

Character Tables for Representations of Finite Groups

Jared Stewart

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Motivation

Groups arise naturally as sets of symmetries of some object which are closed under composition and taking inverses. For example,

- 1 The **symmetric group** of degree n , S_n , is the group of all symmetries of the set $\{1, \dots, n\}$.
- 2 The **dihedral group** of order $2n$, D_n , is the group of all symmetries of the regular n -gon in the plane.

In these two examples, S_n acts on the set $\{1, \dots, n\}$ and D_n acts on the regular n -gon in a natural manner. One may wonder more generally: Given an abstract group G , which objects X does G act on? This is the basic question of representation theory, which attempts to classify all such X up to isomorphism.

Group Actions

Definition

A **group action** of a group G on a set X is a map $\rho: G \times X \rightarrow X$ (written as $g \cdot x$, for all $g \in G$ and $x \in X$) that satisfies the following two axioms:

$$1 \cdot x = x \qquad \qquad \qquad \forall x \in X \qquad \qquad \qquad (1)$$

$$(gh) \cdot x = g \cdot (h \cdot x) \qquad \qquad \forall g, h \in G, x \in X \qquad \qquad (2)$$

The Definition of a Representation

Definition

Let G be a group, let F be a field, and let V be a vector space over F . A **linear representation** of G is an action of G on V that preserves the linear structure of V , i.e. an action of G on V such that

$$g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2 \quad \forall g \in G, v_1, v_2 \in V \quad (3)$$

$$g \cdot (kv) = k(g \cdot v) \quad \forall g \in G, v \in V, k \in F \quad (4)$$

Definition (Alternative definition)

Let G be a group, let F be a field, and let V be a vector space over F . A **linear representation** of G is any group homomorphism

$$\rho: G \rightarrow GL(V).$$

Proposition

The two definitions we have given of a linear representation are equivalent.

Proof.

- (\rightarrow) Suppose that we have a homomorphism $\rho: G \rightarrow GL(V)$. We can obtain a linear action of G on V by defining $g \cdot v = \rho(g)(v)$.
- (\leftarrow) Suppose that we have a linear action of G on V . We obtain a homomorphism $\rho: G \rightarrow GL(V)$ by defining $\rho(g)(v) = g \cdot v$.



The Dimension of a Representation

Definition

Let $\rho: G \rightarrow GL(V)$ be a representation of G . The **dimension** of the representation is the dimension of the vector space V .

Examples of Representations

Example

Let V be an n -dimensional vector space. The map $\rho: G \rightarrow GL(V)$ defined by $\rho(g) = \text{Id}_V$ for all $g \in G$ is a representation of G called the **trivial representation** of dimension n .

Examples of Representations

Example

If G is a finite group that acts on a finite set X , and F is any field, then there is an associated **permutation representation** on the vector space V over F with basis $\{e_x : x \in X\}$. We let G act on the basis elements by the permutation $g \cdot e_x = e_{gx}$ for all $x \in X$ and $g \in G$. This representation has dimension $|X|$.

Examples of Representations

Example

A special case of a permutation representation is that when a finite group acts on itself by left multiplication. We take the vector space V_{reg} which has a basis given by the formal symbols $\{e_g | g \in G\}$, and let $h \in G$ act by

$$\rho_{\text{reg}}(h)(e_g) = e_{hg}.$$

This representation is called the **regular representation** of G , and has dimension $|G|$.

Examples of Representations

Example

For any symmetric group S_n , the **alternating representation** on \mathbb{C} is given by the map

$$\begin{aligned}\rho: S_n &\rightarrow GL(\mathbb{C}) = \mathbb{C}^\times \\ \sigma &\mapsto \operatorname{sgn}(\sigma).\end{aligned}$$

More generally, for any group G with a subgroup H of index 2, we can define an **alternating representation** $\rho: G \rightarrow GL(\mathbb{C})$ by letting $\rho(g) = 1$ if $g \in H$ and $\rho(g) = -1$ if $g \notin H$. (We recover our original example by taking $G = S_n$ and $H = A_n$.)

G -linear maps

Definition

A **homomorphism** between two representations $\rho_1: G \rightarrow GL(V)$ and $\rho_2: G \rightarrow GL(W)$ is a linear map $\psi: V \rightarrow W$ that intertwines with the action of G , i.e. such that

$$\psi \circ \rho_1(g) = \rho_2(g) \circ \psi \quad \forall g \in G.$$

In this case, we also refer to ψ as a **G -linear map**.

Definition

An **isomorphism** of representations is a G -linear map that is also invertible.

Representations as matrices

Example

Given any representation (ρ, V) , where V is a vector space of dimension n over the field K , we can fix a basis for V to obtain an isomorphism of vector spaces $\psi: V \rightarrow K^n$. This yields a representation ϕ of G on K^n by defining

$$\phi(g) = \psi \circ \rho(g) \circ \psi^{-1}$$

for all $g \in G$. This representation is isomorphic to our original representation (ρ, V) . In particular, we can always choose to view complex n -dimensional representations of G as representations on \mathbb{C}^n , where each $\phi(g)$ is given by an $n \times n$ matrix with entries in \mathbb{C} .

Representations as matrices

Example

Let $G = \{(1), (123), (132)\} \subset S_3$. Let $V = \mathbb{C}^3$. Then G acts on the standard basis by $g \cdot e_i = e_{gi}$. Thus, the permutation representation of G (with respect to the standard basis) is given by:

$$\rho((1)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rho((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\rho((132)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Example

Let $G = C_2 \times C_2 = \langle \sigma, \tau | \sigma^2 = \tau^2 = e, \sigma\tau = \tau\sigma \rangle$ be the Klein four-group. Let V be the vector space with basis $\{b_e, b_\sigma, b_\tau, b_{\sigma\tau}\}$. Left multiplication by σ gives a permutation

$$b_e \mapsto b_\sigma$$

$$b_\sigma \mapsto b_e$$

$$b_\tau \mapsto b_{\sigma\tau}$$

$$b_{\sigma\tau} \mapsto b_\tau.$$

We can similarly compute $\rho_{\text{reg}}(\tau)$. Thus, in our basis, the regular representation $\rho_{\text{reg}}: G \rightarrow GL(V)$ is given by the matrices

$$\rho_{\text{reg}}(\sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \rho_{\text{reg}}(\tau) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Example

Let $G = D_4 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square.

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$$\rho(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\rho(\sigma) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\rho(\sigma\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\rho(\sigma^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\rho(\sigma^2\tau) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho(\sigma^3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\rho(\sigma^3\tau) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Subrepresentations

Definition

A **subrepresentation** of V is a G -invariant subspace $W \subseteq V$; that is, a subspace $W \subseteq V$ with the property that $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$. Note that W itself is a representation of G under the action $\rho(g) \upharpoonright_W$.

Representations of C^2

Example

Let $G = C_2 = \langle \tau | \tau^2 = e \rangle$ be the cyclic group of order 2. The regular representation of G written in the standard basis is given by

$$\rho_{\text{reg}}(\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $\rho_{\text{reg}}(e) = \text{Id}_2$. Let ρ_{sgn} be the alternating representation of G on \mathbb{C} , i.e.

$$\rho_{\text{sgn}}: G \rightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$$

$$\tau \mapsto -1$$

$$e \mapsto 1.$$

Representations of C^2

Example (Cont.)

Let $f: \mathbb{C}^2 \rightarrow \mathbb{C}$ be the linear map represented by the matrix $\begin{bmatrix} 1 & -1 \end{bmatrix}$. Then for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$, we have

$$\begin{aligned} f \circ \rho_{\text{reg}}(\tau)(x) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \rho_{\text{sgn}}(\tau) \circ f(x). \end{aligned}$$

Also note that $f \circ \rho_{\text{reg}}(e) = \rho_{\text{sgn}}(e) \circ f$. Thus f is a G -linear map from ρ_{reg} to ρ_{sgn} (i.e. a homomorphism of representations).

Representations of C^2

Example (Cont.)

Now let W be the subspace of \mathbb{C}^2 spanned by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Then

$$\rho_{\text{reg}}(\tau) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $\rho_{\text{reg}}(e) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so W is a G -invariant subspace, i.e. W is a subrepresentation of ρ_{reg} . Note that W is precisely equal to the kernel of the map f , and that W is isomorphic to the 1-dimensional trivial representation of G .

Example

We can generalize the G -invariant subspace from the previous example. Suppose we have a representation $\rho: G \rightarrow GL_n(\mathbb{C})$. If we can find a vector $x \in \mathbb{C}^n$ which is an eigenvector for every matrix $\rho(g)$, $g \in G$, i.e. an $x \in \mathbb{C}^n$ such that

$$\rho(g)(x) = \lambda_g(x) \quad \forall g \in G$$

for some eigenvalues $\lambda_g \in \mathbb{C}$, then the span of x is a 1-dimensional G -invariant subspace of \mathbb{C}^n . It is isomorphic to the 1-dimensional representation

$$\begin{aligned} \rho_2: G &\rightarrow GL_1(\mathbb{C}) \\ g &\mapsto \lambda_g. \end{aligned}$$

Proposition

Let $f: V \rightarrow W$ be a homomorphism of representations of G . Then $\text{Ker}(f)$ is a subrepresentation of V and $\text{Im}(f)$ is a subrepresentation of W .

Proof.

- Let $x \in \text{Ker}(f)$. Then $0 = g0 = gf(x) = f(gx)$ for every $g \in G$. So $gx \in \text{Ker}(f)$ and $\text{Ker}(f)$ is G -invariant.
- Now let $w \in \text{Im}(f)$. There exists $v \in V$ such that $w = f(v)$, so $gw = gf(v) = f(gv)$ for every $g \in G$. Thus $gw \in \text{Im}(f)$, and $\text{Im}(f)$ is G -invariant.



The direct sum of representations

Note

We know from linear algebra that given two vector spaces V and W , we can form the **direct sum** $V \oplus W$ consisting of ordered pairs (v, w) where $v \in V, w \in W$.

The direct sum of representations

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Definition

Let V and W be representations of G . Then $V \oplus W$ admits a natural representation of G , called the **direct sum representation** of V and W , which we define by

$$\begin{aligned}\rho_{V \oplus W}: G &\rightarrow GL(V \oplus W) \\ \rho_{V \oplus W}(g): (x, y) &\mapsto (\rho_V(g)(x), \rho_W(g)(y)).\end{aligned}$$

Irreducible representations and complete reducibility

Definition

A representation is said to be **irreducible** if it has no subrepresentations other than the trivial subrepresentations $0 \subset V$ and $V \subset V$. A representation is called **completely reducible** if it decomposes into a direct sum of irreducible representations.

Note

- 1 Any 1-dimensional representation V has no subspaces other than 0 and V itself, and is thus irreducible.
- 2 Any irreducible representation is, in particular, completely reducible.

Example (A 2-dimensional irreducible representation)

Let $G = D_3 = \langle \sigma, \tau | \sigma^3 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$. (Note that $D_3 \cong S_3$). Consider the regular triangle centered at the origin with vertices

$$(1, 0), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

We can let σ act as rotation by $\frac{2\pi}{3}$ and let τ act as reflection over the x -axis to obtain an action of G on \mathbb{C}^2 given (under the standard basis) by the matrices

$$\rho(\sigma) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$
$$\rho(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Example (A 2-dimensional irreducible representation cont.)

Suppose ρ has a non-trivial subrepresentation W . We must have $\dim W = 1$. Since W is invariant under the action of both $\rho(\sigma)$ and $\rho(\tau)$, there must be some mutual eigenvector for $\rho(\sigma)$ and $\rho(\tau)$ that spans W . The eigenvectors of $\rho(\sigma)$ are

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} \quad (\lambda_1 = e^{\frac{2\pi i}{3}}) \quad \text{and} \quad \begin{bmatrix} 1 \\ i \end{bmatrix} \quad (\lambda_2 = e^{-\frac{2\pi i}{3}}).$$

The eigenvectors of $\rho(\tau)$ are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\lambda_1 = 1) \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\lambda_2 = -1).$$

Thus we see that there is no such W , and our representation is irreducible.

Representations of finite abelian groups

Theorem

If A_1, A_2, \dots, A_r are linear operators on V and each A_i is diagonalizable, then $\{A_i\}$ are simultaneously diagonalizable if and only if they commute.

Representations of finite abelian groups

Theorem

Every complex representation of a finite abelian group is completely reducible into irreducible representations of dimension 1.

Proof.

Take an arbitrary element $g \in G$. Since G is finite, we can find an integer n such that $g^n = 1$ and $\rho(g)^n = Id$. The minimal polynomial of $\rho(g)$ divides $x^n - 1$, which has n distinct roots over \mathbb{C} . So the minimal polynomial of $\rho(g)$ factors into linear factors only over \mathbb{C} , i.e. $\rho(g)$ is diagonalizable. We conclude that each $\rho(g)$ is (separately) diagonalizable since $g \in G$ was arbitrary.

Now, given any two elements $g_1, g_2 \in G$ we have

$\rho(g_1)\rho(g_2) = \rho(g_2)\rho(g_1)$. Since the matrices $\{\rho(g)\}$ commute, $\{\rho(g)\}$ are simultaneously diagonalizable, say with basis

$\{e_1, \dots, e_k\}$. Then we have $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \dots \oplus \mathbb{C}e_n$, with each subspace $\mathbb{C}e_1$ invariant under the action of G . □

Definition

Let W be a subspace of V . A **linear projection** V onto W is a linear map $f: V \rightarrow W$ such that $f|_W = \text{Id}_W$. If W is a subrepresentation of V and the map f is G -invariant, then we say that f is a **G -linear projection**.

Lemma

Let $\rho: G \rightarrow GL(V)$ be a representation, and $W \subset V$ be a subrepresentation. Suppose we have a G -linear projection

$$f: V \rightarrow W.$$

Then $\text{Ker}(f)$ is a complementary subrepresentation to W , i.e. $\text{Ker}(f)$ is a G -invariant subspace of V such that

$$V = \text{Ker}(f) \oplus W$$

Maschke's Theorem

Theorem (Maschke's Theorem)

Let G be a finite group and let F be a field such that $\text{char}(F) \nmid |G|$. If V is any finite dimensional representation of G over F , and $W \subset V$ is a subrepresentation of V , then there exists a complementary subrepresentation $U \subset V$ to W , i.e. there is a G -invariant subspace $U \subset V$ such that

$$V = W \oplus U.$$

Maschke's Theorem

Proof.

It will suffice to find a G -linear projection from V onto W . Fix a basis $\{b_1, \dots, b_m\}$ for W and extend it to a basis $\{b_1, \dots, b_m, b_{m+1}, \dots, b_n\}$ for V . Let $U = \langle b_{m+1}, \dots, b_n \rangle$. Then U is certainly a complementary subspace to W , and we have a natural projection $f: W \oplus U \rightarrow W$ of V onto W with kernel U . There is no reason to think that f should be G -linear, but we can fix this by averaging over G . Define $\tilde{f}: V \rightarrow V$ by

$$\tilde{f}(x) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}))(x).$$

We claim that \tilde{f} is a G -linear projection from V onto W .

Maschke's Theorem

Proof.

First we check that $\text{Im}(\tilde{f}) \subset W$. If $x \in V$ and $g \in G$, then

$$f(\rho(g^{-1})(x)) \in W$$

and so

$$\rho(g)(f(\rho(g^{-1})(x))) \in W$$

since W is G -invariant. Thus

$$\tilde{f}(x) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}))(x) \in W.$$

Maschke's Theorem

Proof.

Next we check that $\tilde{f}|_W = \text{Id}_W$. Let $y \in W$. For any $g \in G$, we know that $\rho(g^{-1})(y)$ is also in W , so $f(\rho(g^{-1})(y)) = \rho(g^{-1})(y)$. Then

$$\begin{aligned}\tilde{f}(y) &= \frac{1}{|G|} \sum_{g \in G} \rho(g)(f(\rho(g^{-1})(y))) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho(g)(\rho(g^{-1})(y)) \\ &= \frac{1}{|G|} \sum_{g \in G} (y) = \frac{|G|y}{|G|} = y\end{aligned}$$

so indeed \tilde{f} is a linear projection of V onto W .

Maschke's Theorem

Proof.

Finally, we check that \tilde{f} is G -linear. If $x \in V$ and $h \in G$, then

$$\begin{aligned}(\tilde{f} \circ \rho(h))(x) &= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}) \circ \rho(h))(x) \\&= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}h))(x) \\&= \frac{1}{|G|} \sum_{g \in G} (\rho(hg) \circ f \circ \rho(g^{-1}))(x) \quad (g \mapsto hg) \\&= (\rho(h) \circ \tilde{f})(x).\end{aligned}$$

