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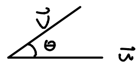
10.1

$\mathbb{R}^n = \{ \vec{u} = (u_1, \dots u_n \mid u_i \in \mathbb{R} \}$

$\vec{u} + \vec{v} = (u_1 + v_1, \dots, u_n + v_n)$

$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle} = \sqrt{u_1^2 + \dots + u_n^2}$



Define : $\vec{u} \perp \vec{v}$ if $\langle \vec{u}, \vec{v} \rangle = 0$

prop: $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{u}\|^2$ iff $\vec{u} \perp \vec{v}$

pf: $\|\vec{u} + \vec{v}\|^2 = \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle = \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle + \langle \vec{v} + \vec{v} \rangle$

$= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\langle \vec{u}, \vec{v} \rangle$

$= \|\vec{u}\|^2 + \|\vec{v}\|^2$ iff $\langle \vec{u}, \vec{v} \rangle = 0$

properties of $\langle \ , \ \rangle$: $\forall \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, $\forall \alpha, \beta \in \mathbb{R}$

1) $\langle \alpha \vec{u} + \beta \vec{v}, \vec{w} \rangle = \alpha \langle \vec{u}, \vec{w} \rangle + \beta \langle \vec{v}, \vec{w} \rangle$

2) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

3) $\langle \vec{u}, \vec{u} \rangle \geq 0$, $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = 0$

Theorem : The Cauchy - Schwarz Inequality

$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\|\|\vec{v}\|$

remark : the proof works for any Cauchy product

pf: let $\vec{v} \neq 0$, assume $\|\vec{u} - \lambda \vec{v}\|^2 \geq 0$ $\forall \lambda \in \mathbb{R}$

$\varphi(\lambda) = \|\vec{u} - \lambda \vec{v}\|^2 = \|\vec{u}\|^2 - 2\lambda \langle u, \vec{v} \rangle + \lambda^2 \|\vec{v}\|^2 \geq 0$

$\varphi'(\lambda) = -2\langle \vec{u}, \vec{v} \rangle + 2\lambda \|\vec{v}\|^2 = 0$

minimizia $\lambda = \langle \vec{u}, \vec{v} \rangle / \langle \vec{v}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle / \|\vec{v}\|^2$

plug in λ in $\|\vec{u} - \lambda \vec{v}\|^2 \geq 0$

$\|\vec{u}\|^2 - 2 \frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{v}\|^2} + \left(\frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \right)^2 \|\vec{v}\|^2 \geq 0$

$\frac{\|\vec{u}\|^2 \|\vec{v}\|^2 - 2\langle \vec{u}, \vec{v} \rangle^2 + \langle \vec{u}, \vec{v} \rangle^2}{\|\vec{v}\|^2} \geq 0$

$\|\vec{u}\|^2 \|\vec{v}\|^2 - \langle \vec{u}, \vec{v} \rangle^2 \geq 0$

Prop: Triangle Inequality

$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ $\forall \vec{u}, \vec{v} \in \mathbb{R}^n$

pf: square LHS, RHS.

$\cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} + 2\langle \vec{u}, \vec{v} \rangle \leq \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} + 2\|\vec{u}\|\|\vec{v}\|$

triangle inequality equal to. $\langle \vec{u}, \vec{v} \rangle \leq \|\vec{u}\|\|\vec{v}\|$ — C-S

example of inner product.

$\|f\| = \left(\int_a^b f^2(t)\, dt \right)^{\frac{1}{2}}$

$\left| \int_a^b f(t) g(t)\, dt \right| \leq \left(\int_a^b |f|^2 \right)^{\frac{1}{2}} \left(\int_a^b |g|^2 \right)^{\frac{1}{2}}$

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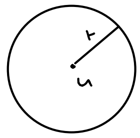
10.3



Open ball of radius r centered at $u \in \mathbb{R}^n$ $B_r(u) = \{ v \in \mathbb{R}^n \mid \|u-v\| < r \}$

Def: $A \subseteq \mathbb{R}^n$ is open if $\forall u \in A$ $\exists r > 0$ s.t $B_r(u) \subseteq A$.

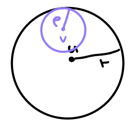
r depents on u

Q: What set in A is defined by " $\exists r > 0$, s.t $\forall u \in A$, $B_r(u) \subseteq A$"

$A = \mathbb{R}^n$ works. $A = \emptyset$ works.

$\exists r > 0$ positive value

Prop: Let $r > 0$, $u \in \mathbb{R}^n$, $A = B_r(u)$ is Open



pf: need to check.

$\forall$ $v \in B_r(u)$ $\exists$ $\rho > 0$ s.t $B_\rho(v) \subseteq B_r(u)$

$\rho = r - \|u - v\|$

Hope to show $B_\rho(v) \subseteq B_r(u)$, i.e. if $w \in B_\rho(v)$, then $w \in B_r(u)$

Let $\|w - v\| < \rho = r - \|u - v\|$

Hope to show $\|w - u\| < r$

Indeed $\|w-u\| = \|w - v + v - u\|$

by triagle inequality $\leq \|w-v\| + \|v - u\| < r - \|u - v\| + \|u - v\| = r$

Def: $A \subseteq \mathbb{R}^n$ is closed if the following is always true:

If $\forall \{u_k\}_{k \in \mathbb{N}} \subseteq A$, and $\{u_k\} \to u \in \mathbb{R}^n$, then $u \in A$

Ex. $\mathbb{R}^n$ open & closed

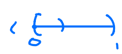
$\emptyset$ open & closed

No other sets in $\mathbb{R}^n$ are Both open and closed.

because $\mathbb{R}^n$ is "connected"

Ex. in $\mathbb{R}$. $A = [0, 1)$ neither open nor closed.

$\nexists r > 0$ s.t. $B_r(0) \subseteq A$



not open.

$\exists$ sequence $u_k \in [0, 1)$, $u_k = 1 - \frac{1}{k} \in A$,

$u_k \to 1$, but $1 \notin A$ not closed.

Def: if $A \subseteq \mathbb{R}^n$, $\mathbb{R}^n \setminus A = A^c =$ complement of A $\overset{\text{def}}{=} \{ u \in \mathbb{R}^n \mid u \notin A \}$

thm: $A \subseteq \mathbb{R}^n$ is open iff A^c is closed.

pf: $\Rightarrow$ assume A is open. want to show A^c closed.

let $\{u_k\} \subseteq A^c$, assume $\{u_k\} \to u \in \mathbb{R}^n$, want to show $u \in A^c$.

this is true, because $u \notin A$. if $u \in A$, A is open, then $\exists r > 0$. s.t $B_r(u) \subseteq A$.

and no elements of $\{u_k\}$ can be in $B_r(u)$ because $\{u_k\} \subseteq A^c$.

so $\{u_k\}$ cannot converge to u.

$\Leftarrow$ Conversly: assume A^c is closed, want to show A is open.

pick up, fix, $u \in A$. want to show $\exists r > 0$, s.t. $B_r(u) \subseteq A$.

assume by contradiction, $\forall r > 0$, $B_r(u) \cap A^c \neq \emptyset$

Specialize to $r = \frac{1}{k}$, $k \in \mathbb{N}$. $\exists$ $u_k \in B_{\frac{1}{k}}(u)$ $u_k \in A$

i.e. $\|u_k - u\| < \frac{1}{k}$ but $u_k \in A^c$.

But $\{u_k\} \subseteq A^c$, $\{u_k\} \to u$, and $u \notin A^c$

Thm. a) $\{V_s\}_{s \in S}$ be a positively infinite family of open sets

then $\bigcup_{s \in S} V_s$ is open.

pf: pick $u \in \bigcup_{s \in S} V_s$

$\exists\ s_0 \in S$ s.t $u \in V_{s_0}$ open.

thus $\exists\ r > 0$. s.t $B_r(u) \subseteq V_{s_0}$

But then $B_r(u) \in \bigcup V_s$

b) Let $V_1 \ldots, V_k$ finitely many open sets, then $V_1 \cap \cdots \cap V_k$ is open.

pf. Let $u \in V_1 \cap \cdots \cap V_k$, $u \in V_1$

hence $\exists\ r_1 > 0$. s.t $B_{r_1}(u) \subseteq V_1$.

..... $\exists\ r_k > 0$ s.t $B_{r_k}(u) \subseteq V_k$.

pick $r = \min\{r_1 \ldots r_k\} > 0$

Remark: the intersection of infinitely many open sets will not be open.

try $V_k = B_{\frac{1}{k}}(0)$

$\bigcap_{k \in K} V_k = \{0\}$ - not open

9/4.

10.3 - 11.1

Thm. a) let $\{F_s\}_{s \in S}$ be a paity infinite family of closed sets. then $\bigcap_{s \in S} F_s$ is closed.

pf. assume all F_s are closed, to conclude $\bigcap_{s \in S} F_s$ is closed,

iff to show $(\bigcap_{s \in S} F_s)^c$ is open.

recall $(\bigcap_{s \in S} F_s)^c = \bigcup_{s \in S} F_s^c$, each F_s^c is open. b/c all F_s are closed.

then $\bigcup_{s \in S} F_s^c$ is open

b) let $F_1 \ldots F_k$ be finitely many closed sets, then $F_1 \cup \cdots \cup F_k$ is closed

pf: Assume $F_1 \ldots F_k$ are closed, sufficient to show $(F_1 \cup \cdots \cup F_k)^c$ is open.

but $(F_1 \cup \cdots \cup F_k)^c = F_1^c \cap \cdots \cap F_k^c$, and each F_i^c is open,

so $F_1^c \cap \cdots \cap F_k^c$ is open. and each F_i^c is open.

show $F_1^c \cap \cdots \cap F_k^c$ is open.

Remark. another proof of b

Let $F_1 \ldots F_k$ closed, want $F_1 \cup \cdots \cup F_k$ to be closed.

apply the definition of "closed".

let $\{x_i\} \subseteq F_1 \cup \cdots \cup F_k$, assume $\{x_i\} \to x \in \mathbb{R}^n$

For ∞ many i, $x_i \in F_1 \cup \cdots \cup F_k$

It must be that $\exists$ some $1 \le k_0 \le k$

s.t ∞ many $x_i \in F_{k_0}$

$\underbrace{\quad}_{F_1}$ $\underbrace{\quad}_{F_2}$ ..... $\underbrace{\quad}_{F_k}$

then $\exists$ subsequence $\{x_{ij}\} \subseteq F_{k_0}$, and $\{x_{ij}\} \to x$, hence $x \in F_{k_0}$

thus $x \in \bigcup_{i=1}^{k} F_i$

Let $A \subseteq \mathbb{R}^n$.

Def. $\text{int } A = \{x \in \mathbb{R}^n \mid \exists\, r>0,\ \text{s.t } B_r(x) \subseteq A\} \subseteq A$

$\text{bd } A = \{x \in \mathbb{R}^n \mid \forall\, r>0,\ B_r(x) \cap A^c \neq \emptyset,\ \forall\, r>0\ B_r(x) \cap A \neq \emptyset\}$

$\text{ext } A = \{x \in \mathbb{R}^n \mid \exists\, r>0\ \text{ s.t } B_r(x) \subseteq A^c\} \subseteq A^c$

$\mathbb{R}^n = \text{int } A \cup \text{bd } A \cup \text{ext } A$, position of $\mathbb{R}^n$

no point is in 2 of the other sets

Remark.

$A \cap \text{boundary } A = \phi$ iff $\forall x \in A,\ \exists\, r>0$ s.t $B_r(x) \subseteq A$

$A \cap \text{boundary } A = \phi$ iff A is open iff $A = \text{int } A$

Remark. $A = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$

$\text{bd } A = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$

A closed iff $\text{bd } A \subseteq A$

Chapter 11.

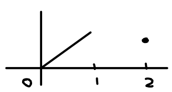
let $A \subseteq \mathbb{R}^n$, $F: A \to \mathbb{R}^n$.

let $u \in A$.

Defn: F is continuous at u if $\forall \{u_k\} \subseteq A$, s.t $\{u_k\} \to u$

it follow that $\{F(u_k)\} \to F(u)$

Q: $f: [0,1] \cup \{2\}$



$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 2 & \text{if } x = 2. \end{cases}$$

Yes f continuous,

continuity at 2. if $\{x_k\} \subseteq [0,1] \cup \{2\}$

$\{x_k\} \to 2$

then $\exists K$ s.t. $x_k = 2\ \ \forall k \geq K$

Let $A \subseteq \mathbb{R}^n$, $f, g: A \to \mathbb{R}$, let $u \in A$, assume f, g are continuous at u.

then $\alpha f + \beta g$ is continuous at u. $(\alpha, \beta \in \mathbb{R})$

fg is continuous at u.

in addition. if $g(x) \neq 0\ \ \forall x \in A$

then f/g is also continuous at u.

pf. use the definition and defn of sequences

Def: $A \subseteq \mathbb{R}^n$. $B \subseteq A$. $F: A \to \mathbb{R}^m$

Def: $F(B) = \{y \in \mathbb{R}^m \mid \exists\, x \in B,\ F(x) = y\}$

$\cap\!|$

$\mathbb{R}^m$

Def: $\underset{\mathbb{R}^n}{\overset{}{A}} \xrightarrow{F} \underset{\mathbb{R}^m}{B} \xrightarrow{G} \mathbb{R}^k$ (with $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$)

let $F: A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$

$G: B \to \mathbb{R}^k$, $B \subseteq \mathbb{R}^m$

assume $F(A) \subseteq B$, let $u \in A$.

assume F continuous at u.

G continuous at $F(u)$

then $G \circ F$ is continuous at u.

pf: check the definition of continuity.

let $\{u_k\} \subseteq A$, $\{u_k\} \to u$

then $\{F(u_k)\} \to F(u)$ b/c F contin. at u.

then $G(F(u_k)) \to G(F(u))$ b/c G cont. at $F(u)$

So $G \circ F(u_k) \to G \circ F(u)$

9/9

Review triangle inequality

$\|x+y\| \leq \|x\| + \|y\| \quad \forall\ x, y \in \mathbb{R}^n$

equivalent to the reverse triangle inequality

$\|x \pm y\| \geq |\|x\| - \|y\||$

pf of rev. tri. ineq. $\|x+y\| \geq \|x\| - \|y\|$

$\|x\| = \|x+y-y\| \leq \|x+y\| + \underbrace{\|-y\|}_{} = \|y\|$

replace y by $-y$ get $\|x-y\| \geq \|x\| - \|y\|$

reverse the role of x and y.

get $\|x \pm y\| \geq \|y\| - \|x\|$ i.e. $\|x \pm y\| \geq |\|x\| - \|y\||$

11.1 .

Let $A \subseteq \mathbb{R}^n$, $u \in A$

Thm. $F: A \to \mathbb{R}^m$ is continuous (with sequence definition) at u.

iff $\forall\ \varepsilon > 0$. $\exists\ \delta > 0$. s.t $\|F(u) - F(v)\| < \varepsilon$ $\quad \forall\ v \in A$ and $\|u-v\| < \delta$

Defn: $F: A \to \mathbb{R}^m$ let $V \subseteq \mathbb{R}^m$

$F^{-1}(V) = \{x \in A \mid F(x) \in V\}$

note $F^{-1}(V)$ is defined as a set even if F^{-1} is not well defined as a function.

F 1 to 1 on A, onto $\mathbb{R}^n$

Remark. Fix $\varepsilon > 0$, $\delta > 0$. $u \in A$. $F: A \to \mathbb{R}^m$ $\|F(v) - F(u)\| < \varepsilon$ if $\|u-v\| < \delta$, $v \in A$

$\Leftrightarrow$ $F(B_\delta(u) \cap A) \subseteq B_\varepsilon(F(u))$

$\Leftrightarrow$ $F^{-1}(B_\varepsilon(F(u))) \supseteq B_\delta(u) \cap A$

i.e. $B_\delta(u) \cap A \subseteq F^{-1}(B_\varepsilon(F(u)))$

ex. $f: \mathbb{R} \to \mathbb{R}$. $f(x) = x^2$.

f^{-1} not well defined as a function.

$f^{-1}(x) = \pm\sqrt{x}$ (not 1-1, it is 2 to 1)

But $f^{-1}([1,4])$ is well defined. as set $f^{-1}([1,4]) = \{x \in \mathbb{R} \mid f(x) \in [1,4]\}$

$= [-1,-2] \cup [1,2]$

ε-δ def of continuity of $u \in A$

equivalent to $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t $B_\delta(u) \cap A \subseteq F^{-1}(B_\varepsilon(F(u)))$

Thm: let $O \subseteq \mathbb{R}^n$ open, $F: O \to \mathbb{R}^n$.

(i) then F is continuous on O (at all $u \in O$)

iff (ii) $F^{-1}(U)$ is open $\forall\, U \subseteq \mathbb{R}^m$ open

pf: (ii) $\overset{\text{implies.}}{\Rightarrow}$ (i)

Assume $F^{-1}(V)$ is open $\forall\, V \in \mathbb{R}^m$

check continuity at fixed $u \in O$

let $\varepsilon > 0$, we want $\delta > 0$ s.t $B_\delta(u) \cap O \subseteq F^{-1}(B_\varepsilon(F(u)))$

pick $V = B_\varepsilon(F(u))$ open.

know $F^{-1}(B_\varepsilon(F(u)))$ is open

For sure $u \in F^{-1}(B_\varepsilon(F(u)))$ b/c $F(u) \in B_\varepsilon(F(u))$

hence $\exists\, \delta > 0$, s.t $B_\delta(u) \subseteq F^{-1}(B_\varepsilon(F(u)))$

and then $B_\delta(u) \cap O \subseteq B_\delta(u) \subseteq F^{-1}(B_\varepsilon(F(u)))$

we didn't use O open in this part.

if (ii) holds, O has to be open. b/c $O = F^{-1}(\mathbb{R}^m)$ ← open

(i) $\Rightarrow$ (ii)

assume F continuous at all $u \in O$

i.e. $\forall\, \varepsilon > 0$. $\exists\, \delta > 0$ s.t $B_\delta(u) \cap O \subseteq F^{-1}(B_\varepsilon(F(u)))$

let $V \subseteq \mathbb{R}^m$ open.

want to show $F^{-1}(V)$ is open.

pick $u \in F^{-1}(V)$ ($F(u) \in V$)

Want $\delta > 0$ s.t $B_\delta(u) \subseteq F^{-1}(V)$

since V is open, $F(u) \in V$. $\exists\, \varepsilon > 0$ s.t $B_\varepsilon(F(u)) \subseteq V$.

then $\exists\, \delta > 0$, s.t. $B_\delta(u) \cap O \subseteq F^{-1}(B_\varepsilon(F(u))) \subseteq F^{-1}(V)$

Since O is open, $\exists\, \delta_2 > 0$ s.t $B_{\delta_2}(u) \subseteq O$

Let $\delta = \min\{\delta_1, \delta_2\}$

then $B_\delta(u) \subseteq O$.

$\subseteq B_{\delta_1}(u) \cap O$

then $B_\delta(u) \subseteq F^{-1}(B_\varepsilon(F(u)))$

Remark: $F: \mathbb{R}^n \to \mathbb{R}^m$ is continuous

iff $F^{-1}(V)$ is open $\forall\, V \subseteq \mathbb{R}^m$ open

iff $F^{-1}(B)$ is closed $\forall\, B \subseteq \mathbb{R}^m$ closed.

pf: $F^{-1}(B)$ is closed iff $(F^{-1}(B))^c$ is open $= F^{-1}(B^c)$ is open

Also B is closed iff B^c is open.

ex. if $f: \mathbb{R}^m \to \mathbb{R}$ is continuous,

$\{x \in \mathbb{R}^n \mid f(x) > a\}$ is open

$\{x \in \mathbb{R}^n \mid f(x) \geq a\}$ is close

$\{x \in \mathbb{R}^n \mid f(x) = a\}$ is closed $\forall\, a \in \mathbb{R}$

$\{a\}$ is closed

9/11 11.2

Def: $A \subseteq \mathbb{R}^n$ is sequentially compact

if $\forall\, \{x_k\} \subseteq A$ $\exists\, \{x_{k_j}\}$ subsequence and $x \in A$ s.t $\{x_{k_j}\} \to x$

prop: If A is sequentially compact, then A is bounded.

(A is bounded if $\exists\, M > 0$ s.t $\|x\| \leq M$ $\forall\, x \in A$)

pf: Let A seq. compact.
Assume by contradiction, A is not bounded
Thus $\forall M>0$, $\exists x_M \in A$ s.t $\|x_M\| > M$
Take $M = k \in \mathbb{N}$, then we have $\{x_k\} \subseteq A$ s.t $\|x_k\| > k$
But $\{x_k\}$ has no convergent subsequence

prop: If A is sequentially compact, then A is closed.

pf: Let A seq compact. to prove A is closed.
take $\{x_k\} \subseteq A$, $\{x_k\} \to x \in \mathbb{R}^m$.
know $\exists \{x_{k_j}\}$ and $y \in A$, s.t $\{x_{k_j}\} \to y \in A$.
but $\{x_{k_j}\} \to x$. so $x = y \in A$.

prop: If $\{x_k\}$ is a bounded sequence in $\mathbb{R}^n$, then $\exists \{x_{k_j}\}$ convergent.

pf: know true for $n=1$
note $P_1(x_k)$, $P_2(x_k)$ are bounded.
preliminy case $n=2$. $\{x_k\} \in \mathbb{R}^2$, $x_k = (P_1(x_k), P_2(x_k))$
we know $\exists$ subsequence s.t $P_1(x_{k_j})$ converges
Look at $P_2(x_{k_j})$ bounded seq of real numbers,
so $\exists$ $P_2(x_{k_{j_\ell}})$ convergent.
then $\{x_{k_{j_\ell}}\} = (P_1(x_{k_{j_\ell}}), P_2(x_{k_{j_\ell}}))$ converges componentwise.
thus convergens.
By induction, assume true for $n=1$,
Look at $\{x_k\} \subseteq \mathbb{R}^m$ bounded
Denote $\underset{\mathbb{R}^n}{x_k} = (\underbrace{P_{1 \le 0 \le n-1}\, x_k}_{\mathbb{R}^{n-1}}, \underset{\mathbb{R}^n}{P_n(x_k)})$ each bounded.

thus $\exists \{P_{1 \le 0 \le n-1}\, x_{k_j}\}$ converges, and $\exists$ $P_n(x_{k_{j_\ell}})$ converges.

Thm: If A is closed and bounded, then A is sequentially compact.

pf: Let $\{x_k\} \subseteq A$, Since A is bounded $\exists$ $\{x_{k_j}\} \to x \in \mathbb{R}^n$
Since A is closed, $x \in A$.

Thm: If A is seq. compact, and $F: A \to \mathbb{R}^m$ continuous, then $F(A)$ is sequentially compact.

pf: Let $\{y_k\} \subseteq F(A)$, want $\{y_{k_j}\} \to y \in F(A)$
But $\exists$ $\{x_k\} \subseteq A$ s.t $F(x_k) = y_k$.
Since A is seq. compact. $\exists$ $\{x_{k_j}\} \to x \in A$.
since F is continuous, $\{F(x_{k_j})\} \to F(x) \in F(A)$
Thus, if $F: A \to \mathbb{R}^n$ is continuous, and A is closed and bounded,
$F(A)$ is closed and bounded.

Q: If A is closed, is $F(A)$ closed?
A: No, $e^x : (-\infty, \infty) \to (0, \infty)$ onto

Q: If A is bounded, is $F(A)$ bounded?
A: No: $1/x : (0,1) \to (1, \infty)$ onto

Thm: let A seq. compact. $f: A \to \mathbb{R}$ continuous.

then f attains a mimimum and maximum value on A.

pf: since $f(A)$ is bounded, $\inf_{x\in A} f(x)$, $\sup_{x\in A} f(x)$ exists.

How to find $x_0 \in A$ s.t $\inf_{x\in A} f(x) = f(x_0)$

know $\exists$ sequence $\{x_k\} \subseteq A$ s.t $\{f(x_k)\} \to \inf_{x\in A} f(x)$

since A is seq. comp.

$\exists$ $\{x_{k_j}\} \to x_0 \in A$ and then $\{f(x_{k_j})\} \to \inf_{x\in A} f(x)$

$$\begin{array}{ccc} \{f(x_{k_j})\} & \to & \inf_{x\in A} f(x) \\ \downarrow & & // \\ f(x_0) & & \end{array}$$

prop: Let $A \subseteq \mathbb{R}^n$ s.t $\forall f: A \to \mathbb{R}$ continuous, f attains both minimum and maximum values, then A has to be sequentially compact

pf: First, show A is bounded.

Suppose not, then $\exists \{x_k\} \subseteq A$, s.t $\|x_k\| > k$.

then $f(x) = \|x\|$ continuous on A, does not have a maximum

Next, show A is closed

Let $\{x_k\} \subseteq A$, $\{x_k\} \to x_0 \in \mathbb{R}^n$

want to show $x_0 \in A$

suppose not . $x_0 \notin A$

then $f(x) = \|x - x_0\|$ does not attain its minium on A.

$\inf_{x\in A} f = 0$. but $\nexists$ point in A s.t $f(\text{point}) = 0$

Let $F: A \to \mathbb{R}^m$.

Def: F is uniformely continuous if for any two sequence $\{u_k\}, \{v_k\} \subseteq A$.

if $\|u_k - v_k\| \to 0$, then $\|F(u_k) - F(v_k)\| \to 0$

Remark: F continuous $\not\Rightarrow$ uniformely continuous

ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

Let $u_k = k$, $v_k = k + \frac{1}{k}$, $|u_k - v_k| \to 0$

$f(u_k) - f(v_k) = k^2 + 2 \cdot k \cdot \frac{1}{k} + \frac{1}{k^2} - k^2$

$= 2 + \frac{1}{k^2} \to 2 \neq 0$

Thm: If $A \subseteq \mathbb{R}^n$ seq. compact, then $F: A \to \mathbb{R}^m$ is continuous, then F is uni. cont.

pf: Let A, F as above, suppose F is not uniform. cont.

then $\exists \{u_k\}, \{v_k\} \subseteq A$. $\|u_k - v_k\| \to 0$ and $\|F(u_k) - F(v_k)\| \not\to 0$

$\exists$ $\varepsilon > 0$. s.t $\forall K \in \mathbb{N}$, $\exists$ $k \geq K$ s.t $\|F(u_k) - F(v_k)\| \geq \varepsilon$

$\exists \{u_{k_j}\} \to u \in A$. then $\{v_{k_j}\} \to u$.

then $\|F(u_{k_j}) - F(v_{k_j})\| \to 0$, a contradiction

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Recall last time we defined $F: A \to \mathbb{R}^m$ to be uniformly continous

if $\forall \{u_k\}, \{v_k\} \subseteq A$, if $\|u_k - v_k\| \to 0$

then $\|F(u_k) - F(v_k)\| \to 0$

Thm: If $F: A \to \mathbb{R}^m$ is continuous and A is sequentially compact, then F is uniformly continuous

$\varepsilon - \delta$ def of uniformly continuity:

$F: A \to \mathbb{R}^m$ is uniformly continuous if $\forall \varepsilon > 0 \ \exists \delta > 0$. δ independent of u, v.

s.t $\|F(u) - F(v)\| < \varepsilon \quad \forall u, v \in A$ with $\|u - v\| < \delta$

Compare with $F: A \to \mathbb{R}^m$ continuous at u:

$\forall \varepsilon > 0 \quad \exists \delta$ (δ depents on u)

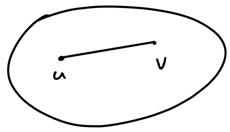
s.t $\|F(u) - F(v)\| < \varepsilon \quad \forall v \in A \quad \|u - v\| < \delta$

Thm: seq. def. of uniformly continuties is equivalent to $\varepsilon - \delta$ def

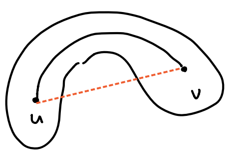
11.3

Def: $A \subseteq \mathbb{R}^n$ is convex if $\forall u, v \in A$ the line segment from u to v is contained in A.

ie. $\forall t \in [0,1] \quad tu + (1-t)v \in A$



convex



not convex

but pathwise connected.

Def: $\gamma: [a,b] \to A$ is continuous, is called a parametrized path from $\gamma(a)$ to $\gamma(b)$

Def: $A \subseteq \mathbb{R}^n$ is pathwise connected if $\forall u, v \in A$, $\exists \gamma: [a,b] \to A$, continuous, such that $\gamma(a) = u$, $\gamma(b) = v$

proposition: If A is pathwise connected, and $F: A \to \mathbb{R}^m$ is continuous, then $F(A)$ is pathwise connected.

pf: pick $u, v \in F(A)$, $\exists x, y \in A$, s.t $u = F(x)$ $v = F(y)$

since A is pathwise connected, $\exists \gamma: [a,b] \to A$, continuous, s.t $\gamma(a) = u$, $\gamma(b) = v$,

then $F \circ \gamma: [a,b] \to F(A)$ is a pathwise path from u to v.

Remark: $A \subseteq \mathbb{R}$ is pathwise connected iff A is an interval.

Def: $A \subseteq \mathbb{R}^n$ has the intermediate value property if $\forall f: A \to \mathbb{R}$, continuous, it follows that $f(A)$ is an interval.

thus A pathwise connected $\Rightarrow$ A has IVP.

11.4

Def: Let $A \subseteq \mathbb{R}^n$ two open sets U, V separate A if

1) $A \cap U \neq \emptyset$, $A \cap V \neq \emptyset$

2) $A = (A \cap U) \cup (A \cap V)$

3) $(A \cap U) \cap (A \cap V) = \emptyset$

Remark: $A \cap U$ called relatively open in A if U is open

1), 2), 3) say A is partitioned into two disjointed non-empty relatively open sets ($A \cap U$, $A \cap U$)

Def: $A \subseteq \mathbb{R}^n$ is connected if $\nexists$ U, V open s.t U, V seperate A.

Thm: A is connected iff A has IVP.

Corollary: A pathwise connected $\Rightarrow$ A has IVP $\Leftrightarrow$ A is connected.

A pathwise connected $\Rightarrow$ A connected.

Remark. converse is not true.

pf: will show A not connected $\Leftrightarrow$ A does not have IVP

$\Rightarrow$ First, assume A is not connected. $\exists$ U, V open, separating A.

will construct $f: A \to \mathbb{R}$, continuous, s.t $f(A)$ is not an interval

$$f(x) = \begin{cases} 1 & \text{if } x \in U \cap A \\ 0 & \text{if } x \in V \cap A \end{cases}$$

check. $\forall x \in A$, $\exists$ at most one value of $f(x)$ (from 3))

$\forall x \in A$, $\exists$ at least one value of $f(x)$ (from 2))

$f(A) = \{0, 1\}$ not an interval (from 1))

Need to show f is continuous, pick $x_0 \in A$, WLOG, $x_0 \in U \cap A$.

Let $\varepsilon > 0$, since U is open, $\exists$ $\delta > 0$. s.t $B_\delta(x_0) \subseteq U$

then $|f(x) - f(x_0)| < \varepsilon$ $\forall$ $x \in A$, $\|x - x_0\| < \delta$

because $f(x) = f(x_0)$ in this case, $x_0 \in U \cap A$, $x \in U \cap A$.

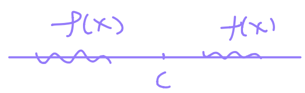
$(x \in B_\delta(x_0) \cap A)$

$\Leftarrow$ Assume $\exists$ $f: A \to \mathbb{R}$, continuous, s.t $f(A)$ is not an interval.

will construct U, V open seperating A.

since $f(A)$ is not an interval, $\exists$ $c \in \mathbb{R}$, $c \notin f(A)$

$\exists$ $x_0, y_0 \in A$, $f(x) < c < f(y)$



Define $\tilde{U} = f^{-1}(-\infty, c)$, $\tilde{V} = f^{-1}(c, \infty)$

notice: 1) $x_0 \in f^{-1}(-\infty, c) = \tilde{U}$

$y_0 \in f^{-1}(c, \infty) = \tilde{V}$

2) $A = \tilde{U} \cup \tilde{V}$ because $c \notin f(A)$

3) $\tilde{U} \cap \tilde{V} = \emptyset$

$\forall x \in A$, $f(x) < c$ if $x \in \tilde{U}$

or $f(x) = c$ there are no such point

$f(x) > c$ if $x \in \tilde{V}$

construct U open s.t $\tilde{U} = U \cap A$. $\forall x \in \tilde{U}$. $f(x) < c$

then $\exists$ $\delta > 0$ ($\delta = \delta_x$) s.t $f(y) < c$ if $y \in A$ $\|y - x\| < \delta$

then define $U = \bigcup_{x \in \tilde{U}} B_{\delta(x)}(x)$ open and $\tilde{U} = U \cap A$

similar for V

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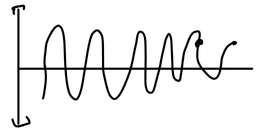
If A is pathwise connected $\Rightarrow$ A is an interval $\Leftrightarrow$ A connected

A connected $\not\Rightarrow$ A pathwise connect

$A = (\{0\} \times [-1,1]) \cup \{(x, \lim(\frac{1}{x})) \mid x > 0\}$



A not pathwise connected.

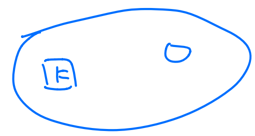
A is connected. b/c $\nexists$ open U.V separate A

Remark. If A is open.

then A connected $\Leftrightarrow$ A pathwise connected

11.4 ex: $K \subseteq O$, O open, K seq. compact

show $\exists$ $\delta > 0$ s.t $B_\delta(x) \subseteq O$ $\forall x \in K$



Is this correct? No. δ should be ind of x.

Let $x \in K$, then $x \in O$, so $\exists$ $\delta > 0$ s.t $B_\delta(x) \subseteq O$

dep on x.

Practice Problem.

1) Let $f: \mathbb{R} \to \mathbb{R}$ continuous, Prove that the graph $G = \{(x, f(x)) \mid x \in \mathbb{R}\}$ is closed.

pf: Let $y_n = f(x_n)$

Let $\{(x_n, y_n)\} \subseteq G$, assume $\{(x_n, y_n)\} \to (x_0, y_0)$

want to conduct $(x_0, y_0) \in G$. i.e. $f(x_0) = y_0$

Since $\{(x_n, y_n)\} \to (x_0, y_0)$, $\{x_n\} \to x_0$.

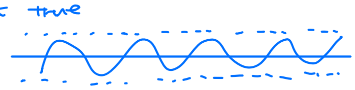
Since f is continuous, $\{f(x_n)\} \to f(x_0)$ so $f(x_0) = y_0$

2) If $f: \mathbb{R} \to \mathbb{R}$ is s.t G is closed, does it follow f is continuous.

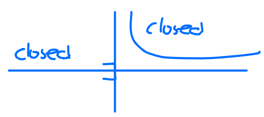
① Try to show $P_2(G)$ is closed

not true

$P_2(G) = (-1, 1)$ - not closed

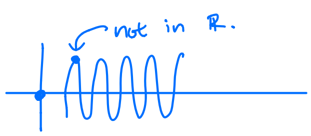


② $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{x} & \text{if } x > 0 \end{cases}$



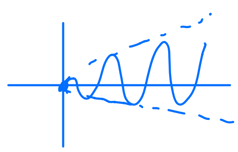
4) If $f: [0,1] \to \mathbb{R}$ is such that G is seq. compact, does it follow that f is continuous? true

① try $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sin(1/x) & 0 < x \le 1 \end{cases}$



not seq. compact, not closed,

② try $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin(1/x) & 0 < x \le 1 \end{cases}$



✓ f continuous

✓ G seq. compact

pf: assume G seq, compact.

Let $\{x_n\} \to x_0$ want $\{f(x_n)\} \to f(x_0)$

$\{x_n, f(x_n)\} \in G$, then $\exists$ subsequence $\{(x_{n_k}), f(x_{n_k})\} \to (x_0, f(x_0)) \in G$

thus $f(x_{n_k}) \to f(x_0)$

Assume, by continuous

$\{f(x_n)\} \to f(x_0)$

$\forall \varepsilon > 0$ $\exists N$ s.t $|f(x_n) - f(x_0)| < \varepsilon$ $\forall n_k \geq N$ not true.

thus $\exists \varepsilon > 0$. s.t $\forall N$ $\exists n \geq N$ s.t $|f(x_{n_k}) - f(x_0)| \geq \varepsilon$

Get a subsequence x_{n_k} s.t $|f(x_{n_k}) - f(x_0)| \geq \varepsilon$ $\forall k$

9/18.

12.1 Metric Space

Def: (X, d) is a metric space if X is a set. and $d: X \times X \to [0, \infty)$

satisfies 1) $d(p,q) = d(q,p)$

2) $d(p,q) \geq 0$, $d(p,q) = 0$ iff $p = q$

3) $d(p,q) \leq d(p,r) + d(r,q)$ $\forall p,q,r \in X$.

Def: if V is a real vector space, $\|\cdot\|$ is a norm of V

if 1) $\|v\| \geq 0$, $\|v\| = 0$ iff $v = 0$

2) $\|v + w\| \leq \|v\| + \|w\|$

3) $\|\alpha v\| = |\alpha| \|v\|$ $\forall v, w \in V$ $\forall \alpha \in \mathbb{R}$

Remark: If $\|\cdot\|$ is a norm on V, then $X = V$ is a metric space with $d(v,w) = \|v - w\|$ $\forall v, w \in V$.

1) $d(p,q) = \|p - q\| = \|q - p\| = d(q,p)$

2) $\|p - q\| \geq 0$, $\|p - q\| = 0$ iff $p - q = 0$

3) $d(p,q) = \|p - q\| = \|p - r + r - q\| \leq \|p - r\| + \|r - q\| = d(p,r) + d(r,q)$

Ex: for $X = V = \mathbb{R}^n$

$\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ usual norm

$d_2(x,y) = \|x - y\|$ $x, y \in \mathbb{R}^n$ is a metric

$\|x\|_1 = \sum_{i=1}^n |x_i|$

check $\|x\|_1 \geq 0$ ✓

$\|x\|_1 = 0$ iff $x = 0$ ✓

$\|x + y\|_1 = \sum_{i=1}^n |x_i + y_i| \leq \sum_{i=1}^n (|x_i| + |y_i|) = \|x\|_1 + \|y\|_1$ ✓

$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

check $\|x + y\|_\infty = \max_{1 \leq i \leq n} |x_i + y_i| \leq \max_{1 \leq i \leq n} (|x_i| + |y_i|) \leq \max |x_i| + \max |y_i|$ ✓

the triangle inequality for $\|x\|_1$, $\|x\|_\infty$ was trival to prove.

Main example:

$V = C([a,b] \to \mathbb{R}) = \{f: [a,b] \to \mathbb{R}\}$ continuous.

main norm $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$, $d(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|$

check the metric space axiom. Suff. to check the norm axiom.

$f \in C([a,b], \mathbb{R})$

$\|f\|_\infty \geq 0$, $\|f\|_\infty = 0$ iff $f(x) = 0 \quad \forall x$

$\|f+g\|_\infty = \max_{x\in[a,b]} |f(x)+g(x)| = |f(x_0)+g(x_0)|$

$\exists \; x_0 \in [a,b]$ $\quad \leq |f(x_0)| + |g(x_0)|$

$\leq \max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} |g(x)| = \|f\|_\infty + \|g\|_\infty$

$\|\alpha f\|_\infty = \max_{x\in[a,b]} |\alpha f(x)| = |\alpha| \max_{x\in[a,b]} |f(x)| = |\alpha| \, \|f\|_\infty$

Example: $V = C([a,b], \mathbb{R})$, $\|f\|_1 = \int_a^b |f(x)| dx$

check triangle inequality:

$\|f+g\|_1 = \int_a^b |f(x)+g(x)| dx \leq \int_a^b (|f(x)| + |g(x)|) dx \leq \int_a^b |f(x)| dx + \int_a^b |g(x)| dx$

Example: $V = C([a,b], \mathbb{R})$. $\|f\|_2 = \left(\int_a^b |f(x)|^2 dx \right)^{1/2}$

check:

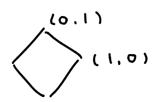
$$\|f+g\|^2 = \int_a^b (f(x)+g(x))^2 dx$$
$$= \int_a^b f^2(x) dx + 2\int_a^b f(x)g(x) dx + \int_a^b g^2(x) dx$$
$$\leq \int_a^b f^2(x) dx + 2\left(\int_a^b f^2(x) dx\right)^{1/2} \left(\int_a^b g^2(x) dx\right)^{1/2} + \int_a^b g^2(x) dx$$
$$= \|f\|_2^2 + 2\|f\|_2 \|g\|_2 + \|g\|_2^2 = (\|f\|_2 + \|g\|_2)^2$$

Remark: $\|f\|_\infty$, $\|f\|_1$, $\|f\|_2$ are example of L^p norms $(1 \leq p \leq \infty)$

Def: Let X be a metric space, $r>0$, $p \in X$, $B_r(p) = \{ q \in X \mid d(p,q) < r \}$

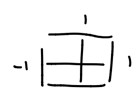
Draw unit ball $B_1(0)$ with respect to $d(x,y) = \begin{cases} \|x-y\|_1 \\ \|x-y\|_2 \\ \|x-y\|_\infty \end{cases}$

$\|y\|_1 = |y_1| + |y_2| < 1$, $y = (y_1, y_2) \in \mathbb{R}^2$



$\{ \|y\|_2 < 1 \}$ 

$\{ \|y\|_\infty < 1 \} = \{ (y_1, y_2) \mid \max\{ |y_1|, |y_2| \} < 1 \}$



Defn: Same as in $\mathbb{R}^n$.

Let (X, d) metric space.

Defn: $A \subseteq X$ is open if $\forall p \in A \;\; \exists r > 0$ s.t. $B_r(p) \subseteq A$.

Ex: $X = [0,1] \subseteq \mathbb{R}$. $d(x,y) = |x-y|$, $B_{\frac{1}{2}}(0) = [0, \frac{1}{2})$

prop: $B_r(u) \subseteq X$ is an open set in X

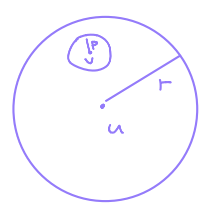
Corollary: $[0, \frac{1}{2})$ is an open set in $[0,1]$

pf. Let $v \in B_r(u)$ want $\rho > 0$ s.t. $B_\rho(v) \subseteq B_r(u)$

choose $\rho = r - d(u,v)$

want to show $B_{r-d(u,v)}(v) \subseteq B_r(u)$

pick $w \in B_{r-(u,v)}(v)$, want $w \in B_r(u)$



Look at $d(w,u) \le d(w,v) + d(v,u) < r-(u-v) + d(v,u) = r$

Let $\{u_k\}$ be a sequence in X

Def: $\{u_k\} \to u \in X$ if $\forall \varepsilon > 0$ $\exists \bar{k}$ s.t $d(u_k, u) < \varepsilon$ $\forall k \geq \bar{k}$.

equivalently $\{d(u_k, u)\} \to 0$

Def: $A \subseteq X$ is closed if whenever $\{u_k\} \subseteq A$ and $\{u_k\} \to u \in X$ it follow $u \in A$

Thm: $A \subseteq X$ is open if $X \setminus A$ (the complement of A in X) is closed.

corollary: If $X = [0,1)$, then $B_{\frac{1}{2}}(0) = [0, \frac{1}{2})$ is open in X.
and the complement $X \setminus B_{\frac{1}{2}}(0) = [\frac{1}{2}, 1)$ is closed in X.

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12.1

Thm.

let X be a metric space, $\{V_\alpha\}$ a positive infinite family of open set. then $\bigcup_\alpha V_\alpha$ is open

If $V_1 \dots V_k$ finite many open sets, then $V_1 \cap \dots \cap V_k$ is open

let $\{F_\alpha\}$ be a positive infinite family of closed sets, then $\bigcap_\alpha F_\alpha$ is closed

If $F_1 \dots F_k$ are closed set, then $F_1 \cup \dots \cup F_k$ is also closed.

12.2

Let (X, d) a metric space. Recall $\{P_k\} \subseteq X$

$\{P_k\} \to P$ in X if $\forall \varepsilon > 0$ $\exists \bar{k}$ s.t $d(P_k, P) < \varepsilon$ $\forall k \geq \bar{k}$

Recall if $X = C([a,b], \mathbb{R})$ with $d(f,g) = \max_{x \in [a,b]} (f(x) - g(x))$,

then "convergence" as define above

($\forall \varepsilon > 0$ $\exists \bar{k}$ s.t $\max_{x \in [a,b]} (f(x) - g(x)) < \varepsilon$ $\forall k \geq \bar{k}$) means "uniform convergence".

Defn. let (X, d) be a metric space. $\{P_k\} \subseteq X$ is Cauchy if

$\forall \varepsilon > 0$ $\exists \bar{k}$ s.t $d(P_k, P_l) < \varepsilon$ $\forall k \geq \bar{k}$

ex. If $X = C([a,b], \mathbb{R})$, $\{f_k\}$ cauchy means $\forall \varepsilon > 0$ $\exists \bar{k}$ s.t $\max_{x \in [a,b]} |f_k(x) - f_l(x)| < \varepsilon$ $\forall k \geq \bar{k}$

this is "uniformly Cauchy"

Prop: If $\{P_k\} \to P$ in (X, d), then $\{P_k\}$ is Cauchy.

pf: let $\varepsilon > 0$ $\exists \bar{k}$ s.t $d(P_k, P) < \varepsilon/2$ $\forall k \geq \bar{k}$

then if $k, l \geq \bar{k}$,

$d(P_k, P_l) \le d(P_k, P) + d(P, P_l) < \varepsilon$

the converse is not true.

Def. If X is such that every Cauchy sequence in X is convergent to a point in X,
then X is complete.

ex: $\mathbb{R}$ is complete.

ex: $X = \mathbb{Q}$ with $d(x,y) = (x-y)$ is not complete

let $\{x_k\} \to \sqrt{2} \notin \mathbb{Q}$

then $\{x_k\}$ is Cauchy, but does not converge to a point in $\mathbb{Q}$..

ex: $\mathbb{R}^n$ is complete.

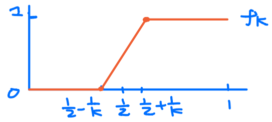
ex: $C([a,b],\mathbb{R})$ with $d(f,g) = \max_{x\in[a,b]} |f(x)-g(x)|$ is complete

If $\{f_k\}$ is Cauchy, it is uniformly Cauchy. $\Rightarrow$ $\{f_k\}$ is uniformly convergent

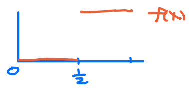
$\Leftrightarrow$ $\{f_k\}$ convergent in X

However, $C([a,b],\mathbb{R})$ with $d(f,g) = \int_a^b |f(x)-g(x)|$ is not complete.

ex:



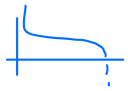
$\{f_k\} \to$



$\int_0^1 |f_k(x) - f(x)| = 0 \le \frac{2}{k}$

But $\nexists$ $f \in C([0,1],\mathbb{R})$ s.t $f_k \to f$ with resp. to d_1

ex: $f: [0,1] \to \mathbb{R}$. $\quad f(x) = \begin{cases} 0 & \text{if } x=0 \\ \frac{1}{\sqrt{x}} & \text{if } 0 < x \le 1 \end{cases}$



let $f_k(x) = \begin{cases} 0 & \text{if } x \le x \le 1/k \\ 1/\sqrt{x} & \text{if } 1/k \le x \le 1 \end{cases}$

recall $\int_0^1 1/\sqrt{x}\, dx < \infty$

$\{f_k\} \to f$ i.e $\int_0^1 |f_k(x) - f(x)|\, dx = \int_0^{\frac{1}{k}} 1/\sqrt{x}\, dx \to 0$

But $\nexists$ $f \in C([0,1],\mathbb{R})$, $\{f_k\} \to f$ with resp to d_1.

Thm: let X be a complete metric space,
then $F \subseteq X$ is complete iff F is closed in X.

pf. Say F is complete, to prove F closed, set $\{x_k\} \subseteq F$, $\{x_k\} \to x \in X$
since $\{x_k\}$ converge in X it is Cauchy.
Since F is complete, $\{x_k\} \to y \in F$
But limit is unique, $x = y \in F$, so $x \in F$.
Conversely.
If F is closed in X, then let $\{x_k\}$ be Cauchy sequence in F.
then $\{x_k\} \to x \in X$, because X is complete.
But since F is closed, $x \in F$, so F is complete

Def: Let X, Y be metric spaces, $T: X \to Y$ is Lipschitz
if $\exists\, c \ge 0$ s.t $d_Y(T(u), T(v)) \le c\, d_X(u,v)$ $\forall u, v \in X$

Def: If $\exists$ $0 \le c \le 1$ as above, T is a contraction

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ differentiable, then f is Lipschitz with constant c iff $|f'(x)| \le c$ $\forall x$.

pf: If f is Lip. then $|f(x) - f(y)| \le c|x-y|$ $\forall x, y \in \mathbb{R}$

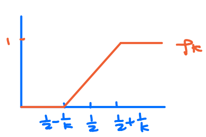
so $\frac{|f(x+h) - f(x)|}{|h|} \le c$ $\quad \forall x, h.$ $\quad (h = y - x)$

Let $h \to 0$ $\quad |f'(x)| \le c$

Conversely,
If $|f'(x)| \le c$ $\forall x \in \mathbb{R}$
let $a, b \in \mathbb{R}$, then $\frac{f(b) - f(a)}{b-a} = f'(x)$ $\quad$ [$\exists$ x between a, b MVT]

thus $\frac{|f(b) - f(a)|}{|b-a|} \le |f'(x)| \le c$.

ex.



$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$ $\quad g(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

$f = g$ except at $x = 1/2$

then $\int_0^1 |f_k - f|\,dx \to 0$, $\int_0^1 |f_k - g|\,dx \to 0$.

f, g seem to be non-unique limits,

But $f, g \notin C([0,1], \mathbb{R})$

9/23.

Thm. let X be a complete metric space, and $T: X \to X$ a contraction.

$\exists\ 0 \le c < 1$ s.t $d(T(p), T(q)) \le c\, d(p, q)$ $\forall p, q \in X$.

Thm: $\exists!\ P \in X$ s.t $T(P) = P$, p is called a fixed point of T

pf. uniqueness:

let $p, q \in X$ s.t $T(P) = P$. $T(q) = q$.

Have to show $p = q$

$d(T(P), T(q)) = d(p, q) \le c\, d(p, q)$

$(1-c)\, d(p, q) \le 0$

$1 - c > 0$ so $d(p, q) = 0$, $p = q$

Existence:

start with any $P_1 \in X$.

Define $P_2 = T(P_1)$ ..... $P_{k+1} = T(P_k)$

claim $\{P_k\} \to P$ and $T(P) = P$.

Notice $d(P_3, P_2) = d(T(P_2), T(P_1)) \le c\, d(P_2, P_1)$

$d(P_4, P_3) = d(T(P_3), T(P_2)) \le c\, d(P_3, P_2) \le c^2 d(P_2, P_1)$

$d(P_{k+1}, P_k) \le c\, d(P_k, P_{k-1}) \le \cdots \le c\, d(P_2, P_1)$ prove by induction

To show $\{P_k\}$ is Cauchy, we need $d(P_{k+1}, P_k) \to 0$ as $k \to \infty$

But not enough to granntee that $\{P_k\}$ is Cauchy.

Counterexample in $\mathbb{R}$:

× ① $P_k = k + \frac{1}{k}$

$P_{k+1} - P_k = (k+1) + \frac{1}{k+1} - (P_k + \frac{1}{k}) \to 1$

② $P_k = \sum_{i=1}^{k} \frac{1}{i} \to \infty$

But $P_{k+1} - P_k = \frac{1}{k+1} \to 0$

To show P_k is Cauchy, we have to show $d(P_{k+l}, P_k) \to \infty$ as $k \to \infty$ uniformly in P.

$d(P_{k+l}, P_k) \le d(P_{k+l}, P_{k+l-1}) + d(P_{k+l-1}, P_k)$

$\le d(P_{k+l}, P_{k+l-1}) + d(P_{k+l-1}, P_{k+l-2}) + \cdots + d(P_{k+1}, P_k)$

$= (c^{k+l-2} + \cdots + c^{k-1})\, d(P_2, P_1)$

$\le (c^{k-1} \sum_{i=0}^{\infty} c^i)\, d(P_2, P_1) = \frac{c^{k-1}}{1-c} d(P_2, P_1) \to 0$

as $k \to \infty$ uniformly in P. so $\{P_k\}$ is Cauchy

since X is complete, $\exists\ P \in X$ s.t $\{P_k\} \to P$

Now will show $T(P) = P$

If $\{P_k\} \to P$, then

① $\{T(P_k)\} \to T(P)$ (Lipschitz maps are continuous)

$0 \le d(T(P_k), T(P)) \le c\, d(P_k, P) \to 0$

② $\{T(P_k)\} = \{P_{k+1}\} \to P$

①+②. $\{T(P_k)\} \to T(P)$.

$\{T(P_k)\} \to P$

since Limits are unique in a metric space, so $T(P) = P$

Applications :

Let O be an open set in $\mathbb{R}^2$, $g: O \to \mathbb{R}$, $(x_0, y_0) \in O$, continuous.
Let $f: I \to \mathbb{R}$, $f(x_0) = y_0$, then the following are equivalent.

1) f is differentiable on I, and $\begin{cases} f'(x) = g(x, f(x)) & \forall x \in I \\ f(x_0) = y_0 \end{cases}$

2) f is continuous on I, and $f(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt$

pf. Assume 2) holds,
then $t \longmapsto g(t, f(t))$ is continuous, $t \in I$ (composition of continuous function)
so $x \longmapsto \int_{x_0}^{x} g(t, f(t))\, dx$ is differentiable,
and $\frac{d}{dx}\left[y_0 + \int_{x_0}^{x} g(t, f(t))\, dt\right] = g(x, f(x))$
and $y_0 + \int_{x_0}^{x} g(t, f(t))\, dt$ elvatuated at $x = x_0$ is y_0
thus $f(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt$ is diff and satisfies 1)
Conversely, Assume 1) holds.
so f is diff and $f'(x) = g(x, f(x))$, $f(x_0) = y_0$.
then f is continuous, diff $\Rightarrow$ continuous.
and f' is continuous $= g(x, f(x))$.
Integrate $f'(x)$.
$$\int_{x_0}^{x} f'(x)\, dt = \int_{x_0}^{x} g(t, f(t))\, dt$$
$$= f(x) - f(x_0) = f(x) - y_0$$
so $f(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt$.

If g satisfies additional conditions, the equation $\begin{cases} f'(x) = g(x, f(x)) \\ f(x_0) = y_0 \end{cases}$

has unique solution on some time interval
i.e. $f(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt$ has unique solution on some I.

Thm: Let O open in $\mathbb{R}^2$, $g: O \to \mathbb{R}$ continuous, assume that $\exists M$
s.t $|g(x_1, y_1) - g(x_2, y_2)| \le M |y_1 - y_2|$
Let $(x_0, y_0) \in O$.
then $\exists I$ open interval containing x_0.
s.t $\begin{cases} f'(x) = g(x, f(x)) \\ f(x_0) = y_0 \end{cases}$ has a unique solution in I.

9/25

Thm: Let $(x_0, y_0) \in \mathbb{R}^2$, $g: \overbrace{[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]}^{B} \to \mathbb{R}$
g is continuous, and assume $\exists M \ge 0$
s.t $|g(x, y_1) - g(x, y_2)| \le M |y_1 - y_2|$ $\forall (x, y_1), (x, y_2) \in B$
then $\exists \delta_0 > 0$ s.t $0 \le l \le \delta_0$
the equation $f'(x) = g(x, f(x))$
$f(x_0) = y_0$
has a unique solution $f: [x_0 - l, x_0 + l] \to [y_0 - b, y_0 + b]$

pf: Let $X_l = \{ f: [x_0 - l, x_0 + l] \to [y_0 - b, y_0 + b],\ f \text{ continuous} \}$
will show if $0 < l \le \delta_0$ effi small.
the integral equation $f(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt$ has a unique solution in X_l
To do this, define
$$T(f)(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt$$
show $\exists \delta_0 > 0$ s.t if $0 < l \le \delta_0$ $T(f): X_l \to X_l$ and T is a contraction

once we prove this, the contraction principle implies $\exists !$ fixed point f.

(since X_ℓ is a complete metric space)

Recall we are using the metric in X_ℓ

$$d(f,g) = \max_{|x-x_0|\le \ell} |f(x)-g(x)|$$

To insure $T: X_\ell \to X_\ell$, let $f \in X_\ell$,

$$T(f)(x) = y_0 + \int_{x_0}^{x} g(t,f(t))\,dt \quad \text{is continuous}$$

Check the range of $T(f)$ $(x\in[x_0-\ell, x_0+\ell])$ is $[y_0-\ell, y_0+\ell]$

$$|T(f)(x) - y_0| = \left|\int_{x_0}^{x} g(t,f(t))\,dt\right|$$
$$\le |x-x_0| \max_{x\in[x_0-\ell,\, x_0+\ell]} |g(t, f(t))\,dt|$$

since g is continuous on the box $B = [x_0-a, x_0+a]\times[y_0-b, y_0+b]$, $\exists\, c>0$

s.t. $|g(t,f(t))| \le c \quad \forall t \in [x_0-\ell, x_0+\ell]$

thus if $x\in[x_0-\ell, x_0+\ell]$

$$|T(f)(x) - y_0| = \left|\int_{x_0}^{x} g(t,f(t))\,dt\right|$$
$$\le |x-x_0| \max_{x\in[x_0-\ell,\, x_0+\ell]} |g(t, f(t))\,dt| \le |x-x_0|\, c \le \ell\cdot c$$

thus we require $\ell\cdot c \le b$ i.e. $\delta_0\cdot c \le b$.

$T: X_\ell \to X_\ell$ ✓

To insure $T: X_\ell \to X_\ell$ is a contraction, let $f_1, f_2 \in X_\ell$

Let $0 \le c < 1$

$$d(T(f_1), T(f_2)) = \max_{x\in[x_0-\ell,\, x_0+\ell]} |T(f_1)(x) - T(f_2)(x)| \le c\, d(f_1,f_2) = \max_{x\in[x_0-\ell,\, x_0+\ell]} |f_1(x)-f_2(x)|$$

look at

$$T(f_1)(x) - T(f_2)(x) = \int_{x_0}^{x} [g(t,f_1(t)) - g(t,f_2(t))]\,dt$$
$$|T(f_1)(x) - T(f_2)(x)| \le |x-x_0|\, M \max_{x\in[x_0-\ell,\, x_0+\ell]} |f_1(t)-f_2(t)|$$

we need.

$$|g(t,f_1(t)) - g(t,f_2(t))| \le M \max_{t\in[x_0-\ell,\, x_0+\ell]} |f_1(t)-f_2(t)|\cdot \ell$$

thus

$$|T(f_1)(x) - T(f_2)(x)| \le |x-x_0|\, M \max_{x\in[x_0-\ell,\, x_0+\ell]} |f_1(t)-f_2(t)| \le \ell\cdot M\cdot \max_{x\in[x_0-\ell,\, x_0+\ell]} |f_1(t)-f_2(t)|$$

for all $x\in[x_0-\ell, x_0+\ell]$

thus

$d(T(f_1), T(f_2)) \le \ell\cdot M\cdot d(f_1,f_2)$ impose a second condition on ℓ,

$c = \ell\cdot M < 1 \quad (\delta_0\cdot M < 1)$

thus.

If $\delta_0 c \le b$ and $\delta_0 M < 1$, and $0 \le \ell \le \delta_0$, $T: X_\ell \to X_\ell$ is a contraction,

and $\exists !$ fixed point which solves the integral equation.

Ex: $g: \mathbb{R}^2 \to \mathbb{R}$, look at $f'(x) = g(x, f(x))$

$f(x_0) = y_0$

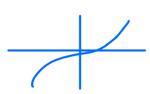
A global solution may not exist

$f'(x) = f^2(x)$, $f(0) = 1$

solution : $f(x) = \dfrac{1}{1-x}$ exists on $(-\infty, 1)$

If g is not Lipschitz continuous, the solution may not be unique

ex: $f(t) = t^3$



$f'(t) = 3t^2 = 3f(t)^{\frac{2}{3}}$

$$\begin{cases} f'(t) = 3f(t)^{\frac{2}{3}} \\ f(0) = 0 \end{cases}$$

has $f(t) = t^3$ one solution, $f(t) = 0 \;\forall t$

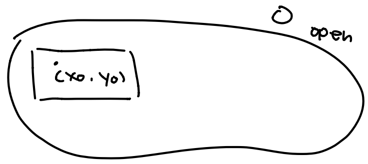
All $f(t) = \begin{cases} 0 & \text{if } t \le c \\ (t-c)^3 & \text{if } t \ge c \end{cases} \quad (c>0)$

there are infinite many solutions. and $g(y) = 3y^{\frac{2}{3}}$ not Lipschitz cont.

$g'(y)$ blows up as $y \to 0$

However if g is Lipschitz, continuous, uniqueness holds globally as long as sol. exists

In the textbook, $g: O \to \mathbb{R}$.



the first step in the proof is to find $a>0$, $b>0$ s.t $[x_0-a, x_0+a] \times [y_0-b, y_0+b] \subseteq O$

By using the contraction principle to prove uniqueness, we only proved uniqueness for $f \in X_\ell$, $f: [x_0-\ell, x_0+\ell] \to [y_0-b, y_0+b]$

9/28.

Uniqueness for solutions to an IVP is a global property.

Thm: Let O open in $\mathbb{R}^2$, $g: O \to \mathbb{R}$ continuous, assume $\exists M \geq 0$ s.t $|g(t,y_1) - g(t,y_2)| \leq M|y_1 - y_2|$

let (t_0, y_0), let f_1, f_2 differentiable: $(t_0-a, t_0+b) \to \mathbb{R}$

s.t $(t, f_1(t)) \in O$ $\quad (t, f_2(t)) \in O \quad \forall t \in (t_0-a, t_0+b)$

and $f_1'(t) = g(t, f_1(t)) \quad f_2'(t) = g(t, f_2(t)) \quad \forall t \in (t_0-a, t_0+b)$

Initial conditions $f_1(t_0) = f_2(t_0) = y_0$

then $f_1(t) = f_2(t) \quad \forall t \in (t_0-a, t_0+a)$

pf. Will show $f_1(t) = f_2(t) \quad \forall t \in [t_0, t_0+b)$

(the argument can be adopted to $(t_0-a, t_0])$)

Define $E(t) = (f_1(t) - f_2(t))^2$ will setup a differentiable inequality for $E(t)$

$$
\begin{aligned}
E'(t) &= 2(f_1(t) - f_2(t)) \cdot (f_1'(t) - f_2'(t)) \\
&= 2(f_1(t) - f_2(t)) [g(t, f_1(t)) - g(t, f_2(t))] \\
&\leq 2(f_1(t) - f_2(t)) |g(t, f_1(t)) - g(t, f_2(t))| \\
&\leq 2M |f_1(t) - f_2(t)|^2 \\
&= 2M E(t)
\end{aligned}
$$

thus $E'(t) - 2M E(t) \leq 0$

Equivalent (using the integrating fraction e^{-2Mt})

$\frac{d}{dt}[e^{-2Mt} E(t)] \leq 0$

$[-2M E(t) + E'(t)] e^{-2M(t)} \leq 0$

thus $e^{-2Mt} E(t)$ is a monotonoly decreasing function.

At 0, its value is 0 and $E(t) = (\quad)^2 \geq 0 \quad \forall t$

thus $e^{-2Mt} E(t) = 0 \quad \forall t \in [t_0, t_0+b)$

and $f_1(t) = f_2(t)$ in that interval

Existence may no be global.

ex. $f'(t) = f^2(t)$, $f(0) = 1$ has solution $f(t) = \frac{1}{1-t}$ in $(-\infty, 1)$

Thm: Let $g: \mathbb{R}^2 \to \mathbb{R}$, continuous, Assume $\exists K, M$ s.t $|g(t,y)| \leq K \quad \forall (t,y) \in \mathbb{R}^2$

and $|g(t,y_1) - g(t,y_2)| \leq M|y_1 - y_2| \quad \forall (t,y_1), (t,y_2) \in \mathbb{R}^2$

then $\forall (t_0, y_0) \in \mathbb{R}^2$, $\exists$ global solution $f: (-\infty, \infty) \to \mathbb{R}$ diff

$$
* \begin{cases} f'(t) = g(t, f(t)) \\ f(t_0) = y_0 \end{cases}
$$

pf: Recall $\exists \ell$ (depending only on K and M)

s.t a solution to * exists on $(t_0 - \ell, t_0 + \ell)$

Show same ℓ works for all points $(t_1, y_1) \in \mathbb{R}^2$

Look at all intervals (a, b), $t_0 \in (a, b)$ such that a solution to * in (a, b)

If $\{a \mid$ such (a, b) exists$\}$ is unbounded below, then the solution exists in $(-\infty, t_0]$

If $\{b \mid$ such (a, b) exists$\}$ is unbounded above, the solution exists in $[t_0, \infty)$

Assume by contradiction $\{b \mid$ solution exists in $[t_0, b)\}$ is bounded above.

let $b_{sup} = \sup\{b \mid$ solution in $[t_0, b)\} < \infty$

then solution exists in $[t_0, b_{sup} - \frac{\ell}{4})$ (ℓ is given by the local existence thm)

Apply our local existence thm, to the "initial time" $b_{sup} - \frac{\ell}{2}$,

with initial condition $f(b_{sup} - \frac{\ell}{2})$

the local existence thm insure that the solution exists on $[b_{sup} - \frac{\ell}{2}, \underbrace{b_{sup} - \frac{\ell}{2} + \ell}_{> b_{sup}})$

Contradiction

ex: $* \begin{cases} f'(t) = \sin(t) + \sin(f(t)) \\ f(0) = 1 \end{cases}$

Here $g(t, y) = \sin(t) + \sin(y)$

$|g(t, y)| \leq 2 \quad \forall (t, y) \in \mathbb{R}^2$

$|g(t, y_1) - g(t, y_2)| \leq |y_1 - y_2|$

hence * has solution in $(-\infty, \infty)$

In general, if $g: \mathbb{R}^2 \to \mathbb{R}$ has partial dirivite with respect to y. $\forall y_1, y_2 \in \mathbb{R}$

$|g(t, y_1) - g(t, y_2)| = \frac{\partial g}{\partial y}(t, z) \, |y_1 - y_2|$ $\quad \exists z$ between y_1 and y_2

thus, if $\exists M$ s.t $\left|\frac{\partial g}{\partial y}(t, y)\right| \leq M$ $\quad \forall (t, y) \in \mathbb{R}^2$

then M works as a Lipschitz constant

Corollary: If $g: \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivative,

then for any bounded open set O.

$\exists M$. s.t $|g(t, y_1) - g(t, y_2)| \leq M |y_1 - y_2|$

9/30.

8) $X = \{ f \in C([0,1],\mathbb{R}),\ |f(x)| \le 1 \}$

$T: X \to C([0,1],\mathbb{R})$, defined by $(T(f))(x) = \int_0^x \cos(f(t))\,dt$

a) is T a contraction?

pf: let $f, g : [0,1] \to [-1,1]$ continuous

$d(T(f), T(g)) = \max_{x\in[0,1]} \left| \int_0^x [\cos(f(t)) - \cos(g(t))]dt \right|$

$\le \max_{x\in[0,1]} \int_0^x |\cos f(t) - \cos g(t)| dt$

$= \int_0^1 |\cos(f(t)) - \cos(g(t))|\, dt$

If $f(t), g(t) \in [-1,1]$. $\exists\, \theta$ between $f(t)$ and $g(t)$, $\theta \in [-1,1]$

then $\cos(f(t)) - \cos(g(t)) = -\sin(\theta)[f(t) - g(t)]$

thus $|\cos(f(t)) - \cos(g(t))| \le \underset{0<\sin 1<1}{\sin(1)} \max_{t\in[0,1]} |f(t) - g(t)| = \underset{c}{\sin(1)}\, d(f,g)$

thus $d(T(f), T(g)) \le \int_0^1 |\cos(f(t)) - \cos(g(t))|\, dt \le \sin(1) \int_0^1 d(f,g)\,dt = \sin(1)\, d(f,g)$

Yes T is a contraction, domain of f, g is $[0,1]$

think. if $X = \{ f \in C([0,10],\mathbb{R}),\ |f(x)| \le 1 \}$, $T f(x) = \int_0^x \cos(f(t))\,dt$,

is T a contraction? find a counterexample.

let $f, g : [0,10] \to [-1,1]$, continuous.

s.t $d(T(f), T(g)) \ge d(f,g)$, f, g constant function works?

try $f: [0,10] \to [-1,1]$ $\quad f(t) = 0\ \ \forall t$, $\cos(f(t)) = 1$

$T f(x) = \int_0^x 1\, dt = x$

try $g: [0,10] \to [-1,1]$, $\quad g(x) = \frac{\pi}{4}$, $\cos(g(t)) = \sqrt{2}/2$

$T g(x) = \int_0^x \sqrt{2}/2\, dt = \frac{\sqrt{2}}{2} x$

$d(f,g) = \frac{\pi}{4}$

$d(T(f), T(g)) = \max_{x\in[0,10]} \left| x - \frac{\sqrt{2}}{2} x \right| = 10 \left| 1 - \frac{\sqrt{2}}{2} \right| \approx 10 \frac{3}{10} \approx 3 \gg \frac{\pi}{4}\ d(f,g)$

so not contraction

b) Assume f is a fixed point of T. (i.e $f(x) = \int_0^x \cos(f(t))\,dt$)

Write down a differential equation and initial condition satisfied by f.

$f'(x) = \cos(f(x))$

$f(0) = 0$

this is a global solution, but the range of the global solution may not be $f'(t) = g(t, f(t))$.

40) $X = \{ f \in C([0,1],\mathbb{R}),\ 0 \le f(t) \le 1 \}$, $T: X \to C([0,1],\mathbb{R})$.

$T f(x) = \int_0^x f(x)^{\frac{2}{3}}\, dt$

a) Does $T: X \to X$? ✓

b) Is T a contraction on X?

Here $g(t,y) = y^{\frac{2}{3}}$ unbounded.

a counterexample: $f = 0$, $g = 1$ $\quad d(f,g) = 1$, $Tf(x) = 0$ $\quad Tg(x) = x$

$d(Tf, Tg) = 1$ not contraction

Think $\exists\, l > 0$ s.t if $X_l = \{ f \in C([0,l],\mathbb{R}),\ 0 \le f \le 1 \}$

then $T: X_l \to X_l$ is a contraction?

No. $\quad f(x) = x$ $\qquad Tf = 0$

$g(x) = \varepsilon$ $\qquad Tg = \int_0^x \varepsilon^{\frac{2}{3}}\, dt = \varepsilon^{\frac{2}{3}} \cdot x$

$d(f,g) = \varepsilon$ $\qquad d(Tf, Tg) = l \cdot \varepsilon^{\frac{2}{3}}$ on $[0,l]$

Fix $l > 0$, small.

think $\exists\, c$ $\quad 0 \le c < 1$ s.t $\quad l \cdot \varepsilon^{\frac{2}{3}} \le c \cdot \varepsilon \quad \forall \varepsilon > 0$

try $c = 1$ $\quad \nexists\, l > 0$ s.t $\quad l \varepsilon^{\frac{2}{3}} \le \varepsilon \quad \forall \varepsilon > 0$

$\ell \le \varepsilon^{\frac{1}{3}} \quad \forall \varepsilon > 0 \Rightarrow \ell = 0$

so T not a contraction on any X_ℓ. $\ell > 0$.

A fixed point T would satisfy $f(x) = \int_0^x f(t)^{\frac{2}{3}}\,dt$

$f'(x) = f(x)^{\frac{2}{3}}, \quad f(0) = 0$

this has non-unique solutions. ($f \equiv 0$, $f = x^3$)

10/2.

Def: $K \subseteq \mathbb{R}^n$ is compact if for every cover of K the open set V_α

$K \subseteq \bigcup_\alpha V_\alpha$

there exists finitely many $V_{\alpha_1}, \dots, V_{\alpha_m}$ which cover K

$K \subseteq V_{\alpha_1} \cup \dots \cup V_{\alpha_m}$

Thm: If $K \subseteq \mathbb{R}^n$, K is compact iff K is closed and bounded iff K is sequentially compact.

① If K is compact, then K is bounded.

pf: choose $V_i = B_i(0)$, $i \in \mathbb{N}$

then $K \subseteq \mathbb{R}^n = \bigcup_{i=1}^{\infty} B_i(0)$

since K is compact, $\exists$ finite many $i_1, \dots, i_m$, $i_1 < i_2 \dots < i_m$.

s.t $K \subseteq B_{i_1}(0) \cup \dots \cup B_{i_m}(0)$

thus $\|x\| < i_m \quad \forall x \in K$, so K is bounded

② If K is compact, then K is closed.

pf: Let $\{x_i\} \subseteq K$, assume that $\{x_i\} \to x \in \mathbb{R}^n$

want to show $x \in K$. Assume by contradiction $x \notin K$

Choose $V_i = \{ y \in \mathbb{R}^n \mid \|x - y\| > \frac{1}{i} \}$

then V_i are open $\bigcup_{i=1}^{\infty} V_i = \mathbb{R}^n \setminus \{x\}$

thus $K \subseteq \mathbb{R}^n \setminus \{x\} \subseteq \bigcup_{i=1}^{\infty} V_i$

since K compact, $\exists\ i_1 < \dots < i_m$ s.t $K \subseteq V_{i_1} \cup \dots \cup V_{i_m} = V_{i_m}$

thus $\|x - y\| > \frac{1}{i_m} \quad \forall y \in K$

this contradict $\{x_j\} \subseteq K$, $\{x_j\} \to x$

③ If K is a closed subset of a compact set L, then K is compact

pf: Let V_α be a family of open sets, $K \subseteq \bigcup_\alpha V_\alpha$.

$L \nsubseteq \bigcup_\alpha V_\alpha$ but $L \subseteq \bigcup_\alpha V_\alpha \cup K^c$ cover of L by open sets

If $x \in L$ is also in K, then $x \in \bigcup_\alpha V_\alpha$

If $x \in L$ and $x \notin K$, then $x \in K^c$

Remark K^c is open

Using the componess of L. $\exists$ finite many $\alpha_1 \dots \alpha_m$.

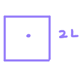
s.t $L \subseteq V_{\alpha_1} \cup \dots \cup V_{\alpha_m} \cup K^c$

thus $K \subseteq L \subseteq V_{\alpha_1} \cup \dots \cup V_{\alpha_m} \cup K^c$

but then no element of K is in K^c.

thus $K \subseteq V_{\alpha_1} \cup \dots \cup V_{\alpha_m}$, thus K compact

④ If K is closed and bounded, implies K compact.

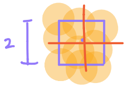


pf: Since K is bounded, $\exists$ cube C $(-L \le x_1 \le L,\ -L \le x_2 \le L \ \dots\ -L \le x_n \le L)$

s.t $K \subseteq C$. WLOG. Take $L = 1$

assume K is closed. $K \subseteq \underbrace{[-1,1] \times [-1,1]}_{C}$

To show C is compact



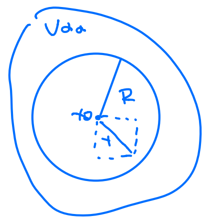
Let V_α be the cover of C by open sets,
assume by contraction, no finite set of V_α covers C
Divide C into 2^n closed subcute of side 1
Since we assume C can not be cover by finite V_α.
it follows that at least of 1 of the subcube C_1 is not cover by finite V_α
Divide C_1 int 2^n subcube of size $1/2$.
At least 1 of them will not be cover by finite V_α. . . . . .

$C \supseteq C_1 \supseteq C_2 \supseteq \cdots \supseteq C_m$ C_m not covered by finite many V_α.

size   2   1 $\frac{1}{2}$ $\frac{1}{2^{-m+1}} \to 0$

Nested family of closed cubes.
$\exists!\ x_0 \in C_i \quad \forall i$ in particular
$x_0 \in C$. and $\exists\ \alpha_0$ s.t $x_0 \in V_{\alpha_0}$
since the side of $C_m \to 0$, $x_0 \in C_m \subseteq V_{\alpha_0}$ for m suff large
thus just V_{α_0} covers C_m
which contradiction. TO

QED



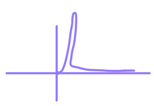
$x_0 \in V_{\alpha_0}$ open $\exists\ r > 0$ s.t $B_r(x_0) \subseteq V_{\alpha_0}$
If the digonal of C_m $Y_0 < R$
$x_0 \in C_m \Rightarrow$ $C_m \subseteq B_r(x_0)$

10/5

$\|f\| = \max_{x \in [0,1]} |f(x)|$

$\|f\|_1 = \int_0^1 |f(x)|\,dx$

Q: is the unit ball with respect to $\|\cdot\|$ open with respect to $\|\cdot\|_1$.
For sure the unit ball B is open with respect to $\|\ \|$
pick the zero function, centered in B.
$\exists ?\ r > 0$ s.t $\{f \in C([0,1], \mathbb{R}) \mid \|f\|_1 < r\} \subseteq B$?
If $r = \frac{1}{2}$, then $\int_0^1 |f|\,dx < \frac{1}{2}$



does not implies $\max |f(x)| < 1$
so $\forall r > 0$, $\exists f$ with $\|f - 0\|_1 < r$ mand $\max |f| > 1$

38. Q: $X = \{f \in C([0,1], \mathbb{R}),\ 0 \le f \le 2\}$, $Tf = \int_0^x t f^2(t)\,dt$
1) $T: X \to X$?
Let $0 \le f \le 2$. $0 \le \int_0^x t \underbrace{f^2(t)}_{\le 4}\,dt \le 1 \cdot 4$
$\hookrightarrow \int_0^x t f^2(t)\,dt \le \int_0^1 t \cdot 4\,dt = \frac{1}{2} \cdot 4 = 2$.
So Yes. $T: X \to X$.
2) Is it a contraction?
$(Tf_1 - Tf_2)(x) = \int_0^x t\,(f_1^2(t) - f_2^2(t))\,dt \le \int_0^1 t\,\underbrace{|f_1(t) - f_2(t)|}_{\le d(f_1, f_2)}\,\underbrace{|f_1(t) + f_2(t)|}_{\le 4}\,dt \le d(f_1, f_2) \int_0^1 t \cdot 4\,dt = 2\,d(f_1, f_2)$

fail to prove T is a contraction.
So try to disprove T a contraction.
let $f_1 = 0$ $f_2 = 2$
$d(f_1, f_2) = 2$, $Tf_1 = 0$, $(Tf_2)(x) = \int_0^x t \cdot 4\,dt = 2x^2$
$d(T(f_1), T(f_2)) = \max_{0 \le x \le 1} |2x^2 - 0| = 2$
So T not a contraction

In a general metric space, a closed & bounded set need not to be sequentiall compact.

ex: $X = C([0,1], \mathbb{R})$ with $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$

let $\{f_n\} \subseteq X$, $f_n(x) = x^n$

$\|f_n\| = \max_{x \in [0,1]} |x^n| = 1$

Let $K = \{ f \in X \mid \|f\| \le 1 \}$ is closed & bdd.

but $\{f_n\}$ does not have any subsequences which converges. with resp. to the given metric.

because $f_n(x) \to 0$ if $0 \le x < 1$ $\quad f_n(x) \to 1$ if $x = 1$

and any subsequences has the same pointwise limits.

Convergence in $X \Leftrightarrow$ uniform convergence.

If $\{f_{n_k}\}$ converges to converges uniformly, it will converge to a continuous function.

Chapter 13. — limits

Def: x^* is a limit point of $A \subseteq \mathbb{R}^n$ if $\exists \{x_k\} \subseteq A \setminus \{x^*\}$ s.t $\{x_k\} \to x^*$

Def: let $f: A \to \mathbb{R}$, let x^* be the limit point of A.

$\lim_{x \to x^*} f(x) = L$ if for any sequence $\{x_k\} \subseteq A \setminus \{x^*\}$

if $\{x_k\} \to x^*$ then $\{f(x_k)\} \to L$

ex: $f: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$, $f(x,y) = \dfrac{xy}{x^2+y^2}$

$\lim_{(x,y) \to (0,0)} f(x,y)$ DNE

Because:

$\lim_{(\frac{1}{k}, 0) \to (0,0)} f(\frac{1}{k}, 0) = 0$

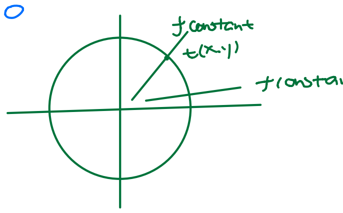
$\lim_{(\frac{1}{k}, \frac{1}{k}) \to (0,0)} f(\frac{1}{k}, \frac{1}{k}) = \dfrac{\frac{1}{k^2}}{\frac{2}{k^2}} = \frac{1}{2}$

Since limits are unique, so the limit DNE

Def: $f: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is homogeneous of degree k.

if $f(tx) = t^k f(x)$ $\quad \forall t > 0$, $\forall x \in \mathbb{R}^n \setminus \{0\}$

note that $f(x,y) = \dfrac{xy}{x^2+y^2}$ $\quad \dfrac{(tx)(ty)}{(tx)^2 + (ty)^2} = \dfrac{xy}{x^2+y^2}$ is homogeneous of deg 0



Remark: If $f: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is homog. of deg 0. and $\lim_{x \to 0} f(x)$ exists,

then f is constant $= \lim_{x \to 0} f(x)$

ex: $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$ $\quad |f(x,y)| \le \left|\dfrac{x^2}{x^2+y^2}\right| |y|$

as $(x,y) \to (0,0)$. $\quad f(x,y) \to 0$

$\lim_{(x,y) \to (0,0)} f(x,y) = 0$

Generlized: let $f \in C(\mathbb{R}^n \setminus \{0\})$ homog. of deg k, $k > 0$.

then $\lim_{x \to 0} f(x) = 0$

let S^{n-1} = unit shere in $\mathbb{R}^n$

$S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$ is seq. compact.

then $\exists\ c$ s.t $|f(\hat{x})| \le c$ $\quad \forall \hat{x} \in S^{n-1}$

then let $x \ne 0$

$|f(x)| = |f(\underbrace{\|x\|}_{t} \cdot \underbrace{\tfrac{x}{\|x\|}}_{\hat{x}})| = \|x\|^k |f(\hat{x})| \le \|x\|^k c \to 0$

10/9

3. $f_k(x) = x^k \in C([0,1], \mathbb{R})$ $d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$

f_k is not Cauchy

Pf: by contradiction. $\{f_k\}$ is not Cauchy.

then $\{f_k\} \to f$ continuous $(C([0,1],\mathbb{R}),$ with d is complete.

but $\{f_k\} \to \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

pointwise discontinuous.

Direct proof : $\forall k$

$d(f_k, f_{2k}) \geq |f_k(\frac{1}{2^{\frac{1}{k}}}) - f_{2k}(\frac{1}{2^{\frac{1}{k}}})| = (\frac{1}{2^{\frac{1}{k}}})^k - (\frac{1}{2^{\frac{1}{k}}})^{2k} = |\frac{1}{2} - \frac{1}{4}| = \frac{1}{4}$

$d(f,g) = \int_0^1 |f(x) - g(x)|\,dx$, $f_k(x) = x^k$ is Cauchy

f_k convergen to $0 \in C([0,1], \mathbb{R})$ with respect to d,

$\int_0^1 (x^k - 0)\,dx = \frac{1}{k+1} \to 0$

4. $X = \{f: [0,\frac{1}{2}] \to [0,2]\}$ with $d(f,g) = \max_{x\in[0,\frac{1}{2}]} |f(x) - g(x)|$, $Tf(x) = 1 + \int_0^x f(t)\,dt$.

$0 \le Tf(x) \le 2$ $\forall x \in [0,\frac{1}{2}]$

$1 \le Tf(x) = 1 + \int_0^x f(t)dt \le 1 + \int_0^{\frac{1}{2}} f(t)\,dt \le 1 + 1$

Yes T a contraction.

$d(Tf, Tg) = \max_{x\in[0,\frac{1}{2}]} |\int_0^x (f(t) - g(t))\,dt| \le \int_0^{\frac{1}{2}} \max_{x\in[0,1]} |f(x) - g(t)|\,dt = \frac{1}{2} d(f,g)$

Fixed point. $f(x) = 1 + \int_0^x f(t)\,dt$

$\begin{cases} f'(x) = f(x) \\ f(0) = 1 \end{cases}$

$\Rightarrow f(x) = e^x$

d. $f_1 = 1$

$f_2 = T(f_1) = 1 + \int_0^x 1\,dt = 1 + x$

$f_3 = T(f_2) = 1 + \int_0^x t\,dt = 1 + x + \frac{1}{2}x^2$

$f_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$ Taylor polynomial of e^x

3a. f_k Cauchy :

$\forall \varepsilon > 0$ $\exists K$ s.t $d(f_k, f_l) < \varepsilon$ $\forall k, l \geq K$

Negation :

$\exists \varepsilon > 0$ s.t $\forall K$, $\exists k, l \geq K$ s.t $d(f_k, f_l) \geq \varepsilon$

$\exists \varepsilon > 0$ s.t $\forall K$, $\exists k \geq K$, $\exists l \geq 1$ s.t $d(f_k, f_{k+l}) \geq \varepsilon$

10/9 13.2

Let $\mathcal{O}$ open in $\mathbb{R}^n$, $x \in \mathcal{O}$

$\frac{\partial f}{\partial x_i}(x) = \lim_{t \to 0} \frac{f(x+te_1) - f(x)}{t}$ provided the limit exists

ex: $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Recall $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

However $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exists at all $(x,y) \neq (0,0)$

Also, $\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{\overset{0}{f(t,0)} - \overset{0}{f(0,0)}}{t} = 0 = \frac{\partial f}{\partial y}(0,0)$$

So the first order partical diretives exists at all x and y, but f is not continuous at $(0,0)$

Later will see that if all $\frac{\partial f}{\partial x_i}(x)$ exists and continuous $\forall x$, $\forall i \in \{1,\dots,n\}$ then f is continuous

Def: Let $f: \mathcal{O} \to \mathbb{R}$, $\mathcal{O}$ open in $\mathbb{R}^n$. if all $\frac{\partial f}{\partial x_i}(x)$ exists and are continuous, f is called continnouslly differentiable $f \in C'(\mathcal{O})$

Later will see C' function are continnous $C(\mathcal{O})$, $C'(\mathcal{O}) \subseteq C(\mathcal{O})$

Def: $f: \mathcal{O} \to \mathbb{R}$ is C^2 if all $\frac{\partial f}{\partial x_i}(x)$ exists $\forall x \in \mathcal{O}$ and $i \in \{1,\dots,n\}$ and all $\frac{\partial}{\partial x_i}\left[\frac{\partial f}{\partial x_j}\right](x)$ exists and are continuous.

Later will see $C^2(\mathcal{O}) \subseteq C'(\mathcal{O}) \subseteq C(\mathcal{O})$

Thm: If $f \in C^2(\mathcal{O})$, then $\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(x) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(x)$

mixed particals are equal denoted $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$

pf: WLOG. will assume $f \in C^2(\mathbb{R}^2)$ Fixed $(x_0, y_0) \in \mathbb{R}^2$

$A = f(x_0+r, y_0+r) - f(x_0+r, y_0) - f(x_0, y_0+r) + f(x_0, y_0)$

Define $\varphi: \mathbb{R} \to \mathbb{R}$, $\varphi(x) = f(x, y_0+r) - f(x, y_0)$

then $A = \varphi(x_0+r) - \varphi(x_0)$

If $\frac{\partial f}{\partial x}$ exists, φ is differentiable and $\exists |r_1| \leq |r|$

then $A = \varphi(x_0+r) - \varphi(x_0) = \varphi'(x_0+r_1) \cdot r = \frac{\partial f}{\partial x}(x+r_1, y_0+r) - \frac{\partial f}{\partial x}(x_0+r_1, y_0)$

we know $\frac{\partial f}{\partial x}(x_0+r_1, y)$ is differentiable as a function of y.

thus MVT with respect to y, $A = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right](x_0+r_1, y_0+r_2) r^2$ for some $|r_2| < r$

by symmetric in x, y, for some $|r_3|, |r_4| < r$

$A = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right](x_0+r_3, y_0+r_4) r^2$

Thus, $\exists |r_1| \dots |r_4| < r$ s.t $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x_0+r_1, y_0+r_2) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x_0+r_3, y_0+r_4)$

let $r \to 0$ then $r_1, \dots, r_4 \to 0$ using the assumption.

that $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ are continuous,

we conduct they are equal at (x_0, y_0)

ex: $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$f \notin C^2$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) \neq \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0)$

for LHS. need $\frac{\partial f}{\partial y}(x,0)$ use calculus formalus for $x \neq 0$.

10/12 13.3

Remark. recall the formula

$$\left.\frac{d}{dt}\right|_{t=0} [f(x+tp)] = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) \cdot p_i$$

essentially a particular case of the chain rule

For $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.

but $\left.\frac{d}{dt}\right|_{t=0} f(t(1,1)) = \lim_{t\to 0} \frac{f(t,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{1}{2} - 0}{0}$ DNE

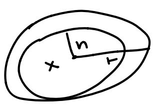
$\left.\frac{d}{dt}\right|_{t=0} [f(x+tp)] = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) \cdot p_i$ for it to be true, we must make additional assumptions on f. For instance, assume all $\frac{\partial f}{\partial x_i}(x)$ exist, and are continuous,

Recall the definition : let $f: O \to \mathbb{R}$, O open in $\mathbb{R}^n$.

If all $\frac{\partial f}{\partial x_i}(x)$ exists and are continuous $\forall x \in O$.

then $f \in C^1(O)$ (continuously differentialle)

Later today will show $f \in C^1(O) \Rightarrow f$ continuous in O



prop : Mean value theorem :

let $f: B_r(x) \to \mathbb{R}$, $B_r(x) \in \mathbb{R}^n$, assume all $\frac{\partial f}{\partial x_i}$ exist.

let $\|h\| < r$. then $\exists z_1, \dots, z_n$, $\|x - z_i\| < \|h\|$

s.t $f(x+h) - f(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(z_i) h_i$

pf: Assume $n = 2$.

$f(x_1+h_1, x_2+h_2) - f(x_1, x_2)$

$= f(x_1+h_1, x_2+h_2) - f(x_1, x_2+h_2) + f(x_1, x_2+h_2) - f(x_1, x_2)$

By the one dimentional MVT.

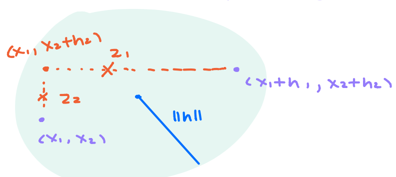
$\exists\ 0 < \theta_1 < 1$, s.t $f(x_1+h_1, x_2+h_2) - f(x_1, x_2+h_2) = \frac{\partial f}{\partial x_1}\underbrace{(x_1+\theta_1 h_1, x_2+h_2)}_{z_1} h_1$

Similiary

$\exists\ 0 < \theta_2 < 1$. s.t $f(x_1, x_2+h_2) - f(x_1, x_2) = \frac{\partial f}{\partial x_2}\underbrace{(x_1, x_2+\theta_2 h_2)}_{z_2} h_2$

Thus $f(x_1+h_1, x_2+h_2) - f(x_1, x_2) = \frac{\partial f}{\partial x_1}(z_1) h_1 + \frac{\partial f}{\partial x_2}(z_2) h_2$

QED



Def : let $f: \mathbb{R}^n \to \mathbb{R}$, let $x \in \mathbb{R}^n$, $p \neq 0$, $p \in \mathbb{R}^n$

$\frac{\partial}{\partial p} f(x) = \lim_{t\to 0} \frac{f(x+tp) - f(x)}{t} = \left.\frac{d}{dt}\right|_{t=0} f(x+tp)$ provided the limit exists

this may also be denoted $\nabla_p f(x)$

Def: $\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)$

Thm: let $f:\mathbb{R}^n \to \mathbb{R}$, assume $\frac{\partial f}{\partial x_i}$ exists $\forall$ pts in $\mathbb{R}^n$,
assume they are cont. at some $x \in \mathbb{R}^n$. let $p \neq 0$.
Then $\frac{d}{dt}\Big|_{t=0} f(x+tp)$ exists and $\frac{d}{dt}\Big|_{t=0} f(x+tp) = \langle \nabla f(x), p \rangle = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) p_i$

pf. look at $\underbrace{f(x+tp)}_{k} - f(x)$, by MVT $\exists z_1, \dots z_n$ s.t $\|x - z_i\| < \|tp\|$

s.t $\frac{f(x+tp) - f(x)}{t} = \frac{\sum_{i=1}^n \frac{\partial f}{\partial x_i}(z_i)\cancel{t}p_i}{\cancel{t}}$

let $t \to 0$. if $\frac{\partial f}{\partial x_i}$ are continuous at x, since $z_i \to x$ as $t \to 0$.

we get $\lim_{t\to 0} \frac{f(x+tp) - f(x)}{t} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) p_i$

Thm: MVT: let $f:\mathbb{R}^n \to \mathbb{R}$, assume $f \in C^1$ (i.e. all $\frac{\partial f}{\partial x_i}$ exists and cont. in $\mathbb{R}^n$)
Given $x, h \in \mathbb{R}^n$, $h \neq 0$. $\exists\ 0 < \theta < 1$
s.t $f(x+h) - f(x) = \langle \nabla f(x+\theta h), h \rangle$

pf: let $\varphi: \mathbb{R} \to \mathbb{R}$, $\varphi(t) = f(x+th)$, φ is differentiable,
then $\varphi(1) = f(x+h)$
$\varphi(0) = f(x)$
$f(x+h) - f(x) = \varphi(1) - \varphi(0)$
$\frac{d\varphi}{dt}(t) = \frac{d}{dt} f(x+th) = \langle \nabla f(x+th), h \rangle$
By the 1 dimentional MVT, $\exists\ 0 < \theta < 1$ s.t $\varphi(1) - \varphi(0) = \varphi'(\theta) \cdot (1-0)$
thus $f(x+h) - f(x) = \langle \nabla f(x+\theta h), h \rangle$

Remark: let $\|p\| = 1$, let $f: \mathbb{R}^n \to \mathbb{R}$, $f \in C^1$, $\nabla f(x) \neq 0$
the direction p for which the rate of change (in the direction of p)
$\frac{d}{dt}\Big|_{t=0} f(x+tp)$ is largest is $p = \nabla f(x) / \|\nabla f(x)\|$

pf: $\frac{d}{dt}\Big|_{t=0} f(x+tp) = \langle \nabla f(x), p \rangle \leq \|\nabla f(x)\| \cdot \|p\| = \|\nabla f(x)\|$

the inequality becomes equality if $p = \nabla f(x) / \|\nabla f(x)\|$
so $\left\langle \nabla f(x), \frac{\nabla f(x)}{\|\nabla f(x)\|} \right\rangle = \frac{\|\nabla f(x)\|^2}{\|\nabla f(x)\|} = \|\nabla f(x)\|$

Thm: let $f:\mathbb{R}^n \to \mathbb{R}$, assume $\frac{\partial f}{\partial x_i}$ exists and are cont. then f is cont. on $\mathbb{R}^n$
pf: $f(x+h) - f(x)$ with $h \to 0$ $\exists\ 0 < \theta < 1$
$|f(x+h) - f(x)| = |\langle \nabla f(x+\theta h), h \rangle|$
$\leq \|\nabla f(x+\theta h)\| \cdot \|h\|$
let $h \to 0$, then $\|\nabla f(x+\theta h)\| \to \|\nabla f(x)\|$ & $\|h\| \to 0$
thus $\lim_{h\to 0} |f(x+h) - f(x)| = 0$, f is continuous at x.

Ch. 14

Thm: First order approximation.
let $f: \mathbb{R}^n \to \mathbb{R}$, $f \in C^1$, then $\lim_{h\to 0} \frac{f(x+h) - f(x) - \langle \nabla f(x), h \rangle}{\|h\|} = 0$

Remark: $|f(x+h) - f(x) - \langle \nabla f(x), h \rangle|$ approximate 0 faster than $\|h\|$,
$|f(x+h) - f(x) - \langle \nabla f(x), h \rangle| = o(\|h\|)$

pf. $\exists\, \theta \in (0,1)$ s.t. $\left|\frac{f(x+th)-f(x)-\langle \nabla f(x), h\rangle}{\|h\|}\right|$

$$= \left|\frac{\langle \nabla f(x+\theta h), h\rangle - \langle \nabla f(x), h\rangle}{\|h\|}\right|$$

$$= \left|\frac{\langle \nabla f(x+\theta h) - \nabla f(x), h\rangle}{\|h\|}\right|$$

$\leq \quad \|\nabla f(x+\theta h) - \nabla f(x)\| \to 0$ as $h \to 0$ because $f \in C^1$

10/14

Recall if $f: \mathbb{R}^n \to \mathbb{R}$, is C^1 then the first order approximation formulas holds

$$\lim_{x\to x_0} \frac{f(x) - [f(x_0) + \langle \nabla f(x_0), x - x_0\rangle]}{\|x - x_0\|} = 0$$

Remark $f(x_0) + \langle \nabla f(x_0), (x-x_0)\rangle = f(x_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0)(x_i - (x_0)_i)$

the first order Taylor formula of f.

Geometry of the graph $z = f(x,y)$, $n = 2$

$G = \{(x, y, f(x,y)\}$

$z = f(x_0, y_0) + \langle \nabla f(x_0, y_0), (x - x_0)(y - y_0)\rangle$ is the tangent plane to the graph of f.

To find tangent directions to the graph of f, consider the following parametrized curves,

$\gamma_1 : \mathbb{R} \to \mathbb{R}^3$, $\gamma_1(t) = (x_0 + t, y_0, f(x_0 + t, y_0))$, $\gamma_1(t) \in G \ \forall t$

$\frac{d}{dt}\gamma_1(0) = (1, 0, \frac{\partial f}{\partial x}(x_0, y_0)) = T_1$ is a tangent direction. at $(x_0, y_0, f(x_0, y_0)) \in G$

$\gamma_2 : \mathbb{R} \to \mathbb{R}^3$, $\gamma_2(t) = (x_0, y_0 + t, f(x_0, y_0 + t))$

$\gamma_2'(0) = (0, 1, \frac{\partial f}{\partial y}(x_0, y_0)) = T_2$ another tangent direction to G at $(x_0, y_0, f(x_0, y_0))$

T_1, T_2 form a basis for the tangent plane at $(x_0, y_0, f(x_0, y_0))$

N, a normal (perpendicular) to T_1 T_2 is given by

$N = (\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0), -1)$

14.2

Let A be a symmetric matrix $n \times n$ matrix. $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}$ $a_{ij} = a_{ji}$

Define matrix times vector

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & & a_{nn}\end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n\end{pmatrix} = \begin{pmatrix} \langle \text{row } 1, x\rangle \\ \vdots \\ \langle \text{row } n, x\rangle\end{pmatrix}$$

The quadratic formula associated with A is

$Q = \langle Ax, x\rangle = \sum_{ij=1}^n a_{ij}x_i x_j = a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots$

ex: If $f: \mathbb{R}^n \to \mathbb{R}$ is C^2

$$\nabla^2 f(x_0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \cdots & & \frac{\partial^2 f}{\partial x_n^2}(x)\end{pmatrix}$$ is symmetric

Prop: let $f: \mathbb{R}^n \to \mathbb{R}$, C^2, let $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $h \neq 0$.

then

$$\frac{d}{dt}[f(x+th)] = \langle \nabla f\langle x+th\rangle, h\rangle$$
$$\frac{d^2}{dt^2}[f(x+th)] = \langle \nabla^2 f\langle x+th\rangle h, h\rangle = \sum_{ij=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x+th)h_i h_j$$

pf: $\frac{d}{dt}\left(\frac{d}{dt} f(x+th)\right) = \sum_{i=1}^n \frac{d}{dt}\left(\frac{\partial f}{\partial x_i}(x+th)h_i\right) = \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(x+th)h_j\right)h_i = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x+th)h_i h_j$

Remark: By the same argument, if $f: \mathbb{R}^n \to \mathbb{R}$ is C^k $(k \in \mathbb{N})$

then $\frac{d^k}{dt^k}[f(x+th)] = \sum_{i_1,\dots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} (f(x+th))\, h_{i_1} \cdots h_{i_k}$

Def: If A is $n \times n$ matrix, not necessary symmetic

define $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

define $\|A\| = \sqrt{\sum_{ij=1}^{n} a_{ij}^2} = \sqrt{\|\text{row}1\|^2 + \cdots + \|\text{row}\, n\|^2}$

Prop: Generalized Cauchy-Schwarz Inequality: $\|Ax\| \le \|A\|\|x\|$

pf: $Ax = \begin{pmatrix} \langle \text{row}1, x \rangle \\ \vdots \\ \langle \text{row}\, n, x \rangle \end{pmatrix}$

$\|Ax\|^2 = \langle \text{row}1, x\rangle^2 + \cdots + \langle \text{row}\, n, x \rangle^2 \le \|\text{row}1\|^2 \cdot \|x\|^2 + \cdots + \|\text{row}\, n\|^2 \cdot \|x\|^2$

$= (\|\text{row}1\|^2 + \cdots + \|\text{row}\, n\|^2)\|x\|^2$

$= \|A\|^2 \|x\|^2$

Remark: there are other useful norms on the space of matrices:

For instance: $\|A\|_{\text{operator}} = \max_{\|x\|=1} \|Ax\|$

$\|Ax\| \le \|A\| \quad \forall\ \|x\| = 1$

$\|A\|_{\text{operator}} \le \|A\|$

Defn: Let A symmetric, A is called positive definite if $\langle Au, u \rangle > 0 \quad \forall u \neq 0$

negative definite if $\langle Au, u \rangle < 0 \quad \forall u \neq 0$

Prop: If A is positive definite, then $\exists\, c > 0$,

s.t $\langle Au, u \rangle \ge c\|u\|^2 \quad \forall u \in \mathbb{R}^n$

note. Both RHS, LHS are homogeneous of degree 2.

pf: If the result is true for a unit vector $\hat{u}$, $\|\hat{u}\| = 1$.

then it is true for all multiples of $\hat{u}$

$\underbrace{\langle A(t\hat{u}), t\hat{u} \rangle}_{t^2 \langle A\hat{u}, \hat{u} \rangle} \ge \underbrace{\|t\hat{u}\|^2}_{t^2 \|\hat{u}\|^2}$

know $\langle A\hat{u}, \hat{u} \rangle > 0 \quad \forall \|\hat{u}\| = 1$

this is a continuous function defined on the compact set of unit vectors,

it has a positive min c.

Remark. If A $n \times n$ symmetic matrix. $\min_{\|x\|=1} \langle Ax, x \rangle =$ lowest eigenvalue of A.

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14.3

Recall if $\varphi: \mathbb{R} \to \mathbb{R}$ is C^2. Then $\forall x, h \in \mathbb{R}$ $\exists$ $\theta \in (0,1)$

s.t $\varphi(x+h) = \varphi(x) + \varphi'(x)h + \frac{1}{2}\varphi''(x+\theta h)h^2$

we will generilize the above formula to $f: \mathbb{R}^n \to \mathbb{R}$, C^2

Let $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$. define $\varphi: \mathbb{R} \to \mathbb{R}$, $\varphi(t) = f(x+th)$

thus $\varphi(0) = f(x)$, $\varphi(1) = f(x+h)$ Recall $\varphi'(t) = \langle \nabla f(x+th), h \rangle = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x+th)h_i$

thus $\varphi''(t) = \langle \nabla^2 f(x+th)h, h \rangle$

$= \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x+th)h_i h_j$

thus with $x = 0 \in \mathbb{R}$, $h = 1 \in \mathbb{R}$, let $f: \mathbb{R}^n \to \mathbb{R}$, C^2, $\exists$ $\theta \in (0,1)$

s.t $f(x+h) = f(x) + \langle \nabla f(x), h \rangle + \frac{1}{2}\langle \nabla^2 f(x+\theta h)h, h \rangle$

$= f(x) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)h_i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x+\theta h)h_i h_j$

Generilization to kth order Talyor polynomial:

Recall if $\varphi: \mathbb{R} \to \mathbb{R}$, C^{k+1}, and if $x \in \mathbb{R}$, $h \in \mathbb{R}$, then $\exists$ $\theta \in (0,1)$

s.t $\varphi(x+h) = \varphi(x) + \varphi'(x)h + \frac{1}{2}\varphi''(x)h^2 + \cdots + \frac{1}{k!}\varphi^{(k)}(x)h^k + \frac{1}{(k+1)!}\varphi^{(k+1)}(x+\theta h)h^{k+1}$

Apply this to $f: \mathbb{R}^n \to \mathbb{R}$, C^{k+1}, $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $\varphi: \mathbb{R} \to \mathbb{R}$, $\varphi(t) = f(x+th)$ $\exists$ $\theta \in (0,1)$

$\varphi(1) = \varphi(0) + \varphi'(0) + \cdots + \frac{1}{k!}\varphi^{(k)}(0) + \frac{1}{(k+1)!}\varphi^{(k+1)}(\theta)$

Thm: If f, x, $h \in \mathbb{R}^n$ on the above,

then $f(x+h) = f(x) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)h_i + \cdots + \frac{1}{k!}\sum_{i_1,\dots,i_k=1}^n \frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}(x)h_{i_1}\cdots h_{i_k}$

$+ \frac{1}{(k+1)!}\sum_{i_1,\dots,i_{k+1}=1}^n \frac{\partial^{k+1} f}{\partial x_{i_1}\cdots\partial x_{i_{k+1}}}(x+\theta h)h_{i_1}\cdots h_{i_{k+1}}$

Remark: for $k=3$, $\frac{\partial^3 f}{\partial x_1 \partial x_1 \partial x_2}$, $\frac{\partial^3 f}{\partial x_1 \partial x_2 \partial x_1}$, $\frac{\partial^3 f}{\partial x_2 \partial x_1 \partial x_1}$ are equal

To write the perivious formula without repetition, use mult-index notation:

$h = (h_1, \dots, h_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{Z}_+$

Define $\alpha! = \alpha_1! \cdots \alpha_n!$

$|\alpha| = \alpha_1 + \cdots + \alpha_n$

$h^\alpha = h_1^{\alpha_1} \cdots h_n^{\alpha_n}$

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

Then Taylor formula becomes:

$$f(x+h) = f(x) + \sum_{|\alpha|=1} \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} h^\alpha + \cdots + \sum_{|\alpha|=k} \frac{1}{\alpha!}\frac{\partial^{|\alpha|} f(x)}{\partial x^\alpha} h^\alpha + \sum_{|\alpha|=k+1} \frac{1}{\alpha!}\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x+\theta h)h^\alpha$$

k^{th} order Taylor polynomial of f at x. denote $T_x^k(h)$

Thm: second-order approximation formula

let $f: \mathbb{R}^n \to \mathbb{R}$, C^2. let $x \in \mathbb{R}^n$. then $\lim_{h \to 0} \frac{f(x+h) - [f(x) + \langle \nabla f(x), h \rangle + \frac{1}{2}\langle \nabla^2 f(x)h, h \rangle]}{\|h\|^2} = 0$

let it be A.

pf: let $f(x+h) = f(x) + \langle \nabla f(x), h \rangle + \frac{1}{2}\langle \nabla^2 f(x+\theta h)h, h \rangle$

then $|A| = \frac{|\frac{1}{2}\langle \nabla^2 f(x+\theta h)h, h \rangle - \frac{1}{2}\langle \nabla^2 f(x)h, h \rangle|}{\|h\|^2} = \frac{1}{2}\frac{|\langle (\nabla^2 f(x+\theta h) - \nabla^2 f(x))h, h \rangle|}{\|h\|^2}$

$$\le \frac{1}{2}\frac{\|(\nabla^2 f(x+\theta h)-\nabla^2 f(x))h\|\,\cancel{\|h\|}}{\|h\|^{\cancel{2}}} \quad \text{by C-S}$$

$$\le \frac{1}{2}\frac{\|(\nabla^2 f(x+\theta h)-\nabla^2 f(x))\|\,\cancel{\|h\|}}{\cancel{\|h\|}} \to 0 \quad \text{by generilized C-S}$$

since $f \in C^2$ all $\frac{\partial^2 f}{\partial x_i \partial x_j}(x+\theta h) \to \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ as $h \to 0$. $0<\theta<1$

Generilization : let $f: \mathbb{R}^n \to \mathbb{R}$, C^k. let $x \in \mathbb{R}^n$

then $\lim_{h\to 0} \frac{f(x+h) - T_x^k(h)}{\|h\|^k} = 0$

Remark : let $f \in C^\infty(\mathbb{R}^n)$ i.e. $f \in C^k(\mathbb{R}^n)$ $\forall k \in \mathbb{N}$

then the above is true $\forall$ fixed k

$f(x+h) = f(x) + \cdots + \sum_{|\alpha|=k} \frac{1}{\alpha!}\frac{\partial^{|\alpha|} f}{\partial x_\alpha}(x) h^\alpha + \cdots$ True ?

if $f: \mathbb{R} \to \mathbb{R}$ is C^∞, and $h \in \mathbb{R}$. $f(x+h) = \sum_{k=0}^{n} \frac{f^k(x)}{k!} h^k$ not always true

Not True when $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ e^{-\frac{1}{x^2}} & \text{if } x > 0 \end{cases}$

$f(x) = \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$

$\frac{d^k}{dx^k}(e^{-\frac{1}{x}}) = P(\frac{1}{x})e^{-\frac{1}{x}} \to 0$ as $x \to 0$, $x > 0$

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14.3

let $f: \mathcal{O} \to \mathbb{R}$, $\mathcal{O}$ open, $x_0 \in \mathcal{O}$

Def : x_0 is a strict local minimizer for f if $\exists \delta > 0$

s.t $f(x_0+h) > f(x_0)$ if $0 < \|h\| < \delta$

Thm : let $f: \mathbb{R}^n \to \mathbb{R}$, C^2, let $x_0 \in \mathbb{R}^n$, assume $\nabla f(x_0) = 0$, $\nabla^2 f(x_0)$ is positive definite.

then x_0 is a strict local minimizer for f.

Remark : $\mathbb{R}^n$ can be replaced by any open $\mathcal{O}$. $x_0 \in \mathcal{O}$

pf: recall $\lim_{h\to 0} \frac{f(x_0+h) - [f(x_0) + \langle \nabla f(x_0), h\rangle + \frac{1}{2}\langle \nabla^2 f(x_0)h, h\rangle]}{\|h\|^2} = 0$

in another words, $f(x_0+h) = f(x_0) + \langle \nabla f(x_0), h\rangle + \frac{1}{2}\langle \nabla^2 f(x_0)h, h\rangle + R(h)$

where $\lim_{h\to 0} \frac{R(h)}{\|h\|^2} = 0$

If $\nabla f(x_0) = 0$. $f(x_0+h) = f(x_0) + \frac{1}{2}\langle \nabla^2 f(x)h, h\rangle + R(h)$

If $\nabla^2 f(x_0)$ is positive definite, then $\exists c > 0$ s.t $\langle \nabla^2 f(x)h, h\rangle \ge c\|h\|^2$ $\forall h \in \mathbb{R}^n$

Since $\lim_{h\to 0}\frac{R(h)}{\|h\|^2} = 0$, $\exists \delta > 0$ s.t $\left|\frac{R(h)}{\|h\|^2}\right| < \frac{c}{4}$ $\forall 0 < \|h\| < \delta$

$|R(h)| < \frac{c}{4}\|h\|^2$ if $0 \le \|h\| < \delta$

$\frac{1}{2}\langle \nabla^2 f(x)h, h\rangle \ge \frac{1}{2}c\|h\|^2$ $\quad R(h) \ge -\frac{c}{4}\|h\|^2$ if $0 < \|h\| < \delta$

then $f(x_0+h) \ge f(x_0) + \frac{c}{4}\|h\|^2 > f(x_0)$ $\forall 0 \le \|h\| < \delta$

thus x_0 is a strict local minimizer.

What about the converse?

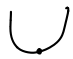
let $f: \mathbb{R}^n \to \mathbb{R}$, C^2, assume x_0 is a strict local minimizer of f.

① is $\nabla f(x_0) = 0$?

Yes, each $t \to f(x_0 + te_i)$ has a min at $t=0$

then $\frac{d}{dt}\Big|_{t=0} f(x_0 + te_i) = \frac{\partial f}{\partial x_i}(x_0) = 0$

② is $\nabla^2 f(x_0)$ positive definite?



$f(0)=0$
$f''(0)=0$

No. look at $f(x) = x^4$

$f(x,y) = x^4 + y^4$.

$\nabla^2 f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ not positive definite. it is positive semi-definite.

note A nxn matrix is positive semi-definite if $\langle Ah, h \rangle \geq 0$ $\forall h$

Try weaker converse.

If $f: \mathbb{R}^n \to \mathbb{R}$, C^2, x_0 is a local minimizer, then $\nabla^2 f(x_0)$ is positive semi-definite. ✓

pf: Assume x_0 is a local minimizer.

Assume by contradiction, $\nabla^2 f(x_0)$ is not positive semi-definite,

i.e. $\exists\, h \neq 0$ s.t $\langle \nabla^2 f(x_0) h, h \rangle < 0$

then let $\varphi(t) = f(x_0 + th)$

$\varphi'(0) = \langle \nabla f(x_0), h \rangle = 0$

$\varphi''(0) = \langle \nabla^2 f(x_0) h, h \rangle < 0$

0 is a local max for φ

thus $\forall t \neq 0$, t suffi close to 0, $\varphi(t) < \varphi(0)$

$f(x_0 + th) < f(x_0)$ $\forall t \neq 0$, t suffi close to 0

then x_0 can not be a local minimizer.

Question: If $f: \mathbb{R}^n \to \mathbb{R}$, C^2, $\nabla f(x_0) = 0$, $\nabla^2 f(x_0)$ is positive semi-definite,

does that follow that x_0 is a local minimizer of f?

NO, $f(x,y) = x^4 - y^4$ at $(0,0)$

Application to the weak maximum principle:

thm: let O open in $\mathbb{R}^n$, let $f: O \to \mathbb{R}$, C^2, assume $\Delta f := \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} < 0$ $\forall x \in O$

then f has no local (interior) minimizer

pf: Assume by contradiction, $x \in O$ is a local minimizer

then $\nabla f(x) = 0$ and $\frac{\partial^2 f}{\partial x_1^2}(x) \geq 0$ .... $\frac{\partial^2 f}{\partial x_n^2}(x) \geq 0$

then $\frac{\partial^2 f}{\partial x_1^2}(x) + \cdots + \frac{\partial^2 f}{\partial x_n^2}(x) < 0$ impossible.

Also, if $f: O \to \mathbb{R}$, C^2, satisfy $\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} > 0$ $\forall x \in O$.

$\nexists$ interior maximizer

because at maximizer x, $\nabla f(x) = 0$ all $\frac{\partial^2 f}{\partial x_i^2}(x) \leq 0$.

Thm: let $\overline{B_1(0)} = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, let $f \in C(\overline{B_1(0)})$, $f \in C^2(B_1(0))$

assume $\Delta f = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} = 0$ i.e. f is harmonic

then $\max_{\overline{B_1(0)}} f = \max_{\text{boundry } \overline{B_1(0)}} f$

pf: $\max_{\overline{B_1(0)}} f \geq \max_{\text{boundry } \overline{B_1(0)}} f$

let $\varepsilon > 0$, look at $f(x) + \varepsilon \|x\|^2 = f_\varepsilon$

then f_ε : $\frac{\partial^2 f_\varepsilon}{\partial x_1^2} + \cdots + \frac{\partial^2 f_\varepsilon}{\partial x_n^2} > 0$

$\Delta f_\varepsilon = 2n\varepsilon > 0$ thus f_ε has no interior maximizer.

f_ε conti. and sequti. compact. set, so $\exists$ maximizer at boundary

$\max_{B_1(0)} f \le \max_{B_1(0)} f_\varepsilon = \max_{\text{boundary } B_1(0)} f_\varepsilon = \max_{\|x\|=1} (f(x)+\varepsilon) = \max_{bd\, B_1(0)} f + \varepsilon$

let $\varepsilon \to 0$. then $\max_{B_1(0)} f \le \max_{bd\, B_1(0)} f$

works for all open set.

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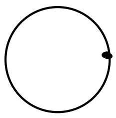
Application for min/max thm.

let $f \in C^2(\mathbb{R}^2)$, assume $f^2(x_1, x_2) = \frac{\partial^2 f}{\partial x_1^2}(x_1,x_2) + \frac{\partial^2 f}{\partial x_2^2}(x_1,x_2) + \frac{\partial f}{\partial x_1}(x_1,x_2)$



$f=0$

assume $f(x_1,x_2) = 0$ if $x_1^2 + x_2^2 = 1$

prove $f(x_1,x_2) \le 0$ $\forall$ $x_1^2 + x_2^2 \le 1$

pf: Assume by contraction, $\exists (x_1^\circ, x_2^\circ)$ with $x_1^{\circ 2} + x_2^{\circ 2} < 1$ s.t $f(x_1^\circ, x_2^\circ) > 0$

So f has a minimizer (x_1, x_2) in $\{x_1^2 + x_2^2 \le 1\}$ and $f(x_1,x_2) \ge f(x_1^\circ, x_2^\circ) > 0$

Also (x_1,x_2) not on boundry because $f=0$ on the boundry.

then

$$\underbrace{\frac{\partial^2 f}{\partial x_1^2}(x_1,x_2)}_{\le 0} + \underbrace{\frac{\partial^2 f}{\partial x_2^2}(x_1,x_2)}_{\le 0} + \underbrace{\frac{\partial f}{\partial x_1}(x_1,x_2)}_{0} = \underbrace{f^2(x_1,x_2)}_{>0}$$

contradiction.

Review of Linear Algebra. 15.1

Def: $T: \mathbb{R}^n \to \mathbb{R}^n$ is linear if $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$ $\forall \alpha, \beta \in \mathbb{R}$, $\forall u, v \in \mathbb{R}^n$

Ex: let $Q = (a_1, \dots, a_n)$, $Q \in \mathbb{R}^n$,

then $T(x) = \langle a, x \rangle = (a_1 \dots a_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is linear

Prop: let $T: \mathbb{R}^n \to \mathbb{R}$ linear, then $\exists!$ $a \in \mathbb{R}^n$ s.t $T(x) = \langle a, x \rangle$

pf: let $x = x_1 e_1 + \cdots + x_n e_n$.

$T(x) = T(x_1 e_1 + \cdots + x_n e_n) = x_1 \underbrace{T(e_1)}_{a_1} + \cdots + x_n \underbrace{T(e_n)}_{a_n \in \mathbb{R}} = a_1 x_1 + \cdots + a_n x_n = \langle a, x \rangle$

To prove a is unique

If $T(x) = \langle a, x \rangle = \langle b, x \rangle$ $\forall x$

want to show $a = b$.

look at $\langle a - b, x \rangle = 0$ $\forall x$

use $x = a - b$ get $\|a-b\|^2 = 0$, thus $a - b = 0$

let A be an $m \times n$ matric $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{mn} \end{pmatrix}$

Define $T: \mathbb{R}^n \to \mathbb{R}^n$ by $T(x) = Ax$ where x is a column vector, then T is linear.

Prop: If $T: \mathbb{R}^n \to \mathbb{R}^n$ is linear, then $\exists!$ A as above s.t $T(x) = Ax$

pf: let $x = x_1 e_1 \dots x_n e_n$, $T(x) = T(x_1 e_1 \dots x_n e_n) = x T(e_1) + \cdots + x_n T(e_n)$

Define $\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} = T(e_1) \quad \cdots \quad \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = T(e_n)$ (we define a_{ij})

Then $T(x) = x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ proved the existance

To prove uniqueness. If $A = \begin{pmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{1n} \\ b_{m1} & b_{mn} \end{pmatrix}$, $Ax = Bx \quad \forall x \in \mathbb{R}^n$

then $(A-B)x = 0 \quad \forall x \in \mathbb{R}^n$

$\begin{pmatrix} a_{11}-b_{11} & \cdots & a_{1n}-b_{1n} \\ \vdots & & \\ a_{m1}-b_{m1} & \cdots & a_{mn}-b_{mn} \end{pmatrix} (x) = 0 \quad \forall x$ pick $x = (a_{11}-b_{11}, \dots, a_{mn}-b_{mn})$

$\begin{pmatrix} (a_{11}-b_{11})^2 + \cdots + (a_{1n}-b_{1n})^2 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ then $a_{1i} = b_{1i} \quad \forall\ 1 \le i \le n$, repeat for other rows

let $\mathbb{R}^n \xrightarrow{S} \mathbb{R}^m \longrightarrow \mathbb{R}^k$, let T linear, $S(x) = Ax \quad \forall x \in \mathbb{R}^n$, $T(x) = Bx \quad \forall x \in \mathbb{R}^m$

then $T \circ S$ is linear and is given by the matrix BA

pf: check $(T \circ S)(x) = (BA)x \quad \forall x \in \mathbb{R}^n$, for basic vector $e_1 \dots e_n$.

To show $T \circ S(e_i) = (BA)(e_i) = i^{th}$ column of BA

$i = 1$ $\quad (T \circ S)(e_1) = T(S(e_1)) = T(Ae_1) = T(a_{11}e_1 + \cdots + a_{m1}e_m)$

$= a_{11} T(e_1) + \cdots + a_{m1} T(e_1)$

This is the 1^{st} column of BA $\quad = \underbrace{a_{11} \begin{pmatrix} b_{11} \\ \vdots \\ b_{k1} \end{pmatrix} + \cdots + a_{m1} \begin{pmatrix} b_{1m} \\ \vdots \\ b_{km} \end{pmatrix}}$

To check $\begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \\ b_{k1} & \cdots & b_{km} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

$= \begin{pmatrix} b_{11}a_{11} + \cdots + b_{1m}a_{m1} \\ \vdots \\ b_{k1}a_{11} + \cdots + b_{km}a_{m} \end{pmatrix}$

Corollary: $T: \mathbb{R}^n \to \mathbb{R}^n$ linear is invertiable as a function iff the corresponding matrix A is invertiable as a matrix iff $\det A \neq 0$.

Remark: If $T: \mathbb{R}^n \to \mathbb{R}^n$ is linear, then T is 1-1 iff T is onto

Let $A_{n \times n}$ matrix, then the following are equivalent.

1) A is inverviable

2) $\exists c > 0$ s.t $\|Ah\| \geq c\|h\|$

pf: 2) $\Rightarrow$ 1

If 2 holds, then A is 1-1

if $Ah_1 = Ah_2$ then $A(h_1 - h_2) = 0$, $\underbrace{\|A(h_1-h_2)\|}_{=0} \geq c\|h_1 - h_2\|$, hence $h_1 = h_2$

Conversely. If A is 1-1 write $h = A^{-1}Ah$

$\|h\| = \|A^{-1}Ah\| \leq \|A^{-1}\|\|Ah\|$

$\|Ah\| \geq \frac{1}{\|A^{-1}\|}\|h\|$

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let $F:\mathbb{R}^n \to \mathbb{R}^m$, assume all $\frac{\partial F_i}{\partial x_i}(x)$ exists $\forall x \in \mathbb{R}^n$.

Define
$$DF(x) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(x) & \cdots & \frac{\partial F_1}{\partial x_n}(x) \\ \vdots & & \\ \frac{\partial F_m}{\partial x_1}(x) & \cdots & \frac{\partial F_m}{\partial x_n}(x) \end{pmatrix}$$
For fixed x, as a matrix.

$DF(x): \mathbb{R}^n \to \mathbb{R}^m$

Ex: If $f:\mathbb{R}^n \to \mathbb{R}$, all $\frac{\partial F_i}{\partial x_i}$ exists, $Df(x) = \left(\frac{\partial f}{\partial x_1}(x) \cdots \frac{\partial f}{\partial x_n}(x)\right) = \nabla f(x)$

So ∇f is a row vector.

Ex: If $\gamma: \mathbb{R} \to \mathbb{R}^m$, $\gamma(t) = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_m \end{pmatrix}$ all $\gamma'(t)$ exists, then $D\gamma(t) = \begin{pmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_m'(t) \end{pmatrix} = \gamma'(t)$ column vector

Recall if $f:\mathbb{R}^n \to \mathbb{R}$, C^1, then $\forall x, h \in \mathbb{R}^n$ $\exists \theta \in (0,1)$

s.t $f(x+h) - f(x) = \langle \nabla f(x+\theta h), h \rangle = \underbrace{\nabla f(x+\theta h)}_{\text{row vector}} \cdot \underbrace{h}_{\text{column vector}}$

Generalize MVT to $F:\mathbb{R}^n \to \mathbb{R}^m$. $F = \begin{pmatrix} F_1 \\ \vdots \\ F_m \end{pmatrix}$ the perivious result applied to F_i,

gives : $\forall F:\mathbb{R}^n \to \mathbb{R}^m$, C^1, $\exists \theta_1, \dots, \theta_m \in (0,1)$

s.t $F_1(x+h) - F_1(x) = \nabla F_1(x+\theta_1 h) \cdot h$
$$\vdots \qquad\qquad \vdots$$
$$F_m(x+h) - F_m(x) = \nabla F_m(x+\theta_m h) \cdot h$$

Thm: let $F:\mathbb{R}^n \to \mathbb{R}^m$, C^1, $\forall x, h \in \mathbb{R}^n$ $\exists \theta_1, \dots, \theta_m \in (0,1)$

s.t $F(x+h) - F(x) = \begin{pmatrix} \nabla F_1(x+\theta_1 h) \\ \vdots \\ \nabla F_m(x+\theta_m h) \end{pmatrix} h$

Thm (First-order approximation formula): let $F:\mathbb{R}^n \to \mathbb{R}^m$, C^1.

then $\lim_{h \to 0} \frac{F(x+h) - F(x) - DF(x)h}{\|h\|} = 0$

pf: the i^{th} component of ↑ $\frac{F_i(x+h) - F_i(x) - \nabla F_i(x)h}{\|h\|} \to 0$

Since $F_i: \mathbb{R}^n \to \mathbb{R}$, C^1

Thm: let $F:\mathbb{R}^n \to \mathbb{R}^m$, fix $x \in \mathbb{R}^n$, assume A is $m \times n$ matrix

s.t $\lim_{h \to 0} \frac{F(x+h) - F(x) - Ah}{\|h\|} = 0$

then all $\frac{\partial F_i}{\partial x_j}(x)$ exists, $DF(x) = A$, $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

pf: By the componentwise convergence criterion, for each $1 \le i \le m$

$$\lim_{h \to 0} \frac{F_i(x+h) - F_i(x) - \langle (a_{i1}, \dots, a_{in}), h \rangle}{\|h\|} = 0$$

let $h = te_j$, where $e_j = j^{th}$ basic vector.

$$\lim_{h \to 0} \frac{F_i(x+te_j) - F_i(x) - ta_{ij}}{|t|} = 0 \Rightarrow \lim_{h \to 0} \frac{F_i(x+te_j) - F_i(x) - ta_{ij}}{t} = 0$$

also $\lim_{h \to 0} \frac{F_i(x+te_j) - F_i(x)}{t} = a_{ij}$, i.e $a_{ij} = \frac{\partial F_i}{\partial x_j}(x)$, $A = DF(x)$

Remark: One can define "F differentiable at x" if $A_{m \times n}$ providing a first order approximation formula : $\lim_{h \to 0} \frac{F(x+h) - F(x) - Ah}{\|h\|} = 0$

then $F \in C^1(\mathbb{R}^n) \Rightarrow F$ diff on $\mathbb{R}^n \Rightarrow DF(x)$ exists.

$\Leftarrow$ $n=1$ $\qquad$ $\not\Leftarrow$ $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (a,b) \neq (0,0) \\ 0 & (a,b) = (0,0) \end{cases}$

Remark: $DF(x) = \begin{pmatrix} \nabla F_1(x) \\ \vdots \\ \nabla F_m(x) \end{pmatrix} = \begin{pmatrix} | & & | \\ \frac{d}{dx_1} F(x) & \cdots & \frac{d}{dx_n} F(x) \\ | & & | \end{pmatrix}$

$\frac{d}{dx_1} F(x_1, \underbrace{x_2 \dots x_n}_{\text{Fixed}})$

15.3 Chain rule #1

let $\mathbb{R} \xrightarrow{\gamma} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$, assume γ, f are C^1, then $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is C^1 and

$\frac{d}{dt}(f \circ \gamma)(t) = \underbrace{\nabla f(\gamma(t))}_{\text{row}} \cdot \underbrace{\gamma'(t)}_{\text{column}} = \langle \nabla f(\gamma(t)), \gamma'(t) \rangle$

pf: depends only on $f \in C^1$ look at $\frac{f(\gamma(t+h)) - f(\gamma(t))}{h}$

By the MVT for f, $\exists$ P on the line segment from $\gamma(t+h)$ to $\gamma(t)$

s.t $\frac{f(\gamma(t+h)) - f(\gamma(t))}{h} = \nabla f(P(t,h)) \cdot \frac{\gamma(t+h) - \gamma(t)}{h}$

Recall that for $\gamma: \mathbb{R} \to \mathbb{R}^n$, $\gamma'(t)$ exists $\forall t \Rightarrow \gamma$ continuous

thus $P(t,h) \to \gamma(t)$ as $h \to 0$

and $\lim_{h\to 0} \underbrace{\nabla f(P(t,h))}_{\downarrow \, (\nabla f)(\gamma(t))} \cdot \underbrace{\frac{\gamma(t+h)-\gamma(t)}{h}}_{\downarrow \, \gamma'(t)} = \nabla f(\gamma(t)) \gamma'(t)$

Chain rule #2

let $\mathbb{R}^n \xrightarrow{G} \mathbb{R}^m \xrightarrow{f} \mathbb{R}$, assume $f, G \in C^1$, then $\nabla(f \circ G)(x) = \nabla f(G(x)) DG(x)$

pf: We have to show

$\left(\frac{\partial}{\partial x_1}(f \circ G)(x) \; \cdots \; \frac{\partial}{\partial x_n}(f \circ G)(x) \right) = \left(\left(\frac{\partial f}{\partial x_1}\right)(G(x)) \; \cdots \; \left(\frac{\partial f}{\partial x_m}\right)(G(x)) \right) \begin{pmatrix} \frac{\partial G_1}{\partial x_1}(x) & \cdots & \frac{\partial G_1}{\partial x_n}(x) \\ \vdots & & \\ \frac{\partial G_m}{\partial x_1}(x) & \cdots & \frac{\partial G_m}{\partial x_n}(x) \end{pmatrix}$

Fix $x_2 \dots x_n$.

$\gamma(x_1) = G(x_1, \underbrace{x_2 \dots x_n}_{\text{fixed}})$

then

$\frac{\partial}{\partial x_1}(f \circ \gamma)(x_1) = (\nabla f)(\gamma(x_1)) \cdot \gamma'(x_1)$

the same argument works for the other components.

Chain rule #3.

$\mathbb{R}^n \xrightarrow{G} \mathbb{R}^m \xrightarrow{F} \mathbb{R}^k$. let F, G of C^1, then $F \circ G$ is C^1,

$D(F \circ G)(x) = (DF)(G(x)) \cdot DG(x)$

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Practice problem :

17) let $f:\mathbb{R}^2\to\mathbb{R}$ of class C^2, and assume $x\in\mathbb{R}^2$ s.t $\lim_{h\to(0,0)} \frac{f(x+h)-f(x)-\langle \nabla f(x),h\rangle}{\|h\|^2}=0$

a) Prove $\lim_{h\to(0,0)} \frac{\langle \nabla^2 f(x)h,h\rangle}{\|h\|^2}=0$

pf: Since $\lim_{h\to 0} \frac{f(x+h)-f(x)-\langle \nabla f(x)h,h\rangle-\frac{1}{2}\langle \nabla^2 f(x)h,h\rangle}{\|h\|^2}=0$

b) Is the previous statement true without taking limits ?

Is $\frac{\langle \nabla^2 f(x)h,h\rangle}{\|h\|^2}=0$ for all $h\neq(0,0)$ true ?

Yes, fix $h\neq 0$. look at $t\neq 0$, $t\in\mathbb{R}$

$\lim_{t\to 0} \frac{\langle \nabla^2 f(x)(th),(th)\rangle}{\|th\|^2}=0$

but $\frac{\langle \nabla^2 f(x)(\cancel{t}h),(\cancel{t}h)\rangle}{\|\cancel{t}h\|^2} = \frac{\langle \nabla^2 f(x)h,h\rangle}{\|h\|^2}$ independent of t, hence it is 0.

show if $\langle \nabla^2 f(x)h,h\rangle=0$ $\forall h$, then $\nabla^2 f(x)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

Q: If A is 2×2 matrix and $\langle Ah,h\rangle=0$ $\forall h$,

Does it follow that $A=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

In general no.

$\langle \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}h_1\\h_2\end{pmatrix},\begin{pmatrix}h_1\\h_2\end{pmatrix}\rangle = \langle \begin{pmatrix}h_2\\-h_1\end{pmatrix},\begin{pmatrix}h_1\\h_2\end{pmatrix}\rangle = 0$ $\forall h$

However if $\nabla^2 f(x)$ is symmetric,

Lemma, if $A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$ and $a_{12}=a_{21}$ and $\langle Ah,h\rangle=0$ $\forall h$.

then $A=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, $\Sigma a_{ij}h_ih_j=0$ $\forall h$

try $h=e_1=\begin{pmatrix}1\\0\end{pmatrix}$, $a_{11}=0$

$h=e_2=\begin{pmatrix}0\\1\end{pmatrix}$, $a_{22}=0$

$h=\begin{pmatrix}1\\1\end{pmatrix}$, $\cancel{a_{11}}+2a_{12}+\cancel{a_{22}}=0$ so $a_{12}=0$

18) let $f\in C^2(\mathbb{R}^2)$, $f:\mathbb{R}^2\to\mathbb{R}$, assume $f(x,y)=0$, if $x^2+y^2=1$

and $\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{\partial^2 f}{\partial y^2}(x,y)+\frac{\partial f}{\partial y}(x,y)=f(x,y)$, if $x^2+y^2<1$.

True or false ?

True $f(x,y)\leq 0$ $\forall (x,y)$, $x^2+y^2<1$

assume, by contradiction, f takes some positive values in $B_1(0,0)$

then $\exists$ maximizer $(x_M,y_M)\in B_1(0,0)$, $f(x_M,y_M)>0$

and $\underbrace{\frac{\partial^2 f}{\partial x^2}(x,y)}_{\leq 0}+\underbrace{\frac{\partial^2 f}{\partial y^2}(x,y)}_{\leq 0}+\underbrace{\frac{\partial f}{\partial y}(x,y)}_{=0}=f(x,y)$

thus $f(x,y)$ must less or equal to 0

True $f(x,y)\geq 0$ $\forall (x,y)$, $x^2+y^2<1$

assume, by contradiction, f takes some positive values in $B_1(0,0)$

then $\exists$ minimizer $(x_m,y_m)\in B_1(0,0)$, and $\underbrace{\frac{\partial^2 f}{\partial x^2}(x,y)}_{\geq 0} + \underbrace{\frac{\partial^2 f}{\partial y^2}(x,y)}_{\geq 0} + \underbrace{\frac{\partial f}{\partial y}(x,y)}_{=0} = f(x,y)$

thus $f(x,y)$ must greater or equal to 0

Chain Rule (calculation with chain rule)

$$F(x,y) = (u(x,y), v(x,y))$$

recall $\mathbb{R}^2 \xrightarrow{F} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$, $F, f \in C^1$,

then $\underset{\llcorner\, \nabla(f\circ F)(x,y)}{D(f\circ F)(x,y)} = \underset{\llcorner\, \nabla f(u,v)}{Df(u(x,y), v(x,y))} \cdot \begin{pmatrix} \partial u/\partial x\,(x,y) & \partial u/\partial y\,(x,y) \\ \partial v/\partial x\,(x,y) & \partial v/\partial y\,(x,y) \end{pmatrix}$

$$\left(\frac{\partial}{\partial x}(\underset{\llcorner\, u = u(x,y)}{f(u,v)}) , \frac{\partial}{\partial y}(f(u,v)) \right) = \left(\frac{\partial f}{\partial u} \quad \frac{\partial f}{\partial v} \right) \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}$$

i.e $\frac{\partial}{\partial x}[f(u(x,y), v(x,y))] = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$

Homework 15.3 #5.

Let O be an open subset of the plane $\mathbb{R}^2$, and let the mapping $F: O \to \mathbb{R}^2$ be represented by $F(x,y) = (u(x,y), v(x,y))$ for (x,y) in O. Then the mapping $F: O \to \mathbb{R}^2$ is called a Cauchy - Riemann mapping provided that each of the function $u: O \to \mathbb{R}$ and $v: O \to \mathbb{R}$ has continuous second-order partial derivatives and

$$\partial u/\partial x\,(x,y) = \partial v/\partial y\,(x,y) \quad \text{and} \quad \partial u/\partial y\,(x,y) = -\partial v/\partial x\,(x,y) \qquad \forall (x,y) \in O$$

Prove that if the function $w: \mathbb{R}^2 \to \mathbb{R}$ is harmonic and the mapping $F: O \to \mathbb{R}^2$ is a Cauchy - Riemann mapping, then the function $w \circ F: O \to \mathbb{R}$ is also harmonic.

$$\underset{\substack{\text{in } \mathbb{R}^2 \\ (x,y)}}{O} \xrightarrow{F = (u(x,y),\, v(x,y))} \underset{(u,v)}{\mathbb{R}^2} \xrightarrow{w} \mathbb{R} \qquad u, v, w \in C^2$$

assume w is harmonic $\frac{\partial^2 w}{\partial u^2}(u,v) + \frac{\partial^2 w}{\partial v^2}(u,v) = 0 \quad \forall u, v$

Definition of harmonic.

If w harmonic. $\Delta w = 0$,

$\partial^2 w/\partial u^2 + \partial^2 w/\partial v^2 = 0$

Given F Cauchy - Riemann

so $\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y)$, $\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y)$

so $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2}$ so u, v harmonic

then $w(u(x,y), v(x,y))$ is harmonic.

Prove $\frac{\partial^2}{\partial x^2}[w(u(x,y), v(x,y))] + \frac{\partial^2}{\partial y^2}[w(u(x,y), v(x,y))] = 0 \quad \forall x, y$

$\frac{\partial}{\partial x}[w(u(x,y), v(x,y))] = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x}\,(u,v)$

$\frac{\partial^2}{\partial x^2}[w(u(x,y), v(x,y))]$

$$= \left[\frac{\partial^2 w}{\partial u^2}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial u \partial v}\cdot\frac{\partial v}{\partial x}\right]\frac{\partial u}{\partial x} + \frac{\partial w}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2} + \left[\frac{\partial^2 w}{\partial u \partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2}\cdot\frac{\partial v}{\partial x}\right]\frac{\partial v}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2}$$

$\frac{\partial^2}{\partial y^2}[w(u(x,y), v(x,y))]$

$$= \left[\frac{\partial^2 w}{\partial u^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial u \partial v}\cdot\frac{\partial v}{\partial y}\right]\frac{\partial u}{\partial y} + \frac{\partial w}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2} + \left[\frac{\partial^2 w}{\partial u \partial v}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2}\cdot\frac{\partial v}{\partial y}\right]\frac{\partial v}{\partial y} + \frac{\partial w}{\partial v}\cdot\frac{\partial^2 v}{\partial y^2}$$

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Calculations using the chain rule

$$\underset{(x,y)}{\mathbb{R}^2} \xrightarrow{R} \underset{(u,v)}{\mathbb{R}^2} \xrightarrow{f} \mathbb{R}$$

let $R: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ so R is a rotation.

let $f: \mathbb{R}^2 \to \mathbb{R}$ be C^2. recall $\Delta f(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y)$

Show $\Delta(f \circ R) = (\Delta f) \circ R$

let $u = \cos\theta \cdot x + \sin\theta \cdot y$, $v = -\sin\theta \cdot x + \cos\theta \cdot y$

We have to show

$$\frac{\partial^2}{\partial x^2}(f(u(x,y),v(x,y))) + \frac{\partial^2}{\partial y^2}(f(u(x,y),v(x,y))) = \left(\frac{\partial^2}{\partial u^2} f\right)(u(x,y),v(x,y)) + \left(\frac{\partial^2}{\partial v^2} f\right)(u(x,y),v(x,y))$$

$$\frac{\partial}{\partial x}(f(u,v)) = \frac{\partial f}{\partial u}(u,v) \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}(u,v) \cdot \frac{\partial v}{\partial x}$$
$$= \frac{\partial f}{\partial u}(u,v) \cdot \cos\theta + \frac{\partial f}{\partial v}(u,v) \cdot (-\sin\theta)$$

$$\frac{\partial^2}{\partial x^2}[f(u,v)] = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}(u,v)\right) \cdot \cos\theta - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}(u,v)\right) \cdot \sin\theta$$

$$= \left[\frac{\partial^2 f}{\partial u^2}(u,v) \cdot \cos\theta - \frac{\partial^2 f}{\partial u \partial v}(u,v) \cdot \sin\theta\right] \cdot \cos\theta - \left[\frac{\partial^2 f}{\partial u \partial v}(u,v) \cdot \cos\theta - \frac{\partial^2 f}{\partial v^2}(u,v) \cdot \sin\theta\right] \sin\theta$$

$$\frac{\partial}{\partial y}(f(u,v)) = \frac{\partial f}{\partial u}(u,v) \cdot \sin\theta + \frac{\partial f}{\partial v}(u,v) \cdot \cos\theta$$

$$\frac{\partial^2}{\partial y^2}(f(u,v)) = \left[\frac{\partial^2 f}{\partial u^2}(u,v) \cdot \sin\theta + \frac{\partial^2 f}{\partial u \partial v}(u,v) \cdot \cos\theta\right] \cdot \sin\theta + \left[\frac{\partial^2 f}{\partial u \partial v}(u,v) \cdot \sin\theta + \frac{\partial^2 f}{\partial v^2}(u,v) \cdot \cos\theta\right] \cos\theta$$

the cross terms cancel,

$$\text{sum} = \frac{\partial^2 f}{\partial u^2}(u(x,y),v(x,y))\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + \frac{\partial^2 f}{\partial v^2}(u,v)\underbrace{(\sin^2\theta + \cos^2\theta)}_{=1}$$

Given u, v are C-R function. u, v are harmonic.

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \qquad \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y)$$

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) \end{aligned}\right\} \text{sum} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

① Describe all functions $f: \mathbb{R}^2 \setminus \{0\}$, $f \in C^2$. s.t $f(x,y) = \varphi(\sqrt{x^2+y^2})$ ($\varphi \in C^2$) and f is harmonic.

Assume : $f(x,y) = c_1 + c_2 \log(\sqrt{x^2+y^2})$

② Describe all functions $f: \mathbb{R}^2 \setminus \{0\}$, $f \in C^2$. s.t $\frac{\partial^2 f}{\partial x^2}(x,y) - \frac{\partial^2 f}{\partial y^2}(x,y) = 0$

Assume : $f(x,y) = A(x+y) + B(x-y)$, $A, B \in C^2$

① $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \setminus \{0\} \xrightarrow{\varphi} \mathbb{R}$

$(x,y) \to r = \sqrt{x^2+y^2}$ $\qquad \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r}$

$f(x,y) = \varphi(r(x,y))$ $\qquad \frac{\partial f}{\partial x}(x,y) = \varphi'(r(x,y)) \cdot \frac{\partial r}{\partial x} = \varphi'(r) \cdot \frac{x}{r}$

$$\frac{\partial f}{\partial x^2}(x,y) = \left(\varphi''(r) \cdot \frac{x}{r}\right) \cdot \frac{x}{r} + \varphi'(r) \cdot \frac{r - \frac{x}{r} \cdot x}{r^2} = \frac{r^2 - x^2}{r^3}$$

$$= \varphi''(r) \cdot \frac{x^2}{r^2} + \varphi'(r) \cdot \frac{y^2}{r^3}$$

$$\frac{\partial f}{\partial y^2}(x,y) = \varphi''(r) \cdot \frac{y^2}{r^2} + \varphi'(r) \frac{x^2}{r^3}$$

$$(\Delta f)(x,y) = \varphi''(r(x,y)) + \frac{1}{r} \cdot \varphi'(r(x,y)) = \frac{\partial}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} \quad \text{for } f(x,y) = \varphi(r)$$

solve $\varphi'' + \frac{1}{r}\varphi' = 0$

let $g(r) = \varphi'(r)$, then $g' + \frac{1}{r} g = 0$, $g(r) = \frac{c}{r}$ general solution

then solve $\varphi'(r) = \frac{c}{r}$, $\varphi(r) = c \log(n) + D$

② let $u = x + y$, $v = x - y$

look at $f(u(x,y), v(x,y))$

$$\frac{\partial}{\partial x}(f(u,v)) = \frac{\partial f}{\partial u}(u,v) \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$\frac{\partial^2 f}{\partial x^2}(u,v) = \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial u \partial v}\right) + \left(\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}\right)$$

$$\frac{\partial}{\partial y}(f(u,v)) = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v}$$

$$\frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial^2 f}{\partial u^2} - \frac{\partial^2 f}{\partial u \partial v}\right) - \left(\frac{\partial^2 f}{\partial u \partial v} - \frac{\partial^2 f}{\partial v^2}\right) = \frac{\partial^2 f}{\partial u^2} - 2 \frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}$$

thus $\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) f(u,v) = 4 \frac{\partial^2 f}{\partial u \partial v}(u(x,y), v(x,y))$

the general solution to $\frac{\partial^2 f}{\partial u \partial v}(u,v)$ is $A(u) + B(v)$, $A, B \in C^2$

because look at $\frac{\partial f}{\partial v}$ satisfy $\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right) = 0$

thus

$$\frac{\partial f}{\partial v}(u,v) = l(v) \qquad \exists\, l \in C^1$$

then

$$\frac{\partial f}{\partial v} = l(v) \text{ has solution } f(u,v) = B(v) + C(u)$$

10/30 . 16.1 Inverse function theorem, $n=1$ easy case.

Thm: $f:\mathbb{R}\to\mathbb{R}$, C^1, let $x_0\in\mathbb{R}$. assume $f'(x_0)\neq 0$.
then $\exists\ I, J$ open intervals with $x_0\in I$, $f(x_0)\in J$.
s.t $f: I\to J$ is one to one, onto.
and $f^{-1}: J\to I$ is C^1.

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \qquad \forall y\in J$$

Remark: once we know f^{-1} is C^1, $f(f^{-1}(y)) = y \quad \forall y\in J$

$f'(f^{-1}(y))\cdot(f^{-1})'(y) = 1 \quad \forall y\in J$

pf: since f' is continuous, and $f'(x_0)\neq 0$ (WLOG $f'(x_0)>0$).
$\exists\ r>0$ and $c>0$ s.t $f'(x)\geq c \quad \forall x\in[x_0-r, x_0+r]$
then f is strictly increasing on $[x_0-r, x_0+r]$ thus 1-1
Also f restricted to $[x_0-r, x_0+r]$ is onto $[f(x_0-r), f(x_0+r)]$
$\forall y\in[f(x_0-r), f(x_0+r)]$, $\exists x\in[x_0-r, x_0+r]$ s.t $f(x)=y$ by the IVT.
Also, f^{-1} is differentiable and $(f^{-1})' = 1/f'(f^{-1}(y))$
thus f^{-1} is C^1

The inverse function theorem in $\mathbb{R}^2$

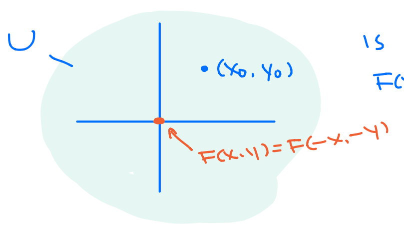
Thm: let $F:\mathbb{R}^2\to\mathbb{R}^2$. C^1, let $(x_0,y_0)\in\mathbb{R}^2$, assume $DF(x_0,y_0)$ is invertible
then $\exists\ U$ neighborhood of (x_0,y_0) (i.e. an open set containing (x_0,y_0)),
$\exists\ V$ neighborhood of $F(x_0,y_0)$
s.t $F: U\to V$ is 1-1, onto,
$F^{-1}: V\to U$ is C^1
and $DF^{-1}(y) = (DF(F^{-1}(y)))^{-1} \quad \forall y\in U$.

Remark: once we know F^{-1} is C^1, use $F(F^{-1}(y)) = y \quad \forall y\in V$
and the chain rule $DF(F^{-1}(y))\cdot(DF^{-1})(y) = I$

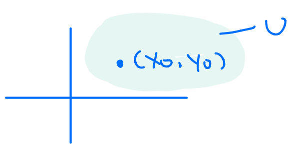
ex: $z = x+iy$, $F(x,y) = (x^2-y^2, 2xy)$, $F(z)=z^2$

$DF(x_0,y_0) = \begin{pmatrix} 2x_0 & -2y_0 \\ 2y_0 & 2x_0\end{pmatrix}$ $\quad \det DF(x_0,y_0) = 4(x_0^2+y_0^2)\neq 0$ iff $(x_0,y_0)\neq 0$

thus the hypothesis of the inverse function theorem holds at all $(x_0,y_0)\neq 0$
thus if $(x_0,y_0)\neq 0$, $\exists\ U$ nbhd of (x_0,y_0) s.t $F: U\to\mathbb{R}^2$ is 1-1 and $F(U)=V$ is open



is F 1-1 on U? No.
$F(x,y) = F(-x,-y)$



F is 1-1 on this U

F is 1-1 on some U. iff $(x_0,y_0)\in U \Rightarrow (-x,-y)\notin U$

It will turn out that $F(U)$ is open for any $U \subseteq \mathbb{R}^2$.

let $U = B_r(0,0)$.

the hypothesis of the IFT does not hold at $(0,0)$,

but $F(U) = B_{r^2}(0,0)$ which is open

In polar coordinates : $(x,y) = (\rho\cos\theta, \rho\sin\theta)$

$F(x,y) = (x^2-y^2, 2xy) = (\rho^2(\cos^2\theta - \sin^2\theta), \rho^2 2\sin\theta\cos\theta) = \rho^2(\cos(2\theta), \sin(2\theta))$ $\quad 0 \le \rho \le r$

ex. e^{x+iy}

$F(x,y) = (e^x\cos(y), e^x\sin(y))$

$$DF(x,y) = \begin{pmatrix} e^x\cos(y) & -e^x\sin(y) \\ e^x\sin(y) & e^x\cos(y) \end{pmatrix}$$

$\det DF(x,y) = e^{2x}(\cos^2(y) + \sin^2(y)) = e^{2x} \neq 0 \quad \forall (x,y)$

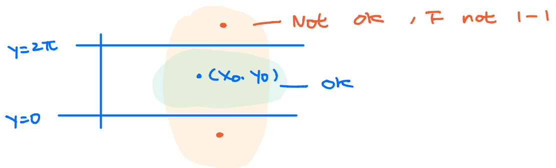
thus the hypothesis of IFT holds at all $(x,y) \in \mathbb{R}^2$

can we conclude $F: \mathbb{R}^2 \to \mathbb{R}^2$ is 1-1 ?

No. $F(x,y) = F(x, y+2k\pi) \quad \forall k \in \mathbb{Z},\ x,y \in \mathbb{R}^2$

can we conclude F is onto $\mathbb{R}^2$?

No. $F(x,y) \neq 0 \quad \forall x,y$



ex. let $f: \mathbb{R} \to \mathbb{R}$, C^1,

define $F(x,y) = (f(x+y), f(x+y))$

$$DF(x,y) = \begin{pmatrix} f'(x+y) & f'(x+y) \\ f'(x+y) & f'(x+y) \end{pmatrix}$$

$\det DF(x,y) = 0 \quad \forall (x,y)$

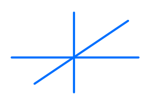
thus the hypothosis of the IFU fails at all (x,y)

the conclusion also fails at all (x,y)

Fix (x_0, y_0)

$\nexists$ U nbhd of (x_0,y_0), and V nbhd of $F(x_0,y_0)$ s.t $F: U \to V$ is 1-1 and onto.

In geometry : F maps $\mathbb{R}^2$ onto the line $\{(x,y) \mid x=y\}$



Not onto. $\mathbb{R}^2$

11/2

10.2 $F:\mathbb{R}^n \to \mathbb{R}^n$, C^1, $x^* \in \mathbb{R}^n$, assume $DF(x^*)$ is invertiable.

$\exists$ U nbhd of x^* s.t $F: U \to \mathbb{R}^n$ is 1-1, and $DF(x)$ is invertiable $\forall x \in U$.

Also F is stable on U.

Recall : If A is an invertiable n×n matrix.

then $\exists\, c > 0$, s.t $\|Ah\| \geq c\|h\|$ $\forall h \in \mathbb{R}^n$

pf: $h = A^{-1}Ah$, $\|h\| = \|A^{-1}Ah\| \underset{C\text{-}S}{\leq} \|A^{-1}\| \cdot \|Ah\|$, $c = \frac{1}{\|A^{-1}\|}$,

$\Rightarrow \|Ax - Ay\| \geq c\|x-y\|$, $h = x-y$

Def: $F: U \to \mathbb{R}^n$, U open in $\mathbb{R}^n$, F is stable if $\exists\, c > 0$

s.t $\|F(x) - F(y)\| \geq c\|x-y\|$ $\forall x, y \in U$

Remark : F stable $\Rightarrow$ F 1-1

pf: let $x, y \in U$, s.t $F(x) = F(y)$

then $\|F(x) - F(y)\| \geq c\|x-y\|$ thus $x=y$

the converse is not true F 1-1 $\not\Rightarrow$ F stable

ex. $F:\mathbb{R}\to\mathbb{R}$, $f(x) = x^3$ is 1-1, not stable.

let $y=0$

$|x^3 - 0| \geq c|x|$ not true

$\nexists\, c > 0$, s.t $x^2 \geq c$ $\forall x$, $c=0$

Remark : $F: U \to \mathbb{R}^n$ is stable iff $F^{-1}: F(U) \to U$ is Lipschitz continuous.

Prop: Let A be an n×n matrix.

Assume $\exists\, c > 0$ s.t $\|Ah\| \geq c\|h\|$ $\forall h \in \mathbb{R}^n$

let B n×n matrix be such that $\|A - B\| \leq c/2$

then $\|Bh\| \geq \frac{c}{2}\|h\|$ $\forall h \in \mathbb{R}^n$

pf: $\|Bh\| = \|Ah - (A-B)h\| \underset{\text{reverse tri-ineq}}{\geq} \|Ah\| - \|(A-B)h\| \geq c\|h\| - \|A-B\|\|h\| \geq c\|h\| - \frac{c}{2}\|h\| = \frac{c}{2}\|h\|$

Thm: Non-linear Stability theorem

let $F:\mathbb{R}^n \to \mathbb{R}^n$, C^1, let $x^* \in \mathbb{R}^n$ s.t $DF(x^*)$ is invertiable.

Then $\exists$ U nbhd of x^*

s.t 1) $DF(x)$ is invertiable $\forall x \in U$

2) F is stable on U

pf:

1) look at $\det DF(x)$, $F \in C^1$ and $\det DF(x^*) \neq 0$

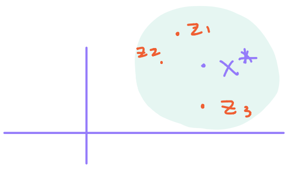
then $\exists\, U_1$ nbhd of x^* s.t $\det DF(x) \neq 0$ then $DF(x)$ is invertiable.

2)

$$F(x) - F(y) = \begin{pmatrix} \nabla F_1(z_1) \\ \vdots \\ \nabla F_n(z_n) \end{pmatrix} (x-y) \quad \text{for some } z_1, \dots, z_n \text{ btw } x, y$$

Next $DF(x^*)$ is invertiable, $\exists\, c > 0$ s.t $\|DF(x^*)h\| > c\|h\|$

pick U_2 nbhd of x^*, U_2 could be $B_r(x^*)$



s.t

$$\left\| DF(x^*) - \begin{pmatrix} \nabla F_1(z_1) \\ \vdots \\ \nabla F_n(z_n) \end{pmatrix} \right\| \le \frac{c}{2} \qquad \forall z_1, \dots, z_n \in U_2$$

then

$$\| F(x) - F(y) \| = \left\| \begin{pmatrix} \nabla F_1(z_1) \\ \vdots \\ \nabla F_n(z_n) \end{pmatrix} (x-y) \right\| \le \frac{c}{2}$$

1) + 2) let $U = U_1 \cap U_2$

Thus we have U open s.t $F: U \to \mathbb{R}^n$ is 1-1.

Next, want to show $\exists\, x^* \in U_1 \subseteq U$ open set s.t $F(U_1)$ is open.

Have to show that if y is sufficiently close to $y^* = F(x^*)$

then the equation $F(x) = y$ has a solution $x \in U$.

the construction principle can be applies. or use the "mimization principle"

16.3

Thm: let $F: U \to \mathbb{R}^n$, C^1, U open in $\mathbb{R}^n$

assume $DF(x)$ is invertiable $\forall x \in U$. Fix $y \in \mathbb{R}^n$

Define $E(x) = \| F(x) - y \|^2$, $E: U \to \mathbb{R}$.

then if E has a local interior minimizer x_0, then $F(x_0) = y$

let x_0 be such a minimizer,

then $(\nabla E)(x_0) = 0$, Compute ∇E using the chain rule

$$\underset{x}{U} \xrightarrow{G} \underset{F(x)-y=z}{\mathbb{R}^n} \xrightarrow{g} \underset{\|z\|^2}{\mathbb{R}}$$

$E = g \circ G$

$$0 = \nabla E(x_0) = DE(x_0) = \underbrace{(Dg)(F(x_0 - y))}_{\text{row}} \cdot \underbrace{DG(x_0)}_{n \times n \text{ invertiable}}$$

$$Dg(z) = \nabla g(z) = \underbrace{2z}_{\text{row vector}}$$

thus $0 = \underbrace{(F(x_0) - y)^T}_{\text{row}} \cdot DG(x_0)$ OR $DG(x_0)^T \underbrace{(F(x_0) - y)}_{\text{column}} = 0$

then $F(x_0) - y \in N(DG(x_0)) = \{0\}$

null space

thus $F(x_0) = y$

11/19

Recall the IFU: let $F: \mathbb{R}^n \to \mathbb{R}^n$, C^1, let $x^* \in \mathbb{R}^n$, s.t $DF(x^*)$ is invertiable.

then $\exists\, U, V$ nbhd of x^*, respectly $F(x^*)$ s.t $F: U \to V$ is 1-1 onto.

$F^{-1}: V \to U$ is C^1, $(DF^{-1})(F(x))$

we have shown $\exists\, U$ nbhd of x^* s.t $F: U \to \mathbb{R}^n$ is 1-1 in fact F is stable

i.e. $\exists\, c > 0$, s.t $\|F(x) - F(z)\| \geq c\|x - z\|$ $\forall x, z \in U$.

and $\det DF(x) \neq 0$ $\forall x \in U$

Also we have shown that if $y \in \mathbb{R}^n$ is such that $E(x) = \|F(x) - y\|$

we have local minimizer $x_m \in U$, $E(x_m) = y$

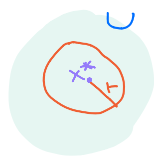
Open image lemma: $\exists\, \rho > 0$. s.t $B_\rho(F(x^*)) \subseteq F(U)$

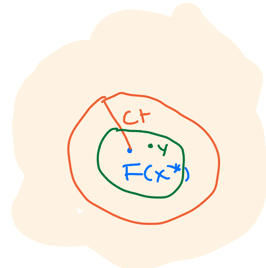
thus $F(U)$ contains an open nbhd of $F(x^*)$

pf: let $r > 0$ s.t $\overline{B_r(x^*)} \subseteq U$

then if $\|x^* - x\| = r$

$\|F(x) - F(x^*)\| \geq cr$





Will show if $\|F(x^*) - y\| < \frac{cr}{2}$

then $y \in F(\overline{B_r(x^*)}) \subseteq F(U)$

will show $E(x) = \|F(x) - y\|$ has a local interior minimizer in $B_r(x^*)$

look at $E: \overline{B_r(x^*)} \to \mathbb{R}$

a continuous function on a sequentially compact set has a minimizer $x_m \in \overline{B_r(x^*)}$

$\|x - x_m\| = r$ then x_m will be a minimizer.

notice: if $\|x - x_m\| = r$ then

① $E(x_m) = \|F(x_m) - y\| \geq \underbrace{\|F(x_m) - F(x^*)\|}_{\geq cr} - \underbrace{\|F(x^*) - y\|}_{< \frac{cr}{2}} \geq cr - \frac{cr}{2} = \frac{cr}{2}$

$E(x_m) \geq \frac{cr}{2}$

② $E(x_*) = \|F(x_*) - y\| < \frac{cr}{2}$

thus $E(x_m) > \frac{cr}{2} > E(x_*)$

thus $x_m \in \operatorname{bd} B_r(x^*)$ can not be a minimizer for $E: \overline{B_r(x^*)} \to \mathbb{R}$.

thus the minimizer x_m is an interior minizer, and therefore $F(x_m) = y$

thus $B_{\frac{cr}{2}}(F(x^*)) \subseteq F(B_r(x^*))$

Show $F^{-1}: V \to U$ is C^1.

from the formula $(DF^{-1})(y) = ((DF)(x))^{-1}$ if $F(x) = y$, $x \in U$, $y \in V$. $F: U \to V$ 1-1 onto stable.

$$\lim_{k \to 0} \frac{\|F^{-1}(y+k) - F^{-1}(y) - ((DF)(x))^{-1} k\|}{\|k\|} = 0$$

notation: $F(x) = y$, $\|k\| = \|F(x+h) - F(x)\| \geq c\|h\|$

$F(x+h) = y + k$ $\quad k = F(x+h) - y$

Then $\frac{\|F^{-1}(y+k) - F^{-1}(y) - ((DF)(x))^{-1} k\|}{\|k\|} = \frac{\|(DF(x))^{-1}[DF(x)(h) - [F(x+h) - F(x)]]\|}{\|k\|} \overset{C-S}{\leq} \frac{\|DF(x)^{-1}\| \, \|F(x+h) - F(x) - DF(x)h\|}{c\|h\|}$

$= \frac{\|(DF(x))^{-1}[DF(x)(F^{-1}(y+k) - F^{-1}(y)) - k]\|}{\|k\|} \leq \frac{\|(DF(x))^{-1}\| \|DF(x)(x+h-x) - (F(x+h) - F(x))\|}{c\|h\|}$

$\to 0$ as $\|k\| \to 0$ and thus $\|h\| \to 0$

second proof:

let $F: \mathbb{R}^n \to \mathbb{R}^n$, C^1, $DF(x^*)$ is invertiable. $F(x^*) = y^*$

Show $\exists\ \delta_0 > 0$ s.t if $\| y - F(x^*) \| <$ some positive number

then $\exists\ x \in \mathbb{R}^n$, $\| x - x^* \| < \delta$, s.t $F(x) = y$.

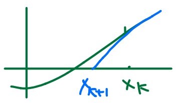
Want to show $F(x) = y$ iff $DF(x^*)^{-1}(F(x) - y) = 0$ iff $\underbrace{x - DF(x^*)^{-1}[F(x) - y]}_{T(x)} = x$

Will show that $\exists\ \delta_0 > 0$ s.t $\forall\ 0 < \delta \le \delta_0$, $T : \overline{B_\delta(x^*)} \to \overline{B_\delta(x^*)}$ is a construction.

and thus the fixed point x can be obtained as a limit of

$$x_{k+1} = \underbrace{x_k - (DF(x^*))^{-1}[F(x_k) - y]}_{T(x_k)}$$

Anology



linear approximation of f.

$y = f(x_k) + f'(x_k)[x - x_k]$

x_{k+1} determined from $0 = f(x_k) + f'(x_k)(x_{k+1} - x_k)$

i.e $x_{k+1} = x_k - (f'(x_k))^{-1} f(x_k)$

we this with $f(x) = F(x) - y$ y fixed

Main step :

$\exists\ \delta_0 > 0$ s.t $\| x - z - DF(x^*)^{-1}(F(x) - F(z)) \| \le \frac{1}{2} \| x - z \|$ $\forall\ x, z \in B_{\delta_0}(x^*)$

pf : $\| x - z - DF(x^*)^{-1}(F(x) - F(z)) \|$

$= \| (DF(x^*))^{-1} [DF(x^*)(x - z) - F(x) - F(z)] \| \le \| DF(x^*)^{-1} \| \, \| F(x) - F(z) - DF(x^*)(x - z) \|$

By the mean value theorem , if $x, z \in B_{\delta_0}(x^*)$

$\| F(x) - F(z) - DF(x^*)(x - z) \|$

$= \left\| \begin{pmatrix} \nabla F_1(z_1) \\ \nabla F_n(z_n) \end{pmatrix} (x - z) - DF(x^*)(x - z) \right\|$ $\exists\ z_1, \dots, z_n \in B_{\delta_0}(x^*)$

$= \left\| \left[\begin{pmatrix} \nabla F_1(z_1) \\ \nabla F_n(z_n) \end{pmatrix} - DF(x^*) \right] (x - z) \right\| \le \left\| \begin{pmatrix} \nabla F_1 \\ \nabla F_n \end{pmatrix} - DF(x^*) \right\| \| x - z \|$

pick δ_0 s.t $\left\| \begin{pmatrix} \nabla F_1(z_1) \\ \nabla F_n(z_n) \end{pmatrix} - DF(x^*) \right\| \le \dfrac{1}{2 \| DF(x^*)^{-1} \|}$

$\forall\ z_1, \dots, z_n \in B_{\delta_0}(x^*)$

$\| x - z - DF(x^*)^{-1}(F(x) - F(z)) \| \le \frac{1}{2} \| x - z \|$ $\forall\ x, z \in B_{\delta_0}(x^*)$

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chapter 17.

let $f:\mathbb{R}^2 \to \mathbb{R}$, $f \in C^1$,

what can we say about the level set $C = \{(x,y) \in \mathbb{R}^2 \mid f(x,y)=0\}$, C has to be closed.

Fact:

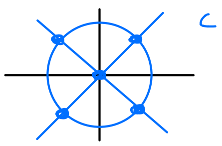
$\forall C \subseteq \mathbb{R}^n$ closed, $\exists f:\mathbb{R}^n \to \mathbb{R}$, $f \in C^1$, s.t $C = \{x \in \mathbb{R}^n \mid f(x) = 0\}$

A sufficient condition for $C = \{(x,y) \in \mathbb{R}^2 \mid f(x,y)=0\}$ to be a C^1 curve (locally the graph of a C^1 function) is $\nabla f(x,y) \neq 0$ $\forall (x,y)$ s.t $f(x,y)=0$

Ex: $f(x,y) = x^2+y^2-1$

$f=0$ on  the unit circle, $\nabla f(x,y) = (2x, 2y) \neq 0$ $\forall x^2+y^2=1$

Ex: $f(x,y) = (x^2+y^2-2)(x^2-y^2)$, $C = \{(x,y) \mid f(x,y) = 0\}$



C is not locally the graph of $y = \varphi(x)$ on $x = \varphi(y)$ with $\varphi \in C^1$ near 5 points.

need to check $\nabla f(x,y) \neq 0$ if $x, y \in C$, (x,y) not one of the five points.

Theorem: Dini's theorem

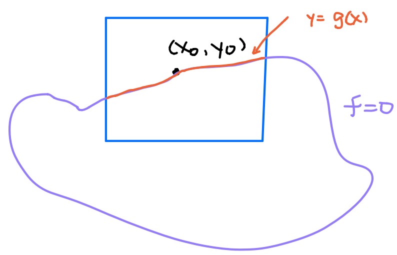
let $f:\mathbb{R}^2 \to \mathbb{R}$, C^1, let (x_0, y_0) s.t $f(x_0,y_0)=0$, assume $\frac{\partial f}{\partial y}(x_0,y_0) \neq 0$.

then $\exists r_1, r_2 > 0$ and $g:(x_0-r_1, x_0+r_1) \to (y_0 - r_2, y_0+r_2)$, $g \in C^1$

$g(x_0) = y_0$, $f(x, g(x)) = 0$ $\forall |x-x_0| < r_1$

In addition, If $|x-x_0| < r_1$ and $|y-y_0| < r_2$ and $f(x,y)=0$

then $y = g(x)$



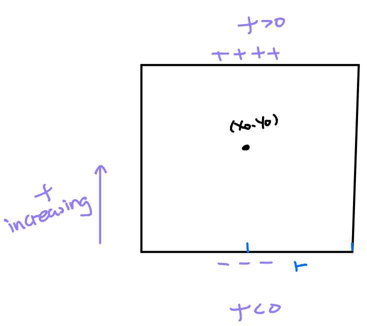
Also $\frac{d}{dx}[f(x, g(x))] = 0$

i.e $\frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x)) g'(x) = 0$ $\forall |x-x_0| < r_1$

pf: $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ WLOG. assume $\frac{\partial f}{\partial y}(x_0,y_0) > 0$

since $f \in C^1$, $\exists r > 0$, $c > 0$

s.t $\frac{\partial f}{\partial y}(x,y) \geq c > 0$ $\forall |x-x_0| \leq r$, $|y-y_0| \leq r$



$y \to f(x,y)$ is strictly increasing in the square

$f(x_0,y_0) = 0$, $f(x_0, y_0 - r) < 0$, $f(x_0, y_0 + r) > 0$

$\exists r_1 > 0$ s.t $f(x, y_0 - r) < 0$ $\forall |x-x_0| < r_1$

$f(x, y_0 + r) > 0$ $\forall |x - x_0| < r_1$

Fix $|x-x_0| < r_1$

since $y \to f(x,y)$ is strictly increasing from negative values to positive values.

$\exists y = y(x)$, $y \in (y_0 - r, y_0 + r)$. s.t $f(x, y(0)) = 0$.

Define $g:(x_0-r_1, x_0+r_1) \to (y_0-r, y_0+r)$ to be that $y(x)$
thus $f(x, g(x))$ satisfied, and the only solution to $f(x,y)=0$
in $(x_0-r_1, x_0+r_1)\times(y_0-r, y_0+r)$, take $r_2=r$

Next, show $g\in C^1$.
First, show g is continuous, take $x,y\in(x_0-r_1, x_0+r_1)$
Apply the MVT to $0 = f(y,g(y)) - f(x,g(x))$
$$= \langle \nabla f(p), (y-x, g(y)-g(x)) \rangle$$
for some p on the line from $(x, g(x))$ to $(y, g(y))$.
Also ∇f is continuous on $[x_0-r_1, x_0+r_1]\times[y_0-r_2, y_0+r_2]$
thus it is bounded. $(\nabla f(p)) \in C$
Get $\frac{\partial f}{\partial x}(p)(y-x) + \frac{\partial f}{\partial y}(p)(g(y)-g(x)) = 0$

$$g(y)-g(x) = \frac{-\frac{\partial f}{\partial x}(p)}{\underbrace{\frac{\partial f}{\partial y}(p)}_{\geq c}}(y-x)$$

$$|g(y)-g(x)| \leq \frac{C}{c}|y-x|$$

thus g is continuous,

Next, let $y=x+h$, p from $(x,g(x))$ to $(x+h, g(x+h))$
$$\frac{g(x+h)-g(x)}{h} = -\frac{\frac{\partial f}{\partial x}(p)}{\frac{\partial f}{\partial y}(p)}$$

as $h\to 0$, $g(x+h)\to g(x)$, $p\to(x, g(x))$
so $$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \frac{-\frac{\partial f}{\partial x}(x,g(x))}{\frac{\partial f}{\partial y}(x,g(x))}$$

thus $g'(x)$ exists, is continuous, and the formula agree with implicit differentiation.

Remark.
the exact same proof shows that
if $f:\mathbb{R}^{n+1}\to\mathbb{R}$. C^1, $f(\underset{\mathbb{R}^n}{x_0}, \underset{\mathbb{R}}{y_0})=0$, $\frac{\partial f}{\partial y}(x_0,y_0)\neq 0$.

then $\exists\ r_1, r_2>0$, $g\in C^1$. $g: B_{r_1}(x_0)\to(y_0-r_2, y_0+r_2)$
s.t $f(x, g(x))=0$ $\forall x\in B_{r_1}(x_0)$
and there are all solution to $f(x,y)=0$, with $(x,y)\in B_{r_1}(x_0)\times(y_0-r_2, y_0+r_2)$

thus the level set $\{(x,y) \mid f(x,y)=0\}\subseteq\mathbb{R}^{n+1}$ agree with the graph of g in
$B_{r_1}(x_0)\times(y_0-r_2, y_0+r_2)$
$g: \underset{\mathbb{R}^n}{B_{r_1}(x_0)} \to \mathbb{R}$ and is an n-dimentional C^1 hypersurface in $\mathbb{R}^{n+1}$

thus if $f:\mathbb{R}^{n+1}\to\mathbb{R}$, and $\nabla f(x)\neq 0$ $\forall x$, $f(x)=0$.
then $S=\{x \mid f(x)=0\}$ is an n-dimentional surface in $\mathbb{R}^{n+1}$
locally after re-labelling coordinates, it is the graph of a C^1 function.

17.2 General implicit function theorem.

let $F: \mathbb{R}^{n+k} \to \mathbb{R}^k$, C^1, let $(\underset{\mathbb{R}^n}{\underset{\cap}{x}}, \underset{\mathbb{R}^k}{\underset{\cap}{y}})$,

assume $F(x_0, y_0) = 0$, and $D_y F(x_0, y_0) = \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(x_0, y_0) & \cdots & \frac{\partial F_1}{\partial y_k}(x_0, y_0) \\ \vdots & & \\ \frac{\partial F_k}{\partial y_1}(x_0, y_0) & \cdots & \frac{\partial F_k}{\partial y_k}(x_0, y_0) \end{pmatrix}$ is invertiable.

then $\exists\ r_1, r_2 > 0$.
and $G: B_{r_1}(x_0) \to B_{r_2}(y_0)$, C^1
s.t $F(x, G(x)) = 0$ $\forall x \in B_{r_1}(x_0)$ and if $F(x, y) = 0$. $(x, y) \in B_{r_1}(x_0) \times B_{r_2}(y_0)$
then $y = G(x)$

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The implicit function Theorem

let $F: \mathbb{R}^{n+k} \to \mathbb{R}^k$, C^1, $(x, y) \in \mathbb{R}^n \times \mathbb{R}^k$

let $(x_*, y_*) \in \mathbb{R}^{n+k}$, $F(x_*, y_*) = 0$. assume $D_y F(x_*, y_*)$ is invertiable. ($k \times k$ matrix)

then $\exists\ r_1, r_2 > 0$ and $G: B_{r_1}(x_*) \to B_{r_2}(y_*)$, C^1 with $G(x_*) = y_*$

s.t $F(x, G(x)) = 0$ $\forall x \in B_{r_1}(x_*)$

In addition if $(x, y) \in B_{r_1}(x_*) \times B_{r_2}(y_*)$ and $F(x, y) = 0$. then $y = G(x)$

pf: Define $H: \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$, $H(x, y) = (x, F(x, y))$ then $H \in C^1$,
$H(x_*, y_*) = (x_*, 0)$

$$DH(x_*, y_*) = \left(\begin{array}{ccc|cc} 1 & \cdots & 0 & 0 & 0 \\ 0 & & 1 & 0 & 0 \\ \hline & D_x F(x, y) & & D_y F(x, y) & \end{array}\right) = \left(\begin{array}{c|c} I & 0 \\ \hline ? & D_y F(x, y) \end{array}\right)$$

this is invertiable - by row reducing to $\left(\begin{array}{c|c} I & 0 \\ \hline 0 & D_y F(x, y) \end{array}\right)$

By the inverse function therom, $\exists$ a nbhd of (x_*, y_*) which can be taken $B_r(x_*) \times B_r(y_*)$ and W a nbhd of $(x_*, 0)$ s.t $H: B_r(x_*) \times B_r(y_*) \to W$ Is 1-1 and onto. have C^1 inverse. $H^{-1} = (M, N): W \to B_r(x_*) \times B_r(y_*)$

write $H(\overbrace{M(x, y), N(x, y)}^{H^{-1}}) = (x, y)$ $\forall (x, y) \in B_{r_1}(x_*) \times B_{r_1}(0)$

$(M(x, y), F(M(x, y), N(x, y)) = (x, y)$ B/c $H(x, y) = (x, F(x, y))$

Thus $M(x, y) = x$ $\forall (x, y)$ $F(x, N(x, y)) = y$

Define $G(x) = N(x, 0)$, then $F(x, G(x)) = 0$ $\forall$ $x \in B_{r_1}(x_*)$
$(F(x, N(x, 0)) = 0$ $\forall x$

Then $G \in C^1$, $G: B_r(x_*) \times B_r(y_*)$ (take $r_2 = r$)

To show that the only solution to $F(x, y) = 0$ in $B_{r_1}(x_*) \times B_{r_2}(x_*)$ are of the form $y = G(x)$, assume $F(x, y) = 0$, $(x, y) \in B_{r_1}(x_*) \times B_{r_2}(y_*)$

$H^{-1} \cdot H(x, y) = (x, y)$ $\forall (x, y) \in B_{r_1}(x_*) \times B_{r_2}(x_*)$

$(M(x, F(x, y)), N(x, F(x, y))) = (x, y)$ $\forall (x, y)$

If we assume $F(x, y) = 0$, then $N(x, 0) = y$

Corollary : under the same assumption, if $F(x, G(x)) = 0 \quad \forall x \in B_{r_1}(x_*)$

then $D_xF(x, G(x)) + (D_yF(x, G(x)))DG(x) = 0$.

this defines $DG(x)$ implicity (in ~~terms~~ of G)

pf: $\underset{\mathbb{R}^n}{x} \longrightarrow \underset{\mathbb{R}^n \times \mathbb{R}^k}{(x, G(x))} \longrightarrow F(x, G(x)) = 0$

Apply the chain rule $\quad 0 = (DF)(x, G(x)) \cdot \begin{pmatrix} I \\ DG(x) \end{pmatrix}$

Explicity $\quad 0 = (D_xF(x, G(x)),\ D_yF(x, G(x))) \begin{pmatrix} I \\ DG(x) \end{pmatrix}$

Remark : If $F: \mathbb{R}^{n+k} \to \mathbb{R}^k$, C^1, $DF(\underset{\mathbb{R}^{n+k}}{P})$ has maximum rank k.

then one can choose k variables playing the role of y in the above theorem

Example . Homework . 17.2 #1

Apply the implicit Function theorem to

$$\begin{cases} (x^2+y^2+z^2)^3 - x + z = 0 \\ \cos(x^2+y^4) + e^z - z = 0 \end{cases}$$

near $(0, 0, 0)$

$F(x, y, z) = ((x^2+y^2+z^2)^3 - x + z,\ \cos(x^2+y^4) + e^z - z)$

$F(0, 0, 0) = (0, 0)$

$DF(0, 0, 0) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ so (x, z) play the role of "y"

$D_{x,z}F(0, 0, 0) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible

By the implicit function theorem $\exists\, r_1$ and $G: (-r_1, r_1) \to \mathbb{R}^2$

s.t $F(G_1(y), y, G_2(y)) = 0 \quad \forall y$ and $y \to (G_1(y), y, G_2(y))$

parametizes the solution set of eqn near $(0, 0, 0)$

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17.3 Surfaces in $\mathbb{R}^3$

Let $f: \mathbb{R}^3 \to \mathbb{R}$, C^1

let $S = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$

Assume $\nabla f(x, y, z) \neq 0 \quad \forall (x, y, z) \in S$, Then S is a C^1 surface

let $(x_0, y_0, z_0) \in S$ WLOG, $\partial f/\partial z\,(x_0, y_0, z_0) \neq 0$

Apply the implicit function theorem,

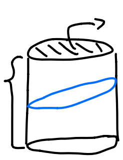
$\exists\, r_1, r_2 > 0$, $g: B_{r_1}(x_0, y_0) \to (z_0 - r_2, z_0 + r_2)$, C^1

s.t $f(x, y, g(x, y)) = 0 \quad \forall (x, y) \in B_{r_1}(x_0, y_0)$

and if $(x, y, z) \in B_{r_1}(x_0, y_0) \times (z_0 - r_2, z_0 + r_2)$, and $f(x, y, z) = 0$.

then $z = g(x, y)$

Thus S agrees with the graph of g in $B_{r_1}(x_0, y_0) \times (z_0 - r_2, z_0 + r_2)$



S defined locally as a graph

Tangent, normal vectors at (x_0, y_0, z_0)

S is parametrized locally, by $(x, y) \to (x, y, g(x,y)) \in S$

In particular $\gamma_1(x) \to (x, y_0, g(x, y_0)) \in S$ is a paramterized path in S.

$\gamma_1'(x_0) = (1, 0, \frac{\partial g}{\partial x}(x_0, y_0)) = T_1$ tangent direction

$\gamma_2(y) \to (x_0, y, g(x_0, y_0))$ is also another paramterized path in S

$\gamma_2'(y_0) = (0, 1, \frac{\partial g}{\partial y}(x_0, y_0)) = T_2$ another tangent direction / vector

T_1 and T_2 are linearly independent. span the tangent space of S at (x_0, y_0, z_0)

These formula depend on g, which is implicitly defined.

To describe the tangent plane (through $(0,0,0)$) in term of the defining function f.

will show $T_1 \perp \nabla f(x_0, y_0, z_0)$

$T_2 \perp \nabla f(x_0, y_0, z_0)$

Writing $f(x, y, g(x,y)) = 0$ $\forall (x,y)$ in a nbhd of (x_0, y_0) taking $\frac{\partial}{\partial x}$;

using the chain rule, $\frac{\partial f}{\partial x}(x_0, y_0, g(x_0, y_0)) + \frac{\partial f}{\partial z}(x_0, y_0, g(x_0, y_0)) \cdot \frac{\partial g}{\partial x}(x_0, y_0) = 0$

i.e. $\langle (\frac{\partial f}{\partial x}(x_0, y_0, g(x_0, y_0)), \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}), (1, 0, \frac{\partial g}{\partial x}(x_0, y_0)) \rangle = 0$

$\langle (\nabla f(x_0, y_0, z_0)), (T_1(x_0, y_0)) \rangle = 0$

taking $\frac{\partial}{\partial y}[f(x, y, g(x,y))]\big|_{(x_0, y_0)} = 0$ get $\langle \nabla f(x_0, y_0, z_0), T_2 \rangle = 0$

thus T_1, T_2 span the plane through $(0,0,0)$ which is // to the tangent plane of S.

at (x_0, y_0, z_0), $\nabla f(x_0, y_0, z_0)$ is normal to S.

Remark: $N(Df(x_0, y_0, z_0))$ is the tangent plane (through $(0,0,0)$) of S at (x_0, y_0, z_0)

C^1 curves in $\mathbb{R}^3$ defined as intersections of 2 C^1 surfaces.

let $g, h : \mathbb{R}^3 \to \mathbb{R}$, C^1,

define $C = \{(x, y, z) \in \mathbb{R}^3 \mid g(x,y,z) = h(x,y,z) = 0\}$

If we assume $\nabla g(x,y,z) \neq 0$, $\nabla h(x,y,z) \neq 0$ $\forall x, y, z \in C$

and $\nabla g(x,y,z)$ and $\nabla h(x,y,z)$ are linearly independent $\forall (x,y,z) \in C$

then C is a C^1 curve in $\mathbb{R}^3$

Apply the implicit function theorem to $F: \mathbb{R}^3 \to \mathbb{R}^2$.

$F(x,y,z) = \begin{pmatrix} g(x,y,z) \\ h(x,y,z) \end{pmatrix}$ $\quad DF(x,y,z) = \begin{pmatrix} \nabla g(x,y,z) \\ \nabla h(x,y,z) \end{pmatrix}$

It is assumed that the rank of $DF(x,y,z)$ is 2 $\forall (x,x,z) \in C$.

pick $(x_0, y_0, z_0) \in C$.

after relabeling coordinates, we can assume

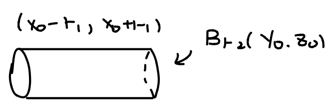
$(D_{y,z} F)(x_0, y_0, z_0) = \begin{pmatrix} \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial y & \partial h/\partial z \end{pmatrix}$ is invertiable

thus (y, z) play the roal of "y" in the implicit function theorem.

thus $\exists\ r_1, r_2 > 0$, $\gamma: (x_0 - r_1, x_0 + r_1) \to B_{r_2}(y_0, z_0) \subseteq \mathbb{R}^2$.

s.t $F(x, \gamma(x)) = 0$ $\forall x (x_0 - r_1, x_0 + r_1)$

on if $(x,y,z) \in (x_0 - r_1, x_0 + r_1) \times B_{r_2}(y_0, z_0)$ one s.t $F(x,y,z) = 0$



then $(y, z) = \gamma(x)$

C agree with $(x, \gamma(x))$

inside the cylinder

Tangent vector to C at (x_0, y_0, z_0) is $T = (1, \gamma'(x_0))$

know $g(x, \gamma(x)) = 0 \quad \forall x \in (x_0 - r_1, x_0 + r_1)$

take $\partial/\partial x$, $\frac{\partial g}{\partial x}(\ ) + \frac{\partial g}{\partial y}(\ )\gamma_1'(x) + \frac{\partial g}{\partial z}(\ )\gamma_2'(x) = 0$

$\langle \nabla g(x_0, y_0, z_0), (1, \gamma'(x_0)) \rangle = 0$

similarly. $\langle \nabla h(x_0, y_0, z_0), T \rangle = 0$.

thus T is the tangent direction

$N_1 = \nabla g(x_0, y_0, z_0)$ $N_2 = \nabla h(x_0, y_0, z_0)$ are normal direction

the tangent direction is // to $\nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0) = \lambda T \qquad \lambda \neq 0$