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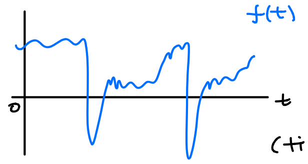
Applied Harmonic Analysis

Background :



(time space)

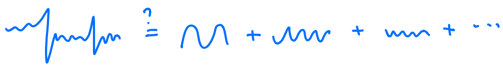
signal { voice, sound
images (2D, 3D)
radio waves (FM)

Fourier analysis :

Fourier series (periodic f)

$f(t) \overset{?}{=} a_0/2 + \sum_{n=1}^{\infty} a_n \cos(w_n t) + b_n \sin(w_n t)$

amplitude — frequency



Fourier transform (nonperiodic f)

$\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-ikt}\, dt$ where $i = \sqrt{-1}$

Function $f \in V$ ← some vector space (say $L^2(\mathbb{R})$ or $\ell^2(\mathbb{R})$)

↓

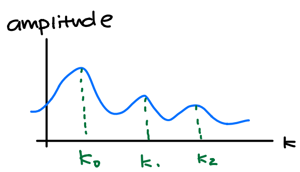
Decompose f (into say sines/cosines or wavelets)

↓

In frequency space, can manipulate f : compression, denoising, apply other filters ..

ex: low-pass filter

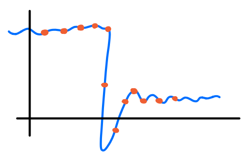
$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(w_n t) + b_n \sin(w_n t)$



(frequency)

In practice, need to discretize !

- DFT (Discrete Fourier Transform)
- DWT (Discrete Wavelet Transform)



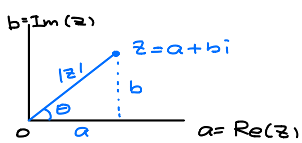
need to sample points of f.

Number Systems.

countable { $\mathbb{N} = \{0, 1, 2, 3 \dots\}$ naturals
$\mathbb{Z} = \{\dots -1, 0, 1, 2 \dots\}$ integers
$\mathbb{Q} = \{ p/q \mid p, q \in \mathbb{Z},\ q \neq 0 \}$ rationals.

uncountable { $\mathbb{R} = \{$ real numbers $\}$ ex: $\pi, -\frac{1}{2}, 0, e, \dots$
$\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}$ complex.

Complex numbers



$|z| = z\bar{z} = \sqrt{a^2 + b^2}$ (modulus)

conjugate $\bar{z} = a - bi$

polar form : $z = re^{i\theta}$ where $r = |z|$, $\theta = \tan^{-1}(\frac{b}{a})$

$e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's Formula)

Complex Fourier series :

$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$ where n = frequency

Roots of complex $z \in \mathbb{C}$

if $w^n = z$, then $w = n^{th}$ root of z

$w = \rho e^{i\alpha}$, $z = re^{i\theta}$

$w^n = \rho^n e^{in\alpha}$

$\Rightarrow \boxed{w = \sqrt[n]{r}\ \exp\left(i\,\frac{\theta + 2k\pi}{n}\right) \quad (k = 0, 1, 2 \ldots n-1)}$

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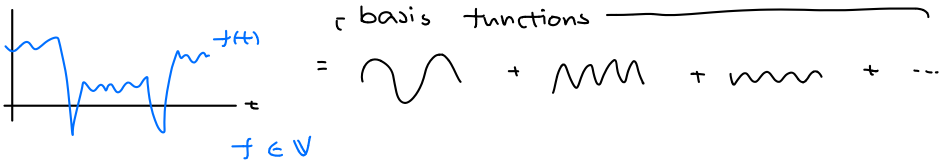
Vector (linear) spaces.



Examples of vector spaces : $\mathbb{R}^n$ finite dimensional

$\mathbb{C}^n$ $2n$ dimensional

$\mathbb{P}_n = \{n^{th}$ degree polynomials$\}$ = poly

infinite dimensional $\begin{cases} C[0,1] = \{u : [0,1] \to \mathbb{R} \mid u \text{ is continuous}\} \\ L^2(a,b) = \{u : (a,b) \to \mathbb{R} \mid \int_a^b |u(t)|^2\, dt < \infty\} \end{cases}$

Vector Space Axioms : let $u, v, w \in \mathbb{V}$ = some set and let $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$

linearity — vectors — fields

1) $u + (v + w) = (u + v) + w$
2) $u + v = v + u$
3) $\exists\, 0 \in \mathbb{V}$ s.t $u + 0 = u$
4) $\exists (-u) \in \mathbb{V}$ s.t $u + (-u) = 0$
5) $\alpha(\beta u) = (\alpha\beta) u$
6) $(\alpha + \beta) u = \alpha u + \beta u$
7) $\alpha(u + v) = \alpha u + \alpha v$

Defn : If $\{v_1, \ldots, v_n\} \subset \mathbb{V}$ and $\{a_1, \ldots, a_n\} \subset \mathbb{R}$ (or $\mathbb{C}$)

then $\sum_{i=1}^n a_i v_i = a_1 v_1 + \cdots + a_n v_n$ is a linear combination of the $\{v_i\}_{i=1}^n$

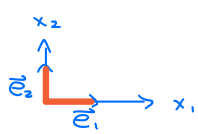
note : n has to be finite

ex : $\sum_{k=1}^{10} a_k \cos(kt)$ is a lin. comb. of $\{\cos(kt)\}_{k=1}^{10}$

Defn : The span of $\{v_1, \ldots, v_n\} \subset \mathbb{V}$ is the set

$\text{span}\{v_1, \ldots, v_n\} = \{\sum_{i=1}^n a_i v_i \mid v_i \in \mathbb{V},\ a_i \in \mathbb{K}\}$ $\quad \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

ex : $\mathbb{R}^2$, $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$\text{span}\{\vec{e}_1\}$ = entire x_1 axis $= \{a\vec{e}_1 \mid a \in \mathbb{R}\}$

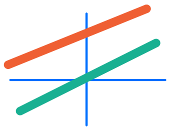
$\text{span}\{\vec{e}_1, \vec{e}_2\} = \mathbb{R}^2 = \{a_1\vec{e}_1 + a_2\vec{e}_2 \mid a_1, a_2 \in \mathbb{R}\} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

Subspaces :

ex : span $\{v_i\}_{i=1}^n$ is a subspace of $\mathbb{V}$.

A subspaces of $\mathbb{V}$ is a set $\mathbb{X} \subset \mathbb{V}$ s.t $u + cv \in \mathbb{V}$ $\forall c \in \mathbb{R}$ or $\mathbb{C}$, $u, v \in \mathbb{X}$

- we say $\mathbb{X} \subset \mathbb{V}$ is closed under scalar multiplication and addition.
- A subspace is itself a vector space.

ex: 

← not a subspace - doesn't go through 0.

← Subspace of $\mathbb{R}^2$

ex: $\mathbb{P}_2$ is a subspace of $C[0,1]$

Defn: A set $\{v_1, \dots, v_n\} \subset \mathbb{V}$ is linearly independent if $\sum_{i=1}^{n} a_i v_i = 0$, then $a_i = 0$ for $i = 1, \dots, n$. note: n is finite.

ex: $\binom{1}{0}, \binom{2}{0}, \binom{4}{0}$ are linearly independent.
since $\binom{1}{0} = \frac{1}{2}\binom{2}{0} + 0\binom{4}{0}$

Defn: A basis B of $\mathbb{V}$ is a linearly independent set of vectors that span $\mathbb{V}$.

ex: $B = \{1, t, t^2\}$ form a basis of $\mathbb{P}_2$.

Defn: The dimension of $\mathbb{V}$ is the number of vectors in any basis.

ex: $\dim(\mathbb{R}^2) = 2$, $\dim(\mathbb{P}_2) = 3$, $\dim(C[0,1]) = \infty$
note: need to be careful in $C[0,1]$
since $\sum_{i=1}^{\infty} a_i v_i$ not allowed in vector space axiom.

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Basis in ∞-dim spaces.

Hamel Basis (ignore) - uses finite # of vectors.

Schauder Basis:

B is a schauder basis if

i) Any finite set of vectors of B is linearly independent.

ii) $\forall \vec{v} \in \mathbb{V}$, $\vec{v} = \sum_{i=1}^{\infty} c_i \vec{v}_i$ where $B = \{\vec{v}_i\}_{i=1}^{\infty}$ (completeness)

interpreted as $\lim_{n\to\infty} \| \vec{v} - \sum_{i=1}^{\infty} c_i \vec{v}_i \|_{\mathbb{V}} = 0$

iii) The coefficients c_i are unique.

example of Schauder basis: Fourier series in $L^2(0, 2\pi) = \mathbb{V}$

$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$

↳ form a Schauder basis $\{1, e^{it}, e^{-it}, e^{2it}, e^{-2it} \dots\}$

Inner-Product Spaces: $\mathbb{K}^n = \mathbb{R}^n$ or $\mathbb{C}^n$

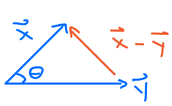
Define inner-product $\langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y}_i$ $x, y \in \mathbb{K}^n$ note $z = x + iy$, $\bar{z} = x - iy$

i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

ii) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ $a, b \in \mathbb{K}$

iii) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$

note: $\langle x, ay \rangle = \overline{\langle ay, x \rangle} = \overline{a\langle y, x \rangle} = \bar{a}\overline{\langle y, x \rangle} = \bar{a}\langle x, y \rangle$

ex: $\mathbb{R}^2$ 

$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cos\theta$

pf: law of cosines $\|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$

$\langle \vec{x} - \vec{y}, \vec{x} - \vec{y} \rangle = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$

$\cancel{\|\vec{x}\|^2} - \langle \vec{x}, \vec{y} \rangle - \langle \vec{y}, -\vec{x} \rangle + \cancel{\|\vec{y}\|^2} = \cancel{\|\vec{x}\|^2} + \cancel{\|\vec{y}\|^2} - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$

Normed spaces

$(\mathbb{V}, \|\cdot\|)$ is a normed space where $\|\cdot\| : \mathbb{V} \to \mathbb{R}$ and satisfies

i) $\|u + v\| \le \|u\| + \|v\| \quad \forall u, v \in \mathbb{V}$. (triangle inequality)

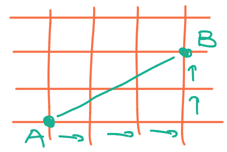
ii) $\|au\| = |a| \|u\| \quad a \in \mathbb{K}$

iii) $\|u\| \ge 0$ and $\|u\| = 0$ iff $u = 0$

All inner-products generate a norm: $\|x\| = \sqrt{\langle x, x\rangle}$

Other norms on $\mathbb{R}^n$: let $p \ge 1$, and set $\|\vec{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$

$\|\vec{x}\|_1 = \sum_{i=1}^n |x_i|$ (1-norm, taxicab norm)



Dist btw. A, B in $\|\vec{x}\|_1$ is 5

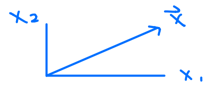
$\|\vec{x}\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$ (2-norm)

generated by $\langle x, y\rangle = \sum_{i=1}^n x_i \bar{y}_i$

$\|\vec{x}\|_\infty = \max_{i=1,\dots,n} |x_i|$ (∞-norm, max norm)

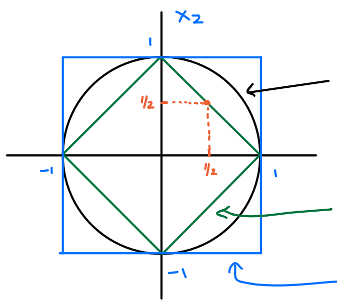
note $\|\vec{x}\|_\infty = \lim_{p\to\infty} \|\vec{x}\|_p$

Ex:



$\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2} = \left(\sum_{i=1}^2 x_i^2\right)^{1/2}$ (Euclidian norm)

Unit Spheres in various norms $\mathbb{R}^2$



$\|\vec{x}\|_2 = 1 \implies \|x\|_2^2 = 1$

$x_1^2 + x_2^2 = 1$

$\|\vec{x}\|_1 = \sum_{i=1}^2 |x_i| = |x_1| + |x_2| = 1$

$\|\vec{x}\|_\infty = \max_{i=1,2} |x_i| = 1$

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Equivalent Norms

we say any two norms $\|x\|_\alpha$, $\|x\|_\beta$ are equivalent if $\exists A, B > 0$ s.t $A\|x\|_\alpha \le \|x\|_\beta \le B\|x\|_\alpha$ (norm are comparable)

Theorem: In finite dimensional spaces, ALL norms are equivalent.

Cauchy-Schwarz Inequality

For any $x, y \in \mathbb{V}$ ($\mathbb{V}$ inner-product space)

we have $|\langle x, y\rangle| \le \|x\|\|y\|$ recall $\|x\| = \sqrt{\langle x, x\rangle}$

pf: If $x = 0$ or $y = 0$, done.

so, suppose $x, y \ne 0$. suppose $\mathbb{V} = \mathbb{R}$

$0 \le \underbrace{\|x - ty\|^2}_{\text{constant}} = \langle x - ty, x - ty\rangle = \underbrace{\|x\|^2}_{c} - \underbrace{2t\langle x, y\rangle}_{-2tb} + \underbrace{t^2\|y\|^2}_{t^2 a}$

$= c - 2tb + at^2$ (quadratic in t)

$t = \dfrac{2b \pm \sqrt{(-2b)^2 - 4ac}}{2a}$ want $(-2b)^2 - 4ac \le 0$

$b^2 \le ac$ (b^2: $|\langle x, y\rangle|^2$, a: $\|y\|^2$, c: $\|x\|^2$) $\implies |\langle x, y\rangle| \le \|x\|\|y\|$

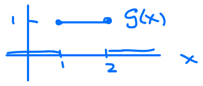
The vector space $L^2[a,b]$

$$L^2[a,b] := \{ f : [a,b] \to \mathbb{C} \mid \int_a^b |f(x)|^2 \, dx < \infty \}$$

this is an inner-product space with $\langle f, g \rangle = \langle f, g \rangle_{L^2[a,b]} := \int_a^b f(x)\overline{g(x)}\,dx$

an norm $\|f\|_{L^2[a,b]} = \|f\|_{L^2} = \left(\int_a^b |f(x)|^2 dx \right)^{1/2}$

ex: $g \in L^2(\mathbb{R})$



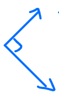
$$\|g\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |g(x)|^2 dx \right)^{1/2} = \left(\int_1^2 1^2 \, dx \right)^{1/2} = 1$$

Cauchy - Schwarz in L^2: $|\langle f, g \rangle| \le \|f\| \|g\|$

$$\left| \int_a^b f(x)\overline{g(x)}\,dx \right| \le \left(\int_a^b |f|^2 dx \right)^{1/2} \left(\int_a^b |g|^2 dx \right)^{1/2}$$

Orthogonality

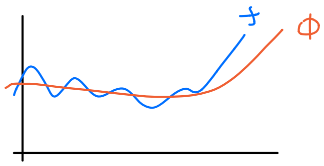
let $u, v \in V$, we say u, v are orthogonal if $\langle u, v \rangle = 0$

ex:  $\vec{x} = \binom{1}{1}$, $\vec{y} = \binom{1}{-1}$

$\langle \vec{x}, \vec{y} \rangle = \langle \binom{1}{1}, \binom{1}{-1} \rangle = 1 - 1 = 0$

note $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|_2 \|\vec{y}\|_2 \cos\theta = 0$ $\quad \theta = \frac{\pi}{2}$

plus if $\|u\|, \|v\| = 1$, then u, v are orthonormal.



- $\|f - \phi\|_2$ is the average distance btw f, ϕ.
- $\|f - \phi\|_\infty$ is a distance that tries to minimize the maximum dist.

note $L^3[a,b] = \{ f : [a,b] \to \mathbb{R} \mid \int_a^b |f(x)|^3 dx < \infty \}$

$\|f\|_{L^3} = \left(\int_a^b |f(x)|^3 dx \right)^{1/3}$

while $\|\vec{x}\|_3 = \left(\sum_{i=1}^n |x_i|^3 \right)^{1/3}$

Orthonormal Bases

Suppose $B = \{e_1, e_2, \dots, e_n\} \subset V$ is an othonormal bass of V.

Have $\langle e_i, e_j \rangle = 0$ unless $i = j$ in which case $\langle e_i, e_j \rangle = \|e_i\|^2 = 1$

Kronecker Delta: $\langle e_i, e_j \rangle = \delta_{ij} = \delta(i-j) = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases}$

and $\forall u \in V$, $u = \sum_{i=1}^n c_i e_i$

use orthogonality: $\langle u, e_j \rangle = \langle \sum_{i=1}^n c_i e_i, e_j \rangle = \sum_{i=1}^n c_i \underbrace{\langle e_i, e_j \rangle}_{= \delta_{ij}} = c_j$

so $u = \sum_{i=1}^n \langle u, e_i \rangle e_i$

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Recall:

Orthonormal basis $\{e_k\}_{k=1}^n \subset V$.

$u \in V$, $u = \sum_{k=1}^n \langle u, e_k \rangle e_k$

$\langle u, u \rangle = \langle \sum_{k=1}^n \langle u, e_k \rangle e_k, u \rangle$

$\|u\|^2 = \sum_{k=1}^n \langle u, e_k \rangle \underbrace{\langle e_k, u \rangle}_{= \overline{\langle u, e_k \rangle}} = \sum_{k=1}^n |\langle u, e_k \rangle|^2 = \|u\|^2$

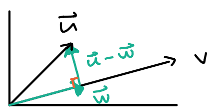
Each $\langle u, e_k \rangle e_k$ is an orthogonal projection of u onto e_k

$P_{e_k} u = \mathrm{Proj}_{e_k} u = \langle u, e_k \rangle e_k$ $\quad$ - $\{e_k\}_{k=1}^n$ is orthonormal here

If $M = \text{span}\{e_1, \dots, e_m\}$ then $P_M u = \text{Proj}_M u = \sum_{k=1}^{m} \langle u, e_k \rangle e_k$

set $\quad m \in \mathbb{Z}^+ = \mathbb{N}$ (index)

In $\mathbb{R}^2$



$\vec{w} = \text{proj}_{\vec{v}} \vec{u}$ have $\vec{w} = \lambda \vec{v}$ for some $\lambda \in \mathbb{R}$

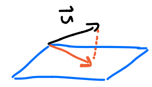
$0 = \langle \vec{u} - \vec{w}, \vec{v} \rangle = \langle \vec{u} - \lambda\vec{v}, \vec{v} \rangle = \langle u, \vec{v} \rangle - \underbrace{\langle \lambda\vec{v}, \vec{v} \rangle}_{\lambda \|\vec{v}\|^2}$

$\lambda = \langle \vec{u}, \vec{v} \rangle / \|\vec{v}\|^2$

$\vec{w} = \lambda\vec{v} = \dfrac{\langle \vec{u}, \vec{v} \rangle \vec{v}}{\|\vec{v}\|^2}$ when $\|\vec{v}\| = 1$, $\vec{w} = \langle \vec{u}, \vec{v} \rangle \vec{v}$

If $\{e_k\}_{k=1}^{n}$ is an orthogonal basis then $u = \sum_{k=1}^{n} \dfrac{\langle u, e_k \rangle}{\|e_k\|^2} e_k$

P_m is a projection if $P_m(P_m) = P_m^2 = P_m$



let's check for $P_M u = \sum_{k=1}^{m} \langle u, e_k \rangle e_k$

$P_M(P_M u) = \sum_{k=1}^{m} \langle P_m u, e_k \rangle e_k$

$= \sum_{k=1}^{m} \underbrace{\langle \sum_{j=1}^{m} \langle u, e_j \rangle e_j, e_k \rangle}_{= \sum_{j=1}^{m} \langle u, e_j \rangle \underbrace{\langle e_j, e_k \rangle}_{= \delta_{jk}} = \langle u, e_k \rangle} e_k$

$= \sum_{k=1}^{m} \langle u, e_k \rangle e_k$

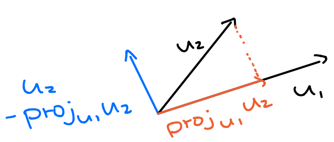
$= P_M u$

Consider $\{1, t\} \subset L^2[0,1] = \mathbb{V}$

$\langle 1, t \rangle = \int_0^1 1 \cdot t \, dt = 1/2 \neq 0 \implies 1, t$ not orthogonal

How to orthogonalize ?

$\{u_1, u_2, \dots, u_n\}$ non-orthogonal want to turn into $\{v_1, v_2 \dots v_n\}$ orthogonal.



Now $\{\overbrace{u_1}^{= v_1}, \overbrace{u_2 - \text{proj}_{u_1} u_2}^{= v_2}\}$ is an orthogonal set spanning the same set as $\{u_1, u_2\}$

Gram - Schmidt Process

To turn $\{u_i\}_{i=1}^{n}$ non-orthogonal set into $\{v_i\}_{i=1}^{n}$ orthogonal.

$v_1 = u_1$

$v_2 = u_2 - \text{proj}_{v_1} u_2$

$v_3 = u_3 - \text{proj}_{\text{span}(v_1, v_2)} u_3 = u_3 - \text{proj}_{v_1} u_3 - \text{proj}_{v_2} u_3$

$\vdots$

$v_n = u_n - \text{proj}_{\text{span}(v_1 \dots v_n)} u_n$

we have $\text{span}\{u_1, \dots, u_n\} = \text{span}\{v_1 \dots v_n\}$, $\{v_1 \dots v_n\}$ is orthogonal,

we can normalize by $\left\{ \dfrac{v_1}{\|v_1\|}, \dfrac{v_2}{\|v_2\|}, \dots, \dfrac{v_n}{\|v_n\|} \right\}$

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Linear Operators (map , function)

$T: \mathbb{V} \to \mathbb{W}$ is a linear map if $\forall c \in \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\forall u, v \in \mathbb{V}$.

i) $T(u+v) = T(u) + T(v)$

ii) $T(cu) = cT(u)$

Ex: $T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(\vec{x}) = A\vec{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\quad A =$ linear operator (Matrix)

Ex: $T: L^2(0,1) \to \mathbb{R}$

$T(u) = \int_0^1 u(t) e^t \, dt$

$T(au+bv) = \int_0^1 (au+bv)(t) e^t \, dt = a T(u) + b T(v)$

Ex: $T: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $x \in \ell^2(\mathbb{N})$, $x = \{x_1, x_2 \dots\}$

$T(x) = \{ \frac{1}{2} x_1, \frac{1}{2^2} x_2, \frac{1}{2^3} x_3, \dots, \frac{1}{2^n} x_n, \dots \}$

$T(ax+by) = \{ \frac{1}{2}(ax_1+by_1), \frac{1}{2^2}(ax_2+by_2) \dots \}$

$= a\{ \frac{1}{2} x_1, \frac{1}{2^2} x_2, \dots \} + b\{ \frac{1}{2} y_1, \frac{1}{2^2} y_2, \dots \}$

$= a T(x) + b T(y)$

In finite dimensions, any linear map $T: V \to W$ can be represented as matrix multiplication. where V, W are vector spaces.

Let $\{V_1 \dots V_n\}$ be a basis of V and let $\{w_1 \dots w_m\}$ be a basis of W.

Note $T(V_j) \in W \quad \forall j = 1, \dots, n$

$\Rightarrow T(V_j) = \sum_{i=1}^m a_{ij} w_i$

let $v = \sum_{j=1}^n x_j V_j \in V$

Then $T(v) = T\left(\sum_{j=1}^n x_j v_j\right) = \sum_{j=1}^n x_j T(v_j) = \sum_{j=1}^n x_j \left(\sum_{i=1}^m a_{ij} w_i\right) = \sum_{i=1}^m \overbrace{\sum_{j=1}^n a_{ij} x_j}^{= c_i} w_i = \sum_{i=1}^m c_i w_i$

Here $c_i = \sum_{j=1}^n a_{ij} x_j$ is precisely matrix-vector multiplication : $\vec{z} = A\vec{x}$

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Think of $T(\vec{v}) = A\vec{v}$

The column of A store the coefficients of $T(V_j)$ $j = 1 \dots n$

where $\{V_j\}_{j=1}^n$ is a basis of V.

Ex: Find the matrix representation of $T: V \to W$ where $V = \mathbb{R}^2$, $W = \text{span}\{\cos t, \sin t\}$. and $T(\vec{v}) = (v_1 + 2v_2)\cos t$, $v_2 \sin t$

basis for V let's use $\{\vec{e}_1, \vec{e}_2\} = \{\binom{1}{0}, \binom{0}{1}\}$

and for W let's use $\{w_1, w_2\} = \{\cos t, \sin t\}$

Have $T(\vec{e}_1) = T\binom{1}{0} = 1 \cdot \cos t + 0 \cdot \sin t$

$T(\vec{e}_2) = T\binom{0}{1} = 2 \cdot \cos t + 1 \cdot \sin t$

$\Rightarrow A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \vec{v} = x_1\binom{1}{0} + x_2\binom{0}{1} \in V = \mathbb{R}^2$, $\vec{z} = \binom{c_1}{c_2}$, $\vec{z} = A\vec{x}$.

basis of $\mathbb{R}^2$

So $T(\vec{v}) = c_1 w_1 + c_2 w_2 = c_1 \cos t + c_2 \sin t$

And if used basis $\{\binom{1}{0}, \binom{1}{1}\}$ for $V = \mathbb{R}^2$.

$T\binom{1}{0} = 1 \cdot \cos t + 0 \cdot \sin t$

$T\binom{1}{1} = 3 \cdot \cos t + 1 \cdot \sin t$

$\Rightarrow A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

Ex: if basis $\{\binom{1}{0}, \binom{0}{1}\}$ of $V = \mathbb{R}^2$, basis $\{w_1, w_2\} = \{\cos t + \sin t, \sin t\}$ of W.

then $A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ $\vec{v} = x_1\binom{1}{0} + x_2\binom{0}{1}$, $T(\vec{v}) = c_1(\cos t + \sin t) + c_2 \sin t$

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Adjoint Operators

$T: V \to W$ linear operator (map)

inner-product spaces with $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$

Can we write $\langle Tv, w\rangle_W$ as $\langle v, T^*w\rangle_V$ for some new operator T^* the adjoint of T.

Tv or $T(v)$ denotes action of T on v

i.e. $T^*: W \to V$ s.t $\langle Tv, w\rangle_W = \langle v, T^*w\rangle_V$

Goal: Try to write $\langle Tv, w\rangle_W = \langle v, z\rangle_V$ for some $z \in V$ and define $T^*w = z$.

Theorem: If V, W are finite-dimensional and $T: V \to W$ is linear, then $T^*: W \to V$ exists and is unique

If $T = T^*$ we call T self-adjoint, $\langle Tv, w\rangle = \langle v, T^*w\rangle = \langle v, Tw\rangle$

Ex: The Fourier transform $F[f] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{ix^3} dx$ is self-adjoint with the $L^2(\mathbb{R})$ inner-product $\langle f, g\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)g(x)\,dx$

so $\langle Ff, g\rangle_{L^2} = \langle f, Fg\rangle_{L^2}$

Ex: $T = I$ $I: V \to V$ (identity map) $Iv = v$

$\langle Iv, w\rangle = \langle v, Iw\rangle$

so $I^*w = Iw \Rightarrow I^* = I$ by definition

Ex: Let $a(t)$ be bounded. $\langle f, g\rangle_{L^2} = \int_a^b f(x)\overline{g(x)}\,dx$

Define $T: L^2(a,b) \to L^2(a,b)$ by $(Tf)(t) = a(t)f(t)$

Find T^*

Have $\langle Tf, g\rangle = \langle af, g\rangle = \int_a^b a(t)f(t)\overline{g(t)}\,dt = \int_a^b f(t)\cdot \underline{a(t)\overline{g(t)}}\,dt$

$= \int_a^b f(t)\overline{\overline{a(t)}g(t)}\,dt$ $= \overline{\overline{a(t)}\,g(t)}$

$= \langle f, \bar{a}g\rangle$ so $(T^*g)(t) = \overline{a(t)}\,g(t)$

If $a(t)$ real-valued, then $T^* = T$ and T self-adjoint.

Integration by Parts:

$$\int_a^b f(t)g'(t)\,dt = f(t)g(t)\Big|_{t=a}^{t=b} - \int_a^b f'(t)g(t)\,dt$$

Ex: $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = A\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Fact: $A^* = \bar{A}^T$ so $A^* = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$ $A^*: \mathbb{R}^2 \to \mathbb{R}^3$

$\langle A\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\rangle_2 = \langle \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\rangle_2 = \langle \begin{pmatrix} x+y \\ x+2z \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\rangle_2 = (x+y)a + (x+2z)b$

$= x(a+b) + ya + z(2b) = \langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a+b \\ a \\ 2b \end{pmatrix}\rangle_3$

we define $A^*\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a \\ 2b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$ $A: \mathbb{R}^3 \to \mathbb{R}^2$, $A^*: \mathbb{R}^2 \to \mathbb{R}^3$

$\langle A\vec{x}, \vec{y}\rangle_2 = \langle \vec{x}, A^*\vec{y}\rangle_3$

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Intro to Frames

V with orthonormal basis $\{v_1, v_2, v_3, \dots\}$ (ONB)

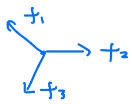
- can be restrictive in practice
- easy to compute c_i s.t $x = \sum c_i v_i$, $c_i = \langle x_i, v_i \rangle$

$\{f_1, f_2, f_3 \dots\}$ is a frame of V if the $\{f_i\}_{i \in I}$ span V and satisfy some other conditions.

- frame can be linearly dependent.

ex: 

a basis of $\mathbb{R}^2$ (orthonormal)

ex: 

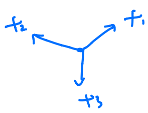
a frame for $\mathbb{R}^2$ (contains redundancies)

ex: 

another frame

ex: Reconstruction, orthonormal basis $\{e_1, e_2\}$

person A, $\{1, 2\}$, $x = 1 \cdot e_1 + 2 \cdot e_2$.
↓ "erasure" "information loss", lose all info. in the e_2 dire.
person B, $\{1, ?\}$, $x = 1 \cdot e_1 + ?$



person A, $\{1, 2, 3\}$, $x = 1 \cdot f_1 + 2 \cdot f_2 + 3 \cdot f_3$
↓ retains more info. than with ONB and sometimes can get "perfect reconstruction"
person B, $\{1, 2, ?\}$, $x = 1 \cdot f_1 + 2 \cdot f_2 + ?$

Primer on Hilbert Spaces

A Hilbert space $\mathbb{H}$ is a complete inner-product space.

- also normed $\|x\|_{\mathbb{H}} = \sqrt{\langle x, x \rangle_H}$

Defn: $\{x_n\}_{n=1}^{\infty} \subset \mathbb{H}$ is a Cauchy-sequence if
$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ s.t $\forall n, m \geq N$, $\|x_n - x_m\|_H < \varepsilon$

Defn: $\mathbb{H}$ is complete if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point $x \in \mathbb{H}$.
i.e. $\lim_{n\to\infty} x_n = x \in \mathbb{H}$, $\|x_n - x\|_H \to 0$ as $n \to \infty$

Hilbert spaces: $\mathbb{R}, \mathbb{R}^n, \mathbb{C}^n, \ell^2(\mathbb{Z}), L^2(a,b), L^2(\mathbb{R})$
any finite dimensional inner-product space i.e. $\mathbb{P}_n$

Non-Hilbert spaces: $S = (0,1) \subseteq \mathbb{R}$, $x_n = 1/n$ for $n \geq 2$ converges to $0 \notin S$

Defn: H is separable if it contains a countable basis.

- All examples above are separable.

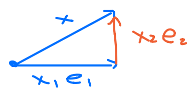
For ONB, of a Hilbert space H, $x = \sum_{i \in I} \langle x, e_i \rangle e_i \quad \forall x \in H$

Dotting with x we get

$\|x\|^2 = \langle x, \sum_{i\in I} \langle x, e_i\rangle e_i \rangle = \sum_{i\in I} \overline{\langle x, e_i\rangle} \langle x, e_i\rangle = \sum_{i\in I} |\langle x, e_i\rangle|^2$

└ parseval equality

ex:



$\mathbb{R}^2$-case: $\|x\|^2 = x_1^2 + x_2^2 = |\langle x, e_1\rangle|^2 + |\langle x, e_2\rangle|^2 = \sum_{i=1}^{2} |\langle x, e_i\rangle|^2$

pythegorean theorem

Frames generzlize Parseval equality:

Defn: $\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$ if I is countable and if $\exists\, 0 < A \le B < \infty$

s.t $\forall x \in \mathbb{H}$ $\quad A\|x\|^2 \le \sum_{i\in I} |\langle x, f_i\rangle|^2 \le B\|x\|^2$

frame bounds.

- largest A and smallest B that work are called optimal frame bounds.
- For $A = B = 1$, equality holds and we call $\{f_i\}_{i\in I}$ a Parseval frame
  i.e. an ONB is parseval
- $\{f_i\}_{i\in I}$ is a tight frame if $A = B$

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Recall: $\{e_i\}_{i\in I}$ ONB: $\quad x = \sum_{i\in I} \langle x, e_i\rangle e_i$

Parseval equality: $\|x\|^2 = \sum_{i\in I} |\langle x, e_i\rangle|^2$

frames $\{f_i\}_{i\in I}$ satisfy $\quad A\|x\|^2 \le \sum_{i\in I} |\langle x, f_i\rangle|^2 \le B\|x\|^2$

note: An OBE is a frame, Let $A = B = 1$

union of N orthonormal bases is a frame

call these $\{e_i^{(j)}\}_{i\in I}$ $\quad j = 1, \dots, N$ every j is a different ONB

bases of $\mathbb{R}^2$, union is a frame of $\mathbb{R}^2$

note by Perseval, $\forall j = 1, \dots, n$. $\quad \|x\|^2 = \sum_{i\in I} |\langle x, e_i^{(j)}\rangle|^2$

$\Rightarrow \sum_{j=1}^{N} \|x\|^2 = \sum_{j=1}^{N} \sum_{i\in I} |\langle x, e_i^{(j)}\rangle|^2$

$\Rightarrow N\|x\|^2 = \sum_{j=1}^{N} \sum_{i\in I} |\langle x, e_i^{(j)}\rangle|^2$

$A = B = N$ and $\{e_i^{(j)}\}$ is a tight frame

If we have just $\{f_1\} = \{\binom{1}{0}\}$ for $\mathbb{R}^2$,

then no A exists s.t $A\|x\|^2 \le \sum_{i=1}^{1} |\langle x, f_i\rangle|^2 = |\langle x, f_1\rangle|^2$

because $|\langle x, f_1\rangle|^2 = x_1^2 \overset{?}{\ge} A\|x\|^2 = A(x_1^2 + x_2^2)$

If such an A existed, let $x_2 \to \infty$ to get contradiction

Lemma: Young's Inequality:

if $a \ge 0$, $b \ge 0$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

then $ab \le \frac{a^p}{p} + \frac{b^q}{q}$

If $p = q = 2$, then $ab \le \frac{a^2}{2} + \frac{b^2}{2} \Rightarrow 2ab \le a^2 + b^2$

i.e. $(x_1 \pm x_2)^2 \ge 0 \Rightarrow x_1^2 + x_2^2 \pm 2x_1x_2 \ge 0$

$\Rightarrow \|x\|^2 \pm 2x_1x_2 \ge 0$

$\|x\|^2 \ge 2x_1x_2 \quad \& \quad 2x_1x_2 \ge -\|x\|^2$

Ex: $\{f_i\}_{i=1}^3 = \{ f_1 = \binom{1}{0},\ f_2 = \binom{0}{1},\ f_3 = \binom{1}{1} \} \subset \mathbb{R}^2$

Goal: $\exists\, A, B$ s.t $A\|x\|^2 \leq \sum_{i=1}^3 |\langle x, f_i\rangle|^2 \leq B\|x\|^2$, $\forall x \in \mathbb{R}^2$

$$\sum_{i=1}^3 |\langle x, f_i\rangle|^2 = x_1^2 + x_2^2 + (x_1+x_2)^2 = 2(x_1^2+x_2^2) + 2x_1x_2$$
$$= 2\|x\|^2 + \underbrace{2x_1x_2}_{\leq \|x\|^2} \leq 3\|x\|^2$$

$$\sum_{i=1}^3 |\langle x, f_i\rangle|^2 = 2\|x\|^2 + \underbrace{2x_1x_2}_{\geq -\|x\|^2} \geq \|x\|^2$$

thus $\|x\|^2 \leq \sum_{i=1}^3 |\langle x, f_i\rangle|^2 \leq 3\|x\|^2$

Bounds $A=1$, $B=3$ are optimal,

to get equality, choose $x = \binom{x_1}{x_2}$ s.t $\{f_i\}_{i=1}^3$ is not tight

Now will develop theory about frames:

ONB: $x = \sum_{i\in I} \langle x, e_i\rangle e_i$ $\quad \{e_i\}_{i\in I} \subset H$

define $\langle x, e_i\rangle := T(x)$ takes x and produces $\langle x, e_i\rangle \in \mathbb{C}$, $T: \mathbb{H} \to \ell^2(I)$

$\sum_{i\in I} \langle x, e_i\rangle e_i := S(\langle x, e_i\rangle) = S(c_i)$ if $c_i := \langle x, e_i\rangle$, $S: \ell^2(I) \to \mathbb{H}$

we call $T(x) = \{\langle x, e_i\rangle\}_{i\in I}$ the analysis operator, $T: \mathbb{H} \to \ell^2(I)$

we call $S(c) = T^*(c) = \sum_{i\in I} c_i e_i$ the synthesis operator

$c = \{c_i\}_{i\in I}$ and $T^*: \ell^2(I) \to \mathbb{H}$

Same definition holds for frames $\{f_i\}_{i\in I}$

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Check $T: \mathbb{H} \to \ell^2(I)$ $\quad T(x) = \{\langle x, f_i\rangle\}_{i\in I}$

Have $\|\{\langle x, f_i\rangle\}_{i\in I}\|^2_{\ell^2(I)} := \sum_{i\in I} |\langle x, f_i\rangle|^2 \leq B\|x\|^2_{\mathbb{H}} < \infty$

└ since $\{f_i\}$ are frame

$\Rightarrow T(x) \in \ell^2(I)$

Defn: A linear operator $T: X \to Y$ is bounded if $\exists\, c > 0$

s.t $\|T(x)\|_Y \leq c\|x\|_X$ $\quad \forall x \in X$.

thus the analysis operator is bounded $\|T(x)\|_{\ell^2(I)} \leq \sqrt{B}\|x\|_{\mathbb{H}}$

Adjoint of Analysis operator T

$$\langle \underbrace{T(x)}_{=\{\langle x, f_i\rangle\}_{i\in I}}, c\rangle_{\ell^2(I)} = \sum_{i\in I} \langle x, f_i\rangle_{\mathbb{H}} \bar{c}_i = \langle x, \underbrace{\sum_{i\in I} f_i c_i}_{=:\, T^* c}\rangle_{\mathbb{H}}$$

$T^* c = \sum_{i\in I} f_i c_i$ $\quad$ synthesis operator

Frame Operator: $S := T^*T : \mathbb{H} \to \mathbb{H}$, $\quad T^*: \ell^2 \to \mathbb{H}$, $T: \mathbb{H} \to \ell^2$

$S(x) = \sum_{i\in I} \langle x, f_i\rangle_{\mathbb{H}} f_i$

Now $\langle S(x), x\rangle_{\mathbb{H}} = \sum_{i\in I} \langle x, f_i\rangle \underbrace{\langle f_i, x\rangle}_{=\overline{\langle x, f_i\rangle}} = \sum_{i\in I} |\langle x, f_i\rangle|^2$

$A\|x\|^2 \leq \langle S(x), x\rangle \leq B\|x\|^2$

Remark: $S = T^*T$ is self-adjoint since

$S^* = (T^*T)^* = T^*(T^*)^* = T^*T = S$

Ex: 2.1.4 in Czaja's notes

finite frame $\{f_i\}_{i=1,\dots,n} \subset \mathbb{R}^d$ $\quad f_1 = \binom{a}{b}, \dots f_n$

Analysis operator $T(x) = \{\langle x, f_i \rangle\}_{i=1}^n$ where $x \in \mathbb{R}^d$

we have $T: \mathbb{R}^d \to \mathbb{R}^n$ and T has an $n \times d$ matrix representation $T(x) = Ax$ $\forall x \in \mathbb{R}^d$.

Let's find A:

suppose x, f_i are all column vectors.

T takes $x \in \mathbb{R}^d$ and maps it to $\{\langle x, f_i \rangle\}_{i=1}^n \in \mathbb{R}^n$

note $\langle x, f_i \rangle = f_i^* x$

$*$ denotes conjugate transpose $A^* := \bar{A}^T$ $\quad$ recall $\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$

$$\binom{1}{0} \cdot \binom{2}{3} = (1 \;\; 0)\binom{2}{3}$$
$$\quad \parallel \qquad\qquad \parallel$$
$$\quad 2 \qquad\qquad 2$$

So $$T(x) = \begin{pmatrix} f_1^* x \\ f_2^* x \\ \vdots \\ f_n^* x \end{pmatrix} = \underbrace{\begin{pmatrix} f_1^* \\ f_2^* \\ \vdots \\ f_n^* \end{pmatrix}}_{:= A_{n \times d}} x = Ax$$

and $T = \begin{pmatrix} f_1^* \\ \vdots \\ f_n^* \end{pmatrix}$ analysis operator

Ex: $\{f_1, f_2, f_3\} = \{\binom{1}{0}, \binom{1}{1}, \binom{4}{2}\} \subset \mathbb{R}^2$ $\quad$ Have $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 4 & 2 \end{pmatrix}$

$$\forall x \in \mathbb{R}^2, \quad T(x) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 4 & 2 \end{pmatrix}\binom{x_1}{x_2} = \begin{pmatrix} (1\;0)\binom{x_1}{x_2} \\ (1\;1)\binom{x_1}{x_2} \\ (4\;1)\binom{x_1}{x_2} \end{pmatrix} = \begin{pmatrix} \langle x, f_1 \rangle \\ \langle x, f_2 \rangle \\ \langle x, f_3 \rangle \end{pmatrix}$$

Frame operator: $$\begin{aligned} S &= T^* T \\ &= (f_1, f_2 \dots f_n)\begin{pmatrix} f_1^* \\ \vdots \\ f_n^* \end{pmatrix} \\ &= f_1 f_1^* + f_2 f_2^* + \dots + f_n f_n^* \\ &= \textstyle\sum_{k=1}^n f_k f_k^* \end{aligned}$$

componentwise, $S_{ij} = \sum_{k=1}^n f_k(i) \overline{f_k(j)}$

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Ex. $\{f_1, f_2\} = \{\binom{1}{0}, \binom{1}{1}\} \subset \mathbb{R}^2$

Find A, B s.t $A\|x\|^2 \le \underbrace{\sum_{i \in I} |\langle x, f_i \rangle|^2}_{\langle S(x), x \rangle_H} \le B\|x\|^2$

Let $\{\lambda_i\}_{i=1}^2$ be the eigenvalue of $S = T^* T$

Then $A = \min_i \lambda_i$ $\quad B = \max_i \lambda_i$

$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\quad T^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad S = T^* T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ frame operator

Recall the eigenvalue of S is a number λ_i s.t $Se_i = \lambda_i e_i$,

and eigenvector e_i to

In this question $\lambda_1, \lambda_2 = \frac{3 \pm \sqrt{5}}{2}$ $\quad A = \frac{3 - \sqrt{5}}{2}$, $\quad B = \frac{3 + \sqrt{5}}{2}$

note: Eigenvalues always positive since S is a positive operator meaning

$S \geq A$ or $\langle S(x), x \rangle \geq A\langle x, x \rangle = A\|x\|^2 > 0$ $\quad \forall x \in H$

Let $\{(\lambda_i, e_i)\}_{i=1}^n$ be eigen pairs of S. let $\lambda_{min} \le \lambda_i \le \lambda_{max}$ $\forall i$

Since S is self-adjoint, we can choose $\{e_i\}$ to be orthonormal.

$\forall x \in \mathbb{R}^n$, $x = \sum_{i=1}^n \langle x, e_i \rangle e_i$

$$S(x) = \sum_{i=1}^n \langle x, e_i \rangle \underbrace{S(e_i)}_{\lambda_i e_i}$$ since S is linear. since e_i are eigenvectors

$$S(x) = \sum_{i=1}^n \langle x, e_i \rangle \lambda_i e_i$$

$$\langle S(x), x \rangle = \sum_{i=1}^n \langle x, e_i \rangle \lambda_i \langle e_i, x \rangle$$
$$= \underbrace{\sum_{i=1}^n |\langle x, e_i \rangle|^2}_{\|x\|^2} \lambda_i$$ parsevals equality

$$\lambda_{min} \|x\|^2 \le \langle S(x), x \rangle \le \lambda_{max} \|x\|^2$$

Theorem: (Tight Frames) $A = B$

let $\{f_i\}_{i \in I} \subset \mathbb{H}$ be a frame for $\mathbb{H}$.

then $S = A \cdot \underline{Id}$ iff $\{f_i\}_{i \in I}$ is a tight frame w/ A frame bound.

identity operator

frame operator

Lemma: let $\|A\|^2 \le \underbrace{\langle S(x), x \rangle}_{\sum_{i \in I} |\langle x, f_i \rangle|^2} \le B\|x\|^2$

then S^{-1} satisfies $B^{-1} \cdot Id \le S^{-1} \le A^{-1} \cdot Id$

meaning $\langle B^{-1} \cdot Id, x \rangle \le \langle S^{-1}x, x \rangle \le \langle A^{-1} \cdot Id, x \rangle$

or $B^{-1} \|x\|^2 \le \langle S^{-1}x, x \rangle \le A^{-1} \|x\|^2$

so S^{-1} generates a frame

idea: Have $A \cdot Id \le S \le B \cdot Id$

$$S^{-1} \cdot A \cdot Id \le S^{-1} \cdot S \le S^{-1} \cdot B \cdot Id$$

$S^{-1} \le A \cdot Id$ $\qquad$ $S^{-1} \ge B^{-1} \cdot Id$

Reconstruction + Dual Frames.

Given $\{\langle x, f_i \rangle\}_{i \in I} \subset \ell^2(I)$, reconstruction of x is possible.

Is $x = \sum_{i \in I} \langle x, f_i \rangle f_i^*$ ← dual of f_i

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Dual Frames + Reconstruction

Given $\{\langle x, f_i \rangle\}_{i \in I}$, how to reconstruct $x \in \mathbb{H}$?

If $\{e_i\}_{i \in I}$ is an ONB, then $x = \sum_{i \in I} \langle x, e_i \rangle e_i$

reconstructed x $\qquad$ out of these coefficients.

let $\{f_i\}_{i \in I} \subset \mathbb{H}$ be a frame of $\mathbb{H}$.

A dual frame of $\{f_i\}_{i \in I}$ is a set of vectors $\{f_i^*\}_{i \in I} \subset \mathbb{H}$

s.t $\forall x \in \mathbb{H}$, $x = \sum_{i \in I} \langle x, f_i \rangle f_i^*$ reconstruction formula.

note, f_i^* is not conjugate transpose.

If $\{f_i\}$ is an ONB, then $f_i^* = f_i$ $\forall i \in I$.

To find f_i^*, using S^{-1} (inverse of the frame operator)

we call $\{S^{-1} f_i\}_{i \in I}$ the canonical dual frame of $\{f_i\}_{i \in I}$

Have $S(x) = (T^*T)(x) = \sum_{i\in I} \langle x, f_i \rangle f_i$

let $y = S(x)$ then $x = S^{-1}(y)$

then $y = \sum_{i\in I} \langle S^{-1}(y), f_i \rangle f_i = \sum_{i\in I} \langle y, S^{-1}(f_i) \rangle f_i$

Because S^{-1} is self-adjoint, let $u = Sx$, $v = Sy$

then $\langle S^{-1}u, v \rangle = \langle S^{-1}(Sx), Sy \rangle = \langle x, Sy \rangle = \langle Sx, y \rangle$
$= \langle S(S^{-1}u), S^{-1}v \rangle = \langle u, S^{-1}v \rangle$

then $S(x) = \sum_{i\in I} \langle x, f_i \rangle f_i$

$x = S^{-1}\left(\sum_{i\in I} \langle x, f_i \rangle f_i\right) = \sum_{i\in I} \langle x, f_i \rangle \underbrace{S^{-1}(f_i)}_{= f_i^*}$

thus $y = \sum_{i\in I} \langle y, S^{-1}f_i \rangle f_i$ solves the representation problem.

$x = \sum_{i\in I} \langle x, f_i \rangle S^{-1} f_i$ solves the reconstruction problem.

For Parseval frames $(A = B = 1)$ we have $x = \sum_{i\in I} \langle x, f_i \rangle f_i \quad \forall x \in \mathbb{H}$.

so $f_i^* = f_i$ in this case.

This is a tight frame, so $S = A \cdot \mathrm{Id} = 1 \cdot \mathrm{Id}$.

so $S^{-1} = \mathrm{Id}$, $S^{-1} f_i = (\mathrm{Id}) \cdot f_i = f_i$

we say Parseval frames are self-dual.

Theorem: If $\mathbb{H}$ is a Hilbert space. $\dim(\mathbb{H}) = n$,
and $\{e_i\}_{i\in I}^{n} \subseteq \mathbb{H}$, with $\|e_i\|_{\mathbb{H}} = 1$.
Then $\{e_i\}_{i\in I}^{n}$ is a Parseval frame iff $\{e_i\}_{i=1}^{n}$ is an ONB

Ex: Consider $\{f_1, f_2, f_3\} = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}4\\2\end{pmatrix}\right\} \subset \mathbb{R}^2$

To compute $\{S^{-1} f_i\}_{i=1}^{3}$

$S = T^*T = \begin{pmatrix} 18 & 9 \\ 9 & 5 \end{pmatrix}$ frame operator $\qquad S^{-1} = \begin{pmatrix} 5/9 & -1 \\ -1 & 2 \end{pmatrix}$

$T(x) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 4 & 2 \end{pmatrix}$, $T^* = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \end{pmatrix}$

$f_1^* = S^{-1} f_1 = \begin{pmatrix} 5/9 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} 5/9 \\ -1 \end{pmatrix}$

$f_2^* = S^{-1} f_2 = \begin{pmatrix} 5/9 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix} -4/9 \\ 1 \end{pmatrix}$

$f_3^* = S^{-1} f_3 = \begin{pmatrix} 2/9 \\ 0 \end{pmatrix}$

$x = \sum_{i=1}^{3} \langle x, f_i \rangle f^*$

Ex: Suppose $\langle x, f_3 \rangle$ is lost, use dual vectors $\{f_1, f_2\}$ for reconstruction.

$f_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $f_2 = \begin{pmatrix}1\\1\end{pmatrix}$

$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $T^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = T^*T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $S^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

so $f_1^* = S^{-1} f_1 = \begin{pmatrix}1\\-1\end{pmatrix}$ and $f_2^* = S^{-1} f_2 = \begin{pmatrix}0\\1\end{pmatrix}$

Get $x = \langle x, f_1 \rangle f_1^* + \langle x, f_2 \rangle f_2^*$

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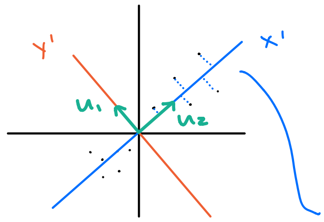
Principal Component Analysis (PCA)



data can be points, vectors, images, sounds...

Compression, dimensionality reduction

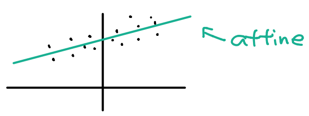
best 1-d subspace (or k-dimensional in general) representing the points.

data can be compressed by projecting onto the x' axis. $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto x'$

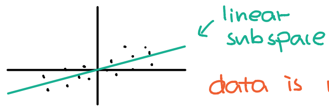
- u_1, u_2 are the principal components, ("left-singular vectors" from the SVD)

PCA $\begin{cases} \text{covariance matrix} \\ \text{SVD} \end{cases}$

Data + Mean Centering



$\longrightarrow$



data is mean-centered

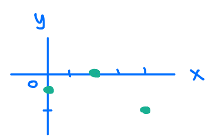
let $x_i \in \mathbb{R}^m$ $(i = 1, \dots, n)$ be some data points.
Often $m \leq n$. Store in   assume x_i column vectors.

$$X = [x_1 \mid x_2 \mid \cdots \mid x_n] \in \mathbb{R}^{m \times n}$$

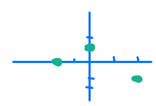
We suppose the $\{x_i\}$ is mean-centered.

Ex: $X = \begin{bmatrix} 2 & 0 & 4 \\ 0 & -1 & -2 \end{bmatrix}$

let $\bar{x}$ = mean of x-values $= \frac{1}{3}[2+0+4] = 2$
$\bar{y}$ = mean of y-values $= \frac{1}{3}[0-1-2] = -1$



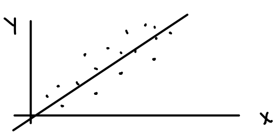
Replace $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$, $X_{\text{centered}} = \begin{bmatrix} 0 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix}$



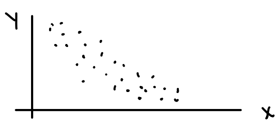
In matrix form, $X_{\text{centered}} = X - (\underbrace{\tfrac{1}{n} X \mathbb{1}_n}_{=:\, \bar{x}}) \mathbb{1}_n^T = X - \bar{x}\mathbb{1}_n^T$ $\mathbb{1}_n = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1}$

Data Covariance

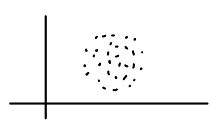
= data spread



- large positive covariance
- larger x → larger y



- large negative covariance
- larger x → smaller y



- low covariance (≈ 0)
- no correlation

Defn: The (sample) covariance matrix is $C_{xx} := \frac{1}{n-1} XX^T \in \mathbb{R}^{m \times m}$
$X \in \mathbb{R}^{m \times n}$

This definition holds if X is mean-centered.

properties: ① C_{xx} is symmetric.
② C_{xx} is positive semi-definite

Ex: $X = \begin{bmatrix} 0 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix}$ ← centered $\qquad C_{XX} = \frac{1}{3-1} XX^T = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$

Interpretation of $C_{XX} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$

Suppose we have $X = [x_1 \;\; x_2]$. let $x_i^{(k)}$ denote the k^{th} component of x_i

then $C_{XX} = \frac{1}{n-1} XX^T$

$$= \frac{1}{n-1}[x_1 \, x_2] \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} = \frac{1}{n-1}[x_1 x_1^T + x_2 x_2^T]$$

$$= \frac{1}{n-1} \begin{pmatrix} {x_1^1}^2 + {x_2^1}^2 & x_1^2 x_1^1 + x_2^2 x_2^1 \\ x_1^1 x_1^2 + x_2^1 x_2^2 & {x_1^2}^2 + {x_2^2}^2 \end{pmatrix}$$

the diagonal terms are variances of the k^{th} components of x_i.

Defn: the (sample) variance of the k^{th} components of x_i is

$$\mathrm{Var}(\{x_i^k\}) := \frac{1}{n-1} \sum_{i=1}^{n} (x_i^k)^2$$

If $\mathrm{Var}(\{x_i^k\})$ is large, then the k^{th} components of x_i are spread out more.

For $C_{XX} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$ know spread (variance) of x_i^1,
is greater than the spread (variance) of x_i^2.

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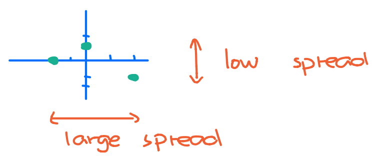
Ex: $X = \begin{bmatrix} 0 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix}$ $\quad C_{XX} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$



For the off-diagonal terms in C_{XX},
these measure the correlation between x_i^1 and x_i^2
(weak correlation in this case)
The closer to 0 that the off-diagonal terms are, the less correlation
there is between those components.

Recall $\mathrm{Var}\{x_i^k\} := \frac{1}{n-1} \sum_{i=1}^{n} (x_i^k)^2$

Defn: $\mathrm{Var}(\{x_i\})$ = total variance of the points $\{x_i\}$

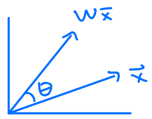
$$:= \frac{1}{n-1} \sum_{i=1}^{n} \|x_i\|_2^2 = \sum_{k=1}^{n} \mathrm{Var}\{x_i^k\}$$

Defn: $\mathrm{tr}(A) = \sum_{i=1}^{n} (A)_{ii}$ $\qquad$ Ex: $\mathrm{tr}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} = 5$

So $\mathrm{Var}(\{x_i\}) = \mathrm{tr}(C_{XX})$ = total variance

Goal: We seek a new representation Y of X with the least correlation amongst the data components.
we will do this by an orthogonal linear transformation W,
so that $X = WY$

Defn: A square matrix $W \in \mathbb{R}^{n \times n}$ is orthogonal if $W^{-1} = W^T$ (so $W^T W = Id$)
- Equivalently, the columns (and rows) of W form an orthonormal basis of $\mathbb{R}^n$.

Ex: 

$W = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal, $\|Wx\|_2 = \|x\|$ $\forall x \in \mathbb{R}^n$

$$C_{XX} = \frac{1}{n-1} \underbrace{X}_{WY} \underbrace{X^T}_{Y^TW^T} = \underbrace{\boxed{\frac{1}{n-1}} W \boxed{YY^T}}_{= C_{YY}} W^T = W C_{YY} W^T$$

$C_{YY} = W^T C_{XX} W$

want a Y that s.t points in Y have the least corr. possible.

Goal now is to choose a W that causes components of points in Y to have the least correlation. Best possible case would be to make C_{YY} diagonal.

Theorem: (Diagonalization)

let $A \in \mathbb{R}^{n\times n}$, TFAE

(i) A is symmetric

(ii) A has an orthonormal set of e-vectors

(iii) A is orthogonally diagonalizable.

i.e. $\exists \Lambda$ diagonal matrix and an orthogonal matrix V s.t. $V^TAV = \Lambda$

note: $V = [v_1 \mid v_2 \mid \dots \mid v_n]$ where v_i are unit e-vectors of A,

and $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$ where $\{\lambda_i\}$ are the corresponding e-values.

Now recall C_{XX} is symmetric, so we can orthogonally diagonalize C_{XX}.

$\exists$ V orthogonal matrix and Λ diagonal matrix

s.t $V^T C_{XX} V = \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$

suppose $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ are the e-values of C_{XX}

and $V = [v_1 \mid v_2 \mid \dots \mid v_n]$

corresponding unit e-vectors of C_{XX}.

these are the principal components of X

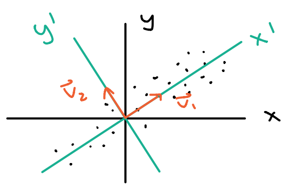
For $C_{YY} = W^T C_{XX} W$, if we let $W = V$

then $C_{YY} = V^T C_{XX} V = \Lambda$ = diagonal matrix

covariance matrix, where the correlation amongst point components is minimal.

The change of variables. $X = WY$ is now $X = VY$ or $Y = V^TX$

now Y containing points that represent the initial data from X in a better coordinate system.



X contains $\begin{pmatrix} x \\ y \end{pmatrix}$ points

Y contains $\begin{pmatrix} x' \\ y' \end{pmatrix}$ points.

$\vec{v}_1, \vec{v}_2$ are the principal components.

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Recall $X = [x_1 \vdots \cdots \vdots x_n]$ and letting $V = [v_1 \vdots \cdots \vdots v_n]$ be the matrix consisting of e-vector of C_{xx}, then $X = VY$ (or $Y = V^T X$) gives an orthogonal linear transformation that better represents points in the new coordinates in Y.

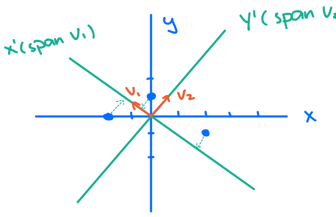
Remarks :

① $C_{YY} = \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, and the eigenvalues $\{\lambda_k\}$ of C_{YY} represent the variances of $\{y_i^k\}$ (the k^{th} components of the data points $\{y_i\}$)

$tr(C_{xx}) = tr(C_{YY})$ total variance is preserved.

② The first principal component v_1 corresponds to the largest e-value of Λ (i.e λ_1) i.e the direction v_1 captures the most variance of the data points.

Ex : $X = \begin{bmatrix} 0 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix}$ with $C_{xx} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$



e-pairs of C_{xx} are $(\lambda_1, v_1) = \left(4.3, \begin{pmatrix} -3.3 \\ 1 \end{pmatrix}\right)$

$(\lambda_2, v_2) = \left(0.7, \begin{pmatrix} 0.3 \\ 1 \end{pmatrix}\right)$

total variance $= 4.3 + 0.7 = 5$.

$4.3/5 = 0.86$, 86% of data is captured by the 1^{st} principal component v_1 direction.

Recall $Y = V^T X$, $V = [v_1 \ v_2] = \begin{bmatrix} -0.957 & 0.29 \\ 0.29 & 0.957 \end{bmatrix}$, $C_{YY} = \Lambda = \begin{pmatrix} 4.3 & 0 \\ 0 & 0.7 \end{pmatrix}$

normalize

note $tr(C_{xx}) = tr(C_{YY})$

so $Y = \begin{bmatrix} -0.957 & 0.29 \\ 0.29 & 0.957 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0.29 & 1.914 & -2.2 \\ 0.957 & -0.58 & -0.38 \end{bmatrix}$

Compression :

- let's project the data points to the line span(v_1)
- we will do this by zeroing out v_2 in $V = [v_1 \ \cancel{v_2}^{0}]$

Define $\tilde{V} = [v_1 \ 0] = \begin{bmatrix} -0.957 & 0 \\ 0.29 & 0 \end{bmatrix}$

$\tilde{Y} = \tilde{V}^T X = \begin{bmatrix} -0.957 & 0.29 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0.29 & 1.914 & -2.204 \\ 0 & 0 & 0 \end{bmatrix}$

To convert the projected points from $\tilde{Y}$ back to the original X coordinate system.

use $V\tilde{Y} = V\tilde{V}^T X =: \underline{\tilde{X}}$ contains points projected to span(v_1) with coordinates in the original X coord. system

note $Y = V^T X$, $X = VY$

$\tilde{X} = V\tilde{Y} = \begin{bmatrix} -0.957 & 0.29 \\ 0.29 & 0.957 \end{bmatrix} \begin{bmatrix} 0.29 & 1.914 & -2.204 \\ 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} -0.27 & -1.8 & 2.1 \\ 0.08 & 0.55 & -0.6 \end{bmatrix}$

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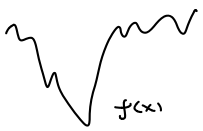
Fourier Series

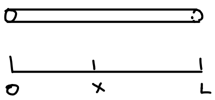
Intro



$\overset{?}{=}$ 

$\overset{?}{=} \sin(x) + \sin(5x) + \sin(7x) + \cdots$ (Fourier series expansion of f)

Motivation



suppose a rod is heated

$u(x,t)$ = heat at location x at time t

Heat equation
$$\begin{cases} u_t - k u_{xx} = f(x,t) & 0 < x < L \\ u(x,0) = \phi(x) & \text{initial condition} \\ u(0,t) = u(L,t) = 0 & \text{boundary condition} \end{cases}$$

Ansatz

Suppose $u(x,t) = X(x)T(t)$ separation of variables

Plug into PDE and rewrite :

$$u_t = \underbrace{X(x)T'(t) = k X''(x)T(t)} = k\, u_{xx}$$

$$\downarrow$$

$$\frac{1}{k}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{constant} = -\lambda \quad \text{while } \lambda > 0$$

Get i) $T'(t) = -k\lambda T(t) \Rightarrow T(t) = T(0)e^{-k\lambda t} = Ae^{-k\lambda t}$

ii) $X''(x) = -\lambda X(x) \Rightarrow X(x) = B\sin(\sqrt{\lambda}x) + C\cos(\sqrt{\lambda}x)$

Thus $u(x,t) = X(x)T(t) = e^{-k\lambda t}(B\sin(\sqrt{\lambda}x) + C\cos(\sqrt{\lambda}x))$

Consider boundary conditions : $u(0,t) = 0$, $u(L,t) = 0$

$u(0,t) = 0 = Ce^{-k\lambda t} \Rightarrow C = 0$

$u(L,t) = 0 = Be^{-k\lambda t}\sin(\sqrt{\lambda}L)$ need $\sin(\sqrt{\lambda}L) = 0$ for nontrivial solns.

$\sqrt{\lambda}L = n\pi \quad (n = 0, 1, 2, \dots)$

$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$

Then $u_n(x,t) = B_n e^{-k\lambda_n t}\sin(\sqrt{\lambda_n}x)$

$= B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right) \quad n = 0, 1, 2, \dots$

satisfies PDE + boundary conditions but not yet the initial cond. $u(x,0) = \phi(x)$

Try to enforce the initial condition : $u_n(x,0) = B_n\sin\left(\frac{n\pi x}{L}\right)$

Have $\sum_{n=0}^{N} u_n(x,t)$ is also a solution (since PDE is linear)

$\phi(x) = \sum_{n=0}^{N} B_n\sin\left(\frac{n\pi x}{L}\right)$? no, unless $\phi \in \text{span}\{\sin\left(\frac{n\pi x}{L}\right)\}_{n=0}^{N}$.

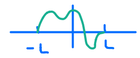
Think $\phi(x) \overset{?}{=} \lim_{N\to\infty}\sum_{n=0}^{N} B_n\sin\left(\frac{n\pi x}{L}\right) = \underline{\sum_{n=0}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right)}$ Forier sine series for $\phi(x)$

Generalizations :

$\phi(x) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)$ Forier cosine series

$\phi(x) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)$ Full Forier series

Complex Form of Fourier Series for $-L < x < L$



$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)$ possible form for All fourier series ($\frac{1}{2}A_0$ is for converience)

Use Euler formula, $e^{i\theta} = \cos\theta + i\sin\theta$

Have $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

Then $\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{A_n - iB_n}{2}\right)e^{i\frac{n\pi x}{L}} + \left(\frac{A_n+iB_n}{2}\right)e^{-i\frac{n\pi x}{L}}\right] = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{L}}$

where $C_n = \begin{cases} \frac{1}{2}A_0 & n=0 \\ \frac{1}{2}(A_n - iB_n) & n>0 \\ \frac{1}{2}(A_n+iB_n) & n<0 \end{cases}$

To find $\{C_n\}_{n=-\infty}^{\infty}$:

$\int_{-L}^{L}\phi(x)e^{-i\frac{m\pi x}{L}}dx = \int_{-L}^{L}\sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{L}} e^{-i\frac{m\pi x}{L}}dx$

$= \sum_{n=-\infty}^{\infty} C_n \underbrace{\int_{-L}^{L} e^{i\frac{n\pi x}{L}}e^{-i\frac{m\pi x}{L}}dx}_{=\begin{cases}0 & \text{if } n\neq m\\ 2L & \text{if } n=m\end{cases} = 2L\delta_{nm}}$ assume can interchange the integral

$= C_m(2L)$

$\Rightarrow C_m = \frac{1}{2L}\int_{-L}^{L}\phi(x)e^{-i\frac{m\pi x}{L}}dx$

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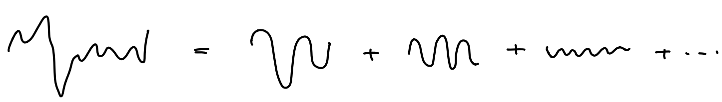
Fourier Series II



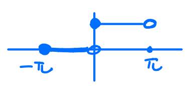
$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)$ $\quad -L<x<L$

Complex form: $\phi(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{L}}$ where $C_n = \begin{cases}\frac{1}{2}A_0 & n=0\\ \frac{1}{2}(A_n-iB_n) & n>0\\ \frac{1}{2}(A_n+iB_n) & n<0\end{cases}$

OR by orthogonality with $e^{-i\frac{m\pi x}{L}}$, $C_n = \frac{1}{2L}\int_{-L}^{L}\phi(x)e^{-i\frac{n\pi x}{L}}dx$

$\left\{\frac{1}{\sqrt{2a}}e^{i\frac{n\pi x}{a}} \mid n\in\mathbb{Z}\right\}$ is orthonormal in $L^2(-a,a)$.

Ex: $f(t) = \begin{cases}1 & 0\le t<\pi\\ 0 & -\pi\le t<0\end{cases}$



$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}1\cdot e^{-i\frac{n\pi x}{L}}dx = \begin{cases}\frac{1}{2} & n=0\\ -\frac{1}{2\pi i n}e^{-inx}\Big|_0^{\pi} = \frac{i}{2\pi n}[e^{-in\pi}-1] = \frac{i}{2\pi n}[(-1)^n-1] & \text{if } n\neq 0\end{cases}$

$= \cos(n\pi) - i\,\cancel{\sin(n\pi)}^{=0} = (-1)^n$

So the complex Fourier series of f is $\sum_{n=-\infty}^{\infty} C_n e^{inx} =: \tilde{f}(x)$ with $C_n = \begin{cases}\frac{1}{2} & n=0\\ \frac{i}{2\pi n}((-1)^n-1) & n\neq 0\end{cases}$

Orthogonality on $-L < x < L$ (real case) $m, n \in \mathbb{Z}$.

Lemma : $\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L & n=m\neq 0 \\ 0 & n\neq m \\ 2L & n=m=0 \end{cases}$

$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L & n=m\neq 0 \\ 0 & n\neq m \end{cases}$

$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = 0 \quad \forall\, n,m \in \mathbb{Z}$

Consider $\Phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)$

$$\int_{-L}^{L} \Phi(x)\sin\left(\frac{m\pi x}{L}\right) dx = \int_{-L}^{L} \frac{A_0}{2} \underbrace{\boxed{\sin\left(\frac{m\pi x}{L}\right) dx}}_{=0} + \sum_{n=1}^{\infty}\int_{-L}^{L} A_n \underbrace{\boxed{\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)}}_{=0} + \sum_{n=1}^{\infty}\int_{-L}^{L} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)$$

assume we can interchange the series + integral

$= 0 + 0 + B_m L$

$B_m = \frac{1}{L}\int_{-L}^{L} \Phi(x)\sin\left(\frac{m\pi x}{L}\right) dx$

$A_n = \frac{1}{L}\int_{-L}^{L} \Phi(x)\cos\left(\frac{n\pi x}{L}\right) dx$ hold for $n = 0, 1, 2 \dots$ note this holds for $n=0$ which is the reason in using $\frac{1}{2}A_0$ rather than A_0

recall $\text{proj}_W x = \sum_i \langle x, \varphi_i\rangle \varphi_i$ where $\{\varphi_i\}$ are an orthonormal basis of W

Fourier series is this BUT with φ_i = sines and cosines

Ex : $f(t) = 1$ on $-1 < t < 1$, $L = 1$

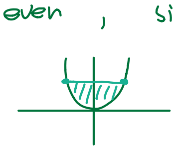
$A_n = \int_{-1}^{1} 1\cdot\cos(n\pi x)\, dx = 2\cdot\int_0^1 \cos(n\pi x)\, dx = \begin{cases} 0, & n\neq 0 \\ 2, & n=0 \end{cases}$

$B_n = \int_{-1}^{1} 1\cdot \sin(n\pi x)\, dx = 0$

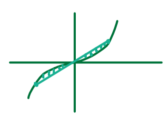
So the Fourier Series of $f(t)=1$ on $-1 < t < 1$ is $\frac{A_0}{2} = \frac{2}{2} = 1$

Note : $\cos(n\pi x)$ is even, $\sin(n\pi x)$ is odd.

even means $f(x) = f(-x)$



odd means $f(-x) = -f(x)$



Even / odd functions & Fourier Series

- If Φ is odd, $A_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\Phi(x)\cos\left(\frac{n\pi x}{L}\right)}_{\text{odd}\cdot\text{even} = \text{odd}} dx = 0 \quad \forall n = 0, 1, 2 \dots$

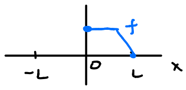
  $\Phi(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$ on $-L < x < L$

- If Φ is even, $B_n = \frac{1}{L}\int_{-L}^{L} \Phi(x)\sin\left(\frac{n\pi x}{L}\right) dx = 0 \quad \forall n = 1, 2 \dots$

  $\Phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$ on $-L < x < L$

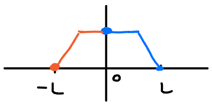
The half-interval $0 < x < L$

consider



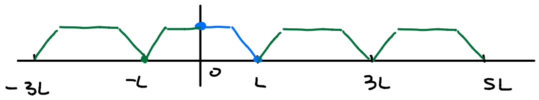
many ways to extend to $-L < x < L$

①

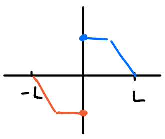


- even extension $f_{even}(x)$
- will get Forier cosine series

periodic extension

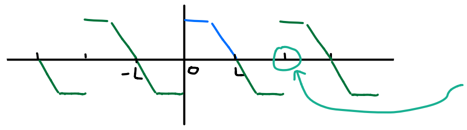


②



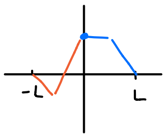
- odd extension $f_{odd}(x)$
- will get Forier sine series

periodic extension



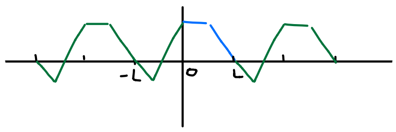
it converges to the average of the jump

③



- arbitrary extension
- will get full Fourier series

periodic extension



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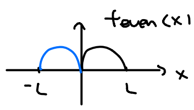
Fourier Series III :

even, odd extensions for f defined on $[0, L]$

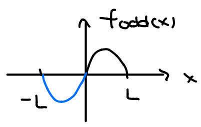
$$f_{even}(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x < 0 \end{cases}$$



cosine series

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

$$f_{odd}(x) = \begin{cases} f(x) & 0 < x \le L \\ -f(-x) & -L \le x < 0 \\ 0 & x = 0 \end{cases}$$



sine series

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

the coefficients are

$$A_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f_{even}(x)\cos\left(\frac{n\pi x}{L}\right)}_{\text{even} \cdot \text{even} = \text{even}} dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f_{odd}\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd} \cdot \text{odd} = \text{even}} dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Ex: find a Fourier sine series for $f(x) = 1$ valid on $(0, 1)$

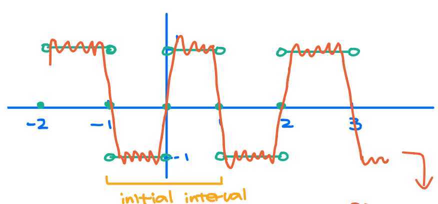
need odd extension

$$f_{odd}(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & x = 0 \\ -1 & x \in (-1, 0) \end{cases}$$

$$B_n = 2\int_0^1 1 \cdot \sin(n\pi x)\,dx = 2\left[-\cos(n\pi x)\frac{1}{n\pi}\right]_0^1$$

$$= 2\left[\frac{-\cos(n\pi)}{n\pi} + \frac{1}{n\pi}\right] = 2\left[\frac{-(-1)^n + 1}{n\pi}\right] = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$$

So the Fourier Sine series is $\displaystyle\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi}\sin(n\pi x)$



$\displaystyle\sum_{n=1}^{20} \frac{4}{n\pi}\sin(n\pi x)$ n odd

Notions of Convergence

Pointwise Convergence

$\sum_{n=1}^{\infty} f_n(x) = \lim_{N\to\infty} \underline{\sum_{n=1}^{N} f_n(x)} = \lim_{N\to\infty} \underline{S_N(x)}$ partial sum , $f_n(x) = \cos(nx)$ OR $\sin(nx)$

Defn:

We say $\{S_N(x)\}_{N=1}^{\infty}$ converges pointwise to $f(x)$ on $a \le x \le b$ if for any fixed $x \in [a,b]$, we have $\lim_{N\to\infty} S_N(x) = f(x)$

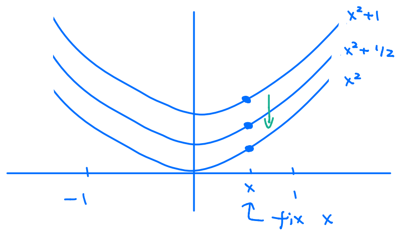
Idea: Fix x, then consider convergence of y-values.

Ex: let $S_N(x) = x^2 + \frac{1}{N}$ on $-1 < x < 1$

let $f(x) = x^2$. Fix x in $(-1, 1)$

then $\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \left(x^2 + \frac{1}{N}\right) = x^2 = f(x)$

so S_N converges to f pointwise.



Uniform Convergence

Defn: We say $\{S_N(x)\}_{N=1}^{\infty}$ converges uniformly to $f(x)$ on $a \le x \le b$

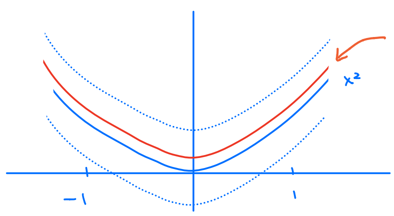
if $\max_{a\le x\le b} |S_N(x) - f(x)| \to 0$ as $N \to \infty$

first take max over all $x \in [a,b]$, then take the limit.

Ex: $S_N(x) = x^2 + \frac{1}{N} \longrightarrow x^2 = f(x)$ pointwise on $[-1,1]$

Have $\max_{-1\le x\le 1} |S_N(x) - f(x)| = \max_{-1\le x\le 1} \left|\left(x^2 + \frac{1}{N}\right) - x^2\right| = \frac{1}{N} \to 0$ as $N \to \infty$

So $\{S_N(x)\} \to x^2$ uniformly on $[-1,1]$



for larger enough N, the sequence is eventually "stuck" in this small neighborhood no matter how small

Note: Uniform convergence implies pointwise.

Mean-Square (L^2) Convergence

Def: $S_N(x)$ converges to $f(x)$ in the mean-square since on $[a,b]$ provided

$$\underbrace{\int_a^b |S_N(x) - f(x)|^2 dx}_{} \to 0 \quad \text{as} \quad N\to\infty$$

L^2 norm, equivalent to $\lim_{n\to\infty} \| S_N - f \|_{L^2(a,b)} = 0$

Ex: $S_N(x) = x^2 + \frac{1}{N} \to x^2 = f(x)$ pointwise on $(0,1)$

Have $\int_0^1 |S_N(x) - f(x)|^2 dx = \int_0^1 |(x^2 - \frac{1}{N}) - x^2|^2 dx = \int_0^1 \frac{1}{N^2} dx = \frac{1}{N^2} \to 0$ as $N \to \infty$

So $S_N \to f$ in $L^2(0,1)$

Note: L^2 is a weak notion of convergence, It does not imply pointwise or uniform convergence.

Convergence of Fourier Series

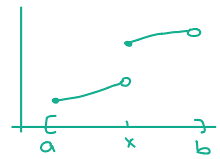
Thm 1: Let $f \in C(\mathbb{R})$ and periodic.

then $\forall x \in \mathbb{R}$ s.t $f'(x)$ exists, the Fourier series at x converges to $f(x)$.

- pointwise converges to $f(x)$

Thm 2: Suppose f is periodic on $\mathbb{R}$ and piecewise continuous,

a finite # of jump discontinuities are allowed.



Suppose f is left and right differentiable at x.

i.e. $\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$ and $\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$ both exist (possibly not equal)

Then the Fourier series of f at x converges to $\frac{1}{2}[f(x+) + f(x-)]$ (average)

- pointwise convergence with jumps allowed
- If f continuous at x, then $\frac{1}{2}[f(x+) + f(x-)] = f(x)$

Thm 3: uniform convergence

Suppose $f \in C(\mathbb{R})$, periodic, and f' is piecewise continuous.

Then the Fourier series of f converges uniformly.

Thm 4: uniform convergence

If the Fourier coefficients of f, call them A_n, B_n are in ℓ^1.

(i.e. $\sum_{n=1}^{\infty} |A_n| + |B_n| < \infty$) then the Fourier series of f converges uniformly.

Thm 5: L^2 convergence

If $f \in L^2(-a,a)$, then $\lim_{N \to \infty} \|f_N - f\|_{L^2(-a,a)} = 0$

where $f_N(x) = \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cos(\frac{n\pi x}{a}) + B_n \sin(\frac{n\pi x}{a})$

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Riemann - Lebesgue lemma

let f be piecewise continuous on $[a,b]$.

Then $\lim_{n \to \infty} \int_a^b f(x)\cos(nx)\,dx = 0$ and $\lim_{n \to \infty} \int_a^b f(x)\sin(nx)\,dx = 0$

→ Fourier coefficients. A_n and B_n

So $\lim_{n \to \infty} A_n = 0$, $\lim_{n \to \infty} B_n = 0$

For any $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t $\forall n \geq N$, have $|A_n|, |B_n| < \varepsilon$

→ let ε be a "threshold", then we can compress / filter by throwing away Fourier modes with low amplitude.

Parseval's Identity

Recall for V finite dimensional that if $\{e_k\}_{k=1}^{n}$ is an ONB,

then $\|u\|_V^2 = \sum_{k=1}^{n} |\langle u, e_k\rangle|^2$

let $f(x) = A_0/2 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{a}) + B_n \sin(\frac{n\pi x}{a})$ and suppose $f \in L^2(-a,a)$.

OR let $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{a}}$ (complex Fourier series)

Then i) $\frac{1}{a}\|f\|^2_{L^2(-a,a)} = \frac{A_0^2}{2} + \sum_{n=1}^{\infty} |A_n|^2 + |B_n|^2$

ii) $\frac{1}{2a}\|f\|^2_{L^2(-a,a)} = \sum_{n=-\infty}^{\infty} |C_n|^2$

"energy" of a signal f = total energy from each frequency component

To show ii) holds:

let $f_N(x) := \sum_{n=-N}^{N} C_n e^{i\frac{n\pi y}{a}}$ be a truncated Fourier series

$\|f_N\|_2^2 = \langle f_N, f_N\rangle = \int_{-a}^{a} \sum_{n=-N}^{N} C_n e^{i\frac{n\pi x}{a}} \sum_{m=-N}^{N} C_m e^{i\frac{m\pi x}{a}}$

$= \sum_{n=-N}^{N} \sum_{m=-N}^{N} C_n C_m \underbrace{\int_{-a}^{a} e^{i\frac{n\pi x}{a}} e^{i\frac{m\pi x}{a}} dx}$ $= 2a \cdot \delta_{nm}$ by orthogonality

$= \sum_{n=-N}^{N} \sum_{m=-N}^{N} C_n C_m \, 2a \, \underline{\delta_{nm}}$ $\begin{cases} \delta = 0 & n = m \\ \delta = 1 & n \neq m \end{cases}$

$= 2a \sum_{n=-N}^{N} C_n^2$

So $\frac{1}{2a}\underbrace{\|f_N\|_2^2}_{\to \|f\|_2^2 \text{ as } n\to\infty} = \underbrace{\sum_{n=-N}^{N} |C_n|^2}_{\to \sum_{n=-\infty}^{\infty} |C_n|^2 \text{ as } N\to\infty}$ $\Rightarrow \frac{1}{2a}\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |C_n|^2$

Recall for Fourier series of a function in $L^2(-a,a)$

that $\lim_{N\to\infty} \|f_N - f\|^2_{L^2(-a,a)} = 0$ (L^2 convergence)

$0 \le \big| \|f_N\| - \|f\| \big| \le \|f_N - f\|$ by the squeeze theorem $\lim_{N\to\infty} \|f_N\| = \|f\|$

Ex: (Basel problem, solved by Euler) To show $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

let $f(x) = x$ on $-\pi \le x \le \pi$

the Fourier coefficients are $A_n = 0$ $\forall n$, $B_n = \frac{2(-1)^{n+1}}{n}$

By the real Parseval's identity

$$\frac{1}{\pi}\|f\|^2_{L^2(-\pi,\pi)} = \sum_{n=1}^{\infty} |B_n|^2 = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\|f\|^2_{L^2(-\pi,\pi)} = \int_{-\pi}^{\pi} x^2 dx = 2\int_0^{\pi} x^2 dx = 2\tfrac{1}{3}x^3\Big|_0^{\pi} = \tfrac{2}{3}\pi^3$$

$$\frac{1}{\pi}\cdot\frac{2}{3}\pi^3 = 4\sum_{n=1}^{\infty} 1/n^2$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} 1/n^2$$

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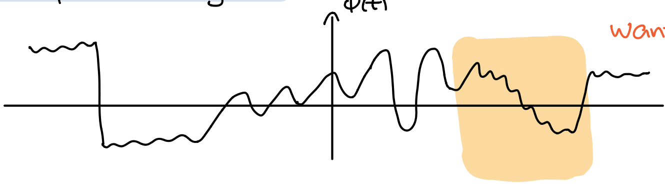
non-periodic signal:



want to analyze frequency here

- has no Fourier series

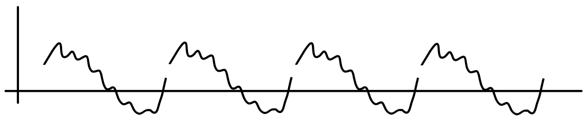
Localization

consider a "window function",

For example we can use $\mathbb{1}_{[t_1,t_2]} = \begin{cases} 1 & t\in[t_1,t_2] \\ 0 & \text{otherwise} \end{cases}$

let $\phi(t) = \mathbb{1}_{[t_1,t_2]}\,\Phi(t) = \begin{cases} \Phi(t) & t\in[t_1,t_2] \\ 0 & \text{otherwise} \end{cases}$

Now extend $\phi(t)$ periodically with period $t_2 - t_1$



and find its Fourier series, and analyze for frequency content
different windows will get diff. responses.

Fourier Transforms

Recall on $(-a,a)$ we have the Fourier series $\phi(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{a}}$

where $C_n = \frac{1}{2a}\int_{-a}^{a} \phi(y) e^{-i\frac{n\pi y}{a}}\,dy$

- as $a \to \infty$

$k = k_n = \frac{n\pi}{a}$

$$\phi(x) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2a}\int_{-a}^{a}\phi(y)e^{-i\frac{n\pi y}{a}}dy\right)e^{i\frac{n\pi x}{a}} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_{-a}^{a}\phi(y)e^{-iky}dy\right)e^{ikx}\frac{\pi}{a}$$

so as $a\to\infty$, $\int_{-a}^{a} \to \int_{-\infty}^{\infty}$, k becomes a continuous variable on $\mathbb{R}$

$\frac{\pi}{a} = \frac{(n+1)\pi}{a} - \frac{n\pi}{a} = k_{n+1} - k_n = \Delta k \to dk$ differential

Passing to the limit we get

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\underbrace{\int_{-\infty}^{\infty}\phi(y)e^{-iky}dy}_{=:\,F[\phi](k)\text{ is the Fourier transform of }\phi}\Big)e^{ikx}\,dk$$

Defn:

$F[\phi](k) = \int_{-\infty}^{\infty}\phi(y)e^{-iky}\,dy$ (Fourier Transform)

$F^{-1}[\Phi(k)](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(k)e^{ikx}\,dk$ (Inverse Fourier Transform)

note $F^{-1}[F[\phi]](x) = \phi(x)$

$F[F^{-1}[\Phi]](k) = \Phi(k)$

Remarks:

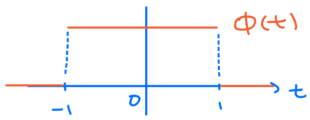
① There are different conventions: $F[\phi] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(y)e^{-iky}dy$

OR $F[\phi] = \int_{-\infty}^{\infty}\phi(y)e^{-2\pi iky}\,dy$

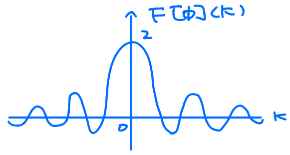
② We call k the frequency variable

③ In n-dimensions, $F[\phi] = \int_{\mathbb{R}^n} e^{-2\pi i\vec{k}\cdot\vec{y}}\phi(\vec{y})\,d\vec{y}$

Ex: square pulse

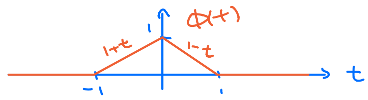


$$F[\phi] = \int_{-\infty}^{\infty} \phi(t) e^{-ikt}\,dt = \int_{-1}^{1} 1 \cdot e^{-ikt}\,dt = -\frac{1}{ik}\left[e^{-ik} - e^{ik}\right] = \frac{2}{k}\underbrace{\frac{e^{ik}-e^{-ik}}{2i}}_{=\sin(k)} = \begin{cases} \frac{2\sin(k)}{k} & k \neq 0 \\ 2 & k = 0 \end{cases}$$



frequency space

Ex:



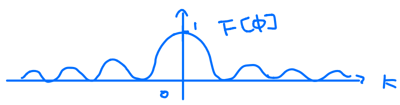
$$F[\phi] = \int_{-\infty}^{\infty} \phi(t)\, e^{-ikt} = 2\int_0^1 \phi(t)\cos(kt)\,dt = 2\int_0^1 (1-t)\cos(kt)\,dt = 2\int_0^1 (1-t)\frac{d}{dt}\frac{\sin(kt)}{k}\,dt$$

$\phi(t)$ even

$\cos(kt)$ even
$\sin(kt)$ odd

$$= 2(1-t)\frac{\sin(kt)}{k}\Big|_{t=0}^{t=1} - 2\int_0^1 \underbrace{\frac{d}{dt}(1-t)}_{=-1}\frac{\sin(kt)}{k}\,dt$$

$$= \frac{2}{k}\int_0^1 \sin(kt)\,dt = -\frac{2}{k^2}\left[\cos(k)-1\right] = \frac{2}{k^2}\left[1-\cos(k)\right] \text{ if } k \neq 0$$



Ex: Suppose $\phi(y) \to 0$ as $|y| \to \infty$

$$F[\phi'] = \int_{-\infty}^{\infty} \phi'(y) e^{-iky}\,dy = \underbrace{\phi(y)e^{-iky}\Big|_{y=-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} \phi(y)\underbrace{\frac{d}{dy}e^{-iky}}_{=ik e^{-iky}}\,dy$$

$$= ik\int_{-\infty}^{\infty} \phi(y) e^{-iky}\,dy = ik\,F[\phi]$$

so $F[\phi'] = ik\,F[\phi]$

So F converts differentiations into multiplication by ik
useful for studying linear ODE/PDE

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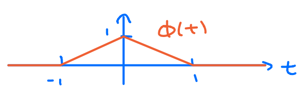
$$F[f](k) = \int_{\mathbb{R}} f(x)\, e^{-2\pi ikx}\,dx = \hat{f}(x)$$

"operator notation" "functional notation"

Ex: Compute $F[\phi(2t-4)]$ where

$$\phi(t) = \begin{cases} 1+t & -1 \leq t < 0 \\ 1-t & 0 \leq t < 1 \\ 0 & \text{else} \end{cases}$$



know $F[\phi](k) = \hat{\phi}(k) = \frac{\sin^2(\pi k)}{(\pi k)^2}$

$$F[\phi(2t-4)] = F[\underbrace{\phi(2(t-2))}_{=:\,\psi(t-2)}] = F[\psi(t-2)] = e^{-2\pi i(2)k}\underbrace{\hat{\psi}(k)}_{\substack{=F[\phi(2t)](k) \\ =\frac{1}{2}\hat{\phi}(\frac{k}{2})}} = \frac{1}{2}e^{-4\pi ik}\hat{\phi}(\tfrac{k}{2}) = \frac{1}{2}e^{-4\pi ik}\frac{\sin^2(\pi\frac{k}{2})}{(\pi k/2)^2}$$

where $\psi(t) = \phi(2t)$

Fact:

i) $F[f(ax)] = \frac{1}{|a|}\hat{f}(\frac{k}{a})$

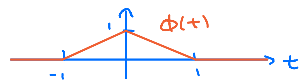
ii) $F[f(x-a)] = e^{-2\pi i a k}\hat{f}(k)$

iii) $F[e^{-t^2}](k) = \sqrt{\pi}\, e^{-(\pi k)^2}$

iv) $F[f(t)g(t)] = (\hat{f} * \hat{g})(k)$

The convolution of f and g, $(f*g)(t) := \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$

Ex: Compute $F[e^{-t^2}\phi(2t-4)]$



know $F[\phi(2t-4)](k) = 2e^{-4\pi i k}\frac{\sin^2(\pi k/2)}{(\pi k)^2}$

Let $\psi(t) = e^{-t^2}$, $\Phi(t) = \phi(2t-4)$

then

$$F[\psi(t)\Phi(t)] = (\hat{\psi} * \hat{\Phi})(k) = \int_{-\infty}^{\infty}\hat{\psi}(k-\tau)\hat{\Phi}(\tau)\,d\tau = \int_{-\infty}^{\infty} \underbrace{\sqrt{\pi}\,e^{-(\pi(k-\tau))^2}}_{\hat{\psi}(k-\tau)}\ \underbrace{2e^{-4\pi i\tau}\cdot\frac{\sin^2(\pi\tau/2)}{(\pi\tau)^2}}_{\hat{\Phi}(\tau)}\,d\tau$$

Remarks on Convulution

let $F[f](k) = \int_{\mathbb{R}} f(x)\,e^{-2\pi i x k}\,dx$

i) $F[f(x)g(x)](k) = (\hat{f} * \hat{g})(k)$

ii) $F[(f*g)(x)](k) = \hat{f}(k)\hat{g}(k)$

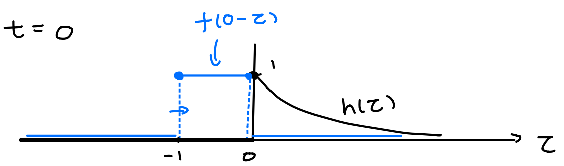
Convolution graphically

let $h(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & \text{else}\end{cases}$ and $f(t) = \mathbb{1}_{[0,1]}(t) = \begin{cases} 1 & t\in[0,1] \\ 0 & \text{else}\end{cases}$

$(f*h)(t) = \int_{-\infty}^{\infty} \underbrace{f(t-\tau)}_{= f(-(\tau-t))} h(\tau)\,dt$

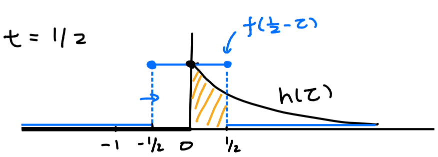
graph of $f(-\tau)$ shifted to the right as t increases

$t=0$



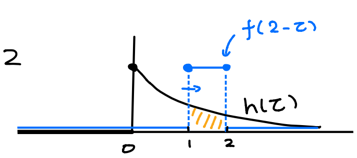
$(f*h)(0) = 0$

$t = 1/2$



$(f*h)(1/2) = $ Area of /// $= \int_0^{1/2} f(\frac{1}{2}-\tau)h(\tau)\,d\tau$

$t = 2$



$(f*h)(2) = $ Area of /// $= \int_1^2 f(2-\tau)h(\tau)\,d\tau$

Plancherel Formula : $\langle F[f], F[g] \rangle_{L^2(\mathbb{R})} = \langle f, g \rangle_{L^2(\mathbb{R})}$

let $f \in L^2(\mathbb{R})$

$$\langle F[f], F[g] \rangle_{L^2} = \langle f, \underline{F^*} F[g] \rangle_{L^2} = \langle f, F^{-1} F[g] \rangle_{L^2} = \langle f, F^{-1}F[g] \rangle_{L^2} = \langle f, g \rangle_{L^2}$$

$F^* = F^{-1}$

$F[f] = \int_{\mathbb{R}} f(x)\, e^{-2\pi i k x}\, dx$

Corollary : Let $f = g$. Then $\| F[f] \|_{L^2(\mathbb{R})} = \| f \|_{L^2(\mathbb{R})}$

means the signal "energy" is equal to the frequency "energy".

It is similar to the Parseval identity for Fourier series.

Ex : Compute $\int_0^\infty \frac{\sin^4(\pi t)}{(\pi t)^4}\, dt = \frac{1}{2} \int_{-\infty}^{\infty} |\hat{\phi}(k)|^2\, dk$ by symmetry

Recall if $\phi(t) = \begin{cases} 1+t & -1 < t < 0 \\ 1-t & 0 < t < 1 \\ 0 & \text{else} \end{cases}$ then $\hat{\phi}(k) = \frac{\sin^2(\pi k)}{(\pi k)^2}$

By Plancherel,

$$\|\hat{\phi}\|^2_{L^2(\mathbb{R})} = \|\phi\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \phi^2(t)\, dt = \int_{-1}^{0} (1+t)^2 dt + \int_0^1 (1-t)^2 dt$$

$$= \int_{-1}^{0} s^2 ds + \int_1^0 s^2 (-ds) = 2\int_0^1 s^2 ds = 2 \tfrac{1}{3} s^3 \Big|_0^1 = \tfrac{2}{3}$$

then $\int_0^\infty \frac{\sin^4(\pi t)}{(\pi t)^4}\, dt = \frac{1}{2} \|\hat{\phi}\|^2_{L^2(\mathbb{R})} = \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}$

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Linear Filters

Fourier series $\phi(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nt) + B_n \sin(nt)$

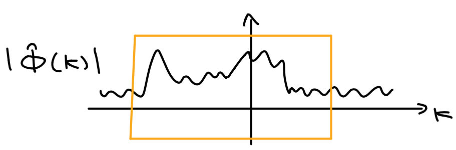
Let L be an operator / map / filter that cuts off n after some frequency N.

i.e. $L[\phi](t) = \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cos(nt) + B_n \sin(nt)$

a "noise filter"

$\phi(t)$  $L[\phi](t)$ 

Can do the same with Fourier Transforms :



cut off frequency after some fixed value and can reconstruct ϕ by taking inverse Fourier transforms.

A linear filter, $L : L^2(\mathbb{R}) \to L^2(\mathbb{R})$,

anything

satisfies $L(af + bg) = a \cdot L(f) + b \cdot L(g) \quad \forall f, g \in L^2(\mathbb{R}), \ \forall a, b \in \mathbb{R} \text{ or } \mathbb{C}$

LTI = linear time- variant

Def: L is time-invariant ($f_a(t) := f(t-a)$ time shift).
if $\forall f$, $\forall a \in \mathbb{R}$ we have $L[f_a](t) = (Lf)(t-a)$

$L[f_a](t)$ filltered time-shifted input signal.
$(Lf)(t-a)$ time-shifted output signal

Ex: $(Lf)(t) := (h * f)(t) = \int_{-\infty}^{\infty} h(t-\tau) f(\tau)\, d\tau$ is a time-invariant filter

Theorem: let L be an LTI filter on $\mathbb{V} = \{ f \mid f$ is piecewise continuous$\}$.
then $\exists h \in L^1(\mathbb{R})$ s.t $\forall f \in \mathbb{V}$,
we have $L[f] = f * h$, where $*$ is convolution

- h is called the impulse response function
- $\hat{h}$ is called the system function

Remark:
The impulse function (Dirac delta) denoted δ, formally satisfies

$$\delta(t) = \begin{cases} \infty & t=0 \\ 0 & \text{else} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0) \quad \forall f \in C(\mathbb{R})$$

note: δ is actually a distribution, not a function

Formally, have

$$L[\delta](t) = (\delta * h)(t) = \int_{-\infty}^{\infty} \delta(\tau) \underbrace{h(t-\tau)}_{=:\, g(\tau)} d\tau = g(0) = h(t)$$

so h is the response to δ.

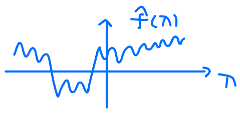
Ex: $L[f] = f * h$, L is the LTI filter, f is the signal.
let's design L to remove frequencies larger than $\underline{\lambda_c}$. cutoff frequency

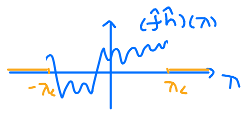
Have $\widehat{L[f]}(\lambda) = \widehat{(f*h)}(\lambda) = \hat{f}(\lambda)\hat{h}(\lambda)$

let $\hat{h}(\lambda) = \begin{cases} 1 & |\lambda| \le \lambda_c \\ 0 & \text{else} \end{cases} =: \mathbb{1}_{[-\lambda_c, \lambda_c]}(\lambda) = \text{rect}\left(\frac{\lambda}{2\lambda_c}\right)$

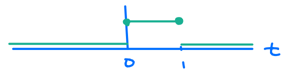
Using Fourier tables, we get

$$h(t) = F^{-1}[\hat{h}] = F^{-1}\left[\text{rect}\left(\frac{\lambda}{2\lambda_c}\right)\right] = 2\lambda_c \,\text{sinc}(2\lambda_c t) = \underbrace{\frac{\sin(2\lambda_c \pi t)}{\pi t}}_{\text{impulse response}}$$





Ex: let's apply L to $f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{else} \end{cases}$

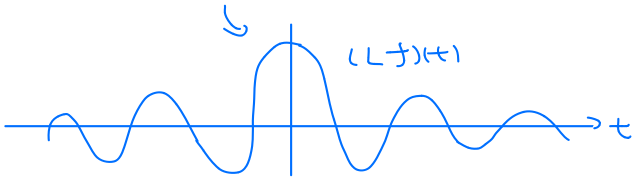


"on-off signal"

$$(Lf)(t) = (f * h)(t)$$
$$= \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$
$$= \int_0^1 \frac{\sin(2\pi_c \pi (t-\tau))}{\pi(t-\tau)}\, d\tau \qquad \text{let } u = 2\pi_c\pi(t-\tau)$$
$$= \frac{1}{\pi} \int_{2\pi_c\pi(t-1)}^{2\pi_c\pi t} \frac{\sin u}{u}\, du \qquad \text{note } S_i(z) := \int_0^z \frac{\sin u}{u}\, du$$

" sine-integral function"

$$= \frac{1}{\pi} \{ S_i(2\pi_c\pi t) - S_i(2\pi_c\pi(t-1)) \}$$



the output signal is nonzero for $t<0$
output signal occurs before input signal arrives (non-physical)

Def: A causal filter is one for which the output signal begins after the input signal has started to arrive.

Theorem: Let $Lf = f * h$ be an LTI filter,
Then L is causal iff $h(t) = 0 \quad \forall\, t < 0$

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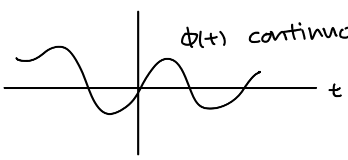
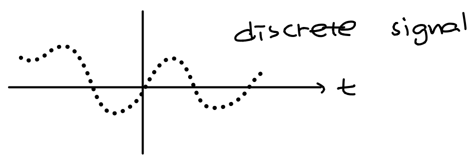
Discrete Fourier Transform





Can compute $F[\phi]$ Need discrete analog of $F[\phi]$

let $V = (V_0, V_1, \dots, V_{N-1}) \in \mathbb{C}^N$ be a discrete signal $(V_i \in \mathbb{C})$
$N \in \mathbb{N}$ fixed, and lets define

$$F_N(m,n) = e^{-2\pi i \frac{mn}{N}} \qquad (m, n = 0, 1, \dots, N-1)$$

F_N is an $N \times N$ complex matrix (the Fourier matrix)

$$F_N = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-2\pi i \frac{1}{N}} & e^{-2\pi i \frac{2}{N}} & \cdots & e^{-2\pi i \frac{N-1}{N}} \\ 1 & \vdots & \vdots & \ddots & \vdots \\ \vdots & & & & \\ 1 & e^{-2\pi i \frac{N-1}{N}} & e^{-2\pi i \frac{2(N-1)}{N}} & \cdots & e^{-2\pi i \frac{(N-1)^2}{N}} \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{column } n=0 & \text{column } n=1 & \text{column } n=N-1 \end{array} \end{array} \begin{array}{l} \text{row } 0=m \\ \text{row } 1=m \\ \\ \\ \text{row } m=N-1 \end{array}$$

Note: multiplying by F_N requires $O(N^2)$ computations.
Later we'll learn FFT (fast Fourier transform)

Define now $F_N : \mathbb{C}^N \to \mathbb{C}^N$ a linear map

s.t. $\underbrace{\hat{v}_k}_{\text{discrete signal}} = [F_N(v)](k) = \underbrace{\frac{1}{\sqrt{N}}}_{\substack{\text{a normalization} \\ \text{so that } F_N \text{ is} \\ \text{unitary}}} \sum_{j=0}^{N-1} v_j e^{-2\pi i \frac{jk}{N}}$ for $k = 0, \dots, N-1$

$i = \sqrt{-1}$

if we define $\omega = e^{-\frac{2\pi i}{N}}$, then

$$\hat{v}_k = [F_N(v)](k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j \omega^{jk}$$ this is the DFT (discrete Fourier transform)

Note: $\omega = \omega_N$ are complex N^{th} roots of unity

Motivation for DFT definition:

Recall $F[\phi](k) = \int_{-\infty}^{\infty} \phi(t) e^{-2\pi i t k} \, dt$ (need to discretize)

consider $\int_0^1 \phi(t) e^{-2\pi i t k} \, dt \approx \frac{1}{N} \sum_{j=0}^{N-1} \boxed{\phi(\tfrac{j}{N})} e^{-2\pi i \frac{j}{N} k}$ vector components of a sampling of a signal $\phi(t)$

our DFT uses $\frac{1}{\sqrt{N}}$ as a normalization

$\frac{0}{N} = 0, \quad \frac{1}{N}, \quad \frac{2}{N}, \quad \frac{3}{N}, \quad \dots, \quad 1 = \frac{N}{N} \quad t$

Lemma: Let $\omega = e^{-2\pi i / N}$. Then $\sum_{j=0}^{N-1} \omega^{jk} = \begin{cases} N & \text{if } \frac{k}{N} \in \mathbb{Z} \\ 0 & \text{else} \end{cases}$

Remark: Because of this, each row / column of F_N sums to zero (except for the 1^{st} row/column)

Ex: Let $v = (1, 1, 1, \dots, 1) \in \mathbb{C}^N$

then $[F_N(v)](k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j \omega^{jk} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \omega^{jk} = \frac{1}{\sqrt{N}} \begin{cases} N & \text{if } \frac{k}{N} \in \mathbb{Z} \\ 0 & \text{else} \end{cases}$ is an integer only for $k=0$ from the range $0, 1, \dots, N-1$

so $\hat{v} = (\underbrace{\sqrt{N}}_{k=0 \text{ component}}, 0, 0, \dots, 0)$

Inverse DFT: Getting from $\hat{v}$ to v, $v = F_N^{-1}(\hat{v})$

Note $F_N(m,n) = e^{-2\pi i \frac{mn}{N}}$ is a symmetric matrix, so $F_N = F_N^T$
It is also unitary, i.e. $F_N^{-1} = F_N^* = \overline{F_N}^T$

so $F_N^{-1} = \overline{F_N}^T = \overline{F_N} \Rightarrow F_N^{-1}(m,n) = e^{2\pi i \frac{mn}{N}} = \overline{\omega}^{mn}$

Finally $[F_N^{-1}(v)](k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j e^{2\pi i \frac{jk}{N}}$ is the inverse DFT.

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DFT II

Recall given $v \in \mathbb{C}^N$ that

$$\hat{v}_k = [F_N(v)](k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j e^{-2\pi i \frac{kj}{N}} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j w^{kj} \qquad \text{where } w = e^{-2\pi i/N}$$

for $k = 0, 1, \dots, N-1$, and

$$v_k = [F_N^{-1}(\hat{v})](k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \hat{v}_j e^{2\pi i \frac{jk}{N}}$$

Periodicity: $\hat{v} \in \mathbb{C}^N$

$$\hat{v}_{k+N} = [F_N(v)](k+N) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j \underline{e^{-2\pi i \frac{(k+N)j}{N}}} \to e^{-2\pi i \frac{kj}{N}} \underline{e^{-2\pi i j}} = 1 \text{ by Euler's formular}$$

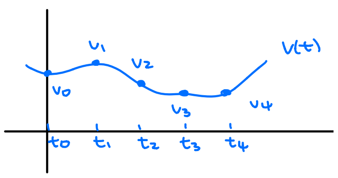
$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j e^{-2\pi i \frac{kj}{N}}$$

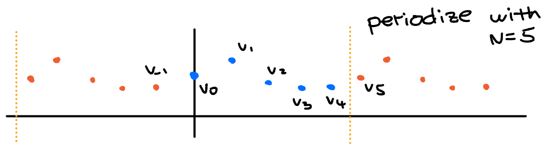
$$= [F_N(v)](k) = \hat{v}_k$$

So $\hat{v}_{k+N} = \hat{v}_k \quad \forall k \in \mathbb{Z}$, $v \in \mathbb{C}^N$, N is the length of discrete signal

$\Rightarrow$ $\hat{v}$ is periodic with period N

if we consider extending it beyond length N.





Ex: $v = (v_0, v_1, \dots, v_{63})$, $\hat{v}_1 = 5$

then $\hat{v}_{65} = \hat{v}_{1+64} = \hat{v}_1 = 5$

Symmetry: Denote $y_k = \hat{v}_k$ for a signal $v \in \mathbb{C}^N$ (so $y = F_N(v)$)

let $\{y_k\}_{k=0}^{N-1}$ be the DFT of $v \in \underline{\mathbb{R}^N}$ real-valued signal

then y_0 is real-valued and $y_{N-k} = \overline{y_k}$ $\forall k = 1, \dots, N-1$

pf: (i) $y_0 = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j \underbrace{e^{-2\pi i \frac{kj}{N}}}_{=1}$ ($k=0$) $= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \underbrace{v_j}_{\in \mathbb{R}} \in \mathbb{R}$

(ii) $y_{N-k} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j \underline{w^{(N-k)j}}$ $\qquad w^{N-k} = \overline{w^k}$, $w = e^{-2\pi i/N}$

① $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

② $\bar{\bar{z}} = z$

③ $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j (\overline{w^k})^j$$

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \overline{v_j w^{kj}}$$

$$= \overline{y_k}$$

since $v_j \in \mathbb{R}$, so $v_j = \overline{v_j}$

Ex: $v = (1, 1, -1, 0)$, $\hat{v} = (\frac{1}{2}, 1 - \frac{1}{2}i, -\frac{1}{2}, 1 + \frac{1}{2}i)$, $N = 4$.

Have $\hat{v}_0 = \frac{1}{2} \in \mathbb{R}$. and $\hat{v}_3 = \hat{v}_{4-1} = 1 + \frac{1}{2}i = \overline{1 - \frac{1}{2}i} = \overline{\hat{v}_1}$

$\hat{v}_2 = \hat{v}_{4-2} = -\frac{1}{2} = \overline{-\frac{1}{2}} = \overline{\hat{v}_2}$

Note: if N = even, then $y_{N-\frac{N}{2}} = y_{\frac{N}{2}} = \overline{y_{\frac{N}{2}}}$ $\Rightarrow$ $y_{\frac{N}{2}}$ is real-valued

DFT and trig interpolation

v is defined on time-interval $[c, d]$ at time $\{t_j\}$

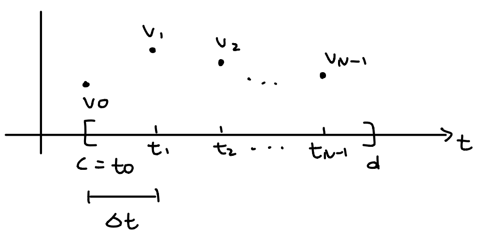
Suppose $v \in \mathbb{R}^N$ is a real-valued signal.

Let $\vec{y} = F_N(v) = \vec{a} + \vec{b}i$ be the DFT of v.

Then

$$P(t) = P_N(t) = \frac{a_0}{\sqrt{N}} + \frac{a_{N/2}}{\sqrt{N}} \cos\frac{N\pi(t-c)}{d-c} + \frac{2}{\sqrt{N}} \sum_{j=1}^{\frac{N}{2}-1} \left(a_j \cos\frac{2j\pi(t-c)}{d-c} - b_j \sin\frac{2j\pi(t-c)}{d-c}\right)$$

interpolates $\{(t_k, v_k)\}_{k=0}^{N-1}$ (i.e. $P(t_k) = v_k$, $\forall k = 0, 1, \dots, N-1$)



let $N \in \mathbb{N}$ and $\delta t = \frac{d-c}{N}$

so $t_k = c + k\delta t$ for $k = 0, 1, \dots, N-1$

write $y = F_N(v)$, so that

$$v_k = [F_N^{-1}(y)](k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} y_j \omega^{-kj}$$

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} y_j e^{2\pi i \frac{kj}{N}}$$ ← note $\frac{k}{N} = \frac{t_k - c}{N\delta t} = \frac{t_k - c}{d-c}$

$$= \sum_{j=0}^{N-1} \underbrace{y_j}_{\text{coefficients are given by the DFT}} \underbrace{\frac{e^{2\pi i j \frac{t_k - c}{d-c}}}{\sqrt{N}}}_{\text{trig basis functions evaluated at } t = t_k}$$

So the data points $\{(t_k, v_k)\}_{k=0}^{N-1}$ is interpolated by the function

$$Q(t) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} y_j \exp\left(\frac{2\pi i j(t-c)}{d-c}\right) \quad \text{i.e } v_k = Q(t_k) \quad \forall k = 0, 1, \dots, N-1$$

Ex: $v = (1, 1, -1, 0)$ on $[0, 1] = [c, d]$

DFT is $\vec{y} = (\frac{1}{2}, 1 - \frac{1}{2}i, -\frac{1}{2}, 1 + \frac{1}{2}i)$

$$= \underbrace{\begin{pmatrix} 1/2 \\ 1 \\ -1/2 \\ 1 \end{pmatrix}}_{\vec{a}} + \underbrace{\begin{pmatrix} 0 \\ -1/2 \\ 0 \\ 1/2 \end{pmatrix}}_{\vec{b}} i$$

So $P(t) = P_4(t) = \frac{1/2}{\sqrt{4}} + \frac{-1/2}{\sqrt{4}} \cos\frac{4\pi(t-0)}{1-0} + \frac{2}{\sqrt{4}} \sum_{j=1}^{1} a_j \cos\frac{2j\pi t}{1} - b_j \sin\frac{2j\pi t}{1}$

$$= \frac{1}{4} - \frac{1}{4}\cos(4\pi t) + \cos(2\pi t) + \frac{1}{2}\sin(2\pi t)$$

4/2/21

Fast Fourier Transform (FFT) – an algorithm to compute the DFT

Recall for $k = 0, 1, \dots, N-1$

have $[F_N(v)](k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j w_N^{jk}$ where $w_N = e^{-2\pi i/N}$

requires $N+1$ multiplications, and $N-1$ additions. for every $k = 0, 1, \dots, N-1$

$\Rightarrow$ total operation cost is $N((N+1)+(N-1)) = 2N^2 = O(N^2)$

Ex: Suppose $N = 10^6 \Rightarrow \approx (10^6)^2 = 10^{12}$ operations needed to construct the DFT

FFT requires only $O(N \log_2 N)$ operations to construct the DFT

If $N = 10^6$, then $\approx N \log_2 N = 10^6 \log_2 10^6 \approx 10^7$

Main Idea behind FFT

let F_N be the $N \times N$ DFT and let $N = 2^q$ $(q \in \mathbb{Z}_+)$

$$[F_N(v)](k) = \sum_{j=0}^{N-1} v_j w_N^{jk} = \underbrace{\sum_{j=0}^{\frac{N}{2}-1} v_{2j} w_N^{k(2j)}}_{\text{even indices}} + \underbrace{\sum_{j=0}^{\frac{N}{2}-1} v_{2j+1} w_N^{k(2j+1)}}_{\text{odd indices}} = w_N^k \sum_{j=0}^{\frac{N}{2}-1} v_{2j+1} w_N^{k(2j)}$$

Ignore $\frac{1}{\sqrt{N}}$ for now

$$= \sum_{j=0}^{\frac{N}{2}-1} \underbrace{v_{2j}}_{=: v_j^e \text{ (even sequence)}} \underbrace{e^{-2\pi i \frac{kj}{N/2}}}_{= w_{\frac{N}{2}}^{kj}} + w_N^k \sum_{j=0}^{\frac{N}{2}-1} \underbrace{v_{2j+1}}_{=: v_j^o \text{ (odd sequence)}} \underbrace{e^{-2\pi i \frac{kj}{N/2}}}_{= w_{\frac{N}{2}}^{kj}}$$

$$= \underbrace{[F_{\frac{N}{2}}(v^e)](k)}_{\text{cost: } 2\left(\frac{N}{2}\right)^2 = \frac{N^2}{2}} + w_N^k \underbrace{[F_{\frac{N}{2}}(v^o)](k)}_{\text{cost: } 2\left(\frac{N}{2}\right)^2 = \frac{N^2}{2}}$$

another add/mult cost $2N$

Total cost to compute is $\frac{N^2}{2} + \frac{N^2}{2} + 2N = N^2 + 2N$ less than $2N^2$

- So each F_N can be computed with two $F_{\frac{N}{2}}$'s and the $F_{\frac{N}{2}}$'s can be computed with $F_{\frac{N}{4}}$'s and so on until we get to F_1.
  F_1, means get a 1×1 matrix at the lowest level.

- It can be shown that iterating this process results in an $O(N \log_2 N)$ algorithm to compute DFT

Matlab: let $x \in \mathbb{C}^N$ and $y = F_N(x)$

`>> fft(x)/sqrt(N);` → to account for our DFT normalization

`>> ifft(y)*sqrt(N);` inverse DFT

Understanding frequencies

Recall if $v = [1, 1, -1, 0]$ then $N = 4$
and $\hat{v} = fft(v)/\sqrt{N} = [\frac{1}{2}, 1 - \frac{1}{2}i, -\frac{1}{2}, 1 + \frac{1}{2}i]$

Supposing we sampled on $[c, d] = [0, 1]$ (so sampling frequency is $F_s = 4$ samples/sec)

a trig interpolant is

$$p(t) = \frac{a_0}{\sqrt{N}} + \frac{2}{\sqrt{N}} \sum_{j=1}^{\frac{N}{2}-1} \left(a_j \cos \frac{2j\pi(t-c)}{d-c} - b_j \sin \frac{2j\pi(t-c)}{d-c} \right) + \frac{a_{N/2}}{\sqrt{N}} \cos \frac{2\frac{N}{2}\pi(t-c)}{d-c}$$

$$= \underbrace{\frac{1}{4}}_{\text{freq } 0} + \underbrace{\cos(2\pi t)}_{\text{freq } 1} + \underbrace{\frac{1}{2}\sin(2\pi t)}_{\text{freq } 1} - \underbrace{\frac{1}{4}\cos(4\pi t)}_{\text{freq } 2}$$

$\cos(2\pi f t)$ ordinary frequency

So for entry $j = \{0, 1, \ldots\}$ in the DFT vector,

we have $f_j = \frac{j}{d-c} = \frac{j}{T} = \frac{j}{N/F_s}$

T: total time

since $F_s \cdot T = N$

samples/second $\cdot$ Total time (sec) = Total # of samples

$\Rightarrow f_j = F_s \cdot \frac{j}{N}$

is the frequency corresponding to the j^{th} slot in the DFT for $j = 0, 1, \ldots, \frac{N}{2}$

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Some more DFT properties

(DFT): $\hat{v}_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j w_N^{jk}$, $v \in \mathbb{C}^N$,

$w = w_N = e^{-2\pi i/N}$

Circular shift (translation)

$[T_n(v)](m) := v(m-n \mod N)$

T_n is a time-shift by n

Congruence Modulo N

$a \equiv b \pmod{N}$ if N divides $a-b$

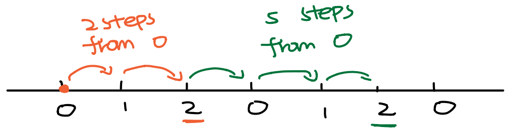
- equivalently $a - b = kN$ for some $k \in \mathbb{Z}$

Ex: $2 \equiv 5 \pmod 3$ since $5 - 2 = k(3)$ for $k = 1 \in \mathbb{Z}$

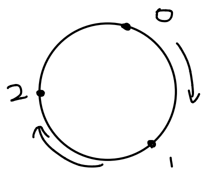
Least residue system module N

$\{0, 1, \dots, N-1\}$:

case $N=3$: $\{0, 1, 2\}$



end up at the same place



$2 \equiv 5 \pmod 3$

note : $0 \equiv N \pmod N$

Ex : $N=3$, $2 \equiv 2+0 \equiv 2+3 \equiv 5 \pmod 3$

Ex: $v = (v_0, v_1, v_2, v_3)$ $N=4$

$[T_2(v)](m) := v(m-2 \mod 4)$

$[T_2(v)](0) = v(\underline{0}-2 \mod 4) = v(2 \mod 4) = v_2$

$0 \equiv 4 \pmod 4$

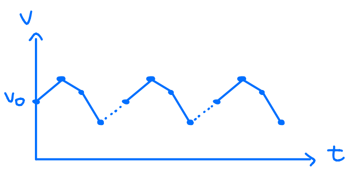
$[T_2(v)](1) = v(1-2 \mod 4) = v(-1+4 \mod 4) = v(3 \mod 4) = v_3$

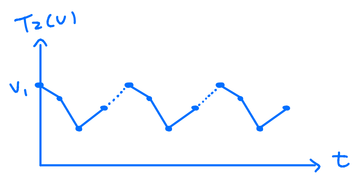
Similarly $[T_2(v)](2) = v_0$, $[T_2(v)](3) = v_1$

So $T_2(v) = (v_2, v_3, v_0, v_1)$

This is equivlent to $(v_0, v_1, v_2, v_3) \xmapsto{T_1(v)} (v_3, v_0, v_1, v_2) \xmapsto{T_1(v_2)} (v_2, v_3, v_0, v_1)$

$T_2(v)$





DFT of time shift ?

$v \in \mathbb{C}^N$

$$[T_n(v)](k) = v(k-n \mod N) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \hat{v}(j) w_N^{-j(k-n)}$$ ← using inverse DFT

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} [\underbrace{w_N^{jn} \hat{v}(j)}_{=: [M_{-n}(\hat{v})](j)}] w_N^{-jk}$$

where $[M_m(v)](j) := v(j) w_N^{-jm}$ is the moduation by m operator

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} [M_{-n}(\hat{v})](j) w_N^{-jk}$$

$$= F_N^{-1} (M_{-n}(\hat{v}))(k)$$

Apply F_N to both sides to get $F_N(T_n(v)) = M_{-n}(\hat{v}) = M_{-n}(F_N(v))$

Similarly $F_N(M_m(v)) = T_m(F_N(v))$

Circular convolution (for periodic sequences)

Recall $(f * g)(t) = \int_{\mathbb{R}} f(\tau) g(t-\tau)\, d\tau$

Defn: $v, w \in \mathbb{R}^N$, $(v * w)(n) = \sum_{m=0}^{N-1} v(m) w(n-m \mod N)$

$$[\mathcal{F}_N(v * m)](k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \underbrace{(v * w)(n)}_{:= \sum_{m=0}^{N-1} v(m) w(n-m \mod N)} \omega_N^{nk}$$

$$= \sum_{m=0}^{N-1} v(m) \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \underbrace{w(n-m \mod N)}_{= [T_m(w)](n) \quad (\text{translation})} \omega_N^{nk}$$

$$= \sum_{m=0}^{N-1} v(m) \underbrace{\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} [T_m(w)](n) \omega_N^{nk}}_{= [\mathcal{F}_N(T_m(w))](k) = [M_{-m}(\hat{w})](k) = \hat{w}(k)\omega_N^{km}}$$

$$= \sum_{m=0}^{N-1} v(m) \hat{w}(k) \omega_N^{km}$$

$$= \hat{w}(k) \underbrace{\sum_{m=0}^{N-1} v(m) \omega_N^{km}}_{} = \sqrt{N} \cdot \hat{v}(k) \text{ by def. of DFT}$$

$$= \sqrt{N}\, \hat{w}(k) \cdot \hat{v}(k)$$

So $\mathcal{F}_N(v * w) = \sqrt{N}\, \hat{w} \cdot \hat{v} = \sqrt{N}\, \mathcal{F}_N(w) \cdot \mathcal{F}(v)$

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Discrete Sine & Cosine Transforms

DCT-II: $C_N^{\mathrm{II}} : \mathbb{R}^N \to \mathbb{R}^N$, $\mathbb{R}^N$ real valued

$C_N^{\mathrm{II}}(m, n) = b(m) \cos \frac{\pi(n+\frac{1}{2})m}{N}$ $\quad (m, n = 0, 1, \dots, N-1)$

where $b(m) := \begin{cases} 1/\sqrt{2} & \text{if } m = 0 \\ 1 & \text{if } 0 < m < N \end{cases}$

- The matrix $C := \sqrt{\frac{2}{N}} C_N^{\mathrm{II}}$ is orthogonal.

Ex: 2×2, $N=2$ $\quad C = \sqrt{\frac{2}{2}} C_2^{\mathrm{II}} = C_2^{\mathrm{II}} = \begin{array}{cc} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} & \begin{array}{l} \leftarrow m=0 \\ \leftarrow m=1 \end{array} \\ \begin{array}{cc} \uparrow\, n=0 & \uparrow\, n=1 \end{array} & \end{array}$

to check C is orthogonal $(C^{-1} = C^T)$

$$C^{-1} = \frac{1}{-\frac{1}{2} - \frac{1}{2}} \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = C^T$$

Signal $v = (v_0, v_1, \dots, v_{N-1})^T \in \mathbb{R}$, then $y = Cv$ is the DCT-II transform of v.

- y contains the (cosine) frequency information for v.

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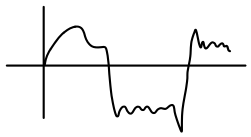
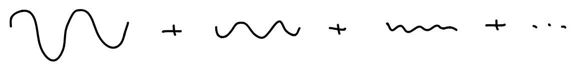
Wavelets :

$s(t)$ continuous, $\longrightarrow$ $\mathcal{F}[s(t)](k)$ Fourier transform

$\vec{s}$ discrete $\longrightarrow$ $F_N(\vec{s})$ DFT

time domain $\longrightarrow$ frequency domain

 $\longrightarrow$ 

$\sin(t)$ + $\sin(2t)$ + $\sin(100t)$ + ...

get no time localization / resolution
(i.e. don't know when the frequencies occur)

Fourier tranform (or DFT) uses basis functions $e^{i\theta}$ (sines / cosines)

Roughly speaking, wavelets use a basis that looks like

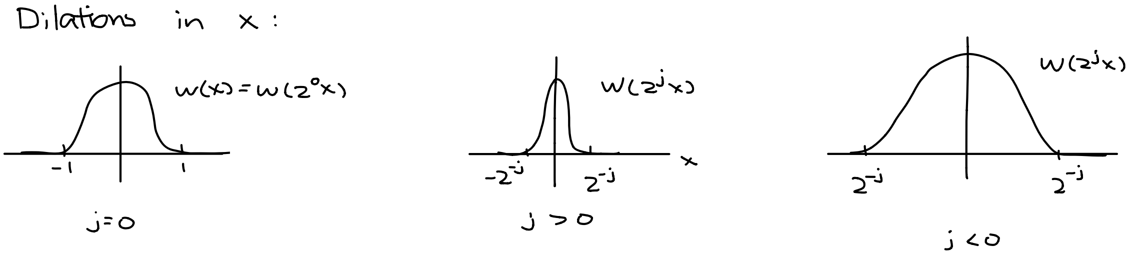
$\{ C_{jk} W(2^j x - k) \}_{jk}$ $(j, k \in \mathbb{Z})$ where $W(\cdot)$ is some fixed function

dilation in height

dilation in width

translation of W

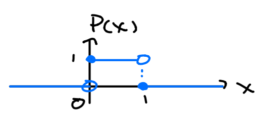
Dilations in x :



Haar Scaling functions

Let $\mathbb{1}_A(x) := \begin{cases} 1 & x \in A \\ 0 & \text{else} \end{cases}$ (characteristic function of A)

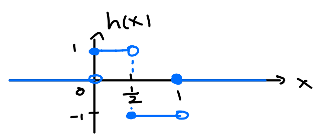
$P(x) := \mathbb{1}_{[0,1)}(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & \text{else} \end{cases}$



(Haar scaling function)

and $h(x) := \mathbb{1}_{[0,\frac{1}{2})}(x) - \mathbb{1}_{[\frac{1}{2},1)}(x)$

$= \begin{cases} 1 & x \in [0, \frac{1}{2}) \\ -1 & x \in [\frac{1}{2}, 1) \\ 0 & \text{else} \end{cases}$



(Haar wavelet)

- We say that a function has compact support if it vanishes outside of a bounded set.
- The support of f is the closure of the set on which f is nonzero.

Ex: $P(x)$ is supported on $\overline{[0,1)} = [0,1]$

Dilation / Translation operators

Let $a > 0$, $b \in \mathbb{R}$.

① $D_a f(x) := \sqrt{a} f(ax)$ (dilation)

② $T_b f(x) := f(x-b)$ (translation)

Note: $\sqrt{a}$ ensures D_a preserves $L^2(\mathbb{R})$ norms!

$$\int_{\mathbb{R}} |D_a f(x)|^2 \, dx = \int_{\mathbb{R}} |\sqrt{a} f(ax)|^2 \, dx = \int_{\mathbb{R}} |f(y)|^2 \, dy$$

$y = ax$

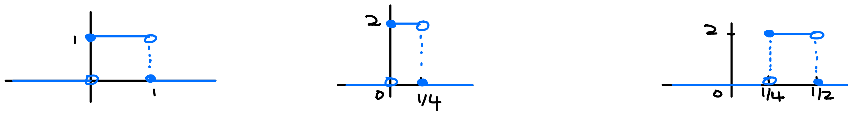
$dy = a\,dx$

$\Rightarrow \|D_a f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$

Definition: $\{P_{j,k}(x)\}_{j,k \in \mathbb{Z}} = D_{2^j} T_k\, p(x) = 2^{j/2} p(2^j x - k)$

is the system of Haar scaling functions.

For j fixed, $\{P_{j,k}(x)\}_{k \in \mathbb{Z}}$ defines the scale j Haar scaling functions.



$P_{0,0} = P(x)$ $\quad$ $P_{2,0} = 2p(4x-0)$ $\quad$ $P_{2,1} = 2p(4x-1) = 2p(4(x-\frac{1}{4}))$

We can write $\quad P_{j,k}(x) = 2^{j/2} \mathbb{1}_{I_{j,k}}(x) = 2^{j/2} \begin{cases} 1 & x \in I_{j,k} \\ 0 & \text{else} \end{cases}$

where $I_{j,k} := [2^{-j}k, 2^{-j}(k+1))$ (dyadic intervals)

rescaling (j) and shifts (k) of $[0,1)$

Ex: $I_{0,0} = [0,1)$, $I_{1,0} = [0, 2^{-1})$, $I_{0,1} = [1,2)$

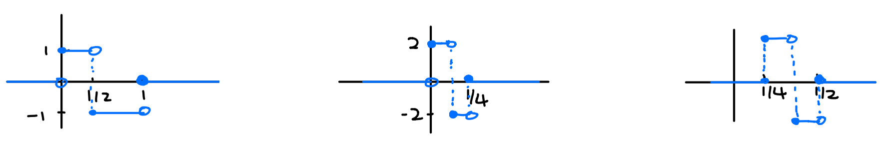
The Haar scaling functions are normalized in $L^2(\mathbb{R})$: $\|P_{j,k}\|_{L^2(\mathbb{R})} = 1 \quad \forall j,k \in \mathbb{Z}$

Haar Wavelet Functions

$\{h_{j,k}\}_{j,k \in \mathbb{Z}}$ is the Haar wavelet system in $\mathbb{R}$

where $\quad h_{j,k}(x) = 2^{j/2} h(2^j x - k) = D_{2^j} T_k h(x)$

For j fixed, $\{h_{j,k}\}_{k \in \mathbb{Z}}$ are the scal j Haar wavelets.



$h_{0,0} = h(x)$ $\quad$ $h_{2,0} = 2h(4x)$ $\quad$ $h_{2,1} = 2h(4x-1)$

Have $\quad h_{j,k}(x) = 2^{j/2}\left(\mathbb{1}_{I_{j+1,2k}}(x) - \mathbb{1}_{I_{j+1,2k+1}}(x)\right)$

The Haar wavelets are also normalized in $L^2(\mathbb{R})$: $\|h_{j,k}\|_{L^2(\mathbb{R})} = 1 \quad \forall j,k \in \mathbb{Z}$

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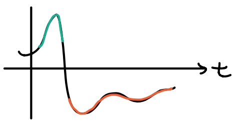
Recall

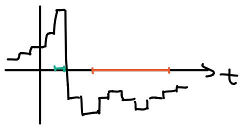
Haar scaling functions $P_{j,k}(x) = D_{2^j} T_k \underline{p(x)} = 2^{j/2} p(2^j x - k)$

$\quad\quad \llcorner = \mathbb{1}_{[0,1)}(x)$

Haar wavelet functions $h_{j,k}(x) = D_{2^j} T_k \underline{h(x)} = 2^{j/2} h(2^j x - k)$

$\quad\quad \llcorner = \begin{cases} 1 & [0, 1/2) \\ -1 & [1/2, 1) \\ 0 & \text{else} \end{cases}$

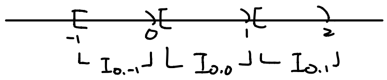


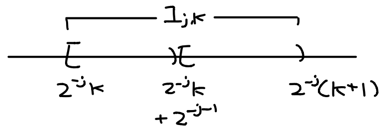


$h_{j,k}$, j have ability to scale in time

Dyadic Intervals

$I_{j,k} = [2^{-j}k, 2^{-j}(k+1))$





$I^{l}_{j,k} = [2^{-j}k, 2^{-j}k + 2^{-j-1})$ $\quad I^{r}_{j,k} = [2^{-j}k + 2^{-j-1}, 2^{-j}(k+1))$

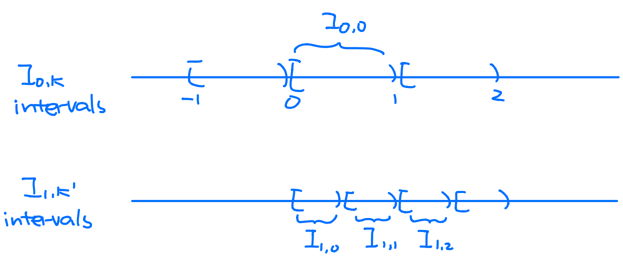
Lemma: Suppose $(j,k) \neq (j',k')$ (so either $j \neq j'$, or $k \neq k'$, or both are not equal)

Then either i) $I_{j,k} \cap I_{j',k'} = \emptyset$

OR ii) $I_{j,k} \subset I_{j',k'}$

OR iii) $I_{j',k'} \subset I_{j,k}$

Ex: $I_{0,k}$ and $I_{1,k'}$. so $j \neq j'$. different scales



Have $I_{1,0} \subset I_{0,0}$

$I_{1,1} \subset I_{0,0}$

$I_{1,2} \cap I_{0,0} = \emptyset$

note $I_{1,0} \subset I^{l}_{0,0}$

$I_{1,1} \subset I^{r}_{0,0}$

So if $I_{j,k} \subset I_{j',k'}$, then either i) $I_{j,k} \subset I^{l}_{j',k}$

ii) $I_{j,k} \subset I^{r}_{j',k'}$

Theorem: $\{h_{j,k} : j,k \in \mathbb{Z}\}$ on $\mathbb{R}$ is an orthonormal basis of $L^2(\mathbb{R})$

means any $f \in L^2(\mathbb{R})$ can be written uniquely as

$$f(x) = \sum_{j,k=-\infty}^{\infty} c_{jk} h_{j,k}(x) \quad \text{for some } c_{jk} \in \mathbb{R}$$

If we insist that $j \geq \underline{J}$ some fixed largest scale, then

Theorem: $\{P_{J,k}, h_{j,k} : \underline{k} \in \mathbb{Z}, j \geq J\}$ form an orthonormal basis of $L^2(\mathbb{R})$

$\llcorner$ but we restrict the scale

$\llcorner$ we use all translates

Let's check the orthogonality in $L^2(\mathbb{R})$

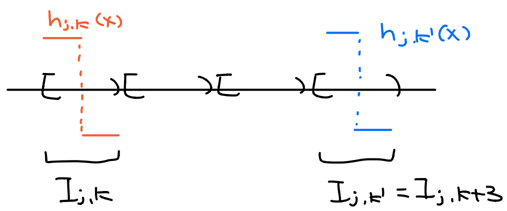
Case ① $j \in \mathbb{Z}$ fixed, $k \neq k'$

② $j \in \mathbb{Z}$ fixed, $k = k'$

③ $j \neq j'$ (suppose WLOG that $j > j'$)

- i) $I_{j,k} \cap I_{j',k'} = \phi$
- ii) $I_{j,k} \subset I^{\ell}_{j',k'}$
- iii) $I_{j,k} \subset I^{r}_{j',k'}$

Case ①,②: $\langle h_{j,k}, h_{j,k'} \rangle = \int_{\mathbb{R}} \underbrace{h_{j,k}(x)}_{\text{supported on } I_{j,k}} \underbrace{h_{j,k'}(x)}_{\text{supported on } I_{j,k'}} dx = \begin{cases} 0 & \text{if } k \neq k' \\ \|h_{j,k}\|^2_{L^2(\mathbb{R})} = 1 & \text{if } k = k' \end{cases}$

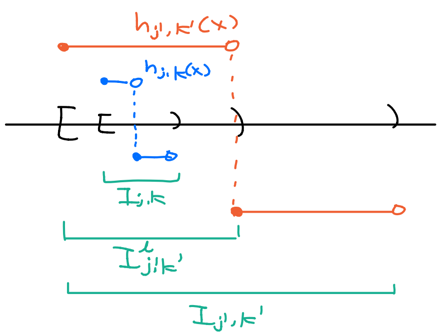


Case ③i: $j \neq j'$ and $I_{j,k} \cap I_{j',k'} = \phi$

$\int_{\mathbb{R}} h_{j,k}(x) h_{j',k'}(x)\, dx = \int_{\mathbb{R}} 0\, dx = 0$ since $h_{j,k}(x) h_{j',k'}(x)$ doesn't have a common support.

Case ③ii: $j \neq j'$ and $I_{j,k} \subset I^{\ell}_{j',k'}$

$\int_{\mathbb{R}} h_{j,k}(x) h_{j',k'}(x)\, dx = \int_{I_{j,k}} h_{j,k}(x) \cdot 2^{j'/2}\, dx = 0$ since $\int_{I_{j,k}} h_{j,k}(x) = 0$



Case ③iii: note $\langle h_{j,k}, h_{j',k} \rangle = 0$

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Haar wavelets on $[0,1]$, and projections

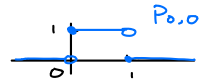
$\{ P_{J,k}, h_{j,k} : k \in \mathbb{Z}, j \geq J \}$ is an ONB for $L^2(\mathbb{R})$

fixing a scale J

Let's restrict so $P_{J,k}$ and $h_{j,k}$ are supported only on $[0,1]$

Need $J \geq 0$ (otherwise $P_{J,k}$ is supported on a set with length >1)

$\text{Supp}(P_{J,k}) = I_{J,k} := [2^{-J}k,\ 2^{-J}(k+1))$



Want $\text{supp}(P_{J,k}) \subset [0,1]$ So want $2^{-J}k \geq 0$ and $2^{-J}(k+1) \leq 1$

$k \geq 0$ $\quad$ $k \leq 2^J - 1$

So we get $0 \leq k \leq 2^J - 1$

- Similarly, we want $j \geq J \geq 0$ and $0 \leq k \leq 2^j - 1$ so $\text{supp}(h_{j,k}) \subset [0,1]$

The scale J Haar system on [0,1]

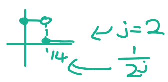
$$\{ P_{J,k}(x) \mid 0 \leq k \leq 2^J - 1\} \cup \{ h_{j,k}(x) \mid j \geq J,\ 0 \leq k \leq 2^j - 1 \}$$

- If $J=0$ we call this the Haar system on $[0,1]$.

Thm: The scale J Haar system is an ONB of $L^2([0,1])$

L^2- projections

$\phi(2^j x - k)$



Fix a scale $j \in \mathbb{Z}$

Haar scaling functions

Define $P_j f(x) = \sum_{k \in \mathbb{Z}} \langle f, P_{j,k} \rangle_{L^2(\mathbb{R})} P_{j,k}(x)$

"Approximation Operator"

This is an orthogonal projection of $f \in L^2(\mathbb{R})$

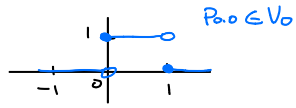
onto $V_j := \overline{\text{span}}\{P_{j,k}\}_{k \in \mathbb{Z}} := \{ v \in L^2(\mathbb{R}) \mid v(x) = \sum_{k \in \mathbb{Z}} c_k P_{j,k}(x),\ c_k \in \mathbb{R} \}$

$= \{ v \in L^2(\mathbb{R}) \mid v(x) = \sum_{k \in \mathbb{Z}} d_k P(\boxed{2^j}x - k),\ d_k \in \mathbb{R} \}$

V_0 is the space of piecewise constants with jumps at the integers.

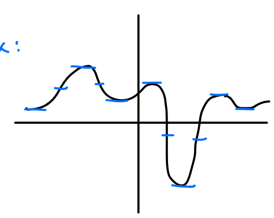
we fix a resolution $\frac{1}{2^j}$, then V_j consists of all "infinite linear combinations" of $\{P_{j,k}\}_{k \in \mathbb{Z}}$

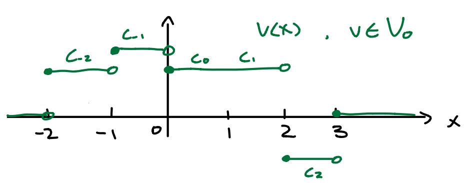
Space V_0



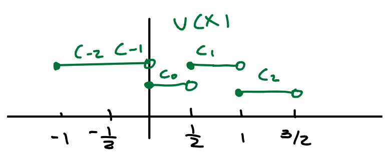
$\frac{1}{2^0} = 1$

Ex:





Space V_1, scale $\frac{1}{2^1} = \frac{1}{2}$, $v(x) = c_0 P_{1,0}(x) + c_1 P_{1,1}(x) + c_2 P_{1,3}(x)$

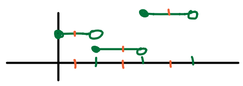


$v \in V_1 = \{$ set of piecewise constants with jumps only allowed at half-integers $\}$

Note if $g \in V_j$ and let $j \le j'$

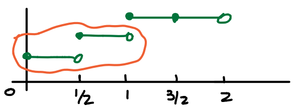
↳ index at a fixer scale

Then $g \in V_j$



so $P_{j'}\, g(x) = g(x)$

The reverse claim is false



$v \in V_1$ but $v \in V_0$

We get a containment $V_j \subset V_{j'}$ if $j \le j'$

and $V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset V_3 \subset \cdots$

Theorem: As $j \to \infty$, we have $\lim_{j\to\infty} P_j f(x) = f(x)$ in the sense that

$$\lim_{j\to\infty} \| P_j f(x) - f(x) \|_{L^2(\mathbb{R})} = 0$$

coarse scale   fine scale

Detail Operator: $Q_j f(x) = P_{j+1} f(x) - P_j f(x) \quad \forall f \in L^2(\mathbb{R})$

fix $j \in \mathbb{Z}$

① we project f to scale $j+1$

② then remove everything that can be represented at scale j

③ so only the scale $j+1$ "detail" is left

The wavelet space at scale j is defined as

$$W_j := \overline{\operatorname{span}}\{ h_{j,k} \mid k \in \mathbb{Z} \}$$
$$= \{ w \in L^2(\mathbb{R}) \mid w(x) = \sum_{k\in\mathbb{Z}} c_k h_{j,k}(x) \;, c_k \in \mathbb{R} \}$$

Thm: Q_j is an orthogonal projection onto W_j

Pf idea: write $V_{j+1} = V_j \oplus W_j$

meaning every $v_{j+1} \in V_{j+1}$ can be written

uniquely as $v_{j+1} = \underset{\in V_j}{v_j} + \underset{\in W_j}{w_j}$ and $\langle v_j, w_j \rangle = 0$

So define $Q_j f(x) = w_j = v_{j+1} - v_j = P_{j+1} f(x) - P_j f(x)$

Thm: If $f \in C_c(\mathbb{R})$

↳ means continous functions that vanish outside a bounded set

then $Q_j f(x) = \sum_{k\in\mathbb{Z}} \langle f, h_{j,k} \rangle h_{j,k}$

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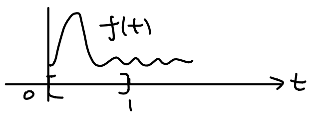
Discrete Haar Transform (DHT)

$f \in L^2[0,1]$



The scale J Haar System is

$\{ p_{J,k} : 0 \le k \le 2^J - 1 \} \cup \{ h_{j,k} : j \ge J,\ 0 \le k \le 2^j - 1 \}$

this forms an ONB of $L^2[0,1]$

Expand f as

$$f(x) = \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \underbrace{\langle f, h_{j,k} \rangle}_{\text{Haar coefficients}} h_{j,k}(x) + \sum_{k=0}^{2^J-1} \underbrace{\langle f, p_{j,k} \rangle}_{\text{Haar coefficients}} p_{J,k}$$

Idea: The DHT is a discrete version of this expansion.

First, remove the ∞ in the first sum

motivation: Approximete f by $P_N f$, where $N > J$

$$f(x) \approx P_N f(x) = \sum_{k=0}^{2^N-1} \langle f, p_{N,k} \rangle p_{N,k}$$

$P_N f$ approximation operator (projection onto scale N Haar scaling functions)

Note: If $f \in V_j$, $j \le N$ then $f = P_N f$

then we can approximate

$\langle f, h_{j,k} \rangle \approx \langle P_N f, h_{j,k} \rangle$ and $\langle f, p_{j,k} \rangle \approx \langle P_N f, p_{J,k} \rangle$

the approximations converge as $N \to \infty$

Also $\displaystyle\sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \langle P_N f, h_{j,k} \rangle h_{j,k}$ reduces to $\displaystyle\sum_{j=J}^{N-1} \sum_{k=0}^{2^j-1} \langle P_N f, h_{j,k} \rangle h_{j,k}$

this is since $\langle \underline{P_N f}, h_{j,k} \rangle = 0$ for $j \ge N$, $0 \le k \le 2^j - 1$

is a linear combination of Haar scaling functions $\{p_{N,k}\}_{k=0}^{2^N-1}$

since scale N Haar system is orthogonal

Next let's discretize f:

$$f(x) \approx P_N f(x) = \sum_{k=0}^{2^N-1} \underline{\langle f, p_{N,k} \rangle} p_{N,k}$$

Define $\{c_0(k)\}_{k=0}^{2^N-1}$ as $c_0(k) := \langle f, p_{N,k} \rangle$

note $\{c_0(k)\}$ is a sequence of length 2^N that we'll use as our "discrete f"

Recall $\displaystyle f(x) \approx \sum_{j=J}^{N-1} \sum_{k=0}^{2^j-1} \underbrace{\langle f, h_{j,k} \rangle}_{= d_{N-j}(k)} h_{j,k}(x) + \sum_{k=0}^{2^J-1} \underbrace{\langle f, p_{J,k} \rangle}_{= \langle f, p_{N-k,k} \rangle = c_{N-J}(k) = c_k(k)} p_{J,k}(x)$

Definition: $c_j(k) := \langle f, p_{N-j,k} \rangle$

$d_j(k) := \langle f, h_{N-j,k} \rangle$

for $1 \le j \le K := N-J$ and $0 \le k \le 2^{N-j}-1$

$\llcorner$ since $N > J$, we can pick some $K \in \mathbb{N}$, s.t $N-K=J$

note for $j=0$, this coincides with the definition of $c_0(k) := \langle f, p_{N,k} \rangle$

Now the c_j, d_j can be computed recursively! (with base case c_0)

$$c_j(k) = \frac{1}{\sqrt{2}}\left(c_{j-1}(2k) + c_{j-1}(2k+1) \right)$$

$$d_j(k) = \frac{1}{\sqrt{2}}\left(c_{j-1}(2k) - c_{j-1}(2k+1) \right)$$

or as a system $\begin{pmatrix} c_j(k) \\ d_j(k) \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{} \begin{pmatrix} c_{j-1}(2k) \\ c_{j-1}(2k+1) \end{pmatrix}$

an orthogonal matrix!

so easy to invert $Q^{-1} = Q^T$

proof: ① $p_{\ell,k} = \frac{1}{\sqrt{2}}\left(p_{\ell+1,2k} + p_{\ell+1,2k+1} \right)$

② $h_{\ell,k} = \frac{1}{\sqrt{2}}\left(p_{\ell+1,2k} - p_{\ell+1,2k+1} \right)$

to check $c_j(k) = \langle f, p_{N-j,k} \rangle$

$= \frac{1}{\sqrt{2}} \langle f, p_{N-j+1,2k} + p_{N-j+1,2k+1} \rangle$

$= \frac{1}{\sqrt{2}} \left(c_{j-1}(2k) + c_{j-1}(2k+1) \right)$

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Matrix Implementation

Let $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{pmatrix}$ an orthogonal matrix.

Now c_j is a vector of length 2^{N-j}, so c_{j-1} has length 2^{N-j+1}

Let Q have the dimension $2^{N-j+1} \times 2^{N-j+1}$, note $\dim(Q)$ changes with j $(Q = Q_j)$

so it can act on c_{j-1}

Then $Qc_{j-1} = [c_j(0), d_j(0), c_j(1), d_j(1), \dots, c_j(2^{N-j}-1), d_j(2^{N-j}-1)]^T$

Intertwined, let's "unmix" these

Fact: Swapping rows of Q preserves orthogonality.

Ex: $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ still orthogonal

Ex: $Q \in \mathbb{R}^{3\times3}$ be orthogonal.

i) PQ is orthogonal (P as above)

ii) PQ results in Q with the 2nd and 3rd rows swapped.

Definition: Let H_j, G_j be $(\frac{L}{2} \times L)$ matrices where $L = 2^{N-j+1}$ $(j=1,\dots,k=N-J)$

$$H_j := \frac{1}{\sqrt{2}} \begin{pmatrix} 1\,1\,0\,0 & & & 0 \\ 0\,0\,1\,1 & & & \\ & \ddots & & \\ 0 & & & 1\,1 \end{pmatrix}$$ H_j will pick out the c_j coefficients

$$G_j := \frac{1}{\sqrt{2}} \begin{pmatrix} 1\,{-1}\,0\,0 & & & 0 \\ 0\,0\,1\,{-1} & & & \\ & \ddots & & \\ 0 & & & 1\,{-1} \end{pmatrix}$$ G_j will pick out the d_j terms

So $\underline{\begin{pmatrix} H_j \\ G_j \end{pmatrix}} \in (2^{N-j+1} \times 2^{N-j+1})$ $(N > J \geq 0 \quad j=1,\dots,k=N-J)$

the matrix for a "single level" (the j^{th}) of the DHT

Note $\begin{pmatrix} H_j \\ G_j \end{pmatrix} c_{j-1} = \begin{pmatrix} c_j \\ d_j \end{pmatrix}$ $\quad c_j = \begin{pmatrix} c_j(0) \\ c_j(1) \\ \vdots \\ c_j(2^{N-j}-1) \end{pmatrix}$ $\quad d_j = \begin{pmatrix} d_j(0) \\ d_j(1) \\ \vdots \\ d_j(2^{N-j}-1) \end{pmatrix}$

Often $J=0$, so H_j, G_j are sized $\frac{L}{2} \times L = 2^{N-j} \times 2^{N-j+1}$ for $j=1,\dots,k=N-J$
So H_1, G_1 are size $2^{N-1} \times 2^N$ and H_{N-J}, G_{N-J} are size 1×2

Discrete Haar Transform

Fix $N > J \geq 0$ and $c_0 := \{c_0(k)\}_{k=0}^{2^N-1}$ a discrete sequence/signal.
$c_0(k) := \langle f, p_{N,k} \rangle$

The DHT of c_0 is the collection
$\{c_{N-J}(k) : 0 \leq k \leq 2^J-1\} \cup \{d_j(k) : 1 \leq j \leq N-J,\ 0 \leq k \leq 2^{N-j}-1\}$

where $\begin{pmatrix} c_j \\ d_j \end{pmatrix} = \begin{pmatrix} H_j \\ G_j \end{pmatrix} c_{j-1}$ and where c_j, d_j are as defined before

Recall the point is that $f(x) \approx \sum_{j=J}^{N-1} \sum_{k=0}^{2^j-1} \underbrace{\langle f, h_{j,k} \rangle}_{= d_{N-j}(k)} h_{j,k}(x) + \sum_{k=0}^{2^J-1} \underbrace{\langle f, p_{J,k} \rangle}_{= c_{N-J}(k)} p_{J,k}(x)$

$- c_{\tilde{k}}(k)$

Case $J=0$:

DHT gives
so $k=0$
$\{c_N(k) : 0 \leq k \leq 2^0-1\} \cup \{d_j(k) : 1 \leq j \leq N,\ 0 \leq k \leq 2^{N-j}-1\}$

$\Rightarrow \quad (c_N, d_N, d_{N-1}, \dots, d_1)^T$

scalar $\quad$ vectors since k has multiple values

Remark:

Once we have $(c_N, d_N, d_{N-1}, \dots, d_1)^T$ we can do some "signal processing" on the cofizients and then have to invert the DHT to get the modified signal in time space (i.e. original domain)

Inverting the DHT

Recall $\begin{pmatrix} H \\ G \end{pmatrix}$ is an orthogonal matrix so $\begin{pmatrix} H \\ G \end{pmatrix}^{-1} = \begin{pmatrix} H \\ G \end{pmatrix}^{*} = (H^* \quad G^*)$

note $A^T = A^* := \bar{A}^T$ so T and $*$ are equivalent if A is real

Ex: $\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{pmatrix}^* = \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 2 & 3 \end{array}\right)$

Recall also $\begin{pmatrix} c_j \\ d_j \end{pmatrix} = \begin{pmatrix} H \\ G \end{pmatrix} c_{j-1} \Rightarrow c_{j-1} = (H^* \;\; G^*)\begin{pmatrix} c_j \\ d_j \end{pmatrix} = H^* c_j + G^* d_j$

and we can iterate back to $j=1$ (i.e. to c_0)

$c_{j-1} = H^* c_j + G^* d_j$ inverse DHT

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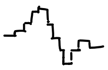
Filtering Transforms

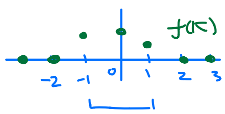
MRA (multiresolution analysis)

$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$ (approximation spaces at different scales)

$L^2(\mathbb{R}) = \overline{\bigcup_{j\in\mathbb{Z}} V_j} = \{ f \in L^2(\mathbb{R}) \mid f = \sum_{i\in\mathbb{Z}} c_i \phi_i \text{ s.t } \phi_i \in V_i \}$

$f = \{f(k)\}_{k\in\mathbb{Z}}$ "filter" (assume f has finite support)

Ex: 

f is supported on indices in $[-1, 1]$

$[F(u)](n) := \sum_{k\in\mathbb{Z}} f(k-2n)u(k) \quad \forall n \in \mathbb{Z}$

"filtering transform"

signal $u = \{u(k)\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$

$2n$ shift related to two-scale relation $\phi(x) = \phi(2x) + \phi(2x-1)$

Haar scaling-function

Remark: $F(u)$ is essentially a convolution.

Recall LTI filters: $(Lu)(t) := (h * u)(t)$

impulse response function

and $(v * w)(n) := \sum_{m=0}^{N-1} v(m) w(n-m \bmod N)$

"discrete circular convolution" v, w periodic in $\mathbb{R}^N$

By setting $k' = k - 2n$, we can write $[F(u)](n)$ as

$[F(u)](n) = \sum_{k'\in\mathbb{Z}} f(k') u(k' + 2n)$

Adjoints: low-pass filter: $[H(u)](n) = \sum_{k\in\mathbb{Z}} h(k-2n) u(k)$

high-pass filter: $[G(u)](n) = \sum_{k\in\mathbb{Z}} g(k-2n) u(k)$

where h, g are some filters.

these satisfy $H^*H + G^*G = I$

$H^*(\underline{Hu}) + G^*(\underline{Gu}) = Iu = u$ ← our signal

used to reconstruct u from the filtered versions

filtered signal components

Recall in $\ell^2(\mathbb{Z})$ that

$\langle F(u), v\rangle = \langle u, F^*(v)\rangle$ where $\langle u, v\rangle_{\ell^2} = \sum_{n\in\mathbb{Z}} u(n)\overline{v(n)}$

compute

$$\langle F(u), v\rangle = \sum_{n\in\mathbb{Z}} \underbrace{[F(u)](n)}_{= \sum_{k\in\mathbb{Z}} f(k-2n)u(k)} \overline{v(n)}$$

note the sums → are finite if supp $f < \infty$

$$= \sum_{k\in\mathbb{Z}} \underbrace{\sum_{n\in\mathbb{Z}} f(k-2n)\overline{v(n)}}_{=:[F^*(v)](k)} u(k) = \langle u, F^*(v)\rangle$$

So $[F^*(v)](k) = \sum_{n\in\mathbb{Z}} f(k-2n)\bar{v}(n)$

Also
$$[F^*v](k) = \begin{cases} \sum_{n\in\mathbb{Z}} f(2n)\overline{v(\tfrac{k}{2}-n)} & \text{if } k \text{ even} \\ \sum_{n\in\mathbb{Z}} f(2n+1)\overline{v(\tfrac{k-1}{2}-n)} & \text{if } k \text{ odd}\end{cases}$$

Let's show the even case: Suppose $k = 2m$ for some $m\in\mathbb{Z}$

Then $\sum_{n\in\mathbb{Z}} f(k-2n)\bar{v}(n) = \sum_{n\in\mathbb{Z}} f(2\underbrace{(m-n)}_{\text{let } m-n=\ell})\bar{v}(n)$

$$= \sum_{\ell\in\mathbb{Z}} f(2\ell)\overline{v(\tfrac{k}{2}-n)}$$ $n = m-\ell = \frac{k}{2}-\ell$

Ceiling function: $\lceil x\rceil := \min\{n\in\mathbb{Z} \mid x \le n\}$ Ex: $\lfloor \pi \rfloor = 3$

floor function: $\lfloor x\rfloor := \max\{n\in\mathbb{Z} \mid x \ge n\}$ Ex: $\lceil \pi\rceil = 4$

Support Lemma

If $\underline{\text{supp } f} = [a,b]$, supp $u = [x, y]$

means $f(n) = 0$, if $n < a$ or $n > b$.

Ex: $f = \{\dots 0, 0, f(a), f(a+1), \dots, f(b-1), f(b), 0, 0, \dots\}$

i) $\text{Supp}(F(u)) = \left[\left\lceil \frac{x-b}{2}\right\rceil, \left\lfloor \frac{y-a}{2}\right\rfloor\right]$

ii) $\text{Supp}(F^*(u)) = [a+2x, b+2y]$

Let's verify i): $[F(u)](n) = \sum_{k\in\mathbb{Z}} f(k)u(k+2n)$

since $\operatorname{supp} f = [a,b]$ $\rightarrow$ $= \sum_{k=a}^{b} f(k)\underbrace{u(k+2n)}_{\text{supported on } [x,y]}$

$\Rightarrow$ Sum is nonzero provided $b+2n \geq x$ and $a+2n \leq y$
otherwise $k+2n$ is outside of $[x,y]$ and the sum is zero

$\Rightarrow$ $n \geq \lceil \frac{x-b}{2} \rceil$ and $n \leq \lfloor \frac{y-a}{2} \rfloor$
so $n \geq \lceil \frac{x-b}{2} \rceil$ and $n \leq \lfloor \frac{y-a}{2} \rfloor$ since n is an integer.

For periodic $u_P = \{u_P(k)\}$ with period P.

we define $[F_P(u_P)](n) = \sum_{k=0}^{P-1} f_P(k-2n)\, u_P(k)$

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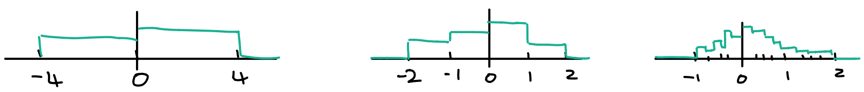
Haar System

$p_{j,k}(x) = 2^{j/2} p(2^j x - k)$

Approximation Spaces

lower resolution $\leftarrow$ $\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$ $\rightarrow$ higher resolution

$\{p_{-2,k}\}_k$ $\{p_{0,k}\}_k$ $\{p_{2,k}\}_k$



$f \in L^2(\mathbb{R})$ $\rightarrow$ Haar representation of f $\rightarrow$ processing

Goal: To generalize, and obtain "better" wavelets / scaling functions.

Multiresolution Analysis (MRA) [of $L^2(\mathbb{R})$]

let $V_j \subset L^2(\mathbb{R})$ $V_j \in \mathbb{Z}$ and fix a scaling function φ.

we say $\{V_j\}_j$ is a MRA of $L^2(\mathbb{R})$ if

i) $V_j \subset V_{j-1}$ $\forall j$

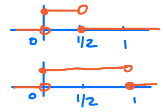
ii) $L^2(\mathbb{R}) = \overline{\bigcup_{j\in\mathbb{Z}} V_j}$ (i.e. $f(x) = \sum_{j\in\mathbb{Z}} c_j \varphi_j(x)$, and the limit is taken in $L^2(\mathbb{R})$)

iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

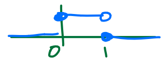
iv) $f(x) \in V_j \Leftrightarrow f(2^{-j}x) \in V_0$

v) $\{\varphi\} \cup \{\varphi(x-k) : k \in \mathbb{Z}\}$ forms an ONB of V_0
$\quad \in V_0$

Ex: $\mathbb{1}_{[0,1/2)}(x) \in V_1$
$\mathbb{1}_{[0,1/2)}(2^{-1}x) \in V_0$



note: For the Haar scaling system, we have scaling function $\varphi = p = p_{0,0}$



and the usual V_j from an MRA of $L^2(\mathbb{R})$

Scaling Relation

$$\varphi(x) = \underbrace{\sum_{k \in \mathbb{Z}}}_{\text{sum is finite if } \operatorname{supp} \varphi < \infty} \underbrace{h(k)}_{\text{some coefficients (Not Haar wavelets)}} \underbrace{\sqrt{2}}_{\text{some normalization}} \varphi(2x-k)$$

Ex: Haar scaling function $p = p_{0,0} = \varphi$

satisfies $p_{0,0}(x) = \frac{1}{\sqrt{2}}(p_{1,0}(x) + p_{1,1}(x))$

$$= \frac{1}{\sqrt{2}}\left[2^{1/2} p(2x-0) + 2^{1/2} p(2x-1)\right]$$

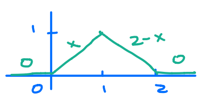
$$= p(2x) + p(2x-1)$$

so scaling relation hold with $h(0) = h(1) = \frac{1}{\sqrt{2}}$ and $h(k) = 0$ for $k \neq 0, 1$

Note: $h = \{h(k)\}_k$ is called a low-pass filter

Different scaling functions give different scaling relations (i.e. different $h(k)$)

Ex: ① "Hat scaling function"



$h(0) = \frac{1}{2\sqrt{2}}$, $h(1) = 1$, $h(2) = \frac{1}{2\sqrt{2}}$