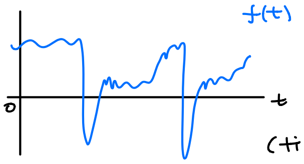
11/25/21

Applied Harmonic Analysis

Background:



signal { voice, sound
images (2D, 3D)
radio waves (FM)

Fourier analysis:

Fourier series (periodic f)

$$f(t) \stackrel{?}{=} a_0/2 + \sum_{n=1}^{\infty} a_n \cos(w_n t) + b_n \sin(w_n t)$$

amplitude — frequency

$\stackrel{?}{=}$ + + + ...

Fourier transform (nonperiodic f)

$$\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-ikt}\, dt \quad \text{where } i = \sqrt{-1}$$

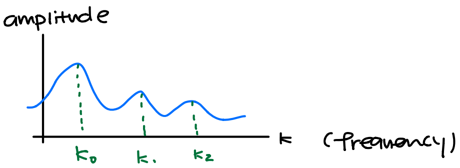
Function $f \in V$ ← some vector space (say $L^2(\mathbb{R})$ or $\ell^2(\mathbb{R})$)

↓

Decompose f (into say sines/cosines or wavelets)

↓

In frequency space, can manipulate f: compression, denoising, apply other filters ..

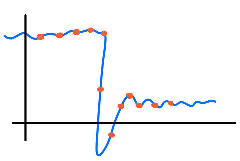


ex: low-pass filter

$$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(w_n t) + b_n \sin(w_n t)$$

In practice, need to discretize!

- DFT (Discrete Fourier Transform)
- DWT (Discrete Wavelet Transform)



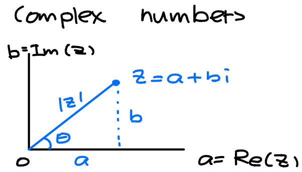
need to sample points of f.

Number Systems.

countable {
$\mathbb{N} = \{0, 1, 2, 3 \ldots\}$ naturals
$\mathbb{Z} = \{\ldots -1, 0, 1, 2 \ldots\}$ integers
$\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z},\ q \neq 0\}$ rationals.

uncountable {
$\mathbb{R} = \{\text{real numbers}\}$ ex: $\pi, -\frac{1}{2}, 0, e, \ldots$
$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ complex.

Complex numbers



$|z| = z\bar{z} = \sqrt{a^2 + b^2}$ (modulus)

conjugate $\bar{z} = a - bi$

polar form: $z = re^{i\theta}$ where $r = |z|$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's Formula)

Complex Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int} \quad \text{where } n = \text{frequency}$$

Roots of complex $z \in \mathbb{C}$

if $w^n = z$, then $w = n^{th}$ root of z

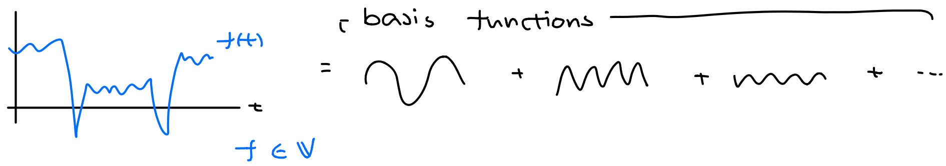
$w = \rho e^{i\alpha}$, $z = re^{i\theta}$

$w^n = \rho^n e^{in\alpha}$

$$\Rightarrow w = \sqrt[n]{r}\,\exp\left(i\,\frac{\theta + 2k\pi}{n}\right) \quad (k = 0, 1, 2 \ldots n-1)$$

1/27/21

Vector (linear) spaces.



Examples of vector spaces : $\mathbb{R}^n$ finite dimensional

$\mathbb{C}^n$ 2n dimensional

$\mathbb{P}_n = \{ n^{th}$ degree polynomials $\}$ = poly

infinite dimensional $\begin{cases} C[0,1] = \{ u : [0,1] \to \mathbb{R} \mid u \text{ is continuous} \} \\ L^2(a,b) = \{ u : (a,b) \to \mathbb{R} \mid \int_a^b |u(t)|^2\, dt < \infty \} \end{cases}$

Vector Space Axioms : let $u, v, w \in \mathbb{V}$ = some set and let $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$

linearity   vectors   fields

1) $u + (v + w) = (u + v) + w$
2) $u + v = v + u$
3) $\exists\, 0 \in \mathbb{V}$ s.t $u + 0 = u$
4) $\exists (-u) \in \mathbb{V}$ s.t $u + (-u) = 0$
5) $\alpha(\beta u) = (\alpha\beta)u$
6) $(\alpha + \beta)u = \alpha u + \beta u$
7) $\alpha(u + v) = \alpha u + \alpha v$

Defn : If $\{v_1, \ldots, v_n\} \subset \mathbb{V}$ and $\{a_1, \ldots, a_n\} \subset \mathbb{R}$ (or $\mathbb{C}$)

then $\sum_{i=1}^n a_i v_i = a_1 v_1 + \cdots + a_n v_n$ is a linear combination of the $\{v_i\}_{i=1}^n$

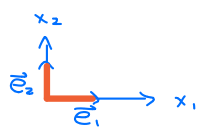
note : n has to be finite

ex : $\sum_{k=1}^{10} a_k \cos(kt)$ is a lin. comb. of $\{\cos(kt)\}_{k=1}^{10}$

Defn : The span of $\{v_1 \ldots v_n\} \subset \mathbb{V}$ is the set

$\text{span}\{v_1 \ldots v_n\} = \{ \sum_{i=1}^n a_i v_i \mid v_i \in \mathbb{V}, a_i \in \mathbb{K} \}$ $\quad \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

ex : $\mathbb{R}^2$, $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$\text{span}\{\vec{e}_1\}$ = entire x_1 axis $= \{ a\vec{e}_1 \mid a \in \mathbb{R} \}$

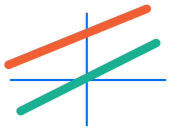
$\text{span}\{\vec{e}_1, \vec{e}_2\} = \mathbb{R}^2 = \{ a_1\vec{e}_1 + a_2\vec{e}_2 \mid a_1, a_2 \in \mathbb{R} \} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

Subspaces :

ex : $\text{span}\{v_i\}_{i=1}^n$ is a subspace of $\mathbb{V}$.

A subspaces of $\mathbb{V}$ is a set $\mathbb{X} \subset \mathbb{V}$ s.t $u + cv \in \mathbb{V}$ $\forall c \in \mathbb{R}$ or $\mathbb{C}$, $u, v \in \mathbb{X}$

- we say $\mathbb{X} \subset \mathbb{V}$ is closed under scalar multiplication and addition.
- A subspace is itself a vector space.

ex: 

← not a subspace – doesn't go through 0.

← Subspace of $\mathbb{R}^2$

ex: $\mathbb{P}_2$ is a subspace of $C[0,1]$

Defn: A set $\{v_1, \dots, v_n\} \subset \mathbb{V}$ is linearly independent if $\sum_{i=1}^{n} a_i v_i = 0$, then $a_i = 0$ for $i = 1, \dots, n$. note: n is finite.

ex: $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}4\\0\end{pmatrix}$ are linearly independent.
since $\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3\\0\end{pmatrix} + 0\begin{pmatrix}4\\0\end{pmatrix}$

Defn: A basis B of $\mathbb{V}$ is a linearly independent set of vectors that span $\mathbb{V}$.

ex: $B = \{1, t, t^2\}$ form a basis of $\mathbb{P}_2$.

Defn: The dimension of $\mathbb{V}$ is the number of vectors in any basis.

ex: $\dim(\mathbb{R}^2) = 2$, $\dim(\mathbb{P}_2) = 3$, $\dim(C[0,1]) = \infty$
note: need to be careful in $C[0,1]$
since $\sum_{i=1}^{\infty} a_i v_i$ not allowed in vector space axiom.

1/29/21

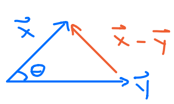
Basis in ∞-dim spaces.

Hamel Basis (ignore) – uses finite # of vectors.
Schauder Basis:
B is a schauder basis if
i) Any finite set of vectors of B is linearly independent.
ii) $\forall \vec{v} \in \mathbb{V}$, $\vec{v} = \sum_{i=1}^{\infty} c_i \vec{v}_i$ where $B = \{\vec{v}_i\}_{i=1}^{\infty}$ (completeness)
interpreted as $\lim_{n\to\infty} \| \vec{v} - \sum_{i=1}^{\infty} c_i \vec{v}_i \|_{\mathbb{V}} = 0$
iii) The coefficients c_i are unique.

example of Schauder basis: Fourier series in $L^2(0, 2\pi) = \mathbb{V}$
$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$
↳ form a Schauder basis $\{1, e^{it}, e^{-it}, e^{2it}, e^{-2it} \dots\}$

Inner-Product Spaces: $\mathbb{K}^n = \mathbb{R}^n$ or $\mathbb{C}^n$

Define inner-product $\langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y}_i$ $\quad x, y \in \mathbb{K}^n$ $\quad$ note $z = x + iy$, $\bar{z} = x - iy$
i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$
ii) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ $\quad a, b \in \mathbb{K}$
iii) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$
note: $\langle x, ay \rangle = \overline{\langle ay, x \rangle} = \overline{a\langle y, x \rangle} = \bar{a}\,\overline{\langle y, x \rangle} = \bar{a}\langle x, y \rangle$

ex: $\mathbb{R}^2$  $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cos\theta$

pf: law of cosines $\quad \|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$
$\langle \vec{x} - \vec{y}, \vec{x} - \vec{y} \rangle = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$
$\cancel{\|\vec{x}\|^2} - \langle \vec{x}, \vec{y} \rangle - \langle \vec{y}, -\vec{x} \rangle + \cancel{\|\vec{y}\|^2} = \cancel{\|\vec{x}\|^2} + \cancel{\|\vec{y}\|^2} - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$

Normed spaces

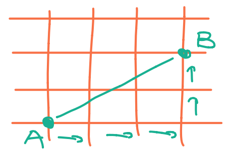
$(\mathbb{V}, \|\cdot\|)$ is a normed space where $\|\cdot\| : \mathbb{V} \to \mathbb{R}$

and satisfies
i) $\|u+v\| \leq \|u\| + \|v\| \quad \forall\, u,v \in \mathbb{V}$. (triangle inequality)
ii) $\|au\| = |a|\,\|u\| \quad a \in \mathbb{K}$
iii) $\|u\| \geq 0$ and $\|u\| = 0$ iff $u = 0$

All inner-products generate a norm: $\|x\| = \sqrt{\langle x, x\rangle}$

Other norms on $\mathbb{R}^n$: let $p \geq 1$, and set $\|\vec{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$

$\|\vec{x}\|_1 = \sum_{i=1}^n |x_i|$ (1-norm, taxicab norm)



Dist btw. A, B in $\|\vec{x}\|_1$ is 5

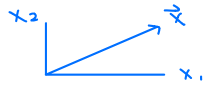
$\|\vec{x}\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$ (2-norm)

generated by $\langle x, y\rangle = \sum_{i=1}^n x_i \bar{y}_i$

$\|\vec{x}\|_\infty = \max\limits_{i=1,\dots,n} |x_i|$ (∞-norm, max norm)

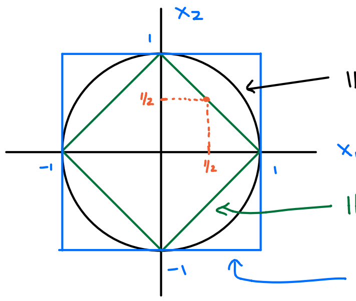
note $\|\vec{x}\|_\infty = \lim\limits_{p\to\infty} \|\vec{x}\|_p$

Ex:



$\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2} = \left(\sum_{i=1}^2 x_i^2\right)^{1/2}$ (Euclidian norm)

Unit Spheres in various norms $\mathbb{R}^2$



$\|\vec{x}\|_2 = 1 \implies \|x\|_2^2 = 1$

$x_1^2 + x_2^2 = 1$

$\|\vec{x}\|_1 = \sum_{i=1}^2 |x_i| = |x_1| + |x_2| = 1$

$\|\vec{x}\|_\infty = \max\limits_{i=1,2} |x_i| = 1$