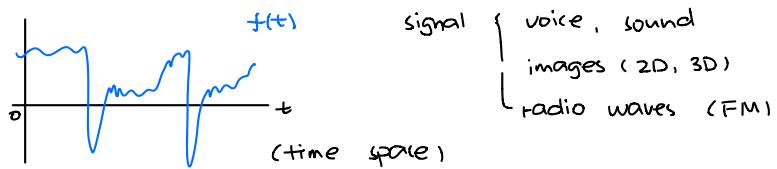


Applied Harmonic Analysis

Background :



Fourier analysis :

Fourier series (periodic f)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$$

amplitude frequency

$$\text{waveform} = N + \text{nr} + \text{nn} + \dots$$

Fourier transform (non-periodic f)

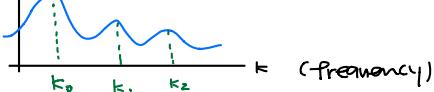
$$F(f) = \frac{1}{2\pi} \int_{\mathbb{R}} f(t) e^{-ikt} dt \quad \text{where } i = \sqrt{-1}$$

Function $f \in V \leftarrow$ some vector space (say $L^2(\mathbb{R})$ or $\ell^2(\mathbb{R})$) \downarrow Decompose f (into say sines/cosines or wavelets) \downarrow In frequency space, can manipulate f : compression, denoising, apply other filters ..

amplitude

ex: low-pass filter

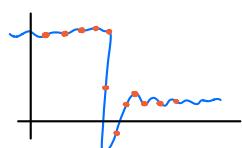
$$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^N a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$$



In practice, need to discretize!

- DFT (Discrete Fourier Transform)

- DWT (Discrete Wavelet Transform)

need to sample points of f .

Number Systems.

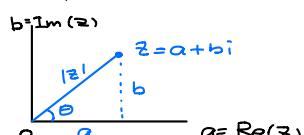
Countable

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$	naturals
$\mathbb{Z} = \{\dots, -1, 0, 1, 2, \dots\}$	integers
$\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$	rationals.

Uncountable

$\mathbb{R} = \{\text{real numbers}\}$	ex: $\pi, -\frac{1}{2}, 0, e, \dots$
$\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$	complex.

(Complex numbers)



$$|z| = z\bar{z} = \sqrt{a^2 + b^2} \quad (\text{modulus})$$

[conjugate $\bar{z} = a - bi$]

Polar form: $z = r e^{i\theta}$ where $r = |z|$, $\theta = \tan^{-1}(\frac{b}{a})$

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (\text{Euler's Formula})$$

Complex Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int} \quad \text{where } n = \text{frequency}$$

Roots of complex $z \in \mathbb{C}$

if $w^n = z$, then $w = n^{\text{th}}$ root of z

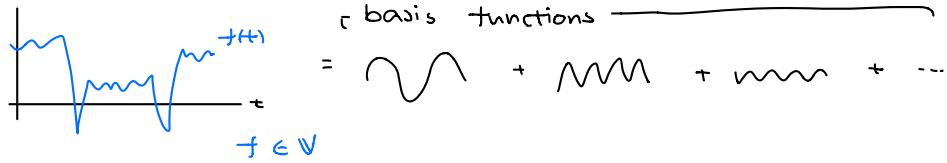
$$w = pe^{i\alpha}, z = re^{i\theta}$$

$$w^n = p^n e^{in\alpha}$$

$$\Rightarrow w = \sqrt[n]{r} \exp(i \frac{\theta + 2k\pi}{n}) \quad (k=0, 1, 2 \dots n-1)$$

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Vector (linear) spaces.



Examples of vector spaces : \mathbb{R}^n finite dimensional

\mathbb{C}^n $2n$ dimensional

$P_n = \{ n^{\text{th}} \text{ degree polynomials} \} = \text{poly}$

infinite dimensional $\begin{cases} C[0,1] = \{ u: [0,1] \rightarrow \mathbb{R} \mid u \text{ is continuous} \} \\ L^2(a,b) = \{ u: (a,b) \rightarrow \mathbb{R} \mid \int_a^b |u(t)|^2 dt < \infty \} \end{cases}$

Vector Space Axioms: let linear $U, V, W \in \mathbb{V}$ = some set and let $\alpha, \beta \in \mathbb{R}$ or fields

- 1) $u + (v + w) = (u + v) + w$
- 2) $u + v = v + u$
- 3) $\exists 0 \in \mathbb{V}$ s.t. $u + 0 = u$
- 4) $\exists (-u) \in \mathbb{V}$ s.t. $u + (-u) = 0$
- 5) $\alpha(\beta u) = (\alpha\beta)u$
- 6) $(\alpha + \beta)u = \alpha u + \beta u$
- 7) $\alpha(u + v) = \alpha u + \alpha v$

Defn: If $\{v_1, \dots, v_n\} \subset \mathbb{V}$ and $\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{R}$ (or \mathbb{C})

then $\sum_{i=1}^n \alpha_i v_i = \alpha_1 v_1 + \dots + \alpha_n v_n$ is a linear combination of the $\{v_i\}_{i=1}^n$

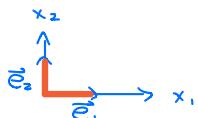
note: n has to be finite

ex: $\sum_{k=1}^{10} \alpha_k \cos(kt)$ is a lin. comb. of $\{\cos(kt)\}_{k=1}^{10}$

Defn: The span of $\{v_1, \dots, v_n\} \subset \mathbb{V}$ is the set

$$\text{span}\{v_1, \dots, v_n\} = \left\{ \sum_{i=1}^n \alpha_i v_i \mid v_i \in \mathbb{V}, \alpha_i \in \mathbb{K} \right\} \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}$$

ex: \mathbb{R}^2 , $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$\text{span}(\vec{e}_1) = \text{entire } x_1 \text{ axis} = \{a\vec{e}_1 \mid a \in \mathbb{R}\}$

$$\text{span}(\vec{e}_1, \vec{e}_2) = \mathbb{R}^2 = \{a_1\vec{e}_1 + a_2\vec{e}_2 \mid a_1, a_2 \in \mathbb{R}\} = \left(\begin{matrix} a_1 \\ a_2 \end{matrix} \right)$$

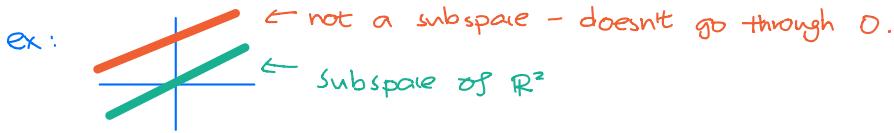
Subspaces:

ex: $\text{span}\{v_i\}_{i=1}^n$ is a subspace of \mathbb{V} .

A subspace of \mathbb{V} is a set $\mathbb{X} \subset \mathbb{V}$ s.t. $u + v \in \mathbb{X} \quad \forall c \in \mathbb{R} \text{ or } \mathbb{C}, u, v \in \mathbb{X}$

- we say $\mathbb{X} \subset \mathbb{V}$ is closed under scalar multiplication and addition.

- A subspace is itself a vector space.



ex: P_2 is a subspace of $C[0,1]$

Defn: A set $\{v_1, \dots, v_n\} \subset V$ is linearly independent

if $\sum_{i=1}^n c_i v_i = 0$, then $c_i = 0$ for $i = 1, \dots, n$. note: n is finite.

ex: $(1, 0), (0, 1), (0, 0)$ are linearly independent.
since $(0, 0) = \frac{1}{2}(0, 1) + 0(1, 0)$

Defn: A basis B of V is a linearly independent set of vectors that span V .

ex: $B = \{1, t, t^2\}$ form a basis of P_2 .

Defn: The dimension of V is the number of vectors in any basis.

ex: $\dim(R^2) = 2$, $\dim(P_2) = 3$, $\dim(C[0,1]) = \infty$

note: need to be careful in $C[0,1]$

since $\sum_{i=1}^{\infty} c_i v_i$ not allowed in vector space axiom.

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Basis in ∞ -dim spaces

Hamel Basis (ignore) - uses finite # of vectors.

Schauder Basis:

B is a schauder basis if

- Any finite set of vectors of B is linearly independent.
- $\forall \vec{v} \in V$, $\vec{v} = \sum_{i=1}^{\infty} c_i \vec{v}_i$ where $B = \{\vec{v}_i\}_{i=1}^{\infty}$ (completeness)
interpreted as $\lim_{n \rightarrow \infty} \|\vec{v} - \sum_{i=1}^n c_i \vec{v}_i\|_V = 0$
- The coefficients c_i are unique.

example of Schauder basis: Fourier series in $L^2(0, 2\pi) = V$

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$$

form a Schauder basis $\{1, e^{it}, e^{-it}, e^{2it}, e^{-2it}, \dots\}$

Inner-Product Spaces: $\mathbb{R}^n = \mathbb{R}^n$ or \mathbb{C}^n

Define inner-product $\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i$ $x, y \in \mathbb{R}^n$ note $z = x+iy$, $\bar{z} = x-iy$

$$\text{i)} \quad \langle x, y \rangle = \overline{\langle y, x \rangle}$$

$$\text{ii)} \quad \langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle \quad a, b \in \mathbb{R}$$

$$\text{iii)} \quad \langle x, x \rangle \geq 0 \text{ and } \langle x, x \rangle = 0 \text{ iff } x = 0$$

$$\text{note: } \langle x, ay \rangle = \overline{a \langle y, x \rangle} = \overline{a} \overline{\langle y, x \rangle} = \bar{a} \langle x, y \rangle$$

ex: \mathbb{R}^2



pj: law of cosines $||\vec{x} - \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2 - 2||\vec{x}|| ||\vec{y}|| \cos \theta$

$$\langle \vec{x} - \vec{y}, \vec{x} - \vec{y} \rangle = ||\vec{x}||^2 + ||\vec{y}||^2 - 2||\vec{x}|| ||\vec{y}|| \cos \theta$$

$$\cancel{||\vec{x}||^2} - \langle \vec{x}, \vec{y} \rangle - \langle \vec{y}, -\vec{x} \rangle + \cancel{||\vec{y}||^2} = ||\vec{x}||^2 + ||\vec{y}||^2 - 2||\vec{x}|| ||\vec{y}|| \cos \theta$$

Normed spaces $(V, \|\cdot\|)$ is a normed space where $\|\cdot\| : V \rightarrow \mathbb{R}$ and satisfies

- $\|u+v\| \leq \|u\| + \|v\| \quad \forall u, v \in V$. (triangle inequality)
- $\|au\| = |a|\|u\| \quad a \in \mathbb{K}$
- $\|u\| \geq 0$ and $\|u\| = 0 \iff u = 0$

All inner-products generate a norm: $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$

Other norms on \mathbb{R}^n : let $p \geq 1$, and set $\|\vec{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$

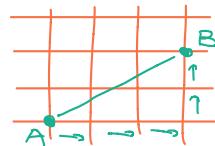
$$\|\vec{x}\|_1 = \sum_{i=1}^n |x_i| \quad (1\text{-norm, taxicab norm})$$

$$\Rightarrow \|\vec{x}\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2} \quad (2\text{-norm})$$

$$\hookrightarrow \text{generated by } \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$$

$$\Rightarrow \|\vec{x}\|_\infty = \max_{i=1,\dots,n} |x_i| \quad (\infty\text{-norm, max norm})$$

$$\hookrightarrow \text{note } \|\vec{x}\|_\infty = \lim_{p \rightarrow \infty} \|\vec{x}\|_p$$

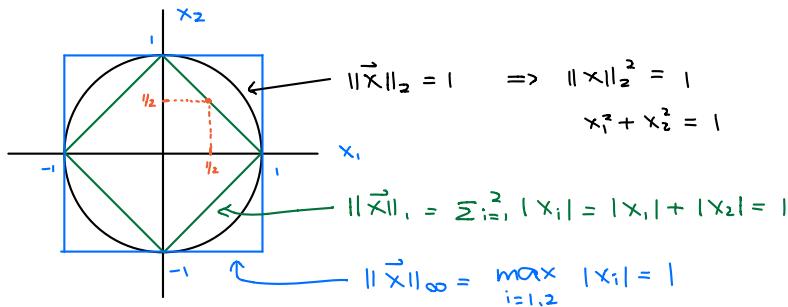


Dist btw. A, B
in $\|\vec{x}\|_1$ is 5

Ex:

$$\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2} = (\sum_{i=1}^2 x_i^2)^{1/2} \quad (\text{Euclidean norm})$$

Unit Spheres in various norms \mathbb{R}^2



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Equivalent Norms

We say any two norms $\|\cdot\|_\alpha, \|\cdot\|_\beta$ are equivalent

If $\exists A, B > 0$ s.t. $A\|\mathbf{x}\|_\alpha \leq \|\mathbf{x}\|_\beta \leq B\|\mathbf{x}\|_\alpha$ (norm are comparable)

Theorem: In finite dimensional spaces, All norms are equivalent.

Cauchy-Schwarz Inequality

For any $\mathbf{x}, \mathbf{y} \in V$ (V inner-product space)

We have $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ recall $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$

Pf: If $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$, done.

So, suppose $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$. Suppose $V = \mathbb{R}$

$$\begin{aligned} 0 \leq \|\mathbf{x} - t\mathbf{y}\|^2 &= \langle \mathbf{x} - t\mathbf{y}, \mathbf{x} - t\mathbf{y} \rangle = \|\mathbf{x}\|^2 - 2t\langle \mathbf{x}, \mathbf{y} \rangle + t^2\|\mathbf{y}\|^2 \\ &\quad \text{constant} \quad \text{"c"} \quad \text{"-2tb"} \quad \text{"t^2a"} \\ &= c - 2tb + at^2 \quad (\text{quadratic in } t) \end{aligned}$$

$$t = \frac{2b \pm \sqrt{(-2b)^2 - 4ac}}{2a} \quad \text{want } (-2b)^2 - 4ac \leq 0$$

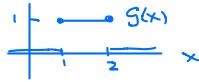
$$\begin{aligned} b^2 \leq ac &\quad \|\mathbf{x}\|^2 \quad \|\mathbf{y}\|^2 \\ |\langle \mathbf{x}, \mathbf{y} \rangle|^2 &\leq \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 \quad \Rightarrow |\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\| \end{aligned}$$

The vector space $L^2[a, b]$

$$L^2[a, b] := \{ f: [a, b] \rightarrow \mathbb{C} \mid \int_a^b |f(x)|^2 dx < \infty \}$$

this is an inner-product space with $\langle f, g \rangle = \langle f, g \rangle_{L^2[a, b]} := \int_a^b f(x) \overline{g(x)} dx$
 an norm $\|f\|_{L^2[a, b]} = \|f\|_{L^2} = (\int_a^b |f(x)|^2 dx)^{1/2}$

ex: $g \in L^2(\mathbb{R})$



$$\|g\|_{L^2(\mathbb{R})} = (\int_{\mathbb{R}} |g(x)|^2 dx)^{1/2} = (\int_1^2 1^2 dx)^{1/2} = 1$$

Cauchy-Schwarz in L^2 : $|\langle f, g \rangle| \leq \|f\| \|g\|$

$$|\int_a^b f(x) \overline{g(x)} dx| \leq (\int_a^b |f|^2 dx)^{1/2} (\int_a^b |g|^2 dx)^{1/2}$$

Orthogonality

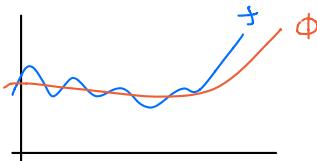
let $u, v \in V$, we say u, v are orthogonal if $\langle u, v \rangle = 0$

ex: $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\langle \vec{x}, \vec{y} \rangle = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle = 1 - 1 = 0$$

$$\text{note } \langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|_2 \|\vec{y}\|_2 \cos \theta = 0 \quad \theta = \frac{\pi}{2}$$

plus if $\|u\|_2, \|v\|_2 = 1$, then u, v are orthonormal.



- $\|f - \phi\|_2$ is the average distance btwn f, ϕ .
- $\|f - \phi\|_\infty$ is a distance that tries to minimize the maximum dist.

note $L^3[a, b] = \{ f: [a, b] \rightarrow \mathbb{R} \mid \int_a^b |f(x)|^3 dx < \infty \}$

$$\|f\|_{L^3} = (\int_a^b |f(x)|^3 dx)^{1/3}$$

$$\text{while } \|\vec{x}\|_3 = (\sum_{i=1}^n |x_i|^3)^{1/3}$$

Orthonormal Bases

Suppose $B = \{e_1, e_2, \dots, e_n\} \subset V$ is an orthonormal basis of V .

I have $\langle e_i, e_j \rangle = 0$ unless $i = j$ in which case $\langle e_i, e_j \rangle = \|e_i\|^2 = 1$

$$\text{Kronecker Delta: } \langle e_i, e_j \rangle = \delta_{ij} = \delta(i-j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

and $\forall u \in V, u = \sum_{i=1}^n c_i e_i$

$$\text{use orthogonality: } \langle u, e_j \rangle = \langle \sum_{i=1}^n c_i e_i, e_j \rangle = \sum_{i=1}^n c_i \underbrace{\langle e_i, e_j \rangle}_{=\delta_{ij}} = c_j$$

so $u = \sum_{i=1}^n \langle u, e_i \rangle e_i$

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Recall:

Orthonormal basis $\{e_k\}_{k=1}^n \subset V$.

$$u \in V, u = \sum_{k=1}^n \langle u, e_k \rangle e_k$$

$$\langle u, u \rangle = \langle \sum_{k=1}^n \langle u, e_k \rangle e_k, u \rangle$$

$$\|u\|^2 = \sum_{k=1}^n \langle u, e_k \rangle \underbrace{\langle e_k, u \rangle}_{=\langle u, e_k \rangle} = \sum_{k=1}^n |\langle u, e_k \rangle|^2 = \|u\|^2$$

Each $\langle u, e_k \rangle e_k$ is an orthogonal projection of u onto e_k

$$P_{e_k} u = \text{Proj}_{e_k} u = \langle u, e_k \rangle e_k$$

- $\{e_k\}_{k=1}^n$ is orthonormal here

If $M = \text{Span}\{e_1, \dots, e_m\}$ then $P_M u = \text{Proj}_M u = \sum_{k=1}^m \langle u, e_k \rangle e_k$
 ↳ set $m \in \mathbb{Z}^+ = \mathbb{N}$ (index)

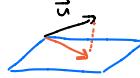
$$\begin{aligned} \vec{w} &= \text{proj}_{\vec{v}} \vec{u} \quad \text{have} \quad \vec{w} = \lambda \vec{v} \quad \text{for some } \lambda \in \mathbb{R} \\ 0 &= \langle \vec{u} - \vec{w}, \vec{v} \rangle = \langle \vec{u} - \lambda \vec{v}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle - \underbrace{\lambda \langle \vec{v}, \vec{v} \rangle}_{\lambda \|\vec{v}\|^2} \\ \lambda &= \langle \vec{u}, \vec{v} \rangle / \|\vec{v}\|^2 \\ \vec{w} &= \lambda \vec{v} = \frac{\langle \vec{u}, \vec{v} \rangle \vec{v}}{\|\vec{v}\|^2} \quad \text{when } \|\vec{v}\|=1, \quad \vec{w} = \langle \vec{u}, \vec{v} \rangle \vec{v} \end{aligned}$$

If $\{e_k\}_{k=1}^n$ is an orthogonal basis then $u = \sum_{k=1}^n \frac{\langle u, e_k \rangle}{\|e_k\|^2} e_k$

P_m is a projection if $P_m(P_m) = P_m^2 = P_m$

lets check for $P_M u = \sum_{k=1}^m \langle u, e_k \rangle e_k \in \pi$

$$\begin{aligned}
 P_M(P_M u) &= \sum_{k=1}^m \langle P_M u, e_k \rangle e_k \\
 &= \sum_{k=1}^m \left(\underbrace{\sum_{j=1}^m \langle u, e_j \rangle e_j}_{= \sum_{j=1}^m \langle u, e_j \rangle \underbrace{\langle e_j, e_k \rangle}_{= \delta_{jk}} } \right) e_k \\
 &= \sum_{j=1}^m \langle u, e_j \rangle \underbrace{\langle e_j, e_k \rangle}_{= \delta_{jk}} = \langle u, e_k \rangle \\
 &= \sum_{k=1}^m \langle u, e_k \rangle e_k \\
 &= P_M u
 \end{aligned}$$

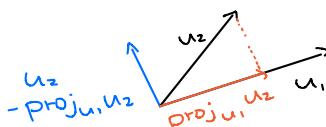


Consider $\{1, t\} \subset L^2[0, 1] = \mathbb{V}$

$$\langle 1, t \rangle = \int_0^1 1 \cdot t \, dt = \frac{1}{2} \neq 0 \quad \Rightarrow \quad 1, t \text{ not orthogonal}$$

How to Orthogonalize?

$\{u_1, u_2, \dots, u_n\}$ non-orthogonal want to turn into $\{v_1, v_2, \dots, v_n\}$ orthogonal.



Now $\{U_1, U_2 - \text{proj}_{U_1} U_2\}$ is an orthogonal set spanning the same set as $\{U_1, U_2\}$

Gram - Schmidt Process

To turn $\{u_i\}_{i=1}^n$ non-orthogonal set into $\{v_i\}_{i=1}^n$ orthogonal.

$$V_1 = U_1$$

$$v_2 = u_2 - \text{proj}_{u_1} u_2$$

$$U_3 = U_3 - \text{Proj}_{\text{span}(U_1, U_2)}, U_3 = U_3 - \text{Proj}_{U_1} U_3 - \text{Proj}_{U_2} U_3$$

⋮

$$v_n = u_n - \text{proj}_{\text{span}\{v_1, \dots, v_{n-1}\}} u_n$$

we have $\text{span}\{u_1, \dots, u_n\} = \text{span}\{v_1, \dots, v_n\}$, $\{v_1, \dots, v_n\}$ is orthogonal,

we can normalize by $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \dots, \frac{v_n}{\|v_n\|} \right\}$

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Linear Operators (map, function)

$T: V \rightarrow W$ is a linear map if $\lambda \in \mathbb{K}$ (\mathbb{R} or \mathbb{C}) and $\lambda u, v \in V$.

- $$\text{i) } T(u+v) = T(u) + T(v)$$

$$U_X : \mathbb{H} \rightarrow \mathcal{P}_n^{\mathbb{R}}$$

$$T(\vec{x}) = A\vec{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad A = \text{linear operator (Matrix)}$$

Ex: $T: L^2(0,1) \rightarrow \mathbb{R}$

$$T(u) = \int_0^1 u(t)e^t dt$$

$$T(au+bu) = \int_0^1 (au+bu)(t)e^t dt = aT(u) + bT(v)$$

Ex: $T: l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$, $x \in l^2(\mathbb{N})$, $x = \{x_1, x_2, \dots\}$

$$T(x) = \{\frac{1}{2}x_1, \frac{1}{2^2}x_2, \frac{1}{2^3}x_3, \dots, \frac{1}{2^n}x_n, \dots\}$$

$$T(ax+by) = \{\frac{1}{2}(ax_1+bx_1), \frac{1}{2^2}(ax_2+bx_2), \dots\}$$

$$= a\{\frac{1}{2}x_1, \frac{1}{2^2}x_2, \dots\} + b\{\frac{1}{2}x_1, \frac{1}{2^2}x_2, \dots\}$$

$$= aT(x) + bT(y)$$

In finite dimensions, any linear map $T: V \rightarrow W$ can be represented as matrix multiplication. where V, W are vector spaces.

Let $\{v_1, \dots, v_n\}$ be a basis of V and let $\{w_1, \dots, w_m\}$ be a basis of W .

Note $T(v_j) \in W \quad \forall j=1, \dots, n$

$$\Rightarrow T(v_j) = \sum_{i=1}^m a_{ij} w_i$$

$$\text{let } v = \sum_{j=1}^n x_j v_j \in V$$

$$= c_i$$

$$\text{Then } T(v) = T\left(\sum_{j=1}^n x_j v_j\right) = \sum_{j=1}^n x_j T(v_j) = \sum_{j=1}^n x_j \left(\sum_{i=1}^m a_{ij} w_i\right) = \sum_{i=1}^m \sum_{j=1}^n a_{ij} x_j w_i = \sum_{i=1}^m c_i w_i$$

Here $c_i = \sum_{j=1}^n a_{ij} x_j$ is precisely matrix-vector multiplication: $\vec{z} = A\vec{x}$

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\text{think of } T(\vec{v}) = A\vec{v}$$

The column of A store the coefficients of $T(v_j) \quad j=1 \dots n$

where $\{v_j\}_{j=1}^n$ is a basis of V .

Ex: Find the matrix representation of $T: V \rightarrow W$ where $V = \mathbb{R}^2$, $W = \text{span}\{\text{cost}, \text{sint}\}$.
and $T(\vec{v}) = (v_1 + 2v_2)\text{cost}, v_2 \text{sint}$

basis for V let's use $\{\vec{e}_1, \vec{e}_2\} = \{(1, 0), (0, 1)\}$

and for W let's use $\{w_1, w_2\} = (\text{cost}, \text{sint})$

$$\text{Have } T(\vec{e}_1) = T(1) = 1 \cdot \text{cost} + 0 \cdot \text{sint} \Rightarrow A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$T(\vec{e}_2) = T(0) = 2 \cdot \text{cost} + 1 \cdot \text{sint}$$

$$\Rightarrow \vec{v} = x_1(1, 0) + x_2(0, 1) \in V = \mathbb{R}^2, \vec{z} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \vec{z} = A\vec{x}.$$

$$\text{basis of } \mathbb{R}^2$$

$$\text{so } T(\vec{v}) = c_1 w_1 + c_2 w_2 = c_1 \text{cost} + c_2 \text{sint}$$

And if used basis $\{(1, 0), (0, 1)\}$ for $V = \mathbb{R}^2$.

$$T(1) = 1 \cdot \text{cost} + 0 \cdot \text{sint} \Rightarrow A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$T(0) = 3 \cdot \text{cost} + 1 \cdot \text{sint}$$

Ex: if basis $\{(1, 0), (0, 1)\}$ of $V = \mathbb{R}^2$, basis $\{w_1, w_2\} = (\text{cost} + \text{sint}, \text{sint})$ of W .

$$\text{then } A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \quad \vec{v} = x_1(1, 0) + x_2(0, 1), T(\vec{v}) = c_1(\text{cost} + \text{sint}) + c_2 \text{sint}$$

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Adjoint Operators

 $T: V \rightarrow W$ linear operator (map)inner-product spaces with $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ Can we write $\langle Tv, w \rangle_W$ as $\langle v, T^*w \rangle_V$ for some new operator T^* the adjoint of T .[Tv or $T(v)$ denotes action of T on v]i.e. $T^*: W \rightarrow V$ s.t. $\langle Tv, w \rangle_W = \langle v, T^*w \rangle_V$ Goal: Try to write $\langle Tv, w \rangle_W = \langle v, z \rangle_V$ for some $z \in V$ and define $T^*w = z$.Theorem: If V, W are finite-dimensional and $T: V \rightarrow W$ is linear, then $T^*: W \rightarrow V$ exists and is uniqueIf $T = T^*$ we call T self-adjoint, $\langle Tv, w \rangle = \langle v, T^*w \rangle = \langle v, Tw \rangle$ Ex: The Fourier transform $F[f] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{ix^2} dx$ is self-adjoint with the $L^2(\mathbb{R})$ inner-product $\langle f, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)g(x) dx$

$$\text{so } \langle Ff, g \rangle_{L^2} = \langle f, Fg \rangle_{L^2}$$

Ex: $T = I$ $I: V \rightarrow V$ (identity map) $Iv = v$

$$\langle Iv, w \rangle = \langle v, Iw \rangle$$

$$\text{so } I^*w = Iw \Rightarrow I^* = I \text{ by definition}$$

Ex: Let $a(t)$ be bounded. $\langle f, g \rangle_{L^2} = \int_a^b f(x)g(x) dx$ Define $T: L^2(a, b) \rightarrow L^2(a, b)$ by $(Tf)(t) = a(t)f(t)$ Find T^*

$$\begin{aligned} \text{Have } \langle Tf, g \rangle &= \langle af, g \rangle = \int_a^b a(t)f(t)\overline{g(t)} dt = \int_a^b f(t) \cdot \underbrace{\overline{a(t)g(t)}}_{} dt \\ &= \int_a^b f(t)\overline{a(t)g(t)} dt \\ &= \langle f, \bar{a}g \rangle \quad \text{so } (T^*g)(t) = \overline{a(t)}g(t) \end{aligned}$$

If $a(t)$ real-valued, then $T^* = T$ and T self-adjoint.

Integration by Parts:

$$\int_a^b f(t)g'(t) dt = -f(t)g(t) \Big|_{t=a}^{t=b} + \int_a^b f'(t)g(t) dt$$

Ex: $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $T(x, y, z) = A\vec{x} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\text{Fact: } A^* = \bar{A}^T \text{ so } A^* = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad A^*: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$\begin{aligned} \langle A \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \rangle_2 &= \langle \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \rangle_2 = \langle \begin{pmatrix} x+y \\ x+2z \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \rangle_2 = (x+y)a + (x+2z)b \\ &= x(a+b) + ya + za + zb = \langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a+b \\ a \\ 2b \end{pmatrix} \rangle_3 \end{aligned}$$

$$\text{we define } A^* \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a \\ 2b \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad A: \mathbb{R}^3 \rightarrow \mathbb{R}^2, A^*: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$\langle A\vec{x}, \vec{y} \rangle_2 = \langle \vec{x}, A^*\vec{y} \rangle_3$$

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Intro to Frames

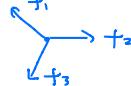
\mathbb{V} with orthonormal basis $\{v_1, v_2, v_3, \dots\}$ (ONB)

- can be restrictive in practice
- easy to compute c_i s.t. $x = \sum c_i v_i$, $c_i = \langle x, v_i \rangle$

$\{f_1, f_2, f_3, \dots\}$ is a frame of \mathbb{V} if the $\{f_i\}_{i \in I}$ span \mathbb{V} and satisfy some other conditions.

- frame can be linearly dependent.

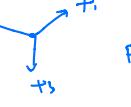
ex:  a basis of \mathbb{R}^2 (orthonormal)

ex:  a frame for \mathbb{R}^2 (contains redundancies)

ex:  another frame

ex: Reconstruction, orthonormal basis $\{e_1, e_2\}$

 person A, $\{1, 2\}$, $x = 1 \cdot e_1 + 2 \cdot e_2$.
 ↓ "erasure" "information loss", lose all info. in the e_2 dire.
 person B, $\{1, ?\}$, $x = 1 \cdot e_1 + ?$

 person A, $\{1, 2, 3\}$, $x = 1 \cdot f_1 + 2 \cdot f_2 + 3 \cdot f_3$
 ↓ retains more info. than with ONB and sometimes can get
 person B, $\{1, 2, ?\}$, $x = 1 \cdot f_1 + 2 \cdot f_2 + ?$ "perfect reconstruction"

Primer on Hilbert Spaces

A Hilbert space H is a complete inner-product space.

- also normed $\|x\|_H = \sqrt{\langle x, x \rangle_H}$

Defn: $\{x_n\}_{n=1}^{\infty} \subset H$ is a Cauchy-sequence if

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \text{ s.t. } \forall n, m \geq N, \quad \|x_n - x_m\|_H < \epsilon$$

Defn: H is complete if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point $x \in H$.

$$\text{i.e. } \lim_{n \rightarrow \infty} x_n = x \in H, \quad \|x_n - x\|_H \rightarrow 0 \text{ as } n \rightarrow \infty$$

Hilbert spaces: $\mathbb{R}, \mathbb{R}^n, \mathbb{C}^n, L^2(\mathbb{Z}), L^2(a, b), L^2(\mathbb{R})$

any finite dimensional inner-product space i.e. \mathbb{P}_n

Non-Hilbert spaces: $S = (0, 1) \subseteq \mathbb{R}$, $x_n = 1/n$ for $n \geq 2$ converges to $0 \notin S$

Defn: H is separable if it contains a countable basis.

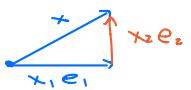
- All examples above are separable.

For ONB, of a Hilbert space H , $x = \sum_{i \in I} \langle x, e_i \rangle e_i \quad \forall x \in H$

Dotting with x we get

$$\|x\|^2 = \langle x, \sum_{i \in I} \langle x, e_i \rangle e_i \rangle = \sum_{i \in I} \langle x, e_i \rangle \langle x, e_i \rangle = \sum_{i \in I} |\langle x, e_i \rangle|^2$$

↳ Parseval equality

ex:  \mathbb{R}^2 -case: $\|x\|^2 = x_1^2 + x_2^2 = |\langle x, e_1 \rangle|^2 + |\langle x, e_2 \rangle|^2 = \sum_{i \in I} |\langle x, e_i \rangle|^2$
Pythagorean theorem

Frames generalize Parseval equality:

Defn: $\{f_i\}_{i \in I}$ is a frame for H if I is countable and if $\exists 0 < A \leq B < \infty$
s.t. $\forall x \in H$ $A \|x\|^2 \leq \sum_{i \in I} |\langle x, f_i \rangle|^2 \leq B \|x\|^2$
frame bounds.

- largest A and smallest B that work are called optimal frame bounds.
- For $A=B=1$, equality holds and we call $\{f_i\}_{i \in I}$ a Parseval frame
i.e. an ONB is parseval
- $\{f_i\}_{i \in I}$ is a tight frame if $A=B$

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Recall: $\{e_i\}_{i \in I}$ ONB: $x = \sum_{i \in I} \langle x, e_i \rangle e_i$

Parseval equality: $\|x\|^2 = \sum_{i \in I} |\langle x, e_i \rangle|^2$

frames $\{f_i\}_{i \in I}$ satisfy $A \|x\|^2 \leq \sum_{i \in I} |\langle x, f_i \rangle|^2 \leq B \|x\|^2$

note: An OBE is a frame. Let $A=B=1$

union of N orthonormal bases is a frame

call these $\{e_i^{(j)}\}_{i \in I}$ $j=1, \dots, N$ every j is a different ONB

 bases of \mathbb{R}^2 , union is a frame of \mathbb{R}^2

note by Parseval, $\forall j=1, \dots, N$. $\|x\|^2 = \sum_{i \in I} |\langle x, e_i^{(j)} \rangle|^2$

$$\Rightarrow \sum_{j=1}^N \|x\|^2 = \sum_{j=1}^N \sum_{i \in I} |\langle x, e_i^{(j)} \rangle|^2$$

$$\Rightarrow N \|x\|^2 = \sum_{j=1}^N \sum_{i \in I} |\langle x, e_i^{(j)} \rangle|^2$$

$A=B=N$ and $\{e_i^{(j)}\}$ is a tight frame

If we have just $\{f_i\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$ for \mathbb{R}^2 ,

then no A exists s.t. $A \|x\|^2 \leq \sum_{i=1}^1 |\langle x, f_i \rangle|^2 = |\langle x, f_1 \rangle|^2$

because $|\langle x, f_1 \rangle|^2 = x_1^2 \geq A \|x\|^2 = A(x_1^2 + x_2^2)$

If such an A existed, let $x_2 \rightarrow \infty$ to get contradiction

Lemma: Young's Inequality:

if $a \geq 0$, $b \geq 0$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$\text{then } ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$$\text{if } p=q=2, \text{ then } ab \leq \frac{a^2}{2} + \frac{b^2}{2} \Rightarrow 2ab \leq a^2 + b^2$$

$$\text{i.e. } (x_1 + x_2)^2 \geq 0 \Rightarrow x_1^2 + x_2^2 \geq 2x_1 x_2 \geq 0$$

$$\Rightarrow \|x\|^2 \geq 2x_1 x_2$$

$$\|x\|^2 \geq 2x_1 x_2 - \|x\|^2$$

$$\text{Ex: } \{f_i\}_{i=1}^3 = \{f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, f_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\} \subset \mathbb{R}^2$$

$$\text{Goal: } \exists A, B. \text{ s.t. } \forall \|x\|^2 \leq \sum_{i=1}^3 |\langle x, f_i \rangle|^2 \leq B \|x\|^2, \quad \forall x \in \mathbb{R}^2$$

$$\begin{aligned} \sum_{i=1}^3 |\langle x, f_i \rangle|^2 &= x_1^2 + x_2^2 + (x_1 + x_2)^2 = 2(x_1^2 + x_2^2) + 2x_1 x_2 \\ &= 2\|x\|^2 + 2x_1 x_2 \leq 3\|x\|^2 \end{aligned}$$

$$\sum_{i=1}^3 |\langle x, f_i \rangle|^2 = 2\|x\|^2 + 2x_1 x_2 \geq \|x\|^2$$

$$\text{thus } \|x\|^2 \leq \sum_{i=1}^3 |\langle x, f_i \rangle|^2 \leq 3\|x\|^2$$

Bounds $A=1, B=3$ are optimal,

to get equality, choose $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ s.t. $\{f_i\}_{i=1}^3$ is not tight

Now will develop theory about frames:

$$\text{ONB: } x = \sum_{i \in I} \langle x, e_i \rangle e_i \quad \{e_i\}_{i \in I} \subset H$$

define $\langle x, e_i \rangle := T(x)$ takes x and produces $\langle x, e_i \rangle \in \mathbb{C}$, $T: H \rightarrow \ell^2(I)$

$$\sum_{i \in I} \langle x, e_i \rangle e_i := S(\langle x, e_i \rangle) = S(c_i) \text{ if } c_i := \langle x, e_i \rangle, \quad S: \ell^2(I) \rightarrow H$$

We call $T(x) = \{\langle x, e_i \rangle\}_{i \in I}$ the analysis operator, $T: H \rightarrow \ell^2(I)$

We call $S(c) = T^*(c) = \sum_{i \in I} c_i e_i$ the synthesis operator

$$c = \{c_i\}_{i \in I} \text{ and } T^*: \ell^2(I) \rightarrow H$$

Same definition holds for frames $\{f_i\}_{i \in I}$

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$$\text{Check } T: H \rightarrow \ell^2(I) \quad T(x) = \{\langle x, f_i \rangle\}_{i \in I}$$

$$\text{Have } \|\{\langle x, f_i \rangle\}_{i \in I}\|_{\ell^2(I)}^2 := \sum_{i \in I} |\langle x, f_i \rangle|^2 \leq B \|x\|_H^2 < \infty$$

Since $\{f_i\}$ are frame

$$\Rightarrow T(x) \in \ell^2(I)$$

Defn: A linear operator $T: X \rightarrow Y$ is bounded if $\exists C > 0$

$$\text{s.t. } \|T(x)\|_Y \leq C \|x\|_X, \quad \forall x \in X.$$

thus the analysis operator is bounded $\|T(x)\|_{\ell^2(I)} \leq \sqrt{B} \|x\|_H$

Adjoint of Analysis operator T

$$\begin{aligned} \langle T(x), c \rangle_{\ell^2(I)} &= \sum_{i \in I} \langle x, f_i \rangle_H \bar{c}_i = \langle x, \sum_{i \in I} f_i c_i \rangle_H \\ &= \langle x, f_i \rangle_{\ell^2(I)} \end{aligned}$$

$$T^* c = \sum_{i \in I} f_i c_i \quad \text{synthesis operator}$$

Frame Operator: $S := T^* T : H \rightarrow H, \quad T^*: \ell^2 \rightarrow H, \quad T: H \rightarrow \ell^2$

$$S(x) = \sum_{i \in I} \langle x, f_i \rangle_H f_i$$

$$\begin{aligned} \text{Now } \langle S(x), x \rangle_H &= \sum_{i \in I} \langle x, f_i \rangle_H \underbrace{\langle f_i, x \rangle_H}_{=\langle x, f_i \rangle} = \sum_{i \in I} |\langle x, f_i \rangle|^2 \\ &\leq B \|x\|^2 \end{aligned}$$

$$A \|x\|^2 \leq \langle S(x), x \rangle \leq B \|x\|^2$$

Remark: $S = T^* T$ is self-adjoint since

$$S^* = (T^* T)^* = T^* (T^*)^* = T^* T = S$$

Ex: 2.1.4 in Coja's notes
 finite frame $\{f_i\}_{i=1 \dots n} \subset \mathbb{R}^d$
 Analysis operator $T(x) = \{\langle x, f_i \rangle\}_{i=1}^n$ where $x \in \mathbb{R}^d$

we have $T: \mathbb{R}^d \rightarrow \mathbb{R}^n$ and T has an $n \times d$ matrix representation $T(x) = Ax \quad \forall x \in \mathbb{R}^d$.

Let's find A :

Suppose x, f_i are all column vectors.

T takes $x \in \mathbb{R}^d$ and maps it to $\{\langle x, f_i \rangle\}_{i=1}^n \in \mathbb{R}^n$
 note $\langle x, f_i \rangle = f_i^* x$

* denotes conjugate transpose $A^* = \bar{A}^T$

recall $\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = (1 \ 0 \ 1) \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\text{So } T(x) = \begin{pmatrix} f_1^* x \\ f_2^* x \\ \vdots \\ f_n^* x \end{pmatrix} = \underbrace{\begin{pmatrix} f_1^* \\ f_2^* \\ \vdots \\ f_n^* \end{pmatrix}}_{=: A_{n \times d}} x = Ax$$

and $T = \begin{pmatrix} f_1^* \\ \vdots \\ f_n^* \end{pmatrix}$ analysis operator

Ex: $\{f_1, f_2, f_3\} = \{(1, 0), (1, 1), (4, 2)\} \subset \mathbb{R}^2$ Have $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 4 & 2 \end{pmatrix}$

$$\forall x \in \mathbb{R}^2, T(x) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \langle x, f_1 \rangle \\ \langle x, f_2 \rangle \\ \langle x, f_3 \rangle \end{pmatrix} = \begin{pmatrix} \langle x, f_1 \rangle \\ \langle x, f_2 \rangle \\ \langle x, f_3 \rangle \end{pmatrix}$$

$$\begin{aligned} \text{Frame operator: } S &= T^* T \\ &= (f_1, f_2, \dots, f_n) \begin{pmatrix} f_1^* \\ f_2^* \\ \vdots \\ f_n^* \end{pmatrix} \\ &= f_1 f_1^* + f_2 f_2^* + \dots + f_n f_n^* \\ &= \sum_{k=1}^n f_k f_k^* \end{aligned}$$

$$\text{Componentwise, } S_{ij} = \sum_{k=1}^n f_k(i) \overline{f_k(j)}$$

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Ex. $\{f_1, f_2\} = \{(1, 0), (1, 1)\} \subset \mathbb{R}^2$

$$\text{Find } A, B \text{ s.t. } A\|x\|^2 \leq \sum_{i=1}^2 |\langle x, f_i \rangle|^2 \leq B\|x\|^2$$

Let $\{\gamma_i\}_{i=1}^2$ be the eigenvalue of $S = T^* T$

$$\text{Then } A = \min_i \gamma_i \quad B = \max_i \gamma_i$$

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad T^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad S = T^* T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ frame operator}$$

Recall the eigenvalue of S is a number γ_i s.t. $Se_i = \gamma_i e_i$,
 and eigenvector $e_i \neq 0$

$$\text{In this question } \gamma_1, \gamma_2 = \frac{3 \pm \sqrt{5}}{2} \quad A = \frac{3 - \sqrt{5}}{2}, \quad B = \frac{3 + \sqrt{5}}{2}$$

Note: Eigenvalues always positive since S is a positive operator meaning
 $S \geq A$ or $\langle Sx, x \rangle \geq A \langle x, x \rangle = A\|x\|^2 \geq 0 \quad \forall x \in \mathbb{H}$

Let $\{(\gamma_i, e_i)\}_{i=1}^n$ be eigen pairs of S . Let $\gamma_{\min} \leq \gamma_i \leq \gamma_{\max}$ $\forall i$.
 Since S is self-adjoint, we can choose $\{e_i\}$ to be orthonormal.

$\forall x \in \mathbb{R}^n$, $x = \sum_{i=1}^n \langle x, e_i \rangle e_i$.

$$S(x) = \sum_{i=1}^n \langle x, e_i \rangle \underbrace{S(e_i)}_{\gamma_i e_i} \quad \text{since } S \text{ is linear.}$$

$\gamma_i e_i$ since e_i are eigenvectors

$$\begin{aligned} S(x) &= \sum_{i=1}^n \langle x, e_i \rangle \gamma_i e_i \\ \langle S(x), x \rangle &= \sum_{i=1}^n \langle x, e_i \rangle \gamma_i \langle e_i, x \rangle \\ &= \underbrace{\sum_{i=1}^n |\langle x, e_i \rangle|^2 \gamma_i}_{\|x\|^2} \quad \text{Parseval's equality} \end{aligned}$$

$$\gamma_{\min} \|x\|^2 \leq \langle S(x), x \rangle \leq \gamma_{\max} \|x\|^2$$

Theorem : (Tight Frames) $A = B$

let $\{f_i\}_{i \in I} \subset H$ be a frame for H .

then $S = A \cdot \underline{\text{Id}}$ iff $\{f_i\}_{i \in I}$ is a tight frame w/ A frame bound.

$\begin{cases} \text{identity operator} \\ \text{frame operator} \end{cases}$

Lemma : let $\|A\|^2 \leq \underbrace{\langle S(x), x \rangle}_{\sum_{i \in I} |\langle x, f_i \rangle|^2} \leq B \|x\|^2$

then S^{-1} satisfies $B^{-1} \cdot \text{Id} \leq S^{-1} \leq A^{-1} \cdot \text{Id}$
 meaning $\langle B^{-1} \cdot \text{Id}, x \rangle \leq \langle S^{-1} x, x \rangle \leq \langle A^{-1} \cdot \text{Id}, x \rangle$
 or $B^{-1} \|x\|^2 \leq \langle S^{-1} x, x \rangle \leq A^{-1} \|x\|^2$
 so S^{-1} generates a frame

Idea : Have $A \cdot \text{Id} \leq S \leq B \cdot \text{Id}$

$$\underbrace{S^{-1} \cdot A \cdot \text{Id}}_{S^{-1} \leq A \cdot \text{Id}} \leq \underbrace{S^{-1} \cdot S}_{S^{-1} \geq S^{-1} \cdot \text{Id}} \leq \underbrace{S^{-1} \cdot B \cdot \text{Id}}_{S^{-1} \geq B^{-1} \cdot \text{Id}}$$

Reconstruction + Dual Frames.

Given $\{\langle x, f_i \rangle\}_{i \in I} \subset l^2(I)$, reconstruction of x is possible.

Is $x = \sum_{i \in I} \langle x, f_i \rangle f_i^*$ $\stackrel{*}{\text{dual of }} f_i$