Lecture 1

27 August 2020 11:00

Linear Programming Problem

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0$$

1) $f(x+y) = f(x) + f(y) \to$ additivity
2) $f(cx) = c f(x) \to$ return to the scale

1) Applicability
2) Solvability — Simplex method.

Optimization (maximization or minimization) Linear objective function subject to some linear equalities or inequalities constraints on the decision variables.

When we will use LPP?

n — variables : n no of equations
no of relations < no of variables
no of " > " "
1) no of variables $\neq$ no of relations
2) relations are of the form inequalities $\to$ Use LPP.

1) Decision variables : $x_j \to$ amount of product j during some month
2) objective function : to be optimized

$$Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \to \text{Cost vector.}$$

Constraints : equalities or inequalities
restrictions on the decision variables.

Linear Prog. Prob.

Maximize (or Minimize) $Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

Maximize (or Minimize) $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ (\leq, =, \geq)\ b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ (\leq, =, \geq)\ b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ (\leq, =, \geq)\ b_m$$

$x_j \ (j = 1, \ldots, n) \rightarrow$ decision variables

$a_{ij},\ b_i,\ c_j \longrightarrow$ constants.

EX 1 2 items – table – chair

Rs 50,000 to invest. Storage capacity of atmost 60 pieces. Production cost of one table is given Rs 2500 and that of a chair is Rs 500. Profit from selling of one table Rs 250 and one chair Rs 75.

How many tables and chairs should produce to make his profit maximum from the available money.

formulation of LPP:

Let x be the no of tables and y be the no of chairs the firm produce.

Maximize $Z = 250x + 75y$

s.t $2500x + 500y \leq 50000 \rightarrow$ financial constraint

$5x + y \leq 100$

$x + y \leq 60 \longrightarrow$ storage constraint

$x \geq 0, y \geq 0 \longrightarrow$ non-negativity constraint

$$\left.\begin{aligned} &\text{Max } Z = 250x + 75y \\ &\text{s.t } \quad 5x + y \leq 100 \\ &\qquad\quad x + y \leq 60 \\ &\qquad\quad x \geq 0, y \geq 0 \end{aligned}\right\} \longrightarrow (1)$$

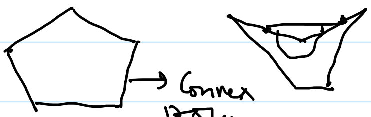
Feasible soln: Set of values of the decision variables which satisfies all the constraints of the LPP.

Feasible soln : set of values of the decision variables which satisfies all the constraints of the LPP.

Optimal soln : A feasible soln which optimizes the objective function is called an Optimal solution

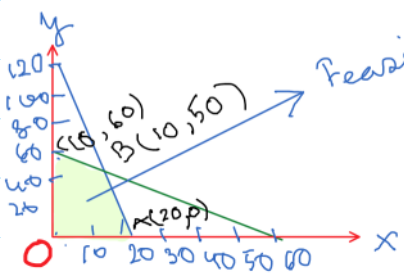
Graphical method of solving LPP with 2 variables.

Result 1: Let R be the feasible region (convex polygon — A convex polygon is a simple (not self intersecting) polygon in which no line segment between 2 points on the boundary ever goes outside the polygon) of a LPP and let $Z = ax + by$ be the objective function. When Z has an optimal value, it occurs at a corner point (vertex) of the feasible region.



Result 2 : If R is bounded, Z has both max and min value in R and each of these occurs at a corner point of R.

Note : If R is unbounded, max or min of objective fn may or may not exist.



Corner points

O(0,0), A(20,0), B(10,50)

C(0,60)

at O, $Z = 0$

at A, $Z = 5000$

at C, $Z = 4500$

at B, $Z = 6250$

$x = 10, y = 50$ ← optimal soln

$Z_{max} = 6250$ → optimal value of objective fn.

Lecture 2

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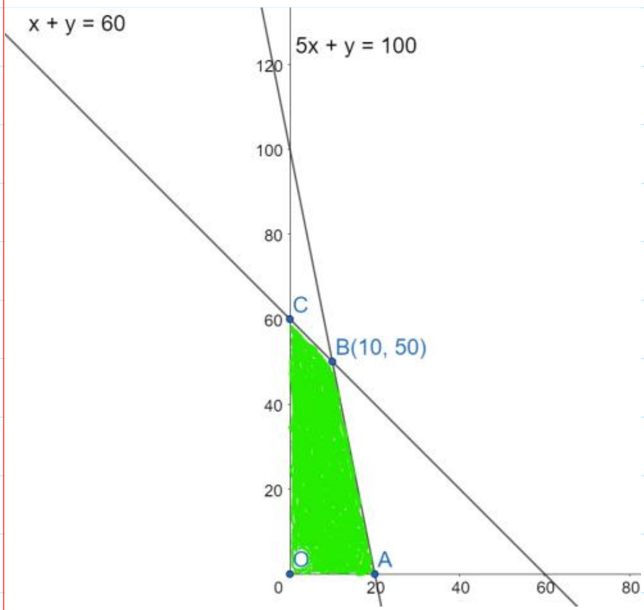


Alternative method

$Z = Z_1$ profit line – max

Cost line – min

$Z = 25$, $10x + 3y = 1$

unbounded feasible region

$$\text{Min } Z = -50x + 20y \to (1)$$

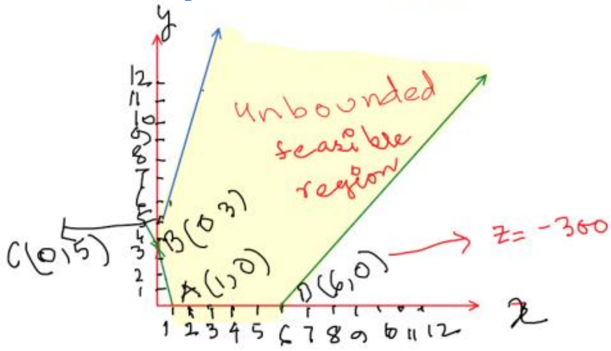
s.t

$$2x - y \geq -5 \to (2)$$

$$3x + y \geq 3 \to (3)$$

$$2x - 3y \leq 12 \to (4)$$

$$x \geq 0, y \geq 0 \to (5)$$



Remarks: Unbounded feasible region

1) M is max Z, if the open half plane $ax + by > M$ has no common pt with the feasible region. Otherwise it has no maximum.

2) Similarly, m is the minimum value of Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise it has no minimum.

$-50x + 20y = -300$ passes through $(6, 0)$

$-5x + 2y = -30$

$-50x + 20y < -300$ The open half plane has common points with the feasible region. The LPP has no optimal solution.

Infinite no. of optimal solution

$$\text{Max } Z = 18x + 6y$$

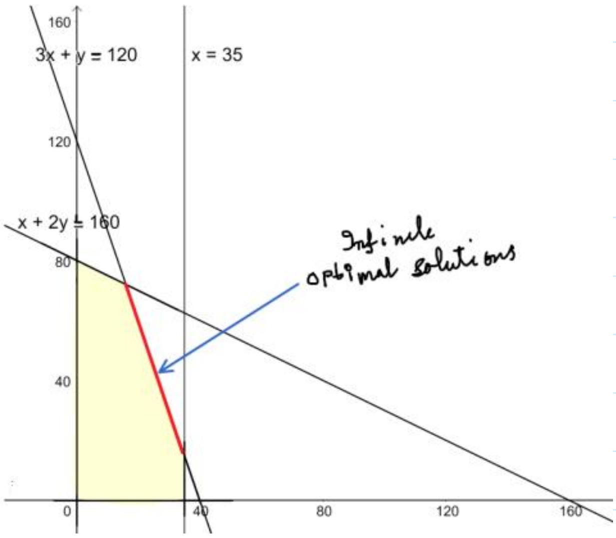
s.t

$$3x + y \leq 120$$

$$x + 2y \leq 160$$

$$x \leq 35$$

$$x \geq 0, y \geq 0$$

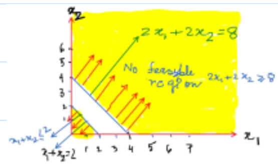


$z = 6 \qquad 3x + y = 1 \qquad 3x + y = 120$

infinite no of optimal solution

No feasible Region : $\text{Max } z = 2x_1 - 3x_2$

$$\text{s.t} \quad x_1 + x_2 \le 2$$
$$2x_1 + 2x_2 \ge 8$$
$$x_1, x_2 \ge 0$$



$\text{Min } z = x_1 + x_2$

$$\text{s.t} \quad x_1 + 2x_2 \le 8$$
$$3x_1 + 2x_2 \le 12$$
$$x_1 + 3x_2 \ge 13$$

Standard form of L.P.P

$\text{Max } z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

$$\text{s.t} \quad a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m$$
$$x_j \ge 0 \quad (j = 1, 2, \ldots, n)$$

$$\text{Max } z = c^T x$$
$$\text{s.t} \quad Ax = b, \; x \ge 0$$

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad A = [a_{ij}]_{m \times n}$$

Transformation to standard LPP form.

1) If the obj. fn is to minimize $z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

Max $\overset{*}{Z} = -Z = -c_1x_1 - c_2x_2 - c_3x_3 \cdots\cdots - c_nx_n$

2) inequality constraint of 'less than type'

say $\quad a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$

introduce slack variable say S

s.t. $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + S = b_i$ and $S \geq 0$

3) inequality constraint 'greater than type'

by subtracting 'surplus variable' say S

s.t. $a_{i1}x_1 + a_{i2}x_2 \cdots + a_{in}x_n - S = b_i$

$S \geq 0$

Remark: Cost of surplus and slack variables are considered as zero in the objective function.

4) x_j — unrestricted in sign $x_j = x_j^+ - x_j^-$ where

$x_j^+ \geq 0, \; x_j^- \geq 0$

Lecture 3

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Note :- Cost of surplus and slack variables are 0.

Ex. – Min $Z = 2x_1 - x_2$

s.t $x_1 + x_2 \geq 2$

$3x_1 + 2x_2 \leq 4$

$x_1 + 2x_2 = 3$

$x_2 \geq 0$, $x_1 \leftarrow$ unrestricted in sign

The standard LPP is given by $[\, x_1 = x_1^+ - x_1^- \,]$

Max $\overset{*}{Z} = -Z = -2x_1^+ + 2x_1^- + x_2 + 0 \cdot x_3 + 0 \cdot x_4$

s.t $x_1^+ - x_1^- + x_2 - x_3 = 2$

$3x_1^+ - 3x_1^- + 2x_2 + x_4 = 4$

$x_1^+ - x_1^- + 2x_2 = 3$

where $x_1^+ \geq 0, x_2 \geq 0, x_1^- \geq 0, x_3 \geq 0, x_4 \geq 0$

and x_3 and x_4 are surplus and slack variables.

Euclidean Space :– E.S of n dim denoted by E^n, is the set of n component real vectors $x = (x_1, x_2, \ldots, x_n)$ and addition and scalar multiplication of vectors are defined. There exists a +ve no. called distance between two vectors x and y where $y = (y_1, y_2, \ \ , y_n)$, given by

$$d = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2}$$

Linearly independence and dependence : A set of vectors $\{x_1, x_2, \ldots, x_n\} \in E^n$ is said to be lin. dependent, if $\exists$ a set of scalars λ_i, $i = 1, 2, \ldots, n$, not all zero, s.t $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \bar{0}$.

If $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \bar{0} \Rightarrow \lambda_i = 0$ for all $i = 1, \ldots, n$ then the set $\{x_1, x_2, \ldots, x_n\}$ is called lin. independent.

$x = $ L.C of $\{x_1, \ldots, x_n\} \rightarrow x \leftarrow$ lin. dependent.

Spanning set :– A set of n-component vectors (B) is said to be a spanning set or generator E^n, if every vector of E^n can be represented as linear combination of vectors of the set B.

Ex: $(a, b, c) \in E^3$, $(a, b, c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$\{(1,0,0), (0,1,0), (0,0,1)\}$ is a spanning set of E^3.

Basis: Any linearly independent set of vectors from E^n, which spans E^n is called a basis for E^n.

$$\lambda_1 (1,0,0) + \lambda_2 (0,1,0) + \lambda_3 (0,0,1) = \bar{0} = (0,0,0)$$

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$$

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis for E^3.

dimension ← no of elements in the basis.

Result — 1) Every basis of E^n contains precisely n no of vectors.

2) Representation of a vector in terms of a basis is unique. (Co-ordinates).

Ex:- Show that the vectors $B = \{(3,0,2), (7,0,9), (4,1,2)\}$ forms a basis of E^3.

$$\begin{vmatrix} 3 & 0 & 2 \\ 7 & 0 & 9 \\ 4 & 1 & 2 \end{vmatrix} = -13 \neq 0 \qquad B \text{ is lin independent.}$$

Let $(a,b,c) \in E^3$ $\quad (a,b,c) = c_1 (3,0,2) + c_2 (7,0,9) + c_3 (4,1,2)$

$$a = 3c_1 + 7c_2 + 4c_3$$
$$b = c_3$$
$$c = 2c_1 + 9c_2 + 2c_3$$

B spans the space E^3.

B is a basis of E^3.

Replacement theorem:- Let $\{x_1, x_2, \ldots, x_n\}$ is a basis of E^n, and a non-zero vector x of E^n can be expressed as a linear combination of these vectors as

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$$

where λ_i's are scalars. If $\lambda_i \neq 0$, x can replace x_i in the basis.

Extension theorem:- Any lin. independent set of E^n, can be extended to a basis.

Prob: $B = \{(1,2,1), (2,1,0), (1,1,-2)\}$ forms a basis of R^3.

$\boxed{\text{Prob}}$: Find a basis for E^3 that contains the vectors $(1,2,2)$ and

$(2,0,1)$.

Soln Consider $\{e_1, e_2, e_3\}$ of E^3. $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$a_1 = (1,2,2)$, $a_2 = (2,0,1)$

$(1,2,2) = 1(1,0,0) + 2(0,1,0) + 2(0,0,1)$

$(1,2,2)$ can replace any vector $\{e_1, e_2, e_3\}$

$\{(1,2,2), (0,1,0), (0,0,1)\}$

$a_2 = (2,0,1) = \lambda_1 (1,2,2) + \lambda_2 (0,1,0) + \lambda_3 (0,0,1)$

$\lambda_1 = 2, \lambda_2 = -4, \lambda_3 = -3$

$\{(1,2,2), (2,0,1), (0,0,1)\}$ is a new basis involving the given vectors a_1 and a_2.

$\boxed{\text{Prob}}$:- Given $A = \{(1,2,1,3), (1,0,0,0), (0,0,1,0), (0,1,0,1)\}$ is a basis of R^4, that $B = \{(1,0,1,0), (0,2,0,3)\}$ is lin. ind. You have to extend B to a basis of R^4.

Lecture 4

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$\boxed{\text{Prob 4}}$:- $A = \{(1,2,1,3), (1,0,0,0), (0,0,1,0), (0,1,0,1)\}$ basis of $\mathbb{R}^4$. $B = \{\overset{u}{(1,0,1,0)}, \overset{v}{(0,2,0,3)}\}$ ← linearly independent set. We have to extend B to a basis of $\mathbb{R}^4$.

$$(1,2,1,3) = (1,0,1,0) + (0,2,0,3)$$

$$(1,2,1,3) \in LS\{(1,0,1,0), (0,2,0,3)\}$$

$$(1,0,0,0) \notin L.S\{(1,0,1,0), (0,2,0,3)\}$$

Using Extension theorem → B is extended to $\{(1,0,1,0), (0,2,0,3), (1,0,0,0)\}$

$$(0,0,1,0) \in LS\{u, v, (1,0,0,0)\}$$

$$(0,1,0,1) \notin LS\{u, v, (1,0,0,0)\}$$

$\{u, v, (1,0,0,0), (0,1,0,1)\}$ → is a basis of $\mathbb{R}^4$

Rank

rank of a non zero matrix say A, is the order of the largest square submatrix in A whose determinant does not vanish.

rank ⟵ maximum no of linearly independent row vectors or column vectors of the matrix.

If A is nonsingular then all the rows and columns are linearly independent.

Basic solution: Let us consider a LPP $\max z = c^T x$

s.t $Ax = b$, $x \geq 0$, $A = (a_{ij})_{m\times n} \to \in \mathbb{R}^{m\times n}$, $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$

$b = \begin{pmatrix} b_1 \\ b_2 \\ b_m \end{pmatrix} \in \mathbb{R}^m$ and here x is to be optimized.

The set of eqns can be written in the form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

$$a_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \quad a_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \quad \dots \quad a_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Assume Rank$(A) = m$ ⟹ m equations are independent.

From n column vectors, we select m no of linearly independent vectors which constitute a basis B of the space $\mathbb{R}^m$.

The vectors not included in the selected set are called non-basic vectors. Assume that all the $(n-m)$ variables associated to the non-basic vectors are zero, we get a system of m equations with m variables. The coefficient matrix B is said to be the basis matrix which is non-singular.

There exists a unique soln to the system of m equations with m variables. This solution is called the basic solution. Vectors associated with the basis vectors are called basic variables. no of basic variables $= m$

no of non-basic variables $= n-m$

Partition A into B and R and write $A = [B \mid R]$

$$Ax = b \Rightarrow Bx_B + Rx_R = b$$

Letting $x_R = 0$, $x_B = B^{-1}b$

Out of n vectors, m vectors are chosen to constitute the basis, the maximum no of basic solution $= {}^nC_m$

Basic feasible solution. (B.F.S) If a feasible solution is basic, then it is called Basic feasible solution.

Degenerate Basic solution: A basic solution is called degenerate if one or more basic variable(s) vanish.

Non-degenerate basic soln: — A basic soln is said to to be non-degenerate if all the basic variables are non-zero.

Ex:- Max $Z = 5x_1 + 4x_2$

s.t $6x_1 + 4x_2 \le 24$

$x_1 + 2x_2 \le 6$

$x_1, x_2 \ge 0$

Find the basic feasible solns.

Reduced to standard form

Max $Z = 5x_1 + 4x_2 + 0.x_3 + 0.x_4$

s.t $6x_1 + 4x_2 + x_3 = 24$

$x_1 + 2x_2 + x_4 = 6$

where x_3, x_4 are slack variables and $x_i \ge 0$, $i = 1, \ldots, 4$.

$n = 4$, $m = 2$, max no of basic soln $= 6$

$n - m = 4 - 2 = 2$ variables are set to zero.

1) $x_1 = 0, x_2 = 0 \Rightarrow x_3 = 24, x_4 = 6$

$X_1 = (0, 0, 24, 6) \rightarrow$ is a basic feasible soln. Corr. $Z = 0$

2) $x_2 = 0, x_4 = 0$, $x_1 = 6$, $x_3 = -12$

$X_2 = (6, 0, -12, 0)$ is a basic soln but not basic feasible soln.

3) $x_3 = 0, x_4 = 0, \ x_1 = 3, x_2 = 1.5$

$X_3 = (3, 1.5, 0, 0)$ is a basic feasible soln $z = 21$

4) $x_1 = 0, \ x_3 = 0, \ x_2 = 6; x_4 = -6$

$X_4 = (0, 6, 0, -6)$ is a basic soln but not basic feasible solution.

5) $x_2 = 0, x_3 = 0, \Rightarrow \ x_1 = 4, \ x_4 = 2$

$X_5 = (4, 0, 0, 2)$ is a b.f.s.

$z = 20$

6) $x_1 = 0, \ x_4 = 0 \Rightarrow x_2 = 3, \ x_3 = 12$

$X_6 = (0, 3, 12, 0)$ is a b.f.s

$z = 12$

The optimal soln is $(3, 1.5, 0, 0)$ and optimal value of the obj. function $z = 21$.

Ex. Find the b.f.s of the following set

$$2x_1 + 5x_2 - x_3 + 4x_4 = 8 \qquad x_i \geq 0$$
$$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$$

Lecture 5

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$\boxed{\text{Solution}}$:

We write the system as

$x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4 = b$

where $a_1 = [2, 1]$, $a_2 = [3, -2]$, $a_3 = [-1, 6]$, $a_4 = [4, -7]$, $b = [8, -3]$

Here max no of basic solution $= {}^4C_2 = 6$ since $m = 2$, $n = 4$.

The six sets of 2 vectors out of 4 are

$B_1 = [a_1, a_2]$, $B_2 = [a_1, a_3]$, $B_3 = [a_1, a_4]$, $B_4 = [a_2, a_3]$, $B_5 = [a_2, a_4]$,

$B_6 = [a_3, a_4]$

$|B_1| = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7$, $|B_2| = \begin{vmatrix} 2 & -1 \\ 1 & 6 \end{vmatrix} = 13$, $|B_3| = -18$, $|B_4| = 16$, $|B_5| = -13$,

$|B_6| = -17$.

$$x_{B_1} = B_1^{-1} b = -\frac{1}{7}\begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$X_1 = [1, 2, 0, 0]$ is a b.f. solution.

$x_{B_2} = B_2^{-1} b = \frac{1}{13}\begin{bmatrix} 6 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{45}{13} \\ -\frac{14}{13} \end{bmatrix}$ — basic but not b.f.s.

$x_{B_3} = B_3^{-1} b = \begin{bmatrix} \frac{22}{9} \\ \frac{7}{9} \end{bmatrix}$ $X_2 = [\frac{22}{9}, 0, 0, \frac{7}{9}]$ is a b.f.s.

$x_{B_4} = B_4^{-1} b = \begin{bmatrix} 45/16 \\ 7/16 \end{bmatrix}$ $X_3 = [0, \frac{45}{16}, \frac{7}{16}, 0]$ is a b.f.S

$x_{B_5} = B_5^{-1} b = \begin{bmatrix} \frac{44}{13} \\ -7/13 \end{bmatrix}$ but not b.f.S.

$x_{B_6} = B_6^{-1} b = \begin{bmatrix} 44/17 \\ 45/17 \end{bmatrix}$ $X_4 = [0, 0, \frac{44}{17}, \frac{45}{17}]$ is a b.f.s.

Prob: Show that the feasible solution $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 2$ to the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 + x_2 - 3x_3 &= 2 \\ 2x_1 + 4x_2 + 3x_3 - x_4 &= 4 \end{aligned}$$

is not basic.

Soln: There are 4 variables and 3 equations. Hence a non-degenerate basic feasible solution of the given system will contain 3 non zero variables.

The vectors associated to the non zero variables ($x_3 = 0$) in the given feasible solution are

$a_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, $a_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

If $(1, 1, 0, 2)$ is a non-degenerate b.f.s then a_1, a_2, a_4 must be linearly independent

$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_4 = 0$

$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 4 & -1 \end{vmatrix} = 0$ The given solution is not basic.

Fundamental Theorem of Linear Programming Problem.

Consider the LPP in its standard form

$$\text{Max } Z = CX \quad , \quad X \geq 0$$
$$\text{s.t } AX = b$$

$A = [a_{ij}]_{m \times n}$ and $A = [a_1, a_2, \dots, a_n]$

$a_j = [a_{1j}, a_{2j}, \dots, a_{mj}] \qquad j = 1, \dots, n.$

The f.T states that if the LPP admits of an optimal solution then the optimal soln will coincide with at least one of the basic feasible solns of the problem.

Reduction of a feasible solution to a basic feasible solution.

Statement: If there be a feasible solution to a set of m simultaneous linear equations $Ax = b$, $x \geq 0$ in n no of unknowns $(n > m)$ and if $\text{rank}(A) = m$ then there is a basic feasible solution to the set of eqns.

Proof: – Let us assume that a feasible solution to the given system of equations with p no of positive variables.

Thus let the feasible soln be

$$\overset{*}{x} = [x_1, x_2, \dots, x_p, 0, 0, \dots, 0]$$

Therefore $\overset{*}{x}$ has $(n-p)$ components zero.

Now, as $\overset{*}{x}$ is a feasible solution,

$$A\overset{*}{x} = b.$$

$$\sum_{j=1}^{p} a_j x_j = b \quad \rightarrow (1)$$

where $A = [a_1, a_2, \dots a_p, a_{p+1}, \dots, a_n] \quad \rightarrow (2)$

The vectors a_j associated with the components of $\overset{*}{x}$ may be either lin. independent or dependent.

If linearly independent then $\overset{*}{x}$ is a basic feasible solution and will be non degenerate if $p = m$ and degenerate if $p < m$.

But if it is linearly dependent, then we can reduce the no of positive components of $\overset{*}{x}$ step by step until the vectors associated with the variables are lin. independent.

Continuation of the theorem: if linearly dependent then

$\sum_{j=1}^{p} \lambda_j a_j = 0 \Rightarrow$ there exists at least one λ_j, not equal to zero. $\rightarrow (3)$

say $\lambda_r \neq 0$.

then we can write $a_r = -\sum_{j=1}^{p} \frac{\lambda_j}{\lambda_r} a_j$ where $j \neq r$

Substituting it in (1) [i.e. $\sum_{j=1}^{p} a_j x_j = b$] we get,

$$\sum_{\substack{j=1 \\ j\neq r}}^{p} a_j \left(x_j - x_r \frac{\lambda_j}{\lambda_r}\right) = b \quad \rightarrow (4)$$

$\therefore \left(x_j - x_r \frac{\lambda_j}{\lambda_r}\right)$ for $j = 1, 2, \cdots, (r-1), (r+1), \cdots, p$ together with $(n-p+1)$ number of zeroes is a solution to $Ax = b$.

Let the solution be denoted by $x_1 = \left[\left(x_1 - x_r \frac{\lambda_1}{\lambda_r}\right), \left(x_2 - x_r\frac{\lambda_2}{\lambda_r}\right), \cdots, \left(x_{r-1} - x_r\frac{\lambda_{r-1}}{\lambda_r}\right), 0, \left(x_{r+1} - x_r \frac{\lambda_{r+1}}{\lambda_r}\right), \cdots, \left(x_p - x_r \frac{\lambda_p}{\lambda_r}\right), 0, 0, 0 \cdots, 0\right]$

Thus we get a solution with not more than $(p-1)$ nonzero variables.
We choose a_r s.t. $(p-1)$ nonzero components of x_1 is non-negative, i.e. X_1 is a feasible solution. $x_j - x_r \frac{\lambda_j}{\lambda_r} \geq 0 \rightarrow (5)$

i.e. $\frac{x_j}{\lambda_j} - \frac{x_r}{\lambda_r} \geq 0$ if $\lambda_j > 0$
and $\frac{x_j}{\lambda_j} - \frac{x_r}{\lambda_r} \leq 0$ if $\lambda_j < 0$ $\Big\} \rightarrow (6)$

For any j, for which $\lambda_j = 0$, (5) is automatically satisfied.

Thus our choice of a_r should be such that $\frac{x_r}{\lambda_r} \leq \frac{x_j}{\lambda_j}$ if $\lambda_j > 0$
and $\frac{x_r}{\lambda_r} \geq \frac{x_j}{\lambda_j}$ if $\lambda_j < 0$ $\rightarrow (7)$

So we choose $\frac{x_r}{\lambda_r}$ such that $\boxed{\max_{\lambda_j < 0} \left\{\frac{x_j}{\lambda_j}\right\} \leq \frac{x_r}{\lambda_r} \leq \min_{\lambda_j > 0} \left\{\frac{x_j}{\lambda_j}\right\}}$

Hence the new solution is feasible satisfying (7)

Thus we get a solution with not more than $(p-1)$ positive variables and the remaining variables being zero. If the vectors associated with the non-zero variables in the solution x_1 are linearly independent then x_1 is a basic feasible solution. Again if this is not lin. independent, the above procedure can be repeated and can reduce one more variable to zero. Continuing this process, ultimately we get the basic feasible solution.

Prob 1: $x_1 = 2, x_2 = 3, x_3 = 1$ is a feasible solution of the following LPP

Max $z = x_1 + 2x_2 + 4x_3$

s.t.
$$\left.\begin{aligned} 2x_1 + x_2 + 4x_3 &= 11 \\ 3x_1 + x_2 + 5x_3 &= 14 \end{aligned}\right\} \rightarrow (1)$$

s.t $\left.\begin{array}{l} 2x_1 + x_2 + 4x_3 = 11 \\ 3x_1 + x_2 + 5x_3 = 14 \end{array}\right\} \to (1)$

$x_1, x_2, x_3 \geq 0$. Find a basic feasible solution.

Soln: – Since $(2,3,1)$ is a feasible solution,

$2a_1 + 3a_2 + 1 \cdot a_3 = b \longrightarrow (2)$

where $a_1 = \binom{2}{3}$, $a_2 = \binom{1}{1}$, $a_3 = \binom{4}{5}$, $b = \binom{11}{14}$

But $(2,3,1)$ is not basic soln. Since here no of nonzero variables is three. We have to reduce this to basic solution where the maximum number of nonzero variables will be 2. Here the vectors a_1, a_2, a_3 are linearly dependent. i.e. $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = \bar{0}$ $\to (3)$

$\Rightarrow$ there exist at least one $\lambda_i \neq 0$.

$\left.\begin{array}{l} 2\lambda_1 + \lambda_2 + 4\lambda_3 = 0 \\ 3\lambda_1 + \lambda_2 + 5\lambda_3 = 0 \end{array}\right\} \to (4)$

$\frac{\lambda_1}{5-4} = \frac{\lambda_2}{12-10} = \frac{\lambda_3}{2-3} = k$ (say)

$k=1 \Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1 \longrightarrow (5)$

$a_1 + 2a_2 - a_3 = \bar{0}$

To reduce the no of positive variables from the feasible solution we find $\frac{x_r}{\lambda_r} = \min_j \left\{ \frac{x_j}{\lambda_j}, \lambda_j > 0 \right\}$

$= \min \left\{ \frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2} \right\}$ $[\lambda_3 < 0]$

$= \min \left\{ \frac{2}{1}, \frac{3}{2} \right\} = \frac{3}{2} = \frac{x_2}{\lambda_2} = \frac{x_r}{\lambda_r}$ $r=2$

min occurs at $j=2$. We will eliminate a_2 from (2)

i.e. we will reduce x_2 to be zero.

new variable $= x_j - \frac{x_r}{\lambda_r} \lambda_j$

new $x_1 = 2 - \frac{3}{2} \cdot 1 = \frac{1}{2}$

new $x_2 = 3 - \frac{3}{2} \cdot 2 = 0 \leftarrow$

new $x_3 = 1 - \frac{3}{2} \cdot (-1) = \frac{5}{2}$

Hence the new solution is $\left(\frac{1}{2}, 0, \frac{5}{2}\right)$. a_1, a_3 are linearly independent, this soln is basic feasible solution.

If again, $\frac{x_r}{\lambda_r} = \max_j \left\{ \frac{x_j}{\lambda_j}, \lambda_j < 0 \right\} = \max \left\{ \frac{x_3}{\lambda_3} \right\} = -1$

then we get another basic feasible solution from the following new $x_1 = 2 - (-1)\cdot 1 = 3$

new $x_2 = 3 - (-1)\cdot 2 = 5$

new $x_3 = 1 - (-1)(-1) = 0$

$\therefore$ $(3, 5, 0)$ is the second basic feasible solution of the given system.

Prob 2 :- Given $x_1 = 1,\ x_2 = 1,\ x_3 = 1,\ x_4 = 0$ is a feasible solution of the system of equations

$$x_1 + 2x_2 + 4x_3 + x_4 = 7$$
$$2x_1 - x_2 + 3x_3 - 2x_4 = 4$$

Reduce the feasible solution to one basic feasible solution.

Prob 3 : $x_1 = 1, x_2 = 2, x_3 = 1, x_4 = 0$ is a feasible soln to the system

$$11x_1 + 2x_2 - 9x_3 + 4x_4 = 6$$
$$15x_1 + 3x_2 - 12x_3 + 6x_4 = 9$$

Reduce the feasible solution to more than one Basic feasible solution and prove That one of them is non degenerate and the other is degenerate.

Prob 4: Show that $x_1 = 5,\ x_2 = 0,\ x_3 = -1$ is a basic solution of the system of eqns

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5$$

Find the other basic solution if there be any.

Prob 5: The 2 linearly independent equations with 3 variables are given as

$$2x_1 - 3x_2 + 5x_3 = 10$$
$$4x_1 + x_2 + 10x_3 = 20$$

Find if possible a basic solution with x_2 as a nonbasic variable.

Prob 6 : Reduce the feasible soln $x_1 = 2, x_2 = 1, x_3 = 1$ of the given system

$$x_1 + 4x_2 - x_3 = 5$$
$$2x_1 + 3x_2 + x_3 = 8$$

to a basic feasible solution.

Lecture 7 — Convex set

08 October 2020 08:51

Convex combination :- If a point x can be expressed as

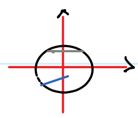
$$x = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_p x_p$$ where $\lambda_i \geq 0$ and x_i are finite number of points of E^n and $\sum_{i=1}^{p} \lambda_i = 1$, then x is said to be a convex combination of the points $x_1, x_2, \dots, x_p$

So for 2 points x_1, x_2, the convex combination can be written as

$$x = \lambda x_2 + (1-\lambda) x_1 \quad \text{when } 0 \leq \lambda \leq 1$$

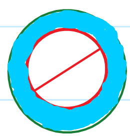
Convex set. A set X is said to be a convex set, if for any 2 points $x_1, x_2 \in X$, their convex combination $x = \lambda x_1 + (1-\lambda) x_2$, $0 \leq \lambda \leq 1$ also belongs to the set X.

Ex:-



The circle $x^2 + y^2 \leq 1$ with its interior is a convex set. But the boundary only $\{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$ is not convex.

A triangle is also an example of convex set.



A ring is not a convex set.

Hyperplane A line is a set of points in E^2, satisfying $c_1 x_1 + c_2 x_2 = z$. Similarly a plane is a set of points in E^3 satisfying $c_1 x_1 + c_2 x_2 + c_3 x_3 = z$.

Generalizing this we can say a set of n points in E^n, satisfying the linear constraints $c_1 x_1 + c_2 x_2 + \dots + c_n x_n = z$ is hyperplane for a particular z.

Result 1 Hyperplane is a convex set.

Let us consider the hyperplane $X = \{x : cx = z\}$ and $x_1, x_2 \in X$ are 2 points of X. So, $cx_1 = z$, $cx_2 = z$ $\therefore$ the convex combination of x_1, x_2 can be written as $x_3 = \lambda x_1 + (1-\lambda) x_2$, $0 \leq \lambda \leq 1$

$\therefore$ $cx_3 = \lambda c x_1 + (1-\lambda) c x_2 = \lambda z + (1-\lambda) z = z$

$\therefore$ $x_3 \in X$. $\therefore$ Convex combination of any 2 pts of hyperplane belongs to the hyperplane. $\therefore$ X is a convex set.

Result 2: Intersection of two convex sets is a convex set.

Let $X = X_1 \cap X_2$. If $x_1, x_2 \in X = X_1 \cap X_2$ then $x_1 \& x_2 \in X_1$ and $x_1, x_2 \in X_2$

Since X_1 and X_2 are convex sets,

$\therefore$ $\lambda x_2 + (1-\lambda) x_1 \in X_1$ and $\lambda x_2 + (1-\lambda) x_1 \in X_2$ for $0 \leq \lambda \leq 1$

$\therefore$ $\lambda x_2 + (1-\lambda) x_1 \in X_1 \cap X_2 = X$. Therefore X is a convex set.

Note :- 1) Union of convex sets may not be convex.

2) The above result can be generalized for n

open half space, closed half space

If the whole space is partitioned into 3 sets X_1, X_2, X_3 given by

$X_1 = \{x : cx < z\}$ → open half space

$X_2 = \{x : cx = z\}$ → Hyperplane

$X_3 = \{x : cx > z\}$ → open half space

Where $\bar{X}_1 = \{x : cx \le z\}$ is a closed half space.

$\bar{X}_2 = \{x : cx \ge z\}$ " " " "

In LPP the objective function is a hyper plane, which is a Convex Set.
The set of feasible solutions is intersections of hyperplane or half space and is a convex set.

Result 3: The set of all feasible solutions of a LPP is a convex set.

Consider any LPP, where the constraints are given by $Ax = b$, $x \geqslant 0$.

Let X be the set of all feasible solutions

If x_1 and x_2 are 2 feasible solutions then $Ax_1 = b$, $Ax_2 = b$ & $x_1 \geqslant 0$ & $x_2 \geqslant 0$.

Let x_3 be the convex combination of x_1 and x_2 as given by

$$x_3 = \lambda x_1 + (1-\lambda) x_2 \quad , \quad 0 \le \lambda \le 1$$

$$\therefore \quad Ax_3 = \lambda A x_1 + (1-\lambda) A x_2 = \lambda b + (1-\lambda) b = b$$

and $x_3 \geqslant 0$ since $x_1, x_2 \geqslant 0$ and $0 \le \lambda \le 1$.

$\therefore$ x_3 is also a feasible solution.

$\therefore$ X is a convex set.

Prob 1: Show that in E^2, the set $X = \{(x_1, x_2) \mid x_1 - 2x_2 = 2\}$ is a convex set.

Solution:- Here we see that the set X is not empty. Let (x_{11}, x_{21}) and (x_{12}, x_{22}) be any 2 pts of the set.

Then
$$\left.\begin{aligned} x_{11} - 2x_{21} &= 2 \\ x_{12} - 2x_{22} &= 2 \end{aligned}\right\} \rightarrow (1)$$

The convex combination of 2 points $x_3 = \lambda(x_{11}, x_{21}) + (1-\lambda)(x_{12}, x_{22})$

$$(x_{13}, x_{23}) = (\lambda x_{11} + (1-\lambda) x_{12} \, , \, \lambda x_{21} + (1-\lambda) x_{22})$$

Consider $x_{13} - 2x_{23} = \lambda x_{11} + (1-\lambda) x_{12} - 2(\lambda x_{21} + (1-\lambda) x_{22})$

$= \lambda(x_{11} - 2x_{21}) + (1-\lambda)(x_{12} - 2x_{22})$ [by 1]

$= \lambda \cdot 2 + 2 - 2\lambda = 2$

Hence the pt x_3 belongs to the given set.

Thus the set is convex.

Prob 2: Prove that the set defined by $X = \{x : |x| \leq 2\}$ is a convex set.

Soln: Let x_1 and x_2 be any 2 points of the set X.

i.e. $|x_1| \leq 2$, $|x_2| \leq 2 \longrightarrow (1)$

Let the convex combination of x_1, x_2 is given by

$x_3 = \lambda x_1 + (1-\lambda) x_2, \quad 0 \leq \lambda \leq 1$

Now $|\lambda x_1 + (1-\lambda) x_2| \leq |\lambda x_1| + |(1-\lambda) x_2|$

$= \lambda |x_1| + (1-\lambda)|x_2| \leq 2\lambda + (1-\lambda)2 = 2$

$|x_3| \leq 2$. $x_3 \in X$. $\therefore X$ is convex set.

Prob 3: Prove that in E^2, the set $X = \{(x,y) : x^2 + y^2 \leq 4\}$ is a convex set.

i) geometrical approach: — The set of points represents a circle of radius 2 with the boundary and all of its interior points. Any line segment joining any 2 points of the set belongs to the set. So the set is convex.

ii) Analytically: — Let (x_1, y_1) and (x_2, y_2) be 2 points of the set X, $\therefore$

$$\left.\begin{aligned} x_1^2 + y_1^2 &\leq 4 \\ x_2^2 + y_2^2 &\leq 4 \end{aligned}\right\} \longrightarrow (1)$$

Any convex combination of 2 points is given by

$(\lambda x_1 + (1-\lambda) x_2, \ \lambda y_1 + (1-\lambda) y_2)$ where $0 \leq \lambda \leq 1$

Now consider, $(\lambda x_1 + (1-\lambda) x_2)^2 + (\lambda y_1 + (1-\lambda) y_2)^2$

$= \lambda^2 (x_1^2 + y_1^2) + (1-\lambda)^2 (x_2^2 + y_2^2) + 2 \cdot \lambda(1-\lambda)(x_1 x_2 + y_1 y_2)$

$\leq \lambda^2 (x_1^2 + y_1^2) + (1-\lambda)^2 (x_2^2 + y_2^2) + 2\lambda(1-\lambda)\left(\frac{x_1^2 + x_2^2}{2} + \frac{y_1^2 + y_2^2}{2}\right)$

$\leq 4\lambda^2 + 4(1-\lambda)^2 + 8\lambda(1-\lambda)$

$= 4$

$\therefore$ Convex combination of 2 pts belong to the given set.

$\therefore$ The set is convex.

Prob 4: Prove that in E^2, the set $X = \{(x,y) : y^2 \leq x\}$ is convex.

Proof: Let (x_1, y_1), (x_2, y_2) be any 2 points of the set then

$y_1^2 \le x_1$, $y_2^2 \le x_2$ $\longrightarrow$ (1)

The convex combination of the points is given by

$$\left(\lambda x_1 + (1-\lambda) x_2 \, , \, \lambda y_1 + (1-\lambda) y_2 \right), \quad 0 \le \lambda \le 1$$

Consider $(\lambda y_1 + (1-\lambda) y_2)^2 = \lambda^2 y_1^2 + (1-\lambda)^2 y_2^2 + 2\lambda(1-\lambda) y_1 y_2$

$$\le \lambda^2 y_1^2 + (1-\lambda)^2 y_2^2 + \lambda(1-\lambda)(y_1^2 + y_2^2)$$

$$= \lambda y_1^2 + (1-\lambda) y_2^2 \le \lambda x_1 + (1-\lambda) x_2$$

$[y_1^2 + y_2^2 \ge 2y_1 y_2]$
[by (1)]

$\therefore$ The convex combination belongs to the set X.

$\therefore$ X is convex.

Prob 4:— Prove that in E^2, the set $X = \{(x, y) \,|\, y^2 \ge 4x\}$ is not convex.

Soln: Consider $(0,0)$ and $(1,2)$ are 2 pts of X.

Convex combination for particular value of $\lambda = \frac{1}{2}$

$$\left(\tfrac{1}{2} \cdot 0 + \tfrac{1}{2} \cdot 1 \, , \, \tfrac{1}{2} \cdot 0 + \tfrac{1}{2} \cdot 2 \right) = \left(\tfrac{1}{2}, 1\right) \notin X$$

as $1^2 \not\ge 4 \cdot \frac{1}{2}$

The set is not convex.

Lecture 8

15 October 2020 10:53

Extreme point of Convex Set.

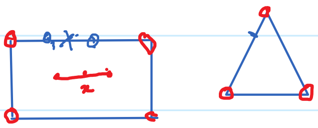
A point x is said to be an extreme point of the convex set C if it can not be expressed as the convex combination of any other two distinct points x_1 and x_2 of the set C.

i.e. x can not be written as $x = \lambda x_1 + (1-\lambda) x_2$ where $0<\lambda<1$.

Note: All extreme points of the convex set are boundary points of the convex set but all boundary points are not extreme points in general.

For example:- 1) Every point on the boundary of a circle is an extreme point of the convex set containing the boundary and interior of the circle.

2) 4 vertices of a rectangle or 3 vertices of a triangle are extreme points.



3) A straight line has no extreme point.

Convex hull :- If X be a point set, then the convex hull of X is the set of all convex combinations of the set of points from X. It is denoted by $C(X)$.

Ex: i) Let X be the set of points lying on the boundary of a circle, then the circle with its interior is its convex hull.

ii) The whole cube is the convex hull of the set of points consisting of eight vertices of the cube in E^3.

Property: $C(X)$ is the smallest convex set containing the set of pts of X.

The set of all convex combinations of a finite number of points is called convex polyhedron generated by the points.

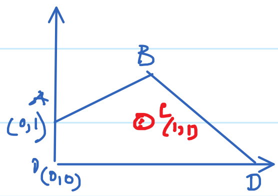
The convex hull of 8 vertices of a cube is an example of convex polyhedron but the circle is not convex polyhedron.

An n dimensional convex polyhedron with $(n+1)$ no. of vertices is called a simplex.

For example → 1 dim line is a simplex
2 dim triangle is a simplex
3 dim tetrahedron is a simplex.

(b1)

Ex. Draw the convex hull of the points $(0,0)$, $(0,1)$, $(1,2)$, and $(4,0)$. Which of these points is an interior point of the convex hull? Express it as a convex combination of the extreme points.



The region bounded by AB, BD, DO, OA gives the convex hull. Here $C(1,1)$ is an interior point of the convex hull.

$$(1,1) = \lambda_1(0,0) + \lambda_2(0,1) + \lambda_3(1,2) + \lambda_4(4,0)$$

$$(1,1) = \frac{1}{16}(0,0) + \frac{1}{2}(0,1) + \frac{1}{4}(1,2) + \frac{3}{16}(4,0)$$

Result 1 All the basic feasible solution to a set of eqns $Ax=b$, $x \geq 0$ are extreme points of the convex set of feasible solutions of the eqns and conversely.

Result 2 : If a LPP admits an optimal solution, then the objective function assumes the optimal value at an extreme point of the convex set generated by set of all feasible solns.

Prob : Find out the extreme points of the set $S=\{(x,y) : |x| \leq 9, |y| \leq 6\}$

Note : Each B.F.S is an extreme point and vice versa. There is a one-to-one correspondence between a b.f.s and extreme point in absence of degeneracy.

Simplex Method

Simplex method is an iterative procedure by which a new basic feasible solution can be obtained from a given basic feasible solution which improves the value of the objective function.

· Introducing slack and surplus variables, we get the LPP in the form —

Optimize $z = cx$

s.t $Ax = b$, $x \geq 0$ and $A = [a_{ij}]_{m\times n}$ and

$c = (c_1, c_2, \ldots, c_r, 0, 0 \ldots 0)$

$x = (x_1, x_2, \ldots, x_r, x_{r+1}, \ldots x_n)$ where $x_{r+1}, \ldots, x_n$ are either slack or surplus variables. We assume that $m < n$ and all components of b are non-negative by some adjustment. $A = (a_1, a_2, \ldots, a_j, \ldots, a_n)$ where a_j is called the activity vectors. There exists atleast one set of m-column vectors from the coefficient matrix A, which is linearly independent. Let $\beta_1, \beta_2, \ldots, \beta_m$ be a set of m linearly independent column vectors, the basis matrix $B = B(\beta_1, \beta_2, \ldots, \beta_m)$ where $x_{B_1}, x_{B_2}, \ldots, x_{B_m}$ be the basic variables associated with the column vectors $\beta_1, \beta_2, \ldots, \beta_m$.

$\boxed{x_B = B^{-1} b}$ is the soln corresponding to the basic variables.

Note: If the coefficient matrix A contains m unit column vectors which are linearly independent, then it can make the basis matrix.
and $x_B = I_m^{-1} b = b \geq 0$

Any column a_j of A may be expressed as linear combination of $\beta_1, \beta_2, \ldots, \beta_m$ (basic vectors).

$$a_j = y_{1j}\beta_1 + y_{2j}\beta_2 + y_{3j}\beta_3 + \cdots + y_{mj}\beta_m = \sum y_{ij}\beta_i = By_j$$

where $y_j = [y_{1j}, y_{2j}, \ldots, y_{mj}]$ and $\boxed{y_j = B^{-1}a_j}$ [When $B = I$ then $y_j = a_j$]

For optimality test, we calculate net evaluation $= z_j - c_j$

where $z_j = c_B y_j$ [$c_B = (c_{B1}, c_{B2}, \ldots, c_{Bm})$ ← Cost of basic variables]

$$= c_{B1}y_{1j} + c_{B2}y_{2j} + \cdots + c_{Bm}y_{mj} = c_B B^{-1}a_j$$ [When $B = I$, $z_j = c_B a_j$]

Calculating net evaluation $z_j - c_j$, for maximization problem $z_j - c_j \geq 0$ gives the optimality test condition. But if atleast one $z_j - c_j < 0$, optimality condn is not satisfied and further iteration is required.

i) If at least one $z_j - c_j < 0$ and at least one $y_{ij} > 0$, then the value of objective function can be improved or remains same.

ii) If $z_j - c_j < 0$ for any j and all $y_{ij} \leq 0$ the objective function has no optimal value and the problem has unbounded solution.

When (i) occurs, To find a new basis, we select new vector to enter in the basis using the method — if $z_k - c_k = \min_j (z_j - c_j, \text{ when } z_j - c_j < 0)$, a_k will be considered as 'entering vector' and the k^{th} column is called key column. (If it is not unique, we can select any vector arbitrarily). a_k will replace the vector which will leave from the current basis as follows.

Let $\min_i \left(\frac{x_{Bi}}{y_{ik}}, y_{ik} > 0 \right) = y_{rk}$, then the vector in the rth position will be replaced by a_k and rth row is called key row. y_{rk} is the key element. (If value of r is not unique, then we can consider any value of such i as r and in that case we may get more than one basis giving basic feasible soln)

Nature of the problem: The following cases may arise.

i) The problem results with finite value of the objective function with finite solution set.

ii) Unbounded value of objective function results. To solve maximization problem, if at any stage atleast one $z_j - c_j < 0$ and all $y_{ij} \leq 0$ [for $i = 1, 2, \ldots, m$] finite value of objective function does not exist.

iii) If the optimal value remaining same, there exists more than one solution set, we say that alternative optimal soln exist. The condn is at optimal stage at least one $z_j - c_j = 0$ corresponding to a non-basic vector.

iv) The problem has no feasible solution.

Problem 1: Apply the simplex process to solve the LPP

Maximize $z = 2x_1 - 3x_2$

subject to $2x_1 + 5x_2 \geq 10$

$3x_1 + 8x_2 \leq 24$

and $x_1, x_2 \geq 0$.

To put the problem in standard form, we introduce surplus variable x_3 and slack variable x_4.

So we get Maximize $z = 2x_1 - 3x_2 + 0 \cdot x_3 + 0 \cdot x_4$

s.t. $2x_1 + 5x_2 - x_3 + 0 \cdot x_4 = 10$

$3x_1 + 8x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 24$

where $x_1, x_2, x_3, x_4 \geq 0$

In matrix form, we have

$$\begin{bmatrix} 2 & 5 & -1 & 0 \\ 3 & 8 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \begin{bmatrix} 10 \\ 24 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & -1 & 0 \\ 3 & 8 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 24 \end{bmatrix}$$

so that, $A = \begin{bmatrix} 2 & 5 & -1 & 0 \\ 3 & 8 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 10 \\ 24 \end{bmatrix}$, $a_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $a_2 \begin{bmatrix} 5 \\ 8 \end{bmatrix}$,

$a_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $a_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Rank$(A) = 2$

Let $B = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ ($\because$ a_1, a_4 are linearly independent)

$B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$ $\therefore$ the b.f.s $x_B = B^{-1} b = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

The initial basic feasible solution is $x_1 = 5$, $x_2 = 0 = x_3$, $x_4 = 9$

In objective function, $c_{B_1} = $ coef. of $x_1 = c_1 = 2$

$c_{B_2} = $ coef of $x_4 = c_4 = 0$ $\quad \therefore c_B = [2, 0]$

Hence value of the objective function for this b.f.s is

$$Z = c_B X_B = [2 \quad 0] \begin{bmatrix} 5 \\ 9 \end{bmatrix} = 10$$

For the non-basis vectors we compute Y_2 and Y_3, i.e.

$$Y_2 = B^{-1} a_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} y_{12} \\ y_{42} \end{bmatrix}$$

$$Y_3 = B^{-1} a_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 3/2 \end{bmatrix} = \begin{bmatrix} y_{13} \\ y_{43} \end{bmatrix}$$

$$z_2 - c_2 = c_B Y_2 - c_2 = [2 \quad 0] \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix} - (-3) = 8$$

$$z_3 - c_3 = c_B Y_3 - c_3 = [2 \quad 0] \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix} - 0 = -1$$

Since atleast one $(z_j - c_j)$ ie $z_3 - c_3$ is negative, optimality condn is not satisfied and a_3 will be the entering vector.

To find the departing vector, we calculate $\min_i \left\{ \frac{x_{Bi}}{y_{i3}} \;\middle|\; y_{i3} > 0 \right\}$

ie $\min \left\{ \frac{x_{B1}}{y_{13}}, \frac{x_{B2}}{y_{43}} \right\}$ But here $y_{13} < 0$ and only choice of leaving vector is a_4. Hence new basis is $[a_1, a_3]$. Thus new basis $= \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$

and $B^{-1} = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix}$. new soln $x_B = B^{-1} b = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$

Hence the new basic feasible solution is $x_1 = 8$, $x_2 = 0$, $x_3 = 6$, $x_4 = 0$

The new value of the objective function is $z = c_B x_B = [2, 0] \begin{bmatrix} 8 \\ 6 \end{bmatrix} = 16$

The value of the objective function has increased from 10 to 16.

Corresponding to non basic vectors a_2, a_4, new $Y_2 = B^{-1} a_2 = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 1/3 \end{bmatrix}$

Here $z_2 - c_2 = c_B Y_2 = [2 \quad 0] \begin{bmatrix} 8/3 \\ 1/3 \end{bmatrix} = \frac{16}{3}$

$z_4 - c_4 = c_B Y_4$

new $Y_4 = B^{-1} a_4 = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} y_{14} \\ y_{34} \end{bmatrix}$

$= [2 \quad 0] \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = \frac{2}{3}$

Since both these are true, optimal solution is obtained as
$x_1 = 8$, $x_2 = 0$, $Z_{max} = 16$.

Prob 2 Show by simplex process that the following LPP admits of an unbounded solution.

Maximize $z = 3x_1 + 4x_2$

subject to $x_1 - x_2 \geq 0$
$-x_1 + 3x_2 \leq 3$

$x_1, x_2 \geq 0$

Prob 3 Solve the LPP, $\max z = x_1 - x_2 + 2x_3 + 3x_4$

subject to $2x_1 + x_2 + 3x_3 + 2x_4 = 11$
$3x_1 - 3x_2 + 5x_3 + x_4 = 17$
$x_j \geq 0$ for $j = 1, 2, 3, 4$

Simplex table:

			c_j	c_1	c_2	c_3	c_4	....	c_k	....	c_n	Remark
B	c_B	x_B	b	a_1	a_2	a_3	a_4	...	a_k	....	a_n	
β_1	c_{B_1}	x_{B1}	b_1	y_{11}	y_{12}	y_{13}	y_{14}	....	y_{1k}	......	y_{1n}	
β_2	c_{B2}	x_{B2}	b_2	y_{21}	y_{22}	y_{23}	y_{24}	.....	y_{2k}	......	y_{2n}	
β_3	c_{B_3}	x_{B_3}	b_3	y_{31}	y_{32}	y_{33}	y_{34}	....	y_{3k}	....	y_{3n}	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$ ...	$\vdots$	.....	$\vdots$	
β_r	c_{Br}	x_{Br}	b_r	y_{r1}	y_{r2}	y_{r3}	y_{r4}	....	y_{rk}	......	y_{rn}	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				$\vdots$				
β_m	c_{Bm}	x_{Bm}	b_m	y_{m1}	y_{m2}	y_{m3}	y_{m4}	.....	y_{mk}	....	y_{mn}	
$z = c_B x_B$			$z_j - c_j$	$z_1 - c_1$	$z_2 - c_2$	$z_3 - c_3$	$z_4 - c_4$	.....	$z_k - c_k$	......	$z_n - c_n$	

Solution of Problem 3:

Step 1: search for a basis which will produce feasible solution.

$$A = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 3 & -3 & 5 & 1 \end{bmatrix} \qquad R(A) = 2$$

$\therefore$ The 2 eqns are linearly independent and consistent.

$$a_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \; a_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \; a_3 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \; a_4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 11 \\ 17 \end{pmatrix}$$

$$c = (c_1, c_2, c_3, c_4) = (1, -1, 2, 3)$$

$$B = (a_1, a_2) = \begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix} \quad \det B = -9 \neq 0$$

B can be considered as basis and $B^{-1} = -\frac{1}{9}\begin{bmatrix} -3 & -1 \\ -3 & 2 \end{bmatrix}$

The basic solution corresponding to the basis B is

$\cdot\; \begin{bmatrix} -3 & 2 \end{bmatrix}$

The basic solution corresponding to the basis B is

$$x_B = B^{-1} b = -\frac{1}{9}\begin{bmatrix} -3 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 11 \\ 17 \end{bmatrix} = \begin{bmatrix} 58/9 \\ -1/9 \end{bmatrix}$$

Hence the solution is not feasible. ∴ B can not be considered as initial basis for the simplex procedure.

Considering $B = (a_1, a_3) = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ $\quad |B| = 1 \neq 0$

$$x_B = B^{-1} b = \frac{1}{1}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 11 \\ 17 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$ which is feasible.

Then $x_B = [x_{B1}, x_{B2}] = [x_1, x_3] = [4 \;\; 1]$ and $B = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ is initial basis.

Step 2: Calculation of y_j and $z_j - c_j$

We know that $[y_1, y_2, y_3, y_4] = B^{-1}(a_1, a_2, a_3, a_4)$

$$\therefore y_1 = \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\; y_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} 14 \\ -9 \end{bmatrix},\; y_3 = \begin{bmatrix} y_{13} \\ y_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y_4 = \begin{bmatrix} y_{14} \\ y_{24} \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix},\quad c_B = (c_1, c_3) = (1, 2)$$

$$z_1 - c_1 = c_B y_1 - c_1 = c_{B1} y_{11} + c_{B2} y_{21} - c_1 = 1\times 1 + 2\times 0 - 1 = 0$$

$$z_2 - c_2 = c_B y_2 - c_2 = c_{B1} y_{12} + c_{B2} y_{22} - c_2 = 1\times 14 + 2(-9) - (-1) = -3$$

Similarly

$$z_3 - c_3 = 1\times 0 + 2\times 1 - 2 = 0$$

$$z_4 - c_4 = 1\times 7 + 2(-4) - 3 = -4$$

$$z_0 = c_B x_B = c_{B1} x_{B1} + c_{B2} x_{B2} = 1\times 4 + 2\times 1 = 6$$

Here since $z_2 - c_2$ and $z_4 - c_4$ are both negative with atleast one of y_{i2} or $y_{i4} > 0$. Thus the basic feasible solution $x_B = [x_1, x_3] = [4, 1]$ corresponding to the basis $B = (a_1, a_3) = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ is not optimal soln. We have to proceed further.

Step 3: Search for a new vector entering in the basis and the vector which leaves the basis.

Since $\min_j (z_j - c_j, z_j - c_j < 0) = -4$ occurring for $j = 4$,

a_4 is the entering vector. Thus 4th column is the key column.

Again $\min_i \left(\frac{x_{Bi}}{y_{i4}}, y_{i4} > 0\right) = \min\left(\frac{x_{B1}}{y_{14}}, \frac{x_{B2}}{y_{24}}, y_{i4} > 0\right)$

$= \min\left(\frac{4}{7}, \ldots\right)$, [here $\frac{x_{B2}}{y_{24}}$ is not considered as $y_{24} = -4 < 0$]

Which occurs for $i = 1$.

∴ The 1st row is key row and $y_{14} = 7$ is the key element.

$\beta_1 = a_1$ is the departing vector

So the new basis is $B = (a_4, a_3) = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and $y_{14} = 7$ is key element.

Simplex Table

	c	1	-1	2	3			Remark

			c	1	-1	2	3			Remark
B	c_B	x_B	b	a_1	a_2	a_3	a_4			
a_1	1	x_1	4	$y_{11}=1$	$y_{12}=14$	$y_{13}=0$	$y_{14}=7$			$x_{B1}/y_{14} = 4/7$
a_3	2	x_3	1	$y_{21}=0$	$y_{22}=-9$	$y_{23}=1$	$y_{24}=-4$			
		z_j-c_j	6	0	-3	0	-4			
a_4	3	x_4	4/7	1/7	2	0	1			
a_3	2	x_3	23/7	4/7	-1	1	0			
		z_j-c_j	$\frac{58}{7}$	4/7	5	0	0			

Net evaluation $z_1 - c_1 = 3 \times \frac{1}{7} + 2 \times \frac{4}{7} - 1 = \frac{4}{7}$

$z_2 - c_2 = 2 \times 3 + 2(-1) - (-1) = 5$

$z_3 - c_3 = 0$

$z_4 - c_4 = 0$

$\therefore$ all $z_j - c_j \geq 0$ for $j = 1, 2, 3, 4$. $\therefore$ The optimality condition is satisfied. The max $z = \frac{58}{7}$ at $x_1 = 0$, $x_2 = 0$, $x_3 = \frac{23}{7}$ and $x_4 = \frac{4}{7}$.

Problem: Solve the following LPP,

$$\text{Max } z = 60x_1 + 50x_2$$

$$\text{s.t. } x_1 + 2x_2 \leq 40$$

$$3x_1 + 2x_2 \leq 60$$

$$x_1, x_2 \geq 0$$

Soln: Introduce slack variables x_3, x_4 we rewrite the prob in standard form.

$$\text{Max } z = 60x_1 + 50x_2 + 0 \cdot x_3 + 0 \cdot x_4$$

$$\text{s.t. } x_1 + 2x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 40$$

$$3x_1 + 2x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 60$$

where $x_1, x_2, x_3, x_4 \geq 0$

Here, we can consider the identity matrix I_2 given by the coefficients of x_3 and x_4.

$(c_1, c_2, c_3, c_4) = (60, 50, 0, 0)$

$a_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $a_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $a_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 40 \\ 60 \end{bmatrix}$

We see that the vectors a_3 and a_4 form the initial basis,

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\therefore x_B = b$, $x_{B1} = 40$, $x_{B2} = 60$, $c_B = (c_{B_1}, c_{B_2}) = (0, 0)$

	c_j	60	50	0	0

			C_j	60	50	0	0	
C_B	B	X_B	b	a_1	a_2	a_3	a_4	Since atleast one $z_j - c_j < 0$,
0	a_3	x_3	40	1	2	1	0	this table does not give
0	a_4	x_4	60	$\boxed{3}$	2	0	1	optimal soln.
			$z_j - c_j$	-60	-50	0	0	Now $\min\left(\frac{x_{Bi}}{y_{ik}}, y_{ik} > 0\right)$
				↑			↓	$= \min\left(\frac{40}{1}, \frac{60}{3}\right) = 20$ $\therefore$ 2nd row is key row. $\therefore$ a_1 is entering vector and a_4 is departing vector.

			c_j	60	50	0	0	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	Here also atleast one
0	a_3	x_3	20	0	$\boxed{1\frac{1}{3}}$	1	$-\frac{1}{3}$	$z_j - c_j < 0$, so
60	a_1	x_1	20	1	$\frac{2}{3}$	0	$\frac{1}{3}$	optimality condn is not
			$z_j - c_j$	0	-10	0	20	satisfied
					↑ ↓			$z_2 - c_2$ is negative most minimum $\therefore$ a_2 is entering vector

$$\text{Min}\left\{\frac{20}{1\frac{1}{3}}, \frac{20}{\frac{2}{3}}\right\} = \frac{20}{1\frac{1}{3}}$$

Corresponding

			C_j	60	50	0	0	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	
50	a_2	x_2	15	0	1	$\frac{3}{4}$	$-\frac{1}{4}$	
60	a_1	x_1	10	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	
			$z_j - c_j$	0	0	$\frac{15}{2}$	$\frac{35}{2}$	

To get new first row divide the elements of 1st row of old table by key element $\left(1\frac{1}{3}\right)$

To get new 2nd row, multiply 1st row elements of new one by $\frac{2}{3}$ and subtract it from the second row of old table.

Since all $z_j - c_j \geq 0$, the optimality condition is satisfied.

and $x_1 = 10$, $x_2 = 15$, $z_{max} = 1350$

Home work problem: solve the LPP: Max $z = 5x_1 + 2x_2 + 2x_3$

Subject to $x_1 + 2x_2 - 2x_3 \leq 30$

$x_1 + 3x_2 + x_3 \leq 36$

$x_1, x_2, x_3 \geq 0$

Lecture 10

4.11.2020

Alternative Optimal soln

If in a LPP, the set of decision variables giving the optimal value of the function to be optimized be not unique, then we say that there are alternative optimal. We consider the two following cases, with alternative basic optimal and alternative non basic optimal respectively.

(i) If $z_j - c_j = 0$ for some nonbasic vector a_j and $y_{ij} > 0$ for at least one $i = r$, then the vector β_r in B is replaced by a_j and we get a new basic feasible solution. Then if z' be the value of objective function with the new B.F.S. and z_b be the corresponding quantity for the old B.F.S z_B which gives the optimal value, we have

$$z' = z_b - \frac{x_{Br}}{y_{rj}}(z_j - c_j)$$, since $z_j - c_j = 0$ for some non basic vector, we get $z' = z_b$. So z' is also optimal and is given by a new BFS.

Thus, if there be an optimum basic feasible solution, to a linear programming problem and if $z_j - c_j = 0$ for some non basic vector a_j and $y_{ij} > 0$ for at least one i, then $\exists$ an alternative basic optimal solution

(ii) If there be an optimal basic feasible solution to a linear programming problem and if $z_j - c_j = 0$ for some non basic vector and $y_{ij} \le 0$ for all $i = 1, 2, \ldots, m$ then there will exist an alternative non-basic optimal solution.

Artificial variable

If considering slack and surplus variables, we don't get identity matrix initially as basis matrix, we then consider Artificial variable to get rid of this problem. The vectors of the co-efficient matrix associated with the artificial variables are called the artificial vectors. The artificial variables themselves or along with slack variables will give us the identity matrix. The cost of each artificial variable is considered as $(-M)$, M being much larger than any other price.

While examining the largeness of the terms like $(pM+q)$ in the index row $z_j - c_j$, only terms containing M is counted. If the terms pM is for two columns are found same, then the q terms are verified.

Since artificial variables are introduced to get initial basis matrix, and they have no physical significance, we have to remove the artificial variables from the soln. As they will be removed from the basis, a large $-ve$ price $(-M)$ is assigned in maximization problem and a large positive price (M) for the minimization problem.

The idea of large price of artificial variables was introduced by A. Charnes and hence is known as Charnes-M-method (Big-M method) or

The method of Penalty.

Problem :- Solve the following problem.

$$\text{Maximize } z = 2x_1 + 3x_2$$

s.t.

$$x_1 + x_2 \le 8$$
$$x_1 + 2x_2 = 5$$
$$2x_1 + x_2 \le 8$$
$$x_1 \ge 0,\ x_2 \ge 0.$$

Let us put in LPP standard form introducing slack variable x_3, x_5 and one artificial variable x_4 to the second constraint with cost $(-M)$ where M is a very large no.

$$\text{Max } z = 2x_1 + 3x_2 + 0\cdot x_3 + (-M)x_4 + 0\cdot x_5$$

s.t.

$$x_1 + x_2 + x_3 = 8$$
$$x_1 + 2x_2 + x_4 = 5$$
$$2x_1 + x_2 + x_5 = 8$$

$$x_1, x_2, x_3, x_4 \ge 0$$

We construct the table

			c_j	2	3	0	$-M$	0	Remarks
C_B	B	X_B	b	a_1	a_2	a_3	a_4	a_5	
0	a_3	x_3	8	1	1	1	0	0	for departing vector
$-M$	a_4	x_4	5	1	$\boxed{2}$	0	1	0	$\min\left\{\frac{8}{1}, \frac{5}{2}, \frac{8}{1}\right\} = \frac{5}{2}$
0	a_5	x_5	8	2	1	0	0	1	$\therefore$ a_4 leaves the basis.
			$z_j - c_j$	$-M-2$	$-2M-3$	0	0	0	
0	a_3	x_3	$11/2$	$\frac{1}{2}$	0	1		0	In the 2nd table, a_1 enters
3	a_2	x_2	$5/2$	$\frac{1}{2}$	1	0		0	the basis. a_5 leaves basis
0	a_5	x_5	$11/2$	$\boxed{\frac{3}{2}}$	0	0		1	in the same way.
			$z_j - c_j$	$-\frac{1}{2}$	0	0		0	In the last one,
0	a_3	x_3	$11/3$	0	0	1		$-\frac{1}{3}$	all $z_j - c_j$ are +ve or
3	a_2	x_2	$2/3$	0	1	0		$-\frac{1}{3}$	zero. so we
2	a_1	x_1	$11/3$	1	0	0		$2/3$	get optimal soln
			$z_j - c_j$	0	0	0		$\frac{1}{3}$	$x_1 = \frac{11}{3}$, $x_2 = \frac{2}{3}$, $z_{max} = 9\frac{1}{3}$

Problem 2 Solve the LPP

$$\text{Maximize } Z = 2x_1 + 3x_2 + x_3$$

subject to

$$-3x_1 + 2x_2 + 3x_3 = 8$$

$$-3x_1 + 4x_2 + 2x_3 = 7$$

where $x_1, x_2, x_3 \geq 0$

The problem is restated after introducing artificial variable x_4, x_5 as both the constraints are equations. In the objective function the price of these variables are considered as $(-M)$, where M is a very large no. The resultant problem in the standard form is

$$\text{Max } Z = 2x_1 + 3x_2 + x_3 - Mx_4 - Mx_5$$

s.t $-3x_1 + 2x_2 + 3x_3 + x_4 + 0\cdot x_5 = 8$

$-3x_1 + 4x_2 + 2x_3 + 0\cdot x_4 + x_5 = 7$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Table

			c_j	2	3	1	$-M$	$-M$		
C_B	B	X_B	b	a_1	a_2	a_3	a_4	a_5		
$-M$	a_4	x_4	8	-3	2	3	1	0		Since $-6M-3$ is the min. index no, a_2 will enter the basis. So find
$-M$	a_5	x_5	7	-3	$\boxed{4}$	2	0	1	$\rightarrow$	the leaving vector $\min\left\{\frac{8}{2}, \frac{7}{4}\right\} = \frac{7}{4}$
			$z_j - c_j$	$6M-2$	$-6M-3$ ↑	$-5M-1$	0	0		a_5 is the leaving vector.
$-M$	a_4	x_4	$9/2$	$-\frac{3}{2}$	0	$\boxed{2}$	1		$\rightarrow$	To get the new 2nd row, we divide
3	a_2	x_2	$7/4$	$-3/4$	1	$\frac{1}{2}$	0			the old 2nd row by key element.
			$z_j - c_j$	$\frac{3M}{2} - \frac{17}{4}$	0	$-2M + \frac{1}{2}$ ↑	0			To get the new 1st row, multiply the new second row by the corresponding element
										in the old first row i.e 2 and subtract
1	a_3	x_3	$9/4$	$-3/4$	0	1				it from the old first row.
3	a_2	x_2	$5/8$	$-3/8$	1	0				i.e new 1st row = old first row $- 2\times$ new 2nd row.
			$z_j - c_j$	$-31/8$	0	0				Similarly for the next table a_3 will enter the basis and since
Here a_1 will be the entering vector but both y_{11} and y_{21} are $-$ve $(-3/4, -3/8)$.										$\min\left\{\frac{9}{2}/2, \frac{7}{4}\times 2\right\} = \frac{9}{4}$ implies that a_4 will leave the basis

So the solution is unbounded.

Problem 3: Solve the LPP

$$\text{Max } Z = 4x_1 + 14x_2$$

s.t $2x_1 + 7x_2 \leq 21$

$7x_1 + 2x_2 \leq 21$

$x_1, x_2 \geq 0$

admits of an infinite no of solution.

Soln:– Adding slack variables x_3 and x_4 to the constraints. Thus the problem becomes

$$\text{Max } z = 4x_1 + 14x_2 + 0\cdot x_3 + 0\cdot x_4$$

$$\text{s.t} \quad 2x_1 + 7x_2 + 1\cdot x_3 + 0\cdot x_4 = 21$$
$$7x_1 + 2x_2 + 0\cdot x_3 + 1\cdot x_4 = 21$$

Table

			c_j	4	14	0	0
C_B	B	X_B	b	a_1	a_2	a_3	a_4
0	a_3	x_3	21	2	$\boxed{7}$	1	0
0	a_4	x_4	21	7	2	0	1
			$z_j - c_j$	-4	-14	0	0
14	a_2	x_2	3	$\frac{2}{7}$	1	$\frac{1}{7}$	0
0	a_4	x_4	15	$\frac{45}{7}$	0	$\frac{-2}{7}$	1
			$z_j - c_j$	0	0	2	0
14	a_2	x_2	$7/3$	0	1	$7/45$	$-2/45$
4	a_1	x_1	$7/3$	1	0	$-2/45$	$7/45$
			$z_j - c_j$	0	0	2	0

$z_2 - c_2 = -14 \Rightarrow a_2$ will enter the basis and $\min\left\{\frac{21}{7}, \frac{21}{2}\right\} = \frac{21}{7}$ $\therefore a_3$ leaves the basis. 7 is key element.

The next table is formed by the previous method.

$z_j - c_j \geq 0$ for all j.

$x_1 = 0,\ x_2 = 3,\ z_{max} = 42$

Here the net evaluation corresponding to the non basic vector a_1 is zero and atleast one y_{i1} (here the both) > 0. This indicates the existence of alternative optimal solution.

So we a_1 to the basis and drop a_4 from the basis. Here also all $z_j - c_j \geq 0$, new optimal soln is

$x_1 = 7/3,\ x_2 = 7/3,\ z_{max} = 42.$

We have two alternative optimal solns given by $x_1 = 0, x_2 = 3$ and $x_1 = x_2 = 7/3$. We may get an infinite no. of optimal soln $(\bar{x}_1, \bar{x}_2)$ where $\bar{x}_1 = \lambda\cdot 0 + (1-\lambda)\frac{7}{3}$, $\bar{x}_2 = \lambda\cdot 3 + (1-\lambda)\,7/3$ for $0 \leq \lambda \leq 1$.

Lecture 11

rules of artificial variables

(i) If the basis does not contain any artificial variable, then the constraints equations are all consistent and the solution obtained obtained is an optimal basic solution.

(ii) If the basis contains one or more artificial variables at zero level, (the variables corresponding to these vectors have the value zero), then the solution obtained is an optimal soln and the constraints equations are consistent although there may be a redundancy for some of them.

(iii) If the basis contains one or more artificial variables at positive level, then the original problem will have no feasible soln.

$\boxed{\text{Prob 1}}$ Solve the LPP by Big M method.

$$\text{Max } z = -2x_1 + x_2 + 3x_3$$

$$\text{s.t } x_1 - 2x_2 + 3x_3 = 2$$

$$3x_1 + 2x_2 + 4x_3 = 1$$

where $x_1, x_2, x_3 \geq 0$.

Soln: $\text{Max } z = -2x_1 + x_2 + 3x_3 + (-M)x_4 + (-M)x_5$

			c_j	-2	1	3	$-M$	$-M$
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
$-M$	a_4	x_4	2	1	-2	3	1	0
$-M$	a_5	x_5	1	3	2	$\boxed{4}$	0	1 →
			$z_j - c_j$	$-4M+2$	-1	$-7M-3$ ↑	0	0
$-M$	a_4	x_4	$5/4$	$-5/4$	$-7/2$	0	1	
3	a_3	x_3	$1/4$	$3/4$	$1/2$	1	0	
			$z_j - c_j$	$\frac{5M}{4} + \frac{17}{4}$	$\frac{7M}{2} + \frac{1}{2}$	0	0	

a_3 is entering vector.

$\text{Min}\left(2/3, 1/4\right) = \frac{1}{4}$, a_5 is leaving vector

Here $z_j - c_j \geq 0$ for all j, but the artificial vector a_4 appears in the basis at the +ve level (i.e. $x_4 = 5/4 > 0$). Hence there is no feasible soln to the given problem.

Problem 2: Solve LPP,

$$\text{Max } Z = 3x_1 - x_2$$

$$\text{s.t.} \quad -x_1 + x_2 \geq 2$$
$$5x_1 - 2x_2 \geq 2$$

$$-x_1 + x_2 - x_3 = 2$$
$$5x_1 - 2x_2 - x_4 = 2$$

We consider artificial variables x_5 and x_6 are added to the LHS of the equations and then we write

$$-x_1 + x_2 - x_3 + x_5 = 2$$
$$5x_1 - 2x_2 - x_4 + x_6 = 2$$

The adjusted objective function z is given by

$$z = 3x_1 - x_2 + 0 \cdot x_3 + 0 \cdot x_4 - Mx_5 - Mx_6$$

Table

			c_j	3	-1	0	0	$-M$	$-M$
B	C_B	X_B	b	a_1	a_2	a_3	a_4	a_5	a_6
a_5	$-M$	x_5	2	-1	1	-1	0	1	0
a_6	$-M$	x_6	2	$\boxed{5}$	-2	0	-1	0	1
			$z_j - c_j$	$-4M-3$	$M+1$	M	M	0	0
a_5	$-M$	x_5	$12/5$	0	$3/5$	-1	$-1/5$	1	
a_1	3	x_1	$2/5$	1	$-2/5$	0	$-\frac{1}{5}$	0	
			$z_j - c_j$	0	$-3/5 M - \frac{1}{5}$	M	$M/5 - \frac{3}{5}$	0	
a_2	-1	x_2	4	0	1	$-5/3$	$-1/3$		
a_1	3	x_1	2	1	0	$-2/3$	$-1/3$		
$z_j - c_j$				0	0	$-1/3$	$-2/3$		

$\min\left\{\frac{2}{5}\right\} = \frac{2}{5}$

Here all the artificial variables are driven out from the basis. Now in the 3rd column, $z_3 - c_3 = -\frac{1}{3} < 0$, with $y_{i3} \leq 0$ for $i = 1, 2$. Hence the problem has no finite optimal soln. (unbounded soln)

$\boxed{\text{Problem}}$ Solve by Big M method :-

$$\text{Minimize } z = 2x_1 + 3x_2$$

$$\text{s.t.} \quad 2x_1 + 7x_2 \geq 22$$

$$x_1 + x_2 \geq 6$$

$$5x_1 + x_2 \geq 10 \quad , x_1, x_2 \geq 0$$

Introducing surplus and artificial variables,

$$\text{Max } z^* = -2x_1 - 3x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 - Mx_6 - Mx_7 - Mx_8$$

where $\text{Min } z = \text{Max } z^* = \text{Max}(-z)$

s.t. (x_3, x_4, x_5 are surplus variables and x_6, x_7, x_8 are artificial variables)

$$2x_1 + 7x_2 - x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 1 \cdot x_6 + 0 \cdot x_7 + 0 \cdot x_8 = 22$$
$$x_1 + x_2 + 0 \cdot x_3 - x_4 + 0 \cdot x_5 + 0 \cdot x_6 + 1 \cdot x_7 + 0 \cdot x_8 = 6$$
$$5x_1 + x_2 + 0 \cdot x_3 + 0 \cdot x_4 - x_5 + 0 \cdot x_6 + 0 \cdot x_7 + 1 \cdot x_8 = 10$$

			c_j	-2	-3	0	0	0	$-M$	$-M$	$-M$	
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	
$-M$	a_6	x_6	22	2	$\boxed{7}$	-1	0	0	1	0	0	$\rightarrow$
$-M$	a_7	x_7	6	1	1	0	-1	0	0	1	0	
$-M$	a_8	x_8	10	5	1	0	0	-1	0	0	1	
			$z_j - c_j$	$-8M+2$	$-9M+3$ $\uparrow$	M	M	M	0	0	0	

			c_j	-2	-3	0	0	0	$-M$	$-M$	$-M$	
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	
-3	a_2	x_2	$22/7$	$2/7$	1	$-1/7$	0	0		0	0	
$-M$	a_7	x_7	$20/7$	$5/7$	0	$1/7$	-1	0		1	0	
$-M$	a_8	x_8	$48/7$	$\boxed{33/7}$	0	$1/7$	0	-1		0	1	$\rightarrow$
			$z_j - c_j$	$-38/7 M + 8/7$ $\uparrow$	0	$-\frac{2M}{7} + \frac{3}{7}$	M	M		0	0	

			c_j	-2	-3	0	0	0	$-M$	$-M$	$-M$	
C_B	B	X_B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	
-3	a_2	x_2	$30/11$	0	1	$-5/33$	0	$2/33$		0		
$-M$	a_7	x_7	$20/11$	0	0	$4/33$	-1	$5/33$		1		
-2	a_1	x_1	$16/11$	1	0	$1/33$	0	$-7/33$		0		
			$z_j - c_j$	0	0	$-4M/33 + 13/33$	M	$-5M/33 + 8/33$ ↑		0		
-3	a_2	x_2	2	0	1	$-\frac{1}{5}$	$2/5$	0				
0	a_5	x_5	12	0	0	$\frac{4}{5}$	$-33/5$	1				
-2	a_1	x_1	4	1	0	$1/5$	$-7/5$	0				
				0	0	$1/5$	$8/5$	0				

Since all $z_j - c_j \geq 0$ optimal soln $x_1 = 4, x_2 = 2, x_5 = 12$

Min $z = 14$

Problem 1 H.A

Solve LPP by Big-M method and prove that the problem has finite optimal soln.

$$\text{Min } z = 3x_1 + 5x_2$$

s.t.

$$x_1 + 2x_2 \geq 8$$
$$3x_1 + 2x_2 \geq 12 \qquad x_1, x_2 \geq 0$$
$$5x_1 + 6x_2 \leq 60$$

Prob 2 Solve the following LPP by Big M method, and prove that the problem has no feasible soln.

$$\text{Max } z = 5x_1 + 11x_2$$

s.t.

$$2x_1 + x_2 \leq 4$$
$$3x_1 + 4x_2 \geq 24$$
$$2x_1 - 3x_2 \geq 6, \quad x_1, x_2 \geq 0$$

Prob 3 Use simplex method to solve LPP

$2x_1 - 5x_2 \geq 6, \; x_1, x_2 \geq 0$

Prob 3 Use simplex method to solve LPP

$$\text{Max } z = 2x_2 + x_3$$

s.t $\quad x_1 + x_2 - 2x_3 \leq 7$

$$-3x_1 + x_2 + 2x_3 \leq 3$$

$$x_1, x_2, x_3 \geq 0$$

(Prob 4) Solve LPP and P.T alternating optimal soln exist.

$$\text{Max } z = 2x_1 - x_2 + 3x_3 + x_4$$

s.t $\quad 2x_1 + x_2 + 3x_3 + 5x_4 \leq 12$

$$3x_1 + 2x_2 + x_3 + 4x_4 \leq 15$$

$$x_1, x_2, x_3, x_4 \geq 0$$