Indian Institute of Engineering Science and Technology, Shibpur

Five Year Dual Degree (B. Tech.-M. Tech.) 3rd Semester Final Examination, 2017

Subject: Mathematics III (MA-301)

Full Marks: 70

Time: 3 hours

Use a separate answerscript for each half

1st Half (Full Marks - 40)

This half consists of questions worth 60 marks. Answer as much as you can or want to. The maximum you can score in this half is 40.

1. (a) A random variable X has probability density function (pdf) given by

$$f(x) = \begin{cases} kx^2 & 0 \le x < 6 \\ k(12-x)^2 & 6 \le x < 12 \\ 0 & \text{otherwise.} \end{cases}$$

Evaluate the constant k and find $P(X \le 9 | X \ge 6)$.

(b) Derive the normal equations for the simple linear regression model :

$$y_i = \alpha + \beta x_i + e_i, \quad i = 1, 2, \ldots, n.$$

$6 + 4 = 10$

2. (a) State and prove Bayes' Theorem.

(b) Suppose that $X \sim U(-\pi/2, \pi/2)$. Find the pdf of $Y = \tan X$.

$6 + 4 = 10$

3. (a) The probability that at least one of the events A and B occurs is 3/5. If A and B occur simultaneously with probability 1/5, then find $P(\bar{A}) + P(\bar{B})$, where $\bar{A}$, $\bar{B}$ are the complementary events of A, B respectively.

(b) If the random variable X has a Binomial distribution with parameters n and p, find its mean and variance.

$4 + 6 = 10$

0·607

13/32

4. (a) Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys and 1 girl and 3 boys. One child is selected at random from each group. Find the probability that the three selected children consist of 1 girl and 2 boys.

(b) State and prove Tchebycheff's inequality.

$5 + 5 = 10$

5. (a) The following is a random sample of size 10 from a uniform distribution on $(0, \theta)$, $\theta > 0$: $5.6, 3.9, 3.2, 4.8, 7.8, 6.1, 10.2, 8.4, 7.1, 9.8$. Find the Maximum Likelihood Estimator (MLE) of θ. Is the MLE an unbiased estimator for the parameter θ? Justify your answer.

(b) State the Weak Law of Large Numbers.

$8 + 2 = 10$

6. (a) Prove that the correlation coefficient between two random variables always lies between -1 and 1.

(b) Define Mean Square Error (MSE) of an estimator. Establish the bias-variance decomposition formula for MSE.

$5 + 5 = 10$

$s^2 - s$

$\frac{S}{S(S^2-2S-5)}$

$\frac{A}{S} + \frac{B}{(S^2-2S-5)}$

$AS^2 - 2SA - 5A - SB$

$A = 0$

$A+B=0$

$-5A = 1$

$\frac{S}{S^2(S^2-2S-5)}$

$\frac{A}{S^2} + \frac{B}{(S^2-2S-5)}$

$AS^2 - 2SA - 5A + SB$

SECOND HALF

ANSWER ANY ***THREE QUESTIONS*** IN THIS HALF

7. (a) Define Laplace transform of a function and find the Laplace transforms of the following functions:

   i) $f(t) = te^{-t}\sin(3t)$ ii) $f(t) = (t+3)^2 e^t$

   (b) If $L[f(t)] = F(s)$, then show that $L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}F(s)$. **7+3=10**

8. (a) State and prove convolution theorem for Laplace transform.

   $L(t^n) = \frac{n!}{s^{n+1}}$

   (b) Find the inverse Laplace transforms of the following functions:

   i) $F(s) = \dfrac{1}{(s^2+1)(s^2+9)}$ ii) $F(s) = \log\left(1+\dfrac{a^2}{s^2}\right)$ **4+6=10**

9. (a) Solve the differential equation using Laplace transform

$$\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} = 0, \text{ with } y(0)=0, y'(0)=0, y''(0)=1.$$

   (b) Show that the set of all feasible solutions of a linear programming problem is an convex set. **7+3=10**

10. (a) Let $x_1 = 1$, $x_2 = 1$, $x_3 = 1$ and $x_4 = 0$ be a feasible solution(F.S.) of the system

$$x_1 + 2x_2 + 4x_3 + x_4 = 7$$
$$2x_1 - x_2 + 3x_3 - 2x_4 = 4.$$

   Reduce the above F.S. into two different B.F.S.

   (b) Solve the following L.P.P. graphically:

   Minimize $Z = x_1 + x_2$
   subject to $5x_1 + 9x_2 \le 45$,
   $x_1 + x_2 \ge 2$,
   $x_1 \le 4$,
   $x_1, x_2 \ge 0$.

   6+4=10

$\frac{dE}{d\alpha} = ?$

11. Using Charne's Big M method, solve the following L.P.P. :

Minimize $Z = 4x_1 + x_2$

subject to

$$3x_1 + x_2 = 3,$$
$$4x_1 + 3x_2 \geq 6,$$
$$x_1 + 2x_2 \leq 4,$$
$$x_1, x_2 \geq 0.$$

10

OR

Apply two phase method to show that the following L.P.P. has unbounded solution:

Maximize $Z = 2x_1 - x_2 + 2x_3$

subject to

$$x_1 + x_2 - 3x_3 \leq 8,$$
$$4x_1 - x_2 + x_3 \geq 2,$$
$$2x_1 + 3x_2 - x_3 \geq 4,$$
$$x_1, x_2, x_3 \geq 0.$$

10

$\int_0^t e^{2t} \sin t$

$L(t^n) = \frac{n!}{s^{n+1}}$

$\frac{1}{(x-1)} \times \frac{1}{(x-1)^2} \times \frac{1}{(x-2)}$

$y = a + bx$

$\sum y_i \; a \; b x_i$

$\frac{dE}{da} = 0$

$-\sum y_i - a \quad -\sum x_i = 0$

$\frac{1}{(x-1)^2(x-2)}$

$= {}^nC_2 (p)^2 (q)^{n-2}$

$n(n-1) = 1$

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Five Year Integrated Dual Degree (B. Tech- M. Tech) Programme
3rd Semester Mid-Term Examination, 2017

$n^2 - n - 1 = 0$

Subject: Mathematics – III

Subject Code: MA-301

$\frac{1 \pm \sqrt{5}}{2}$

$\frac{3 \cdot 4}{2}$

Time: 2 hours **Full Marks: 100**

$n = \frac{1 \pm \sqrt{1+4}}{2}$ ${}^nC_r p^r q^{n-r}$

ANSWER ANY *FOUR* QUESTIONS

1. a) Each coefficient in the equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have real roots. $b^2 - 4ac \geq 0$

$b^2 \geq 4ac$

b) Two friends decided to meet at Central Library in between 2 PM to 3 PM on a given day with the condition that whoever arrives first will wait for 10 minutes or at the end of the hour. Find out the probability that they will not meet on that day.

$1 + 9$

c) If $B \subset A$, then show that $P(A \cap B^c) = P(A) - P(B)$ and $P(B) \leq P(A)$.

10+8+7=25

2. a) State and prove Baye's theorem.

b) In a precision bombing attack there is 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% or better chance of completely destroying the target?

$P(x = 99)$

$1 - P(x = 100)$

12+13=25

3. a) Prove that under certain conditions (to be stated by you), Binomial distribution approximates to the Poisson distribution.

2^{101}

b) 300000 candidates appear in a public examination. Let the marks obtained by the students' follows normal distribution with mean 55 with standard deviation10. How many students would you expect below 35 and above 85? What is the minimum score of best 2.5% candidates? [Given that $P(0 < Z < 1.96) = 0.475$, where $Z \sim N(0,1^2)$]

12+13=25

$n(n-1)(n-2)$

$1 = {}^nC_2 p^2 q^{n-2}$

$3(2)$

100×99

6×5 990

$9(8) =$

7×6

$8 \times 9 = 72$

4. a) If f(t) is a periodic function of period T then show that Laplace transform of f(t) is

$$\frac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}}.$$

b) Evaluate the Laplace transform of the following functions:

i) $f(t)=e^{-t}\int_0^t \frac{\sin t}{t}dt$ ii) $f(t)=e^{-2t}(4\cos 5t-2\sin 4t)$ iii) $f(t)=\frac{\cos\sqrt{t}}{\sqrt{t}}$

7+18=25

5. a) Find the inverse Laplace transform of the following functions:

i) $F(s)=\log\frac{s+1}{s-1}$ ii) $F(s)=\frac{s+3}{s^2-4s+13}$

b) Food X contains 6 units of vitamin A and 7 units of vitamin B per gram and costs 12 p./gm. Food Y contains 8 units of vitamin A and 12 units of vitamin B per gram and costs 20 p./gm. The daily requirements of vitamin A and B are at least 100 units and 120 units respectively. Formulate the above as a LPP to minimize the cost. Hence solve the LPP graphically.

12+13=25

6. a) Define convex set. Prove that the set of all feasible solutions of a LPP is a convex set. Check whether the following sets are convex or not.

(I) $S_1=\{(x_1,x_2):(x_1-3)^2+x_2^2\le 5\}$

(II) $S_2=\{(x_1,x_2):|x_1-5|\le 1,|x_2|\le 1\}$

(III) $S_3=\{(x_1,x_2):x_1^2+x_2^2=1,\ x_1\ge 0,\ x_2\ge 0\}$

b) Find the extreme points of the convex set of feasible solution of the LPP:

Minimize $Z=2x_1+3x_2+4x_3+5x_4$

Subject to $2x_1+3x_2+5x_3+6x_4=16$

$x_1+2x_2+2x_3+3x_4=9$

$x_1,x_2,x_3,x_4\ge 0$.

Hence find the minimum value of Z.

10+15=25