FOURIER SERIES

(Study material)

Periodic function: A function $f(x)$ which satisfies the relation $f(x+T) = f(x)$ $\forall$ x and some fixed T is called a periodic function of period T.

i.e. if T is the period of $f(x)$, then

$$f(x) = f(x+T) = f(x+2T) = \cdots = f(x+nT) = \cdots$$

$$\& \quad f(x) = f(x-T) = f(x-2T) = \cdots = f(x-nT) = \cdots$$

$$\therefore \quad f(x) = f(x \pm nT) \text{ where } n \in \mathbb{N}.$$

Eg: (i) $\sin x$, $\cos x$, $\sec x$ are periodic function of period 2π.

(ii) $\tan x$, $\cot x$ are periodic function of period π.

(iii) $\sin nx$, $\cos nx$ are periodic function of period $\frac{2\pi}{n}$.

(1) what is the period of the function $\sin x + \sin 2x$?

Fourier series: (Euler's formulae)

Let $f(x)$ be a periodic function of period 2π defined in the interval $[c, c+2\pi]$. Then the Fourier series of $f(x)$ is given by,

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx} \rightarrow (1)$$

where,

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\, dx$$

To determine a_0: Integrating both sides of (1),

$$\int_c^{c+2\pi} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2\pi} dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{a_0}{2}\cdot 2\pi + 0 + 0$$

$$= a_0\pi$$

$$\Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx}$$

To determine a_n: Multiplying by $\cos nx$ of (1) and integrating w.r.t. x between the limits $[c, c+2\pi]$ we get,

$$\int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi}\cos nx\,dx$$

$$+\int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx\right)\cos nx\,dx$$

$$= 0 + a_n\pi + 0 \qquad \left[\text{as } \int_c^{c+2\pi}\sin nx\cos nx\,dx = 0 \text{ check}\right]$$

$$\Rightarrow \boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx}$$

To determine b_n: Multiplying by $\sin nx$ of (1) & integrating between the limits $[c, c+2\pi]$, we get,

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi} \sin nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + b_n\pi$$

$$\Rightarrow \boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx}$$

(1) Let $f(x)$ be a function (of period 2π) defined by,

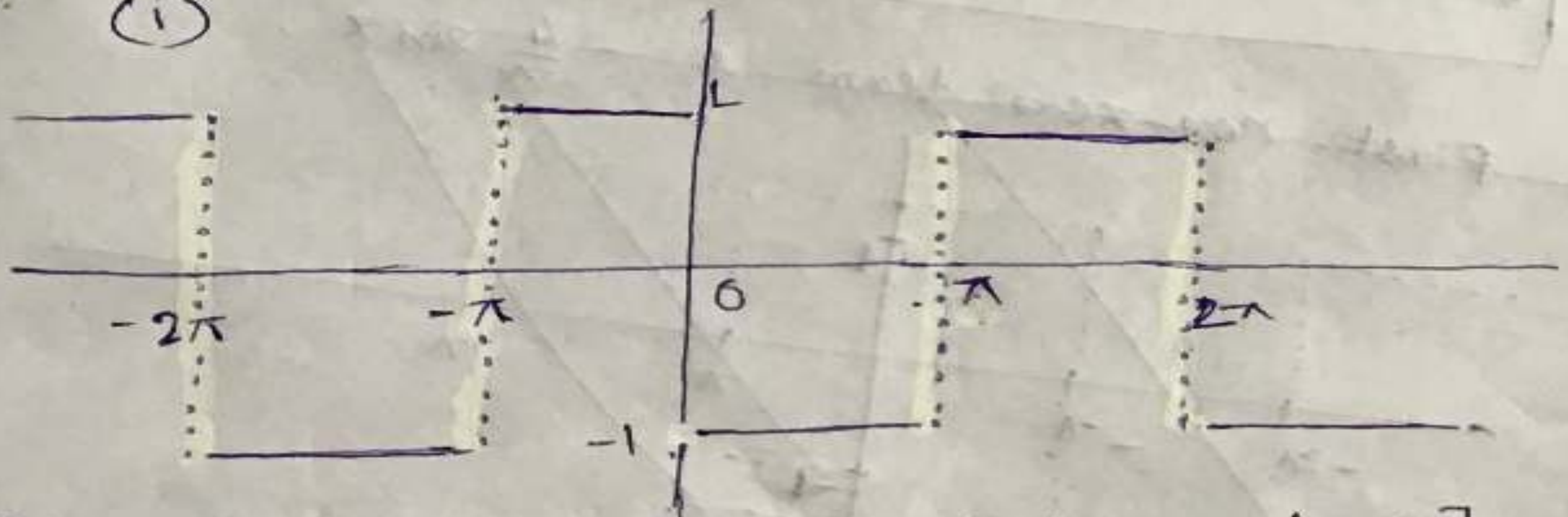
$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 < x < \pi \end{cases}$$

(i) Sketch $f(x)$

(ii) Compute the Fourier series of $f(x)$.

(iii) Sketch 1st non-zero term of the Fourier series.

Ans. (i)



[Note: In every 1 & -1, it has a jump]

(ii) Let the Fourier series of $f(x)$ be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

[Here $c = -\pi$]

Then, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 1 \cdot dx + \int_{0}^{\pi} (-1)\,dx\right]$$

$$= \frac{1}{\pi}(+\pi - \pi) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(-\cos nx)\,dx$$

$$= 0 \quad [\text{check}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\sin nx\,dx + \int_{0}^{\pi}(-\sin nx)\,dx\right]$$

$$= \frac{-1}{n\pi}\left[\cos nx\right]_{-\pi}^{0} + \frac{1}{n\pi}\left[\cos nx\right]_{0}^{\pi}$$

$$= \frac{-2}{n\pi}\left[1-\cos n\pi\right]$$

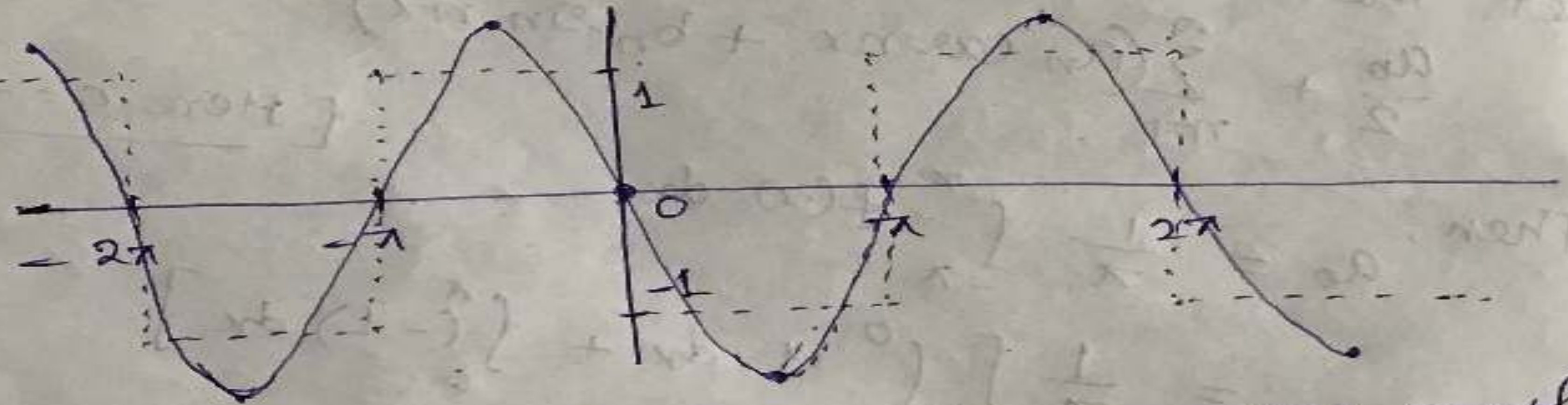
as $\cos n\pi = (-1)^n$

$$\therefore \quad b_n = -\frac{2}{n\pi}\left[1-(-1)^n\right] = \begin{cases} -\frac{4}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

i.e. $b_1 = -\frac{4}{\pi}$, $b_2 = 0$, $b_3 = -\frac{4}{3\pi}$, $b_4 = 0, \ldots$

$$\therefore \boxed{f(x) = -\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)}$$

(c) First non-zero term is $-\frac{4}{\pi}\sin x$



[Note: The dotted one is the previous graph. Notice that the first non-zero term is a good approximation of the given function. If we consider the more terms of the series, it will be more closer to the original function]

H.W. Obtain the Fourier series to represent

(i) $f(x) = \frac{1}{4}(\pi - x)^2, \quad 0 < x < 2\pi$

(ii) $f(x) = x^2, \quad -\pi \le x \le \pi$