Matrix II

2.1 DETERMINANT

Matrices are, in general, symbols (arrays of numbers) and do not bear any numerical values, that is, these are not scalars. But certain numerical values can be assigned to square matrices, called determinants. However, such an assignment does not uniquely identify a square matrix. Determinant can be regarded as a scalar valued function over the domain of all square matrices but it is not a one-one correspondence since distinct matrices may have same determinant. Theory of determinant is a powerful technique for mathematical investigations and has tremendous applications in many spheres of mathematical activities.

Permutation

Permutation of a given collection of n distinct objects is an arrangement of these objects and we know that the total number of possible permutations of n distinct objects is $\lfloor n$.

For example, let there be three distinct objects denoted by the numbers 1, 2, 3, then there are exactly $\lfloor 3 = 6$ permutations, namely (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1).

If in a permutation a larger number preceeds a smaller one, we say that there is an *inversion*. A permutation is said to be even or odd according as the total number of inversions is even or odd.

For example, there is no inversion in the permutation (1, 2, 3) and there are two inversions in the permutation (3, 1, 2), as 3 preceeds 1, 2 and so these permutations are even. Also, in the permutation (2, 1, 3) only one inversion occurred, as 2 preceeds 1 and so it is an odd permutation.

Definition

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \hline a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (a_{ij})_{n\times n}$ be a square matrix of order n. Then the determinant of

A is denoted and defined by $\det A = |a_{ij}|_{n\times n} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \hline a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum P(j_1, j_2, \ldots, j_n)\, a_{1j_1} a_{2j_2} \ldots a_{nj_n}$,

where the summation is extended over all $\lfloor n$ possible permutations $(j_1, j_2, \ldots, j_n)$ of $(1, 2, \ldots, n)$ and

$$P(j_1, j_2, \ldots, j_n) = \begin{cases} 1, \text{ if } (j_1, j_2, \ldots, j_n) \text{ is an even permutation} \\ -1, \text{ if } (j_1, j_2, \ldots, j_n) \text{ is an odd permutation} \end{cases}$$

Therefore there are $\lfloor n$ number of terms in the summation and $P(j_1, j_2, \ldots, j_n)$ determines only the sign of each term. It is noted that each term is the product of elements of A such that exactly one element from each row and exactly one element from each column have been taken.

The term + $a_{11}, a_{22} \ldots a_{nn}$ formed by the elements of the diagonal drawn from the left hand top corner to the right hand bottom corner is the *leading term*. The elements $a_{11}, a_{22}, \ldots, a_{nn}$ of the leading term are known as the *leading elements*.

The determinant of $A = (a_{ij})_{n \times n}$, *i.e.*, det A is known as determinant of order n or nth order determinant.

Note: Determinant of a square matrix A is also denoted by $|A|$.

Determinant of order 2 or second order determinant

Let us consider a second order matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Here 1, 2 can be permuted in exactly two ways given by (1, 2) and (2, 1). So $P(1, 2) = 1$, since there is no inversion in the permutation (1, 2), *i.e.*, an even permutation and the corresponding term is 1. $a_{11}\, a_{22} = a_{11}\, a_{22}$.

Next, $P(2, 1) = -1$, since there is only one inversion in the permutation (2, 1) *i.e.*, an odd permutation and therefore the corresponding term is (-1). $a_{12}\, a_{21} = -a_{12}\, a_{21}$.

$$\therefore \qquad \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}.$$

Determinant of order 3 or third order determinant

Let us consider a third order matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Here 1, 2, 3 can be permuted in exactly six ways, namely (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1). Now, when $(j_1, j_2, j_3) = (1, 2, 3)$, we have $P(1, 2, 3) = 1$, since there is no inversion, *i.e.*, the permutation is even. Therefore, the corresponding term of det A is 1. $a_{11}\, a_{22}\, a_{33} = a_{11}\, a_{22}\, a_{33}$.

Next, when $(j_1, j_2, j_3) = (1, 3, 2)$, we have $P(1, 3, 2) = -1$, since there is only one inversion, *i.e.*, an odd permutation. Hence, the corresponding term of det A is $(-1)\, a_{11}\, a_{23}\, a_{32} = -a_{11}\, a_{23}\, a_{32}$.

Proceeding in this way we get

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

The right hand side expression is known as the expansion of det A. Here this expression may be further rearranged and written as

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad \ldots(2.1.1)$$

This is known as the expansion of det A by the first row. The general rule for expanding a determinant of any order by any particular row (or column) is stated below:

Step 1: Take the elements of the stated row (or column) one by one and multiply it by the determinant obtained by deleting the row and the column containing the corresponding element.

Step 2: Attach to each product a sign given by $(-1)^{i+j}$, where the leading element is lying at the junction of ith row and jth column.

Example: Let $$D = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 6 & 7 & 2 \end{vmatrix}.$$

The expansion of D with respect to first row is

$$D = 1\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 6 & 2 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 6 & 7 \end{vmatrix} = 1(8-7) - 2(0-6) + 3(0-24)$$

$$= 1 + 12 - 72 = -59.$$

The expansion of D with respect to second column is

$$D = -2\begin{vmatrix} 0 & 1 \\ 6 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 3 \\ 6 & 2 \end{vmatrix} - 7\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = -2(0-6) + 4(2-18) - 7(1-0)$$

$= 12 - 64 - 7 = -59$, which is same as above.

2.2 MINOR AND COFACTOR OF AN ELEMENT IN A DETERMINANT

The *cofactor* of an element a_{ij} in the determinant $D = |a_{ij}|_{n\times n}$ is denoted by A_{ij} and is defined as

$$A_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the determinant obtained from D by deleting its ith row and jth column and is called the *minor* of the element a_{ij} in D.

Thus in $$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\, a_{33} - a_{23}\, a_{32},$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{12}\,a_{31} - a_{11}\,a_{32}.$$

Expansion of determinant by cofactors

The expansion of a determinant $D = |\,a_{ij}\,|_{n\times n}$ in terms of cofactors can be expressed as follows:

$$D = \sum_{j=1}^{n} a_{ij}\,A_{ij}, \text{ where } i = 1, 2, \ldots, n \qquad \ldots(2.2.1)$$

or

$$D = \sum_{i=1}^{n} a_{ij}\,A_{ij}, \text{ where } j = 1, 2, \ldots, n \qquad \ldots(2.2.2)$$

where A_{ij} is the cofactor of a_{ij} in D, $i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n$.

The expansion of D with respect to any row is represented by (2.2.1) and the same with respect to any column is represented by (2.2.2).

From (2.1.1), we have

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

Similarly,

$$\begin{aligned} D &= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \\ &= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \\ &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \\ &= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} \\ &= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}. \end{aligned}$$

2.3 PROPERTIES OF DETERMINANTS

Here we state properties of determinant with illustrative proofs using only third order determinants. These proofs can be extended to determinant of any order.

Property 1: The value of a determinant is unchanged if rows and columns are interchanged.

Proof: We know that the value of a determinant is same if it is expanded with respect to any row or any column.

Therefore if D_1 is the determinant obtained from D by changing its rows into columns then expansion of D_1 with respect to first column will have the same result as in the expansion of D with respect to first row.

$\therefore \qquad D_1 = D.$

Example: $\begin{vmatrix} 2 & 3 & 4 \\ -1 & -2 & 0 \\ 3 & 6 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 3 & -2 & 6 \\ 4 & 0 & -1 \end{vmatrix}.$

Property 2: If two rows (or columns) of a determinant are interchanged, the sign of the determinant is changed but its numerical value remains unaltered.

Proof: Let $D_1 = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ be the determinant obtained from D by interchanging its first row (R_1) and second row (R_2).

Expanding D_1 with respect to second row we get

$$D_1 = -a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -(a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}) = -D.$$

Similarly the other cases may be proved.

Example: $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 5 & 3 & -1 \end{vmatrix} = -\begin{vmatrix} 3 & 2 & 1 \\ 4 & 0 & 2 \\ -1 & 3 & 5 \end{vmatrix}$ (interchanging C_1 and C_3).

Property 3: If two rows (or two columns) of a determinant are identical then the value of the determinant is zero.

Proof: Let D_1 be the determinant having two identical rows (or columns), then by Property 2 the value of the determinant will be changed to $-D_1$, if its two identical rows (or columns) are interchanged, but the interchange of two identical rows (or columns) does not alter the determinant.

$$\therefore \qquad D_1 = -D_1, \text{ or } 2D_1 = 0, \text{ or } D_1 = 0.$$

Example: $\begin{vmatrix} 1 & 4 & 5 \\ 2 & 3 & 0 \\ 1 & 4 & 5 \end{vmatrix} = 0.$

Property 4: If every element of any row (or any column) be multiplied by a constant, the determinant is then multiplied by the same constant.

Proof: Let $D_1 = \begin{vmatrix} k\,a_{11} & k\,a_{12} & k\,a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

be the determinant obtained from D by multiplying each element of its first row by a constant k.

Expanding D_1 with respect to first row, we get

$$D_1 = k\,a_{11}A_{11} + k\,a_{12}A_{12} + k\,a_{13}A_{13}$$

$$= k(a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}) = kD.$$

Similarly the other cases may be proved.

Note: (*i*) If sign of every element of a row (or column) is changed, then the sign of the determinant will be changed.

(*ii*) If the elements of any row (or column) are constant multiples of the corresponding elements of any other row (or column), then the value of the determinant is zero.

Example: $\begin{vmatrix} 4 & 2 & 3 \\ 3 & 6 & 0 \\ 1 & 8 & 9 \end{vmatrix} = 3 \begin{vmatrix} 2 & 2 & 3 \\ 1 & 2 & 0 \\ 1 & 8 & 9 \end{vmatrix}$.

Property 5: If every element of any row (or column) of a determinant be expressed as a sum of two numbers, then the determinant can be expressed as a sum of two determinants of same order.

Proof: Let
$$D_1 = \begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Expanding D_1 with respect to first row, we get

$$D_1 = (a_{11}+b_{11})\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - (a_{12}+b_{12})\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (a_{13}+b_{13})\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \left\{a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right\}$$

$$+ \left\{b_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - b_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + b_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right\}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Similarly the other cases may be proved.

Note: (*i*) If every element of any row (or column) of a determinant be the sum of p numbers, then the determinant can be expressed as a sum of p determinants of same order.

(*ii*) If the elements of the three rows (or columns) be the sums of p, q, r numbers respectively, then the determinant can be expressed as the sum of pqr determinants.

Example: $\begin{vmatrix} 2 & 3 & -4 \\ 1+x & 5+y & 0+z \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -4 \\ 1 & 5 & 0 \\ 2 & -3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 & -4 \\ x & y & z \\ 2 & -3 & 1 \end{vmatrix}$.

Property 6: The value of a determinant remains unchanged if each element of any row (or column) is added with a constant multiple of the corresponding element of another row (or column).

Proof: Follows from Properties 3, 4 and 5.

Example: $\begin{vmatrix} 2 & 3 & -1 \\ 8 & 7 & -2 \\ 4 & 6 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & -5 & -2 \\ 4 & 6 & 0 \end{vmatrix}$ $(R_2 - 2R_3 \to R'_2)$

Property 7: If the elements of a determinant are polynomials in x and any two rows (or columns) become identical when x is replaced by a, then $(x - a)$ is a factor of the determinant.

Proof: Here the determinant can be expressed as a polynomial in x. If two rows (or two columns) become identical when x is replaced by a, then by Property 3, the value of the determinant is zero.

Hence by factor theorem, $(x-a)$ is a factor of the underlying determinant.

Note: In general, if r rows (or r columns) become identical when x is replaced by a in a determinant whose elements are polynomials in x, then $(x-a)^{r-1}$ is a factor of that determinant.

Example: Let $D = \begin{vmatrix} 2x^2 & 2x & 1 \\ x^2+x & x+1 & 1 \\ 2x & 2 & 1 \end{vmatrix} = f(x).$

Obviously $f(1) = \begin{vmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 2 & 2 & 1 \end{vmatrix}$ has three identical rows, so $(x-1)^2$ is a factor of D.

Property 8: If the elements of an nth order determinant D are functions of x, then

$$\frac{dD}{dx} = \sum_{i=1}^{n} D_i,$$

where D_i is same as D except that the elements of its ith row (or ith column) are differential coefficients of the corresponding elements of the ith row (or ith column) of D.

Example: $\dfrac{d}{dx}\begin{vmatrix} x^4 & 5x^3 & 2x+1 \\ x^2 & 3x+1 & x \\ x^3+x^2 & 2x^2+x & 4x \end{vmatrix}$

$$= \begin{vmatrix} 4x^3 & 15x^2 & 2 \\ x^2 & 3x+1 & x \\ x^3+x^2 & 2x^2+x & 4x \end{vmatrix} + \begin{vmatrix} x^4 & 5x^3 & 2x+1 \\ 2x & 3 & 1 \\ x^3+x^2 & 2x^2+x & 4x \end{vmatrix}$$

$$+ \begin{vmatrix} x^4 & 5x^3 & 2x+1 \\ x^2 & 3x+1 & x \\ 3x^2+2x & 4x+1 & 4 \end{vmatrix}$$

ILLUSTRATIVE EXAMPLES

Example 1: Without expanding prove the followings:

(*i*) $\begin{vmatrix} 2000 & 2001 & 2002 \\ 2003 & 2004 & 2005 \\ 2006 & 2007 & 2008 \end{vmatrix} = 0$ (*W.B.U.T. 2007, 2011*) (*ii*) $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix} = 0$

(iv) $\begin{vmatrix} 100 & 101 & 102 \\ 105 & 106 & 107 \\ 110 & 111 & 112 \end{vmatrix} = \begin{vmatrix} 100 & 101 & 102 \\ 5 & 5 & 5 \\ 10 & 10 & 10 \end{vmatrix}$ $(R_2 - R_1 \to R_2', R_3 - R_1 \to R_3')$

$= 2\begin{vmatrix} 100 & 101 & 102 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{vmatrix}$ (taking out 2 from third row)

$= 2 \times 0$ (since two rows are identical)

$= 0.$

Example 2: Prove that

(i) $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(ii) $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = 0$ (iii) $\begin{vmatrix} 1 & \alpha & \alpha^2-\beta\gamma \\ 1 & \beta & \beta^2-\gamma\alpha \\ 1 & \gamma & \gamma^2-\alpha\beta \end{vmatrix} = 0$ (W.B.U.T. 2005, 2012)

(iv) $\begin{vmatrix} -bc & bc+b^2 & bc+c^2 \\ ca+a^2 & -ca & ca+c^2 \\ ab+a^2 & ab+b^2 & -ab \end{vmatrix} = (bc+ca+ab)^3$

(v) $\begin{vmatrix} a^2+x & ab & ac & ad \\ ab & b^2+x & bc & bd \\ ac & bc & c^2+x & cd \\ ad & bd & cd & d^2+x \end{vmatrix} = x^3(a^2+b^2+c^2+d^2+x).$

Solution: (i) $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}$ [multiplying first, second and third rows by a, b, c respectively and taking out $\frac{1}{abc}$]

$= \frac{abc}{abc}\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

(iv) $\begin{vmatrix} 100 & 101 & 102 \\ 105 & 106 & 107 \\ 110 & 111 & 112 \end{vmatrix} = \begin{vmatrix} 100 & 101 & 102 \\ 5 & 5 & 5 \\ 10 & 10 & 10 \end{vmatrix}$ $(R_2 - R_1 \to R_2', R_3 - R_1 \to R_3')$

$= 2\begin{vmatrix} 100 & 101 & 102 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{vmatrix}$ (taking out 2 from third row)

$= 2 \times 0$ (since two rows are identical)

$= 0.$

Example 2: Prove that

(i) $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(ii) $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = 0$ (iii) $\begin{vmatrix} 1 & \alpha & \alpha^2-\beta\gamma \\ 1 & \beta & \beta^2-\gamma\alpha \\ 1 & \gamma & \gamma^2-\alpha\beta \end{vmatrix} = 0$ (W.B.U.T. 2005, 2012)

(iv) $\begin{vmatrix} -bc & bc+b^2 & bc+c^2 \\ ca+a^2 & -ca & ca+c^2 \\ ab+a^2 & ab+b^2 & -ab \end{vmatrix} = (bc+ca+ab)^3$

(v) $\begin{vmatrix} a^2+x & ab & ac & ad \\ ab & b^2+x & bc & bd \\ ac & bc & c^2+x & cd \\ ad & bd & cd & d^2+x \end{vmatrix} = x^3(a^2+b^2+c^2+d^2+x).$

Solution: (i) $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}$ [multiplying first, second and third rows by a, b, c respectively and taking out $\frac{1}{abc}$]

$= \frac{abc}{abc}\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$ (W.B.U.T. 2007)

(iv) $\begin{vmatrix} 100 & 101 & 102 \\ 105 & 106 & 107 \\ 110 & 111 & 112 \end{vmatrix} = 0.$ (W.B.U.T. 2005, 2013)

Solution: (*i*) $\begin{vmatrix} 2000 & 2001 & 2002 \\ 2003 & 2004 & 2005 \\ 2006 & 2007 & 2008 \end{vmatrix} = \begin{vmatrix} 2000 & 1 & 2 \\ 2003 & 1 & 2 \\ 2006 & 1 & 2 \end{vmatrix}$ $(C_2 - C_1 \to C_2', C_3 - C_1 \to C_3')$

$$= 2\begin{vmatrix} 2000 & 1 & 1 \\ 2003 & 1 & 1 \\ 2006 & 1 & 1 \end{vmatrix} \text{ (taking out 2 from third column)}$$

$= 2 \times 0$ (since two columns are identical)

$= 0.$

(*ii*) $D = \begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix} = (-1)^3 \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$ (taking out (–1) from each row)

$$= -\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix} \text{ (interchanging rows and columns)}$$

$= -D$

$\therefore \quad 2D = 0, \text{ or } D = 0.$

(*iii*) Let $D = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$

$\therefore \quad D = (-1)^3 \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$

$$= -\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} \text{ (interchanging rows and columns)}$$

$= -D$

$\therefore \quad 2D = 0, \text{ or } D = 0.$

(*iv*)
$$\begin{vmatrix} 100 & 101 & 102 \\ 105 & 106 & 107 \\ 110 & 111 & 112 \end{vmatrix} = \begin{vmatrix} 100 & 101 & 102 \\ 5 & 5 & 5 \\ 10 & 10 & 10 \end{vmatrix} \quad (R_2 - R_1 \to R_2', R_3 - R_1 \to R_3')$$

$$= 2\begin{vmatrix} 100 & 101 & 102 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{vmatrix} \quad \text{(taking out 2 from third row)}$$

$= 2 \times 0$ (since two rows are identical)

$= 0.$

Example 2: Prove that

(*i*)
$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

(*ii*)
$$\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = 0$$

(*iii*)
$$\begin{vmatrix} 1 & \alpha & \alpha^2-\beta\gamma \\ 1 & \beta & \beta^2-\gamma\alpha \\ 1 & \gamma & \gamma^2-\alpha\beta \end{vmatrix} = 0$$ (*W.B.U.T. 2005, 2012*)

(*iv*)
$$\begin{vmatrix} -bc & bc+b^2 & bc+c^2 \\ ca+a^2 & -ca & ca+c^2 \\ ab+a^2 & ab+b^2 & -ab \end{vmatrix} = (bc+ca+ab)^3$$

(*v*)
$$\begin{vmatrix} a^2+x & ab & ac & ad \\ ab & b^2+x & bc & bd \\ ac & bc & c^2+x & cd \\ ad & bd & cd & d^2+x \end{vmatrix} = x^3(a^2+b^2+c^2+d^2+x).$$

Solution: (*i*)
$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}$$ [multiplying first, second and third rows by a, b, c respectively and taking out $\frac{1}{abc}$]

$$= \frac{abc}{abc}\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a^2 - c^2 & a^3 - c^3 \\ 0 & b^2 - c^2 & b^3 - c^3 \\ 1 & c^2 & c^3 \end{vmatrix} \quad (R_1 - R_3 \to R_1',\ R_2 - R_3 \to R_2')$$

$$= \begin{vmatrix} a^2 - c^2 & a^3 - c^3 \\ b^2 - c^2 & b^3 - c^3 \end{vmatrix} = (a - c)(b - c) \begin{vmatrix} a + c & a^2 + ac + c^2 \\ b + c & b^2 + bc + c^2 \end{vmatrix}$$

$$= (a - c)(b - c)\{(a + c)(b^2 + bc + c^2) - (b + c)(a^2 + ac + c^2)\}$$

$$= (a - c)(b - c)(b - a)(ab + bc + ca) = (a - b)(b - c)(c - a)(ab + bc + ca).$$

(*ii*) Let
$$D = \begin{vmatrix} 1 & a & a^2 & a^3 + bcd \\ 1 & b & b^2 & b^3 + cda \\ 1 & c & c^2 & c^3 + dab \\ 1 & d & d^2 & d^3 + abc \end{vmatrix}$$

$$\therefore \quad D = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix} = D_1 + D_2 \text{ (say)}$$

Now, $$D_2 = \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix} = \frac{1}{abcd} \begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix}$$

$$= \frac{abcd}{abcd} \begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix} \quad \text{(taking out } abcd \text{ from fourth column)}$$

$$= - \begin{vmatrix} a & a^2 & 1 & a^3 \\ b & b^2 & 1 & b^3 \\ c & c^2 & 1 & c^3 \\ d & d^2 & 1 & d^3 \end{vmatrix} \quad \text{(interchanging third and fourth columns)}$$

$$= (-1)^2 \begin{vmatrix} a & 1 & a^2 & a^3 \\ b & 1 & b^2 & b^3 \\ c & 1 & c^2 & c^3 \\ d & 1 & d^2 & d^3 \end{vmatrix} \quad \text{(interchanging second and third columns)}$$

$$= (-1)^3 \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$$ (interchanging first and second columns)

$= -D_1$.

$\therefore \quad D_2 = -D_1$, or $D_1 + D_2 = 0$

Hence $\quad D = D_1 + D_2 = 0$.

(iii) Here $$\begin{vmatrix} 1 & \alpha & \alpha^2-\beta\gamma \\ 1 & \beta & \beta^2-\gamma\alpha \\ 1 & \gamma & \gamma^2-\alpha\beta \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} - \begin{vmatrix} 1 & \alpha & \beta\gamma \\ 1 & \beta & \gamma\alpha \\ 1 & \gamma & \alpha\beta \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} - \frac{1}{\alpha\beta\gamma}\begin{vmatrix} \alpha & \alpha^2 & \alpha\beta\gamma \\ \beta & \beta^2 & \alpha\beta\gamma \\ \gamma & \gamma^2 & \alpha\beta\gamma \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} - \frac{\alpha\beta\gamma}{\alpha\beta\gamma}\begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} - (-1)\begin{vmatrix} \alpha & 1 & \alpha^2 \\ \beta & 1 & \beta^2 \\ \gamma & 1 & \gamma^2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} - (-1)^2\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} = 0.$$

(iv) Here $$\begin{vmatrix} -bc & bc+b^2 & bc+c^2 \\ ca+a^2 & -ca & ca+c^2 \\ ab+a^2 & ab+b^2 & -ab \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} -abc & abc+ab^2 & abc+ac^2 \\ abc+a^2b & -abc & abc+bc^2 \\ abc+a^2c & abc+b^2c & -abc \end{vmatrix}$$

(multiplying 1st, 2nd, 3rd rows by a, b, c respectively)

$$= \frac{1}{abc}\begin{vmatrix} a(ab+bc+ca) & b(ab+bc+ca) & c(ab+bc+ca) \\ abc+a^2b & -abc & abc+bc^2 \\ abc+a^2c & abc+b^2c & -abc \end{vmatrix} \quad (R_1+R_2+R_3 \to R_1')$$

$$= \frac{(ab+bc+ca)\,abc}{abc}\begin{vmatrix} 1 & 1 & 1 \\ bc+ab & -ca & ab+bc \\ bc+ca & ca+bc & -ab \end{vmatrix}$$ (taking out $(ab + bc + ca)$ from 1st row and a, b, c from 1st, 2nd, 3rd columns respectively)

$$= (ab+bc+ca)\begin{vmatrix} 1 & 0 & 0 \\ bc+ab & -(ab+bc+ca) & 0 \\ bc+ca & 0 & -(ab+bc+ca) \end{vmatrix} \quad (C_2 - C_1 \to C_2', C_3 - C_1 \to C_3')$$

$$= (ab+bc+ca)\begin{vmatrix} -(ab+bc+ca) & 0 \\ 0 & -(ab+bc+ca) \end{vmatrix} = (ab+bc+ca)^3.$$

(v) Here $\begin{vmatrix} a^2+x & ab & ac & ad \\ ab & b^2+x & bc & bd \\ ac & bc & c^2+x & cd \\ ad & bd & cd & d^2+x \end{vmatrix}$

$$= \frac{1}{abcd}\begin{vmatrix} a^3+ax & ab^2 & ac^2 & ad^2 \\ a^2b & b^3+bx & bc^2 & bd^2 \\ a^2c & b^2c & c^3+cx & cd^2 \\ a^2d & b^2d & c^2d & d^3+dx \end{vmatrix} \quad (aC_1 \to C_1', bC_2 \to C_2', cC_3 \to C_3', dC_4 \to C_4')$$

$$= \frac{abcd}{abcd}\begin{vmatrix} a^2+x & b^2 & c^2 & d^2 \\ a^2 & b^2+x & c^2 & d^2 \\ a^2 & b^2 & c^2+x & d^2 \\ a^2 & b^2 & c^2 & d^2+x \end{vmatrix}$$ (taking out a, b, c, d from 1st, 2nd, 3rd, 4th rows respectively)

$$= \begin{vmatrix} a^2+b^2+c^2+d^2+x & b^2 & c^2 & d^2 \\ a^2+b^2+c^2+d^2+x & b^2+x & c^2 & d^2 \\ a^2+b^2+c^2+d^2+x & b^2 & c^2+x & d^2 \\ a^2+b^2+c^2+d^2+x & b^2 & c^2 & d^2+x \end{vmatrix} \quad (C_1 + C_2 + C_3 + C_4 \to C_1')$$

$$= (a^2+b^2+c^2+d^2+x)\begin{vmatrix} 1 & b^2 & c^2 & d^2 \\ 1 & b^2+x & c^2 & d^2 \\ 1 & b^2 & c^2+x & d^2 \\ 1 & b^2 & c^2 & d^2+x \end{vmatrix}$$

$$= (a^2+b^2+c^2+d^2+x)\begin{vmatrix} 1 & b^2 & c^2 & d^2 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix} \quad (R_2 - R_1 \to R_2', R_3 - R_1 \to R_3', R_4 - R_1 \to R_4')$$

$$\ldots^2+b^2+c^2+d^2+x)\begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = x^3(a^2+b^2+c^2+d^2+x).$$

...mple 3: Show that

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$ (W.B.U.T. 2002, 2010)

$$\begin{vmatrix} a+1 & a & a & a \\ a & a+2 & a & a \\ a & a & a+3 & a \\ a & a & a & a+4 \end{vmatrix} = 24\left(1+\frac{a}{1}+\frac{a}{2}+\frac{a}{3}+\frac{a}{4}\right)$$ (W.B.U.T. 2004)

...on: Here $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{1}{d}+1 \end{vmatrix}$$

(taking out a, b, c, d from 1st, 2nd, 3rd and 4th row respectively)

$$\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

$(R_1+R_2+R_3+R_4 \to R_1')$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{b} & 1 & 0 & 0 \\ \frac{1}{c} & 0 & 1 & 0 \\ \frac{1}{d} & 0 & 0 & 1 \end{vmatrix} \quad (C_2 - C_1 \to C_2', C_3 - C_1 \to C_3', C_4 - C_1 \to C_4')$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \text{ (expanding with respect to 1st row)}$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) 1 \text{ (expanding with respect to 1st row)}$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

(*ii*) Here $\begin{vmatrix} a+1 & a & a & a \\ a & a+2 & a & a \\ a & a & a+3 & a \\ a & a & a & a+4 \end{vmatrix} = 2.3.4 \begin{vmatrix} a+1 & \frac{a}{2} & \frac{a}{3} & \frac{a}{4} \\ a & \frac{a}{2}+1 & \frac{a}{3} & \frac{a}{4} \\ a & \frac{a}{2} & \frac{a}{3}+1 & \frac{a}{4} \\ a & \frac{a}{2} & \frac{a}{3} & \frac{a}{4}+1 \end{vmatrix}$

(taking out 2, 3, 4 from 2nd, 3rd, 4th column respectively)

$$= 24\begin{vmatrix} 1+a+\frac{a}{2}+\frac{a}{3}+\frac{a}{4} & \frac{a}{2} & \frac{a}{3} & \frac{a}{4} \\ 1+a+\frac{a}{2}+\frac{a}{3}+\frac{a}{4} & 1+\frac{a}{2} & \frac{a}{3} & \frac{a}{4} \\ 1+a+\frac{a}{2}+\frac{a}{3}+\frac{a}{4} & \frac{a}{2} & 1+\frac{a}{3} & \frac{a}{4} \\ 1+a+\frac{a}{2}+\frac{a}{3}+\frac{a}{4} & \frac{a}{2} & \frac{a}{3} & 1+\frac{a}{4} \end{vmatrix} \quad (C_1+C_2+C_3+C_4 \to C_1')$$

$$= 24\left(1+a+\frac{a}{2}+\frac{a}{3}+\frac{a}{4}\right)\begin{vmatrix} 1 & \frac{a}{2} & \frac{a}{3} & \frac{a}{4} \\ 1 & 1+\frac{a}{2} & \frac{a}{3} & \frac{a}{4} \\ 1 & \frac{a}{2} & 1+\frac{a}{3} & \frac{a}{4} \\ 1 & \frac{a}{2} & \frac{a}{3} & 1+\frac{a}{4} \end{vmatrix}$$

$$= 24\left(1+a+\frac{a}{2}+\frac{a}{3}+\frac{a}{4}\right)\begin{vmatrix} 1 & \frac{a}{2} & \frac{a}{3} & \frac{a}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$(R_2 - R_1 \to R_2',\ R_3 - R_1 \to R_3',\ R_4 - R_1 \to R_4')$

$$= 24\left(1+a+\frac{a}{2}+\frac{a}{3}+\frac{a}{4}\right)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{(expanding with respect to 1st column)}$$

$$= 24\left(1+\frac{a}{1}+\frac{a}{2}+\frac{a}{3}+\frac{a}{4}\right)$$

Example 4: Show that

(*i*) $$\begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (z+x)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix} = 2xyz\,(x+y+z)^3$$ (*W.B.U.T. 2008*)

(*ii*) $$\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2\,(ab+bc+ca)^3$$

(iii) $\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = 2s^3(s-a)(s-b)(s-c)$ where $2s = a+b+c$

(iv) $\begin{vmatrix} (a+b)^2 & ca & cb \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3.$

Solution: (i) $\begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (z+x)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix} = \begin{vmatrix} (y+z)^2-x^2 & 0 & x^2 \\ 0 & (z+x)^2-y^2 & y^2 \\ z^2-(x+y)^2 & z^2-(x+y)^2 & (x+y)^2 \end{vmatrix}$

$(C_1 - C_3 \to C_1', C_2 - C_3 \to C_2')$

$$= \begin{vmatrix} (y+z+x)(y+z-x) & 0 & x^2 \\ 0 & (z+x+y)(z+x-y) & y^2 \\ (z+x+y)(z-x-y) & (z+x+y)(z-x-y) & (x+y)^2 \end{vmatrix}$$

$$= (x+y+z)^2 \begin{vmatrix} y+z-x & 0 & x^2 \\ 0 & z+x-y & y^2 \\ z-x-y & z-x-y & (x+y)^2 \end{vmatrix}$$

(taking out $(x+y+z)$ from 1st and 2nd columns)

$$= (x+y+z)^2 \begin{vmatrix} y+z-x & 0 & x^2 \\ 0 & z+x-y & y^2 \\ -2y & -2x & 2xy \end{vmatrix} \quad (R_3 - (R_1+R_2) \to R_3')$$

$$= \frac{2(x+y+z)^2}{xy} \begin{vmatrix} xy+xz-x^2 & 0 & x^2 \\ 0 & yz+yx-y^2 & y^2 \\ -xy & -xy & xy \end{vmatrix}$$

$$= 2(x+y+z)^2 \begin{vmatrix} xy+xz-x^2 & 0 & x^2 \\ 0 & yz+yx-y^2 & y^2 \\ -1 & -1 & 1 \end{vmatrix}$$

$$= 2(x+y+z)^2 \begin{vmatrix} xy+xz & x^2 & x^2 \\ y^2 & yz+yx & y^2 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{(adding 3rd column to 1st and 2nd columns)}$$

$$= 2(x+y+z)^2\{(xy+xz)(yz+yx)-x^2y^2\} \quad \text{(expanding with respect to 3rd row)}$$

$$= 2xy(x+y+z)^2\{(y+z)(z+x)-xy\}$$

$$= 2xy(x+y+z)^2(yz+z^2+zx) = 2xyz(x+y+z)^3.$$

$$\therefore \quad \begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (z+x)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3. \qquad ...(1)$$

(*ii*) Now, putting $x = bc$, $y = ca$ and $z = ab$ in (1), we get

$$\begin{vmatrix} a^2(b+c)^2 & b^2c^2 & b^2c^2 \\ c^2a^2 & b^2(c+a)^2 & c^2a^2 \\ a^2b^2 & a^2b^2 & c^2(a+b)^2 \end{vmatrix} = 2a^2b^2c^2(ab+bc+ca)^3$$

Taking out a^2, b^2, c^2 from 1st, 2nd and 3rd column respectively, we get

$$a^2b^2c^2 \begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2a^2b^2c^2(ab+bc+ca)^3$$

$$\therefore \quad \begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2(ab+bc+ca)^3.$$

(*iii*) Let $D = \begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix}$

Let us put $s - a = x$, $s - b = y$ and $s - c = z$, then $x + y + z = 3s - (a + b + c) = 3s - 2s = s$, $y + z = 2s - (b + c) = a$, $z + x = 2s - (c + a) = b$, $x + y = 2s - (a + b) = c$.

$$\therefore \quad D = \begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (z+x)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3 \qquad \text{[by (1)]}$$

$$= 2s^3(s-a)(s-b)(s-c).$$

(iv) Let
$$D = \begin{vmatrix} (a+b)^2 & ca & cb \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix}.$$

Let us put $a = y$, $b = z$ and $c = x$.

$$\therefore \quad D = \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (z+x)^2 & yz \\ zx & yz & (x+y)^2 \end{vmatrix}$$

$$= \frac{1}{xyz}\begin{vmatrix} x(y+z)^2 & x^2y & zx^2 \\ xy^2 & y(z+x)^2 & y^2z \\ z^2x & yz^2 & z(x+y)^2 \end{vmatrix}$$

$(xR_1 \to R_1', yR_2 \to R_2', zR_3 \to R_3')$

$$= \frac{xyz}{xyz}\begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (z+x)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix}$$

(taking out x, y, z from 1st, 2nd, 3rd columns respectively)

$$= 2xyz(x+y+z)^3 \qquad \text{[by (1)]}$$

$$= 2abc(a+b+c)^3 \qquad (\because a = y, b = z \text{ and } c = x)$$

2.4 MINORS AND COMPLEMENTARY MINORS OF A DETERMINANT

Minor

Let $D = |a_{ij}|_{n\times n}$ be a determinant of order n. If r number of rows and r number of columns $(r < n)$ be selected from D, then the rth order determinant formed by the elements which are lying at the junctions of these selected rows and columns is called *a minor* of *order r* of $D = |a_{ij}|_{n\times n}$.

Example: Let $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = |a_{ij}|_{4\times4}$

If first and third rows and second and fourth columns are selected from $D = | a_{ij} |_{4\times4}$, then we get

a minor $M_{13,\,24} = \begin{vmatrix} a_{12} & a_{14} \\ a_{32} & a_{34} \end{vmatrix}$ of order 2 of $D = | a_{ij} |_{n\times n}$.

Similarly, $M_{234,\,124} = \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$ is a minor of order 3 of $D = | a_{ij} |_{4\times4}$.

Complementary minor

Let M be a minor of order $r\ (< n)$ of $D = | a_{ij} |_{n\times n}$. If all the rows and columns of M are deleted from $D = | a_{ij} |_{n\times n}$, then the remaining determinant of order $(n - r)$ is said to be a *complementary minor of M*.

Thus in $D = | a_{ij} |_{4\times4}$, $M_{24,13} = \begin{vmatrix} a_{21} & a_{23} \\ a_{41} & a_{43} \end{vmatrix}$ is the complementary minor of $M_{13,24}$.

Also, the element a_{13} is the complementary minor of $M_{234,124}$.

Algebraic complement of a minor

Let M be a minor of order $r\ (< n)$ obtained from i_1th, i_2th, ..., i_rth rows and j_1th, j_2th, ..., j_rth columns of $D = | a_{ij} |_{n\times n}$ and let M' be its complementary minor. Then *algebraic complement* of M is defined as $(-1)^{i_1+i_2+\ldots+i_r+j_1+j_2+\ldots+j_r} \times M'$.

Example: In $D = | a_{ij} |_{4\times4}$, algebraic complement of the minor

$$M_{13,\,24} = (-1)^{1+3+2+4} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{41} & a_{43} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{23} \\ a_{41} & a_{43} \end{vmatrix}.$$

Algebraic complement of the minor $M_{23,13} = (-1)^{2+3+1+3} \begin{vmatrix} a_{12} & a_{14} \\ a_{42} & a_{44} \end{vmatrix} = -\begin{vmatrix} a_{12} & a_{14} \\ a_{42} & a_{44} \end{vmatrix}$.

2.5 LAPLACE'S METHOD OF EXPANSION OF A DETERMINANT

Theorem

If any $r\ (< n)$ rows be selected in an nth order determinant $D = | a_{ij} |_{n\times n}$ and each minor of order r which can be formed from these r rows be multiplied by its algebraic complement, then D can be expressed as the sum of all such products.

Example: Let $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$.

According to Laplace's method of expansion, we get on selecting first two rows,

$$D = (-1)^{1+2+1+2}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} + (-1)^{1+2+1+3}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}\begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}$$

$$+(-1)^{1+2+1+4}\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}\begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} + (-1)^{1+2+2+3}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}\begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix}$$

$$+(-1)^{1+2+2+4}\begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix}\begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + (-1)^{1+2+3+4}\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}\begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$$

Note: In any determinant, cofactor of an element (considering it as a minor of order one) is its algebraic complement and therefore, the expansion of a determinant in terms of cofactors is an application of Laplace's method of expansion.

Example: Expanding by Laplace's method show that

$$\begin{vmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix} = 4(a^2+b^2)(c^2+d^2).$$

Solution: Applying Laplace's method of expansion, we get on selecting first two rows,

$$\begin{vmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix} = (-1)^{1+2+1+2}\begin{vmatrix} a & -b \\ b & a \end{vmatrix}\begin{vmatrix} c & -d \\ d & c \end{vmatrix} + (-1)^{1+2+1+3}\begin{vmatrix} a & -a \\ b & -b \end{vmatrix}\begin{vmatrix} -d & -d \\ c & c \end{vmatrix}$$

$$+(-1)^{1+2+1+4}\begin{vmatrix} a & b \\ b & -a \end{vmatrix}\begin{vmatrix} -d & c \\ c & d \end{vmatrix} + (-1)^{1+2+2+3}\begin{vmatrix} -b & -a \\ a & -b \end{vmatrix}\begin{vmatrix} c & -d \\ d & c \end{vmatrix}$$

$$+(-1)^{1+2+2+4}\begin{vmatrix} -b & b \\ a & -a \end{vmatrix}\begin{vmatrix} c & c \\ d & d \end{vmatrix} + (-1)^{1+2+3+4}\begin{vmatrix} -a & b \\ -b & -a \end{vmatrix}\begin{vmatrix} c & -d \\ d & c \end{vmatrix}$$

$$= (a^2+b^2)(c^2+d^2) - 0 + (-a^2-b^2)(-d^2-c^2) + (b^2+a^2)(c^2+d^2)$$

$$-0+(a^2+b^2)(c^2+d^2) = 4(a^2+b^2)(c^2+d^2).$$

2.6 SOLUTION OF A SYSTEM OF LINEAR SIMULTANEOUS EQUATIONS: CRAMER'S RULE

Let us consider a system of n linear simultaneous equations with n number of variables $x_1, x_2, \ldots, x_n$ as stated below:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \hline a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{array}\right\} \quad \ldots (2.6.1)$$

The above system of equations is said to be homogeneous or non-homogeneous according as the constants $b_1, b_2, ..., b_n$ are all zero or at least one is non-zero.

Cramer's rule

If the determinant of the coefficients

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \hline a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0,$$

then the unique solution of the system (2.6.1) is given by $x_j = \dfrac{D_j}{D}$, $j = 1, 2, ..., n$, where D_j is an nth order determinant obtained from D by replacing the elements of its jth column by $b_1, b_2, ..., b_n$.

We have
$$x_1 D = \begin{vmatrix} a_{11}x_1 & a_{12} & \cdots & a_{1n} \\ a_{21}x_1 & a_{22} & \cdots & a_{2n} \\ \hline a_{n1}x_1 & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Multiplying 2nd, 3rd, ..., nth columns by $x_2, x_3, ..., x_n$ respectively and then adding to 1st column, we get

$$x_1 D = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & a_{12} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & a_{22} & \cdots & a_{2n} \\ \hline a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} b_1 & a_{12} \ldots a_{1n} \\ b_2 & a_{22} \ldots a_{2n} \\ \hline b_n & a_{n1} \ldots a_{nn} \end{vmatrix} \qquad \text{[using (2.6.1)]}$$

$$= D_1$$

$$\therefore \quad x_1 = \frac{D_1}{D} \qquad (\because D \neq 0).$$

Proceeding similarly, we get $x_2 = \dfrac{D_2}{D}, x_3 = \dfrac{D_3}{D}, \ldots, x_n = \dfrac{D_n}{D}$.

Homogeneous system of linear equations

If the system of equations (2.6.1) be homogeneous (*i.e.*, $b_1 = b_2 = \ldots = b_n = 0$) with $D \neq 0$, then by Cramer's rule the unique solution is $x_1 = x_2 = \ldots = x_n = 0$, because each of $D_j = 0$ as all the elements of its jth column are zero. This solution is known as *trivial solution*.

Theorem

The necessary and sufficient condition that the system of homogeneous equations

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = 0$$

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may have non-trivial solution, *i.e.*, infinitely many solutions is that

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \hline a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0.$$

ILLUSTRATIVE EXAMPLES

Example 1: Solve by Cramer's rule the following system of equations:

$$3x+y+z=4$$
$$x-y+2z=6$$
$$x+2y-z=-3.$$

(*W.B.U.T. 2009*)

Solution: Here the determinant of coefficients,

$$D=\begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 3(1-4)-1(-1-2)+1(2+1) = -3 \neq 0.$$

Therefore, Cramer's rule is applicable.

Now, $$D_1 = \begin{vmatrix} 4 & 1 & 1 \\ 6 & -1 & 2 \\ -3 & 2 & -1 \end{vmatrix} = 4(1-4)-1(-6+6)+1(12-3) = -3,$$

$$D_2 = \begin{vmatrix} 3 & 4 & 1 \\ 1 & 6 & 2 \\ 1 & -3 & -1 \end{vmatrix} = 3(-6+6)-4(-1-2)+1(-3-6) = 3,$$

$$D_3 = \begin{vmatrix} 3 & 1 & 4 \\ 1 & -1 & 6 \\ 1 & 2 & -3 \end{vmatrix} = 3(3-12)-1(-3-6)+4(2+1) = -6.$$

Therefore, by Cramer's rule:

$$x = \frac{D_1}{D} = \frac{-3}{-3} = 1,\ y = \frac{D_2}{D} = \frac{3}{-3} = -1,\ z = \frac{D_3}{D} = \frac{-6}{-3} = 2.$$

Example 2: Given the equations $x = cy + bz$, $y = az + cx$, $z = bx + ay$, where x, y, z are not all zero, show that $a^2+b^2+c^2+2abc = 1$.

Solution: The given system of equations are rearranged as

$$x-cy-bz=0$$
$$cx-y+az=0$$
$$bx+ay-z=0$$

which is a homogeneous system of equations. Since by question all x, y, z are not zero, the above system of equations has non-trivial solutions.

$$\text{Therefore, we have } \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

or $$\begin{vmatrix} 1 & -c & -b \\ 0 & c^2-1 & a+bc \\ 0 & a+bc & b^2-1 \end{vmatrix} = 0 \quad (R_2 - cR_1 \to R_2',\ R_3 - bR_1 \to R_3')$$

or $$\begin{vmatrix} c^2-1 & a+bc \\ a+bc & b^2-c \end{vmatrix} = 0, \text{ or } (c^2-1)(b^2-1)-(a+bc)^2 = 0$$

or $$b^2c^2 - b^2 - c^2 + 1 - a^2 - b^2c^2 - 2abc = 0$$

$$\therefore \quad a^2 + b^2 + c^2 + 2abc = 1.$$

2.7 PRODUCT OF DETERMINANTS

The product of two determinants of same order is another determinant of same order.

We now show that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} = \begin{vmatrix} a_1c_1+b_1d_1 & a_1c_2+b_1d_2 \\ a_2c_1+b_2d_1 & a_2c_2+b_2d_2 \end{vmatrix}$$

$$\text{Here } \begin{vmatrix} a_1c_1+b_1d_1 & a_1c_2+b_1d_2 \\ a_2c_1+b_2d_1 & a_2c_2+b_2d_2 \end{vmatrix} = \begin{vmatrix} a_1c_1 & a_1c_2+b_1d_2 \\ a_2c_1 & a_2c_2+b_2d_2 \end{vmatrix} + \begin{vmatrix} b_1d_1 & a_1c_2+b_1d_2 \\ b_2d_1 & a_2c_2+b_2d_2 \end{vmatrix}.$$

(using property 5 of art. 2.3)

$$= \begin{vmatrix} a_1c_1 & a_1c_2 \\ a_2c_1 & a_2c_2 \end{vmatrix} + \begin{vmatrix} a_1c_1 & b_1d_2 \\ a_2c_1 & b_2d_2 \end{vmatrix} + \begin{vmatrix} b_1d_1 & a_1c_2 \\ b_2d_1 & a_2c_2 \end{vmatrix} + \begin{vmatrix} b_1d_1 & b_1d_2 \\ b_2d_1 & b_2d_2 \end{vmatrix}$$

$$= a_1a_2\begin{vmatrix} c_1 & c_2 \\ c_1 & c_2 \end{vmatrix} + c_1d_2\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + d_1c_2\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} + d_1d_2\begin{vmatrix} b_1 & b_1 \\ b_2 & b_2 \end{vmatrix}$$

$$= c_1d_2\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} - c_2d_1\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}(c_1d_2 - c_2d_1)$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}$$

$$\text{Similarly, we have } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1l_1 + b_1m_1 + c_1n_1 & a_1l_2 + b_1m_2 + c_1n_2 & a_1l_3 + b_1m_3 + c_1n_3 \\ a_2l_1 + b_2m_1 + c_2n_1 & a_2l_2 + b_2m_2 + c_2n_2 & a_2l_3 + b_2m_3 + c_2n_3 \\ a_3l_1 + b_3m_1 + c_3n_1 & a_3l_2 + b_3m_2 + c_3n_2 & a_3l_3 + b_3m_3 + c_3n_3 \end{vmatrix} \quad \ldots (2.7.1)$$

The product of two determinants in (2.7.1) is known as row-by-row multiplication in which the (i, j)th element of the product determinant is obtained by adding the products of the elements of the ith row of the first determinant with the corresponding elements of the jth row of the second determinant.

Since a determinant remains unaltered by the interchange of rows and columns, there may be multiplication of row-by-column, column-by-row, column-by-column.

Notes: (*i*) Unless otherwise stated, we generally apply row-by-row multiplication.

(*ii*) The product of a determinant of mth order and a determinant of nth order $(m < n)$ can be expressed as a determinant of order n, since we can express a determinant of order m $(< n)$ as a determinant of order n as illustrated below:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & b_1 & c_1 \\ 0 & a_2 & b_2 & c_2 \\ 0 & a_3 & b_3 & c_3 \end{vmatrix}.$$

(*iii*) The above procedure can be generalized for multiplication of two determinants of more than three order.

ILLUSTRATIVE EXAMPLES

Example 1: Prove that

(*i*) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

(*ii*) The product of two expressions of the form $x^3 + y^3 + z^3 - 3xyz$ is of the same form.

(*iii*) $\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc)^2$

(*iv*) $\begin{vmatrix} a^2 + b^2 + c^2 & bc + ca + ab & bc + ca + ab \\ bc + ca + ab & a^2 + b^2 + c^2 & bc + ca + ab \\ bc + ca + ab & bc + ca + ab & a^2 + b^2 + c^2 \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc)^2.$

Solution: (*i*) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix} \quad (C_1 + C_2 + C_3 \to C_1)$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \quad (R_2 - R_1 \to R_2',\ R_3 - R_1 \to R_3')$$

$$= (a+b+c)\begin{vmatrix} c-b & a-c \\ a-b & b-c \end{vmatrix} = (a+b+c)\{(c-b)(b-c)-(a-c)(a-b)\}$$

$$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$\therefore \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc) \qquad \ldots(i)$$

Note: The determinant in (*i*) is known as circulant determinant.

(*ii*) Let us consider two forms of $x^3+y^3+z^3-3xyz$, namely $x_1^3+y_1^3+z_1^3-3x_1y_1z_1$ and $x_2^3+y_2^3+z_2^3-3x_2y_2z_2$.

The product of these two forms is $(x_1^3+y_1^3+z_1^3-3x_1y_1z_1)(x_2^3+y_2^3+z_2^3-3x_2y_2z_2)$

$$=(-1)^2\begin{vmatrix} x_1 & y_1 & z_1 \\ y_1 & z_1 & x_1 \\ z_1 & x_1 & y_1 \end{vmatrix}\begin{vmatrix} x_2 & y_2 & z_2 \\ y_2 & z_2 & x_2 \\ z_2 & x_2 & y_2 \end{vmatrix} \qquad \text{[using (i)]}$$

$$= \begin{vmatrix} x_1x_2+y_1y_2+z_1z_2 & x_1y_2+y_1z_2+z_1x_2 & x_1z_2+y_1x_2+z_1y_2 \\ y_1x_2+z_1y_2+x_1z_2 & y_1y_2+z_1z_2+x_1x_2 & y_1z_2+z_1x_2+x_1y_2 \\ z_1x_2+x_1y_2+y_1z_2 & z_1y_2+x_1z_2+y_1x_2 & z_1z_2+x_1x_2+y_1y_2 \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

(where $a = x_1x_2+y_1y_2+z_1z_2$, $b = x_1y_2+y_1z_2+z_1x_2$ and $c = x_1z_2+y_1x_2+z_1y_2$)

$$= -\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \qquad \text{(interchanging 2nd and 3rd rows)}$$

$$= a^3+b^3+c^3-3abc. \qquad \text{[using (i)]}$$

(*iii*) Now $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} \quad \ldots(2)$$

(using row-by-row multiplication)

Here $\begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} = -\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} = (-1)^2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ ...(3)

From (2) and (3), we have

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = (a^3+b^3+c^3-3abc)^2. \quad \text{[using (1)]}$$

(iv) Here $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= \begin{vmatrix} a^2+b^2+c^2 & ab+bc+ca & bc+ca+ab \\ ba+cb+ac & b^2+c^2+a^2 & bc+ca+ab \\ ca+ab+bc & cb+ac+ba & c^2+a^2+b^2 \end{vmatrix}$$

(applying row-by-row multiplication)

$$\therefore \begin{vmatrix} a^2+b^2+c^2 & bc+ca+ab & bc+ca+ab \\ bc+ca+ab & a^2+b^2+c^2 & bc+ca+ab \\ bc+ca+ab & bc+ca+ab & a^2+b^2+c^2 \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = (a^3+b^3+c^3-3abc)^2 \quad \text{[using (1)]}$$

Example 2: Evaluate $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ and hence prove that

(i) $\begin{vmatrix} b^2c^2+a^2d^2 & bc+ad & 1 \\ c^2a^2+b^2d^2 & ca+bd & 1 \\ a^2b^2+c^2d^2 & ab+cd & 1 \end{vmatrix} = (a-b)(b-c)(c-a)(a-d)(b-d)(c-d)$

(ii) $\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$

(iii) $\begin{vmatrix} 0 & (a-b)^2 & (a-c)^2 \\ (b-a)^2 & 0 & (b-c)^2 \\ (c-a)^2 & (c-b)^2 & 0 \end{vmatrix} = 2(a-b)^2(b-c)^2(c-a)^2$

(iv) $\begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} = (a-b)^2(b-c)^2(c-a)^2$ where $s_r = a^r + b^r + c^r$.

Solution: $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$ $(R_2 - R_1 \to R_2',\ R_3 - R_1 \to R_3')$

$$= \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (b-a)(c-a)(c+a-b-a) = (a-b)(b-c)(c-a)$$

$\therefore$ $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$ ...(1)

(i) Here $\begin{vmatrix} b^2c^2 + a^2d^2 & bc+ad & 1 \\ c^2a^2 + b^2d^2 & ca+bd & 1 \\ a^2b^2 + c^2d^2 & ab+cd & 1 \end{vmatrix}$

$$= \begin{vmatrix} (bc+ad)^2 & bc+ad & 1 \\ (ca+bd)^2 & ca+bd & 1 \\ (ab+cd)^2 & ab+cd & 1 \end{vmatrix} \quad (C_1 + 2abcd\ C_3 \to C_1')$$

$$= \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}$$ (putting $\alpha = bc + ad$, $\beta = ca + bd$, $\gamma = ab + cd$)

$$= -\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} \quad \text{(interchanging 1st and 3}^{\text{rd}}\text{ columns)}$$

$$= -(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha) \qquad \text{[using (1)]}$$

$$= -(bc+ad-ca-bd)(ca+bd-ab-cd)(ab+cd-bc-ad)$$

$$= -(b-a)(c-d)(c-b)(a-d)(b-d)(a-c)$$

$$= (a-b)(b-c)(c-a)(a-d)(b-d)(c-d).$$

(*ii*) Now $\begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2-2ax+x^2 & a^2-2ay+y^2 & a^2-2az+z^2 \\ b^2-2bx+x^2 & b^2-2by+y^2 & b^2-2bz+z^2 \\ c^2-2cx+x^2 & c^2-2cy+y^2 & c^2-2cz+z^2 \end{vmatrix}$$

(applying row-by-row multiplication)

or $$-2\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$$

or $$2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$$

$$\therefore \quad \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x).$$

(*iii*) $$\begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} a^2-2a^2+a^2 & a^2-2ab+b^2 & a^2-2ac+c^2 \\ b^2-2ab+a^2 & b^2-2b^2+b^2 & b^2-2bc+c^2 \\ c^2-2ca+a^2 & c^2-2cb+b^2 & c^2-2c^2+c^2 \end{vmatrix}$$

(applying row-by-row multiplication)

or $$-2\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & (a-b)^2 & (a-c)^2 \\ (b-a)^2 & 0 & (b-c)^2 \\ (c-a)^2 & (c-b)^2 & 0 \end{vmatrix},$$

$$\therefore \begin{vmatrix} 0 & (a-b)^2 & (a-c)^2 \\ (b-a)^2 & 0 & (b-c)^2 \\ (c-a)^2 & (c-b)^2 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= 2\,(a-b)^2\,(b-c)^2\,(c-a)^2 \qquad \text{[by (1)]}$$

(*iv*) Now $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$$= \begin{vmatrix} 1+1+1 & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{vmatrix}$$

(applying row-by-row multiplication)

$$\therefore \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)^2(b-c)^2(c-a)^2 \qquad \text{[by (1)]}$$

Example 3: Prove that $\begin{vmatrix} a_1\alpha_1+b_1\beta_1 & a_2\alpha_1+b_2\beta_1 & a_3\alpha_1+b_3\beta_1 \\ a_1\alpha_2+b_1\beta_2 & a_2\alpha_2+b_2\beta_2 & a_3\alpha_2+b_3\beta_2 \\ a_1\alpha_3+b_1\beta_3 & a_2\alpha_3+b_2\beta_3 & a_3\alpha_3+b_3\beta_3 \end{vmatrix} = 0.$

Solution: $\begin{vmatrix} a_1\alpha_1+b_1\beta_1 & a_2\alpha_1+b_2\beta_1 & a_3\alpha_1+b_3\beta_1 \\ a_1\alpha_2+b_1\beta_2 & a_2\alpha_2+b_2\beta_2 & a_3\alpha_2+b_3\beta_2 \\ a_1\alpha_3+b_1\beta_3 & a_2\alpha_3+b_2\beta_3 & a_3\alpha_3+b_3\beta_3 \end{vmatrix}$

$$= \begin{vmatrix} \alpha_1 a_1+\beta_1 b_1+0\cdot 0 & \alpha_1 a_2+\beta_1 b_2+0\cdot 0 & \alpha_1 a_3+\beta_1 b_3+0\cdot 0 \\ \alpha_2 a_1+\beta_2 b_1+0\cdot 0 & \alpha_2 a_2+\beta_2 b_2+0\cdot 0 & \alpha_2 a_3+\beta_2 b_3+0\cdot 0 \\ \alpha_3 a_1+\beta_3 b_1+0\cdot 0 & \alpha_3 a_2+\beta_3 b_2+0\cdot 0 & \alpha_3 a_3+\beta_3 b_3+0\cdot 0 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & 0 \end{vmatrix}\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = 0.$$

2.8 ADJOINT OF A DETERMINANT

Let $D = |a_{ij}|_{n\times n}$ be a determinant of order n and A_{ij} be the cofactor of a_{ij} in D, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$. Then the determinant obtained by replacing each element of D by its cofactor in D is said to be the *adjoint* or *adjugate of the determinant* D and it is denoted by D'. Therefore $D' = |A_{ij}|_{n\times n}$.

Example: Let $D = \begin{vmatrix} 1 & 2 & 0 \\ -1 & 4 & 2 \\ 0 & 1 & 3 \end{vmatrix}$

The adjoint of D is $D' = \begin{vmatrix} \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} -1 & 4 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} 10 & 3 & -1 \\ -6 & 3 & -1 \\ 4 & -2 & 6 \end{vmatrix}$

Theorem 1: Let $D = |a_{ij}|_{n \times n}$ and A_{ij} is the cofactor of a_{ij} in D, $i = 1, 2, ..., n$; $j = 1, 2, ..., n$.

Then $\left.\begin{aligned}\sum_{j=1}^{n} a_{rj}A_{sj} &= D, \text{ if } r = s \\ &= 0, \text{ if } r \neq s\end{aligned}\right\}$ and $\begin{aligned}\sum_{i=1}^{n} a_{ip}A_{iq} &= D, \text{ if } p = q \\ &= 0, \text{ if } p \neq q\end{aligned}$

Proof: From (2.2.1) and (2.2.2) of art. 2.2, we have

$\sum_{j=1}^{n} a_{rj}A_{sj} = D$ if $r = s$ and $\sum_{i=1}^{n} a_{ip}A_{iq} = D$ if $p = q$.

This means that if we choose any row (or column) and find the sum of products of all its elements and their corresponding cofactor, we obtain the value of the determinant.

Let us choose any row, say first row, and find the sum of products of its elements and the cofactors of corresponding elements of any other row, say second row.

Then the sum $\sum_{j=1}^{n} a_{1j}A_{2j}$ is the result of replacing $a_{21}, a_{22}, ..., a_{2n}$ by $a_{11}, a_{12}, ..., a_{1n}$ in D and it is, therefore, equal to a determinant whose first two rows are identical.

Therefore, $\sum_{j=1}^{n} a_{1j}A_{2j} = 0$ (applying property 3 of art. 2.3). This holds good for any two rows or any two columns. Thus, $\sum_{j=1}^{n} a_{rj}A_{sj} = 0$ if $r \neq s$ and $\sum_{i=1}^{n} a_{ip}A_{iq} = 0$ if $p \neq q$.

Theorem 2 (Jacobi's Theorem): If D be an nth order determinant and D' be its adjoint, then $D' = D^{n-1}$.

Proof: Let $D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$ and let A_{ij} be the cofactor of a_{ij} in D, $i = 1, 2, ..., n$; $j = 1, 2, ..., n$.

Case 1: When $D \neq 0$.

Now
$$DD' = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \hline a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \hline A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} \sum_{j=1}^{n} a_{1j}A_{1j} & \sum_{j=1}^{n} a_{1j}A_{2j} & \cdots & \sum_{j=1}^{n} a_{1j}A_{nj} \\ \sum_{j=1}^{n} a_{2j}A_{1j} & \sum_{j=1}^{n} a_{2j}A_{2j} & \cdots & \sum_{j=1}^{n} a_{2j}A_{nj} \\ \hline \sum_{j=1}^{n} a_{nj}A_{1j} & \sum_{j=1}^{n} a_{nj}A_{2j} & \cdots & \sum_{j=1}^{n} a_{nj}A_{nj} \end{vmatrix}$$

$$= \begin{vmatrix} D & 0 & \cdots & 0 \\ 0 & D & \cdots & 0 \\ \hline 0 & 0 & \cdots & D \end{vmatrix} \qquad \text{(applying Theorem 1)}$$

$$= D^n.$$

$$\therefore \qquad D' = D^{n-1}. \qquad (\because D \neq 0)$$

Case 2: When $D = 0$

If each element $a_{ij} = 0$, then each $A_{ij} = 0$, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$. Therefore $D' = D^{n-1} = 0$.

Let there exists at least one non-zero element in D. Without loss of generality, let $a_{11} \neq 0$, then by expanding D in terms of cofactors, we have

$$\left.\begin{array}{l} a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} = \det D = 0 \\ a_{11}A_{21} + a_{12}A_{22} + \ldots + a_{1n}A_{2n} = 0 \\ \hline a_{11}A_{n1} + a_{12}A_{n2} + \ldots + a_{1n}A_{nn} = 0 \end{array}\right\} \qquad \ldots(1)$$

Now
$$a_{11}\, D' = \begin{vmatrix} a_{11}A_{11} & A_{12} & \cdots & A_{1n} \\ a_{11}A_{21} & A_{22} & \cdots & A_{2n} \\ \hline a_{11}A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} & A_{12} & \cdots & A_{1n} \\ a_{11}A_{21} + a_{12}A_{22} + \ldots + a_{1n}A_{2n} & A_{22} & \cdots & A_{2n} \\ \hline a_{11}A_{n1} + a_{12}A_{n2} + \ldots + a_{1n}A_{nn} & A_{n2} & \cdots & A_{nn} \end{vmatrix}$$

$$(C_1 + a_{12}C_2 + \ldots + C_{1n}C_n \rightarrow C_1')$$

$$= \begin{vmatrix} 0 & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \cdots & A_{2n} \\ \hline 0 & A_{n2} & \cdots & A_{nn} \end{vmatrix} \qquad \text{[using (1)]}$$

$$= 0$$

$\therefore \qquad D' = 0. \qquad (\because a_{11} \neq 0)$

Hence $D' = D^{n-1}$, since in this case $D = D' = 0$.

Therefore $D' = D^{n-1}$ for any nth order determinant D.

Note: If D is a third order determinant, then $D' = D^2$.

2.9 RECIPROCAL OF A DETERMINANT

Let $D = |a_{ij}|$ be an nth order determinant and A_{ij} be the cofactor of a_{ij} in D, $i = 1, 2, ..., n; j = 1, 2, ..., n$. If $D \neq 0$, then the determinant obtained by dividing each element of adjoint of D by the value of D, is said to be the *reciprocal* of D.

$$\text{Therefore, reciprocal of } D = \begin{vmatrix} \frac{A_{11}}{D} & \frac{A_{12}}{D} & \cdots & \frac{A_{1n}}{D} \\ \frac{A_{21}}{D} & \frac{A_{22}}{D} & \cdots & \frac{A_{2n}}{D} \\ \hline \frac{A_{n1}}{D} & \frac{A_{n2}}{D} & \cdots & \frac{A_{nn}}{D} \end{vmatrix}$$

$$= \frac{1}{D^n} \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \hline A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix} = \frac{D'}{D^n} = \frac{D^{n-1}}{D^n} = \frac{1}{D}. \qquad \text{[by Jacobi's Theorem]}$$

2.10 SYMMETRIC AND SKEW-SYMMETRIC DETERMINANTS

A determinant $D = |a_{ij}|_{n \times n}$ is said to be *symmetric* if $a_{ij} = a_{ji}$ for $i = 1, 2, ..., n; j = 1, 2, ..., n$, and D is said to be *skew-symmetric* if $a_{ij} = -a_{ji}$ for $i = 1, 2, ..., n; j = 1, 2, ..., n$.

Therefore, for a symmetric determinant, the elements equidistant from the principal diagonal are equal in magnitude and sign.

For a skew-symmetric determinant, the elements equidistant from the principal diagonal are equal in magnitude but opposite in sign and all the diagonal elements (or leading elements) are zero, since for diagonal elements $a_{ii} = -a_{ii}$, or $2a_{ii} = 0$, or $a_{ii} = 0$.

Notes: (*i*) If A is a symmetric matrix, then det A is a symmetric determinant.

(*ii*) If A is a skew-symmetric matrix, then det A is a skew-symmetric determinant.

Example: $\begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ 1 & 2 & 6 \end{vmatrix}$ is a symmetric determinant and $\begin{vmatrix} 0 & 2 & -5 \\ -2 & 0 & 6 \\ 5 & -6 & 0 \end{vmatrix}$ is a skew-symmetric determinant.

Properties

1. The square of any determinant is symmetric.
2. The adjoint of a symmetric determinant is a symmetric determinant.
3. Every skew-symmetric determinant of odd order is zero.
4. Every skew-symmetric determinant of even order is a perfect square.
5. The adjoint of a skew-symmetric determinant is symmetric or skew-symmetric if its order be odd or even respectively.

ILLUSTRATIVE EXAMPLES

Example 1: Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$. (*W.B.U.T. 2006*)

Solution: Now $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$ ...(1)

(see Ex. 1(*i*) of art. 2.7)

$$\therefore \quad \text{Adjoint of } D = D' = \begin{vmatrix} \begin{vmatrix} c & a \\ a & b \end{vmatrix} & -\begin{vmatrix} b & a \\ c & b \end{vmatrix} & \begin{vmatrix} b & c \\ c & a \end{vmatrix} \\ -\begin{vmatrix} b & c \\ a & b \end{vmatrix} & \begin{vmatrix} a & c \\ c & b \end{vmatrix} & -\begin{vmatrix} a & b \\ c & a \end{vmatrix} \\ \begin{vmatrix} b & c \\ c & a \end{vmatrix} & -\begin{vmatrix} a & c \\ b & a \end{vmatrix} & \begin{vmatrix} a & b \\ b & c \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$$

By Jacobi's theorem, $D' = D^2$

$$\therefore \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = D^2 = (a^3+b^3+c^3-3abc)^2. \qquad \text{[using (1)]}$$

Example 2: If A, B, C are cofactors of a, b, c in $D = \begin{vmatrix} a & b & c \\ b & c & a \\ a & b & c \end{vmatrix}$, then prove that

$$(a+b+c)^2(a+b\omega+c\omega^2)^2(a+b\omega^2+c\omega)^2$$
$$= -(A+B+C)(A+B\omega+C\omega^2)(A+B\omega^2+C\omega),$$

where ω is an imaginary cube root of unity.

$$= \begin{vmatrix} 0 & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \cdots & A_{2n} \\ \hdotsfor{4} \\ 0 & A_{n2} & \cdots & A_{nn} \end{vmatrix} \qquad \text{[using (1)]}$$

$$= 0$$

$\therefore \qquad D' = 0. \qquad (\because a_{11} \neq 0)$

Hence $D' = D^{n-1}$, since in this case $D = D' = 0$.

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$$= \frac{1}{D^n} \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \hdotsfor{4} \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix} = \frac{D'}{D^n} = \frac{D^{n-1}}{D^n} = \frac{1}{D}. \qquad \text{[by Jacobi's Theorem]}$$

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Solution: Now $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$ ...(1)

(see Ex. 1(*i*) of art. 2.7)

$$\therefore \quad \text{Adjoint of } D = D' = \begin{vmatrix} \begin{vmatrix} c & a \\ a & b \end{vmatrix} & -\begin{vmatrix} b & a \\ c & b \end{vmatrix} & \begin{vmatrix} b & c \\ c & a \end{vmatrix} \\ -\begin{vmatrix} b & c \\ a & b \end{vmatrix} & \begin{vmatrix} a & c \\ c & b \end{vmatrix} & -\begin{vmatrix} a & b \\ c & a \end{vmatrix} \\ \begin{vmatrix} b & c \\ c & a \end{vmatrix} & -\begin{vmatrix} a & c \\ b & a \end{vmatrix} & \begin{vmatrix} a & b \\ b & c \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$$

By Jacobi's theorem, $D' = D^2$

$$\therefore \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = D^2 = (a^3+b^3+c^3-3abc)^2. \qquad \text{[using (1)]}$$

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$$(a+b+c)^2(a+b\omega+c\omega^2)^2(a+b\omega^2+c\omega)^2$$
$$= -(A+B+C)(A+B\omega+C\omega^2)(A+B\omega^2+C\omega),$$

where ω is an imaginary cube root of unity.

Solution: Now $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$ (see Ex. 1 (i) of art. 2.7)

$$\therefore \qquad D = -(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega) \qquad ...(1)$$

$$\text{Adjoint of } D = D' = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix}$$

By Jacobi's theorem, $D' = D^2$

$$\therefore \qquad \begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix} = (a+b+c)^2(a+b\omega+c\omega^2)^2(a+b\omega^2+c\omega)^2 \qquad \text{[by (1)]}$$

$$\therefore \ (a+b+c)^2(a+b\omega+c\omega^2)^2(a+b\omega^2+c\omega)^2$$

$$= -(A+B+C)(A+B\omega+C\omega^2)(A+B\omega^2+C\omega) \qquad \text{[applying (1)]}$$

Example 3: Show that

$$\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}^2 = \begin{vmatrix} bc & -ca & ab \\ bc & ca & -ab \\ -bc & ca & ab \end{vmatrix} = \begin{vmatrix} c^2+a^2 & a^2 & c^2 \\ a^2 & a^2+b^2 & b^2 \\ c^2 & b^2 & b^2+c^2 \end{vmatrix} = 4a^2b^2c^2.$$

Solution: Let $$D = \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix} \qquad ...(1)$$

$$\therefore \qquad \text{Adjoint of } D = D' = \begin{vmatrix} \begin{vmatrix} b & 0 \\ b & c \end{vmatrix} & -\begin{vmatrix} a & 0 \\ 0 & c \end{vmatrix} & \begin{vmatrix} a & b \\ 0 & b \end{vmatrix} \\ -\begin{vmatrix} 0 & c \\ b & c \end{vmatrix} & \begin{vmatrix} a & c \\ 0 & c \end{vmatrix} & -\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \\ \begin{vmatrix} 0 & c \\ b & 0 \end{vmatrix} & -\begin{vmatrix} a & c \\ a & 0 \end{vmatrix} & \begin{vmatrix} a & 0 \\ a & b \end{vmatrix} \end{vmatrix} = \begin{vmatrix} bc & -ca & ab \\ bc & ca & -ab \\ -bc & ca & ab \end{vmatrix} \qquad ...(2)$$

Now $$\begin{vmatrix} bc & -ca & ab \\ bc & ca & -ab \\ -bc & ca & ab \end{vmatrix} = a^2b^2c^2 \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -2 \\ -1 & 0 & 2 \end{vmatrix} (C_2 + C_1 \to C_2, C_3 - C_1 \to C_3)$$

$$= 4a^2b^2c^2 \qquad \text{...(3)}$$

Also
$$D^2 = \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix} = \begin{vmatrix} a^2+c^2 & a^2 & c^2 \\ a^2 & a^2+b^2 & b^2 \\ c^2 & b^2 & b^2+c^2 \end{vmatrix} \qquad \text{...(4)}$$

By Jacobi's theorem $\qquad D' = D^2 \qquad$ ...(5)

From (1) – (5), we get

$$\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}^2 = \begin{vmatrix} bc & -ca & ab \\ bc & ca & -ab \\ -bc & ca & ab \end{vmatrix} = \begin{vmatrix} c^2+a^2 & a^2 & c^2 \\ a^2 & a^2+b^2 & b^2 \\ c^2 & b^2 & b^2+c^2 \end{vmatrix} = 4a^2b^2c^2.$$

Example 4: If $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and its adjugate $D' = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$, then prove that

(i) $\dfrac{BC-F^2}{a} = \dfrac{CA-G^2}{b} = \dfrac{AB-H^2}{h} = D$

(ii) $\dfrac{GH-AF}{f} = \dfrac{HF-BG}{g} = \dfrac{FG-CH}{h} = D$

where all the elements of D are non-zero and $D \neq 0$.

Solution: (i) Let us consider the product

$$\begin{vmatrix} 1 & 0 & 0 \\ H & B & F \\ G & F & C \end{vmatrix}\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} a & h & g \\ aH+hB+gF & hH+bB+fF & gH+fB+cF \\ aG+hF+gC & hG+bF+fC & gG+fF+cC \end{vmatrix}$$

$$= \begin{vmatrix} a & h & g \\ 0 & D & 0 \\ 0 & 0 & D \end{vmatrix} = aD^2. \qquad \text{...(1)}$$

or
$$(BC-F^2)\,D = aD^2 \text{ or } \frac{BC-F^2}{a} = D$$

Now
$$\begin{vmatrix} A & H & G \\ 0 & 1 & 0 \\ G & F & C \end{vmatrix}\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} aA+hH+gG & hA+bH+fG & gA+fH+cG \\ h & b & f \\ aG+hF+gC & hG+bF+fC & gG+fF+cC \end{vmatrix}$$

$$= \begin{vmatrix} D & 0 & 0 \\ h & b & f \\ 0 & 0 & D \end{vmatrix} = bD^2$$

or $$(AC-G^2)D = bD^2 \text{ or } \frac{AC-G^2}{b} = D \qquad \ldots(2)$$

Also $$\begin{vmatrix} A & H & G \\ H & B & F \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} aA+hH+gG & hA+bH+fG & gA+fH+cG \\ aH+hB+gF & hH+bB+fF & gH+fB+cF \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} D & 0 & 0 \\ 0 & D & 0 \\ g & f & c \end{vmatrix} = cD^2.$$

or $$(AB-H^2)D = cD^2 \text{ or } \frac{AB-H^2}{c} = D \qquad \ldots(3)$$

From (1)–(3), we get

$$\frac{BC-F^2}{a} = \frac{CA-G^2}{b} = \frac{AB-H^2}{c} = D.$$

(*ii*) Now $$\begin{vmatrix} 0 & 1 & 0 \\ H & B & F \\ G & F & C \end{vmatrix} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} h & b & f \\ aH+hB+gF & hH+bB+fF & gH+fB+cF \\ aG+hF+gC & hG+bF+fC & gG+fF+cC \end{vmatrix}$$

$$= \begin{vmatrix} h & b & f \\ 0 & D & 0 \\ 0 & 0 & D \end{vmatrix} = hD^2$$

or $$(FG-CH)D = hD^2 \text{ or } \frac{FG-CH}{h} = D \qquad \ldots(4)$$

Also $$\begin{vmatrix} A & H & G \\ 0 & 0 & 1 \\ G & F & C \end{vmatrix} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} aA+hH+gG & hA+bH+fG & gA+fH+cG \\ g & f & c \\ aG+hF+gC & hG+bF+fC & gG+fF+cC \end{vmatrix}$$

$$= \begin{vmatrix} D & 0 & 0 \\ g & f & c \\ 0 & 0 & D \end{vmatrix} = fD^2$$

or $$(GH-AF)D = fD^2 \text{ or } \frac{GH-AF}{f} = D \qquad \ldots(5)$$

Again $$\begin{vmatrix} A & H & G \\ H & B & F \\ 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} aA+hH+gG & hA+bH+fG & gA+fH+cG \\ aH+hB+gF & hH+bB+fF & gH+fB+cF \\ a & h & g \end{vmatrix}$$

$$= \begin{vmatrix} D & 0 & 0 \\ 0 & D & 0 \\ a & h & g \end{vmatrix} = gD^2$$

or $$(HF - BG)\, D = gD^2 \text{ or } \frac{HF - BG}{g} = D \qquad \ldots(6)$$

From (4) – (6), we get

$$\frac{GH - AF}{f} = \frac{HF - BG}{g} = \frac{FG - CH}{h} = D.$$

Example 5: Prove that

$$\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix} = \lambda^3(a^2+b^2+c^2+\lambda^2)^3.$$

Solution: Let $$D = \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$$

$\therefore$ Adjoint of $$D = D' = \begin{vmatrix} \begin{vmatrix} \lambda & a \\ -a & \lambda \end{vmatrix} & -\begin{vmatrix} -c & a \\ b & \lambda \end{vmatrix} & \begin{vmatrix} -c & \lambda \\ b & -a \end{vmatrix} \\ -\begin{vmatrix} c & -b \\ -a & \lambda \end{vmatrix} & \begin{vmatrix} \lambda & -b \\ b & \lambda \end{vmatrix} & -\begin{vmatrix} \lambda & c \\ b & -a \end{vmatrix} \\ \begin{vmatrix} c & -b \\ \lambda & a \end{vmatrix} & -\begin{vmatrix} \lambda & -b \\ -c & a \end{vmatrix} & \begin{vmatrix} \lambda & c \\ -c & \lambda \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} = D^2 \qquad \ldots(1)$$

(by Jacobi's Theorem)

Now $$D = \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda & a \\ -a & \lambda \end{vmatrix} - c \begin{vmatrix} -c & a \\ b & \lambda \end{vmatrix} - b \begin{vmatrix} -c & \lambda \\ b & -a \end{vmatrix}$$

$$= \lambda(\lambda^2 + a^2) + c(c\lambda + ab) - b(ac - b\lambda) = \lambda^3 + \lambda a^2 + \lambda c^2 + \lambda b^2$$
$$= \lambda(a^2 + b^2 + c^2 + \lambda) \qquad \ldots(2)$$

$$\therefore \begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix} = D^2 \cdot D \qquad \text{[by (1)]}$$

$$= D^3 = \lambda^3(a^2+b^2+c^2+\lambda^2)^3. \qquad \text{[by (2)]}$$

Example 6: If A_i, B_i, C_i be the respective cofactors of a_i, b_i, c_i $(i = 1, 2, 3)$ in

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & b_3 \end{vmatrix}$, then show that $\begin{vmatrix} B_1+C_1 & C_1+A_1 & A_1+B_1 \\ B_2+C_2 & C_2+A_2 & A_2+B_2 \\ B_3+C_3 & C_3+A_3 & A_3+B_3 \end{vmatrix} = 2\Delta^2.$ (*W.B.U.T. 2003*)

Solution: Applying property 5 of art. 2.3, we get

$$\begin{vmatrix} B_1+C_1 & C_1+A_1 & A_1+B_1 \\ B_2+C_2 & C_2+A_2 & A_2+B_2 \\ B_3+C_3 & C_3+A_3 & A_3+B_3 \end{vmatrix} = \begin{vmatrix} B_1 & C_1+A_1 & A_1+B_1 \\ B_2 & C_2+A_2 & A_2+B_2 \\ B_3 & C_3+A_3 & A_3+B_3 \end{vmatrix} + \begin{vmatrix} C_1 & C_1+A_1 & A_1+B_1 \\ C_2 & C_2+A_2 & A_2+B_2 \\ C_3 & C_3+A_3 & A_3+B_3 \end{vmatrix}$$

$$= \begin{vmatrix} B_1 & C_1+A_1 & A_1 \\ B_2 & C_2+A_2 & A_2 \\ B_3 & C_3+A_3 & A_3 \end{vmatrix} + \begin{vmatrix} C_1 & A_1 & A_1+B_1 \\ C_2 & A_2 & A_2+B_2 \\ C_3 & A_3 & A_3+B_3 \end{vmatrix} \qquad \begin{bmatrix} \text{in the 1st det } C_3 - C_1 \to C_3' \\ \text{in the 2nd det } C_2 - C_1 \to C_2' \end{bmatrix}$$

$$= \begin{vmatrix} B_1 & C_1 & A_1 \\ B_2 & C_2 & A_2 \\ B_3 & C_3 & A_3 \end{vmatrix} + \begin{vmatrix} C_1 & A_1 & B_1 \\ C_2 & A_2 & B_2 \\ C_3 & A_3 & B_3 \end{vmatrix} \qquad \begin{bmatrix} \text{in the 1st det } C_2 - C_3 \to C_2' \\ \text{in the 2nd det } C_3 - C_1 \to C_3' \end{bmatrix}$$

$$= -\begin{vmatrix} B_1 & A_1 & C_1 \\ B_2 & A_2 & C_2 \\ B_3 & A_3 & C_3 \end{vmatrix} - \begin{vmatrix} A_1 & C_1 & B_1 \\ A_2 & C_2 & B_2 \\ A_3 & C_3 & B_3 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} + \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \Delta^2 + \Delta^2 = 2\Delta^2.$$

(by Jacobi's Theorem).

Example 7: Find the adjoint and the reciprocal determinants of $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 4 & 2 \end{vmatrix}$.

Solution: Let $D = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 4 & 2 \end{vmatrix}$

The adjoint of D is
$$D' = \begin{vmatrix} \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 & -1 \\ -8 & 3 & -2 \\ 3 & -1 & 1 \end{vmatrix}$$

Now
$$D = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix} \quad (R_3 - R_1 \to R_3')$$

$$= \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1.$$

Dividing each element of D' by the value of D, *i.e.*, by 1, we get the reciprocal of

$$D = \begin{vmatrix} -2 & 1 & -1 \\ -8 & 3 & -2 \\ 3 & -1 & 1 \end{vmatrix}.$$

Example 8: Prove that every skew-symmetric determinant of odd order is zero.

(*W.B.U.T. 2005*)

Solution: Let us consider the following skew-symmetric determinant of order n, where n is an odd positive integer.

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} 0 & -a_{21} & \cdots & -a_{n1} \\ -a_{12} & 0 & \cdots & -a_{n2} \\ -a_{1n} & -a_{2n} & \cdots & 0 \end{vmatrix}$$

(since for a skew-symmetric determinant $a_{ij} = -a_{ji}$; $\therefore$ $a_{ii} = -a_{ii}$ or $2a_{ii} = 0$ or $a_{ii} = 0$)

$$= (-1)^n \begin{vmatrix} 0 & a_{21} & \cdots & a_{n1} \\ a_{12} & 0 & \cdots & a_{n2} \\ a_{1n} & a_{2n} & \cdots & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & 0 \end{vmatrix}$$

($\because$ n is odd and interchanging rows and columns)

$$= -D \qquad (\because a_{ii} = 0)$$

or $\quad 2D = 0 \quad \therefore D = 0.$

2.11 SINGULAR AND NON-SINGULAR MATRICES

A square matrix $A = (a_{ij})_{n \times n}$ is called a *singular matrix* if det $A = |a_{ij}|_{n \times n} = 0$.

If det $A \neq 0$ then A is said to be *non-singular* (or *regular*) matrix.

Example: The matrix $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$ is singular since det $A = \begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0.$

The matrix $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is non-singular because $\det I = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$.

2.12 ADJOINT OF A MATRIX

Let $A = (a_{ij})_{n\times n}$ be a square matrix and A_{ij} be the cofactor of a_{ij} in det A, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$.

Then the **adjoint** or *adjugate* of the square matrix A is denoted by adj (A) and defined as

$$adj\,(A) = (A_{ij})^T = (A_{ji})_{n\times n}.$$

Therefore adj (A) is the square matrix obtained from a square matrix by replacing its elements by the corresponding cofactors and then taking transpose of the resulting matrix.

Example: Let us consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

$$\therefore\ adj\,(A) = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 3 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \end{pmatrix}^T = \begin{pmatrix} 2 & 0 & -2 \\ -2 & -1 & 1 \\ -4 & 1 & 3 \end{pmatrix}^T = \begin{pmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{pmatrix}$$

Properties

1. $\text{adj}(A^T) = [\text{adj}(A)]^T$, where A is a square matrix.
2. If A be an nth order square matrix and c be any number, then $\text{adj}(cA) = c^{n-1}\text{adj}(A)$.
3. For an nth order square matrix A, $A\cdot\text{adj}(A) = \text{adj}(A)\cdot A = |A|\cdot I_n$, where $|A| = \det A$.

Proof: Let $A = (a_{ij})_{n\times n}$ and A_{ij} be the cofactor of a_{ij} in $|A|$, where $i, j = 1, 2, \ldots, n$.

$$\therefore\qquad A\cdot\text{adj}(A) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{r=1}^{n} a_{1r}A_{1r} & \sum_{r=1}^{n} a_{1r}A_{2r} & \cdots & \sum_{r=1}^{n} a_{1r}A_{nr} \\ \sum_{r=1}^{n} a_{2r}A_{1r} & \sum_{r=1}^{n} a_{2r}A_{2r} & \cdots & \sum_{r=1}^{n} a_{2r}A_{nr} \\ \hline \sum_{r=1}^{n} a_{nr}A_{1r} & \sum_{r=1}^{n} a_{nr}A_{2r} & \cdots & \sum_{r=1}^{n} a_{nr}A_{nr} \end{pmatrix}$$

$$= \begin{pmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \hline 0 & 0 & \cdots & |A| \end{pmatrix} \qquad \left[\because \sum_{r=1}^{n} a_{ir}A_{jr} = \begin{cases} |A|, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}\right]$$

$$\therefore \qquad A \cdot \operatorname{adj}(A) = |A| \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 1 \end{pmatrix} = |A| \cdot I_n$$

Similarly, it can be proved that $\operatorname{adj}(A) \cdot A = |A| \cdot I_n$.

Therefore, $A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = |A| \cdot I_n$.

4. For any nth order square matrix A, $|\operatorname{adj}(A)| = |A|^{n-1}$, if A is non-singular, *i.e.*, $|A| \neq 0$.

Proof: We have $A \cdot \operatorname{adj}(A) = |A| I_n = dI_n$, where $d = |A|$.

$\therefore$ $|A \cdot \operatorname{adj}(A)| = |dI_n| = d^n |I_n|$ (by property 4 of art. 2.3).

Using row-by-column multiplication of two determinants, we get

$$|A||\operatorname{adj}(A)| = d^n = |A|^n \qquad (\because |I_n| = \det I_n = 1)$$

$$\therefore \qquad |\operatorname{adj}(A)| = |A|^{n-1}. \qquad (\because |A| \neq 0).$$

2.13 INVERSE OF A MATRIX

An nth order square matrix A is said to be *invertible* if there exists a matrix B of same order such that $AB = BA = I_n$, where I_n is identity matrix of order n. Then B is called the **inverse** of A and is denoted by A^{-1}.

$$\therefore \qquad AA^{-1} = A^{-1}A = I_n.$$

Example: $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the inverse of itself, because $I \cdot I = I \cdot I = I$, therefore, $I^{-1} = I$.

Properties

1. An nth order square matrix is invertible if and only if it is non-singular.

Also $\quad A^T \cdot (A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$

and $\quad (A^{-1})^T \cdot A^T = (AA^{-1})^T = I_n^T = I_n$

$[\because (AB)^T = B^T A^T \text{ and } AA^{-1} = A^{-1}A = I_n, \text{by (1)}].$

$\therefore \quad A^T \cdot (A^{-1})^T = (A^{-1})^T \cdot A^T = I_n.$

Using the definition of inverse, we have $(A^T)^{-1} = (A^{-1})^T$.

4. If A and B are two invertible matrices of same order, then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: Let A, B are two nth order invertible matrices.

Therefore, $\quad |A| \neq 0, AA^{-1} = A^{-1}A = I_n$ $\quad$ ... (1)

and $\quad |B| \neq 0, BB^{-1} = B^{-1}B = I_n.$

Here, $|AB| = |A||B| \neq 0$, since $|A| \neq 0, |B| \neq 0$. $\quad$ ...(2)

Hence, AB is invertible, *i.e.*, $(AB)^{-1}$ exists.

Now, $\quad (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$

and $\quad (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$ $\quad$ [by (1)]

$\therefore \quad (AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I_n.$ $\quad$ [by (2)]

Using the definition of inverse, we have $(AB)^{-1} = B^{-1}A^{-1}$.

2.14 ORTHOGONAL MATRIX

A square matrix A of order n is said to be orthogonal if $AA^T = I$, the identity matrix of order n. (W.B.U.T. 2010, 2012)

Example: $\quad A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal, since

$$AA^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Properties

1. If A is an orthogonal matrix of order n, then
   (i) $|A| = \pm 1$, *i.e.*, A is non-singular
   (ii) $A^{-1} = A^T$, so that , $A^TA = I$. (W.B.U.T. 2003)

Proof: (i) Since A is an orthogonal matrix of order n, we have $AA^T = I$.

$\therefore \quad |AA^T| = |I| \quad$ or $\quad |A||A^T| = 1$

or $\quad |A|^2 = 1 \quad (\because |A^T| = |A|)$

or $\quad |A| = \pm 1$

Therefore, A is non-singular, since $|A| \neq 0$.

Proof: Step 1: Let A be an invertible matrix of order n. Therefore, there exists a matrix B of order n, such that $AB = BA = I_n$.

Now $|AB| = |I_n|$, or $|A||B| = 1$ (applying row-by-column multiplication of determinant)

$\therefore$ $|A| \neq 0$, *i.e.*, A is non-singular.

Step 2: Let A be *a* non-singular matrix of order n, *i.e.*, $|A| \neq 0$.

Now, we know that $A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = |A| I_n$

$$\therefore \quad A \cdot \frac{\text{adj}(A)}{|A|} = \frac{\text{adj}(A)}{|A|} \cdot A = I_n \quad (\because |A| \neq 0).$$

According to the definition of inverse, $\dfrac{\text{adj}(A)}{|A|}$ is the inverse of A. Therefore A is invertible.

Note: $A^{-1} = \dfrac{\text{adj}(A)}{|A|}$, provided $|A| \neq 0$.

Example: Let
$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\text{adj}(A) = \begin{pmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{pmatrix} \qquad \text{(see art. 2.12)}$$

Here $|A| = 2\times2 - 1\times0 + 3\times(-2) = -2 \neq 0$.

$$\therefore A^{-1} \text{ exists and } A^{-1} = \frac{\text{adj}(A)}{|A|} = \begin{pmatrix} -1 & 1 & 2 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}.$$

Verification: $\quad AA^{-1} = A^{-1}A = I.$

2. If A is invertible, then A^{-1} is also invertible and $(A^{-1})^{-1} = A$.

Proof: Let A be a square matrix of order n. Since A is invertible, therefore

$$AA^{-1} = A^{-1}A = I_n \qquad \text{...(1)}$$

$$|AA^{-1}| = |I_n| = 1 \text{ or } |A||A^{-1}| = 1.$$

Therefore, $|A^{-1}| \neq 0$ and hence A^{-1} is invertible. From (1) and using the definition of inverse, it follows that $(A^{-1})^{-1} = A$.

3. If A is an invertible matrix, then A^T is also and $(A^T)^{-1} = (A^{-1})^T$.

Proof: Since A is invertible, therefore $|A| \neq 0$ and $AA^{-1} = A^{-1}A = I_n$. ...(1)

Now, $|A^T| = |A| \neq 0$ ($\because$ the value of a determinant remains unchanged if rows and columns are interchanged).

Therefore A^T is invertible, *i.e.*, $(A^T)^{-1}$ exists.

Also $A^T \cdot (A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$

and $(A^{-1})^T \cdot A^T = (AA^{-1})^T = I_n^T = I_n$

$[\because (AB)^T = B^T A^T \text{ and } AA^{-1} = A^{-1}A = I_n, \text{by (1)}].$

$\therefore \quad A^T \cdot (A^{-1})^T = (A^{-1})^T \cdot A^T = I_n.$

Using the definition of inverse, we have $(A^T)^{-1} = (A^{-1})^T$.

4. If A and B are two invertible matrices of same order, then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: Let A, B are two nth order invertible matrices.

Therefore, $|A| \neq 0, AA^{-1} = A^{-1}A = I_n$ ... (1)

and $|B| \neq 0, BB^{-1} = B^{-1}B = I_n$. ...(2)

Here, $|AB| = |A||B| \neq 0$, since $|A| \neq 0, |B| \neq 0$.

Hence, AB is invertible, *i.e.*, $(AB)^{-1}$ exists.

Now, $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$

and $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$ [by (1)]

$\therefore \quad (AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I_n.$ [by (2)]

Using the definition of inverse, we have $(AB)^{-1} = B^{-1}A^{-1}$.

2.14 ORTHOGONAL MATRIX

A square matrix A of order n is said to be orthogonal if $AA^T = I$, the identity matrix of order n.

(W.B.U.T. 2010, 2012)

Example: $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal, since

$$AA^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Properties

1. If A is an orthogonal matrix of order n, then
   (*i*) $|A| = \pm 1$, *i.e.*, A is non-singular
   (*ii*) $A^{-1} = A^T$, so that, $A^TA = I$.

(W.B.U.T. 2003)

Proof: (*i*) Since A is an orthogonal matrix of order n, we have $AA^T = I$.

$\therefore \quad |AA^T| = |I|$ or $|A||A^T| = 1$

or $|A|^2 = 1 \quad (\because |A^T| = |A|)$

or $|A| = \pm 1$

Therefore, A is non-singular, since $|A| \neq 0$.

(ii) Since A is non-singular, therefore A^{-1} exists and $AA^{-1} = A^{-1}A = I$ ...(1)

Now, $$AA^T = I \therefore A^{-1}(AA^T) = A^{-1}I$$

or $$(A^{-1}A)A^T = A^{-1} \text{ or } IA^T = A^{-1}$$ [by (1)]

$\therefore$ $$A^{-1} = A^T$$

Therefore, $$A^TA = A^{-1}A = I.$$

Note: If A is an orthogonal matrix, then $AA^T = A^TA = I$.

2. If A is an orthogonal matrix, then A^T and A^{-1} are also orthogonal matrices.

Proof: Since A is an orthogonal matrix, therefore

$$AA^T = A^TA = I \quad ...(1)$$

$$A^{-1} = A^T \quad ...(2)$$

$\therefore$ $$A^T(A^T)^T = A^TA = I$$ [$\because (A^T)^T = A$ and by (1)]

Hence A^T is orthogonal.

Also, $$A^{-1}(A^{-1})^T = A^{-1}(A^T)^T$$ [by (2)]

$$= A^{-1}A = I, \text{ therefore } A^{-1} \text{ is orthogonal.}$$

3. If A and B are orthogonal matrices of same order, then AB is also orthogonal.

Proof: Since A and B are orthogonal matrices of same order, therefore,

$$AA^T = BB^T = I \quad ...(1)$$

Now, $$(AB)(AB)^T = (AB)(B^TA^T) = A(BB^T)A^T$$

$$= AIA^T$$ [by (1)]

$$= AA^T = I$$ [by (1)]

Hence AB is orthogonal.

2.15 TRACE OF A MATRIX

The sum of the diagonal elements of a square matrix $A = (a_{ij})_{n \times n}$ of order n is said to be its trace and is denoted by $Tr(A)$.

$\therefore$ $$Tr(A) = \sum_{i=1}^{n} a_{ii}$$

Using the definition, we easily get the following properties of trace.

Properties

If $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ are two square matrices of order n, then we have

(i) $$Tr(A+B) = \sum_{i=1}^{n}(a_{ii}+b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$$
$$= Tr(A) + Tr(B)$$

(ii) $$Tr(kA) = \sum_{i=1}^{n} k\,a_{ii} = k\sum_{i=1}^{n} a_{ii} = kTr(A), \text{ where } k \text{ is any number.}$$

(iii) $Tr(AB) = \sum_{i=1}^{n} [AB]_{ii} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij} b_{ji}\right)$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} b_{ji} a_{ij}\right) = \sum_{j=1}^{n}[BA]_{jj} = Tr(BA).$$

ILLUSTRATIVE EXAMPLES

Example 1: For a square matrix A of order n, prove that

$$\text{adj}(A^T) = [\text{adj}(A)]^T$$

If further A is a symmetric matrix, then show that adj (A) is also symmetric.

Solution: Let
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \hline a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

then
$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \hline a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

Now,
$$\text{adj }(A) = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \hline A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

and
$$\text{adj}(A^T) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \hline A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

$\therefore$ $$\text{adj}(A^T) = [\text{adj }(A)]^T \qquad ...(1)$$

If further A is symmetric, *i.e.*, $A = A^T$, then from (1), adj (A) = [adj $(A)]^T$, *i.e.*, adj (A) is a symmetric matrix.

Example 2: Prove that the inverse of a square matrix is unique.

Solution: Let A be a square matrix. If possible, let B and C are two inverse matrices of A.

$\therefore$ $$AB = BA = I \;(\because B \text{ is inverse of } A) \qquad ...(1)$$

and $$AC = CA = I \;(\because C \text{ is inverse of } A) \qquad ...(2)$$

Now, $$C(AB) = (CA)B \text{ (by associative law)}$$

or $$C \cdot I = I \cdot B \text{ [by (1) and (2)]}$$

or $$C = B.$$

Hence the inverse of a square matrix is unique.

Example 3: If A, B, C are square matrices of order n, such that $AB = I$ and $CA = I$, then prove that $B = C$.

Solution: Here $AB = I$, therefore $|AB| = |I|$ or $|A||B| = 1$.

$\therefore |B| \neq 0$ and hence B^{-1} exists.

Now $AB = I$ ...(1)

$CA = I$ ...(2)

Post-multiplying (1) by B^{-1} we get

$$(AB)B^{-1} = IB^{-1} \text{ or } A(BB^{-1}) = B^{-1} \text{ or } AI = B^{-1}$$

$\therefore A = B^{-1}$.

Replacing A by B^{-1} in (2), we get $CB^{-1} = I$.

Post-multiplying by B, we get $(CB^{-1})B = IB$ or $C(B^{-1}B) = B$ or $CI = B$. $\therefore C = B$.

Example 4: If A be non-singular and $AB = AC$, then show that $B = C$, where B and C are square matrices of same order.

Solution: Since A is non-singular, therefore, A^{-1} exists. It is given that $AB = AC$.

$$\therefore \quad A^{-1}(AB) = A^{-1}(AC)$$

$$\text{or} \quad (A^{-1}A)B = (A^{-1}A)C$$

$$\text{or} \quad IB = IC \quad (\because A^{-1}A = I)$$

$$\therefore \quad B = C.$$

Example 5: If a matrix A satisfies a relation $A^2 + 2A - I = O$, prove that A^{-1} exists and $A^{-1} = A + 2I$, I being an identity matrix.

Solution: Here $A^2 + 2A - I = O$ or $A^2 + 2A = I$ ...(1)

$$\text{or} \quad A(A + 2I) = I.$$

Now the determinant of the product of two square matrices is equal to the product (row-by-column wise) of their determinants.

This gives $|A(A+2I)| = |I|$ or $|A||A+2I| = 1$.

Therefore, $|A| \neq 0$ and hence A^{-1} exists. Pre-multiplying (1) by A^{-1}, we get

$$A^{-1}(A^2+2A) = A^{-1}I,$$

$$\text{or} \quad (A^{-1}A)A + 2A^{-1}A = A^{-1}$$

$$\text{or} \quad IA + 2I = A^{-1}$$

$$\therefore \quad A^{-1} = A + 2I.$$

Example 6: If $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, then find A^2 and show that $A^2 = A^{-1}$.

Solution: Here

$$A^2 = A \cdot A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2+1 & -1+1+0 & 1+0+0 \\ 2-2+0 & -2+1+0 & 2+0+0 \\ 1+0+0 & -1+0+0 & 1+0+0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{pmatrix}.$$

Now, $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0$, therefore A^{-1} exists.

$$A^3 = A \cdot A^2 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$\therefore \quad A^{-1}A^3 = A^{-1}I$ or $(A^{-1}A)A^2 = A^{-1}$ or $IA^2 = A^{-1}$.

$\therefore \quad A^{-1} = A^2.$

Example 7: If the product of two non-zero square matrices is a null matrix, then prove that both of them are singular matrices.

Solution: Let A and B are two non-zero square matrices each of order n and AB is a null matrix. Now suppose B is non-singular and hence B^{-1} exists. Post-multiplying both sides of $AB = O$ by B^{-1}, we get

$$(AB)\,B^{-1} = OB^{-1} \text{ or } A\,(BB^{-1}) = O$$

or $$AI = O \text{ or } A = O.$$

But A is a non-null matrix and hence B must be a singular matrix. Again, let A is a non-singular matrix and hence A^{-1} exists.

Pre-multiplying both sides of $AB = O$ by A^{-1}, we get

$$A^{-1}\,(AB) = A^{-1}O \text{ or } (A^{-1}A)\,B = O$$

or $$IB = O \text{ or } B = O$$

But B is a non-zero matrix and hence A must be a singular matrix.

Example 8: If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, then show that $A^2 - 4A - 5I = O$, where I, O are the identity matrix and the null matrix of order 3 respectively. Deduce that A is non-singular and find A^{-1}.

$$= \frac{1}{81}\begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Hence A is an orthogonal matrix.

Example 10: Given $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$, find C such that $BCA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Solution: $$BCA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} C \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

or $$C\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\therefore \quad C\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix} = -\begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = -\begin{pmatrix} 4 & -3 & 4 \\ -3 & 2 & -3 \end{pmatrix}$$

$$\therefore \quad C = -\begin{pmatrix} 4 & -3 & 4 \\ -3 & 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix}^{-1} \qquad \text{...(1)}$$

Now $$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = -1$$

$$\therefore \quad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix}^{-1} = -\begin{pmatrix} 1 & 0 & -2 \\ 2 & -1 & -1 \\ -3 & 1 & 2 \end{pmatrix}^{T} = -\begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -2 & -1 & 2 \end{pmatrix}$$

From (1),

$$C = \begin{pmatrix} 4 & -3 & 4 \\ -3 & 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -2 & -1 & 2 \end{pmatrix}$$

$$A^2 = A\cdot A = \begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix} = \begin{pmatrix}1+4+4&2+2+4&2+4+2\\2+2+4&4+1+4&4+2+2\\2+4+2&4+2+2&4+4+1\end{pmatrix} = \begin{pmatrix}9&8&8\\8&9&8\\8&8&9\end{pmatrix}$$

$$\therefore \quad A^2-4A-5I = \begin{pmatrix}9&8&8\\8&9&8\\8&8&9\end{pmatrix} - 4\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix} - 5\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$= \begin{pmatrix}9-4-5&8-8-0&8-8-0\\8-8-0&9-4-5&8-8-0\\8-8-0&8-8-0&9-4-5\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\therefore \quad A^2-4A-5I = O \text{ or } A(A-4I) = 5I$$

$$\therefore \quad |A(A-4I)| = 5|I| \text{ or } |A||A-4I| = 5$$

Thus $|A| \neq 0$ and hence A is non-singular. Pre-multiplying both sides of (1) by A^{-1}, we get ...(1)

$$A^{-1}A(A-4I) = 5A^{-1}I \text{ or } I(A-4I) = 5A^{-1}.$$

$$\therefore \quad A^{-1} = \frac{1}{5}(A-4I) = \frac{1}{5}\begin{pmatrix}-3&2&2\\2&-3&2\\2&2&-3\end{pmatrix}.$$

Example 9: Show that matrix $A = \frac{1}{9}\begin{pmatrix}-8&1&4\\4&4&7\\1&-8&4\end{pmatrix}$ is orthogonal.

Solution: We have

$$AA^T = \frac{1}{9}\begin{pmatrix}-8&1&4\\4&4&7\\1&-8&4\end{pmatrix}\frac{1}{9}\begin{pmatrix}-8&1&4\\4&4&7\\1&-8&4\end{pmatrix}^T$$

$$= \frac{1}{81}\begin{pmatrix}-8&1&4\\4&4&7\\1&-8&4\end{pmatrix}\begin{pmatrix}-8&4&1\\1&4&-8\\4&7&4\end{pmatrix}$$

$$= \frac{1}{81}\begin{pmatrix}64+1+16&-32+4+28&-8-8+16\\-32+4+28&16+16+49&4-32+28\\-8-8+16&4-32+28&1+64+16\end{pmatrix}$$

$$= \frac{1}{81}\begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Hence A is an orthogonal matrix.

Example 10: Given $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$, find C such that $BCA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Solution:
$$BCA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} C \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

or
$$C \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\therefore \quad C \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix} = -\begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = -\begin{pmatrix} 4 & -3 & 4 \\ -3 & 2 & -3 \end{pmatrix}$$

$$\therefore \quad C = -\begin{pmatrix} 4 & -3 & 4 \\ -3 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix}^{-1} \qquad \ldots(1)$$

Now
$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = -1$$

$$\therefore \quad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix}^{-1} = -\begin{pmatrix} 1 & 0 & -2 \\ 2 & -1 & -1 \\ -3 & 1 & 2 \end{pmatrix}^{T} = -\begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -2 & -1 & 2 \end{pmatrix}$$

From (1),
$$C = \begin{pmatrix} 4 & -3 & 4 \\ -3 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -2 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+0-8 & 8+3-4 & -12-3+8 \\ -3+0+6 & -6-2+3 & 9+2-6 \end{pmatrix} = \begin{pmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{pmatrix}.$$

Example 11: Let A and B are any two nth order square matrices and I is an nth order identity matrix. Prove that $AB - BA = I$ cannot hold under any circumstances.

Solution: If possible, let $AB - BA = I$.

Therefore, $\quad Tr(AB - BA) = Tr(I)$

or $\quad Tr(AB) - Tr(BA) = n \quad (\because Tr(I) = n)$

But $Tr(AB) = Tr(BA)$ and $n \neq 0$, hence this cannot hold. Therefore $AB - BA = I$ cannot hold.

MISCELLANEOUS EXAMPLES

Example 1: Show that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$.

Solution: Here $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -b(1+a^2+b^2) \\ 0 & 1+a^2+b^2 & a(1+a^2+b^2) \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} \quad (R_1 + bR_3 \to R_1', R_2 - aR_3 \to R_2')$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 2b & -2a & 1-a^2+b^2 \end{vmatrix} \quad (C_3 + bC_1 \to C_3')$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & a \\ -2a & 1-a^2+b^2 \end{vmatrix} = (1+a^2+b^2)^3.$$