APPLICATIONS OF LAPLACE TRANSFORM TO PARTIAL DIFFERENTIAL EQUATIONS

The Laplace transform may be used in solving various partial differential equations subject to boundary conditions.

Let us first compute the Laplace transforms of 1st and 2nd order partial derivatives.

Problem: Let the function U(x, t) be defined for $a \leq x \leq b$, $t > 0$. Find

$$\text{(a) } L\left\{\frac{\partial \mathrm{U}}{\partial t}\right\}, \quad \text{(b) } L\left\{\frac{\partial \mathrm{U}}{\partial x}\right\}, \quad \text{(c) } L\left\{\frac{\partial^2 \mathrm{U}}{\partial t^2}\right\}, \quad \text{(d) } L\left\{\frac{\partial^2 \mathrm{U}}{\partial x^2}\right\},$$

assuming that U(x, t) satisfies the suitable restrictions of theorem 1, when regarded as a function of t.

Solution:

(a) $$L\left\{\frac{\partial \mathrm{U}}{\partial t}\right\} = \int_0^\infty \frac{\partial \mathrm{U}}{\partial t} \,.\, e^{-st}\, dt$$

$$= \lim_{P \to \infty} \int_0^P \frac{\partial \mathrm{U}}{\partial t} \,.\, e^{-st}\, dt$$

$$= \lim_{P \to \infty} \left\{[U(x,t).\, e^{-st}]_0^P + s \int_0^P U(x,t).\, e^{-st}\, dt\right\},$$

integrating by parts

$$= s \int_0^\infty U(x,t).\, e^{-st}\, dt - U(x,0)$$

$$= s\, u(x,s) - U(x,0)$$

$= s\, u - U(x,0)$, where $u = u(x,s) = L\{U(x,t)\}$.

(b) $L\left\{\frac{\partial \mathrm{U}}{\partial x}\right\} = \int_0^\infty \frac{\partial \mathrm{U}}{\partial x} \,.\, e^{-st}\, dt$

$= \frac{d}{dx}\int_0^\infty U.\, e^{-st}\, dt$, differentiating under the integral sign

$= \frac{du}{dx}$.

(c) $L\left\{\frac{\partial^2 \mathrm{U}}{\partial t^2}\right\} = L\left\{\frac{\partial \mathrm{V}}{\partial t}\right\}$, where $V = \frac{\partial \mathrm{U}}{\partial t}$

$= s\, L\{V\} - V(x,0)$, using (a)

$= s\, L\left\{\frac{\partial \mathrm{U}}{\partial t}\right\} - U_t(x,0)$

$=$ s $[s\, L\{U\} - U(x,0)] - U_t(x,0)$, using (a)

$= s^2\, u - s\, U(x,0) - U_t(x,0)$.

(d) $L\left\{\frac{\partial^2 \mathrm{U}}{\partial x^2}\right\} = \int_0^\infty \frac{\partial^2 \mathrm{U}}{\partial x^2} \,.\, e^{-st}\, dt$

$= \frac{d^2}{dx^2}\int_0^\infty U(x,t).\, e^{-st}\, dt$

$= \frac{d^2 u}{dx^2}$.

Problem: Solve: $\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}$, x > 0, t > 0,

under the conditions

(i) $U(x, 0) = 0$,

(ii) $U_t(x, 0) = 0$,

(iii) $U(0, t) = A_0 \sin wt$,

(iv) $|U(x, t)| < M$, that is, U(x, t) is bounded.

Solution: Taking Laplace transform of the given partial differential equation, we get

$$s^2 u - s\,U(x,0) - U_t(x,0) = c^2 \frac{d^2u}{dx^2},$$

where $u(x,s) = L\{U(x,t)\}$

or, $\frac{d^2u}{dx^2} - \frac{s^2}{c^2} u = 0$, using the conditions (i) , (ii).

The general solution of this differential equation is

$$u(x,s) = c_1 e^{\frac{s}{c}x} + c_2 e^{-\frac{s}{c}x}.$$

The condition (iv) implies that $u(x,s)$ is bounded.

$$\therefore c_1 = 0$$

$$\therefore u(x,s) = c_2 e^{-\frac{s}{c}x} \qquad (1)$$

By taking Laplace transform, we get from (iii),

$$u(0, s) = A_0 \frac{w}{s^2 + w^2} \left[\because L\{\sin wt\} = \frac{w}{s^2 + w^2}\right]$$

Using this condition we get from (1),

$$c_2 = A_0 \frac{w}{s^2 + w^2}.$$

$\therefore$ (1) becomes

$$u(x, s) = A_0 \frac{w}{s^2 + w^2} e^{-\frac{s}{c}x}$$

$$\therefore U(x, t) = L^{-1}\{u(x, s)\}$$

$$= L^{-1}\left\{A_0 \, e^{-\frac{x}{c}s} \frac{w}{s^2 + w^2}\right\}$$

$$\therefore U(x, t) = A_0 \sin\left[w\left(t - \frac{x}{c}\right)\right], \qquad t > \frac{x}{c}$$

$$= 0, \qquad t < \frac{x}{c}$$

$$\left[\because L^{-1}\{e^{-as} f(s)\} = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}, \text{where } f(s) = L\{F(t)\}\right].$$