ILLUSTRATIVE EXAMPLES-I

Example 1: Given the following bivariate probability distribution:

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Find:

(i) $P(X \le 1, Y=2)$,

(ii) $P(X \le 1)$,

(iii) $P(Y \le 3)$,

(iv) $P(X < 3, Y \le 4)$.

Solution: Marginal Distribution

X \ Y	1	2	3	4	5	6	Marginal $p_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
Marginal $P_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	$\sum P_X(x)=1$ $\sum P_Y(y)=1$

(i) $P(X \le 1, Y=2) = P(X=0, Y=2) + P(X=1, Y=2) = 0 + \frac{1}{16} = \frac{1}{16}$

(ii) $P(X \le 1) = P(X=0) + P(X=1) = \frac{8}{32} + \frac{10}{16} = \frac{7}{8}$.

(iii) $P(Y \le 3) = P(Y=1) + P(Y=2) + P(Y=3) = \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$.

(iv) $P(X<3, Y \le 4) = P(X=0, Y \le 4) + P(X=1, Y \le 4) + P(X=2, Y \le 4)$

$$= \left(\frac{1}{32} + \frac{2}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{32} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64}\right)$$

$$= \frac{3}{32} + \frac{6}{16} + \frac{6}{64} = \frac{36}{64} = \frac{9}{16}.$$

Example 2: ~~Ex. 7.~~ Given the following bivariate probability distribution. Obtain (i) marginal distribution of X and Y, (ii) the conditional distribution of X given Y=2.

$Y \downarrow$ \ $X \rightarrow$	-1	0	1
0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
1	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
2	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$

Solution:

$Y \downarrow$ \ $X \rightarrow$	-1	0	1	$\sum_x p(x,y) = p_{\cdot j}$
0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{4}{15}$ $(= p_{\cdot 1})$
1	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{6}{15}$ $(= p_{\cdot 2})$
2	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{5}{15}$ $(= p_{\cdot 3})$
$\sum_y p(x,y) = p_{i\cdot}$	$\frac{6}{15}$ $(= p_{1\cdot})$	$\frac{5}{15}$ $(= p_{2\cdot})$	$\frac{4}{15}$ $(= p_{3\cdot})$	1

(i) Marginal distribution of X

$P(X=-1) = \frac{6}{15}$, $P(X=0) = \frac{5}{15}$, $P(X=1) = \frac{4}{15}$.

Marginal distribution of Y

$P(Y=0) = \frac{4}{15}$, $P(Y=1) = \frac{6}{15}$, $P(Y=2) = \frac{5}{15}$.

(ii) $P(X=x \mid Y=2) = \dfrac{P(X=x, Y=2)}{P(Y=2)}$.

$$p_{-1/2} = P(X=-1 \mid Y=2) = \frac{P(X=-1, Y=2)}{P(Y=2)} = \frac{2/15}{5/15} = \frac{2}{5}.$$

$$p_{0/2} = P(X=0 \mid Y=2) = \frac{P(X=0, Y=2)}{P(Y=2)} = \frac{1/15}{5/15} = \frac{1}{5}.$$

$$p_{1/2} = P(X=1 \mid Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{2/15}{5/15} = \frac{2}{5}.$$

$$\therefore \sum_i p_{i/2} = p_{-1/2} + p_{0/2} + p_{1/2} = \frac{2}{5} + \frac{1}{5} + \frac{2}{5} = 1.$$

Example 3: The joint probability distribution of two random variables X and Y is given by

$P(X=0, Y=1) = \frac{1}{3}$, $P(X=1, Y=-1) = \frac{1}{3}$ and $P(X=1, Y=1) = \frac{1}{3}$.

Find (i) marginal distributions of X and Y,

(ii) the conditional probability distribution of X given $Y=1$.

Solution:

(i) $P(X=-1) = \sum_{y} P(X=-1, Y=y)$

$= P(X=-1, Y=-1) + P(X=-1, Y=0) + P(X=-1, Y=1)$

$= 0 + 0 + 0 = 0.$

Similarly, $P(X=0) = \frac{1}{3}$

and $P(X=1) = \frac{2}{3}$.

Y \ X	-1	0	1	Marginal (Y)
-1	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0	0	0
1	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
Marginal (X)	0	$\frac{1}{3}$	$\frac{2}{3}$	1

Marginal distribution of X				Marginal distribution of Y			
Values of X (x):	-1	0	1	Values of Y (y):	-1	0	1
$P(X=x)$:	0	$\frac{1}{3}$	$\frac{2}{3}$	$P(Y=y)$:	$\frac{1}{3}$	0	$\frac{2}{3}$

(ii) The conditional probability distribution of X given Y:

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}.$$

$$\therefore P(X=-1 \mid Y=1) = \frac{P(X=-1, Y=1)}{P(Y=1)} = 0,$$

$$P(X=0 \mid Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{1/3}{2/3} = \frac{1}{2},$$

$$P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{1/3}{2/3} = \frac{1}{2}.$$

Example 7: Let X be uniformly distributed in $(0,1)$ and let the conditional distribution of Y on the hypothesis $X=x$ be uniform in $(0,x)$. Determine the distribution of the two dimensional random variable (X,Y) and the marginal distribution of Y.

Solution: The marginal density function $f_X(x)$ of X:

$$f_X(x) = \begin{cases} 1, & \text{for } 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

The conditional density function $f_Y(y|x)$ of Y, given $X=x$:

$$f_Y(y|x) = \begin{cases} \frac{1}{x}, & \text{for } 0<y<x<1 \\ 0, & \text{elsewhere} \end{cases}$$

Since $f_Y(y|x) = \dfrac{f(x,y)}{f_X(x)} \Rightarrow f(x,y) = f_X(x)\, f_Y(y|x)$, the probability density function (distribution) of (X,Y) is given by

$$f(x,y) = \begin{cases} \frac{1}{x}, & \text{for } 0<y<x<1 \\ 0, & \text{elsewhere} \end{cases}$$

The marginal density function (marginal distribution) of Y is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{y}^{1} \frac{1}{x}\,dx = \log\frac{1}{y},\ 0<y<1.$$

$$\therefore f_Y(y) = \begin{cases} -\log y, & 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

Example Ex) 8: The probability density function (p.d.f.) of a two-dimensional random variable (X, Y) is

$$f(x,y) = \begin{cases} \frac{1}{8}(x+y), & 0 \le x, y \le 2 \\ 0 & , \text{elsewhere} \end{cases}$$

(i) Are X and Y independent?

(ii) Evaluate $P(X-Y>1)$.

Solution

Now, $$\int_0^2\int_0^2 \frac{1}{8}(x+y)\,dx\,dy = \frac{1}{8}\int_0^2 \left[xy+\frac{y^2}{2}\right]_{y=0}^{2} dx = \frac{1}{8}\int_0^2 (2x+2)\,dx$$

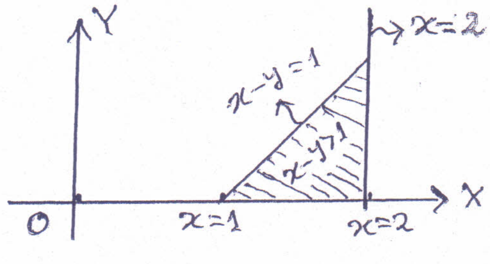
$$= \frac{1}{4}\left[\frac{x^2}{2}+x\right]_0^2 = 1.$$

So, the given $f(x,y)$ is a valid p.d.f.

Here, $$f_X(x) = \frac{1}{8}\int_0^2 (x+y)\,dy = \begin{cases} \frac{1}{4}(x+1) ; & 0 \le x \le 2 \\ 0 & ; \text{elsewhere} \end{cases}$$

Similarly, $$f_Y(y) = \begin{cases} \frac{1}{4}(y+1) ; & 0 \le y \le 2 \\ 0 & ; \text{elsewhere} \end{cases}$$

$\therefore$ X and Y are not independent since $f(x,y) \neq f_X(x) f_Y(y)$.



$$P(X-Y>1) = \int_{x=1}^{2}\left[\int_{y=0}^{x-1} f(x,y)\,dy\right]dx$$

$$= \frac{1}{8}\int_{x=1}^{2}\left[\int_{y=0}^{x-1}(x+y)\,dy\right]dx$$

$$= \frac{1}{8}\int_{x=1}^{2}\left[xy+\frac{y^2}{2}\right]_{y=0}^{x-1} dx = \frac{1}{8}\int_1^2 \left[x(x-1)+\frac{1}{2}(x-1)^2\right]dx$$

$$= \frac{1}{8}\int_1^2 \left[\frac{3}{2}x^2-2x+\frac{1}{2}\right]dx = \frac{1}{8}\left[\frac{1}{2}x^3-x^2+\frac{x}{2}\right]_{x=1}^{2} = (4-4+1)-\left(\frac{1}{2}-1+\frac{1}{2}\right) = 1$$

Note: Random variables X and Y in R^2 are said to be independent if

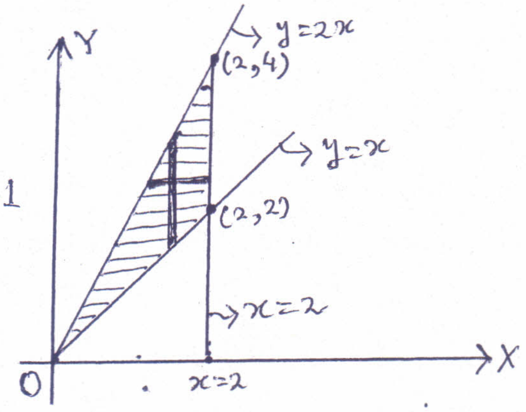
$$f(x,y) = f_X(x)\, f_Y(y) \quad [\text{or, } p_{ij} = p_{i\cdot}\, p_{\cdot j}].$$

Example 59: The bivariate random variable (X,Y) has the p.d.f.

$$f(x,y) = \begin{cases} kx^2(8-y), & x<y<2x,\ 0\le x\le 2 \\ 0 & ,\ \text{elsewhere.} \end{cases}$$

Determine (i) k (ii) $f_X(x)$, $f_Y(y)$

(iii) conditional probability density functions $f_{X/Y}(x,y)$ and $f_{Y/X}(x,y)$.



Solution:

(i) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = k\int_0^2 x^2\left[\int_x^{2x}(8-y)\,dy\right]dx = 1$$

$$\Rightarrow k\int_0^2 x^2\left[8y-\frac{y^2}{2}\right]_x^{2x} dx = 1$$

$$\Rightarrow k\int_0^2\left(8x^3-\frac{3}{2}x^4\right)dx = 1 \Rightarrow k\left[2x^4-\frac{3}{10}x^5\right]_0^2 = 1 \Rightarrow k = \frac{5}{112}.$$

(ii) $$f_X(x) = k\int_x^{2x} x^2(8-y)\,dy = \begin{cases}\frac{5}{112}\left(8x^3-\frac{3}{2}x^4\right); & 0\le x\le 2 \\ 0 & ;\ \text{elsewhere}\end{cases}$$

$$f_Y(y) = k\int_{y/2}^{y} x^2(8-y)\,dx = \frac{5}{112}\left[\frac{8x^3}{3}-\frac{x^3 y}{3}\right]_{x=y/2}^{y} = \begin{cases}\frac{7\times 5}{112\times 24}(8-y)y^3 & ;\ 0\le y\le 4 \\ 0 & ;\ \text{elsewhere.}\end{cases}$$

(iii) Conditional density functions are as follows:

$$f_{X/Y}(x,y) = \frac{f(x,y)}{f_Y(y)} = \frac{Kx^2(8-y)}{\frac{7K}{24}(8-y)y^3} = \begin{cases} \frac{24x^2}{7y^3} ; & 0 \le x \le 2; x<y<2x \\ 0 & , \text{elsewhere} \end{cases}$$

$$f_{Y/X}(x,y) = \frac{f(x,y)}{f_X(x)} = \frac{Kx^2(8-y)}{K(8x^3-\frac{3}{2}x^4)} = \begin{cases} \frac{2(8-y)}{x(16-3x)} & ; 0 \le x \le 2; x<y<2x \\ 0 & ; \text{elsewhere.} \end{cases}$$

Example ~~Ex 2~~ 10: The joint p.d.f. of two random variables X and Y is given by

$$f(x,y) = e^{-(x+y)} I_{(0,\infty)}(x) I_{(0,\infty)}(y).$$

Find (i) $P(X>1)$, (ii) $P(X<Y|X<2Y)$ and (iii) $P(1<X+Y<2)$.

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Solution:

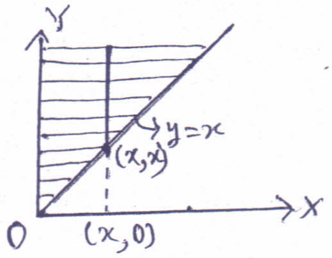
$$f_X(x) = \int_0^\infty f(x,y)\,dy = \int_0^\infty e^{-x}e^{-y}dy = e^{-x}\lim_{B\to\infty}\int_0^B e^{-y}dy$$

$$= e^{-x}\lim_{B\to\infty}\left[-e^{-y}\right]_0^B = e^{-x}, \quad x>0.$$

(i) $$P(X>1) = \int_1^\infty f_X(x)\,dx = \int_1^\infty e^{-x}dx = \lim_{B\to\infty}\int_1^B e^{-x}dx$$

$$= \lim_{B\to\infty}\left[-e^{-x}\right]_1^B = e^{-1}.$$

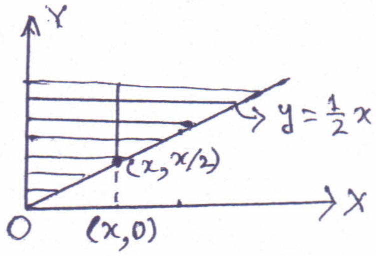
(ii) $$P(X<Y|X<2Y) = \frac{P\{(X<Y)\cap(X<2Y)\}}{P(X<2Y)} = \frac{P(X<Y)}{P(X<2Y)} \quad \cdots (1)$$



$$P(X<Y)=\int_0^\infty e^{-x}\left\{\int_x^\infty e^{-y}\,dy\right\}dx$$

$$=\int_0^\infty e^{-2x}\,dx=\lim_{B\to\infty}\int_0^B e^{-2x}\,dx$$

$$=\lim_{B\to\infty}\left[-\tfrac{1}{2}e^{-2x}\right]_0^B=\frac{1}{2}$$



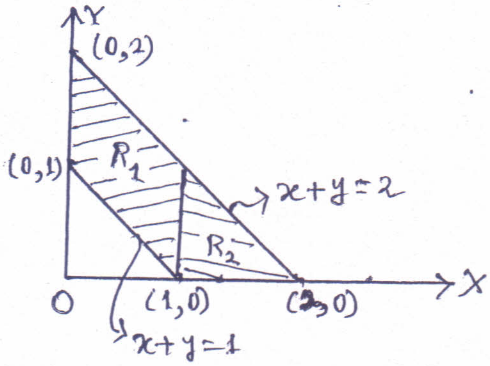
$$P(X<2Y)=\int_0^\infty e^{-x}\left\{\int_{x/2}^\infty e^{-y}\,dy\right\}dx$$

$$=\int_0^\infty e^{-3x/2}\,dx=\lim_{B\to\infty}\int_0^B e^{-3x/2}\,dx$$

$$=\lim_{B\to\infty}\left[-\tfrac{2}{3}e^{-3x/2}\right]_0^B=\frac{2}{3}.$$

$\therefore$ From (1): Required probability $=\dfrac{1/2}{2/3}=\dfrac{3}{4}$.

(iii) $P(1<X+Y<2)=P(1-X<Y<2-X)$



$$=P\{(X,Y)\in R_1\cup R_2\}$$

$$=\int_0^1 e^{-x}\left\{\int_{1-x}^{2-x} e^{-y}\,dy\right\}dx+\int_1^2 e^{-x}\left\{\int_0^{2-x} e^{-y}\,dy\right\}dx$$

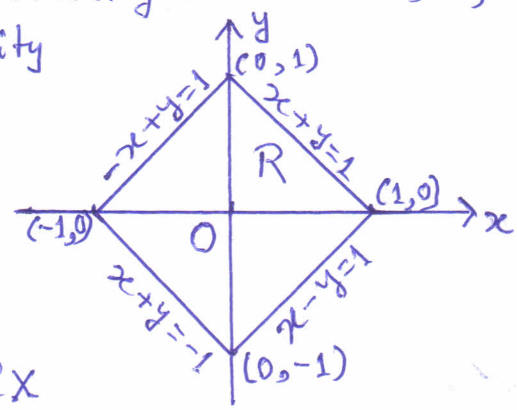
$$=\int_0^1 e^{-x}\left(e^{-1+x}-e^{-2+x}\right)dx+\int_1^2 e^{-x}\left(1-e^{-2+x}\right)dx$$

$$=\int_0^1\left(e^{-1}-e^{-2}\right)dx+\int_1^2\left(e^{-x}-e^{-2}\right)dx=e^{-1}-e^{-2}+\left[-e^{-x}-xe^{-2}\right]_1^2$$

$$=e^{-1}-e^{-2}-e^{-2}-2e^{-2}+e^{-1}+e^{-2}=2e^{-1}-3e^{-2}$$

Example 11: Raindrops fall at random on a square R with vertices (1, 0), (0, 1), (−1, 0), (0, −1). An outcome is the point (x, y) in R struck by a particular raindrop. Let $X(x,y) = x$, $Y(x,y) = y$ and assume (X, Y) has uniform distribution over R. Determine the joint and marginal distributions of X and Y. Are the random variables X and Y independent?

Solution: Here the two-dimensional random variable (X, Y) is uniform over the region R which is a square region of area 2 square units having vertices (1, 0), (0, 1), (−1, 0), (0, −1). The probability density function of the joint distribution of X and Y is:



$$f(x,y) = \begin{cases} \frac{1}{2}, & \text{for } (x,y) \in R \\ 0, & \text{otherwise} \end{cases}$$

Marginal density function $f_X(x)$ of X

When $0 < x \le 1$: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x-1}^{1-x} \frac{1}{2}\,dy = 1 - x$

When $-1 \le x \le 0$: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-x-1}^{x+1} \frac{1}{2}\,dy = 1 + x.$

Hence the marginal density function of X is given by

$$f_X(x) = \begin{cases} 1+x, & \text{for } -1 \le x \le 0 \\ 1-x, & \text{for } 0 < x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, the marginal density function of Y is given by

$$f_Y(y) = \begin{cases} 1+y, & \text{for } -1 \le y \le 0 \\ 1-y, & \text{for } 0 < y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Since $f(x,y) \ne f_X(x) f_Y(y)$, X and Y are not independent.