Fourier Series

Fouries series is an infinite representation of periodic function in terms of the trigonometric sine and cosine functions. In many engineering problems, especially in the study of periodic phenomenae in conduction of heat electro-dynamics and acoustics, it is necessary to express a function in a series of sines and co-sines.

Fourier series is a very powerful method to solve ordinary and partial differential equations, particularly with periodic functions appearing as non-homogeneous terms.

We know that, Taylor's series expansion is valid only for functions which are continuous and differentiable. Fourier series is possible not only for continuous function but also for the function having finite no. of jump discontinuity

Periodic function:

A function $f(x)$ is said to be periodic function with period $T>0$ if for all x, $f(x+T)=f(x)$ and T is the least of such values.

Ex. 1) $\sin x, \cos x$ are periodic function with period 2π.

2) $\tan x, \cot x$ are periodic function with period π.

Let a function $f(x)$ ~~is said to be~~ has period 2π. In this case, it is enough to consider behavior of the function on the interval $[-\pi, \pi]$.

Euler's Formulae:

The Fourier Series for the function $f(x)$ in the interval $c \le x \le c+2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, $$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx.$$

Conditions for Fourier Expansion (Dirichlet condition):

The reader must not be misled by the ~~to~~ belief that the Fourier series of $f(x)$ in each case shall be valid. The above discussion has merely shown that if $f(x)$ has an expansion, then the co-effecients are given by Euler's formulae.

A function $f(x)$ defined in $[0, 2\pi]$ has a valid Fourier series expansion of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ where a_0, a_n, b_n are constants, provided

1) $f(x)$ is well defined and single-valued, except possibly at a finite no. of point in the interval $[0, 2\pi]$.
2) $f(x)$ has finite no. of discontinuity in the interval $[0, 2\pi]$.
3) $f(x)$ has ~~a~~ finite no. of finite maxima and minima.

Definition of Fourier Series

Let $f(x)$ be a function defined in $[0, 2\pi]$. Let $f(x+2\pi) = f(x)$ $\forall x$, then the Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

These values a_0, a_n, b_n are called as Fourier co-effecients of $f(x)$ in $[0, 2\pi]$.

- Let $f(x)$ be a function defined in $[-\pi, \pi]$.
  Let $f(x+2\pi) = f(x)\ \forall x$, the Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad \text{— (1)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

These values a_0, a_n, b_n are called as Fourier co-effecient of $f(x)$ in $[-\pi, \pi]$

- For a function $f(x)$ periodic on an interval $[-L, L]$ instead of $[-\pi, \pi]$, a simple change of variables can be used to transform the interval of integration from $[-\pi, \pi]$ to $[-L, L]$. Let,

$$x = \frac{\pi x'}{L} \Rightarrow x' = \frac{Lx}{\pi}.$$

$$dx = \frac{\pi\,dx'}{L}$$

Plugging this in (1) gives –

$$f(x') = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x'}{L}\right) + b_n \sin\left(\frac{n\pi x'}{L}\right)\right]$$

and,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x')\,dx'$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x')\cos\left(\frac{n\pi x'}{L}\right)dx'$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x')\sin\left(\frac{n\pi x'}{L}\right)dx'$$

Similarly; the function is instead defined on the interval $[0, 2L]$, the above equation simply become

$$a_0 = \frac{1}{L}\int_0^{2L} f(x')\,dx'$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x') \cos\left(\frac{n\pi x'}{L}\right) dx'$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x') \sin\left(\frac{n\pi x'}{L}\right) dx'.$$

Convergence of Fourier series:

We introduce the Fourier partial sum $f_N(x)$ of the function $f(x)$ defined on $[-\pi, \pi]$ as

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx).$$

where a_0, a_n, b_n are Fourier co-effecients of $f(x)$ in $[-\pi, \pi]$.

Let $f(x)$ be a piecewise smooth function on the interval $[-\pi, \pi]$. Then for any $x_0 \in [-\pi, \pi]$

$$\lim_{N\to\alpha} f_N(x_0) = f(x_0), \text{ if } f(x) \text{ is continuous at } x_0$$

$$= \frac{f(x_0-0) + f(x_0+0)}{2}, \text{ if } f(x) \text{ has a}$$

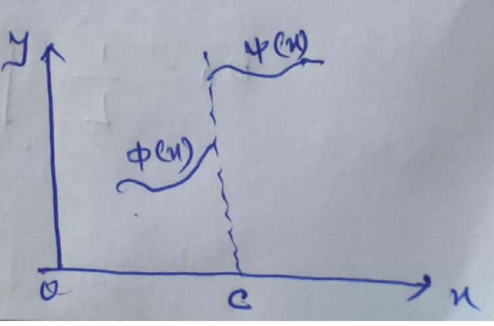
jump discontinuity at x_0,

where $f(x_0-0)$ and $f(x_0+0)$ represents the left limit and the right limit at the point x_0.

ie, For instance, if in the interval $[\alpha, \alpha+2\pi]$, $f(x)$ is defined by

$$f(x) = \phi(x), \ \alpha \leq x < c$$

$$= \psi(x), \ c < x \leq \alpha+2\pi$$



c is a point of jump discontinuity

Then,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\alpha}\left[a_n \cos nx + b_n \sin nx\right]$$

~~$\forall x \in [\alpha, \alpha+2\pi]$~~

$\forall x \in [\alpha, c) \cup (c, \alpha+2\pi]$

and at the point of discontinuity $x=c$, then Fourier series at this point gives

$$\frac{1}{2}\left[f(c-0) + f(c+0)\right].$$

where,
$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\,dx + \int_{c}^{\alpha+2\pi} \psi(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x) \cos nx\,dx + \int_{c}^{\alpha+2\pi} \psi(x) \cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x) \sin nx\,dx + \int_{c}^{\alpha+2\pi} \psi(x) \sin nx\,dx\right].$$

Fourier Series for even and odd function:

(1) When $f(x)$ is an even function

then, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$

Since $\cos nx$ is an even function, $f(x)$ is an even function

$f(x)$ is an even function.

$\Rightarrow$ product of two even function is an even function.

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

($\because$ Integrand is even)

Now, $\sin nx$ is an odd function, $f(x)$ is even function

$\Rightarrow$ product of odd and even is odd.

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \quad (\because \text{Integrand is odd})$$

Thus, if a function $f(x)$ is even in $[-\pi, \pi]$, its Fourier series expansion contains only cosine terms.
Hence Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx .$$

where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$, $n = 0, 1, 2, \dots$

(2) when $f(x)$ is an odd function

then, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0$$

[∵ $\cos nx$ is even and $f(x)$ is odd; product of even and odd function is odd]

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

[∵ $\sin nx$ is odd and $f(x)$ is odd; product of ~~odd~~ two odd function is even function]

Thus, if a function $f(x)$ is odd in $[-\pi, \pi]$, its Fourier series expansion contains only sine terms.
Fourier series expansion contains only sine terms, Fourier series is given by,
Hence Fouries series is given by

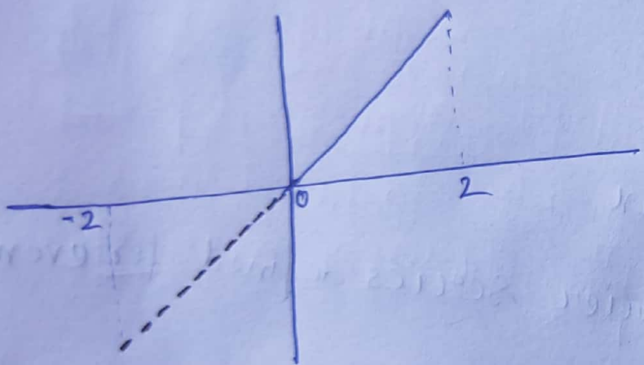
$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx.$

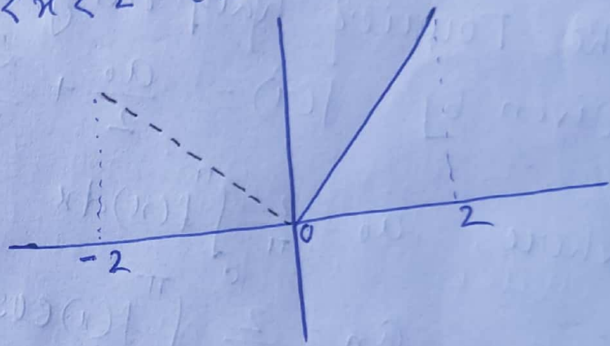
Half Range Series

Many a time it is required to obtain a Fourier expansion of a function $f(x)$ for the range $(0, c)$; which is half the period of the Fourier series. As it is immaterial whatever the function may be outside the range $0<x<c$, we extend the function to cover the range $-c<x<c$ so that the new function may be odd or even. In such cases the graphs for the values of x in $(0, c)$ are the same but outside $(0, c)$ are different for odd or even functions.

Example:- Extend $f(x) = x$, $0<x<2$ in $-2<x<2$.



Here we ~~exp~~ extend the given function in the interval $-2<x<2$ as the new function is an odd function.



Here we extend the give function in the interval $-2<x<2$ as the new function is an even function.

Half range Fourier Sine Series:

If it is required to expand $f(x)$ as a sine series in $0<x<c$; then we extend the function reflecting it in the origin, so that $f(x) = -f(-x)$.

Then the extended function is odd in $(-c, c)$ and the expansion will give the desired Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

where $$b_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} \, dx$$

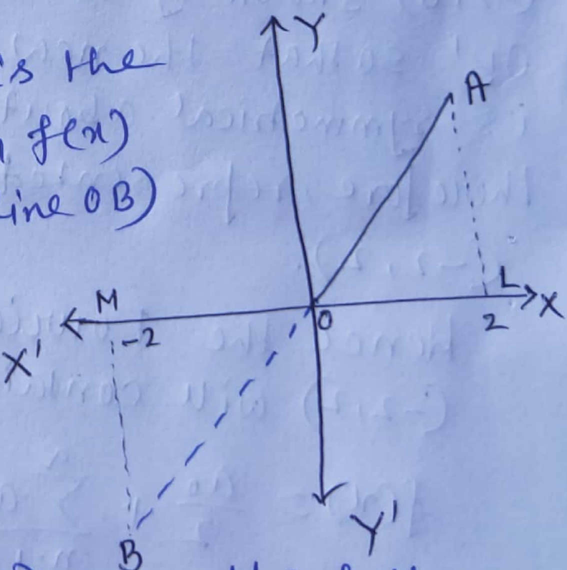
Half range Fourier cosine series:

If it be required to express $f(x)$ as a cosine series $0<x<c$, we extend the function reflecting it in the y-axis, so that $f(-x)=f(x)$.
Then the extended function is even in $(-c,c)$, and its expansion will give the required Fourier cosine series:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{c}$$

where
$$a_0=\frac{2}{c}\int_0^c f(x)\,dx$$
$$a_n=\frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

Example 1. Express $f(x)=x$ as a half-range sine series in $0<x<2$.



The graph of $f(x)=x$ in $0<x<2$ is the line OA. Let us extend the function $f(x)$ in the interval $-2<x<0$ (shown by line OB) so that new function is symmetrical about origin and, therefore, represented as an odd function in $(-2,2)$.
Hence the Fourier series for $f(x)$ over the full period $(-2,2)$ will contain sine terms only and given by,

$$f(x)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{2}.$$

where $$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \int_0^2 x \sin\frac{n\pi x}{2}\,dx$$

$$= \left[-x\frac{\cos\frac{n\pi x}{2}}{n\pi/2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2 = -\frac{4\cdot(-1)^n}{n\pi}.$$

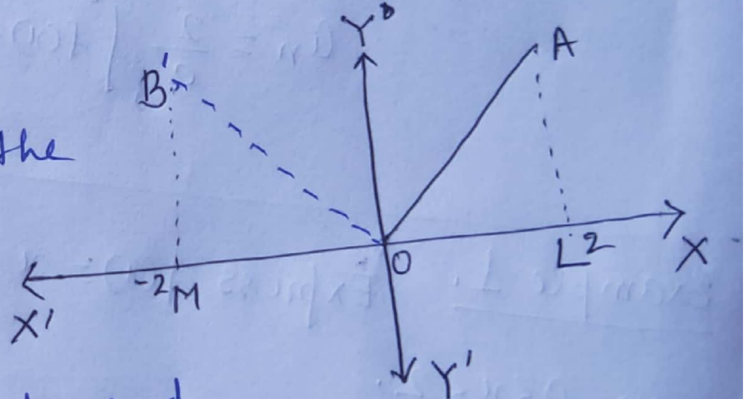
Thus $b_1 = 4/\pi$, $b_2 = -4/2\pi$, $b_3 = \frac{4}{3\pi}$, ...

Hence the Fourier sine series ~~is given by~~ for $f(x)$ over the half-range $(0,2)$ is

$$f(x) = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

<u>Example 2</u>

Express $f(x) = x$ as a half-range cosine series in $0<x<2$.



The graph of $f(x) = x$ in $(0,2)$ is the line OA. Let us extend the function $f(x)$ in the interval $(-2,0)$ shown by the line OB' so that the new function is symmetrical about the y axis and therefore, ~~representeds~~ represents an even function in $(-2,2)$.

Hence the Fourier series for $f(x)$ over the full period $(-2,2)$ will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

where $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 x\,dx = 2$

and $a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$

$= \int_0^2 x\cos\frac{n\pi x}{2}\,dx$

$= \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_0^2 = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right].$

Thus $a_1 = -8/\pi^2,\ a_2 = 0,\ a_3 = -\frac{8}{3^2\pi^2},\ a_4 = 0,\ a_5 = -\frac{8}{5^2\pi^2}$ etc.

Hence the desired Fourier cosine series for $f(x)$ over the half range $(0,2)$ is

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos \pi x/2}{1^2} + \frac{\cos 3\pi x/2}{3^2} + \frac{\cos 5\pi x/2}{5^2} + \cdots\right]$$

Example 3

Find the Fourier Series of the periodic function defined as $f(x) = \begin{cases} -\pi, & : -\pi < x < 0 \\ x; & 0 < x < \pi. \end{cases}$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

We know that, the Fourier series of $f(x)$ defined in the interval $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right]$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_0^{\pi} x\,dx\right]$

$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = -\pi/2$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_0^{\pi} f(x)\cos nx\,dx\right]$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)\Big|_{-\pi}^{0} + \left\{\frac{x\sin nx}{n} - \frac{1}{n}\frac{(-\cos nx)}{n}\right\}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}(0) + \left\{\left(\frac{\pi(0)}{n} + \frac{1}{n^2}(-1)^n\right) - \frac{1}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} + \frac{1}{n^2}\right] = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\Rightarrow a_n = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

(Routine calculation)

$= \dots\dots\dots$

$$= \frac{1}{\pi}(1 - 2\cos n\pi)$$

Hence, the Fourier series for given $f(x)$ is given by

$$f(x) = \frac{-\pi/2}{2} + \sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi}\left[(-1)^n - 1\right]\cos nx + \frac{1}{n}(1 - 2\cos n\pi)\sin nx\right\}$$

$$\Rightarrow f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right)$$
$$+ \left(3\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right)$$

Deduction!

Put $x=0$ in the above function $f(x)$ we get

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

Since, $f(x)$ is discontinuous at $x=0$,

$$\lim_{x\to 0^-} f(x) = f(0-0) = -\pi$$

$$\lim_{x\to 0^+} f(x) = f(0+0) = 0.$$

Now, $$f(0) = \frac{1}{2}\left[f(0+0)+f(0-0)\right]$$

$$\Rightarrow f(0) = \frac{1}{2}(-\pi) = -\pi/2$$

$$\therefore -\frac{\pi}{2} = -\frac{\pi}{4} + \frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$\Rightarrow \boxed{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}}.$$

Hence the result.