Cardinality of sets

Monday, September 28, 2020 10:40 AM

Prob. $S = \{0, 2, 4, 6, \ldots\ldots\}$

We need to find a bijective function $f: Z^+ \to S$.

$Z^+ = \{1, 2, 3, 4, \ldots\}$

We define a function $f: x \to (2x-2)$

$1 \to 2.(1) - 2 = 0$

$2 \to 2.(2) - 2 = 4 - 2 = 2$

$3 \to 2.(3) - 2 = 6 - 2 = 4$

$4 \to 2.(4) - 2 = 8 - 2 = 6.$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

One-to-one : $2x - 2 = 2y - 2 \Rightarrow 2x = 2y \Rightarrow x = y$

Onto : ~~$\forall s \in S$~~ $\forall s \in S, \left(\frac{s+2}{2}\right)$ is the preimage in Z^+.

Therefore, $|S| = |Z^+|$. Hence, the set S is Countable.

Prob. $Z = \{-\infty, \ldots\ldots 0, \ldots\ldots +\infty\}$

We define a bijection $f: Z^+ \to Z$.

We define a bijection $f: Z^+ \to Z$.

$f(n) = n/2$, when n is even.

$f(n) = \cancel{(n+1)} - \left(\frac{n-1}{2}\right)$ when n is odd.

$|Z| = |Z^+|$. $\Rightarrow$ the set of integers is countable.

Prob: We define a function $f: Z^+ \to Q^+$ by starting at $1/1$ and following the arrows as directed, skipping over any number that has already been counted.

Set $F(1) = 1/1$, $F(2) = 1/2$, $F(3) = 2/1$, $F(4) = 3/1$. Then we skip $2/2$ as $2/2 = 1$ is counted first.

After that set $F(5) = 1/3$, $F(6) = 1/4$, $F(7) = 2/3$, $F(8) = 3/2$, $F(9) = 4/1$ and $F(10) = 5/1$. We skip $4/2$, $3/3$, and $2/4$ and $F(11) = 1/5$.

Because all positive rational numbers are listed once, ~~as~~ the set of positive rational numbers is countable.

Prob:

$$r_1 = 0.d_{11}d_{12}d_{13}\ldots$$
$$r_2 = 0.d_{21}d_{22}d_{23}\ldots$$
$$r_3 = 0.d_{31}d_{32}d_{33}\ldots$$

$$\left.\begin{array}{l}
0.\,2\,0\,1\,4\,8\,8\,0\,2\ldots \\
0.\,1\,1\,6\,6\,6\,0\,2\ldots \\
0.\,0\,3\,3\,5\,3\,3\,2\ldots \\
0.\,9\,6\,7\,7\,6\,8\,0\ldots \\
0.\,0\,0\,0\,3\,1\,0\,0\ldots
\end{array}\right\}$$

Construct a diagonal along all $d_{i,i}$'s.

We want to form a new number, say s

$$s = 0.d_1d_2d_3d_4\ldots$$

$s = 0.322\ldots$

$$d_i = \begin{cases} 2, & \text{if } d_{ii} \neq 2 \\ 3, & \text{if } d_{ii} = 2 \end{cases}$$

Since, $s \in \mathbb{R}^*$, $\exists t$ such that $s = r_t$.

Therefore,

$$0.d_1d_2d_3d_4\ldots d_t = 0.d_{t1}d_{t2}d_{t3}d_{t4}\ldots d_{tt}\ldots$$

$$\Rightarrow \quad d_t = d_{tt}$$

However, by construction d_t and d_{tt} cannot be equal.

$$\Rightarrow r_t \neq s.$$

$\therefore d_t \neq d_{tt} \Rightarrow r_t \neq S.$

Every real number has a unique decimal expansion. Therefore, the real number S is not equal to any of $r_1, r_2, r_3 \ldots$ because the decimal expansion of S is different from that of r_i in the $i^{\underline{th}}$ place to the right of the decimal point, for each i.

But $S \in \mathbb{R}^*$ and it is not in the list.

Hence, this is a contradiction to our initial supposition that $\mathbb{R}^*$ is countable.

$\therefore \mathbb{R}^*$ is not countable (listable).

Likewise, we conclude $\mathbb{R}$ is also not countable.