Character Tables for Representations of Finite Groups

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Table of contents

Motivation

Groups arise naturally as sets of symmetries of some object which are closed under composition and taking inverses. For example,

1. The **symmetric group** of degree n, S_n, is the group of all symmetries of the set $\{1, \ldots, n\}$.
2. The **dihedral group** of order $2n$, D_n, is the group of all symmetries of the regular n-gon in the plane.

In these two examples, S_n acts on the set $\{1, \ldots, n\}$ and D_n acts on the regular n-gon in a natural manner. One may wonder more generally: Given an abstract group G, which objects X does G act on? This is the basic question of representation theory, which attempts to classify all such X up to isomorphism.

Group Actions

Definition

A **group action** of a group G on a set X is a map $\rho\colon G \times X \to X$ (written as $g \cdot x$, for all $g \in G$ and $x \in X$) that satisfies the following two axioms:

$$1 \cdot x = x \qquad \forall x \in X \tag{1}$$

$$(gh) \cdot x = g \cdot (h \cdot x) \qquad \forall g, h \in G, x \in X \tag{2}$$

The Definition of a Representation

Definition

Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is an action of G on V that preserves the linear structure of V, i.e. an action of G on V such that

$$g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2 \qquad \forall g \in G, v_1, v_2 \in V \tag{3}$$

$$g \cdot (kv) = k(g \cdot v) \qquad \forall g \in G, v \in V, k \in F \tag{4}$$

Definition (Alternative definition)

Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is any group homomorphism

$$\rho\colon G \to GL(V).$$

Proposition

The two definitions we have given of a linear representation are equivalent.

Proof.

($\rightarrow$) Suppose that we have a homomorphism $\rho\colon G \to GL(V)$. We can obtain a linear action of G on V by defining $g \cdot v = \rho(g)(v)$.

($\leftarrow$) Suppose that we have a linear action of G on V. We obtain a homomorphism $\rho\colon G \to GL(V)$ by defining $\rho(g)(v) = g \cdot v$.

$\square$

The Dimension of a Representation

Definition

Let $\rho\colon G \to GL(V)$ be a representation of G. The **dimension** of the representation is the dimension of the vector space V.

Examples of Representations

Example

Let V be an n-dimensional vector space. The map $\rho\colon G \to GL(V)$ defined by $\rho(g) = \mathrm{Id}_V$ for all $g \in G$ is a representation of G called the **trival representation** of dimension n.

Examples of Representations

Example

If G is a finite group that acts on a finite set X, and F is any field, then there is an associated **permutation representation** on the vector space V over F with basis $\{e_x \colon x \in X\}$. We let G act on the basis elements by the permutation $g \cdot e_x = e_{gx}$ for all $x \in X$ and $g \in G$. This representation has dimension $|X|$.

Examples of Representations

Example

A special case of a permutation representation is that when a finite group acts on itself by left multiplication. We take the vector space V_{reg} which has a basis given by the formal symbols $\{e_g | g \in G\}$, and let $h \in G$ act by

$$\rho_{\text{reg}}(h)(e_g) = e_{hg}.$$

This representation is called the **regular representation** of G, and has dimension $|G|$.

Examples of Representations

Example

For any symmetric group S_n, the **alternating representation** on $\mathbb{C}$ is given by the map

$$\begin{aligned} \rho\colon S_n &\to GL(\mathbb{C}) = \mathbb{C}^\times \\ \sigma &\mapsto \mathrm{sgn}(\sigma). \end{aligned}$$

More generally, for any group G with a subgroup H of index 2, we can define an **alternating representation** $\rho\colon G \to GL(\mathbb{C})$ by letting $\rho(g) = 1$ if $g \in H$ and $\rho(g) = -1$ if $g \notin H$. (We recover our original example by taking $G = S_n$ and $H = A_n$.)

G-linear maps

Definition

A **homomorphism** between two representations $\rho_1 \colon G \to GL(V)$ and $\rho_2 \colon G \to GL(W)$ is a linear map $\psi \colon V \to W$ that interwines with the action of G, i.e. such that

$$\psi \circ \rho_1(g) = \rho_2(g) \circ \psi \quad \forall g \in G.$$

In this case, we also refer to ψ as a **G-linear map**.

Definition

An **isomorphism** of representations is a G-linear map that is also invertible.

Representations as matrices

Example

Given any representation (ρ, V), where V is a vector space of dimension n over the field K, we can fix a basis for V to obtain an isomorphism of vector spaces $\psi\colon V \to K^n$. This yields a representation ϕ of G on K^n by defining

$$\phi(g) = \psi \circ \rho(g) \circ \psi^{-1}$$

for all $g \in G$. This representation is isomorphic to our original representation (ρ, V). In particular, we can always choose to view complex n-dimensional representations of G as representations on $\mathbb{C}^n$, where each $\phi(g)$ is given by an $n \times n$ matrix with entries in $\mathbb{C}$.

Representations as matrices

Example

Let $G = \{(1), (123), (132)\} \subset S_3$. Let $V = \mathbb{C}^3$. Then G acts on the standard basis by $g \cdot e_i = e_{gi}$. Thus, the permutation representation of G (with respect to the standard basis) is given by:

$$\rho((1)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rho((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\rho((132)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Example

Let $G = C_2 \times C_2 = \langle \sigma, \tau | \sigma^2 = \tau^2 = e, \sigma\tau = \tau\sigma \rangle$ be the Klein four-group. Let V be the vector space with basis $\{b_e, b_\sigma, b_\tau, b_{\sigma\tau}\}$. Left multiplication by σ gives a permutation

$$\begin{aligned} b_e &\mapsto b_\sigma \\ b_\sigma &\mapsto b_e \\ b_\tau &\mapsto b_{\sigma\tau} \\ b_{\sigma\tau} &\mapsto b_\tau. \end{aligned}$$

We can similarly compute $\rho_{\text{reg}}(\tau)$. Thus, in our basis, the regular representation $\rho_{\text{reg}} \colon G \to GL(V)$ is given by the matrices

$$\rho_{\text{reg}}(\sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \rho_{\text{reg}}(\tau) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Example

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square.

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Example

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$$\rho(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\rho(\sigma) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \rho(\sigma\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\rho(\sigma^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \rho(\sigma^2\tau) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho(\sigma^3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \rho(\sigma^3\tau) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Subrepresentations

Definition

A **subrepresentation** of V is a G-invariant subspace $W \subseteq V$; that is, a subspace $W \subseteq V$ with the property that $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$. Note that W itself is a representation of G under the action $\rho(g) \restriction_W$.

Representations of C^2

Example

Let $G = C_2 = \langle \tau | \tau^2 = e \rangle$ be the cyclic group of order 2. The regular representation of G written in the standard basis is given by

$$\rho_{\text{reg}}(\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $\rho_{\text{reg}}(e) = \text{Id}_2$. Let ρ_{sgn} be the alternating representation of G on $\mathbb{C}$, i.e.

$$\begin{aligned} \rho_{\text{sgn}} \colon G &\to GL_1(\mathbb{C}) = \mathbb{C}^\times \\ \tau &\mapsto -1 \\ e &\mapsto 1. \end{aligned}$$

Representations of C^2

Example (Cont.)

Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be the linear map represented by the matrix $\begin{bmatrix} 1 & -1 \end{bmatrix}$. Then for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$, we have

$$\begin{aligned} f \circ \rho_{\mathsf{reg}}(\tau)(x) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \rho_{\mathsf{sgn}}(\tau) \circ f(x). \end{aligned}$$

Also note that $f \circ \rho_{\mathsf{reg}}(e) = \rho_{\mathsf{sgn}}(e) \circ f$. Thus f is a G-linear map from ρ_{reg} to ρ_{sgn} (i.e. a homomorphism of representations).

Representations of C^2

Example (Cont.)

Now let W be the subspace of $\mathbb{C}^2$ spanned by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Then

$$\rho_{\text{reg}}(\tau)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $\rho_{\text{reg}}(e)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so W is a G-invariant subpace, i.e. W is a subrepresentation of ρ_{reg}. Note that W is precisely equal to the kernel of the map f, and that W is isomorphic to the 1-dimensional trivial representation of G.

Example

We can generalize the G-invariant subspace from the previous example. Suppose we have a representation $\rho\colon G \to GL_n(\mathbb{C})$. If we can find a vector $x \in \mathbb{C}^n$ which is an eigenvector for every matrix $\rho(g), g \in G$, i.e. an $x \in \mathbb{C}^n$ such that

$$\rho(g)(x) = \lambda_g(x) \quad \forall g \in G$$

for some eigenvalues $\lambda_g \in \mathbb{C}$, then the span of x is a 1-dimensional G-invariant subspace of $\mathbb{C}^n$. It is isomorphic to the 1-dimensional representation

$$\begin{aligned} \rho_2\colon G &\to GL_1(\mathbb{C}) \\ g &\mapsto \lambda_g. \end{aligned}$$

Proposition

Let $f : V \to W$ be a homomorphism of representations of G. Then Ker(f) is a subrepresentation of V and Im(f) is a subrepresentation of W.

Proof.

- Let $x \in \mathrm{Ker}(f)$. Then $0 = g0 = gf(x) = f(gx)$ for every $g \in G$. So $gx \in \mathrm{Ker}(f)$ and $\mathrm{Ker}(f)$ is G-invariant.
- Now let $w \in \mathrm{Im}(f)$. There exists $v \in V$ such that $w = f(v)$, so $gw = gf(v) = f(gv)$ for every $g \in G$. Thus $gw \in \mathrm{Im}(f)$, and $\mathrm{Im}(f)$ is G-invariant.

□

The direct sum of representations

Note

We know from linear algebra that given two vector spaces V and W, we can form the **direct sum** $V \oplus W$ consisting of ordered pairs (v, w) where $v \in V, w \in W$.

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Definition

Let V and W be representations of G. Then $V \oplus W$ admits a natural representation of G, called the **direct sum representation** of V and W, which we define by

$$\rho_{V \oplus W} \colon G \to GL(V \oplus W)$$
$$\rho_{V \oplus W}(g) \colon (x, y) \mapsto (\rho_V(g)(x), \rho_W(g)(y)).$$

Irreducible representations and complete reducibility

Definition

A representation is said to be **irreducible** if it has no subrepresentations other than the trivial subrepresentations $0 \subset V$ and $V \subset V$. A representation is called **completely reducible** if it decomposes into a direct sum of irreducible subrepresentations.

Note

1. Any 1-dimensional representation V has no subspaces other than 0 and V itself, and is thus irreducible.
2. Any irreducible representation is, in particular, completely reducible.