Lecture : Eigenvalues and Eigenvectors

1. Introduction:

To explain eigenvalues, we first explain eigen vectors. Let, A be a square matrix. Almost all the vectors change direction when they are multiplied by A.

Certain exceptional vectors $\underset{\sim}{x}$ are in the same direction as $A\underset{\sim}{x}$. Those special type of vectors are called eigen vectors.

Multiply an eigen vector by A, and the vector $A\underset{\sim}{x}$ is a number λ times the original $\underset{\sim}{x}$.

The ~~Basic~~ basic equation $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$. The number λ is an eigenvalue of A.

The eigen value λ tells whether the special vector $\underset{\sim}{x}$ is streched or shrunk or reversed or left unchanged — when it is multiplied by A. we may find $\lambda = 2$ or $\frac{1}{2}$ or 1 or (-1).

The eigen value λ could be zero! Then $A\underset{\sim}{x} = 0\underset{\sim}{x}$ means the eigen vector $\underset{\sim}{x}$ is in the nullspace.

2. Definition:

Let A be a square matrix. A number λ is called an eigenvalue of A if there exists a non-zero vector $\underset{\sim}{x}$ such that $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$. ...... (1)

In the above definition, the vector $\underset{\sim}{x}$ is called eigen vector associated to the eigen value λ.

Remarks:

(1) $\underset{\sim}{x} \neq \underset{\sim}{0}$ is crucial, since $\underset{\sim}{x} = \underset{\sim}{0}$ always satisfy equation (1).

(2) If $\underset{\sim}{x}$ is an eigen vectors for λ, then so is $c\underset{\sim}{x}$ for any constant c.

(3) Geometrically, in 3D, eigen vectors of A are ~~those~~ whose directions are unchanged under linear transformation A.

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3. Characteristic Equation:

We observe from equation (1), that λ is an eigen value iff $(A-\lambda I)\underset{\sim}{x} = \underset{\sim}{0}$ has a nontrivial solution.
By the inverse matrix theorem,

$$\det(A-\lambda I) = 0 .$$

If $A = (a_{ij})_{n\times n}$ then,

$\det(A-\lambda I) = 0$ gives

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

This is called the characteristic equation of A.
On expanding the determinant, the characteristic equation takes the form –

$(-1)^n \lambda^n + K_1\lambda^{n-1} + K_2\lambda^{n-2} + \cdots + K_n = 0$, where each K_i $(i=1,2,\ldots,n)$ are expressible in terms of a_{ij}.

The roots of this equation is called the eigen values or characteristic roots of the matrix A.

For each eigen value λ, we need to find a basis for the eigenspace i.e, Null$(A-\lambda I)$ for finding eigen vectors.

i.e, we have to solve: $(A-\lambda I)\underset{\sim}{x} = \underset{\sim}{0}$, for fixed λ.

Summary: To solve eigen value problem for an $n\times n$ matrix

1. compute the determinant of $A-\lambda I$, which is a polynomial of degree n.
2. Find the roots of this polynomial.
3. For each eigen value λ, solve $(A-\lambda I)\underset{\sim}{x} = \underset{\sim}{0}$ to find an eigen vector $\underset{\sim}{x}$.

Example 1:

Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Solution:— The characteristic equation is

$$|A - \lambda I| = 0$$

ie,
$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(2-\lambda)(5-\lambda) = 0$$

Thus the eigen values of A are 2, 3, 5

If $\underset{\sim}{x} = [x, y, z]^T$ be the eigen vector corresponds to the eigen value λ then we have

$$[A - \lambda I]\underset{\sim}{x} = \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting, $\lambda = 2$, we have $x + y + 4z = 0$ ~~And~~, $6z = 0$ and $3z = 0$

ie, $x + y = 0$ and $z = 0$

$$\therefore \frac{x}{1} = \frac{y}{-1} = \frac{z}{0} = k_1 \text{ (say)}$$

Hence, the eigenvectors corresponding the eigen value $\lambda = 2$, ~~we have~~ $k_1(1, -1, 0)$.

Putting $\lambda = 3$, we have $y + 4z = 0$, $-y + 6z = 0$ and $2z = 0$.

ie, $y = 0$, $z = 0$.

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0} = k_2 \text{ (say)}$$

Hence, the eigen vectors corresponding to the eigen value $\lambda = 3$ are $k_2(1, 0, 0)$.

Similarly, the eigen vectors corresponding to $\lambda = 5$
are $k_3(3,2,1)$.

4. Cayley-Hamilton theorem and its applications :

We now introduce a very important theorem in matrix theory; i.e., Cayley-Hamilton theorem.

Statement: Every square matrix satisfies its own Characteristic equation.

If $A_{n\times n}$ be a square matrix, then its characteristic equation is $\det(A-\lambda I_n)=0$. Let us denote $\det(A-\lambda I_n)$ as $p(\lambda)$. Then according to Cayley-Hamilton theorem

$$p(A)=0.$$

Applications:

Example 2

Verify Cayley-Hamilton theorem for $A=\begin{bmatrix}1&4\\2&3\end{bmatrix}$. Find A^{-1}.

Solution: We know that,

$$\det(A-\lambda I)=0$$

$$\left|\begin{bmatrix}1&4\\2&3\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}\right|=0$$

$$\Rightarrow \begin{vmatrix}1-\lambda&4\\2&3-\lambda\end{vmatrix}=0$$

$$\Rightarrow \lambda^2-4\lambda-5=0.$$

The characteristic equation is $\underbrace{\lambda^2-4\lambda-5}_{p(\lambda)}=0$.

Now, $A^2=\begin{bmatrix}1&4\\2&3\end{bmatrix}\begin{bmatrix}1&4\\2&3\end{bmatrix}=\begin{bmatrix}9&16\\8&17\end{bmatrix}$

$$A^2-4A-5I=\begin{bmatrix}9&16\\8&17\end{bmatrix}-4\begin{bmatrix}1&4\\2&3\end{bmatrix}-5\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}=0$$

Hence $p(A)=0$ and Cayley-Hamiton theorem is verified.

Now we have

$$A^2 - 4A - 5I = 0$$

Multiplying both side by A^{-1}

$$(A^{-1}A^2) - 4(A^{-1}A) - 5(A^{-1}I) = (A^{-1}0) = 0$$

$$\Rightarrow \quad A - 4I - 5A^{-1} = 0$$

$$\Rightarrow \quad A - 4I = 5A^{-1}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}(A - 4I)$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

<u>Note</u> Without calculating we can find inverse of A using Cayley-Hamilton theorem.

<u>Example 2:</u>

Let, $A = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$. Find A^6.

<u>Solution:</u> $p(\lambda)$ = characteristic polynomial of A

$$= \det(A - \lambda I)$$

$$= \lambda^2 - 2\lambda - 1 \quad \text{(check!)}$$

By Cayley-Hamilton theorem,

$p(A) = 0$ i.e., $A^2 - 2A - I = 0$

$$A^2 = 2A + I$$

$$A^3 = 2A^2 + A = 2(2A + I) + A = 5A + 2I$$

$$A^4 = 5A^2 + 2A = 5(2A + I) + 2A = 12A + 5I$$

$$A^5 = 12A^2 + 5A = 12(2A + I) + 5A = 29A + 12I$$

$$A^6 = 29A^2 + 12A = 29(2A + I) + 12A = 70A + 29I$$

$$= 70\begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix} + 29\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 239 & -70 \\ 140 & -41 \end{bmatrix}$$

Note: The Cayley-Hamilton theorem is useful for calculating A^n for any square matrix.

5. Properties of Eigen Values

Theorem 1: Any square matrix A and its transpose A^T have the same eigen values.

Proof: We have, $(A-\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$

$$|(A-\lambda I)^T| = |A^T - \lambda I|$$

$$\Rightarrow |A-\lambda I| = |A^T - \lambda I| \quad [\because |B^T| = |B|]$$

$\therefore |A-\lambda I| = 0$ iff $|A^T - \lambda I| = 0$.

ie, λ is an eigen value of A iff it is an eigen value of A^T.

Theorem 2: The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ be a triangular matrix of order n.

Then, $|A-\lambda I| = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$

Roots of $|A-\lambda I| = 0$ are $a_{11}, a_{22}, \ldots, a_{nn}$.

Hence the eigen values of A are the diagonal elements of A, ie, $a_{11}, a_{22}, \ldots, a_{nn}$.

Corollary: The eigen values of a diagonal matrix are just the diagonal elements.

Theorem 3: The eigen values of an idempotent matrix are either zero or one.

Proof: Let A be an idempotent matrix so that $A^2 = A$.

If λ be an eigen value of A, then there exists a non-zero vector X such that $AX = \lambda X$.

$\therefore A(AX) = A(\lambda X)$ ie, $A^2X = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X$

ie, $AX = \lambda^2 X$

ie, $\lambda X = \lambda^2 X$

ie, $(\lambda - \lambda^2)X = 0$

$X \neq 0 \Rightarrow \lambda - \lambda^2 = 0 \Rightarrow \lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0$ or 1. Hence the result.

Theorem 4:

The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.

Proof: consider the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

So that $|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$

$$= -\lambda^3 + \lambda^2 (a_{11} + a_{22} + a_{33}) - \lambda (\cdots\cdots) + \cdots\cdots \quad \text{—— (i)}$$

If $\lambda_1, \lambda_2, \lambda_3$ are eigen values of A, then

$$|A - \lambda I| = (-1)^3 (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$= -\lambda^3 + \lambda^2 (\lambda_1 + \lambda_2 + \lambda_3) - \lambda (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) + \lambda_1\lambda_2\lambda_3 \quad \text{—— (ii)}$$

Equating (i) & (ii) we get $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$.

Hence the result.

[This property is proved for a matrix of order 3, but method will be capable of easy extension to matrices of any order]

Theorem 5:

The product of the eigen values of a matrix is equal to its determinant.

Proof:— putting $\lambda = 0$ in (ii) we get the desired result.

Theorem 6:

If λ is an eigen value of a matrix A, is then $1/\lambda$ is the eigen value of A^{-1}.

Proof:— If X be the eigen vector corresponding to λ,

$$AX = \lambda X$$

premultiplying both side by A^{-1}

$$A^{-1}(AX) = A^{-1}(\lambda X) \Rightarrow (A^{-1}A)X = \lambda (A^{-1}X)$$

$$\Rightarrow \quad A^{-1}X = \frac{1}{\lambda} X$$

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This proves that $1/\lambda$ is an eigen value of A^{-1}.

Theorem 7:

If λ is an eigen value of ~~a matrix A~~ an orthogonal matrix A, then $1/\lambda$ is an eigen value of A.

Proof:- We know that if λ is an eigen value of a matrix A then $1/\lambda$ is an eigen value of A^{-1} (Th. 6).

Since A is an orthogonal matrix, A^{-1} is same as its transpose A^T. But A and A^T has same set of eigenvalues.

$\therefore$ $1/\lambda$ is also an eigen value of A.

Theorem 8:

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$. ($m$ being positive integer).

Proof:- Let λ_i be the eigenvalue and X_i be the corresponding eigen vector of A $(i = 1, 2, \ldots, n)$.

Then,
$$AX_i = \lambda_i X_i$$
$$A^2 X_i = \lambda_i (A_i X_i) = \lambda_i (\lambda_i X_i) = \lambda_i^2 X_i$$

Similarly, $A^3 X_i = \lambda_i^2 (AX_i) = \lambda_i^2 (\lambda_i X_i) = \lambda_i^3 X_i$

Proceeding similarly we get,
$$A^m X_i = \lambda_i^m X_i$$

which tells us λ_i^m is an eigen value of A^m and X_i is the eigenvector corresponding to λ_i^m.

Example 4:

If λ is an eigen value of a non-singular matrix, Show that $|A|/\lambda$ is an eigen value of adj A.

→ we know that, if λ is an eigen value of the matrix A, $\frac{1}{\lambda}$ is an eigen value of A^{-1}. Then ∃ an eigen vector X_1 such that,

$$A^{-1}X_1 = \frac{1}{\lambda}X_1$$

$$\Rightarrow \frac{\text{adj} A}{|A|}X_1 = \frac{1}{\lambda}X_1 \quad \left[\because A^{-1} = \frac{1}{|A|}\text{adj} A\right]$$

$$\Rightarrow (\text{adj} A)X_1 = \frac{|A|}{\lambda}X_1$$

This implies $\frac{|A|}{\lambda}$ is an eigen value of adj A.

Example 5:

Find eigen values of adj A and of $A^2 - 2A + I$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

→ First we find the eigen values of A.

$|A - \lambda I| = 0$ gives $\begin{vmatrix} 2-\lambda & 3 & 4 \\ 0 & 4-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda = 2, 4, 3.$$

det A = product of eigen values $= 2 \cdot 4 \cdot 3 = 24$

∴ Eigen values of adj A are $\frac{24}{2}, \frac{24}{4}, \frac{24}{3}$ i.e, 12, 6, 8.

If λ is an eigen value of A, $\lambda^2 - 2\lambda + 1$ is eigen value of $A^2 - 2A + I$.

∴ Eigen values of $A^2 - 2A + I$ are

$$2^2 - 2 \cdot 2 + 1,\ 4^2 - 2 \cdot 4 + 1,\ 3^2 - 2 \cdot 3 + 1$$

i.e, 1, 9, 4.

Example 6:

Find all of the eigen values and eigenvectors of

$$A = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix}$$

Compute the characteristic polynomial $-(\lambda-2)^2(\lambda+1)$.

$-(\lambda-2)^2(\lambda+1)=0$ gives $\lambda = 2, 2, -1$

Definition

If A is a matrix with characteristic equation $p(\lambda)=0$, the multiplicity of a root λ is called the algebric multiplicity of the eigen value λ.

Here in the above example, $\lambda=2$ has algebric multiplicity 2, while $\lambda=-1$ has algebric multiplicity 1.

The eigen value $\lambda=2$ gives us two linearly independent eigen vectors $(-4,1,0)$ and $(2,0,1)$ and $\lambda=-1$, we obtain the single eigen vector $(-1,1,1)$ (Check!)

Definition

The number of linearly independent eigenvectors corresponding to a single eigenvalue is its geometric multiplicity.

In the above example, $\lambda=2$ has geometric multiplicity 2, while $\lambda=-1$ has geometric multiplicity 1.

Theorem:

The geometric multiplicity of an eigenvalue is less than or equal to its algebric muliplicity.

6. Reduction To Diagonal Form

Definition :-

A ~~square matrix A is~~ of order n is said to be a similar matrix of order n if $\exists$

A square matrix A is called similar to a square matrix B iff there exists a non-singular matrix P such that $A = P^{-1}BP$.

Definition :-

A square matrix A is said to be diagonalizable iff there exists a diagonal matrix D such that A is similar to D. ie,

$A = P^{-1}DP$ for some non singular matrix P.

Theorem

If a square matrix of order n has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

[This result will be proved for a square matrix of order 3 but method will be capable of easy extension to matrices of any order]

Proof :- Let A be a square matrix of order 3. Let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values and $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$, $X_3 = \begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix}$ be the corresponding eigen vectors.

Denoting the square matrix $[X_1\ X_2\ X_3] = \begin{bmatrix} x_1 & x_1' & x_1'' \\ x_2 & x_2' & x_2'' \\ x_3 & x_3' & x_3'' \end{bmatrix}$ by P, we have

$$AP = A[X_1\ X_2\ X_3] = [AX_1\ AX_2\ AX_3] = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_1' & \lambda_3 x_1'' \\ \lambda_1 x_2 & \lambda_2 x_2' & \lambda_3 x_2'' \\ \lambda_1 x_3 & \lambda_2 x_3' & \lambda_3 x_3'' \end{bmatrix}$$

$$= [\lambda_1 X_1\ \lambda_2 X_2\ \lambda_3 X_3]$$

$$= \begin{bmatrix} x_1 & x_1' & x_1'' \\ x_2 & x_2' & x_2'' \\ x_3 & x_3' & x_3'' \end{bmatrix} \times \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$= PD$$

$\therefore P^{-1}AP = D$, D is diagonal matrix

and this proves the theorem.

Working procedure

1. Find the eigen values of the square matrix A.
2. Find the corresponding eigen vectors and write P.
3. Find the diagonal matrix D from $D = P^{-1}AP$.

Example

Reduce the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to the diagonal form.

Solution: The characteristic equation of A is

$$\begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{vmatrix} = 0 \text{ or, } \lambda^3 - \lambda^2 - 5\lambda + 5 = 0$$

Solving we get $\lambda_1 = 1,\ \lambda_2 = \sqrt{5},\ \lambda_3 = -\sqrt{5}$ as the eigen values of A.

when $\lambda = 1$ corresponding eigen vector is $(1, 0, -1)$
when $\lambda = \sqrt{5}$ corresponding eigen vector is $(\sqrt{5}-1, 1, -1)$
when $\lambda = -\sqrt{5}$ corresponding eigen vector is $(\sqrt{5}+1, -1, 1)$

$$P = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Finding $P^{-1}AP$ =

$$\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1.1 & \sqrt{5}.(\sqrt{5}-1) & -\sqrt{5}.(\sqrt{5}+1) \\ 1.0 & \sqrt{5}.1 & -\sqrt{5}.(-1) \\ 1.1 & \sqrt{5}.(-1) & -\sqrt{5}.1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

$$= PD$$

$$\therefore D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}.$$

Note:

(1) When there is repeated eigen values diagonalisation is possible only when eigen vectors are linearly independent.

(2) Diagonalisation of a matrix is quite useful for obtaining powers of a matrix.

→ Let A be the square matrix and diagonalisable. Then $\exists$ P such that

$$D = P^{-1}AP$$

$$D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}A^2P$$

Similarly, $\boxed{D^n = P^{-1}A^nP} \Rightarrow A^n = P^{-1}D^nP.$