Basic Discrete Structures

Sets, Functions, Sequences, Matrices, and Relations
(Lecture – 10)

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Partial Orderings

Definition: A relation R on a set S is called a *partial ordering*, or *partial order*, if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

Example: Assume R denotes the "greater than or equal" relation ($\geq$) on the set S={1,2,3,4,5}.

- Is the relation reflexive? Yes
- Is it antisymmetric? Yes
- Is it transitive? Yes
- **Conclusion:** R is a partial ordering.

Partial Orderings

- Customarily, the notation $a \preccurlyeq b$ is used to denote that ***(a, b)* ∈ *R* in an arbitrary poset *(S,R)***.
- This notation is used because the "less than or equal to" relation on the set of real numbers is the most familiar example of a partial ordering and the symbol is similar to the $\leq$ symbol.
- The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$. Also, we say "a is less than b" or "b is greater than a" if $a \prec b$.

Definition 1: The elements a and b of a poset $(S, \preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$ holds, then a and b are called ***incomparable***.

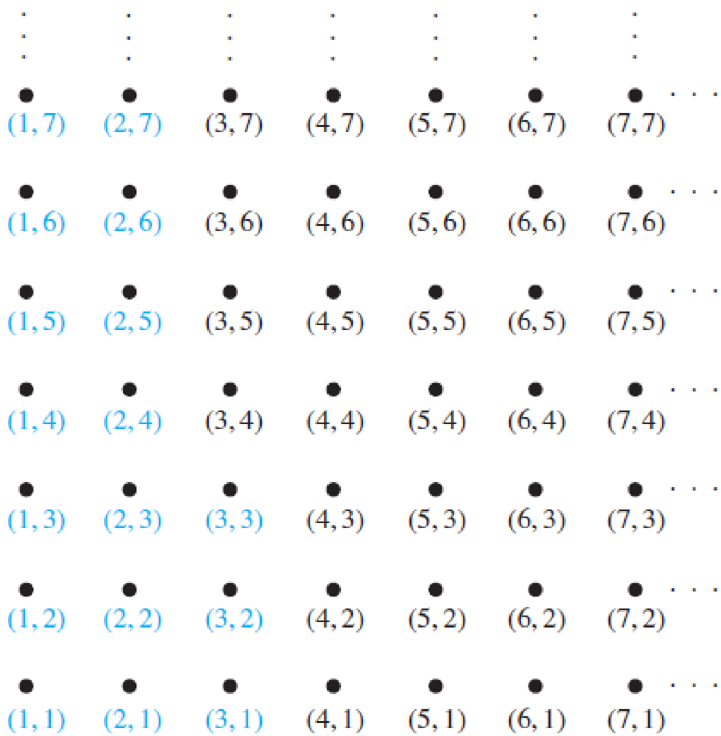
Partial Orderings

- The adjective "partial" is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a **total ordering**.

Definition 2: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

Lexicographic Ordering

Definition: Given two posets $(A_1,\preccurlyeq_1)$ and $(A_2,\preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1,b_2), that is, $(a_1, a_2) \prec (b_1,b_2)$, either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ then $a_2 \prec_2 b_2$.



- Ordered pairs less that (3,4) in $\boldsymbol{Z}^+ x\ \boldsymbol{Z}^+$
- The definition can be extended to a lexicographic ordering on strings.
- Consider the strings $a_1a_2 \ldots a_m$ and $b_1b_2 \ldots b_n$ on a partially ordered set S. Suppose these strings are not equal.
- Let t be the minimum of m and n. The definition of lexicographic ordering is that the string $a_1a_2 \ldots a_m$ is less than $b_1b_2 \ldots b_n$ if and only if
  - $(a_1, a_2, \ldots, a_t) \prec (b_1, b_2, \ldots, b_t)$, or
  - $(a_1, a_2, \ldots, a_t) = (b_1, b_2, \ldots, b_t)$ and $m < n$,

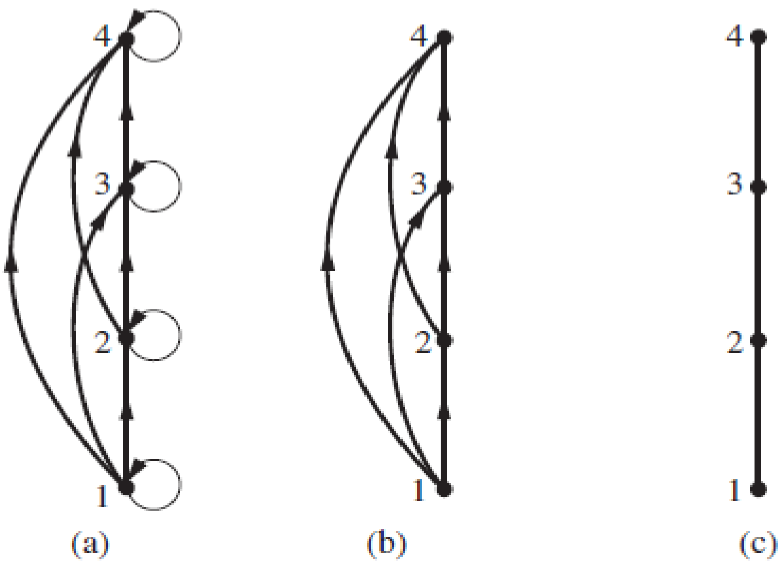
where $\prec$ in this inequality represents the lexicographic ordering of S_t.

Lexicographic Ordering

- To determine the ordering of two different strings, the longer string is truncated to the length of the shorter string, namely, to $t = \min(m, n)$ terms.
- The t-tuples made up of the first t terms of each string are compared using the lexicographic ordering on S_t .
- One string is less than another string:
  - if the t-tuple corresponding to the first string is less than the t-tuple of the second string,
  - or if these two t-tuples are the same, but the second string is longer.

Hasse Diagram

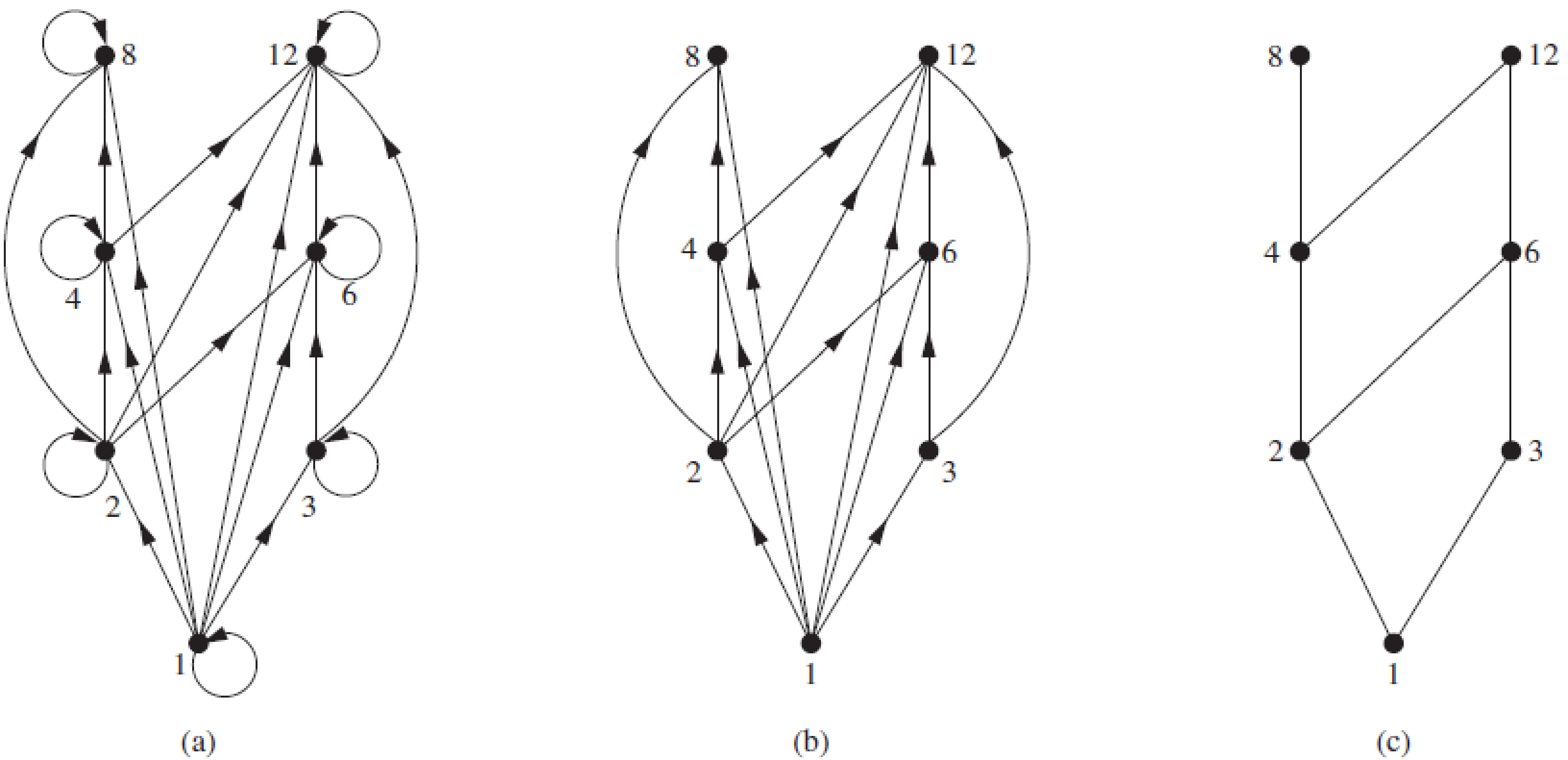
- Minimizes the directed graph for finite poset $(S, \preccurlyeq)$ on a set.
- Consider a partial ordering $\{(a, b) \mid a \leq b\}$ on the set $\{1, 2, 3, 4\}$



- Start with the directed graph for this relation.
- Because a partial ordering is reflexive, a loop (a, a) is present at every vertex a. Remove these loops.
- Next, remove all edges that must be in the partial ordering because of the presence of other edges and transitivity. That is, remove all edges (x, z) for which there is an element $y \in S$ such that $x \prec y$ and $y \prec z$.
- Finally, arrange each edge so that its initial vertex is below its terminal vertex.
- Remove all the arrows on the directed edges, because all edges point "upward" toward their terminal vertex.

Hasse Diagram

- Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid a$ divides $b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.



Maximal and Minimal Elements

- Maximal element:
  - An element of a poset is called maximal if it is not less than any element of the poset to *which it is comparable*.
  - Element a is **maximal** in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$ and *a and b are comparable*.
  - Basic idea: element which is not preceded by any other element
- Minimal element:
  - An element of a poset is called minimal if it is not greater than any element of the poset to *which it is comparable*.
  - Element a is **minimal** if there is no element $b \in S$ such that $b \prec a$ and *a and b are comparable*.
  - Basic idea: element which precedes every other elements
- In a Hasse diagram, maximal and minimal elements are the "top" and "bottom" elements in the diagram.
- *Note: A poset can have more than one maximal element and more than one minimal element.*

Greatest and Least Elements

- Greatest element:
  - If an element in a poset that is greater than every other element, it is called the greatest element.
  - Element a is the **greatest element** of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$.
  - The greatest element is unique when it exists
- Least element:
  - An element is called the least element if it is less than all the other elements in the poset.
  - Element a is the **least element** of $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$.
  - The least element is unique when it exists.

Upper & Lower Bounds

- Upper bound
  - Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset A of a poset $(S, \preccurlyeq)$.
  - If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an **upper bound** of A.
- Least upper bound/Supremum/Join
  - An element x is called the **least upper bound** of the subset A if x is an upper bound that is less than every other upper bound of A.
  - Because there is only one such element, if it exists, it makes sense to call this element *the* least upper bound
  - x is the least upper bound of A if $a \preccurlyeq x$ whenever $a \in A$, and $x \preccurlyeq z$ whenever z is an upper bound of A.
  - The least upper bound of set A is denoted as *lub* (A).

Upper & Lower Bounds

- Lower bound
  - Sometimes it is possible to find an element that is less than or equal to all the elements in a subset A of a poset $(S, \preccurlyeq)$.
  - If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a **lower bound** of A.
- Greatest lower bound/Infimum/Meet
  - The element y is called the **greatest lower bound** of the subset A if y is an lower bound that is greater than every other lower bound of A.
  - Because there is only one such element, if it exists, it makes sense to call this element *the* greatest lower bound
  - y is the greatest lower bound of A if $z \preccurlyeq y$ whenever z is an lower bound of A.
  - The greatest lower bound of set A is denoted as $glb\ (A)$.