(*iii*) $$P(X \le 2) = P(X = -2) + P(X = -1) + P(X = 0) + P(X = 1) + P(X = 2)$$
$$= 0.1 + k + 0.2 + 2k + 0.3 = 0.6 + 3k$$

$$\therefore \quad P(X \le 2) \ge 0.8 \Rightarrow 0.6 + 3k \ge 0.8 \Rightarrow k \ge \frac{1}{15}.$$

Hence, the minimum value of k is $\frac{1}{15}$.

(*iv*) The distribution function $F(x)$ of X is given below:

$$F(x) = \begin{cases} 0 & , -\infty < x < -2 \\ 0.1 & , -2 \le x < -1 \\ 0.1 + \frac{1}{15} = \frac{1}{6} & , -1 \le x < 0 \\ 0.1 + \frac{1}{15} + 0.2 = \frac{11}{30} & , 0 \le x < 1 \\ \frac{11}{30} + \frac{2}{15} = \frac{1}{2} & , 1 \le x < 2 \\ \frac{1}{2} + 0.3 = 0.8 & , 2 \le x < 3 \\ 0.8 + \frac{3}{15} = 1 & , x \ge 3 \end{cases}.$$

Example 7: *A random variable X has the following probability mass function:*

x	0	1	2	3	4	5	6
$P(X = x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

(*a*) *Find the value of k*

(*b*) *Find* $P(X < 4)$, $P(X \ge 5)$, $P(3 < X \le 5)$

(*c*) *Obtain the distribution function* $F(x)$

(*d*) *What is the smallest value of x for which* $P(X \le x) > 0.5$? (W.B.U.T. 2011)

Solution: (*a*) From the properties of probability mass function we know that $\Sigma P(X = x) = 1$

$$\Rightarrow \quad k + 3k + 5k + 7k + 9k + 11k + 13k = 1$$

$$\Rightarrow \quad 49k = 1 \Rightarrow k = \frac{1}{49}.$$

Obviously, $P(X = x) \ge 0$, $\forall x$ and $k = \frac{1}{49}$.

(*b*) $$P(X < 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49} = \frac{16}{49}$$

$$P(X \geq 5) = P(X = 5) + P(X = 6) = \frac{11}{49} + \frac{13}{49} = \frac{24}{49}$$

$$P(3 < X \leq 5) = P(X = 4) + P(X = 5) = \frac{9}{49} + \frac{11}{49} = \frac{20}{49}.$$

(*c*) The distribution function $F(x)$ of the random variable X is given below:

$$F(x) = \begin{cases} 0 & , \quad -\infty < x < 0 \\ \frac{1}{49} & , \quad 0 \leq x < 1 \\ \frac{4}{49} & , \quad 1 \leq x < 2 \\ \frac{9}{49} & , \quad 2 \leq x < 3 \\ \frac{16}{49} & , \quad 3 \leq x < 4 \\ \frac{25}{49} & , \quad 4 \leq x < 5 \\ \frac{36}{49} & , \quad 5 \leq x < 6 \\ 1 & , \quad x \geq 6 \end{cases}.$$

(*d*) From the distribution function we observe that

$$P(X \leq 3) = F(3) = \frac{16}{49} < 0.5$$

and $$P(X \leq 4) = F(4) = \frac{25}{49} > 0.5.$$

So, the smallest value of x for which $P(X \leq x) > 0.5$ is 4.

Example 8: *The probability mass function of a discrete random variable X is defined as* $P(X = 0) = 3k^2$, $P(X = 1) = 4k - 10k^2$, $P(X = 2) = 5k - 1$ *and* $P(X = x) = 0$ *if* $x \neq 0, 1, 2$.

(*i*) *Find the value of k*

(*ii*) *Find* $P(0 < X < 2/X > 0)$

(*iii*) *Obtain the distribution function F(x) of X*

(*iv*) *Find the smallest value of x for which* $F(x) = P(X \leq x) > \frac{1}{2}$.

Solution: (*i*) From the properties of probability mass function we know that

$$\Sigma P(X = x) = 1$$

$$\Rightarrow \quad 3k^2 + 4k - 10k^2 + 5k - 1 = 1$$

$$\Rightarrow \quad 7k^2 - 9k + 2 = 0$$

$$\Rightarrow \quad 7k^2 - 7k - 2k + 2 = 0$$

$$\Rightarrow \quad (k - 1)(7k - 2) = 0 \qquad \text{...(1)}$$

Now, $k = 1 \Rightarrow P(X = 1) = 4 - 10 = -6 < 0$, which is not possible.

$\therefore$ $k \neq 1$. So, from (1), $k = \frac{2}{7}$.

Obviously, $P(X = x) \geq 0, \forall x$ and $k = \frac{2}{7}$.

(*ii*)
$$P(0 < X < 2/ X > 0) = \frac{P\{(0 < X < 2) \cap (X > 0)\}}{P(X > 0)}$$
$$= \frac{P(0 < X < 2)}{P(X > 0)} = \frac{P(X = 1)}{P(X = 1) + P(X = 2)}$$

Now, $$P(X = 1) = 4 \cdot \frac{2}{7} - 10\left(\frac{2}{7}\right)^2 = \frac{16}{49}$$

and $$P(X = 2) = 5 \cdot \frac{2}{7} - 1 = \frac{3}{7}$$

$\therefore$ $$P(0 < X < 2 / X > 0) = \frac{16}{37}.$$

(*iii*) The distribution function $F(x)$ of the random variable X is given below:

$$F(x) = P(X \leq x) = \begin{cases} 0 & , -\infty < x < 0 \\ 3k^2 = 3\left(\frac{2}{7}\right)^2 = \frac{12}{49} & , 0 \leq x < 1 \\ \frac{12}{49} + \frac{16}{49} = \frac{28}{49} & , 1 \leq x < 2 \\ 1 & , x \geq 2 \end{cases}$$

(*iv*) From the distribution function it is obvious that the smallest value of x for which

$$F(x) = P(X \leq x) > \frac{1}{2} \text{ is } x = 1.$$

Example 9: *A random variable X has the following p.d.f.:*

$$f(x) = \begin{cases} cx^2 & , 0 \leq x \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

Find: (*i*) *c* (*ii*) $P\left(0 \leq X \leq \frac{1}{2}\right)$. *(W.B.U.T. 2009, 2011)*

Solution: (*i*) The given function $f(x)$ is a possible probability density function (p.d.f.) if $f(x) \geq 0$, $\forall x$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Now, $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} cx^2 dx + \int_{1}^{\infty} 0\,dx = 1$$

$$\Rightarrow \qquad c\left[\frac{x^3}{3}\right]_0^1 = 1$$

$$\Rightarrow \qquad c = 3.$$

$$\therefore \qquad f(x) = \begin{cases} 3x^2 & , \ 0 \le x \le 1 \\ 0 & , \ \text{otherwise} \end{cases}.$$

Obviously, $f(x) \ge 0, \forall x$. So, $f(x)$ is a possible p.d.f. for $c = 3$.

(*ii*) $$P\left(0 \le X \le \frac{1}{2}\right) = \int_0^{1/2} f(x)dx = \int_0^{1/2} 3x^2 dx = \left[x^3\right]_0^{1/2} = \frac{1}{8}$$

Example 10: *A continuous random variable X has p.d.f.:*

$$f(x) = \begin{cases} 3x^2 & , \ 0 \le x \le 1 \\ 0 & , \ \text{elsewhere} \end{cases}$$

Find 'a' and 'b' such that

(i) $P(X \le a) = P(X > a)$ *(ii) $P(X > b) = 0.973$.*

Solution: (*i*) We know that $P(X \le a) + P(X > a) = 1$.

Also, by question: $P(X \le a) = P(X > a)$.

$$\therefore \qquad P(X \le a) = \frac{1}{2}$$

$$\Rightarrow \qquad \int_{-\infty}^{a} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \qquad \int_0^a 3x^2 dx = \frac{1}{2}$$

$$\Rightarrow \qquad \left[x^3\right]_0^a = \frac{1}{2}$$

$$\Rightarrow \qquad a^3 = \frac{1}{2}$$

$$\Rightarrow \qquad a = 2^{-1/3}.$$

(*ii*) Given, $$P(X > b) = 0.973$$

$$\Rightarrow \qquad \int_b^{\infty} f(x)dx = 0.973$$

$$\Rightarrow \qquad \int_b^1 3x^2 dx = 0.973$$

$$\Rightarrow \qquad \left[x^3\right]_b^1 = 0.973$$

$$\Rightarrow \qquad 1 - b^3 = 0.973$$

$$\Rightarrow \qquad b^3 = 1 - 0.973 = 0.027$$

$$\Rightarrow \qquad b = 0.3.$$

Example 11: *(i) Is the function defined as follows a density function?*

$$f(x) = \begin{cases} e^{-x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

(ii) If so, determine the probability that the variate having this density will fall in the interval (1, 2)?

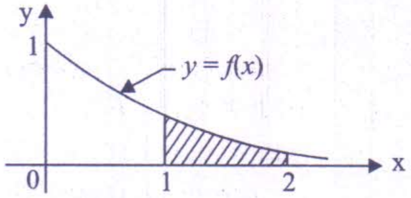
(iii) Also find the cumulative probability function F(2)?

Solution: *(i)* Here, $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{\infty} e^{-x} dx = \lim_{B \to \infty} \int_{0}^{B} e^{-x} dx$

$$= \lim_{B \to \infty} [-e^{-x}]_0^B = \lim_{B \to \infty} (1 - e^{-B}) = 1.$$

Also, $f(x) \geq 0, \forall x$.

Hence, $f(x)$ is a possible probability density function of a random variable X.



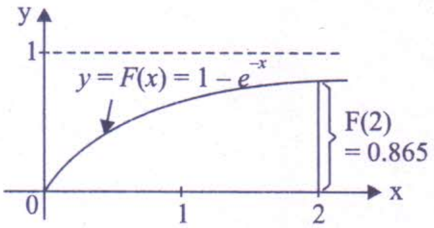
(ii) Required probability $= P(1 < X < 2)$

$$= \int_1^2 e^{-x} dx = [-e^{-x}]_1^2$$

$$= e^{-1} - e^{-2} = 0.368 - 0.135$$

$$= 0.233$$

This probability is equal to the shaded area as shown in the adjacent figure.



(iii) Cumulative probability function $F(2)$

$$= P(X \leq 2) = \int_{-\infty}^{2} f(x)dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{2} e^{-x} dx$$

$$= [-e^{-x}]_0^2 = 1 - e^{-2} = 1 - 0.135 = 0.865$$

It is shown in the adjacent figure.

Example 12: *Show that the function:*

$$f(x) = \begin{cases} \dfrac{1}{4} & , -2 < x < 2 \\ 0 & , \text{elsewhere} \end{cases}$$

is a possible probability density function. Construct the distribution function and compute

(i) $P(X < 1)$ *(ii)* $P(|X| > 1)$ *(iii)* $P(|X - 1| \geq 2)$ *(iv)* $P(2X + 3 > 5)$

Solution: Observe that $f(x) \geq 0, \forall x$

and $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{-2} 0\, dx + \int_{-2}^{2} \frac{1}{4} dx + \int_{2}^{\infty} 0\, dx = \int_{-2}^{2} \frac{1}{4} dx = \frac{1}{4}[x]_{-2}^{2} = 1.$

Therefore, $f(x)$ is a possible probability density function. The distribution function $F(x)$ is defined by

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} 0\, dt = 0, \text{ for } -\infty < x \leq -2$$

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$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{-2} 0\,dt + \int_{-2}^{x} \frac{1}{4}dt = \frac{1}{4}[t]_{-2}^{x} = \frac{1}{4}(x+2), \text{ for } -2 < x \le 2$$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{-2} 0\,dt + \int_{-2}^{2} \frac{1}{4}dt + \int_{2}^{x} 0\,dt = \frac{1}{4}[t]_{-2}^{2} = 1, \text{ for } x > 2$$

So, the distribution function $F(x)$ is given by:

$$F(x) = \begin{cases} 0 & , \text{ for } -\infty < x \le -2 \\ \frac{1}{4}(x+2) & , \text{ for } -2 < x \le 2 \\ 1 & , \text{ for } x > 2 \end{cases}$$

(i) $$P(X < 1) = \int_{-\infty}^{1} f(x)dx = \int_{-2}^{1} \frac{1}{4}dx = \frac{1}{4}[x]_{-2}^{1} = \frac{1}{4}(1+2) = \frac{3}{4}.$$

(ii) $$P(|X| > 1) = 1 - P(|X| \le 1) = 1 - P(-1 \le X \le 1)$$

$$= 1 - \int_{-1}^{1} f(x)dx = 1 - \int_{-1}^{1} \frac{1}{4}dx = 1 - \frac{1}{4}[x]_{-1}^{1} = 1 - \frac{1}{2} = \frac{1}{2}.$$

(iii) $P(|X-1| \ge 2) = 1 - P(|X-1| < 2)$ ...(1)

When $X - 1 \ge 0$: $|X-1| < 2 \Rightarrow X - 1 < 2 \Rightarrow X < 3$.

When $X - 1 < 0$: $|X-1| < 2 \Rightarrow -(X-1) < 2 \Rightarrow X - 1 > -2 \Rightarrow X > -1$.

So, from (1), we get

$$P(|X-1| \ge 2) = 1 - P(-1 < X < 3) = 1 - \left\{\int_{-1}^{2} \frac{1}{4}dx + \int_{2}^{3} 0\,dx\right\}$$

$$= 1 - \frac{1}{4}[x]_{-1}^{2} = 1 - \frac{1}{4}(2+1) = \frac{1}{4}.$$

(iv) $$P(2X + 3 > 5) = P\left(X > \frac{5-3}{2}\right) = P(X > 1) = \int_{1}^{\infty} f(x)dx$$

$$= \int_{1}^{2} f(x)dx = \int_{1}^{2} \frac{1}{4}dx = \frac{1}{4}[x]_{1}^{2} = \frac{1}{4}.$$

Example 13: *Show that the function f(x) given by*

$$f(x) = \begin{cases} x & , 0 \le x < 1 \\ k - x & , 1 \le x \le 2 \\ 0 & , \text{elsewhere} \end{cases}$$

is a probability density function for a suitable value of the constant k. Construct the distribution function of the random variable X and compute the probability that the random variable X lies between $\frac{1}{2}$ *and* $\frac{3}{2}$.

(W.B.U.T. 2003, 2005, 2006, 2009, 2010)

Solution: The given function $f(x)$ is a possible probability density function (p.d.f.) if $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

Now,
$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \quad \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x\,dx + \int_{1}^{2} (k-x)dx + \int_{2}^{\infty} 0\,dx = 1$$

$$\Rightarrow \quad \left[\frac{x^2}{2}\right]_0^1 + \left[kx - \frac{x^2}{2}\right]_1^2 = 1$$

$$\Rightarrow \quad \frac{1}{2} + 2k - 2 - k + \frac{1}{2} = 1$$

$$\Rightarrow \quad k = 2.$$

$$\therefore \quad f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 2-x, & 1 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

Obviously, $f(x) \geq 0$, $\forall x$. So, $f(x)$ is a possible p.d.f. for $k = 2$.

The distribution function $F(x)$ is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

For $-\infty < x < 0$: $F(x) = \int_{-\infty}^{x} f(t)dt = 0.$

For $0 \leq x < 1$: $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} t\,dt = \left[\frac{t^2}{2}\right]_0^x = \frac{x^2}{2}$

For $1 \leq x < 2$: $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{1} t\,dt + \int_{1}^{x} (2-t)dt$

$$= \left[\frac{t^2}{2}\right]_0^1 + \left[2t - \frac{t^2}{2}\right]_1^x = \frac{1}{2} + \left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)$$

$$= 2x - \frac{x^2}{2} - 1.$$

For $x \geq 2$: $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{1} t\,dt + \int_{1}^{2} (2-t)dt + \int_{2}^{x} 0\,dt$

$$= \left[\frac{t^2}{2}\right]_0^1 + \left[2t - \frac{t^2}{2}\right]_1^2 = \frac{1}{2} + (4-2) - \left(2 - \frac{1}{2}\right) = 1.$$

So, the distribution function $F(x)$ of the random variable X is given by:

$$F(x) = \begin{cases} 0 & , \ -\infty < x < 0 \\ \dfrac{x^2}{2} & , \ 0 \le x < 1 \\ 2x - \dfrac{x^2}{2} - 1 & , \ 1 \le x < 2 \\ 1 & , \ x \ge 2 \end{cases}$$

Now, $P\left(\frac{1}{2} < X < \frac{3}{2}\right) = \int_{1/2}^{3/2} f(x)\,dx = \int_{1/2}^{1} x\,dx + \int_{1}^{3/2} (2-x)\,dx$

$$= \left[\frac{x^2}{2}\right]_{1/2}^{1} + \left[2x - \frac{x^2}{2}\right]_{1}^{3/2}$$

$$= \left(\frac{1}{2} - \frac{1}{8}\right) + \left(3 - \frac{9}{8}\right) - \left(2 - \frac{1}{2}\right) = \frac{3}{4}.$$

Example 14: *Show that the function*

$$f(x) = \begin{cases} |x| & , \ -1 < x < 1 \\ 0 & , \ \text{elsewhere} \end{cases}$$

is a possible probability density function and hence find the corresponding distribution function.

(*W.B.U.T. 2006*)

Solution: Observe that $f(x) \ge 0, \forall x$.

Also, $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} |x|\,dx + \int_{1}^{\infty} 0\,dx = \int_{-1}^{1} |x|\,dx$

$$= 2\int_0^1 x\,dx \qquad (\because \ |x| \text{ is an even function})$$

$$= 2\left[\frac{x^2}{2}\right]_0^1 = 2 \cdot \frac{1}{2} = 1.$$

Hence $f(x)$ is a possible probability density function.

The distribution function $F(x)$ is defined by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt.$$

For $-\infty < x \le -1$: $F(x) = \int_{-\infty}^{x} f(t)dt = 0.$

For $-1 < x < 0$: $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{x} (-t)dt \qquad [\because \ |t| = -t, \text{ for } t < 0]$

$$= -\left[\frac{t^2}{2}\right]_{-1}^{x} = -\frac{1}{2}(x^2 - 1) = \frac{1}{2}(1 - x^2).$$

For $0 \le x < 1$: $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{0} (-t)dt + \int_{0}^{x} t\,dt$

$$= -\left[\frac{t^2}{2}\right]_{-1}^{0} + \left[\frac{t^2}{2}\right]_{0}^{x} = \frac{1}{2} + \frac{x^2}{2} = \frac{1}{2}(1 + x^2).$$

For $x \geq 1$: $$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{1} |t|\,dt + \int_{1}^{x} 0\,dx = 2\int_{0}^{1} t\,dt$$

$$= 2\left[\frac{t^2}{2}\right]_0^1 = 2\cdot\frac{1}{2} = 1.$$

So, the distribution function $F(x)$ of the random variable X is given by:

$$F(x) = \begin{cases} 0 & , -\infty < x \leq -1 \\ \frac{1}{2}(1-x^2) & , -1 < x < 0 \\ \frac{1}{2}(1+x^2) & , 0 \leq x < 1 \\ 1 & , x \geq 1 \end{cases}$$

Example 15: *For what value of k, the function*

$$f(x) = \begin{cases} ke^{-b(x-a)} & , a \leq x < \infty,\ a, b, x > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

will be a probability density function? (*W.B.U.T. 2003*)

Solution: The given function $f(x)$ is a possible probability density function if

$$f(x) \geq 0, \forall x \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)dx = 1$$

Now, $$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_{-\infty}^{a} 0\,dx + \int_{a}^{\infty} ke^{-b(x-a)}dx = 1$$

$$\Rightarrow k\lim_{B\to\infty}\int_{a}^{B} e^{-b(x-a)}dx = 1$$

$$\Rightarrow k\lim_{B\to\infty}\left[\frac{e^{-b(x-a)}}{-b}\right]_a^B = 1$$

$$\Rightarrow k\lim_{B\to\infty}\frac{1}{b}\left\{1-e^{-b(B-a)}\right\} = 1$$

$$\Rightarrow \frac{k}{b} = 1 \quad [\because\ e^{-b(B-a)} \to 0 \text{ as } B \to \infty, \text{ since } b > 0]$$

$$\Rightarrow k = b.$$

$$\therefore \quad f(x) = \begin{cases} be^{-b(x-a)}, & a \leq x < \infty,\ a, b > 0 \\ 0 \quad , & \text{elsewhere} \end{cases}$$

Obviously, $f(x) \geq 0, \forall x$. So, $f(x)$ will be a probability density function for $k = b$.

Example 16: *The random variable X has the probability function*

$$f(x) = k \quad , \quad \text{if } x = 0$$
$$= 2k \quad , \quad \text{if } x = 1$$

$$= 3k \quad , \quad if \;\; x = 2$$
$$= 0 \quad , \quad elsewhere$$

Determine (i) value of k (ii) $P(X < 2)$ (iii) $P(X \le 2)$ (iv) the smallest value of k for which $P(X \le 1) \ge \frac{1}{2}$. (W.B.U.T. 2004)

Solution: Here the random variable X has the following probability distribution:

$X = x$	0	1	2
$f(x) = P(X = x)$	k	$2k$	$3k$

(i) $$\Sigma f(x) = 1 \;\Rightarrow\; k + 2k + 3k = 1 \;\Rightarrow\; k = \frac{1}{6}$$

So, for $k = \frac{1}{6}$, $f(x) \ge 0$ and $\Sigma f(x) = 1$

(ii) $$P(X < 2) = P(X = 0) + P(X = 1) = k + 2k = 3k = \frac{3}{6} = \frac{1}{2}$$

(iii) $$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$$
$$= k + 2k + 3k = 6k = 1$$

(iv) $$P(X \le 1) \ge \frac{1}{2}$$

$$\Rightarrow \quad P(X = 0) + P(X = 1) \ge \frac{1}{2}$$

$$\Rightarrow \quad k + 2k \ge \frac{1}{2}$$

$$\Rightarrow \quad k \ge \frac{1}{6}.$$

Hence the smallest value of k is $\frac{1}{6}$.

Example 17: *The length of the life of a tyre manufactured by a company follows a continuous distribution given by the density function*

$$f(x) = \begin{cases} \dfrac{k}{x^3} & , \quad 1000 \le x \le 1500 \\ 0 & , \quad elsewhere \end{cases}$$

Find k and find the probability that a randomly selected tyre would function for at least 1200 hours. (W.B.U.T. 2005)

Solution: The given function $f(x)$ is a possible probability density function if $f(x) \ge 0$, $\forall\, x$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

Now, $$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_{-\infty}^{1000} 0\,dx + \int_{1000}^{1500} \frac{k}{x^3}dx + \int_{1500}^{\infty} 0\,dx = 1$$

$$\Rightarrow \qquad k\left[\frac{x^{-3+1}}{-3+1}\right]_{1000}^{1500} = 1$$

$$\Rightarrow \qquad \frac{k}{2}\cdot\frac{1}{10^4}\left(\frac{1}{100}-\frac{1}{225}\right) = 1$$

$$\Rightarrow \qquad k = 36 \times 10^5$$

$$\therefore \qquad f(x) = \begin{cases} 36\times10^5 x^{-3} & , \quad 1000 \le x \le 1500 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Obviously, $f(x) \geq 0, \forall x$.

$$\therefore \qquad P(X \geq 1200) = \int_{1200}^{\infty} f(x)dx = \int_{1200}^{1500} 36\times10^5 x^{-3}dx + \int_{1500}^{\infty} 0\,dx$$

$$= 36 \times 10^5\left[\frac{x^{-2}}{-2}\right]_{1200}^{1500} = 180\left(\frac{1}{144}-\frac{1}{225}\right) = \frac{9}{20}$$

This is the required probability.

Example 18: *The life in hours of a certain type of electronic component of a computer follows a continuous distribution given by the density function:*

$$f(x) = \begin{cases} \dfrac{k}{x^2} & , \quad x \geq 100 \\ 0 & , \quad x < 100 \end{cases}$$

Find k and determine the probability that all four such components in a computer will have to be replaced in the first 250 hours of its operation.

Solution: The given function $f(x)$ is a possible probability density function if $f(x) \geq 0$, $\forall\, x$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

$$\text{Now,} \qquad \int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \qquad \int_{-\infty}^{100} 0\,dx + \int_{100}^{\infty} kx^{-2}dx = 1$$

$$\Rightarrow \qquad k\lim_{B\to\infty}\int_{100}^{B} x^{-2}dx = 1$$

$$\Rightarrow \qquad k\lim_{B\to\infty}\left[-\frac{1}{x}\right]_{100}^{B} = 1$$

$$\Rightarrow \qquad k\lim_{B\to\infty}\left(\frac{1}{100}-\frac{1}{B}\right) = 1$$

$$\Rightarrow \qquad k = 100 \qquad \left[\because \frac{1}{B} \to 0 \text{ as } B \to \infty\right]$$

$$\therefore \qquad f(x) = \begin{cases} 100x^{-2} & , \quad x \geq 100 \\ 0 & , \quad x < 100 \end{cases}$$

Obviously, $f(x) \geq 0$, $\forall\, x$. So, $f(x)$ is a possible probability density function for $k = 100$.

Let A_i denote the event 'the ith electronic component will last 250 hours or more', $i = 1, 2, 3, 4$.

$$\therefore \quad P(A_i) = \int_{250}^{\infty} f(x)dx = 100 \lim_{B\to\infty} \int_{250}^{B} x^{-2}dx$$

$$= 100 \lim_{B\to\infty} \left[-\frac{1}{x}\right]_{250}^{B} = 100 \lim_{B\to\infty} \left(\frac{1}{250} - \frac{1}{B}\right)$$

$$= \frac{100}{250} = \frac{2}{5}.$$

$$\therefore \quad P(\overline{A_i}) = 1 - P(A_i) = 1 - \frac{2}{5} = \frac{3}{5} \qquad (i = 1, 2, 3, 4)$$

Since the events $\overline{A}_1, \overline{A}_2, \overline{A}_3$ and $\overline{A}_4$ are independent, the required probability is

$$P(\overline{A}_1\, \overline{A}_2\, \overline{A}_3\, \overline{A}_4) = P(\overline{A}_1)P(\overline{A}_2)P(\overline{A}_3)P(\overline{A}_4)$$

$$= \left(\frac{3}{5}\right)^4 = \frac{81}{625}.$$

Example 19: *The p.d.f. of a random variable X is f(x) = k(x − 1) (2 − x), for 1 ≤ x ≤ 2. Determine (i) the value of k (ii) the distribution function F(x) (iii)* $P\left(\frac{5}{4} \leq X \leq \frac{3}{2}\right)$. *(W.B.U.T. 2007, 2010)*

Solution: (*i*) The given function $f(x)$ is a possible probability density function (p.d.f.) if $f(x) \geq 0$, $\forall\, x$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

Now,
$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \quad \int_{-\infty}^{1} 0\,dx + \int_{1}^{2} k(x-1)(2-x)dx + \int_{2}^{\infty} 0\,dx = 1$$

$$\Rightarrow \quad k\int_{1}^{2} (3x - x^2 - 2)dx = 1$$

$$\Rightarrow \quad k\left[\frac{3x^2}{2} - \frac{x^3}{3} - 2x\right]_1^2 = 1$$

$$\Rightarrow \quad k\left[\left(6 - \frac{8}{3} - 4\right) - \left(\frac{3}{2} - \frac{1}{3} - 2\right)\right] = 1$$

$$\Rightarrow \quad \frac{k}{6} = 1 \Rightarrow k = 6$$

$$\therefore \quad f(x) = \begin{cases} 6(x-1)(2-x), & \text{for } 1 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

Obviously, $f(x) \geq 0$, $\forall\, x$. So, $f(x)$ is a possible probability density function for $k = 6$.

(*ii*) The distribution function $F(x)$ of the random variable X is defined by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$$

For $-\infty < x < 1$: $\quad F(x) = \int_{-\infty}^{x} f(t)dt = 0.$

For $1 \le x < 2$: $\quad F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{1} 0\,dt + \int_{1}^{x} 6(t-1)(2-t)dt = 6\int_{1}^{x}(3t - t^2 - 2)dt$

$$= 6\left[\frac{3t^2}{2} - \frac{t^3}{3} - 2t\right]_1^x = 6\left[\frac{3x^2}{2} - \frac{x^3}{3} - 2x - \left(\frac{3}{2} - \frac{1}{3} - 2\right)\right]$$

$$= 5 - 12x + 9x^2 - 2x^3.$$

For $x \ge 2$: $\quad F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{1} 0\,dt + \int_{1}^{2} 6(t-1)(2-t)dt + \int_{2}^{\infty} 0\,dt = 1.$

So, the distribution function $F(x)$ of the random variable X is given by

$$F(x) = \begin{cases} 0 & , \quad -\infty < x < 1 \\ 5 - 12x + 9x^2 - 2x^3 & , \quad 1 \le x < 2 \\ 1 & , \quad x \ge 2 \end{cases}$$

(*iii*) Using p.d.f.:

$$P\left(\frac{5}{4} \le X \le \frac{3}{2}\right) = \int_{5/4}^{3/2} f(x)dx = 6\int_{5/4}^{3/2}(x-1)(2-x)dx$$

$$= 6\left[\frac{3x^2}{2} - \frac{x^3}{3} - 2x\right]_{5/4}^{3/2}$$

$$= 6\left[\frac{27}{8} - \frac{27}{24} - 3 - \left(\frac{75}{32} - \frac{125}{192} - \frac{5}{2}\right)\right] = \frac{11}{32}.$$

Alternative using d.f.:

$$P\left(\frac{5}{4} \le X \le \frac{3}{2}\right) = F\left(\frac{3}{2}\right) - F\left(\frac{5}{4}\right)$$

$$= 5 - 12\cdot\frac{3}{2} + 9\cdot\left(\frac{3}{2}\right)^2 - 2\left(\frac{3}{2}\right)^3 - \left\{5 - 12\cdot\frac{5}{4} + 9\left(\frac{5}{4}\right)^2 - 2\left(\frac{5}{4}\right)^3\right\}$$

$$= \frac{11}{32}.$$

Example 20: *For what values of a the function $f(x) = ax^{-2}u(x - a)$ is a possible probability density function of a r.v. X, where $u(x - a)$ is a unit step function. Construct the corresponding distribution function and compute $P(X \ge 10)$.*

Solution: Since $u(x - a)$ is a unit step function, we have

$$u(x-a) = \begin{cases} 0 & , \quad \text{for } x < a \\ 1 & , \quad \text{for } x \ge a \end{cases} \qquad \ldots(1)$$

The given function $f(x)$ is a possible probability density function if $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

Now,
$$\int_{-\infty}^{\infty} f(x)dx = 1$$
$$\Rightarrow \quad \int_{-\infty}^{\infty} ax^{-2}u(x-a)dx = 1$$
$$\Rightarrow \quad a\int_{a}^{\infty} x^{-2}dx = 1 \qquad \text{[By (1)]}$$
$$\Rightarrow \quad a\lim_{B\to\infty}\int_{a}^{B} x^{-2}dx = 1$$
$$\Rightarrow \quad a\lim_{B\to\infty}\left[\frac{x^{-1}}{-1}\right]_a^B = 1$$
$$\Rightarrow \quad a\lim_{B\to\infty}\left(\frac{1}{a}-\frac{1}{B}\right) = 1$$
$$\Rightarrow \quad a\cdot\frac{1}{a} = 1 \qquad \left[\because \frac{1}{B}\to 0 \text{ as } B\to\infty\right]$$
$$\Rightarrow \quad a > 0 \qquad [\because F(x) \geq 0, \forall x]$$

So, $f(x)$ is a possible probability density function for all $a > 0$. The distribution function $F(x)$ is given by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} at^{-2}u(t-a)dt$$
$$= a\int_{a}^{x} t^{-2}dt \qquad \text{[By (1)]}$$
$$= a\left[-\frac{1}{t}\right]_a^x = a\left(\frac{1}{a}-\frac{1}{x}\right) = 1-\frac{a}{x}, \quad \text{where } x \geq a > 0.$$
$$F(x) = 0, \text{ elsewhere}$$

$$\therefore \quad P(X \geq 10) = 1 - P(X < 10) = 1-\left(1-\frac{a}{10}\right) = \frac{a}{10}.$$

Example 21: (i) *Let $f(x) = ke^{-\alpha x}(1 - e^{-\alpha x})$, $x \geq 0$ where $\alpha > 0$. Find k such that $f(x)$ is a possible probability density function and determine the corresponding distribution function.* (*) Place E-6

Solution: (ii) The given function $f(x)$ is a possible probability density function if $f(x) \geq 0$, $\forall x$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

Now,
$$\int_{-\infty}^{\infty} f(x)dx = 1$$
$$\Rightarrow \quad \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} ke^{-\alpha x}(1-e^{-\alpha x})dx = 1$$
$$\Rightarrow \quad k\lim_{B\to\infty}\int_{0}^{B}(e^{-\alpha x}-e^{-2\alpha x})dx = 1$$
$$\Rightarrow \quad k\lim_{B\to\infty}\frac{1}{2\alpha}\left[-2e^{-\alpha x}+e^{-2\alpha x}\right]_0^B = 1$$
$$\Rightarrow \quad k\lim_{B\to\infty}\frac{1}{2\alpha}\left[-2e^{-\alpha B}+e^{-2\alpha B}+1\right] = 1$$

(C5) (E-6)

Place in Example 2.1 (ii), page ~~105~~ 107 & 108 .

(ii) The probability density function of a random variable Z is given by

$$f(z) = \begin{cases} kze^{-z^2}, & \text{for } z > 0 \\ 0, & \text{for } z \le 0. \end{cases}$$

Find the value of k and find out the corresponding distribution function of Z. (WBUT-2014)

Solution (ii) The given function $f(z)$ is a possible p.d.f. if $f(z) \geq 0, \forall z$ and $\int_{-\infty}^{\infty} f(z)\,dz = 1$.

Now, $\int_{-\infty}^{\infty} f(z)\,dz = \int_0^{\infty} kze^{-z^2}dz = \lim_{B\to\infty} k\int_0^B ze^{-z^2}dz$

$= \lim_{B\to\infty} \frac{k}{2}\int_0^{B^2} e^{-u}\,du$ (setting $z^2 = u \Rightarrow z\,dz = \frac{du}{2}$)

$= \lim_{B\to\infty} \frac{k}{2}\left[-e^{-u}\right]_0^{B^2} = \lim_{B\to\infty} \frac{k}{2}\left[1 - e^{-B^2}\right] = \frac{k}{2}.$

$\therefore \int_{-\infty}^{\infty} f(z)\,dz = 1 \Rightarrow k = 2.$

$$\therefore f(z) = \begin{cases} 2ze^{-z^2}, & \text{for } z > 0 \\ 0, & \text{for } z \le 0. \end{cases}$$

Obviously, $f(z) \geq 0, \forall z$. So, $f(z)$ is a possible p.d.f. for $k = 2$.

Next, if $z > 0$, then

$F(z) = \int_{-\infty}^{z} f(t)\,dt = \int_0^z 2te^{-t^2}dt = 1 - e^{-z^2}$

Hence the distribution function of Z is given by

$$F(z) = \begin{cases} 1 - e^{-z^2}, & \text{if } z > 0 \\ 0, & \text{if } z \le 0. \end{cases}$$

$$\Rightarrow \qquad \frac{k}{2\alpha} = 1 \qquad \left[\begin{array}{l} \because e^{-\alpha B}, e^{-2\alpha B} \to 0 \text{ as } B \to \infty \\ \text{where } \alpha > 0 \end{array}\right]$$

$$\Rightarrow \qquad k = 2\alpha$$

$\therefore \ f(x) = 2\alpha e^{-\alpha x}(1 - e^{-\alpha x}),\ x \geq 0$, where $\alpha > 0$.

Obviously, $f(x) \geq 0, \forall\ x$. So, $f(x)$ is a possible probability density function for $k = 2\alpha$.

The distribution function $F(x)$ of the random variable X is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt.$$

For $-\infty < x < 0$: $\quad F(x) = \int_{-\infty}^{x} f(t)dt = 0.$

For $x \geq 0$:

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} 0\, dt + \int_{0}^{x} 2\alpha(e^{-\alpha t} - e^{-2\alpha t})dt$$

$$= 2\alpha\left[\frac{e^{-\alpha t}}{-\alpha} - \frac{e^{-2\alpha t}}{-2\alpha}\right]_0^x = \left[-2e^{-\alpha t} + e^{-2\alpha t}\right]_0^x$$

$$= 1 - 2e^{-\alpha x} + e^{-2\alpha x}$$

So, the distribution function $F(x)$ of the random variable X is given by:

$$F(x) = \begin{cases} 0 & , \quad -\infty < x < 0 \\ 1 - 2e^{-\alpha x} + e^{-2\alpha x} & , \quad x \geq 0,\ \alpha > 0 \end{cases}.$$

(ii) From E-6

Example 22: *Suppose that the amount of money (in rupees) that a person has saved (negative savings indicate debt) is found to be a random phenomena described by the distribution function $F(x)$ as*

$$F(x) = \begin{cases} \frac{1}{2}e^{-(x/5000)^2}, & x \leq 0 \\ 1 - \frac{1}{2}e^{-(x/5000)^2}, & x > 0 \end{cases}.$$

(*i*) *Find the probability density function (p.d.f.) if it exists.*

(*ii*) *What is the probability that the amount of savings possessed by a person will be* (*a*) *more than ₹ 5000* (*b*) *less than – ₹ 5000* (*c*) *between – ₹ 5000 and ₹ 5000* (*d*) *equal to ₹ 5000?*

(*iii*) *What is the conditional probability that the amount of savings possessed by a person will be* (*a*) *less than ₹ 10000 given that it is more than ₹ 5000* (*b*) *more than ₹ 5000 given that it is less than ₹ 10000?*

Solution: (*i*) The probability density function is given by:

$$f(x) = \frac{dF}{dx} = \begin{cases} \dfrac{-x}{25(10)^6}e^{-(x/5000)^2}, & x \leq 0 \\ \dfrac{x}{25(10)^6}e^{-(x/5000)^2}, & x > 0 \end{cases}$$

Here the random variable X denotes the savings of the person.

(*ii*) (*a*) P(amount of savings > ₹ 5000)

$$= P(X > 5000) = 1 - P(X \leq 5000)$$

$$= 1 - F(5000)$$

$$= 1 - \left\{1 - \frac{1}{2} e^{-(5000/5000)^2}\right\} = \frac{1}{2e}.$$

(*b*) P(Savings < – ₹ 5000) = P (X < –5000)

$$= F(-5000) = \frac{1}{2} e^{-(-5000/5000)^2} = \frac{1}{2e}.$$

(*c*) P(Savings lies between – 5000 and 5000)

$$= P(-5000 < X < 5000) = F(5000) - F(-5000)$$

$$= \left(1 - \frac{1}{2e}\right) - \frac{1}{2e} \qquad \text{[By (a) and (b)]}$$

$$= 1 - \frac{1}{e}.$$

(*d*) P(Savings = ₹ 5000) = $P(X = 5000) = 0$, since for a continuous random variable the probability that it can take a particular value is zero.

(*iii*) (*a*) P(Savings is less than ₹ 10000 given that it is more than ₹ 5000)

$$= P(X < 10000/X > 5000)$$

$$= \frac{P\{(X < 10000) \cap (X > 5000)\}}{P(X > 5000)} \qquad \text{[By definition]}$$

$$= \frac{P(5000 < X < 10000)}{P(X > 5000)}$$

Now, $P(5000 < X < 10000)$

$$= F(10000) - F(5000)$$

$$= \left(1 - \frac{1}{2e^4}\right) - \left(1 - \frac{1}{2e}\right) = \frac{1}{2e}\left(1 - \frac{1}{e^3}\right)$$

and $\quad P(X > 5000) = \dfrac{1}{2e}.$

$\therefore$ Required probability $= \dfrac{1}{2e}\left(1 - \dfrac{1}{e^3}\right) \Big/ \left(\dfrac{1}{2e}\right) = 1 - \dfrac{1}{e^3}.$

(*b*) P(Savings is more than ₹ 5000 given that it is less than ₹ 10000)

$$= P(X > 5000/X < 10000) = \frac{P\{(X > 5000) \cap (X < 10000)\}}{P(X < 10000)}$$

$$= \frac{P(5000 < X < 10000)}{P(X < 10000)}$$

Now, $P(5000 < X < 10000) = \frac{1}{2e}\left(1-\frac{1}{e^3}\right)$

and $P(X < 10000) = F(10000) = 1-\frac{1}{2e^4}$.

$\therefore$ Required probability $= \frac{1}{2e}\left(1-\frac{1}{e^3}\right)\Big/\left(1-\frac{1}{2e^4}\right) = \frac{e^3-1}{2e^4-1}$.

Example 23: *Determine the expected value of the number on a die when thrown.*

Solution: Let X denotes the random variable which represents the number on a die when thrown. So, the probability distribution of X is

x_i	:	1	2	3	4	5	6
$P(x_i) = P(X = x_i)$	:	1/6	1/6	1/6	1/6	1/6	1/6

$$\therefore \quad E(X) = \sum x_i p(x_i) = 1\left(\frac{1}{6}\right)+2\left(\frac{1}{6}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{1}{6}\right)+5\left(\frac{1}{6}\right)+6\left(\frac{1}{6}\right)$$

$$= \frac{1}{6}(1+2+3+4+5+6)$$

$$= \frac{1}{6}\cdot\frac{1}{2}\cdot 6(6+1) = \frac{7}{2}.$$

This is the required expected value.

Example 24: *An unbiased coin is tossed three times. Let X denotes the random variable which represents the number of tails appearing. Determine the probability distribution of X and hence find E(X), Var (X).*

Solution: When a coin is tossed three times, then the sample space is

$$S = \{HHH, THH, HTH, HHT, HTT, THT, TTH, TTT\}.$$

Here X represents the random variable denoting the number of tails appearing. So, the probability distribution of X is

x_i	:	0	1	2	3
$P(x_i) = P(X = x_i)$	:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\therefore \quad E(X) = \sum x_i p(x_i) = 0\times\frac{1}{8}+1\times\frac{3}{8}+2\times\frac{3}{8}+3\times\frac{1}{8} = \frac{12}{8} = \frac{3}{2}$$

$$\text{Var}(X) = \sum x_i^2 p(x_i) - \{E(X)\}^2$$

$$= 0^2\times\frac{1}{8}+1^2\times\frac{3}{8}+2^2\times\frac{3}{8}+3^2\times\frac{1}{8}-\left(\frac{3}{2}\right)^2 = \frac{3}{4}.$$

Example 25: *A die is tossed thrice. A success is 'getting 1 or 6' on a toss. Find the mean and variance of the number of successes.*

Solution: Let S and F denote respectively the events 'one success' and 'one failure'.

$$\therefore \quad P(S) = P(\text{face 1 or 6}) = P(\text{face 1}) + P(\text{face 6})$$

$$= \frac{1}{6}+\frac{1}{6} = \frac{1}{3}$$

$$P(F) = 1 - P(S) = 1 - \frac{1}{3} = \frac{2}{3}$$

Here the event space contains $2^3 = 8$ event points: $\{SSS, FSS, SFS, SSF, SFF, FSF, FFS, FFF\}$.

Let X denotes the random variable representing the number of successes for tossing a die thrice.

$\therefore$ $$P(X = 0) = P(FFF) = P(F)\,P(F)\,P(F) \quad \text{(Since the events are independent)}$$

$$= \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{8}{27}$$

$$P(X = 1) = P\{(SFF) \cup (FSF) \cup (FFS)\}$$

$$= P(SFF) + P(FSF) + P(FFS) \quad \text{(Since the events are m.e.)}$$

$$= P(S)\,P(F)\,P(F) + P(F)\,P(S)\,P(F) + P(F)\,P(F)\,P(S)$$

$$(\because \text{ The events are independent})$$

$$= 3\times\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3} = \frac{4}{9}.$$

$$P(X = 2) = P\{(FSS) \cup (SFS) \cup (SSF)\}$$

$$= P(FSS) + P(SFS) + P(SSF) \quad \text{(Since the events are m.e.)}$$

$$= P(F)\,P(S)\,P(S) + P(S)\,P(F)\,P(S) + P(S)\,P(S)\,P(F)$$

$$(\because \text{ The events are independent})$$

$$= 3\times\frac{2}{3}\times\frac{1}{3}\times\frac{1}{3} = \frac{2}{9}$$

$$P(X = 3) = P(SSS) = P(S)\,P(S)\,P(S) \quad (\because \text{ The events are independent})$$

$$= \frac{1}{3}\times\frac{1}{3}\times\frac{1}{3} = \frac{1}{27}.$$

The probability distribution of X is given below:

x_i	:	0	1	2	3
$p_i = P(X = x_i)$	:	8/27	4/9	2/9	1/27

$\therefore$ $$\text{Mean} = E(X) = \sum x_i p_i = 0\times\frac{8}{27} + 1\times\frac{4}{9} + 2\times\frac{2}{9} + 3\times\frac{1}{27} = 1,$$

$$\text{Variance} = \text{Var}(X) = \sum x_i^2 p_i - \{E(X)\}^2 = 0^2\times\frac{8}{27} + 1^2\times\frac{4}{9} + 2^2\times\frac{2}{9} + 3^2\times\frac{1}{27} - 1^2$$

$$= \frac{4}{9} + \frac{8}{9} + \frac{3}{9} - 1 = \frac{2}{3}.$$

Example 26: *Four defective bulbs are accidentally mixed with ten good bulbs. It is not possible to just look at a bulb and tell whether or not it is defective. Determine the probability distribution of the number of defective bulbs, if three bulbs are drawn at random without replacement from this lot. Also find the mean, variance and coefficient of variation of defective bulbs.*

Solution: Let X denotes the random variable representing the number of defective bulbs in 3 drawn bulbs. Clearly X can take the values 0, 1, 2 or 3.

Number of defective bulbs $= 4$

Number of good bulbs = 10

Total number of bulbs = 14

$$P(X=0) = P(\text{no defective}) = P(\text{all 3 good ones})$$

$$= {}^{10}C_3/{}^{14}C_3 = \frac{10!}{3!7!}.\frac{3!11!}{14!} = \frac{10\times9\times8}{14\times13\times12}$$

$$= \frac{30}{91}$$

$$P(X=1) = P(\text{1 defective and 2 good ones})$$

$$= {}^4C_1\,{}^{10}C_2/{}^{14}C_3$$

$$= 4.\frac{10!}{2!8!}.\frac{3!11!}{14!} = 4.\frac{10\times9}{2}.\frac{6}{14\times13\times12} = \frac{45}{91}$$

$$P(X=2) = P(\text{2 defectives and 1 good one})$$

$$= {}^4C_2\,{}^{10}C_1/{}^{14}C_3$$

$$= \frac{4!}{2!2!}\cdot10\cdot\frac{3!11!}{14!} = \frac{6\times10\times6}{14\times13\times12} = \frac{15}{91}$$

$$P(X=3) = P(\text{all 3 defectives}) = {}^4C_3/{}^{14}C_3 = 4.\frac{3!11!}{14!}$$

$$= \frac{4\times6}{14\times13\times12} = \frac{1}{91}.$$

Thus the probability distribution of the random variable X is

x_i	:	0	1	2	3
$p_i = P(X=x_i)$	:	$\frac{30}{91}$	$\frac{45}{91}$	$\frac{15}{91}$	$\frac{1}{91}$

$\therefore$ $$\text{Mean} = E(X) = \sum x_i p_i$$

$$= 0\times\frac{30}{91}+1\times\frac{45}{91}+2\times\frac{15}{91}+3\times\frac{1}{91} = \frac{78}{91}$$

$$\text{Variance} = \text{Var}(X) = \sum x_i^2 p_i - \{E(X)\}^2$$

$$= 0^2\times\frac{30}{91}+1^2\times\frac{45}{91}+2^2\times\frac{15}{91}+3^2\times\frac{1}{91}-\left(\frac{78}{91}\right)^2$$

$$= \frac{114}{91}-\left(\frac{78}{91}\right)^2 = \frac{4290}{8281}.$$

$\therefore$ $$\text{S.D.} = +\sqrt{\text{Var}(X)} = 0.72$$

$$\text{Coefficient of variation} = \frac{\text{S.D.}}{\text{Mean}}\times100 = 0.72\times\frac{91}{78}\times100 = 84\%.$$