Character Tables for Representations of Finite Groups

Jared Stewart

April 4, 2016

Table of contents

Motivation

Groups arise naturally as sets of symmetries of some object which are closed under composition and taking inverses. For example,

1. The **symmetric group** of degree n, S_n, is the group of all symmetries of the set $\{1, \ldots, n\}$.
2. The **dihedral group** of order $2n$, D_n, is the group of all symmetries of the regular n-gon in the plane.

In these two examples, S_n acts on the set $\{1, \ldots, n\}$ and D_n acts on the regular n-gon in a natural manner. One may wonder more generally: Given an abstract group G, which objects X does G act on? This is the basic question of representation theory, which attempts to classify all such X up to isomorphism.

Group Actions

Definition

A **group action** of a group G on a set X is a map $\rho\colon G \times X \to X$ (written as $g \cdot x$, for all $g \in G$ and $x \in X$) that satisfies the following two axoims:

$$1 \cdot x = x \qquad \forall x \in X \tag{1}$$

$$(gh) \cdot x = g \cdot (h \cdot x) \qquad \forall g, h \in G, x \in X \tag{2}$$

The Definition of a Representation

Definition

Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is an action of G on V that preserves the linear structure of V, i.e. an action of G on V such that

$$g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2 \qquad \forall g \in G, v_1, v_2 \in V \tag{3}$$

$$g \cdot (kv) = k(g \cdot v) \qquad \forall g \in G, v \in V, k \in F \tag{4}$$

Definition (Alternative definition)

Let G be a group, let F be a field, and let V be a vector space over F. A **linear representation** of G is any group homomorphism

$$\rho \colon G \to GL(V).$$

Proposition

The two definitions we have given of a linear representation are equivalent.

Proof.

($\rightarrow$) Suppose that we have a homomorphism $\rho\colon G \to GL(V)$. We can obtain a linear action of G on V by defining $g \cdot v = \rho(g)(v)$.

($\leftarrow$) Suppose that we have a linear action of G on V. We obtain a homomorphism $\rho\colon G \to GL(V)$ by defining $\rho(g)(v) = g \cdot v$.

$\square$

The Dimension of a Representation

Definition

Let $\rho\colon G \to GL(V)$ be a representation of G. The **dimension** of the representation is the dimension of the vector space V.

Examples of Representations

Example

Let V be an n-dimensional vector space. The map $\rho\colon G \to GL(V)$ defined by $\rho(g) = \mathsf{Id}_V$ for all $g \in G$ is a representation of G called the **trival representation** of dimension n.

Examples of Representations

Example

If G is a finite group that acts on a finite set X, and F is any field, then there is an associated **permutation representation** on the vector space V over F with basis $\{e_x : x \in X\}$. We let G act on the basis elements by the permutation $g \cdot e_x = e_{gx}$ for all $x \in X$ and $g \in G$. This representation has dimension $|X|$.

Examples of Representations

Example

A special case of a permutation representation is that when a finite group acts on itself by left multiplication. We take the vector space V_{reg} which has a basis given by the formal symbols $\{e_g | g \in G\}$, and let $h \in G$ act by

$$\rho_{\text{reg}}(h)(e_g) = e_{hg}.$$

This representation is called the **regular representation** of G, and has dimension $|G|$.

Examples of Representations

Example

For any symmetric group S_n, the **alternating representation** on $\mathbb{C}$ is given by the map

$$\begin{aligned} \rho\colon S_n &\to GL(\mathbb{C}) = \mathbb{C}^\times \\ \sigma &\mapsto \mathsf{sgn}(\sigma). \end{aligned}$$

More generally, for any group G with a subgroup H of index 2, we can define an **alternating representation** $\rho\colon G \to GL(\mathbb{C})$ by letting $\rho(g) = 1$ if $g \in H$ and $\rho(g) = -1$ if $g \notin H$. (We recover our original example by taking $G = S_n$ and $H = A_n$.)

G-linear maps

Definition

A **homomorphism** between two representations $\rho_1 \colon G \to GL(V)$ and $\rho_2 \colon G \to GL(W)$ is a linear map $\psi \colon V \to W$ that interwines with the action of G, i.e. such that

$$\psi \circ \rho_1(g) = \rho_2(g) \circ \psi \quad \forall g \in G.$$

In this case, we also refer to ψ as a **G-linear map**.

Definition

An **isomorphism** of representations is a G-linear map that is also invertible.

Representations as matrices

Example

Given any representation (ρ, V), where V is a vector space of dimension n over the field K, we can fix a basis for V to obtain an isomorphism of vector spaces $\psi\colon V \to K^n$. This yields a representation ϕ of G on K^n by defining

$$\phi(g) = \psi \circ \rho(g) \circ \psi^{-1}$$

for all $g \in G$. This representation is isomorphic to our original representation (ρ, V). In particular, we can always choose to view complex n-dimensional representations of G as representations on $\mathbb{C}^n$, where each $\phi(g)$ is given by an $n \times n$ matrix with entries in $\mathbb{C}$.

Representations as matrices

Example

Let $G = \{(1), (123), (132)\} \subset S_3$. Let $V = \mathbb{C}^3$. Then G acts on the standard basis by $g \cdot e_i = e_{gi}$. Thus, the permutation representation of G (with respect to the standard basis) is given by:

$$\rho((1)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\rho((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$\rho((132)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Example

Let $G = C_2 \times C_2 = \langle \sigma, \tau | \sigma^2 = \tau^2 = e, \sigma\tau = \tau\sigma \rangle$ be the Klein four-group. Let V be the vector space with basis $\{b_e, b_\sigma, b_\tau, b_{\sigma\tau}\}$. Left multiplication by σ gives a permutation

$$\begin{aligned} b_e &\mapsto b_\sigma \\ b_\sigma &\mapsto b_e \\ b_\tau &\mapsto b_{\sigma\tau} \\ b_{\sigma\tau} &\mapsto b_\tau. \end{aligned}$$

We can similarly compute $\rho_{\text{reg}}(\tau)$. Thus, in our basis, the regular representation $\rho_{\text{reg}} \colon G \to GL(V)$ is given by the matrices

$$\rho_{\text{reg}}(\sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \rho_{\text{reg}}(\tau) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Example (2-dim rep of D_4.)

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square.

Example (2-dim rep of D_4.)

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square. Consider a square in the plane with vertices at $(1, 1), (1, -1), (-1, -1)$, and $(-1, 1)$.

Example (2-dim rep of D_4.)

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square. Consider a square in the plane with vertices at $(1,1), (1,-1), (-1,-1)$, and $(-1,1)$. We let σ act on the square as a rotation by $\frac{\pi}{2}$, and let τ act by reflection over the x-axis. This naturally gives rise to a linear action of G on all of $\mathbb{C}^2$.

Example (2-dim rep of D_4.)

Let $G = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ be the symmetry group of the square. Consider a square in the plane with vertices at $(1,1), (1,-1), (-1,-1)$, and $(-1,1)$. We let σ act on the square as a rotation by $\frac{\pi}{2}$, and let τ act by reflection over the x-axis. This naturally gives rise to a linear action of G on all of $\mathbb{C}^2$. Under the standard basis, we get the matrices:

$$\rho(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\rho(\sigma) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \rho(\sigma\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\rho(\sigma^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \rho(\sigma^2\tau) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho(\sigma^3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \rho(\sigma^3\tau) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Subrepresentations

Definition

A **subrepresentation** of V is a G-invariant subspace $W \subseteq V$; that is, a subspace $W \subseteq V$ with the property that $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$. Note that W itself is a representation of G under the action $\rho(g) \restriction_W$.

Representations of C^2

Example

Let $G = C_2 = \langle \tau | \tau^2 = e \rangle$ be the cyclic group of order 2. The regular representation of G written in the standard basis is given by

$$\rho_{\mathsf{reg}}(\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $\rho_{\mathsf{reg}}(e) = \mathsf{Id}_2$. Let ρ_{sgn} be the alternating representation of G on $\mathbb{C}$, i.e.

$$\begin{aligned} \rho_{\mathsf{sgn}} : G &\to GL_1(\mathbb{C}) = \mathbb{C}^\times \\ \tau &\mapsto -1 \\ e &\mapsto 1. \end{aligned}$$

Representations of C^2

Example (Cont.)

Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be the linear map represented by the matrix $\begin{bmatrix} 1 & -1 \end{bmatrix}$. Then for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$, we have

$$\begin{aligned} f \circ \rho_{\mathsf{reg}}(\tau)(x) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \rho_{\mathsf{sgn}}(\tau) \circ f(x). \end{aligned}$$

Also note that $f \circ \rho_{\mathsf{reg}}(e) = \rho_{\mathsf{sgn}}(e) \circ f$. Thus f is a G-linear map from ρ_{reg} to ρ_{sgn} (i.e. a homomorphism of representations).

Representations of C^2

Example (Cont.)

Now let W be the subspace of $\mathbb{C}^2$ spanned by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Then

$$\rho_{\text{reg}}(\tau) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $\rho_{\text{reg}}(e) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so W is a G-invariant subpace, i.e. W is a subrepresentation of ρ_{reg}. Note that W is precisely equal to the kernel of the map f, and that W is isomorphic to the 1-dimensional trivial representation of G.

Example

We can generalize the G-invariant subspace from the previous example. Suppose we have a representation $\rho\colon G \to GL_n(\mathbb{C})$. If we can find a vector $x \in \mathbb{C}^n$ which is an eigenvector for every matrix $\rho(g), g \in G$, i.e. an $x \in \mathbb{C}^n$ such that

$$\rho(g)(x) = \lambda_g(x) \quad \forall g \in G$$

for some eigenvalues $\lambda_g \in \mathbb{C}$, then the span of x is a 1-dimensional G-invariant subspace of $\mathbb{C}^n$. It is isomorphic to the 1-dimensional representation

$$\begin{aligned} \rho_2\colon G &\to GL_1(\mathbb{C}) \\ g &\mapsto \lambda_g. \end{aligned}$$

Proposition

Let $f : V \to W$ be a homomorphism of representations of G. Then Ker(f) is a subrepresentation of V and Im(f) is a subrepresentation of W.

Proof.

- Let $x \in \text{Ker}(f)$. Then $0 = g0 = gf(x) = f(gx)$ for every $g \in G$. So $gx \in \text{Ker}(f)$ and $\text{Ker}(f)$ is G-invariant.
- Now let $w \in \text{Im}(f)$. There exists $v \in V$ such that $w = f(v)$, so $gw = gf(v) = f(gv)$ for every $g \in G$. Thus $gw \in \text{Im}(f)$, and $\text{Im}(f)$ is G-invariant.

$\square$

The direct sum of representations

Note

We know from linear algebra that given two vector spaces V and W, we can form the **direct sum** $V \oplus W$ consisting of ordered pairs (v, w) where $v \in V, w \in W$.

The direct sum of representations

Note

We know from linear algebra that given two vector spaces V and W, we can form the **direct sum** $V \oplus W$ consisting of ordered pairs (v, w) where $v \in V, w \in W$.

Definition

Let V and W be representations of G. Then $V \oplus W$ admits a natural representation of G, called the **direct sum representation** of V and W, which we define by

$$\rho_{V \oplus W} \colon G \to GL(V \oplus W)$$
$$\rho_{V \oplus W}(g) \colon (x, y) \mapsto (\rho_V(g)(x), \rho_W(g)(y)).$$

Irreducible representations and complete reducibility

Definition

A representation is said to be **irreducible** if it has no subrepresentations other than the trivial subrepresentations $0 \subset V$ and $V \subset V$. A representation is called **completely reducible** if it decomposes into a direct sum of irreducible representations. We sometimes write **irrep** as shorthand for irreducible representation.

Note

1. Any 1-dimensional representation V has no subspaces other than 0 and V itself, and is thus irreducible.
2. Any irreducible representation is, in particular, completely reducible.

Example (A 2-dimensional irrep)

Let $G = D_3 = \langle \sigma, \tau | \sigma^3 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$. (Note that $D_3 \cong S_3$). Consider the regular triangle centered at the origin with vertices

$$(1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2}).$$

We can let σ act as rotation by $\frac{2\pi}{3}$ and let τ act as reflection over the x-axis to obtain an action of G on $\mathbb{C}^2$ given (under the standard basis) by the matrices

$$\rho(\sigma) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\rho(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Example (A 2-dimensional irrep cont.)

Suppose ρ has a non-trivial subrepresentation W. We must have dim $W = 1$. Since W is invariant under the action of both $\rho(\sigma)$ and $\rho(\tau)$, there must be some mutual eigenvector for $\rho(\sigma)$ and $\rho(\tau)$ that spans W. The eigenvectors of $\rho(\sigma)$ are

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} \quad (\lambda_1 = e^{\frac{2\pi i}{3}}) \quad \text{and} \quad \begin{bmatrix} 1 \\ i \end{bmatrix} \quad (\lambda_2 = e^{-\frac{2\pi i}{3}}).$$

The eigenvectors of $\rho(\tau)$ are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\lambda_1 = 1) \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\lambda_2 = -1).$$

Thus we see that there is no such W, and our representation is irreducible.

Representations of finite abelian groups

Theorem

If $A_1, A_2, \ldots, A_r$ are linear operators on V and each A_i is diagonalizable, then $\{A_i\}$ are simultaneously diagonalizable if and only if they commute.

Representations of finite abelian groups

Theorem

Every complex representation of a finite abelian group is completely reducible into irreducible representations of dimension 1.

Proof.

Take an arbitrary element $g \in G$. Since G is finite, we can find an integer n such that $g^n = 1$ and $\rho(g)^n = Id$. The minimal polynomial of $\rho(g)$ divides $x^n - 1$, which has n distinct roots over $\mathbb{C}$, so it factors into linear factors only over $\mathbb{C}$, i.e. $\rho(g)$ is diagonalizable. We conclude that each $\rho(g)$ is (separately). Now, given any two elements $g_1, g_2 \in G$ we have $\rho(g_1)\rho(g_2) = \rho(g_2)\rho(g_1)$. Since the matrices $\{\rho(g)\}$ commute, $\{\rho(g)\}$ are simultaneously diagonalizable, say with basis $\{e_1, ..., e_k\}$. Then we have $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \ldots \oplus \mathbb{C}e_n$, with each subspace $\mathbb{C}e_i$ invariant under the action of G. $\square$

Definition

Let W be a subspace of V. A **linear projection** V onto W is a linear map $f\colon V \to W$ such that $f\restriction_W = \mathrm{Id}_W$. If W is a subrepresentation of V and the map f is G-invariant, then we say that f is a **G-linear projection**.

Lemma

Let $\rho\colon G \to GL(V)$ be a representation, and $W \subset V$ be a subrepresentation. Suppose we have a G-linear projection

$$f\colon V \to W.$$

Then Ker(f) is a complementary subrepresentation to W, i.e. Ker(f) is a G-invariant subspace of V such that

$$V = \mathit{Ker}(f) \oplus W$$

Maschke's Theorem

Theorem (Maschke's Theorem)

Let G be a finite group and let F be a field such that char$(F) \nmid |G|$. If V is any finite dimensional representation of G over F, and $W \subset V$ is a subrepresentation of V, then there exists a complementary subrepresentation $U \subset V$ to W, i.e. there is a G-invariant subspace $U \subset V$ such that

$$V = W \oplus U.$$

Maschke's Theorem

Proof.

It will suffice to find a G-linear projection from V onto W. Fix a basis $\{b_1, \ldots, b_m\}$ for W and extend it to a basis $\{b_1, \ldots, b_m, b_{m+1}, \ldots, b_n\}$ for V. Let $U = \langle b_{m+1}, \ldots, b_n \rangle$. Then U is certainly a complementary subspace to W, and we have a natural projection $f\colon W \oplus U \to W$ of V onto W with kernel U. There is no reason to think that f should be G-linear, but we can fix this by averaging over G. Define $\widetilde{f}\colon V \to V$ by

$$\widetilde{f}(x) = \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}))(x).$$

We claim that $\widetilde{f}$ is a G-linear projection from V onto W.

Maschke's Theorem

Proof.

First we check that $\mathrm{Im}(\tilde{f}) \subset W$. If $x \in V$ and $g \in G$, then

$$f(\rho(g^{-1})(x)) \in W$$

and so

$$\rho(g)(f(\rho(g^{-1})(x))) \in W$$

since W is G-invariant. Thus

$$\widetilde{f}(x) = \frac{1}{|G|}\sum_{g\in G}(\rho(g)\circ f\circ\rho(g^{-1}))(x) \in W.$$

Maschke's Theorem

Proof.

Next we check that $\widetilde{f}\restriction_W = \mathrm{Id}_W$. Let $y \in W$. For any $g \in G$, we know that $\rho(g^{-1})(y)$ is also in W, so $f(\rho(g^{-1})(y)) = \rho(g^{-1})(y)$. Then

$$\begin{aligned}
\widetilde{f}(y) &= \frac{1}{|G|}\sum_{g\in G}\rho(g)(f(\rho(g^{-1})(y))) \\
&= \frac{1}{|G|}\sum_{g\in G}\rho(g)(\rho(g^{-1})(y)) \\
&= \frac{1}{|G|}\sum_{g\in G}(y) = \frac{|G|y}{|G|} = y
\end{aligned}$$

so indeed $\widetilde{f}$ is a linear projection of V onto W.

Maschke's Theorem

Proof.

Finally, we check that $\widetilde{f}$ is G-linear. If $x \in V$ and $h \in G$, then

$$\begin{aligned}
(\widetilde{f} \circ \rho(h))(x) &= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1}) \circ \rho(h))(x) \\
&= \frac{1}{|G|} \sum_{g \in G} (\rho(g) \circ f \circ \rho(g^{-1} h))(x) \\
&= \frac{1}{|G|} \sum_{g \in G} (\rho(hg) \circ f \circ \rho(g^{-1}))(x) \quad (g \mapsto hg) \\
&= (\rho(h) \circ \widetilde{f})(x).
\end{aligned}$$

$\square$

Corollary

Let G be a finite group and let F be a field such that $\text{char}(F) \nmid |G|$. then any finite-dimensional representation of G over F is completely reducible.

Proof.

Let V be a representation of G over F of dimension n. If V is irreducible, then V is, in particular, completely reducible. If not, then V contains a proper subrepresentation $W \subset V$. From Maschke's Theorem, we know there exists a subrepresentation $U \subset V$ such that

$$V = W \oplus U. \tag{5}$$

Both W and U have dimension less than n, so by induction we know that W and U are completely reducible. We deduce that V is completely reducible. □

Example

Recall that for $G = C_2$, we found a 1-dim subrepresentation

$$W = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle \subset V_{\text{reg}} = \mathbb{C}^2.$$

We know a complementary subrepresentation to W exists by Machke's Theorem, so let's try to find one. Consider

$$U = \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle \subset V_{\text{reg}}.$$

Then

$$\rho_{\text{reg}}(\tau) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so U is G-invariant. We see that $V = W \oplus U$, since $W \cap U = \{0\}$ and dim $U +$ dim $W = 2 =$ dim V. (Note U is isomorphic to the alternating representation ρ_{sgn}.)

Definition

Let V and W be vector spaces. Recall that the set **Hom**$(\mathbf{V}, \mathbf{W})$ of linear maps from V to W itself form a vector space where we define the addition of vectors by

$$\begin{aligned}(f_1 + f_2)\colon V &\to W\\ x &\mapsto f_1(x) + f_2(x)\end{aligned}$$

for $f_1, f_2 \in \mathrm{Hom}(V, W)$ and scalar multiplication for $\lambda \in \mathbb{C}$ by

$$\begin{aligned}(\lambda f_1)\colon V &\to W\\ x &\mapsto \lambda f_1(x).\end{aligned}$$

Proposition

Suppose we have representations $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ of G. Then there is a natural representation of G on the vector space Hom(V, W) given for all $g \in G$ by

$$\begin{aligned}\rho_{Hom(V,W)}(g) \colon Hom(V,W) &\to Hom(V,W) \\ f &\mapsto \rho_W(g) \circ f \circ \rho_V(g^{-1}).\end{aligned}$$

Proof (sketch).

1. $\rho_{\mathrm{Hom}(V,W)}(g)(f) \in \mathrm{Hom}(V,W)$ since the composition of linear maps is linear.
2. For every $g \in G$, $\rho_{\mathrm{Hom}(V,W)}(g)$ is invertible.
3. The map $g \mapsto \rho_{\mathrm{Hom}(V,W)}(g)$ is a homomorphism.

$\square$

Definition

Let V and W be two representations of G. The set of G-linear maps from V to W forms a subspace of $\mathrm{Hom}(V, W)$, which we denote by $\mathbf{Hom_G(V, W)}$. In other words, $\mathrm{Hom}_G(V, W)$ is the vector space consisting of all *homomorphisms of representations* between V and W.

Definition

Let $\rho\colon G \to GL(V)$ be a representation. We define the **invariant subrepresentation** $V^G \subset V$ to be the set

$$\{v \in V \mid \rho(g)(v) = v, \quad \forall g \in G\}.$$

Proposition

Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be representations of G. Then the subrepresentation

$$Hom_G(V, W) \subset Hom(V, W)$$

is precisely the invariant subrepresentation $Hom(V, W)^G$ of $Hom(V, W)$.

Proof.

Let $f \in \text{Hom}(V, W)$. Then $f \in \text{Hom}(V, W)^G$ iff we have

$$\begin{aligned} f &= \rho_{\text{Hom}(V,W)}(g)(f) \quad \forall g \in G \\ \iff f &= \rho_W(g) \circ f \circ \rho_V(g^{-1}) \quad \forall g \in G \\ \iff f \circ \rho_V(g) &= \rho_W(g) \circ f \quad \forall g \in G \end{aligned}$$

which is exactly the condition that f is G-linear. □

Theorem (Schur's Lemma over $\mathbb{C}$.)

If V is an irreducible representation of G over $\mathbb{C}$, then evey linear operator $\phi \colon V \to V$ commuting with G is a scalar.

Proof.

Let $\phi \colon V \to V$ be a linear operator commuting with G, and let λ be an eigenvalue of ϕ. Observe that the eigenspace E_λ is G-invariant: If $v \in E_\lambda$, then $\phi(v) = \lambda v$. This implies that $\phi(gv) = g\phi(v) = g(\lambda v) = \lambda(gv)$, i.e. $gv \in E_\lambda$. Since g was arbitrary, E_λ is indeed G-invariant. Now $E_\lambda \neq 0$, so since V is irreducible, $E_\lambda = V$. Thus $\phi = \lambda \mathrm{Id}$. □

Corollary

Suppose V and W are irreducible. The space $\mathrm{Hom}_G(V, W)$ is 1-dimensional if the representations are isomorphic, and in this case any non-zero map is an isomorphism. Otherwise, $\mathrm{Hom}_G(V, W) = \{0\}$.

Proof.

Suppose $\mathrm{Hom}_G(V, W) \neq \{0\}$ and let $\phi \in \mathrm{Hom}_G(V, W)$. We have seen $\ker(\phi)$ and $\mathrm{im}(\phi)$ are both G-invariant. Irreducibility yields $\ker(\phi) = 0$ or V and $\mathrm{im}(\phi) = 0$ or W as the only possibilities. Since $\phi \neq 0$, then $\ker(\phi) = 0$, $\mathrm{im}(\phi) = W$, and ϕ is an isomorphism. Let ψ be another nonzero interwining operator from V to W. Then $\phi^{-1} \circ \psi \in \mathrm{Hom}_G(V, V)$. We can apply Schur's Lemma over $\mathbb{C}$ to see that $\phi^{-1} \circ \psi = \lambda \mathrm{Id}$, hence $\psi = \lambda\phi$. So ϕ spans $\mathrm{Hom}_G(V, W)$. □

Proposition

Let V and W be irreducible representations of G. Then

$$\textit{dim } \textit{Hom}_G(V,W) = \begin{cases} 1 & \textit{if } V \textit{ and } W \textit{ are isomorphic} \\ 0 & \textit{if } V \textit{ and } W \textit{ are not isomorphic} \end{cases}$$

Proof.

Suppose V and W are not isomorphic. Then the Corollary to Schur's Lemma states that the only G-linear map from V to W is the zero map, hence $\text{Hom}_G(V,W) = \{0\}$.
On the other hand, suppose that $f\colon V \to W$ is an isomorphism. Then for any $h \in \text{Hom}_G(V,W)$, we have $f^{-1} \circ h \in \text{Hom}_G(V,V)$. By Schur's Lemma, $f^{-1} \circ h = \lambda \text{Id}_V$ for some $\lambda \in \mathbb{C}$, i.e. $h = \lambda f$. Thus f spans $\text{Hom}_G(V,W)$. □

Proposition

Let $\rho\colon G \to GL(V)$ be a representation, let

$$V = U_1 \oplus \ldots \oplus U_s$$

be a decomposition of V into irreps, and let W be any irrep of G. Then the number of irreps in the set $\{U_1, \ldots, U_s\}$ which are isomorphic to W equals the dimension of $\mathrm{Hom}_G(V, W)$.

Lemma

If U, V, and W are representations of G, then there are natural isomorphisms

- $\mathrm{Hom}_G(V, U \oplus W) = \mathrm{Hom}_G(V, U) \oplus \mathrm{Hom}_G(V, W)$
- $\mathrm{Hom}_G(U \oplus W, V) = \mathrm{Hom}_G(U, V) \oplus \mathrm{Hom}_G(W, V)$

Proof.

We use the previous proposition to see that t he number of irreps in the set $\{U_1, \ldots, U_s\}$ which are isomorphic to W is equal to

$$\sum_{i=1}^{s} \text{dim Hom}_G(U_i, W).$$

Then

$$\text{Hom}_G(V, W) = \bigoplus_{i=1}^{s} \text{Hom}_G(U_i, W).$$

by our lemma, so taking the dimension of both sides yields

$$\text{dim Hom}_G(V, W) = \sum_{i=1}^{s} \text{dim Hom}_G(U_i, W).$$

□

Theorem (Uniqueness of decomposition into irreducibles.)

Let $\rho\colon G \to GL(V)$ be a representation, and let

$$V = U_1 \oplus \ldots \oplus U_s$$
$$V = \widetilde{U_1} \oplus \ldots \oplus \widetilde{U_r}$$

be two decompositions of V into irreducible subrepresentations. Then $s = r$, and (after reordering if necessary) U_i and $\widetilde{U}_i$ are isomorphic for every $i \in \{1, \ldots, s\}$.

Proof.

For any irrep W of G, the number of irreps in either decomposition that are isomorphic to W is equal to $\dim \mathrm{Hom}_G(V, W)$. So the two decompositions contain the same number of factors isomorphic to W for any irrep W of G. □

The Dual Space

Definition

Let V be a vector space. Recall that we define the **dual vector space** to be

$$V^* = \mathrm{Hom}(V, \mathbb{C}).$$

If we fix a basis $\{b_1, \ldots, b_n\}$ for V, then the **dual basis**$\{f_1, \ldots, f_n\}$ for V^* is defined by

$$f_i(b_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The Dual Representation

Definition

Let $\rho_V : G \to GL(V)$ be a representation of G. Then we have seen that V^* carries a representation of G defined by

$$\rho_{\mathsf{Hom}(V,\mathbb{C})}(g)(f) = f \circ \rho_V(g^{-1})$$

We call this the **dual representation** to ρ_V, and denote it by ρ_V^*.

Proposition

If we fix a basis for V, then $\rho_{V^*}(g)$ is given by the matrix

$$(\rho_V(g^{-1}))^T$$

with respect to the dual basis.

Definition

Suppose V and W are two vector spaces over a field K. Then we define a new vector space called the **tensor product** of V and W, denoted by $V \otimes_K W$. This space is the quotient of the free vector space on $V \times W$ (with basis given by formal symbols $v \otimes w, v \in V, w \in W$), by the subspace D spanned by all elements of the form

$$\begin{aligned}(v_1 + v_2, w) - (v_1, w) - (v_2, w)\\(v, w_1 + w_2) - (v, w_1) - (v, w_2)\\(k \cdot v, w) - (v, k \cdot w)\end{aligned}$$

for $v, v_1, v_2 \in V, w, w_1, w_2 \in W$, and $k \in K$. When the ground field K is clear it can be omitted from the notation. The elements of $V \otimes W$ are called **tensors**, and the coset $v \otimes w$ of (v, w) in $V \otimes W$ is called a **simple tensor**.

Definition

We can define a representation of G on $V \otimes W$ called the **tensor product representation**. We define

$$\rho_{V\otimes W}(g) \colon V \otimes W \to V \otimes W$$

to be the linear map given by

$$\rho_{V\otimes W}(g) \colon a_i \otimes b_j \mapsto \rho_V(g)(a_i) \otimes \rho_W(g)(b_j).$$

Proposition

Let V and W be representations of G. Then $V \otimes W$ is isomorphic to Hom(V^*, W).

The definition of a Character

Definition

The **character** of a representation $\rho\colon G \to GL(V)$ is the function

$$\chi_V : G \to \mathbb{C}$$

defined by

$$\chi_V(g) = \mathrm{Tr}(\rho(g)).$$

Note

The character is of a representation is not a homomorphism in general, since $\mathrm{Tr}(MN) \neq \mathrm{Tr}(M)\mathrm{Tr}(N)$ in general.

Basic properties of Characters

Proposition

Let V be a representation of G.

- χ_V *is conjugation invariant:* $\chi_V(hgh^{-1}) = \chi_V(g) \quad \forall g, h \in G$.
- $\chi_V(e) =$ *dim* V.
- $\chi_V(g^{-1}) = \overline{\chi_V(g)} \quad \forall g \in G$.
- $\chi_{V^*}(g) = \overline{\chi_V(g)} \quad \forall g \in G$.

Proposition

Let V and W be representations of G.

- $\chi_{V \oplus W} = \chi_V + \chi_W$.
- $\chi_{V \otimes W} = \chi_V \cdot \chi_W$.

Proposition

Isomorphic representations have the same character.

Proof.

Isomorphic representations can be described by the same set of matrices with the right choice of bases. Thus each $\rho(g)$ has the same trace. $\square$

Definition

Let $\mathbb{C}^G$ denote the vector space of all functions from G to $\mathbb{C}$. A basis for $\mathbb{C}^G$ is given by the set of functions

$$\{\delta_g | g \in G\}$$

defined by

$$\delta_g : h \mapsto \begin{cases} 1 & \text{if } h = g \\ 0 & \text{if } h \neq g. \end{cases}$$

Definition

Let $\varphi, \psi \in \mathbb{C}^G$. We define a **hermetian inner product** on $\mathbb{C}^G$ by

$$\langle \varphi | \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}.$$

Inner product of Characters

Theorem

Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be representations of G, and let χ_V, χ_W be their characters. Then

$$\langle \chi_W | \chi_V \rangle = \textit{dim } \textit{Hom}_G(V, W).$$

Corollary

Let $\chi_1, \ldots, \chi_r$ be characters of pairwise non-isomorphic irreducible representations of G. Then

$$\langle \chi_i | \chi_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Proof.

Let χ_i and χ_j be the characters of the irreducible representations U_i, U_j. Then

$$\langle \chi_i | \chi_j \rangle = \dim \mathrm{Hom}_G(U_i, U_j) = \begin{cases} 1 & \text{if } U_i, U_j \text{ are isomorphic} \\ 0 & \text{if } U_i, U_j \text{ are not isomorphic.} \end{cases}$$

□

Corollary

Let χ be any character of G. Then χ is irreducible if and only if

$$\langle\chi|\chi\rangle = 1$$

Proof.

Write χ as a linear combination of irreducible characters

$$\chi = m_1\chi_1 + \ldots + m_k\chi_k$$

where each m_i is a non-negative integer. Then

$$\begin{aligned}\langle\chi|\chi\rangle &= \sum_{i,j\in[1,k]} m_i m_j \langle\chi_i|\chi_j\rangle \\ &= m_1^2 + \ldots + m_k^2.\end{aligned}$$

So $\langle\chi|\chi\rangle = 1$ if and only if exactly one of the $m_i = 1$ and the rest are 0. $\square$

Proof that $\langle \chi_W | \chi_V \rangle = \dim \operatorname{Hom}_G(V, W)$

Lemma

Let $\rho\colon G \to GL(V)$ be any representation. Consider the linear map

$$\begin{aligned} \Psi\colon V &\to V \\ x &\mapsto \frac{1}{|G|} \sum_{g \in G} \rho(g)(x). \end{aligned}$$

Then Ψ is a projection from V onto the invariant subspace V^G.

Lemma

Let V be a vector space with subspace $U \subset V$, and let $\pi\colon V \to V$ be a projection onto U. Then

$$\mathit{Tr}(\pi) = \mathit{dim}\ U.$$

Proof that $\langle \chi_W | \chi_V \rangle = \dim \mathrm{Hom}_G(V, W)$

Proof.

We have seen that

$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G \subset \mathrm{Hom}(V, W).$$

By the first Lemma, we have a projection

$$\begin{aligned} \Psi \colon \mathrm{Hom}(V, W) &\to \mathrm{Hom}(V, W)^G \\ f &\mapsto \frac{1}{|G|} \sum_{g \in G} \rho_{\mathrm{Hom}(V,W)}(g)(f). \end{aligned}$$

We claim that

$$\mathrm{Tr}(\Psi) = \langle \chi_W | \chi_V \rangle.$$

Once this claim is established, then the theorem will follow from our second Lemma, since $\mathrm{Tr}(\Psi) = \dim \mathrm{Hom}_G(V, W)$.

Proof that $\langle \chi_W | \chi_V \rangle = \dim \mathrm{Hom}_G(V, W)$

Proof.

We proceed by calculating $\mathrm{Tr}(\Psi)$. Fix bases $\{a_1, \ldots, a_n\}$ for V and $\{b_1, \ldots, b_m\}$ for W. Then $\mathrm{Hom}(V, W)$ has an associated basis

$$\{f_{ji} | 1 \leq i \leq n, 1 \leq j \leq m\}$$

where

$$f_{ji}(a_i) = \begin{cases} b_j & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

We may calculate $\mathrm{Tr}(\Psi)$ as follows: For each i, j, compute the expression of $\Psi(f_{ji})$ in this basis, and take the coefficient of the basis element f_{ji}. This is a diagonal entry in the matrix for Ψ. Summing these values over all i and j will gives us $\mathrm{Tr}(\Psi)$.

Proof that $\langle \chi_W | \chi_V \rangle = \dim \mathrm{Hom}_G(V, W)$

Proof.

Let $\widetilde{\rho_V}, \widetilde{\rho_W}$ be the matrix representations obtained by writing ρ_V and ρ_W in the given bases. We know that

$$\mathrm{Hom}(V, W) = V^* \otimes W$$

so if we write $\rho_{\mathrm{Hom}(V,W)}$ in the basis $\{f_{ji}\}$ then we get the tensor product of $\widetilde{\rho_{V^*}}$ and $\widetilde{\rho_W}$. Thus

$$\begin{aligned} \rho_{\mathrm{Hom}(V,W)}(g)(f_{ji}) &= \rho_W(g) \circ f_{ji} \circ \rho_V(g^{-1}) \\ &= \sum_{\substack{k \in [1,n] \\ t \in [1,m]}} \widetilde{\rho_V}(g^{-1})_{ik} \widetilde{\rho_W}(g)_{tj} f_{kt}. \end{aligned}$$

Show another step?

Proof that $\langle \chi_W | \chi_V \rangle = \dim \mathrm{Hom}_G(V, W)$

Proof.

$\square$