kt}.\end{aligned}$$

The coefficient of f_{ji} in this expression is

$$\frac{1}{|G|}\sum_{g\in G}\widetilde{\rho_V}(g^{-1})_{ii}\widetilde{\rho_W}(g)_{jj}.$$

Therefore

$$\begin{aligned}\mathrm{Tr}(\Psi) &= \sum_{\substack{k\in[1,n]\\ t\in[1,m]}}\frac{1}{|G|}\sum_{g\in G}\widetilde{\rho_V}(g^{-1})_{ii}\widetilde{\rho_W}(g)_{jj}\\ &= \frac{1}{|G|}\sum_{g\in G}\left(\sum_{i=1}^{n}\widetilde{\rho_V}(g^{-1})_{ii}\right)\left(\sum_{j=1}^{m}\widetilde{\rho_W}(g)_{jj}\right)\\ &= \frac{1}{|G|}\sum_{g\in G}\chi_V(g^{-1})\chi_W(g)\\ &= \frac{1}{|G|}\sum_{g\in G}\chi_W(g)\overline{\chi_V}(g) \quad \text{(by Proposition 3)}\\ &= \langle\chi_W|\chi_V\rangle.\end{aligned}$$

□

Corollary 1.67. *Let $\chi_1,\dots,\chi_r$ be the irreducible characters of G. Then*

$$\langle\chi_i|\chi_j\rangle = \begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$$

Proof. Let χ_i and χ_j be the characters of the irreducible representations U_i, U_j. Then by Proposition 1.43,

$$\langle\chi_i|\chi_j\rangle = \dim \mathrm{Hom}_G(U_i,U_j) = \begin{cases}1 & \text{if } U_i, U_j \text{ are isomorphic}\\ 0 & \text{if } U_i, U_j \text{ are not isomorphic.}\end{cases}$$

□

Corollary 1.68. *Let χ be any character of G. Then χ is irreducible if and only if*

$$\langle \chi | \chi \rangle = 1$$

Proof. Write χ as a linear combination of irreducible characters

$$\chi = m_1\chi_1 + \ldots + m_k\chi_k$$

where each m_i is a non-negative integer. Then by Lemma 1.67,

$$\begin{aligned} \langle \chi | \chi \rangle &= \sum_{i,j \in [1,k]} m_i m_j \langle \chi_i | \chi_j \rangle \\ &= m_1^2 + \ldots + m_k^2. \end{aligned}$$

So $\langle \chi | \chi \rangle = 1$ if and only if exactly one of the $m_i = 1$ and the rest are 0. □

Corollary 1.69. *Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be representations of G. Then V and W are isomorphic if and only if $\chi_V = \chi_W$.*

Proof. We have already seen that isomorphic representations have the same character by Proposition . On the other hand, suppose $\chi_V = \chi_W$. Let $U_1, \ldots, U_r$ be the irreducible representations of G, and let $\chi_1, \ldots, \chi_r$ be their characters. We can write

$$V = U_1^{m_1} \oplus \ldots \oplus U_r^{m_r}$$

for some non-negative integers $m_1, \ldots, m_r$, and

$$W = U_1^{l_1} \oplus \ldots \oplus U_r^{l_r}$$

for some non-negative integers $l_1, \ldots, l_r$. So

$$\chi_V = m_1\chi_1 + \ldots + m_r\chi_r$$

and

$$\chi_W = l_1\chi_1 + \ldots + l_r\chi_r$$

. Thus we have

$$m_i = \langle \chi_V | \chi_i \rangle = \langle \chi_W | \chi_i \rangle = l_i$$

for all $i \in \{1, \ldots, r\}$ since $\chi_V = \chi_W$. This proves V and W are isomorphic. □

[6] [4] [1] [2] [5] [3]

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