UNIVERSITY OF MISSOURI

MASTER'S PROJECT

A Survey on Character Tables for Representations of Finite Groups

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"Thanks to my solid academic training, today I can write hundreds of words on virtually any topic without possessing a shred of information, which is how I got a good job in journalism."

Dave Barry

UNIVERSITY OF MISSOURI

Abstract

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Masters of Arts

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by Jared Stewart

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too…

Acknowledgements

The acknowledgements and the people to thank go here, don't forget to include your project advisor…

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Chapter 1

Basics of Representation Theory

1.1 Group Actions

Definition 1.1. A ***(left)* group action** of a group G on a set X is a map $\varphi\colon G \times X \to X$ (written as $g \cdot a$, for all $g \in G$ and $a \in A$) that satisfies the following two axoims:

$$id_G \cdot x = x \qquad \forall x \in X \tag{1.1.1}$$
$$(gh) \cdot x = g \cdot (h \cdot x) \qquad \forall g, h \in G, x \in X \tag{1.1.2}$$

Note. We could likewise define the concept of a *right* group action, where the set elements would be multiplied by group elements on the right instead of on the left. Throughout we shall use the term *group action* to mean a *left* group action.

Proposition 1.2. *Let G act on the set X. For any fixed $g \in G$, the map σ_g from X into X defined by $\sigma_g(x) = g \cdot x$ is a permutation of the set X, i.e. $\sigma_g \in S_X$.*

Proof. We show that σ_g is a permutation of X by finding a two-sided inverse map, namely $\sigma_{g^{-1}}$. Observe that for any $x \in X$, we have

$$\begin{aligned}
(\sigma_{g^{-1}} \circ \sigma_g)(x) &= \sigma_{g^{-1}}(\sigma_g(x)) && \text{(by definition of function composition)}\\
&= g^{-1} \cdot (g \cdot x) && \text{(by definition of } \sigma_g \text{ and } \sigma_{g^{-1}})\\
&= (g^{-1}g) \cdot x && \text{(by axiom 1.1.1 of an action)}\\
&= id_G \cdot x &&\\
&= x && \text{(by axiom 1.1.2 of an action).}
\end{aligned}$$

Thus $\sigma_{g^{-1}} \circ \sigma_g$ is the identity map on X. We can reverse the roles of g and g^{-1} to see that $\sigma_g \circ \sigma_{g^{-1}}$ is also the identity map on X. Having a two-sided inverse, we conslude that σ_g is a permutation of X. □

Proposition 1.3. *Let G act on the set X. The map from G to the symmetric group S_X defined by $g \mapsto \sigma_g(x) = g \cdot x$ is a group homomorphism.*

Proof. Define the map $\varphi\colon G \to S_X$ by $\varphi(g) = \sigma_g$. We have seen from Proposition 1.2 that σ_g is indeed an element of S_X. It remains to show that $\varphi(g_1 g_2) = \varphi(g_1) \circ \varphi(g_2)$ for any $g_1, g_2 \in G$. Observe that

$$\begin{aligned}
\varphi(g_1g_2)(x) &= \sigma_{g_1g_2}(x) && \text{(by definition of } \varphi\text{)}\\
&= (g_1g_2)\cdot x && \text{(by definition of } \sigma_{g_1g_2}\text{)}\\
&= g_1\cdot(g_2\cdot x) && \text{(by axiom 1.1.1 of an action)}\\
&= \sigma_{g_1}(\sigma_{g_2}(x)) && \text{(by definition of } \sigma_{g_1} \text{ and } \sigma g_2\text{)}\\
&= \varphi(g_1)(\varphi(g_2)(x)) && \text{(by definition of } \varphi\text{)}\\
&= (\varphi(g_1)\circ\varphi(g_2))(x) && \text{(by definition of function composition).}
\end{aligned}$$

Since the values of $\varphi(g_1g_2)$ and $\varphi(g_1)\circ\varphi(g_2)$ agree on every element $x\in X$, these two permutations are equal. We conclude that φ is a homomorphism, since g_1 and g_2 were arbitrary elements of G. □

Proposition 1.4. *Any homomorphism ψ from the group G into the symmetric group on S_X on a set X gives rise to an action of G on X, defined by taking $g\cdot x=\psi(g)(x)$.*

Proof. Suppose that we have a homomorphism ψ from G into S_X. We can define a map from $G\times X$ to X by $g\cdot x=\psi(g)(x)$. We verify that this map satisfies the definition of a group action of G on X:
(axiom 1.1.1) $id_G\cdot x=\psi(id_G)(x)=id_X(x)=x$
(axiom 1.1.2) $(gh)\cdot x=\psi(gh)(x)=(\psi(g)\psi(h))(x)=\psi(g)(\psi(h)(x))=g\cdot(h\cdot x)$ □

Proposition 1.5. *The actions of G on the set X are in bijective correspondence with the homomorphisms from G into the symmetric group S_X.*

Proof. By Proposition 1.3, any action of G on X yields a homomorphism from G into S_X. Conversely, any homomorphism from G into S_X establishes an action of G on X by Proposition 1.4. □

1.2 The Definition of a Representation

Definition 1.6. Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is any group homomorphism $\varphi\colon G\to GL(V)$.

Definition 1.7. Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is any action of G on V which preserves the linear structure of V, that is,

$$g\cdot(v_1+v_2)=g\cdot v_1+g\cdot v_2 \qquad \forall g\in G, v_1,v_2\in V \tag{1.7.1}$$
$$g\cdot(kv)=k(g\cdot v) \qquad \forall g\in G, v\in V, k\in F \tag{1.7.2}$$

Proposition 1.8. *The definitions of a linear representation given in 1.6 and 1.7 above are equivalent.*

Proof. $(\rightarrow)$ Suppose that we have a homomorphism $\varphi\colon G\to GL(V)$. Note that $GL(V)$ is a subgroup of the symmetric group S_V on V, so we can apply Proposition 1.4 to obtain an action of G on V by $g\cdot v=\varphi(g)(v)$. We check that this action preserves the linear structure of V.
1.7.1 For any $g\in G, v_1,v_2\in V$ we have $g\cdot(v_1+v_2)=\varphi(g)(v_1+v_2)=\varphi(g)(v_1)+\varphi(g)(v_2)=g\cdot v_1+g\cdot v_2$.
1.7.2 For any $g\in G, v\in V, k\in F$ we have $g\cdot(kv)=\varphi(g)(kv)=k(\varphi(g)(v))=k(g\cdot v)$.

($\leftarrow$) Suppose that we have an action of G on V which preserves the linear structure of V in the sense of Definition 1.7. We can apply Proposition 1.3 to obtain a homomorphism $\varphi\colon G \to S_V$ given by $\varphi(g) = \sigma_g$ where $\sigma_g(v) = g \cdot v$. It remains to show that the image $\varphi(G)$ of G under φ is actually contained in $GL(V)$, i.e. that for each $g \in G$ the map σ_g is linear. Fix an element $g \in G$. For any $k \in F$ and $v \in V$ we have

$$\begin{aligned} \sigma_g(kv) &= g \cdot (kv) && \text{(by definition of } \sigma_g\text{)} \\ &= k(g \cdot v) && \text{(by property 1.7.1)} \\ &= k(\sigma_g(v)) && \text{(by definition of } \sigma_g\text{).} \end{aligned}$$

Also, for any $v_1, v_2 \in V$ we have

$$\begin{aligned} \sigma_g(v_1 + v_2) &= g \cdot (v_1 + v_2) && \text{(by definition of } \sigma_g\text{)} \\ &= g \cdot v_1 + g \cdot v_2 && \text{(by property 1.7.2)} \\ &= \sigma_g(v_1) + \sigma_g(v_2) && \text{(by definition of } \sigma_g\text{).} \end{aligned}$$

Thus σ_g is linear, and $\varphi(G) \subset GL(V)$ proves that we have a homomorphism $\varphi\colon G \to GL(V)$.

$\square$

Definition 1.9. Let G be a group, let F be a field, let V be a vector space over F, and let $\varphi\colon G \to GL(V)$ be a representation of G. The **dimension** of the representation is the dimension of V over F.

Example 1.10.

1. Let V be a 1-dimensional vector space over the field F. The map $\varphi\colon G \to GL(V)$ defined by $\varphi(g) = 1$ for all $g \in G$ is a representation called the *trival representation* of G. The trivial representation has dimension 1.

2. If a finite group G acts on a finite set X and F is any field, then there is an associated *permutation representation* on the vector space V over F with basis $\{e_x\colon x \in X\}$. We let G act on the basis elements by $g \cdot e_x = e_{gx}$ for all $x \in X$ and $g \in G$. Note that G permutes the basis elements of V.

3. A fundamental special case of a permutation representation is given by a finite group acting on itself by left multiplication. In this case, the elements of G form a basis for V, and each $g \in G$ permutes the basis elements by $g \cdot g_i = gg_i$. This is called the *regular representation* of G and has dimension $|G|$. We shall see that this representation encodes information about all other representations of G.

4. For any symmetric group S_n the *alternating representation* on $V = \mathbb{C}$ is given by the map $\varphi\colon S_n \to GL(\mathbb{C}) = \mathbb{C}^\times$ defined by $\varphi(\sigma) = \mathrm{sgn}(\sigma)$. More generally, for any group G with a subgroup H of index 2, we can define an *alternating representation* $\varphi\colon G \to GL(\mathbb{C})$ by letting $\varphi(g) = 1$ if $g \in H$ and $\varphi(g) = -1$ if $g \notin H$. (We recover our original example by taking $G = S_n$ and $H = A_n$.)

Definition 1.11. A **homomorphism** between two representations $\varphi_1\colon G \to GL(V)$ and $\varphi_2\colon G \to GL(W)$ is a linear map $\psi\colon V \to W$ that interwines with (respects) the G-action, i.e. such that

$$\psi(\varphi_1(g)(v)) = \varphi_2(g)(\psi(v)) \quad \forall v \in V, g \in G$$

An **isomorphism** of representations is a homomorphism of representations that is also an invertible map.

Note. If we have an isomorphism ψ between representations (φ_1, V) and (φ_2, W) we can rewrite the compatibility relation as $\varphi_2(g) = \psi \circ \varphi_1(g) \circ \psi^{-1}$.

Note. Given any representation φ of G on a vector space V over a field F of dimension n, we can fix a basis for V to obtain an isomorphism $\psi \colon V \to F^n$.

Chapter 2

Spaghetti

2.1 Definition of a Representation AGAIN

Definition 2.1. A **linear representation** of a group G on a vector space V is a group homomorphism from G to $GL(V)$, the general linear group on V.

More explicitly, a representation is a map $\rho : G \to GL(V)$ such that

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2) \quad \forall g_1, g_2 \in G.$$

Definition 2.2. A **linear representation** ρ of a group G on a vector space V over a field K is a group action of G on V which preserves the linear structure of V. That is,

1. $\rho(g)(v_1 + v_2) = \rho(g)(v_1) + \rho(g)(v_2) \quad \forall g \in G, v_1, v_2 \in V$
2. $\rho(g)(kv) = k \cdot \rho(g)v \quad \forall g \in G, v \in V, k \in K$

2.1.1 Subsection 1

Definition 2.3. Here is a new definition.

$$E = mc^2 \tag{2.3.1}$$

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2.1.2 Subsection 2

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Definition 2.4. A **linear representation** ρ of a group G on a vector space V over a field K is a group action of G on V which preserves the linear structure of V. That is,

$$\rho(g)(v_1 + v_2) = \rho(g)(v_1) + \rho(g)(v_2) \quad \forall g \in G, \forall v_1, v_2 \in V \tag{2.4.1}$$

$$\rho(g)(kv) = k \cdot \rho(g)v \quad \forall g \in G, v \in V, k \in K$$

2.2 Main Section 2

Definition 2.5. Here is a new definition.

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Appendix A

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Bibliography

[1] Alex Bartel. *Introduction to representation theory of finite groups*. https://math.berkeley.edu/~teleman/math/RepThry.pdf. Online; accessed 11 November 2015. 2014.

[2] D.S. Dummit and R.M. Foote. *Abstract Algebra*. Wiley, 2003. ISBN: 9780471433347. URL: https://books.google.com/books?id=KJDBQgAACAAJ.

[3] Constantin Teleman. *Representation Theory*. https://math.berkeley.edu/~teleman/math/RepThry.pdf. Online; accessed 11 November 2015. 2005.