Character Tables for Representations of Finite Groups

Jared Stewart

April 3, 2016

Table of contents

Motivation

Groups arise naturally as sets of symmetries of some object which are closed under composition and taking inverses. For example,

1. The **symmetric group** of degree n, S_n, is the group of all symmetries of the set $\{1, \dots, n\}$.
2. The **dihedral group** of order $2n$, D_n, is the group of all symmetries of the regular n-gon in the plane.

In these two examples, S_n acts on the set $\{1, \dots, n\}$ and D_n acts on the regular n-gon in a natural manner. One may wonder more generally: Given an abstract group G, which objects X does G act on? This is the basic question of representation theory, which attempts to classify all such X up to isomorphism.

Group Actions

Definition

A **group action** of a group G on a set X is a map $\rho\colon G \times X \to X$ (written as $g \cdot x$, for all $g \in G$ and $x \in X$) that satisfies the following two axioms:

$$1 \cdot x = x \qquad \forall x \in X \tag{1}$$

$$(gh) \cdot x = g \cdot (h \cdot x) \qquad \forall g, h \in G, x \in X \tag{2}$$

The Definition of a Representation

Definition

Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is an action of G on V that preserves the linear structure of V, i.e. an action of G on V such that

$$g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2 \qquad \forall g \in G, v_1, v_2 \in V \tag{3}$$

$$g \cdot (kv) = k(g \cdot v) \qquad \forall g \in G, v \in V, k \in F \tag{4}$$

Definition (Alternative definition)

Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is any group homomorphism

$$\rho\colon G \to GL(V).$$

Proposition

The two definitions we have given of a linear representation are equivalent.

Proof.

($\rightarrow$) Suppose that we have a homomorphism $\rho\colon G \to GL(V)$. We can obtain a linear action of G on V by defining $g \cdot v = \rho(g)(v)$.

($\leftarrow$) Suppose that we have a linear action of G on V. We obtain a homomorphism $\rho\colon G \to GL(V)$ by defining $\rho(g)(v) = g \cdot v$.

$\square$

The Dimension of a Representation

Definition

Let $\rho\colon G \to GL(V)$ be a representation of G. The **dimension** of the representation is the dimension of the vector space V.

Examples of Representations

Example

Let V be an n-dimensional vector space. The map $\rho\colon G \to GL(V)$ defined by $\rho(g) = \mathrm{Id}_V$ for all $g \in G$ is a representation of G called the **trival representation** of dimension n.

Examples of Representations

Example

If G is a finite group that acts on a finite set X, and F is any field, then there is an associated **permutation representation** on the vector space V over F with basis $\{e_x : x \in X\}$. We let G act on the basis elements by the permutation $g \cdot e_x = e_{gx}$ for all $x \in X$ and $g \in G$. This representation has dimension $|X|$.

Examples of Representations

Example

A special case of a permutation representation is that when a finite group acts on itself by left multiplication. We take the vector space V_{reg} which has a basis given by the formal symbols $\{e_g | g \in G\}$, and let $h \in G$ act by

$$\rho_{\text{reg}}(h)(e_g) = e_{hg}.$$

This representation is called the **regular representation** of G, and has dimension $|G|$.

Examples of Representations

Example

For any symmetric group S_n, the **alternating representation** on $\mathbb{C}$ is given by the map

$$\rho\colon S_n \to GL(\mathbb{C}) = \mathbb{C}^\times$$
$$\sigma \mapsto \mathrm{sgn}(\sigma).$$

More generally, for any group G with a subgroup H of index 2, we can define an **alternating representation** $\rho\colon G \to GL(\mathbb{C})$ by letting $\rho(g) = 1$ if $g \in H$ and $\rho(g) = -1$ if $g \notin H$. (We recover our original example by taking $G = S_n$ and $H = A_n$.)

G-linear maps

Definition

A **homomorphism** between two representations $\rho_1 \colon G \to GL(V)$ and $\rho_2 \colon G \to GL(W)$ is a linear map $\psi \colon V \to W$ that interwines with the action of G, i.e. such that

$$\psi \circ \rho_1(g) = \rho_2(g) \circ \psi \quad \forall g \in G.$$

In this case, we also refer to ψ as a **G-linear map**.

Definition

An **isomorphism** of representations is a G-linear map that is also invertible.

Representations as matrices

Example

Given any representation (ρ, V), where V is a vector space of dimension n over the field K, we can fix a basis for V to obtain an isomorphism of vector spaces $\psi\colon V \to K^n$. This yields a representation ϕ of G on K^n by defining

$$\phi(g) = \psi \circ \rho(g) \circ \psi^{-1}$$

for all $g \in G$. This representation is isomorphic to our original representation (ρ, V). In particular, we can always choose to view complex n-dimensional representations of G as representations on $\mathbb{C}^n$, where each $\phi(g)$ is given by an $n \times n$ matrix with entries in $\mathbb{C}$.

Representations as matrices

Example

Let $G = \{(1), (123), (132)\} \subset S_3$. Let $V = \mathbb{C}^3$. Then G acts on the standard basis by $g \cdot e_i = e_{gi}$. Thus, the permutation representation of G (with respect to the standard basis) is given by:

$$\rho((1)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rho((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\rho((132)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Example

Let $G = C_2 \times C_2 = \langle \sigma, \tau | \sigma^2 = \tau^2 = e, \sigma\tau = \tau\sigma \rangle$ be the Klein four-group. Let V be the vector space with basis $\{b_e, b_\sigma, b_\tau, b_{\sigma\tau}\}$. Left multiplication by σ gives a permutation

$$\begin{aligned} b_e &\mapsto b_\sigma \\ b_\sigma &\mapsto b_e \\ b_\tau &\mapsto b_{\sigma\tau} \\ b_{\sigma\tau} &\mapsto b_\tau. \end{aligned}$$

We can similarly compute $\rho_{\text{reg}}(\tau)$. Thus, in our basis, the regular representation $\rho_{\text{reg}} \colon G \to GL(V)$ is given by the matrices

$$\rho_{\text{reg}}(\sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \rho_{\text{reg}}(\tau) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Example

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square.

Example

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square. Consider a square in the plane with vertices at $(1,1), (1,-1), (-1,-1)$, and $(-1,1)$.

Example

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square. Consider a square in the plane with vertices at $(1,1), (1,-1), (-1,-1)$, and $(-1,1)$. We let σ act on the square as a rotation by $\frac{\pi}{2}$, and let τ act by reflection over the x-axis. This naturally gives rise to a linear action of G on all of $\mathbb{C}^2$.

Example

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square. Consider a square in the plane with vertices at $(1,1), (1,-1), (-1,-1)$, and $(-1,1)$. We let σ act on the square as a rotation by $\frac{\pi}{2}$, and let τ act by reflection over the x-axis. This naturally gives rise to a linear action of G on all of $\mathbb{C}^2$. Under the standard basis, we get the matrices:

$$\rho(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\rho(\sigma) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \rho(\sigma\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\rho(\sigma^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \rho(\sigma^2\tau) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho(\sigma^3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \rho(\sigma^3\tau) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Subrepresentations

Definition

A **subrepresentation** of V is a G-invariant subspace $W \subseteq V$; that is, a subspace $W \subseteq V$ with the property that $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$. Note that W itself is a representation of G under the action $\rho(g) \restriction_W$.

Representations of C^2

Example

Let $G = C_2 = \langle \tau | \tau^2 = e \rangle$ be the cyclic group of order 2. The regular representation of G written in the standard basis is given by

$$\rho_{\text{reg}}(\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $\rho_{\text{reg}}(e) = \mathrm{Id}_2$. Let ρ_{sgn} be the alternating representation of G on $\mathbb{C}$, i.e.

$$\begin{aligned} \rho_{\text{sgn}} \colon G &\to GL_1(\mathbb{C}) = \mathbb{C}^\times \\ \tau &\mapsto -1 \\ e &\mapsto 1. \end{aligned}$$

Representations of C^2

Example (Cont.)

Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be the linear map represented by the matrix $\begin{bmatrix} 1 & -1 \end{bmatrix}$. Then for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$, we have

$$\begin{aligned} f \circ \rho_{\mathsf{reg}}(\tau)(x) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \rho_{\mathsf{sgn}}(\tau) \circ f(x). \end{aligned}$$

Also note that $f \circ \rho_{\mathsf{reg}}(e) = \rho_{\mathsf{sgn}}(e) \circ f$. Thus f is a G-linear map from ρ_{reg} to ρ_{sgn} (i.e. a homomorphism of representations).

Representations of C^2

Example (Cont.)

Now let W be the subspace of $\mathbb{C}^2$ spanned by the vector $\begin{bmatrix}1\\1\end{bmatrix}$.

Then

$$\rho_{\text{reg}}(\tau)\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

and $\rho_{\text{reg}}(e)\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$, so W is a G-invariant subpace, i.e. W is a subrepresentation of ρ_{reg}. Note that W is precisely equal to the kernel of the map f, and that W is isomorphic to the 1-dimensional trivial representation of G.

Example

We can generalize the G-invariant subspace from the previous example. Suppose we have a representation $\rho\colon G \to GL_n(\mathbb{C})$. If we can find a vector $x \in \mathbb{C}^n$ which is an eigenvector for every matrix $\rho(g), g \in G$, i.e. an $x \in \mathbb{C}^n$ such that

$$\rho(g)(x) = \lambda_g(x) \quad \forall g \in G$$

for some eigenvalues $\lambda_g \in \mathbb{C}$, then the span of x is a 1-dimensional G-invariant subspace of $\mathbb{C}^n$. It is isomorphic to the 1-dimensional representation

$$\begin{aligned} \rho_2\colon G &\to GL_1(\mathbb{C}) \\ g &\mapsto \lambda_g. \end{aligned}$$

Proposition

Let $f : V \to W$ be a homomorphism of representations of G. Then Ker(f) is a subrepresentation of V and Im(f) is a subrepresentation of W.

Proof.

- Let $x \in \text{Ker}(f)$. Then $0 = g0 = gf(x) = f(gx)$ for every $g \in G$. So $gx \in \text{Ker}(f)$ and Ker(f) is G-invariant.
- Now let $w \in \text{Im}(f)$. There exists $v \in V$ such that $w = f(v)$, so $gw = gf(v) = f(gv)$ for every $g \in G$. Thus $gw \in \text{Im}(f)$, and Im(f) is G-invariant.

$\square$

The direct sum of representations

Note

We know from linear algebra that given two vector spaces V and W, we can form the **direct sum** $V \oplus W$ consisting of ordered pairs (v, w) where $v \in V, w \in W$.

The direct sum of representations

Note

We know from linear algebra that given two vector spaces V and W, we can form the **direct sum** $V \oplus W$ consisting of ordered pairs (v, w) where $v \in V, w \in W$.

Definition

Let V and W be representations of G. Then $V \oplus W$ admits a natural representation of G, called the **direct sum representation** of V and W, which we define by

$$\rho_{V\oplus W} \colon G \to GL(V \oplus W)$$
$$\rho_{V\oplus W}(g) \colon (x, y) \mapsto (\rho_V(g)(x), \rho_W(g)(y)).$$

Irreducible representations and complete reducibility

Definition

A representation is said to be **irreducible** if it has no subrepresentations other than the trivial subrepresentations $0 \subset V$ and $V \subset V$. A representation is called **completely reducible** if it decomposes into a direct sum of irreducible representations.

Note

1. Any 1-dimensional representation V has no subspaces other than 0 and V itself, and is thus irreducible.
2. Any irreducible representation is, in particular, completely reducible.

Example (A 2-dimensional irreducible representation)

Let $G = D_3 = \langle \sigma, \tau | \sigma^3 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$. (Note that $D_3 \cong S_3$). Consider the regular triangle centered at the origin with vertices

$$(1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2}).$$

We can let σ act as rotation by $\frac{2\pi}{3}$ and let τ act as reflection over the x-axis to obtain an action of G on $\mathbb{C}^2$ given (under the standard basis) by the matrices

$$\rho(\sigma) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\rho(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Example (A 2-dimensional irreducible representation cont.)

Suppose ρ has a non-trivial subrepresentation W. We must have dim $W = 1$. Since W is invariant under the action of both $\rho(\sigma)$ and $\rho(\tau)$, there must be some mutual eigenvector for $\rho(\sigma)$ and $\rho(\tau)$ that spans W. The eigenvectors of $\rho(\sigma)$ are

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} \quad (\lambda_1 = e^{\frac{2\pi i}{3}}) \quad \text{and} \quad \begin{bmatrix} 1 \\ i \end{bmatrix} \quad (\lambda_2 = e^{-\frac{2\pi i}{3}}).$$

The eigenvectors of $\rho(\tau)$ are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\lambda_1 = 1) \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\lambda_2 = -1).$$

Thus we see that there is no such W, and our representation is irreducible.

Representations of finite abelian groups

Theorem

If $A_1, A_2, \ldots, A_r$ *are linear operators on* V *and each* A_i *is diagonalizable, then* $\{A_i\}$ *are simultaneously diagonalizable if and only if they commute.*

Representations of finite abelian groups

Theorem

Every complex representation of a finite abelian group is completely reducible into irreducible representations of dimension 1.

Proof.

Take an arbitrary element $g \in G$. Since G is finite, we can find an integer n such that $g^n = 1$ and $\rho(g)^n = Id$. The minimal polynomial of $\rho(g)$ divides $x^n - 1$, which has n distinct roots over $\mathbb{C}$. So the minimal polynomial of $\rho(g)$ factors into linear factors only over $\mathbb{C}$, i.e. $\rho(g)$ is diagonalizable. We conclude that each $\rho(g)$ is (separately) diagonalizable since $g \in G$ was arbitrary. Now, given any two elements $g_1, g_2 \in G$ we have $\rho(g_1)\rho(g_2) = \rho(g_2)\rho(g_1)$. Since the matrices $\{\rho(g)\}$ commute, $\{\rho(g)\}$ are simultaneously diagonalizable, say with basis $\{e_1, ..., e_k\}$. Then we have $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \ldots \oplus \mathbb{C}e_n$, with each subspace $\mathbb{C}e_1$ invariant under the action of G. $\square$

Definition

Let W be a subspace of V. A **linear projection** V onto W is a linear map $f\colon V \to W$ such that $f\restriction_W = \mathrm{Id}_W$. If W is a subrepresentation of V and the map f is G-invariant, then we say that f is a **G-linear projection**.

Lemma

Let $\rho\colon G \to GL(V)$ be a representation, and $W \subset V$ be a subrepresentation. Suppose we have a G-linear projection

$$f\colon V \to W.$$

Then $Ker(f)$ is a complementary subrepresentation to W, i.e. $Ker(f)$ is a G-invariant subspace of V such that

$$V = Ker(f) \oplus W$$

Maschke's Theorem

Theorem (Maschke's Theorem)

Let G be a finite group and let F be a field such that char$(F) \nmid |G|$. If V is any finite dimensional representation of G over F, and $W \subset V$ is a subrepresentation of V, then there exists a complementary subrepresentation $U \subset V$ to W, i.e. there is a G-invariant subspace $U \subset V$ such that

$$V = W \oplus U.$$

Maschke's Theorem

Proof.

It will suffice to find a G-linear projection from V onto W. Fix a basis $\{b_1, \ldots, b_m\}$ for W and extend it to a basis $\{b_1, \ldots, b_m, b_{m+1}, \ldots, b_n\}$ for V. Let $U = \langle b_{m+1}, \ldots, b_n \rangle$. Then U is certainly a complementary subspace to W, and we have a natural projection $f \colon W \oplus U \to W$ of V onto W with kernel U. There is no reason to think that f should be G-linear, but we can fix this by averaging over G. Define $\widetilde{f} \colon V \to V$ by

$$\widetilde{f}(x) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}))(x).$$

We claim that $\widetilde{f}$ is a G-linear projection from V onto W.

Maschke's Theorem

Proof.

First we check that $\mathrm{Im}(\tilde{f}) \subset W$. If $x \in V$ and $g \in G$, then

$$f(\rho(g^{-1})(x)) \in W$$

and so

$$\rho(g)(f(\rho(g^{-1})(x))) \in W$$

since W is G-invariant. Thus

$$\widetilde{f}(x) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}))(x) \in W.$$

Maschke's Theorem

Proof.

Next we check that $\widetilde{f} \restriction_W = \mathrm{Id}_W$. Let $y \in W$. For any $g \in G$, we know that $\rho(g^{-1})(y)$ is also in W, so $f(\rho(g^{-1})(y)) = \rho(g^{-1})(y)$. Then

$$\begin{aligned}\widetilde{f}(y) &= \frac{1}{|G|}\sum_{g\in G}\rho(g)(f(\rho(g^{-1})(y)))\\ &= \frac{1}{|G|}\sum_{g\in G}\rho(g)(\rho(g^{-1})(y))\\ &= \frac{1}{|G|}\sum_{g\in G}(y) = \frac{|G|y}{|G|} = y\end{aligned}$$

so indeed $\widetilde{f}$ is a linear projection of V onto W.

Maschke's Theorem

Proof.

Finally, we check that $\widetilde{f}$ is G-linear. If $x \in V$ and $h \in G$, then

$$\begin{aligned}
(\widetilde{f} \circ \rho(h))(x) &= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}) \circ \rho(h))(x) \\
&= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}h))(x) \\
&= \frac{1}{|G|} \sum_{g \in G} (\rho(hg) \circ f \circ \rho(g^{-1}))(x) \quad (g \mapsto hg) \\
&= (\rho(h) \circ \widetilde{f})(x).
\end{aligned}$$

$\square$

Corollary

Let G be a finite group and let F be a field such that $\text{char}(F) \nmid |G|$. then any finite-dimensional representation of G over F is completely reducible.

Proof.

Let V be a representation of G over F of dimension n. If V is irreducible, then V is, in particular, completely reducible. If not, then V contains a propositioner subrepresentation $W \subset V$. From Maschke's Theorem (**??**), we know there exists a subrepresentation $U \subset V$ such that

$$V = W \oplus U. \tag{5}$$

Both W and U have dimension less than n, so by induction we know that W and U are completely reducible. We deduce from 5 that V is completely reducible. □