$r(t) = u(t) - 2e \quad u(t)$

$t = 1.5$ sec $\quad r(1.5) = 1 - 2e^{-1.5}$

$= 1 - 0.446 = \underline{0.554}$

P (5) The laplace transform of $f(t) = e^{2t} \cdot \sin 5t \cdot u(t)$ is

$$F(s) = \frac{5}{(s-2)^2 + 25} = \frac{5}{s^2 - 4s + 29}$$

21.09.2020

Fourier Series

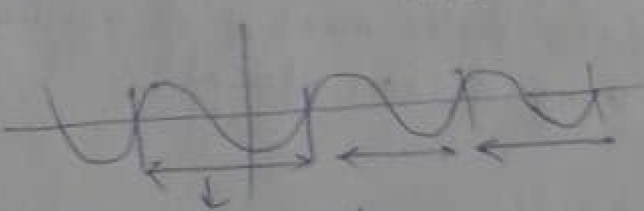
Fourier series expansion (FSE) is used for periodic signals for expand them in terms of their harmonics which are sinusoidal and orthogonal to one another.

FSE is used for analysis purpose for particularly periodic signals

CTFS $\quad$ DTFS

For nonperiodic signals → Fourier Transform

laplace transform — CTS } design purpose
z-transform — DTS



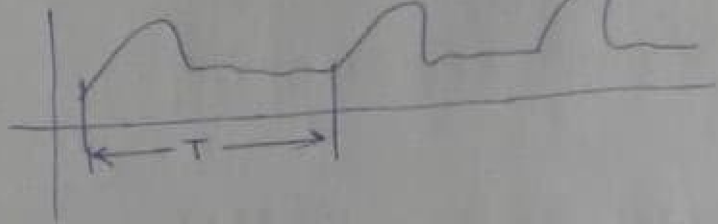
Time period

$x(t) \longrightarrow x(t \pm T) = x(t)$

T is the time period $= nT_0$

ntimes fundamental time period.

$$f = \frac{\text{cycle}}{\text{sec}} = \frac{1}{T}$$



$$x(t) = 2\sin\omega t + \sin 2\omega t + 7\sin 3\omega t + \ldots$$

Depending on the integer value two types of harmonics are there

- even harmonics
- odd harmonics

even → $2\omega t, 4\omega t, 6\omega t \ldots \ldots 12\omega t$

odd → $3\omega t, 5\omega t, 7\omega t$

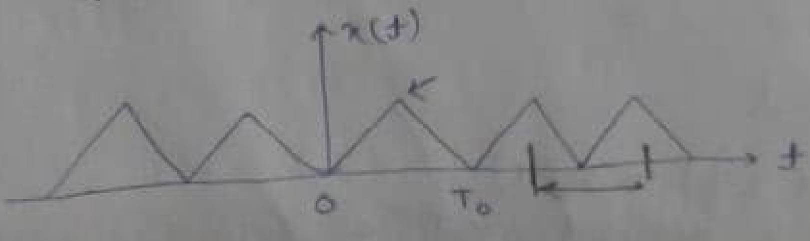
In any signal different frequency componet along with fundamental freq. componet, we say harmonics are present in the signal.

3 types Fourier series —

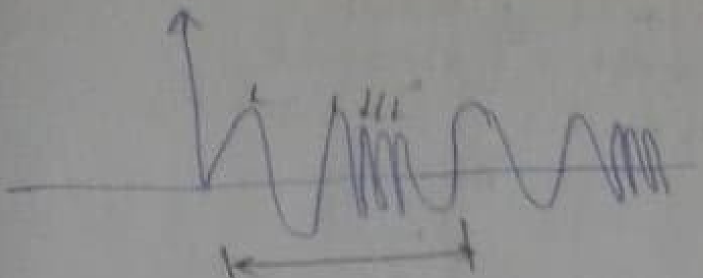
1. Trigonometric FSEx
2. Complex exponential FSEx
3. Polar or harmonic FSEx

Conditions for existance of Fourier series — (Dirichelet conditions)

1. signal should have finite number of maxima and minima over the range of time period.

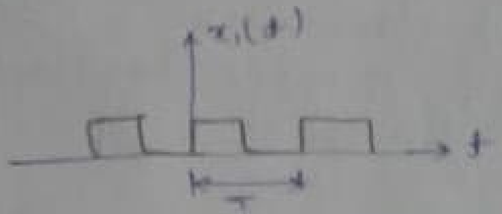


FSEx → possible



FSEx → not possible (infinite number of maxima and minima)

(ii) Signal should have finite number of discontinuites over the range of time period.



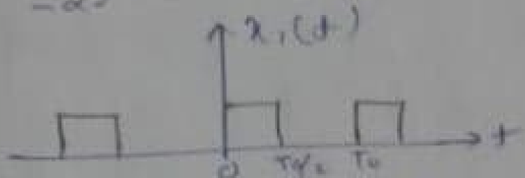
FSEx → possible

infite number of discont-inuity

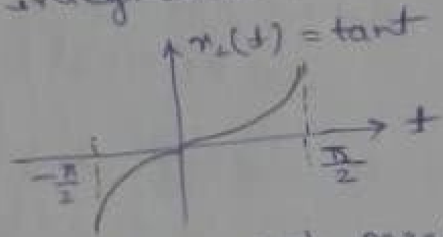
FSEx → not posible

(iii) Signal should be absolutely integrable over the range of time period.

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty \rightarrow x(t) \text{ is absolutely integrable}$$



FSEx → possible



FSEx → not possible

(1) Trigonometric FSEx

$x(t)$ = dc or Avg value of $x(t)$
+ Cosine terms
+ Sine terms

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t}$$

a_n & b_n → Fourier coefficients

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{T_0}\int_{T_0} x(t)\,dt \rightarrow \text{DC value or avg value}$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\,dt$$

ex-

$$x(t) = 4 + \underbrace{5}_{a_1}\cos wt + \underbrace{4}_{b_1}\sin wt + \underbrace{3}_{a_2}\cos 2w_0t + \underbrace{5}_{b_2}\sin 2w_0t + \underbrace{4}_{a_3}\cos 3w_0t + \underbrace{2}_{b_3}\sin 3w_0t + \dots$$

$a_0 = 4$

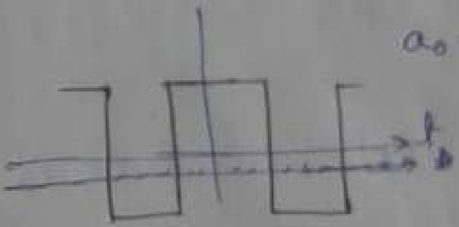
$a_2 = 3 < b_2 = 5$ → In 2nd harmonic sin term is more dominent than cos term

$a_3 = 4 > b_3 = 2$ → In 3rd harmonic cos term is more dominent than sin term

- Signal is symetrical about the time axis then average value is zero
  $a_0 = 0$

- $x(t)$ → even signal → $b_n = 0$
  $x(-t) = x(t)$

- $x(t)$ → odd signal → $a_n = 0$
  $x(-t) = -x(t)$



$a_0 = 0$

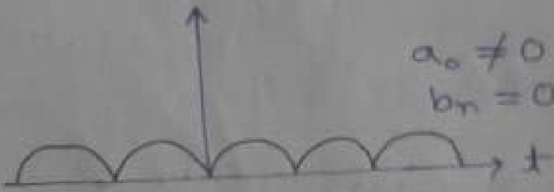
$b_n = 0$

$a_n = ?$

$a_0 = 0$

$a_n = 0$

$b_n = ?$



$a_0 \neq 0$

$b_n = 0$

$a_n \neq 0$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nW_0t + \sum_{n=1}^{\infty} b_n \sin W_0t$$

prove $x(t)$ is periodic.

$$x(t) = x(t \pm T)$$

$$x(t + T_0) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 (t + T_0) + \sum_{n=1}^{\infty} b_n \sin n\omega_0 (t + T_0)$$

$$= a_0 + \sum_{n=1}^{\alpha} a_n \cos n(\omega_0 t + \omega_0 T_0) + \sum_{n=1}^{\alpha} b_n \sin n(\omega_0 t + \omega_0 T_0)$$

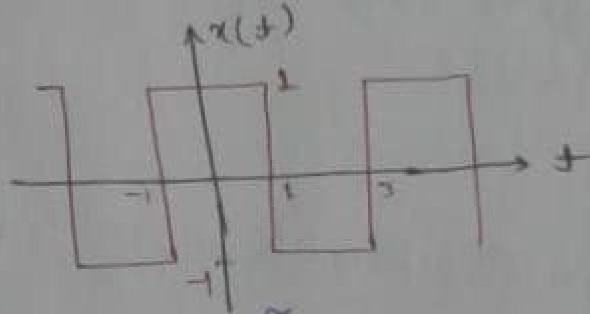
$$\omega_0 = \frac{2\pi}{T_0} \qquad \omega_0 T_0 = 2\pi$$

$$= a_0 + \sum_{n=1}^{\alpha} a_n \cos n(\omega_0 t + 2\pi) + \sum_{n=1}^{\alpha} b_n \sin n(\omega_0 t + 2\pi)$$

$$= a_0 + \sum_{n=1}^{\alpha} a_n \cos n\omega_0 t + \sum_{n=1}^{\alpha} b_n \sin \omega_0 t$$

$x(t + T) = x(t) \Rightarrow x(t)$ is periodic

ex-1



Find fourier series expansion of $x(t)$

$$x(t) = a_0 + \sum_{n=1}^{\alpha} a_n \cos n\omega_0 t + \sum_{n=1}^{\alpha} b_n \sin n\omega_0 t$$

$a_0 = 0$
$b_n = 0$

$$x(t) = \sum_{n=1}^{\alpha} a_n \cos n\omega_0 t$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cdot \cos n\omega_0 t \, dt$$

$T_0 = 4$ sec, $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \frac{\pi}{2}$ rad/se

$$a_n = \frac{2}{4} \int_{-1}^{3} x(t) \cos n \frac{\pi}{2} \cdot t \; dt$$

$$= \frac{1}{2} \left[\int_{-1}^{1} 1 \cdot \cos\left(n \frac{\pi}{2} t\right) dt + \int_{1}^{3} (-1) \cdot \cos\left(\frac{n\pi}{2} t\right) dt \right]$$

$$= \frac{1}{2} \left[\int_{-1}^{1} \cos \frac{n\pi}{2} t \; dt - \int_{1}^{3} \cos \frac{n\pi}{2} t \; dt \right]$$

assume, $\frac{n\pi}{2} t = \theta$

$$\frac{n\pi}{2} dt = d\theta$$

$$dt = \frac{2}{n\pi} d\theta$$

$t = -1 \rightarrow \theta = -\frac{n\pi}{2}$
$t = 1 \rightarrow \theta = \frac{n\pi}{2}$
$t = 3 \rightarrow \theta = \frac{3n\pi}{2}$

$$a_n = \frac{1}{2} \left[\int_{-\frac{n\pi}{2}}^{\frac{n\pi}{2}} (\cos\theta) \cdot \left(\frac{2}{n\pi} d\theta\right) - \int_{\frac{n\pi}{2}}^{\frac{3n\pi}{2}} (\cos\theta) \left(\frac{2}{n\pi} d\theta\right) \right]$$

$$= \frac{1}{2} \cdot \frac{2}{n\pi} \left[\sin\theta \Big|_{-\frac{n\pi}{2}}^{\frac{n\pi}{2}} - \sin\theta \Big|_{\frac{n\pi}{2}}^{\frac{3n\pi}{2}} \right]$$

$$= \frac{1}{n\pi} \left[\sin\frac{n\pi}{2} - \sin\left(-\frac{n\pi}{2}\right) - \sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2} \right]$$

$$\sin\frac{3n\pi}{2} = \sin\left(n\pi + \frac{n\pi}{2}\right) \quad \sin\frac{n\pi}{2}$$

Case 1 $\rightarrow$ n is even
$2\pi, 4\pi, 6\pi$

Case-2 n is odd $\Rightarrow -\sin\frac{n\pi}{2}$
1, 3, 5 7

n → even

$$a_n = \frac{1}{n\pi} \left[\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} - \frac{\sin n\pi}{2} + \sin\frac{n\pi}{2} \right]$$

$$= \frac{2}{n\pi} \sin\frac{n\pi}{2} = 0$$

n = 2, $a_2 = \frac{2}{2\pi} \sin\frac{2\pi}{2} = \frac{1}{\pi} \sin\pi = 0$

n → odd

$$a_n = \frac{1}{n\pi} \left[\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} \right]$$

$$= \frac{4}{n\pi} \sin\frac{n\pi}{2}$$

Case A. $n = 1, 5, 9, 13$

$a_1 = \frac{4}{\pi} \sin\frac{\pi}{2} = \frac{4}{\pi}$

$a_5 = \frac{4}{5\pi} \sin\frac{5\pi}{2} = \frac{4}{5\pi}$

$a_n = \frac{4}{n\pi}$

Case B. $n = 3, 7, 11, 15$

$a_3 = \frac{4}{3\pi} \sin\frac{3\pi}{2} = -\frac{4}{3\pi}$

$a_7 = -\frac{4}{7\pi}$

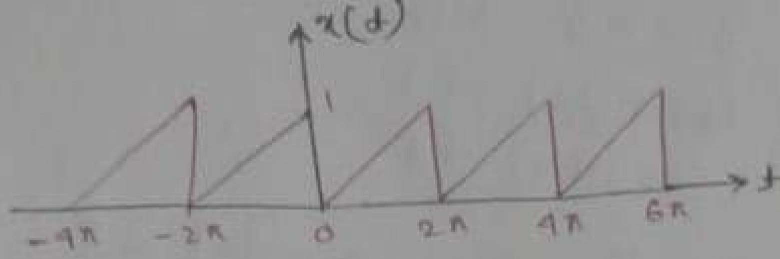
$a_n = -\frac{4}{n\pi}$

$$x(t) = \sum_{n=1}^{\infty} a_n \cos n \frac{\pi}{2} t$$

$$x(t) = \frac{4}{\pi} \cos \frac{\pi}{2} t - \frac{4}{3\pi} \cos \frac{3\pi}{2} t + \frac{4}{5\pi} \cos \frac{5\pi}{2} t$$

$$- \frac{4}{7\pi} \cos \frac{7\pi}{2} t + \cdots \infty$$

Ex-2



Express this signal in its harmonic using FSEx

$a_0 \neq 0$, $b_n \neq 0$, $a_n \neq 0$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$T_0 = 2\pi$

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) \cdot dt = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{t}{2\pi}\right) dt$$

$$= \frac{1}{4\pi^2} \left[\frac{t^2}{2}\right]_0^{2\pi} = \frac{1}{2}$$

$$\omega_0 = \frac{2\pi}{2\pi} = 1 \text{ rad/sec}$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cdot \cos n\omega_0 t \, dt$$

$$= \frac{2}{2\pi} \int_0^{2\pi} \left(\frac{t}{2\pi}\right) \cdot \cos nt \, dt$$

$$= \frac{1}{2\pi^2} \int_0^{2\pi} t \cos nt \, dt$$

$$= \quad \int t \cos nt = t \int \cos nt \, dt - \int \left(1 \int \cos nt \, dt\right) dt$$

$$= t \frac{\sin nt}{n} - \int \frac{\sin nt}{n} dt$$

$$= t \frac{\sin nt}{n} - \left[- \frac{\cos nt}{n^2}\right]$$

$$= \frac{t \sin nt}{n} + \frac{\cos nt}{n^2}$$

$$a_n = \frac{1}{2\pi^2}\left[\frac{t \sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi^2}\left[(2\pi)\frac{\sin n.2\pi}{n} + \frac{\cos n(2\pi)}{n^2} - 0 - \frac{\cos n\,0}{n^2}\right]$$

$= \quad \sin n\,2\pi \rightarrow 0$

$\sin n.0 \rightarrow 0$

$\cos n(2\pi) \rightarrow 1$

$\cos (n.0) \rightarrow 1$

$$a_n = \frac{1}{2\pi^2}\left[0 + \frac{1}{n^2} - 0 - \frac{1}{n^2}\right] = 0$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t) \sin n\omega_0 t \, dt$$

$$= -\frac{1}{n\pi}$$

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\alpha} -\left(\frac{1}{n\pi}\right) \sin nt$$

$$= \frac{1}{2} - \sum_{n=1}^{\alpha} \frac{1}{n\pi} \sin nt$$

$$= \frac{1}{2} - \frac{1}{\pi}\sin t - \frac{1}{2\pi}\sin 2t - \frac{1}{3\pi}\sin 3t -$$

23.07.2019

Complex Exponential Fourier Series.

$x(t)$ is a periodic signal

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

c_n is the complex exponential fourier co-efficient

$$c_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt \quad \text{——— (1)}$$

$\downarrow$ Reversal

$x(t) \to x(-t)$

$n \longrightarrow -n$

$-n \longrightarrow n$

$$c_{-n} = \frac{1}{T_0}\int_{T_0} x(t)\, e^{jn\omega_0 t}\, dt \quad \text{——— (2)}$$

$\downarrow$ Conjugate

$$c_{-n}^{*} = \frac{1}{T_0}\int_{T_0} x^{*}(t)\, e^{-jn\omega_0 t}\, dt$$

If c_n is conjugate symm. this means, $c_n = c_{-n}^{*}$

$\downarrow \qquad \downarrow$

Real in nature. $\rightarrow$ $x(t) = x^{*}(t)$

there is no imaginary Part

c_n is complex co-efficient.

$$c_n = |c_n|\, e^{j\angle c_n} \quad \text{——— (3)}$$

$\downarrow$ Reversal

$n \to -n$

$$c_{-n} = |c_{-n}|\, e^{j\angle c_{-n}}$$

$\downarrow$ Conjugate

$$c_{-n}^{*} = |c_{-n}|\, e^{-j\angle c_{-n}} \quad \text{——— (4)}$$

$c_n = c_{-n}^*$

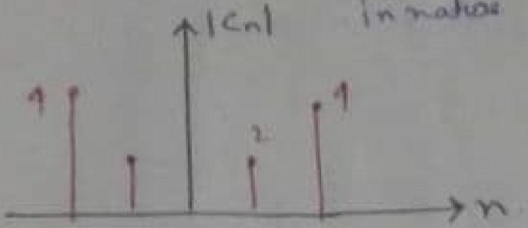
$|c_n| e^{j\angle c_n} = |c_{-n}| e^{-j\angle c_n}$

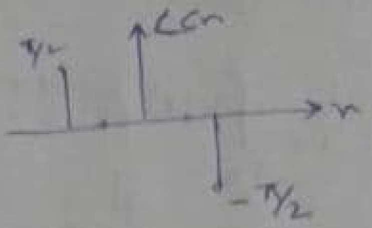
$\boxed{|c_n| = |c_{-n}|}$ and $\angle c_n = -\angle c_{-n}$

even in nature

$-\angle c_n = \angle c_n$

odd in nature.

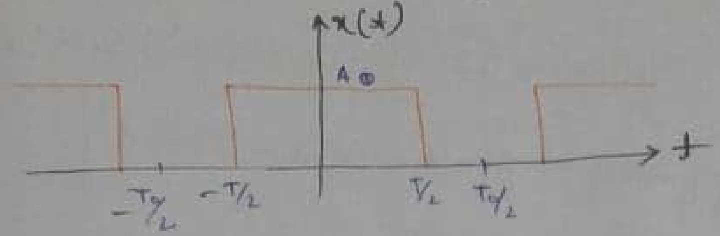




Note:

For ~~every~~ any real time domain signal the exponential Fourier series coefficient (c_n) should be conjugate symmetry and vice versa also true.

Ex-1 Find c_n for signal $x(t)$



$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\left[\int_{-\frac{T_0}{2}}^{-\tau/2} 0 + \int_{-\tau/2}^{\tau/2} A_0\, e^{-jn\omega_0 t}\, dt + \int_{\tau/2}^{\frac{T_0}{2}} 0\right]$$

$$= \frac{A_0}{T_0}\int_{-\tau/2}^{\tau/2} e^{-jn\omega_0 t}\, dt$$

Let $-jn\omega_0 t = \theta$ when $t = -\tau/2$, $\theta = jn\omega_0 \tau/2$

$-jn\omega_0\, dt = d\theta$ when $t = \tau/2$, $\theta = -jn\omega_0 \tau/2$

$$dt = \frac{d\theta}{-jn\omega_0}$$

$$= \frac{A_0}{T_0}\int_{jn\omega_0\tau/2}^{-jn\omega_0\tau/2} e^{\theta}\left(-\frac{1}{jn\omega_0}\right) d\theta$$

$$= -\frac{A_0}{jn\omega_0 T_0}\int_{jn\omega_0\tau/2}^{-jn\omega_0\tau/2} e^{\theta}\, d\theta$$

$$= -\frac{A_0}{jn\omega_0 T_0}\left[e^{-jn\omega_0\tau/2} - e^{jn\omega_0\tau/2}\right]$$

$e^{jx} = \cos x + j\sin x$

$e^{-jx} = \cos x - j\sin x$

$$= \frac{-A_0}{jn\omega_0 T_0}\left[e^{-jx} - e^{jx}\right]$$

$$= -\frac{A_0}{jn\omega_0 T_0}\left[\cos(-x) + j\sin(-x) - \cos x - j\sin(x)\right]$$

$$= -\frac{A_0}{jn\omega_0 T_0}\left[\cos x - j\sin x - \cos x - j\sin x\right]$$

$$= -\frac{A_0}{jn\omega_0 T_0}\left(-2j\sin x\right)$$

$$= \frac{A_0}{jn\omega_0 T_0} \cdot 2j\sin n\omega_0 T/2$$

$$C_n = \frac{2A_0}{n\omega_0 T_0}\sin n\omega_0 T/2$$

n_0 $\quad C_0$

$n=1$ $\quad C_1$

$n=-1$ $\quad C_{-1}$

$n=2$ $\quad C_2$

$\cancel{C_0} = \alpha$

$$C_0 = \frac{2A}{0 \cdot \omega_0 T_0} \cdot \sin 0 \cdot \omega_0 T/2$$

$$= \frac{0}{0}$$

ex-2 Find C_n for the signal given below,

$$x(t) = 3 + 2\sin\omega_0 t + \cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

$$x(t) = \sum_{n=-\alpha}^{\alpha} C_n e^{jn\omega_0 t}$$

$$x(t) = \ldots + C_{-2}e^{-j2\omega_0 t} + C_1 e^{-j\omega_1 t} + C_0 + C_1 e^{+j\omega_1 t} + C_2 e^{+j\omega_2 t} + \ldots$$

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

$$e^{-i\alpha} = \cos\alpha - i\sin\alpha$$

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

$$\sin\alpha = \frac{e^{i\alpha} - e^{i\alpha}}{2i}$$

$$x(t) = 3 + 2 \cdot \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$
$$+ \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$
$$+ \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right]$$

$$x(t) = 3 + \frac{1}{j} e^{j\omega_0 t} - \frac{1}{j} e^{-j\omega_0 t}$$
$$+ \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$
$$+ \frac{1}{2}\left[e^{j2\omega_0 t}\, e^{j\pi/4}\right] + \frac{1}{2}\left[e^{-j2\omega_0 t} \cdot e^{-j\pi/2}\right]$$

$$e^{j\pi/4} = \cos \pi/4 + j \sin \pi/4 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} = \frac{1+j}{\sqrt{2}}$$

$$e^{-j\pi/4} = \frac{1-j}{\sqrt{2}}$$

$$x(t) = 3 + \left(\frac{1}{j} + \frac{1}{2}\right) e^{j\omega_0 t} + \left(\frac{1}{2} - \frac{1}{j}\right) e^{-j\omega_0 t}$$
$$+ \frac{1+j}{2\sqrt{2}} e^{j2\omega_0 t} + \frac{1-j}{2\sqrt{2}} e^{-j2\omega_0 t}$$

$$x(t) = 3 + \left(\frac{1}{2} - j\right) e^{j\omega_0 t} + \left(\frac{1}{2} + j\right) e^{-j\omega_0 t}$$
$$+ \frac{1+j}{2\sqrt{2}} e^{j2\omega_0 t} + \frac{1-j}{2\sqrt{2}} e^{-j2\omega_0 t}$$

$C_0 = 3$

$C_1 = \frac{1}{2} - j$

$C_{-1} = \frac{1}{2} + j$

$C_2 = \frac{1+j}{2\sqrt{2}}$

$C_{-2} = \frac{1-j}{2\sqrt{2}}$

(1) The signal $x(t)$ has period $= 1$ and the following fourier co-efficient

$$c_n = \left(-\frac{1}{3}\right)^n, \quad n \geq 0$$
$$= 0 \quad n < 0$$

what is $x(t)$

(A) $x(t) = \dfrac{1}{1-\frac{1}{3}e^{j2\pi t}}$ ✓ (B) $x(t) = \dfrac{1}{1+\frac{1}{3}e^{j2\pi t}}$

(C) $x(t) = \dfrac{1}{1+\frac{1}{3}e^{-j2\pi t}}$ (D) $x(t) = \dfrac{1}{1-\frac{1}{3}e^{-j2\pi t}}$

$$x(t) = \sum_{n=-\alpha}^{\alpha} C_n \cdot e^{jn\omega_0 t}$$

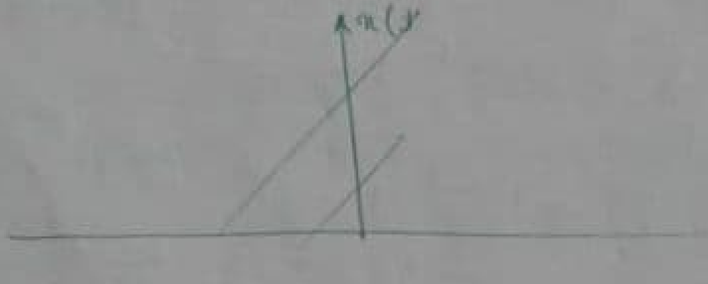
$T^0 = 1$
$\omega_0 = 2\pi$

$$= \sum_{-\alpha}^{0} 0 \, e^{jn \cdot 2\pi t} + \sum_{n=0}^{\alpha} \left(-\frac{1}{3}\right)^n e^{jn2\pi t}$$

$$= 1 + \left(-\frac{1}{3}e^{j2\pi t}\right) + \left(-\frac{1}{3}e^{j2\pi t}\right)^2 +$$

$$S_\alpha = \frac{a}{1-r}, \quad r = -\frac{1}{3} \cdot e^{j2\pi t}$$

$$x(t) = \frac{1}{1-\left(-\frac{1}{3}e^{j2\pi t}\right)} = \frac{1}{1+\frac{1}{3}e^{j2\pi t}}$$



15-09-2020

Properties of Fourier series.

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T_0} \int_{T_0} x(t) \cdot e^{-jn\omega_0 t} \, dt$$

① Linearity: Let $x_1(t) \longrightarrow C_{1n}$

$x_2(t) \longrightarrow C_{2n}$

Time period of signal is T_0

Scaling operation

$$\underset{\downarrow \, T_1}{\alpha x_1(t)} + \underset{\downarrow \, T_2}{\beta x_2(t)} = \underset{\downarrow \, C_n'}{x(t)} \rightarrow \text{Time } LCM(T_1, T_2) = T_0$$

$$C_n' = \frac{1}{T_0} \int_{T_0} x(t) \, e^{-jn\omega_0 t} \, dt$$

$$= \frac{1}{T_0} \int_{T_0} [\alpha x_1(t) + \beta x_2(t)] e^{-jn\omega_0 t} \, dt$$

$$= \alpha \frac{1}{T_0} \int_0^{T_0} x_1(t) \cdot e^{-jn\omega_0 t} \, dt + \beta \frac{1}{T_0} \int_0^{T_0} x_2(t) \cdot e^{-jn\omega_0 t} \, dt$$

$$C_n' = \alpha C_{1n} + \beta \cdot C_{2n}$$

$$\boxed{\alpha x_1(t) + \beta x_2(t) \rightleftharpoons \alpha C_{1n} + \beta C_{2n}}$$

② Conjugation: $x(t) \longrightarrow C_n$

$$C_n = \frac{1}{T_0} \int_0^{T_0} x(t) \, e^{-jn\omega_0 t} \, dt$$

$\downarrow$ conjugate

$$C_n^* = \frac{1}{T_0} \int_0^{T_0} x^*(t) \cdot e^{jn\omega_0 t} \, dt$$

$\Big\downarrow$ reversal $(n \to -n)$

$$c_{-n}^{*} = \frac{1}{T_0}\int_0^{T_0} x^*(t)\, e^{-jn\omega_0 t}\, dt$$

$x^*(t) \Rightarrow c^*_{-n}$

(i) If $x(t)$ is real and even then c_n is real & even

(ii) If $x(t)$ is real and odd then c_n is imaginary and odd. $a_0 = 0$

(3) Time reversal

$$x(t) \longrightarrow c_n \quad (\text{period } T_0)$$

$x(t) \Rightarrow x(-t)$

$$c_n' = \frac{1}{T_0}\int_0^{T_0} x(-t)\, e^{-jn\omega_0 t}\, dt$$

$-t = \tau \qquad t = 0 \to \tau = 0$

$-dt = d\tau \qquad t = T_0 \quad \tau = -T_0$

$$c_n' = \frac{1}{T_0}\int_0^{-T_0} x(\tau)\, e^{-jn\omega_0(-\tau)}\,(-d\tau)$$

$$= \frac{1}{T_0}\int_{-T_0}^{0} x(\tau)\, e^{-j(-n)\omega_0 \tau}\, d\tau$$

$c_n' = c_{-n}$

$$\begin{array}{ccc} x(t) & \longrightarrow & x(-t) \\ \downarrow & & \downarrow \\ c_n & & c_{-n} \end{array}$$

(4) Time scaling

$x(t) \longrightarrow c_n$

$x(t) \Rightarrow x(at)$ $\qquad |a| > 1$ compressing of signal

reducing Time period $\uparrow$ $\downarrow$ freq $\uparrow$

$$T_0' = \frac{T_0}{a}$$

$$\omega_0' = \frac{2\pi}{T_0'} = \frac{2\pi}{T_0} \times a = a.\omega_0$$

$$C_n' = \frac{1}{T_0'} \int_0^{T_0'} x(at).e^{-jn\omega_0' t}\, dt$$

$$= \frac{1}{T_0/a} \int_0^{T_0/a} x(at)\, e^{-jn\omega_0 . a . t}\, dt$$

Let $at = \tau \Rightarrow a\,dt = d\tau$

$t = 0, \tau = 0$

$t = \frac{T_0}{a} \quad \tau = T_0$

$$= \frac{a}{T_0} \int_0^{T_0} x(\tau)\, e^{-jn\omega_0 \tau}\, \frac{d\tau}{a}$$

$$= \frac{1}{T_0} \int_0^{T_0} x(\tau)\, e^{-jn\omega_0 \tau}.\, d\tau$$

$$\underline{C_n' = C_n}$$

T_0', ω_0' will change

⑤ Time shifting

$$x(t) \rightarrow C_n \ (T_0)$$

$$C_n = \frac{1}{T_0} \int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$x(t) \rightleftharpoons x(t - t_0) \rightarrow$ Left
$\quad\quad x(t + T_0) \rightarrow$ Right

$$C_n' = \frac{1}{T_0} \int_0^{T_0} x(t - t_0).e^{-jn\omega_0 t}.\, dt$$

$t - t_0 = \tau \quad\quad dt = d\tau$

$t = \tau + t_0$

$t = 0, \tau = -t_0$

$t = T_0, \tau = T_0 - t_0$

$$= \frac{1}{T_0} \int_{-t_0}^{T_0-t_0} x(\tau) . e^{-jn\omega_0(\tau+t_0)} . d\tau$$

$$= \frac{1}{T_0} \int_{-t_0}^{T_0-t_0} x(\tau) . e^{-jn\omega_0\tau} . e^{-jn\omega_0 t_0} . d\tau$$

$$= e^{-jn\omega_0 t_0} \frac{1}{T_0} \int_{-t_0}^{T_0-t_0} x(\tau) . e^{-jn\omega_0\tau} . d\tau .$$

$$\boxed{C_n' = e^{-jn\omega_0 t_0} . C_n}$$

(6) frequency shifting

$$x(t) \longrightarrow C_n \quad (\text{period } T_0)$$

$$C_n \longrightarrow C_{n-m}$$

m is the amount by which frequency will change.

new signal $x'(t) = ??$

$$C_{n-m} = \frac{1}{T_0} \int_0^{T_0} x(t) . e^{-j(n-m)\omega_0 t} . dt$$

$$= \frac{1}{T} \int_0^{T_0} x(t) . e^{-jn\omega_0 t} e^{+jm\omega_0 t} . dt$$

$$= \frac{1}{T_0} \int_0^{T_0} \underbrace{x(t) . e^{jm\omega_0 t}}_{x'(t)} \cancel{dt} . e^{-jn\omega_0 t} . dt$$

$$x'(t) = x(t) . e^{jm\omega_0 t}$$

7. Convolution in time

$$\left.\begin{array}{l} x_1(t) \longrightarrow c_{1n} \\ x_2(t) \longrightarrow c_{2n} \end{array}\right\} \text{ Same } T_0$$

$$x_1(t) * x_2(t) \rightleftharpoons T_0(c_{1n} \cdot c_{2n})$$

proof: $x_1(t) * x_2(t) = x(t) \rightarrow c_n'$

$$c_n' = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \left[x_1(t) * x_2(t)\right] e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \left[\int_0^{T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau\right] e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \left[\int_0^{T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau\right] \cdot \frac{e^{-jn\omega_0 t}\, e^{-jn\omega_0 \tau}}{e^{-jn\omega_0 \tau}}\, dt$$

$$= \left(\frac{1}{T_0}\int_0^{T_0} x_1(\tau)\, e^{-jn\omega_0 \tau}\, d\tau\right)\left(\int_0^{T_0} x_2(t-\tau)\, e^{-jn\omega_0 (t-\tau)}\, dt\right)$$

$$= \left(\frac{1}{T_0}\int_0^{T_0} x_1(\tau)\, e^{-jn\omega_0 \tau}\, d\tau\right)\left(\int_{-\tau}^{T_0-\tau} x_2(\alpha)\, e^{-jn\omega_0 \alpha}\, d\alpha\right)$$

$t-\tau=\alpha$, $dt=d\alpha$

$t=0,\ \alpha=-\tau$

$t=T_0\ \alpha=T_0-\tau$

$$= \left(\frac{1}{T_0}\int_0^{T_0} x_1(\tau)\, e^{-jn\omega_0 \tau}\, d\tau\right) \cdot T_0 \left(\frac{1}{T_0}\int_{-\tau}^{T_0-\tau} x_2(\alpha)\, e^{-jn\omega_0 \alpha}\, d\alpha\right)$$

$$= c_{1n} \cdot T_0 \cdot c_{2n} \qquad \boxed{c_n' = T_0(c_{1n} \cdot c_{2n})}$$

(8) Multiplication in time

$$x_1(t) \longrightarrow C_{1n}$$
$$x_2(t) \longrightarrow C_{2n}$$

$$x_1(t)\, x_2(t) \rightleftharpoons C_{1n} * C_{2n}$$

~~Differ~~

(9) Differenciation in time

$$x(t) \longrightarrow C_n \quad (\text{period } T_0)$$

$$\frac{d}{dt} x(t) \rightarrow (jn\omega_0)\, C_n$$

$$\frac{d^2}{dt^2} x(t) \longrightarrow (jn\omega_0)^2\, C_n$$

$$\frac{d^k}{dt^k} x(t) \longrightarrow (jn\omega_0)^k . C_n$$

(10) Integration in time

$$x(t) \longrightarrow C_n \quad (\text{period } T_0)$$

$$\int x(t) = \frac{C_n}{jn\omega_0}$$

(11) Parseval's Power theorem

$$x(t) \longrightarrow C_n \; (\text{Period } T_0)$$

Average power, $P_{x(t)} = \sum_{n=-\alpha}^{\alpha} |C_n|^2$

Proof: $$x(t) = \sum_{n=-\alpha}^{\alpha} C_n\, e^{jn\omega_0 t}$$

$$x^*(t) = \sum_{n=-\alpha}^{\alpha} C_n^* . e^{-jn\omega_0 t}$$

$$x(t) . x^*(t) = |x(t)|^2$$

$$z = a + ib \qquad |z| = \sqrt{a^2+b^2}$$
$$z^* = a - jb \qquad |z|^2 = a^2+b^2$$
$$z . z^* = a^2 - j^2 b^2 = a^2+b^2 = |z|^2$$

$$P_{x(t)} = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 \, dt$$

$$= \frac{1}{T_0}\int_0^{T_0} x(t) \cdot x^*(t) \cdot dt$$

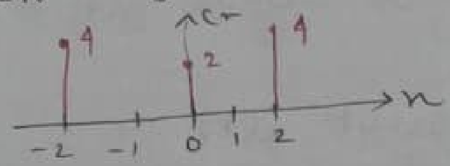
$$= \frac{1}{T_0}\int_0^{T_0} x(t) \cdot \sum_{n=-\alpha}^{\alpha} C_n^* \cdot e^{-jnw_0 t} \, dt$$

$$= \sum_{n=-\alpha}^{\alpha} C_n^* \frac{1}{T_0}\int_0^{T_0} x(t) \cdot e^{-jnw_0 t} \cdot dt$$

$$= \sum_{n=-\alpha}^{\alpha} C_n^* \, C_n$$

$$\boxed{P_{x(t)} = \sum_{n=-\alpha}^{\alpha} |C_n|^2}$$

ex-1: Find average power of signal $x(t)$ when C_n is given by



method 1

$$P_{x(t)} = \sum_{n=-\alpha}^{\alpha} |C_n|^2 = |C_{-2}|^2 + |C_0|^2 + |C_2|^2$$

$$= |4|^2 + |2|^2 + |4|^2$$

$$= 36$$

method 2

$$x(t) = \sum_{n=-\alpha}^{\alpha} C_n \cdot e^{jnw_0 t}$$

$$= C_{-2} \, e^{-j2w_0 t} + C_0 e^0 + C_2 \, e^{j2w_0 t}$$

$$= 4 \, e^{-j2w_0 t} + 2 + 4 \, e^{j w w_0 t}$$

$$= 2 + 4\left[e^{-j2w_0 t} + e^{j2w_0 t}\right]$$

$$= 2 + 4 \cdot 2 \cos 2w_0 t$$

$$x(t) = 2 + 8 \cos 2 w_0 t$$

$$P_{av} = 2^2 + \frac{8^2}{2} = 36 \text{ Watts}$$

ex-2: Fin c_n' in terms of C_n where

$x(t) \rightarrow C_n$
$Y(t) \rightarrow C_n'$

(i) $y(t) = x(t+1) + x(t-1)$

$x(t)$ having co-efficient C_n

$x(t \pm t_0) \rightleftharpoons C_n e^{\pm jn\omega_0 t_0}$

$x(t+1) \rightleftharpoons C_n e^{jn\omega_0}$

$x(t-1) \rightleftharpoons C_n e^{-jn\omega_0}$

$$x(t+1) + x(t-1) = C_n e^{jn\omega_0} + C_n e^{-jn\omega_0}$$
$$= C_n (e^{jn\omega_0} + e^{-jn\omega_0})$$
$$= C_n \cdot 2 \cdot \cos n\omega_0$$

Co-efficient of signal $y(t)$ is $c_n' = 2C_n \cos n\omega_0$

(ii) $y(t) = e^{-j2\omega_0 t} \cdot x(t)$

this is properties of frequency shifting

$$e^{jm\omega_0 t} x(t) \rightleftharpoons C_{n-m}$$

$$y(t) = e^{-j2\omega_0 t} \cdot x(t)$$
$$= e^{j(-2)\omega_0 t} x(t) \rightleftharpoons C_{n-(-2)}$$
$$= C_{n+2}$$

the coefficient of signal $y(t)$ is

$$c_n' = C_{n+2}.$$

(iii) $y(t) = \dfrac{d^2 x(t)}{dt^2}$

$$C_n' = (jn\omega_0)^2 C_n$$
$$= -n^2 \omega_0^2 C_n$$

(iv) $y(t) = \text{odd}[x(t)]$

$$= \frac{x(t) - x(-t)}{2}$$

$$C_n' = \frac{C_n - C_{-n}}{2}$$

(v) $y(t) = \text{Real}[x(t)]$

$$= \frac{x(t) + x^*(t)}{2}$$

$$C_n' = \frac{C_n + \cancel{C_n}\, C_{-n}^*}{2}$$

Magnitude and phase spectrum of Fourier series co-efficient (C_n)

ex-1 $\quad x(t) = 2 + 3\sin 2t + 4\cos 3t$

$$T_1 = \frac{2\pi}{2} \qquad T_2 = \frac{2\pi}{3}$$

$$T_0 = LCM(T, T_2) = 2\pi$$

$$\omega_0 = \frac{2\pi}{T_0} = 1$$

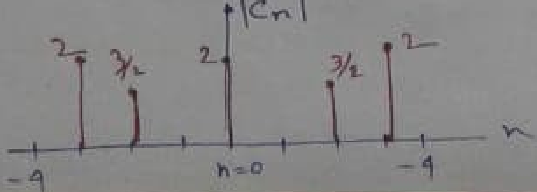
$$x(t) = 2 + 3\sin 2\omega_0 t + 4\cos 3\omega_0 t$$

$$= 2 + \frac{2}{2j}\left(e^{j2\omega_0 t} - e^{-j2\omega_0 t}\right)$$

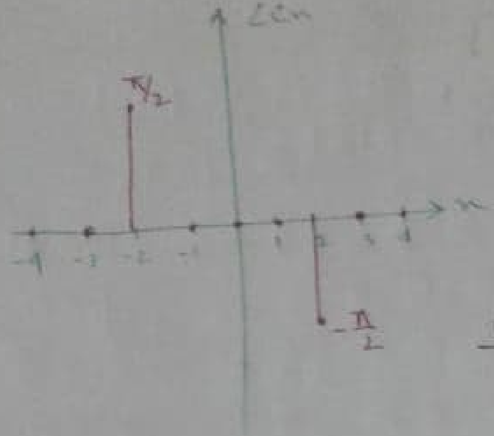
$$+ \frac{4}{2}\left(e^{j3\omega_0 t} + e^{-j3\omega_0 t}\right)$$

$C_0 = 2$, $\quad C_2 = \frac{3}{2j}$, $\quad C_{-2} = -\frac{3}{2j}$, $\quad C_3 = 2$, $C_{-3} = 2$

$C_0 = 2\angle 0 \quad C_2 = \frac{3}{2}\angle(-90^\circ) \quad C_{-2} = \frac{3}{2}\angle 90^\circ \quad C_3 = 2\angle 0^\circ \quad C_{-3} = 2\angle 0$



magnitude spectrum



phase spectrum

* If the signal is Real then magnitude spectrum is even symmetry and phase spectrum is odd symmetry

ex-2 $\quad x(t) = e^{j3t} + 2\sin 3t + 8\cos 2t$

$\downarrow T_1 \qquad \downarrow T_2 \qquad \downarrow T_3$

$T_0 = LCM(T_1, T_2, T_3) = 2\pi$

$\omega_0 = 1$

$x(t) = e^{j3\omega_0 t} + 2\sin 3\omega_0 t + 8\cos 2\omega_0 t$

$= e^{j3\omega_0 t} + \frac{2}{2j}\left(e^{j3\omega_0 t} - e^{-j3\omega_0 t}\right)$

$+ \frac{8}{2}\left(e^{j2\omega_0 t} + e^{-j2\omega_0 t}\right)$

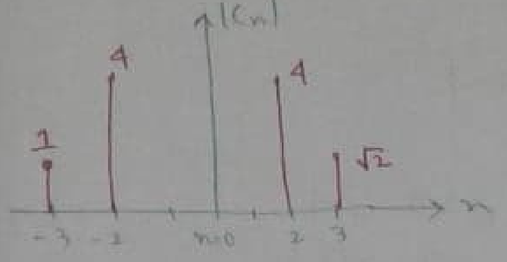
$= \left(1 + \frac{1}{j}\right) e^{j3\omega_0 t} - \frac{1}{j} e^{-j3\omega_0 t}$

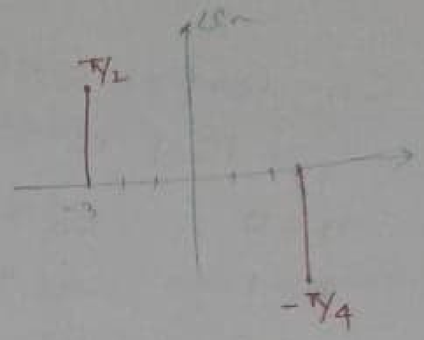
$+ 4 e^{j2\omega_0 t} + 4 e^{-j2\omega_0 t}$

$C_3 = 1 + \frac{1}{j} = 1 - j \qquad C_{-3} = j \qquad C_2 = 4 \qquad C_{-2} = 4$

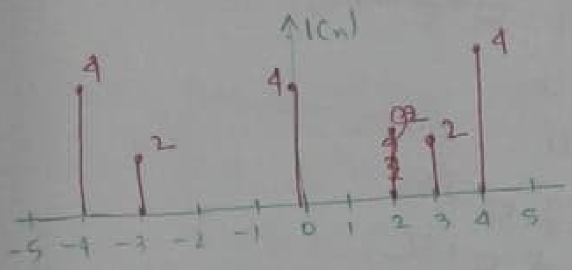
$|C_3| = \sqrt{2} \qquad |C_{-3}| = 1 \qquad |C_2| = 4 \qquad |C_2| = 4$

$\angle C_3 = -45^\circ \qquad C_{-3} = 90^\circ \qquad \angle C_2 = 0 \qquad C_{-2} = 0$

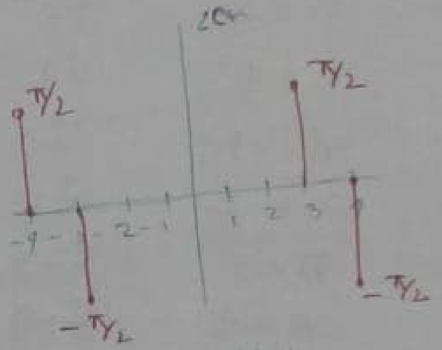




ex-3 Find $x(t)$



magnitude plot



phase plot

$c_0 = 4 \qquad c_3 = 2j \qquad c_{-3} = -2j \qquad c_4 = -4j$

$\angle c_0 = 0 \qquad \angle c_3 = \frac{\pi}{2} \qquad \angle c_{-3} = -\frac{\pi}{2} \qquad \angle c_4 = -\frac{\pi}{2}$

$c_{-4} = 4j$

$\angle_{-4} = \frac{\pi}{2}$

$$x(t) = 4 + 2j\, e^{j3\omega_0 t} - 2j\, e^{-j3\omega_0 t} + 4j\, e^{j4\omega_0 t} - 4j\, e^{-j4\omega_0 t}$$

$$= 4 + 2j\left(e^{j3\omega_0 t} - e^{-j3\omega_0 t}\right) + 4j\left(e^{j4\omega_0 t} - e^{-j4\omega_0 t}\right)$$

$$= 4 - \frac{4}{2j}\left(e^{j3\omega_0 t} - e^{-j3\omega_0 t}\right) - \frac{8}{2j}\left(e^{j4\omega_0 t} - e^{-j4\omega_0 t}\right)$$

$$= 4 - 4\sin 3\omega_0 t - 8\sin 4\omega_0 t$$

$$\cancel{= 4 - 4\sin 3t} - 8\sin 4t$$

c_n = Complex coefficient

$= |c_n| \cdot e^{j\angle c_n}$

$n = 0, \quad c_0 = 4 \cdot e^{j \cdot 0} = 4$

$n = 1 \quad c_1 = 0 \cdot e^{j} = 0$

$n = 2 \quad c_2 = 0 \; e^{j2} = 0$

$n = 3. \quad c_3 = 2 \cdot e^{j\pi/2}$

$n = 4 \quad c_4 = 4 \; e^{j(-\pi/2)}$

$n = -1 \quad c_{-1} = 0$

$n = -2 \quad c_{-2} = 0$

$n = -3 \quad c_{-3} = 2 \; e^{-j\pi/2}$

$n = -4 \quad c_{-4} = 4 \; e^{j\pi/2}$

$$x(t) = \sum_{n=-\alpha}^{\alpha} c_n \, e^{jn\omega_0 t}$$

$$\cancel{= c_0 + 2 \cdot e^{j\pi/2} + 4 \cdot e^{j(-}}$$

$$= c_0 + c_3 \cdot e^{j3\omega_0 t} + c_4 e^{j4\omega_0 t} + c_{-3} \cdot e^{-j3\omega_0 t} + c_4 \; e^{-j4\omega_0 t}$$

$$= 4 + 2 \cdot e^{j\pi/2} \cdot e^{j3\omega_0 t} + 4 \cdot e^{-j\pi/2} e^{j4\omega_0 t} + 2 e^{-j\pi/2} e^{-j3\omega_0 t} + 4 e^{+j\pi/2} \cdot e^{-j4\omega_0 t}$$

$$= 4 + 2 e^{j(3\omega_0 t + \pi/2)} + 4 e^{j(4\omega_0 t - \pi/2)} + 2^{-j(3\omega_0 t + \pi/2)} + 4 \cdot e^{-j(4\omega_0 t - \pi/2)}$$

$$= 4 + 2 \cdot 2 \cos\left(3\omega_0 t + \frac{\pi}{2}\right) + 4 \cdot 2 \cdot \cos\left(4\omega_0 t - \frac{\pi}{2}\right)$$

$$x(t) = 4 + 4 \cos\left(3\omega_0 t + \frac{\pi}{2}\right) + 8 \cos\left(4\omega_0 t - \frac{\pi}{2}\right)$$

$x(t)$	C_n
Real	Conjugate symmetry
C. Sym	Real.
Img	Conjugate anti symmetry
Conjugate anti symmetry	Img
Real + even symetry	Real + even symetry
Img + even	Img + even
Read + Odd	Img + Odd.
Img + odd	Read + Odd

ex-1 $x(t) = \sum_{n=-\alpha}^{\alpha} j\cos n\pi \, e^{jn\pi t}$

$\downarrow$

Img + even.

$x(t) = x(-t)$

$\cos \omega t = \cos(-\omega t)$

$\cos \rightarrow$ even

$n = 1, -1$

$x(t) = \dots + j\cos\pi . e^{j\pi t} + j\cos(-\pi)\, e^{-j\pi t} + \dots$

$= \dots - je^{j\pi t} - je^{-j\pi t} + \dots$

$= \dots - j[e^{j\pi t} + e^{-j\pi t}] + \dots$

$= \dots \underbrace{-j \, 2\cos\pi t}_{\text{Img + even}}$

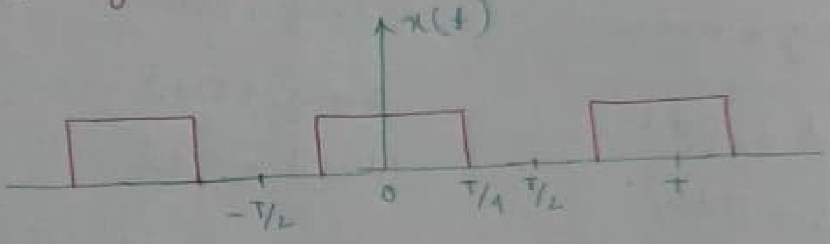
ex-2: $x(t) = \sum_{n=-\alpha}^{\alpha} j \sin\frac{n\pi}{2} \; e^{jn\pi t}$

$c_n \rightarrow$ Img + odd.

$x(t) = \cdots\cdots \; 2j^2 \sin \pi t - \cdots$

$= \cdots \underbrace{-2 \sin \pi t}_{\text{Real + odd}}$

ex. Determine the Fourier ~~series~~ co-efficient for given periodic signal $x(t)$ is



ⓐ $\frac{A}{j\pi K} \sin(\frac{\pi}{2}k)$

ⓑ $\frac{A}{\pi j K} \cos(\frac{\pi}{2}k)$

ⓒ $\frac{A}{\pi K} \sin(\frac{\pi}{2}k)$

ⓓ $\frac{A}{\pi K} \cos(\frac{\pi}{2}k)$

$x(t)$ is real and even.

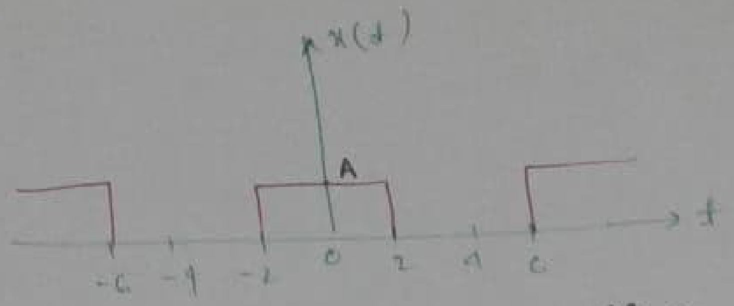
$C_K = \frac{A}{\pi K} \sin(\frac{\pi}{2} \cdot k)$ Real

$C_{-K} = \frac{A}{-\pi K} \cdot \sin(-\frac{\pi}{2} \cdot k)$

$= \frac{A}{\pi K} \sin(\frac{\pi}{2}k)$

$C_K = C_{-K} \rightarrow$ even

Q.



Find fourier series co-efficients of the web form shown in fig

$T_0 = 8$

$\omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}$

$$C_n = \frac{2A_0}{n\omega_0 T_0} \sin n\omega_0 \tau/2$$

$$= 2A$$

$$C_n = \frac{1}{T_0}\int_{-4}^{4} f(t) \cdot e^{-jn\omega_0 t} \cdot dt \qquad \int_{-4}^{-2} \quad \int_{-2}^{2} A \quad \int_{2}^{4}$$

$$C_n = \frac{A}{2} \cdot \frac{\sin n\pi/2}{n\,\pi/2}$$