Character Tables for Representations of Finite Groups

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Motivation

Groups arise naturally as sets of symmetries of some object which are closed under composition and taking inverses. For example,

1. The **symmetric group** of degree n, S_n, is the group of all symmetries of the set $\{1, \ldots, n\}$.
2. The **dihedral group** of order $2n$, D_n, is the group of all symmetries of the regular n-gon in the plane.

In these two examples, S_n acts on the set $\{1, \ldots, n\}$ and D_n acts on the regular n-gon in a natural manner. One may wonder more generally: Given an abstract group G, which objects X does G act on? This is the basic question of representation theory, which attempts to classify all such X up to isomorphism.

Group Actions

Definition

A **group action** of a group G on a set X is a map $\rho\colon G \times X \to X$ (written as $g \cdot x$, for all $g \in G$ and $x \in X$) that satisfies the following two axoims:

$$1 \cdot x = x \qquad \forall x \in X \tag{1}$$

$$(gh) \cdot x = g \cdot (h \cdot x) \qquad \forall g, h \in G, x \in X \tag{2}$$

The Definition of a Representation

Definition

Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is an action of G on V that preserves the linear structure of V, i.e. an action of G on V such that

$$g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2 \qquad \forall g \in G, v_1, v_2 \in V \tag{3}$$

$$g \cdot (kv) = k(g \cdot v) \qquad \forall g \in G, v \in V, k \in F \tag{4}$$

Definition (Alternative definition)

Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is any group homomorphism

$$\rho \colon G \to GL(V).$$

Proposition

The two definitions we have given of a linear representation are equivalent.

Proof.

($\rightarrow$) Suppose that we have a homomorphism $\rho\colon G \to GL(V)$. We can obtain a linear action of G on V by defining $g \cdot v = \rho(g)(v)$.

($\leftarrow$) Suppose that we have a linear action of G on V. We obtain a homomorphism $\rho\colon G \to GL(V)$ by defining $\rho(g)(v) = g \cdot v$.

$\square$

The Dimension of a Representation

Definition

Let $\rho \colon G \to GL(V)$ be a representation of G. The **dimension** of the representation is the dimension of the vector space V.

Examples of Representations

Example

Let V be an n-dimensional vector space. The map $\rho\colon G \to GL(V)$ defined by $\rho(g) = \mathsf{Id}_V$ for all $g \in G$ is a representation of G called the **trival representation** of dimension n.

Examples of Representations

Example

If G is a finite group that acts on a finite set X, and F is any field, then there is an associated **permutation representation** on the vector space V over F with basis $\{e_x : x \in X\}$. We let G act on the basis elements by the permutation $g \cdot e_x = e_{gx}$ for all $x \in X$ and $g \in G$. This representation has dimension $|X|$.

Examples of Representations

Example

A special case of a permutation representation is that when a finite group acts on itself by left multiplication. We take the vector space V_{reg} which has a basis given by the formal symbols $\{e_g | g \in G\}$, and let $h \in G$ act by

$$\rho_{\text{reg}}(h)(e_g) = e_{hg}.$$

This representation is called the **regular representation** of G, and has dimension $|G|$.

Examples of Representations

Example

For any symmetric group S_n, the **alternating representation** on $\mathbb{C}$ is given by the map

$$\rho\colon S_n \to GL(\mathbb{C}) = \mathbb{C}^\times$$
$$\sigma \mapsto \mathsf{sgn}(\sigma).$$

More generally, for any group G with a subgroup H of index 2, we can define an **alternating representation** $\rho\colon G \to GL(\mathbb{C})$ by letting $\rho(g) = 1$ if $g \in H$ and $\rho(g) = -1$ if $g \notin H$. (We recover our original example by taking $G = S_n$ and $H = A_n$.)

G-linear maps

Definition

A **homomorphism** between two representations $\rho_1 \colon G \to GL(V)$ and $\rho_2 \colon G \to GL(W)$ is a linear map $\psi \colon V \to W$ that interwines with the action of G, i.e. such that

$$\psi \circ \rho_1(g) = \rho_2(g) \circ \psi \quad \forall g \in G.$$

In this case, we also refer to ψ as a **G-linear map**.

Definition

An **isomorphism** of representations is a G-linear map that is also invertible.

Representations as matrices

Example

Given any representation (ρ, V), where V is a vector space of dimension n over the field K, we can fix a basis for V to obtain an isomorphism of vector spaces $\psi\colon V \to K^n$. This yields a representation ϕ of G on K^n by defining

$$\phi(g) = \psi \circ \rho(g) \circ \psi^{-1}$$

for all $g \in G$. This representation is isomorphic to our original representation (ρ, V). In particular, we can always choose to view complex n-dimensional representations of G as representations on $\mathbb{C}^n$, where each $\phi(g)$ is given by an $n \times n$ matrix with entries in $\mathbb{C}$.

Representations as matrices

Example

Let $G = \{(1), (123), (132)\} \subset S_3$. Let $V = \mathbb{C}^3$. Then G acts on the standard basis by $g \cdot e_i = e_{gi}$. Thus, the permutation representation of G (with respect to the standard basis) is given by:

$$\rho((1)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rho((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\rho((132)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Example

Let $G = C_2 \times C_2 = \langle \sigma, \tau | \sigma^2 = \tau^2 = e, \sigma\tau = \tau\sigma \rangle$ be the Klein four-group. Let V be the vector space with basis $\{b_e, b_\sigma, b_\tau, b_{\sigma\tau}\}$. Left multiplication by σ gives a permutation

$$\begin{aligned} b_e &\mapsto b_\sigma \\ b_\sigma &\mapsto b_e \\ b_\tau &\mapsto b_{\sigma\tau} \\ b_{\sigma\tau} &\mapsto b_\tau. \end{aligned}$$

We can similarly compute $\rho_{\mathsf{reg}}(\tau)$. Thus, in our basis, the regular representation $\rho_{\mathsf{reg}} \colon G \to GL(V)$ is given by the matrices

$$\rho_{\mathsf{reg}}(\sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \rho_{\mathsf{reg}}(\tau) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Example (2-dim rep of D_4.)

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square.

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$$\rho(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\rho(\sigma) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \rho(\sigma\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\rho(\sigma^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \rho(\sigma^2\tau) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho(\sigma^3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \rho(\sigma^3\tau) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Subrepresentations

Definition

A **subrepresentation** of V is a G-invariant subspace $W \subseteq V$; that is, a subspace $W \subseteq V$ with the property that $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$. Note that W itself is a representation of G under the action $\rho(g) \restriction_W$.

Representations of C^2

Example

Let $G = C_2 = \langle \tau | \tau^2 = e \rangle$ be the cyclic group of order 2. The regular representation of G written in the standard basis is given by

$$\rho_{\text{reg}}(\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $\rho_{\text{reg}}(e) = \text{Id}_2$. Let ρ_{sgn} be the alternating representation of G on $\mathbb{C}$, i.e.

$$\begin{aligned} \rho_{\text{sgn}} \colon G &\to GL_1(\mathbb{C}) = \mathbb{C}^\times \\ \tau &\mapsto -1 \\ e &\mapsto 1. \end{aligned}$$

Representations of C^2

Example (Cont.)

Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be the linear map represented by the matrix $\begin{bmatrix} 1 & -1 \end{bmatrix}$. Then for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$, we have

$$\begin{aligned} f \circ \rho_{\text{reg}}(\tau)(x) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \rho_{\text{sgn}}(\tau) \circ f(x). \end{aligned}$$

Also note that $f \circ \rho_{\text{reg}}(e) = \rho_{\text{sgn}}(e) \circ f$. Thus f is a G-linear map from ρ_{reg} to ρ_{sgn} (i.e. a homomorphism of representations).

Representations of C^2

Example (Cont.)

Now let W be the subspace of $\mathbb{C}^2$ spanned by the vector $\begin{bmatrix}1\\1\end{bmatrix}$.

Then

$$\rho_{\text{reg}}(\tau)\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

and $\rho_{\text{reg}}(e)\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$, so W is a G-invariant subpace, i.e. W is a subrepresentation of ρ_{reg}. Note that W is precisely equal to the kernel of the map f, and that W is isomorphic to the 1-dimensional trivial representation of G.

Example

We can generalize the G-invariant subspace from the previous example. Suppose we have a representation $\rho\colon G \to GL_n(\mathbb{C})$. If we can find a vector $x \in \mathbb{C}^n$ which is an eigenvector for every matrix $\rho(g), g \in G$, i.e. an $x \in \mathbb{C}^n$ such that

$$\rho(g)(x) = \lambda_g(x) \quad \forall g \in G$$

for some eigenvalues $\lambda_g \in \mathbb{C}$, then the span of x is a 1-dimensional G-invariant subspace of $\mathbb{C}^n$. It is isomorphic to the 1-dimensional representation

$$\rho_2\colon G \to GL_1(\mathbb{C})$$
$$g \mapsto \lambda_g.$$

Proposition

Let $f : V \to W$ be a homomorphism of representations of G. Then Ker(f) is a subrepresentation of V and Im(f) is a subrepresentation of W.

Proof.

- Let $x \in \text{Ker}(f)$. Then $0 = g0 = gf(x) = f(gx)$ for every $g \in G$. So $gx \in \text{Ker}(f)$ and Ker(f) is G-invariant.
- Now let $w \in \text{Im}(f)$. There exists $v \in V$ such that $w = f(v)$, so $gw = gf(v) = f(gv)$ for every $g \in G$. Thus $gw \in \text{Im}(f)$, and Im(f) is G-invariant.

$\square$

The direct sum of representations

Note

We know from linear algebra that given two vector spaces V and W, we can form the **direct sum** $V \oplus W$ consisting of ordered pairs (v, w) where $v \in V, w \in W$.

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Definition

Let V and W be representations of G. Then $V \oplus W$ admits a natural representation of G, called the **direct sum representation** of V and W, which we define by

$$\rho_{V \oplus W} \colon G \to GL(V \oplus W)$$
$$\rho_{V \oplus W}(g) \colon (x, y) \mapsto (\rho_V(g)(x), \rho_W(g)(y)).$$

Irreducible representations and complete reducibility

Definition

A representation is said to be **irreducible** if it has no subrepresentations other than the trivial subrepresentations $0 \subset V$ and $V \subset V$. A representation is called **completely reducible** if it decomposes into a direct sum of irreducible representations. We sometimes write **irrep** as shorthand for irreducible representation.

Note

1. Any 1-dimensional representation V has no subspaces other than 0 and V itself, and is thus irreducible.
2. Any irreducible representation is, in particular, completely reducible.

Example (A 2-dimensional irrep)

Let $G = D_3 = \langle \sigma, \tau | \sigma^3 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$. (Note that $D_3 \cong S_3$). Consider the regular triangle centered at the origin with vertices

$$(1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2}).$$

We can let σ act as rotation by $\frac{2\pi}{3}$ and let τ act as reflection over the x-axis to obtain an action of G on $\mathbb{C}^2$ given (under the standard basis) by the matrices

$$\rho(\sigma) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\rho(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Example (A 2-dimensional irrep cont.)

Suppose ρ has a non-trivial subrepresentation W. We must have dim $W = 1$. Since W is invariant under the action of both $\rho(\sigma)$ and $\rho(\tau)$, there must be some mutual eigenvector for $\rho(\sigma)$ and $\rho(\tau)$ that spans W. The eigenvectors of $\rho(\sigma)$ are

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} \quad (\lambda_1 = e^{\frac{2\pi i}{3}}) \quad \text{and} \quad \begin{bmatrix} 1 \\ i \end{bmatrix} \quad (\lambda_2 = e^{-\frac{2\pi i}{3}}).$$

The eigenvectors of $\rho(\tau)$ are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\lambda_1 = 1) \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\lambda_2 = -1).$$

Thus we see that there is no such W, and our representation is irreducible.

Representations of finite abelian groups

Theorem

If $A_1, A_2, \ldots, A_r$ are linear operators on V and each A_i is diagonalizable, then $\{A_i\}$ are simultaneously diagonalizable if and only if they commute.

Representations of finite abelian groups

Theorem

Every complex representation of a finite abelian group is completely reducible into irreducible representations of dimension 1.

Proof.

Take an arbitrary element $g \in G$. Since G is finite, we can find an integer n such that $g^n = 1$ and $\rho(g)^n = Id$. The minimal polynomial of $\rho(g)$ divides $x^n - 1$, which has n distinct roots over $\mathbb{C}$. So the minimal polynomial of $\rho(g)$ factors into linear factors only over $\mathbb{C}$, i.e. $\rho(g)$ is diagonalizable. We conclude that each $\rho(g)$ is (separately) diagonalizable since $g \in G$ was arbitrary. Now, given any two elements $g_1, g_2 \in G$ we have $\rho(g_1)\rho(g_2) = \rho(g_2)\rho(g_1)$. Since the matrices $\{\rho(g)\}$ commute, $\{\rho(g)\}$ are simultaneously diagonalizable, say with basis $\{e_1, ..., e_k\}$. Then we have $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \ldots \oplus \mathbb{C}e_n$, with each subspace $\mathbb{C}e_1$ invariant under the action of G. □

Definition

Let W be a subspace of V. A **linear projection** V onto W is a linear map $f\colon V \to W$ such that $f\restriction_W = \mathrm{Id}_W$. If W is a subrepresentation of V and the map f is G-invariant, then we say that f is a **G-linear projection**.

Lemma

Let $\rho\colon G \to GL(V)$ be a representation, and $W \subset V$ be a subrepresentation. Suppose we have a G-linear projection

$$f\colon V \to W.$$

Then Ker(f) is a complementary subrepresentation to W, i.e. Ker(f) is a G-invariant subspace of V such that

$$V = \mathit{Ker}(f) \oplus W$$

Maschke's Theorem

Theorem (Maschke's Theorem)

Let G be a finite group and let F be a field such that char$(F) \nmid |G|$. If V is any finite dimensional representation of G over F, and $W \subset V$ is a subrepresentation of V, then there exists a complementary subrepresentation $U \subset V$ to W, i.e. there is a G-invariant subspace $U \subset V$ such that

$$V = W \oplus U.$$

Maschke's Theorem

Proof.

It will suffice to find a G-linear projection from V onto W. Fix a basis $\{b_1, \ldots, b_m\}$ for W and extend it to a basis $\{b_1, \ldots, b_m, b_{m+1}, \ldots, b_n\}$ for V. Let $U = \langle b_{m+1}, \ldots, b_n \rangle$. Then U is certainly a complementary subspace to W, and we have a natural projection $f\colon W \oplus U \to W$ of V onto W with kernel U. There is no reason to think that f should be G-linear, but we can fix this by averaging over G. Define $\widetilde{f}\colon V \to V$ by

$$\widetilde{f}(x) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}))(x).$$

We claim that $\widetilde{f}$ is a G-linear projection from V onto W.

Maschke's Theorem

Proof.

First we check that $\text{Im}(\tilde{f}) \subset W$. If $x \in V$ and $g \in G$, then

$$f(\rho(g^{-1})(x)) \in W$$

and so

$$\rho(g)(f(\rho(g^{-1})(x))) \in W$$

since W is G-invariant. Thus

$$\widetilde{f}(x) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}))(x) \in W.$$

Maschke's Theorem

Proof.

Next we check that $\widetilde{f} \restriction_W = \mathrm{Id}_W$. Let $y \in W$. For any $g \in G$, we know that $\rho(g^{-1})(y)$ is also in W, so $f(\rho(g^{-1})(y)) = \rho(g^{-1})(y)$. Then

$$\begin{aligned}
\widetilde{f}(y) &= \frac{1}{|G|} \sum_{g \in G} \rho(g)(f(\rho(g^{-1})(y))) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho(g)(\rho(g^{-1})(y)) \\
&= \frac{1}{|G|} \sum_{g \in G} (y) = \frac{|G| y}{|G|} = y
\end{aligned}$$

so indeed $\widetilde{f}$ is a linear projection of V onto W.

Maschke's Theorem

Proof.

Finally, we check that $\widetilde{f}$ is G-linear. If $x \in V$ and $h \in G$, then

$$\begin{aligned}
(\widetilde{f} \circ \rho(h))(x) &= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}) \circ \rho(h))(x) \\
&= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}h))(x) \\
&= \frac{1}{|G|} \sum_{g \in G} (\rho(hg) \circ f \circ \rho(g^{-1}))(x) \quad (g \mapsto hg) \\
&= (\rho(h) \circ \widetilde{f})(x).
\end{aligned}$$

$\square$

Corollary

Let G be a finite group and let F be a field such that $\mathrm{char}(F) \nmid |G|$. then any finite-dimensional representation of G over F is completely reducible.

Proof.

Let V be a representation of G over F of dimension n. If V is irreducible, then V is, in particular, completely reducible. If not, then V contains a proper subrepresentation $W \subset V$. From Maschke's Theorem, we know there exists a subrepresentation $U \subset V$ such that

$$V = W \oplus U. \tag{5}$$

Both W and U have dimension less than n, so by induction we know that W and U are completely reducible. We deduce that V is completely reducible. $\square$

Example

Recall that for $G = C_2$, we found a 1-dim subrepresentation

$$W = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle \subset V_{\text{reg}} = \mathbb{C}^2.$$

We know a complementary subrepresentation to W exists by Machke's Theorem, so let's try to find one. Consider

$$U = \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle \subset V_{\text{reg}}.$$

Then

$$\rho_{\text{reg}}(\tau) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so U is G-invariant. We see that $V = W \oplus U$, since $W \cap U = \{0\}$ and $\dim U + \dim W = 2 = \dim V$. (Note U is isomorphic to the alternating representation ρ_{sgn}.)

Definition

Let V and W be vector spaces. Recall that the set **Hom**$(\mathbf{V}, \mathbf{W})$ of linear maps from V to W itself form a vector space where we define the addition of vectors by

$$\begin{aligned}(f_1 + f_2)\colon V &\to W \\ x &\mapsto f_1(x) + f_2(x)\end{aligned}$$

for $f_1, f_2 \in \mathrm{Hom}(V, W)$ and scalar multiplication for $\lambda \in \mathbb{C}$ by

$$\begin{aligned}(\lambda f_1)\colon V &\to W \\ x &\mapsto \lambda f_1(x).\end{aligned}$$

Proposition

Suppose we have representations $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ of G. Then there is a natural representation of G on the vector space Hom(V, W) given for all $g \in G$ by

$$\begin{aligned} \rho_{\mathit{Hom}(V,W)}(g) \colon \mathit{Hom}(V, W) &\to \mathit{Hom}(V, W) \\ f &\mapsto \rho_W(g) \circ f \circ \rho_V(g^{-1}). \end{aligned}$$

Proof (sketch).

1. $\rho_{\mathrm{Hom}(V,W)}(g)(f) \in \mathrm{Hom}(V, W)$ since the composition of linear maps is linear.
2. For every $g \in G$, $\rho_{\mathrm{Hom}(V,W)}(g)$ is invertible.
3. The map $g \mapsto \rho_{\mathrm{Hom}(V,W)}(g)$ is a homomorphism.

$\square$

Definition

Let V and W be two representations of G. The set of G-linear maps from V to W forms a subspace of Hom(V, W), which we denote by **Hom**$_{\mathbf{G}}(\mathbf{V}, \mathbf{W})$. In other words, Hom$_G(V, W)$ is the vector space consisting of all *homomorphisms of representations* between V and W.

Definition

Let $\rho \colon G \to GL(V)$ be a representation. We define the **invariant subrepresentation** $V^G \subset V$ to be the set

$$\{v \in V \mid \rho(g)(v) = v, \quad \forall g \in G\}.$$

Proposition

Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be representations of G. Then the subrepresentation

$$Hom_G(V,W) \subset Hom(V,W)$$

is precisely the invariant subrepresentation $Hom(V,W)^G$ of $Hom(V,W)$.

Proof.

Let $f \in \mathrm{Hom}(V,W)$. Then $f \in \mathrm{Hom}(V,W)^G$ iff we have

$$\begin{aligned} & f = \rho_{\mathrm{Hom}(V,W)}(g)(f) \quad \forall g \in G \\ \iff & f = \rho_W(g) \circ f \circ \rho_V(g^{-1}) \quad \forall g \in G \\ \iff & f \circ \rho_V(g) = \rho_W(g) \circ f \quad \forall g \in G \end{aligned}$$

which is exactly the condition that f is G-linear. □

Theorem (Schur's Lemma over $\mathbb{C}$.)

If V is an irreducible representation of G over $\mathbb{C}$, then evey linear operator $\phi \colon V \to V$ commuting with G is a scalar.

Proof.

Let $\phi \colon V \to V$ be a linear operator commuting with G, and let λ be an eigenvalue of ϕ. Observe that the eigenspace E_λ is G-invariant: If $v \in E_\lambda$, then $\phi(v) = \lambda v$. This implies that $\phi(gv) = g\phi(v) = g(\lambda v) = \lambda(gv)$, i.e. $gv \in E_\lambda$. Since g was arbitrary, E_λ is indeed G-invariant. Now $E_\lambda \neq 0$, so since V is irreducible, $E_\lambda = V$. Thus $\phi = \lambda \mathrm{Id}$. $\square$

Corollary

Suppose V and W are irreducible. The space $\mathrm{Hom}_G(V, W)$ is 1-dimensional if the representations are isomorphic, and in this case any non-zero map is an isomorphism. Otherwise, $\mathrm{Hom}_G(V, W) = \{0\}$.

Proof.

Suppose $\mathrm{Hom}_G(V, W) \neq \{0\}$ and let $\phi \in \mathrm{Hom}_G(V, W)$. We have seen $\ker(\phi)$ and $\mathrm{im}(\phi)$ are both G-invariant. Irreducibility yields $\ker(\phi) = 0$ or V and $\mathrm{im}(\phi) = 0$ or W as the only possibilities. Since $\phi \neq 0$, then $\ker(\phi) = 0$, $\mathrm{im}(\phi) = W$, and ϕ is an isomorphism. Let ψ be another nonzero interwining operator from V to W. Then $\phi^{-1} \circ \psi \in \mathrm{Hom}_G(V, V)$. We can apply Schur's Lemma over $\mathbb{C}$ to see that $\phi^{-1} \circ \psi = \lambda \mathrm{Id}$, hence $\psi = \lambda \phi$. So ϕ spans $\mathrm{Hom}_G(V, W)$. □

Proposition

Let V and W be irreducible representations of G. Then

$$\textit{dim } \mathrm{Hom}_G(V,W) = \begin{cases} 1 & \textit{if } V \textit{ and } W \textit{ are isomorphic} \\ 0 & \textit{if } V \textit{ and } W \textit{ are not isomorphic} \end{cases}$$

Proof.

Suppose V and W are not isomorphic. Then the Corollary to Schur's Lemma states that the only G-linear map from V to W is the zero map, hence $\mathrm{Hom}_G(V,W) = \{0\}$.
On the other hand, suppose that $f : V \to W$ is an isomorphism. Then for any $h \in \mathrm{Hom}_G(V,W)$, we have $f^{-1} \circ h \in \mathrm{Hom}_G(V,V)$. By Schur's Lemma, $f^{-1} \circ h = \lambda \mathrm{Id}_V$ for some $\lambda \in \mathbb{C}$, i.e. $h = \lambda f$. Thus f spans $\mathrm{Hom}_G(V,W)$. □

Proposition

Let $\rho \colon G \to GL(V)$ be a representation, let

$$V = U_1 \oplus \ldots \oplus U_s$$

be a decomposition of V into irreps, and let W be any irrep of G. Then the number of irreps in the set $\{U_1, \ldots, U_s\}$ which are isomorphic to W equals the dimension of $\mathrm{Hom}_G(V, W)$.

Lemma

If U, V, and W are representations of G, then there are natural isomorphisms

- $\mathrm{Hom}_G(V, U \oplus W) = \mathrm{Hom}_G(V, U) \oplus \mathrm{Hom}_G(V, W)$
- $\mathrm{Hom}_G(U \oplus W, V) = \mathrm{Hom}_G(U, V) \oplus \mathrm{Hom}_G(W, V)$

Proof.

The number of irreps in the set $\{U_1, \ldots, U_s\}$ which are isomorphic to W is equal to

$$\sum_{i=1}^{s} \dim \mathrm{Hom}_G(U_i, W).$$

Then

$$\mathrm{Hom}_G(V, W) = \bigoplus_{i=1}^{s} \mathrm{Hom}_G(U_i, W).$$

so that

$$\dim \mathrm{Hom}_G(V, W) = \sum_{i=1}^{s} \dim \mathrm{Hom}_G(U_i, W).$$

□

Theorem (Uniqueness of decomposition into irreducibles.)

Let $\rho\colon G \to GL(V)$ be a representation, and let

$$V = U_1 \oplus \ldots \oplus U_s$$
$$V = \widetilde{U_1} \oplus \ldots \oplus \widetilde{U_r}$$

be two decompositions of V into irreducible subrepresentations. Then $s = r$, and (after reordering if necessary) U_i and $\widetilde{U}_i$ are isomorphic for every $i \in \{1, \ldots, s\}$.

Proof.

For any irrep W of G, the number of irreps in either decomposition that are isomorphic to W is equal to dim $\mathrm{Hom}_G(V, W)$. So for any irrep W, the two decompositions contain the same number of factors isomorphic to W. $\square$