

UNIVERSITY OF MISSOURI

MASTER'S PROJECT

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# A Survey on Character Tables for Representations of Finite Groups

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# Chapter 1

## Basic Notions of Representation Theory

### 1.1 Group Actions

**Definition 1.1.** A *(left) group action* of a group  $G$  on a set  $X$  is a map  $\rho: G \times X \rightarrow X$  (written as  $g \cdot a$ , for all  $g \in G$  and  $a \in A$ ) that satisfies the following two axioms:

$$1 \cdot x = x \quad \forall x \in X \quad (1.1.1)$$

$$(gh) \cdot x = g \cdot (h \cdot x) \quad \forall g, h \in G, x \in X \quad (1.1.2)$$

*Note.* We could likewise define the concept of a *right* group action, where the set elements would be multiplied by group elements on the right instead of on the left. Throughout we shall use the term *group action* to mean a *left* group action.

**Proposition 1.2.** Let  $G$  act on the set  $X$ . For any fixed  $g \in G$ , the map  $\sigma_g$  from  $X$  into  $X$  defined by  $\sigma_g(x) = g \cdot x$  is a permutation of the set  $X$ . That is,  $\sigma_g \in S_X$ .

*Proof.* We show that  $\sigma_g$  is a permutation of  $X$  by finding a two-sided inverse map, namely  $\sigma_{g^{-1}}$ . Observe that for any  $x \in X$ , we have

$$\begin{aligned} (\sigma_{g^{-1}} \circ \sigma_g)(x) &= \sigma_{g^{-1}}(\sigma_g(x)) \\ &= g^{-1} \cdot (g \cdot x) && \text{(by definition of } \sigma_g \text{ and } \sigma_{g^{-1}}) \\ &= (g^{-1}g) \cdot x && \text{(by axiom 1.1.1 of an action)} \\ &= 1 \cdot x \\ &= x && \text{(by axiom 1.1.2 of an action).} \end{aligned}$$

Thus  $\sigma_{g^{-1}} \circ \sigma_g$  is the identity map on  $X$ . We can reverse the roles of  $g$  and  $g^{-1}$  to see that  $\sigma_g \circ \sigma_{g^{-1}}$  is also the identity map on  $X$ . Having a two-sided inverse, we conclude that  $\sigma_g$  is a permutation of  $X$ .  $\square$

**Proposition 1.3.** Let  $G$  act on the set  $X$ . The map from  $G$  into the symmetric group  $S_X$  defined by  $g \mapsto \sigma_g(x) = g \cdot x$  is a group homomorphism.

*Proof.* Define the map  $\rho: G \rightarrow S_X$  by  $\rho(g) = \sigma_g$ . We have seen from Proposition 1.2 that  $\sigma_g$  is indeed an element of  $S_X$ . It remains to show that  $\rho(g_1g_2) = \rho(g_1) \circ \rho(g_2)$  for any  $g_1, g_2 \in G$ . Observe that

$$\begin{aligned}
\rho(g_1 g_2)(x) &= \sigma_{g_1 g_2}(x) && \text{(by definition of } \rho) \\
&= (g_1 g_2) \cdot x && \text{(by definition of } \sigma_{g_1 g_2}) \\
&= g_1 \cdot (g_2 \cdot x) && \text{(by axiom 1.1.1 of an action)} \\
&= \sigma_{g_1}(\sigma_{g_2}(x)) && \text{(by definition of } \sigma_{g_1} \text{ and } \sigma_{g_2}) \\
&= \rho(g_1)(\rho(g_2)(x)) && \text{(by definition of } \rho) \\
&= (\rho(g_1) \circ \rho(g_2))(x) && \text{(by definition of function composition).}
\end{aligned}$$

Since the values of  $\rho(g_1 g_2)$  and  $\rho(g_1) \circ \rho(g_2)$  agree on every element  $x \in X$ , these two permutations are equal. We conclude that  $\rho$  is a homomorphism, since  $g_1$  and  $g_2$  were arbitrary elements of  $G$ .  $\square$

**Proposition 1.4.** Any homomorphism  $\psi$  from the group  $G$  into the symmetric group  $S_X$  on a set  $X$  gives rise to an action of  $G$  on  $X$ , defined by taking  $g \cdot x = \psi(g)(x)$ .

*Proof.* Suppose that we have a homomorphism  $\psi$  from  $G$  into  $S_X$ . We can define a map from  $G \times X$  to  $X$  by  $g \cdot x = \psi(g)(x)$ . We verify that this map satisfies the definition of a group action of  $G$  on  $X$ :

$$\text{(axiom 1.1.1)} \quad 1 \cdot x = \psi(1)(x) = \text{id}_X(x) = x$$

$$\text{(axiom 1.1.2)} \quad (gh) \cdot x = \psi(gh)(x) = (\psi(g)\psi(h))(x) = \psi(g)(\psi(h)(x)) = g \cdot (h \cdot x) \quad \square$$

**Corollary 1.5.** The actions of  $G$  on the set  $X$  are in bijective correspondence with the homomorphisms from  $G$  into the symmetric group  $S_X$ .

*Proof.* By Proposition 1.3, any action of  $G$  on  $X$  yields a homomorphism from  $G$  into  $S_X$ . Conversely, any homomorphism from  $G$  into  $S_X$  establishes an action of  $G$  on  $X$  by Proposition 1.4.  $\square$

## 1.2 The Definition of a Representation

**Definition 1.6.** Let  $G$  be a group. A **representation** of  $G$  is a homomorphism  $\rho: G \rightarrow GL_n(\mathbb{C})$  for some positive integer  $n$ .

**Definition 1.7.** Two representations  $\rho_1: G \rightarrow GL_n(\mathbb{C})$  and  $\rho_2: G \rightarrow GL_n(\mathbb{C})$  of  $G$  are **equivalent** if there exists  $P \in GL_n(\mathbb{C})$  such that  $\rho_2 = P^{-1} \rho_1 P$ .

Equivalent representations are fundamentally "the same" in some sense, but to make this precise we need to shift our thinking to linear maps instead of matrices.

**Definition 1.8.** Let  $G$  be a group, let  $F$  be a field, and let  $V$  be a vector space over  $F$ . A **linear representation** of  $G$  is any group homomorphism  $\rho: G \rightarrow GL(V)$ . If we fix a basis for  $V$ , we get a representation in the previous sense.

**Definition 1.9** (Alternative definition). Let  $G$  be a group, let  $F$  be a field, and let  $V$  be a vector space over  $F$ . A **linear representation** of  $G$  is an action of  $G$  on  $V$  which preserves the linear structure of  $V$ , i.e. an action of  $G$  on  $V$  such that

$$g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2 \quad \forall g \in G, v_1, v_2 \in V \quad (1.9.1)$$

$$g \cdot (kv) = k(g \cdot v) \quad \forall g \in G, v \in V, k \in F \quad (1.9.2)$$



*Note.* Unless otherwise specified, we use *representation* to mean *finite-dimensional complex representation*.

**Proposition 1.10.** *The definitions of a linear representation given in 1.8 and 1.9 above are equivalent.*

*Proof.* ( $\rightarrow$ ) Suppose that we have a homomorphism  $\rho: G \rightarrow GL(V)$ . Note that  $GL(V)$  is a subgroup of the symmetric group  $S_V$  on  $V$ , so we can apply Proposition 1.4 to obtain an action of  $G$  on  $V$  by  $g \cdot v = \rho(g)(v)$ . We check that this action preserves the linear structure of  $V$ .

**1.9.1** For any  $g \in G, v_1, v_2 \in V$  we have  $g \cdot (v_1 + v_2) = \rho(g)(v_1 + v_2) = \rho(g)(v_1) + \rho(g)(v_2) = g \cdot v_1 + g \cdot v_2$ .

**1.9.2** For any  $g \in G, v \in V, k \in F$  we have  $g \cdot (kv) = \rho(g)(kv) = k(\rho(g)(v)) = k(g \cdot v)$ .

( $\leftarrow$ ) Suppose that we have an action of  $G$  on  $V$  which preserves the linear structure of  $V$  in the sense of Definition 1.9. We can apply Proposition 1.3 to obtain a homomorphism  $\rho: G \rightarrow S_V$  given by  $\rho(g) = \sigma_g$  where  $\sigma_g(v) = g \cdot v$ . It remains to show that the image  $\rho(G)$  of  $G$  under  $\rho$  is actually contained in  $GL(V)$ , i.e. that for each  $g \in G$  the map  $\sigma_g$  is linear. Fix an element  $g \in G$ . For any  $k \in F$  and  $v \in V$ , we have

$$\begin{aligned} \sigma_g(kv) &= g \cdot (kv) && \text{(by definition of } \sigma_g) \\ &= k(g \cdot v) && \text{(by property 1.9.1)} \\ &= k(\sigma_g(v)) && \text{(by definition of } \sigma_g). \end{aligned}$$

Also, for any  $v_1, v_2 \in V$  we have

$$\begin{aligned} \sigma_g(v_1 + v_2) &= g \cdot (v_1 + v_2) && \text{(by definition of } \sigma_g) \\ &= g \cdot v_1 + g \cdot v_2 && \text{(by property 1.9.2)} \\ &= \sigma_g(v_1) + \sigma_g(v_2) && \text{(by definition of } \sigma_g). \end{aligned}$$

Thus  $\sigma_g$  is linear, and  $\rho(G) \subset GL(V)$  proves that we have a homomorphism  $\rho: G \rightarrow GL(V)$ . □

**Definition 1.11.** Let  $G$  be a group, let  $F$  be a field, let  $V$  be a vector space over  $F$ , and let  $\rho: G \rightarrow GL(V)$  be a representation of  $G$ . The **dimension** of the representation is the dimension of  $V$  over  $F$ .

**Example 1.12.** 1. Let  $V$  be a 1-dimensional vector space over the field  $F$ . The map  $\rho: G \rightarrow GL(V)$  defined by  $\rho(g) = 1$  for all  $g \in G$  is a representation called the *trivial representation* of  $G$ . The trivial representation has dimension 1.

2. If  $G$  is a finite group that acts on a finite set  $X$ , and  $F$  is any field, then there is an associated *permutation representation* on the vector space  $V$  over  $F$  with basis  $\{e_x: x \in X\}$ . We let  $G$  act on the basis elements by the permutation  $g \cdot e_x = e_{gx}$  for all  $x \in X$  and  $g \in G$ . This representation has dimension  $|X|$ .
3. A fundamental special case of a permutation representation that we shall return to later on is that when a finite group acts on itself by left multiplication. In this case, the elements of  $G$  form a basis for  $V$ , and each  $g \in G$  permutes the basis

elements by  $g \cdot g_i = gg_i$ . This representation is called the *regular representation* of  $G$  and has dimension  $|G|$ . We shall see later that this representation encodes information about all other representations of  $G$ .

4. For any symmetric group  $S_n$ , the *alternating representation* on  $V = \mathbb{C}$  is given by the map  $\rho: S_n \rightarrow GL(\mathbb{C}) = \mathbb{C}^\times$  defined by  $\rho(\sigma) = \text{sgn}(\sigma)$ . More generally, for any group  $G$  with a subgroup  $H$  of index 2, we can define an *alternating representation*  $\rho: G \rightarrow GL(\mathbb{C})$  by letting  $\rho(g) = 1$  if  $g \in H$  and  $\rho(g) = -1$  if  $g \notin H$ . (We recover our original example by taking  $G = S_n$  and  $H = A_n$ .)

**Definition 1.13.** A **homomorphism** between two representations  $\rho_1: G \rightarrow GL(V)$  and  $\rho_2: G \rightarrow GL(W)$  is a linear map  $\psi: V \rightarrow W$  that intertwines with (respects) the  $G$ -action, i.e. a map  $\psi$  such that

$$\psi(\rho_1(g)(v)) = \rho_2(g)(\psi(v)) \quad \forall v \in V, g \in G$$

An **isomorphism** of representations is a homomorphism of representations that is also an invertible map.

*Note.* If we have representations  $(\rho_1, V)$  and  $(\rho_2, W)$  and an isomorphism of vector spaces  $\psi: V \rightarrow W$  then we can rewrite the compatibility requirement above as  $\rho_2(g) = \psi \circ \rho_1(g) \circ \psi^{-1}$  for all  $g \in G$ .

Given any representation  $(\rho, V)$  of a group  $G$  on a vector space  $V$  over a field  $F$  of dimension  $n$ , we can fix a basis for  $V$  to obtain an isomorphism of vector spaces  $\psi: V \rightarrow F^n$ . This yields a representation  $\phi$  of  $G$  on  $F^n$  by defining  $\phi(g) = \psi \circ \rho(g) \circ \psi^{-1}$  for all  $g \in G$ . Clearly, this representation is isomorphic to our original representation  $(\rho, V)$ . In particular, this means we can always choose to view  $n$ -dimensional complex representations as representations on  $\mathbb{C}^n$  where each  $\phi(g)$  is given by an  $n \times n$  matrix with entries in  $\mathbb{C}$ .

Suppose that we have two representations  $\rho_1: G \rightarrow GL_n(F)$  and  $\rho_2: G \rightarrow GL_m(F)$  given by  $\rho_1(g) = X_g$  and  $\rho_2(g) = Y_g$ . A homomorphism between these representations is then an  $m \times n$  matrix  $A$  such that  $AX_g = Y_gA$  for all  $g \in G$ . An isomorphism is given precisely when such  $A$  is square and invertible. Thus, two representations  $\rho_1: G \rightarrow GL_n(F)$  and  $\rho_2: G \rightarrow GL_n(F)$  are isomorphic if and only if there exists  $A \in GL_n(F)$  such that  $\rho_1(g) = A\rho_2(g)A^{-1}$  for all  $g \in G$ . This establishes the following proposition:

**Proposition 1.14.** *The isomorphism classes of  $n$ -dimensional representations of  $G$  on  $\mathbb{C}$  are in bijection with the quotient  $\text{Hom}(G; GL_n(\mathbb{C}))/GL_n(\mathbb{C})$  of group homomorphisms  $G \rightarrow GL_n(\mathbb{C})$  modulo the conjugation action of  $GL_n(\mathbb{C})$ .*

### 1.3 Representations of Cyclic Groups

**Example 1.15** (Representations of  $\mathbb{Z}$ ). We want to classify all representations of the group  $\mathbb{Z}$  under addition. Consider an  $n$ -dimensional representation  $\rho: \mathbb{Z} \rightarrow GL_n(\mathbb{C})$ . For  $\rho$  to be a group homomorphism requires that  $\rho(0) = \text{Id}$ . Observe that for any  $0 \neq n \in \mathbb{Z}$ , we have  $\rho(n) = \rho(1 + \dots + 1) = \rho(1)^n$ . Thus  $\rho$  is completely determined by the matrix  $\rho(1) \in GL_n(\mathbb{C})$ , and any such matrix determines a representation of  $\mathbb{Z}$ . It follows that the  $n$ -dimensional isomorphism classes of representations of  $\mathbb{Z}$  are in bijection with the conjugacy classes in  $GL_n(\mathbb{C})$ . These conjugacy classes can be parameterized by the *Jordan canonical form*.

**Example 1.16** (Representations of the cyclic group of order  $n$ ). We shall classify all representations of the cyclic group  $G = \{g, g^2, \dots, g^{n-1}, g^n = 1\}$  of order  $n$ . Consider a representation  $\rho: G \rightarrow GL(V)$ . As in the previous example, we know that  $\rho(1) = \text{Id}$  and  $\rho(g^k) = \rho(g)^k$ . Thus our representation  $\rho$  is determined completely by the linear transformation  $\rho(g)$ . It will be helpful to fix a basis of  $V$  so that we may view  $\rho(g)$  as a matrix. Recall from linear algebra that there exists a basis in which  $\rho(g)$  takes the *Jordan canonical form*

$$\rho(g) = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_m \end{bmatrix}$$

where each *Jordan block*  $J_k$  is of the form

$$J_k = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \ddots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix}.$$

Now  $I = \rho(g)^n$  is a block-diagonal matrix with diagonal blocks  $J_k^n$ , so we must have that each block  $J_k^n = \text{Id}$ . Observe that we can write each block  $J_k$  as  $J_k = \lambda \text{Id} + N$  where  $N$  is the Jordan block with  $\lambda = 0$ . Thus we have

$$\text{Id} = J_k^n = (\lambda \text{Id} + N)^n = \lambda^n \text{Id} + \binom{n}{1} \lambda^{n-1} N + \binom{n}{2} \lambda^{n-2} N^2 + \dots + \binom{n}{n-1} \lambda N^{n-1} + N^n$$

. The following lemma will show that in fact  $N = 0$ .

**Lemma 1.17.** *Let  $N$  be the Jordan block with  $\lambda = 0$  of size  $n \times n$ . For any integer  $p$  with  $1 \leq p \leq n-1$ , then  $N^p$  is the matrix with ones in the positions  $(i, j)$  where  $j = i + p$  and zeroes everywhere else. (The ones lie along a line parallel to the diagonal,  $p$  steps above it.)*

*Proof.* (By induction.)

*Base case:* This is simply the definition of  $N$ .

*Inductive step:* Suppose that the lemma holds for  $N^p$ . We compute the  $(i, j)$  entry of  $N^{p+1}$ :

$$(N^{p+1})_{i,j} = \sum_{k=1}^n (N^p)_{i,k} N_{k,j} = (N^p)_{i,i+p} N_{i+p,j} = N_{i+p,j} = \begin{cases} 1 & \text{if } j = i + (p+1) \\ 0 & \text{otherwise} \end{cases}$$

□

Now, if  $N \neq 0$  then each term  $\binom{n}{k} \lambda^{n-k} N^k$  for  $k > 0$  would yield some non-zero non-diagonal entries (in the positions  $(i, j)$  where  $j = i + k$ ) and hence our sum could not equal the identity matrix. We must conclude that  $N = 0$ ,  $J_k = \lambda \text{Id}$  is a  $1 \times 1$  block, and  $J_k^n = \lambda^n \text{Id}$ . Thus  $\rho(g)$  is a diagonal matrix with  $n$ th roots of unity as diagonal entries.

To summarize, every  $m$ -dimensional representation  $\rho$  of the cyclic group  $G = \langle g \rangle$  of order  $n$  can be seen to act (with the right choice of basis) as  $m \times m$  diagonal matrices all with  $n$ th roots of unity along the diagonal. In particular, these representations are determined completely by the value of  $\rho(g)$  and are classified up to isomorphism by unordered  $m$ -tuples of  $n$ th roots of unity.

## 1.4 Constructions from Linear Algebra

**Definition 1.18.** A **subrepresentation** of  $V$  is a  $G$ -invariant subspace  $W \subseteq V$ ; that is, a subspace  $W \subseteq V$  with the property that  $\rho(g)(w) \in W$  for all  $g \in G$  and  $w \in W$ . Note that  $W$  itself is a representation of  $G$  under the action  $\rho(g)|_W$ .

From elementary linear algebra, we know that given a subspace  $W \subseteq V$ , we can form the **quotient space**  $V/W$  consisting of cosets  $v + W$  in  $V$ . If  $W$  is a subrepresentation of  $V$ , we would like to define an action of  $G$  on  $V/W$  by the natural choice of  $g(v + W) = \rho(g)(v) + W$ . It remains to verify that this action is well defined. If we choose another  $v' \in v + W$ , then  $v - v' \in W$ , so that  $\rho(g)(v - v') \in W$  since  $W$  is  $G$ -invariant. Thus, the cosets  $\rho(g)(v) + W$  and  $\rho(g)(v') + W$  agree and this action is indeed well defined. This justifies the following definition:

**Definition 1.19.** Let  $W$  be a  $G$ -subrepresentation of  $V$ . Then  $V/W$  forms a representation of  $G$  called the **quotient representation** of  $V$  under  $W$  with the action  $g(v + W) = \rho(g)(v) + W$ .

Recall also from linear algebra that given two vector spaces  $V_1$  and  $V_2$ , we can form the **direct sum**  $V_1 \oplus V_2$  consisting of ordered pairs  $(v_1, v_2)$  where  $v_1 \in V_1, v_2 \in V_2$ .

**Definition 1.20.** Let  $V_1$  and  $V_2$  be representations of  $G$ . Then  $V_1 \oplus V_2$  forms a representation of  $G$  called the **direct sum representation** of  $V_1$  and  $V_2$  with the action  $g(v_1, v_2) = (g \cdot v_1, g \cdot v_2)$ .

## 1.5 Complete Reducibility and Unitarity

**Definition 1.21.** A representation is said to be **irreducible** if it contains no proper invariant subspaces. It is called **completely reducible** if it decomposes into a direct sum of irreducible subrepresentations.

**Example 1.22.** 1. Any irreducible representation is completely reducible.

2. Any 1-dimensional representations has no proper subspaces, and is thus irreducible.

**Theorem 1.23.** If  $A_1, A_2, \dots, A_r$  are linear operators on  $V$  and each  $A_i$  is diagonalizable, they are simultaneously diagonalizable if and only if they commute.

*Proof.* See Conrad [2, Theorem 5.1]. □

**Theorem 1.24.** Every complex representation of a finite abelian group is completely reducible into irreducible representations of dimension 1.

*Proof.* Take an arbitrary element  $g \in G$ . Since  $G$  is finite, we can find an integer  $n$  such that  $g^n = 1$  and  $\rho(g)^n = Id$ . Hence the minimal polynomial of  $\rho(g)$  divides  $x^n - 1$ . Recall that  $x^n - 1$  has  $n$  distinct roots over  $\mathbb{C}$ , which are generated by taking powers of  $\xi = e^{\frac{2\pi i}{n}}$ . This means that the minimal polynomial  $\rho(g)$  factors into linear factors only over  $\mathbb{C}$  so that  $\rho(g)$  is diagonalizable. We conclude that each  $\rho(g)$  is (separately) diagonalizable since  $g \in G$  was arbitrary.

Now, given any two elements  $g_1, g_2 \in G$  we have

$$\begin{aligned} \rho(g_1)\rho(g_2) &= \rho(g_1g_2) && \text{(since } \rho \text{ is a homomorphism)} \\ &= \rho(g_2g_1) && \text{(since } G \text{ is abelian)} \\ &= \rho(g_2)\rho(g_1) && \text{(since } \rho \text{ is a homomorphism).} \end{aligned}$$

Thus the matrices  $\{\rho(g)\}$  commute, so we may apply theorem 1.23 to conclude that  $\{\rho(g)\}$  are simultaneously diagonalizable, say with basis  $\{e_1, \dots, e_k\}$ . Then we have  $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \dots \oplus \mathbb{C}e_n$ , with each subspace  $\mathbb{C}e_i$  invariant under the action of  $G$ .  $\square$

We recall the following definition from linear algebra:

**Definition 1.25.** Let  $V$  be a complex vector space. A **Hermitian inner product** on  $V$  is a map  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{C}$  that satisfies the following properties for all  $u, v, w \in V$  and  $c \in \mathbb{C}$ :

1.  $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ .
2.  $\langle cu, v \rangle = c\langle u, v \rangle$ .
3.  $\langle u, v \rangle = \overline{\langle v, u \rangle}$ .
4.  $\langle v, v \rangle \geq 0$  with equality if and only if  $v = 0$ .

**Definition 1.26.** A representation  $\rho$  of  $G$  on a complex vector space  $V$  is **unitary** if  $V$  has been equipped with a hermetian inner product  $\langle \cdot, \cdot \rangle$  which is preserved by the action of  $G$ , that is,

$$\langle v, w \rangle = \langle \rho(g)(v), \rho(g)(w) \rangle \quad \forall v, w \in V, g \in G.$$

A representation is said to be **unitarisable** if it can be equipped with such a product (even without one being specified).

**Theorem 1.27.** [Weyl's unitary trick] *Finite-dimensional representations of finite groups are unitarisable.*

*Proof.* Take any Hermitian inner product on  $V$ , say  $\langle \cdot, \cdot \rangle'$ . We define a new inner product on  $V$  by averaging over  $G$ :

$$\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle'.$$

This new inner product satisfies properties 1, 2, and 3 of Definition 1.25 by linearity. It remains to check positivity (4). Clearly  $\langle v, v \rangle = 0$  when  $v = 0$ , since each term of the sum will equal zero. In the case where  $v \neq 0$ , observe that

$$\langle v, v \rangle = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, \rho(g)v \rangle' \geq 0$$

since each term of the sum is non-negative by the positivity of  $\langle \cdot, \cdot \rangle'$ . The only problem would occur if each term of this sum is equal to zero. But  $\langle \rho(e)v, \rho(e)v \rangle' = \langle v, v \rangle' > 0$ . Thus  $\langle v, v \rangle > 0$ .

Finally, we show that our new inner product is  $G$ -invariant. For any  $h \in G$ , we have

$$\begin{aligned} \langle \rho(h)v, \rho(h)w \rangle &= \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)\rho(h)v, \rho(g)\rho(h)w \rangle' \\ &= \frac{1}{|G|} \sum_{g \in G} \langle \rho(gh)v, \rho(gh)w \rangle' && \text{(since } \rho \text{ is a homomorphism)} \\ &= \frac{1}{|G|} \sum_{k \in G} \langle \rho(k)v, \rho(k)w \rangle' && \text{(by a change of variables)} \\ &= \langle v, w \rangle. \end{aligned}$$

□

**Lemma 1.28.** *Let  $V$  be a unitary representation of  $G$  and let  $W \subseteq V$  be a  $G$ -invariant subspace. Then the orthogonal complement  $W^\perp$  is also  $G$ -invariant.*

*Proof.* Choose arbitrary elements  $v \in W^\perp$  and  $g \in G$ . We need to show that  $gv \in W^\perp$ . Now for any  $w \in W$ , we have  $\langle v, w \rangle = 0$ . Thus  $\langle gv, gw \rangle = g\bar{g}\langle v, w \rangle = 0$  for any  $w \in W$ . Notice that  $w' = gw \in W$  since  $W$  is  $G$ -invariant. This implies that  $\langle gv, w' \rangle = 0$ , i.e.  $gv \in W^\perp$ . □

**Theorem 1.29.** *A finite-dimensional unitary representation of a group is fully reducible into unitary irreducible subrepresentations.*

*Proof.* Let  $V$  be a finite dimensional unitary representation of  $G$ . We proceed by induction on the dimension of  $V$ . If  $\dim(V) = 1$ , then  $V$  is necessarily irreducible. Now, suppose the theorem holds for all  $W$  with  $\dim(V) \leq n - 1$  and suppose  $\dim(V) = n$ . If  $V$  is irreducible, we are done. Otherwise, there exists a proper  $G$ -invariant subspace  $W (\neq 0, V)$ . We can write  $V = W \oplus W^\perp$  by Lemma 1.28. Applying the inductive hypothesis to  $W$  and  $W^\perp$ , we obtain a decomposition into irreducibles

$$V = (W_1 \oplus \dots \oplus W_j) \oplus (W_{j+1} \oplus \dots \oplus W_k).$$

□

**Corollary 1.30.** *Every complex representation of a finite group is completely reducible.*

*Proof.* Any such representation is unitarisable by Theorem 1.27. We can then apply Theorem 1.29 to obtain full reducibility. □

## 1.6 Schur's Lemma

**Theorem 1.31.** *[Schur's Lemma over  $\mathbb{C}$ .] If  $V$  is an irreducible  $G$ -representation over  $\mathbb{C}$ , then every linear operator  $\phi: V \rightarrow V$  commuting with  $G$  is a scalar.*

*Proof.* Let  $\lambda$  be an eigenvalue of  $\phi$ . Observe that the eigenspace  $E_\lambda$  is  $G$ -invariant: If  $v \in E_\lambda$ , then  $\phi(v) = \lambda v$ . This implies that  $\phi(gv) = g\phi(v) = g(\lambda v) = \lambda(gv)$ , i.e.  $gv \in E_\lambda$ . Since  $g$  was arbitrary,  $E_\lambda$  is indeed  $G$ -invariant. Now  $E_\lambda \neq 0$ , so by irreducibility  $E_\lambda = V$ . Thus  $\phi = \lambda \text{Id}$ . □

**Definition 1.32.** Let  $V$  and  $W$  be vector spaces. Recall that the set  $\mathbf{Hom}(V, W)$  of linear maps from  $V$  to  $W$  is itself a vector space. If  $f_1, f_2$  are two linear maps from  $V$  to  $W$ , then we define their sum by

$$(f_1 + f_2): V \rightarrow W \\ x \mapsto f_1(x) + f_2(x)$$

and we define scalar multiplication of  $\lambda \in \mathbb{C}$  by

$$(\lambda f_1): V \rightarrow W \\ x \mapsto \lambda f_1(x).$$

Now suppose we have representations  $\rho_V: G \rightarrow GL(V)$  and  $\rho_W: G \rightarrow GL(W)$  of  $G$ . Then there is a natural representation of  $G$  on the vector space  $\mathbf{Hom}(V, W)$  given by

$$\rho_{\mathbf{Hom}(V, W)}(g): \mathbf{Hom}(V, W) \rightarrow \mathbf{Hom}(V, W) \\ f \mapsto \rho_W(g) \circ f \circ \rho_V(g^{-1})$$

for all  $g \in G$ .

**Proposition 1.33.**  $\rho_{\mathbf{Hom}(V, W)}$  is a representation of  $G$ . That is, the map

$$\rho_{\mathbf{Hom}(V, W)}: G \rightarrow GL(\mathbf{Hom}(V, W)) \\ g \mapsto \rho_{\mathbf{Hom}(V, W)}(g).$$

is a homomorphism.

*Proof.* We must check two things:

1. For all  $g \in G$ ,  $\rho_{\mathbf{Hom}(V, W)}(g)$  is invertible.
2. The map  $g \mapsto \rho_{\mathbf{Hom}(V, W)}(g)$  is a homomorphism.

□

**Definition 1.34.** Let  $V$  and  $W$  be two representations of  $G$ . The set of  $G$ -linear maps from  $V$  to  $W$  forms a subspace of  $\mathbf{Hom}(V, W)$ , which we denote by  $\mathbf{Hom}_G(V, W)$ . In other words,  $\mathbf{Hom}_G(V, W)$  is the vector space consisting of all *homomorphisms of representations* between  $V$  and  $W$ .

**Corollary 1.35.** If  $V$  and  $W$  are irreducible, the space  $\mathbf{Hom}_G(V, W)$  is 1-dimensional if the representations are isomorphic, and in this case any non-zero map is an isomorphism. Otherwise,  $\mathbf{Hom}_G(V, W) = \{0\}$ .

*Proof.* Let  $0 \neq \phi \in \mathbf{Hom}_G(V, W)$ . If  $v \in \ker(\phi)$ , then  $\phi(v) = 0$  implies that  $\phi(gv) = g\phi(v) = g0 = 0$ , i.e.  $gv \in \ker(\phi)$ . Similarly, if  $v \in \text{im}(\phi)$ , then  $v = \phi(w)$  implies that  $\phi(gw) = g\phi(w) = gv$ , i.e.  $gv \in \text{im}(\phi)$ . Thus  $\ker(\phi)$  and  $\text{im}(\phi)$  are both  $G$ -invariant.

Irreducibility yields  $\ker(\phi) = 0$  or  $V$  and  $\text{im}(\phi) = 0$  or  $W$  as the only possibilities. If  $\phi \neq 0$ , then  $\ker(\phi) = 0$ . This means that  $\phi$  is injective,  $\text{im}(\phi) = W$ , and  $\phi$  is an isomorphism.

Let  $\psi$  be another intertwining operator from  $V$  to  $W$ . Then  $\phi^{-1} \circ \psi$  is also an intertwining operator from  $V$  to  $V$ . We can apply Schur's Lemma over  $\mathbb{C}$  to see that  $\phi^{-1} \circ \psi = \lambda \text{Id}$ , hence  $\psi = \lambda\phi$ . □

More definitions are required before we can state a more general Schur's Lemma (not restricted to just  $\mathbb{C}$ ).

**Definition 1.36.** An **algebra** over a field  $K$  is a ring with unit, containing a distinguished copy of  $K$  that commutes with every algebra element, and with  $1 \in K$  being the algebra unit. A **division ring** is a ring where every non-zero element is invertible, and a **division algebra** is a division ring which is also a  $K$ -algebra.

**Definition 1.37.** Let  $V$  be a representation of  $G$  over  $K$ . The **endomorphism algebra**  $\text{End}^G(V)$  is the space of linear self-maps  $\phi: V \rightarrow V$  which commute with the group action, that is,  $\rho(g) \circ \phi = \phi \circ \rho(g) \quad \forall g \in G$ . The addition is the usual addition of linear maps (pointwise), and the multiplication is function composition. The distinguished copy of  $K$  is given by  $K\text{Id}$ .

**Theorem 1.38.** [Schur's Lemma] If  $V$  is an irreducible finite-dimensional representation of  $G$  over  $K$ , then  $\text{End}^G(V)$  is a finite-dimensional division algebra over  $K$ .

## 1.7 Tensor Product

Let  $V$  and  $W$  be two vector spaces over  $K$ , and assume we have bases  $\{a_1, \dots, a_n\}$  for  $V$  and  $\{b_1, \dots, b_m\}$  for  $W$ .

**Definition 1.39.** The **tensor product**  $V \otimes_K W$  of  $V$  and  $W$  is the  $K$ -vector space which has a basis given by the set of symbols

$$\{a_i \otimes b_j | 1 \leq i \leq n, 1 \leq j \leq m\}.$$

When the ground field  $K$  is clear, it can be omitted from the notation. If we have vectors  $x \in V$  and  $y \in W$ , we can define a vector  $x \otimes y \in V \otimes W$  as follows. Write  $x$  and  $y$  in the given bases with coefficients  $\lambda_i, \mu_j \in K$ , so

$$x = \lambda_1 a_1 + \dots + \lambda_n a_n$$

$$y = \mu_1 b_1 + \dots + \mu_m b_m.$$

Then we define

$$x \otimes y = \sum_{\substack{i \in [1, n] \\ j \in [1, m]}} \lambda_i \mu_j a_i \otimes b_j.$$

Now let  $V$  and  $W$  be two representations of  $G$ .

**Definition 1.40.** We can define a representation of  $G$  on  $V \otimes W$  called the **tensor product representation**. We define

$$\rho_{V \otimes W}(g): V \otimes W \rightarrow V \otimes W$$

to be the linear map given by

$$\rho_{V \otimes W}(g): a_i \otimes b_j \mapsto \rho_V(g)(a_i) \otimes \rho_W(g)(b_j).$$



## 1.8 Isotypical Decomposition

## 1.9 Character Theory

**Definition 1.41.** The **character** of a representation  $\rho: G \rightarrow GL(V)$  is the function  $\chi_V: G \rightarrow \mathbb{C}$  defined by  $\chi_V(g) = \text{Tr}(\rho(g))$ .

*Note.* The character of a representation is not a homomorphism in general, since  $\text{Tr}(MN) \neq \text{Tr}(M)\text{Tr}(N)$  in general.

**Proposition 1.42.** (*Basic Properties*)

1.  $\chi_V$  is conjugation invariant:  $\chi_V(hgh^{-1}) = \chi_V(g)$  for all  $g, h \in G$ .
2.  $\chi_V(1) = \dim V$ .
3.  $\chi_V(g^{-1}) = \overline{\chi_V(g)}$  for all  $g \in G$ .

*Proof.* 1.  $\chi_V(hgh^{-1}) = \text{Tr}(hgh^{-1}) = \text{Tr}(ghh^{-1}) = \text{Tr}(g) = \chi_V(g)$  for any  $g, h \in G$ .

2.  $\chi_V(1) = \text{Tr}(\text{Id}_V) = \dim V$ .

3. Since  $G$  is finite, we have seen that  $\rho(g)$  is a diagonal matrix with roots of unity along the diagonal with the right choice of basis. The inverse of a root of unity is given by its complex conjugate, so under this same basis,  $\rho(g)^{-1}$  is clearly given by  $\overline{\rho(g)}$ . Thus,  $\chi_V(g^{-1}) = \text{Tr}(\rho(g^{-1})) = \text{Tr}(\rho(g)^{-1}) = \text{Tr}(\overline{\rho(g)}) = \overline{\text{Tr}(\rho(g))} = \overline{\chi_V(g)}$ .  $\square$

**Definition 1.43.** A **class function** on  $G$  is a function on  $G$  whose values are invariant by conjugation of elements in  $G$ . The value of a class function at an element  $g \in G$  depends only on the conjugacy class of  $g$ . We may therefore view class functions as functions on the set of conjugacy classes of  $G$ .

*Note.* The character  $\chi_V$  of a representation  $V$  of  $G$  is a class function on  $G$ .

**Proposition 1.44.** *Isomorphic representations have the same character.*

*Proof.* We have seen (CITE ME!!!) that isomorphic representations can be described by the same matrices in the right choice of basis.  $\square$

We will see later that the converse is true - if two representations have the same character, then they are isomorphic.

**Proposition 1.45.** Let  $\rho_V: G \rightarrow GL(V)$  and  $\rho_W: G \rightarrow GL(W)$  be representations of  $G$  with characters  $\chi_V$  and  $\chi_W$ .

1.  $\chi_{V \oplus W} = \chi_V + \chi_W$ .
2.  $\chi_{V \otimes W} = \chi_V \cdot \chi_W$ .

*Proof.* 1. Pick bases for  $V$  and  $W$ , so that  $\rho_V(g)$  and  $\rho_W(g)$  are described by matrices  $M$  and  $N$ . Then  $\rho_{V \oplus W}(g)$  is described by the block-diagonal matrix

$$\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

So we have  $\text{Tr}(\rho_{V \oplus W}(g)) = \text{Tr}(M) + \text{Tr}(N) = \text{Tr}(\rho_V(g)) + \text{Tr}(\rho_W(g))$ .

2.  $\rho_{V \otimes W}$  is given by the matrix

$$[M \otimes N]_{js,it} = M_{ji}N_{st}$$

□

**Proposition 1.46.** 1. Let  $\{V_i\}$  be the irreducible representations of  $G$ , with  $d_i$  the dimension of  $V_i$  and  $\chi_i$  the corresponding irreducible character. Then

$$\chi_{\text{reg}} = d_1\chi_1 + \dots + d_r\chi_r$$

- 2.

$$\chi_{\text{reg}}(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$$

[6] [4] [1] [2] [5] [3]

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