Basic Discrete Structures

Sets, Functions, Sequences, Matrices, and Relations

(Lecture – 3)

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Floor & Ceiling Functions

The *floor function* assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The *ceiling function* assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

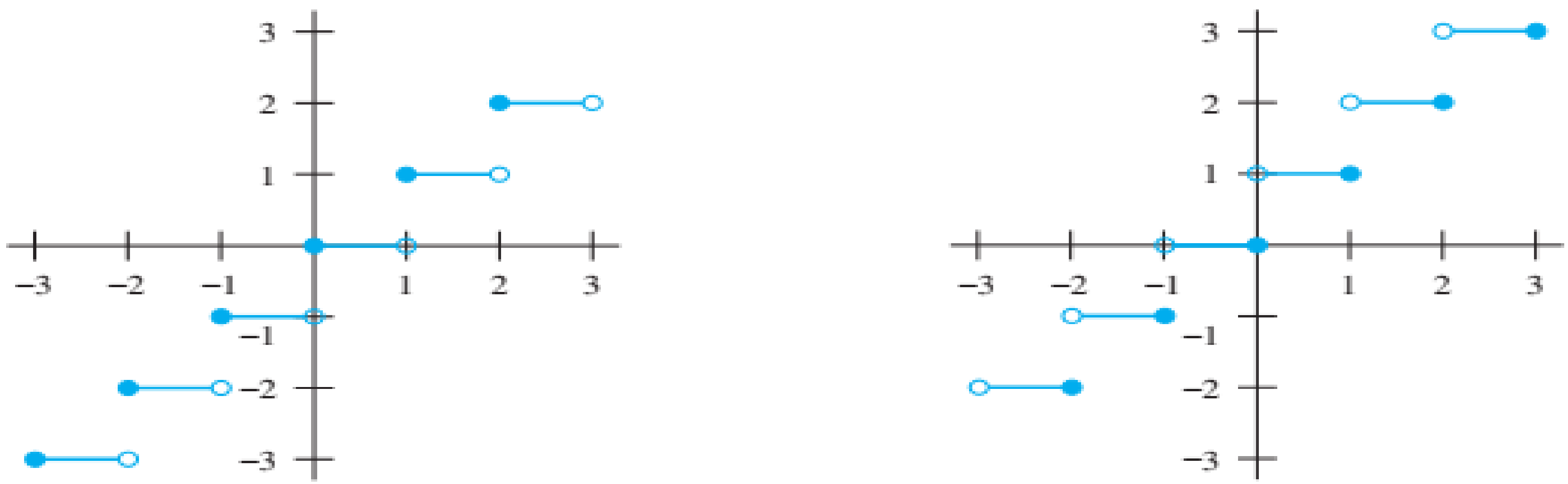
- Floor function: same value throughout the interval $[n, n+1)$, namely n, and then it jumps up to $n+1$ when $x = n+1$.
- Ceiling function: same value throughout the interval $(n, n+1]$, namely $n+1$, and then jumps to $n+2$ when x is a little larger than $n+1$.
- A useful approach for considering statements about the floor function is to let $x = n + \varepsilon$,where n is the integer, and ε is the fractional part of x, satisfies the inequality $0 \leq \varepsilon < 1$.
- Similarly, when considering statements about the ceiling function, it is useful to write $x = n - \varepsilon$, where n is an integer and $0 \leq \varepsilon < 1$.

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x+1$
(2)	$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

Floor & Ceiling Functions

- In Figure 10(a), the floor function is shown. Note that this function has the same value throughout the interval *[n, n + 1)*, namely n, and then it jumps up to *n + 1* when *x* = *n + 1*.
- In Figure 10(b), the graph of the ceiling function is shown. Note that this function has the same value throughout the interval *(n, n + 1]*, namely *n + 1*, and then jumps to *n + 2* when *x* is a little larger than *n + 1*.



(a) $y = \lfloor x \rfloor$ (b) $y = \lceil x \rceil$

FIGURE 10 **Graphs of the (a) Floor and (b) Ceiling Functions.**

$\lfloor \frac{1}{2} \rfloor = 0$, $\lceil \frac{1}{2} \rceil = 1$, $\lfloor -\frac{1}{2} \rfloor = -1$, $\lceil -\frac{1}{2} \rceil = 0$, $\lfloor 3.1 \rfloor = 3$, $\lceil 3.1 \rceil = 4$, $\lfloor 7 \rfloor = 7$, $\lceil 7 \rceil = 7$.

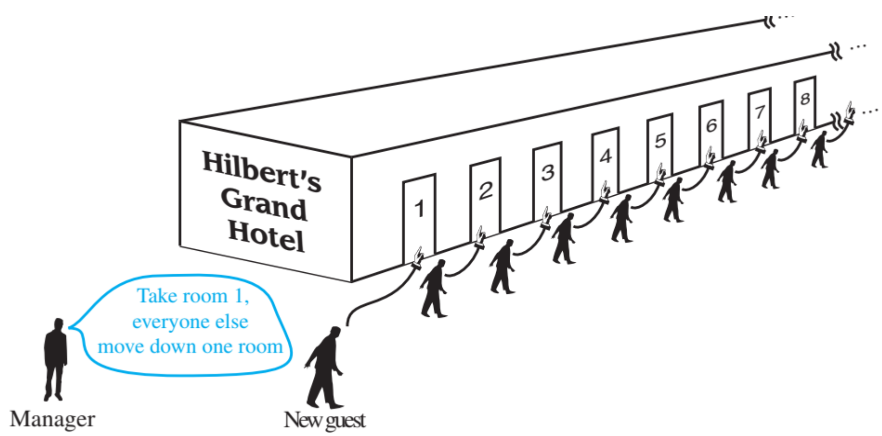
Cardinality of Sets

- **Recall:** The cardinality of a finite set is defined by the number of elements in the set.
- **Definition 1:** The sets A and B have **the same cardinality** if there is a one-to-one correspondence between elements in A and B. When A and B have the same cardinality, we say $|A| = |B|$.
  - In other words if there is a *bijection* from A to B.
  - Recall bijection is *one-to-one* and *onto*.
- **Definition 2:**

If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.

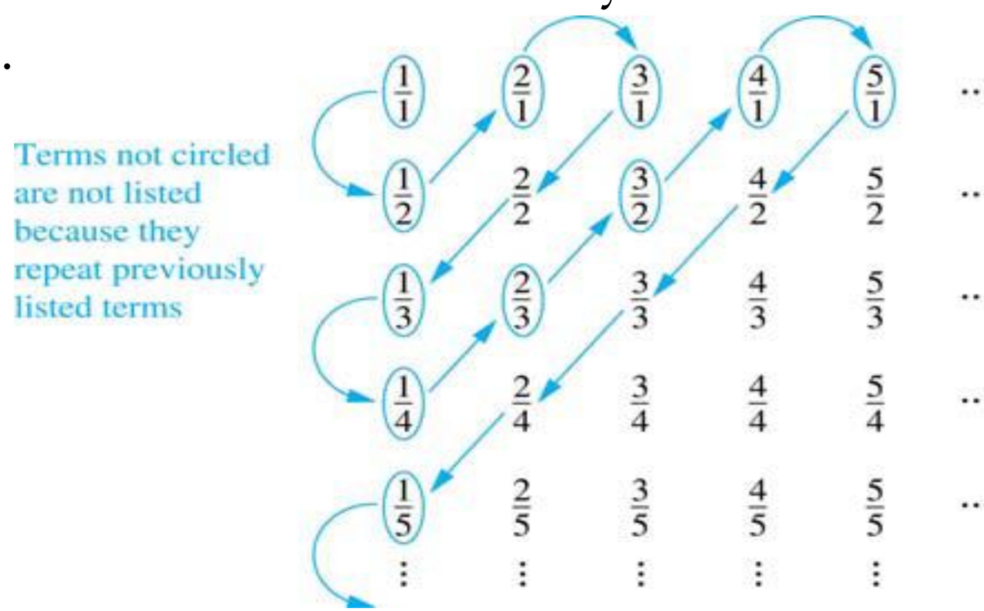
Cardinality of Sets

- **Definition:** A set that is either finite or has the same cardinality as the set of positive integers Z^+ is called **countable.** A set that is not countable is called **uncountable** or **infinite.**
  - An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
  - One-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$, where $a_1 = f(1), a_2 = f(2), \ldots, a_n = f(n), \ldots$.
  - <u>Hilbert's Paradox</u>: *something impossible with finite sets may be possible with infinite sets.*



Cardinality of Sets

- **Theorem:** The set of integers $\boldsymbol{Z}$ is countable.
- **<u>Definition</u>**: A *rational number* can be expressed as the ratio of two integers p and q such that $q \neq 0$.
  - ¾ is a rational number
  - $\sqrt{2}$ is not a rational number.
- **<u>Theorem</u>**: The positive rational numbers are countable.
- Proof: The positive rational numbers are countable since they can be arranged in a sequence: $r_1, r_2, r_3, .$
  - First row: q = 1
  - Second row: q = 2, etc.
  - <u>Constructing the list</u>:
    - First list p/q with $p + q = 2$.
    - Next list p/q with $p + q = 3$ and so on.



Cardinality of Sets

- **<u>Theorem</u>:** The set of real numbers is an uncountable set.
- **<u>Proof</u>**: We will be using proof by contradiction. Suppose that the real numbers are countable. Then every subset of the reals is countable, in particular, the interval [0,1] is countable. This implies the elements of this set can be listed say r_1, r_2, r_3, … where
  - $r_1 = 0.d_{11}d_{12}d_{13}d_{14} \ldots$
  - $r_2 = 0.d_{21}d_{22}d_{23}d_{24} \ldots$
  - $r_3 = 0.d_{31}d_{32}d_{33}d_{34} \ldots..$

  Where, the $d_{ij}\, \varepsilon\, \{0,1,2,3,4,5,6,7,8,9\}$.
  - Use *Cantor's diagonalization argument* to contradict the supposition!