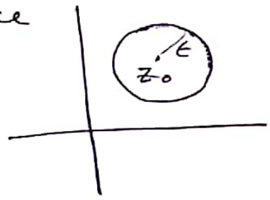
① # Complex Analysis

Argand plane: There is a one-one relation between each complex numbers and the points in a plane called complex plane or argand plane.

For example: $2+3i$ corresponds to the point $(2,3)$
$-5+i$ " " " " $(-5,1)$
2 " " " " $(2,0)$.

that is for each complex number, there is only one point in the complex plane.

Neighbourhood of a point: Let z_0 be a point in the complex plane. Then $|z-z_0|<\epsilon$ represents all the points within the circle with z_0 as centre and ϵ as radius, and is called a neighbourhood of z_0.



Complex function: Let $z=x+iy$ be a complex variable. Then $f(z)$ represents a complex function.

Ex $f(z)=z^2+2z$ also represents a complex

Ex $f(z)=u(x,y)+iv(x,y)$ are real valued function. Here $u(x,y)$ & $v(x,y)$ are real valued function of x & y.

Ex $f(z)=x^2+2xy+i(x^3y+x\cos y)$ is a complex function.

Limit & Continuity:

l is said to be the limit of $f(z)$ at $z=z_0$, if for any positive ϵ, there exists a positive δ, such that $|f(z)-l|<\epsilon$ whenever $0<|z-z_0|<\delta$ and we write

$$\lim_{z\to z_0} f(z)=l.$$

$f(z)$ is said to be continuous at z_0 if $\lim_{z\to z_0} f(z)=f(z_0)$.

② ## Derivative of a complex function.

A complex function $w=f(z)$ defined in a domain D is said to be derivable at $z=z_0$, if $\lim\limits_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists and the limit is called the derivative of $f(z)$ at $z=z_0$, denoted by $f'(z_0)$.

Ex1 Show that $f(z)=\bar{z}$ is continuous at $z=0$ but ~~not~~ $f'(0)$ does not exist.

Soln: $f(0)=0$. Now $|f(z)-0|=|\bar{z}-0|=|\bar{z}|=|z|=|z-0|$

Thus $|f(z)-0|<\epsilon$ whenever $|z-0|<\epsilon$.

Hence 0 is the limit of $f(z)$ at $z=0$ ie $\lim\limits_{z\to 0} f(z)=0=f(0)$

$\therefore f(z)$ is continuous at $z=0$.

Again $\lim\limits_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim\limits_{z\to 0} \frac{\bar{z}-0}{z-0} = \lim\limits_{z\to 0} \frac{\bar{z}}{z}$

Now when $z\to 0$ along the line $y=0$,

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x-iy}{x+iy} = \lim_{x\to 0} \frac{x}{x} \text{ along } y=0$$
$$=1$$

~~And~~ $$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x-iy}{x+iy} = \lim_{y\to 0} \frac{-iy}{iy} \text{ along } x=0$$
$$=-1$$

Thus $\lim\limits_{z\to 0} \frac{\bar{z}}{z}$ does not exist and hence $f(z)=\bar{z}$ is not derivable at $z=0$.

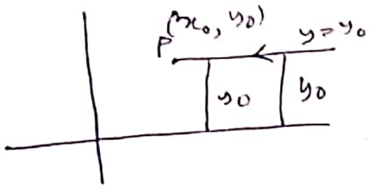
Analytic function

A function $f(z)$ is said to be analytic at a point $z=z_0$, if there exists a neighbourhood $|z-z_0|<\delta$ at all points of which $f'(z)$ exists. If $f(z)$ is analytic at each points of the domain D, then $f(z)$ is said to be analytic in D.

③ Cauchy-Riemann Equations (C-R equations)

The necessary condition for a complex valued function $W = f(z) = u(x,y) + iv(x,y)$ to be differentiable at $z_0 = x_0 + iy_0$ is that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ should exist at (x_0, y_0) and satisfies $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at (x_0, y_0).

Proof: Let $f(z)$ is differentiable at $z = z_0$.



Then the limit
$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

Let $f(z) = u(x,y) + iv(x,y)$

Then $$f'(z_0) = \lim_{(x,y) \to (x_0,y_0)} \frac{u(x,y) + iv(x,y) - u(x_0,y_0) - iv(x_0,y_0)}{x + iy - x_0 - iy_0} \text{ exists}$$

Let $z \to z_0$ along the line $y = y_0$. Then on the line $y = y_0$

$$f'(z_0) = \lim_{(x,y) \to (x_0,y_0)} \frac{u(x,y_0) + iv(x,y_0) - u(x_0,y_0) - iv(x_0,y_0)}{x + iy_0 - x_0 - iy_0}$$

$$= \lim_{x \to x_0} \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \text{ at } (x_0, y_0) \quad \text{---} (1), \text{ as } f'(z_0) \text{ exists.}$$

Again let ~~f(z)~~ $z \to z_0$ along the line $x = x_0$.

Then on the line $x = x_0$,

$$f'(z_0) = \lim_{y \to y_0} \frac{u(x_0,y) + iv(x_0,y) - u(x_0,y_0) - iv(x_0,y_0)}{x_0 + iy - x_0 - iy_0}$$

$$= \lim_{y \to y_0} \frac{u(x_0,y) - u(x_0,y_0)}{i(y - y_0)} + i \lim_{y \to y_0} \frac{v(x_0,y) - v(x_0,y_0)}{i(y - y_0)}$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \text{ at } (x_0, y_0) \quad \text{---} (2) \text{ as } f(z_0) \text{ exists}$$

From (1) and (2) we get $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ at (x_0, y_0)

The above two equations are called Cauchy-Riemann differential equations or shortly C-R equations.

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④

Ex1 Show that $f(z) = |z|^2$ is not derivable any where except at $z=0$.

$f(z) = |z|^2 = x^2+y^2$

$\therefore u(x,y) = x^2+y^2, \quad v(x,y) = 0$

$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$

Thus $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ except at $z=0$ i.e. C-R equations are not satisfied. Hence $f(z)$ is not ~~analytic~~ derivable anywhere except at $z=0$.

Ex2 Show that $f(z) = x+2i$ is nowhere differentiable.

Here $u(x,y) = x, \quad v(x,y) = 2$

$\therefore \frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 0$ i.e. $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ any where in the complex plane.

Thus $f(z) = x+2i$ is nowhere differentiable.

Ex3 Show that for the function $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \; z \neq 0$

$= 0, \quad z = 0$

$f'(0)$ does not exist, though C-R equations are satisfied at $(0,0)$.

Sol $u(x,y) = \frac{x^3-y^3}{x^2+y^2}, \; (x,y) \neq (0,0)$ & $v(x,y) = \frac{x^3+y^3}{x^2+y^2}, \; (x,y) \neq (0,0)$

$= 0, \quad (x,y) = (0,0) \qquad = 0, \quad (x,y) = (0,0).$

At $(0,0)$, $\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x-0} = \lim_{x\to 0} \frac{x-0}{x} = 1$

$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y-0} = \lim_{y\to 0} \frac{-y-0}{y} = -1$

$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x-0} = \lim_{x\to 0} \frac{x-0}{x} = 1$

$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y-0} = \lim_{y\to 0} \frac{y-0}{y} = 1$

Thus at $(0,0)$, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 1, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ i.e. C-R equations are satisfied.

Now $f'(0) = \lim_{(x,y)\to(0,0)} \frac{\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} - 0}{x+iy - 0} = \lim_{(x,y)\to(0,0)} \frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)(x+iy)}$

Now $\lim_{(x,y)\to(0,0)} \frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)(x+iy)} = 1+i$ along the line $y=0$

$= \frac{1+i-(1-i)}{2(1+i)}$ along $y=x$

$= \frac{i}{1+i}$ along $y=x$

$\therefore f'(0)$ does not exist.

⑤

Ex 1 Show that the function $f(z)$ defined by $f(z) = \frac{(\bar{z})^2}{z}$, $z \neq 0$ & $f(0) = 0$ is not differentiable at the origin though the C-R equations are satisfied at that point.

Soln $f(z) = \frac{(\bar{z})^2}{z} = \frac{(\bar{z})^3}{z\bar{z}} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3 - 3x^2(iy) + 3x(iy)^2 - (iy)^3}{x^2+y^2}$

$= \frac{x^3 - 3xy^2}{x^2+y^2} + i\frac{(y^3 - 3x^2y)}{x^2+y^2}$ for $(x,y) \neq (0,0)$

$\therefore u(x,y) = \frac{x^3-3xy^2}{x^2+y^2}$ & $v(x,y) = \frac{y^3-3x^2y}{x^2+y^2}$ for $(x,y) \neq (0,0)$

and both are 0 for $(x,y) = (0,0)$.

At $(0,0)$, $\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{\frac{h^3-0}{h^2+0}}{h-0} = \lim_{h\to 0} \frac{h}{h} = 1$

$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{\frac{0}{k^2}}{k-0} = 0$

$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{\frac{0}{h^2}}{h-0} = 0$

$\frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{\frac{k^3-0}{0+k^2}}{k-0} = 1$

$\therefore$ At $(0,0)$, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

ie C-R equations are satisfied.

Now $f'(0) = \lim_{z\to 0} \frac{\frac{\bar{z}^2}{z} - 0}{z - 0} = \lim_{z\to 0} \frac{\bar{z}^2}{z^2} = \lim_{(x,y)\to(0,0)} \frac{(x-iy)^2}{(x+iy)^2}$

Along the path $y = mx$, $f'(0) = \lim_{x\to 0} \frac{(x-imx)^2}{(x+imx)^2} = \frac{(1-im)^2}{(1+im)^2}$

Which is different for different values of m.

Thus $f'(0)$ does not exist.

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