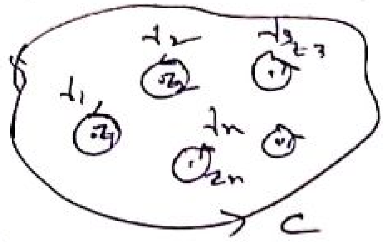
Cauchy's Residue theorem:

Statement: Let $f(z)$ be analytic within and on a positively oriented closed contour C apart from a finite number of singular points within C, Then

$$\oint_C f(z)\,dz = 2\pi i \,[\text{Sum of the residues of } f(z) \text{ at the singularities}]$$

Proof: Let $z_1, z_2, \ldots, z_n$ be the singularities of $f(z)$ within the positively oriented closed contour C. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the circles with centre at $z_1, z_2, \ldots z_n$ respectively such that these circles do not intersect and that none of them intersect C.



Then since $f(z)$ is analytic within and on the region bounded by $C, \gamma_1, \gamma_2, \ldots, \gamma_n$, by deformation of contours

$$\oint_C f(z)\,dz = \oint_{\gamma_1} f(z)\,dz + \cdots + \oint_{\gamma_n} f(z)\,dz \quad \text{—— (1)}$$

[see page no. 17]

Now since $f(z)$ has an isolated singularity at z_1, it can expanded in Laurent's series in the circle γ_1 in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_1)^n + \sum_{n=1}^{\infty} b_n (z-z_1)^{-n}$$

and the residue of $f(z)$ at z_1 is

$$b_1 = \frac{1}{2\pi i} \oint_{\gamma_1} f(z)\,dz \quad \text{—— (2)}$$

[See page no. 28].

(37) Thus $\oint_{\gamma_1} f(z)dz = 2\pi i \times$ residue of $f(z)$ at $z=z_1$

Similarly $\int_{\gamma_2} f(z)dz = 2\pi i \times$ residue of $f(z)$ at $z=z_2$

and so on.

Hence from (1) we get

$$\oint_c f(z)dz = 2\pi i \times (\text{sum of the residues at } z_1, z_2 \cdots z_n)$$
$$= 2\pi i \times (\text{sum of the residues at the singularities}).$$

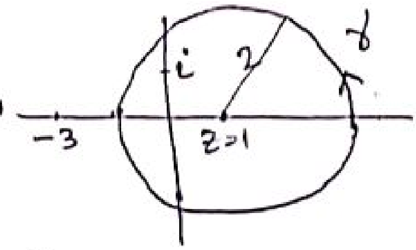
Ex 1 Evaluate the following integrals using the residue theorem:

(a) $\int_\gamma \frac{e^{-iz}}{(z+3)(z-i)^2} dz$, $\gamma = \{z : z = 1 + 2e^{i\theta}, \theta \text{ varies from } 0 \text{ to } 2\pi\}$

(b) $\int_\gamma \frac{z}{(z^2+1)(z-3)^2} dz$, $\gamma = \{z : |z| = 2\}$, positively oriented

(c) $\int_\gamma \frac{\sin 3z}{(z-\frac{\pi}{4})^4} dz$, $\gamma = \{(x,y), |x| \le 2, |y| \le 2\}$, positively oriented.

Soln (a) Here the contour is $z - 1 = 2e^{i\theta}$, θ varies from 0 to 2π, is a circle (positively oriented) with centre at $z=1$ and radius 2 and $z=i$ is within c.



Thus $f(z) = \frac{e^{-iz}}{(z+3)(z-i)^2}$ is analytic within and on c except at $z=i$ which is a pole type singularity of order 2.

Residue of $f(z)$ at $z=i$ is $b_1 = \frac{1}{(2-1)!} \frac{d}{dz}\left[(z-i)^2 f(z)\right]_{z=i}$

(See page no. 31) $= \frac{d}{dz}\left[\frac{e^{-iz}}{z+3}\right]_{z=i}$

(3a) $$= \left[\frac{(z+3)(-ie^{-iz}) - e^{-iz}}{(z+3)^2}\right]_{z=i}$$ (Derivative rule is as good as real analysis)

$$= \frac{(i+3)(-ie) - e}{(i+3)^2} = \frac{e[1-3i-1]}{(3+i)^2} = \frac{-3ie}{(3+i)^2}$$

$$= \frac{-3ie(3-i)^2}{[(3+i)(3-i)]^2} = \frac{-3ie(9-1-6i)}{(9+1)^2}$$

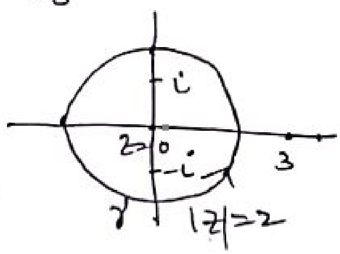
$$= \frac{-3ei(8-6i)}{100} = -\frac{3e(4i+3)}{50}$$

Hence $$\int_C \frac{e^{-iz}}{(z+3)(z-i)^2}\,dz = 2\pi i\,[\text{Residue of } f(z) \text{ at } z=i]$$

$$= 2\pi i\left[\frac{-3e(4i+3)}{50}\right]$$

$$= \frac{3\pi e(4-3i)}{25}$$

(b) Here ~~for~~ γ is a circle (positively oriented) with centre at $z=0$ and radius 2 ~~and $z=i$~~



$$f(z) = \frac{z}{(z^2+1)(z-3)^2}$$ possess three singularities at $z=i$, $z=-i$ (simple poles) and at $z=3$ (pole of order 2) out of which $z=i$ and $z=-i$ lies within γ.

Now residue of $f(z)$ at $z=i$ is

$$\lim_{z\to i}\left[(z-i)f(z)\right] = \lim_{z\to i}\left[\frac{z}{(z+i)(z-3)^2}\right] = \frac{i}{(2i)(i-3)^2}$$

$$= \frac{1}{2(i-3)^2}$$

Residue of $f(z)$ at $z=-i$ is

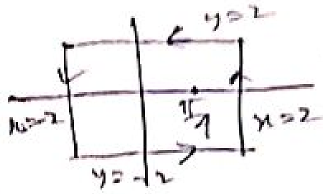
$$\lim_{z\to -i}\left[(z+i)f(z)\right] = \lim_{z\to -i}\left[\frac{z}{(z-i)(z-3)^2}\right] = \frac{-i}{-2i(-i-3)^2}$$

$$= \frac{1}{2(-i-3)^2}$$

Hence $$\int_C \frac{z}{(z^2+1)(z-3)^2}\,dz = 2\pi i\left[\frac{1}{2(i-3)^2} + \frac{1}{2(-i-3)^2}\right] = \pi i\left[\frac{2(9-1)}{[(-3)^2-(i)^2]^2}\right]$$

$$= \frac{4\pi i}{25}$$

(39)(c) $\displaystyle\int_\gamma \frac{\sin 3z}{(z-\frac{\pi}{4})^4}\,dz$



Here $f(z) = \dfrac{\sin 3z}{(z-\frac{\pi}{4})^4}$ is analytic within and on c except $z=\frac{\pi}{4}$ within c which is a pole of order 4.

$\therefore$ Residue of $f(z)$ at $z = \frac{\pi}{4}$

$$\lim_{z\to\frac{\pi}{4}} \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-\tfrac{\pi}{4})^4 f(z)\right\} = \lim_{z\to\frac{\pi}{4}}\left(\frac{1}{6}\frac{d^3}{dz^3}(\sin 3z)\right)$$

$$= \lim_{z\to\frac{\pi}{4}} \frac{1}{6}\cdot\left[-27\cos 3z\right] = -\frac{9}{2}\cos\frac{3\pi}{4} = -\frac{9}{2}\left(-\frac{1}{\sqrt{2}}\right)$$

Hence $$\int_\gamma \frac{\sin 3z}{(z-\frac{\pi}{4})^4}\,dz = 2\pi i\left(\frac{9}{2\sqrt{2}}\right) = \frac{9\pi i}{\sqrt{2}}$$

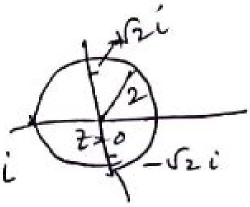
Ex2 Evaluate the following using Cauchy's residue theorem:

(i) $\displaystyle\int_C \frac{dz}{z^2+2}$ on circle $|z|=2$, positively oriented

(ii) $\displaystyle\int_C \frac{z}{(2z+1)(z-3)}\,dz$, $C = \{z : |z-1| = 5$, in the counter clockwise direction$\}$

(iii) $\displaystyle\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$, $C = \{z : |z| = 4\}$

Soln (i) $f(z) = \dfrac{1}{z^2+2} = \dfrac{1}{(z+\sqrt{2}i)(z-\sqrt{2}i)}$



has two simple poles $z=\sqrt{2}i$ & $z=-\sqrt{2}i$ within c. Except those two points $f(z)$ is analytic within and on c.

Now $$\operatorname{Res} f(z)\Big|_{z=\sqrt{2}i} = (z-\sqrt{2}i)f(z)\Big|_{z=\sqrt{2}i} = \frac{1}{z+\sqrt{2}i}\Big|_{z=\sqrt{2}i} = \frac{1}{2\sqrt{2}i}$$

$$\operatorname{Res} f(z)\Big|_{z=-\sqrt{2}i} = (z+\sqrt{2}i)f(z)\Big|_{z=-\sqrt{2}i} = \frac{1}{z-\sqrt{2}i}\Big|_{z=-\sqrt{2}i} = \frac{1}{-2\sqrt{2}i}$$

$$\therefore \int_C \frac{dz}{z^2+2} = 2\pi i\left[\frac{1}{2\sqrt{2}i} + \frac{1}{-2\sqrt{2}i}\right] = 0$$

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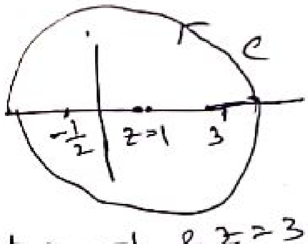
(ii) $\displaystyle\int_C \frac{z}{(2z+1)(z-3)}\,dz$



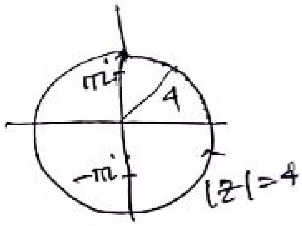
$f(z) = \dfrac{z}{(2z+1)(z-3)}$ is analytic within and on C except two point $z=-\frac{1}{2}$ & $z=3$ within C which are simple poles of $f(z)$.

Now $\displaystyle \operatorname{Res} f(z)\Big|_{z=-\frac{1}{2}} = (z+\tfrac{1}{2})f(z)\Big|_{z=-\frac{1}{2}} = \frac{z}{2(z-3)}\Big|_{z=-\frac{1}{2}} = \frac{-\frac{1}{2}}{2(-\frac{1}{2}-3)} = \frac{1}{14}$

$\displaystyle \operatorname{Res} f(z)\Big|_{z=3} = (z-3)f(z)\Big|_{z=3} = \frac{z}{(2z+1)}\Big|_{z=3} = \frac{3}{7}$

$\displaystyle \therefore \int_C \frac{z}{(2z+1)(z-3)}\,dz = 2\pi i\left[\frac{1}{14}+\frac{3}{7}\right] = \frac{2\pi i}{14}\times 7 = \pi i$

(iii) $\displaystyle\int_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$



$\displaystyle \frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{[(z+\pi i)(z-\pi i)]^2}$

$\displaystyle = \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}$

$\therefore z=\pi i$ and $z=-\pi i$ are two poles of order 2 of $f(z) = \dfrac{e^z}{(z^2+\pi^2)^2}$ which lies within C. Except those two poles, $f(z)$ is analytic within and on C.

Now $\displaystyle \operatorname{Res} f(z)\Big|_{z=\pi i} = \frac{d}{dz}\Big[(z-\pi i)^2 f(z)\Big]_{z=\pi i} = \frac{d}{dz}\left[\frac{e^z}{(z+\pi i)^2}\right]_{z=\pi i}$

$\displaystyle = \frac{(z+\pi i)^2 e^z - e^z\cdot 2(z+\pi i)}{(z+\pi i)^4}\Bigg|_{z=\pi i} = \frac{(z+\pi i)e^z - 2e^z}{(z+\pi i)^3}\Bigg|_{z=\pi i}$

$\displaystyle = \frac{2\pi i\, e^{\pi i} - 2e^{\pi i}}{(2\pi i)^3} = \frac{-2\pi i+2}{-8\pi^3 i}$, as $e^{\pi i} = \cos\pi + i\sin\pi = -1$

Also $\displaystyle \operatorname{Res} f(z)\Big|_{z=-\pi i} = \frac{d}{dz}\Big[(z+\pi i)^2 f(z)\Big]_{z=-\pi i} = \frac{d}{dz}\left[\frac{e^z}{(z-\pi i)^2}\right]_{z=-\pi i}$

$\displaystyle = \frac{(z-\pi i)^2 e^z - 2(z-\pi i)e^z}{(z-\pi i)^4}\Bigg|_{z=-\pi i} = \frac{(z-\pi i)e^z - 2e^z}{(z-\pi i)^3}\Bigg|_{z=-\pi i}$

$\displaystyle = \frac{-2\pi i\, e^{-\pi i} - 2e^{-\pi i}}{(-2\pi i)^3} = \frac{2\pi i+2}{8\pi^3 i}$

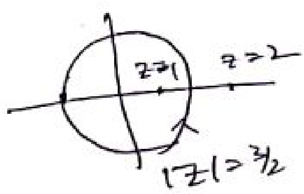
(4i) Using Cauchy's residue theorem we get

$$\int_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = 2\pi i\left[\frac{-2\pi i+2}{-8\pi^3 i} + \frac{2\pi i+2}{8\pi^3 i}\right]$$

$$= \frac{1}{4\pi^2}\left[\frac{-2\pi i+2}{-1} + \frac{2\pi i+2}{1}\right]$$

$$= \frac{1}{4\pi^2} \times 4\pi i = \frac{i}{\pi}$$

Ex3 Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz$, where C is the positively oriented circle $|z| = \frac{3}{2}$



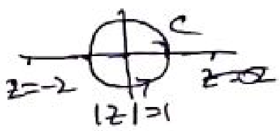
$f(z) = \frac{e^{2z}}{(z-1)(z-2)}$ is analytic within and on C except at the point $z=1$ within C.

$$\therefore \int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = 2\pi i\left[\operatorname{Res} f(z)\right]_{z=1}$$

$$= 2\pi i\left[(z-1)f(z)\right]_{z=1} = 2\pi i\left[\frac{e^z}{z-2}\right]_{z=1} = 2\pi i\,\frac{e^1}{1-2}$$

$$= -2\pi e i$$

Ex4 Evaluate $\oint_{C:|z|=1} \frac{dz}{z+2}$ and deduce that $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$



Soln $f(z) = \frac{1}{z+2}$ is analytic within and on C

$\therefore$ By Cauchy-Goursat theorem $\oint_C \frac{dz}{z+2} = 0$

Now on C, $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$, θ varies from $-\pi$ to π [or 0 to 2π]

$$\therefore \int_C \frac{dz}{z+2} = \int_{-\pi}^{\pi} \frac{ie^{i\theta}\,d\theta}{e^{i\theta}+2} = \int_{-\pi}^{\pi} \frac{i(\cos\theta + i\sin\theta)\,d\theta}{(\cos\theta+2)+i\sin\theta}$$

$-[\sin\theta\cos\theta + 2\sin\theta - \sin\theta\cos\theta]$

$$= \int_{-\pi}^{\pi} \frac{i(\cos\theta+i\sin\theta)[\cos\theta+2-i\sin\theta]\,d\theta}{(\cos\theta+2)^2+\sin^2\theta} = \int_{-\pi}^{\pi} \frac{i[\cos^2\theta+2\cos\theta+\sin^2\theta]}{5+4\cos\theta}\,d\theta$$

$$= \int_{-\pi}^{\pi} \frac{i(1+2\cos\theta)\,d\theta}{5+4\cos\theta} - \int_{-\pi}^{\pi} \frac{2\sin\theta}{5+4\cos\theta}\,d\theta = 2i\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta + 0$$

As $\int_C \frac{dz}{z+2} = 0$, Hence $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$ [1st integrand is even fn. & 2nd is odd fn.]

(42) Application of contour integration for evaluating definite integrals of the type $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$ or $\int_{-\pi}^{\pi} f(\sin\theta, \cos\theta)\, d\theta$.

Ex1 Evaluate $\int_0^{\pi} \frac{1}{(a+\cos\theta)^2}\, d\theta$, $a^2>1$.

Solution: $\int_0^{\pi} \frac{1}{(a+\cos\theta)^2}\, d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1}{(a+\cos\theta)^2}\, d\theta$, as the integrand is even function

We consider the unit circle $C: |z|=1$ or $z=e^{i\theta}, -\pi\le\theta\le\pi$

Then $dz = ie^{i\theta}d\theta$ or $d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$.

~~With this~~ Further $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$

With this transformation, the real integral transforms to contour integration

$$\int_0^{\pi} \frac{1}{(a+\cos\theta)^2}\, d\theta = \frac{1}{2}\oint_{C:|z|=1} \frac{1}{\left[a+\frac{1}{2}\left(\frac{z^2+1}{z}\right)\right]^2}\cdot\frac{dz}{iz} = \frac{1}{2i}\int_C \frac{4z^2\, dz}{z\left[z^2+1+2az\right]^2}$$

$$= \frac{2}{i}\int_C \frac{z\, dz}{\left[z^2+1+2az\right]^2}.$$

Now roots of $z^2+1+2az=0$ are $z = \frac{-2a\pm\sqrt{4a^2-4}}{2} = -a\pm\sqrt{a^2-1}$

i.e. $z_1 = -a+\sqrt{a^2-1}$ & $z_2 = -a-\sqrt{a^2-1}$

As $z_1z_2=1$ and $|z_2|>1$ (since $a^2>1$), we must have $|z_1|<1$ so that z_1 lies within C.

$\therefore$ $f(z) = \frac{z}{(z^2+1+2az)^2}$ has a pole of order 2 at $z=z_1$ and $f(z)$ is analytic everywhere within except that point and on C.

$$\therefore \int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2}{i}\times 2\pi i \times \operatorname{Res} f(z)\Big|_{z=z_1}.$$

$$\text{Now } \operatorname{Res} f(z)\Big|_{z=z_1} = \frac{d}{dz}\left((z-z_1)^2 f(z)\right)\Big|_{z=z_1} = \frac{d}{dz}\left[\frac{z}{(z-z_2)^2}\right]_{z=z_1}$$

$$= \frac{(z-z_2)^2 - 2z(z-z_2)}{(z-z_2)^4}\Bigg|_{z=z_1} = \frac{z_1-z_2-2z_1}{(z_1-z_2)^3} = \frac{-(z_1+z_2)}{(z_1-z_2)^3} = \frac{-(-2a)}{\left(2\sqrt{a^2-1}\right)^3}$$

(43) $\therefore \int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} \quad (a^2>1) = \frac{4\pi \times 2a}{8(a^2-1)^{3/2}} = \frac{\pi a}{(a^2-1)^{3/2}}$

Ex2 Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ by contour integration

Soln We make the transformation $z = e^{i\theta}$, ~~then~~ so that $dz = ie^{i\theta}d\theta$ or $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z+\frac{1}{z})$ and the real integral reduces to contour integration

$$\oint_C \frac{1}{2+\frac{1}{2}(z+\frac{1}{z})} \cdot \frac{dz}{iz}$$, where c is the unit circle $c: |z|=1$

Now $$\int_C \frac{dz}{2+\frac{1}{2}(z+\frac{1}{z})} \cdot \frac{dz}{iz} = \int_C \frac{dz}{\left(2+\frac{z^2+1}{2z}\right)iz} = \int_C \frac{2}{i} \cdot \frac{dz}{z^2+4z+1}$$

$$= \frac{2}{i}\int_C \frac{dz}{z^2+4z+1}.$$

Now $z^2+4z+1=0 \Rightarrow z = \frac{-4 \pm \sqrt{16-4}}{2}$

or $z = \frac{-4 \pm 2\sqrt{3}}{2} = -2+\sqrt{3}, -2-\sqrt{3}$
$= z_1, z_2$ say

Now $|z_2|>1$ and as $z_1 z_2 = 1$, $|z_1|$ must be less than 1.

$$\therefore \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\int_C \frac{dz}{(z-z_1)(z-z_2)},$$ where z_1 lies within C

and $f(z) = \frac{2}{i} \cdot \frac{1}{z^2+4z+1}$ is analytic within and on C except at the point $z = z_1$. Hence by Cauchy's residue theorem we get

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = 2\pi i \times \left[\operatorname{Res} f(z)\right]_{z=z_1} = 2\pi i\left[(z-z_1)f(z)\right]_{z=z_1}$$

$$= 2\pi i\left[\frac{2}{i} \cdot \frac{1}{z-z_2}\right]_{z=z_1} = 4\pi \times \frac{1}{z_1-z_2}$$

$$= 4\pi \times \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$