UNIVERSITY OF MISSOURI

MASTER'S PROJECT

A Survey on Character Tables for Representations of Finite Groups

Author:
Jared Stewart

Supervisor:
Dr. Calin Chindris

A project submitted in fulfilment of the requirements for the degree of Masters of Arts

in the

Department of Mathematics

January 16, 2016

Acknowledgements

The acknowledgements and the people to thank go here, don't forget to include Professor Chindris…

Contents

Chapter 1

Basic Notions of Representation Theory

1.1 Group Actions

Definition 1.1. A ***(left)* group action** of a group G on a set X is a map $\varphi\colon G \times X \to X$ (written as $g \cdot a$, for all $g \in G$ and $a \in A$) that satisfies the following two axioms:

$$1 \cdot x = x \qquad \forall x \in X \tag{1.1.1}$$

$$(gh) \cdot x = g \cdot (h \cdot x) \qquad \forall g, h \in G, x \in X \tag{1.1.2}$$

Note. We could likewise define the concept of a *right* group action, where the set elements would be multiplied by group elements on the right instead of on the left. Throughout we shall use the term *group action* to mean a *left* group action.

Proposition 1.2. *Let G act on the set X. For any fixed $g \in G$, the map σ_g from X into X defined by $\sigma_g(x) = g \cdot x$ is a permutation of the set X, i.e. $\sigma_g \in S_X$.*

Proof. We show that σ_g is a permutation of X by finding a two-sided inverse map, namely $\sigma_{g^{-1}}$. Observe that for any $x \in X$, we have

$$\begin{aligned}
(\sigma_{g^{-1}} \circ \sigma_g)(x) &= \sigma_{g^{-1}}(\sigma_g(x)) && \text{(by definition of function composition)}\\
&= g^{-1} \cdot (g \cdot x) && \text{(by definition of } \sigma_g \text{ and } \sigma_{g^{-1}})\\
&= (g^{-1}g) \cdot x && \text{(by axiom 1.1.1 of an action)}\\
&= 1 \cdot x\\
&= x && \text{(by axiom 1.1.2 of an action).}
\end{aligned}$$

Thus $\sigma_{g^{-1}} \circ \sigma_g$ is the identity map on X. We can reverse the roles of g and g^{-1} to see that $\sigma_g \circ \sigma_{g^{-1}}$ is also the identity map on X. Having a two-sided inverse, we conslude that σ_g is a permutation of X. □

Proposition 1.3. *Let G act on the set X. The map from G to the symmetric group S_X defined by $g \mapsto \sigma_g(x) = g \cdot x$ is a group homomorphism.*

Proof. Define the map $\varphi\colon G \to S_X$ by $\varphi(g) = \sigma_g$. We have seen from Proposition 1.2 that σ_g is indeed an element of S_X. It remains to show that $\varphi(g_1 g_2) = \varphi(g_1) \circ \varphi(g_2)$ for any $g_1, g_2 \in G$. Observe that

$$\begin{aligned}
\varphi(g_1g_2)(x) &= \sigma_{g_1g_2}(x) && \text{(by definition of } \varphi) \\
&= (g_1g_2)\cdot x && \text{(by definition of } \sigma_{g_1g_2}) \\
&= g_1\cdot(g_2\cdot x) && \text{(by axiom 1.1.1 of an action)} \\
&= \sigma_{g_1}(\sigma_{g_2}(x)) && \text{(by definition of } \sigma_{g_1} \text{ and } \sigma g_2) \\
&= \varphi(g_1)(\varphi(g_2)(x)) && \text{(by definition of } \varphi) \\
&= (\varphi(g_1)\circ\varphi(g_2))(x) && \text{(by definition of function composition).}
\end{aligned}$$

Since the values of $\varphi(g_1g_2)$ and $\varphi(g_1)\circ\varphi(g_2)$ agree on every element $x \in X$, these two permutations are equal. We conclude that φ is a homomorphism, since g_1 and g_2 were arbitrary elements of G. □

Proposition 1.4. *Any homomorphism ψ from the group G into the symmetric group on S_X on a set X gives rise to an action of G on X, defined by taking $g \cdot x = \psi(g)(x)$.*

Proof. Suppose that we have a homomorphism ψ from G into S_X. We can define a map from $G \times X$ to X by $g \cdot x = \psi(g)(x)$. We verify that this map satisfies the definition of a group action of G on X:
(axiom 1.1.1) $1 \cdot x = \psi(1)(x) = id_X(x) = x$
(axiom 1.1.2) $(gh)\cdot x = \psi(gh)(x) = (\psi(g)\psi(h))(x) = \psi(g)(\psi(h)(x)) = g\cdot(h\cdot x)$ □

Proposition 1.5. *The actions of G on the set X are in bijective correspondence with the homomorphisms from G into the symmetric group S_X.*

Proof. By Proposition 1.3, any action of G on X yields a homomorphism from G into S_X. Conversely, any homomorphism from G into S_X establishes an action of G on X by Proposition 1.4. □

1.2 The Definition of a Representation

Definition 1.6. Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is any group homomorphism $\varphi\colon G \to GL(V)$.

Definition 1.7. Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is any action of G on V which preserves the linear structure of V, that is,

$$g\cdot(v_1+v_2) = g\cdot v_1 + g\cdot v_2 \qquad \forall g \in G, v_1, v_2 \in V \tag{1.7.1}$$

$$g\cdot(kv) = k(g\cdot v) \qquad \forall g \in G, v \in V, k \in F \tag{1.7.2}$$

Note. Unless otherwise specificed, we use *representation* to mean *finite-dimensional complex representation*.

Proposition 1.8. *The definitions of a linear representation given in 1.6 and 1.7 above are equivalent.*

Proof. $(\to)$ Suppose that we have a homomorphism $\varphi\colon G \to GL(V)$. Note that $GL(V)$ is a subgroup of the symmetric group S_V on V, so we can apply Proposition 1.4 to obtain an action of G on V by $g \cdot v = \varphi(g)(v)$. We check that this action preserves the linear structure of V.

1.7.1 For any $g \in G, v_1, v_2 \in V$ we have $g \cdot (v_1 + v_2) = \varphi(g)(v_1 + v_2) = \varphi(g)(v_1) + \varphi(g)(v_2) = g \cdot v_1 + g \cdot v_2$.
1.7.2 For any $g \in G, v \in V, k \in F$ we have $g \cdot (kv) = \varphi(g)(kv) = k(\varphi(g)(v)) = k(g \cdot v)$.

($\leftarrow$) Suppose that we have an action of G on V which preserves the linear structure of V in the sense of Definition 1.7. We can apply Proposition 1.3 to obtain a homomorphism $\varphi\colon G \to S_V$ given by $\varphi(g) = \sigma_g$ where $\sigma_g(v) = g \cdot v$. It remains to show that the image $\varphi(G)$ of G under φ is actually contained in $GL(V)$, i.e. that for each $g \in G$ the map σ_g is linear. Fix an element $g \in G$. For any $k \in F$ and $v \in V$ we have

$$\begin{aligned} \sigma_g(kv) &= g \cdot (kv) && \text{(by definition of } \sigma_g) \\ &= k(g \cdot v) && \text{(by property 1.7.1)} \\ &= k(\sigma_g(v)) && \text{(by definition of } \sigma_g). \end{aligned}$$

Also, for any $v_1, v_2 \in V$ we have

$$\begin{aligned} \sigma_g(v_1 + v_2) &= g \cdot (v_1 + v_2) && \text{(by definition of } \sigma_g) \\ &= g \cdot v_1 + g \cdot v_2 && \text{(by property 1.7.2)} \\ &= \sigma_g(v_1) + \sigma_g(v_2) && \text{(by definition of } \sigma_g). \end{aligned}$$

Thus σ_g is linear, and $\varphi(G) \subset GL(V)$ proves that we have a homomorphism $\varphi\colon G \to GL(V)$.

□

Definition 1.9. Let G be a group, let F be a field, let V be a vector space over F, and let $\varphi\colon G \to GL(V)$ be a representation of G. The **dimension** of the representation is the dimension of V over F.

Example 1.10. 1. Let V be a 1-dimensional vector space over the field F. The map $\varphi\colon G \to GL(V)$ defined by $\varphi(g) = 1$ for all $g \in G$ is a representation called the *trival representation* of G. The trivial representation has dimension 1.

2. If a finite group G acts on a finite set X and F is any field, then there is an associated *permutation representation* on the vector space V over F with basis $\{e_x \colon x \in X\}$. We let G act on the basis elements by $g \cdot e_x = e_{gx}$ for all $x \in X$ and $g \in G$. Note that G permutes the basis elements of V.

3. A fundamental special case of a permutation representation is given by a finite group acting on itself by left multiplication. In this case, the elements of G form a basis for V, and each $g \in G$ permutes the basis elements by $g \cdot g_i = gg_i$. This is called the *regular representation* of G and has dimension $|G|$. We shall see later that this representation encodes information about all other representations of G.

4. For any symmetric group S_n the *alternating representation* on $V = \mathbb{C}$ is given by the map $\varphi\colon S_n \to GL(\mathbb{C}) = \mathbb{C}^\times$ defined by $\varphi(\sigma) = \text{sgn}(\sigma)$. More generally, for any group G with a subgroup H of index 2, we can define an *alternating representation* $\varphi\colon G \to GL(\mathbb{C})$ by letting $\varphi(g) = 1$ if $g \in H$ and $\varphi(g) = -1$ if $g \notin H$. (We recover our original example by taking $G = S_n$ and $H = A_n$.)

Definition 1.11. A **homomorphism** between two representations $\varphi_1 \colon G \to GL(V)$ and $\varphi_2 \colon G \to GL(W)$ is a linear map $\psi \colon V \to W$ that interwines with (respects) the G-action, i.e. such that

$$\psi(\varphi_1(g)(v)) = \varphi_2(g)(\psi(v)) \quad \forall v \in V, g \in G$$

An **isomorphism** of representations is a homomorphism of representations that is also an invertible map.

Note. If we have representations (φ_1, V) and (φ_2, W) and an isomorphism of vector spaces $\psi \colon V \to W$ then we can rewrite the compatibility requirement above as $\varphi_2(g) = \psi \circ \varphi_1(g) \circ \psi^{-1}$ for all $g \in G$.

Given any representation (φ, V) of G on a vector space V over a field F of dimension n, we can fix a basis for V to obtain an isomorphism of vector spaces $\psi \colon V \to F^n$. We obtain a representation ϕ of G on F^n by defining $\phi = \psi \circ \varphi(g) \circ \psi^{-1}$ for all $g \in G$. Clearly, this representation is isomorphic to the original representation (φ, V). In particular we can always choose to view n-dimensional complex representations as representations on $\mathbb{C}^n$ where each $\phi(g)$ is given by an $n \times n$ matrix with entries in $\mathbb{C}$.

Suppose that we have two representations $\varphi \colon G \to GL_n(F)$ and $\phi \colon G \to GL_m(F)$ given by $\varphi(g) = X_g$ and $\phi(g) = Y_g$. A homomorphism between these representations is then an $m \times n$ matrix A such that $AX_g = Y_g A$ for all $g \in G$. An isomorphism is given precisely when such A is square and invertible. Thus, two representations $\varphi \colon G \to GL_n(F)$ and $\phi \colon G \to GL_n(F)$ are isomorphic if and only if there exists $A \in GL_n(F)$ such that $\varphi(g) = A\phi(g)A^{-1}$ for all $g \in G$. This establishes the following proposition:

Proposition 1.12. *The isomorphism classes of n-dimensional representations of G on $\mathbb{C}$ are in bijection with the quotient $Hom(G; GL_n(\mathbb{C}))/GL_n(\mathbb{C})$ of group homomorphisms $G \to GL_n(\mathbb{C})$ modulo the conjugation action of $GL_n(\mathbb{C})$.*

1.3 Representations of Cyclic Groups

Example 1.13 (Representations of $\mathbb{Z}$)**.** We want to classify all representations of the group $\mathbb{Z}$ under addition. Consider an n-dimensional representation $\varphi \colon \mathbb{Z} \to GL_n$. For φ to be a group homomorphism requires that $\varphi(0) = \mathrm{Id}$. Observe that for any $0 \neq n \in \mathbb{Z}$, we have $\varphi(n) = \varphi(1 + \ldots + 1) = \varphi(1)^n$. Thus φ is completely determined by the matrix $\varphi(1) \in GL_n(\mathbb{C})$, and any such matrix determines a representation of $\mathbb{Z}$. It follows that the n-dimensional isomorphism classes of representations of $\mathbb{Z}$ are in bijection with the conjugacy classes in $GL_n(\mathbb{C})$. These conjugacy classes can be parameterized by the *Jordan canonical form*.

Example 1.14 (Representations of the cyclic group of order n)**.** We shall classify all representations of the cyclic group $G = 1 = g^n, g, \ldots, g^{n-1}$ of order n. Consider a representation $\varphi \colon G \to GL(V)$. As in the previous example, we know that $\varphi(1) = \mathrm{Id}$ and $\varphi(g^k) = \varphi(g)^k$. Thus our representation φ is determined completely by the linear transformation $\varphi(g)$. It will be helpful to fix a basis of V so that we may view $\varphi(g)$ as a matrix A. Recall from linear algebra that there exists a basis in which $\varphi(g)$ takes the *Jordan normal form*.

$$A = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_m \end{bmatrix}$$

where each *Jordan block* J_k takes the form

$$J_k = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \ddots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix}.$$

Now $I = A^n$ is a block-diagonal matrix with diagonal blocks J_k^n, so we must have that each block $J_k^n = \mathrm{Id}$. Observe that we can write each block as $J_k = \lambda\mathrm{Id} + N$ where N is the Jordan block with $\lambda = 0$. Thus we have

$$\mathrm{Id} = J_k^n = (\lambda\mathrm{Id}+N)^n = \lambda^n\mathrm{Id} + \binom{n}{1}\lambda^{n-1}N + \binom{n}{2}\lambda^{n-2}N^2 + \ldots + \binom{n}{n-1}\lambda N^{n-1} + N^n$$

.

Lemma 1.15. *Let N be the Jordan block with $\lambda = 0$ of size $n \times n$. For any integer p with $1 \leq p \leq n-1$, then N^p is the matrix with ones in the positions (i, j) where $j = i + p$ and zeroes everywhere else. (The ones lie along a line parallel to the diagonal, p steps above it.)*

Proof. (By induction.)

- *Base case* This is simply the definition of N.
- *Inductive step* Suppose that the lemma holds for N^p. We compute the (i, j) entry of N^{p+1}:

$$(N^{p+1})_{i,j} = \sum_{k=1}^{n} (N^p)_{i,k} N_{k,j} = (N^p)_{i,i+p} N_{i+p,j} = N_{i+p,j} = \begin{cases} 1 & \text{if} j = i + (p+1) \\ 0 & \text{otherwise} \end{cases}$$

□

Now, if $N \neq 0$ then each term $\binom{n}{k}\lambda^{n-k}N^k$ for $k > 0$ would yield some non-zero non-diagonal entries (in the positions (i, j) where $j = i + k$) and hence our sum could not equal the identity matrix. We must conclude that $N = 0$, and $J_k = \lambda^n$ is a 1×1 block. Thus $\varphi(g)$ is a diagonal matrix with nth roots of unity as diagonal entries.

To summarize, every m-dimensional representation φ of the cyclic group $G = \langle g \rangle$ of order n can be seen to act (in the right choice of basis) as $m \times m$ diagonal matrices with nth roots of unity along the diagonal. In particular, these representations are determined completely by the value of $\varphi(g)$ and are classified up to isomorphism by unordered m-tuples of nth roots of unity.

1.4 Constructions from Linear Algebra

Definition 1.16. A **subrepresentation** of V is a G-invariant subspace $W \subseteq V$; that is a subspace $W \subseteq V$ with the property that $\varphi(g)(w) \in W$ for all $g \in G, w \in W$. Note that W effects a representation of G under the action $\varphi(g)\restriction_W$.

From elementary linear algebra, we know that given a subspace $W \subseteq V$, we can form the **quotient space** V/W consisting of cosets $v + W$ in V. If W is a subrepresentation of V, we would like to define an action of G on V/W by the natural choice of $g(v+W) = \varphi(g)(v) + W$. We must that this action is well defined. If we choose another $v' \in v + W$, then $v - v' \in W$ so that $\varphi(g)(v - v') \in W$ since W is G-invariant. Thus, the cosets $\varphi(g)(v) + W$ and $\varphi(g)(v') + W$ agree and this action is indeed well defined.

Definition 1.17. Let W be a G-subrepresentation of V. Then V/W forms a representation of G called the **quotient representation** of V under W, with the action $g(v+W) = \varphi(g)(v) + W$.

We recall also from linear algebra that given two vector spaces V_1 and V_2, we can form the **direct sum** $V_1 \oplus V_2$ consisting of ordered pairs (v_1, v_2) where $v_1 \in V_1, v_2 \in V_2$.

Definition 1.18. Let V_1 and V_2 be representations of G. Then $V_1 \oplus V_2$ forms a representation of G called the **direct sum representation**, with the action $g(v_1, v_2) = (g \cdot v_1, g \cdot v_2)$.

1.5 Complete Reducibility and Unitarity

Definition 1.19. A representation is called **irreducible** if it contains no proper invariant subspaces. It is called **completely reducible** if it decomposes into a direct sum of irreducible subrepresentations.

Example 1.20.
1. Any irreducible representation is completely reducible.
2. Any 1-dimensional representations has no proper subspaces, and is thus irreducible.

Theorem 1.21. *If $A_1, A_2, \ldots, A_r$ are linear operators on V and each A_i is diagonalizable, they are simultaneously diagonalizable if and only if they commute.*

Proof. See [1, Theorem 5.1]. □

Theorem 1.22. *Every complex representation of a finite abelian group is completely reducible, and every irreducible representation is 1-dimensional.*

Proof. Take an arbitrary element $g \in G$. Since G is finite, we can find an integer n such that $g^n = 1$ and $\varphi(g)^n = Id$. Hence the minimal polynomial of $\varphi(g)$ divides $x^n - 1$. Recall that $x^n - 1$ has n distinct roots over $\mathbb{C}$, which are generated by taking powers of $\xi = e^{\frac{2\pi i}{n}}$. This means that the minimal polynomial $\varphi(g)$ factors into linear factors only over $\mathbb{C}$ so that $\varphi(g)$ is diagonalizable. We conclude that each $\varphi(g)$ is (separately) diagonalizable since $g \in G$ was arbitrary.

Now, given any two elements $g_1, g_2 \in G$ we have

$$\begin{aligned}
\varphi(g_1)\varphi(g_2) &= \varphi(g_1 g_2) && \text{(since } \varphi \text{ is a homomorphism)} \\
&= \varphi(g_2 g_1) && \text{(since } G \text{ is abeilian)} \\
&= \varphi(g_2)\varphi(g_1) && \text{(since } \varphi \text{ is a homomorphism).}
\end{aligned}$$

Thus the matrices $\{\varphi(g)\}$ commute, so we can apply 1.21 to conclude that $\{\varphi(g)\}$ are simultaneously diagonalizable. This basis $\{e_1, ..., e_k\}$ yields the decomposition $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \ldots \oplus \mathbb{C}e_n$. □

We recall the following definition from linear algebra:

Definition 1.23. Let V be a complex vector space. A **Hermitian inner product** on V is a map $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ that satisfies the following properties for all $u, v, w \in V$ and $c \in \mathbb{C}$:

1. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$.
2. $\langle cu, v\rangle = c\langle u, v\rangle$.
3. $\langle u, v\rangle = \overline{\langle v, u\rangle}$.
4. $\langle v, v\rangle \geq 0$ with equality if and only if $v = 0$.

Definition 1.24. A representation φ of G on a complex vector space V is **unitary** if V has been equipped with a hermetian inner product $\langle\cdot,\cdot\rangle$ which is preserved by the action of G, that is,

$$\langle v, w\rangle = \langle \varphi(g)(v), \varphi(g)(w)\rangle \quad \forall v, w \in V, g \in G.$$

A representation is said to be **unitarisable** if it can be equipped with such a product (even without one being specified).

Theorem 1.25. *[Weyl's unitary trick] Finite-dimensional represenations of finite groups are unitarisable.*

Proof. Take any Hermetian inner product on V, say $\langle\cdot,\cdot\rangle'$. We define a new inner product on V by *averaging over G*:

$$\langle v,w\rangle := \frac{1}{|G|}\sum_{g\in G}\langle\varphi(g)v,\varphi(g)w\rangle'.$$

This new inner product satifies properties 1, 2, and 3 of Definition 1.23 by linearity. It remains to check positivity (4). Clearly $\langle v,v\rangle = 0$ when $v = 0$, since each term of the sum will equal zero. In the case where $v \neq 0$, observe that

$$\langle v,v\rangle = \frac{1}{|G|}\sum_{g\in G}\langle\varphi(g)v,\varphi(g)v\rangle' \geq 0$$

since each term of the sum is non-negative by the positivity of $\langle\cdot,\cdot\rangle'$. The only problem would occur if each term of this sum is equal to zero. But $\langle\varphi(e)v,\varphi(e)v\rangle' = \langle v,v\rangle' > 0$. Thus $\langle v,v\rangle > 0$.

Finally, we show that our new inner product is G-invariant. For any $h \in G$, we have

$$\begin{aligned}
\langle\varphi(h)v,\varphi(h)w\rangle &= \frac{1}{|G|}\sum_{g\in G}\langle\varphi(g)\varphi(h)v,\varphi(g)\varphi(h)w\rangle' \\
&= \frac{1}{|G|}\sum_{g\in G}\langle\varphi(gh)v,\varphi(gh)w\rangle' && \text{(since } \varphi \text{ is a homomorphism)} \\
&= \frac{1}{|G|}\sum_{k\in G}\langle\varphi(k)v,\varphi(k)w\rangle' && \text{(by a change of variables)} \\
&= \langle v,w\rangle.
\end{aligned}$$

□

Lemma 1.26. *Let V be a unitary representation of G and let $W \subseteq V$ be a G-invariant subspace. Then the orthogonal complement $W^\perp$ is also G-invariant.*

Proof. Choose arbitrary elements $v \in W^\perp$ and $g \in G$. We need to show that $gv \in W^\perp$. Now for any $w \in W$, we have $\langle v, w\rangle = 0$. Thus $\langle gv, gw\rangle = g\bar{g}\langle v, w\rangle = 0$ for any $w \in W$. Notice that $w' = gw \in W$ since W is G-invariant. This implies that $\langle gv, w'\rangle = 0$, i.e. $gv \in W^\perp$. □

Theorem 1.27. *A finite-dimensional unitary representation of a group is fully reducible into unitary irreducible subrepresentations.*

Proof. Let V be a finite dimensional unitary representation of G. We proceed by induction on the dimension of V. If $\dim(V) = 1$, then V is necessarily irreducible. Now, suppose the theorem holds for all W with $\dim(V) \leq n-1$ and suppose $\dim(V) = n$. If V is irreducible, we are done. Otherwise, there exists a proper G-invariant subspace $W(\neq 0, V)$. We can write $V = W \oplus W^\perp$ by Lemma 1.26. Applying the inductive hypothesis to W and $W^\perp$, we obtain a decomposition into irreducibles

$$V = (W_1 \oplus \ldots \oplus W_j) \oplus (W_{j+1} \oplus \ldots \oplus W_k).$$

□

Corollary 1.28. *Every complex representation of a finite group is completely reducible.*

Proof. Any such representation is unitarisable y by Theorem 1.25. We can then apply Theorem 1.27 to obtain full reduciblility. □

Bibliography

[1] Keith Conrad. *The Minimal Polynomial and Some Applications*. http://www.math.uconn.edu/~kconrad/blurbs/linmultialg/minpolyandappns.pdf. Online; accessed 12 December 2015. 2014.