CHAPTER 6

LAPLACE TRANSFORM

6.1 INTRODUCTION

Laplace transform method is very useful since it provides easy and effective ways for the solution of many problems arising in various fields of science and technology. The crucial idea which make the Laplace transform a very powerful method is that it replaces operations of calculus by operations of algebra.

In this chapter, we study Laplace transform, its properties as well as its inverse and finally its application in solving linear ordinary differential equations with constant coefficients.

6.2 LAPLACE TRANSFORM

Definition: Let $f(t)$ be a given function which is defined for all $t \geq 0$. We multiply $f(t)$ by e^{-st}, where s is a parameter (real or complex) independent of t, and integrate with respect to t from 0 to ∞. Then if the resulting integral exists (*i.e.*, has some finite value), it is a function of s, say, $F(s)$:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

This function $F(s)$ of the variable s is called the **Laplace transform** of the function $f(t)$ and is also denoted by $L\{f(t)\}$ or $\bar{f}(s)$.

If the above improper integral does not exist, we say the Laplace transform of $f(t)$ does not exist.

Note: In this chapter, we always take $s > 0$.

Linearity Property of Laplace Transform

A transformation T is called linear if for every pair of functions $f_1(t)$ and $f_2(t)$ and for every pair of constants c_1 and c_2, we have

$$T\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 T\{f_1(t)\} + c_2 T\{f_2(t)\}.$$

Theorem 1: Laplace transform is a linear transformation, *i.e.*,

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\},$$ where c_1, c_2 are constants.

Proof: We have
$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

$$\therefore \quad L\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}\{c_1 f_1(t) + c_2 f_2(t)\}\,dt$$

$$= c_1 \int_0^\infty e^{-st} f_1(t)\,dt + c_2 \int_0^\infty e^{-st} f_2(t)\,dt$$

$$= c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}.$$

Note: This result can be easily generalised.

Existence of Laplace Transform

Theorem 2: Let a function $f(t)$ satisfies the following two conditions:

(*i*) For every $b > 0$, the interval $[0, b]$ can be broken up into a finite number of subintervals in each of which $f(t)$ is continuous and $f(t)$ approaches definite limit as t approaches either end points of these subintervals from the interior.

(*ii*) $|e^{-at} f(t)| \leq M$, or $|f(t)| \leq M e^{at}$, for all $t \geq 0$ and for some constants a and $M (> 0)$.

Then the Laplace transform of $f(t)$ exists (*i.e.*, convergent) in the domain $Re(s) > a$.

Proof: We have
$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

Now,
$$\left| \int_0^\infty e^{-st} f(t)\,dt \right| \leq \int_0^\infty |e^{-st} f(t)|\,dt = \int_0^\infty |e^{-st}|\,|f(t)|\,dt$$

$$\leq \int_0^\infty |e^{-(x+iy)t}|\,M e^{at}\,dt \qquad \text{[Putting } s = x + iy]$$

$$= M \int_0^\infty e^{-t(x-a)}\,dt,$$

which is convergent if $x > a$, *i.e.*, $Re(s) > a$.

6.3 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

(*i*) $L(1) = \dfrac{1}{s}, \; s > 0$

(*ii*) $L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}, \; n > -1, \; s > 0$

(iii) $L(e^{at}) = \dfrac{1}{s-a}, \; s > a$

(iv) $L(\sin at) = \dfrac{a}{s^2+a^2}, \; s > 0$

(v) $L(\cos at) = \dfrac{s}{s^2+a^2}, \; s > 0$

(vi) $L(\sinh at) = \dfrac{a}{s^2-a^2}, \; s > |a|$

(vii) $L(\cosh at) = \dfrac{s}{s^2-a^2}, \; s > |a|$

Proof: (i)
$$L(1) = \int_0^{\infty} e^{-st} \cdot 1 dt = \lim_{B\to\infty} \int_0^{B} e^{-st}\, dt$$
$$= \lim_{B\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^B = \lim_{B\to\infty} \frac{1}{s}(1-e^{-sB})$$
$$= \frac{1}{s} \qquad (\because s>0)$$

(ii)
$$L(t^n) = \int_0^{\infty} e^{-st}\, t^n\, dt = \int_0^{\infty} e^{-st}\, t^{(n+1)-1}\, dt$$
$$= \frac{\Gamma(n+1)}{s^{n+1}}, \; n > -1, \; s > 0$$
$$\left[\because \int_0^{\infty} e^{-ax} x^{n-1} dx = \frac{\Gamma(n)}{a^n}, \; a > 0, \; n > 0. \text{ See Relation 1, Art. 5.5, Chapter-5}\right]$$

(iii)
$$L(e^{at}) = \int_0^{\infty} e^{-st} e^{at}\, dt = \int_0^{\infty} e^{-(s-a)t}\, dt = \lim_{B\to\infty} \int_0^{B} e^{-(s-a)t}\, dt$$
$$= \lim_{B\to\infty} \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^B = \lim_{B\to\infty} \frac{1}{s-a}\left[1-e^{-(s-a)B}\right]$$
$$= \frac{1}{s-a}, \text{ for } s > a.$$

(iv)
$$L(\sin at) = \int_0^{\infty} e^{-st} \sin at\, dt = \lim_{B\to\infty} \int_0^{B} e^{-st} \sin at\, dt$$

$$= \lim_{B\to\infty}\left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^B$$

$$= \lim_{B\to\infty}\left[\frac{a}{s^2+a^2} - \frac{e^{-sB}}{s^2+a^2}(s\sin aB + a\cos aB)\right]$$

$$= \frac{a}{s^2+a^2}$$

$[\because e^{-sB} \to 0 \text{ as } B\to\infty, \text{ because } s>0]$

$$L(\cos at) = \int_0^\infty e^{-st}\cos at\, dt = \lim_{B\to\infty}\int_0^B e^{-st}\cos at\, dt$$

$$= \lim_{B\to\infty}\left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^B$$

$$= \lim_{B\to\infty}\left[\frac{s}{s^2+a^2} - \frac{e^{-sB}}{s^2+a^2}(s\cos aB - a\sin aB)\right]$$

$$= \frac{s}{s^2+a^2} \qquad [\because e^{-sB} \to 0 \text{ as } B\to\infty, \text{ because } s>0]$$

(vi)

$$L(\sinh at) = \int_0^\infty e^{-st}\sinh at\, dt = \int_0^\infty e^{-st}\frac{1}{2}(e^{at}-e^{-at})\,dt$$

$$= \lim_{B\to\infty}\frac{1}{2}\int_0^B\{e^{-(s-a)t} - e^{-(s+a)t}\}\,dt$$

$$= \lim_{B\to\infty}\frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^B$$

$$= \lim_{B\to\infty}\frac{1}{2}\left[\frac{1}{s-a}\{1-e^{-(s-a)B}\} - \frac{1}{s+a}\{1-e^{-(s+a)B}\}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$$

$[\because e^{-(s-a)B} \to 0,\ e^{-(s+a)B} \to 0 \text{ as } B\to\infty, \text{ because } s>|a|]$

$$= \frac{a}{s^2-a^2}, \text{ for } s>|a|.$$

(vii)

$$L(\cosh at) = \int_0^\infty e^{-st}\cosh at\, dt = \int_0^\infty e^{-st}\frac{1}{2}(e^{at}+e^{-at})\,dt$$

$$= \lim_{B\to\infty} \frac{1}{2}\int_0^B \{e^{-(s-a)t} + e^{-(s+a)t}\}\,dt$$

$$= \lim_{B\to\infty} \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^B$$

$$= \lim_{B\to\infty} \frac{1}{2}\left[\frac{1}{s-a}\{1-e^{-(s-a)B}\} + \frac{1}{s+a}\{1-e^{-(s+a)B}\}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)$$

[$\because e^{-(s-a)B} \to 0,\ e^{-(s+a)B} \to 0$ as $B \to \infty$, because $s > |a|$]

$$= \frac{s}{s^2-a^2},\ \text{for } s > |a|$$

Theorem 3: (First shifting property)

If $L\{f(t)\} = \bar{f}(s)$ then $L\{e^{at} f(t)\} = \bar{f}(s-a)$.

Proof: By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$$

$$\therefore \quad L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= \int_0^\infty e^{-pt} f(t)\,dt \qquad \text{[where } p = s-a]$$

$$= \bar{f}(p) = \bar{f}(s-a) \qquad [\because p = s-a]$$

Note: If we know the Laplace transform $\bar{f}(s)$ of $f(t)$, we can write the Laplace transform of $e^{at} f(t)$ simply replacing s by $s-a$ to get $\bar{f}(s-a)$.

Applying this property to (*ii*), (*iv*) – (*vii*) we get the following useful results:

(*viii*) $L(e^{at} t^n) = \dfrac{\Gamma(n+1)}{(s-a)^{n+1}},\ s-a>0,\ n>-1$

(*ix*) $L(e^{at} \sin bt) = \dfrac{b}{(s-a)^2+b^2},\ s-a>0$

(x) $L(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2},\ s-a>0$

(xi) $L(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2-b^2},\ s-a>|b|$

(xii) $L(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2-b^2},\ s-a>|b|.$

Table of Laplace Transforms

No.	$f(t)$	$L\{f(t)\}$
1.	1	$\frac{1}{s},\ s>0$
2.	$t^n,\ n>-1$	$\frac{\Gamma(n+1)}{s^{n+1}},\ n>-1,\ s>0$
3.	e^{at}	$\frac{1}{s-a},\ s>a$
4.	$e^{at}t^n,\ n>-1$	$\frac{\Gamma(n+1)}{(s-a)^{n+1}},\ n>-1,\ s>a$
5.	$\sin bt$	$\frac{b}{s^2+b^2},\ s>0$
6.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2},\ s-a>0$
7.	$\cos bt$	$\frac{s}{s^2+b^2},\ s>0$
8.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2},\ s-a>0$
9.	$\sinh bt$	$\frac{b}{s^2-b^2},\ s>|b|$
10.	$e^{at}\sinh bt$	$\frac{b}{(s-a)^2-b^2},\ s-a>|b|$
11.	$\cosh bt$	$\frac{s}{s^2-b^2},\ s>|b|$
12.	$e^{at}\cosh bt$	$\frac{s-a}{(s-a)^2-b^2},\ s-a>|b|$

Using the Laplace transforms of the functions listed in the previous table, nearly all the Laplace transforms can be obtained by applying the general theorems which we shall consider later on.

Two useful standard integrals

1. $\int e^{ax} \cos bx\, dx = \dfrac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

2. $\int e^{ax} \sin bx\, dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

Theorem 4: (Second shifting property)

If $L\{f(t)\} = \bar{f}(s)$ and $g(t) = \begin{cases} f(t-a), & \text{for } t > a \\ 0, & \text{for } t < a \end{cases}$ then $L\{g(t)\} = e^{-as}\bar{f}(s)$.

Proof: By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$$

$$\therefore \quad L\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt = \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt = \lim_{B\to\infty}\int_a^B e^{-st} f(t-a)\,dt$$

$$= \lim_{B\to\infty}\int_0^{B-a} e^{-s(x+a)} f(x)\,dx$$

[Putting $t-a=x$, so, $t=x+a$ and $dt=dx$]

$$= e^{-as}\int_0^\infty e^{-sx} f(x)\,dx = e^{-as}\int_0^\infty e^{-st} f(t)\,dt$$

$$= e^{-as}\bar{f}(s).$$

Theorem 5: (Change of scale property)

If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \dfrac{1}{a}\bar{f}\left(\dfrac{s}{a}\right)$, $a > 0$.

Proof: By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s).$$

$$\therefore \quad L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt = \lim_{B\to\infty}\int_0^B e^{-st} f(at)\,dt$$

$$\lim_{B\to\infty}\int_0^{aB} e^{-\frac{s}{a}x} f(x)\frac{dx}{a} \qquad \left[\text{Put } at = x, \text{ so, } dt = \frac{dx}{a}\right]$$

$$= \frac{1}{a}\int_0^{\infty} e^{-\frac{s}{a}x} f(x)\,dx \qquad [\text{Here } a > 0, \text{ so, } aB \to \infty \text{ as } B \to \infty]$$

$$= \frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

ILLUSTRATIVE EXAMPLES

Example 1: Find the Laplace transforms of

(i) $t^k, k > 0$ (W.B.U.T. 2002)

(ii) $f(t) = \begin{cases} 1, & \text{if } 0 < t < 2 \\ 2, & \text{if } t > 2 \end{cases}$ (W.B.U.T. 2002)

(iii) $f(t) = \begin{cases} e^t, & 0 < t \le 1 \\ 0, & t > 1 \end{cases}$ (W.B.U.T. 2003)

(iv) $f(t) = \begin{cases} \sin\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t \le \frac{\pi}{3} \end{cases}$ (W.B.U.T. 2004)

(v) $f(t) = at + b$ (W.B.U.T. 2005)

(vi) $f(t) = \begin{cases} 0, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}$ (W.B.U.T. 2005)

(vii) $f(t) = e^t \sin t \cos t$ (W.B.U.T. 2006)

(viii) $f(t) = \begin{cases} 1, & \text{if } t > \alpha \\ 0, & \text{if } t < \alpha \end{cases}$ (W.B.U.T. 2006)

where α is any positive real number.

(ix) $f(t) = \dfrac{\sin(at)}{t}$, given that $L\left\{\dfrac{\sin t}{t}\right\} = \tan^{-1}\left(\dfrac{1}{s}\right)$ (W.B.U.T. 2007)

(x) $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$ (W.B.U.T. 2008)

Solution: (*i*) By definition,

$$L\{t^k\} = \int_0^\infty e^{-st}\, t^k\, dt = \lim_{B\to\infty}\int_0^B e^{-st}\, t^k\, dt$$

$$= \lim_{B\to\infty}\int_0^{sB} e^{-x}\,\frac{x^k}{s^k}\,\frac{dx}{s} \qquad \left[\text{Putting } st = x, \text{ so } t = \frac{x}{s},\ dt = \frac{dx}{s}\right]$$

$$= \frac{1}{s^{k+1}}\int_0^\infty e^{-x}\, x^{(k+1)-1}\, dx \qquad [\text{Here } sB \to \infty \text{ as } B \to \infty \text{ since } s > 0]$$

$$= \frac{\Gamma(k+1)}{s^{k+1}} = \frac{k!}{s^{k+1}}, \text{ if } k \text{ is a positive integer.}$$

(*ii*) By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st}\cdot 1\,dt + \int_2^\infty e^{-st}\cdot 2\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^2 + \lim_{B\to\infty} 2\int_2^B e^{-st}\,dt$$

$$= \frac{e^{-2s}}{-s} + \frac{1}{s} + 2\lim_{B\to\infty}\left[\frac{e^{-st}}{-s}\right]_2^B$$

$$= -\frac{e^{-2s}}{s} + \frac{1}{s} + 2\lim_{B\to\infty}\left[\frac{e^{-sB}}{-s} + \frac{e^{-2s}}{s}\right]$$

$$= -\frac{e^{-2s}}{s} + \frac{1}{s} + 2\left[0 + \frac{e^{-2s}}{s}\right] \qquad [\text{Here } e^{-sB} \to 0 \text{ as } B \to \infty \text{ since } s > 0]$$

$$= \frac{1}{s} + \frac{e^{-2s}}{s}.$$

(*iii*) By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}\cdot e^t\,dt + \int_1^\infty e^{-st}\cdot 0\,dt$$

$$= \int_0^1 e^{(1-s)t}\,dt = \left[\frac{e^{(1-s)t}}{1-s}\right]_0^1 = \frac{e^{(1-s)}}{1-s} - \frac{1}{1-s}$$

$$= \frac{e^{(1-s)} - 1}{1-s}.$$

(*iv*) Let $\varphi(t) = \sin t.$

$$\therefore \quad f(t) = \begin{cases} \varphi\left(t-\frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t \le \frac{\pi}{3} \end{cases}$$

We know that

$$L\{\varphi(t)\} = L\{\sin t\} = \frac{1}{s^2+1} = \bar{\varphi}(s).$$

Therefore, using second shifting property, we have

$$L\{f(t)\} = e^{-\left(\frac{\pi}{3}\right)s}\bar{\varphi}(s) = \frac{e^{-\pi s/3}}{s^2+1},\ s>0.$$

(*v*)
$$L\{f(t)\} = L\{at+b\} = aL\{t\}+bL\{1\}$$

$$= a\frac{\Gamma(2)}{s^2}+\frac{b}{s} \qquad \left[\because L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}, n>-1 \text{ and } L\{1\} = \frac{1}{s}\right]$$

$$= \frac{a}{s^2}+\frac{b}{s} \qquad [\because \Gamma(2)=\Gamma(1+1)=1!=1]$$

$$= \frac{1}{s^2}(a+bs)$$

(*vi*) By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^1 e^{-st}\cdot 0\,dt + \int_1^2 e^{-st}\cdot t\,dt + \int_2^\infty e^{-st}\cdot 0\,dt$$

$$= \int_1^2 te^{-st}\,dt = \left[-\frac{t}{s}e^{-st}\right]_1^2 + \frac{1}{s}\int_1^2 e^{-st}\,dt \qquad \text{[By integration by parts]}$$

$$= -\frac{1}{s}(2e^{-2s}-e^{-s})+\frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_1^2$$

$$= -\frac{1}{s}(2e^{-2s}-e^{-s})-\frac{1}{s^2}(e^{-2s}-e^{-s})$$

$$= \left(\frac{1}{s}+\frac{1}{s^2}\right)e^{-s}-\left(\frac{1}{s^2}+\frac{2}{s}\right)e^{-2s}.$$

(*vii*)
$$L\{e^t \sin t\cos t\} = \frac{1}{2}L\{e^t \sin 2t\}$$

We know that

$$L\{\sin 2t\} = \frac{2}{s^2+2^2}$$

Using first shifting property, we have

$$L\{e^t \sin 2t\} = \frac{2}{(s-1)^2+4}.$$

$$\therefore \quad L\{e^t \sin t \cos t\} = \frac{1}{2} L\{e^t \sin 2t\} = \frac{1}{(s-1)^2+4}$$

(*viii*) By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\alpha e^{-st}\cdot 0\,dt + \int_\alpha^\infty e^{-st}\cdot 1\,dt$$

$$= \lim_{B\to\infty} \int_\alpha^B e^{-st}\,dt = \lim_{B\to\infty}\left[\frac{e^{-st}}{-s}\right]_\alpha^B$$

$$= -\frac{1}{s}\lim_{B\to\infty}(e^{-sB} - e^{-s\alpha})$$

$$= -\frac{1}{s}(0 - e^{-s\alpha}) \qquad [\text{Here } e^{-sB} \to 0 \text{ as } B\to\infty \text{ since } s>0]$$

$$= \frac{1}{se^{s\alpha}}.$$

(*ix*) 'Change of scale property' states that if $L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$.

Since $\quad L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s},$

therefore $\quad L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\tan^{-1}\frac{1}{\frac{s}{a}} = \frac{1}{a}\tan^{-1}\frac{a}{s}$

or $\quad \frac{1}{a}L\left\{\frac{\sin at}{t}\right\} = \frac{1}{a}\tan^{-1}\frac{a}{s}.$

$$\therefore \quad L\left\{\frac{\sin(at)}{t}\right\} = \tan^{-1}\frac{a}{s}.$$

(*x*) By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\pi e^{-st}\sin t\,dt + \int_\pi^\infty e^{-st}\cdot 0\,dt$$

$$= \int_0^\pi e^{-st}\sin t\,dt = I \text{ (say)}.$$

$$\therefore \quad I = [-e^{-st}\cos t]_0^{\pi} - \int_0^{\pi} e^{-st}(-s)(-\cos t)\,dt \qquad \text{[By integration by parts]}$$

$$= e^{-s\pi} + 1 - s\int_0^{\pi} e^{-st}\cos t\,dt$$

$$= e^{-s\pi} + 1 - s\left[\{e^{-st}\sin t\}_0^{\pi} - \int_0^{\pi} e^{-st}(-s)\sin t\,dt\right]$$

[By integration by parts]

$$= e^{-s\pi} + 1 - s^2\int_0^{\pi} e^{-st}\sin t\,dt$$

$$\therefore \quad I = e^{-s\pi} + 1 - s^2 I$$

$$(1+s^2)I = 1 + e^{-s\pi}$$

$$\therefore \quad I = L\{f(t)\} = \frac{1+e^{-s\pi}}{1+s^2}.$$

Example 2: Find the Laplace transforms of

(*i*) $\sin 3t \cos 2t$

(*ii*) $\sin^2 2t$

(*iii*) $\cos^3 2t$

Solution: (*i*) Since

$$\sin 3t \cos 2t = \frac{1}{2}(\sin 5t + \sin t),$$

Therefore
$$L(\sin 3t \cos 2t) = \frac{1}{2}\{L(\sin 5t) + L(\sin t)\}$$

$$= \frac{1}{2}\left\{\frac{5}{s^2+5^2} + \frac{1}{s^2+1^2}\right\} = \frac{3s^2+15}{(s^2+25)(s^2+1)}$$

(*ii*) Since

$$\sin^2 2t = \frac{1}{2}(1-\cos 4t),$$

Therefore
$$L(\sin^2 2t) = \frac{1}{2}\{L(1) - L(\cos 4t)\}$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4^2}\right\}$$

(*iii*) Now,

$$\cos 6t = 4\cos^3 2t - 3\cos 2t$$

$$\therefore \quad \cos^3 2t = \frac{1}{4}\cos 6t + \frac{3}{4}\cos 2t$$

$$\therefore \quad L(\cos^3 2t) = \frac{1}{4}L(\cos 6t) + \frac{3}{4}L(\cos 2t)$$

$$= \frac{1}{4}\cdot\frac{s}{s^2+6^2} + \frac{3}{4}\cdot\frac{s}{s^2+2^2}$$

Example 3: Find the Laplace transforms of

(*i*) $e^{-3t}(2\cos 5t - 3\sin 5t)$

(*ii*) $e^{2t}\cos^2 t$

(*iii*) $e^{4t}\sin 3t\cos t$

(*iv*) $(t+3)^2 e^t$

(*v*) $e^{-t}(3\sinh 2t - 5\cosh 2t)$

(*vi*) $\cosh 2t \sin t$

Solution: (*i*) We have

$$L\{2\cos 5t - 3\sin 5t\} = 2L(\cos 5t) - 3L(\sin 5t)$$

$$= 2\frac{s}{s^2+5^2} - 3\cdot\frac{5}{s^2+5^2}$$

$$= \frac{2s-15}{s^2+25}.$$

Therefore, by first shifting property, we have

$$L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = \frac{2(s+3)-15}{(s+3)^2+25}$$

$$= \frac{2s-9}{s^2+6s+34}.$$

(*ii*) Now,

$$L(\cos^2 t) = \frac{1}{2}L(1+\cos 2t)$$

$$= \frac{1}{2}\{L(1) + L(\cos 2t)\}$$

$$= \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2+2^2}\right\}$$

Therefore, by first shifting property, we have

$$L\{e^{2t}\cos^2 t\} = \frac{1}{2}\left\{\frac{1}{s-2} + \frac{s-2}{(s-2)^2+4}\right\}$$

(*iii*) Now,

$$L\{\sin 3t\cos t\} = \frac{1}{2}L\{\sin 4t+\sin 2t\}$$
$$= \frac{1}{2}\{L(\sin 4t)+L(\sin 2t)\}$$
$$= \frac{1}{2}\left\{\frac{4}{s^2+4^2}+\frac{2}{s^2+2^2}\right\}$$

Therefore, by first shifting property, we have

$$L\{e^{4t}\sin 3t\cos t\} = \frac{1}{2}\left\{\frac{4}{(s-4)^2+16}+\frac{2}{(s-4)^2+4}\right\}.$$

(*iv*) Now,

$$L\{(t+3)^2\} = L\{t^2+6t+9\}$$
$$= L(t^2)+6L(t)+9L(1)$$
$$= \frac{\Gamma(3)}{s^3}+6\frac{\Gamma(2)}{s^2}+9\cdot\frac{1}{s} \qquad \left[\because L(t^n)=\frac{\Gamma(n+1)}{s^{n+1}},\ n>-1\right]$$
$$= \frac{2!}{s^3}+\frac{6}{s^2}+\frac{9}{s} = \frac{1}{s^3}(2+6s+9s^2).$$

Therefore, by first shifting property, we have

$$L\{(t+3)^2e^t\} = \frac{1}{(s-1)^3}\{2+6(s-1)+9(s-1)^2\}$$
$$= \frac{9s^2-12s+5}{(s-1)^3}.$$

(*v*) Now,

$$L\{3\sinh 2t-5\cosh 2t\} = 3L(\sinh 2t)-5L(\cosh 2t)$$
$$= 3\cdot\frac{2}{s^2-2^2}-5\cdot\frac{s}{s^2-2^2} = \frac{6-5s}{s^2-4}.$$

Therefore, by first shifting property, we have

$$L\{e^{-t}(3\sinh 2t-5\cosh 2t)\} = \frac{6-5(s+1)}{(s+1)^2-4} = \frac{1-5s}{s^2+2s-3}.$$

(*vi*)
$$L\{\cosh 2t\sin t\} = L\left\{\frac{1}{2}(e^{2t}+e^{-2t})\sin t\right\}$$
$$= \frac{1}{2}\{L(e^{2t}\sin t)+L(e^{-2t}\sin t)\}$$

We know that

$$L\{\sin t\} = \frac{1}{s^2+1^2}.$$

Therefore, by first shifting property, we have

$$L\{\cosh 2t \sin t\} = \frac{1}{2}\left\{\frac{1}{(s-2)^2+1} + \frac{1}{(s+2)^2+1}\right\}.$$

Example 4: Show that

(*i*) $L(t \sin at) = \dfrac{2as}{(s^2+a^2)^2}$

(*ii*) $L(t \cos at) = \dfrac{s^2-a^2}{(s^2+a^2)^2}$

Solution: We know that

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}, n > -1$$

$$\therefore \quad L(t) = \frac{\Gamma(2)}{s^2} = \frac{1}{s^2}.$$

Therefore, by first shifting property, we have

$$L(te^{iat}) = \frac{1}{(s-ia)^2} = \frac{(s+ia)^2}{\{(s-ia)(s+ia)\}^2}$$

or $$L\{t(\cos at + i \sin at)\} = \frac{(s^2-a^2)+i(2as)}{(s^2+a^2)^2}.$$

Equating real and imaginary parts from both sides, we get

$$L(t \cos at) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

and $$L(t \sin at) = \frac{2as}{(s^2+a^2)^2}.$$

Example 5: Find the Laplace transform of $f(t)$ defined by

$$f(t) = \begin{cases} t/\tau, & \text{when } 0 < t < \tau \\ 1, & \text{when } t > \tau \end{cases}.$$

Solution: By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\tau e^{-st}\frac{t}{\tau}\,dt + \int_\tau^\infty e^{-st}\cdot 1\,dt$$

$$= \frac{1}{\tau}\left[\left\{t\cdot\frac{e^{-st}}{-s}\right\}_0^\tau - \int_0^\tau 1\cdot\frac{e^{-st}}{-s}\,dt\right] + \lim_{B\to\infty}\int_\tau^B e^{-st}\,dt$$

$$= \frac{1}{\tau}\left[\frac{\tau e^{-s\tau}}{-s} - \left\{\frac{e^{-st}}{s^2}\right\}_0^\tau\right] + \lim_{B\to\infty}\left[\frac{e^{-st}}{-s}\right]_\tau^B$$

$$= -\frac{e^{-s\tau}}{s} - \frac{(e^{-s\tau}-1)}{\tau s^2} + \lim_{B\to\infty} \frac{1}{s}(e^{-s\tau} - e^{-sB})$$

$$= -\frac{e^{-s\tau}}{s} + \frac{1-e^{-s\tau}}{\tau s^2} + \frac{e^{-s\tau}}{s} \quad [\text{Here } e^{-sB} \to 0 \text{ as } B\to\infty, \text{ since } s>0]$$

$$= \frac{1-e^{-s\tau}}{\tau s^2}.$$

Example 6: If $L\{f(t)\} = \dfrac{p^2-p+1}{(2p+1)^2(p-1)}$, apply the change of scale property to show that

$$L\{f(2t)\} = \frac{p^2-2p+4}{4(p+1)^2(p-2)}.$$

(W.B.U.T. 2002)

Solution: 'Change of scale property' states that if

$$L\{f(t)\} = \bar{f}(s), \text{ then } L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

Since
$$L\{f(t)\} = \frac{p^2-p+1}{(2p+1)^2(p-1)},$$

therefore
$$L\{f(2t)\} = \frac{1}{2}\,\frac{\left(\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right) + 1}{\left(2\cdot\frac{p}{2}+1\right)^2\left(\frac{p}{2}-1\right)}$$

$$= \frac{p^2-2p+4}{4(p+1)^2(p-2)}.$$

6.4 LAPLACE TRANSFORM OF $t^n f(t)$, WHERE n IS A POSITIVE INTEGER

Theorem 6: If $L\{f(t)\} = \bar{f}(s)$, then $L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n}\bar{f}(s)$, where n is a positive integer.

(W.B.U.T. 2005)

Proof: By definition,

$$\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \quad \frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

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$$= \int_0^\infty \frac{\partial}{\partial s}\{e^{-st} f(t)\}\, dt \qquad \text{[By Leibnitz's rule]}$$

$$= \int_0^\infty - te^{-st} f(t)\, dt$$

$$\therefore \quad \int_0^\infty e^{-st}\{tf(t)\}\, dt = -\frac{d}{ds}\bar{f}(s).$$

Therefore, the theorem is true for $n = 1$. Let us assume that the theorem is true for a positive integer m.

$$\therefore \quad \int_0^\infty e^{-st}\{t^m f(t)\}\, dt = (-1)^m \frac{d^m}{ds^m}\bar{f}(s) \qquad \text{[By assumption]}$$

Then
$$\frac{d}{ds}\int_0^\infty e^{-st}\{t^m f(t)\}\, dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}\bar{f}(s)$$

or
$$\int_0^\infty \frac{\partial}{\partial s}\{e^{-st}\, t^m f(t)\}\, dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}\bar{f}(s) \qquad \text{[By Leibnitz's rule]}$$

or
$$\int_0^\infty (-te^{-st})\, t^m f(t)\, dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}\bar{f}(s)$$

$$\therefore \quad \int_0^\infty e^{-st}\{t^{m+1} f(t)\}\, dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}\bar{f}(s).$$

Therefore, if the theorem is true for a positive integer m, then it is also true for $m + 1$. But the theorem is true for $n = 1$, hence it is true for $1 + 1 = 2$, $2 + 1 = 3$, $3 + 1 = 4$ and so on.

Thus the theorem is true for all positive integral values of n.

ILLUSTRATIVE EXAMPLES

Example 1: Find the Laplace transforms of

(*i*) $t \cos at$

(*ii*) $t^2 \sin at$

(*iii*) $t^4 e^{-3t}$

(*iv*) $t e^{-2t} \sin 3t$

Solution: (*i*) We know that

$$L(\cos at) = \frac{s}{s^2 + a^2}.$$

$$\therefore \quad L(t \cos at) = -\frac{d}{ds}\left(\frac{s}{s^2 + a^2}\right) = -\frac{(s^2 + a^2 - s \cdot 2s)}{(s^2 + a^2)^2}$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}.$$

(ii) We know that

$$L(\sin at) = \frac{a}{s^2 + a^2}.$$

$$\therefore \quad L(t^2 \sin at) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2 + a^2}\right) = \frac{d}{ds}\left\{\frac{-2as}{(s^2 + a^2)^2}\right\}$$

$$= \frac{-2a(s^2 + a^2)^2 + 8as^2(s^2 + a^2)}{(s^2 + a^2)^4}$$

$$= \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}.$$

(iii) We know that

$$L(e^{-3t}) = \frac{1}{s+3}.$$

$$\therefore \quad L(t^4 e^{-3t}) = (-1)^4 \frac{d^4}{ds^4}(s+3)^{-1} = (-1)^4 \frac{(-1)^4\, 4!}{(s+3)^5}$$

$$= \frac{24}{(s+3)^5}.$$

(iv) We know that

$$L(\sin 3t) = \frac{3}{s^2 + 3^2}.$$

$$\therefore \quad L(t \sin 3t) = -\frac{d}{ds}\left(\frac{3}{s^2 + 3^2}\right) = \frac{6s}{(s^2 + 9)^2}.$$

Therefore, by first shifting property, we get

$$L(t e^{-2t} \sin 3t) = \frac{6(s+2)}{\{(s+2)^2 + 9\}^2} = \frac{6(s+2)}{(s^2 + 4s + 13)^2}.$$

Example 2: Find the Laplace transforms of

(i) $t(3\sin 2t - 2\cos 2t)$ (ii) $\sin at - at\cos at$.

Solution: (i) We have

$$L(3\sin 2t - 2\cos 2t) = 3L(\sin 2t) - 2L(\cos 2t)$$

$$= 3 \cdot \frac{2}{s^2 + 2^2} - 2 \cdot \frac{s}{s^2 + 2^2} = \frac{6 - 2s}{s^2 + 4}.$$

$$\therefore \quad L\{t(3\sin 2t - 2\cos 2t)\} = -\frac{d}{ds}\left(\frac{6 - 2s}{s^2 + 4}\right) = \frac{8 + 12s - 2s^2}{(s^2 + 4)^2}.$$

(ii) $$L(\sin at - at\cos at) = L(\sin at) - aL(t\cos at)$$

$$= \frac{a}{s^2+a^2} - a\cdot(-1)\frac{d}{ds}\{L(\cos at)\}$$

$$= \frac{a}{s^2+a^2} + a\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{a}{s^2+a^2} + \frac{a(a^2-s^2)}{(s^2+a^2)^2} = \frac{2a^3}{(s^2+a^2)^2}.$$

Example 3: Find the Laplace transform of

$$(1+te^{-t})^3.$$

Solution:

$$L\{(1+te^{-t})^3\} = L\{1+3te^{-t}+3t^2e^{-2t}+t^3e^{-3t}\}$$

$$= L(1)+3L(te^{-t})+3L(t^2e^{-2t})+L(t^3e^{-3t})$$

$$= L(1)+3(-1)\frac{d}{ds}\{L(e^{-t})\}+3(-1)^2\frac{d^2}{ds^2}\{L(e^{-2t})\}+(-1)^3\frac{d^3}{ds^3}\{L(e^{-3t})\}$$

$$= \frac{1}{s}-3\frac{d}{ds}\left(\frac{1}{s+1}\right)+3\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right)-\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)$$

$$= \frac{1}{s}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}.$$

Example 4: Evaluate $\int_0^\infty e^{-3t}t\sin t\,dt$.

Solution: We have,

$$\int_0^\infty e^{-st}t\sin t\,dt = L(t\sin t) = -\frac{d}{ds}\{L(\sin t)\}$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}.$$

Putting $s=3$, we have

$$\int_0^\infty e^{-3t}t\sin t\,dt = \frac{6}{(3^2+1)^2} = \frac{3}{50}.$$

Example 5: Evaluate $\int_0^\infty t^3e^{-t}\sin t\,dt$.

Solution: We have,

$$\int_0^\infty e^{-st}t^3\sin t\,dt = L\{t^3\sin t\} = (-1)^3\frac{d^3}{ds^3}L(\sin t)$$

$$= -\frac{d^3}{ds^3}\left(\frac{1}{s^2+1^2}\right) = \frac{d^2}{ds^2}\left\{\frac{2s}{(s^2+1)^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{(s^2+1)^2 \cdot 2 - 2s \cdot 2(s^2+1)2s}{(s^2+1)^4}\right\}$$

$$= \frac{d}{ds}\left\{\frac{2(1-3s^2)}{(s^2+1)^3}\right\}$$

$$= 2 \cdot \frac{(s^2+1)^3 \cdot (-6s) - (1-3s^2)3(s^2+1)^2 \cdot 2s}{(s^2+1)^6}$$

$$= 2 \cdot \frac{-6s(s^2+1) - 6s(1-3s^2)}{(s^2+1)^4} = \frac{24s(s^2-1)}{(s^2+1)^4}$$

Putting $s = 1$, we get

$$\int_0^\infty e^{-t}\, t^3 \sin t \, dt = 0.$$

6.5 LAPLACE TRANSFORM OF $\frac{1}{t} f(t)$

Theorem 7: If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty \bar{f}(s)\, ds$, provided the integral exists.

Proof: By definition,

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Integrating both sides with respect to s from s to ∞, we get

$$\int_s^\infty \bar{f}(s)\, ds = \int_s^\infty \{\int_0^\infty e^{-st} f(t)\, dt\}\, ds$$

$$= \int_0^\infty \{\int_s^\infty e^{-st} f(t)\, ds\}\, dt \quad \text{[Changing the order of integration]}$$

$$= \int_0^\infty \{\int_s^\infty e^{-st}\, ds\} f(t)\, dt \quad [\because t \text{ is independent of } s]$$

Now,
$$\int_s^\infty e^{-st}\, ds = \lim_{B\to\infty} \int_s^B e^{-st}\, ds = \lim_{B\to\infty} \left[\frac{e^{-st}}{-t}\right]_s^B$$

$$= \lim_{B\to\infty} \frac{1}{t}(e^{-st} - e^{-Bt})$$

$$= \frac{e^{-st}}{t} \qquad \text{(Here } e^{-Bt} \to 0 \text{ as } B\to\infty, \text{ since } t>0)$$

$$\therefore \quad \int_s^{\infty} \bar{f}(s)\,ds = \int_0^{\infty} e^{-st}\frac{f(t)}{t}\,dt = L\left\{\frac{1}{t}f(t)\right\}.$$

ILLUSTRATIVE EXAMPLES

Example 1: Find the Laplace transforms of

(W.B.U.T. 2004)

L (*i*) $\frac{1}{t}(1-e^t)$

L (*ii*) $\frac{1}{t}(\cos at - \cos bt)$

Solution: (*i*) We have

$$L(1-e^t) = L(1) - L(e^t)$$

$$= \frac{1}{s} - \frac{1}{s-1}.$$

$$\therefore \quad L\left(\frac{1-e^t}{t}\right) = \int_s^{\infty}\left(\frac{1}{s}-\frac{1}{s-1}\right)ds = \lim_{B\to\infty}\int_s^{B}\left(\frac{1}{s}-\frac{1}{s-1}\right)ds$$

$$= \lim_{B\to\infty}\left[\log|s| - \log|s-1|\right]_s^B$$

$$= \lim_{B\to\infty}\left[\log\left|\frac{s}{s-1}\right|\right]_s^B$$

$$= \lim_{B\to\infty}\left[\log\left|\frac{B}{B-1}\right| - \log\left|\frac{s}{s-1}\right|\right]$$

$$= -\log\left|\frac{s}{s-1}\right| \qquad \left[\because \left|\frac{B}{B-1}\right| = \frac{1}{\left|1-\frac{1}{B}\right|} \to 1 \text{ as } B\to\infty\right]$$

$$= \log\left|\frac{s-1}{s}\right|.$$

(*ii*) We have

$$L(\cos at - \cos bt) = L(\cos at) - L(\cos bt)$$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}.$$

$$\therefore \quad L\left\{\frac{1}{t}(\cos at - \cos bt)\right\} = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \lim_{B\to\infty} \int_s^B \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \lim_{B\to\infty} \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^B$$

$$= \lim_{B\to\infty} \left[\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^B$$

$$= \lim_{B\to\infty} \left[\frac{1}{2}\log\left(\frac{B^2+a^2}{B^2+b^2}\right) - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \lim_{B\to\infty} \left[\frac{1}{2}\log\left(\frac{1+\frac{a^2}{B^2}}{1+\frac{b^2}{B^2}}\right) - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= -\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) \qquad \left[\because \frac{a^2}{B^2}, \frac{b^2}{B^2} \to 0 \text{ as } B\to\infty\right]$$

$$= \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2}.$$

Example 2: Find the Laplace transform of $\frac{\sin at}{t}$. Hence show that $\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$.

(W.B.U.T. 2005, 2007)

Solution: We have

$$L(\sin at) = \frac{a}{s^2+a^2}.$$

$$\therefore \quad L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2+a^2}\, ds = \lim_{B\to\infty} \int_s^B \frac{a}{s^2+a^2}\, ds$$

$$= \lim_{B\to\infty} \left[\tan^{-1}\frac{s}{a}\right]_s^B = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right).$$

Putting $a = 1$, we get

$$L\left(\frac{\sin t}{t}\right) = \cot^{-1}(s) \qquad \dots(1)$$

By definition of Laplace transform, equation (1) means

$$\int_0^\infty e^{-st}\left(\frac{\sin t}{t}\right) dt = \cot^{-1}(s)$$

As $s \to 0+$, we have

$$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}.$$

Example 3: Evaluate $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt.$

Solution: We have

$$L(e^{-at} - e^{-bt}) = L(e^{-at}) - L(e^{-bt})$$

$$= \frac{1}{s+a} - \frac{1}{s+b}.$$

$$\therefore \quad L\left\{\frac{1}{t}(e^{-at} - e^{-bt})\right\} = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$\therefore \quad \int_0^\infty e^{-st} \cdot \frac{1}{t}(e^{-at} - e^{-bt})\, dt = \lim_{B\to\infty} \int_s^B \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \lim_{B\to\infty} [\log(s+a) - \log(s+b)]_s^B$$

$$= \lim_{B\to\infty} \left[\log \frac{s+a}{s+b}\right]_s^B = \lim_{B\to\infty} \left[\log \frac{B+a}{B+b} - \log \frac{s+a}{s+b}\right]$$

$$= -\log\left(\frac{s+a}{s+b}\right) \qquad \left[\because \log \frac{B+a}{B+b} = \log \frac{1+\frac{a}{B}}{1+\frac{b}{B}} \to \log 1 = 0 \text{ as } B\to\infty\right]$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

As $s \to 0+$, we have

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt = \log\left(\frac{b}{a}\right).$$

Example 4: Show that Laplace transform of $\frac{\cos at}{t}$ does not exist.

Solution: We have

$$L(\cos at) = \frac{s}{s^2+a^2}.$$

$$\therefore \quad L\left(\frac{\cos at}{t}\right) = \int_s^\infty \frac{s}{s^2+a^2}\,ds = \lim_{B\to\infty}\int_s^B \frac{s}{s^2+a^2}\,ds$$

$$= \lim_{B\to\infty}\left[\frac{1}{2}\log(s^2+a^2)\right]_s^B$$

$$= \lim_{B\to\infty}\left[\frac{1}{2}\log(B^2+a^2)-\frac{1}{2}\log(s^2+a^2)\right].$$

This limit does not exist since $\log(B^2+a^2)\to\infty$ as $B\to\infty$.

Hence, Laplace transform of $\frac{\cos at}{t}$ does not exist.

Example 5: Evaluate $L\left\{e^{-3t}\,\frac{\sin 2t}{t}\right\}$.

Solution: We have

$$L(\sin 2t) = \frac{2}{s^2+2^2}.$$

$$\therefore \quad L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty \frac{2}{s^2+2^2}\,ds = \lim_{B\to\infty} 2\int_s^B \frac{ds}{s^2+2^2}$$

$$= \lim_{B\to\infty}\left[\frac{2}{2}\tan^{-1}\frac{s}{2}\right]_s^B = \lim_{B\to\infty}\left(\tan^{-1}\frac{B}{2}-\tan^{-1}\frac{s}{2}\right)$$

$$= \frac{\pi}{2}-\tan^{-1}\frac{s}{2} = \cot^{-1}\frac{s}{2}.$$

Using first shifting property, we have

$$L\left\{e^{-3t}\,\frac{\sin 2t}{t}\right\} = \cot^{-1}\left(\frac{s+3}{2}\right).$$

Example 6: Evaluate $\int_0^\infty \frac{e^{-4t}\sin^2 t}{t}\,dt.$

Solution: Now,

$$\int_0^\infty e^{-st}\frac{1}{t}\sin^2 t\,dt = L\left\{\frac{1}{t}\sin^2 t\right\}$$

$$\sin^2 t = \frac{1}{2}(1-\cos 2t)$$

$$\therefore \quad L(\sin^2 t) = \frac{1}{2}L(1-\cos 2t) = \frac{1}{2}\{L(1)-L(\cos 2t)\}$$

$$= \frac{1}{2}\left\{\frac{1}{s}-\frac{s}{s^2+2^2}\right\}$$

$$\therefore \quad L\left\{\frac{1}{t}\sin^2 t\right\} = \int_s^\infty \frac{1}{2}\left\{\frac{1}{s}-\frac{s}{s^2+4}\right\}ds$$

$$= \lim_{B\to\infty}\int_s^B \frac{1}{2}\left\{\frac{1}{s}-\frac{s}{s^2+4}\right\}ds$$

$$= \lim_{B\to\infty}\frac{1}{2}\left[\log s-\frac{1}{2}\log(s^2+4)\right]_s^B$$

$$= \lim_{B\to\infty}\frac{1}{4}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^B$$

$$= \lim_{B\to\infty}\frac{1}{4}\left[\log\left(\frac{B^2}{B^2+4}\right)-\log\left(\frac{s^2}{s^2+4}\right)\right]$$

$$= -\frac{1}{4}\log\left(\frac{s^2}{s^2+4}\right)$$

$$\left[\because \log\left(\frac{B^2}{B^2+4}\right) = \log\left(\frac{1}{1+\frac{4}{B^2}}\right) \to \log 1 = 0 \text{ as } B\to\infty\right]$$

$$= \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right).$$

$$\therefore \quad \int_0^\infty e^{-st}\frac{1}{t}\sin^2 t\,dt = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

Putting $s = 4$, we get

$$\int_0^\infty e^{-4t}\frac{1}{t}\sin^2 t\,dt = \frac{1}{4}\log\left(\frac{4^2+4}{4^2}\right) = \frac{1}{4}\log\left(\frac{5}{4}\right).$$

6.6 LAPLACE TRANSFORMS OF DERIVATIVES

Theorem 8:

(*i*) If $f(t)$ be continuous and $L\{f(t)\} = \bar{f}(s)$, then $L\{f'(t)\} = -f(0) + s\bar{f}(s)$.

(*ii*) If $f(t)$ and its first $(n-1)$ derivatives be continuous, then

$$L\{f^n(t)\} = -f^{n-1}(0) - sf^{n-2}(0) - \ldots - s^{n-1}f(0) + s^n\bar{f}(s).$$

(It is assumed that $e^{-sB}f^m(B) \to 0$ as $B \to \infty$, for $m = 0, 1, 2, \ldots, n-1$)

Proof: (*i*) By definition,

$$L\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt = \lim_{B\to\infty}\int_0^B e^{-st}f'(t)\,dt$$

$$= \lim_{B\to\infty}\left[\{e^{-st}f(t)\}_0^B + s\int_0^B e^{-st}f(t)\,dt\right] \qquad \text{[Integrating by parts]}$$

$$= \lim_{B\to\infty}\{e^{-sB}f(B) - f(0)\} + s\int_0^\infty e^{-st}f(t)\,dt$$

Now assume $f(t)$ to be such that

$$\lim_{B\to\infty} e^{-sB}f(B) = 0.$$

$$\therefore \quad L\{f'(t)\} = -f(0) + s\bar{f}(s).$$

(*ii*) Applying the result of (*i*), we have

$$L\{f''(t)\} = L\left\{\frac{d}{dt}f'(t)\right\} = -f'(0) + sL\{f'(t)\}$$

$$= -f'(0) + s\{-f(0) + s\bar{f}(s)\}$$

$$= -f'(0) - sf(0) + s^2\bar{f}(s).$$

Similarly,

$$L\{f'''(t)\} = -f''(0) - sf'(0) - s^2f(0) + s^3\bar{f}(s).$$

In general,

$$L\{f^n(t)\} = -f^{n-1}(0) - sf^{n-2}(0) - \ldots - s^{n-1}f(0) + s^n\bar{f}(s),$$

where we assume that

$$\lim_{t\to\infty} e^{-st}f^m(t) = 0,$$

for $m = 0, 1, 2, \ldots, n-1$.

Note: (*i*) If $f(0) = f'(0) = f''(0) = \ldots = 0$, we get

$$L\{f'(t)\} = sL\{f(t)\},\ L\{f''(t)\} = s^2L\{f(t)\},\ L\{f'''(t)\}$$

$$= s^3L\{f(t)\}, \ldots, L\{f^n(t)\} = s^nL\{f(t)\}.$$

(*ii*) These results are very useful to solve differential equations.

(*iii*) Laplace transforms of functions can also be derived by the application of Theorem 8 (see Example 2).

ILLUSTRATIVE EXAMPLES

Example 1: Given $f(t) = t+1, 0 \le t \le 2$ and $f(t) = 3,\ t > 2$, find $L\{f(t)\}$ and $L\{f'(t)\}$.

Solution: By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st}(t+1)\,dt + \int_2^\infty e^{-st}\cdot 3\,dt$$

$$= \left[(t+1)\left(\frac{e^{-st}}{-s}\right) - \frac{e^{-st}}{s^2}\right]_0^2 + \lim_{B\to\infty} 3\int_2^B e^{-st}\,dt$$

$$= -\frac{3e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s} + \frac{1}{s^2} + \lim_{B\to\infty} 3\left[\frac{e^{-st}}{-s}\right]_2^B$$

$$= -\frac{3e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s} + \frac{1}{s^2} + \lim_{B\to\infty} 3\left[-\frac{e^{-Bs}}{s} + \frac{e^{-2s}}{s}\right]$$

$$= -\frac{3e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s} + \frac{1}{s^2} + \frac{3e^{-2s}}{s}$$

[Here $e^{-Bs} \to 0$ as $B \to \infty$, since $s>0$]

$$= \frac{1}{s} + \frac{1}{s^2}(1-e^{-2s}).$$

Now we know that

$$L\{f'(t)\} = -f(0) + sL\{f(t)\}.$$

Here $f(0) = 1$.

$$\therefore \quad L\{f'(t)\} = -1 + s\left\{\frac{1}{s} + \frac{1}{s^2}(1-e^{-2s})\right\}$$

$$= \frac{1}{s}(1-e^{-2s}).$$

Example 2: Find Laplace transform of $f(t) = t\sin at$ using the theorem of Laplace transform of derivative.

Solution: Here

$$f'(t) = \sin at + at\cos at$$

$$f''(t) = a\cos at + a(\cos at - at\sin at)$$

Also

$$= 2a\cos at - a^2 t\sin at.$$

Using the result

$$f(0) = 0,\ f'(0) = 0.$$

$$L\{f''(t)\} = -f'(0) - sf(0) + s^2 L\{f(t)\},$$

Theorem 9: If $L\{f(t)\} = \overline{f}(s)$, then $L\{\int_0^t f(u)\,du\} = \frac{1}{s}\overline{f}(s)$.

Proof: Let $g(t) = \int_0^t f(u)\,du$, then

$$g'(t) = f(t) \text{ and } g(0) = 0 \qquad \ldots(1)$$

We know that

$$L\{g'(t)\} = -g(0) + sL\{g(t)\}$$

$\therefore$

$$L\{f(t)\} = sL\{\int_0^t f(u)\,du\} \qquad \text{[By (1)]}$$

i.e.,

$$L\{\int_0^t f(u)\,du\} = \frac{1}{s}L\{f(t)\} = \frac{1}{s}\overline{f}(s).$$

ILLUSTRATIVE EXAMPLES

Theorem 9: If $L\{f(t)\} = \overline{f}(s)$, then $L\{\int_0^t f(u)\,du\} = \frac{1}{s}\overline{f}(s)$.

Proof: Let $g(t) = \int_0^t f(u)\,du$, then

$$g'(t) = f(t) \text{ and } g(0) = 0 \qquad \ldots(1)$$

We know that

$$L\{g'(t)\} = -g(0) + sL\{g(t)\}$$

$$\therefore \quad L\{f(t)\} = sL\{\int_0^t f(u)\,du\} \qquad \text{[By (1)]}$$

i.e.,
$$L\{\int_0^t f(u)\,du\} = \frac{1}{s}L\{f(t)\} = \frac{1}{s}\overline{f}(s).$$

Solution: (*i*) We know that

$$L(\sin 2t) = \frac{2}{s^2+2^2}.$$

$$\therefore \quad L\{\int_0^t \sin 2u\,du\} = \frac{1}{s}\cdot\frac{2}{s^2+4} = \frac{2}{s(s^2+4)}.$$

(*ii*) We know that

$$L(\cosh t) = \frac{s}{s^2-1^2}$$

$$\therefore \quad L(t\cosh t) = -\frac{d}{ds}\left(\frac{s}{s^2-1}\right) = \frac{s^2+1}{(s^2-1)^2}$$

$$\therefore \quad L\{\int_0^t u\cosh u\,du\} = \frac{1}{s}L(t\cosh t) = \frac{s^2+1}{s(s^2-1)^2}.$$

(*iii*) We know that

$$L(\sin t) = \frac{1}{s^2+1^2}$$

$$\therefore \quad L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}\,ds = \lim_{B\to\infty}\int_s^B \frac{1}{s^2+1}\,ds$$

$$= \lim_{B\to\infty}[\tan^{-1}s]_s^B = \lim_{B\to\infty}\{\tan^{-1}B - \tan^{-1}s\}$$

$$= \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s.$$

$$\therefore \quad L\left(\frac{\sin t}{t}\right) = \cot^{-1}s. \qquad \text{...(I)}$$

Using first shifting property, we have

$$L\left(e^t\,\frac{\sin t}{t}\right) = \cot^{-1}(s-1).$$

$$\therefore \quad L\left\{\int_0^t e^u\,\frac{\sin u}{u}\,du\right\} = \frac{1}{s}\cot^{-1}(s-1).$$

(*iv*) We have

$$L\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}L\left\{\frac{\sin t}{t}\right\}$$

$$= \frac{1}{s}\cot^{-1}s \qquad \text{[By (I)]}$$

Using first shifting property, we have

$$L\left\{e^{-t}\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s+1}\cot^{-1}(s+1).$$

(*v*) From (1), we have

$$L\left(\frac{\sin t}{t}\right) = \cot^{-1} s.$$

Therefore, by first shifting property, we have

$$L\left(e^{-t}\frac{\sin t}{t}\right) = \cot^{-1}(s+1).$$

$$\therefore \quad L\left\{\int_0^t e^{-u}\frac{\sin u}{u}\,du\right\} = \frac{1}{s}L\left(e^{-t}\frac{\sin t}{t}\right) = \frac{1}{s}\cot^{-1}(s+1).$$

Hence $\quad L\left\{t\int_0^t e^{-u}\frac{\sin u}{u}\,du\right\} = -\frac{d}{ds}\left\{\frac{1}{s}\cot^{-1}(s+1)\right\}$

$$= -\frac{s\cdot\left\{\dfrac{-1}{1+(s+1)^2}\right\}-\cot^{-1}(s+1)}{s^2}$$

$$= \frac{s+(s^2+2s+2)\cot^{-1}(s+1)}{s^2(s^2+2s+2)}.$$

(*vi*) We have

$$L(1-e^{-t}) = L(1)-L(e^{-t}) = \frac{1}{s}-\frac{1}{s+1}$$

$$\therefore \quad L\left(\frac{1-e^{-t}}{t}\right) = \int_s^\infty\left(\frac{1}{s}-\frac{1}{s+1}\right)ds = \lim_{B\to\infty}\int_s^B\left(\frac{1}{s}-\frac{1}{s+1}\right)ds$$

$$= \lim_{B\to\infty}[\log s-\log(s+1)]_s^B = \lim_{B\to\infty}\left[\log\frac{s}{s+1}\right]_s^B$$

$$= \lim_{B\to\infty}\left[\log\frac{B}{B+1}-\log\frac{s}{s+1}\right]$$

$$\left[\because \frac{B}{B+1}=\frac{1}{1+\frac{1}{B}}\to 1 \text{ as } B\to\infty\right]$$

$$= -\log\frac{s}{s+1}$$

$$= \log\frac{s+1}{s} = \log\left(1+\frac{1}{s}\right).$$

$$\therefore \quad L\left\{\int_0^t \frac{1-e^{-u}}{u}\,du\right\} = \frac{1}{s}L\left(\frac{1-e^{-t}}{t}\right) = \frac{1}{s}\log\left(1+\frac{1}{s}\right).$$

(*vii*) We know that

$$L(\sin t) = \frac{1}{s^2+1^2}$$

$$\therefore \quad L(t\sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$$\therefore \quad L\{\int_0^t (u\sin u)\,du\} = \frac{1}{s}L(t\sin t) = \frac{1}{s}\cdot\frac{2s}{(s^2+1)^2}.$$

So,

$$L\{\int_0^t\int_0^t (u\sin u)\,du\,du\} = L[\int_0^t\{\int_0^t (u\sin u)\,du\}\,du]$$

$$= \frac{1}{s}L\{\int_0^t (u\sin u)\,du\} = \frac{1}{s}\cdot\frac{1}{s}\cdot\frac{2s}{(s^2+1)^2}$$

$$= \frac{2}{s(s^2+1)^2}.$$

Hence,

$$L\{\int_0^t\int_0^t\int_0^t (u\sin u)\,du\,du\,du\} = L[\int_0^t\{\int_0^t\int_0^t (u\sin u)\,du\,du\}\,du]$$

$$= \frac{1}{s}L\{\int_0^t\int_0^t (u\sin u)\,du\,du\}$$

$$= \frac{1}{s}\cdot\frac{2}{s(s^2+1)^2} = \frac{2}{s^2(s^2+1)^2}.$$

Example 2: Prove that

$$\int_{t=0}^{\infty}\int_{u=0}^{t}\frac{e^{-t}\sin u}{u}\,du\,dt = \frac{\pi}{4}.$$

Solution: From (1) of Example 1, (*iii*), we get

$$L\left(\frac{\sin t}{t}\right) = \cot^{-1} s.$$

$$\therefore \quad L\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}L\left(\frac{\sin t}{t}\right) = \frac{1}{s}\cot^{-1} s. \quad ...(1)$$

By definition,

$$L\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \int_{t=0}^{\infty} e^{-st}\left\{\int_{u=0}^{t}\frac{\sin u}{u}\,du\right\}dt$$

$$= \int_{t=0}^{\infty}\int_{u=0}^{t}\frac{e^{-st}\sin u}{u}\,du\,dt \qquad \ldots(2)$$

From (1) and (2), we get

$$\int_{t=0}^{\infty}\int_{u=0}^{t}\frac{e^{-st}\sin u}{u}\,du\,dt = \frac{1}{s}\cot^{-1} s$$

Putting $s = 1$, we get

$$\int_{t=0}^{\infty}\int_{u=0}^{t}\frac{e^{-t}\sin u}{u}\,du\,dt = \cot^{-1} 1 = \frac{\pi}{4}.$$

6.8 LAPLACE TRANSFORMS OF PERIODIC FUNCTIONS

Theorem 10: If $f(t)$ is a periodic function with period $T(>0)$, *i.e.*, $f(t+T) = f(t)$, then

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt.$$

Proof: By definition,

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \ldots \text{ to } \infty.$$

In the second integral put $t = u + T$, in the third integral put $t = u + 2T$ and so on, then

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du + \ldots \text{ to } \infty.$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du + \ldots \text{ to } \infty$$

$[\because\ f(u) = f(u+T) = f(u+2T) \text{ etc.}]$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-2sT}\int_0^T e^{-st} f(t)\,dt + \ldots \text{ to } \infty$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots \text{ to } \infty)\int_0^T e^{-st} f(t)\,dt = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

ILLUSTRATIVE EXAMPLES

Example 1: Find the Laplace transform of a periodic function $f(t)$ given by

$$f(t) = t \quad \text{for} \quad 0 < t < c$$
$$= 2c - t \quad \text{for} \quad c < t < 2c$$

(W.B.U.T. 2003)

Solution: Here $f(t)$ is a periodic function with period $2c$.

$$\therefore \quad L\{f(t)\} = \frac{1}{1-e^{-2sc}}\int_0^{2c} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2sc}}\left\{\int_0^c t\,e^{-st}\,dt + \int_c^{2c}(2c-t)e^{-st}\,dt\right\}$$

$$= \frac{1}{1-e^{-2sc}}\left[\left\{\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right\}_0^c + \left\{(2c-t)\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2}\right\}_c^{2c}\right]$$

$$= \frac{1}{1-e^{-2sc}}\left\{\frac{ce^{-sc}}{-s} - \frac{e^{-sc}}{s^2} + \frac{1}{s^2} + \frac{e^{-2sc}}{s^2} + \frac{ce^{-sc}}{s} - \frac{e^{-sc}}{s^2}\right\}$$

$$= \frac{(1-e^{-sc})^2}{s^2(1-e^{-2sc})} = \frac{1-e^{-sc}}{s^2(1+e^{-sc})}.$$

Example 2: Find the Laplace transform of the function

$$f(t) = \sin\omega t, \quad 0 < t < \pi/\omega$$
$$= 0, \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}.$$

Solution: Since $f(t)$ is a periodic function with period $2\pi/\omega$, we have

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\cdot 0\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}(-s\sin\omega t - \omega\cos\omega t)}{s^2+\omega^2}\right]_0^{\pi/\omega}$$

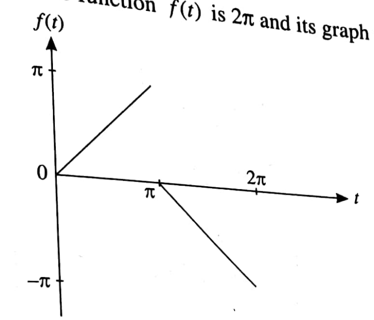
$$= \frac{\omega e^{-\pi s/\omega} + \omega}{(1-e^{-2\pi s/\omega})(s^2+\omega^2)} = \frac{\omega}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}.$$

Example 3: Draw the graph of the periodic function

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$$

and find its Laplace transform.

Solution: Here the period of the function $f(t)$ is 2π and its graph is shown below:



$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st}\, t\,dt + \int_{\pi}^{2\pi} e^{-st}\,(\pi - t)\,dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\left\{ \frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2} \right\}_0^{\pi} + \left\{ (\pi - t)\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2} \right\}_{\pi}^{2\pi} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[-\frac{\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \frac{\pi}{s}(e^{-2\pi s} - e^{-\pi s}) + \frac{1}{s^2}(1-e^{-\pi s})^2 \right\}$$

$$= \frac{1}{1+e^{-\pi s}} \left\{ \frac{1}{s^2}(1-e^{-\pi s}) - \frac{\pi}{s} e^{-\pi s} \right\}$$

Example 4: Find the Laplace transform of

(*i*) $f(t) = k\dfrac{t}{T}$ for $0 < t < T$ and $f(t+T) = f(t)$.

(*ii*) $f(t) = \begin{cases} 1, & \text{for } 0 \le t < a \\ -1, & \text{for } a < t < 2a \end{cases}$

and $f(t)$ is periodic with period $2a$.

Solution: (*i*) Since $f(t)$ is periodic with period T, we have

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\frac{kt}{T}\,dt$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{k}{T}\int_0^T t\,e^{-st}\,dt$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{k}{T}\left[\frac{te^{-st}}{-s}-\frac{e^{-st}}{s^2}\right]_0^T$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{k}{T}\left[-\frac{Te^{-sT}}{s}-\frac{e^{-sT}}{s^2}+\frac{1}{s^2}\right]$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{k}{T}\left[\frac{1}{s^2}(1-e^{-sT})-\frac{Te^{-sT}}{s}\right]$$

$$= k\left\{\frac{1}{Ts^2}-\frac{e^{-sT}}{s(1-e^{-sT})}\right\}$$

(*ii*) Since $f(t)$ is periodic with period $2a$, we have

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\cdot 1\,dt + \int_a^{2a} e^{-st}\cdot(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left\{\frac{e^{-st}}{-s}\right\}_0^a + \left\{\frac{e^{-st}}{s}\right\}_a^{2a}\right]$$

$$= \frac{1}{s(1-e^{-2as})}(-e^{-as}+1+e^{-2as}-e^{-as})$$

$$= \frac{(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})} = \frac{1}{s}\left(\frac{1-e^{-as}}{1+e^{-as}}\right)$$

$$= \frac{1}{s}\left(\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right) = \frac{1}{s}\tanh\left(\frac{as}{2}\right).$$

Example 5: If $f(t) = t^2, 0 < t < 1$ and $f(t+1) = f(t)$, find $L\{f(t)\}$.

Solution: Here $f(t)$ is a periodic function with period $T = 1$, therefore

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-s}} \int_0^1 e^{-st}\, t^2\, dt$$

$$= \frac{1}{1-e^{-s}} \left[\left\{ \frac{t^2 e^{-st}}{-s} \right\}_0^1 + \frac{2}{s} \int_0^1 t e^{-st}\, dt \right]$$

$$= \frac{1}{1-e^{-s}} \left[-\frac{1}{s} e^{-s} + \frac{2}{s} \left\{ \left(\frac{t e^{-st}}{-s} \right)_0^1 - \left(\frac{e^{-st}}{s^2} \right)_0^1 \right\} \right]$$

$$= \frac{1}{1-e^{-s}} \left(-\frac{1}{s} e^{-s} - \frac{2}{s^2} e^{-s} - \frac{2}{s^3} e^{-s} + \frac{2}{s^3} \right)$$

$$= \frac{-(s^2+2s+2)e^{-s}+2}{s^3(1-e^{-s})}.$$

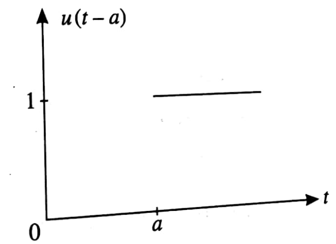
6.9 LAPLACE TRANSFORM OF UNIT STEP FUNCTION

Definition: The function u defined as

$$u(t-a) = \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t \geq a, \end{cases}$$

where a is always positive, is called **unit step function** or **Heaviside's unit step function.**

Its graph is shown below:



$\therefore$
$$f(t)u(t-a) = \begin{cases} 0, & \text{for } t < a \\ f(t), & \text{for } t \geq a. \end{cases}$$

The function $f(t)u(t-a)$ represents the graph of $f(t)$ shifted through a distance a to the right and it is of special importance.

Theorem 11: If $u(t-a)$ is a unit step function, then $L\{u(t-a)\} = \dfrac{e^{-as}}{s}$.

Proof:

$$L\{u(t-a)\} = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} .1 dt = \lim_{B\to\infty} \int_a^B e^{-st}\, dt$$

$$= \lim_{B\to\infty} \left[\frac{e^{-st}}{-s}\right]_a^B = \lim_{B\to\infty} \frac{1}{s}(e^{-sa} - e^{-sB})$$

$$= \frac{e^{-as}}{s} \qquad \text{[Here } e^{-sB} \to 0 \text{ as } B\to\infty, \text{ since } s>0]$$

Theorem 12: The function

$$f(t) = \begin{cases} f_1(t), & t<a \\ f_2(t), & t>a \end{cases}$$

can be expressed as

$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\}u(t-a),$$

where $u(t-a)$ is a unit step function.

Proof: By the definition of unit step function,

$$f(t) = \begin{cases} f_1(t) + \{f_2(t) - f_1(t)\}\cdot 0, & t<a \\ f_1(t) + \{f_2(t) - f_1(t)\}\cdot 1, & t>a \end{cases}$$

$$\therefore \quad f(t) = \begin{cases} f_1(t), & t<a \\ f_2(t), & t>a. \end{cases}$$

The above theorem can be extended which are given below without proofs.

Theorem 13: The function

$$f(t) = \begin{cases} f_1(t), & t<a_1 \\ f_2(t), & a_1<t<a_2 \\ f_3(t), & t>a_2 \end{cases}$$

can be expressed as $f(t) = f_1(t) + \{f_2(t) - f_1(t)\}u(t-a_1) + \{f_3(t) - f_2(t)\}u(t-a_2)$, where $u(t-a_1)$, $u(t-a_2)$ are unit step functions.

Theorem 14: The function

$$f(t) = \begin{cases} f_1(t), & t<a_1 \\ f_2(t), & a_1<t<a_2 \\ f_3(t), & a_2<t<a_3 \\ \vdots & \\ f_{n-1}(t), & a_{n-2}<t<a_{n-1} \\ f_n(t), & t>a_{n-1} \end{cases}$$

can be expressed as $f(t) = f_1(t) + \{f_2(t) - f_1(t)\}u(t-a_1) + \{f_3(t) - f_2(t)\}u(t-a_2) + \{f_4(t) - f_3(t)\}$ $u(t-a_3) + \dots + \{f_n(t) - f_{n-1}(t)\}u(t-a_{n-1})$, where $u(t-a_1), u(t-a_2), u(t-a_3), \dots, u(t-a_{n-1})$ are unit step functions.

Theorem 15: (Alternative form of second shifting property)

If $L\{f(t)\} = \overline{f}(s)$, then $L\{f(t-a)u(t-a)\} = e^{-as}\overline{f}(s)$.

Corollary

$$L\{f(t)u(t-a)\} = e^{-as} L\{f(t+a)\}.$$

Proof:

$$L\{f(t)u(t-a)\} = \int_0^\infty e^{-st} f(t)u(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t)\cdot 0\,dt + \int_a^\infty e^{-st} f(t)\cdot 1\,dt$$

$$= \lim_{B\to\infty} \int_a^B e^{-st} f(t)\,dt$$

$$= \lim_{B\to\infty} \int_0^{B-a} e^{-s(x+a)} f(x+a)\,dx \qquad \text{[Putting } x = t-a, \text{ so } dx = dt]$$

$$= \lim_{B\to\infty} e^{-sa} \int_0^{B-a} e^{-sx} f(x+a)\,dx$$

$$= e^{-as} \int_0^\infty e^{-sx} f(x+a)\,dx$$

$$= e^{-as} \int_0^\infty e^{-st} f(t+a)\,dt = e^{-as} L\{f(t+a)\}$$

Note: This is an useful result.

Solution: Using Theorem 13, we have $f(t) = 0 + \{(t-1) - 0\}u(t-1) + \{1 - (t-1)\}u(t-2)$

$= (t-1)u(t-1) - (t-2)u(t-2)$, where $u(t-1)$, $u(t-2)$ are unit step functions.

By second shifting property, we have

$$L\{g(t-a)u(t-a)\} = e^{-as} L\{g(t)\}$$

$$\therefore \quad L\{f(t)\} = L\{(t-1)u(t-1) - (t-2)u(t-2)\}$$

$$= L\{(t-1)u(t-1)\} - L\{(t-2)u(t-2)\}$$

$$= e^{-s} L\{t\} - e^{-2s} L\{t\}$$

$$= e^{-s} \cdot \frac{1}{s^2} - e^{-2s} \cdot \frac{1}{s^2} \qquad \left[\because \ L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}, n > -1\right]$$

$$= \frac{1}{s^2}(e^{-s} - e^{-2s}).$$

Example 2: Using unit step function, find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi \\ \sin 3t, & t \ge 2\pi \end{cases}$$

Solution: Using Theorem 13, we have $f(t) = \sin t + (\sin 2t - \sin t)u(t-\pi) + (\sin 3t - \sin 2t)$ $u(t-2\pi)$, where $u(t-\pi)$, $u(t-2\pi)$ are unit step functions.

We know that

$$L\{g(t)u(t-a)\} = e^{-as} L\{g(t+a)\}$$

and

$$L(\sin bt) = \frac{b}{s^2+b^2}.$$

$$\therefore \quad L\{f(t)\} = L(\sin t) + L\{(\sin 2t - \sin t)u(t-\pi)\}$$

$$+ L\{(\sin 3t - \sin 2t)u(t-2\pi)\}$$

$$= \frac{1}{s^2+1^2} + e^{-\pi s} L\{\sin 2(t+\pi) - \sin(t+\pi)\}$$

$$+ e^{-2\pi s} L\{\sin 3(t+2\pi) - \sin 2(t+2\pi)\}$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\{L(\sin 2t) + L(\sin t)\} + e^{-2\pi s}\{L(\sin 3t) - L(\sin 2t)\}$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\left\{\frac{2}{s^2+2^2} + \frac{1}{s^2+1^2}\right\} + e^{-2\pi s}\left\{\frac{3}{s^2+3^2} - \frac{2}{s^2+2^2}\right\}$$

Example 3: Express the following function in terms of unit step function:

$$f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$$

and find its Laplace transform. (W.B.U.T. 2002)

Solution: Using Theorem 12, we have $f(t) = t-1+\{(3-t)-(t-1)\}u(t-2) = t-1+(4-2t)u(t-2)$, where $u(t-2)$ is a step function.

We know that

$$L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}$$

$$L(t) = \frac{\Gamma(1+1)}{s^{1+1}} = \frac{1}{s^2},\ L(1) = \frac{1}{s}.$$

$$\therefore \quad L\{f(t)\} = L(t)-L(1)+L\{(4-2t)u(t-2)\}$$

$$= \frac{1}{s^2}-\frac{1}{s}+e^{-2s}L\{4-2(t+2)\}$$

$$= \frac{1}{s^2}-\frac{1}{s}-2e^{-2s}L(t)$$

$$= \frac{1}{s^2}-\frac{1}{s}-\frac{2e^{-2s}}{s^2} = \frac{1}{s^2}(1-2e^{-2s})-\frac{1}{s}.$$

Example 4: Express the function

$$F(t) = e^{-t},\ 0<t<2$$

$$= 0,\quad t\geq 2$$

in terms of unit step function and hence find $L\{F(t)\}$. (W.B.U.T. 2004)

Solution: Using Theorem 12, we have $F(t) = e^{-t}+(0-e^{-t})u(t-2)$, where $u(t-2)$ is a step function.

We know that

$$L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}$$

and

$$L(e^{-t}) = \frac{1}{s+1}.$$

$$\therefore \quad L\{F(t)\} = L(e^{-t})-L\{e^{-t}u(t-2)\}$$

$$= \frac{1}{s+1}-e^{-2s}L\{e^{-(t+2)}\}$$

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$$= \frac{1}{s+1} - e^{-2s} \cdot e^{-2} L(e^{-t})$$

$$= \frac{1}{s+1} - \frac{e^{-2(s+1)}}{s+1}.$$

Example 5: Find the Laplace transform of $f(t)$ defined as: (M–201/13)

$$f(t) = \begin{cases} \dfrac{t}{k}, & \text{where } 0 < t < k \\ 1, & \text{when } t > k \end{cases}$$

Solution:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^k e^{-st} \frac{t}{k}\,dt + \int_k^\infty e^{-st} \cdot 1\,dt \quad \text{...(1)}$$

Now,

$$\int te^{-st}\,dt = t\int e^{-st}\,dt - \int \left\{\left(\frac{dt}{dt}\right)\int e^{-st}\,dt\right\} dt$$

$$= -\frac{t}{s}e^{-st} + \frac{1}{s}\int e^{-st}\,dt = -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}$$

$$\int_k^\infty e^{-st}\,dt = \lim_{B\to\infty}\int_k^B e^{-st}\,dt = \lim_{B\to\infty}\left[-\frac{1}{s}e^{-st}\right]_k^B$$

$$= \lim_{B\to\infty}\left[-\frac{1}{s}e^{-sB} + \frac{1}{s}e^{-sk}\right] = \frac{1}{s}e^{-sk}$$

Therefore, from (1):

$$L\{f(t)\} = -\frac{1}{k}\left[\frac{t}{s}e^{-st} + \frac{1}{s^2}e^{-st}\right]_0^k + \frac{1}{s}e^{-sk}$$

$$= -\frac{1}{k}\left[\frac{k}{s}e^{-sk} + \frac{1}{s^2}e^{-sk} - \frac{1}{s^2}\right] + \frac{1}{s}e^{-sk}$$

$$= \frac{1}{ks^2}(1 - e^{-sk}).$$

6.10 TABLE OF LAPLACE TRANSFORM THEOREMS

No.	Operation	$f(t)$	$L\{f(t)\} = \bar{f}(s)$
1.	Linearity property	$c_1 f_1(t) + c_2 f_2(t)$	$c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}$
2.	First shifting property	$e^{at} f(t)$	$\bar{f}(s-a)$
3.	Second shifting property	(i) $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$ (ii) $f(t-a)u(t-a)$ (iii) $f(t)u(t-a)$	(i) $e^{-as}\bar{f}(s)$ (ii) $e^{-as}\bar{f}(s)$ (iii) $e^{-as} L\{f(t+a)\}$
4.	Change of scale property	$f(at)$	$\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$, $a > 0$
5.	Multiplication Theorem	(i) $tf(t)$ (ii) $t^n f(t)$	(i) $-\frac{d}{ds}\bar{f}(s)$ (ii) $(-1)^n \frac{d^n}{ds^n}\bar{f}(s)$
6.	Division Theorem	$\frac{1}{t} f(t)$	$\int_s^\infty \bar{f}(s)\,ds$
7.	Differentiation Theorem	(i) $f'(t) = \frac{d}{dt} f(t)$ (ii) $f^n(t) = \frac{d^n}{dt^n} f(t)$	(i) $-f(0) + s\bar{f}(s)$ (ii) $-f^{n-1}(0) - sf^{n-2}(0) - \ldots - s^{n-1}f(0) + s^n \bar{f}(s)$
8.	Integral Theorem	$\int_0^t f(u)\,du$	$\frac{1}{s}\bar{f}(s)$
9.	Periodic function	$f(t)$ is periodic with period T	$\frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

10.	Unit step function	(i) If $f(t) = \begin{cases} f_1(t), & t<a \\ f_2(t), & t>a \end{cases}$ then $f(t) = f_1(t)+\{f_2(t)-f_1(t)\}$ $u(t-a)$ (ii) If $f(t) = \begin{cases} f_1(t), & t<a_1 \\ f_2(t), & a_1<t<a_2 \\ f_3(t), & t>a_2 \end{cases}$ then $f(t) = f_1(t)+\{f_2(t)-f_1(t)\}$ $u(t-a_1)+\{f_3(t)-f_2(t)\}u(t-a_2)$	(i) $L\{f(t)\} = L\{f_1(t)\}+e^{-as}L\{f_2(t+a)$ $-f_1(t+a)\}$ (ii) $L\{f(t)\} = L\{f_1(t)\}+e^{-a_1 s}L\{f_2(t+a_1)$ $-f_1(t+a_1)\}+e^{-a_2 s}L\{f_3(t+a_2)$ $-f_2(t+a_2)\}$

Initial value theorem: $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\bar{f}(s)$, provided the limits exist.

Final value theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\bar{f}(s)$, provided the limits exist.

6.11 INVERSE LAPLACE TRANSFORM

Definition: If $L\{f(t)\} = \bar{f}(s)$ then $f(t)$ is called the **inverse Laplace transform** of the function $\bar{f}(s)$ and is written as

$$f(t) = L^{-1}\{\bar{f}(s)\}.$$

Note: L^{-1} is known as the **inverse Laplace transformation operator.**

Use of standard results

(i) $L(1) = \dfrac{1}{s}$ $\quad\therefore\quad L^{-1}\left(\dfrac{1}{s}\right) = 1$

(ii) $L(e^{at}) = \dfrac{1}{s-a}$ $\quad\therefore\quad L^{-1}\left(\dfrac{1}{s-a}\right) = e^{at},\ s>a$

(iii) $L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$ $\quad\therefore\quad L^{-1}\left(\dfrac{1}{s^{n+1}}\right) = \dfrac{t^n}{\Gamma(n+1)},\ n>-1$

$$= \frac{t^n}{n!},\ \text{if } n \text{ is a positive integer}$$

(iv) $L(\sin at) = \dfrac{a}{s^2+a^2}$ $\quad\therefore\quad L^{-1}\left(\dfrac{1}{s^2+a^2}\right) = \dfrac{1}{a}\sin at$

(v) $L(\cos at) = \dfrac{s}{s^2+a^2}$ $\quad\therefore\quad L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \cos at$

$\frac{1}{s}$	1
$\frac{1}{s-a}$, $s>a$	e^{at}
$\frac{1}{s^{n+1}}$, $n>-1$	(*i*) $\frac{t^n}{\Gamma(n+1)}$ (*ii*) $\frac{t^n}{n!}$, if n is a positive integer
$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$

$$L^{-1}\{c_1\bar{f}_1(s)+c_2\bar{f}_2(s)\} = c_1 f_1(t)+c_2 f_2(t)$$

$\therefore$
$$= c_1L^{-1}\{\bar{f}_1(s)\}+c_2L^{-1}\{\bar{f}_2(s)\} \qquad \text{[By (1)]}$$

Note: This result can be easily generalised.

Illustration:

$$L^{-1}\left\{\frac{4}{2s-3}-\left(\frac{3+4s}{9s^2-16}\right)+\frac{8-6s}{16s^2+9}\right\} = 2L^{-1}\left\{\frac{1}{s-\dfrac{3}{2}}\right\}-3L^{-1}\left\{\frac{1}{9s^2-16}\right\}-4L^{-1}\left\{\frac{s}{9s^2-16}\right\}$$

$$+8L^{-1}\left\{\frac{1}{16s^2+9}\right\}-6L^{-1}\left\{\frac{s}{16s^2+9}\right\}$$

$$= 2e^{3t/2}-\frac{1}{3}L^{-1}\left\{\frac{1}{s^2-\left(\dfrac{4}{3}\right)^2}\right\}-\frac{4}{9}L^{-1}\left\{\frac{s}{s^2-\left(\dfrac{4}{3}\right)^2}\right\}$$

$$+\frac{1}{2}L^{-1}\left\{\frac{1}{s^2+\left(\dfrac{3}{4}\right)^2}\right\}-\frac{3}{8}L^{-1}\left\{\frac{s}{s^2+\left(\dfrac{3}{4}\right)^2}\right\}$$

$$= 2e^{3t/2}-\frac{1}{3}\cdot\frac{3}{4}\sinh\left(\frac{4t}{3}\right)-\frac{4}{9}\cosh\left(\frac{4t}{3}\right)$$

$$+\frac{1}{2}\cdot\frac{4}{3}\sin\left(\frac{3t}{4}\right)-\frac{3}{8}\cos\left(\frac{3t}{4}\right)$$

$$= 2e^{3t/2}-\frac{1}{4}\sinh\left(\frac{4t}{3}\right)-\frac{4}{9}\cosh\left(\frac{4t}{3}\right)+\frac{2}{3}\sin\left(\frac{3t}{4}\right)-\frac{3}{8}\cos\left(\frac{3t}{4}\right)$$

Shifting Property of Inverse Laplace Transform

Theorem 17: (First shifting property)

$$L^{-1}\{\bar{f}(s-a)\} = e^{at}L^{-1}\{\bar{f}(s)\}.$$

Proof: Let $L^{-1}\{\bar{f}(s)\} = f(t)$, therefore by the first shifting property of Laplace transform, we have

$$L\{e^{at}f(t)\} = \bar{f}(s-a).$$

$\therefore$
$$L^{-1}\{\bar{f}(s-a)\} = e^{at}f(t) = e^{at}L^{-1}\{\bar{f}(s)\}.$$

Illustration:

$$L^{-1}\left\{\frac{1}{(s-2)^2}\right\} = e^{2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{2t}\,t \qquad \left[\because\ L^{-1}\left(\frac{1}{s^{n+1}}\right)=\frac{t^n}{n!},\ n \text{ is a positive integer}\right]$$

Theorem 18: (Second shifting property)

If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\{e^{-as}\bar{f}(s)\} = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$

$= f(t-a)u(t-a),$

where $u(t-a)$ is a unit step function.

Proof: Since $L^{-1}\{\bar{f}(s)\} = f(t)$, so $L\{f(t)\} = \bar{f}(s)$.

Let $$g(t) = f(t-a)u(t-a) = \begin{cases} f(t-a), & t>a \\ 0, & t<a. \end{cases}$$

Therefore, by second shifting property of Laplace transform, we have

$$L\{g(t)\} = e^{-as}\bar{f}(s)$$

$$\therefore \quad L^{-1}\{e^{-as}\bar{f}(s)\} = g(t) = f(t-a)u(t-a)$$

$$= \begin{cases} f(t-a), & t>a \\ 0, & t<a. \end{cases}$$

Illustration: Evaluate $L^{-1}\left\{\dfrac{e^{-s}-3e^{-3s}}{s^2}\right\}$.

We know that

$$L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}, \text{ if } n \text{ is a positive integer.}$$

$$\therefore \quad L^{-1}\left(\frac{1}{s^2}\right) = t.$$

By second shifting property of Inverse Laplace transform, we get

$$L^{-1}\left(e^{-s}\frac{1}{s^2}\right) = \begin{Bmatrix} t-1, & t>1 \\ 0, & t<1 \end{Bmatrix} = (t-1)u(t-1)$$

$$L^{-1}\left(e^{-3s}\frac{1}{s^2}\right) = \begin{Bmatrix} t-3, & t>3 \\ 0, & t<3 \end{Bmatrix} = (t-3)u(t-3)$$

$$\therefore \quad L^{-1}\left\{\frac{e^{-s}-3e^{-3s}}{s^2}\right\} = L^{-1}\left(e^{-s}\cdot\frac{1}{s^2}\right) - 3L^{-1}\left(e^{-3s}\cdot\frac{1}{s^2}\right)$$

$$= (t-1)u(t-1) - 3(t-3)u(t-3).$$

Change of Scale Property of Inverse Laplace Transform

Theorem 19: If $L\{f(t)\} = \bar{f}(s)$ denotes the Laplace transform of the function $f(t)$, then

(W.B.U.T. 2002)

$$L^{-1}\{\bar{f}(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right), \; a>0.$$

Proof: Now, $$L\left\{\frac{1}{a}f\left(\frac{t}{a}\right)\right\} = \frac{1}{a}L\left\{f\left(\frac{t}{a}\right)\right\}$$

$$= \frac{1}{a} \cdot \frac{1}{\frac{1}{a}} \bar{f}\left(\frac{s}{\frac{1}{a}}\right) \quad \text{[By change of scale property of Laplace transform]}$$

$$= \bar{f}(as)$$

$$\therefore \quad L^{-1}\{\bar{f}(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right), \; a > 0.$$

Illustration: Find $L^{-1}\left(\frac{1}{4s^2+9}\right)$.

We have $\quad L^{-1}\left(\frac{3}{s^2+3^2}\right) = \sin 3t.$

Therefore, by change of scale property,

$$L^{-1}\left\{\frac{3}{(2s)^2+3^2}\right\} = \frac{1}{2}\sin 3\left(\frac{t}{2}\right)$$

or
$$3L^{-1}\left(\frac{1}{4s^2+9}\right) = \frac{1}{2}\sin\left(\frac{3t}{2}\right)$$

$$\therefore \quad L^{-1}\left(\frac{1}{4s^2+9}\right) = \frac{1}{6}\sin\left(\frac{3t}{2}\right).$$

Inverse Laplace Transform on Derivatives of Functions

Theorem 20:

$$L^{-1}\{\bar{f}^n(s)\} = (-1)^n t^n L^{-1}\{\bar{f}(s)\},$$

where
$$\bar{f}^n(s) = \frac{d^n}{ds^n}\{\bar{f}(s)\}.$$

Proof: Let $\quad L^{-1}\{\bar{f}(s)\} = f(t),$

therefore $\quad L\{f(t)\} = \bar{f}(s).$

We know that
$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{\bar{f}(s)\}$$
$$= (-1)^n \bar{f}^n(s).$$

$$\therefore \quad (-1)^n L\{t^n f(t)\} = \bar{f}^n(s)$$

or
$$L\{(-1)^n t^n f(t)\} = \bar{f}^n(s)$$

$$\therefore \quad L^{-1}\{\bar{f}^n(s)\} = (-1)^n t^n f(t) = (-1)^n t^n L^{-1}\{\bar{f}(s)\}.$$

Note: Putting $n = 1$, we get $L^{-1}\{\bar{f}'(s)\} = -tL^{-1}\{\bar{f}(s)\}$.

Illustration: Evaluate $L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}$.

Here
$$\bar{f}(s) = \log\left(\frac{s+a}{s+b}\right) = \log(s+a) - \log(s+b)$$

∴
$$\bar{f}'(s) = \frac{1}{s+a} - \frac{1}{s+b}.$$

∴
$$L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{\frac{1}{s+a}\right\} - L^{-1}\left\{\frac{1}{s+b}\right\} = e^{-at} - e^{-bt}.$$

We know that

$$L^{-1}\{\bar{f}'(s)\} = -tL^{-1}\{\bar{f}(s)\}.$$

∴
$$L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\} = L^{-1}\{\bar{f}(s)\} = -\frac{1}{t}L^{-1}\{\bar{f}'(s)\} = \frac{1}{t}(e^{-bt} - e^{-at})$$

Inverse Laplace transform of a function multiplied by powers of s.

Theorem 21: If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $f(0) = f'(0) = \ldots = f^{n-1}(0) = 0$, then $L^{-1}\{s^n\bar{f}(s)\} = f^n(t)$, where $f^n(t) = \frac{d^n}{dt^n}f(t)$.

Proof: Since $L^{-1}\{\bar{f}(s)\} = f(t)$, therefore $L\{f(t)\} = \bar{f}(s)$.

We know that (see Theorem 8)

$$L\{f^n(t)\} = -f^{n-1}(0) - sf^{n-2}(0) - \ldots - s^{n-1}f(0) + s^n\bar{f}(s).$$
$$= s^n\bar{f}(s) \qquad [\because f(0) = f'(0) = \ldots = f^{n-1}(0) = 0]$$

∴
$$L^{-1}\{s^n\bar{f}(s)\} = f^n(t)$$

Note: Putting $n = 1$, we get $L^{-1}\{s\bar{f}(s)\} = f'(t) = \frac{df}{dt} = \frac{d}{dt}L^{-1}\{\bar{f}(s)\}$, provided $f(0) = 0$.

Illustration: Evaluate $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}$.

Let
$$\bar{f}(s) = \frac{1}{s^2+a^2}.$$

∴
$$\bar{f}'(s) = -\frac{2s}{(s^2+a^2)^2}$$

We know that

$$L^{-1}\{\bar{f}(s)\} = \frac{1}{a}\sin at = f(t),$$ say, and by Theorem 20,

$$L^{-1}\{\bar{f}'(s)\} = -tL^{-1}\{\bar{f}(s)\} \qquad [\because f(0)=0]$$

$$\therefore \quad L^{-1}\left\{-\frac{2s}{(s^2+a^2)^2}\right\} = -t\,\frac{1}{a}\sin at$$

or
$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}\,t\sin at.$$

Therefore, by Theorem 21,

$$L^{-1}\left\{s\cdot\frac{s}{(s^2+a^2)^2}\right\} = \frac{d}{dt}\left\{\frac{1}{2a}\,t\sin at\right\}$$

$$\therefore \quad L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \frac{1}{2a}(\sin at + at\cos at).$$

Division by powers of s

Theorem 22: If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\left\{\dfrac{\bar{f}(s)}{s}\right\} = \int_0^t f(u)\,du.$

Proof: Since $L^{-1}\{\bar{f}(s)\} = f(t)$, therefore $L\{f(t)\} = \bar{f}(s)$.

We know that (see Theorem 9)

$$L\{\int_0^t f(u)\,du\} = \frac{1}{s}\bar{f}(s).$$

$$\therefore \quad L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(u)\,du.$$

Note: In general it can be easily proved that if $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\left\{\dfrac{\bar{f}(s)}{s^n}\right\} = \int_0^t\int_0^t\ldots\int_0^t f(u)\,du\,du\ldots du$, where the integration is taken n times w.r.t. u.

Illustration: Evaluate $L^{-1}\left\{\dfrac{1}{s^3(s^2+1)}\right\}$.

Since $L^{-1}\left\{\dfrac{1}{(s^2+1)}\right\} = \sin t$, therefore

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$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u\, du = [-\cos u]_0^t = 1-\cos t.$$

$\therefore$

$$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos u)\, du = [u-\sin u]_0^t = t-\sin t.$$

$\therefore$

$$L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (u-\sin u)\, du = \left[\frac{u^2}{2}+\cos u\right]_0^t$$

$$= \frac{t^2}{2}+\cos t-1,$$

i.e.,

$$L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t\int_0^t\int_0^t \sin u\, du\, du\, du$$

$$= \frac{t^2}{2}+\cos t-1.$$

Inverse Laplace Transform of Integrals

Theorem 23: If $L^{-1}\{\overline{f}(s)\} = f(t)$, then $L^{-1}\{\int_s^\infty \overline{f}(u)\, du\} = \frac{f(t)}{t}$.

Proof: Since $L^{-1}\{\overline{f}(s)\} = f(t)$, so $L\{f(t)\} = \overline{f}(s)$.

By Theorem 7, we have

$$L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty \overline{f}(u)\, du.$$

$\therefore$

$$L^{-1}\{\int_s^\infty \overline{f}(u)\, du\} = \frac{f(t)}{t}.$$

Convolution

Definition: If $f(t)$ and $g(t)$ are two integrable functions then their convolution is denoted as $f*g$ and defined by

$$f*g = \int_0^t f(u)\, g(t-u)\, du.$$

Theorem 24:

$$f*g = g*f.$$

Proof: By definition,

$$f*g = \int_0^t f(u)\, g(t-u)\, du = -\int_t^0 f(t-v)\, g(v)\, dv$$

[Putting $t-u=v$, so $du = -dv$]

$$= \int_0^t f(t-v)g(v)\,dv = \int_0^t g(v)f(t-v)\,dv = g*f.$$

We state below the **Convolution theorem** without proof.

Theorem 25: (Convolution theorem)

If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $L^{-1}\{\bar{g}(s)\} = g(t)$ then $L^{-1}\{\bar{f}(s)\bar{g}(s)\} = \int_0^t f(u)g(t-u)\,du = f*g.$

(W.B.U.T. 2006)

Illustration: Evaluate $L^{-1}\left\{\dfrac{1}{(s+3)(s-1)}\right\}$.

We know

$$L^{-1}\left(\frac{1}{s+3}\right) = e^{-3t},$$

$$L^{-1}\left(\frac{1}{s-1}\right) = e^{t}.$$

Therefore, by Convolution theorem,

$$L^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\} = \int_0^t e^{-3u}\cdot e^{t-u}\,du$$

$$= e^t\int_0^t e^{-4u}\,du = e^t\left[\frac{e^{-4u}}{-4}\right]_0^t = \frac{e^t}{4}(1-e^{-4t}).$$

Use of Partial Fractions

If $\bar{f}(s)$ is of the form $\dfrac{g(s)}{h(s)}$, where g and h are polynomials in s, then break $\bar{f}(s)$ into partial fractions as described below:

(i) One requirement is that *the degree of the polynomial in the numerator must be less than the degree of the polynomial in the denominator*. If this is not the case, use the process of division so as to obtain the necessary requirement.

(ii) *The next step is to factorise the denominator into its ultimate real facors* and these factors must be one or other of the following four types:

1. Linear but not repeated, of the type $(ax+b)$
2. Linear and repeated, of the type $(ax+b)^n$
3. Quadratic but not repeated, such as (ax^2+bx+c).
4. Repeated quadratic, such as $(ax^2+bx+c)^n$.

(iii) The third step is to write down *the given fraction as the sum of a set of simple fractions* according to the following rules:

(a) For each factor of the type $(ax+b)$, there should be a single fraction of the form $\frac{A}{ax+b}$, A is a constant.

(b) For each factor of the type $(ax+b)^n$, there should be fractions of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_n}{(ax+b)^n},\ A_1, A_2, \ldots, A_n \text{ are constants.}$$

(c) Similarly for ax^2+bx+c, take $\frac{Ax+B}{ax^2+bx+c}$ and for $(ax^2+bx+c)^n$, take

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$

(iv) Next *make the numerator of the sum of component fractions identically equal to the numerator of the given fraction* to find $A, B, A_1, B_1, \ldots$

Inverse Laplace transform can also be evaluated by using this method of partial fraction. See Example 1: (*iv*), (*v*), (*vii*) – (*x*).

Table of Inverse Laplace Theorems

No.	Operation	$\bar{f}(s)$	$L^{-1}\{\bar{f}(s)\}=f(t)$
1.	Linear property	$c_1\bar{f}_1(s)+c_2\bar{f}_2(s)$	$c_1L^{-1}\{\bar{f}_1(s)\}+c_2L^{-1}\{\bar{f}_2(s)\}$
2.	First shifting property	$\bar{f}(s-a)$	$e^{at}L^{-1}\{\bar{f}(s)\}$
3.	Second shifting property	$e^{-as}\bar{f}(s)$	$f(t-a)u(t-a)=\begin{cases}f(t-a), & t>a\\ 0, & t<a\end{cases}$
4.	Change of scale property	$\bar{f}(as),\ a>0$	$\frac{1}{a}f\left(\frac{t}{a}\right)$
5.	Differentiation theorem	$\bar{f}^n(s)$	$(-1)^n t^n f(t)$
6.	Multiplication theorem	(*i*) $s\bar{f}(s)$ (*ii*) $s^n\bar{f}(s)$	(*i*) $f'(t)$ if $f(0)=0$ (*ii*) $f^n(t)$ if $f(0)=f'(0)=\ldots=f^{n-1}(0)=0$

7.	Division theorem	(i) $\dfrac{\bar{f}(s)}{s}$ (ii) $\dfrac{\bar{f}(s)}{s^n}$	(i) $\int_0^t f(u)\,du$ (ii) $\int_0^t\int_0^t\cdots\int_0^t f(u)\,du\,du\ldots du$ (n times integration)
8.	Integral theorem	$\int_s^\infty \bar{f}(u)\,du$	$\dfrac{f(t)}{t}$
9.	Convolution theorem	$\bar{f}(s)\,\bar{g}(s)$	$\int_0^t f(u)\,g(t-u)\,du$

Two useful standard integrals

1. $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

2. $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

ILLUSTRATIVE EXAMPLES

Example 1: Evaluate

(i) $L^{-1}\left\{\dfrac{s+1}{s^2+s+1}\right\}$ (W.B.U.T. 2003)

(ii) $L^{-1}\left\{\dfrac{1}{(s^2+a^2)(s^2+b^2)}\right\},\ a^2 \neq b^2$ (W.B.U.T. 2006)

(iii) $L^{-1}\left\{\dfrac{s^2}{(s+1)^5}\right\}$ (W.B.U.T. 2006)

(iv) $L^{-1}\left\{\dfrac{4p+5}{(p-4)^2(p+3)}\right\}$ (W.B.U.T. 2003)

(v) $L^{-1}\left\{\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$

(vi) $L^{-1}\left\{\dfrac{s+1}{(s^2+2s+2)^2}\right\}$

(vii) $L^{-1}\left\{\dfrac{s+2}{(s+3)(s+1)^3}\right\}$

(viii) $L^{-1}\left\{\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

(ix) $L^{-1}\left\{\dfrac{s^2+16s-24}{s^4+20s^2+64}\right\}$

(x) $L^{-1}\left\{\dfrac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}\right\}$.

Solution: (i) Now,

$$L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = L^{-1}\left\{\frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= L^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= e^{-\frac{1}{2}t}L^{-1}\left\{\frac{s}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}+\frac{1}{2}e^{-\frac{1}{2}t}L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

[Using first shifting property]

$$= e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t+\frac{1}{2}e^{-\frac{t}{2}}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\sin\frac{\sqrt{3}}{2}t$$

$$\left[\because\ L^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos at \text{ and } L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{1}{a}\sin at\right]$$

$$= e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right).$$

(ii) We have

$$L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{a^2-b^2}L^{-1}\left\{\frac{(s^2+a^2)-(s^2+b^2)}{(s^2+a^2)(s^2+b^2)}\right\}$$

$$= \frac{1}{a^2-b^2}L^{-1}\left\{\frac{1}{s^2+b^2}-\frac{1}{s^2+a^2}\right\}$$

$$= \frac{1}{a^2-b^2}\left\{\frac{\sin bt}{b}-\frac{\sin at}{a}\right\} \qquad \left[\because\ L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{1}{a}\sin at\right],$$

where we assume that $a^2 \neq b^2$.

(*iii*) We know

$$L^{-1}\left(\frac{1}{s^5}\right) = \frac{t^4}{4!},$$

since $$L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$$ (if n is a positive integer)

Using first shifting property, we have

$$L^{-1}\left\{\frac{1}{(s+1)^5}\right\} = e^{-t}\frac{t^4}{4!} = f(t), \text{ say.}$$

$$\therefore \quad f'(t) = \frac{df}{dt} = \frac{1}{4!}(-e^{-t}t^4 + 4t^3e^{-t}) = \frac{e^{-t}}{24}t^3(4-t)$$

$$\therefore \quad f(0) = f'(0) = 0$$

$$\therefore \quad L^{-1}\left\{\frac{s^2}{(s+1)^5}\right\} = \frac{d^2 f}{dt^2} \quad \text{(see Theorem 21)}$$

$$= \frac{d}{dt}\left\{\frac{e^{-t}}{24}t^3(4-t)\right\}$$

$$= \frac{1}{24}\{-e^{-t}(4t^3-t^4)+e^{-t}(12t^2-4t^3)\}$$

$$= \frac{t^2e^{-t}}{24}(t^2-8t+12).$$

(*iv*) Let

$$\frac{4p+5}{(p-4)^2(p+3)} = \frac{A}{p-4}+\frac{B}{(p-4)^2}+\frac{C}{p+3}$$

$$\therefore \quad \frac{4p+5}{(p-4)^2(p+3)} = \frac{A(p-4)(p+3)+B(p+3)+C(p-4)^2}{(p-4)^2(p+3)}$$

$$\therefore \quad 4p+5 = A(p-4)(p+3)+B(p+3)+C(p-4)^2.$$

This is an identity and hence it is true for all values of p.

Putting $p = -3$, we get $\quad -7 = 49C$, *i.e.*, $C = -\frac{1}{7}$.

Putting $p = 4$, we get $\quad 21 = 7B$, *i.e.*, $B = 3$.

Putting $p = 0$, we get $\quad 5 = -12A + 3B + 16C$

$$= -12A + 9 - \frac{16}{7}$$

$$\therefore \quad A = \frac{1}{7}.$$

$$\therefore \quad \frac{4p+5}{(p-4)^2(p+3)} = \frac{1}{7}\cdot\frac{1}{p-4}+\frac{3}{(p-4)^2}-\frac{1}{7}\cdot\frac{1}{p+3}.$$

$$\therefore \quad L^{-1}\left\{\frac{4p+5}{(p-4)^2(p+3)}\right\} = \frac{1}{7}L^{-1}\left(\frac{1}{p-4}\right)+3L^{-1}\left\{\frac{1}{(p-4)^2}\right\}-\frac{1}{7}L^{-1}\left(\frac{1}{p+3}\right)$$

$$= \frac{1}{7}e^{4t}+3e^{4t}L^{-1}\left(\frac{1}{p^2}\right)-\frac{1}{7}e^{-3t}$$

$$\left[\because L^{-1}\left(\frac{1}{p-a}\right)=e^{at} \text{ and by first shifting property}\right]$$

$$= \frac{1}{7}e^{4t}+3t\,e^{4t}-\frac{1}{7}e^{-3t}$$

$$\left[\because L^{-1}\left(\frac{1}{p^{n+1}}\right)=\frac{t^n}{n!}, \text{ where } n \text{ is a positive integer}\right]$$

(*v*) Now,

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$$

$$= \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3} \qquad \text{(say)}$$

$$\therefore \quad 2s^2-6s+5 = A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2).$$

This is an identity and hence it is true for all values of s.

Putting $s = 1$, we get $\quad 1 = 2A$, *i.e.*, $A = \frac{1}{2}$.

Putting $s = 2$, we get $\quad 1 = -B$, *i.e.*, $B = -1$.

Putting $s = 3$, we get $\quad 5 = 2C$, *i.e.*, $C = \frac{5}{2}$.

$$\therefore \quad \frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{1}{2}\cdot\frac{1}{s-1}-\frac{1}{(s-2)}+\frac{5}{2}\cdot\frac{1}{s-3}.$$

$$\therefore \quad L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right)-L^{-1}\left(\frac{1}{s-2}\right)+\frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right) \qquad \left[\because L^{-1}\left(\frac{1}{s-a}\right)=e^{at}\right]$$

$$= \frac{1}{2}e^{t}-e^{2t}+\frac{5}{2}e^{3t}$$

(*vi*)

$$L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\} = L^{-1}\left[\frac{s+1}{\{(s+1)^2+1\}^2}\right]$$

$$= e^{-t}L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} \qquad \text{[By first shifting property]}$$

Let
$$\bar{f}(s) = \frac{1}{s^2+1}$$

$\therefore$
$$\bar{f}'(s) = \frac{-2s}{(s^2+1)^2}.$$

Here $L^{-1}\{\bar{f}(s)\} = \sin t$. By Theorem 20, $L^{-1}\{\bar{f}'(s)\} = -t L^{-1}\{\bar{f}(s)\}$

$\therefore$
$$L^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\} = -t\sin t,$$

or
$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$$

$\therefore$
$$L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\} = e^{-t}L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}te^{-t}\sin t.$$

(*vii*) Let
$$\frac{s+2}{(s+3)(s+1)^3} = \frac{A}{s+3}+\frac{B}{s+1}+\frac{C}{(s+1)^2}+\frac{D}{(s+1)^3}$$

$\therefore \quad s+2 = A(s+1)^3+B(s+3)(s+1)^2+C(s+3)(s+1)+D(s+3).$ ...(1)

This is an identity and hence it is true for all values of s.

Putting $s=-1$, we get $1=2D$, *i.e.*, $D=\frac{1}{2}$.

Putting $s=-3$, we get $-1=-8A$, *i.e.*, $A=\frac{1}{8}$.

Equating the coefficient of s^3 from both sides of (1), we get

$0 = A+B$, or $B=-A=-\frac{1}{8}$.

Equating the coefficient of s^2 from both sides of (1), we get
$$0 = 3A+5B+C,$$

i.e.,
$$C=-3A-5B = -\frac{3}{8}+\frac{5}{8} = \frac{1}{4}$$

$\therefore$
$$\frac{s+2}{(s+3)(s+1)^3} = \frac{1}{8}\cdot\frac{1}{s+3}-\frac{1}{8}\cdot\frac{1}{s+1}+\frac{1}{4}\cdot\frac{1}{(s+1)^2}+\frac{1}{2}\cdot\frac{1}{(s+1)^3}$$

$\therefore$
$$L^{-1}\left\{\frac{s+2}{(s+3)(s+1)^3}\right\} = \frac{1}{8}L^{-1}\left(\frac{1}{s+3}\right)-\frac{1}{8}L^{-1}\left(\frac{1}{s+1}\right)+\frac{1}{4}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$
$$= \frac{1}{8}e^{-3t}-\frac{1}{8}e^{-t}+\frac{1}{4}e^{-t}L^{-1}\left(\frac{1}{s^2}\right)+\frac{1}{2}e^{-t}L^{-1}\left(\frac{1}{s^3}\right)$$

$$\left[\because\ L^{-1}\left(\frac{1}{s-a}\right)=e^{at}\ \text{ and by first shifting property}\right]$$

$$=\frac{1}{8}e^{-3t}-\frac{1}{8}e^{-t}+\frac{1}{4}e^{-t}\frac{t}{1!}+\frac{1}{2}e^{-t}\frac{t^2}{2!}$$

$$\left[\because\ L^{-1}\left(\frac{1}{s^{n+1}}\right)=\frac{t^n}{n!},\ n \text{ is a positive integer}\right]$$

$$=\frac{1}{8}e^{-3t}+\frac{1}{8}(2t^2+2t-1)\,e^{-t}.$$

(*viii*) Let

$$\frac{5s+3}{(s-1)(s^2+2s+5)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+2s+5}$$

$$\therefore\qquad 5s+3 = A(s^2+2s+5)+(Bs+C)(s-1). \qquad ...(1)$$

This is an identity and hence it is true for all values of s.
Putting $s = 1$, we get $8 = 8A$, *i.e.*, $A = 1$.
Equating the coefficient of s^2 from both sides of (1), we get

$$0 = A + B,\ \ i.e.,\ \ B = -A = -1.$$

Equating the constant term from both sides of (1), we get

$$3 = 5A - C,\ \ i.e.,\ \ C = 5A - 3 = 5 - 3 = 2.$$

$$\therefore\qquad \frac{5s+3}{(s-1)(s^2+2s+5)}=\frac{1}{s-1}+\frac{-s+2}{s^2+2s+5}$$

$$\therefore\qquad L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}=L^{-1}\left\{\frac{1}{s-1}\right\}+L^{-1}\left\{\frac{-s+2}{s^2+2s+5}\right\}$$

$$=e^{t}+L^{-1}\left\{\frac{-(s+1)+3}{(s+1)^2+4}\right\}\qquad\left[\because\ L^{-1}\left(\frac{1}{s-a}\right)=e^{at}\right]$$

$$=e^{t}-L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}+3L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$=e^{t}-e^{-t}L^{-1}\left(\frac{s}{s^2+2^2}\right)+3e^{-t}L^{-1}\left(\frac{1}{s^2+2^2}\right)\qquad\text{[By first shifting property]}$$

$$=e^{t}-e^{-t}\cos 2t+\frac{3}{2}e^{-t}\sin 2t$$

$$\left[\because\ L^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos at\ \text{ and }\ L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{1}{a}\sin at\right]$$

(ix) Now,

$$\frac{s^2+16s-24}{s^4+20s^2+64} = \frac{s^2+16s-24}{(s^2+4)(s^2+16)} = \frac{As+B}{s^2+4}+\frac{Cs+D}{s^2+16} \quad \text{(say)}$$

$$\therefore \quad s^2+16s-24 = (As+B)(s^2+16)+(Cs+D)(s^2+4) \quad ...(1)$$

This is an identity. Equating the coefficients of s^3, s^2, s and the constant term from both sides of (1), we respectively get

$A+C=0, B+D=1, 16A+4C=16$ and $16B+4D=-24$.

Solving, we get

$$A=\frac{4}{3},\ C=-\frac{4}{3},\ B=-\frac{7}{3},\ D=\frac{10}{3}.$$

$$\therefore \quad \frac{s^2+16s-24}{s^4+20s^2+64} = \frac{1}{3}\frac{4s-7}{s^2+4}-\frac{1}{3}\frac{(4s-10)}{s^2+16}$$

$$\therefore \quad L^{-1}\left\{\frac{s^2+16s-24}{s^4+20s^2+64}\right\} = \frac{1}{3}L^{-1}\left\{\frac{4s-7}{s^2+4}\right\}-\frac{1}{3}L^{-1}\left\{\frac{4s-10}{s^2+16}\right\}$$

$$= \frac{4}{3}L^{-1}\left(\frac{s}{s^2+2^2}\right)-\frac{7}{3}L^{-1}\left(\frac{1}{s^2+2^2}\right)-\frac{4}{3}L^{-1}\left(\frac{s}{s^2+4^2}\right)+\frac{10}{3}L^{-1}\left(\frac{1}{s^2+4^2}\right)$$

$$= \frac{4}{3}\cos 2t-\frac{7}{3}\cdot\frac{1}{2}\sin 2t-\frac{4}{3}\cos 4t+\frac{10}{3}\cdot\frac{1}{4}\sin 4t$$

$$\left[\because \ L^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos at,\ L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{1}{a}\sin at\right]$$

$$= \frac{4}{3}\cos 2t-\frac{7}{6}\sin 2t-\frac{4}{3}\cos 4t+\frac{10}{12}\sin 4t.$$

(x) Let

$$\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)} = \frac{As+B}{s^2+2s+5}+\frac{Cs+D}{s^2+2s+2}.$$

$$\therefore \quad s^2+2s-4 = (As+B)(s^2+2s+2)+(Cs+D)(s^2+2s+5) \quad ...(1)$$

This is an identity. Equating the coefficients of s^3, s^2, s and the constant term from both sides of (1), we respectively get

$A+C=0, B+2A+D+2C=1, 2A+2B+5C+2D=2$ and $2B+5D=-4$.

Solving, we get

$A=0,\ B=3,\ C=0,\ D=-2.$

$$\therefore \quad \frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)} = \frac{3}{s^2+2s+5}-\frac{2}{s^2+2s+2}$$

$$= \frac{3}{(s+1)^2+2^2}-\frac{2}{(s+1)^2+1^2}$$

$$\therefore \quad L^{-1}\left\{\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}\right\} = 3L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}-2L^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\}$$

$$= 3e^{-t}L^{-1}\left(\frac{1}{s^2+2^2}\right)-2e^{-t}L^{-1}\left(\frac{1}{s^2+1^2}\right)$$

[By first shifting property]

$$= 3e^{-t}\cdot\frac{1}{2}\sin 2t-2e^{-t}\sin t$$

$$\left[\because\ L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{1}{a}\sin at\right]$$

$$= \frac{3}{2}e^{-t}\sin 2t-2e^{-t}\sin t.$$

Example 2: Find the inverse Laplace transforms of

(i) $\dfrac{s}{(s^2+a^2)^2}$ (ii) $\dfrac{1}{(s^2+a^2)^2}$

(iii) $\dfrac{s+1}{s^2(s+4)}$

Solution: By Theorem 20,

$$L^{-1}\{\bar{f}'(s)\} = -tL^{-1}\{\bar{f}(s)\}. \qquad \text{...(1)}$$

Take

$$\bar{f}(s) = \frac{1}{s^2+a^2}$$

$$\therefore \quad \bar{f}'(s) = -\frac{2s}{(s^2+a^2)^2}.$$

Using (1), we get

$$L^{-1}\left\{-\frac{2s}{(s^2+a^2)^2}\right\} = -tL^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$\therefore \quad L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2}t\cdot\frac{1}{a}\sin at = \frac{1}{2a}t\sin at.$$

Alternative

By Theorem 23,

$$L^{-1}\left\{\int_s^\infty \bar{f}(u)\,du\right\} = \frac{f(t)}{t} = \frac{1}{t}L^{-1}\{\bar{f}(s)\}$$

$$\therefore \quad L^{-1}\{\bar{f}(s)\} = tL^{-1}\left\{\int_s^\infty \bar{f}(u)\,du\right\}$$

$\therefore$
$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = t\,L^{-1}\left\{\int_s^\infty \frac{u}{(u^2+a^2)^2}\,du\right\} \qquad \ldots(2)$$

Now,
$$\int_s^\infty \frac{u}{(u^2+a^2)^2}\,du = \lim_{B\to\infty}\int_s^B \frac{u}{(u^2+a^2)^2}\,du$$

$$= \lim_{B\to\infty}\frac{1}{2}\int_{s^2+a^2}^{B^2+a^2}\frac{1}{x^2}\,dx$$

[Putting $x = u^2 + a^2$, so $dx = 2u\,du$]

$$= \lim_{B\to\infty}\frac{1}{2}\left[-\frac{1}{x}\right]_{s^2+a^2}^{B^2+a^2} = \lim_{B\to\infty}\frac{1}{2}\left[\frac{1}{s^2+a^2}-\frac{1}{B^2+a^2}\right]$$

$$= \frac{1}{2}\cdot\frac{1}{s^2+a^2} \qquad \left[\because \frac{1}{B^2+a^2}\to 0 \text{ as } B\to\infty\right]$$

Therefore, from (2),
$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = t\,L^{-1}\left\{\frac{1}{2}\cdot\frac{1}{s^2+a^2}\right\}$$

$$= t\cdot\frac{1}{2}L^{-1}\left(\frac{1}{s^2+a^2}\right) = t\cdot\frac{1}{2}\cdot\frac{1}{a}\sin at$$

$$= \frac{1}{2a}t\sin at. \qquad \ldots(3)$$

(*ii*) By Theorem 22,
$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(u)\,du,$$

where
$$f(t) = L^{-1}\{\bar{f}(s)\}$$

Let
$$\bar{f}(s) = \frac{s}{(s^2+a^2)^2}$$

$\therefore$
$$f(t) = L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$= \frac{1}{2a}t\sin at \qquad \text{[By (3) of (i)]}$$

$\therefore$
$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right\}$$

$$= \int_0^t \frac{1}{2a}u\sin au\,du$$

$$= \frac{1}{2a}\left[\left\{u\cdot\left(-\frac{\cos au}{a}\right)\right\}_0^t - \int_0^t 1\cdot\left(-\frac{\cos au}{a}\right)du\right]$$

$$= \frac{1}{2a}\left[-\frac{t}{a}\cos at + \left\{\frac{\sin au}{a^2}\right\}_0^t\right]$$

$$= \frac{1}{2a^3}(\sin at - at\cos at).$$

(iii) Now,

$$L^{-1}\left\{\frac{s+1}{s^2(s+4)}\right\} = L^{-1}\left\{\frac{1}{s(s+4)} + \frac{1}{s^2(s+4)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s(s+4)}\right\} + L^{-1}\left\{\frac{1}{s^2(s+4)}\right\} \quad ...(1)$$

We know that

$$L^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}.$$

Using Theorem 22, we get

$$L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s+4}\right\} = \int_0^t e^{-4u}du = \left[\frac{e^{-4u}}{-4}\right]_0^t = \frac{1}{4}(1-e^{-4t})$$

$$\therefore \quad L^{-1}\left\{\frac{1}{s^2(s+4)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s(s+4)}\right\} = \int_0^t \frac{1}{4}(1-e^{-4u})du$$

$$= \frac{1}{4}\left[u - \frac{e^{-4u}}{-4}\right]_0^t = \frac{1}{4}\left\{t + \frac{1}{4}e^{-4t} - \frac{1}{4}\right\}$$

Therefore, from (1),

$$L^{-1}\left\{\frac{s+1}{s^2(s+4)}\right\} = \frac{1}{4}(1-e^{-4t}) + \frac{1}{4}\left(t + \frac{1}{4}e^{-4t} - \frac{1}{4}\right)$$

$$= \frac{1}{4}\left(\frac{3}{4} + t - \frac{3}{4}e^{-4t}\right).$$

Example 3: Find inverse Laplace transform of

(i) $\log\left(1+\frac{a^2}{s^2}\right)$ (ii) $\tan^{-1}\left(\frac{2}{s}\right)$

(iii) $\cot^{-1}(s+1)$ (iv) $\frac{s^2}{(s+a)^3}$

Solution: (*i*) Let

$$\bar{f}(s) = \log\left(1+\frac{a^2}{s^2}\right) = \log\left(\frac{s^2+a^2}{s^2}\right)$$

$$= \log(s^2+a^2) - 2\log s$$

$$\therefore \quad \bar{f}'(s) = \frac{d}{ds}\bar{f}(s) = \frac{2s}{s^2+a^2} - 2\cdot\frac{1}{s}$$

$$\therefore \quad L^{-1}\{\bar{f}'(s)\} = 2L^{-1}\left(\frac{s}{s^2+a^2}\right) - 2L^{-1}\left(\frac{1}{s}\right)$$

$$= 2\cos at - 2 \qquad \text{...(1)}$$

By Theorem 20, we have

$$L^{-1}\{\bar{f}'(s)\} = -t\,L^{-1}\{\bar{f}(s)\}$$

$$\therefore \quad L^{-1}\left\{\log\left(1+\frac{a^2}{s^2}\right)\right\} = L^{-1}\{\bar{f}(s)\} = -\frac{1}{t}L^{-1}\{\bar{f}'(s)\}$$

$$= \frac{2}{t}(1-\cos at) \qquad \text{[By (1)]}$$

(*ii*) Let

$$\bar{f}(s) = \tan^{-1}\left(\frac{2}{s}\right)$$

$$\therefore \quad \bar{f}'(s) = \frac{d}{ds}\bar{f}(s) = \frac{1}{1+\left(\frac{2}{s}\right)^2}\cdot\left(-\frac{2}{s^2}\right) = -\frac{2}{s^2+2^2}$$

$$\therefore \quad L^{-1}\{\bar{f}'(s)\} = L^{-1}\left(-\frac{2}{s^2+2^2}\right) = -2L^{-1}\left(\frac{1}{s^2+2^2}\right)$$

$$= -2\cdot\frac{1}{2}\sin 2t = -\sin 2t \qquad \text{...(1)}$$

By Theorem 20, we have

$$L^{-1}\{\bar{f}'(s)\} = -t\,L^{-1}\{\bar{f}(s)\}$$

$$\therefore \quad L^{-1}\left\{\tan^{-1}\left(\frac{2}{s}\right)\right\} = L^{-1}\{\bar{f}(s)\} = -\frac{1}{t}L^{-1}\{\bar{f}'(s)\}$$

$$= \frac{1}{t}\sin 2t \qquad \text{[By (1)]}$$

(*iii*) Let

$$\bar{f}(s) = \cot^{-1}(s+1) = \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$\therefore \quad \bar{f}'(s) = \frac{d}{ds}\bar{f}(s) = -\frac{1}{1+(s+1)^2}$$

$$\therefore \quad L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{-\frac{1}{1+(s+1)^2}\right\} = -L^{-1}\left\{\frac{1}{1+(s+1)^2}\right\}$$

$$= -e^{-t}L^{-1}\left(\frac{1}{1+s^2}\right) \qquad \text{[By first shifting property]}$$

$$= -e^{-t}L^{-1}\left(\frac{1}{s^2+1^2}\right) = -e^{-t}\sin t$$

By Theorem 20, we have

$$L^{-1}\{\bar{f}'(s)\} = -t\,L^{-1}\{\bar{f}(s)\}$$

$$\therefore \quad L^{-1}\{\cot^{-1}(s+1)\} = L^{-1}\{\bar{f}(s)\} = -\frac{1}{t}L^{-1}\{\bar{f}'(s)\}$$

$$= \frac{e^{-t}}{t}\sin t.$$

(*iv*) Let

$$\bar{f}(s) = \frac{1}{s+a}$$

$$\therefore \quad \bar{f}'(s) = -\frac{1}{(s+a)^2},$$

$$\bar{f}''(s) = \frac{2}{(s+a)^3}.$$

By Theorem 20, we have

$$L^{-1}\{\bar{f}''(s)\} = (-1)^2 t^2 L^{-1}\{\bar{f}(s)\}$$

$$\therefore \quad L^{-1}\left\{\frac{2}{(s+a)^3}\right\} = t^2 L^{-1}\left\{\frac{1}{s+a}\right\} = t^2 e^{-at}$$

$$\therefore \quad L^{-1}\left\{\frac{1}{(s+a)^3}\right\} = \frac{t^2}{2}e^{-at} = g(t), \text{ say.}$$

[**Alternative**

$$L^{-1}\left\{\frac{1}{(s+a)^3}\right\} = e^{-at}L^{-1}\left(\frac{1}{s^3}\right), \text{ by first shifting property}$$

$$= e^{-at}\frac{t^2}{2}, \text{ since } L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!},$$

when n is a positive integer]

$$g'(t) = t\,e^{-at} - \frac{at^2}{2}e^{-at}$$

Here $\qquad g(0) = g'(0) = 0.$

$\therefore$

Therefore by Theorem 21,

$$L^{-1}\left\{\frac{s^2}{(s+a)^3}\right\} = \frac{d^2}{dt^2}L^{-1}\left\{\frac{1}{(s+a)^3}\right\}$$

$$= \frac{d^2}{dt^2}g(t) = e^{-at} - at\,e^{-at} - at\,e^{-at} + \frac{a^2t^2}{2}e^{-at}$$

$$= \frac{1}{2}(a^2t^2 - 4at + 2)\,e^{-at}.$$

Example 4: Find the inverse Laplace transform of the following by Convolution theorem:

(i) $\dfrac{1}{(s^2+1)(s^2+9)}$ (W.B.U.T. 2002) (ii) $\dfrac{1}{(p-2)(p^2+1)}$ (W.B.U.T. 2004)

(iii) $\dfrac{p}{(p^2+1)^2}$ (W.B.U.T. 2005, 2006) (iv) $\dfrac{1}{(s^2+2s+5)^2}$ (W.B.U.T. 2008, 2012)

(v) $\dfrac{1}{(s-1)^2(s-2)^3}$ (W.B.U.T. 2009) (vi) $\dfrac{1}{(s+3)(s-1)}$

(vii) $\dfrac{1}{s(s^2-a^2)}$ (viii) $\dfrac{1}{s\sqrt{s+4}}$

(ix) $\dfrac{s^2}{(s^2+a^2)^2}$ (x) $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$

(xi) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$, $a^2 \neq b^2$ (W.B.U.T. 2013) (xii) $\dfrac{s}{(s^2+1)(s^2+4)(s^2+9)}$

(xiii) $\dfrac{1}{p(p^2+1)^2}$ (xiv) $\dfrac{s}{(s^2+a^2)^2}$. (M–201/11, 13)

Solution: (i) Since

$$L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$$

and

$$L^{-1}\left(\frac{1}{s^2+9}\right) = L^{-1}\left(\frac{1}{s^2+3^2}\right) = \frac{1}{3}\sin 3t,$$

therefore by Convolution theorem

$$L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \int_0^t \sin u \frac{\sin 3(t-u)}{3}\,du$$

$$= \frac{1}{6}\int_0^t \{\cos(4u-3t) - \cos(3t-2u)\}\,du$$

$$= \frac{1}{6}\left[\frac{1}{4}\sin(4u-3t)+\frac{1}{2}\sin(3t-2u)\right]_{u=0}^{t}$$

$$= \frac{1}{6}\left[\frac{1}{4}\sin t+\frac{1}{2}\sin t+\frac{1}{4}\sin 3t-\frac{1}{2}\sin 3t\right]$$

$$= \frac{1}{24}(3\sin t-\sin 3t).$$

(*ii*) Since

$$L^{-1}\left(\frac{1}{p-2}\right) = e^{2t}$$

and

$$L^{-1}\left(\frac{1}{p^2+1}\right) = \sin t,$$

therefore by Convolution theorem

$$L^{-1}\left\{\frac{1}{(p-2)(p^2+1)}\right\} = \int_0^t e^{2(t-u)}\sin u\,du$$

$$= e^{2t}\frac{1}{5}[e^{-2u}\{-2\sin u-\cos u\}]_{u=0}^{t}$$

$$= \frac{1}{5}e^{2t}[e^{-2t}\{-2\sin t-\cos t\}+1]$$

$$= \frac{1}{5}(e^{2t}-2\sin t-\cos t).$$

(*iii*) Since

$$L^{-1}\left\{\frac{p}{p^2+1}\right\} = \cos t$$

and

$$L^{-1}\left\{\frac{1}{p^2+1}\right\} = \sin t,$$

therefore by Convolution theorem

$$L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = L^{-1}\left\{\frac{p}{p^2+1}\cdot\frac{1}{p^2+1}\right\}$$

$$= \int_0^t \cos u\sin(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \{\sin t-\sin(2u-t)\}\,du$$

$$= \frac{1}{2}\left[u\sin t+\frac{1}{2}\cos(2u-t)\right]_{u=0}^{t}$$

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$$= \frac{1}{2}\left[t\sin t + \frac{1}{2}\cos t - \frac{1}{2}\cos t\right] = \frac{1}{2}t\sin t.$$

(*iv*) Here

$$L^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{1}{s^2+2^2}\right\} \qquad \text{[By first shifting property]}$$

$$= e^{-t}\frac{1}{2}\sin 2t = \frac{1}{2}e^{-t}\sin 2t = f(t), \text{ say.}$$

Therefore, by Convolution theorem

$$L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\} = L^{-1}\left\{\frac{1}{(s^2+2s+5)}\cdot\frac{1}{(s^2+2s+5)}\right\}$$

$$= \int_0^t f(u)\, f(t-u)\, du$$

$$= \int_0^t \left(\frac{1}{2}e^{-u}\sin 2u\right)\frac{1}{2}e^{-(t-u)}\sin 2(t-u)\, du$$

$$= \frac{1}{8}e^{-t}\int_0^t 2\sin 2u\,\sin 2(t-u)\, du$$

$$= \frac{1}{8}e^{-t}\int_0^t \{\cos(4u-2t) - \cos 2t\}\, du$$

$$= \frac{1}{8}e^{-t}\left[\frac{1}{4}\sin(4u-2t) - u\cos 2t\right]_{u=0}^{t}$$

$$= \frac{1}{8}e^{-t}\left\{\frac{1}{4}\sin 2t - t\cos 2t + \frac{1}{4}\sin 2t\right\}$$

$$= \frac{1}{16}e^{-t}(\sin 2t - 2t\cos 2t).$$

(*v*) Here

$$L^{-1}\left\{\frac{1}{(s-1)^2}\right\} = e^t L^{-1}\left(\frac{1}{s^2}\right) \qquad \text{[By first shifting property]}$$

$$= e^t \cdot t$$

$$\left[\because\ L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}, \text{ where } n \text{ is a positive integer}\right]$$

$$= g(t), \text{ say.}$$

Similarly,

$$L^{-1}\left\{\frac{1}{(s-2)^3}\right\} = e^{2t}\,L^{-1}\left(\frac{1}{s^3}\right) = e^{2t}\cdot\frac{t^2}{2!}$$

$$= \frac{1}{2}t^2 e^{2t} = f(t), \text{ say.}$$

Therefore, by Convolution theorem

$$L^{-1}\left\{\frac{1}{(s-1)^2(s-2)^3}\right\} = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t \frac{1}{2}u^2 e^{2u}\, e^{t-u}\,(t-u)\,du$$

$$= \frac{1}{2}e^t \int_0^t e^u\,(tu^2-u^3)\,du \qquad \ldots(1)$$

Using integration by parts, we have

$$\int e^u\,(tu^2-u^3)\,du = (tu^2-u^3)\int e^u du - \int\left\{\frac{d}{du}(tu^2-u^3)\right\}\left\{\int e^u\,du\right\}du$$

$$= (tu^2-u^3)\,e^u - \int(2tu-3u^2)\,e^u\,du$$

$$= (tu^2-u^3)\,e^u - \left[(2tu-3u^2)e^u - \int(2t-6u)\,e^u\,du\right]$$

$$= (tu^2-u^3)\,e^u - (2tu-3u^2)e^u + [(2t-6u)e^u + 6e^u]$$

$$= (tu^2-u^3-2tu+3u^2+2t-6u+6)\,e^u$$

$$\therefore \quad \int_0^t e^u\,(tu^2-u^3)\,du = (t^2-4t+6)\,e^t - 2t - 6$$

Therefore, from (1),

$$L^{-1}\left\{\frac{1}{(s-1)^2(s-2)^3}\right\} = \frac{1}{2}e^t\,\{(t^2-4t+6)e^t - 2t - 6\}$$

(*vi*) Since

$$L^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}$$

and

$$L^{-1}\left(\frac{1}{s-1}\right) = e^t,$$

Therefore, by Convolution theorem

$$L^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\} = \int_0^t e^{-3u}\,e^{t-u}\,du = e^t\int_0^t e^{-4u}\,du$$

$$= e^t \left[\frac{e^{-4u}}{-4}\right]_0^t = \frac{e^t}{4}(1 - e^{-4t}).$$

(*vii*) Now,

$$L^{-1}\left(\frac{1}{s^2 - a^2}\right) = \frac{1}{a}\sinh at = f(t), \text{ say.}$$

$$L^{-1}\left(\frac{1}{s}\right) = 1 = g(t), \text{ say.}$$

Therefore, by Convolution theorem

$$L^{-1}\left\{\frac{1}{s(s^2 - a^2)}\right\} = L^{-1}\left\{\frac{1}{s^2 - a^2} \cdot \frac{1}{s}\right\} = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \left(\frac{1}{a}\sinh au\right).1\, du = \frac{1}{a^2}[\cosh au]_0^t$$

$$= \frac{1}{a^2}(\cosh at - 1)$$

(*viii*) Now,

$$L^{-1}\left\{\frac{1}{(s+4)^{1/2}}\right\} = e^{-4t} L\left(\frac{1}{s^{1/2}}\right) \qquad \text{[By first shifting property]}$$

$$= e^{-4t} \frac{t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} \qquad \left[\because L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{\Gamma(n+1)},\ n > -1\right]$$

$$= e^{-4t} \frac{t^{-1/2}}{\sqrt{\pi}} = f(t), \text{ say.}$$

$$L^{-1}\left(\frac{1}{s}\right) = 1 = g(t), \text{ say.}$$

Therefore, by Convolution theorem

$$L^{-1}\left\{\frac{1}{s\sqrt{s+4}}\right\} = L^{-1}\left\{\frac{1}{(s+4)^{1/2}} \cdot \frac{1}{s}\right\}$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t e^{-4u} \frac{u^{-1/2}}{\sqrt{\pi}}\, du.$$

(*ix*) Now,

$$L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at = f(t), \text{ say.}$$

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Therefore, by Convolution theorem

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+a^2}\right\}$$

$$= \int_0^t f(u)\,f(t-u)\,du = \int_0^t \cos au \,\cos a(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \{\cos at + \cos a(2u-t)\}\,du$$

$$= \frac{1}{2}\left[u\cos at + \frac{1}{2a}\sin a(2u-t)\right]_{u=0}^{t}$$

$$= \frac{1}{2}\left\{t\cos at + \frac{1}{2a}\sin at + \frac{1}{2a}\sin at\right\}$$

$$= \frac{1}{2}\left(t\cos at + \frac{1}{a}\sin at\right)$$

(x)

$$L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t = g(t),\ \text{say.}$$

$$L^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} = e^{-t}L^{-1}\left(\frac{s}{s^2+1}\right) \qquad \text{[By first shifting property]}$$

$$= e^{-t}\cos t = f(t),\ \text{say.}$$

Therefore, by Convolution theorem

$$L^{-1}\left\{\frac{s^2+s}{(s^2+1)(s^2+2s+2)}\right\} = L^{-1}\left\{\frac{s+1}{(s+1)^2+1}\cdot\frac{s}{s^2+1}\right\}$$

$$= \int_0^t f(u)\,g(t-u)\,du = \int_0^t e^{-u}\cos u\,\cos(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-u}\{\cos t + \cos(2u-t)\}\,du$$

$$= \frac{1}{2}(\cos t)\int_0^t e^{-u}\,du + \frac{1}{2}\int_0^t e^{-u}\cos(2u-t)\,du \qquad \text{...(1)}$$

Now,

$$\int e^{-u}\cos(2u-t)\,du = \frac{1}{2}\int e^{-\frac{1}{2}(x+t)}\cos x\,dx \qquad \text{[Putting } 2u-t=x, \text{ so } 2du=dx]$$

$$= \frac{1}{2}e^{-t/2}\int e^{-\frac{x}{2}}\cos x\,dx$$

$$= \frac{2}{5} e^{-t/2} e^{-x/2} \left\{ -\frac{1}{2} \cos x + \sin x \right\}$$

$$= \frac{2}{5} e^{-t/2} e^{-u+\frac{t}{2}} \left\{ -\frac{1}{2} \cos(2u-t) + \sin(2u-t) \right\} \qquad [\because \; x = 2u - t]$$

$$= \frac{2}{5} e^{-u} \left\{ -\frac{1}{2} \cos(2u-t) + \sin(2u-t) \right\}$$

Therefore, from (1), we have

$$L^{-1}\left\{\frac{s^2+s}{(s^2+1)(s^2+2s+2)}\right\} = \frac{1}{2}[-e^{-u}]_0^t \cos t + \frac{1}{5}\left[e^{-u}\left\{-\frac{1}{2}\cos(2u-t) + \sin(2u-t)\right\}\right]_{u=0}^t$$

$$= \frac{1}{2}(1-e^{-t})\cos t + \frac{1}{10}[e^{-t}(-\cos t + 2\sin t) + \cos t + 2\sin t]$$

$$= \frac{1}{10}\{e^{-t}(2\sin t - 6\cos t) + 2\sin t + 6\cos t\}$$

(*xi*) Since

$$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

and

$$L^{-1}\left(\frac{s}{s^2+b^2}\right) = \cos bt$$

therefore, by Convolution theorem

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\}$$

$$= \int_0^t \cos au \cos b(t-u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos\{(a-b)u+bt\} + \cos\{(a+b)u-bt\}]\, du$$

$$= \frac{1}{2}\left[\frac{\sin\{(a-b)u+bt\}}{a-b} + \frac{\sin\{(a+b)u-bt\}}{a+b}\right]_{u=0}^t$$

$$= \frac{1}{2}\left\{\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right\}$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}.$$

(*xii*) Now,

$$L^{-1}\left(\frac{s}{s^2+4}\right) = L^{-1}\left(\frac{s}{s^2+2^2}\right) = \cos 2t = g(t), \text{ say.}$$

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$$L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \frac{1}{24}(3\sin t - \sin 3t)$$ [By (*i*)]

$$= f(t), \text{ say.}$$

Therefore, by Convolution theorem

$$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right\} = L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\cdot\frac{1}{(s^2+4)}\right\}$$

$$= \int_0^t f(u)\,g(t-u)\,du$$

$$= \frac{1}{24}\int_0^t (3\sin u - \sin 3u)\cos 2(t-u)\,du$$

$$= \frac{1}{48}\int_0^t [3\{2\sin u\cos 2(t-u)\} - 2\sin 3u\cos 2(t-u)]\,du$$

$$= \frac{1}{48}\int_0^t [3\{\sin(2t-u)+\sin(3u-2t)\} - \{\sin(2t+u)+\sin(5u-2t)\}]\,du$$

$$= \frac{1}{48}\left[3\left\{\cos(2t-u) - \frac{1}{3}\cos(3u-2t)\right\} - \left\{-\cos(2t+u) - \frac{1}{5}\cos(5u-2t)\right\}\right]_{u=0}^{t}$$

$$= \frac{1}{48}\left\{3\cos t - \cos t - 3\cos 2t + \cos 2t + \cos 3t + \frac{1}{5}\cos 3t - \cos 2t - \frac{1}{5}\cos 2t\right\}$$

$$= \frac{1}{48}\left(2\cos t - \frac{16}{5}\cos 2t + \frac{6}{5}\cos 3t\right)$$

$$= \frac{1}{120}(5\cos t - 8\cos 2t + 3\cos 3t).$$

(*xiii*) Now,

$$L^{-1}\left(\frac{1}{p^2}\right) = t = g(t), \text{ say} \qquad \left[\because L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}, \text{ when } n \text{ is a positive integer}\right]$$

$$L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2}t\sin t = f(t), \text{ say.}$$ [By (*iii*)]

Therefore, by Convolution theorem

$$L^{-1}\left\{\frac{1}{p(p^2+1)^2}\right\} = L^{-1}\left\{\frac{p}{(p^2+1)^2}\cdot\frac{1}{p^2}\right\}$$

$$= \int_0^t \left(\frac{1}{2}u\sin u\right)(t-u)\,du$$

$$= \frac{t}{2}\int_0^t u \sin u \, du - \frac{1}{2}\int_0^t u^2 \sin u \, du \quad \ldots(1)$$

Using integration by parts, we get

$$\int u \sin u \, du = u\int \sin u \, du - \int\left(\frac{du}{du}\right)\left(\int \sin u \, du\right) du$$

$$= -u \cos u + \sin u$$

$$\int u^2 \sin u \, du = u^2 \int \sin u \, du + 2\int u \cos u \, du$$

$$= -u^2 \cos u + 2\left[u\int \cos u \, du - \int 1 \cdot \sin u \, du\right]$$

$$= -u^2 \cos u + 2u \sin u + 2\cos u$$

Therefore, from (1), we get

$$L^{-1}\left\{\frac{1}{p(p^2+1)^2}\right\} = \frac{1}{2}[-tu \cos u + t \sin u + (u^2-2)\cos u - 2u \sin u]_{u=0}^{t}$$

$$= \frac{1}{2}\{-2\cos t - t\sin t + 2\}$$

(*xiv*) Since $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$ and $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at,$

therefore by Convolution theorem:

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\}$$

$$= \int_0^t (\cos au)\frac{1}{a}\sin a(t-u)\, du = \frac{1}{2a}\int_0^t \{\sin at - \sin(2au - at)\}\, du$$

$$= \frac{1}{2a}\left[u \sin at + \frac{1}{2a}\cos(2au - at)\right]_{u=0}^{t}$$

$$= \frac{1}{2a}\left[t \sin at + \frac{1}{2a}\cos at - \frac{1}{2a}\cos at\right] = \frac{t \sin at}{2a}.$$

Example 5: Apply Convolution theorem to prove that

$$\int_0^t \sin u \cos(t-u)\, du = \frac{t}{2}\sin t.$$

(*W.B.U.T. 2007*)

Solution: Now,

$$L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t = f(t), \text{ say.}$$

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$$L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t = g(t), \text{ say.}$$

By Convolution theorem

$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{s}{s^2+1}\right\} = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t \sin u \cos(t-u)\,du \qquad \text{...(1)}$$

By Theorem 20, we have

$$L^{-1}\{\bar{f}'(s)\} = -tL^{-1}\{\bar{f}(s)\}$$

Take $\qquad \bar{f}(s) = \dfrac{1}{s^2+1}$

$\therefore \qquad \bar{f}'(s) = \dfrac{d}{ds}\bar{f}(s) = \dfrac{-2s}{(s^2+1)^2}$

$\therefore \qquad L^{-1}\left\{-\dfrac{2s}{(s^2+1)^2}\right\} = -t\,L^{-1}\left\{\dfrac{1}{s^2+1}\right\}$

$\therefore \qquad L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \dfrac{t}{2}L^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \dfrac{t}{2}\sin t \qquad \text{...(2)}$

From (1) and (2), we get

$$\int_0^t \sin u \cos(t-u)\,du = \frac{t}{2}\sin t.$$

Example 6: Evaluate the inverse Laplace transform of

(*i*) $\dfrac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$ $\qquad$ (*ii*) $\dfrac{e^{-cs}}{s^2(s+a)}\ (c>0)$

(*iii*) $\dfrac{e^{-as}}{(s+b)^{5/2}}$ $\qquad$ (*iv*) $e^{-s}\left(\dfrac{1-\sqrt{s}}{s^2}\right)^2$

Solution: By second shifting property,

$$L^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a)u(t-a) = \begin{cases} f(t-a), & t>a \\ 0, & t<a, \end{cases}$$

where $f(t) = L^{-1}\{\bar{f}(s)\}$.

(*i*) We have

$$L^{-1}\left(\frac{s}{s^2+\pi^2}\right) = \cos \pi t$$

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and

$$L^{-1}\left(\frac{\pi}{s^2+\pi^2}\right) = \sin \pi t.$$

Using second shifting property,

$$L^{-1}\left\{\frac{se^{-\frac{s}{2}}+\pi e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-s/2}\cdot\frac{s}{s^2+\pi^2}\right\}+L^{-1}\left\{e^{-s}\cdot\frac{\pi}{s^2+\pi^2}\right\}$$

$$= \left[\cos\left\{\pi\left(t-\frac{1}{2}\right)\right\}\right]u\left(t-\frac{1}{2}\right)+[\sin\{\pi(t-1)\}]u(t-1)$$

$$= (\sin \pi t)\,u\left(t-\frac{1}{2}\right)-(\sin \pi t)\,u(t-1)$$

$$= \left\{u\left(t-\frac{1}{2}\right)-u(t-1)\right\}\sin \pi t,$$

where $u\left(t-\frac{1}{2}\right)$, $u(t-1)$ are unit step functions.

(*ii*) Let

$$\frac{1}{s^2(s+a)} = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+a}$$

$\therefore$

$$1 = As(s+a)+B(s+a)+Cs^2 \qquad ...(1)$$

This is an identity and hence it is true for all values of s.

Putting $s = 0$, we get $\quad 1= aB,\ i.e.,\ B = \frac{1}{a}.$

Putting $s = -a$, we get $\quad 1= Ca^2,\ i.e.,\ C = \frac{1}{a^2}.$

Equating the coefficient of s^2 from both sides of (1), we get

$$0 = A + C,\ i.e.,\ A = -C = -\frac{1}{a^2}.$$

We know that $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$, when n is a positive integer.

$\therefore$

$$L^{-1}\left(\frac{1}{s^2}\right) = \frac{t}{1!} = t.$$

Also

$$L^{-1}\left(\frac{1}{s}\right) = 1,\quad L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}.$$

Therefore, by second shifting property, we have

$$L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = L^{-1}\left[e^{-cs}\left\{-\frac{1}{a^2}\cdot\frac{1}{s}+\frac{1}{a}\cdot\frac{1}{s^2}+\frac{1}{a^2}\cdot\frac{1}{(s+a)}\right\}\right]$$

$$= -\frac{1}{a^2}L^{-1}\left\{e^{-cs}\cdot\frac{1}{s}\right\}+\frac{1}{a}L^{-1}\left\{e^{-cs}\cdot\frac{1}{s^2}\right\}+\frac{1}{a^2}L^{-1}\left\{e^{-cs}\cdot\frac{1}{s+a}\right\}$$

$$= -\frac{1}{a^2}\cdot 1\cdot u(t-c)+\frac{1}{a}\cdot(t-c)u(t-c)+\frac{1}{a^2}e^{-a(t-c)}u(t-c)$$

$$= \frac{1}{a^2}\{-1+a(t-c)+e^{-a(t-c)}\}u(t-c),$$

...ere $u(t-c)$, $c>0$, is a unit step function.

(*iii*) Now,

$$L^{-1}\left\{\frac{1}{(s+b)^{5/2}}\right\} = e^{-bt}L^{-1}\left(\frac{1}{s^{5/2}}\right) \qquad \text{[By first shifting property]}$$

$$= e^{-bt}\frac{t^{3/2}}{\Gamma\left(\frac{5}{2}\right)} \qquad \left[\because\ L^{-1}\left(\frac{1}{s^{n+1}}\right)=\frac{t^n}{\Gamma(n+1)},\ n>-1\right]$$

$$= \frac{e^{-bt}t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{e^{-bt}t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{4}{3\sqrt{\pi}}e^{-bt}t^{3/2}$$

Therefore, by second shifting property, we have

$$L^{-1}\left\{\frac{e^{-as}}{(s+b)^{5/2}}\right\} = \frac{4}{3\sqrt{\pi}}e^{-b(t-a)}(t-a)^{3/2}u(t-a),$$

...here $u(t-a)=\begin{cases}0, & t<a\\ 1, & t\geq a,\end{cases}$ is a unit step function.

(*iv*) Now,

$$L^{-1}\left\{\frac{1-\sqrt{s}}{2}\right\}^2 = L^{-1}\left\{\frac{1-2\sqrt{s}+s}{s^4}\right\}$$

$$= \frac{t^3}{6} - \frac{2t^{5/2}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} + \frac{t^2}{2}$$

$$= \frac{t^3}{6} - \frac{16}{15\sqrt{\pi}} t^{5/2} + \frac{t^2}{2}.$$

Therefore, by second shifting property, we have

$$L^{-1}\left\{e^{-s}\left(\frac{1-\sqrt{s}}{s^2}\right)^2\right\} = \left[\frac{1}{6}(t-1)^3 - \frac{16}{15\sqrt{\pi}}(t-1)^{5/2} + \frac{1}{2}(t-1)^2\right]u(t-1),$$

where $u(t-1) = \begin{cases} 0, & t<1 \\ 1, & t \geq 1, \end{cases}$ is a unit step function.

6.12 SOLUTION OF LINEAR ORDINARY DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS (INITIAL VALUE PROBLEM) USING LAPLACE TRANSFORM

We have already proved that (see Theorem 8)

$$L\{f'(t)\} = -f(0) + sL\{f(t)\}$$

$$L\{f''(t)\} = -f'(0) - sf(0) + s^2L\{f(t)\}$$

$$L\{f'''(t)\} = -f''(0) - sf'(0) - s^2 f(0) + s^3L\{f(t)\}$$

Denoting $L(y)$ by $\bar{y}$, we get

$$\boxed{\begin{aligned} L(y') &= s\bar{y} - y(0) \\ L(y'') &= s^2\bar{y} - sy(0) - y'(0) \\ L(y''') &= s^3\bar{y} - s^2y(0) - sy'(0) - y''(0) \end{aligned}}$$

ILLUSTRATIVE EXAMPLES

Example 1: (*i*) Solve the following differential equation by Laplace transform:

$$(D^2 + 6D + 9)y = 0, \ y(0) = y'(0) = 1.$$ (*M–201/13*)

(*ii*) Solve $(D^3 - 2D^2 + 5D)y = 0$ with $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$.

Solution: (*i*) Let $L(y) = \bar{y}$. Taking Laplace transform of both sides of the given *ODE*, we get

$$L(y'') + 6L(y') + 9L(y) = 0 \quad \ldots(1)$$

We know that $L(y') = s\bar{y} - y(0)$, $L(y'') = s^2\bar{y} - sy(0) - y'(0)$.

Using given conditions:

$$L(y') = s\bar{y} - 1,\ L(y'') = s^2\bar{y} - s - 1.$$

Therefore, equation (1) becomes:

$$s^2\bar{y} - s - 1 + 6s\bar{y} - 6 + 9\bar{y} = 0$$

$$(s^2 + 6s + 9)\bar{y} = 7 + s$$

$$\therefore \quad \bar{y} = \frac{s+7}{s^2+6s+9} = \frac{(s+3)+4}{(s+3)^2} = \frac{1}{s+3} + \frac{4}{(s+3)^2}$$

$$\therefore \quad y = L^{-1}\left\{\frac{1}{s+3}\right\} + L^{-1}\left\{\frac{4}{(s+3)^2}\right\}$$

$$= e^{-3t} + 4e^{-3t}L^{-1}\left\{\frac{1}{s^2}\right\} = e^{-3t} + 4e^{-3t}t = (4t+1)e^{-3t}$$

This is the required solution.

(*ii*) Let $L(y) = \bar{y}$. Taking Laplace transform of both sides of given ODE, we get

$$L(y''') - 2L(y'') + 5L(y') = 0. \quad \text{...(1)}$$

We know that

$$L(y') = s\bar{y} - y(0),\ L(y'') = s^2\bar{y} - sy(0) - y'(0),$$

$$L(y''') = s^3\bar{y} - s^2y(0) - sy'(0) - y''(0).$$

Using given conditions,

$$L(y') = s\bar{y},\ L(y'') = s^2\bar{y},\ L(y''') = s^3\bar{y} - 1.$$

Therefore, equation (1) becomes,

$$s^3\bar{y} - 1 - 2s^2\bar{y} + 5s\bar{y} = 0, \text{ or } (s^3 - 2s^2 + 5s)\bar{y} = 1$$

$$\therefore \quad \bar{y} = \frac{1}{s^3 - 2s^2 + 5s} = \frac{1}{s\{(s-1)^2 + 2^2\}}$$

$$\therefore \quad y = L^{-1}\left[\frac{1}{s\{(s-1)^2 + 2^2\}}\right]$$

We have, $L^{-1}\left\{\dfrac{1}{(s-1)^2+2^2}\right\} = e^tL^{-1}\left\{\dfrac{1}{s^2+2^2}\right\} = e^t\dfrac{1}{2}\sin 2t = f(t)$, say,

and $\quad L^{-1}\left(\dfrac{1}{s}\right) = 1 = g(t)$, say.

Therefore, by Convolution theorem,

$$y = L^{-1}\left[\frac{1}{s\{(s-1)^2+2^2\}}\right] = L^{-1}\left[\frac{1}{(s-1)^2+2^2}\cdot\frac{1}{s}\right] = \int_0^t f(u)g(t-u)du$$

$$= \int_0^t \left(\frac{1}{2}e^u \sin 2u\right)\cdot 1du = \frac{1}{2}\cdot\frac{1}{5}[e^u(\sin 2u - 2\cos 2u)]_0^t$$

$$= \frac{1}{10}\{e^t(\sin 2t - 2\cos 2t)+2\}$$

This is the required solution.

Example 2: Solve $(D^2-3D+2)y = 4e^{2t}$ with $y(0) = -3$ and $y'(0) = 5$.

Solution: Let $L(y) = \bar{y}$. Taking Laplace transform of both sides of given ODE, we get

$$L(y'')-3L(y')+2L(y) = 4L(e^{2t}) = \frac{4}{s-2} \qquad ...(1)$$

We know that

$$L(y') = s\bar{y}-y(0),\ L(y'') = s^2\bar{y}-sy(0)-y'(0)$$

Using given conditions,

$$L(y') = s\bar{y}+3,\ L(y'') = s^2\bar{y}+3s-5$$

Therefore, equation (1) becomes

$$s^2\bar{y}+3s-5-3(s\bar{y}+3)+2\bar{y} = \frac{4}{s-2}$$

or $$(s^2-3s+2)\bar{y} = \frac{4}{s-2}+14-3s$$

$\therefore$ $$\bar{y} = \frac{-3s^2+20s-24}{(s-2)(s^2-3s+2)} = \frac{-3s^2+20s-24}{(s-1)(s-2)^2}$$

Let $$\frac{-3s^2+20s-24}{(s-1)(s-2)^2} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}.$$

$\therefore$ $$-3s^2+20s-24 = A(s-2)^2+B(s-1)(s-2)+C(s-1) \qquad ...(2)$$

This is an identity and hence it is true for all values of s.

Putting $s = 1$, we get $A = -7$

Putting $s = 2$, we get $C = 4$

Equating the coefficient of s^2 from both sides of (2), we get

$$A+B = -3,\ i.e.,\ B = -3-A = -3+7 = 4.$$

$\therefore$ $$\bar{y} = \frac{-7}{s-1}+\frac{4}{s-2}+\frac{4}{(s-2)^2}.$$

$\therefore$ $$y = L^{-1}\left\{\frac{-7}{s-1}+\frac{4}{s-2}+\frac{4}{(s-2)^2}\right\} = -7L^{-1}\left\{\frac{1}{s-1}\right\}+4L^{-1}\left\{\frac{1}{s-2}\right\}+4L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= -7e^t + 4e^{2t} + 4e^{2t} L^{-1}\left(\frac{1}{s^2}\right) \quad \left[\because L^{-1}\left(\frac{1}{s-a}\right) = e^{at} \text{ and by first shifting property}\right]$$

$$= -7e^t + 4e^{2t} + 4te^{2t} \quad \left[\because L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}, n \text{ is a positive integer}\right]$$

This is the required solution.

Example 3: Solve the differential equation using Laplace transform:

$$y'' - 3y' + 2y = 4t + e^{3t},$$

where $y(0) = 1$ and $y'(0) = -1$. *(W.B.U.T. 2002)*

Solution: Let $L(y) = \bar{y}$. Taking Laplace transform of both sides of the given ODE, we get

$$L(y'') - 3L(y') + 2L(y) = 4L(t) + L(e^{3t})$$

or $$s^2\bar{y} - sy(0) - y'(0) - 3\{s\bar{y} - y(0)\} + 2\bar{y} = \frac{4}{s^2} + \frac{1}{s-3}$$

or $$s^2\bar{y} - s + 1 - 3(s\bar{y} - 1) + 2\bar{y} = \frac{s^2 + 4s - 12}{s^2(s-3)} \quad \text{[By given conditions]}$$

or $$(s^2 - 3s + 2)\bar{y} = \frac{(s-2)(s+6)}{s^2(s-3)} + s - 4$$

or $$(s-1)(s-2)\bar{y} = \frac{(s-2)(s+6)}{s^2(s-3)} + s - 4$$

$\therefore$ $$\bar{y} = \frac{s+6}{s^2(s-1)(s-3)} + \frac{s-4}{(s-1)(s-2)} \quad \text{...(1)}$$

Let $$\frac{s+6}{s^2(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-3}$$

$\therefore$ $$s + 6 = As(s-1)(s-3) + B(s-1)(s-3) + Cs^2(s-3) + Ds^2(s-1) \quad \text{...(2)}$$

This is an identity and hence it is true for all values of s. Putting $s = 0, 1, 3$ successively, we get $B = 2$, $C = -\frac{7}{2}$, $D = \frac{1}{2}$.

Equating the coefficient of s^3 from both sides of (2), we get

$$0 = A + C + D, \text{ i.e., } A = -C - D = \frac{7}{2} - \frac{1}{2} = 3.$$

Again let $$\frac{s-4}{(s-1)(s-2)} = \frac{E}{s-1} + \frac{F}{s-2}$$

$\therefore$ $$s - 4 = E(s-2) + F(s-1).$$

Putting $s = 1, 2$ successively, we get

$$E = 3, F = -2$$

$\therefore$ $$\bar{y} = \frac{3}{s} + \frac{2}{s^2} - \frac{7}{2}\cdot\frac{1}{s-1} + \frac{1}{2}\cdot\frac{1}{s-3} + \frac{3}{s-1} - \frac{2}{s-2} \quad \text{[By (1)]}$$

We know that $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}, L^{-1}\left(\frac{1}{s}\right) = 1$ and $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}, n$ is a positive integer.

$$\therefore \quad y = 3L^{-1}\left(\frac{1}{s}\right) + 2L^{-1}\left(\frac{1}{s^2}\right) - \frac{7}{2}L^{-1}\left(\frac{1}{s-1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s-3}\right) + 3L^{-1}\left(\frac{1}{s-1}\right) - 2L^{-1}\left(\frac{1}{s-2}\right)$$

$$= 3 + 2t - \frac{7}{2}e^t + \frac{1}{2}e^{3t} + 3e^t - 2e^{2t}$$

$$= 3 + 2t + \frac{1}{2}(e^{3t} - e^t) - 2e^{2t}.$$

This is the required solution.

Example 4: Solve $\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} - \frac{dy}{dt} + y = 8te^{-t}$

where $y(0) = 0, y'(0) = 1, y''(0) = 0$.

Solution: Let $L(y) = \bar{y}$. Taking Laplace transform of both sides of the given ODE, we get

$$L\left(\frac{d^3y}{dt^3}\right) - L\left(\frac{d^2y}{dt^2}\right) - L\left(\frac{dy}{dt}\right) + L(y) = 8L(te^{-t})$$

or $$s^3\bar{y} - s^2y(0) - sy'(0) - y''(0) - \{s^2\bar{y} - sy(0) - y'(0)\} - \{s\bar{y} - y(0)\} + \bar{y}$$

$$= -8\frac{d}{ds}\{L(e^{-t})\}$$ [By Theorem 6]

or $$s^3\bar{y} - s - s^2\bar{y} + 1 - s\bar{y} + \bar{y} = -8\frac{d}{ds}\left(\frac{1}{s+1}\right)$$ [By given conditions and since $L(e^{at}) = \frac{1}{s-a}$]

or $$(s^3 - s^2 - s + 1)\bar{y} - s + 1 = \frac{8}{(s+1)^2}$$

or $$(s-1)^2(s+1)\bar{y} = s - 1 + \frac{8}{(s+1)^2}$$

$$\therefore \quad \bar{y} = \frac{1}{(s-1)(s+1)} + \frac{8}{(s-1)^2(s+1)^3} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) + \frac{8}{(s-1)^2(s+1)^3} \quad ...(1)$$

Let $$\frac{1}{(s-1)^2(s+1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)^3}$$

$$\therefore \quad 1 = A(s-1)(s+1)^3 + B(s+1)^3 + C(s-1)^2(s+1)^2 + D(s-1)^2(s+1) + E(s-1)^2. \quad ...(2)$$

This is an identity and hence it is true for all values of s.

Putting $s = 1, -1$ successively, we get

$$B = \frac{1}{8},\ E = \frac{1}{4}.$$

Equating the coefficients of s^4, s^3, s^2 successively from both sides of (2), we get

$$A + C = 0,\ 2A + B + D = 0,\ 3B - 2C - D + E = 0$$

Solving: $\quad A = -\frac{3}{16},\quad C = \frac{3}{16},\quad D = \frac{1}{4}.$

Therefore, from (1), we get

$$\bar{y} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) - \frac{3}{2}\cdot\frac{1}{s-1} + \frac{1}{(s-1)^2} + \frac{3}{2}\cdot\frac{1}{s+1} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}$$

Now, we know that $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$ and

$$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!},\ n \text{ is a positive integer.}$$

$$\therefore \quad y = \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}L^{-1}\left(\frac{1}{s+1}\right) - \frac{3}{2}L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$+\frac{3}{2}L^{-1}\left\{\frac{1}{s+1}\right\} + 2L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + 2L^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= \frac{1}{2}e^t - \frac{1}{2}e^{-t} - \frac{3}{2}e^t + e^t L\left(\frac{1}{s^2}\right) + \frac{3}{2}e^{-t}$$

$$+\,2e^{-t}L^{-1}\left(\frac{1}{s^2}\right) + 2e^{-t}L^{-1}\left(\frac{1}{s^3}\right) \quad \text{[By first shifting property]}$$

$$= -e^t + e^{-t} + te^t + 2te^{-t} + t^2e^{-t}$$

$$= (1+t)^2 e^{-t} - (1-t)e^t$$

This is the required solution.

Example 5: Solve by Laplace transform the equation $y''(t) + y(t) = 8\cos t$, where $y(0) = 1$, $y'(0) = -1$. *(W.B.U.T. 2005)*

Solution: Let $L(y) = \bar{y}$. Taking Laplace transform of both sides of the given ODE, we get

$$L(y'') + L(y) = 8L(\cos t)$$

or $$s^2\bar{y} - sy(0) - y'(0) + \bar{y} = 8\frac{s}{s^2+1} \quad \text{[Using given conditions]}$$

or $$s^2\bar{y} - s + 1 + \bar{y} = \frac{8s}{s^2+1}$$

or $$(s^2+1)\bar{y} = \frac{8s}{s^2+1} + s - 1.$$

$$\bar{y} = \frac{8s}{(s^2+1)^2} + \frac{s-1}{s^2+1} = -4\frac{d}{ds}\left(\frac{1}{s^2+1}\right) + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$\therefore$
$$y = -4L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\} + L^{-1}\left(\frac{s}{s^2+1}\right) - L^{-1}\left(\frac{1}{s^2+1}\right)$$

$\therefore$
$$= 4tL^{-1}\left(\frac{1}{s^2+1}\right) + \cos t - \sin t \qquad \left[\because L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = -tL^{-1}\{\bar{f}(s)\}\right]$$

$$= 4t\sin t + \cos t - \sin t$$

$$= (4t-1)\sin t + \cos t$$

This is the required solution.

Example 6: Solve $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 20\sin 2t$ with $y(0) = 1$ and $y'(0) = 2$.

Solution: Let $L(y) = \bar{y}$. Taking Laplace transform of both sides of the given ODE, we get

$$L\left(\frac{d^2y}{dt^2}\right) - L\left(\frac{dy}{dt}\right) - 2L(y) = 20L(\sin 2t)$$

or
$$s^2\bar{y} - sy(0) - y'(0) - \{s\bar{y} - y(0)\} - 2\bar{y} = 20\cdot\frac{2}{s^2+2^2}$$

or
$$s^2\bar{y} - s - 2 - s\bar{y} + 1 - 2\bar{y} = \frac{40}{s^2+4} \qquad \text{[By given conditions]}$$

$\therefore$
$$(s^2 - s - 2)\bar{y} = \frac{40}{s^2+4} + s + 1 = \frac{s^3+s^2+4s+44}{s^2+4}$$

$\therefore$
$$\bar{y} = \frac{s^3+s^2+4s+44}{(s+1)(s-2)(s^2+4)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+4} \text{ (say)} \qquad \text{...(1)}$$

$$\therefore\ s^3+s^2+4s+44 = A(s-2)(s^2+4) + B(s+1)(s^2+4) + (Cs+D)(s+1)(s-2) \qquad \text{...(2)}$$

This is an identity and hence it is true for all values of s. Putting $s = -1, 2$ successively, we get $40 = -15A$, *i.e.*, $A = -\frac{8}{3}$ and $B = \frac{8}{3}$.

Equating the coefficient of s^3 and constant term from both sides of (2), we get $1 = A + B + C$ and $44 = -8A + 4B - 2D$.

Solving: $C = 1, D = -6$, since $A = -\frac{8}{3}$, $B = \frac{8}{3}$

Therefore, from (1),

$$\bar{y} = -\frac{8}{3}\cdot\frac{1}{s+1} + \frac{8}{3}\cdot\frac{1}{s-2} + \frac{s-6}{s^2+4}$$

$$\therefore \quad y = -\frac{8}{3}L^{-1}\left(\frac{1}{s+1}\right)+\frac{8}{3}L^{-1}\left(\frac{1}{s-2}\right)+L^{-1}\left(\frac{s}{s^2+2^2}\right)-6L^{-1}\left(\frac{1}{s^2+2^2}\right)$$

$$= -\frac{8}{3}e^{-t}+\frac{8}{3}e^{2t}+\cos 2t-3\sin 2t$$

This is the required solution.

Example 7: Solve the differential equation by Laplace transform:

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y = e^{-t}\sin t,$$

where $y(0)=0$, $y'(0)=1$. (*W.B.U.T. 2012*)

Solution: Let $L(y) = \overline{y}$. Taking Laplace transform of both sides of the given ODE, we get

$$L\left(\frac{d^2y}{dt^2}\right)+2L\left(\frac{dy}{dt}\right)+5L(y) = L(e^{-t}\sin t)$$

or $$s^2\overline{y}-sy(0)-y'(0)+2\{s\overline{y}-y(0)\}+5\overline{y} = \frac{1}{(s+1)^2+1} \qquad \left[\because L(e^{at}\sin bt)=\frac{b}{(s-a)^2+b^2}\right]$$

or $$s^2\overline{y}-1+2s\overline{y}+5\overline{y} = \frac{1}{s^2+2s+2}$$ [By given conditions]

or $$(s^2+2s+5)\overline{y} = \frac{1}{s^2+2s+2}+1$$

$$\therefore \quad \overline{y} = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{(s+1)^2+2}{\{(s+1)^2+1\}(s+1)^2+4\}}$$

$$\therefore \quad y = L^{-1}\left[\frac{(s+1)^2+2}{\{(s+1)^2+1\}\{(s+1)^2+4\}}\right] = e^{-t}L^{-1}\left\{\frac{s^2+2}{(s^2+1)(s^2+4)}\right\}$$

[By first shifting property]

$$= \frac{e^{-t}}{3}L^{-1}\left\{\frac{3s^2+6}{(s^2+1)(s^2+4)}\right\} = \frac{e^{-t}}{3}L^{-1}\left\{\frac{s^2+4+2(s^2+1)}{(s^2+1)(s^2+4)}\right\}$$

$$= \frac{e^{-t}}{3}\left\{L^{-1}\left(\frac{1}{s^2+1}\right)+L^{-1}\left(\frac{2}{s^2+2^2}\right)\right\}$$

$$= \frac{e^{-t}}{3}(\sin t+\sin 2t)$$

This is the required solution.

Example 8: Solve $y''(t)+y(t) = \sin 2t$, when (*W.B.U.T. 2003*)

(*i*) $y(0)=0$ and $y'(0)=1$

(*ii*) $y(0)=y'(0)=1$.

Solution: Let $L(y) = \overline{y}$. Taking Laplace transform of both sides of the given ODE, we get

$$L(y'')+L(y) = L(\sin 2t)$$

or $$s^2\bar{y} - sy(0) - y'(0) + \bar{y} = \frac{2}{s^2+2^2}$$

or $$(s^2+1)\bar{y} = \frac{2}{s^2+4} + sy(0) + y'(0) \quad ...(1)$$

(*i*) $y(0) = 0$ and $y'(0) = 1$

In this case from (1), we get

$$\bar{y} = \frac{s^2+6}{(s^2+4)(s^2+1)} = \frac{1}{3}\cdot\frac{3s^2+18}{(s^2+4)(s^2+1)}$$

$$= \frac{1}{3}\frac{5(s^2+4)-2(s^2+1)}{(s^2+4)(s^2+1)} = \frac{5}{3}\cdot\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4}.$$

$\therefore$ $$y(t) = \frac{5}{3}L^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}L^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$$= \frac{5}{3}\sin t - \frac{1}{3}\sin 2t = \frac{1}{3}(5\sin t - \sin 2t).$$

This is the required solution.

(*ii*) $y(0) = y'(0) = 1$

In this case from (1), we get

$$\bar{y} = \frac{2}{(s^2+4)(s^2+1)} + \frac{s+1}{s^2+1}$$

$$= \frac{2}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) + \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$= \frac{5}{3}\cdot\frac{1}{s^2+1} - \frac{2}{3}\cdot\frac{1}{s^2+4} + \frac{s}{s^2+1}$$

$\therefore$ $$y(t) = \frac{5}{3}L^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}L^{-1}\left(\frac{2}{s^2+2^2}\right) + L^{-1}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{5}{3}\sin t - \frac{1}{3}\sin 2t + \cos t$$

$$= \frac{1}{3}(5\sin t - \sin 2t + 3\cos t)$$

This is the required solution.

Example 9: Solve $y''(t) + y(t) = t\cos 2t$, where $y(0) = y'(0) = 0$.

Solution: Let $L(y) = \bar{y}$. Taking Laplace transform of both sides of the given ODE, we get

$$L(y'') + L(y) = L(t\cos 2t)$$

or $$s^2\bar{y} - sy(0) - y'(0) + \bar{y} = -\frac{d}{ds}\{L(\cos 2t)\}$$

$\therefore$ $$(s^2+1)\bar{y} = -\frac{d}{ds}\left(\frac{s}{s^2+2^2}\right) = -\frac{1}{s^2+4} + \frac{2s^2}{(s^2+4)^2} \quad [\because y(0) = y'(0) = 0]$$

$\therefore \quad \bar{y} = \dfrac{s^2-4}{(s^2+1)(s^2+4)^2} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+4} + \dfrac{Es+F}{(s^2+4)^2}$ (say).

$\therefore \quad s^2 - 4 = (As+B)(s^2+4)^2 + (Cs+D)(s^2+1)(s^2+4) + (Es+F)(s^2+1)$

This is an identity. Equating the coefficients of equal powers of s, we get

$A + C = 0,\ B + D = 0,\ 8A + 5C + E = 0,\ 8B + 5D + F = 1,$

$16A + 4C + E = 0$ and $16B + 4D + F = -4.$

Solving: $A = C = E = 0,\ B = -\dfrac{5}{9},\ D = \dfrac{5}{9},\ F = \dfrac{8}{3}.$

$$\therefore \quad \bar{y} = -\frac{5}{9}\cdot\frac{1}{s^2+1} + \frac{5}{9}\cdot\frac{1}{s^2+4} + \frac{8}{3}\cdot\frac{1}{(s^2+4)^2}.$$

$$\therefore \quad y(t) = -\frac{5}{9}L^{-1}\left(\frac{1}{s^2+1}\right) + \frac{5}{9}L^{-1}\left(\frac{1}{s^2+2^2}\right) + \frac{8}{3}L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{8}{3}L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} \qquad \text{...(1)}$$

Now, $\quad L^{-1}\left(\dfrac{1}{s^2+4}\right) = L^{-1}\left(\dfrac{1}{s^2+2^2}\right) = \dfrac{1}{2}\sin 2t = f(t)$, say.

Therefore, by Convolution theorem

$$L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+4}\cdot\frac{1}{s^2+4}\right\} = \int_0^t f(u)f(t-u)du$$

$$= \frac{1}{4}\int_0^t \sin 2u \sin 2(t-u)du$$

$$= \frac{1}{8}\int_0^t \{\cos(4u-2t) - \cos 2t\}du$$

$$= \frac{1}{8}\left[\frac{1}{4}\sin(4u-2t) - u\cos 2t\right]_{u=0}^{t}$$

$$= \frac{1}{8}\left[\frac{1}{4}\sin 2t - t\cos 2t + \frac{1}{4}\sin 2t\right] \qquad \text{...(2)}$$

From (1) and (2), we have

$$y(t) = -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{6}\sin 2t - \frac{t}{3}\cos 2t$$

$$= -\frac{5}{9}\sin t + \frac{4}{9}\sin 2t - \frac{t}{3}\cos 2t$$

This is the required solution.

Example 10: Solve the differential equation by Laplace Transform: (W.B.U.T. 2010)

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = t\cos t, \quad y(0) = y'(0) = 0.$$

Solution: Let $L(y) = \bar{y}$. Taking Laplace transform of both sides of the given ODE, we get

$$L\left(\frac{d^2y}{dt^2}\right) - 2L\left(\frac{dy}{dt}\right) - 3L(y) = L(t\cos t)$$

or $$s^2\bar{y} - sy(0) - y'(0) - 2\{s\bar{y} - y(0)\} - 3\bar{y} = -\frac{d}{ds}\{L(\cos t)\}$$

$\therefore$ $$(s^2 - 2s - 3)\,\bar{y} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = -\frac{1}{s^2+1} + \frac{2s^2}{(s^2+1)^2} \qquad [\because y(0) = y'(0) = 0]$$

$\therefore$ $$\bar{y} = \frac{s^2-1}{(s+1)(s-3)(s^2+1)^2} = \frac{s-1}{(s-3)(s^2+1)^2} = \frac{A}{s-3} + \frac{Bs+C}{s^2+1} + \frac{Ds+F}{(s^2+1)^2} \text{ (say)}$$

$\therefore$ $$s - 1 = A(s^2+1)^2 + (Bs+C)(s-3)(s^2+1) + (Ds+E)(s-3) \qquad \text{...(1)}$$

This is an identity and hence it is true for all values of s. Putting $s = 3$, we get $A = \frac{1}{50}$.

Equating the coefficients of s^4, s^3, s^2 and the constant term from both sides of (1), we get
$A + B = 0, -3B + C = 0, 2A - 3C + B + D = 0, A - 3C - 3E = -1$

Solving: $$B = -\frac{1}{50},\ C = -\frac{3}{50},\ D = -\frac{1}{5},\ E = \frac{2}{5}.$$

$\therefore$ $$\bar{y} = \frac{1}{50}\cdot\frac{1}{s-3} - \frac{1}{50}\cdot\frac{s}{s^2+1} - \frac{3}{50}\cdot\frac{1}{s^2+1} - \frac{1}{5}\cdot\frac{s}{(s^2+1)^2} + \frac{2}{5}\cdot\frac{1}{(s^2+1)^2}$$

$\therefore$ $$y(t) = \frac{1}{50}L^{-1}\left(\frac{1}{s-3}\right) - \frac{1}{50}L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{3}{50}L^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{5}\cdot L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} + \frac{2}{5}L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$$

$$= \frac{e^{3t}}{50} - \frac{1}{50}\cos t - \frac{3}{50}\sin t - \frac{1}{5}L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} + \frac{2}{5}L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} \qquad \text{...(2)}$$

By Theorem 20,

$$L^{-1}\{\bar{f}'(s)\} = -tL^{-1}\{\bar{f}(s)\}$$

Put $$\bar{f}(s) = \frac{1}{s^2+1}.$$

$\therefore$ $$L^{-1}\left\{-\frac{2s}{(s^2+1)^2}\right\} = -tL^{-1}\left\{\frac{1}{s^2+1}\right\} = -t\sin t$$

$$\therefore \quad L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$$

Now, $$L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t = f(t), \text{ say}. \qquad \text{...(3)}$$

Therefore, by Convolution theorem

$$L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}\right\} = \int_0^t f(u)f(t-u)du$$

$$= \int_0^t \sin u \sin(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \{\cos(2u-t)-\cos t\}du$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin(2u-t)-u\cos t\right]_{u=0}^{t}$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin t - t\cos t + \frac{1}{2}\sin t\right]$$

$$= \frac{1}{2}(\sin t - t\cos t) \qquad \text{...(4)}$$

From (2) – (4), we have

$$y(t) = \frac{e^{3t}}{50} - \frac{1}{50}\cos t - \frac{3}{50}\sin t - \frac{1}{10}t\sin t + \frac{1}{5}(\sin t - t\cos t)$$

$$= \frac{e^{3t}}{50} - \frac{1}{50}(1+10t)\cos t + \frac{1}{50}(7-5t)\sin t$$

This is the required solution.

Example 11: Solve the following differential equation using Laplace and inverse Laplace transforms: (*W.B.U.T. 2006*)

$$(D^2-1)y = \alpha\cosh nt, \text{ where } y(0)=0, y'(0)=2.$$

Solution: Let $L(y) = \bar{y}$. Taking Laplace transform of both sides of the given ODE, we get

$$L(y'') - L(y) = \alpha L(\cosh nt)$$

$$s^2\bar{y} - sy(0) - y'(0) - \bar{y} = \alpha\frac{s}{s^2-n^2} \qquad [\because y(0)=0, y'(0)=2]$$

$$s^2\bar{y} - 2 - \bar{y} = \alpha\frac{s}{s^2-n^2}$$

$$\therefore \quad \bar{y} = \frac{\alpha s}{(s^2-n^2)(s^2-1)} + \frac{2}{s^2-1} \quad ...(1)$$

Let
$$\frac{s}{(s^2-n^2)(s^2-1)} = \frac{As+B}{s^2-n^2} + \frac{Cs+D}{s^2-1}$$

$$\therefore \quad s = (As+B)(s^2-1) + (Cs+D)(s^2-n^2)$$

This is an identity and hence it is true for all values of s. Putting $s = 1, -1, n, -n$ successively, we get

$$C = \frac{1}{1-n^2},\ D = 0,\ A = \frac{1}{n^2-1},\ B = 0.$$

Therefore, from (1),

$$\bar{y} = \frac{\alpha}{n^2-1}\left\{\frac{s}{s^2-n^2} - \frac{s}{s^2-1}\right\} + \frac{2}{s^2-1}.$$

$$\therefore \quad y(t) = \frac{\alpha}{n^2-1}\left[L^{-1}\left(\frac{s}{s^2-n^2}\right) - L^{-1}\left(\frac{s}{s^2-1}\right)\right] + 2L^{-1}\left(\frac{1}{s^2-1}\right)$$

$$= \frac{\alpha}{n^2-1}(\cosh nt - \cosh t) + 2\sinh t$$

This is the required solution.

Example 12: Solve $y''(t) + 3y'(t) + 2y(t) = 2(t^2+t+1)$ with $y(0) = 2$ and $y'(0) = 0$.

Solution: Let $L(y) = \bar{y}$. Taking Laplace transform of the both sides of the given ODE, we get

$$L(y'') + 3L(y') + 2L(y) = 2\{L(t^2) + L(t) + L(1)\}$$

or
$$s^2\bar{y} - sy(0) - y'(0) + 3\{s\bar{y} - y(0)\} + 2\bar{y} = 2\left(\frac{2}{s^3} + \frac{1}{s^2} + \frac{1}{s}\right) \quad \left[\because L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}},\ L(1) = \frac{1}{s}\right]$$

or
$$s^2\bar{y} - 2s + 3s\bar{y} - 6 + 2\bar{y} = \frac{2}{s^3}(2+s+s^2)$$

or
$$(s^2+3s+2)\bar{y} = \frac{2}{s^3}(2+s+s^2) + 2s + 6$$

or
$$(s+2)(s+1)\bar{y} = \frac{2}{s^3}(s^4+3s^3+s^2+s+2)$$

Now,
$$s^4+3s^3+s^2+s+2 = s^4+s^3+2s^3+2s^2-s^2-s+2s+2$$
$$= s^3(s+1) + 2s^2(s+1) - s(s+1) + 2(s+1)$$
$$= (s+1)(s^3+2s^2-s+2)$$

$$\therefore \quad \bar{y} = \frac{2(s^3+2s^2-s+2)}{s^3(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+2} \text{(say).}$$

$$\therefore \quad 2(s^3+2s^2-s+2) = As^2(s+2) + Bs(s+2) + C(s+2) + Ds^3 \quad ...(1)$$

This is an identity and hence it is true for all values of s. Putting $s = 0, -2$ successively, we get $C = 2, D = -1$.

Equating the coefficients of s^3 and s^2 from both sides of (1), we get

$A + D = 2$, *i.e.*, $A = 2 - D = 3$ $(\because D = -1)$

$2A + B = 4$, *i.e.*, $B = 4 - 2A = 4 - 6 = -2$.

$$\therefore \quad \bar{y} = 3\cdot\frac{1}{s} - 2\cdot\frac{1}{s^2} + 2\cdot\frac{1}{s^3} - \frac{1}{s+2}$$

$$\therefore \quad y(t) = 3L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s^2}\right) + 2L^{-1}\left(\frac{1}{s^3}\right) - L^{-1}\left(\frac{1}{s+2}\right)$$

$$= 3 - 2t + 2\cdot\frac{t^2}{2} - e^{-2t} \quad \left[\because L^{-1}\left(\frac{1}{s}\right) = 1, L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}, n \text{ is a positive integer, and } L^{-1}\left(\frac{1}{s-a}\right) = e^{at}\right]$$

$$= 3 - 2t + t^2 - e^{-2t}$$

This is the required solution.

Example 13: Solve $(D^2 + n^2)y = a\sin(nt + \alpha)$, where $y(0) = y'(0) = 0$.

Solution: Let us rewrite the given ODE as

$$(D^2 + n^2)y = a(\sin nt\cos\alpha + \cos nt\sin\alpha)$$

Taking Laplace transform of both sides, we get

$$L(y'') + n^2L(y) = (a\cos\alpha)L(\sin nt) + (a\sin\alpha)L(\cos nt)$$

or
$$s^2\bar{y} - sy(0) - y'(0) + n^2\bar{y} = (a\cos\alpha)\frac{n}{s^2+n^2} + (a\sin\alpha)\frac{s}{s^2+n^2}$$

or
$$(s^2 + n^2)\bar{y} = \frac{an\cos\alpha}{s^2+n^2} + \frac{as\sin\alpha}{s^2+n^2}$$

$$\therefore \quad \bar{y} = \frac{an\cos\alpha}{(s^2+n^2)^2} + \frac{as\sin\alpha}{(s^2+n^2)^2}$$

$$\therefore \quad y(t) = an\cos\alpha L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\} + a\sin\alpha L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\} \quad \text{...(1)}$$

By Theorem 20, $L^{-1}\{\bar{f}'(s)\} = -tL^{-1}\{\bar{f}(s)\}$.

Put $\bar{f}(s) = \dfrac{1}{s^2+n^2}$.

$$\therefore \quad L^{-1}\left\{-\frac{2s}{(s^2+n^2)^2}\right\} = -tL^{-1}\left\{\frac{1}{s^2+n^2}\right\} = -\frac{t}{n}\sin nt \quad \text{...(2)}$$

$$\therefore \quad L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\} = \frac{t}{2n}\sin nt$$

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Now, $$L^{-1}\left(\frac{1}{s^2+n^2}\right) = \frac{1}{n}\sin nt = f(t), \text{ say.}$$

Therefore, by Convolution theorem

$$L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+n^2}\cdot\frac{1}{s^2+n^2}\right\} = \int_0^t f(u)f(t-u)du$$

$$= \frac{1}{n^2}\int_0^t \sin nu \sin n(t-u)du$$

$$= \frac{1}{2n^2}\int_0^t \{\cos(2nu-nt)-\cos nt\}du$$

$$= \frac{1}{2n^2}\left[\frac{1}{2n}\sin(2nu-nt)-u\cos nt\right]_{u=0}^{t}$$

$$= \frac{1}{2n^2}\left[\frac{1}{2n}\sin nt - t\cos nt + \frac{1}{2n}\sin nt\right]$$

$$= \frac{1}{2n^2}\left(\frac{1}{n}\sin nt - t\cos nt\right) \qquad ...(3)$$

From (1) – (3), we get

$$y(t) = \frac{a}{2n}\left(\frac{1}{n}\sin nt - t\cos nt\right)\cos\alpha + \frac{at}{2n}\sin\alpha\sin nt$$

$$= \frac{a}{2n^2}(\cos\alpha\sin nt - nt\cos\alpha\cos nt + nt\sin\alpha\sin nt)$$

$$= \frac{a}{2n^2}\{\cos\alpha\sin nt - nt\cos(\alpha+nt)\}$$

This is the required solution.

Example 14: Using Laplace transform, find the solution of the initial value problem

$$y''(t)+9y(t) = 9u(t-3), y(0) = y'(0) = 0.$$

Here $u(t-3)$ is a unit step function.

Solution: Let $L(y) = \bar{y}$. Taking Laplace transform of both sides of the given equation, we get

$$L(y'')+9L(y) = 9L\{u(t-3)\}$$

or $$s^2\bar{y} - sy(0) - y'(0) + 9\bar{y} = 9\frac{e^{-3s}}{s} \qquad \text{[By Theorem 11]}$$

or $$(s^2+9)\bar{y} = 9\frac{e^{-3s}}{s} \qquad [\because y(0) = y'(0) = 0]$$

or $$\bar{y} = \frac{9e^{-3s}}{(s^2+9)s}$$

$$\therefore \quad y(t) = 9L^{-1}\left\{\frac{e^{-3s}}{(s^2+9)s}\right\} \quad \text{...(1)}$$

$$L^{-1}\left(\frac{1}{s^2+9}\right) = L^{-1}\left(\frac{1}{s^2+3^2}\right) = \frac{1}{3}\sin 3t = f(t), \text{ say.}$$

$$L^{-1}\left(\frac{1}{s}\right) = 1 = g(t), \text{ say.}$$

By Convolution theorem,

$$L^{-1}\left\{\frac{1}{(s^2+9)s}\right\} = \int_0^t f(u)g(t-u)du$$

$$= \int_0^t \frac{1}{3}\sin 3u\, du$$

$$= \frac{1}{9}[-\cos 3u]_0^t = \frac{1}{9}(1-\cos 3t)$$

Therefore, by second shifting property

$$L^{-1}\left\{e^{-3s}\frac{1}{(s^2+9)s}\right\} = \frac{1}{9}\{1-\cos 3(t-3)\}u(t-3)$$

Therefore, from (1), we get

$$y(t) = \{1-\cos 3(t-3)\}u(t-3),$$

where $u(t-3)$ is a unit step function.

This is the required solution.

Example 15: A resistance r in series with inductance l is connected with e.m.f. $E(t)$ and the current $i(t)$ is given by

$$l\frac{di}{dt}+ri = E(t) = \begin{cases} a, & 0<t<T \\ 0, & t \ge T \end{cases}$$

with $i(0) = 0$. Find $i(t)$.

Solution: Let $L(i) = \bar{i}$. Taking Laplace transform of both sides of the given equation, we get

$$lL\left\{\frac{di}{dt}\right\} + rL(i) = L(E)$$

Or
$$l\{s\bar{i} - i(0)\} + r\bar{i} = \int_0^\infty e^{-st}E(t)dt \quad [\because i(0)=0]$$

$$\therefore \quad (ls+r)\bar{i} = \int_0^T e^{-st}.a\,dt + \int_T^\infty e^{-st}.0\,dt$$

$$= a\left[\frac{e^{-st}}{-s}\right]_{t=0}^T + 0 = \frac{a}{s}(1-e^{-sT})$$

$$\therefore \quad \bar{i} = \frac{a}{(ls+r)s} - \frac{ae^{-sT}}{(ls+r)s}$$

$$\therefore \quad i(t) = aL^{-1}\left\{\frac{1}{(ls+r)s}\right\} - aL^{-1}\left\{\frac{e^{-sT}}{(ls+r)s}\right\} \qquad \ldots(1)$$

Now,
$$\frac{1}{(ls+r)s} = \frac{1}{l}\frac{1}{\left(s+\frac{r}{l}\right)s} = \frac{1}{l}\cdot\frac{l}{r}\frac{s+\frac{r}{l}-s}{\left(s+\frac{r}{l}\right)s}$$

$$= \frac{1}{r}\left\{\frac{1}{s} - \frac{1}{s+\frac{r}{l}}\right\}$$

$$\therefore \quad L^{-1}\left\{\frac{1}{(ls+r)s}\right\} = \frac{1}{r}\left\{L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+\frac{r}{l}}\right)\right\}$$

$$= \frac{1}{r}(1-e^{-rt/l})$$

$$\therefore \quad L^{-1}\left\{\frac{e^{-sT}}{(ls+r)s}\right\} = \frac{1}{r}[1-e^{-\frac{r}{l}(t-T)}]u(t-T) \qquad \text{[By second shifting property]}$$

Therefore, from (1),

$$i(t) = \frac{a}{r}\left\{1-e^{-\frac{r}{l}t}\right\} - \frac{a}{r}\left\{1-e^{-\frac{r}{l}(t-T)}\right\}u(t-T),$$

where $u(t-T) = \begin{cases} 0, & 0<t<T \\ 1, & t \geq T \end{cases}$

When $0 < t < T$

$$i(t) = \frac{a}{r}\{1-e^{-\frac{r}{l}t}\}$$

When $t \geq T$

$$i(t) = \frac{a}{r}\{1-e^{-\frac{r}{l}t}\} - \frac{a}{r}\{1-e^{-\frac{r}{l}(t-T)}\}$$

$$= \frac{a}{r}\{e^{-\frac{r}{l}(t-T)} - e^{-\frac{r}{l}t}\}$$

$$= \frac{a}{r}e^{-\frac{r}{l}t}\{e^{\frac{r}{l}T} - 1\}$$

Note: Here the switch of the circuit is connected at $t = 0$ and disconnected at $t = T$.

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Example 16: Solve the differential equation by Laplace transform:

$$\frac{d^2y}{dt^2}+9y = 1,\ y(0) = 1,\ y\left(\frac{\pi}{2}\right) = -1.$$

(W.B.U.T. 2008)

Solution: Let $L(y) = \overline{y}$. Taking Laplace transform of both sides of the given ODE, we get

$$L\left(\frac{d^2y}{dt^2}\right)+9L(y) = L(1)$$

$$s^2\overline{y}-sy(0)-y'(0)+9\overline{y} = \frac{1}{s}$$

$$(s^2+9)\overline{y} = \frac{1}{s}+s+c \qquad [\because y(0)=1 \text{ and supposing } y'(0)=c]$$

$$\therefore \quad \overline{y} = \frac{1}{s(s^2+9)}+\frac{s}{s^2+9}+\frac{c}{s^2+9}.$$

$$\therefore \quad y(t) = L^{-1}\left\{\frac{1}{s(s^2+9)}\right\}+L^{-1}\left\{\frac{s}{s^2+9}\right\}+cL^{-1}\left\{\frac{1}{s^2+9}\right\} \qquad ...(1)$$

Now,
$$L^{-1}\left\{\frac{s}{s^2+9}\right\} = L^{-1}\left\{\frac{s}{s^2+3^2}\right\} = \cos 3t,$$

$$L^{-1}\left\{\frac{1}{s^2+3^2}\right\} = \frac{1}{3}\sin 3t$$

$$\therefore \quad L^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \int_0^t \frac{1}{3}\sin 3u\,du \qquad \text{[By Theorem 22]}$$

$$= \left[-\frac{1}{9}\cos 3u\right]_0^t = \frac{1}{9}(1-\cos 3t)$$

Therefore, from (1), we get

$$y(t) = \frac{1}{9}(1-\cos 3t)+\cos 3t+\frac{c}{3}\sin 3t \qquad ...(2)$$

$$\therefore \quad y\left(\frac{\pi}{2}\right) = \frac{1}{9}\left(1-\cos\frac{3\pi}{2}\right)+\cos\frac{3\pi}{2}+\frac{c}{3}\sin\frac{3\pi}{2} \qquad \left[\because y\left(\frac{\pi}{2}\right)=-1\right]$$

or
$$-1 = \frac{1}{9}-\frac{c}{3}$$

$$\therefore \quad c = \frac{10}{3}.$$

Therefore, from (2), we get

$$y(t) = \frac{1}{9}(1-\cos 3t)+\cos 3t+\frac{10}{9}\sin 3t$$

$$= \frac{8}{9}\cos 3t+\frac{10}{9}\sin 3t+\frac{1}{9}.$$

This is the required solution.