Logic & Proofs (Lecture – 4)

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Rules of Inference

- Rules of inference are the basic tools for establishing the proofs.
- Proofs in mathematics are valid arguments that establish the truth of mathematical statements.
- By an **argument**, we mean a sequence of statements that end with a conclusion.
- By **valid**, we mean that the conclusion, or final statement of the argument, must follow from the truth of the preceding statements, or **premises**, of the argument.
- To deduce new statements from the available statements, we use *rules of inference* which are templates for constructing valid arguments.

Valid Arguments in Propositional Logic

- Consider the following argument involving propositions (which, by definition, is a sequence of propositions):
  - "If you have a current password, then you can log onto the network." (premise)
  - "You have a current password." (premise)
  - Therefore, "You can log onto the network." (conclusion)
- Determine whether this is a valid argument.
  - Determine whether the conclusion must be true if both the premises are true

> An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*. An argument is *valid* if the truth of all its premises implies that the conclusion is true.
>
> An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

Rules of Inference for Propositional Logic

- Use of truth tables to show that an argument form is valid becomes tedious when the number of propositional variables becomes large.
- Instead, we can first establish the validity of some relatively simple argument forms, called **rules of inference**. These rules of inference can be used as building blocks to construct more complicated valid argument forms.
- The tautology $(p \land (p \rightarrow q)) \rightarrow q$ is the basis of the rule of inference called **modus ponens**, or the **law of detachment**. (Modus ponens is Latin for *mode that affirms*.)
- This tautology leads to the following valid argument form:

$$\begin{array}{rl} & p \\ & p \rightarrow q \\ \hline \therefore & q \end{array}$$

Rules of Inference

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$	$(p \land (p \rightarrow q)) \rightarrow q$	Modus ponens
$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$	$(\neg q \land (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$	$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$	$((p \lor q) \land \neg p) \rightarrow q$	Disjunctive syllogism
$\begin{array}{l} p \\ \hline \therefore p \lor q \end{array}$	$p \rightarrow (p \lor q)$	Addition
$\begin{array}{l} p \land q \\ \hline \therefore p \end{array}$	$(p \land q) \rightarrow p$	Simplification
$\begin{array}{l} p \\ q \\ \hline \therefore p \land q \end{array}$	$((p) \land (q)) \rightarrow (p \land q)$	Conjunction
$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \hline \therefore q \lor r \end{array}$	$((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$	Resolution

Resolution

- Computer programs have been developed to automate the task of reasoning and proving theorems.
- Many of these programs make use of a rule of inference known as resolution: $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$
- Resolution plays an important role in programming languages based on the rules of logic, such as Prolog (where resolution rules for quantified statements are applied).
- To construct proofs in propositional logic using resolution as the only rule of inference, the hypotheses and the conclusion must be expressed as **clauses**, where a clause is a disjunction of variables or negations of these variables.
- We can replace a statement in propositional logic that is not a clause by one or more equivalent statements that are clauses.

Fallacies

- Several common fallacies arise in incorrect arguments. These fallacies resemble rules of inference, but are based on contingencies rather than tautologies.
  - **Fallacy of affirming the conclusion**: They treat the argument with premises $p \rightarrow q$ and q and conclusion p as a valid argument form, which it is not because $((p \rightarrow q) \land q) \rightarrow p$ is not a tautology.
  - **Fallacy of denying the hypothesis**: The proposition $((p \rightarrow q) \land \neg p) \rightarrow \neg q$ is not a tautology, because it is false when p is false and q is true. Many incorrect arguments use this incorrectly as a rule of inference.

Rules of Inference for Quantified Statements

- **<u>Universal instantiation</u>** is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain, given the premise $\forall x P(x)$.
  - Example: Universal instantiation is used when we conclude from the statement "All women are wise" that "Lisa is wise," where Lisa is a member of the domain of all women.
- **<u>Universal generalization</u>** is the rule of inference that states that $\forall x P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain.
  - This is used when we show that $\forall x P(x)$ is true by taking an arbitrary element c from the domain and show that $P(c)$ is true. The element c that we select must be an arbitrary, and not a specific, element of the domain.

Rules of Inference for Quantified Statements

- **Existential instantiation** is the rule that allows us to conclude that there is an element c in the domain for which $P(c)$ is true if we know that $\exists x P(x)$ is true.
  - We cannot select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true. Usually we have no knowledge of what c is, only that it exists. Because it exists, we may give it a name (c) and continue our argument
- **Existential generalization** is the rule of inference that is used to conclude that $\exists x P(x)$ is true when a particular element c with $P(c)$ true is known. That is, if we know one element c in the domain for which $P(c)$ is true, then we know that $\exists x P(x)$ is true.

Rules of Inference for Quantified Statements

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization