Basic Discrete Structures

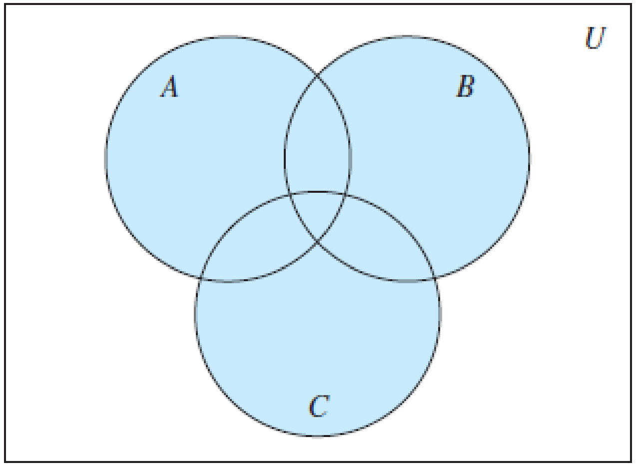
Sets, Functions, Sequences, Matrices, and Relations
(Lecture – 2)

Dr. Nirnay Ghosh

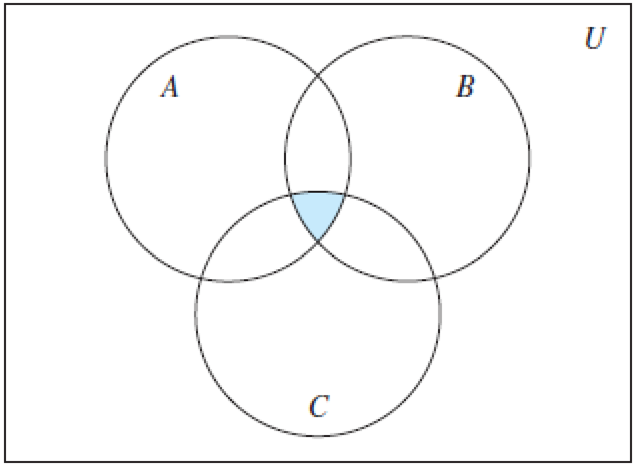
Generalized Unions and Intersections

The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.



(a) $A \cup B \cup C$ is shaded.



(b) $A \cap B \cap C$ is shaded.

Membership Tables

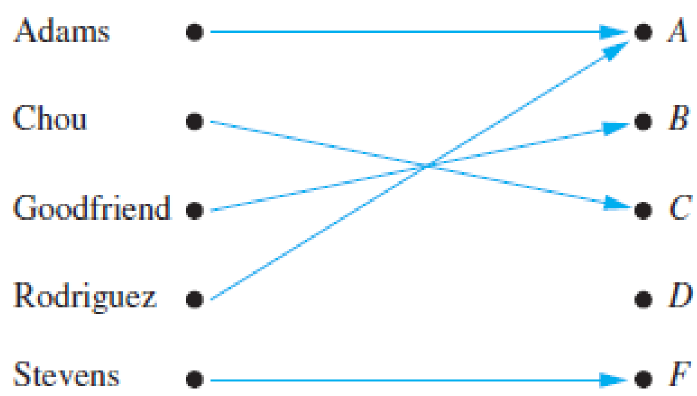
- Set identities can also be proved using **membership tables**.
- We consider each combination of sets that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity.
- To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used.
- Example: Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Functions

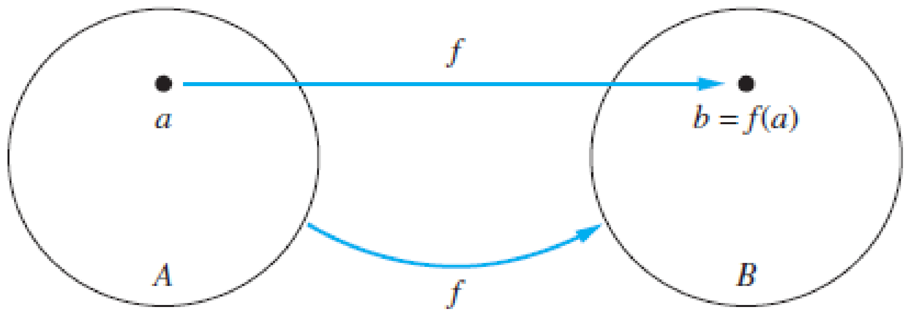
- In many instances we assign to each element of a set a particular element of a second set.
  - Example of a function: Assignment of grades in a particular class



- Functions: important for both mathematics and computer science
  - Used to define discrete structures: sequences and strings
  - How long it takes a computer to solve problems of given size
  - Many computer programs and subroutines are designed to calculate values of functions.
  - Recursive functions, which are functions defined in terms of themselves, are used throughout computer science

Functions

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b. The *range*, or *image*, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f *maps* A to B.



- Two functions are **equal** when they have the same domain, have the same codomain, and map each element of their common domain to the same element in their common codomain.
- A function is called **real-valued** if its codomain is the set of real numbers, and it is called **integer-valued** if its codomain is the set of integers.
- Two real-valued functions or two integer-valued functions with the same domain can be added, as well as multiplied.

Functions

Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$
$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

- Image of a subset can be defined as follows:

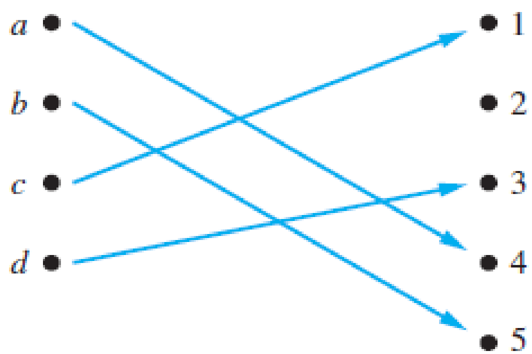
Let f be a function from A to B and let S be a subset of A. The *image* of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S\ (t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

Mappings: One-to-One Functions

A function f is said to be *one-to-one*, or an *injunction*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be *injective* if it is one-to-one.

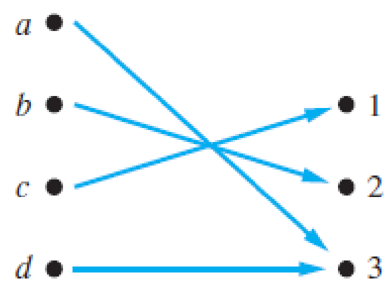


- Conditions that guarantee that a function is one-to-one:
  - Increasing/Strictly increasing, Decreasing/Strictly decreasing functions

A function f whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$, and *strictly increasing* if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called *decreasing* if $f(x) \geq f(y)$, and *strictly decreasing* if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f. (The word *strictly* in this definition indicates a strict inequality.)

Mappings: Onto & One-to-one correspondence Functions

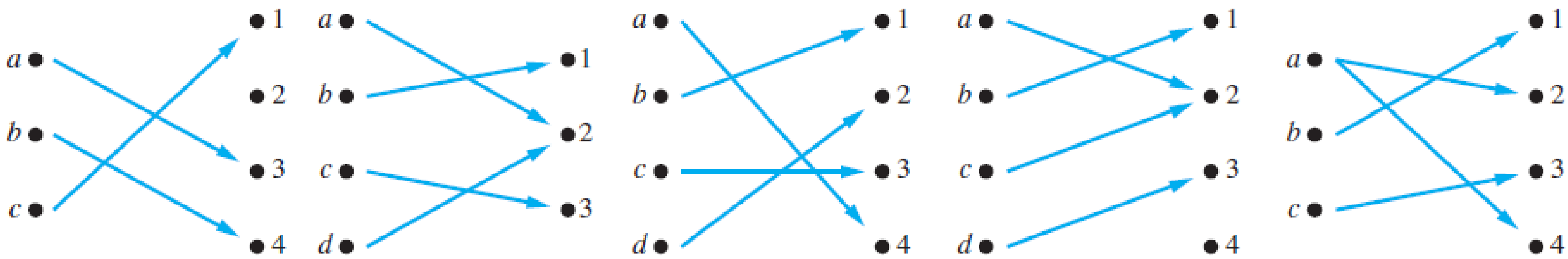
A function f from A to B is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.



- A function f is onto if $\forall y \exists x\, (f(x) = y)$, where the domain for x is the domain of the function and the domain for y is the codomain of the function.

The function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto. We also say that such a function is *bijective*.

Mappings



(a) One-to-one, not onto (b) Onto, not one-to-one (c) One-to-one, and onto (d) Neither one-to-one nor onto (e) Not a function

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

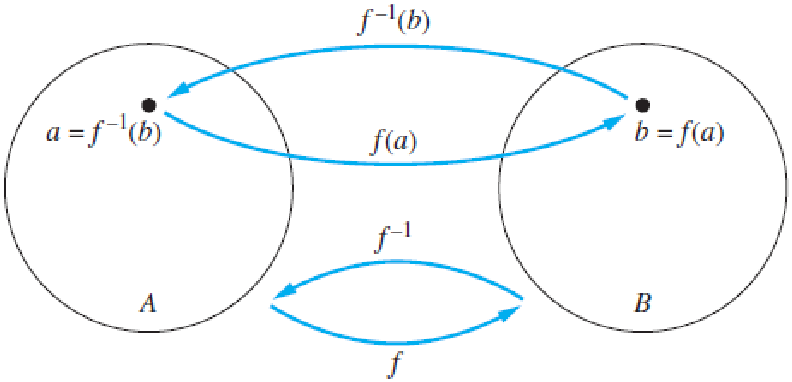
To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Inverse Functions

Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

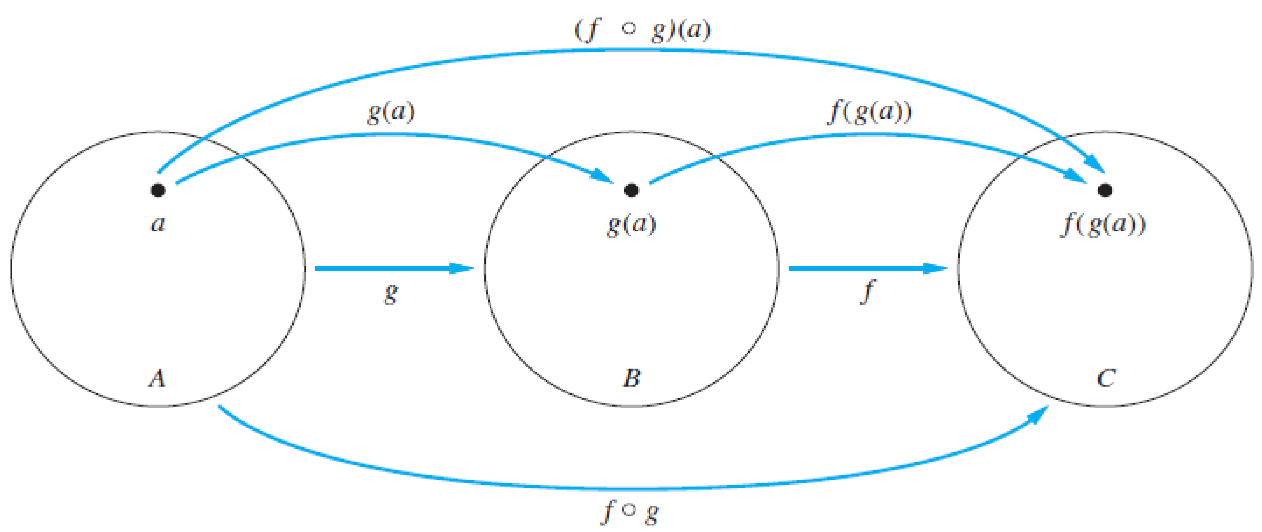


- A one-to-one correspondence is called **invertible** because we can define an **inverse** of this function.
- A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Composition of Functions

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a)).$$



Floor & Ceiling Functions

The *floor function* assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The *ceiling function* assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

- Floor function: same value throughout the interval $[n, n + 1)$, namely n, and then it jumps up to $n + 1$ when $x = n + 1$.
- Ceiling function: same value throughout the interval $(n, n + 1]$, namely $n + 1$, and then jumps to $n + 2$ when x is a little larger than $n + 1$.
- A useful approach for considering statements about the floor function is to let $x = n + \varepsilon$, where n is the integer, and ε is the fractional part of x, satisfies the inequality $0 \leq \varepsilon < 1$.
- Similarly, when considering statements about the ceiling function, it is useful to write $x = n - \varepsilon$, where n is an integer and $0 \leq \varepsilon < 1$.

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Floor & Ceiling Functions

- In Figure 10(a), the floor function is shown. Note that this function has the same value throughout the interval $[n, n+1)$, namely n, and then it jumps up to $n+1$ when $x = n+1$.
- In Figure 10(b), the graph of the ceiling function is shown. Note that this function has the same value throughout the interval $(n, n+1]$, namely $n+1$, and then jumps to $n+2$ when x is a little larger than $n+1$.

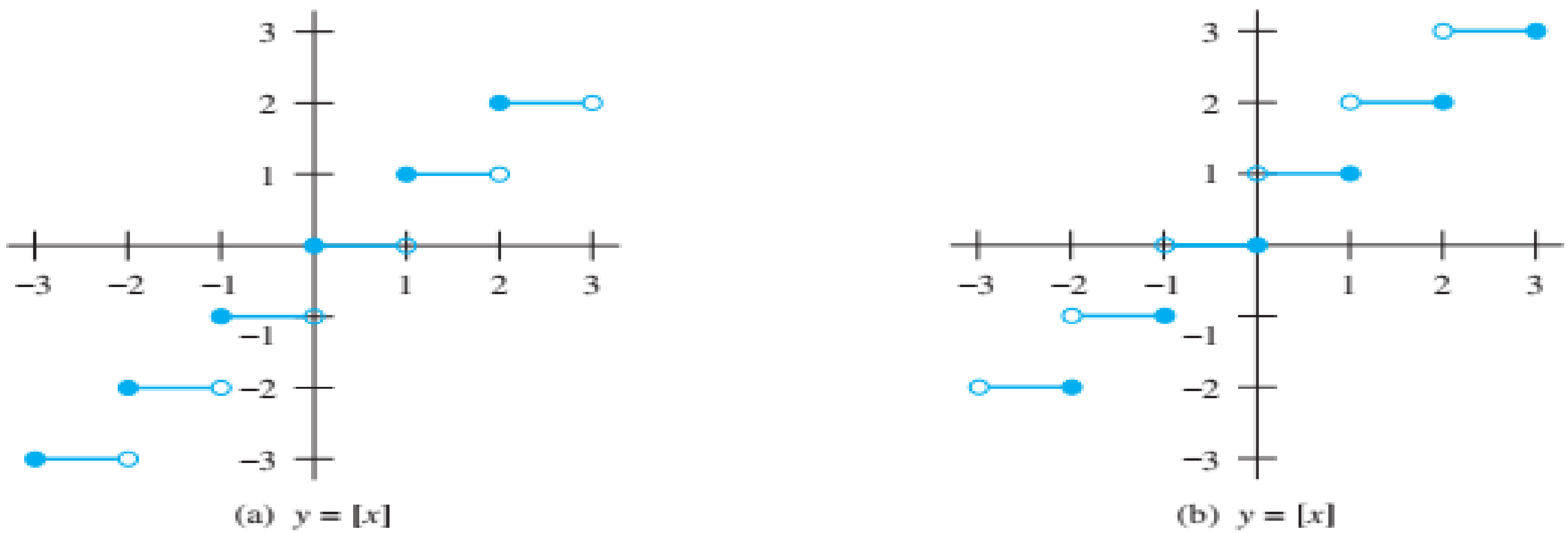


FIGURE 10 **Graphs of the (a) Floor and (b) Ceiling Functions.**

$\lfloor \frac{1}{2} \rfloor = 0$, $\lceil \frac{1}{2} \rceil = 1$, $\lfloor -\frac{1}{2} \rfloor = -1$, $\lceil -\frac{1}{2} \rceil = 0$, $\lfloor 3.1 \rfloor = 3$, $\lceil 3.1 \rceil = 4$, $\lfloor 7 \rfloor = 7$, $\lceil 7 \rceil = 7$.