(viii) $(-1)^{n-1} n!\left\{\dfrac{(n+2)(n+1)}{2(x-1)^{n+3}}+\dfrac{(n+1)}{(x-1)^{n+2}}+\dfrac{1}{(x-1)^{n+1}}-\dfrac{1}{(x-2)^{n+1}}\right\}$

(ix) $(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\theta=\cot^{-1}x$

(x) $(-1)^n(n-1)!\sin^n\theta\sin n\theta$, where $\theta=\cot^{-1}x$

(xi) $\dfrac{9}{2}(-1)^n n!(x-3)^{-(n+1)}+\dfrac{(-1)^n}{2}n!(x-1)^{-(n+1)}-4(-1)^n n!(x-2)^{-(n+1)}$.

(xii) $(-1)^n n!\left\{\dfrac{1}{(x-1)^{n+1}}-\dfrac{5}{(x-2)^{n+1}}+\dfrac{5}{(x-3)^{n+1}}\right\}$

3. (i) $e^{ax}a^{n-3}\{a^3x^3+3na^2x^2+3n(n-1)ax+n(n-1)(n-2)\}$

(ii) $x^3\sin\left(\dfrac{n\pi}{2}+x\right)+3n\,x^2\sin\left\{\dfrac{1}{2}(n-1)\pi+x\right\}$

$$+3n(n-1)x\sin\left\{\frac{1}{2}(n-2)\pi+x\right\}+n(n-1)(n-2)\sin\left\{\frac{1}{2}(n-3)\pi+x\right\}$$

(iii) $e^{ax}\left\{a^n\cos bx+{}^nC_1\,a^{n-1}b\cos\left(bx+\dfrac{\pi}{2}\right)\right.$

$$\left.+{}^nC_2\,a^{n-2}b^2\cos\left(bx+2\cdot\frac{\pi}{2}\right)+\ldots+b^n\cos\left(bx+n\cdot\frac{\pi}{2}\right)\right\}$$

(iv) $(-1)^{n-1}(n-3)!\sin^{n-2}\theta\{(n-1)(n-2)\sin n\theta\cos^2\theta$

$$-2n(n-2)\sin(n-1)\theta\cos\theta+n(n-1)\sin(n-2)\theta\},\ \text{where } \cot\theta=x$$

(v) $n!\{(1-x)^n-({}^nC_1)^2(1-x)^{n-1}x+({}^nC_2)^2(1-x)^{n-2}x^2-\ldots+(-1)^n x^n\}$

(vi) $2^{\frac{1}{2}(n-2)}e^x\left[2x^2\cos\left(x+\dfrac{n\pi}{4}\right)+2^{3/2}nx\cos\left\{x+(n-1)\dfrac{\pi}{4}\right\}+n(n-1)\cos\left\{x+(n-2)\dfrac{\pi}{4}\right\}\right]$.

15. $(y_n)_0=m(m^2+1^2)(m^2+3^2)\ldots\{m^2+(n-2)^2\}$, when n is odd

$=m^2(m^2+2^2)(m^2+4^2)\ldots\{m^2+(n-2)^2\}$, when n is even.

16. $(y_n)_0=0$, if n is even

$=(-1)^{\frac{n-1}{2}}1^2.3^2.5^2\ldots(n-2)^2$ if n is odd.

17. $(y_n)_0=(-1)^n\{1.3.5.\ldots(2n-1)\}$.

20. $(y_n)_0=\begin{cases}0, & \text{if } n \text{ is even}\\ 2(1^2-4)(3^2-4)\ldots\{(n-2)^2-4\}, & \text{if } n \text{ is odd.}\end{cases}$

CHAPTER 6

ω

Mean Value Theorems and Expansions of Functions

6.1 INTRODUCTION

Calculus is one of the most beautiful intellectual achievements of Mathematicians and it deals with the mathematical study of change, motion, growth or decay etc. One of the most important ideas of differential calculus is derivative which measures the rate of change of a given function and concept of derivative is very useful in engineering, science, economics, medicine and computer science. There are many real valued functions of a real variable which are continuous in a finite and closed interval $[a, b]$ and also derivable on the open interval (a, b). Such functions possess some interesting and very useful properties and these properties are formulated in the form of theorems known as mean value theorems. In this chapter we shall deal with Rolle's theorem, Lagrange's mean value theorem which connects the average rate of change of a function over an interval with the instantaneous rate of change of the function at a point within that interval, Taylor's theorem (generalized mean value theorem) which enables us to express any differentiable function in power series, namely Taylor's and Maclaurin's series.

6.2 ROLLE'S THEOREM

Let f be a function defined on a finite closed interval $[a, b]$ satisfying the following conditions:

(i) $f(x)$ is continuous for all x in $a \le x \le b$,

(ii) $f(x)$ is derivable for all x in $a < x < b$ and

(iii) $f(a) = f(b)$.

Then there exists at least one value of x, say c, $a < c < b$, such that $f'(c) = 0$.

Notes: 1. Here we have assumed continuity in a closed interval, we have assumed derivability only in the open interval, *i.e.*, to say for the conclusion to be valid, we do not need derivability at the end points a, b.

2. If $f(x)$ be constant in $a \le x \le b$, then $f(a) = f(b)$ and $f'(x) = 0$ at every point in $a < x < b$.

3. If a, b are two roots of the equation $f(x) = 0$, *i.e.*, $f(a) = f(b) = 0$, then the equation $f'(x) = 0$ will have at least one root between a and b, provided

(i) $f(x)$ is continuous in $a \le x \le b$ and

(ii) $f'(x)$ exists in $a < x < b$.

If $f(x)$ be a polynomial in x, the conditions (i) and (ii) are obviously satisfied.

Geometrical Interpretation of Rolle's Theorem

If the graph of $y = f(x)$ has the ordinates at two points $A \equiv (a, f(a)), B \equiv (b, f(b))$ equal, *i.e.*, $f(a) = f(b)$, and if the graph be continuous (*i.e.*, without break) throughout the interval from A to B and if the curve has a unique tangent at each point on it from A to B except possibly at the two end points A and B, then there must exist at least one point on the arc AB, where the tangent is parallel to the x-axis.

Compare the figures 6.1 and 6.2

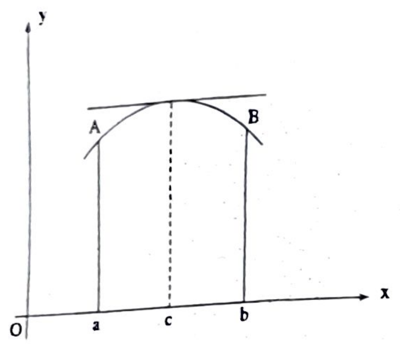


Fig. 6.1

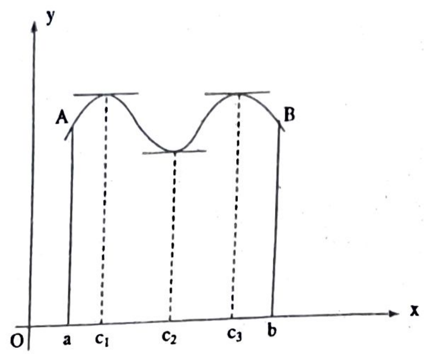


Fig. 6.2

ILLUSTRATIVE EXAMPLES

Example 1: Verify Rolle's theorem in each of the following functions:

(BESUS 2013, W.B.U.T. 2003, 2012)

(*i*) $f(x) = |x|$ in $-1 \le x \le 1$

(*ii*) $f(x) = \cos^2 x$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

(*iii*) $f(x) = 4x^3 + x^2 - 4x - 1$ in $[-1, 1]$

(*iv*) $f(x) = \sin x$ in $[0, \pi]$

(*v*) $f(x) = \frac{1}{x} + \frac{1}{1-x}$ in $[0, 1]$.

Solution: (*i*) Here $f(x) = |x|, -1 \le x \le 1$

$$f(x) = -x, \text{ for } -1 \le x \le 0$$
$$= x, \text{ for } 0 < x \le 1.$$

Obviously $f(x)$ is continuous for all x in $-1 \le x \le 1$ except possibly at $x = 0$.

Now, $$f(0+0) = \lim_{x \to 0+} f(x) = \lim_{x \to 0+} x \qquad [\because x \to 0+ \Rightarrow x > 0]$$
$$= 0$$

$$f(0-0) = \lim_{x \to 0-} f(x) = \lim_{x \to 0-} (-x) \qquad [\because x \to 0- \Rightarrow x < 0]$$
$$= 0$$

Also, $f(0) = 0$. Therefore, $f(0 + 0) = f(0 - 0) = f(0)$ and hence $f(x)$ is continuous at $x = 0$.

Thus, $f(x)$ is continuous for all x in $-1 \le x \le 1$.

Here $f(x)$ is derivable for all x in $-1 < x < 1$ except possibly at $x = 0$.

Now, $$Rf'(0) = \lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0+} \frac{h-0}{h} \qquad [\because h \to 0+ \Rightarrow h > 0]$$
$$= 1$$
$$Lf'(0) = \lim_{h \to 0-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0-} \frac{-h-0}{h} \qquad [\because h \to 0- \Rightarrow h < 0]$$
$$= -1.$$

Therefore, $Rf'(0) \ne Lf'(0)$ and hence $f(x)$ is not derivable in $-1 < x < 1$.

Therefore, we conclude that Rolle's theorem is not applicable to the function $f(x) = |x|$ in $-1 \le x \le 1$.

(*ii*) Here (*a*) $f(x)$ is continuous in $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$.

(*b*) $f'(x) = -\sin 2x$ exists in $-\frac{\pi}{4} < x < \frac{\pi}{4}$.

(*c*) $f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \frac{1}{2}$.

All the conditions of Rolle's theorem are satisfied and as such there exists a c, where $f'(c) = -\sin 2c = 0$, namely $c = 0$, $-\frac{\pi}{4} < 0 < \frac{\pi}{4}$.

Hence Rolle's theorem is verified.

(*iii*) Since every polynomial in x is continuous and derivable for all real values of x, so

(*a*) $f(x)$ is continuous in $-1 \le x \le 1$.

(*b*) $f'(x) = 12x^2 + 2x - 4$ in $-1 < x < 1$.

Also (*c*) $f(-1) = f(1) = 0$.

Thus $f(x)$ satisfies all the conditions of Rolle's theorem. By Rolle's theorem, we have

$$f'(c) = 0, \text{ where } -1 < c < 1.$$

Here $f'(c) = 0$ gives $12c^2 + 2c - 4 = 0$, or, $(3c + 2)(2c - 1) = 0$ whose two solutions are $c = -\frac{2}{3}, \frac{1}{2}$ and $-1 < -\frac{2}{3} < 1$ as well as $-1 < \frac{1}{2} < 1$.

Hence Rolle's theorem is verified for the given function.

re (a) $f(x)$ is continuous in $0 \le x \le \pi$.

(b) $f'(x) = \cos x$ exists in $0 < x < \pi$.

(c) $f(0) = f(\pi) = 0$.

e conditions of Rolle's theorem are satisfied and as such there exists a c, where

$c = 0$, namely $c = \frac{\pi}{2}$, $0 < \frac{\pi}{2} < \pi$.

e Rolle's theorem is verified.

Here (a) $f(x)$ is continuous in $0 < x < 1$ (not in $0 \le x \le 1$)

(b) $f'(x) = \dfrac{1}{(1-x)^2} - \dfrac{1}{x^2}$ exists in $0 < x < 1$

(c) $f(0) \ne f(1)$, both are undefined

hus the conditions of Rolle's theorem do not hold good. But get there exists a c, where $f'(c) =$

ely $c = \frac{1}{2}$, where $0 < \frac{1}{2} < 1$.

Note: The above examples lead to the following conclusion:

If $f(x)$ satisfies all the conditions of Rolle's theorem in $[a, b]$ then the result $f'(c) = 0$, where $c < b$ is assured, but if any of the conditions is not satisfied then Rolle's theorem will not be ..., it may still be true but the truth is not ensured.

Example 2: Show that Rolle's theorem is not applicable to $f(x) = \tan x$ in $[0, \pi]$, although $f(0) = f(\pi)$.

(W.B.U.T. 2004, 2006, 2011)

Solution: The function $f(x) = \tan x$ is not continuous everywhere in $[0, \pi]$ since $\tan x \to \infty$ as $\to \frac{\pi}{2}$ - and $\tan \frac{\pi}{2}$ is undefined, where $0 < \frac{\pi}{2} < \pi$.

Thus the conditions of Rolle's theorem do not hold and hence Rolle's theorem is not applicable to he function $f(x) = \tan x$ in $[0, \pi]$, although $f(0) = f(\pi) = 0$.

Example 3: If $f(x) = (x-a)^m (x-b)^n$ where m, n are positive integers, show that c in Rolle's theorem divides the segment $a \le x \le b$ in the ratio $m : n$.

Solution: Since every polynomial in x is continuous and derivable for all real values of x, so

(i) $f(x) = (x-a)^m (x-b)^n$ is continuous in $a \le x \le b$,

(ii) $f'(x) = (x-a)^{m-1}(x-b)^{n-1}\{m(x-b)+n(x-a)\}$ exists in $a < x < b$.

Also (iii) $f(a) = f(b) = 0$.

Therefore by Rolle's theorem, there exists c, $a < c < b$, such that $f'(c) = (c-a)^{m-1}(c-b)^{n-1}\{m(c-b)+n(c-a)\} = 0$.

Hence $m(c-b)+n(c-a) = 0$ or $c = \dfrac{mb+na}{m+n}$, which divides the segment $a \le x \le b$ in the ratio $m : n$.

Example 4. (a) Verify Rolle's theorem in the following cases:

(i) $$f(x) = |\cos(x)| \text{ in the interval } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

[BESUS (B.Arch.) 2013, W.B.U.T. (B.Arch.) 2013]

(ii) $$f(x) = |\sin(x)| \text{ in the interval } [-\pi, \pi]$$

[BESUS (B.Arch.) 2013]

(b) Let $$f(x) = \begin{vmatrix} \sin(x) & \sin(\alpha) & \sin(\beta) \\ \cos(x) & \cos(\alpha) & \cos(\beta) \\ \tan(x) & \tan(\alpha) & \tan(\beta) \end{vmatrix}, \quad 0 < \alpha < \beta < \frac{\pi}{2}.$$

then show that there exists $c(\alpha < c < \beta)$ such that $f'(c) = 0$.

[BESUS (B. Arch.)2013]

Solution: (a) (i) $f(x) = |\cos(x)| = \cos(x)$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Obviously $f(x) = |\cos(x)|$ is continuous and derivable for all x in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so

(I) $f(x)$ is continuous in $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$,

(II) $f'(x) = -\sin x$ in $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Also (III) $f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = 0$.

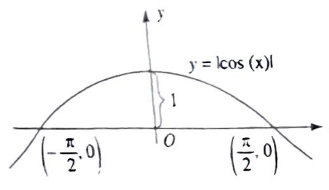


Fig. 6.3

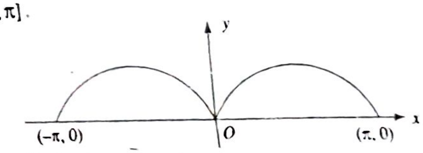
Thus $f(x)$ satisfies all the conditions of Rolle's theorem and so there exists $c = 0\left(-\frac{\pi}{2} < 0 < \frac{\pi}{2}\right)$ such that $f'(0) = -\sin 0 = 0$.

Hence Rolle's theorem is verified for the given function.

(ii) Here $f(x) = |\sin(x)|$ in $[-\pi, \pi]$

$$\therefore \quad f(x) = \begin{cases} -\sin x, & \text{for } -\pi \le x \le 0 \\ \sin x, & \text{for } 0 < x \le \pi \end{cases}$$

Obviously $f(x)$ is continuous for all x in $[-\pi, \pi]$.



Graph of $y = |\sin x|$

Fig. 6.4

Now, $$Rf'(0) = \lim_{h \to 0+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h \to 0+} \frac{\sin h - 0}{h} = 1$$

and $$Lf'(0) = \lim_{h \to 0-} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h \to 0-} \frac{-\sin h - 0}{h} = -1$$

Therefore, $Rf'(0) \neq Lf'(0)$ and hence $f(x)$ is not derivable at $x = 0$. Thus $f(x)$ is not derivable for all x in $-\pi < x < \pi$.

So, we conclude that Rolle's theorem is not applicable to the function $f(x) = |\sin(x)|$ in $[-\pi, \pi]$

(*b*) Obiously $f(x)$ is continuous and derivable for all x in $0 < \alpha < \beta < \frac{\pi}{2}$.

Also, $$f(\alpha) = \begin{vmatrix} \sin(\alpha) & \sin(\alpha) & \sin(\beta) \\ \cos(\alpha) & \cos(\alpha) & \cos(\beta) \\ \tan(\alpha) & \tan(\alpha) & \tan(\beta) \end{vmatrix}$$

$$= \begin{vmatrix} \sin(\beta) & \sin(\alpha) & \sin(\beta) \\ \cos(\beta) & \cos(\alpha) & \cos(\beta) \\ \tan(\beta) & \tan(\alpha) & \tan(\beta) \end{vmatrix}$$

$$= f(\beta) = 0.$$

Thus $f(x)$ satisfies all the conditions of Rolle's theorem in $0 < \alpha < \beta < \frac{\pi}{2}$ and so there exists $c(\alpha < c < \beta)$ such that $f'(c) = 0$.

6.3 LAGRANGE'S MEAN VALUE THEOREM

Let f be a function defined on a finite closed interval $[a, b]$ be such that it is

(*i*) continuous for all values of x in $a \le x \le b$ and

(*ii*) derivable for all values of x in $a < x < b$ then there exists at least one value of x, say c, $a < c < b$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Note:

(*i*) This theorem is also known as 'Formula of finite increment of Lagrange' or 'Average Value Theorem' or 'The Law of Mean'.

(*ii*) If $f(a) = f(b)$ then $f'(c) = 0$, $a < c < b$. Thus Lagrange's Mean Value Theorem becomes Rolle's theorem when $f(a) = f(b)$.

Alternative form of Lagrange's Mean Value Theorem

Let f be a function defined on a finite closed interval $[a, a+h]$ be such that it is

(*i*) continuous for all values of x in $a \le x \le a + h$ and

(*ii*) derivable for all values of x in $a < x < a + h$ then $f(a+h) = f(a) + h\,f'(a+\theta h)$, $0 < \theta < 1$.

Notes:

(*i*) For the interval $[0, h]$, the above form reduces to the Maclaurin's Formula,

$$f(h) = f(0) + h\,f'(\theta h),\ 0 < \theta < 1.$$

(*ii*) Here $\frac{f(b)-f(a)}{b-a}$ measures the mean (or average) rate of increase of the function f in the interval $[a, b]$. Therefore the theorem expresses the fact that, under the stated conditions, the mean rate of increase in any interval is equal to the actual rate of increase at some point within the interval. For example, the mean velocity of a moving point in any interval of time is equal to the actual velocity at some instant within the interval. This is the justification of the name Mean Value Theorem.

Geometrical Interpretation: Consider the following Fig. 6.5

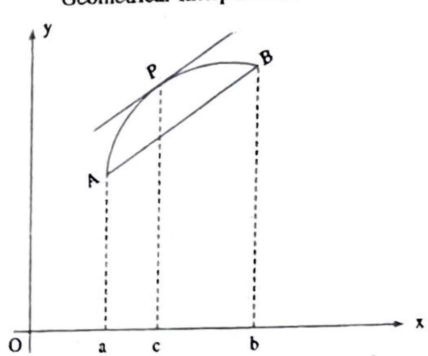


Fig. 6.5

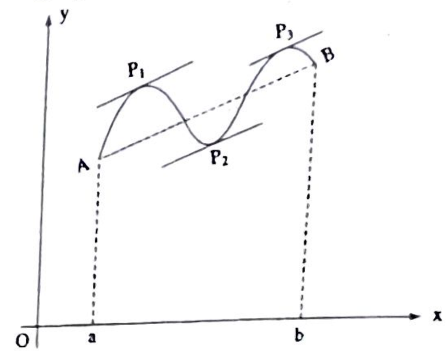


Fig. 6.6

Let $y = f(x)$ be represented by the curve AB. This curve has a tangent at every point in the interval $[a, b]$.

Slope of the chord $AB = \frac{f(b)-f(a)}{b-a}$.

Here $P(x = c)$ be a point on the curve such that

$$\frac{f(b)-f(a)}{b-a} = f'(c) = \text{slope of the tangent at } P.$$

$\therefore$ Slope of the tangent at P = slope of the chord AB.

Therefore the tangent at P is parallel to the chord AB.

The same is true for Fig. 6.6

Hence we have the following geometrical interpretation of Lagrange's Mean Value Theorem:

"If the graph of $y = f(x)$ be represented by an arc AB without any break on $[a, b]$ having tangent at every point on AB, then there must exist at least one point C between A and B on arc AB at which the tangent is parallel to the chord AB joining the two points $A \equiv (a, f(a))$ and $B \equiv (b, f(b))$."

ILLUSTRATIVE EXAMPLES

Example 1: Verify Lagrange's mean value theorem for the following functions in the specified intervals:

(i) $f(x) = x^3 - x^2 - 5x + 3,\ 0 \le x \le 4$

(ii) $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$ in $[-1, 1]$

(iii) $f(x) = \log_e x,\ 1 \le x \le e$

(iv) $f(x) = x^{1/3},\ -1 \le x \le 1$

Solution: (i) We know that every polynomial in x is continuous and derivable for all real values of x, therefore

$$f(x) = x^3 - x^2 - 5x + 3$$

is continuous on $[0, 4]$ and derivable on $(0, 4)$.

Also $\qquad f'(x) = 3x^2 - 2x - 5, 0 < x < 4.$

Therefore $f(x)$ satisfies all conditions of Lagrange's mean value theorem and hence there exists c, $0 < c < 4$, such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}, \text{ or } 3c^2 - 2c - 5 = \frac{31 - 3}{4},$$

or $\qquad 3c^2 - 2c - 12 = 0.$

It has two roots, namely $c = \dfrac{1 \pm \sqrt{37}}{3}$, of which $\dfrac{1 + \sqrt{37}}{3}$ lies between 0 and 4.

Thus, Lagrange's mean value theorem is verified for the given function in the interval $[0, 4]$.

(ii) Since x and $\sin \dfrac{1}{x}$ are derivable for all $x \neq 0$, so $f(x)$ is derivable for all $x \neq 0$.

$$\therefore \quad f'(x) = \frac{d}{dx}\left(x \sin \frac{1}{x}\right) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}, \text{ when } x \neq 0.$$

But $$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin \frac{1}{h} - 0}{h} = \lim_{h \to 0} \sin \frac{1}{h},$$

which does not exist.

Hence $f(x)$ is not derivable in $(-1, 1)$.

Therefore Lagrange's mean value theorem is not applicable for the given function in $[-1, 1]$.

(iii) Since $\log_e x$ is continuous and derivable for all x in $(0, \infty)$, so $f(x) = \log_e x$ is continuous on $1 \le x \le e$ and derivable on $1 < x < e$.

Also $\qquad f'(x) = \dfrac{1}{x}, 1 < x < e.$

Therefore $f(x)$ satisfies all conditions of Lagrange's mean value theorem and hence there exists c, $1 < c < e$, such that

$$f'(c) = \frac{f(e) - f(1)}{e - 1}, \text{ or } \frac{1}{c} = \frac{1 - 0}{e - 1}, \text{ or } c = e - 1$$

which lies in $[1, e]$, since $2 < e < 3$.

Thus, Lagrange's mean value theorem is verified for the given function in $[1, e]$.

(iv) Here $f(x) = x^{1/3}$. Therefore, $f'(x) = \dfrac{1}{3} \cdot x^{-2/3} = \dfrac{1}{3x^{2/3}}$.

Thus $f'(x)$ does not exist at $x = 0$ and hence $f(x)$ is not derivable in $(-1, 1)$.

Therefore the Lagrange's mean value theorem is not applicable here.

Example 2: If $f'(x) = 0$ in $[a, b]$, then by using Lagrange's Mean Value Theorem prove that $f(x)$ is constant in $[a, b]$.

Solution: Suppose x_1, x_2 are two arbitrary points in $[a, b]$ such that $a \le x_1 < x_2 \le b$. By Lagrange's mean value theorem, we get

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c),\ x_1 < c < x_2$$
$$= 0 \text{ (it is given)}$$

Therefore, $\quad f(x_2) - f(x_1) = 0$, or $f(x_2) = f(x_1)$.

Since x_1, x_2 are any two arbitrary points in $[a, b]$ therefore it follows that $f(x)$ is constant in $[a, b]$.

Example 3: If $f(x)$ is continuous in $a \le x \le b$ and $f'(x) > 0$ in $a < x < b$, then by using mean value theorem show that $f(x)$ is a strictly increasing function in $[a, b]$.

Solution: Suppose x_1, x_2 are two arbitrary points in (a, b) such that $a < x_1 < x_2 < b$. By Lagrange's mean value theorem, we get

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c),\ x_1 < c < x_2,$$

or $$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0, \quad \text{since } f'(c) > 0 \text{ by the given condition.}$$

But $x_2 - x_1 > 0$, therefore $f(x_2) - f(x_1) > 0$, *i.e.*, $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

Hence $f(x)$ is a strictly increasing function in (a, b).

Example 4: Deduce Lagrange's Mean Value Theorem from Rolle's Theorem.

Solution: Let us choose $g(x) = f(x) - f(a) - A(x - a)$, where A is a constant, f is continuous in $[a, b]$ and derivable in (a, b). Therefore $g(x)$ is continuous in $[a, b]$ and derivable in (a, b). Also

$g(a) = 0$ and determine A such that

$$g(b) = f(b) - f(a) - A(b-a) = 0$$

$$\Rightarrow \quad A = \frac{f(b)-f(a)}{b-a}.$$

Thus g satisfies all the conditions of Rolle's theorem and therefore there exists a number c, $a < c < b$, such that

$$g'(c) = 0$$

or $$f'(c) - A = 0, \quad \text{since } g(x) = f(x) - f(a) - A(x-a)$$

Therefore, $$f'(c) = A = \frac{f(b)-f(a)}{b-a}, \quad a<c<b,$$

which is the result of Lagrange's Mean Value Theorem.

Example 5: In the mean value theorem $f(h) = f(0) + hf'(\theta h)$, $0<\theta<1$, prove that the limiting value of θ as $h \to 0$ is $\frac{1}{2}$ if $f(x) = \cos x$.

Solution: Here $f(x) = \cos x$, therefore $f(0) = 1$ and $f'(x) = -\sin x$.

From the given relation $f(h) = f(0) + hf'(\theta h)$, we have

$$\cos h = 1 + h(-\sin\theta h), \text{ or } \sin\theta h = \frac{1-\cos h}{h}$$

or $$\theta \cdot \frac{\sin\theta h}{\theta h} = \frac{1}{2}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2$$

$\therefore$ $$\lim_{h\to 0}\left(\theta \cdot \frac{\sin\theta h}{\theta h}\right) = \frac{1}{2}\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2$$

or $$\lim_{h\to 0}\theta \cdot \lim_{\theta h\to 0}\frac{\sin\theta h}{\theta h} = \frac{1}{2}\left(\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2$$

$\therefore$ $$\lim_{h\to 0}\theta = \frac{1}{2}.$$

Example 6: Apply Lagrange's mean value theorem to prove that the chord on the parabola $y = x^2 + 2ax + b$ joining the points at $x = \alpha$ and $x = \beta$ is parallel to its tangent at the point $x = \frac{1}{2}(\alpha+\beta)$.

Solution: Let $f(x) = x^2 + 2ax + b$. Since $f(x)$ is a polynomial in x, therefore $f(x)$ is continuous in $[\alpha, \beta]$ and derivable in (α, β). Hence by Lagrange's mean value theorem, there exists a real number

c, $\alpha < c < \beta$, such that

$$f'(c) = \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \qquad \ldots(1)$$

Here $$f(x) = x^2 + 2ax + b, \quad \text{therefore} \quad f'(x) = 2x + 2a \text{ and}$$

$$f(\beta) - f(\alpha) = \beta^2 + 2a\beta + b - (\alpha^2 + 2a\alpha + b) = (\beta-\alpha)(\beta+\alpha) + 2a(\beta-\alpha)$$
$$= (\beta-\alpha)(\beta+\alpha+2a)$$

$\therefore$ $$\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = \beta+\alpha+2a. \quad \text{Also } f'(c) = 2c + 2a.$$

Hence, from (1), we have

$$2c + 2a = \beta + \alpha + 2a. \quad \therefore \quad c = \frac{\alpha+\beta}{2}.$$

Therefore, the chord joining the points at $x = \alpha$ and $x = \beta$ is parallel to the tangent at the point $\frac{\alpha+\beta}{2}$.

Example 7: Use mean value theorem to prove the following inequalities:

R (i) $0 < \frac{1}{x}\log_e\frac{e^x-1}{x} < 1$ (W.B.U.T. 2002, 2012)

(ii) $\frac{x}{1+x} < \log_e(1+x) < x$ if $x > 0$ (W.B.U.T. 2011)

(iii) $\frac{x}{1+x^2} < \tan^{-1}x < x$ when $0 < x < \frac{\pi}{2}$ (W.B.U.T. 2008, 2010)

(iv) $x < -\ln(1-x) < \frac{x}{1-x}$ when $0 < x < 1$

(v) $\frac{b-a}{\sqrt{(1-a^2)}} < \sin^{-1}b - \sin^{-1}a < \frac{b-a}{\sqrt{(1-b^2)}}$, $0<a<b<1$ (W.B.U.T. 2008)

(vi) $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$, where $0 < a < b$ and hence deduce that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$ (W.B.U.T. 2013)

Solution: (i) Let $f(x) = e^x$, $x > 0$, it is continuous in $[0, x]$ and derivable in $(0, x)$.

From the mean value theorem, $f(x) = f(0) + xf'(\theta x)$, $0<\theta<1$.

Therefore, $$e^x = 1 + xe^{\theta x} \qquad [\text{since } f(0) = 1 \text{ and } f'(x) = e^x]$$

or $$e^{\theta x} = \frac{e^x-1}{x}, \text{ or } \theta x = \log_e\frac{e^x-1}{x}$$

$$\theta = \frac{1}{x}\log_e \frac{e^x - 1}{x}.$$

Hence,
$$0 < \frac{1}{x}\log_e \frac{e^x - 1}{x} < 1 \qquad (\because\ 0<\theta<1)$$

(*ii*) Let $f(x) = \log_e(1+x)$, $x > 0$, it is continuous in $[0, x]$ and derivable in $(0, x)$.

Then from the mean value theorem, we have
$$f(x) = f(0) + xf'(\theta x),\ 0<\theta<1.$$

$$\log(1+x) = \frac{x}{1+\theta x}, \quad \text{since } f(0)=0 \text{ and } f'(x) = \frac{1}{1+x}. \qquad \text{...(1)}$$

Now $0<\theta<1$ and $x>0$, so $0<\theta x<x$.

$$\therefore \quad 1 < 1+\theta x < 1+x, \text{ or } 1 > \frac{1}{1+\theta x} > \frac{1}{1+x},$$

or
$$x > \frac{x}{1+\theta x} > \frac{x}{1+x}, \text{ or } \frac{x}{1+x} < \frac{x}{1+\theta x} < x \qquad \text{...(2)}$$

From (1) and (2), we conclude that
$$\frac{x}{1+x} < \log(1+x) < x.$$

(*iii*) Let $f(x) = \tan^{-1}x$, $0<x<\frac{\pi}{2}$, it is continuous in $[0, x]$ and derivable in $(0, x)$.

Using mean value theorem, we have
$$f(x) = f(0) + xf'(\theta x),\ 0<\theta<1.$$

$$\therefore \quad \tan^{-1}x = \frac{x}{1+\theta^2x^2}, \text{ since } f(0) = 0 \text{ and } f'(x) = \frac{1}{1+x^2}. \qquad \text{...(1)}$$

Now $0<\theta<1$ and $x>0$, so $0<\theta x<x$.

$$\therefore \quad 0<\theta^2x^2<x^2, \text{ or } 1<1+\theta^2x^2<1+x^2,$$

or
$$1 > \frac{1}{1+\theta^2x^2} > \frac{1}{1+x^2}, \text{ or } x > \frac{x}{1+\theta^2x^2} > \frac{x}{1+x^2},$$

or
$$\frac{x}{1+x^2} < \frac{x}{1+\theta^2x^2} < x \qquad \text{...(2)}$$

From (1) and (2), we have
$$\frac{x}{1+x^2} < \tan^{-1}x < x, \text{ when } 0<x<\frac{\pi}{2}.$$

(*iv*) Let $f(x) = \ln(1-x)$, $0<x<1$, it is continuous in $[0, x]$ and derivable in $(0, x)$.

By mean value theorem, we have
$$f(x) = f(0) + xf'(\theta x),\ 0<\theta<1.$$

$$\therefore \quad \ln(1-x) = -\frac{x}{1-\theta x}, \text{ since } f(0)=0 \text{ and } f'(x) = -\frac{1}{1-x}.$$

or
$$-\ln(1-x) = \frac{x}{1-\theta x} \qquad \text{...(1)}$$

Now, $0<\theta<1$ and $0<x<1$, so $0<\theta x<x$.

or $0>-\theta x>-x$, or $1>1-\theta x>1-x>0$, or $1<\frac{1}{1-\theta x}<\frac{1}{1-x}$

$$\therefore \quad x < \frac{x}{1-\theta x} < \frac{x}{1-x} \qquad \text{...(2)}$$

From (1) and (2), we have
$$x < -\ln(1-x) < \frac{x}{1-x}.$$

(*v*) Let $f(x) = \sin^{-1}x$, it is continuous in $[a, b]$ and derivable in (a, b), where $0<a<b<1$.

Therefore, by Lagrange's mean value theorem, there exists at least one value c of x, $a<c<b$, such that
$$f(b) - f(a) = (b-a)f'(c),\ a<c<b$$

or
$$\sin^{-1}b - \sin^{-1}a = \frac{(b-a)}{\sqrt{1-c^2}},\ 0<a<c<b<1 \qquad \text{...(1)}$$
$$\left(\text{since } f'(x) = \frac{1}{\sqrt{1-x^2}}\right)$$

Now, $\quad 0<a<c<b<1 \Rightarrow 1-a^2>1-c^2>1-b^2>0$

$$\Rightarrow \quad \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\Rightarrow \quad \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \quad \frac{b-a}{\sqrt{(1-a^2)}} < \frac{b-a}{\sqrt{(1-c^2)}} < \frac{b-a}{\sqrt{(1-b^2)}} \qquad (\because\ b>a) \qquad \text{...(2)}$$

From (1) and (2), we have
$$\frac{b-a}{\sqrt{(1-a^2)}} < \sin^{-1}b - \sin^{-1}a < \frac{b-a}{\sqrt{(1-b^2)}}$$

(*vi*) Let $f(x) = \tan^{-1}x$, it is continuous in (a, b) and derivable in (a, b), where $0<a<b$.

Therefore, by mean value theorem, there exists at least one value c of x, $a<c<b$, such that
$$f(b) - f(a) = (b-a)f'(c),\ a<c<b$$

or
$$\tan^{-1}b - \tan^{-1}a = \frac{b-a}{1+c^2},\ a<c<b \qquad \text{...(1)}$$
$$\left(\text{since } f'(x) = \frac{1}{1+x^2}\right)$$

Now, $a<c<b \Rightarrow 1+a^2<1+c^2<1+b^2$

$$\Rightarrow \quad \frac{1}{1+a^2}>\frac{1}{1+c^2}>\frac{1}{1+b^2}$$

$$\Rightarrow \quad \frac{1}{1+b^2}<\frac{1}{1+c^2}<\frac{1}{1+a^2}$$

$$\Rightarrow \quad \frac{b-a}{1+b^2}<\frac{b-a}{1+c^2}<\frac{b-a}{1+a^2} \qquad (\because b>a). \qquad \ldots(2)$$

From (1) and (2), we conclude that

$$\frac{b-a}{1+b^2}<\tan^{-1}b-\tan^{-1}a<\frac{b-a}{1+a^2}, \text{ where } 0<a<b. \qquad \ldots(3)$$

If we put $a=1$, $b=\frac{4}{3}$ in (3), we get

$$\frac{3}{25}<\tan^{-1}\frac{4}{3}-\tan^{-1}1<\frac{1}{6}$$

or

$$\frac{\pi}{4}+\frac{3}{25}<\tan^{-1}\frac{4}{3}<\frac{\pi}{4}+\frac{1}{6} \qquad \left(\because \tan^{-1}1=\frac{\pi}{4}\right).$$

Example 8: Using mean value theorem prove the following inequalities:

(i) $1+\frac{x}{2\sqrt{1+x}}<\sqrt{1+x}<1+\frac{x}{2}, -1<x<0$ *(W.B.U.T. 2004)*

(ii) $\frac{\pi}{6}+\frac{\sqrt{3}}{15}<\sin^{-1}\left(\frac{3}{5}\right)<\frac{\pi}{6}+\frac{1}{8}.$ *(W.B.U.T. 2005)*

Solution: (i) Let $f(x)=\sqrt{1+x}$, $-1<x<0$, it is continuous in $[x, 0]$ and derivable in $(x, 0)$. From the Lagrange's mean value theorem, we get

$$\frac{f(0)-f(x)}{-x}=f'(\theta x),\ 0<\theta<1$$

or

$$f(x)=f(0)+xf'(\theta x),\ 0<\theta<1.$$

Here $f(0)=1$ and $f'(x)=\frac{1}{2\sqrt{1+x}}$.

$$\therefore \quad \sqrt{1+x}=1+\frac{x}{2\sqrt{1+\theta x}} \qquad \ldots(1)$$

Now, $0<\theta<1 \Rightarrow 0>\theta x>x \quad (\because x<0)$

$$\Rightarrow \quad 1>\sqrt{1+\theta x}>\sqrt{1+x}>0 \qquad (\because -1<x<0)$$

$$\Rightarrow \quad 1<\frac{1}{\sqrt{1+\theta x}}<\frac{1}{\sqrt{1+x}}$$

$$\Rightarrow \quad \frac{1}{2}x>\frac{1}{2}\frac{x}{\sqrt{1+\theta x}}>\frac{1}{2}\frac{x}{\sqrt{1+x}} \qquad (\because x<0)$$

$$\Rightarrow \quad \frac{1}{2}\frac{x}{\sqrt{1+x}}<\frac{x}{2\sqrt{1+\theta x}}<\frac{x}{2}$$

$$\Rightarrow \quad 1+\frac{1}{2}\frac{x}{\sqrt{1+x}}<1+\frac{x}{2\sqrt{1+\theta x}}<1+\frac{x}{2}$$

$$\Rightarrow \quad 1+\frac{x}{2\sqrt{1+x}}<\sqrt{1+x}<1+\frac{x}{2} \qquad \text{[by (1)]}$$

where $-1<x<0$.

(ii) Let $f(x)=\sin^{-1}x$, it is continuous in $\left[\frac{1}{2},\frac{3}{5}\right]$ and derivable in $\left(\frac{1}{2},\frac{3}{5}\right)$. Using Lagrange's mean value theorem in $\left[\frac{1}{2},\frac{3}{5}\right]$, we get a number c, where $\frac{1}{2}<c<\frac{3}{5}$, such that

$$\frac{f\left(\frac{3}{5}\right)-f\left(\frac{1}{2}\right)}{\frac{3}{5}-\frac{1}{2}}=f'(c)$$

or

$$10\left(\sin^{-1}\frac{3}{5}-\frac{\pi}{6}\right)=\frac{1}{\sqrt{1-c^2}} \qquad \left(\because f'(x)=\frac{1}{\sqrt{1-x^2}}\right)$$

$$\therefore \quad \sin^{-1}\frac{3}{5}-\frac{\pi}{6}=\frac{1}{10\sqrt{1-c^2}} \qquad \ldots(1)$$

Now, $\frac{1}{2}<c<\frac{3}{5}$. $\therefore \quad c^2<\frac{9}{25} \Rightarrow -c^2>-\frac{9}{25}$

$$\Rightarrow \quad 1-c^2>1-\frac{9}{25}=\frac{16}{25} \Rightarrow \sqrt{1-c^2}>\frac{4}{5}$$

$$\Rightarrow \quad \frac{1}{\sqrt{1-c^2}}<\frac{5}{4} \Rightarrow \frac{1}{10\sqrt{1-c^2}}<\frac{5}{4\times 10}=\frac{1}{8} \qquad \ldots(2)$$

Again, $c>\frac{1}{2} \Rightarrow c^2>\frac{1}{4} \Rightarrow -c^2<-\frac{1}{4}$

$$\Rightarrow \quad 1-c^2<1-\frac{1}{4}=\frac{3}{4} \Rightarrow \sqrt{1-c^2}<\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \frac{1}{10\sqrt{1-c^2}}>\frac{2}{10\sqrt{3}}=\frac{\sqrt{3}}{15} \qquad \ldots(3)$$

From (2) and (3), we have

$$\frac{\sqrt{3}}{15} < \frac{1}{10\sqrt{1-c^2}} < \frac{1}{8}, \text{ or } \frac{\sqrt{3}}{15} < \sin^{-1}\frac{3}{5} - \frac{\pi}{6} < \frac{1}{8}, \quad \text{[by (1)]}$$

or

$$\frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}.$$

Example 9: If $f''(x)$ exists for all points in $[a, b]$ and

$$\frac{f(c)-f(a)}{c-a} = \frac{f(b)-f(c)}{b-c}$$

where $a < c < b$, then there is a number ξ such that $a < \xi < b$ and $f''(\xi) = 0$.

Solution: Since $f''(x)$ exists in $[a, b]$, f', f are continuous in $[a, b]$. Applying Lagrange's mean value theorem to the intervals $[a, c]$ and $[c, b]$ respectively, we get

$$\frac{f(c)-f(a)}{c-a} = f'(\xi_1), a < \text{ at least one } \xi_1 < c \quad \text{...(1)}$$

and

$$\frac{f(b)-f(c)}{b-a} = f'(\xi_2), c < \text{ at least one } \xi_2 < b \quad \text{...(2)}$$

From (1) and (2), we get on using the given relation

$$f'(\xi_1) = f'(\xi_2).$$

Now the function f' satisfies all the conditions of Rolle's theorem in $[\xi_1, \xi_2]$.

Therefore, there is a number ξ such that

$$f''(\xi) = 0 \quad \text{where} \quad \xi_1 < \xi < \xi_2 \text{ i.e., } a < \xi < b.$$

Example 10: If $f(x+h) = f(x) + hf'(x+\theta h)$, $0 < \theta < 1$; find the value of θ when $f(x) = x^2$.

Solution: Here $f(x) = x^2$, therefore $f(x+h) = (x+h)^2 = x^2 + 2hx + h^2$.

Now, from $f(x+h) = f(x) + hf'(x+\theta h), 0 < \theta < 1$, we have

$$x^2 + 2hx + h^2 = x^2 + 2h(x+\theta h) \quad (\because f'(x) = 2x)$$

or

$$h^2 = 2h^2\theta$$

$\therefore$

$$\theta = \frac{1}{2} \quad (\because h \neq 0)$$

Example 11: Estimate $\sqrt[3]{65}$ using Lagrange's mean value theorem.

Solution: Let us consider the function $f(x) = x^{1/3}$ in $[64, 65]$. Evidently $f(x)$ is continuous for all values of x in $[64, 65]$ and $f'(x) = \frac{1}{3}x^{-2/3}$ exists for all values of x in $[64, 65]$.

By Lagrange's mean value theorem, there exists a value c, $64 < c < 65$, such that

$$f(65) - f(64) = (65-64)f'(c)$$

or

$$(65)^{1/3} - (64)^{1/3} = \frac{1}{3}(64+\theta)^{-2/3}, \quad \text{where } c = 64+\theta, 0<\theta<1.$$

$\therefore$

$$\sqrt[3]{65} = 4 + \frac{1}{3}\cdot\frac{1}{(64+\theta)^{2/3}}.$$

$\therefore$

$$4 < \sqrt[3]{65} < 4 + \frac{1}{48}, \text{ or } 4 < \sqrt[3]{65} < 4\frac{1}{48}.$$

Example 12: Prove that $\sin 46^\circ \sim \frac{1}{2}\sqrt{2}\left(1+\frac{\pi}{180}\right)$ Is the estimate high or less?

(W.B.U.T. 2003)

Solution: Let $f(x) = \sin x$, which is continuous and derivable for all real values of x and $f'(x) = \cos x$.

By Lagrange's mean value theorem in $[a, a+h]$, we have

$$f(a+h) = f(a) + hf'(a+\theta h), \; 0<\theta<1.$$

Putting $a = 45^\circ$ and $h = 1^\circ$, we get $f(46^\circ) = f(45^\circ) + 1^\circ \cos(45^\circ + \theta \cdot 1^\circ)$

or

$$\sin 46^\circ = \sin 45^\circ + \frac{\pi}{180}\cos(45^\circ + \theta^\circ) \quad \left(\because 1^\circ = \frac{\pi}{180}\text{ radian}\right)$$

$$\sim \sin 45^\circ + \frac{\pi}{180}\cos 45^\circ \quad (\because 0 < \theta^\circ < 1^\circ, \text{ i.e., } \theta^\circ \text{ is very small})$$

$\therefore$

$$\sin 46^\circ \sim \frac{1}{\sqrt{2}}\left(1+\frac{\pi}{180}\right) = \frac{1}{2}\sqrt{2}\left(1+\frac{\pi}{180}\right)$$

This estimate is high since $0 < \theta^\circ < 1^\circ$.

Note: Applying Lagrange's mean value theorem, approximate solution of equation $f(x) = 0$ can be obtained (Newton's method) as follows:

Let $a + h$ be the exact root of $f(x) = 0$, so

$$0 = f(a+h) = f(a) + hf'(a+\theta h), 0<\theta<1.$$

Therefore,

$$h = -\frac{f(a)}{f'(a)}.$$

Hence starting at a guess value 'a', h (correction) can be calculated approximately and by iteraction a better root can be obtained.

Example 13: Calculate approximately the root of the equation $x^4 - 12x + 7 = 0$ near 2 by using Lagrange's mean value theorem.

Solution: Let $f(x) = x^4 - 12x + 7$, which is continuous and derivable for all real values of x and $f'(x) = 4x^3 - 12$.

By Lagrange's mean value theorem, $f(a+h) = f(a) + hf'(a+\theta h)$, $0<\theta<1$ so $h = -\frac{f(a)}{f'(a)}$.

Here $f(2) = 2^4 - 12\times 2 + 7 = -1$ and $f'(2) = 4\times 2^3 - 12 = 20.$

$$\therefore \quad h = -\frac{f(2)}{f'(2)} = \frac{1}{20} = 0.05$$

Therefore, an approximate root is $x = a + h = 2 + 0.05 = 2.05.$

Observations: If we apply Lagrange's mean value theorem to two functions $f(x)$ and $g(x)$, both satisfy the conditions of the theorem in $[a, b]$, we get

$$f(b) - f(a) = (b-a)f'(c_1),\ a < \text{ at least one } c_1 < b$$

and

$$g(b) - g(a) = (b-a)g'(c_2),\ a < \text{at least one } c_2 < b.$$

Dividing we have

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c_1)}{g'(c_2)},\ c_1, c_2 \text{ are, in general different.}$$

Cauchy takes a step further to make $c_1 = c_2$ and establishes a theorem, which we are going to study in the next article.

6.4 CAUCHY'S MEAN VALUE THEOREM

If f and g be two real valued functions of a real variable x defined in the closed interval $[a, b]$ such that

(*i*) $f(x)$ and $g(x)$ both are continuous in $a \le x \le b$,

(*ii*) $f(x)$ and $g(x)$ both are derivable in $a < x < b$ and

(*iii*) $g'(x) \ne 0$ for any value of x in $a < x < b$, then there exists at least one value c of x, where $a < c < b$, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Alternative form of Cauchy's mean value theorem

If we take $b = a + h$, $c = a + \theta h$, $0 < \theta < 1$, Cauchy's Mean Value Theorem takes the form

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)},\ 0<\theta<1,\ h>0,$$

which is the alterntive form for Cauchy's Mean Value Theorem.

Deduction of Lagrange's Mean Value Theorem from Cauchy's Mean Value Theorem

If we take $g(x) = x$, then $g(x)$ satisfies all the stated condition in Cauchy's Mean Value Theorem and we have

(*i*) $f(x)$ is continuous for all x in $a \le x \le b$ and

(*ii*) $f(x)$ is derivable for all x in $a < x < b$, then there exists at least one value c of x, where $a < c < b$, such that

$$\frac{f(b)-f(a)}{b-a} = \frac{f'(c)}{1} \qquad (\because\ g(x) = x \text{ and } g'(x) = 1)$$

or

$$\frac{f(b)-f(a)}{b-a} = f'(c),$$

which is the Lagrange's Mean Value Theorem.

ILLUSTRATIVE EXAMPLES

Example 1: Verify Cauchy's mean value theorem for the following functions:

(*i*) $f(x) = x^4,\ g(x) = x^2$ in the interval $[1, 2]$.

(*ii*) $f(x) = e^x,\ g(x) = e^{-x}$ in the interval $[3, 7]$

(*iii*) $f(x) = \cos x,\ g(x) = \sin x$ in the interval $\left[0, \frac{\pi}{2}\right]$

Solution: (*i*) Here $f(x) = x^4$, $g(x) = x^2$ both are continuous in $[1, 2]$ and derivable in $(1, 2)$.

Now $f'(x) = 4x^3$, $g'(x) = 2x$ and $g'(x) \ne 0$ for any x in $(1, 2)$.

Thus f and g satisfy all the conditions of Cauchy's Mean Value Theorem and therefore there should exist c, $1 < c < 2$, such that

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)},\ \text{or}\ \frac{2^4-1^4}{2^2-1^2} = \frac{4c^3}{2c},$$

or $\quad 2c^2 = 5.\quad$ Hence, $c = \pm\sqrt{\frac{5}{2}}$, of which $\sqrt{\frac{5}{2}}$ lies between 1 and 2.

Therefore, Cauchy's Mean Value Theorem is verified for the given function in the interval $[1, 2]$.

(*ii*) Here $f(x) = e^x$, $g(x) = e^{-x}$ both are continuous in $[3, 7]$ and derivable in $(3, 7)$.

Also $\quad f'(x) = e^x$, $g'(x) = -e^{-x}$ and $g'(x) \ne 0$ for any x in $(3, 7)$.

Thus f and g satisfy all the conditions of Cauchy's Mean Value Theorem and therefore there should exist c, $3 < c < 7$, such that

$$\frac{f(7)-f(3)}{g(7)-g(3)} = \frac{f'(c)}{g'(c)},\ \text{or}\ \frac{e^7-e^3}{e^{-7}-e^{-3}} = -\frac{e^c}{e^{-c}},\ \text{or}\ e^{2c} = e^{10}.$$

Therefore $c = 5$ which lies between 3 and 7.

Hence, Cauchy's Mean Value Theorem is verified for the given function in the interval $[3, 7]$.

(*iii*) Since $\cos x$ and $\sin x$ are both continuous and derivable for all real x, so $f(x) = \cos x$, $g(x) = \sin x$ are continuous in $\left[0, \frac{\pi}{2}\right]$ and derivable in $\left(0, \frac{\pi}{2}\right)$

Also $f'(x) = -\sin x$, $g'(x) = \cos x$ and $g'(x) \ne 0$ for all x in $\left(0, \frac{\pi}{2}\right)$.

Thus f and g satisfy all the conditions of Cauchy's Mean Value Theorem and therefore there should exist c, $0 < c < \frac{\pi}{2}$, such that

$$\frac{f\left(\frac{\pi}{2}\right)-f(0)}{g\left(\frac{\pi}{2}\right)-g(0)} = \frac{f'(c)}{g'(c)},\ \text{or}\ \frac{\cos\frac{\pi}{2}-\cos 0}{\sin\frac{\pi}{2}-\sin 0} = \frac{-\sin c}{\cos c},\ \text{or}\ \frac{-1}{1} = \frac{-\sin c}{\cos c},$$

or $\tan c = 1$, which gives a solution $c = \frac{\pi}{4}$ which lies between 0 and $\frac{\pi}{2}$.

Hence, Cauchy's Mean Value Theorem is verified.

Example 2: In Cauchy's Mean Value Theorem, if $f(x) = e^x$ and $g(x) = e^{-x}$, show that θ is independent of both x and h and is equal to $\frac{1}{2}$.

(W.B.U.T. 2003)

Solution: Since $f(x) = e^x$, $g(x) = e^{-x}$ both are continuous and derivable for all real x and $g'(x) = -e^{-x} \neq 0$ for all real x, therefore by Cauchy's mean value theorem,

$$\frac{f(x+h)-f(x)}{g(x+h)-g(x)} = \frac{f'(x+\theta h)}{g'(x+\theta h)},\ 0<\theta<1.$$

$$\therefore \quad \frac{e^{x+h}-e^x}{e^{-(x+h)}-e^{-x}} = \frac{e^{x+\theta h}}{-e^{-(x+\theta h)}},$$

or $$\frac{e^{x+h}-e^x}{e^{-(x+h)}\cdot e^{-x}(e^x-e^{x+h})} = -e^{2(x+\theta h)}$$

or $$-e^{x+h}\cdot e^x = -e^{2(x+\theta h)}$$

or $$e^{2x+h} = e^{2x+2\theta h}$$

$$\therefore \quad 2x+h = 2x+2\theta h, \text{ or } \theta = \frac{1}{2} \qquad (\because h \neq 0).$$

So, θ is independent of both x and h and is equal to $\frac{1}{2}$.

Example 3: If, in the Cauchy's mean value theorem, we write

$$f(x) = \sqrt{x} \text{ and } g(x) = \frac{1}{\sqrt{x}},$$

then c is the geometric mean between a and b and if we write

$$f(x) = \frac{1}{x^2} \text{ and } g(x) = \frac{1}{x},$$

then c is the harmonic mean between a and b.

Solution: When $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$, we have

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \text{ or } \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2}c^{-1/2}}{-\frac{1}{2}c^{-3/2}}$$

$$\therefore \quad -\sqrt{ab} = -c, \text{ or } c = \sqrt{ab}.$$

Therefore, c is the geometric mean between a and b.

When $$f(x) = \frac{1}{x^2} \text{ and } g(x) = \frac{1}{x}, \text{ we have}$$

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \text{ or } \frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{b}-\frac{1}{a}} = \frac{-2c^{-3}}{-c^{-2}}$$

or $$\left(\frac{a^2-b^2}{a-b}\right)\frac{ab}{a^2b^2} = \frac{2}{c}, \text{ or } \frac{a+b}{ab} = \frac{2}{c}$$

$$\therefore \quad c = \frac{2ab}{a+b}.$$

Therefore, c is the harmonic mean between a and b.

6.5 GENERALIZED MEAN VALUE THEOREM—TAYLOR'S THEOREM

Theorem 1: (Taylor's theorem with Lagrange's form of remainder).

Let f be a function defined on the closed interval $[a, b]$ such that

(*i*) the $(n-1)$th derivative $f^{(n-1)}$ is continuous in the closed interval $[a, b]$ and

(*ii*) the nth derivative $f^{(n)}$ exists in the open interval (a, b),

then there exists at least one value c, $a < c < b$, such that

$$f(b) = f(a)+(b-a)f'(a)+\frac{1}{2!}(b-a)^2 f''(a)+\ldots$$
$$+\frac{1}{(n-1)!}(b-a)^{n-1}f^{(n-1)}(a)+\frac{1}{n!}(b-a)^n f^{(n)}(c).$$

Alternative form of the above theorem

Let f be a function defined on the closed interval $[a, a+h]$, $h > 0$, such that

(*i*) the $(n-1)$th derivative $f^{(n-1)}$ is continuous in $[a, a+h]$ and

(*ii*) the nth derivative $f^{(n)}$ exists in $(a, a+h)$,

then there exists at least one number θ, $0 < \theta < 1$, such that

$$f(a+h) = f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^n(a+\theta h).$$

Note: The last term of the above series, *i.e.*, $(n+1)$th term, is called the Lagrange's form of Remainder after n terms and is denoted by R_n.

$$\therefore \quad R_n = \frac{h^n}{n!}f^n(a+\theta h),\ 0<\theta<1.$$

Theorem 2: (Taylor's theorem with Cauchy's form of remainder)

Let f be a function defined on the closed interval $[a, b]$ such that

(*i*) the $(n-1)$th derivative $f^{(n-1)}$ is continuous in the closed interval $[a, b]$ and

(*ii*) the nth derivative $f^{(n)}$ exists in the open interval (a, b), then there exists at least one value c, $a < c < b$, such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2!}(b-a)^2 f''(a) + \ldots$$

$$+ \frac{1}{(n-1)!}(b-a)^{n-1} f^{(n-1)}(a) + \frac{1}{(n-1)!}(b-a)(b-c)^{n-1} f^{(n)}(c).$$

Alternative form of the above theorem

Let f be a function defined on the closed interval $[a, a+h]$, $h > 0$, such that

(*i*) the $(n-1)$th derivative $f^{(n-1)}$ is continuous in $[a, a+h]$ and

(*ii*) the nth derivative $f^{(n)}$ exists in $(a, a+h)$,

then there exists at least one number θ, $0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h).$$

Note: The last term of the above series, *i.e.*, $(n+1)$th term, is called the Cauchy's form of Remainder after n terms and is denoted by R_n.

$$\therefore \quad R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(a+\theta h), \; 0 < \theta < 1.$$

Remarks: (*i*) The Taylor's theorem also holds if $h < 0$ and in this case the interval $[a, a+h]$ is to be replaced by $[a+h, a]$.

(*ii*) The result of Taylor's theorem is also known as Taylor's formula or Taylor's series for the function $f(x)$.

(*iii*) By taking $b = x$ in Taylor's theorem, we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n,$$

where $R_n = \dfrac{(x-a)^n}{n!}f^{(n)}\{a+\theta(x-a)\}$, $0 < \theta < 1$ [Lagrange's form]

$= \dfrac{(x-a)^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}\{a+\theta(x-a)\}$, $0 < \theta < 1$ [Cauchy's form]

which is called the expansion of $f(x)$ about $x = a$.

(*iv*) Taylor's theorem is also known as the nth order mean value theorem or nth mean value theorem or mean value theorem of the order n. The 1st order mean value theorem is the Lagrange's mean value theorem and the 2nd order mean value theorem is

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta h), \; 0 < \theta < 1.$$

6.6 MACLAURIN'S THEOREM

Let $f(x)$ be a function defined in the closed interval $[0, x]$ such that

(*i*) the $(n-1)$th derivative $f^{(n-1)}$ is continuous in $[0, x]$ and

(*ii*) the nth derivative $f^{(n)}$ exists in $(0, x)$, then there exists at least one number θ, $0 < \theta < 1$, such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n,$$

where $R_n = \dfrac{x^n}{n!}f^{(n)}(\theta x)$, $0 < \theta < 1$ [Lagrange's form]

$= \dfrac{x^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta x)$, $0 < \theta < 1$ [Cauchy's form]

Note: The above series is known as Maclaurin's series in finite form for the function $f(x)$.

6.7 EXPANSION OF FUNCTIONS IN INFINITE SERIES

Taylor's Infinite Series

Theorem: Suppose $f(x)$ be a function possessing derivatives of all orders in $[a, a+h]$, then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^{(n)}(a) + \ldots \text{ to } \infty$$

if $\lim_{n\to\infty} R_n = 0$, where R_n is the remainder after n terms.

Maclaurin's Infinite Series

Theorem: Suppose $f(x)$ be a function possessing derivatives of all orders in $[0, x]$, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots \text{ to } \infty$$

if $\lim_{n\to\infty} R_n = 0$, where R_n is the remainder after n terms.

6.8 SOME USEFUL LIMITS

The following limits are useful for the derivation of many results in this chapter:

(*i*) $\lim_{n\to\infty} nx^n = 0$, for $|x| < 1$ (*ii*) $\lim_{n\to\infty} \dfrac{x^n}{n} = \begin{cases} 0, & \text{for } |x| \le 1 \\ \infty, & \text{for } x > 1 \end{cases}$

(iii) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, for all values of x.

(iv) $\lim_{n\to\infty} \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!} x^n = 0$, for $|x|<1$.

ILLUSTRATIVE EXAMPLES

Example 1: (i) If $f(h) = f(0)+hf'(0)+\frac{h^2}{2!}f''(\theta h),\ 0<\theta<1,\ f(x) = \frac{1}{1+x}$ and $h=7$, find θ.

(W.B.U.T. 2008)

(ii) Show that the Lagrange's remainder after n terms in the expansion of $e^{ax}\cos(bx)$ in powers of x is $\frac{(a^2+b^2)^{n/2}}{n!}x^n e^{a\theta x}\cos\left\{b\theta x+n\tan^{-1}\frac{b}{a}\right\},\ 0<\theta<1.$

[BESUS (B.Arch) 2013]

Solution: (i) Here $f(x) = (1+x)^{-1}, f'(x) = -(1+x)^{-2}, f''(x) = 2(1+x)^{-3}$.

Putting these values in the given expression, we get

$$\frac{1}{1+h} = 1-h+\frac{h^2}{(1+\theta h)^3}. \quad (\because\ f(0) = 1,\ f'(0) = -1)$$

If $h = 7$, the above relation becomes

$$\frac{1}{8} = 1-7+\frac{49}{(1+7\theta)^3}, \text{ or } \frac{49}{8} = \frac{49}{(1+7\theta)^3},$$

or $$(1+7\theta)^3 = 8$$

$\therefore$ $$1+7\theta = 2 \Rightarrow \theta = \frac{1}{7}.$$

(ii) Lagrange's remainder after n terms in the expansion of $f(x)$ in powers of x is

$$\frac{x^n}{n!}f^n(\theta x),\ 0<\theta<1 \text{ (see Theorem 1, art. 6.5)}$$

Here $$f(x) = e^{ax}\cos(bx) \text{ and so}$$

$$f^n(x) = (a^2+b^2)^{\frac{n}{2}}e^{ax}\cos\left(bx+n\tan^{-1}\frac{b}{a}\right) \text{ [see IX, art. 5.2, Chapter -5]}$$

$\therefore$ $$\frac{x^n}{n!}f^n(\theta x) = \frac{(a^2+b^2)^{\frac{n}{2}}}{n!}x^n e^{a\theta x}\cos\left(b\theta x+n\tan^{-1}\frac{b}{a}\right).$$

Example 2: From the relation $f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(\theta x),\ 0<\theta<1$, prove that $\log(1+x) > x-\frac{x^2}{2}$ if $x>0$ and $\cos x > 1-\frac{x^2}{2}$ if $0<x<\frac{\pi}{2}$.

Solution: If $f(x) = \log(1+x)$, then $f(0) = 0, f'(x) = \frac{1}{1+x}$,

$$f'(0) = 1, f''(x) = -\frac{1}{(1+x)^2}, f''(\theta x) = -\frac{1}{(1+\theta x)^2}.$$

Therefore, the given relation becomes,

$$\log(1+x) = x-\frac{x^2}{2!}\cdot\frac{1}{(1+\theta x)^2}$$

Since $x>0,\ \theta>0$, therefore, $\frac{x^2}{(1+\theta x)^2}<x^2$, or $\frac{-x^2}{2(1+\theta x)^2}>-\frac{x^2}{2}$

Hence $$\log(1+x)>x-\frac{x^2}{2} \text{ if } x>0.$$

When $f(x) = \cos x$, we have $f(0) = 1, f'(x) = -\sin x, f'(0) = 0$,

$$f''(x) = -\cos x,\ f''(\theta x) = -\cos\theta x.$$

By the given relation, we get

$$\cos x = 1-\frac{x^2}{2}\cos\theta x.$$

Now, $0<\theta<1$ and $0<x<\frac{\pi}{2}$ gives $0<\cos\theta x<1$.

$\therefore$ $$\frac{x^2}{2}\cos\theta x < \frac{x^2}{2}, \text{ or } -\frac{x^2}{2}\cos\theta x > -\frac{x^2}{2},$$

or $$1-\frac{x^2}{2}\cos\theta x > 1-\frac{x^2}{2}.$$

Hence $$\cos x > 1-\frac{x^2}{2} \text{ if } 0<x<\frac{\pi}{2}.$$

Example 3: Using Maclaurin's series, show that

$$\sin x > x-\frac{1}{6}x^3 \text{ if } 0<x<\frac{\pi}{2}.$$

Solution: Let $f(x) = \sin x$, therefore, $f(0) = 0$, $f'(x) = \cos x$,

$f'(0) = 1, f''(x) = -\sin x, f''(0) = 0, f'''(x) = -\cos x, f'''(\theta x) = -\cos\theta x$

Now the Maclaurin's series for the function $f(x)$ with Lagrange's form of remainder after three terms is

$$f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(\theta x),\quad 0<\theta<1$$

$$\therefore \quad \sin x = x-\frac{x^3}{3!}\cos\theta x = x-\frac{x^3}{6}\cos\theta x \qquad \ldots(1)$$

Now $0<x<\frac{\pi}{2}$ and $0<\theta<1$ give $0<\theta x<\frac{\pi}{2}$,

or $\quad 0<\cos\theta x<1$

$$\therefore \quad \frac{x^3}{6}\cos\theta x < \frac{x^3}{6} \qquad (\because x>0)$$

or $\quad -\frac{x^3}{6}\cos\theta x > -\frac{x^3}{6}$

or $\quad x-\frac{x^3}{6}\cos\theta x > x-\frac{x^3}{6}$

$$\therefore \quad \sin x > x-\frac{1}{6}x^3 \quad \text{[by (1)] if } 0<x<\frac{\pi}{2}.$$

Example 4: Apply Maclaurin's theorem to $f(x) = (1+x)^4$ to deduce that $(1+x)^4 = 1+4x+6x^2+4x^3+x^4$. *(W.B.U.T. 2001)*

Solution: Here $f(x) = (1+x)^4$ is continuous and derivable for all real values of x.

Here $\quad f(0) = 1,\ f'(x) = 4(1+x)^3,\ f'(0) = 4,\ f''(x) = 12(1+x)^2,$

$f''(0) = 12,\ f'''(x) = 24(1+x),\ f'''(0) = 24,\ f^{iv}(x) = 24,\ f^{iv}(\theta x) = 24$

Now the Maclaurin's series for the function $f(x)$ with Lagrange's form of remainder after four terms is

$$f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(\theta x),\quad 0<\theta<1.$$

$$\therefore \quad (1+x)^4 = 1+4x+\frac{x^2}{2}\cdot 12+\frac{x^3}{6}\cdot 24+\frac{x^4}{24}\cdot 24$$

$$= 1+4x+6x^2+4x^3+x^4.$$

Example 5: Expand a^x in a finite series with Lagrange's form of remainder. *(W.B.U.T. 2002)*

Solution: Let $f(x) = a^x = e^{\log_e a^x} = e^{kx}$, where $k = \log_e a = \log a$. Here $f(x)$ is continuous and derivable for all real values of x.

Now $\quad f^n(x) = k^n e^{kx}$ and $f^n(0) = k^n = (\log a)^n,$

$n = 1, 2, 3, 4, \ldots,\ f^n(\theta x) = k^n e^{k\theta x} = (\log a)^n a^{\theta x}$

Using Maclaurin's theorem with Lagrange's form of remainder after n terms, we get

$$a^x = f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+\frac{x^n}{n!}f^{(n)}(\theta x)$$

$$= 1+x(\log a)+\frac{x^2}{2!}(\log a)^2+\ldots+\frac{x^{n-1}}{(n-1)!}(\log a)^{n-1}+\frac{x^n}{n!}a^{\theta x}(\log a)^n,\quad \text{where } 0<\theta<1.$$

R

Example 6: If $f''(x)>0$ for all values of x, prove that

22

$$f\left(\frac{x_1+x_2}{2}\right)<\frac{f(x_1)+f(x_2)}{2}.$$

Solution: Using Taylor's theorem, we have

$$f(x_1) = f\left(\frac{x_1+x_2}{2}+\frac{x_1-x_2}{2}\right) = f\left(\frac{x_1+x_2}{2}\right)+\left(\frac{x_1-x_2}{2}\right)f'\left(\frac{x_1+x_2}{2}\right)$$

$$+\frac{1}{2!}\left(\frac{x_1-x_2}{2}\right)^2 f''\left\{\frac{x_1+x_2}{2}+\theta_1\left(\frac{x_1-x_2}{2}\right)\right\},\quad 0<\theta_1<1 \qquad \ldots(1)$$

and

$$f(x_2) = f\left(\frac{x_1+x_2}{2}+\frac{x_2-x_1}{2}\right) = f\left(\frac{x_1+x_2}{2}\right)+\left(\frac{x_2-x_1}{2}\right)f'\left(\frac{x_1+x_2}{2}\right)$$

$$+\frac{1}{2!}\left(\frac{x_2-x_1}{2}\right)^2 f''\left\{\frac{x_1+x_2}{2}+\theta_2\left(\frac{x_2-x_1}{2}\right)\right\},\quad 0<\theta_2<1 \qquad \ldots(2)$$

Adding (1) and (2), we get

$$f(x_1)+f(x_2) = 2f\left(\frac{x_1+x_2}{2}\right)+\frac{1}{2}\left(\frac{x_1-x_2}{2}\right)^2\left[f''\left\{\frac{x_1+x_2}{2}+\theta_1\left(\frac{x_1-x_2}{2}\right)\right\}\right.$$

$$\left.+f''\left\{\frac{x_1+x_2}{2}+\theta_2\left(\frac{x_2-x_1}{2}\right)\right\}\right] \qquad \ldots(3)$$

Since $\left(\frac{x_1-x_2}{2}\right)^2>0$ and $f''(x)>0$ for all values of x, we get from (3),

$$f(x_1)+f(x_2) > 2f\left(\frac{x_1+x_2}{2}\right)$$

or $\quad f\left(\frac{x_1+x_2}{2}\right)<\frac{f(x_1)+f(x_2)}{2}$

Example 7: Show that $\lim_{h\to 0}\theta = \frac{1}{n+1}$, where θ is given by

$$f(a+h) = f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^{(n)}(a+\theta h)$$

provided that $f^{(n+1)}(x)$ is continuous at a, $f^{(n+1)}(a)\neq 0$.

Solution: Since $f^{(n+1)}(x)$ is continuous at a, therefore there exists $\delta > 0$, such that $f^{(n+1)}(x)$ exists $[a-\delta, a+\delta]$. Also $f(x), f'(x), \ldots, f^{(n)}(x)$ are all continuous in $[a-\delta, a+\delta]$. Taking $a+h$, a point this interval, we have

$$f(a+h) = f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^{(n)}(a+\theta h),\ 0<\theta<1,$$

and

$$f(a+h) = f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^{(n)}(a)+\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta' h),\ 0<\theta'<1.$$

$$\therefore\quad f^{(n)}(a+\theta h) = f^{(n)}(a)+\frac{h}{n+1}f^{(n-1)}(a+\theta' h)$$

Using Lagrange's mean value theorem, we have

$$f^{(n)}(a)+\theta h f^{(n+1)}(a+\theta\theta'' h) = f^{(n)}(a)+\frac{h}{n+1}f^{(n+1)}(a+\theta' h),\ 0<\theta''<1,$$

or $$\theta f^{(n+1)}(a+\theta\theta'' h) = \frac{1}{n+1}f^{(n+1)}(a+\theta' h)$$

or $$\lim_{h\to 0}\theta f^{(n+1)}(a) = \frac{1}{n+1}f^{(n+1)}(a), \text{ since } f^{(n+1)}(x) \text{ is continuous.}$$

or $$\lim_{h\to 0}\theta = \frac{1}{n+1}, \text{ since } f^{(n+1)}(a)\neq 0.$$

Example 8: Expand the polynomial $x^5+2x^4-x^2+x+1$ in power of $(x+1)$.

Solution: We know that

$$f(a+h) = f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\ldots$$

Let $$f(x) = x^5+2x^4-x^2+x+1$$

$$\therefore\quad f(x) = f(-1+x+1) = f(-1)+(x+1)f'(-1)+\frac{(x+1)^2}{2!}f''(-1)+\frac{(x+1)^3}{3!}f'''(-1)+\ldots$$

Here $$f(-1) = (-1)^5+2(-1)^4-(-1)^2+(-1)+1 = 0$$

$$f'(x) = 5x^4+8x^3-2x+1, \qquad f'(-1) = 0$$

$$f''(x) = 20x^3+24x^2-2, \qquad f''(-1) = 2$$

$$f'''(x) = 60x^2+48x, \qquad f'''(-1) = 12$$

$$f^{iv}(x) = 120x+48, \qquad f^{iv}(-1) = -72$$

$$f^{v}(x) = 120$$

$$f^{(n)}(x) = 0, \text{ for } n\geq 6$$

$$\therefore\quad f(x) = f(-1)+(x+1)f'(-1)+\frac{(x+1)^2}{2!}f''(-1)+\frac{(x+1)^3}{3!}f'''(-1)+\frac{(x+1)^4}{4!}f^{(4)}(-1)+\frac{(x+1)^5}{5!}f^{(5)}(-1)\ (\because f^{(n)}(-1)=0, \text{ for } n\geq 6)$$

$$= 0+0+\frac{(x+1)^2}{2!}\cdot 2+\frac{(x+1)^3}{3!}\cdot 12+\frac{(x+1)^4}{4!}\cdot(-72)+\frac{(x+1)^5}{5!}\cdot 120$$

$$= (x+1)^2+2(x+1)^3-3(x+1)^4+(x+1)^5$$

This is the required expansion of the given polynomial.

Example 9: Expand $f(x) = e^{\sin x}$, using Maclaurin's series up to the term containing x^4.

Solution: Here $f(x) = e^{\sin x}, \quad f(0) = 1$

$$f'(x) = e^{\sin x}\cos x = f(x)\cos x, \qquad f'(0) = 1$$

$$f''(x) = f'(x)\cos x - f(x)\sin x, \qquad f''(0) = 1$$

$$f'''(x) = f''(x)\cos x - 2f'(x)\sin x - f(x)\cos x, \qquad f'''(0) = 0$$

$$f^{(4)}(x) = f'''(x)\cos x - 3f''(x)\sin x - 3f'(x)\cos x + f(x)\sin x, \qquad f^{(iv)}(0) = -3$$

$$\therefore\quad e^{\sin x} = f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\ldots$$

$$= 1+x+\frac{x^2}{2}-\frac{x^4}{8}+\ldots \text{to } \infty.$$

This is the required expansion.

Example 10: Expand $\tan^{-1}x$ in powers of x using Maclaurin's expansion.

Solution: Let $\quad y(x) = \tan^{-1}x, \quad \therefore\ y(0) = 0$

$$y_1(x) = \frac{1}{1+x^2}, \qquad \therefore\ y_1(0) = 1$$

or $$(1+x^2)y_1 = 1$$

Differentiating both sides with respect to x, we have ...(1)

$$(1+x^2)y_2+2xy_1 = 0$$

Differentiating both sides n times with respect to x by using Leibnitz's theorem, we have

$$(1+x^2)y_{n+2}+{}^nC_1 2xy_{n+1}+{}^nC_2\cdot 2y_n+2\{xy_{n+1}+{}^nC_1 y_n\} = 0$$

or $$(1+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n = 0.$$

At $x = 0$, $\quad y_{n+2}(0) = -n(n+1)y_n(0), \; n = 1, 2, 3, \ldots$

$n = 1, \quad y_3(0) = -2y_1(0) = -2$

$n = 2, \quad y_4(0) = -6y_2(0) = 0 \quad (\because \text{ from } (1),\ y_2(0) = 0)$

$n = 3, \quad y_5(0) = -12y_3(0) = 24$

$n = 4, \quad y_6(0) = 0$

Therefore, by Maclaurin's series

$$\tan^{-1} x = y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \frac{x^5}{5!}y_5(0) + \ldots \text{ to } \infty$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \text{ to } \infty.$$

Example 11: Expand the following functions in power of x in infinite series: *(W.B.U.T. 2004)*

(i) e^x

(ii) $\sin x$,

(iii) $\log_e(1 + x)$, $-1 < x \le 1$ *(BESUS 2013, W.B.U.T. 2006)*

(iv) $(1 + x)^m$, $-1 < x < 1$, m is any real number.

Solution: *(i)* Let $f(x) = e^x$, which possesses derivatives of every order for all real values of x.

Now $\quad f^{(n)}(x) = e^x \quad$ and $\quad f^{(n)}(0) = 1, \quad n = 1, 2, 3, 4, \ldots$

The Lagrange's form of remainder after n terms is

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x) = \frac{x^n}{n!} e^{\theta x}$$

Now, we know that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, for all values of x.

$$\therefore \quad \lim_{n\to\infty} R_n = e^{\theta x} \lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Therefore, conditions for Maclaurin's infinite series are satisfied and we get

$$e^x = f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots \text{ to } \infty$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{ to } \infty \quad (\text{since } f(0) = 1,\ f^{(n)}(0) = 1,\ n = 1, 2, 3, \ldots)$$

which is valid for every value of x.

(ii) Let $f(x) = \sin x$, which possesses derivatives of every order for all real values of x.

Now $\quad f^{(n)}(x) = \sin\left(\frac{n\pi}{2} + x\right)$ and $f^{(n)}(0) = \sin\frac{n\pi}{2}$,

$$n = 1, 2, 3, \ldots, f(0) = 0,\ f'(0) = \sin\frac{\pi}{2} = 1,\ f''(0) = 0,$$

$$f'''(0) = -1,\ f^{iv}(0) = 0,\ f^{v}(0) = 1, \ldots$$

The Lagrange's form or remainder after n terms is

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x) = \frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right)$$

$$\therefore \quad 0 \le |R_n| = \left|\frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right)\right| = \left|\frac{x^n}{n!}\right|\left|\sin\left(\frac{n\pi}{2} + \theta x\right)\right| \le \left|\frac{x^n}{n!}\right|$$

$[\because |\sin x| \le 1, \text{ for all } x]$

or
$$-\left|\frac{x^n}{n!}\right| \le R_n \le \left|\frac{x^n}{n!}\right|.$$

We know that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, for all values of x, and so $\lim_{n\to\infty}\left|\frac{x^n}{n!}\right| = 0$. Therefore it follows from above that $\lim_{n\to\infty} R_n = 0$, for all real values of x.

Thus, conditions for Maclaurin's infinite series are satisfied and we have

$$\sin x = f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots \text{ to } \infty$$

or
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \text{ to } \infty$$

which is valid for every real value of x.

(iii) Let $f(x) = \log_e(1+x)$, which possesses derivatives of every order for $1 + x > 0$, *i.e.*, $x > -1$.

Also, $\quad f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ and therefore

$$f^{(n)}(0) = (-1)^{n-1}(n-1)!, \quad n = 1, 2, 3, \ldots$$

Case I: Let $0 \le x \le 1$.

The Lagrange's form of remainder after n terms is

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x) = (-1)^{n-1} \cdot \frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n.$$

Here $x/(1+\theta x)$ and, therefore, also $\{x/(1+\theta x)\}^n$ is positive (or zero for $x = 0$) and less than 1, whatever value n may have.