# Geometric Differential Equations

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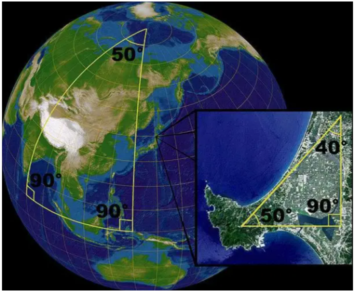

# Overview

# Introduction and Motivation

- Geometric PDEs allow us to generalize differential operators and other important mathematical notions to different ambient spaces
- Geometric PDEs establish are an object that connect analysis, differential equations and geometry together
- Examples:
  - Spectral geometry
  - Differential and Riemannian geometry
  - Study of geometric flows
  - Study of Riemannian submanifolds (curves and surfaces)

# Applications

- PDE on manifolds
  - Geometric modeling
  - Computer graphics
    - Surface design is treated as a boundary value problem by the PDE.
    - The boundary conditions imposed around the edges of the surface control the internal shape of the surface.
- Shape optimization
  - To find the optimal shape that minimizes a certain cost functional while satisfying given constraints
  - To find a bounded set $\Omega$ that minimizes $F(\Omega)$
  - Examples:
    - Determining the shape of a bridge of a given mass that best supports its load
    - Determining the optimal shape of a wing that minimizes the drag coefficient while preserving its lift

# Problems

In this project, we will be taking a look at the following problems:

**Problem**

*Given a smooth, curve* $\gamma \subset \mathbb{R}^2$*, find* $u : \gamma \to \mathbb{R}$ *that satisfies*

$$\begin{cases} -\Delta_\gamma u = f \\ +B.C. \end{cases}$$

*where* $\Delta_\gamma$ *is the Laplace-Beltrami operator.*

# Smooth Curves and its Tangent Vector

We consider a smooth curve $\gamma = \{\varphi(t) : t \in [a, b] \subset \mathbb{R}\}$ parameterized by $\varphi : [a, b] \to \mathbb{R}^2$, where $\varphi'(t) \neq 0$ for all $t \in [a, b]$. A point $\mathbf{x} \in \gamma$ is given by

$$\mathbf{x} = \varphi(t) = \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix}.$$

The tangent vector at a point $\mathbf{x} = \varphi(t)$ is given by the expression

$$\varphi'(t) = \begin{bmatrix} \varphi_1'(t) \\ \varphi_2'(t) \end{bmatrix}.$$

## Definition

Let

$$g(t) = [\varphi'(t)]^{\mathsf{T}}\varphi'(t) = \varphi_1'(t)^2 + \varphi_2'(t)^2$$

be such that

$$\int_\gamma 1 = \int_a^b \sqrt{g(t)}dt.$$

## Definition

Given some function $v : \gamma \to \mathbb{R}$ we consider $\hat{v} : [a, b] \to \mathbb{R}$ such that $\hat{v}(t) = v(\varphi(t)) = v(x)$.

# Tangential Gradient

## Definition

If $\hat{v} : [a, b] \to \mathbb{R} \in C^1(\mathbb{R})$, then we define the tangential gradient of the corresponding $v$ denoted by $\nabla_\gamma v(\mathbf{x})$. This $\nabla_\gamma v(\mathbf{x})$ abides by the relation

$$\hat{v}'(t) = [\varphi'(t)]^{\mathbf{T}} \nabla_\gamma v(\mathbf{x})$$

for all $t \in [a, b]$, $\mathbf{x} = \varphi(t)$.

The tangential gradient $\nabla_\gamma$ of a function $v(\mathbf{x})$ can also be rewritten in the following form:

$$\nabla_\gamma v(\mathbf{x}) = \frac{\hat{v}'(t)}{g(t)} \varphi'(t).$$

# Laplace-Beltrami Operator

Finally, we define the Laplace-Beltrami operator, which can be thought of as the generalization of the Laplacian to any smooth manifold.

**Definition**

If $\gamma$ and $\hat{v} : [a, b] \to \mathbb{R}$ are of class $C^2$, then,

$$\Delta_\gamma v(\mathbf{x}) = \mathrm{div}_\gamma(\nabla_\gamma v(\mathbf{x})) = \frac{1}{\sqrt{g(t)}} \frac{d}{dt} \left( \frac{1}{\sqrt{g(t)}} \hat{v}'(t) \right).$$

$\Delta_\gamma$ is referred to as the Laplace-Beltrami operator.

# Problem Formulation

Strong Form: Find $u : \gamma \to \mathbb{R}$ that satisfies

$$\begin{cases} -\Delta_\gamma u = f \quad \text{on } \gamma \\ u(\varphi(a)) = \alpha \text{ and } u(\varphi(b)) = \beta \end{cases}$$

.

Weak Form: Find $u \in H^1(\gamma)$ that satisfies

$$\begin{cases} \int_\gamma \nabla_\gamma u \cdot \nabla_\gamma v = \int_\gamma fv \\ u(\varphi(a)) = \alpha \text{ and } u(\varphi(b)) = \beta \end{cases}$$

for all $v \in H_0^1(\gamma)$.

# The Finite Element Method

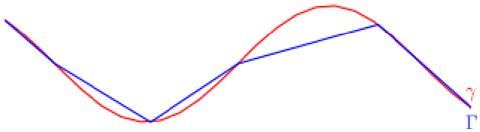


Figure: The Curve $\gamma$ Compared with the Approximate Curve $\Gamma$

We have a parameterization of $\gamma$ that produces a discrete curve we call $\Gamma$. This parameterization is given by $\varphi_\Gamma : [a, b] \to \mathbb{R}^2$ which is a piecewise affine parameter. $\varphi_\Gamma$ is expressed by the following:

$$\varphi_\Gamma(t) = \mathbf{x}_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}}(\mathbf{x}_i - \mathbf{x}_{i-1}).$$

We define the hat function as

$$\phi_i(\mathbf{x}) = \begin{cases} \frac{||\mathbf{x}-\mathbf{x}_{i-1}||}{||\mathbf{x}_i-\mathbf{x}_{i-1}||} & \text{if } \mathbf{x} \in K_i \\ \frac{||\mathbf{x}-\mathbf{x}_i||}{||\mathbf{x}_{i+1}-\mathbf{x}_i||} & \text{if } \mathbf{x} \in K_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

where $\phi_i \in V_N = \{v \in C^1(\Gamma) \colon v|_{K_i} \circ \varphi_{K_i} \in \mathbb{P}_1 \, \forall K_i\}$.
Recall from last Friday

$$V_N = \text{span}(\phi_0, \ldots, \phi_N)$$

# Constructing Our Linear System

Recall that

$$\int_\gamma \nabla_\gamma u \cdot \nabla_\gamma v = \int_\gamma fv$$

for every $v \in H_0^1$. Discretize such that

$$u_h(x) = \sum_{i=0}^{N} u_i \phi_i(x)$$

where $u_h \in V_N$.
Then

$$\sum_{j=0}^{N} u_j \int_\Gamma \nabla_\Gamma \phi_i \cdot \nabla_\Gamma \phi_j = \int_\Gamma f_\Gamma \phi_i$$

where $v_h = \phi_j$ for $j = 0, 1, \ldots, N$.

# Constructing Our Linear System (contd.)

On $\Gamma$, $a_{i,j} = \int_\Gamma \nabla_\Gamma \phi_i \cdot \nabla_\Gamma \phi_j$.

$$a_{i,i-1} = \frac{-1}{\|\mathbf{x}_i - \mathbf{x}_{i-1}\|}$$

$$a_{i,i} = \frac{1}{\|\mathbf{x}_i - \mathbf{x}_{i-1}\|} + \frac{1}{\|\mathbf{x}_{i+1} - \mathbf{x}_i\|}$$

$$a_{i,i+1} = \frac{-1}{\|\mathbf{x}_i - \mathbf{x}_{i-1}\|}$$

# Constructing Our Linear System (contd.)

To approximate $f$,

$$f_i = \int_\Gamma f_\Gamma \phi_i = \int_{K_i} f_\Gamma \phi_i + \int_{K_{i+1}} f_\Gamma \phi_i$$

For $K_i$,

$$\int_{K_i} f_\Gamma \phi_i = \int_0^1 f_\Gamma(\varphi(\mathbf{x}_{i-1} + t(\mathbf{x}_i - \mathbf{x}_{i-1})))t\|\mathbf{x}_i - \mathbf{x}_{i-1}\|dt$$

.

For $K_{i+1}$,

$$\int_{K_{i+1}} f_\Gamma \phi_i = \int_0^1 f_\Gamma(\varphi(\mathbf{x}_i + t(\mathbf{x}_{i+1} - \mathbf{x}_i)))(1-t)\|\mathbf{x}_{i+1} - \mathbf{x}_i\|dt$$

.

# Example 1: $u(\mathbf{x}) = \sin(2\pi x_1)$

- Let our curve $\gamma := \{\varphi(t) \mid t \in [0,1]\}$ be parametrized by $\varphi(t) = \begin{bmatrix} t \\ t \end{bmatrix}$
- We let our solution to the differential equation (with Dirichlet boundary conditions)

$$\begin{cases} -\Delta_\gamma u = 2\pi^2 \sin(2\pi x_1) & \text{on } \gamma \\ u(\varphi(0)) = 0 \quad \text{and} \quad u(\varphi(1)) = 0 \end{cases}$$

be $u : \gamma \to \mathbb{R}$ with analytical solution $u(x_1, x_2) = \sin(2\pi x_1)$

# Defining the $L^2$ error

**Definition ($L^2$ error)**

Let $u : \gamma \to \mathbb{R}$ be our analytic solution, $u_h : \Gamma \to \mathbb{R}$ our finite element approximation, and $P : \Gamma \to \gamma$ be the lifting of points from our discrete "curve" to our continuous curve.

$$L^2 \text{ error} := \|u \circ P - u_h\|_{L^2(\Gamma)}$$

- **Implementation**: We implemented the above formula by computing the error-squared. We essentially reduced it to the sum of line integrals:

$$(L^2 \text{ error})^2 = \|u \circ P - u_h\|^2_{L^2(\Gamma)} = \int_\Gamma (u \circ P - u_h)^2$$

$$(L^2 \text{ error})^2 = \sum_{i=1}^{N} \int_{K_i} (u \circ P - u_h)^2$$

- where $K_i$ is the line segment in $\mathbb{R}^2$ connecting $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$

# Rate of Convergence of the $L^2$ Error (Example 1)

- Reconsider our example 1, $u(x_1, x_2) = \sin(2\pi x_1)$
- We empirically demonstrate that the $L^2$ error converges with order $\mathcal{O}(h^2) = \mathcal{O}(N^{-2})$ where $h$ is the mesh size:

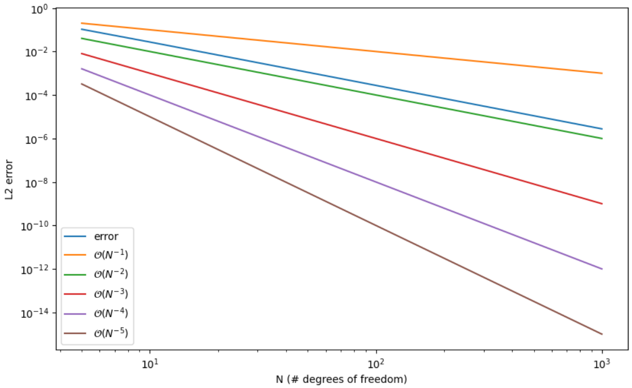


Figure: $L^2$ error for $u(\mathbf{x}) = \sin(2\pi x_1)$

# Defining the $H^1$ error

Generally speaking, the $H^1$ norm of any $f \in H^1$ is defined as:

$$\|f\|_{H^1(\Omega)} := \|f\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega)}$$

For our purposes,

**Definition ($H^1$ error)**

Let $u : \gamma \to \mathbb{R}$ be the analytic solution to our differential equation $-\Delta_\gamma u = f$ with Dirichlet boundary conditions, $u_h : \Gamma \to \mathbb{R}$ our finite element approximation,

$$\|u \circ P - u_h\|_{H^1(\Gamma)} := \|u \circ P - u_h\|_{L^2(\Gamma)} + \|\nabla_\gamma u \circ P - \nabla_\Gamma u_h\|_{L^2(\Gamma)}$$

# $H^1$ error in Example 1: $u(\mathbf{x}) = \sin(2\pi x_1)$

We observe the $H^1$ error converges with order $\mathcal{O}(h) = \mathcal{O}(N^{-1})$:

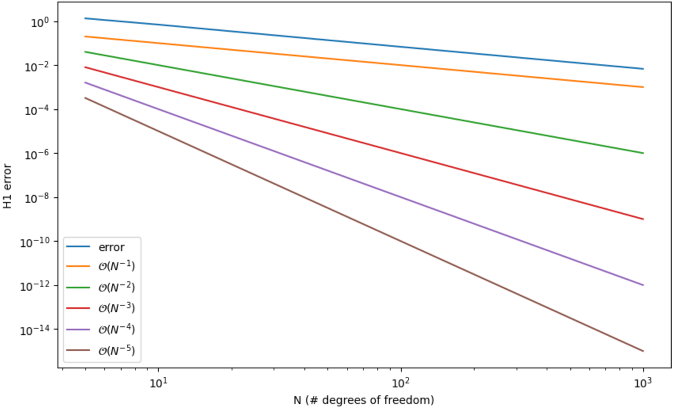


Figure: $H^1$ error

## Example 2: $u(\mathbf{x}) = \sin(2\pi x_1) + \sin(2\pi x_2)$

- Let $\gamma := \{\varphi(t) \mid t \in [0,1]\}$ be parametrized by $\varphi(t) = \begin{bmatrix} t \\ t \end{bmatrix}$.
- $u(x_1, x_2) : \gamma \to \mathbb{R}$ solves

$$\begin{cases} -\Delta_\gamma u = 2\pi^2 \sin(2\pi x_1) + 2\pi^2 \sin(2\pi x_2) & \text{on } \gamma \\ u(\varphi(0)) = 0 \quad \text{and} \quad u(\varphi(1)) = 0 \end{cases}$$

where $u(x_1, x_2) = \sin(2\pi x_1) + \sin(2\pi x_2)$.

# Example 2: $u(\mathbf{x}) = \sin(2\pi x_1) + \sin(2\pi x_2)$

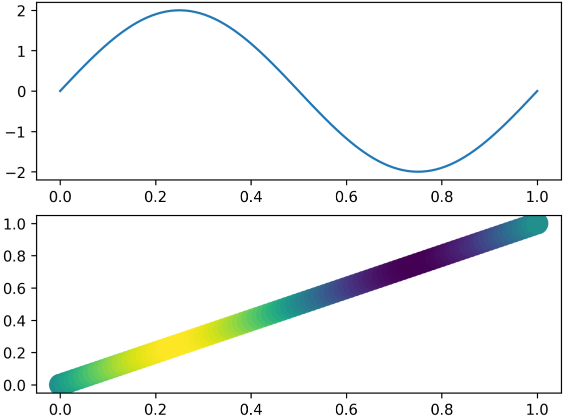


Figure: The approximate solution of the PDE on $\varphi(t) = (t, t)$.

# $L^2$ Error

- The rate of convergence is approximately $\mathcal{O}(h^2)$, similar to the previous example.

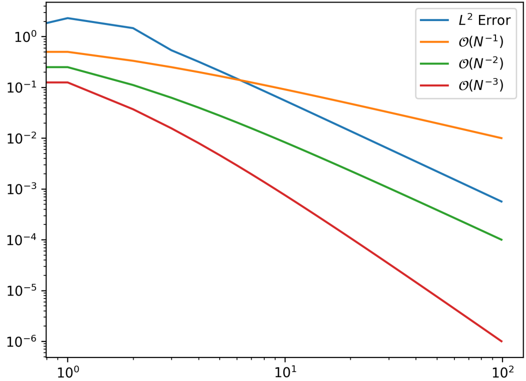


Figure: The rate of convergence of the $L^2$ error

# $H^1$ Error

- The rate of convergence is approximately $\mathcal{O}(h^1)$, again similar to the previous example.

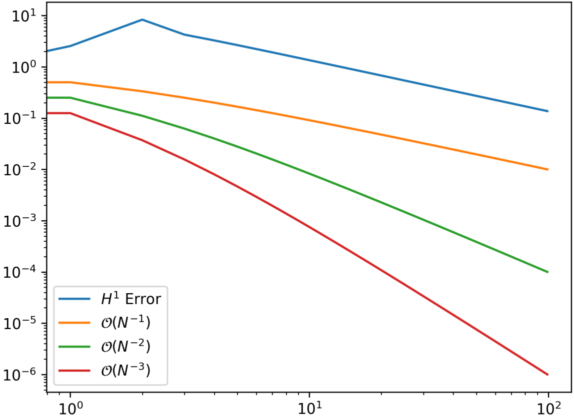


Figure: The rate of convergence of the $H^1$ error

# Neumann Conditions

Also consider the case where we're given mixed boundary conditions. Given $u : \gamma \to \mathbb{R}$, $f : \gamma \to \mathbb{R}$, with the parametrization of $\gamma$ defined as $\varphi : [a, b] \to \mathbb{R}^2$

$$\begin{cases} \Delta_\gamma u = f & \text{on } \gamma \\ u(\varphi(a)) = \alpha & \text{and} \quad \nabla_\gamma u(\varphi(b)) \cdot \tau(b) = \beta \end{cases}$$

where $\tau(t) := \frac{1}{\|\varphi'(t)\|}\varphi'(t)$

## Problem (Weak Form (Neumann boundary condition))

*The weak form is given by*

$$\int_\gamma -(\Delta_\gamma u)v = \int_\gamma \nabla_\gamma u \cdot \nabla_\gamma v - \left[\frac{1}{\sqrt{g(t)}}\hat{u}'(t)\hat{v}(t)\right]_{t=a}^{t=b}$$

$$\int_\gamma \nabla_\gamma u \cdot \nabla_\gamma v = \int_\gamma fv + \frac{1}{\sqrt{g(b)}}\hat{u}'(b)\hat{v}(b)$$

# Implementation of Neumann condition

We amend our matrix formulation to accommodate the Neumann boundary condition:

$$a_{N,N} = \frac{1}{\|\mathbf{x}_N - \mathbf{x}_{N-1}\|}$$

$$a_{N,N-1} = -\frac{1}{\|\mathbf{x}_N - \mathbf{x}_{N-1}\|}$$

$$f_N = \beta + \int_\Gamma f_\Gamma \phi_N$$

# Example 3: Neumann conditions

- $\gamma := \{\varphi(t) \mid t \in [0,1]\}$ is parametrized by $\varphi(t) = \begin{bmatrix} t \\ t + 0.2 * \sin(2\pi t) \end{bmatrix}$
- $u(x_1, x_2) : \gamma \to \mathbb{R}$ solves

$$\begin{cases} -\Delta_\gamma u = 1 \quad \text{on } \gamma \\ u(\varphi(0)) = 1 \quad \text{and } \nabla_\gamma u(\varphi(1)) \cdot \tau(1) = 1 \end{cases}$$

# Example 3: Neumann conditions

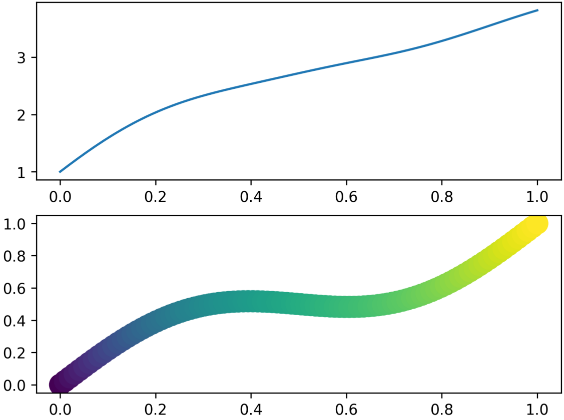


Figure: The approximate solution of the PDE on $\varphi(t) = (t, t + 0.2 * \sin(2\pi t))$ with boundary conditions $u(\varphi(0)) = 1$ and $\nabla_\gamma u(\varphi(1)) = 1$.

# Example 4: Closed Curve (Circle)

- $\gamma := \{\varphi(t) \mid t \in [0,1]\}$ is parametrized by $\varphi(t) = \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{bmatrix}$
- $u(x_1, x_2) : \gamma \to \mathbb{R}$ solves

$$-\Delta_\gamma u = f(x_1, x_2) \quad \text{on } \gamma$$

where

$$f(x_1, x_2) = \begin{cases} 1 & \text{if } x_2 \geq 0 \\ -1 & \text{if } x_2 < 0 \end{cases}$$

# Example 4: Closed Curve (Circle)

There are infinitely many solutions up to a constant without the boundary. So, impose

$$\int_\Gamma u = 0.$$

As a result,

$$\int_\Gamma f = 0.$$

Discretize such that

$$\int_\Gamma u_h = 0.$$

Then $\int_\Gamma \sum_{i=0}^N u_i \phi_i = \sum_{i=0}^N u_i \int_\Gamma \phi_i = 0$. Thus,

$$\left[\int_\Gamma \phi_0 \quad \int_\Gamma \phi_2 \quad \cdots \quad \int_\Gamma \phi_N)\right] \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{bmatrix}.$$

# Example 4: Closed Curve (Circle) (contd.)

Suppose the 1st row is $\int_\Gamma u = 0$. Then

$$F_0 = 0$$

since $\int_\Gamma u_h = 0$ in row 1. But, we cannot neglect the 1st node:

$$a_{N,N} = \frac{1}{\|\mathbf{x}_N - \mathbf{x}_{N-1}\|}$$

$$a_{N,N-1} = -\frac{1}{\|\mathbf{x}_N - \mathbf{x}_{N-1}\|}$$

# Example 4: Closed Curve (Circle) (contd.)

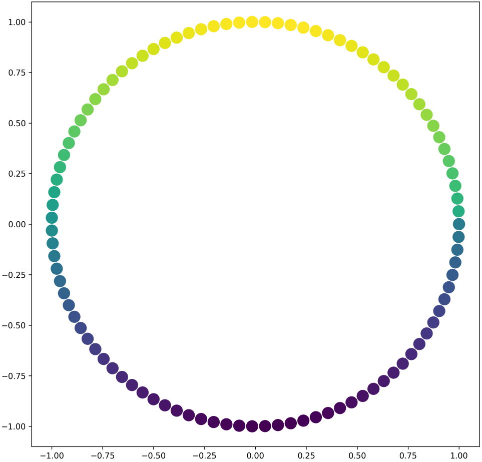


Figure: The approximate solution for the PDE on a closed curve $\varphi(t) = (\cos(2\pi t), \sin(2\pi t))$.