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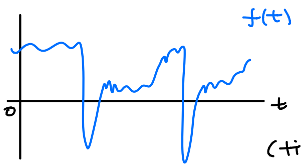
Applied Harmonic Analysis

Background :



(time space)

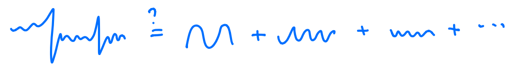
signal { voice, sound
images (2D, 3D)
radio waves (FM)

Fourier analysis :

Fourier series (periodic f)

$f(t) \stackrel{?}{=} a_0/2 + \sum_{n=1}^{\infty} a_n \cos(w_n t) + b_n \sin(w_n t)$

amplitude　　frequency



Fourier transform (non periodic f)

$\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-ikt} \, dt$　where $i = \sqrt{-1}$

Function $f \in V$ ← some vector space (say $L^2(\mathbb{R})$ or $\ell^2(\mathbb{R})$)

↓

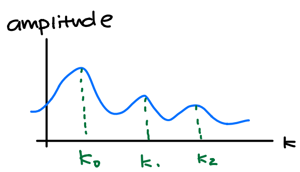
Decompose f (into say sines/cosines or wavelets)

↓

In frequency space, can manipulate f : compression, denoising, apply other filters ..

ex: low-pass filter

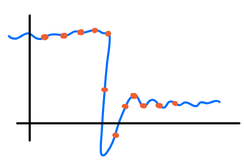
$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(w_n t) + b_n \sin(w_n t)$



(frequency)

In practice, need to discretize !

- DFT (Discrete Fourier Transform)
- DWT (Discrete Wavelet Transform)



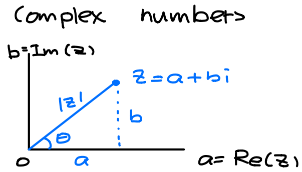
need to sample points of f .

Number Systems.

countable {
$\mathbb{N} = \{0, 1, 2, 3 \ldots\}$ naturals
$\mathbb{Z} = \{\ldots -1, 0, 1, 2 \ldots\}$ integers
$\mathbb{Q} = \{ p/q \mid p, q \in \mathbb{Z},\ q \neq 0 \}$ rationals.

uncountable {
$\mathbb{R} = \{$ real numbers $\}$　ex: $\pi, -\frac{1}{2}, 0, e, \ldots$
$\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}$ complex.

Complex numbers



$|z| = z\bar{z} = \sqrt{a^2 + b^2}$ (modulus)

conjugate $\bar{z} = a - bi$

polar form : $z = re^{i\theta}$ where $r = |z|$, $\theta = \tan^{-1}(\frac{b}{a})$

$e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's Formula)

complex Fourier series :

$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$ where $n =$ frequency

Roots of complex $z \in \mathbb{C}$

if $w^n = z$, then w = n^{th} root of z

$w = \rho e^{i\alpha}$, $z = re^{i\theta}$

$w^n = \rho^n e^{in\alpha}$

$\Rightarrow w = \sqrt[n]{r}\, \exp\left(i\, \frac{\theta + 2k\pi}{n}\right) \quad (k = 0, 1, 2, \dots n-1)$

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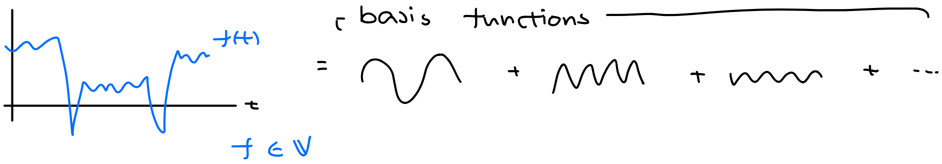
Vector (linear) spaces.



Examples of vector spaces : $\mathbb{R}^n$ finite dimensional

$\mathbb{C}^n$ 2n dimensional

$\mathbb{P}_n = \{n^{th}$ degree polynomials$\}$ = poly

infinite dimensional $\begin{cases} C[0,1] = \{u : [0,1] \to \mathbb{R} \mid u \text{ is continuous}\} \\ L^2(a,b) = \{u : (a,b) \to \mathbb{R} \mid \int_a^b |u(t)|^2\, dt < \infty\} \end{cases}$

Vector Space Axioms : let $u, v, w \in \mathbb{V}$ = some set and let $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$

linearity — vectors — fields

1) $u + (v + w) = (u + v) + w$

2) $u + v = v + u$

3) $\exists\, 0 \in \mathbb{V}$ s.t $u + 0 = u$

4) $\exists (-u) \in \mathbb{V}$ s.t $u + (-u) = 0$

5) $\alpha(\beta u) = (\alpha\beta) u$

6) $(\alpha + \beta) u = \alpha u + \beta u$

7) $\alpha(u + v) = \alpha u + \alpha v$

Defn : If $\{v_1, \dots, v_n\} \subset \mathbb{V}$ and $\{a_1, \dots, a_n\} \subset \mathbb{R}$ (or $\mathbb{C}$)

then $\sum_{i=1}^n a_i v_i = a_1 v_1 + \dots + a_n v_n$ is a linear combination of the $\{v_i\}_{i=1}^n$

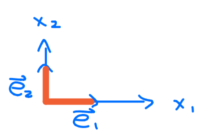
note : n has to be finite

ex: $\sum_{k=1}^{10} a_k \cos(kt)$ is a lin. comb. of $\{\cos(kt)\}_{k=1}^{10}$

Defn : The span of $\{v_1, \dots, v_n\} \subset \mathbb{V}$ is the set

$\text{span}\{v_1, \dots, v_n\} = \{\sum_{i=1}^n a_i v_i \mid v_i \in \mathbb{V},\ a_i \in \mathbb{K}\}$ $\quad \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

ex : $\mathbb{R}^2$, $\vec{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$



$\text{span}\{\vec{e}_1\}$ = entire x_1 axis $= \{a\vec{e}_1 \mid a \in \mathbb{R}\}$

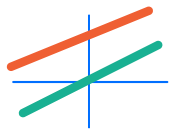
$\text{span}\{\vec{e}_1, \vec{e}_2\} = \mathbb{R}^2 = \{a_1\vec{e}_1 + a_2\vec{e}_2 \mid a_1, a_2 \in \mathbb{R}\} = \begin{pmatrix}a_1\\a_2\end{pmatrix}$

Subspaces :

ex : $\text{span}\{v_i\}_{i=1}^n$ is a subspace of $\mathbb{V}$.

A subspaces of $\mathbb{V}$ is a set $\mathbb{X} \subset \mathbb{V}$ s.t $u + cv \in \mathbb{V}$ $\forall c \in \mathbb{R}$ or $\mathbb{C}$, $u, v \in \mathbb{X}$

- we say $\mathbb{X} \subset \mathbb{V}$ is closed under scalar multiplication and addition.
- A subspace is itself a vector space.

ex: 

ex: $\mathbb{P}_2$ is a subspace of $C[0,1]$

Defn: A set $\{v_1, \dots, v_n\} \subset \mathbb{V}$ is linearly independent if $\sum_{i=1}^{n} a_i v_i = 0$, then $a_i = 0$ for $i = 1, \dots, n$. note: n is finite.

ex: $\binom{1}{0}, \binom{2}{0}, \binom{4}{0}$ are linearly independent.
since $\binom{1}{0} = \frac{1}{2}\binom{2}{0} + 0\binom{4}{0}$

Defn: A basis B of $\mathbb{V}$ is a linearly independent set of vectors that span $\mathbb{V}$.

ex: $B = \{1, t, t^2\}$ form a basis of $\mathbb{P}_2$.

Defn: The dimension of $\mathbb{V}$ is the number of vectors in any basis.

ex: $\dim(\mathbb{R}^2) = 2$, $\dim(\mathbb{P}_2) = 3$, $\dim(C[0,1]) = \infty$
note: need to be careful in $C[0,1]$
since $\sum_{i=1}^{\infty} a_i v_i$ not allowed in vector space axiom.

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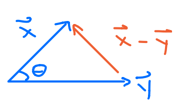
Basis in ∞-dim spaces.

Hamel Basis (ignore) - uses finite # of vectors.
Schauder Basis:
B is a schauder basis if
i) Any finite set of vectors of B is linearly independent.
ii) $\forall\, \vec{v} \in \mathbb{V}$, $\vec{v} = \sum_{i=1}^{\infty} c_i \vec{v}_i$ where $B = \{\vec{v}_i\}_{i=1}^{\infty}$ (completeness)
interpreted as $\lim_{n \to \infty} \| \vec{v} - \sum_{i=1}^{n} c_i \vec{v}_i \|_{\mathbb{V}} = 0$
iii) The coefficients c_i are unique.

example of Schauder basis: Fourier series in $L^2(0, 2\pi) = \mathbb{V}$
$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$
↳ form a Schauder basis $\{1, e^{it}, e^{-it}, e^{2it}, e^{-2it} \dots\}$

Inner-Product Spaces: $\mathbb{K}^n = \mathbb{R}^n$ or $\mathbb{C}^n$

Define inner-product $\langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y}_i$ $\quad x, y \in \mathbb{K}^n$ $\quad$ note $z = x + iy$, $\bar{z} = x - iy$
i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$
ii) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ $\quad a, b \in \mathbb{K}$
iii) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$
note: $\langle x, ay \rangle = \overline{\langle ay, x \rangle} = \overline{a\langle y, x \rangle} = \bar{a}\,\overline{\langle y, x \rangle} = \bar{a}\langle x, y \rangle$

ex: $\mathbb{R}^2$ 

$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cos\theta$

pf: law of cosines $\|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$
$\langle \vec{x} - \vec{y}, \vec{x} - \vec{y} \rangle = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$
$\cancel{\|\vec{x}\|^2} - \langle \vec{x}, \vec{y} \rangle - \langle \vec{y}, -\vec{x} \rangle + \cancel{\|\vec{y}\|^2} = \cancel{\|\vec{x}\|^2} + \cancel{\|\vec{y}\|^2} - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$

Normed spaces $(V, \|\cdot\|)$ is a normed space where $\|\cdot\| : V \to \mathbb{R}$

and satisfies

i) $\|u+v\| \le \|u\| + \|v\| \quad \forall u, v \in V$. (triangle inequality)

ii) $\|au\| = |a|\,\|u\| \quad a \in \mathbb{K}$

iii) $\|u\| \ge 0$ and $\|u\| = 0$ iff $u = 0$

All inner-products generate a norm: $\|x\| = \sqrt{\langle x, x\rangle}$

Other norms on $\mathbb{R}^n$: let $p \ge 1$, and set $\|\vec{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$

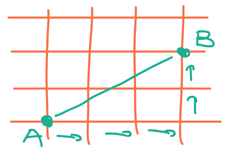
$\|\vec{x}\|_1 = \sum_{i=1}^n |x_i|$ (1-norm, taxicab norm)

$\|\vec{x}\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$ (2-norm)

generated by $\langle x, y\rangle = \sum_{i=1}^n x_i \bar{y}_i$

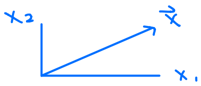
$\|\vec{x}\|_\infty = \max_{i=1,\dots,n} |x_i|$ (∞-norm, max norm)

note $\|\vec{x}\|_\infty = \lim_{p\to\infty} \|\vec{x}\|_p$



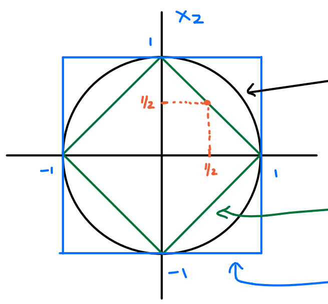
Dist btw. A, B in $\|\vec{x}\|_1$ is 5

Ex:



$\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2} = \left(\sum_{i=1}^2 x_i^2\right)^{1/2}$ (Euclidian norm)

Unit Spheres in various norms $\mathbb{R}^2$



$\|\vec{x}\|_2 = 1 \implies \|x\|_2^2 = 1$

$x_1^2 + x_2^2 = 1$

x_1

$\|\vec{x}\|_1 = \sum_{i=1}^2 |x_i| = |x_1| + |x_2| = 1$

$\|\vec{x}\|_\infty = \max_{i=1,2} |x_i| = 1$

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Equivalent Norms

we say any two norms $\|x\|_\alpha$, $\|x\|_\beta$ are equivalent

if $\exists A, B > 0$ s.t $A\|x\|_\alpha \le \|x\|_\beta \le B\|x\|_\alpha$ (norm are comparable)

Theorem: In finite dimensional spaces, ALL norms are equivalent.

Cauchy-Schwarz Inequality

For any $x, y \in V$ (V inner-product space)

we have $|\langle x, y\rangle| \le \|x\|\|y\|$ recall $\|x\| = \sqrt{\langle x, x\rangle}$

pf: If $x = 0$ or $y = 0$, done.

so, suppose $x, y \ne 0$. suppose $V = \mathbb{R}$

$0 \le \underset{\text{constant}}{\|x - ty\|^2} = \langle x - ty, x - ty\rangle = \underbrace{\|x\|^2}_{c} - \underbrace{2t\langle x, y\rangle}_{-2tb} + \underbrace{t^2\|y\|^2}_{t^2 a}$

$= c - 2tb + at^2$ (quadratic in t)

$t = \frac{2b \pm \sqrt{(-2b)^2 - 4ac}}{2a}$ want $(-2b)^2 - 4ac \le 0$

$b^2 \le ac$ with $b^2 = |\langle x, y\rangle|^2$, $a = \|y\|^2$, $c = \|x\|^2$ $\implies |\langle x, y\rangle| \le \|x\|\|y\|$

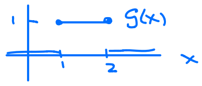
The vector space $L^2[a,b]$

$$L^2[a,b] := \{ f : [a,b] \to \mathbb{C} \mid \int_a^b |f(x)|^2\, dx < \infty \}$$

this is an inner-product space with $\langle f, g \rangle = \langle f, g \rangle_{L^2[a,b]} := \int_a^b f(x)\overline{g(x)}\, dx$

an norm $\|f\|_{L^2[a,b]} = \|f\|_{L^2} = \left(\int_a^b |f(x)|^2 dx \right)^{1/2}$

ex: $g \in L^2(\mathbb{R})$



$$\|g\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |g(x)|^2 dx \right)^{1/2} = \left(\int_1^2 1^2\, dx \right)^{1/2} = 1$$

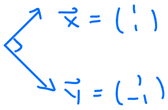
Cauchy-Schwarz in L^2: $|\langle f, g \rangle| \le \|f\| \|g\|$

$$\left| \int_a^b f(x)\overline{g(x)}\, dx \right| \le \left(\int_a^b |f|^2 dx \right)^{1/2} \left(\int_a^b |g|^2 dx \right)^{1/2}$$

Orthogonality

let $u, v \in V$, we say u, v are orthogonal if $\langle u, v \rangle = 0$

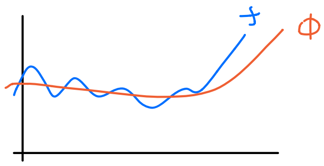
ex:



$\langle \vec{x}, \vec{y} \rangle = \langle \binom{1}{1}, \binom{1}{-1} \rangle = 1 - 1 = 0$

note $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|_2 \|\vec{y}\|_2 \cos\theta = 0$ $\quad \theta = \frac{\pi}{2}$

plus if $\|u\|, \|v\| = 1$, then u, v are orthonormal.



- $\|f - \phi\|_2$ is the average distance btw f, ϕ.
- $\|f - \phi\|_\infty$ is a distance that tries to minimize the maximum dist.

note $L^3[a,b] = \{ f : [a,b] \to \mathbb{R} \mid \int_a^b |f(x)|^3 dx < \infty \}$

$\|f\|_{L^3} = \left(\int_a^b |f(x)|^3 dx \right)^{1/3}$

while $\|\vec{x}\|_3 = \left(\sum_{i=1}^n |x_i|^3 \right)^{1/3}$

Orthonormal Bases

Suppose $B = \{e_1, e_2, \dots, e_n\} \subset V$ is an othonormal bass of V.

Have $\langle e_i, e_j \rangle = 0$ unless $i = j$ in which case $\langle e_i, e_j \rangle = \|e_i\|^2 = 1$

Kronecker Delta: $\langle e_i, e_j \rangle = \delta_{ij} = \delta(i-j) = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases}$

and $\forall u \in V$, $u = \sum_{i=1}^n c_i e_i$

use orthogonality: $\langle u, e_j \rangle = \langle \sum_{i=1}^n c_i e_i, e_j \rangle = \sum_{i=1}^n c_i \underbrace{\langle e_i, e_j \rangle}_{= \delta_{ij}} = c_j$

so $u = \sum_{i=1}^n \langle u, e_i \rangle e_i$

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Recall:

Orthonormal basis $\{e_k\}_{k=1}^n \subset V$.

$u \in V$, $u = \sum_{k=1}^n \langle u, e_k \rangle e_k$

$\langle u, u \rangle = \langle \sum_{k=1}^n \langle u, e_k \rangle e_k, u \rangle$

$\|u\|^2 = \sum_{k=1}^n \langle u, e_k \rangle \underbrace{\langle e_k, u \rangle}_{= \overline{\langle u, e_k \rangle}} = \sum_{k=1}^n |\langle u, e_k \rangle|^2 = \|u\|^2$

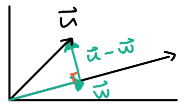
Each $\langle u, e_k \rangle e_k$ is an orthogonal projection of u onto e_k

$P_{e_k} u = \mathrm{Proj}_{e_k} u = \langle u, e_k \rangle e_k$ $\qquad$ - $\{e_k\}_{k=1}^n$ is orthonormal here

If $M = \operatorname{span}\{e_1, \dots, e_m\}$ then $P_M u = \operatorname{Proj}_M u = \sum_{k=1}^{m} \langle u, e_k \rangle e_k$

set $m \in \mathbb{Z}^+ = \mathbb{N}$ (index)

In $\mathbb{R}^2$



$\vec{w} = \operatorname{proj}_{\vec{v}} \vec{u}$ have $\vec{w} = \lambda \vec{v}$ for some $\lambda \in \mathbb{R}$

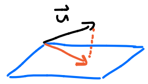
$0 = \langle \vec{u} - \vec{w}, \vec{v} \rangle = \langle \vec{u} - \lambda\vec{v}, \vec{v} \rangle = \langle u, \vec{v} \rangle - \underbrace{\langle \lambda \vec{v}, \vec{v} \rangle}_{\lambda \|\vec{v}\|^2}$

$\lambda = \langle \vec{u}, \vec{v} \rangle / \|\vec{v}\|^2$

$\vec{w} = \lambda \vec{v} = \dfrac{\langle \vec{u}, \vec{v} \rangle \vec{v}}{\|\vec{v}\|^2}$ when $\|\vec{v}\| = 1$, $\vec{w} = \langle \vec{u}, \vec{v} \rangle \vec{v}$

If $\{e_k\}_{k=1}^{n}$ is an orthogonal basis then $u = \sum_{k=1}^{n} \dfrac{\langle u, e_k \rangle}{\|e_k\|^2} e_k$

P_m is a projection if $P_m(P_m) = P_m^2 = P_m$



let's check for $P_M u = \sum_{k=1}^{m} \langle u, e_k \rangle e_k$

$P_M(P_M u) = \sum_{k=1}^{m} \langle P_m u, e_k \rangle e_k$

$= \sum_{k=1}^{m} \underbrace{\langle \sum_{j=1}^{m} \langle u, e_j \rangle e_j, e_k \rangle}_{} e_k$

$= \sum_{j=1}^{m} \langle u, e_j \rangle \underbrace{\langle e_j, e_k \rangle}_{= \delta_{jk}} = \langle u, e_k \rangle$

$= \sum_{k=1}^{m} \langle u, e_k \rangle e_k$

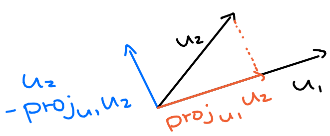
$= P_M u$

Consider $\{1, t\} \subset L^2[0,1] = \mathbb{V}$

$\langle 1, t \rangle = \int_0^1 1 \cdot t \, dt = 1/2 \neq 0 \Rightarrow 1, t$ not orthogonal

How to orthogonalize ?

$\{u_1, u_2, \dots, u_n\}$ non-orthogonal want to turn into $\{v_1, v_2 \dots v_n\}$ orthogonal.



Now $\{\overbrace{u_1}^{=v_1}, \overbrace{u_2 - \operatorname{proj}_{u_1} u_2}^{= v_2}\}$ is an orthogonal set spanning the same set as $\{u_1, u_2\}$

Gram - Schmidt Process

To turn $\{u_i\}_{i=1}^{n}$ non-orthogonal set into $\{v_i\}_{i=1}^{n}$ orthogonal.

$v_1 = u_1$

$v_2 = u_2 - \operatorname{proj}_{v_1} u_2$

$v_3 = u_3 - \operatorname{proj}_{\operatorname{span}(v_1, v_2)} u_3 = u_3 - \operatorname{proj}_{v_1} u_3 - \operatorname{proj}_{v_2} u_3$

$\vdots$

$v_n = u_n - \operatorname{proj}_{\operatorname{span}(v_1 \dots v_n)} u_n$

we have $\operatorname{span}\{u_1, \dots, u_n\} = \operatorname{span}\{v_1 \dots v_n\}$, $\{v_1 \dots v_n\}$ is orthogonal,

we can normalize by $\left\{ \dfrac{v_1}{\|v_1\|}, \dfrac{v_2}{\|v_2\|}, \dots, \dfrac{v_n}{\|v_n\|} \right\}$

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Linear Operators (map, function)

$T: \mathbb{V} \to \mathbb{W}$ is a linear map if $\forall c \in \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\forall u, v \in \mathbb{V}$.

i) $T(u+v) = T(u) + T(v)$

ii) $T(cu) = cT(u)$

Ex: $T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(\vec{x}) = A\vec{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $A =$ linear operator (Matrix)

Ex: $T: L^2(0,1) \to \mathbb{R}$

$T(u) = \int_0^1 u(t) e^t \, dt$

$T(au+bv) = \int_0^1 (au+bv)(t) e^t \, dt = a\,T(u) + b\,T(v)$

Ex: $T: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $x \in \ell^2(\mathbb{N})$, $x = \{x_1, x_2 \ldots\}$

$T(x) = \{\frac{1}{2}x_1, \frac{1}{2^2}x_2, \frac{1}{2^3}x_3, \ldots, \frac{1}{2^n}x_n, \ldots\}$

$T(ax+by) = \{\frac{1}{2}(ax_1+by_1), \frac{1}{2^2}(ax_2+by_2) \ldots\}$

$= a\{\frac{1}{2}x_1, \frac{1}{2^2}x_2, \ldots\} + b\{\frac{1}{2}y_1, \frac{1}{2^2}y_2, \ldots\}$

$= a\,T(x) + b\,T(y)$

In finite dimensions, any linear map $T: V \to W$ can be represented as matrix multiplication. where V, W are vector spaces.

Let $\{v_1 \ldots v_n\}$ be a basis of V and let $\{w_1 \ldots w_m\}$ be a basis of W.

Note $T(v_j) \in W \quad \forall\, j = 1, \ldots, n$

$\Rightarrow T(v_j) = \sum_{i=1}^m a_{ij} w_i$

let $v = \sum_{j=1}^n x_j v_j \in V$

Then $T(v) = T\left(\sum_{j=1}^n x_j v_j\right) = \sum_{j=1}^n x_j T(v_j) = \sum_{j=1}^n x_j \left(\sum_{i=1}^m a_{ij} w_i\right) = \sum_{i=1}^m \overbrace{\sum_{j=1}^n a_{ij} x_j}^{= c_i} w_i = \sum_{i=1}^m c_i w_i$

Here $c_i = \sum_{j=1}^n a_{ij} x_j$ is precisely matrix-vector multiplication : $\vec{c} = A\vec{x}$

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Think of $T(\vec{v}) = A\vec{v}$

The column of A store the coefficients of $T(v_j)$ $j = 1 \ldots n$

where $\{v_j\}_{j=1}^n$ is a basis of V.

Ex: Find the matrix representation of $T: V \to W$ where $V = \mathbb{R}^2$, $W = \text{span}\{\cos t, \sin t\}$. and $T(\vec{v}) = (v_1 + 2v_2)\cos t$, $v_2 \sin t$

basis for V let's use $\{\vec{e}_1, \vec{e}_2\} = \{\binom{1}{0}, \binom{0}{1}\}$

and for W let's use $\{w_1, w_2\} = \{\cos t, \sin t\}$

Have $T(\vec{e}_1) = T\binom{1}{0} = 1 \cdot \cos t + 0 \cdot \sin t$

$T(\vec{e}_2) = T\binom{0}{1} = 2 \cdot \cos t + 1 \cdot \sin t$ $\Rightarrow A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \vec{v} = x_1\binom{1}{0} + x_2\binom{0}{1} \in V = \mathbb{R}^2$, $\vec{c} = \binom{c_1}{c_2}$, $\vec{c} = A\vec{x}$.

basis of $\mathbb{R}^2$

So $T(\vec{v}) = c_1 w_1 + c_2 w_2 = c_1 \cos t + c_2 \sin t$

And if used basis $\{\binom{1}{0}, \binom{1}{1}\}$ for $V = \mathbb{R}^2$.

$T\binom{1}{0} = 1 \cdot \cos t + 0 \cdot \sin t$

$T\binom{1}{1} = 3 \cdot \cos t + 1 \cdot \sin t$ $\Rightarrow A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

Ex: if basis $\{\binom{1}{0}, \binom{0}{1}\}$ of $V = \mathbb{R}^2$, basis $\{w_1, w_2\} = \{\cos t + \sin t, \sin t\}$ of W.

then $A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ $\vec{v} = x_1\binom{1}{0} + x_2\binom{0}{1}$, $T(\vec{v}) = c_1(\cos t + \sin t) + c_2 \sin t$

2/8/21

Adjoint Operators

$T: V \to W$ linear operator (map)

inner-product spaces with $\langle \cdot , \cdot \rangle_V$ and $\langle \cdot , \cdot \rangle_W$

Can we write $\langle Tv, w \rangle_W$ as $\langle v, T^* w \rangle_V$ for some new operator T^* the adjoint of T.

Tv or $T(v)$ denotes action of T on v

i.e. $T^*: W \to V$ s.t $\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V$

Goal: Try to write $\langle Tv, w \rangle_W = \langle v, z \rangle_V$ for some $z \in V$ and define $T^* w = z$.

Theorem: If V, W are finite-dimensional and $T: V \to W$ is linear, then $T^*: W \to V$ exists and is unique

If $T = T^*$ we call T self-adjoint, $\langle Tv, w \rangle = \langle v, T^* w \rangle = \langle v, Tw \rangle$

Ex: The Fourier transform $F[f] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{ix3} dx$ is self-adjoint with the $L^2(\mathbb{R})$ inner-product $\langle f, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x) g(x) dx$

so $\langle Ff, g \rangle_{L^2} = \langle f, Fg \rangle_{L^2}$

Ex: $T = I$ $\quad I: V \to V$ (identity map) $\quad Iv = v$

$\langle Iv, w \rangle = \langle v, Iw \rangle$

so $I^* w = Iw \implies I^* = I$ by definition

Ex: Let $a(t)$ be bounded. $\langle f, g \rangle_{L^2} = \int_a^b f(x) \overline{g(x)} dx$

Define $T: L^2(a,b) \to L^2(a,b)$ by $(Tf)(t) = a(t) f(t)$

Find T^*

Have $\langle Tf, g \rangle = \langle af, g \rangle = \int_a^b a(t) f(t) \overline{g(t)} dt = \int_a^b f(t) \cdot \underline{a(t) \overline{g(t)}} dt$

$= \int_a^b f(t) \overline{\overline{a(t)} g(t)} dt$ $\qquad = \overline{\overline{a(t)} g(t)}$

$= \langle f, \bar{a} g \rangle$ $\qquad$ so $(T^* g)(t) = \overline{a(t)} g(t)$

If $a(t)$ real-valued, then $T^* = T$ and T self-adjoint.

Integration by Parts:

$$\int_a^b f(t) g'(t) dt = f(t) g(t) \Big|_{t=a}^{t=b} - \int_a^b f'(t) g(t) dt$$

Ex: $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = A\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Fact: $A^* = \bar{A}^T$ so $A^* = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\qquad A^*: \mathbb{R}^2 \to \mathbb{R}^3$

$$\left\langle A \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle_2 = \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle_2 = \left\langle \begin{pmatrix} x+y \\ x+2z \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle_2 = (x+y)a + (x+2z)b$$

$$= x(a+b) + ya + z(2b) = \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a+b \\ a \\ 2b \end{pmatrix} \right\rangle_3$$

we define $A^* \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a \\ 2b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$ $\qquad A: \mathbb{R}^3 \to \mathbb{R}^2$, $A^* = \mathbb{R}^2 \to \mathbb{R}^3$

$\langle A\vec{x}, \vec{y} \rangle_2 = \langle \vec{x}, A^* \vec{y} \rangle_3$

2/10/21

Intro to Frames

V with orthonormal basis $\{v_1, v_2, v_3, \dots\}$ (ONB)

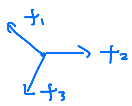
- can be restrictive in practice
- easy to compute c_i s.t $x = \sum c_i v_i$, $c_i = \langle x_i, v_i \rangle$

$\{f_1, f_2, f_3 \dots\}$ is a frame of V if the $\{f_i\}_{i \in I}$ span V and satisfy some other conditions.

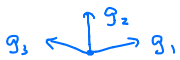
- frame can be linearly dependent.

ex: 

a basis of $\mathbb{R}^2$ (orthonormal)

ex: 

a frame for $\mathbb{R}^2$ (contains redundancies)

ex: 

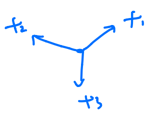
another frame

ex: Reconstruction , orthonormal basis $\{e_1, e_2\}$

person A , $\{1, 2\}$, $x = 1 \cdot e_1 + 2 \cdot e_2$.

$\downarrow$ "erasure" "information loss" , lose all info. in the e_2 dire.

person B , $\{1, ?\}$, $x = 1 \cdot e_1 + ?$



person A , $\{1, 2, 3\}$, $x = 1 \cdot f_1 + 2 \cdot f_2 + 3 \cdot f_3$

$\downarrow$ retains more info. than with ONB and sometimes can get "perfect reconstruction"

person B , $\{1, 2, ?\}$, $x = 1 \cdot f_1 + 2 \cdot f_2 + ?$

Primer on Hilbert Spaces

A Hilbert space $\mathbb{H}$ is a complete inner-product space.

- also normed $\|x\|_{\mathbb{H}} = \sqrt{\langle x, x \rangle_H}$

Defn: $\{x_n\}_{n=1}^{\infty} \subset \mathbb{H}$ is a Cauchy-sequence if

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ s.t $\forall n, m \geq N$, $\| x_n - x_m \|_H < \varepsilon$

Defn: $\mathbb{H}$ is complete if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point $x \in \mathbb{H}$.

i.e. $\lim_{n\to\infty} x_n = x \in \mathbb{H}$, $\|x_n - x\|_H \to 0$ as $n \to \infty$

Hilbert spaces : $\mathbb{R}, \mathbb{R}^n, \mathbb{C}^n, \ell^2(\mathbb{Z}), L^2(a,b), L^2(\mathbb{R})$

any finite dimensional inner-product space i.e. P_n

Non-Hilbert spaces : $S = (0,1) \subseteq \mathbb{R}$, $x_n = 1/n$ for $n \geq 2$ converges to $0 \notin S$

Defn: H is separable if it contains a countable basis.

- All examples above are separable.

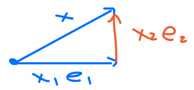
For ONB, of a Hilbert space H, $x = \sum_{i \in I} \langle x, e_i \rangle e_i \quad \forall x \in H$

Dotting with x we get

$$\|x\|^2 = \langle x, \textstyle\sum_{i\in I} \langle x, e_i\rangle e_i \rangle = \textstyle\sum_{i\in I} \overline{\langle x, e_i\rangle} \langle x, e_i\rangle = \textstyle\sum_{i\in I} |\langle x, e_i\rangle|^2$$

└ parseval equality

ex:



$\mathbb{R}^2$-case: $\|x\|^2 = x_1^2 + x_2^2 = |\langle x, e_1\rangle|^2 + |\langle x, e_2\rangle|^2 = \sum_{i\in I}^{2} |\langle x, e_i\rangle|^2$

pythegorean theorem

Frames generzlize Parseval equality:

Defn: $\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$ if I is countable and if $\exists\, 0 < A \le B < \infty$

s.t $\forall x \in \mathbb{H}$ $\quad A\|x\|^2 \le \sum_{i\in I} |\langle x, f_i\rangle|^2 \le B\|x\|^2$

frame bounds.

- largest A and smallest B that work are called optimal frame bounds.
- For $A = B = 1$, equality holds and we call $\{f_i\}_{i\in I}$ a Parseval frame
  i.e. an ONB is parseval
- $\{f_i\}_{i\in I}$ is a tight frame if $A = B$

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