Character Tables for Representations of Finite Groups

Jared Stewart
Advised by Dr. Calin Chindris

April 18, 2016

University of Missouri

Table of contents

Motivation for Representation Theory

Groups arise naturally as sets of symmetries of some object which are closed under composition and taking inverses. For example,

1. The **symmetric group** of degree n, S_n, is the group of all symmetries of the set $\{1, \ldots, n\}$.
2. The **dihedral group** of order $2n$, D_n, is the group of all symmetries of the regular n-gon in the plane.

In these two examples, S_n acts on the set $\{1, \ldots, n\}$ and D_n acts on the regular n-gon in a natural manner. One may wonder more generally: Given an abstract group G, which objects X does G act on? This is the basic question of representation theory, which attempts to classify all such X up to isomorphism.

The Definition of a Representation

Definition

Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is any group homomorphism

$$\rho\colon G \to GL(V).$$

Definition

The **dimension** of a representation $\rho\colon G \to GL(V)$ is the dimension of the vector space V.

Examples of Representations

Example

Let V be an n-dimensional vector space. The map $\rho\colon G \to GL(V)$ defined by $\rho(g) = \mathrm{Id}_V$ for all $g \in G$ is a representation of G called the **trival representation** of dimension n.

Example

If G is a finite group that acts on a finite set X, and F is any field, then there is an associated **permutation representation** on the vector space V over F with basis $\{e_x \colon x \in X\}$. We let G act on the basis elements by the permutation $g \cdot e_x = e_{gx}$ for all $x \in X$ and $g \in G$. This representation has dimension $|X|$.

The Regular Representation

Example

A special case of a permutation representation is that when a finite group acts on itself by left multiplication. We take the vector space V_{reg} which has a basis given by the formal symbols $\{e_g | g \in G\}$, and let $h \in G$ act by

$$\rho_{\text{reg}}(h)(e_g) = e_{hg}.$$

This representation is called the **regular representation** of G, and has dimension $|G|$.

Example

Let $G = \{(1), (123), (132)\} \leq S_3$. Let $V = \mathbb{C}^3$. Then G acts on the standard basis by $g \cdot e_i = e_{gi}$. Thus, the permutation representation of G (with respect to the standard basis) is given by:

$$\rho((1)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rho((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\rho((132)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Example

Let $G = C_2 \times C_2 = \langle \sigma, \tau | \sigma^2 = \tau^2 = e, \sigma\tau = \tau\sigma \rangle$ be the Klein four-group. Let V be the vector space with basis $\{b_e, b_\sigma, b_\tau, b_{\sigma\tau}\}$. Left multiplication by σ gives a permutation

$$\begin{aligned} b_e &\mapsto b_\sigma \\ b_\sigma &\mapsto b_e \\ b_\tau &\mapsto b_{\sigma\tau} \\ b_{\sigma\tau} &\mapsto b_\tau. \end{aligned}$$

We can similarly compute $\rho_{\mathsf{reg}}(\tau)$. Thus, in our basis, the regular representation $\rho_{\mathsf{reg}} \colon G \to GL(V)$ is given by the matrices

$$\rho_{\mathsf{reg}}(\sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \rho_{\mathsf{reg}}(\tau) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The Alternating Representation

Example

For any symmetric group S_n, the **alternating representation** on $\mathbb{C}$ is given by the map

$$\begin{aligned} \rho\colon S_n &\to GL(\mathbb{C}) = \mathbb{C}^\times \\ \sigma &\mapsto \mathsf{sgn}(\sigma). \end{aligned}$$

More generally, for any group G with a subgroup H of index 2, we can define an **alternating representation** $\rho\colon G \to GL(\mathbb{C})$ by letting $\rho(g) = 1$ if $g \in H$ and $\rho(g) = -1$ if $g \notin H$. (We recover our original example by taking $G = S_n$ and $H = A_n$.)

G-linear maps

Definition

A **homomorphism** between two representations $\rho_1\colon G \to GL(V)$ and $\rho_2\colon G \to GL(W)$ is a linear map $\psi\colon V \to W$ that interwines with the action of G, i.e. such that

$$\psi \circ \rho_1(g) = \rho_2(g) \circ \psi \quad \forall g \in G.$$

In this case, we also refer to ψ as a **G-linear map**.

Definition

An **isomorphism** of representations is a G-linear map that is also invertible.

Representations as matrices

Example

Given any representation (ρ, V), where V is a vector space of dimension n over the field K, we can fix a basis for V to obtain an isomorphism of vector spaces $\psi\colon V \to K^n$. This yields a representation ϕ of G on K^n by defining

$$\phi(g) = \psi \circ \rho(g) \circ \psi^{-1}$$

for all $g \in G$. This representation is isomorphic to our original representation (ρ, V). In particular, we can always choose to view complex n-dimensional representations of G as representations on $\mathbb{C}^n$, where each $\phi(g)$ is given by an invertible $n \times n$ matrix with entries in $\mathbb{C}$.

Example (2-dim rep of D_4.)

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square.

Example (2-dim rep of D_4.)

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square. Consider a square in the plane with vertices at $(1,1), (1,-1), (-1,-1)$, and $(-1,1)$.

Example (2-dim rep of D_4.)

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square. Consider a square in the plane with vertices at $(1,1), (1,-1), (-1,-1)$, and $(-1,1)$. We let σ act on the square as a rotation by $\frac{\pi}{2}$, and let τ act by reflection over the x-axis. This naturally gives rise to a linear action of G on all of $\mathbb{C}^2$.

Example (2-dim rep of D_4.)

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square. Consider a square in the plane with vertices at $(1, 1), (1, -1), (-1, -1)$, and $(-1, 1)$. We let σ act on the square as a rotation by $\frac{\pi}{2}$, and let τ act by reflection over the x-axis. This naturally gives rise to a linear action of G on all of $\mathbb{C}^2$. Under the standard basis, we get the matrices:

$$\rho(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\rho(\sigma) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \rho(\sigma\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\rho(\sigma^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \rho(\sigma^2\tau) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho(\sigma^3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \rho(\sigma^3\tau) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

The direct sum of representations

Definition

Let V and W be representations of G. Then $V \oplus W$ admits a natural representation of G, called the **direct sum representation** of V and W, which we define by

$$\rho_{V \oplus W} \colon G \to GL(V \oplus W)$$
$$\rho_{V \oplus W}(g) \colon (x, y) \mapsto (\rho_V(g)(x), \rho_W(g)(y)).$$

Irreducible representations and complete reducibility

Definition

A **subrepresentation** of V is a G-invariant subspace $W \subseteq V$; that is, a subspace $W \subseteq V$ with the property that $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$. Note that W itself is a representation of G under the action $\rho(g) \restriction_W$.

Definition

A representation is said to be **irreducible** if it has no subrepresentations other than the trivial subrepresentations $0 \subset V$ and $V \subset V$. A representation is called **completely reducible** if it decomposes into a direct sum of irreducible representations. We sometimes write **irrep** as shorthand for irreducible representation.

Example (A 2-dimensional irrep)

Let $G = D_3 = \langle \sigma, \tau | \sigma^3 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$. (Note that $D_3 \cong S_3$). Consider the regular triangle centered at the origin with vertices

$$(1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2}).$$

We can let σ act as rotation by $\frac{2\pi}{3}$ and let τ act as reflection over the x-axis to obtain an action of G on $\mathbb{C}^2$ given (under the standard basis) by the matrices

$$\rho(\sigma) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\rho(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Example (A 2-dimensional irrep cont.)

Suppose ρ has a non-trivial subrepresentation W. We must have dim $W = 1$. Since W is invariant under the action of both $\rho(\sigma)$ and $\rho(\tau)$, there must be some mutual eigenvector for $\rho(\sigma)$ and $\rho(\tau)$ that spans W. The eigenvectors of $\rho(\sigma)$ are

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} \quad (\lambda_1 = e^{\frac{2\pi i}{3}}) \quad \text{and} \quad \begin{bmatrix} 1 \\ i \end{bmatrix} \quad (\lambda_2 = e^{-\frac{2\pi i}{3}}).$$

The eigenvectors of $\rho(\tau)$ are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\lambda_1 = 1) \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\lambda_2 = -1).$$

Thus we see that there is no such W, and our representation is irreducible.

Representations of finite abelian groups

Theorem

If $A_1, A_2, \ldots, A_r$ are linear operators on V and each A_i is diagonalizable, then $\{A_i\}$ are simultaneously diagonalizable if and only if they commute.

Representations of finite abelian groups

Theorem

Every complex representation of a finite abelian group is completely reducible into irreducible representations of dimension 1.

Proof.

Take an arbitrary element $g \in G$. Since G is finite, we can find an integer n such that $g^n = 1$ and $\rho(g)^n = Id$. The minimal polynomial of $\rho(g)$ divides $x^n - 1$, which has n distinct roots over $\mathbb{C}$, so it factors into linear factors only over $\mathbb{C}$, i.e. $\rho(g)$ is diagonalizable. Now, given any two elements $g_1, g_2 \in G$ we have $\rho(g_1)\rho(g_2) = \rho(g_2)\rho(g_1)$. Since the matrices $\{\rho(g)\}$ commute, $\{\rho(g)\}$ are simultaneously diagonalizable, say with respect to basis $\{e_1, ..., e_k\}$. Then we have $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \ldots \oplus \mathbb{C}e_k$, with each subspace $\mathbb{C}e_i$ invariant under the action of G since e_i is an eigenvector for every $\rho(g)$. $\square$

Question:

Is every finite dimensional representation completely reducible?

Answer:

No, in general.

Example (Complete reducibility fails in the modular case)

Let F be a field whose characteristic divides $|G|$. Consider the element

$$x = \sum_{g \in G} g \in FG.$$

Then $gx = x$ for every $g \in G$. Moreover

$$x^2 = |G|x = 0.$$

If follows that FG contains nilpotent ideals, so is not semisimple. (Recall that an algebra is semisimple iff it is artinian and contains no nonzero nilpotent ideals.)

Example (Complete reducibility fails when the group is infinite)

Consider the additive group $G = (F, +)$, which we can view as a subgroup of $GL_2(F)$ by identifying $t \in F$ with the matrix

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Then consider the linear action of G on $V = K^2$ given by $t \cdot (x, y) = (x + ty, y)$. Any one-dimensional subspace spanned by a vector $(x_1, y_1) \in V$ is G-invariant precisely when for all $t \in F$ there exist $\lambda_t \in F$ such that

$$t \cdot (x_1, y_1) = \lambda_t (x_1, y_1).$$

But this requires $y_1 = 0$, so that the only one-dimensional G-subrepresentation of V is spanned by $(1, 0)$. This subrepresentation has no G-invariant direct complement.

Maschke's Theorem

Theorem (Maschke's Theorem)

Let G be a finite group and let F be a field such that char$(F) \nmid |G|$. If V is any finite dimensional representation of G over F, and $W \subset V$ is a subrepresentation of V, then there exists a complementary subrepresentation $U \subset V$ to W, i.e. there is a G-invariant subspace $U \subset V$ such that

$$V = W \oplus U.$$

Definition

Let W be a subspace of V. A **linear projection** V onto W is a linear map $f: V \to W$ such that $f \restriction_W = \mathrm{Id}_W$. If W is a subrepresentation of V and the map f is G-invariant, then we say that f is a **G-linear projection**.

Lemma

Let $\rho: G \to GL(V)$ be a representation, and $W \subset V$ be a subrepresentation. Suppose we have a G-linear projection

$$f: V \to W.$$

Then Ker(f) is a complementary subrepresentation to W, i.e. Ker(f) is a G-invariant subspace of V such that

$$V = \mathit{Ker}(f) \oplus W$$

Maschke's Theorem

Proof.

It will suffice to find a G-linear projection from V onto W. Fix a basis $\{b_1, \ldots, b_m\}$ for W and extend it to a basis $\{b_1, \ldots, b_m, b_{m+1}, \ldots, b_n\}$ for V. Let $U = \langle b_{m+1}, \ldots, b_n \rangle$. Then U is certainly a complementary subspace to W, and we have a natural projection $f \colon W \oplus U \to W$ of V onto W with kernel U. There is no reason to think that f should be G-linear, but we can fix this by averaging over G. Define $\widetilde{f} \colon V \to V$ by

$$\widetilde{f}(x) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}))(x).$$

We claim that $\widetilde{f}$ is a G-linear projection from V onto W.

Maschke's Theorem

Proof.

First we check that $\mathrm{Im}(\tilde{f}) \subset W$. If $x \in V$ and $g \in G$, then

$$f(\rho(g^{-1})(x)) \in W$$

and so

$$\rho(g)(f(\rho(g^{-1})(x))) \in W$$

since W is G-invariant. Thus

$$\widetilde{f}(x) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}))(x) \in W.$$

Maschke's Theorem

Proof.

Next we check that $\widetilde{f}\restriction_W = \mathrm{Id}_W$. Let $y \in W$. For any $g \in G$, we know that $\rho(g^{-1})(y)$ is also in W, so $f(\rho(g^{-1})(y)) = \rho(g^{-1})(y)$. Then

$$\begin{aligned}
\widetilde{f}(y) &= \frac{1}{|G|}\sum_{g\in G}\rho(g)(f(\rho(g^{-1})(y))) \\
&= \frac{1}{|G|}\sum_{g\in G}\rho(g)(\rho(g^{-1})(y)) \\
&= \frac{1}{|G|}\sum_{g\in G}(y) = \frac{|G|y}{|G|} = y
\end{aligned}$$

so indeed $\widetilde{f}$ is a linear projection of V onto W.

Maschke's Theorem

Proof.

Finally, we check that $\widetilde{f}$ is G-linear. If $x \in V$ and $h \in G$, then

$$\begin{aligned}(\widetilde{f} \circ \rho(h))(x) &= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}) \circ \rho(h))(x) \\ &= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}h))(x) \\ &= \frac{1}{|G|} \sum_{g \in G} (\rho(hg) \circ f \circ \rho(g^{-1}))(x) \quad (g \mapsto hg) \\ &= (\rho(h) \circ \widetilde{f})(x).\end{aligned}$$

□

Consequences of Machke's Theorem

Corollary

Let G be a finite group and let F be a field such that $\mathrm{char}(F) \nmid |G|$. Then any finite-dimensional representation of G over F is completely reducible.

Proof.

Let V be a representation of G over F of dimension n. If V is irreducible, then V is, in particular, completely reducible. If not, then V contains a proper subrepresentation $W \subset V$. From Maschke's Theorem, we know there exists a subrepresentation $U \subset V$ such that

$$V = W \oplus U. \tag{1}$$

Both W and U have dimension less than n, so by induction we know that W and U are completely reducible. We deduce that V is completely reducible. □

Example

Recall that for $G = C_2$, we found a 1-dim subrepresentation

$$W = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle \subset V_{\text{reg}} = \mathbb{C}^2.$$

We know a complementary subrepresentation to W exists by Machke's Theorem, so let's try to find one. Consider

$$U = \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle \subset V_{\text{reg}}.$$

Then

$$\rho_{\text{reg}}(\tau) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so U is G-invariant. We see that $V = W \oplus U$, since $W \cap U = \{0\}$ and dim $U +$ dim $W = 2 =$ dim V. (Note U is isomorphic to the alternating representation ρ_{sgn}.)

Proposition

Suppose we have representations $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ of G. Then there is a natural representation of G on the vector space Hom(V, W) given for all $g \in G$ by

$$\begin{aligned} \rho_{\mathit{Hom}(V,W)}(g) \colon \mathit{Hom}(V,W) &\to \mathit{Hom}(V,W) \\ f &\mapsto \rho_W(g) \circ f \circ \rho_V(g^{-1}). \end{aligned}$$

Definition

Let V and W be two representations of G. The set of G-linear maps from V to W, which we denote by $\mathbf{Hom_G(V, W)}$, forms a subspace of $\mathrm{Hom}(V, W)$. In other words, $\mathrm{Hom}_G(V, W)$ is the vector space consisting of all *homomorphisms of representations* between V and W.

Definition

Let $\rho \colon G \to GL(V)$ be a representation. We define the **invariant subrepresentation** $V^G \subset V$ to be the set

$$\{v \in V \mid \rho(g)(v) = v, \quad \forall g \in G\}.$$

Remark

$\mathrm{Hom}_G(V, W) = (\mathrm{Hom}(V, W))^G$.

Theorem (Schur's Lemma over $\mathbb{C}$.)

If V is a complex irreducible representation of G, then $\mathit{End}_G(V) = \{\lambda \mathit{Id}_v | \lambda \in \mathbb{C}\}$.

Proof.

Let $\phi\colon V \to V$ be a G-linear endomorphism of V, and let λ be an eigenvalue of ϕ. We claim that the eigenspace E_λ is G-invariant. If $v \in E_\lambda$, then $\phi(v) = \lambda v$. This implies that $\phi(gv) = g\phi(v) = g(\lambda v) = \lambda(gv)$, i.e. $gv \in E_\lambda$. Since g was arbitrary, E_λ is indeed G-invariant. Now $E_\lambda \neq 0$, so since V is irreducible, $E_\lambda = V$. Thus $\phi = \lambda \mathrm{Id}$. □

Corollary

Suppose V and W are irreducible. The space $\text{Hom}_G(V, W)$ is 1-dimensional if the representations are isomorphic, and in this case any non-zero map is an isomorphism. Otherwise, $\text{Hom}_G(V, W) = \{0\}$.

Proof.

Suppose $\text{Hom}_G(V, W) \neq \{0\}$ and let $\phi \in \text{Hom}_G(V, W)$ be a nonzero G-linear map. Since $\ker(\phi)$ and $\text{im}(\phi)$ are both G-invariant, irreducibility yields ($\ker(\phi) = 0$ or V) and ($\text{im}(\phi) = 0$ or W) as the only possibilities. Since $\phi \neq 0$, then $\ker(\phi) = 0$, $\text{im}(\phi) = W$, and ϕ is an isomorphism. Let ψ be another nonzero G-linear map from V to W. Then $\phi^{-1} \circ \psi \in \text{Hom}_G(V, V)$. We can apply Schur's Lemma over $\mathbb{C}$ to see that $\phi^{-1} \circ \psi = \lambda \text{Id}$, hence $\psi = \lambda\phi$. So ϕ spans $\text{Hom}_G(V, W)$. $\square$

Proposition

Let V and W be irreducible representations of G. Then

$$\textit{dim } \textit{Hom}_G(V, W) = \begin{cases} 1 & \textit{if } V \textit{ and } W \textit{ are isomorphic} \\ 0 & \textit{if } V \textit{ and } W \textit{ are not isomorphic} \end{cases}$$

Proposition

Let $\rho\colon G \to GL(V)$ *be a representation, let*

$$V = U_1 \oplus \ldots \oplus U_s$$

be a decomposition of V *into irreps, and let* W *be any irrep of* G*. Then the number of irreps in the set* $\{U_1, \ldots, U_s\}$ *which are isomorphic to* W *equals the dimension of* $\mathrm{Hom}_G(V, W)$*.*

Proof.

Have:

$$\mathrm{Hom}_G(V, W) = \bigoplus_{i=1}^{s} \mathrm{Hom}_G(U_i, W),$$

so taking the dimension of both sides yields

$$\dim \mathrm{Hom}_G(V, W) = \sum_{i=1}^{s} \dim \mathrm{Hom}_G(U_i, W).$$

By previous Proposition, this sum is exactly the $\#$ of irreps in $\{U_1, \ldots, U_s\}$ which are isomorphic to W. $\square$

Theorem (Uniqueness of decomposition into irreducibles.)

Let $\rho\colon G \to GL(V)$ be a representation, and let

$$V = U_1 \oplus \ldots \oplus U_s$$
$$V = \widetilde{U_1} \oplus \ldots \oplus \widetilde{U_r}$$

be two decompositions of V into irreducible subrepresentations. Then $s = r$, and (after reordering if necessary) U_i and $\widetilde{U}_i$ are isomorphic for every $i \in \{1, \ldots, s\}$.

Proof.

For any irrep W of G, the number of irreps in either decomposition that are isomorphic to W is equal to $\dim \operatorname{Hom}_G(V, W)$. So the two decompositions contain the same number of factors isomorphic to W for any irrep W of G. $\square$

The definition of a Character

Definition

The **character** of a representation $\rho\colon G \to GL(V)$ is the function

$$\chi_V : G \to \mathbb{C}$$

defined by

$$\chi_V(g) = \mathrm{Tr}(\rho(g)).$$

Note

The character of a representation is not a homomorphism in general, since $\mathrm{Tr}(MN) \neq \mathrm{Tr}(M)\mathrm{Tr}(N)$ in general.

Basic properties of Characters

Proposition

Let V be a representation of G.

- χ_V *is conjugation invariant:* $\chi_V(hgh^{-1}) = \chi_V(g) \quad \forall g, h \in G$.
- $\chi_V(e) =$ *dim* V.
- $\chi_V(g^{-1}) = \overline{\chi_V(g)} \quad \forall g \in G$.
- $\chi_{V^*}(g) = \overline{\chi_V(g)} \quad \forall g \in G$.

Proposition

Let V and W be representations of G.

- $\chi_{V \oplus W} = \chi_V + \chi_W$.
- $\chi_{V \otimes W} = \chi_V \cdot \chi_W$.

Proposition

Isomorphic representations have the same character.

Proof.

Isomorphic representations can be described by the same set of matrices with the right choice of bases. Thus each $\rho(g)$ has the same trace. □

Definition

Let $\mathbb{C}^G$ denote the vector space of all functions from G to $\mathbb{C}$. A basis for $\mathbb{C}^G$ is given by the set of functions

$$\{\delta_g | g \in G\}$$

defined by

$$\delta_g : h \mapsto \begin{cases} 1 & \text{if } h = g \\ 0 & \text{if } h \neq g. \end{cases}$$

Definition

Let $\varphi, \psi \in \mathbb{C}^G$. We define a **hermetian inner product** on $\mathbb{C}^G$ by

$$\langle \varphi | \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}.$$

Inner product of Characters

Theorem

Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be representations of G, and let χ_V, χ_W be their characters. Then

$$\langle \chi_W | \chi_V \rangle = \textit{dim Hom}_G(V, W).$$

Corollary

Let $\chi_1, \ldots, \chi_r$ be characters of pairwise non-isomorphic irreducible representations of G. Then

$$\langle \chi_i | \chi_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Proof.

Let χ_i and χ_j be the characters of the irreducible representations U_i, U_j. Then

$$\langle \chi_i | \chi_j \rangle = \dim \mathrm{Hom}_G(U_j, U_i) = \begin{cases} 1 & \text{if } U_i, U_j \text{ are isomorphic} \\ 0 & \text{if } U_i, U_j \text{ are not isomorphic.} \end{cases}$$

□

Corollary

Let χ be any character of G. Then χ is irreducible if and only if

$$\langle\chi|\chi\rangle = 1$$

Proof.

Write χ as a linear combination of irreducible characters

$$\chi = m_1\chi_1 + \ldots + m_k\chi_k$$

where each m_i is a non-negative integer. Then

$$\begin{aligned}\langle\chi|\chi\rangle &= \sum_{i,j\in[1,k]} m_i m_j \langle\chi_i|\chi_j\rangle \\ &= m_1^2 + \ldots + m_k^2.\end{aligned}$$

So $\langle\chi|\chi\rangle = 1$ if and only if exactly one of the $m_i = 1$ and the rest are 0. $\square$

Example

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$. Recall the two dimensional representation W of D_4 given earlier. We compute the character of this representation by taking the trace of the matrices from that example:

$$\begin{aligned} \chi_W(e) &= 2 & \chi_W(\tau) &= 0 \\ \chi_W(\sigma) &= 0 & \chi_W(\sigma\tau) &= 0 \\ \chi_W(\sigma^2) &= -2 & \chi_W(\sigma^2\tau) &= 0 \\ \chi_W(\sigma^3) &= 0 & \chi_W(\sigma^3\tau) &= 0. \end{aligned}$$

Then

$$\langle \chi_W | \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_W(g) \overline{\chi_W(g)} = \frac{1}{8}(4+4) = 1$$

so we conclude that W is irreducible.

Corollary

Let V and W be representations of G. Then V and W are isomorphic if and only if $\chi_V = \chi_W$.

Proof.

Suppose $\chi_V = \chi_W$. We can find non-negative integers m_i and l_j such that

$$V = U_1^{m_1} \oplus \ldots \oplus U_r^{m_r} \quad \text{and} \quad W = U_1^{l_1} \oplus \ldots \oplus U_r^{l_r}$$

where $U_1, \ldots, U_r$ are distinct irreps of G. Then

$$\chi_V = m_1\chi_1 + \ldots + m_r\chi_r \quad \text{and} \quad \chi_W = l_1\chi_1 + \ldots + l_r\chi_r.$$

It follows that

$$m_i = \langle \chi_V | \chi_i \rangle = \langle \chi_W | \chi_i \rangle = l_i$$

for all $i \in \{1, \ldots, r\}$ since $\chi_V = \chi_W$. $\square$

Lemma

$$\chi_{\text{reg}}(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$$

Proposition

The multiplicity of any irreducible representation in the regular representation equals its dimension.

Proof.

Let V be an irreducible representation of G. Then

$$\begin{aligned} \langle \chi_{\text{reg}}, \chi_V \rangle &= \frac{1}{|G|} \chi_{\text{reg}}(e)\overline{\chi_V(e)} \\ &= \frac{1}{|G|} |G|(\dim V) = \dim V. \end{aligned}$$

□

Corollary

There are finitely many irreducible representations of G, up to isomorphism.

Corollary

Let $U_1, \ldots, U_r$ be the irreducible representations of G with degrees $d_1, \ldots, d_r$. Then

$$|G| = \sum_{i=1}^{n} d_i^2$$

Definition

We define **the character table of** G to be the table of complex numbers whose:

- rows are index by the isomorphism classes of irreducible representations of G,
- columns are indexed by the conjugacy classes of G,
- i, j entry is given by value of the character corresponding to row i evaluated at the conjugacy class corresponding to column j.

Note

To find the inner product of χ_V and χ_W, we only need to calculate χ_V and χ_W once on each conjugacy class, i.e.

$$\langle \chi_V | \chi_W \rangle = \frac{1}{|G|} \sum_{[g]} |[g]| \chi_V(g) \overline{\chi_W(g)}.$$

Character table of D_3

Example

Consider $G = D_3 = \langle \sigma, \tau | \sigma^3 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$. We have seen three irreducible representations of D_3, namely the 1-dimensional trivial representation, the 1-dimensional alternating representation, and the 2-dimensional irreducible representation W constructed geometrically. Observe that

$$|D_3| = 6 = 1^2 + 1^2 + 2^2$$

so these are all of the irreducible representations of D_3 up to isomorphism.

Character table of D_3

Example

The conjugacy classes of D_3 are $\{e\}$, $\{\sigma, \sigma^2\}$, and $\{\tau, \tau\sigma, \tau\sigma^2\}$. Thus, the character table of D_3 is given by

Character table of D_3			
Conjugacy class representative $[g]$	$[e]$	$[\tau]$	$[\sigma]$
χ_1 (1-d trivial reprn)	1	1	1
χ_{sgn} (1-d sign reprn)	1	-1	1
χ_W (2-d reprn obtained geometrically)	2	0	-1

Character Table of D_4

Example

Let $G = D_4$. Let $U_1, \ldots, U_r$ be the irreducible representations of D_4, with dimensions $d_1, \ldots, d_r$ respectively, and let U_1 be the 1-dimensional trivial representation. Then

$$d_2^2 + \ldots + d_r^2 = |G| - d_1^2 = 8 - 1 = 7.$$

There are two possibilities:
1. $r = 8$, and $d_i = 1$ for all $1 \leq i \leq 8$.
2. or $r = 5$, and $d_2 = d_3 = d_4 = 1$, $d_5 = 2$.

We saw earlier that G has a two-dimensional irreducible representation, so in fact (2) holds.

Character Table of D_4

Example

The remaining 1-dimensional representations are easy to find, since they must satisfy the relations $\rho(\sigma)^2 = 1$ and $\rho(\tau)^2 = 1$. Thus the character table for D_4 is as follows:

Character table of D_4					
Conjugacy class	$\{1\}$	$\{\sigma, \sigma^3\}$	$\{\sigma^2\}$	$\{\tau, \sigma^2\tau\}$	$\{\sigma\tau, \sigma^3\tau\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	1	1	-1
χ_4	1	-1	1	-1	1
χ_W (2-d reprn)	2	0	-2	0	0