Sets & Functions - 2

Wednesday, September 23, 2020 8:48 AM

$A_i = \{1, 2, 3, \ldots\ldots i\}$ for $i = 1, 2, 3, \ldots\ldots$

$\bigcup_{i=1}^{\infty} A_i = \{1, 2, 3, \ldots\ldots\} = Z^+$ ← set of +ve integers.

$\bigcap_{i=1}^{\infty} A_i = \{1\}$

$A_1 = (1)$
$A_2 = (1, 2)$
$A_3 = (1, 2, 3)$
$A_n = (1, 2, 3 \ldots\ldots n)$

$f_1(x) = x^2$; $f_2(x) = x - x^2$ $\quad f_1, f_2 : R \to R.$

$f_1(x) + f_2(x) = (f_1 + f_2)(x)$

$= \cancel{x^2} + x - \cancel{x^2} = x$

$(f_1 f_2) x = f_1(x) . f_2(x) = x^2 . (x - x^2)$
$= x^3 - x^4.$

Prob

$f(x) = x^2 \quad f : Z \to Z.$

$f(1) = 1 = f(-1). \Rightarrow$ The function $f(x) = x^2$ is not one-to-one.

Prob

$f(x) = x+1 \qquad f: R \to R$

The function $f(x) = x+1$ is a one-to-one as $x+1 \neq y+1$ when $x \neq y$.

Prob. $f: \{a, b, c, d\} \to \{1, 2, 3\}$

$f(a) = 3, \; f(b) = 2, \; f(c) = 1, \; f(d) = 3$

All the elements in the co-domain has at least one preimage in the domain $\{a, b, c, d\}$.

Hence, f is onto function.

Prob. $f(x) = x^2 \; ; \; f: Z \to Z$

The function f is not onto because there is no integer with $x^2 = 5$.

Prob. $f(x) = x+1 \; ; \; f: Z \to Z$.

The function f is onto because for every integer y there is an integer x such that $y = x+1$.

Inverse function must be one-to-one correspondence $\Rightarrow$

Otherwise, some element in the codomain will be the image of more than one element in the domain.

'b' in the

image of more than one element in the ...

~~We cannot assign to each element 'b' in the codomain a unique 'a' in the domain s.t. $f(a)=b$.~~

If f is not onto, for some element b in the codomain, no element 'a' in the domain exists for which $f(a)=b$.

Prob. $f: \{a, b, c\} \to \{1, 2, 3\}$

$f(a)=2,\ f(b)=3,\ f(c)=1$

✓ This is one-to-one because $f(a) \neq f(b)$ for $a \neq b$.

✓ This is onto as every image has a corresponding pre-image.

✓ This is one-to-one correspondence and hence inverse function exists.

✓ $f^{-1}(1)=c,\ f^{-1}(2)=a,\ f^{-1}(3)=b.$

$$\boxed{\begin{array}{l} f(x)=y \\ f^{-1}(y)=\underline{x} \\ f^{-1}(y)=y-1 \end{array}}$$

Prob. $f: \mathbb{Z} \to \mathbb{Z},\ f(x)=x+1.$

✓ This function has an inverse because it is one-to-one correspondence. Suppose y is the image of ... Then $x=y-1$

correspondence. Suppose y is the image of x such that $y = x+1$. Then, $x = y-1$. This means that $y-1$ is the unique element of Z that is sent to y by f. Consequently, $f^{-1}(y) = y-1$.

Prob.

$f: R \to R$ and $f(x) = x^2$.

$f(-2) = f(2) = 4$, so f is not one-to-one.

Hence, f is not invertible

f: Set of nonnegative ~~integers~~ real numbers. to set of nonnegative integers

$f(x) = x^2 \longrightarrow$ invertible $\rightarrow$ one-to-one, $\rightarrow$ onto

Prob.

$g: \{a, b, c\} \to \{a, b, c\}$

$g(a) = b$, $g(b) = c$, and $g(c) = a$

$f: \{a, b, c\} \to \{1, 2, 3\}$

$f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

$f(a) = 3,\ f(b) = 2,$ and $f(c) = 1.$

$$f\circ g(a) = f(g(a)) = f(b) = 2$$

$$f\circ g(b) = f(g(b)) = f(c) = 1$$

$$f\circ g(c) = f(g(c)) = f(a) = 3.$$

Note $g\circ f$ is not defined, because the range of f is not a subset of the domain of g.

Prob:

$$f: \mathbb{Z} \to \mathbb{Z},\quad f(x) = 2x+3$$

$$g: \mathbb{Z} \to \mathbb{Z},\quad g(x) = 3x+2$$

(i) $$f\circ g(x) = f(g(x)) = f(3x+2)$$
$$= 2\cdot(3x+2)+3$$
$$= 6x+7.$$

(ii) $$g\circ f(x) = g(f(x)) = g(2x+3)$$
$$= 3(2x+3)+2$$
$$= 6x+11.$$

function and its inverse, in

Composition of a function and its inverse, in either order, generates the identity function.

→ Let $f: A \to B$ is a one-to-one correspondence from the set A to the set B. The inverse function f^{-1} exists and is given as $f^{-1}: B \to A$.

Thus, if $f(a) = b$ then $f^{-1}(b) = a$.

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$$

$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b.$$

Consequently, $f^{-1} \circ f = \iota_A$ and $f \circ f^{-1} = \iota_B$, where ι_A and ι_B are identity functions on the sets A & B respectively.

Prob: We need to determine the smallest integer that is at least as large as the quotient when 100 is divided by 8

Consequently, $\lceil 100/8 \rceil = \lceil 12.5 \rceil = 13$.

bytes are reqd.

Prob. **Proof:** $\lfloor x+n \rfloor = \lfloor x \rfloor + n$. (Property 4a)

Suppose that $\lfloor x \rfloor = m$, where m is a positive integer. By property (1a), $m \leq x < m+1$.

By adding 'n' to all these inequalities we get,

$$m+n \leq x+n < m+n+1.$$

Using property (1a) again, we get,

$$\lfloor x+n \rfloor = m+n.$$

$$\Rightarrow \lfloor x+n \rfloor = \lfloor x \rfloor + n.$$

Proof: $\lceil x+n \rceil = \lceil x \rceil + n$. (property-4b)

Suppose $\lceil x \rceil = m$, where m is a positive integer.

Using property (1b), it follows that,

$$m-1 < x \leq m.$$

Adding 'n' on all sides of the inequalities,

$$m+n-1 < x+n \leq m+n.$$

$$\Rightarrow \lceil x+n \rceil = m+n. \text{ (using property 1b)}$$

$$\Rightarrow \quad \lfloor x+n \rfloor = m+n. \ (\ldots$$

$$\Rightarrow \quad \lceil x+n \rceil = \lceil x \rceil + n.$$

To prove: $\lfloor -x \rfloor = -\lceil x \rceil$ (property 3a)

L.H.S.: Let us take $x = n + \epsilon$, where n is the integer and ϵ is the fractional part, such that $0 < \epsilon < 1$.

Using property 1a we get,

$$\lfloor n+\epsilon \rfloor \longleftrightarrow n \leq (n+\epsilon) < n+1.$$

Taking $-$ve sign on all sides of the inequation

$$-(n+1) \leq -(n+\epsilon) < -n.$$

Therefore,

$$\lfloor -(n+\epsilon) \rfloor = -(n+1)$$

i.e. $\lfloor -x \rfloor = -(n+1)$

R.H.S.:

Since, $x = n+\epsilon$, $0 < \epsilon < 1$, it follows:

$$\lceil n+\epsilon \rceil \longleftrightarrow n < (n+\epsilon) \leq n+1.$$

Therefore, $\lceil n+\epsilon \rceil = n+1.$

or, $\lceil x \rceil = n+1.$

or, $\lceil x \rceil = n+1.$

$$\Rightarrow -\lceil x \rceil = -(n+1).$$

Therefore, $\lfloor -x \rfloor = -\lceil x \rceil$ (property 3a)

To prove: $\lceil -x \rceil = -\lfloor x \rfloor$ (property 3b)

L.H.S: Let us take $x = n - \epsilon$, where n is the integer and ϵ is the fractional part of x such that $0 < \epsilon < 1$.

Therefore,

$$\lceil n - \epsilon \rceil \leftrightarrow n-1 < (n-\epsilon) \leq n.$$

Taking $-$ve sign on all sides of the inequation,

$$-n \leq -(n-\epsilon) < -(n-1)$$

Therefore, $\lceil -(n-\epsilon) \rceil = -(n-1)$

$$\Rightarrow \lceil -x \rceil = -(n-1).$$

R.H.S: Since, $x = n - \epsilon$ and $0 < \epsilon < 1$,

$$\lfloor n-\epsilon \rfloor \leftrightarrow (n-1) \leq (n-\epsilon) < n.$$

$$\Rightarrow \lfloor n-\epsilon \rfloor = (n-1)$$

$$\Rightarrow \lfloor n-\epsilon \rfloor = (n-1)$$

$$\Rightarrow -\lfloor n-\epsilon \rfloor = -(n-1)$$

$$\Rightarrow -\lfloor x \rfloor = -(n-1).$$

Hence, $\lceil -x \rceil = -\lfloor x \rfloor$ (property 3b).

Prob.

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor.$$

Let, $x = n + \epsilon$, where n is the integer and $0 \leq \epsilon < 1$. There are two cases:

Case: 1 $\quad 0 < \epsilon < 1/2$

In this case, $2x = 2n + 2\epsilon$. and $\lfloor 2x \rfloor = 2n$ because, $0 < 2\epsilon < 1$.

Similarly, $x + \frac{1}{2} = n + \epsilon + \frac{1}{2} = n + (\epsilon + \frac{1}{2})$.

Therefore, $\lfloor x + 1/2 \rfloor = n$. because, $0 < \frac{1}{2} + \epsilon < 1$.

Consequently, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = n + n = 2n$. Thus, $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

Case 2: $\quad \frac{1}{2} \leq \epsilon < 1.$

~~Case~~ $\frac{1}{2} \leq \epsilon < 1$.

In this case, $2x = 2n + 2\epsilon = (2n+1) + (2\epsilon - 1)$.

Because, $0 < 2\epsilon - 1 < 1$, it follows that,

$\lfloor 2x \rfloor = (2n+1)$.

Now, $x + \frac{1}{2} = n + \epsilon + \frac{1}{2} = (n+1) + (\epsilon - \frac{1}{2})$.

Consequently, $\lfloor x + \frac{1}{2} \rfloor = n+1$, because

$0 \leq \epsilon - \frac{1}{2} < 1$. ~~Cons~~

Therefore, $\lfloor 2x \rfloor = 2n+1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor =$

$n + n + 1 = 2n+1$. therefore, $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

To show: $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$. for all real numbers x and y.

→ Although this statements ~~seem~~ seems reasonable, it is false.

Let $x = 1/2$ and $y = 1/2$. Thus, $\lceil x+y \rceil =$

$\lceil 1/2 + 1/2 \rceil = \lceil 1 \rceil = 1$.

But, $\lceil x \rceil + \lceil y \rceil = \lceil 1/2 \rceil + \lceil 1/2 \rceil = 1 + 1 = 2$.