JOY NGUYEN - FTEC 6334 ASSIGNMENT 1

(1) Find Metrics : X^*

a) Mean Squared Error (MSE) : $MSE = \sum_{i=1}^{n} (x_i - X^*)^2$

Optimize MSE $\Leftrightarrow \frac{d(MSE)}{d(X^*)} = 0$

$$f'(X^*) = [(x_1 - X^*)^2 + \dots + (x_n - X^*)^2]' = 0$$

$$f'(X^*) = [(x_1^2 + \dots + x_n^2) - 2X^*(x_1 + \dots + x_n) + n(X^*)^2]' = 0$$

$$-2(x_1 + \dots + x_n) + 2n(X^*) = 0$$

$$X^* = \frac{x_1 + \dots + x_n}{n} \quad \text{(mean)}$$

$$f''(X^*) = 2n > 0$$

$\Rightarrow X^* = \frac{\sum_{i=1}^{n} x_i}{n}$ (mean) will minimize MSE

b) Mean Absolute Error (MAE) : $MAE = \sum_{i=1}^{n} |x_i - X^*|$

Assume $x_1 \leq x_2 \leq \dots \leq x_n$

$$f'(X^*) = [|x_1 - X^*| + \dots + |x_n - X^*|]'$$

If $X^* \leq x_1 \Rightarrow f'(X^*) = [(x_1 + \dots + x_n) - nX^*]' = -n < 0$

$X^* \geq x_n \Rightarrow f'(X^*) = [nX^* - (x_1 + \dots + x_n)]' = n > 0$

$\Rightarrow$ To optimize MAE ; $x_1 \leq X^* \leq x_n$

Assume $\underbrace{x_1 \leq x_2 \leq \dots \leq x_l}_{l \text{ elements}} \leq X^* \leq \underbrace{x_{l+1} \leq \dots \leq x_n}_{m \text{ elements}}$

$$\Rightarrow f'(X^*) = [l(X^*) - (x_1 + \dots + x_l) + (x_{l+1} + \dots + x_n) - m(X^*)]'$$

$$= l - m$$

$f'(X^*) = 0 \Rightarrow l - m = 0 \Rightarrow Pr(x \leq X^*) = Pr(x \geq X^*)$

If $n \% 2 = 0 \Rightarrow X^* \in [x_{\frac{n}{2}} ; x_{\frac{n}{2}+1}]$ satisfies $f'(X^*) = 0$ (median)

$n \% 2 = 1 \Rightarrow X^* = x_{\frac{n+1}{2}}$ satisfies $f'(X^*) = 0$

$$f''(X^*) = 0$$

$\Rightarrow X^*$ satisfies $f'(X^*) = 0$ will minimize MAE

c) Mean Squared Percentage Error (MSPE): $\quad MSPE = \sum_{i=1}^{n} \left(\frac{x_i - x^*}{x_i}\right)^2$

$$MSPE = \left(1 - \frac{x^*}{x_1}\right)^2 + \dots + \left(1 - \frac{x^*}{x_n}\right)^2$$

$$= n - 2x^*\left(\frac{1}{x_1} + \dots + \frac{1}{x_n}\right) + (x^*)^2\left(\frac{1}{x_1^2} + \dots + \frac{1}{x_n^2}\right)$$

$$f'(x^*) = -2\left(\frac{1}{x_1} + \dots + \frac{1}{x_n}\right) + 2(x^*)\left(\frac{1}{x_1^2} + \dots + \frac{1}{x_n^2}\right)$$

$$f'(x^*) = 0 \Rightarrow x^* = \frac{1/x_1 + \dots + 1/x_n}{1/x_1^2 + \dots + 1/x_n^2}$$

Assume $\frac{1}{x_1^2} = w_1, \dots, \frac{1}{x_n^2} = w_n \Rightarrow x^* = \frac{w_1 x_1 + \dots + w_n x_n}{w_1 + \dots + w_n}$ (weighted mean)

$$f''(x^*) = 2\left(\frac{1}{x_1^2} + \dots + \frac{1}{x_n^2}\right) > 0$$

$\Rightarrow$ x^* satisfies $f'(x^*) = 0$ will minimize MSPE

d) Mean Absolute Percentage Error (MAPE): $\quad MAPE = \sum_{i=1}^{n} \left|\frac{x_i - x^*}{x_i}\right|$

$$MAPE = \frac{|x_1 - x^*|}{|x_1|} + \frac{|x_2 - x^*|}{|x_2|} + \dots + \frac{|x_n - x^*|}{|x_n|}$$

Assume $x_1 \leq x_2 \leq \dots \leq x_n$

If $x^* \leq x_1 \Rightarrow MAPE = \frac{x_1 - x^*}{|x_1|} + \dots + \frac{x_n - x^*}{|x_n|} = \left(\frac{x_1}{|x_1|} + \dots + \frac{x_n}{|x_n|}\right) - x^*\left(\frac{1}{|x_1|} + \dots + \frac{1}{|x_n|}\right.$

$$\Rightarrow f'(x^*) = -\left(\frac{1}{|x_1|} + \dots + \frac{1}{|x_n|}\right) < 0$$

$$x^* \geq x_n \Rightarrow f'(x^*) = \frac{1}{|x_1|} + \dots + \frac{1}{|x_n|} > 0$$

$\Rightarrow$ To optimize MAPE, $x_1 \leq x^* \leq x_n$

Assume $x_1 \leq x_2 \leq \dots \leq x_k \leq x^* \leq x_{k+1} \leq \dots \leq x_n$

$$\Rightarrow MAPE = \frac{x^* - x_1}{|x_1|} + \dots + \frac{x^* - x_k}{|x_k|} + \frac{x_{k+1} - x^*}{|x_{k+1}|} + \dots + \frac{x_n - x^*}{|x_n|}$$

$$\Rightarrow f'(x^*) = \left(\frac{1}{|x_1|} + \dots + \frac{1}{|x_k|}\right) - \left(\frac{1}{|x_{k+1}|} + \dots + \frac{1}{|x_n|}\right)$$

$$f'(x^*) = 0 \Rightarrow \frac{1}{|x_1|} + \dots + \frac{1}{|x_k|} = \frac{1}{|x_{k+1}|} + \dots + \frac{1}{|x_n|}$$

$$f''(x^*) = 0$$

$\Rightarrow$ x^* satisfies $f'(x^*) = 0$ will minimize MAPE

$\Rightarrow$ $x^* \in [x_k ; x_{k+1}]$ such that $\sum_{i=1}^{k} \frac{1}{|x_i|} = \sum_{i=1}^{n-k} \frac{1}{|x_n|}$