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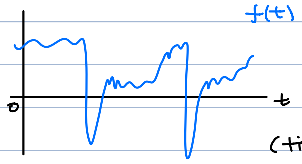
Applied Harmonic Analysis

Background:

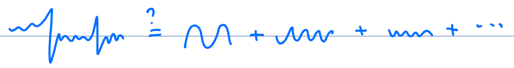


signal { voice, sound
images (2D, 3D)
radio waves (FM)

Fourier analysis:

Fourier series (periodic f)

$f(t) \overset{?}{=} a_0/2 + \sum_{n=1}^{\infty} \underset{\text{amplitude}}{a_n} \cos(w_n t) + b_n \sin(\underset{\text{frequency}}{w_n} t)$



Fourier transform (nonperiodic f)

$\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-ikt}\, dt$ where $i = \sqrt{-1}$

Function $f \in V \leftarrow$ some vector space (say $L^2(\mathbb{R})$ or $\ell^2(\mathbb{R})$)

↓

Decompose f (into say sines/cosines or wavelets)

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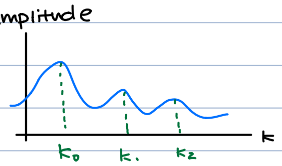
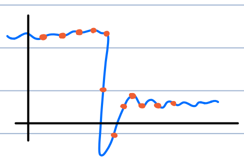
In frequency space, can manipulate f: compression, denoising, apply other filters ..



ex: low-pass filter

$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(w_n t) + b_n \sin(w_n t)$

In practice, need to discretize!
- DFT (Discrete Fourier Transform)
- DWT (Discrete Wavelet Transform)



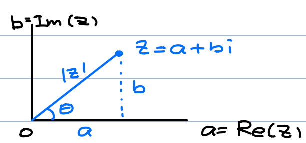
need to sample points of f.

Number Systems.

countable {
$\mathbb{N} = \{0, 1, 2, 3 \ldots\}$ naturals
$\mathbb{Z} = \{\ldots -1, 0, 1, 2 \ldots\}$ integers
$\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z},\ q \neq 0\}$ rationals.

uncountable {
$\mathbb{R} = \{$ real numbers $\}$ ex: $\pi, -\frac{1}{2}, 0, e, \ldots$
$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ complex.

Complex numbers



$|z| = z\bar{z} = \sqrt{a^2 + b^2}$ (modulus)

conjugate $\bar{z} = a - bi$

polar form: $z = re^{i\theta}$ where $r = |z|$, $\theta = \tan^{-1}(\frac{b}{a})$

$e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's Formula)

Complex Fourier series:

$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$ where n = frequency

Roots of complex $z \in \mathbb{C}$

if $w^n = z$, then $w = n^{th}$ root of z

$w = \rho e^{i\alpha}$, $z = re^{i\theta}$

$w^n = \rho^n e^{in\alpha}$

$\Rightarrow w = \sqrt[n]{r} \exp\left(i \frac{\theta + 2k\pi}{n}\right) \quad (k = 0, 1, 2 \dots n-1)$

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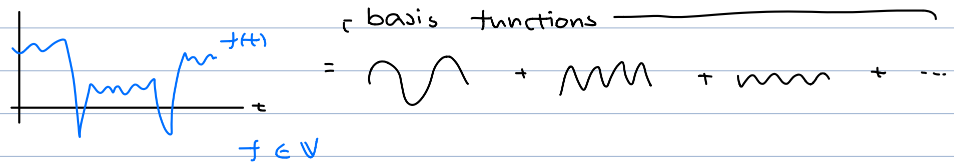
Vector (linear) spaces.



Examples of vector spaces : $\mathbb{R}^n$ finite dimensional

$\mathbb{C}^n$ 2n dimensional

$\mathbb{P}_n = \{n^{th}$ degree polynomials$\}$ = poly

infinite dimensional $\begin{cases} C[0,1] = \{u : [0,1] \to \mathbb{R} \mid u \text{ is continuous}\} \\ L^2(a,b) = \{u : (a,b) \to \mathbb{R} \mid \int_a^b |u(t)|^2 \, dt < \infty\} \end{cases}$

Vector Space Axioms : let $u, v, w \in \mathbb{V}$ = some set and let $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$

linearity   vectors   fields

1) $u + (v + w) = (u + v) + w$
2) $u + v = v + u$
3) $\exists \, 0 \in \mathbb{V}$ s.t $u + 0 = u$
4) $\exists (-u) \in \mathbb{V}$ s.t $u + (-u) = 0$
5) $\alpha(\beta u) = (\alpha\beta) u$
6) $(\alpha + \beta) u = \alpha u + \beta u$
7) $\alpha(u + v) = \alpha u + \alpha v$

Defn : If $\{v_1, \dots, v_n\} \subset \mathbb{V}$ and $\{a_1, \dots, a_n\} \subset \mathbb{R}$ (or $\mathbb{C}$)

then $\sum_{i=1}^n a_i v_i = a_1 v_1 + \cdots + a_n v_n$ is a linear combination of the $\{v_i\}_{i=1}^n$

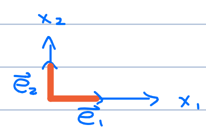
note : n has to be finite

ex : $\sum_{k=1}^{10} a_k \cos(kt)$ is a lin. comb. of $\{\cos(kt)\}_{k=1}^{10}$

Defn : The span of $\{v_1 \dots v_n\} \subset \mathbb{V}$ is the set

$\text{span}\{v_1 \dots v_n\} = \{\sum_{i=1}^n a_i v_i \mid v_i \in \mathbb{V}, a_i \in \mathbb{K}\}$ $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

ex : $\mathbb{R}^2$, $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$\text{span}\{\vec{e}_1\}$ = entire x_1 axis $= \{a\vec{e}_1 \mid a \in \mathbb{R}\}$

$\text{span}\{\vec{e}_1, \vec{e}_2\} = \mathbb{R}^2 = \{a_1\vec{e}_1 + a_2\vec{e}_2 \mid a_1, a_2 \in \mathbb{R}\} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

Subspaces :