

UNIVERSITY OF MISSOURI

MASTER'S PROJECT

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# Character Tables for Representations of Finite Groups

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# Chapter 1

## Basic Notions of Representation Theory

### 1.1 Introduction

Groups arise naturally as sets of symmetries of some object which are closed under composition and taking inverses. For example,

1. The **symmetric group** of degree  $n$ ,  $S_n$ , is the group of all symmetries of the set  $\{1, \dots, n\}$ .
2. The **dihedral group** of order  $2n$ ,  $D_{2n}$ , is the group of all symmetries of the regular  $n$ -gon in the plane.

In these two examples,  $S_n$  acts on the set  $\{1, \dots, n\}$  and  $D_{2n}$  acts on the regular  $n$ -gon in a natural manner. One may wonder more generally: Given an abstract group  $G$ , which objects  $X$  does  $G$  act on? This is the basic question of representation theory, which attempts to classify all such  $X$  up to isomorphism.

### 1.2 Group Actions

**Definition 1.1.** A (*left*) **group action** of a group  $G$  on a set  $X$  is a map  $\rho: G \times X \rightarrow X$  (written as  $g \cdot a$ , for all  $g \in G$  and  $a \in A$ ) that satisfies the following two axioms:

$$1 \cdot x = x \quad \forall x \in X \quad (1.1.1)$$

$$(gh) \cdot x = g \cdot (h \cdot x) \quad \forall g, h \in G, x \in X \quad (1.1.2)$$

*Note.* We could likewise define the concept of a *right* group action, where the set elements would be multiplied by group elements on the right instead of on the left. Throughout we shall use the term *group action* to mean a *left* group action.

**Proposition 1.2.** Let  $G$  act on the set  $X$ . For any fixed  $g \in G$ , the map  $\sigma_g$  from  $X$  into  $X$  defined by  $\sigma_g(x) = g \cdot x$  is a permutation of the set  $X$ . That is,  $\sigma_g \in S_X$ .

*Proof.* We show that  $\sigma_g$  is a permutation of  $X$  by finding a two-sided inverse map, namely  $\sigma_{g^{-1}}$ . Observe that for any  $x \in X$ , we have

$$\begin{aligned} (\sigma_{g^{-1}} \circ \sigma_g)(x) &= \sigma_{g^{-1}}(\sigma_g(x)) \\ &= g^{-1} \cdot (g \cdot x) && \text{(by definition of } \sigma_g \text{ and } \sigma_{g^{-1}}) \\ &= (g^{-1}g) \cdot x && \text{(by axiom 1.1.1 of an action)} \\ &= 1 \cdot x \\ &= x && \text{(by axiom 1.1.2 of an action).} \end{aligned}$$

Thus  $\sigma_{g^{-1}} \circ \sigma_g$  is the identity map on  $X$ . We can reverse the roles of  $g$  and  $g^{-1}$  to see that  $\sigma_g \circ \sigma_{g^{-1}}$  is also the identity map on  $X$ . Having a two-sided inverse, we conclude that  $\sigma_g$  is a permutation of  $X$ .  $\square$

**Proposition 1.3.** *Let  $G$  act on the set  $X$ . The map from  $G$  into the symmetric group  $S_X$  defined by  $g \mapsto \sigma_g(x) = g \cdot x$  is a group homomorphism.*

*Proof.* Define the map  $\rho: G \rightarrow S_X$  by  $\rho(g) = \sigma_g$ . We have seen from Proposition 1.2 that  $\sigma_g$  is indeed an element of  $S_X$ . It remains to show that  $\rho(g_1g_2) = \rho(g_1) \circ \rho(g_2)$  for any  $g_1, g_2 \in G$ . Observe that

$$\begin{aligned}
 \rho(g_1g_2)(x) &= \sigma_{g_1g_2}(x) && \text{(by definition of } \rho) \\
 &= (g_1g_2) \cdot x && \text{(by definition of } \sigma_{g_1g_2}) \\
 &= g_1 \cdot (g_2 \cdot x) && \text{(by axiom 1.1.1 of an action)} \\
 &= \sigma_{g_1}(\sigma_{g_2}(x)) && \text{(by definition of } \sigma_{g_1} \text{ and } \sigma_{g_2}) \\
 &= \rho(g_1)(\rho(g_2)(x)) && \text{(by definition of } \rho) \\
 &= (\rho(g_1) \circ \rho(g_2))(x) && \text{(by definition of function composition).}
 \end{aligned}$$

Since the values of  $\rho(g_1g_2)$  and  $\rho(g_1) \circ \rho(g_2)$  agree on every element  $x \in X$ , these two permutations are equal. We conclude that  $\rho$  is a homomorphism, since  $g_1$  and  $g_2$  were arbitrary elements of  $G$ .  $\square$

**Proposition 1.4.** *Any homomorphism  $\psi$  from the group  $G$  into the symmetric group  $S_X$  on a set  $X$  gives rise to an action of  $G$  on  $X$ , defined by taking  $g \cdot x = \psi(g)(x)$ .*

*Proof.* Suppose that we have a homomorphism  $\psi$  from  $G$  into  $S_X$ . We can define a map from  $G \times X$  to  $X$  by  $g \cdot x = \psi(g)(x)$ . We verify that this map satisfies the definition of a group action of  $G$  on  $X$ :

$$\text{(axiom 1.1.1)} \quad 1 \cdot x = \psi(1)(x) = \text{id}_X(x) = x$$

$$\text{(axiom 1.1.2)} \quad (gh) \cdot x = \psi(gh)(x) = (\psi(g)\psi(h))(x) = \psi(g)(\psi(h)(x)) = g \cdot (h \cdot x) \quad \square$$

**Corollary 1.5.** *The actions of  $G$  on the set  $X$  are in bijective correspondence with the homomorphisms from  $G$  into the symmetric group  $S_X$ .*

*Proof.* By Proposition 1.3, any action of  $G$  on  $X$  yields a homomorphism from  $G$  into  $S_X$ . Conversely, any homomorphism from  $G$  into  $S_X$  establishes an action of  $G$  on  $X$  by Proposition 1.4.  $\square$

### 1.3 The Definition of a Representation

**Definition 1.6.** Let  $G$  be a group, let  $F$  be a field, and let  $V$  be a vector space over  $F$ . A **linear representation** of  $G$  is any group homomorphism  $\rho: G \rightarrow GL(V)$ . If we fix a basis for  $V$ , we get a representation in the previous sense.

**Definition 1.7** (Alternative definition). Let  $G$  be a group, let  $F$  be a field, and let  $V$  be a vector space over  $F$ . A **linear representation** of  $G$  is an action of  $G$  on  $V$  which preserves the linear structure of  $V$ , i.e. an action of  $G$  on  $V$  such that

$$g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2 \quad \forall g \in G, v_1, v_2 \in V \quad (1.7.1)$$

$$g \cdot (kv) = k(g \cdot v) \quad \forall g \in G, v \in V, k \in F \quad (1.7.2)$$

*Note.* Unless otherwise specified, we will restrict our discussion to *finite-dimensional* representations over  $\mathbb{C}$ .

**Proposition 1.8.** *The definitions of a linear representation given in 1.6 and 1.7 above are equivalent.*

*Proof.* ( $\rightarrow$ ) Suppose that we have a homomorphism  $\rho: G \rightarrow GL(V)$ . Note that  $GL(V)$  is a subgroup of the symmetric group  $S_V$  on  $V$ , so we can apply Proposition 1.4 to obtain an action of  $G$  on  $V$  by  $g \cdot v = \rho(g)(v)$ . We check that this action preserves the linear structure of  $V$ .

**1.7.1** For any  $g \in G, v_1, v_2 \in V$  we have  $g \cdot (v_1 + v_2) = \rho(g)(v_1 + v_2) = \rho(g)(v_1) + \rho(g)(v_2) = g \cdot v_1 + g \cdot v_2$ .

**1.7.2** For any  $g \in G, v \in V, k \in F$  we have  $g \cdot (kv) = \rho(g)(kv) = k(\rho(g)(v)) = k(g \cdot v)$ .

( $\leftarrow$ ) Suppose that we have an action of  $G$  on  $V$  which preserves the linear structure of  $V$  in the sense of Definition 1.7. We can apply Proposition 1.3 to obtain a homomorphism  $\rho: G \rightarrow S_V$  given by  $\rho(g) = \sigma_g$  where  $\sigma_g(v) = g \cdot v$ . It remains to show that the image  $\rho(G)$  of  $G$  under  $\rho$  is actually contained in  $GL(V)$ , i.e. that for each  $g \in G$  the map  $\sigma_g$  is linear. Fix an element  $g \in G$ . For any  $k \in F$  and  $v \in V$ , we have

$$\begin{aligned} \sigma_g(kv) &= g \cdot (kv) && \text{(by definition of } \sigma_g) \\ &= k(g \cdot v) && \text{(by property 1.7.1)} \\ &= k(\sigma_g(v)) && \text{(by definition of } \sigma_g). \end{aligned}$$

Also, for any  $v_1, v_2 \in V$  we have

$$\begin{aligned} \sigma_g(v_1 + v_2) &= g \cdot (v_1 + v_2) && \text{(by definition of } \sigma_g) \\ &= g \cdot v_1 + g \cdot v_2 && \text{(by property 1.7.2)} \\ &= \sigma_g(v_1) + \sigma_g(v_2) && \text{(by definition of } \sigma_g). \end{aligned}$$

Thus  $\sigma_g$  is linear, and  $\rho(G) \subset GL(V)$  proves that we have a homomorphism  $\rho: G \rightarrow GL(V)$ . □

**Definition 1.9.** Let  $G$  be a group, let  $F$  be a field, let  $V$  be a vector space over  $F$ , and let  $\rho: G \rightarrow GL(V)$  be a representation of  $G$ . The **dimension** of the representation is the dimension of  $V$  over  $F$ .

**Example 1.10.** 1. Let  $V$  be a 1-dimensional vector space over the field  $F$ . The map  $\rho: G \rightarrow GL(V)$  defined by  $\rho(g) = 1$  for all  $g \in G$  is a representation called the *trivial representation* of  $G$ . The trivial representation has dimension 1.

2. If  $G$  is a finite group that acts on a finite set  $X$ , and  $F$  is any field, then there is an associated *permutation representation* on the vector space  $V$  over  $F$  with basis  $\{e_x: x \in X\}$ . We let  $G$  act on the basis elements by the permutation  $g \cdot e_x = e_{gx}$  for all  $x \in X$  and  $g \in G$ . This representation has dimension  $|X|$ .

3. A fundamental special case of a permutation representation that we shall return to later on is that when a finite group acts on itself by left multiplication. We take the vector space  $V_{\text{reg}}$  which has a basis given by the formal symbols  $\{e_g | g \in G\}$ ,

and let  $G$  act by  $\rho_{\text{reg}}(h)(e_g) = e_{hg}$ . This representation is called the *regular representation* of  $G$ , and has dimension  $|G|$ . We shall see later that this representation encodes information about all other representations of  $G$ .

4. For any symmetric group  $S_n$ , the *alternating representation* on  $V = \mathbb{C}$  is given by the map  $\rho: S_n \rightarrow GL(\mathbb{C}) = \mathbb{C}^\times$  defined by  $\rho(\sigma) = \text{sgn}(\sigma)$ . More generally, for any group  $G$  with a subgroup  $H$  of index 2, we can define an *alternating representation*  $\rho: G \rightarrow GL(\mathbb{C})$  by letting  $\rho(g) = 1$  if  $g \in H$  and  $\rho(g) = -1$  if  $g \notin H$ . (We recover our original example by taking  $G = S_n$  and  $H = A_n$ .)

**Definition 1.11.** A **homomorphism** between two representations  $\rho_1: G \rightarrow GL(V)$  and  $\rho_2: G \rightarrow GL(W)$  is a linear map  $\psi: V \rightarrow W$  that intertwines with (respects) the  $G$ -action, i.e.

$$\psi \circ \rho_1(g) = \rho_2(g) \circ \psi \quad \forall g \in G.$$

An **isomorphism** of representations is a homomorphism of representations that is also an invertible map.

*Note.* If we have representations  $(\rho_1, V)$  and  $(\rho_2, W)$  and an isomorphism of vector spaces  $\psi: V \rightarrow W$  then we can rewrite the compatibility requirement above as  $\rho_2(g) = \psi \circ \rho_1(g) \circ \psi^{-1}$  for all  $g \in G$ .

Given any representation  $(\rho, V)$  of a group  $G$  on a vector space  $V$  over a field  $F$  of dimension  $n$ , we can fix a basis for  $V$  to obtain an isomorphism of vector spaces  $\psi: V \rightarrow F^n$ . This yields a representation  $\phi$  of  $G$  on  $F^n$  by defining  $\phi(g) = \psi \circ \rho(g) \circ \psi^{-1}$  for all  $g \in G$ . Clearly, this representation is isomorphic to our original representation  $(\rho, V)$ . In particular, this means we can always choose to view  $n$ -dimensional complex representations as representations on  $\mathbb{C}^n$  where each  $\phi(g)$  is given by an  $n \times n$  matrix with entries in  $\mathbb{C}$ .

Suppose that we have two representations  $\rho_1: G \rightarrow GL_n(F)$  and  $\rho_2: G \rightarrow GL_m(F)$  given by  $\rho_1(g) = X_g$  and  $\rho_2(g) = Y_g$ . A homomorphism between these representations is then an  $m \times n$  matrix  $A$  such that  $AX_g = Y_gA$  for all  $g \in G$ . An isomorphism is given precisely when such  $A$  is square and invertible. Thus, two representations  $\rho_1: G \rightarrow GL_n(F)$  and  $\rho_2: G \rightarrow GL_n(F)$  are isomorphic if and only if there exists  $A \in GL_n(F)$  such that  $\rho_1(g) = A\rho_2(g)A^{-1}$  for all  $g \in G$ . This establishes the following proposition:

**Proposition 1.12.** *The isomorphism classes of  $n$ -dimensional representations of  $G$  on  $\mathbb{C}$  are in bijection with the quotient  $\text{Hom}(G; GL_n(\mathbb{C}))/GL_n(\mathbb{C})$  of group homomorphisms  $G \rightarrow GL_n(\mathbb{C})$  modulo the conjugation action of  $GL_n(\mathbb{C})$ .*

*Note.* In other words,

**Definition 1.13.** A **subrepresentation** of  $V$  is a  $G$ -invariant subspace  $W \subseteq V$ ; that is, a subspace  $W \subseteq V$  with the property that  $\rho(g)(w) \in W$  for all  $g \in G$  and  $w \in W$ . Note that  $W$  itself is a representation of  $G$  under the action  $\rho(g)|_W$ .

**Example 1.14.** Let  $G = \{(1), (123), (132)\} \subset S_3$ . Note that  $G$  is a subgroup of  $S_3$ , and is isomorphic to the cyclic group of order 3. Let  $V = \mathbb{C}^3$ . Then  $G$  acts on the standard basis by  $g \cdot e_i = e_{gi}$ . Thus, the permutation representation of  $G$  (with respect to the

standard basis) is given by:

$$\begin{aligned}\rho((1)) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \rho((123)) &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ \rho((132)) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.\end{aligned}$$

**Example 1.15.** Let  $G = C_2 \times C_2$  be the Klein four-group, i.e. the group generated by  $\sigma, \tau$  such that  $\sigma^2 = \tau^2 = e$  and  $\sigma\tau = \tau\sigma$ . Let  $V$  be the vector space with basis  $\{b_e, b_\sigma, b_\tau, b_{\sigma\tau}\}$ . Left multiplication by  $\sigma$  gives a permutation

$$\begin{aligned}b_e &\mapsto b_\sigma \\ b_\sigma &\mapsto b_e \\ b_\tau &\mapsto b_{\sigma\tau} \\ b_{\sigma\tau} &\mapsto b_\tau\end{aligned}$$

Thus, with respect to our given basis of  $V$ , the regular representation  $\rho_{\text{reg}}: G \rightarrow GL(V)$  evaluated at  $\sigma$  is given by the matrix

$$\rho_{\text{reg}}(\sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We similarly calculate

$$\begin{aligned}\rho_{\text{reg}}(\tau) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ \rho_{\text{reg}}(\sigma) &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

**Example 1.16.** Let  $G = D_8 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$  be the symmetry group of the square. We can use geometry to construct a representation of  $G$ . Consider a square in the plane with vertices at  $(1, 1)$ ,  $(1, -1)$ ,  $(-1, -1)$ , and  $(-1, 1)$ . We let  $\sigma$  act on the square as a rotation by  $\frac{\pi}{2}$ , and let  $\tau$  act by reflecting over the  $x$ -axis. This naturally gives rise to a linear action of  $G$  on all of  $\mathbb{C}^2$ . Under the standard basis, we get the

matrices:

$$\begin{aligned}
 \rho(e) &= \text{Id}_2 \\
 \rho(\sigma) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
 \rho(\sigma^2) &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\
 \rho(\sigma^3) &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
 \rho(\tau) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
 \rho(\sigma\tau) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
 \rho(\sigma^2\tau) &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
 \rho(\sigma^3\tau) &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}
 \end{aligned}$$

**Example 1.17.** Let  $G = C_2 = \langle \tau | \tau^2 = e \rangle$  be the cyclic group of order 2. The regular representation of  $G$ , written in the standard basis, is

$$\rho_{\text{reg}}(\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and  $\rho_{\text{reg}}(e) = \text{Id}_2$ . Let  $\rho_1$  be the alternating representation of  $G$  on  $\mathbb{C}$ , i.e.

$$\begin{aligned}
 \rho_1: G &\rightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times (= \mathbb{C} \setminus \{0\}) \\
 \tau &\mapsto -1 \\
 e &\mapsto 1
 \end{aligned}$$

Let  $f: \mathbb{C}^2 \rightarrow \mathbb{C}$  be the linear map represented by the matrix  $\begin{bmatrix} 1 & -1 \end{bmatrix}$  in the standard basis. Then for any vector  $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$ , we have

$$\begin{aligned}
 f \circ \rho_{\text{reg}}(\tau)(x) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
 &= - \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
 &= \rho_1(\tau) \circ f(x).
 \end{aligned}$$

Thus,  $f$  is a  $G$ -linear map from  $\rho_{\text{reg}}$  to  $\rho_1$  (i.e. a homomorphism of representations).

Now, let  $W$  be the subspace of  $\mathbb{C}^2$  spanned by the vector  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Then

$$\rho_{\text{reg}}(\tau) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



and  $\rho_{\text{reg}}(e) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , so  $W$  is a subrepresentation of  $\rho_{\text{reg}}$ . (In fact,  $W$  is isomorphic to the 1-dimensional trivial representation of  $G$ .)

**Example 1.18.** We can generalize  $G$ -invariant subspace from the previous example. Suppose we have a representation  $\rho: G \rightarrow GL_n(\mathbb{C})$ . If we can find a vector  $x \in \mathbb{C}^n$  which is an eigenvector for every matrix  $\rho(g)$ ,  $g \in G$ , i.e. an  $x \in \mathbb{C}^n$  such that

$$\rho(g)(x) = \lambda_g(x) \quad \forall g \in G$$

for some eigenvalues  $\lambda_g \in \mathbb{C}$ , then the span of  $x$  is a 1-dimensional  $G$ -invariant subspace of  $\mathbb{C}^n$ . It is isomorphic to the 1-dimensional representation

$$\begin{aligned} \rho_2: G &\rightarrow GL_1(\mathbb{C}) \\ g &\mapsto \lambda_g. \end{aligned}$$

**Proposition 1.19.** Let  $f: V \rightarrow W$  be a homomorphism of representations of  $G$ . Then  $\text{Ker}(f)$  is a subrepresentation of  $V$  and  $\text{Im}(f)$  is a subrepresentation of  $W$ .

*Proof.* Let  $x \in \text{Ker}(f)$ . Then  $0 = g0 = gf(x) = f(gx)$  for every  $g \in G$ , so  $gx \in \text{Ker}(f)$  and  $\text{Ker}(f)$  is  $G$ -invariant.

Now let  $w \in \text{Im}(f)$ . There exists  $v \in V$  such that  $w = f(v)$ , so  $gw = gf(v) = f(gv)$  for every  $g \in G$ . Thus  $gw \in \text{Im}(f)$ , and  $\text{Im}(f)$  is  $G$ -invariant.  $\square$

*Note.* Recall the subrepresentation  $W$  that we constructed in Example 1.17. This subrepresentation is precisely equal to the kernel of the map  $f$  from that example.

## 1.4 Representations of Cyclic Groups

**Example 1.20** (Representations of  $\mathbb{Z}$ ). Let's try to classify all representations of the group  $\mathbb{Z}$  under addition. Consider an  $n$ -dimensional representation  $\rho: \mathbb{Z} \rightarrow GL_n(\mathbb{C})$ . For  $\rho$  to be a group homomorphism requires that  $\rho(0) = \text{Id}$ . Observe that for any  $0 \neq n \in \mathbb{Z}$ , we have  $\rho(n) = \rho(1 + \dots + 1) = \rho(1)^n$ . Then  $\rho$  is completely determined by the matrix  $\rho(1) \in GL_n(\mathbb{C})$ , and any such matrix determines a representation of  $\mathbb{Z}$ . It follows that the  $n$ -dimensional isomorphism classes of representations of  $\mathbb{Z}$  are in bijection with the conjugacy classes in  $GL_n(\mathbb{C})$ . These conjugacy classes can be parameterized by the *Jordan canonical form*.

**Example 1.21** (Representations of the cyclic group of order  $n$ ). We shall classify all representations of the cyclic group  $G = \{g, g^2, \dots, g^{n-1}, g^n = 1\}$  of order  $n$ . Consider a representation  $\rho: G \rightarrow GL(V)$ . As in the previous example, we know that  $\rho(1) = \text{Id}$  and  $\rho(g^k) = \rho(g)^k$ . Thus our representation  $\rho$  is determined completely by the linear transformation  $\rho(g)$ . It will be helpful to fix a basis of  $V$  so that we may view  $\rho(g)$  as a matrix. Recall from linear algebra that there exists a basis in which  $\rho(g)$  takes the *Jordan canonical form*

$$\rho(g) = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_m \end{bmatrix}$$

where each *Jordan block*  $J_k$  is of the form

$$J_k = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \ddots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix}.$$

Now  $I = \rho(g)^n$  is a block-diagonal matrix with diagonal blocks  $J_k^n$ , so we must have that each block  $J_k^n = \text{Id}$ . Observe that we can write each block  $J_k$  as  $J_k = \lambda \text{Id} + N$  where  $N$  is the Jordan block with  $\lambda = 0$ . Thus we have

$$\text{Id} = J_k^n = (\lambda \text{Id} + N)^n = \lambda^n \text{Id} + \binom{n}{1} \lambda^{n-1} N + \binom{n}{2} \lambda^{n-2} N^2 + \dots + \binom{n}{n-1} \lambda N^{n-1} + N^n.$$

The following lemma will show that in fact  $N = 0$ .

**Lemma 1.22.** *Let  $N$  be the Jordan block with  $\lambda = 0$  of size  $n \times n$ . For any integer  $p$  with  $1 \leq p \leq n-1$ , then  $N^p$  is the matrix with ones in the positions  $(i, j)$  where  $j = i + p$  and zeroes everywhere else. (The ones lie along a line parallel to the diagonal,  $p$  steps above it.)*

*Proof.* (By induction.)

*Base case:* This is simply the definition of  $N$ .

*Inductive step:* Suppose that the lemma holds for  $N^p$ . We compute the  $(i, j)$  entry of  $N^{p+1}$ :

$$(N^{p+1})_{i,j} = \sum_{k=1}^n (N^p)_{i,k} N_{k,j} = (N^p)_{i,i+p} N_{i+p,j} = N_{i+p,j} = \begin{cases} 1 & \text{if } j = i + (p+1) \\ 0 & \text{otherwise} \end{cases}$$

□

Now, if  $N \neq 0$  then each term  $\binom{n}{k} \lambda^{n-k} N^k$  for  $k > 0$  would yield some non-zero non-diagonal entries (in the positions  $(i, j)$  where  $j = i + k$ ) and hence our sum could not equal the identity matrix. We must conclude that  $N = 0$ ,  $J_k = \lambda \text{Id}$  is a  $1 \times 1$  block, and  $J_k^n = \lambda^n \text{Id}$ . Thus  $\rho(g)$  is a diagonal matrix with  $n$ th roots of unity as diagonal entries.

To summarize, every  $m$ -dimensional representation  $\rho$  of the cyclic group  $G = \langle g \rangle$  of order  $n$  can be seen to act (with the right choice of basis) as  $m \times m$  diagonal matrices all with  $n$ th roots of unity along the diagonal. In particular, these representations are determined completely by the value of  $\rho(g)$  and are classified up to isomorphism by unordered  $m$ -tuples of  $n$ th roots of unity.

## 1.5 Constructing New Representations from Old

From elementary linear algebra, we know that given a subspace  $W \subseteq V$ , we can form the **quotient space**  $V/W$  consisting of cosets  $v + W$  in  $V$ . If  $W$  is a subrepresentation of  $V$ , we would like to define an action of  $G$  on  $V/W$  by the natural choice of  $g(v + W) = \rho(g)(v) + W$ . It remains to verify that this action is well defined. If we choose another  $v' \in v + W$ , then  $v - v' \in W$ , so that  $\rho(g)(v - v') \in W$  since  $W$  is  $G$ -invariant. Thus, the

cosets  $\rho(g)(v) + W$  and  $\rho(g)(v') + W$  agree and this action is indeed well defined. This justifies the following definition:

**Definition 1.23.** Let  $W$  be a  $G$ -subrepresentation of  $V$ . Then  $V/W$  forms a representation of  $G$  called the **quotient representation** of  $V$  under  $W$  with the action  $g(v + W) = \rho(g)(v) + W$ .

Recall also from linear algebra that given two vector spaces  $V_1$  and  $V_2$ , we can form the **direct sum**  $V_1 \oplus V_2$  consisting of ordered pairs  $(v_1, v_2)$  where  $v_1 \in V_1, v_2 \in V_2$ .

**Definition 1.24.** Let  $V_1$  and  $V_2$  be representations of  $G$ . Then  $V_1 \oplus V_2$  forms a representation of  $G$  called the **direct sum representation** of  $V_1$  and  $V_2$  with the action  $g(v_1, v_2) = (g \cdot v_1, g \cdot v_2)$ .

## 1.6 Complete Reducibility

**Definition 1.25.** A representation is said to be **irreducible** if it contains no proper invariant subspaces. It is called **completely reducible** if it decomposes into a direct sum of irreducible subrepresentations.

**Example 1.26.** 1. Any irreducible representation is, in particular, completely reducible.  
2. Any 1-dimensional representations has no proper subspaces, and is thus irreducible.

**Theorem 1.27.** If  $A_1, A_2, \dots, A_r$  are linear operators on  $V$  and each  $A_i$  is diagonalizable, they are simultaneously diagonalizable if and only if they commute.

*Proof.* See Conrad [3, Theorem 5.1]. □

**Theorem 1.28.** Every complex representation of a finite abelian group is completely reducible into irreducible representations of dimension 1.

*Proof.* Take an arbitrary element  $g \in G$ . Since  $G$  is finite, we can find an integer  $n$  such that  $g^n = 1$  and  $\rho(g)^n = \text{Id}$ . Hence the minimal polynomial of  $\rho(g)$  divides  $x^n - 1$ . Recall that  $x^n - 1$  has  $n$  distinct roots over  $\mathbb{C}$ , which are generated by taking powers of  $\xi = e^{\frac{2\pi i}{n}}$ . This means that the minimal polynomial  $\rho(g)$  factors into linear factors only over  $\mathbb{C}$  so that  $\rho(g)$  is diagonalizable. We conclude that each  $\rho(g)$  is (separately) diagonalizable since  $g \in G$  was arbitrary.

Now, given any two elements  $g_1, g_2 \in G$  we have

$$\begin{aligned} \rho(g_1)\rho(g_2) &= \rho(g_1g_2) && \text{(since } \rho \text{ is a homomorphism)} \\ &= \rho(g_2g_1) && \text{(since } G \text{ is abelian)} \\ &= \rho(g_2)\rho(g_1) && \text{(since } \rho \text{ is a homomorphism).} \end{aligned}$$

Thus the matrices  $\{\rho(g)\}$  commute, so we may apply Theorem 1.27 to conclude that  $\{\rho(g)\}$  are simultaneously diagonalizable, say with basis  $\{e_1, \dots, e_k\}$ . Then we have  $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \dots \oplus \mathbb{C}e_n$ , with each subspace  $\mathbb{C}e_i$  invariant under the action of  $G$ . □

**Definition 1.29.** Let  $W$  be a subspace of  $V$ . A **linear projection**  $V$  onto  $W$  is a linear map  $f: V \rightarrow W$  such that  $f|_W = \text{Id}_W$ . If  $W$  is a subrepresentation of  $V$  and the map  $f$  is  $G$ -invariant, then we say that  $f$  is a  **$G$ -linear projection**.

**Lemma 1.30.** *Let  $\rho: G \rightarrow GL(V)$  be a representation, and  $W \subset V$  be a subrepresentation. Suppose we have a  $G$ -linear projection*

$$f: V \rightarrow W.$$

*Then  $\text{Ker}(f)$  is a complementary subrepresentation to  $W$ , i.e.  $\text{Ker}(f)$  is a  $G$ -invariant subspace of  $V$  such that*

$$V = \text{Ker}(f) \oplus W$$

*Proof.* We have seen that  $\text{Ker}(f)$  is a subrepresentation of  $V$  in Proposition 1.19. Now if  $y \in \text{Ker}(f) \cap W$  then  $y = f(y) = 0$ , so  $\text{Ker}(f) \cap W = 0$ . Finally  $\text{Im}(f) = W$ , so by the Rank-Nullity theorem

$$\dim \text{Ker}(f) + \dim W = \dim V.$$

Thus  $V = \text{Ker}(f) \oplus W$ . □

**Theorem 1.31** (Maschke's Theorem). *Let  $G$  be a finite group and let  $F$  be a field such that  $\text{char}(F) \nmid |G|$ . If  $V$  is any finite-dimensional representation of  $G$  over  $F$ , and  $W \subset V$  is a subrepresentation of  $V$ , then there exists a complementary subrepresentation  $U \subset V$ , i.e. there is a  $G$ -invariant subspace  $U \subset V$  such that*

$$V = W \oplus U.$$

*Proof.* By the previous Lemma 1.30 it will suffice to find a  $G$ -linear projection from  $V$  onto  $W$ . Fix a basis  $\{b_1, \dots, b_m\}$  for  $W$  and extend it to a basis  $\{b_1, \dots, b_m, b_{m+1}, \dots, b_n\}$  for  $V$ . Let  $U = \langle b_{m+1}, \dots, b_n \rangle$ . Then  $U$  is certainly a complementary subspace to  $W$ , and we have a natural projection  $f: W \oplus U \rightarrow W$  of  $V$  onto  $W$  with kernel  $U$ . There is no reason to think that  $f$  should be  $G$ -linear, but we can fix this by averaging over  $G$ . Define  $\tilde{f}: V \rightarrow V$  by

$$\tilde{f}(x) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}))(x).$$

We claim that  $\tilde{f}$  is a  $G$ -linear projection from  $V$  onto  $W$ . First we check that  $\text{Im}(\tilde{f}) \subset W$ . If  $x \in V$  and  $g \in G$ , then

$$f(\rho(g^{-1})(x)) \in W$$

and so

$$\rho(g)(f(\rho(g^{-1})(x))) \in W$$

since  $W$  is  $G$ -invariant. Thus  $\tilde{f}(x) \in W$ . Next we check that  $\tilde{f}|_W = \text{Id}_W$ . Let  $y \in W$ . For any  $g \in G$ , we know that  $\rho(g^{-1})(y)$  is also in  $W$ , so

$$f(\rho(g^{-1})(y)) = \rho(g^{-1})(y).$$

Then

$$\begin{aligned}
 \tilde{f}(y) &= \frac{1}{|G|} \sum_{g \in G} \rho(g)(f(\rho(g^{-1})(y))) \\
 &= \frac{1}{|G|} \sum_{g \in G} \rho(g)(\rho(g^{-1})(y)) \\
 &= \frac{1}{|G|} \sum_{g \in G} \rho(gg^{-1})(y) \\
 &= \frac{1}{|G|} \sum_{g \in G} (y) \\
 &= \frac{|G|y}{|G|}
 \end{aligned}$$

so indeed  $\tilde{f}$  is a linear projection of  $V$  onto  $W$ . Finally, we check that  $\tilde{f}$  is  $G$ -linear. If  $x \in V$  and  $h \in G$ , then

$$\begin{aligned}
 (\tilde{f} \circ \rho(h))(x) &= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}) \circ \rho(h))(x) \\
 &= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}h))(x) \\
 &= \frac{1}{|G|} \sum_{g \in G} (\rho(hg) \circ f \circ \rho(g^{-1}))(x) \quad (\text{relabelling } g \mapsto hg) \\
 &= (\rho(h) \circ \tilde{f})(x).
 \end{aligned}$$

□

**Corollary 1.32.** *Let  $G$  be a finite group and let  $F$  be a field such that  $\text{char}(F) \nmid |G|$ . then any finite-dimensional representation of  $G$  over  $F$  is completely reducible.*

*Proof.* Let  $V$  be a representation of  $G$  over  $F$  of dimension  $n$ . If  $V$  is irreducible, then  $V$  is, in particular, completely reducible. If not, then  $V$  contains a proper subrepresentation  $W \subset V$ . From Maschke's Theorem (1.31), we know there exists a subrepresentation  $U \subset V$  such that

$$V = W \oplus U. \quad (1.32.1)$$

Both  $W$  and  $U$  have dimension less than  $n$ , so by induction we know that  $W$  and  $U$  are completely reducible. We deduce from 1.32.1 that  $V$  is completely reducible. □

## 1.7 Vector Spaces of Linear Maps

**Definition 1.33.** Let  $V$  and  $W$  be vector spaces. Recall that the set  $\mathbf{Hom}(V, W)$  of linear maps from  $V$  to  $W$  is itself a vector space. If  $f_1, f_2$  are two linear maps from  $V$  to  $W$ , then we define their sum by

$$\begin{aligned}
 (f_1 + f_2): V &\rightarrow W \\
 x &\mapsto f_1(x) + f_2(x)
 \end{aligned}$$

and we define scalar multiplication of  $\lambda \in \mathbb{C}$  by

$$\begin{aligned} (\lambda f_1): V &\rightarrow W \\ x &\mapsto \lambda f_1(x). \end{aligned}$$

Now suppose we have representations  $\rho_V: G \rightarrow GL(V)$  and  $\rho_W: G \rightarrow GL(W)$  of  $G$ . Then there is a natural representation of  $G$  on the vector space  $\text{Hom}(V, W)$  given by

$$\begin{aligned} \rho_{\text{Hom}(V, W)}(g): \text{Hom}(V, W) &\rightarrow \text{Hom}(V, W) \\ f &\mapsto \rho_W(g) \circ f \circ \rho_V(g^{-1}) \end{aligned}$$

for all  $g \in G$ . Note that  $\rho_{\text{Hom}(V, W)}(g)(f)$  is certainly a linear map from  $V$  to  $W$  since the composition of linear maps is linear.

**Proposition 1.34.**  $\rho_{\text{Hom}(V, W)}$  is a representation of  $G$ . That is, the map

$$\begin{aligned} \rho_{\text{Hom}(V, W)}: G &\rightarrow GL(\text{Hom}(V, W)) \\ g &\mapsto \rho_{\text{Hom}(V, W)}(g). \end{aligned}$$

is a homomorphism.

*Proof.* We must check two things:

1. For every  $g \in G$ ,  $\rho_{\text{Hom}(V, W)}(g)$  is invertible.
2. The map  $g \mapsto \rho_{\text{Hom}(V, W)}(g)$  is a homomorphism.

First, we check that

$$\begin{aligned} \rho_{\text{Hom}(V, W)}(g) \circ \rho_{\text{Hom}(V, W)}(h): f &\mapsto \rho_{\text{Hom}(V, W)}(g)(\rho_W(h) \circ f \circ \rho_V(h^{-1})) \\ &= \rho_W(g) \circ \rho_W(h) \circ f \circ \rho_V(h^{-1}) \circ \rho_V(g^{-1}) \\ &= \rho_W(gh) \circ f \circ \rho_V(g^{-1}h^{-1}) \\ &= \rho_{\text{Hom}(V, W)}(gh)(f) \end{aligned}$$

so indeed  $\rho_{\text{Hom}(V, W)}$  is a homomorphism. We can use this fact to see that  $\rho_{\text{Hom}(V, W)}(g^{-1})$  is inverse to  $\rho_{\text{Hom}(V, W)}(g)$  as

$$\begin{aligned} \rho_{\text{Hom}(V, W)}(g) \circ \rho_{\text{Hom}(V, W)}(g^{-1}) &= \rho_{\text{Hom}(V, W)}(e) \\ &= \text{Id}_{\text{Hom}(V, W)} \\ &= \rho_{\text{Hom}(V, W)}(g^{-1}) \circ \rho_{\text{Hom}(V, W)}(g). \end{aligned}$$

Thus  $\rho_{\text{Hom}(V, W)}(g)$  is invertible for every  $g \in G$ , and  $\rho_{\text{Hom}(V, W)}$  is a representation of  $G$ .  $\square$

**Definition 1.35.** Let  $V$  and  $W$  be two representations of  $G$ . The set of  $G$ -linear maps from  $V$  to  $W$  forms a subspace of  $\text{Hom}(V, W)$ , which we denote by  $\mathbf{Hom}_G(V, W)$ . In other words,  $\text{Hom}_G(V, W)$  is the vector space consisting of all *homomorphisms of representations* between  $V$  and  $W$ .

**Definition 1.36.** Let  $\rho: G \rightarrow GL(V)$  be a representation. We define the **invariant subrepresentation**  $V^G \subset V$  to be the set

$$\{v \in V \mid \rho(g)(v) = v, \quad \forall g \in G\}.$$

Note that  $V^G$  is a subspace of  $V$ , and is also clearly a subrepresentation. It is isomorphic to a trivial representation of some dimension.

**Proposition 1.37.** *Let  $\rho_V: G \rightarrow GL(V)$  and  $\rho_W: G \rightarrow GL(W)$  be representations of  $G$ . Then the subrepresentation*

$$\text{Hom}_G(V, W) \subset \text{Hom}(V, W)$$

*is precisely the invariant subrepresentation  $\text{Hom}(V, W)^G$  of  $\text{Hom}(V, W)$ .*

*Proof.* Let  $f \in \text{Hom}(V, W)$ . Then  $f$  is an element of the invariant subrepresentation  $\text{Hom}(V, W)^G$  iff we have

$$\begin{aligned} f &= \rho_{\text{Hom}(V, W)}(g)(f) \quad \forall g \in G \\ \iff f &= \rho_W(g) \circ f \circ \rho_V(g^{-1}) \quad \forall g \in G \\ \iff f \circ \rho_V(g) &= \rho_W(g) \circ f \quad \forall g \in G \end{aligned}$$

which is exactly the condition that  $f$  is  $G$ -linear, i.e. that  $f \in \text{Hom}_G(V, W)$ .  $\square$

**Lemma 1.38.** *Let  $A$  and  $B$  be two representations of  $G$ . Then*

$$(A \oplus B)^G = A^G \oplus B^G.$$

*Proof.* Observe that

$$\begin{aligned} (a, b) \in (A \oplus B)^G &\iff \rho_{A \oplus B}(g)(a, b) = (a, b) && \forall g \in G \\ &\iff (\rho_A(g)(a), \rho_B(g)(b)) = (a, b) && \forall g \in G \\ &\iff (a, b) \in A^G \oplus B^G. \end{aligned}$$

$\square$

**Lemma 1.39.** *Let  $\psi: A \rightarrow B$  be an isomorphism between representations of  $G$ . Then  $\psi$  induces an isomorphism between their invariant subrepresentations*

$$\psi|_{A^G}: A^G \rightarrow B^G.$$

*Proof.* Clearly the restriction of  $\psi$  to  $A^G \subset A$  induces an isomorphism to some subrepresentation of  $B$ , but we must check that the image of this restriction actually equals  $B^G$ . We verify that

$$\begin{aligned} a \in A^G &\iff \rho_A(g)(a) = a && \forall g \in G \\ &\iff \psi(\rho_A(g)(a)) = \psi(a) && \forall g \in G \\ &\iff \rho_B(g)\psi(a) = \psi(a) && \forall g \in G \\ &\iff \psi(a) \in B^G. \end{aligned}$$

$\square$

## 1.8 Schur's Lemma

**Theorem 1.40.** *[Schur's Lemma over  $\mathbb{C}$ .] If  $V$  is an irreducible representation of  $G$  over  $\mathbb{C}$ , then every linear operator  $\phi: V \rightarrow V$  commuting with  $G$  is a scalar.*

*Proof.* Let  $\phi: V \rightarrow V$  be a linear operator commuting with  $G$ , and let  $\lambda$  be an eigenvalue of  $\phi$ . Observe that the eigenspace  $E_\lambda$  is  $G$ -invariant: If  $v \in E_\lambda$ , then  $\phi(v) = \lambda v$ . This implies that  $\phi(gv) = g\phi(v) = g(\lambda v) = \lambda(gv)$ , i.e.  $gv \in E_\lambda$ . Since  $g$  was arbitrary,  $E_\lambda$  is indeed  $G$ -invariant. Now  $E_\lambda \neq 0$ , so since  $V$  is irreducible,  $E_\lambda = V$ . Thus  $\phi = \lambda \text{Id}$ .  $\square$

**Corollary 1.41.** *If  $V$  and  $W$  are irreducible, the space  $\text{Hom}_G(V, W)$  is 1-dimensional if the representations are isomorphic, and in this case any non-zero map is an isomorphism. Otherwise,  $\text{Hom}_G(V, W) = \{0\}$ .*

*Proof.* We claim  $\ker(\phi)$  and  $\text{im}(\phi)$  are both  $G$ -invariant. Let  $0 \neq \phi \in \text{Hom}_G(V, W)$ . If  $v \in \ker(\phi)$ , then  $\phi(v) = 0$  implies that  $\phi(gv) = g\phi(v) = g0 = 0$ , i.e.  $gv \in \ker(\phi)$ . Similarly, if  $v \in \text{im}(\phi)$ , then  $v = \phi(w)$  implies that  $\phi(gw) = g\phi(w) = gv$ , i.e.  $gv \in \text{im}(\phi)$ . Irreducibility yields  $\ker(\phi) = 0$  or  $V$  and  $\text{im}(\phi) = 0$  or  $W$  as the only possibilities. If  $\phi \neq 0$ , then  $\ker(\phi) = 0$ ,  $\text{im}(\phi) = W$ , and  $\phi$  is an isomorphism. Let  $\psi$  be another nonzero intertwining operator from  $V$  to  $W$ . Then  $\phi^{-1} \circ \psi \in \text{Hom}_G(V, V)$ . We can apply Schur's Lemma over  $\mathbb{C}$  to see that  $\phi^{-1} \circ \psi = \lambda \text{Id}$ , hence  $\psi = \lambda \phi$ . So  $\phi$  spans  $\text{Hom}_G(V, W)$ .  $\square$

More definitions are required before we can state a more general Schur's Lemma (not restricted to just  $\mathbb{C}$ ).

**Definition 1.42.** An **algebra** over a field  $K$  is a ring with unit, containing a distinguished copy of  $K$  that commutes with every algebra element, and with  $1 \in K$  being the algebra unit. A **division ring** is a ring where every non-zero element is invertible, and a **division algebra** is a division ring which is also a  $K$ -algebra.

**Definition 1.43.** Let  $V$  be a representation of  $G$  over  $K$ . The **endomorphism algebra**  $\text{End}_G(V)$  is the space of linear self-maps  $\phi: V \rightarrow V$  which commute with the group action, that is,  $\rho(g) \circ \phi = \phi \circ \rho(g) \quad \forall g \in G$ . The addition is the usual addition of linear maps (pointwise), and the multiplication is function composition. The distinguished copy of  $K$  is given by  $K\text{Id}$ .

*Note.* You may notice that  $\text{End}_G(V)$  is precisely the space  $\text{Hom}_G(V, V)$  we have already seen. However,  $\text{Hom}_G(V, W)$  is in general only a vector space over the base field, not an algebra.

**Theorem 1.44.** [Schur's Lemma] *If  $V$  is an irreducible finite-dimensional representation of  $G$  over  $K$ , then  $\text{End}_G(V)$  is a finite-dimensional division algebra over  $K$ .*

*Proof.* Let  $0 \neq f \in \text{End}_G(V)$ . Then  $\ker(f)$  and  $\text{im}(f)$  are both  $G$ -invariant subspaces of  $V$ . Since  $V$  is irreducible and  $f \neq 0$ , we must have  $\ker(f) = \{0\}$  and  $\text{im}(f) = V$ , i.e.  $f$  is bijective.  $\square$

*Note.* We recover Schur's Lemma over  $\mathbb{C}$  by the fact that  $\mathbb{C}$  is the only finite-dimensional division algebra over  $\mathbb{C}$ .

## 1.9 Isotypical Decomposition

**Lemma 1.45.** *Let  $U, V, W$  be three vector spaces. Then we have natural isomorphisms*

1.  $\text{Hom}(V, U \oplus W) = \text{Hom}(V, U) \oplus \text{Hom}(V, W)$
2.  $\text{Hom}(U \oplus W, V) = \text{Hom}(U, V) \oplus \text{Hom}(W, V)$ .



Additionally, if  $U, V, W$  carry representations of  $G$ , then (1) and (2) are isomorphisms of representations.

*Proof.* We have inclusion and projection maps

$$U \begin{array}{c} \xrightarrow{\iota_U} \\ \xleftarrow{\pi_U} \end{array} U \oplus W \begin{array}{c} \xrightarrow{\pi_W} \\ \xleftarrow{\iota_W} \end{array} W$$

given by

$$\begin{aligned} \iota_U &: x \mapsto (x, 0) \\ \pi_U &: (x, y) \mapsto x \end{aligned}$$

and similarly for  $\iota_W$  and  $\pi_W$ . It is clear that

$$\text{Id}_{U \oplus W} = \iota_U \circ \pi_U + \iota_W \circ \pi_W.$$

We also note that the four spaces under consideration all have dimension  $(\dim V)(\dim W + \dim U)$ .

(1) We define a map

$$\begin{aligned} \psi &: \text{Hom}(V, U \oplus W) \rightarrow \text{Hom}(V, U) \oplus \text{Hom}(V, W) \\ f &\mapsto (\pi_U \circ f, \pi_W \circ f). \end{aligned}$$

We claim that this map has an inverse given by

$$\begin{aligned} \psi^{-1} &: \text{Hom}(V, U) \oplus \text{Hom}(V, W) \rightarrow \text{Hom}(V, U \oplus W) \\ (f_U, f_W) &\mapsto \iota_U \circ f_U + \iota_W \circ f_W. \end{aligned}$$

Check that

$$\begin{aligned} \psi^{-1} \circ \psi &: f \mapsto \iota_U \circ \pi_U \circ f + \iota_W \circ \pi_W \circ f \\ &= (\iota_U \circ \pi_U + \iota_W \circ \pi_W) \circ f \\ &= \text{Id}_{\text{Hom}(V, U \oplus W)} \circ f = f. \end{aligned}$$

Since both vector spaces have the same dimension,  $\psi \circ \psi^{-1}$  must be the identity map as well, and  $\psi$  is an isomorphism of vector spaces. Now suppose we have representations  $\rho_V, \rho_W, \rho_U$  of  $G$  on  $V, W$  and  $U$ . Then we claim  $\psi$  is  $G$ -linear. Recall that by definition,

$$\rho_{\text{Hom}(V, U \oplus W)}(g)(f) = \rho_{U \oplus W}(g) \circ f \circ \rho_V(g^{-1}).$$

Observe that for any  $g \in G$  and  $f \in \text{Hom}(V, U \oplus W)$ ,

$$\begin{aligned} \pi_U \circ (\rho_{\text{Hom}(V, U \oplus W)}(g)(f)) &= \pi_U \circ \rho_{U \oplus W}(g) \circ f \circ \rho_V(g^{-1}) \\ &= \rho_U(g) \circ \pi_U \circ f \circ \rho_V(g^{-1}) \quad (\text{since } \pi_U \text{ is } G\text{-linear}) \\ &= \rho_{\text{Hom}(U, V)}(g)(f) \end{aligned}$$

and similarly for  $W$ , so that

$$\begin{aligned}\psi(\rho_{\text{Hom}(V, U \oplus W)}(g)(f)) &= (\pi_U \circ \rho_{\text{Hom}(V, U \oplus W)}(g)(f), \pi_W \circ \rho_{\text{Hom}(V, U \oplus W)}(g)(f)) \\ &= (\rho_{\text{Hom}(V, U)}(g)(\pi_U \circ f), \rho_{\text{Hom}(V, W)}(g)(\pi_W \circ f)) \\ &= \rho_{\text{Hom}(V, U) \oplus \text{Hom}(V, W)}(g)(\pi_U \circ f, \pi_W \circ f).\end{aligned}$$

Thus  $\psi$  is  $G$ -linear, and we've proved (1).

(2) Define a map

$$\begin{aligned}\phi: \text{Hom}(U \oplus W, V) &\rightarrow \text{Hom}(U, V) \oplus \text{Hom}(W, V) \\ &= (f \circ \iota_U, f \circ \iota_W).\end{aligned}$$

The proof is similar to (1). □

**Corollary 1.46.** *If  $U, V, W$  are representations of  $G$ , then there are natural isomorphisms*

1.  $\text{Hom}_G(V, U \oplus W) = \text{Hom}_G(V, U) \oplus \text{Hom}_G(V, W)$
2.  $\text{Hom}_G(U \oplus W, V) = \text{Hom}_G(U, V) \oplus \text{Hom}_G(W, V)$

*Proof.* (1). By Lemma (1.45), we have an isomorphism of representations

$$\psi: \text{Hom}(V, U \oplus W) \rightarrow \text{Hom}(V, U) \oplus \text{Hom}(V, W).$$

We can apply Lemma (1.39) to obtain an isomorphism on the invariant subrepresentations

$$\text{Hom}(V, U \oplus W)^G \cong (\text{Hom}(V, U) \oplus \text{Hom}(V, W))^G.$$

Then Lemma (1.38) implies that

$$\text{Hom}(V, U \oplus W)^G \cong \text{Hom}(V, U)^G \oplus \text{Hom}(V, W)^G.$$

The statement now follows from Proposition (1.37).

(2). The argument is similar to the one above. □

**Proposition 1.47.** *Let  $V$  and  $W$  be irreducible representations of  $G$ . Then*

$$\dim \text{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \text{ and } W \text{ are isomorphic} \\ 0 & \text{if } V \text{ and } W \text{ are not isomorphic} \end{cases}$$

*Proof.* Suppose  $V$  and  $W$  are not isomorphic. Then Schur's Lemma Corollary 1.41 states that the only  $G$ -linear map from  $V$  to  $W$  is the zero map, hence  $\text{Hom}_G(V, W) = \{0\}$ .

On the other hand, suppose that  $f: V \rightarrow W$  is an isomorphism. Then for any  $h \in \text{Hom}_G(V, W)$ , we have  $f^{-1} \circ h \in \text{Hom}_G(V, V)$ . By Schur's Lemma,  $f^{-1} \circ h = \lambda \text{Id}_V$  for some  $\lambda \in \mathbb{C}$ , i.e.  $h = \lambda f$ . Thus  $f$  spans  $\text{Hom}_G(V, W)$ . □

**Proposition 1.48.** *Let  $\rho: G \rightarrow GL(V)$  be a representation, and let*

$$V = U_1 \oplus \dots \oplus U_s$$

*be a decomposition of  $V$  into irreducibles. Let  $W$  be any irreducible representation of  $G$ . Then the number of irreducible representations in the set  $\{U_1, \dots, U_s\}$  which are isomorphic to  $W$  is equal to the dimension of  $\text{Hom}_G(V, W)$ , and also equal to the dimension of  $\text{Hom}_G(W, V)$ .*

*Proof.* We know from Proposition (1.47) that the number of irreducible representations in the set  $\{U_1, \dots, U_s\}$  which are isomorphic to  $W$  is equal to

$$\sum_{i=1}^s \dim \operatorname{Hom}_G(U_i, W).$$

By Corollary (1.46),

$$\operatorname{Hom}_G(V, W) = \bigoplus_{i=1}^s \operatorname{Hom}_G(U_i, W)$$

so that

$$\dim \operatorname{Hom}_G(V, W) = \sum_{i=1}^s \dim \operatorname{Hom}_G(U_i, W).$$

The same argument works if we consider  $\operatorname{Hom}_G(W, V)$  and  $\operatorname{Hom}_G(W, U_i)$  in place of  $\operatorname{Hom}_G(V, W)$  and  $\operatorname{Hom}_G(U_i, W)$ .  $\square$

**Theorem 1.49.** Let  $\rho: G \rightarrow GL(V)$  be a representation, and let

$$\begin{aligned} V &= U_1 \oplus \dots \oplus U_s \\ V &= \widetilde{U}_1 \oplus \dots \oplus \widetilde{U}_r \end{aligned}$$

be two decompositions of  $V$  into irreducible subrepresentations. Then  $s = r$ , and (after reordering if necessary)  $U_i$  and  $\widetilde{U}_i$  are isomorphic for every  $i \in \{1, \dots, s\}$ .

*Proof.* Let  $W$  be any irreducible representation of  $G$ . By Proposition (1.48), the number of irreducible subrepresentations in the first decomposition that are isomorphic to  $W$  is equal to  $\dim \operatorname{Hom}_G(V, W)$ . On the other hand, the number of irreducible subrepresentations in the second decomposition that are isomorphic to  $W$  is also equal to  $\dim \operatorname{Hom}_G(V, W)$ . So for any irreducible representation  $W$ , the two decompositions contain the same number of factors isomorphic to  $W$ .  $\square$

**Definition 1.50.** Let  $V$  be a finite-dimensional representation of  $G$ . Decompose  $V$  into a direct sum

$$V = (V_{11} \oplus \dots \oplus V_{1n_1}) \oplus (V_{21} \oplus \dots \oplus V_{2n_2}) \oplus \dots \oplus (V_{k1} \oplus \dots \oplus V_{kn_k})$$

where

1.  $V_{11}, V_{21}, \dots, V_{k1}$  are pairwise non-isomorphic irreducible representations of  $G$ .
2.  $V_{i1} \cong \dots \cong V_{in_i} \quad \forall 1 \leq i \leq k$ .

Now, for any irreducible representation  $S$  of  $G$ , we use the decomposition of  $V$  above to define

$$V_S = \bigoplus_{\{(i,j) | V_{ij} \cong S\}} V_{ij}.$$

Then we have

$$V_S \cong \begin{cases} S^{n_i} & \text{if } S \cong V_{1i} \text{ for some } 1 \leq i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

This leaves us with the canonical decomposition

$$V = \bigoplus_S V_S$$

where  $S$  ranges over a set of representatives of the isomorphism classes of the irreducible representations of  $G$ . We call this the **canonical decomposition of  $V$  into isotypical components**. It follows from Theorem 1.49 that such a decomposition is unique, i.e. it does not depend on our initial decomposition of  $V$  into irreducibles.

## 1.10 Duals and Tensor Products

**Definition 1.51.** Let  $V$  be a vector space. Recall that we define the **dual vector space** to be

$$V^* = \text{Hom}(V, \mathbb{C}).$$

This is a special case of  $\text{Hom}(V, W)$  where  $W = \mathbb{C}$ . We know that if  $\{b_1, \dots, b_n\}$  is a basis for  $V$ , then there is a **dual basis**  $\{f_1, \dots, f_n\}$  for  $V$  defined by

$$f_i(b_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let  $\rho_V: G \rightarrow GL(V)$  be a representation of  $G$ , and let  $\mathbb{C}$  be the 1-dimensional trivial representation of  $G$ . Then we have seen that  $V^*$  carries a representation of  $G$  defined by

$$\rho_{\text{Hom}(V, \mathbb{C})}(g)(f) = f \circ \rho_V(g^{-1})$$

We call this the **dual representation** to  $\rho_V$  and denote it by  $\rho_V^*$ .

**Proposition 1.52.** *If we fix a basis for  $V$ , then  $\rho_V^*(g)$  is given by the matrix*

$$(\rho_V(g^{-1}))^T$$

*with respect to the dual basis.*

*Proof.* Fix a basis  $\{a_1, \dots, a_n\}$  for  $V$ . Let  $\rho_V(g^{-1})$  be described by the matrix  $M$ , so that

$$\rho_V(g^{-1})(a_j) = \sum_{1 \leq i \leq n} M_{ij} a_i.$$

Let  $\rho_V^*(g)$  be described by the matrix  $N$  with respect to the dual basis  $\{\alpha_1, \dots, \alpha_n\}$ , so that

$$\rho_V^*(g)(\alpha_j) = \sum_{1 \leq i \leq n} N_{ij} \alpha_i.$$

Then

$$\begin{aligned}
N_{ji} &= \sum_{1 \leq k \leq n} N_{ki} \delta_{kj} \\
&= \sum_{1 \leq k \leq n} N_{ki} (\alpha_k a_j) \\
&= \left( \sum_{1 \leq k \leq n} N_{ki} \alpha_k \right) a_j \\
&= (\rho_V^*(g)(\alpha_i))(a_j) \\
&= (\alpha_i \circ \rho_V(g^{-1}))(a_j) \quad (\text{by definition of the dual representation}) \\
&= \alpha_i(\rho_V(g^{-1})(a_j)) \\
&= \alpha_i \left( \sum_{1 \leq k \leq n} M_{kj} a_k \right) \\
&= \sum_{1 \leq k \leq n} M_{kj} \alpha_i a_k \\
&= \sum_{1 \leq k \leq n} M_{kj} \delta_{ik} \\
&= M_{ij}.
\end{aligned}$$

That is,  $N = M^T$ . □

**Definition 1.53.** Suppose  $V$  and  $W$  are two vector spaces over a field  $K$ . Then we define a new vector space called the **tensor product** of  $V$  and  $W$ , denoted by  $V \otimes_K W$ . This space is the quotient of the free vector space on  $V \times W$  (with basis given by formal symbols  $v \otimes w, v \in V, w \in W$ ), by the subspace  $D$  spanned by all elements of the form

$$\begin{aligned}
&(v_1 + v_2, w) - (v_1, w) - (v_2, w) \\
&(v, w_1 + w_2) - (v, w_1) - (v, w_2) \\
&(k \cdot v, w) - (v, k \cdot w)
\end{aligned}$$

for  $v, v_1, v_2 \in V, w, w_1, w_2 \in W$ , and  $k \in K$ . When the ground field  $K$  is clear it can be omitted from the notation. The elements of  $V \otimes W$  are called **tensors**, and the coset  $v \otimes w$  of  $(v, w)$  in  $V \otimes W$  is called a **simple tensor**. We have the relations

$$\begin{aligned}
(v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \\
v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2 \\
(k \cdot v) \otimes w &= v \otimes (k \cdot w) = k \cdot (v \otimes w).
\end{aligned}$$

**Definition 1.54.** Let  $V$  and  $W$  be vector spaces over  $K$ . A map  $\phi: V \times W \rightarrow K$  is called **K-balanced** if

$$\begin{aligned}
\phi(v_1 + v_2, w) &= \phi(v_1, w) + \phi(v_2, w) \\
\phi(v, w_1 + w_2) &= \phi(v, w_1) + \phi(v, w_2) \\
\phi(v, kw) &= \phi(kv, w)
\end{aligned}$$

for all  $v \in V, w \in W, k \in K$ .

**Example 1.55.** Mapping  $V \times W$  to the free  $K$ -vector space on  $V \times W$ , and then passing to the quotient defines a map  $\iota: V \times W \rightarrow V \otimes W$  with  $\iota(v, w) = v \otimes w$ . From the relations satisfied by the tensor product, we see that the map  $\iota$  is  $K$ -balanced.

**Theorem 1.56.** [Universal property of the tensor product] Suppose  $V, W$ , and  $U$  are vector spaces over the field  $K$ . Let  $\varphi: V \times W \rightarrow U$  be a  $K$ -balanced map, and let  $\iota$  be the map above. Then there is a unique linear map  $\tilde{\varphi}: V \otimes W \rightarrow U$  such that  $\varphi$  factors through  $\iota$ , i.e.,  $\varphi = \tilde{\varphi} \circ \iota$ .

*Proof.* The map  $\varphi$  extends by linearity to a linear transformation  $\tilde{\varphi}$  from the free vector space on  $V \times W$  to  $U$  such that  $\tilde{\varphi}(v, w) = \varphi(v, w)$  for all  $v \in V, w \in W$ . Since  $\varphi$  is  $K$ -balanced,  $\tilde{\varphi}$  maps each of the elements which span the subspace  $D$  from the definition of the tensor product to 0. For example,

$$\tilde{\varphi}((kv, w) - (v, kw)) = \varphi(kv, w) - \varphi(v, kw) = 0.$$

Thus the kernel of  $\tilde{\varphi}$  contains  $D$ , and so  $\tilde{\varphi}$  induces a linear map  $\varphi: V \otimes W \rightarrow U$ . Then

$$\varphi(v \otimes w) = \tilde{\varphi}(v, w) = \varphi(v, w)$$

i.e.,  $\varphi = \varphi \circ \iota$ . Note that  $\varphi$  is completely determined by this equation since the elements  $v \otimes w$  span  $V \otimes W$ .  $\square$

**Proposition 1.57.** Let  $\{e_i\}_{i \in I}$  and  $\{f_j\}_{j \in J}$  be bases for  $V$  and  $W$ . Then  $\{e_i \otimes f_j \mid i \in I, j \in J\}$  is a basis for  $V \otimes W$ .

*Proof.* An elementary tensor in  $V \otimes W$  has the form  $v \otimes w$ . Write  $v = \sum_i a_i e_i$  and  $w = \sum_j b_j f_j$ , where all but finitely many of  $a_i$  and  $b_j$  are 0. Then

$$v \otimes w = \sum_i a_i e_i \otimes \sum_j b_j f_j = \sum_{i,j} a_i b_j e_i \otimes f_j$$

is a linear combination of the tensors  $e_i \otimes f_j$ . Since every tensor can be written as a sum of elementary tensors, the elements  $e_i \otimes f_j$  span  $V \otimes W$ .

Now, we must show that this spanning set is linearly independent. Suppose that  $\sum_{i,j} c_{ij} e_i \otimes f_j = 0$ , where all but finitely many  $c_{ij}$  are 0. We want to show that  $c_{ij} = 0$  for every  $i \in I, j \in J$ . Fix two elements  $i_0 \in I$  and  $j_0 \in J$ . To show that  $c_{i_0 j_0} = 0$ , consider the  $K$ -balanced map

$$\begin{aligned} V \times W &\rightarrow K \\ (v, w) &\mapsto a_{i_0} b_{j_0} \end{aligned}$$

where  $v = \sum_i a_i e_i$  and  $w = \sum_j b_j f_j$ . By the universal property of tensor products, there is a linear map  $f_0: V \otimes W \rightarrow K$  such that  $f_0(v \otimes w) = a_{i_0} b_{j_0}$  on any elementary tensor  $v \otimes w$ . In particular,  $f_0(e_{i_0} \otimes f_{j_0}) = 1$  and  $f_0(e_i \otimes f_j) = 0$  for  $(i, j) \neq (i_0, j_0)$ . Applying  $f_0$  to our assumption that  $\sum_{i,j} c_{ij} e_i \otimes f_j = 0$  in  $V \otimes W$  tells us that  $c_{i_0 j_0} = 0$  in  $K$ .  $\square$

**Proposition 1.58.** There are natural isomorphisms

1.  $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$
2.  $(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$ .

*Proof.* (1.) For each fixed  $w \in W$ , the mapping  $(u, v) \mapsto u \otimes (v \otimes w)$  is  $K$ -balanced, so by Theorem 1.56 there is a unique linear map from  $U \otimes V$  to  $U \otimes (V \otimes W)$  with  $u \otimes v \mapsto u \otimes (v \otimes w)$ . This shows that the map from  $(U \otimes V) \times W$  to  $U \otimes (V \otimes W)$  given by  $(u \otimes v, w) \mapsto u \otimes (v \otimes w)$  is well defined. This map is also  $K$ -balanced, and thus another application of Theorem 1.56 shows that it induces a linear map  $(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$  such that  $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$ . In a similar manner, we can construct a map  $U \otimes (V \otimes W) \rightarrow (U \otimes V) \otimes W$  with  $u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$  which is inverse to our first map. This proves the isomorphism.

(2.) The map  $(U \oplus V) \times W \rightarrow (U \oplus W) \otimes (V \oplus W)$  defined by  $((u, v), w) \mapsto (u \otimes w, v \otimes w)$  is clearly  $K$ -balanced. Thus it induces a linear map  $f: (U \oplus V) \otimes W \rightarrow (U \otimes W) \oplus (V \otimes W)$  with

$$f((u, v) \otimes w) = (u \otimes w, v \otimes w).$$

In the other direction, we use the  $K$ -balanced maps  $U \times W \rightarrow (U \oplus V) \otimes W$  and  $V \times W \rightarrow (U \oplus V) \otimes W$  given by  $(u, w) \mapsto (u, 0) \otimes w$  and  $(v, w) \mapsto (0, v) \otimes w$  to obtain linear maps from  $U \otimes W$  and  $V \otimes W$  to  $(U \oplus V) \otimes W$ . Together these maps give a linear transformation  $g$  from the direct sum  $(U \otimes W) \oplus (V \otimes W)$  to  $(U \oplus V) \otimes W$  with

$$g(u \otimes w_1, v \otimes w_2) = (u, 0) \otimes w_1 + (0, v) \otimes w_2.$$

It is straightforward to see that  $f$  and  $g$  are inverse linear transformations, and the isomorphism holds.  $\square$

Now let  $V$  and  $W$  be two representations of  $G$ .

**Definition 1.59.** We can define a representation of  $G$  on  $V \otimes W$  called the **tensor product representation**. We let

$$\rho_{V \otimes W}(g): V \otimes W \rightarrow V \otimes W$$

be the linear map given by

$$\rho_{V \otimes W}(g): a_i \otimes b_j \mapsto \rho_V(g)(a_i) \otimes \rho_W(g)(b_j).$$

Suppose  $\rho_V(g)$  is described by the matrix  $M$  and  $\rho_W(g)$  is described by the matrix  $N$  in the given bases  $\{a_1, \dots, a_n\}$  for  $V$  and  $\{b_1, \dots, b_m\}$  for  $W$ . Then

$$\begin{aligned} \rho_{V \otimes W}(g): a_i \otimes b_t &\mapsto \left( \sum_{j=1}^n M_{ji} a_j \right) \otimes \left( \sum_{s=1}^m N_{st} b_s \right) \\ &= \sum_{\substack{j \in [1, n] \\ s \in [1, m]}} M_{ji} N_{st} a_j \otimes b_s. \end{aligned}$$

So  $\rho_{V \otimes W}$  is described by the  $nm \times nm$  matrix  $M \otimes N$  whose entries are

$$[M \otimes N]_{js, it} = M_{ji} N_{st}.$$

This matrix has  $nm$  rows, and to specify a row we need a pair of numbers  $(j, s)$  where  $j \in \{1, \dots, n\}$  and  $s \in \{1, \dots, m\}$ .

**Proposition 1.60.** Let  $V$  and  $W$  be representations of  $G$ . Then  $V \otimes W$  is isomorphic to  $\text{Hom}(V^*, W)$ .

*Proof.* Let  $\{a_1, \dots, a_n\}$  be a basis for  $V$ , let  $\{\alpha_1, \dots, \alpha_n\}$  be the corresponding dual basis for  $V^*$ , and let  $\{b_1, \dots, b_m\}$  be a basis for  $W$ . Then  $\text{Hom}(V^*, W)$  has a basis  $\{f_{it} | 1 \leq i \leq$

$n, 1 \leq t \leq m\}$  where

$$f_{it}(\alpha_j) = \begin{cases} b_t & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

We obtain an isomorphism of vector spaces between  $\text{Hom}(V^*, W)$  and  $V \otimes W$  by the map

$$\psi(f_{it}) = a_i \otimes b_t$$

extended to all of  $\text{Hom}(V^*, W)$  by linearity. It remains to show that this isomorphism of vector spaces yields an isomorphism of representations, i.e. we need to check that

$$\psi \circ \rho_{\text{Hom}(V^*, W)}(g) = \rho_{V \otimes W}(g) \circ \psi$$

for all  $g \in G$ . Fix  $g \in G$ , and let  $M$  and  $N$  denote the matrices which describe  $\rho_V(g)$  and  $\rho_W(g)$  in the given bases. By definition,

$$\rho_{\text{Hom}(V^*, W)}(g)(f_{it}) = \rho_W(g) \circ f_{it} \circ \rho_{V^*}(g^{-1}).$$

Now  $\rho_{V^*}(g^{-1})$  is given by the matrix  $M^T$  in the dual basis by Proposition 1.52, so

$$\rho_{V^*}(g^{-1})(\alpha_k) = \sum_{j=1}^n M_{kj} \alpha_j.$$

Then

$$f_{it} \circ \rho_{V^*}(g^{-1})(\alpha_k) = M_{ki} b_t$$

which means that

$$\rho_W(g) \circ f_{it} \circ \rho_{V^*}(g^{-1})(\alpha_k) = M_{ki} \left( \sum_{s=1}^m N_{st} b_s \right).$$

Thus, if we write  $\rho_W(g) \circ f_{it} \circ \rho_{V^*}(g^{-1})$  in terms of the basis  $\{f_{js}\}$ , we have

$$\rho_W(g) \circ f_{it} \circ \rho_{V^*}(g^{-1}) = \sum_{\substack{j \in [1, n] \\ s \in [1, m]}} M_{ji} N_{st} f_{js}$$

(since both sides agree on every basis vector  $\alpha_k$ ). Therefore,

$$\begin{aligned} \psi \circ \rho_{\text{Hom}(V^*, W)}(g)(f_{it}) &= \psi \left( \rho_W(g) \circ f_{it} \circ \rho_{V^*}(g^{-1}) \right) \\ &= \psi \left( \sum_{\substack{j \in [1, n] \\ s \in [1, m]}} M_{ji} N_{st} f_{js} \right) \\ &= \sum_{\substack{j \in [1, n] \\ s \in [1, m]}} M_{ji} N_{st} a_j \otimes b_s \\ &= \rho_{V \otimes W}(g)(a_i \otimes b_t) \quad (\text{by definition of the tensor product representation}) \\ &= \rho_{V \otimes W}(g) \circ \psi(f_{it}) \end{aligned}$$



□

## 1.11 Character Theory

**Definition 1.61.** The **character** of a representation  $\rho: G \rightarrow GL(V)$  is the function  $\chi_V: G \rightarrow \mathbb{C}$  defined by  $\chi_V(g) = \text{Tr}(\rho(g))$ .

*Note.* The character of a representation is not a homomorphism in general, since  $\text{Tr}(MN) \neq \text{Tr}(M)\text{Tr}(N)$  in general.

**Proposition 1.62.** (*Basic Properties*)

1.  $\chi_V$  is conjugation invariant:  $\chi_V(hgh^{-1}) = \chi_V(g)$  for all  $g, h \in G$ .
2.  $\chi_V(1) = \dim V$ .
3.  $\chi_V(g^{-1}) = \overline{\chi_V(g)}$  for all  $g \in G$ .
4.  $\chi_{V^*}(g) = \overline{\chi_V(g)}$  for all  $g \in G$ .

*Proof.* 1.  $\chi_V(hgh^{-1}) = \text{Tr}(hgh^{-1}) = \text{Tr}(ghh^{-1}) = \text{Tr}(g) = \chi_V(g)$  for any  $g, h \in G$ .

2.  $\chi_V(1) = \text{Tr}(\text{Id}_V) = \dim V$ .

3. Since  $G$  is finite, we have seen that  $\rho(g)$  is a diagonal matrix with roots of unity along the diagonal with the right choice of basis. The inverse of a root of unity is given by its complex conjugate, so under this same basis,  $\rho(g)^{-1}$  is clearly given by  $\overline{\rho(g)}$ . Thus,  $\chi_V(g^{-1}) = \text{Tr}(\rho(g^{-1})) = \text{Tr}(\rho(g)^{-1}) = \text{Tr}(\overline{\rho(g)}) = \overline{\text{Tr}(\rho(g))} = \overline{\chi_V(g)}$ .

4.

$$\begin{aligned}
 \chi_{V^*}(g) &= \text{Tr}(\rho_{V^*}(g)) \\
 &= \text{Tr}(\rho_V(g^{-1})^T) \quad (\text{by Proposition 1.52}) \\
 &= \text{Tr}(\rho_V(g^{-1})) \\
 &= \overline{\chi_V(g)} \quad (\text{by 3})
 \end{aligned}$$

□

**Proposition 1.63.** *Isomorphic representations have the same character.*

*Proof.* We have seen in Proposition 1.12 that isomorphic representations can be described by the same set of matrices with the right choice of bases. Thus they have the same trace. □

We will see later that the converse is true - if two representations have the same character, then they are isomorphic.

**Proposition 1.64.** *Let  $\rho_V: G \rightarrow GL(V)$  and  $\rho_W: G \rightarrow GL(W)$  be representations of  $G$  with characters  $\chi_V$  and  $\chi_W$ .*

1.  $\chi_{V \oplus W} = \chi_V + \chi_W$ .
2.  $\chi_{V \otimes W} = \chi_V \cdot \chi_W$ .

*Proof.* Pick bases for  $V$  and  $W$ , so that  $\rho_V(g)$  and  $\rho_W(g)$  are described by matrices  $M$  and  $N$ .

1.  $\rho_{V \oplus W}(g)$  is described by the block-diagonal matrix

$$\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

So we have  $\text{Tr}(\rho_{V \oplus W}(g)) = \text{Tr}(M) + \text{Tr}(N) = \text{Tr}(\rho_V(g)) + \text{Tr}(\rho_W(g))$ .

2.  $\rho_{V \otimes W}(g)$  is given by the matrix

$$[M \otimes N]_{js,it} = M_{ji}N_{st}$$

so

$$\begin{aligned} \text{Tr}(M \otimes N) &= \sum_{i,t} [M \otimes N]_{ii,tt} \\ &= \sum_{i,t} (M_{ii}N_{tt}) \\ &= \text{Tr}(M)\text{Tr}(N). \end{aligned}$$

Thus  $\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g)$ .

□

**Definition 1.65.** Let  $\mathbb{C}^G$  denote the set of all functions from  $G$  to  $\mathbb{C}$ . Then  $\mathbb{C}^G$  is a vector space with the sum of two functions defined pointwise and with scalar multiplication defined for  $f \in \mathbb{C}^G, \lambda \in \mathbb{C}$  by

$$\begin{aligned} \lambda f &: G \rightarrow \mathbb{C} \\ g &\mapsto \lambda f(g). \end{aligned}$$

A basis for  $\mathbb{C}^G$  is clearly given by the set of functions

$$\{\delta_g | g \in G\}$$

defined by

$$\delta_g: h \mapsto \begin{cases} 1 & \text{if } h = g \\ 0 & \text{if } h \neq g \end{cases}$$

**Definition 1.66.** Let  $\varphi, \psi \in \mathbb{C}^G$ . We define an **inner product** on  $\mathbb{C}^G$  by

$$\langle \varphi | \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}.$$

It is easy to see that  $\langle \varphi | \psi \rangle$  is linear in the first variable, conjugate-linear in the second variable (i.e.,  $\langle \varphi | \lambda \psi \rangle = \overline{\lambda} \langle \varphi | \psi \rangle$ ), and that  $\langle \varphi | \psi \rangle = \overline{\langle \psi | \varphi \rangle}$ . These three properties are the definition of a Hermitian inner product. Note that our basis elements  $\delta_g$  are orthogonal with respect to this inner product, but not orthonormal since

$$\langle \delta_g | \delta_g \rangle = \begin{cases} \frac{1}{|G|} & \text{if } h = g \\ 0 & \text{if } h \neq g \end{cases}.$$

The characters of  $G$  are elements of  $\mathbb{C}^G$ , so we can evaluate this inner product on pairs of characters. The answer turns out to be very useful, but before we can begin the proof we require two quick lemmas:

**Lemma 1.67.** *Let  $\rho: G \rightarrow GL(V)$  be any representation. Consider the linear map*

$$\begin{aligned}\Psi: V &\rightarrow V \\ x &\mapsto \frac{1}{|G|} \sum_{g \in G} \rho(g)(x).\end{aligned}$$

*Then  $\Psi$  is a projection from  $V$  onto the invariant subspace  $V^G$ .*

*Proof.* We need to check that  $\Psi(x) \in V^G$  for all  $x \in V$ . For any  $h \in G$ ,

$$\begin{aligned}\rho(h)(\Psi(x)) &= \frac{1}{|G|} \sum_{g \in G} \rho(h)\rho(g)(x) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho(hg)(x) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho(g)(x) \quad (\text{by relabelling } g \mapsto h^{-1}g) \\ &= \Psi(x).\end{aligned}$$

Thus  $\Psi$  is a linear map  $V \rightarrow V^G$ . Finally we need to check that  $\Psi|_{V^G} = \text{Id}_{V^G}$ . Let  $x \in V^G$ . Then

$$\begin{aligned}\Psi(x) &= \frac{1}{|G|} \sum_{g \in G} \rho(g)(x) \\ &= \frac{1}{|G|} \sum_{g \in G} (x) \\ &= \frac{|G|}{|G|} x = x.\end{aligned}$$

□

**Lemma 1.68.** *Let  $V$  be a vector space with subspace  $U \subset V$ , and let  $\pi: V \rightarrow V$  be a projection onto  $U$ . Then*

$$\text{Tr}(\pi) = \dim U.$$

*Proof.* Recall that  $V = U \oplus \text{Ker}(\pi)$  from Lemma 1.30. If we fix bases for  $U$  and  $\text{Ker}(\pi)$ , which together give a basis for  $V$ , then  $\pi$  is given by the block-diagonal matrix

$$\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{0} \end{bmatrix}$$

where  $\dim U$  is the size of the upper left block and  $\dim \text{Ker}(\pi)$  is the size of the bottom right block. So  $\text{Tr}(\pi) = \text{Tr}(\mathbf{1}_U) = \dim U$ . □

**Theorem 1.69.** *Let  $\rho_V: G \rightarrow GL(V)$  and  $\rho_W: G \rightarrow GL(W)$  be representations of  $G$ , and let  $\chi_V, \chi_W$  be their characters. Then*

$$\langle \chi_W | \chi_V \rangle = \dim \text{Hom}_G(V, W).$$

In particular, the inner product of two characters is always a non-negative integer. (Whereas in general, the inner product of two arbitrary functions can be any complex number.)

*Proof.* We have seen in Proposition 1.37 that

$$\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}(V, W)$$

as the invariant subrepresentation, and by the previous lemma we have a projection

$$\begin{aligned} \Psi: \mathrm{Hom}(V, W) &\rightarrow \mathrm{Hom}(V, W) \\ f &\mapsto \frac{1}{|G|} \sum_{g \in G} \rho_{\mathrm{Hom}(V, W)}(g)(f). \end{aligned}$$

We claim that

$$\mathrm{Tr}(\Psi) = \langle \chi_W | \chi_V \rangle.$$

Once this claim is established, then Lemma 1.68 will prove the theorem. We proceed by calculating  $\mathrm{Tr}(\Psi)$ . Fix bases  $\{a_1, \dots, a_n\}$  for  $V$  and  $\{b_1, \dots, b_m\}$  for  $W$ . Then  $\mathrm{Hom}(V, W)$  has an associated basis

$$\{f_{ji} | 1 \leq i \leq n, 1 \leq j \leq m\}$$

where

$$f_{ji}(a_i) = \begin{cases} b_j & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

We may calculate  $\mathrm{Tr}(\Psi)$  as follows: For each  $i, j$ , compute the expression of  $\Psi(f_{ji})$  in this basis, and take the coefficient of the basis element  $f_{ji}$ . This is a diagonal entry in the matrix for  $\Psi$ . Summing these values over all  $i$  and  $j$  will give us  $\mathrm{Tr}(\Psi)$ .

Let  $\widetilde{\rho}_V, \widetilde{\rho}_W$  be the matrix representations obtained by writing  $\rho_V$  and  $\rho_W$  in the given bases. We know that

$$\mathrm{Hom}(V, W) = V^* \otimes W$$

so if we write  $\rho_{\mathrm{Hom}(V, W)}$  in the basis  $\{f_{ji}\}$  then we get the tensor product of  $\widetilde{\rho}_{V^*}$  and  $\widetilde{\rho}_W$ . Thus

$$\begin{aligned} \rho_{\mathrm{Hom}(V, W)}(g)(f_{ji}) &= \rho_W(g) \circ f_{ji} \circ \rho_V(g^{-1}) \\ &= \sum_{\substack{k \in [1, n] \\ t \in [1, m]}} \widetilde{\rho}_V(g^{-1})_{ik} \widetilde{\rho}_W(g)_{tj} f_{kt}. \quad \text{Show another step?} \end{aligned}$$

Then

$$\begin{aligned} \Psi(f_{ji}) &= \frac{1}{|G|} \sum_{g \in G} \rho_{\mathrm{Hom}(V, W)}(g)(f) \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{k \in [1, n] \\ t \in [1, m]}} \widetilde{\rho}_V(g^{-1})_{ik} \widetilde{\rho}_W(g)_{tj} f_{kt}. \end{aligned}$$

The coefficient of  $f_{ji}$  in this expression is

$$\frac{1}{|G|} \sum_{g \in G} \widetilde{\rho}_V(g^{-1})_{ii} \widetilde{\rho}_W(g)_{jj}.$$

Therefore

$$\begin{aligned} \text{Tr}(\Psi) &= \sum_{\substack{k \in [1, n] \\ t \in [1, m]}} \frac{1}{|G|} \sum_{g \in G} \widetilde{\rho}_V(g^{-1})_{ii} \widetilde{\rho}_W(g)_{jj} \\ &= \frac{1}{|G|} \sum_{g \in G} \left( \sum_{i=1}^n \widetilde{\rho}_V(g^{-1})_{ii} \right) \left( \sum_{j=1}^m \widetilde{\rho}_W(g)_{jj} \right) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g^{-1}) \chi_W(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_W(g) \overline{\chi_V}(g) \quad (\text{by Proposition 1.62.3}) \\ &= \langle \chi_W | \chi_V \rangle. \end{aligned}$$

□

**Corollary 1.70** (Row orthogonality relations). *Let  $\chi_1, \dots, \chi_r$  be characters of pairwise non-isomorphic irreducible representations of  $G$ . Then*

$$\langle \chi_i | \chi_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

*Proof.* Let  $\chi_i$  and  $\chi_j$  be the characters of the irreducible representations  $U_i, U_j$ . Then by Proposition 1.47,

$$\langle \chi_i | \chi_j \rangle = \dim \text{Hom}_G(U_i, U_j) = \begin{cases} 1 & \text{if } U_i, U_j \text{ are isomorphic} \\ 0 & \text{if } U_i, U_j \text{ are not isomorphic.} \end{cases}$$

□

**Corollary 1.71.** *Let  $\chi$  be any character of  $G$ . Then  $\chi$  is irreducible if and only if*

$$\langle \chi | \chi \rangle = 1$$

*Proof.* Write  $\chi$  as a linear combination of irreducible characters

$$\chi = m_1 \chi_1 + \dots + m_k \chi_k$$

where each  $m_i$  is a non-negative integer. Then by Corollary 1.70,

$$\begin{aligned} \langle \chi | \chi \rangle &= \sum_{i, j \in [1, k]} m_i m_j \langle \chi_i | \chi_j \rangle \\ &= m_1^2 + \dots + m_k^2. \end{aligned}$$

So  $\langle \chi | \chi \rangle = 1$  if and only if exactly one of the  $m_i = 1$  and the rest are 0.

□

**Corollary 1.72.** *Let  $\rho_V: G \rightarrow GL(V)$  and  $\rho_W: G \rightarrow GL(W)$  be representations of  $G$ . Then  $V$  and  $W$  are isomorphic if and only if  $\chi_V = \chi_W$ .*

*Proof.* We have already seen that isomorphic representations have the same character by Proposition . On the other hand, suppose  $\chi_V = \chi_W$ . Let  $U_1, \dots, U_r$  be the irreducible representations of  $G$ , and let  $\chi_1, \dots, \chi_r$  be their characters. We can write

$$V = U_1^{m_1} \oplus \dots \oplus U_r^{m_r}$$

for some non-negative integers  $m_1, \dots, m_r$ , and

$$W = U_1^{l_1} \oplus \dots \oplus U_r^{l_r}$$

for some non-negative integers  $l_1, \dots, l_r$ . So

$$\chi_V = m_1\chi_1 + \dots + m_r\chi_r$$

and

$$\chi_W = l_1\chi_1 + \dots + l_r\chi_r.$$

Thus we have

$$m_i = \langle \chi_V | \chi_i \rangle = \langle \chi_W | \chi_i \rangle = l_i$$

for all  $i \in \{1, \dots, r\}$  since  $\chi_V = \chi_W$ . This proves  $V$  and  $W$  are isomorphic.  $\square$

**Lemma 1.73.**  $\chi_{\text{reg}}(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{if } g \neq 1 \end{cases}$

*Proof.* For each  $g \in G$ , we have  $\rho_{\text{reg}}(g)(e_h) = e_{gh}$ . So  $\rho_{\text{reg}}$  maps any basis element to another basis element. Thus

$$\text{Tr}(\rho_{\text{reg}}(g)) = |\{g \in G | h = gh\}| = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{if } g \neq 1 \end{cases}.$$

$\square$

**Proposition 1.74.** *The multiplicity of any irreducible representation in the regular representation equals its dimension.*

*Proof.* Let  $V$  be an irreducible representation of  $G$ . Then

$$\begin{aligned} \langle \chi_{\text{reg}}, \chi_V \rangle &= \frac{1}{|G|} \chi_{\text{reg}}(1) \overline{\chi_V(1)} \\ &= \frac{1}{|G|} |G| (\dim V) \\ &= \dim V. \end{aligned}$$

Therefore the multiplicity of  $V$  in the regular representation is  $\dim V$ .  $\square$

**Corollary 1.75.** *There are finitely many irreducible representations of  $G$ , up to isomorphism.*

*Proof.* Since  $\dim FG = |G|$ , there can be at most  $|G|$  irreducible representations of  $G$  up to isomorphism.  $\square$

**Corollary 1.76.** Let  $\chi_1, \dots, \chi_r$  be the irreducible characters of  $G$  with degrees  $d_1, \dots, d_r$ . Then

$$|G| = \sum_{i=1}^n d_i^2$$

*Proof.* Let  $U_1, \dots, U_r$  be distinct representatives of the isomorphism classes of irreducible representations of  $G$ . Then

$$V_{\text{reg}} = U_1^{d_1} \oplus \dots \oplus U_r^{d_r}$$

by Proposition 1.74. The claim follows from taking dimensions of each side of this equation.  $\square$

**Definition 1.77.** A **class function** on  $G$  is a function on  $G$  whose values are invariant by conjugation of elements in  $G$ . The value of a class function at an element  $g \in G$  depends only on the conjugacy class of  $g$ . We may therefore view class functions as functions on the set of conjugacy classes of  $G$ .

*Note.* The character  $\chi_V$  of a representation  $V$  of  $G$  is a class function on  $G$ .

[8] [5] [1] [3] [6] [4] [2] [7]

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