UNIVERSITY OF MISSOURI

MASTER'S PROJECT

A Survey on Character Tables for Representations of Finite Groups

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Chapter 1

Basic Notions of Representation Theory

1.1 Group Actions

Definition 1.1. A ***(left)* group action** of a group G on a set X is a map $\rho\colon G \times X \to X$ (written as $g \cdot a$, for all $g \in G$ and $a \in A$) that satisfies the following two axioms:

$$1 \cdot x = x \qquad \forall x \in X \tag{1.1.1}$$
$$(gh) \cdot x = g \cdot (h \cdot x) \qquad \forall g, h \in G, x \in X \tag{1.1.2}$$

Note. We could likewise define the concept of a *right* group action, where the set elements would be multiplied by group elements on the right instead of on the left. Throughout we shall use the term *group action* to mean a *left* group action.

Proposition 1.2. *Let G act on the set X. For any fixed $g \in G$, the map σ_g from X into X defined by $\sigma_g(x) = g \cdot x$ is a permutation of the set X. That is, $\sigma_g \in S_X$.*

Proof. We show that σ_g is a permutation of X by finding a two-sided inverse map, namely $\sigma_{g^{-1}}$. Observe that for any $x \in X$, we have

$$\begin{aligned}
(\sigma_{g^{-1}} \circ \sigma_g)(x) &= \sigma_{g^{-1}}(\sigma_g(x)) \\
&= g^{-1} \cdot (g \cdot x) && \text{(by definition of } \sigma_g \text{ and } \sigma_{g^{-1}}) \\
&= (g^{-1}g) \cdot x && \text{(by axiom 1.1.1 of an action)} \\
&= 1 \cdot x \\
&= x && \text{(by axiom 1.1.2 of an action).}
\end{aligned}$$

Thus $\sigma_{g^{-1}} \circ \sigma_g$ is the identity map on X. We can reverse the roles of g and g^{-1} to see that $\sigma_g \circ \sigma_{g^{-1}}$ is also the identity map on X. Having a two-sided inverse, we conclude that σ_g is a permutation of X. □

Proposition 1.3. *Let G act on the set X. The map from G into the symmetric group S_X defined by $g \mapsto \sigma_g(x) = g \cdot x$ is a group homomorphism.*

Proof. Define the map $\rho\colon G \to S_X$ by $\rho(g) = \sigma_g$. We have seen from Proposition 1.2 that σ_g is indeed an element of S_X. It remains to show that $\rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2)$ for any $g_1, g_2 \in G$. Observe that

$$\begin{aligned}
\rho(g_1g_2)(x) &= \sigma_{g_1g_2}(x) && \text{(by definition of } \rho) \\
&= (g_1g_2)\cdot x && \text{(by definition of } \sigma_{g_1g_2}) \\
&= g_1\cdot(g_2\cdot x) && \text{(by axiom 1.1.1 of an action)} \\
&= \sigma_{g_1}(\sigma_{g_2}(x)) && \text{(by definition of } \sigma_{g_1} \text{ and } \sigma g_2) \\
&= \rho(g_1)(\rho(g_2)(x)) && \text{(by definition of } \rho) \\
&= (\rho(g_1)\circ\rho(g_2))(x) && \text{(by definition of function composition).}
\end{aligned}$$

Since the values of $\rho(g_1g_2)$ and $\rho(g_1)\circ\rho(g_2)$ agree on every element $x \in X$, these two permutations are equal. We conclude that ρ is a homomorphism, since g_1 and g_2 were arbitrary elements of G. □

Proposition 1.4. *Any homomorphism ψ from the group G into the symmetric group S_X on a set X gives rise to an action of G on X, defined by taking $g\cdot x = \psi(g)(x)$.*

Proof. Suppose that we have a homomorphism ψ from G into S_X. We can define a map from $G\times X$ to X by $g\cdot x = \psi(g)(x)$. We verify that this map satisfies the definition of a group action of G on X:
(axiom 1.1.1) $1\cdot x = \psi(1)(x) = id_X(x) = x$
(axiom 1.1.2) $(gh)\cdot x = \psi(gh)(x) = (\psi(g)\psi(h))(x) = \psi(g)(\psi(h)(x)) = g\cdot(h\cdot x)$ □

Corollary 1.5. *The actions of G on the set X are in bijective correspondence with the homomorphisms from G into the symmetric group S_X.*

Proof. By Proposition 1.3, any action of G on X yields a homomorphism from G into S_X. Conversely, any homomorphism from G into S_X establishes an action of G on X by Proposition 1.4. □

1.2 The Definition of a Representation

Definition 1.6. Let G be a group. A **representation** of G is a homomorphism $\rho\colon G \to GL_n(\mathbb{C})$ for some positive integer n.

Definition 1.7. Two representations $\rho_1\colon G \to GL_n(\mathbb{C})$ and $\rho_2\colon G \to GL_n(\mathbb{C})$ of G are **equivalent** if there exists $P \in GL_n(\mathbb{C})$ such that $\rho_2 = P^{-1}\rho_1 P$.

Equivalent representations are fundamentally "the same" in some sense, but to make this precise we need to shift our thinking to linear maps instead of matrices.

Definition 1.8. Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is any group homomorphism $\rho\colon G \to GL(V)$. If we fix a basis for V, we get a representation in the previous sense.

Definition 1.9 (Alternative definition)**.** Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is an action of G on V which preserves the linear structure of V, i.e. an action of G on V such that

$$g\cdot(v_1+v_2) = g\cdot v_1 + g\cdot v_2 \qquad \forall g\in G, v_1, v_2 \in V \tag{1.9.1}$$
$$g\cdot(kv) = k(g\cdot v) \qquad \forall g\in G, v\in V, k\in F \tag{1.9.2}$$

Note. Unless otherwise specificed, we use *representation* to mean *finite-dimensional complex representation*.

Proposition 1.10. *The definitions of a linear representation given in 1.8 and 1.9 above are equivalent.*

Proof. ($\rightarrow$) Suppose that we have a homomorphism $\rho\colon G \to GL(V)$. Note that $GL(V)$ is a subgroup of the symmetric group S_V on V, so we can apply Proposition 1.4 to obtain an action of G on V by $g \cdot v = \rho(g)(v)$. We check that this action preserves the linear structure of V.
1.9.1 For any $g \in G$, $v_1, v_2 \in V$ we have $g \cdot (v_1 + v_2) = \rho(g)(v_1 + v_2) = \rho(g)(v_1) + \rho(g)(v_2) = g \cdot v_1 + g \cdot v_2$.
1.9.2 For any $g \in G, v \in V, k \in F$ we have $g \cdot (kv) = \rho(g)(kv) = k(\rho(g)(v)) = k(g \cdot v)$.

($\leftarrow$) Suppose that we have an action of G on V which preserves the linear structure of V in the sense of Definition 1.9. We can apply Proposition 1.3 to obtain a homorphism $\rho\colon G \to S_V$ given by $\rho(g) = \sigma_g$ where $\sigma_g(v) = g \cdot v$. It remains to show that the image $\rho(G)$ of G under ρ is actually contained in $GL(V)$, i.e. that for each $g \in G$ the map σ_g is linear. Fix an element $g \in G$. For any $k \in F$ and $v \in V$, we have

$$\begin{aligned} \sigma_g(kv) &= g \cdot (kv) && \text{(by definition of } \sigma_g) \\ &= k(g \cdot v) && \text{(by property 1.9.1)} \\ &= k(\sigma_g(v)) && \text{(by definition of } \sigma_g). \end{aligned}$$

Also, for any $v_1, v_2 \in V$ we have

$$\begin{aligned} \sigma_g(v_1 + v_2) &= g \cdot (v_1 + v_2) && \text{(by definition of } \sigma_g) \\ &= g \cdot v_1 + g \cdot v_2 && \text{(by property 1.9.2)} \\ &= \sigma_g(v_1) + \sigma_g(v_2) && \text{(by definition of } \sigma_g). \end{aligned}$$

Thus σ_g is linear, and $\rho(G) \subset GL(V)$ proves that we have a homomorphism $\rho\colon G \to GL(V)$.

□

Definition 1.11. Let G be a group, let F be a field, let V be a vector space over F, and let $\rho\colon G \to GL(V)$ be a representation of G. The **dimension** of the representation is the dimension of V over F.

Example 1.12.

1. Let V be a 1-dimensional vector space over the field F. The map $\rho\colon G \to GL(V)$ defined by $\rho(g) = 1$ for all $g \in G$ is a representation called the *trival representation* of G. The trivial representation has dimension 1.

2. If G is a finite group that acts on a finite set X, and F is any field, then there is an associated *permutation representation* on the vector space V over F with basis $\{e_x\colon x \in X\}$. We let G act on the basis elements by the permutation $g \cdot e_x = e_{gx}$ for all $x \in X$ and $g \in G$. This representation has dimension $|X|$.

3. A fundamental special case of a permutation representation that we shall return to later on is that when a finite group acts on itself by left multiplication. In this case, the elements of G form a basis for V, and each $g \in G$ permutes the basis

elements by $g \cdot g_i = gg_i$. This representation is called the *regular representation* of G and has dimension $|G|$. We shall see later that this representation encodes information about all other representations of G.

4. For any symmetric group S_n, the *alternating representation* on $V = \mathbb{C}$ is given by the map $\rho\colon S_n \to GL(\mathbb{C}) = \mathbb{C}^\times$ defined by $\rho(\sigma) = \operatorname{sgn}(\sigma)$. More generally, for any group G with a subgroup H of index 2, we can define an *alternating representation* $\rho\colon G \to GL(\mathbb{C})$ by letting $\rho(g) = 1$ if $g \in H$ and $\rho(g) = -1$ if $g \notin H$. (We recover our original example by taking $G = S_n$ and $H = A_n$.)

Definition 1.13. A **homomorphism** between two representations $\rho_1\colon G \to GL(V)$ and $\rho_2\colon G \to GL(W)$ is a linear map $\psi\colon V \to W$ that interwines with (respects) the G-action, i.e. a map ψ such that

$$\psi(\rho_1(g)(v)) = \rho_2(g)(\psi(v)) \quad \forall v \in V, g \in G$$

An **isomorphism** of representations is a homomorphism of representations that is also an invertible map.

Note. If we have representations (ρ_1, V) and (ρ_2, W) and an isomorphism of vector spaces $\psi\colon V \to W$ then we can rewrite the compatibility requirement above as $\rho_2(g) = \psi \circ \rho_1(g) \circ \psi^{-1}$ for all $g \in G$.

Given any representation (ρ, V) of a group G on a vector space V over a field F of dimension n, we can fix a basis for V to obtain an isomorphism of vector spaces $\psi\colon V \to F^n$. This yields a representation ϕ of G on F^n by defining $\phi(g) = \psi \circ \rho(g) \circ \psi^{-1}$ for all $g \in G$. Clearly, this representation is isomorphic to our original representation (ρ, V). In particular, this means we can always choose to view n-dimensional complex representations as representations on $\mathbb{C}^n$ where each $\phi(g)$ is given by an $n \times n$ matrix with entries in $\mathbb{C}$.

Suppose that we have two representations $\rho_1\colon G \to GL_n(F)$ and $\rho_2\colon G \to GL_m(F)$ given by $\rho_1(g) = X_g$ and $\rho_2(g) = Y_g$. A homomorphism between these representations is then an $m \times n$ matrix A such that $AX_g = Y_g A$ for all $g \in G$. An isomorphism is given precisely when such A is square and invertible. Thus, two representations $\rho_1\colon G \to GL_n(F)$ and $\rho_2\colon G \to GL_n(F)$ are isomorphic if and only if there exists $A \in GL_n(F)$ such that $\rho_1(g) = A\rho_2(g)A^{-1}$ for all $g \in G$. This establishes the following proposition:

Proposition 1.14. *The isomorphism classes of n-dimensional representations of G on $\mathbb{C}$ are in bijection with the quotient $Hom(G; GL_n(\mathbb{C}))/GL_n(\mathbb{C})$ of group homomorphisms $G \to GL_n(\mathbb{C})$ modulo the conjugation action of $GL_n(\mathbb{C})$.*

1.3 Representations of Cyclic Groups

Example 1.15 (Representations of $\mathbb{Z}$)**.** We want to classify all representations of the group $\mathbb{Z}$ under addition. Consider an n-dimensional representation $\rho\colon \mathbb{Z} \to GL_n(\mathbb{C})$. For ρ to be a group homomorphism requires that $\rho(0) = \mathrm{Id}$. Observe that for any $0 \neq n \in \mathbb{Z}$, we have $\rho(n) = \rho(1 + \ldots + 1) = \rho(1)^n$. Thus ρ is completely determined by the matrix $\rho(1) \in GL_n(\mathbb{C})$, and any such matrix determines a representation of $\mathbb{Z}$. It follows that the n-dimensional isomorphism classes of representations of $\mathbb{Z}$ are in bijection with the conjugacy classes in $GL_n(\mathbb{C})$. These conjugacy classes can be parameterized by the *Jordan canonical form*.

Example 1.16 (Representations of the cyclic group of order n)**.** We shall classify all representations of the cyclic group $G = \{g, g^2, \dots, g^{n-1}, g^n = 1\}$ of order n. Consider a representation $\rho \colon G \to GL(V)$. As in the previous example, we know that $\rho(1) = \mathrm{Id}$ and $\rho(g^k) = \rho(g)^k$. Thus our representation ρ is determined completely by the linear transformation $\rho(g)$. It will be helpful to fix a basis of V so that we may view $\rho(g)$ as a matrix. Recall from linear algebra that there exists a basis in which $\rho(g)$ takes the *Jordan canonical form*

$$\rho(g) = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_m \end{bmatrix}$$

where each *Jordan block* J_k is of the form

$$J_k = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \ddots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix}.$$

Now $I = \rho(g)^n$ is a block-diagonal matrix with diagonal blocks J_k^n, so we must have that each block $J_k^n = \mathrm{Id}$. Observe that we can write each block J_k as $J_k = \lambda\mathrm{Id} + N$ where N is the Jordan block with $\lambda = 0$. Thus we have

$$\mathrm{Id} = J_k^n = (\lambda\mathrm{Id} + N)^n = \lambda^n\mathrm{Id} + \binom{n}{1}\lambda^{n-1}N + \binom{n}{2}\lambda^{n-2}N^2 + \dots + \binom{n}{n-1}\lambda N^{n-1} + N^n$$

. The following lemma will show that in fact $N = 0$.

Lemma 1.17. *Let N be the Jordan block with $\lambda = 0$ of size $n \times n$. For any integer p with $1 \le p \le n-1$, then N^p is the matrix with ones in the positions (i, j) where $j = i + p$ and zeroes everywhere else. (The ones lie along a line parallel to the diagonal, p steps above it.)*

Proof. (By induction.)

Base case: This is simply the definition of N.

Inductive step: Suppose that the lemma holds for N^p. We compute the (i, j) entry of N^{p+1}:

$$(N^{p+1})_{i,j} = \sum_{k=1}^{n} (N^p)_{i,k} N_{k,j} = (N^p)_{i,i+p} N_{i+p,j} = N_{i+p,j} = \begin{cases} 1 & \text{if} j = i + (p+1) \\ 0 & \text{otherwise} \end{cases}$$

□

Now, if $N \neq 0$ then each term $\binom{n}{k}\lambda^{n-k}N^k$ for $k > 0$ would yield some non-zero non-diagonal entries (in the positions (i, j) where $j = i + k$) and hence our sum could not equal the identity matrix. We must conclude that $N = 0$, $J_k = \lambda\mathrm{Id}$ is a 1×1 block, and $J_k^n = \lambda^n\mathrm{Id}$. Thus $\rho(g)$ is a diagonal matrix with nth roots of unity as diagonal entries.

To summarize, every m-dimensional representation ρ of the cyclic group $G = \langle g \rangle$ of order n can be seen to act (with the right choice of basis) as $m \times m$ diagonal matrices all with nth roots of unity along the diagonal. In particular, these representations are determined completely by the value of $\rho(g)$ and are classified up to isomorphism by unordered m-tuples of nth roots of unity.

1.4 Constructions from Linear Algebra

Definition 1.18. A **subrepresentation** of V is a G-invariant subspace $W \subseteq V$; that is, a subspace $W \subseteq V$ with the property that $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$. Note that W itself is a representation of G under the action $\rho(g)\restriction_W$.

From elementary linear algebra, we know that given a subspace $W \subseteq V$, we can form the **quotient space** V/W consisting of cosets $v + W$ in V. If W is a subrepresentation of V, we would like to define an action of G on V/W by the natural choice of $g(v + W) = \rho(g)(v) + W$. It remains to verify that this action is well defined. If we choose another $v' \in v + W$, then $v - v' \in W$, so that $\rho(g)(v - v') \in W$ since W is G-invariant. Thus, the cosets $\rho(g)(v) + W$ and $\rho(g)(v') + W$ agree and this action is indeed well defined. This justifies the following definition:

Definition 1.19. Let W be a G-subrepresentation of V. Then V/W forms a representation of G called the **quotient representation** of V under W with the action $g(v + W) = \rho(g)(v) + W$.

Recall also from linear algebra that given two vector spaces V_1 and V_2, we can form the **direct sum** $V_1 \oplus V_2$ consisting of ordered pairs (v_1, v_2) where $v_1 \in V_1, v_2 \in V_2$.

Definition 1.20. Let V_1 and V_2 be representations of G. Then $V_1 \oplus V_2$ forms a representation of G called the **direct sum representation** of V_1 and V_2 with the action $g(v_1, v_2) = (g \cdot v_1, g \cdot v_2)$.

1.5 Complete Reducibility and Unitarity

Definition 1.21. A representation is said to be **irreducible** if it contains no proper invariant subspaces. It is called **completely reducible** if it decomposes into a direct sum of irreducible subrepresentations.

Example 1.22.

1. Any irreducible representation is completely reducible.
2. Any 1-dimensional representations has no proper subspaces, and is thus irreducible.

Theorem 1.23. *If $A_1, A_2, \ldots, A_r$ are linear operators on V and each A_i is diagonalizable, they are simultaneously diagonalizable if and only if they commute.*

Proof. See Conrad [2, Theorem 5.1]. □

Theorem 1.24. *Every complex representation of a finite abelian group is completely reducible into irreducible representations of dimension* 1.

Proof. Take an arbitrary element $g \in G$. Since G is finite, we can find an integer n such that $g^n = 1$ and $\rho(g)^n = Id$. Hence the minimal polynomial of $\rho(g)$ divides $x^n - 1$. Recall that $x^n - 1$ has n distinct roots over $\mathbb{C}$, which are generated by taking powers of $\xi = e^{\frac{2\pi i}{n}}$. This means that the minimal polynomial $\rho(g)$ factors into linear factors only over $\mathbb{C}$ so that $\rho(g)$ is diagonalizable. We conclude that each $\rho(g)$ is (separately) diagonalizable since $g \in G$ was arbitrary.

Now, given any two elements $g_1, g_2 \in G$ we have

$$\begin{aligned} \rho(g_1)\rho(g_2) &= \rho(g_1 g_2) && \text{(since } \rho \text{ is a homomorphism)} \\ &= \rho(g_2 g_1) && \text{(since } G \text{ is abeilian)} \\ &= \rho(g_2)\rho(g_1) && \text{(since } \rho \text{ is a homomorphism).} \end{aligned}$$

Thus the matrices $\{\rho(g)\}$ commute, so we may apply theorem 1.23 to conclude that $\{\rho(g)\}$ are simultaneously diagonalizable, say with basis $\{e_1, ..., e_k\}$. Then we have $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \ldots \oplus \mathbb{C}e_n$, with each subspace $\mathbb{C}e_1$ invariant under the action of G. □

We recall the following definition from linear algebra:

Definition 1.25. Let V be a complex vector space. A **Hermitian inner product** on V is a map $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{C}$ that satisfies the following properties for all $u, v, w \in V$ and $c \in \mathbb{C}$:

1. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.
2. $\langle cu, v \rangle = c\langle u, v \rangle$.
3. $\langle u, v \rangle = \overline{\langle v, u \rangle}$.
4. $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$.

Definition 1.26. A representation ρ of G on a complex vector space V is **unitary** if V has been equipped with a hermetian inner product $\langle \cdot, \cdot \rangle$ which is preserved by the action of G, that is,

$$\langle v, w \rangle = \langle \rho(g)(v), \rho(g)(w) \rangle \quad \forall v, w \in V, g \in G.$$

A representation is said to be **unitarisable** if it can be equipped with such a product (even without one being specified).

Theorem 1.27. *[Weyl's unitary trick] Finite-dimensional represenations of finite groups are unitarisable.*

Proof. Take any Hermetian inner product on V, say $\langle \cdot, \cdot \rangle'$. We define a new inner product on V by *averaging over* G:

$$\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle'.$$

This new inner product satifies properties 1, 2, and 3 of Definition 1.25 by linearity. It remains to check positivity (4). Clearly $\langle v, v \rangle = 0$ when $v = 0$, since each term of the sum will equal zero. In the case where $v \neq 0$, observe that

$$\langle v, v \rangle = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, \rho(g)v \rangle' \geq 0$$

since each term of the sum is non-negative by the positivity of $\langle \cdot, \cdot \rangle'$. The only problem would occur if each term of this sum is equal to zero. But $\langle \rho(e)v, \rho(e)v \rangle' = \langle v, v \rangle' > 0$. Thus $\langle v, v \rangle > 0$.

Finally, we show that our new inner product is G-invariant. For any $h \in G$, we have

$$\begin{aligned} \langle \rho(h)v, \rho(h)w \rangle &= \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)\rho(h)v, \rho(g)\rho(h)w \rangle' \\ &= \frac{1}{|G|} \sum_{g \in G} \langle \rho(gh)v, \rho(gh)w \rangle' && \text{(since } \rho \text{ is a homomorphism)} \\ &= \frac{1}{|G|} \sum_{k \in G} \langle \rho(k)v, \rho(k)w \rangle' && \text{(by a change of variables)} \\ &= \langle v, w \rangle. \end{aligned}$$

□

Lemma 1.28. *Let V be a unitary representation of G and let $W \subseteq V$ be a G-invariant subspace. Then the orthogonal complement $W^\perp$ is also G-invariant.*

Proof. Choose arbitrary elements $v \in W^\perp$ and $g \in G$. We need to show that $gv \in W^\perp$. Now for any $w \in W$, we have $\langle v, w \rangle = 0$. Thus $\langle gv, gw \rangle = g\overline{g}\langle v, w \rangle = 0$ for any $w \in W$. Notice that $w' = gw \in W$ since W is G-invariant. This implies that $\langle gv, w' \rangle = 0$, i.e. $gv \in W^\perp$. □

Theorem 1.29. *A finite-dimensional unitary representation of a group is fully reducible into unitary irreducible subrepresentations.*

Proof. Let V be a finite dimensional unitary representation of G. We proceed by induction on the dimension of V. If $\dim(V) = 1$, then V is necessarily irreducible. Now, suppose the theorem holds for all W with $\dim(V) \leq n - 1$ and suppose $\dim(V) = n$. If V is irreducible, we are done. Otherwise, there exists a proper G-invariant subspace $W(\neq 0, V)$. We can write $V = W \oplus W^\perp$ by Lemma 1.28. Applying the inductive hypothesis to W and $W^\perp$, we obtain a decomposition into irreducibles

$$V = (W_1 \oplus \ldots \oplus W_j) \oplus (W_{j+1} \oplus \ldots \oplus W_k).$$

□

Corollary 1.30. *Every complex representation of a finite group is completely reducible.*

Proof. Any such representation is unitarisable y by Theorem 1.27. We can then apply Theorem 1.29 to obtain full reduciblility. □

1.6 Vector Spaces of Linear Maps

Definition 1.31. Let V and W be vector spaces. Recall that the set $\mathbf{Hom}(\mathbf{V}, \mathbf{W})$ of linear maps from V to W is itself a vector space. If f_1, f_2 are two linear maps from V to W, then we define their sum by

$$\begin{aligned} (f_1 + f_2) \colon V &\to W \\ x &\mapsto f_1(x) + f_2(x) \end{aligned}$$

and we define scalar multiplication of $\lambda \in \mathbb{C}$ by

$$\begin{aligned}(\lambda f_1)\colon V &\to W\\ x &\mapsto \lambda f_1(x).\end{aligned}$$

Now suppose we have representations $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ of G. Then there is a natural representation of G on the vector space $\mathrm{Hom}(V, W)$ given by

$$\begin{aligned}\rho_{\mathrm{Hom}(V,W)}(g)\colon \mathrm{Hom}(V, W) &\to \mathrm{Hom}(V, W)\\ f &\mapsto \rho_W(g) \circ f \circ \rho_V(g^{-1})\end{aligned}$$

for all $g \in G$. Note that $\rho_{\mathrm{Hom}(V,W)}(g)(f)$ is certainly a linear map from V to W since the composition of linear maps is linear.

Proposition 1.32. *$\rho_{Hom(V,W)}$ is a representation of G. That is, the map*

$$\begin{aligned}\rho_{Hom(V,W)}\colon G &\to GL(Hom(V, W))\\ g &\mapsto \rho_{Hom(V,W)}(g).\end{aligned}$$

is a homomorphism.

Proof. We must check two things:

1. The map $g \mapsto \rho_{\mathrm{Hom}(V,W)}(g)$ is a homomorphism.
2. For every $g \in G$, $\rho_{\mathrm{Hom}(V,W)}(g)$ is invertible.

First, we check that

$$\begin{aligned}\rho_{\mathrm{Hom}(V,W)}(g) \circ \rho_{\mathrm{Hom}(V,W)}(h)\colon f &\mapsto \rho_{\mathrm{Hom}(V,W)}(g)(\rho_W(h) \circ f \circ \rho_V(h^{-1}))\\ &= \rho_W(g) \circ \rho_w(h) \circ f \circ \rho_V(h^{-1}) \circ \rho_V(g^{-1})\\ &= \rho_W(gh) \circ f \circ \rho_V(g^{-1}h^{-1})\\ &= \rho_{\mathrm{Hom}(V,W)}(gh)(f)\end{aligned}$$

so indeed $\rho_{\mathrm{Hom}(V,W)}$ is a homomorphism. We can use this fact to see that $\rho_{\mathrm{Hom}(V,W)}(g^{-1})$ is inverse to $\rho_{\mathrm{Hom}(V,W)}(g)$ as

$$\begin{aligned}\rho_{\mathrm{Hom}(V,W)}(g) \circ \rho_{\mathrm{Hom}(V,W)}(g^{-1}) &= \rho_{\mathrm{Hom}(V,W)}(e)\\ &= \mathrm{Id}_{\mathrm{Hom}(V,W)}\\ &= \rho_{\mathrm{Hom}(V,W)}(g^{-1}) \circ \rho_{\mathrm{Hom}(V,W)}(g).\end{aligned}$$

Thus $\rho_{\mathrm{Hom}(V,W)}(g)$ is invertible for every $g \in G$, and $\rho_{\mathrm{Hom}(V,W)}$ is a representation of G. □

Definition 1.33. Let V and W be two representations of G. The set of G-linear maps from V to W forms a subspace of $\mathrm{Hom}(V, W)$, which we denote by $\mathbf{Hom_G(V, W)}$. In other words, $\mathrm{Hom}_G(V, W)$ is the vector space consisting of all *homomorphisms of representations* between V and W.

Definition 1.34. Let $\rho\colon G \to GL(V)$ be a representation. We define the **invariant subrepresentation** $V^G \subset V$ to be the set

$$\{v \in V \mid \rho(g)(v) = v, \quad \forall g \in G\}.$$

Note that V^G is a subspace of V, and is also clearly a subrepresentation. It is isomorphic to a trivial representation of some dimension.

Proposition 1.35. *Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be representations of G. Then the subrepresentation*

$$Hom_G(V, W) \subset Hom(V, W)$$

is precisely the invariant subrepresentation $Hom(V, W)^G$ of $Hom(V, W)$.

Proof. Let $f \in \mathrm{Hom}(V, W)$. Then f is an element of the invariant subrepresentation $\mathrm{Hom}(V, W)^G$ iff we have

$$\begin{aligned} & f = \rho_{\mathrm{Hom}(V,W)}(g)(f) \quad \forall g \in G \\ \iff & f = \rho_W(g) \circ f \circ \rho_V(g^{-1}) \quad \forall g \in G \\ \iff & f \circ \rho_V(g) = \rho_W(g) \circ f \quad \forall g \in G \end{aligned}$$

which is exactly the condition that f is G-linear, i.e. that $f \in \mathrm{Hom}_G(V, W)$. □

Lemma 1.36. *Let A and B be two representations of G. Then*

$$(A \oplus B)^G = A^G \oplus B^G.$$

Proof. Observe that

$$\begin{aligned} (a, b) \in (A \oplus B)^G \iff & \rho_{A \oplus B}(g)(a, b) = (a, b) && \forall g \in G \\ \iff & (\rho_A(g)(a), \rho_B(g)(b)) = (a, b) && \forall g \in G \\ \iff & (a, b) \in A^G \oplus B^G. \end{aligned}$$

□

Lemma 1.37. *Let $\psi \colon A \to B$ be an isomorphism between representations of G. Then ψ induces an isomorphism between their invariant subrepresentations*

$$\psi\restriction_{A^G} \colon A^G \to B^G.$$

Proof. Clearly the restriction of ψ to $A^G \subset A$ induces an isomorphism to some subrepresentation of B, but we must check that the image of this restriction actually equals B^G. We verify that

$$\begin{aligned} a \in A^G \iff & \rho_A(g)(a) = a && \forall g \in G \\ \iff & \psi(\rho_A(g)(a)) = \psi(a) && \forall g \in G \\ \iff & \rho_B(g)\psi(a) = \psi(a) && \forall g \in G \\ \iff & \psi(a) \in B^G. \end{aligned}$$

□

1.7 Schur's Lemma

Theorem 1.38. *[Schur's Lemma over $\mathbb{C}$.] If V is an irreducible G-representation over $\mathbb{C}$, then evey linear operator $\phi \colon V \to V$ commuting with G is a scalar.*

Proof. Let λ be an eigenvalue of ϕ. Observe that the eigenspace E_λ is G-invariant: If $v \in E_\lambda$, then $\phi(v) = \lambda v$. This implies that $\phi(gv) = g\phi(v) = g(\lambda v) = \lambda(gv)$, i.e. $gv \in E_\lambda$. Since g was arbitrary, E_λ is indeed G-invariant. Now $E_\lambda \neq 0$, so by irreducibility $E_\lambda = V$. Thus $\phi = \lambda\mathrm{Id}$. □

Corollary 1.39. *If V and W are irreducible, the space $Hom_G(V, W)$ is 1-dimensional if the representations are isomorphic, and in this case any non-zero map is an isomorphism. Otherwise, $Hom^G(V, W) = \{0\}$.*

Proof. We claim $\ker(\phi)$ and $\mathrm{im}(\phi)$ are both G-invariant. Let $0 \neq \phi \in \mathrm{Hom}_G(V, W)$. If $v \in \ker(\phi)$, then $\phi(v) = 0$ implies that $\phi(gv) = g\phi(v) = g0 = 0$, i.e. $gv \in \ker(\phi)$. Similarly, if $v \in \mathrm{im}(\phi)$, then $v = \phi(w)$ implies that $\phi(gw) = g\phi(w) = gv$, i.e. $gv \in \mathrm{im}(\phi)$.

Irreducibility yields $\ker(\phi) = 0$ or V and $\mathrm{im}(\phi) = 0$ or W as the only possibilities. If $\phi \neq 0$, then $\ker(\phi) = 0$. This means that ϕ is injective, $\mathrm{im}(\phi) = W$, and ϕ is an isomorphism.

Let ψ be another interwining operator from V to W. Then $\phi^{-1} \circ \psi$ is also an intertwining operator from V to V. We can apply Schur's Lemma over $\mathbb{C}$ to see that $\phi^{-1} \circ \psi = \lambda\mathrm{Id}$, hence $\psi = \lambda\phi$. □

More definitions are required before we can state a more general Schur's Lemma (not restricted to just $\mathbb{C}$).

Definition 1.40. An **algebra** over a field K is a ring with unit, containing a distinguished copy of K that commutes with every algebra element, and with $1 \in K$ begin the algebra unit. A **division ring** is a ring where every non-zero element is invertible, and a **division algebra** is a division ring which is also a K-algebra.

Definition 1.41. Let V be a representation of G over K. The **endomorphism algebra** $\mathrm{End}^G(V)$ is the space of linear self-maps $\phi\colon V \to V$ which commute with the group action, that is, $\rho(g) \circ \phi = \phi \circ \rho(g) \quad \forall g \in G$. The addition is the usual addition of linear maps (pointwise), and the multiplication is function composition. The distinguished copy of K is given by $K\mathrm{Id}$.

Theorem 1.42. *[Schur's Lemma] If V is an irreducible finite-dimensional representation of G over K, then $End^G(V)$ is a finite-dimensional division algebra over K.*

1.8 Tensor Product

Let V and W be two vector spaces over K, and assume we have bases $\{a_1, \ldots, a_n\}$ for V and $\{b_1, \ldots, b_m\}$ for W.

Definition 1.43. The **tensor product** $V \bigotimes_K W$ of V and W is the K-vector space which has a basis given by the set of symbols

$$\{a_i \otimes b_j | 1 \leq i \leq n, 1 \leq j \leq m\}.$$

When the ground field K is clear, in can be omitted from the notation. If we have vectors $x \in V$ and $y \in W$, we can define a vector $x \otimes y \in V \bigotimes W$ as follows. Write x and y in the given bases with coefficients $\lambda_i, \mu_j \in K$, so

$$x = \lambda_1 a_1 + \ldots + \lambda_n a_n$$

$$y = \mu_1 b_1 + \ldots + \mu_m b_m.$$

Then we define

$$x \otimes y = \sum_{\substack{i \in [1,n] \\ j \in [1,m]}} \lambda_i \mu_j a_i \otimes b_j.$$

Now let V and W be two representations of G.

Definition 1.44. We can define a representation of G on $V \otimes W$ called the **tensor product representation**. We define

$$\rho_{V \otimes W}(g) \colon V \otimes W \to V \otimes W$$

to be the linear map given by

$$\rho_{V \otimes W}(g) \colon a_i \otimes b_j \mapsto \rho_V(g)(a_i) \otimes \rho_W(g)(b_j).$$

1.9 Isotypical Decomposition

Lemma 1.45. *Let U, V, W be three vector spaces. Then we have natural isomorphisms*

1. $Hom(V, U \oplus W) = Hom(V, U) \oplus Hom(V, W)$
2. $Hom(U \oplus W, V) = Hom(U, V) \oplus Hom(W, V)$.

Additionally, if U, V, W carry representations of G, then (1) and (2) are isomorphisms of representations.

Proof. We have inclusion and projection maps

$$U \underset{\pi_U}{\overset{\iota_U}{\rightleftarrows}} U \oplus W \underset{\iota_W}{\overset{\pi_W}{\rightleftarrows}} W$$

given by

$$\iota_U \colon x \mapsto (x, 0)$$
$$\pi_U \colon (x, y) \mapsto x$$

and similarly for ι_W and π_W. It is clear that

$$\mathrm{Id}_{U \oplus W} = \iota_U \circ \pi_U + \iota_W \circ \pi_W.$$

We also note that the four spaces under consideration all have dimension $(\dim V)(\dim W + \dim U)$.

(1) We define a map

$$\psi \colon \mathrm{Hom}(V, U \oplus W) \to \mathrm{Hom}(V, U) \oplus \mathrm{Hom}(V, W)$$
$$f \mapsto (\pi_U \circ f, \pi_W \circ f).$$

We claim that this map has an inverse given by

$$\psi^{-1} \colon \mathrm{Hom}(V, U) \oplus \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, U \oplus W)$$
$$(f_U, f_W) \mapsto \iota_U \circ f_U + \iota_W \circ f_W.$$

Check that

$$\begin{aligned}\psi^{-1} \circ \psi \colon f &\mapsto \iota_U \circ \pi_U \circ f + \iota_W \circ \pi_W \circ f \\ &= (\iota_U \circ \pi_U + \iota_W \circ \pi_W) \circ f \\ &= \mathrm{Id}_{\mathrm{Hom}(V,W)} \circ f = f.\end{aligned}$$

Since both vector spaces have the same dimension, $\psi \circ \psi^{-1}$ must be the identity map as well, and ψ is an isomorphism of vector spaces. Now suppose we have representations ρ_V, ρ_W, ρ_U of G on V, W and U. Then we claim ψ is G-linear. Recall that by definition,

$$\rho_{\mathrm{Hom}(V,U\oplus W)}(g)(f) = \rho_{U\oplus W}(g) \circ f \circ \rho_V(g^{-1}).$$

Observe that for any $g \in G$ and $f \in \mathrm{Hom}(V, U \oplus W)$,

$$\begin{aligned}\pi_U \circ (\rho_{\mathrm{Hom}(V,U\oplus W)}(g)(f)) &= \pi_U \circ \rho_{U\oplus W}(g) \circ f \circ \rho_V(g^{-1}) \\ &= \rho_U(g) \circ \pi_U \circ f \circ \rho_V(g^{-1}) \quad (\text{since } \pi_U \text{ is } G\text{-linear}) \\ &= \rho_{\mathrm{Hom}(U,V)}(g)(f)\end{aligned}$$

and similarly for W, so that

$$\begin{aligned}\psi(\rho_{\mathrm{Hom}(V,U\oplus W)}(g)(f)) &= (\pi_U \circ \rho_{\mathrm{Hom}(V,U\oplus W)}(g)(f), \pi_W \circ \rho_{\mathrm{Hom}(V,U\oplus W)}(g)(f)) \\ &= (\rho_{\mathrm{Hom}(V,U)}(g)(\pi_U \circ f), \rho_{\mathrm{Hom}(V,W)}(g)(\pi_W \circ f)) \\ &= \rho_{\mathrm{Hom}(V,U)\oplus\mathrm{Hom}(V,W)}(g)(\pi_U \circ f, \pi_W \circ f).\end{aligned}$$

Thus ψ is G-linear, and we've proved (1).

(2) Define a map

$$\begin{aligned}\phi \colon \mathrm{Hom}(U \oplus W, V) &\to \mathrm{Hom}(U, V) \oplus \mathrm{Hom}(W, V) \\ &= (f \circ \iota_U, f \circ \iota_W).\end{aligned}$$

We [finish me]. The book says proof is like (1) □

Corollary 1.46. *If U, V, W are representations of G, then there are natural isomorphisms*

1. *$Hom_G(V, U \oplus W) = Hom_G(V, U) \oplus Hom_G(V, W)$*
2. *$Hom_G(U \oplus W, V) = Hom_G(U, V) \oplus Hom_G(W, V)$*

Proof. (1). By Lemma (1.45), we have an isomorphism of representations

$$\psi \colon \mathrm{Hom}(V, U \oplus W) \to \mathrm{Hom}(V, U) \oplus \mathrm{Hom}(V, W).$$

We can apply Lemma (1.37) to obtain an isomorphism on the invariant subrepresentations

$$\mathrm{Hom}(V, U \oplus W)^G \cong (\mathrm{Hom}(V, U) \oplus \mathrm{Hom}(V, W))^G.$$

Then Lemma (1.36) implies that

$$\mathrm{Hom}(V, U \oplus W)^G \cong \mathrm{Hom}(V, U)^G \oplus \mathrm{Hom}(V, W)^G.$$

The statement now follows from Proposition (1.35).

(2). The argument is similar to the one above. □

Proposition 1.47. *Let V and W be irreducible representations of G. Then*

$$\dim \mathrm{Hom}_G(V,W) = \begin{cases} 1 & \text{if } V \text{ and } W \text{ are isomorphic} \\ 0 & \text{if } V \text{ and } W \text{ are not isomorphic} \end{cases}$$

Proof. Suppose V and W are not isomorphic. Then Schur's Lemma states that the only G-linear map from V to W is the zero map, hence $\mathrm{Hom}_G(V,W) = \{0\}$.

On the other hand, suppose that $f\colon V \to W$ is an isomorphism. Then for any $h \in \mathrm{Hom}_G(V,W)$, we have $f^{-1} \circ h \in \mathrm{Hom}_G(V,V)$. By Schur's Lemma, $f^{-1} \circ h = \lambda \mathrm{Id}_V$, i.e. $h = \lambda f$ for some $\lambda \in \mathbb{C}$. Thus f spans $\mathrm{Hom}_G(V,W)$. □

Proposition 1.48. *Let $\rho\colon G \to GL(V)$ be a representation, and let*

$$V = U_1 \oplus \ldots \oplus U_s$$

be a decomposition of V into irreducibles. Let W be any irreducible representation of G. Then the number of irreducible representations in the set $\{U_1, \ldots, U_s\}$ which are isomorphic to W is equal to the dimension of $\mathrm{Hom}_G(W,V)$, and also equal to the dimension of $\mathrm{Hom}_G(V,W)$.

Proof. □

1.10 Character Theory

Definition 1.49. The **character** of a representation $\rho\colon G \to GL(V)$ is the function $\chi_V : G \to \mathbb{C}$ defined by $\chi_V(g) = \mathrm{Tr}(\rho(g))$.

Note. The character is of a representation is not a homorphism in general, since $\mathrm{Tr}(MN) \neq \mathrm{Tr}(M)\mathrm{Tr}(N)$ in general.

Proposition 1.50. *(Basic Properties)*

1. *χ_V is conjugation invariant: $\chi_V(hgh^{-1}) = \chi_V(g)$ for all $g, h \in G$.*
2. *$\chi_V(1) = \dim V$.*
3. *$\chi_V(g^{-1}) = \overline{\chi_V(g)}$ for all $g \in G$.*

Proof. 1. $\chi_V(hgh^{-1} = \mathrm{Tr}(hgh^{-1}) = \mathrm{Tr}(ghh^{-1}) = \mathrm{Tr}(g) = \chi_V(g)$ for any $g, h \in G$.

2. $\chi_V(1) = \mathrm{Tr}(\mathrm{Id}_V) = \dim V$.
3. Since G is finite, we have seen that $\rho(g)$ is a diagonal matrix with roots of unity along the diagonal with the right choice of basis. The inverse of a root of unity is given by its complex conjugate, so under this same basis, $\rho(g)^{-1}$ is clearly given by $\overline{\rho(g)}$. Thus, $\chi_V(g^{-1}) = \mathrm{Tr}(\rho(g^{-1})) = \mathrm{Tr}(\rho(g)^{-1}) = \mathrm{Tr}(\overline{\rho(g)}) = \overline{\mathrm{Tr}(\rho(g))} = \overline{\chi_V(g)}$. □

Definition 1.51. A **class function** on G is a function on G whose values are invariant by conjugation of elements in G. The value of a class function at an element $g \in G$ depends only on the conjugacy class of g. We may therefore view class functions as functions on the set of conjugacy classes of G.

Note. The character χ_V of a representation V of G is a class function on G.

Proposition 1.52. *Isomorphic representations have the same character.*

Proof. We have seen (CITE ME!!!) that isomorphic representations can be described by the same matrices in the right choice of basis. □

We will see later that the converse is true - if two representations have the same character, then they are isomorphic.

Proposition 1.53. *Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be representations of G with characters χ_V and χ_W.*

1. $\chi_{V\oplus W} = \chi_V + \chi_W$.
2. $\chi_{V\otimes W} = \chi_V \cdot \chi_W$.

Proof.

1. Pick bases for V and W, so that $\rho_V(g)$ and $\rho_W(g)$ are described by matrices M and N. Then $\rho_{V\oplus W}(g)$ is described by the block-diagonal matrix

$$\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

So we have $\mathrm{Tr}(\rho_{V\oplus W}(g)) = \mathrm{Tr}(M) + \mathrm{Tr}(N) = \mathrm{Tr}(\rho_V(g)) + \mathrm{Tr}(\rho_W(g))$.

2. $\rho_{V\otimes W}$ is given by the matrix

$$[M \otimes N]_{js,it} = M_{ji}N_{st}$$

□

Proposition 1.54.

1. *Let $\{V_i\}$ be the irreducible representations of G, with d_i the dimension of V_i and χ_i the corresponding irreducible character. Then*

$$\chi_{reg} = d_1\chi_1 + \ldots + d_r\chi_r$$

2. $$\chi_{reg}(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$$

[6] [4] [1] [2] [5] [3]

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