Relations - 2

Wednesday, October 7, 2020 8:44 AM

Prob-12

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Reflexive: Because all elements along the main diagonal are 1, so R is reflexive.

Symmetric: M_R is symmetric on two sides of the main diagonal. Hence, R is symmetric.

Antisymmetric: M_R is not antisymmetric because $m_{12} = 1$ and $m_{2,1} = 1$ | $m_{2,3} = 1$ and $m_{3,2} = 1$.

Prob-13

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Finding matrix for Composition of relations

- ~~R → A~~ $R: A \to B$, $S: B \times C$.
- A, B, and C have m, n, and p elements respectively.
- $M_R = [r_{i,k}]_{m \times n}$, $M_S = [s_{k,j}]_{n \times p}$
- $M_{S \circ R} = [t_{i,j}]_{m \times p}$
- The ordered pair $(a_i, c_j) \in S \circ R$ iff there is an element b_k such that $(a_i, b_k) \in R$ and $(b_k, c_j) \in S$.
- It follows that $t_{i,j} = 1$ iff $r_{i,k} = s_{k,j} = 1$ for some k.

- $M_{S \circ R} = M_R \odot M_S$ → boolean product (disjunction of conjunctions)

Prob-14.

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{S \circ R} = \begin{bmatrix} (1 \wedge 0) \vee (0 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) & (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) \\ - - - & - - - & - - - \\ - - - & - - - & - - - \end{bmatrix}$$

$$M_{SoR} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Prob-14

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$M_{RoR} = M_R \odot M_R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} m^{(2)}_{1,3} &= 1 \\ m^{(1)}_{1,3} &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} m^{(2)}_{3,1} &= 1 \\ m^{(1)}_{3,1} &= 0 \end{aligned}\right\}$$

R is not transitive.

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$$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Reflexive
Irreflexive
Symmetric
Anti-symmetric
Transitive

Reflexive – X
Irreflexive – X

$$M_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Irreflexive — ✗

Symmetric — ✓

Anti-symmetric — ✗ (because $m_{1,2} = 1$ and $m_{2,1} = 1$ which implies (1, 2) ∈ R and (2, 1) ∈ R but $1 \neq 2$)

Transitive — ✗ (perform boolean multiplication of matrices ~~and check if~~ $(a, b) \in R^2$ ~~then~~ $(a, b) \in R$.)

② $M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$M_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Reflexive — ✓

Irreflexive — ✗

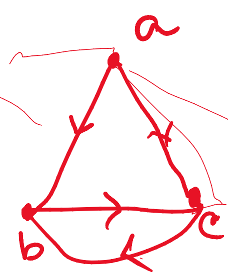
Symmetric — ✗ (matrix is not symmetric ~~along~~ about its main diagonal)

Antisymmetric — ✓

Transitive — ✗ (perform boolean multiplication and check)!

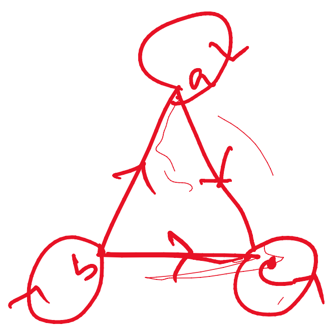
Digraphs

(1)



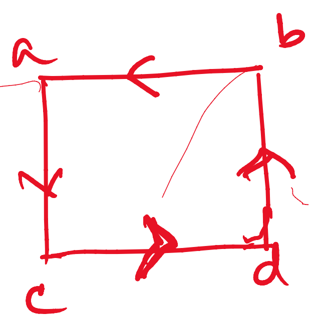
Reflexive – No
Irreflexive – Yes.
Symmetric – No
Antisymmetric – No
Transitive – No.
[(b, c) ∈ R, (c, b) ∈ R, (b, b) ∉ R]

(2)



Reflexive – Yes
Irreflexive – No.
Symmetric – No.
Antisymmetric – Yes
(no two different arrows b/w any pair of points)
Transitive – Yes

(3)



Reflexive – No
Irreflexive – Yes.
Symmetric – No.
Antisymmetric – Yes.
Transitive – No.

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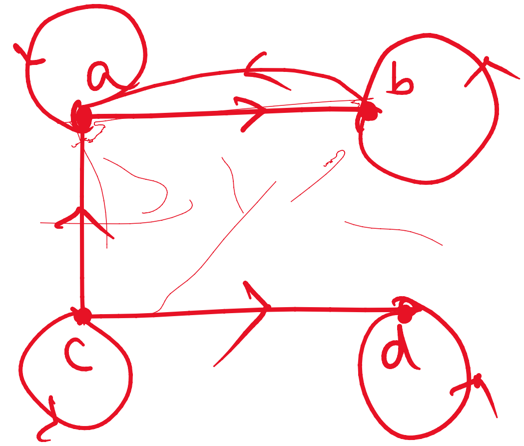

Reflexive – Yes.

Irreflexive – No.

Symmetric – No.

Antisymmetric – No.

Transitive – No

((c, a) ∈ R, (a, b) ∈ R
while (c, b) ∉ R)

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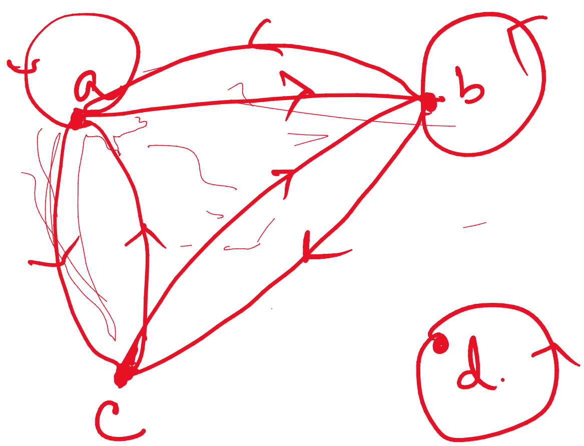

Reflexive – No

Irreflexive – No

Symmetric – Yes.

Anti-symmetric – No

Transitive – No.

((c, a) ∈ R, (a, c) ∈ R,
while (c, c) ∉ R)

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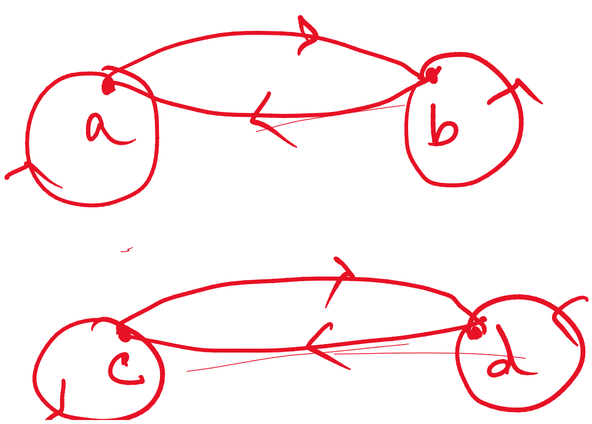

Reflexive – Yes.

Irreflexive – No

Symmetric – Yes.

Antisymmetric – No.

Antisymmetric — No.

Transitive — Yes.

Count the no. of reflexive, symmetric, anti-symmetric and transitive properties on a set with n elements.

→ A relation $R \subseteq A \times A$. No. of ordered pairs in $A \times A$ is n^2.

If R is reflexive, each of the ordered pairs of the form (a,a) $\forall a \in A$ must be in R. Each of the other (n^2-n) are of the form (a,b) where $a \neq b$ and they may or may not be in R. Hence, by product rule for counting, there are $2^{(n^2-n)}$ or $2^{n(n-1)}$ no. of reflexive relations.

A symmetric relation R consists of the elements along the main diagonal of a symmetric matrix and the elements in the upper and lower triangular matrices should be either present or either absent. Therefore, the no. of ordered pairs which may be present in a symmetric relation are $\left(\frac{n^2-n}{2}+n\right)$ i.e. $\left(\frac{n^2+n}{2}\right)$. Hence, by the product rule for counting, there are $2^{\left(\frac{n^2+n}{2}\right)}$ or $2^{\frac{n(n+1)}{2}}$ no. of

Counting, there are $2^{\left(\frac{n^2+n}{2}\right)}$ or $2^{\frac{n(n+1)}{2}}$ no. of Symmetric relations.

An anti-symmetric ~~rel~~ relation R consists of the elements along the main diagonal of the symmetric matrix. and the distinct pairs of a and b can occur in one of the three ways: (i) include (a, b); (ii) include (b, a); (iii) include no relation b/w a and b.

Therefore, the no. of anti-symmetric relations possible =

$$2^n \cdot 3^{\left(\frac{n^2-n}{2}\right)} = 2^n \cdot 3^{\frac{n(n-1)}{2}}$$