Mathematical Induction-2

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Proof by strong induction

Prob-1 Show that if n is an integer greater than 1, then n can be written as the product of primes.

Let $P(n)$ be the proposition that n can be written as the product of primes.

Basis step: $P(2)$ is true, because 2 can be written as the product of one prime.

Inductive step: The inductive hypothesis is the assumption that $P(j)$ is true for all integers j with $2 \leq j \leq k$.

To complete the inductive step, show $P(k+1)$ is true.

Two cases:

(i) $(k+1)$ is prime: We immediately see that $P(k+1)$ is true.

(ii) $(k+1)$ is composite: $(k+1)$ can be factorized into two positive integers a and b with $2 \leq a \leq b \leq k$.

As both a and b are integers and at least 2 and not exceeding k, we can use inductive hypothesis to write that a and b can be expressed as product of primes.

Thus, if $(k+1)$ is composite, it can be written as product of primes which are in the factorization of a and b.

Prob-2

Let n be the no. of matches in each pile.

$P(n)$ is the proposition that the second player wins when there are initially n matches in each pile.

Basis step: When $n=1$, the first player has only one choice of removing the single match from his pile. This enables the second player to remove the single match from his pile to win the game.

Inductive Step: The inductive hypothesis is $P(j)$ is true for all j with $1 \leq j \leq k$. To complete the inductive step we have to show that $P(k+1)$ is also true.

Suppose there are $(k+1)$ matches in each pile and the first player removes r matches such that $1 \leq r \leq k$. No. of matches left in the first pile is $(k+1-r)$. Now, the second player can also remove r matches so that the no. of remaining matches in the second pile is also $(k+1-r)$.

$k+1$
$-r$

$k+1$
$-r$

Because, $1 \leq k+1-r \leq k$, we use inductive hypothesis to conclude that the second player always wins

Prob.

Let $T(n)$ be the proposition that a simple polygon with n sides can be triangulated into $(n-2)$ triangles.

<u>Basis Step</u>: $T(3)$ is true because it is a simple polygon with three sides. We do not need to add any diagonal to triangulate a triangle — it is already triangulated into one triangle.

<u>Inductive step</u>: For the inductive hypothesis we assume that $T(j)$ is true where $3 \leq j \leq k$. To complete the inductive step we have to show $T(k+1)$ is true.

Suppose there is a simple polygon P with $(k+1)$ sides.

Because $k+1 \geq 4$, according to the lemma, P has an interior diagonal ab. Now, the diagonal ab splits P into two smaller regular polygons Q and R with s and t sides, respectively, such that $3 \leq s \leq k$ and $3 \leq t \leq k$.

Furthermore, the no. of sides of P is two less than the sum of the no. of sides of Q and R. This is because, each side of P is a side of either Q or R, but not both, and the diagonal ab is the side of both Q and R. That is, $\boxed{k+1 = s+t-2}$ — (1)

Using inductive hypothesis, we can triangulate polygons Q and R into $(s-2)$ and $(t-2)$ triangles. Triangulation of Q and R together produce the triangulation of P. So, the total no. of triangles formed by triangulation of P

$$
\begin{aligned}
P &= (s-2)+(t-2) \\
&= (s+t-2)-2 \\
&= (k+1)-2 \quad \text{(from (1))}
\end{aligned}
$$

This completes the inductive step showing that every regular polygon with n sides can be triangulated into $(n-2)$ triangles.