5/10/2020

P① A periodic function $f(t)$. with a period of 2π, is represented as its Fourier series.

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

if $f(t) = \begin{cases} A \sin t & 0 \leq t \leq \pi \\ 0 & \pi < t < 2\pi \end{cases}$

The fourier series co-effecients a_1 and b_1 of $f(t)$ are ??

Ⓐ $a_1 = 0,\ b_1 = A/\pi$ Ⓑ $a_1 = 0,\ b_1 = A/2$

Ⓒ $a_1 = A/\pi,\ b_1 = 0$ Ⓓ $a_1 = A/2,\ b_1 = 0$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos n\omega t \, dt$$

$T = 2\pi$, $\omega = 1$

$$a_1 = \frac{2}{2\pi} \int_0^{2\pi} x(t) \cdot \cos t \, dt$$

$$= \frac{2}{2\pi} \int_0^{\pi} A \sin t \cos t \, dt = 0$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin n\omega t \, dt$$

$$= \frac{2}{2\pi} \int_0^{\pi} A \sin t \sin t \, dt = \frac{A}{2}$$

P.2 The fourier transform of a continious time signal $x(t)$ is given by $X(\omega) = \dfrac{1}{(10+j\omega)^2}$ $-\alpha < \omega < +\alpha$

where $j = \sqrt{-1}$ and ω denotes frequency then the value of $|\ln x(t)|_{at\ t=1} = ?$

(ln denotes logarithm to base e)

$$e^{-at}u(t) \rightleftharpoons \frac{1}{a+j\omega}$$

$$a = 10 \quad e^{-10t}\,u(t) = \frac{1}{10+j\omega}$$

$$t\,e^{-10t}\,u(t) = j\frac{d}{d\omega}\left(\frac{1}{10+j\omega}\right)$$

$$= \frac{1}{(10+j\omega)^2}$$

$$|\ln x(t)|_{t=1} = \left|\ln\{t\,e^{-10t}\,u(t)\}\right|_{t=1}$$

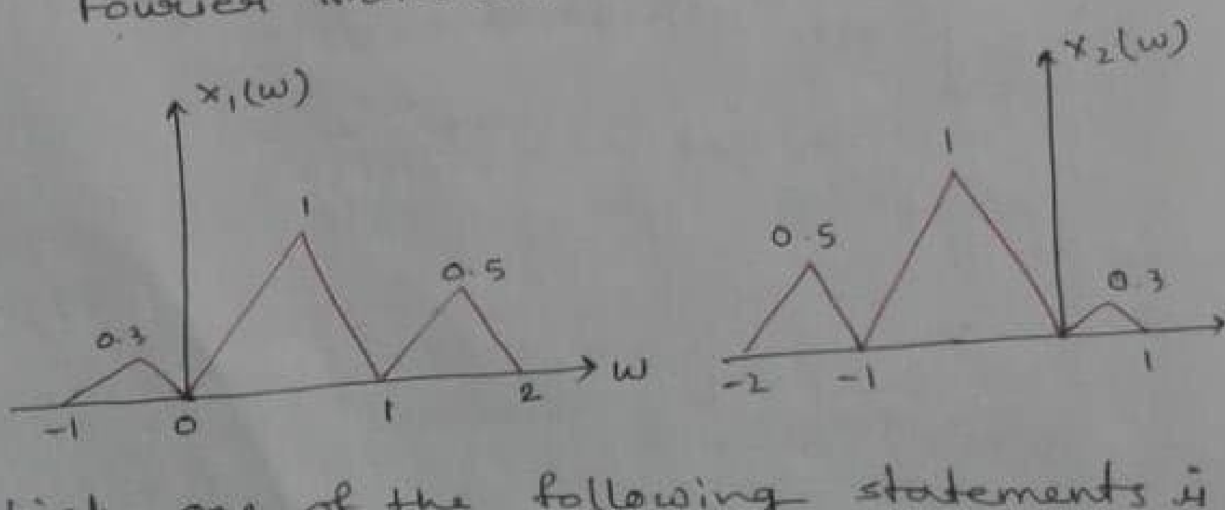
$$= |\ln t + \ln e^{-10t} + \ln u(t)|_{t=1}$$

$$= |0 + -10\times 1 + 0|$$

$$|\ln x(t)|_{t=1} = 10 \quad (\text{Ans})$$

gate 2017

P.3 Suppose $x_1(t)$ and $x_2(t)$ have the Fourier transform as shown bellow



which one of the following statements is true.

(A) $x_1(t)$ and $x_2(t)$ are complex and $x_1(t) \cdot x_2(t)$ is also complex with non zero imaginary part.

(B) $x_1(t)$ and $x_2(t)$ are real and $x_1(t) \cdot x_2(t)$ is also real.

(C) $x_1(t)$ and $x_2(t)$ are complex but $x_1(t)\, x_2(t)$ is real.

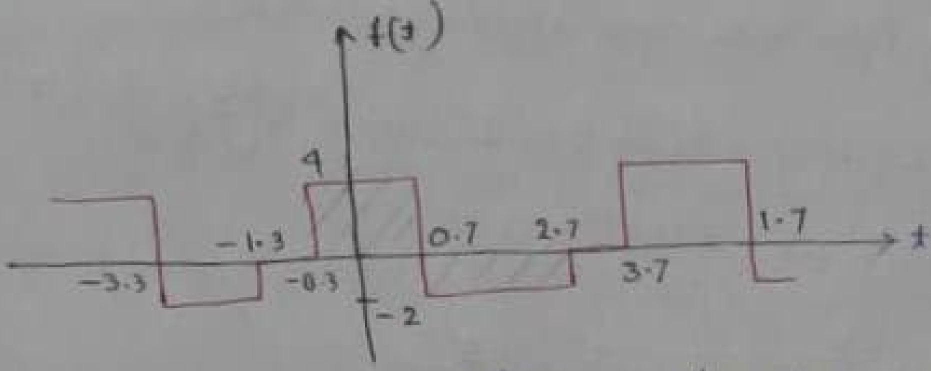
(D) $x_1(t)$ and $x_2(t)$ are imaginary but $x_1(t) \cdot x_2(t)$ is real.

$X_2(\omega) = X_1(-\omega)$

$x_2(t) = x_1(-t)$

$x_1(t)\, x_2(t)$ will be real.

P (4) The mean square value of the given periodic waveform $f(t)$ is ___ ?



$$\text{Mean square value} = \frac{\text{Area under the squared function}}{\text{period of the fuction}}$$

$$= \frac{\overset{4^2(0.7-(-1.3)) + 2^2 \times (2.7-0.7)}{16 \times 1 + 4 \times 2}}{4} = \frac{24}{4} = 6$$

gate 2016

P (5) Let $f(x)$ be a real periodic function satisfying $f(-x) = -f(x)$. the general form of its Fourier series representation would be -

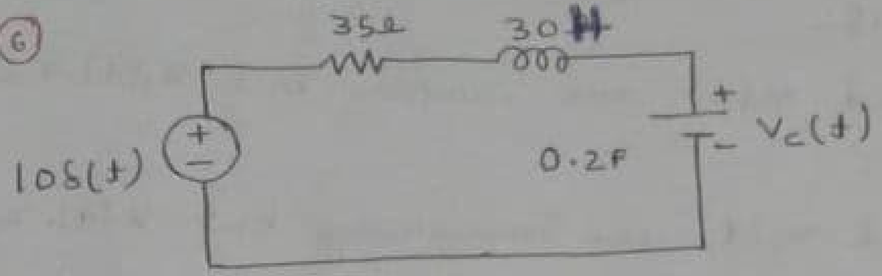
(A) $f(x) = a_0 + \sum_{K=1}^{\infty} a_K \cos(Kx)$

(B) $f(x) = \sum_{K=1}^{\infty} b_K \sin(Kx)$

ⓒ $f(x) = a_0 + \sum_{K=1}^{\alpha} a_{2K} \cos(Kx)$

ⓓ $f(x) = \sum_{K=1}^{\alpha} a_{2K+1} \sin(2K+1).x$

P.⑥



Find $V_c(t)$ for ckt shown in figure

Solution : Let $i(t)$ be the current in the circuit, then applying KVL

$$10\,\delta(t) = 35\,i(t) + 30.\frac{di(t)}{dt} + \frac{1}{0.2}\int_{-\alpha}^{t} i(t)\,dt$$

Taking Fourier transform,

$$10 = 35\,I(j\omega) + 30.j\omega.I(j\omega) + 5\left[\frac{I(j\omega)}{j(\omega)} + \pi I(0)\delta(\omega)\right]$$

$I(0)$ at $I(0^+) = 0$

$$I(j\omega) = \frac{10}{35 + 30j\omega + \frac{5}{j\omega}}$$

$$I(j\omega) = \frac{10\,j\omega}{35\,j\omega + 30(j\omega)^2 + 5}$$

$$= \frac{2\,j\omega}{6(j\omega)^2 + 7\,j\omega + 1} = \frac{2\,j\omega}{(6\,j\omega + 1)(j\omega + 1)}$$

$$V_c(j\omega) = \frac{1}{0.2\,j\omega}.I(j\omega)$$

$$= \frac{5/3}{\left(j\omega + \frac{1}{6}\right)(j\omega + 1)} = \frac{2}{j\omega + \frac{1}{6}} - \frac{2}{j\omega + 1}$$

Taking Inverse Fourier Tranform

$$v_c(t) = 2.\left(e^{-t/6} - e^{-t}\right).u(t)$$

Z - Tranform

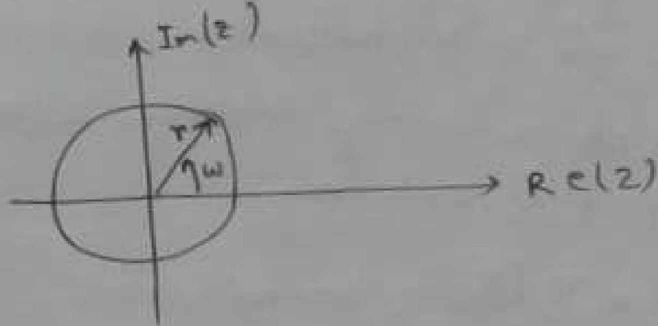
$$x[n] = \{1, 2, \underset{\substack{\uparrow \\ n=0}}{-1}, \quad \}$$

$$x[n] \rightleftharpoons X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n}$$

where n is an integer
z is a complex variable.

$$z = r.e^{j\omega} = |z| \angle\theta$$

$$r = |z| .$$



$$\sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \rightarrow \text{Bidirectional}$$

$$\sum_{n=0}^{\infty} x[n].\, z^{-n} \rightarrow \text{unidirectional}$$

Ex-1 $\quad x[n] = a^n u[n]$

$X[z] = ??$ $\quad$ ROC = ?? $\quad$ Region of Convergence

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\alpha} (az^{-1})^n$$

$$= 1 + az^{-1} + (az^{-1})^2 + \dots$$

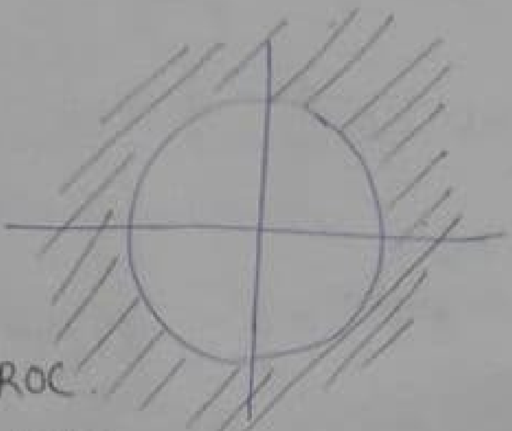
$$x(z) = \frac{1}{a - az^{-1}}$$

$|az^{-1}| < 1$

$\left|\frac{a}{z}\right| < 1$

$\underline{|z| > |a|}$

ROC

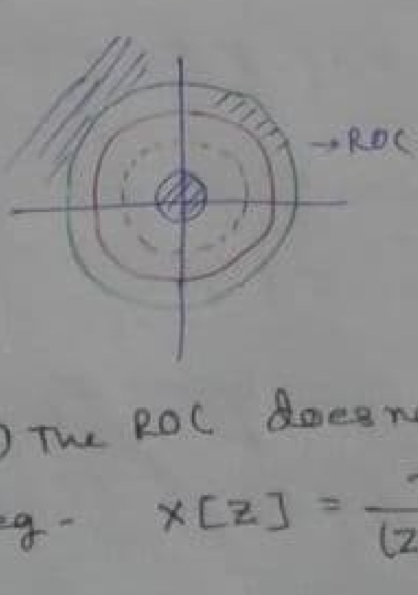


ROC (outside the circle)

Properties of ROC of z-transform.

① The ROC consist of a ring in the z-plane centered about the origin.

$$x[n] \xrightarrow{z} X[z]$$
$\downarrow$
finite.



② The ROC doesnot contain any pole.

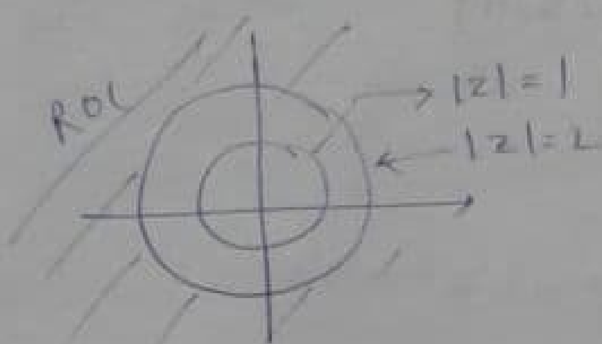
eg- $$X[Z] = \frac{Z^2}{(Z-1)(Z-2)}$$

$X[z]$ is finite except $\cancel{x = \alpha}$ $Z = \alpha$

location of poles of $X(z)$ $\begin{cases} Z-1=0 \Rightarrow Z=1 \\ Z-2=0 \Rightarrow Z=2 \end{cases}$

③ $x[n]$ is a right sided signal then the ROC is the region in the z-plane outside the outermost pole i.e. outside the circle of radi equal to the largest magnitude at the pole of $X(z)$

$$X(z) = \frac{z^2}{(z+1)(z+2)}$$



eg –

$\{1, 2, 3, 4, 5\}$
$\uparrow$

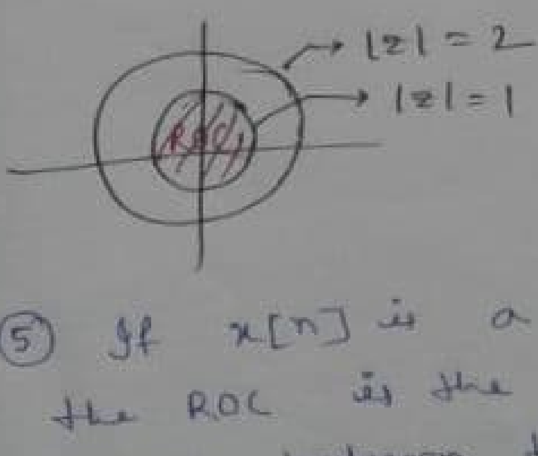
④ If $x[n]$ is left sided signal then the ROC is the region in the z-plane inside the innermost pole ie inside the circle of radi equal to the smallest magnitude at the pole of $X(z)$

$$X[z] = \frac{z^2}{(z+1)(z+2)}$$

eg –

$u[-n-1]$,

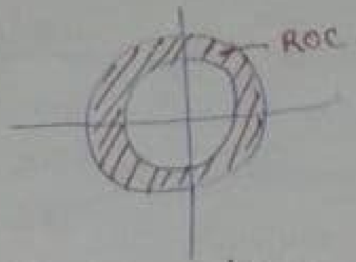
$x[n] = \{1, 2, 4, 6, 7\}$
$\uparrow$



⑤ If $x[n]$ is a both sided signal, the the ROC is the annular region (ring like) region between two circles

$x[n] = \{1, 2, 3, 4, 5\}$
$\uparrow$

$$x(z) = \frac{z^2}{(z+1)(z+2)}$$



⑥ If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibily $z=0$ and/or $z=\infty$

eg- $x[n] = \{1, 2, \underset{\uparrow}{-1}, 3, 4\}$

$$x[z] = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$= 1 \times z^2 + 2z - 1 \cdot z^0 + 3z^{-1} + 4z^{2}$$

$$= \underbrace{z^2 + 2z}_{\substack{z=\infty \\ x(z)=\infty}} - 1 + \underbrace{\frac{3}{z} + \frac{4}{z^2}}_{\substack{z=0 \\ x(z)=\infty}}$$

Condition for existance of z-transform

$$x[n] \rightleftharpoons x(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$|x(z)| < \infty$$

$$\left|\sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}\right| < \infty$$

$$\sum_{n=-\infty}^{\infty} |x[n] \cdot z^{-n}| < \infty$$

$$z = r \cdot e^{j\omega}$$

$$\sum_{n=-\infty}^{\infty} |x[n]\, (r \cdot e^{j\omega})^{-n}| < \infty$$

$$\sum_{n=-\infty}^{\infty} |x[n]\, r^{-n}| \cdot |e^{-j\omega n}| < \infty$$

$$\sum_{n=-\alpha}^{\alpha} |x[n] \cdot r^{-n}| < \alpha \qquad [\text{as } |e^{-j\omega n}| = 1]$$

Condition for existance of z-transform

$$\sum_{n=-\alpha}^{\alpha} |x[n] \cdot r^{-n}| < \alpha$$

when $r = 1$

$$\sum_{n=-\alpha}^{\alpha} |x[n]| < \alpha$$

this is the condition for a signal to be <u>absolutely summable</u>.

Condition for existance of z-transform

$$\sum_{n=-\infty}^{\infty} |x[n] \cdot r^{-n}| < \infty$$

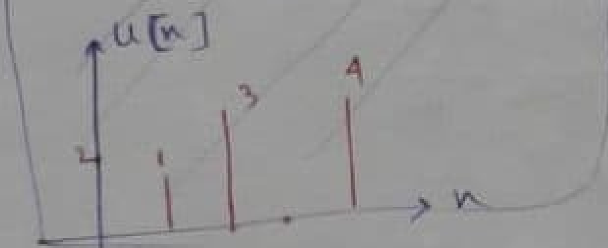
when $r = 1$

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

this is the condition for a signal to be <u>absolutely summable</u>.

07.10.2020

$u[n] = \{\underset{\uparrow}{2}, 1, 0, 3, 0, 4\}$



Properties of z-transform

① Linearity: $x_1[n] \xrightarrow{z} X_1(z)$ ROC $= R_1$

$x_2[n] \longrightarrow X_2(z)$ ROC $= R_2$

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \xrightarrow{z} \alpha_1 X_1(z) + \alpha_2 X_2(z)$$

ROC $= R_1 \cap R_3$

② multiplication by a constant:

$x[n] \xrightarrow{z} X(z)$ ROC $= R$

$\alpha x[n] \xrightarrow{z} \alpha X(z)$ ROC $= R$

$x[n] * h[n] \xrightarrow{z} X(z)\, H(z) \qquad ROC = R_1 \cap R_2$

(18)

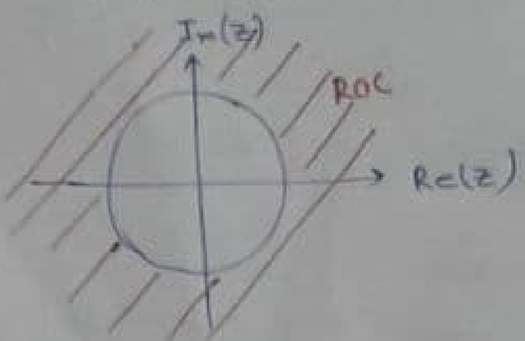
$x[n]$	$X(z)$	ROC
$\delta[n]$	1	all z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 $(m>0)$ or ∞ $(m<0)$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$n\, a^n\, u[n]$	$\frac{1}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-n\, a^n\, u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$(n+1)\, a^n u[n]$	$\frac{1}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$u[n-1]$	$\frac{z^{-1}}{(1-z^{-1})}$	$\lvert z\rvert > 1$
$2u[n-8]$	$2\,\frac{z^{-8}}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\left(\frac{3}{4}\right)^n u[n-4]$	$\frac{\left(\frac{3}{4}\right)^n z^{-4}}{1-\frac{3}{4}z^{-1}}$	$\lvert z\rvert > 3/4$
$e^{-\alpha nT}$ $(\alpha>0, n>0)$	$\frac{1}{1-e^{-\alpha T}z^{-1}}$	$\lvert z\rvert > e^{-\alpha T}$
$\sin\omega nT\, u[nT]$	$\frac{(\sin\omega T).z^{-1}}{1-2.z^{-1}\cos\omega T+z^{-2}}$	$\lvert z\rvert > 1$
$\cos\omega nT\, u[nT]$	$\frac{1-(\cos\omega T)z^{-1}}{1-2z^{-1}\cos\omega T+z^{-2}}$	$\lvert z\rvert > 1$
$nT\, e^{-\alpha nT}$	$\frac{T e^{-\alpha T} z^{-1}}{(1-e^{-\alpha T} z^{-1})^2}$	$\lvert z\rvert > e^{\alpha T}$

ex-1 $x[n] = a^n u[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n].z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1-az^{-1}}$$

ROC $= |az^{-1}| < 1$ or $|z| > |a|$



ex-2 $x[n] = -a^n . u[-n-1] = \begin{cases} 0 & n \geq 0 \\ -a^n & n \leq -1 \end{cases}$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n].z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (-a^n).z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} (a^{-1}z)^{-n}$$

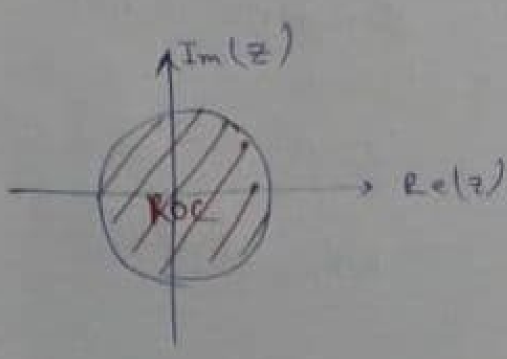
Let $m = -n$

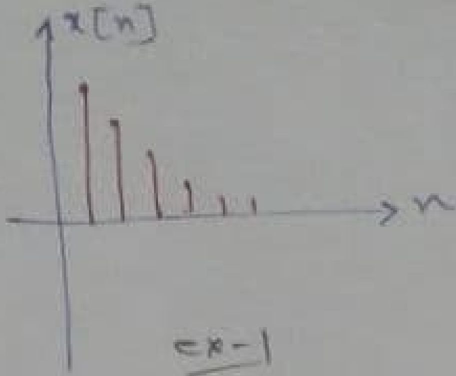
$$X[z] = -\sum_{m=1}^{\infty} (a^{-1}z)^m \qquad \cancel{-}\ a^{-z}$$

$$= -\frac{a^{-1}z}{1-a^{-1}z} = \frac{1}{1-a.z^{-1}}$$

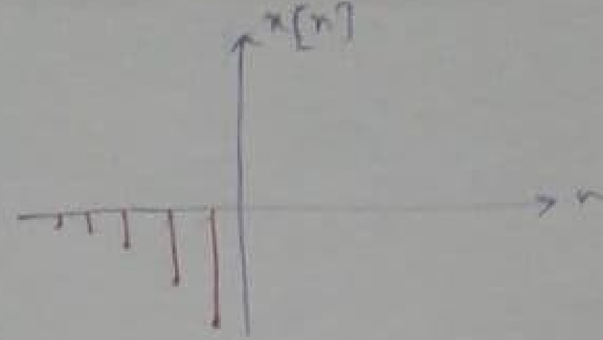
ROC $= |a^{-1}z| < 1$

or $|z| < |a|$





ex-1



ex-2

ex-3 $\quad x[n] = \alpha^n u[n] + \beta^n u[-n-1]$

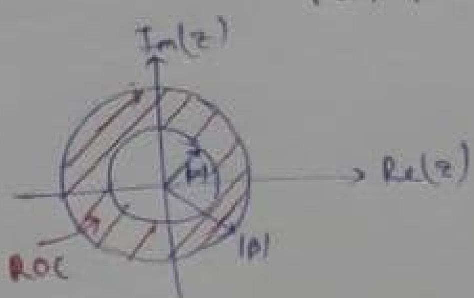
$$X[z] = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} \beta^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{m=1}^{\infty} (\beta^{-1} z)^m$$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$|\alpha z^{-1}| < 1 \qquad |\beta^{-1} z| < 1$

$|z| > |\alpha| \qquad |z| < |\beta|$



$$X[z] = \frac{1}{1-\alpha z^{-1}} + \frac{\beta^{-1} z}{1-\beta^{-1} z}$$

$$= \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\beta z^{-1}}$$

$$= \frac{(-\beta+\alpha) z^{-1}}{(1-\alpha z^{-1})(1-\beta z^{-1})}$$

$$= \frac{(\alpha-\beta) z^{-1}}{1 - z^{-1}(\alpha+\beta) + \alpha\beta . z^{-2}}$$

$$X(z) = \frac{\alpha-\beta}{z - (\alpha+\beta) + \alpha\beta z^{-1}}$$

ROC $= |\alpha| < |z| < |\beta|$

③ Time shifting:

① Time delay: $x[n] \xrightarrow{z} X(z)$ ROC = R

$x[n-k] \xrightarrow{z} z^{-k} X(z)$ ROC = $R \cap \{0<|z|\}$

② Time advance:

$x[n+k] \xrightarrow{z} z^{k} X(z)$ ROC = $R \cap \{|z| < \alpha\}$

④ Scaling in z-domain:

$x[n] \xrightarrow{z} X(z)$ ROC = R

$z_0^n x[n] \xrightarrow{z} X\left(\frac{z}{z_0}\right)$ ROC = $|z_0| R$

⑤ Time expansion:

$x[n] \xrightarrow{z} X(z)$ ROC = R

$x^k[n] \longrightarrow X(z^k)$ ROC = $R^{1/k}$

⑥ Time Reversal:

$x[n] \xrightarrow{z} X(z)$ ROC = R

$x[-n] \longrightarrow X\left(\frac{1}{z}\right)$ ROC = $\frac{1}{R}$

⑦ Conjugation:

$x[n] \xrightarrow{z} X[z]$ ROC = R

$x^*[n] \longrightarrow X^*(z^*)$ ROC = R

⑧ Initial value theorem:

$$x[0] = \lim_{n \to 0} x[n] = \lim_{z \to \alpha} X(z)$$

⑨ Final value theorem:

$$x[\alpha] = \lim_{n \to \alpha} x[n] = \lim_{z \to 1} (1-z^{-1}) \cdot X(z)$$

⑩ Multiplication by n:

$x[n] \xrightarrow{z} X(z)$ ROC = R

$n\,x[n] \longrightarrow -z \frac{d\,X(z)}{dz}$ ROC = R

⑪ Convolution theorem:

$x[n] \xrightarrow{z} X(z)$ ROC = R_1

$h[n] \xrightarrow{z} H(z)$ ROC = R_2

9.10.2020

Ex-1: Consider the z-transform

$$x(z) = \frac{3 - \frac{11}{15} z^{-1}}{\left(1 - \frac{1}{3} z^{-1}\right)\left(1 - \frac{1}{5} z^{-1}\right)}$$

Find the inverse z-transform for different ROC's

(i) $|z| > \frac{1}{3}$

(ii) $\frac{1}{5} < |z| < \frac{1}{3}$

(iii) $|z| < \frac{1}{5}$

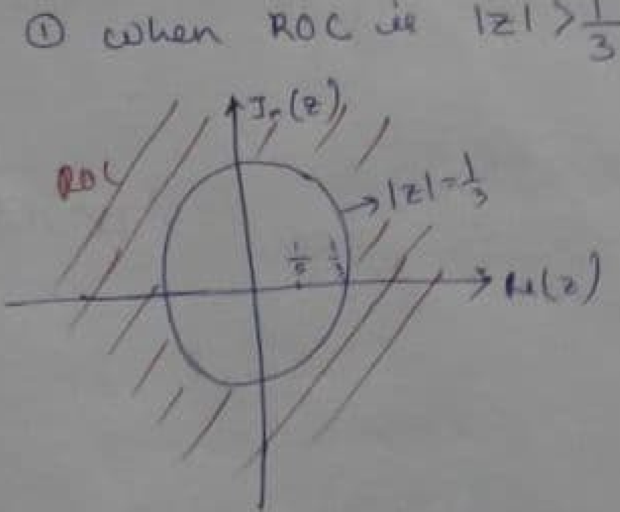
→ $$x(z) = \frac{3 - \frac{11}{15} z^{-1}}{\left(1 - \frac{1}{3} z^{-1}\right)\left(1 - \frac{1}{5} z^{-1}\right)}$$

$$= \frac{z^{-1}\left(3z - \frac{11}{15}\right)}{z^{-2}\left(z - \frac{1}{3}\right)\left(z - \frac{1}{5}\right)}$$

$$= \frac{1}{z^{-1}}\left[\frac{2}{z - \frac{1}{3}} + \frac{1}{z - \frac{1}{5}}\right]$$

$$x(z) = \frac{2}{1 - \frac{1}{3} z^{-1}} + \frac{1}{1 - \frac{1}{5} z^{-1}}$$

(i) when ROC is $|z| > \frac{1}{3}$



$z = \frac{1}{3}$ & $z = \frac{1}{5}$ | ROC lies outside outermost pole.

So Inverse z-transform is a right sided signal

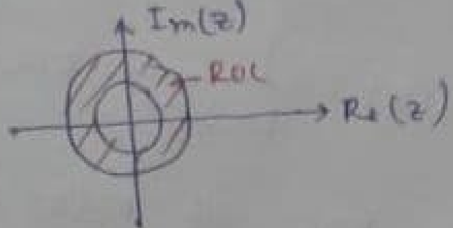
$$\left(\frac{1}{5}\right)^n u[n] \xrightarrow{z} \frac{1}{1-\frac{1}{5}z^{-1}} \quad , \ |z| > \frac{1}{5}$$

$$\left(\frac{1}{3}\right)^n u[n] \xrightarrow{z} \frac{1}{1-\frac{1}{5}z^{-1}} \quad , \ |z| > \frac{1}{3}$$

Combine ROC as $|z| > \frac{1}{3}$

So $x[n] = 2\left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{5}\right)^n u[n]$

(ii) ROC $\rightarrow \frac{1}{5} < |z| < \frac{1}{3}$



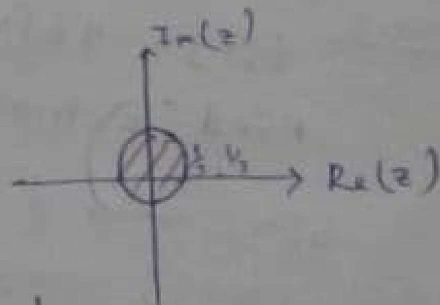
$$\left(\frac{1}{5}\right)^n u[n] \xrightarrow{z} \frac{1}{1-\frac{1}{5}z^{-1}} \quad |z| > \frac{1}{5}$$

$$-\left(\frac{1}{3}\right)^n u[-n-1] \xrightarrow{z} \frac{1}{1-\frac{1}{3}z^{-1}} \quad |z| < \frac{1}{3}$$

By combining two functions

$$x[n] = \left(\frac{1}{5}\right)^n u[n] - 2\cdot\left(\frac{1}{3}\right)^n u[-n-1]$$

(iii) $|z| < \frac{1}{5}$



$$-\left(\frac{1}{3}\right)^n u[-n-1] \longrightarrow \frac{1}{1-\frac{1}{3}z^{-1}} \quad |z| < \frac{1}{3}$$

$$-\left(\frac{1}{5}\right)^n u[n-1] \longrightarrow \frac{1}{1-\frac{1}{5}z^{-1}} \quad |z| < \frac{1}{5}$$

$$x[n] = -2\cdot\left(\frac{1}{3}\right)^n u[-n-1] - \left(\frac{1}{5}\right)^n u[-n-1]$$

Ex. 2 $f[n] = [3 (2^n) - 4 (3^n)]$

Find z-transform and the ROC of the signal.

$$f[n] = [3 \cdot 2^n - 4 \cdot 3^n]\, u[n]$$

$$= \underbrace{3 \cdot 2^n u[n]}_{f_1} - \underbrace{4 \cdot 3^n u[n]}_{f_2}$$

$$f_1 = 2^n u[n] \xrightarrow{Z} F_1(z) = \frac{1}{1 - 2z^{-1}}$$

ROC $|z| > 2$

$$f_2 = 3^n u[n] \xrightarrow{Z} F_2(z) = \frac{1}{1 - 3z^{-1}} \quad \text{ROC } |z| > 3$$

The intersection of the ROC of $F_1(z)$ and $F_2(z)$ is $|z| > 3$

$$F(z) = \frac{3}{1 - 2z^{-1}} - \frac{4}{1 - 3z^{-1}}, \quad |z| > 3$$

$$= \frac{-(1 + z^{-1})}{1 - 5z^{-1} + 6z^{-2}} \quad |z| > 3$$

Ex-3 $f[n] = 4\delta[n+2] + 7\delta[n] + 3\delta[n-1]$

Find z-transform.

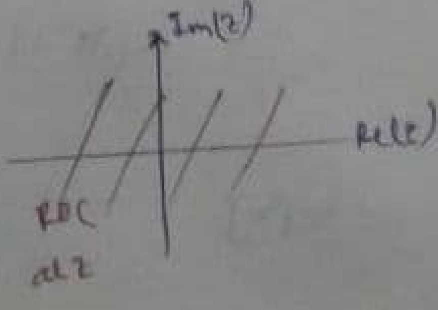
$\delta[n] \xrightarrow{Z} 1$

$x[n+k] \xrightarrow{Z} z^k X(z)$

$\delta[n+2] \longrightarrow z^2$

$\delta[n-1] \longrightarrow z^{-1}$

$F[z] = 4 \cdot z^2 + z + 3z^{-1}$



$$x[n] \longrightarrow X(z)$$

$$n\,x[n] \longrightarrow -z \cdot \frac{d}{dz} X(z)$$

$$4\,n\,\delta[n+2]\,.$$

ex-4 : $X(z) = 5z^{4} + 3z^{2} + 2 + z^{-1} + 4z^{-3}$

$0 < |z| < \alpha$

Find $x[n]$

Inverse z-transform

$$x[n] = 5\,\delta[n+4] + 3\,\delta[n+2] + 2\,\delta[n] + \delta[n-1] + 4\,\delta[n-3]$$

ex-5 :

$$f[n] = \left(\frac{1}{5}\right)^{n} u[n] + \left(\frac{3}{4}\right)^{n} u[n-4] - 2u[n-8]$$

$$F(z) = \frac{1}{1-\frac{1}{5}z^{-1}} + \frac{\left(\frac{3}{4}\right)^{n} z^{-4}}{1-\frac{3}{4}z^{-1}} - \frac{2\,z^{-8}}{1-z^{-1}}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$|z| > \frac{1}{5} \qquad |z| > \frac{3}{4} \qquad |z| > 1$$

Combined ROC $|z| > 1$

10.10.2020

Relationship between Laplace transform and z-transform

$x(t)$ is sample at sampling rate $\frac{1}{T}$ to get discrete value $x(KT)$ which has z-transform

$$X(z) = \sum_{K=-\infty}^{\infty} x(KT)\, z^{-K}$$

The same signal $x(t)$ can be consider as the impulse sampled at the same rate $\frac{1}{T}$ and may be represented as

$$x(t) = \sum_{K=-\infty}^{\infty} x(KT)\, \delta(t-KT)$$

$\Downarrow$ Laplace transform

$$X(s) = \sum_{K=-\infty}^{\infty} x(KT)\, e^{-KST}$$

If $e^{ST} = z$

$$X(s) = \sum_{K=-\infty}^{\infty} x(KT).\, z^{-K} = X(z)$$

$$= X(z)\Big|_{z=e^{Ts}}$$

SYSTEM FUNCTION OF A LINEAR TIME INVARIANT SYSTEM

$$Y(z) = H(z)\, X(z).$$

$Y(z)$ is the z-transform of the output $Y(n)$

$X(z)$ is the " " of " " $x(n)$

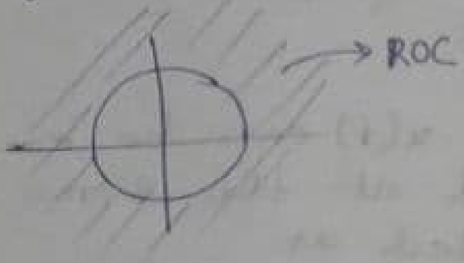
$H(z)$ " " " " $h(n)$

$h(n) \rightarrow$ unit sample response

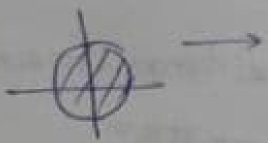
$H(z)$ = system function
or
Transfer function
or
~~pha~~ pulse Transfer function

Casality

A discrete time system is causal if it has an impulse responce $h[n]$, That is zero for $n<0$ i.e $h[n]$ is a right sided signal.



left sided signal



anti causal
non-causal }

Stability:

A DTS is BIBO Stable if its impulse response being absolutly summable;

$$\sum_{n=-\alpha}^{\alpha} |h(n)| < \alpha$$

$H(z)$ include the unit circle, $|z|=1$

Interconections of the system

$h_1(n)$ & $h_2(n)$ are connected in series

The overall impulse responce $h(n)=h_1(n) * h_2(n)$

$\downarrow$ Z-Tr

$$H(z) = H_1(z) \cdot H_2(z)$$

$h_1(n) \xrightarrow{z} H_1(z)$ ROC $= R_1$

$h_2(n) \xrightarrow{z} H_2(z)$ ROC $= R_2$

$h(n) \xrightarrow{z} H(z)$ ROC $= R_1 \cap R_2$

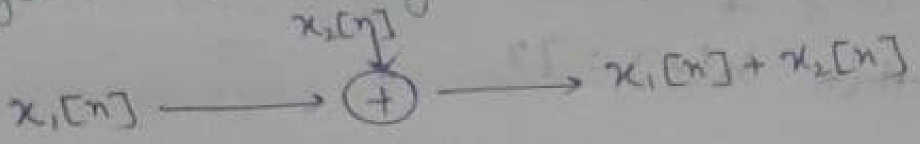
$h_1(n)$ & $h_2(n)$ are connected in parallel
The overall impulse response

$$h(n) = h_1[n] + h_2[n]$$

$$\Downarrow \text{Z-Tr}$$

$$H(z) = H_1(z) + H_2(z) \qquad ROC = R_1 \cap R_2$$

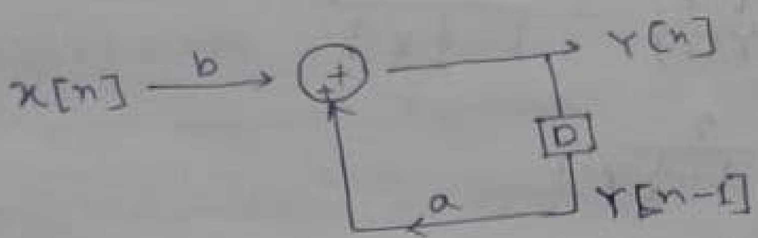
System Block-diagrams



$$x[n] \xrightarrow{a} ax[n]$$

$$x[n] \xrightarrow{D} x[n-1]$$

$$y[n] = -ay[n-1] + bx[n]$$



Ex-1 Draw the block-diagram representation for the causal LTI system with system function,

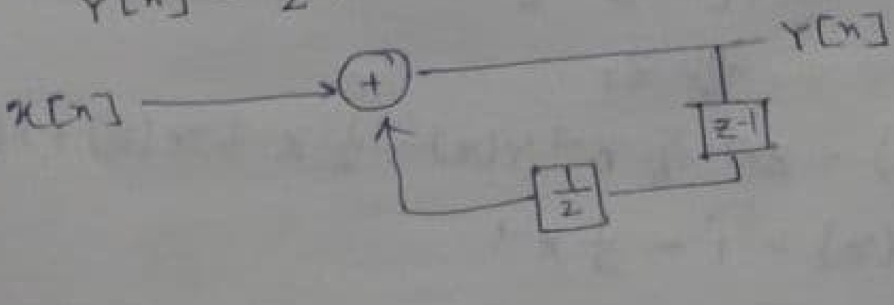
$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}.$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$\downarrow \text{IZT}$$

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

$x[n-k] \rightarrow z^{-k}X(z)$



z^{-1} is the system of a unit delay

ex-2: Determine the system function and unit sample response of the system described by the diff eq.

$$Y[n] = \frac{1}{2} Y[n-1] + 2X[n]$$

~~H(z~~ $h(n) = ??$

$$Y[n] = \frac{1}{2} Y[n-1] + 2x[n]$$

$$\Downarrow$$

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

$$H(z) = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

$$\Downarrow \text{ IZT}$$

$$h(n) = 2\left(\frac{1}{2}\right)^n u(n)$$

Ex-3 Determine the response of the system

$$Y[n] = \frac{5}{6} Y[n-1] - \frac{1}{6} Y[n-2] + x[n]$$

to the input signal $x[n] = \delta[n] - \frac{1}{3}\delta[n-1]$

$$Y[n] = \frac{5}{6} Y[n-1] - \frac{1}{6} Y[n-2] + x[n]$$

$$\Downarrow \text{ ZT}$$

$$Y(z) = \frac{5}{6} z^{-1} Y(z) - \frac{1}{6} z^{-2} Y(z) + X(z)$$

$$X(z) = 1 - \frac{1}{3} z^{-1}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\left(1-\frac{5}{6}z^{-1}\right)+\frac{1}{6}z^{-2}}$$

$$= \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$Y(z) = H(z)\cdot X(z)$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}}$$

$\Downarrow$ IZT

$$Y[n] = \left(\frac{1}{2}\right)^2 u[n]$$

Analitic function

$f(z) = u + iv$

u, v are the functions of (x,y)

$F(z)$ function is said to be analytic if it is differentiable at $z = z_0$ & its neighbourhood values

Sufficient condion for $f(z)$ to be analytic

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\}$$ Cauchy - Riemann equation / C-R equation

ex-1: $f(z) = 2xy + i(x^2 - y^2)$

$f(z) = u + iv$

$u = 2xy$ $\quad v = x^2 - y^2$

(i) $\frac{\partial u}{\partial x} = 2y$ $\quad$ (ii) $\frac{\partial v}{\partial x} = 2x$

$\frac{\partial u}{\partial y} = 2x$ $\quad$ $\frac{\partial v}{\partial y} = -2y$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

not satisfies CR eq

not analitic function

ex-2. $f(z) = z^3$

$f(z) = (x + jy)^3$

$= x^3 + (jy)^3 + 3x \cdot jy\,(x + jy)$

$= x^3 - jy^3 + 3jx^2y - 3xy^2$

$= (x^3 - 3xy^2) + j(3x^2y - y^3)$

(i) $\frac{\partial u}{\partial x} = 3x^2 - 3y^2$

$\frac{\partial u}{\partial y} = -6xy$

(ii) $\frac{\partial v}{\partial x} = 6xy$

$\frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ f(z) is analytic

Cauchy's Integral formula

If f(z) is analytic function within & on a closed contour C, and if 'a' is any point within C

Then $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$

$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i + f(a)$

Cauchy's Residue Theorem

$$\int_c f(z).dz = 2\pi i \times \text{[Sum of all residues at all poles within C]}$$

c Residue → $\lim_{z \to z_0} (z-z_0) f(z)$

P(1) Which ~~of~~ one of the following function is analytic in the region $|z| \leq 1$?

(A) $\frac{z^2-1}{z+j0.5}$ (B) $\frac{z^2-1}{z+2}$ (C) $\frac{z^2-1}{z-0.5}$ (D) $\frac{z^2-1}{z}$

$\oint_c f(z) \cdot dz = 0$ $f(z)$ is analytic

P(2) The closed loop line integral

$\oint_{|z|=5} \frac{z^3+z^2+8}{z+2} dz$ evaluated counter-clockwise is

(A) $+4j\pi$ (B) $-4j\pi$ (C) $+8j\pi$ (D) $-8j\pi$

Singular point of $F(z)$ is $z=-2$ with lies inside c $|z|=5$

$$\oint_c F(z).dz = \oint_c \frac{z^3+z^2+8}{z+2}.dz$$

$$= \oint_c \frac{z^3+z^2+8}{z-(-2)}.dz$$

$$= 2\pi j \times F(-2)$$

$$= 2\pi j \times [(-2)^3+(-2)^2+8]$$

$$= 2\pi j \times [-8+4+8)$$

$$= 8\pi j$$

P (3) The value of the integral $\oint_c \frac{z+1}{z^2-4} \cdot dz$ in counter clockwise direction around a circle c of radius 1 with centre at the point $z=-2$ is

(A) $\frac{\pi i}{2}$ (B) $2\pi i$ (C) $-\frac{\pi i}{2}$ (D) $-2\pi i$

$|z-a|=r$

$|z+2|=1$

$z=-2$

$$\oint_c \frac{z+1}{z^2-4} dz = 2\pi i \times f(z)\Big|_{z=-2}$$

$$= 2\pi i \times \lim_{z\to 2} (z+2)\left[\frac{z+1}{(z+2)(z-2)}\right]$$

$$= \frac{\pi i}{2}$$

P (4) The value of the contour integral in the complex plane.

$\int \frac{z^3-2z+3}{z-2} \cdot dz$ along the contour $|z|=3$, taken counter clockwise is

(A) $-18\pi i$ (B) 0 (C) $14\pi i$ (D) $48\pi i$

P(5) Initial & final value

DTS $x[n] \rightarrow$ whose Z Tr. is given by

$$x(z) = 3 + 5z^{-1} + 7z^{-2}$$

the initial value $x[0] = \lim_{n\to 0} x[n]$

$$= \lim_{z\to \alpha} X(z)$$

$$= \lim_{z\to \alpha} (3+5z^{-1}+7z^{-2})$$

$$= 3$$

the final value $x[\alpha] = \lim_{n\to \alpha} x[n]$

$$= \lim_{z\to 1} (1-z^{-1})\, x(z)$$

$$= \lim_{z\to 1} (1-z^{-1})(3+5z^{-1}+7z^{-2})$$

$$= \lim_{z\to 1} \left(3+2z^{-1}+2z^{-1}-7z^{-3}\right)$$

$$= 0$$

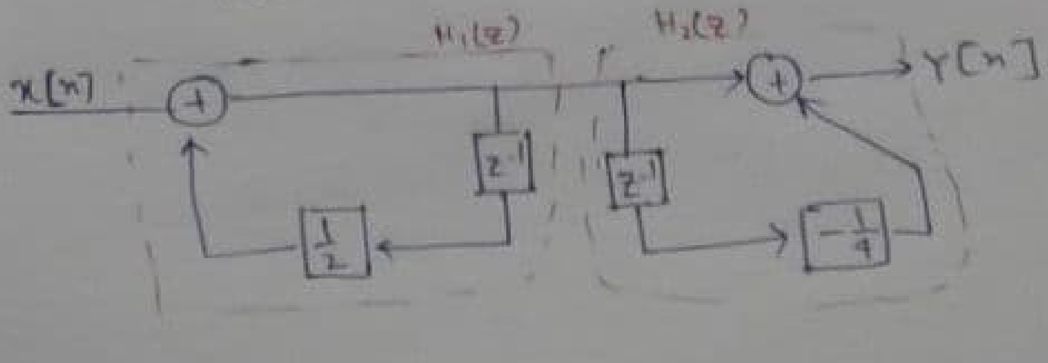
P(6) construct a block-diagram of a LTI discrete-time system with system function,

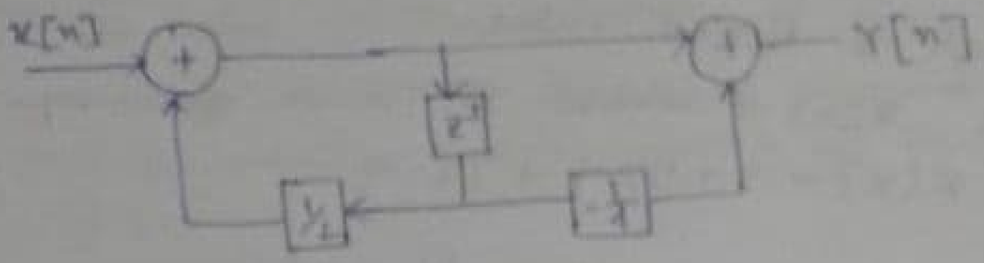
$$H(z) = \frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$H(z) = H_1(z) \cdot H_2(z)$$

where $H_1(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}}$

$$H_2(z) = 1 - \frac{1}{4}z^{-1}$$





equivalent block diagram representation