138

3.5 POISSON DISTRIBUTION

Definition: A discrete random variable X is said to follow a **Poisson distribution** if its probability mass function (p.m.f.) is given by

$$P(r) = P(X = r) = \begin{cases} \dfrac{e^{-\lambda}\lambda^r}{r!}, & r = 0, 1, 2, \ldots \\ 0 & \text{, elsewhere} \end{cases}$$

$\lambda > 0$ is known as the **parameter** of the distribution. A random variable X which follows Poisson distribution is called a **Poisson variate** and is denoted as $X \sim P(\lambda)$.

Obviously:

(*i*) $P(X = r) \geq 0, \forall\, r.$

(*ii*) $\displaystyle\sum_{r=0}^{\infty} P(X = r) = \sum_{r=0}^{\infty} e^{-\lambda}\frac{\lambda^r}{r!} = e^{-\lambda}\sum_{r=0}^{\infty}\frac{\lambda^r}{r!} = e^{-\lambda}\cdot e^{\lambda} = 1.$

So, this is a valid probability distribution.

Occurrence of Poisson Distribution

In some situations where a random experiment results in two outcomes success or failure (*i.e.*, happening or not happening), the number of successes (happenings an event) only can be observed and not the number of failures (not happenings). We can observe how many road accidents occur but we cannot observe how many accidents do not occur. We can observe how many persons die from cancer but we can not observe how many do not die from cancer. In such situations Binomial distribution cannot be used. We use Poisson distribution where the following conditions must be satisfied:

(*i*) $n \to \infty$ (*ii*) $p \to 0$ (*iii*) np = constant.

Following are some instances where Poisson distribution may be successfully used:

- Number of deaths from a disease such as heart-attack, cancer, etc.
- Number of accidents during a week or a month.
- Number of telephone-calls received at a particular telephone exchange during a period of time.
- Number of cars passing a crossing per minute during a period of time.
- Number of defective items in a lot.

Example: *If the chance of being killed by flood during a year is 1/3000, use Poisson distribution to calculate probability that out of 3000 persons living in a village, at least one will die in flood in a year.* (*W.B.U.T. 2009*)

Solution: Let the random variable X corresponds to the number of persons being killed by flood during a year.

Given, p = probability that one person will die in a year due to flood out of 3000 persons = 1/3000 and n = 3000.

So, X follows Poisson distribution with parameter

$\lambda = np = 3000 \times \dfrac{1}{3000} = 1$ = average number of person being killed by flood during a year.

$$\therefore \quad P(X = r) = e^{-\lambda} \frac{\lambda^r}{r!} = \frac{e^{-1}}{r!} \qquad (\because \ \lambda = 1),\ r = 0, 1, 2, 3, \ldots$$

$$\therefore \quad \text{Required probability} = P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{e^{-1}}{0!} = 1 - e^{-1}.$$

3.6 MEAN AND VARIANCE OF THE POISSON DISTRIBUTION

Theorem: *If X is a Poisson variate with parameter $\lambda (> 0)$ then (i) mean = $E(X) = \lambda$ and (ii) $Var(X) = \lambda$.* (*W.B.U.T. 2002, 2007, 2012*)

Proof: Given $X \sim P(\lambda)$

$$\therefore \quad P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!},\ r = 0, 1, 2, \ldots$$
$$= 0, \text{ elsewhere}$$

(*i*)
$$\text{Mean} = E(X) = \sum_{r=0}^{\infty} r\,P(X = r) = \sum_{r=1}^{\infty} r \frac{e^{-\lambda}\lambda^r}{r!} = \lambda e^{-\lambda} \sum_{r=1}^{\infty} \frac{\lambda^{r-1}}{(r-1)!}$$
$$= \lambda e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \qquad \text{[Replacing } (r-1) \text{ by } r\text{]}$$
$$= \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda.$$

(*ii*) Now,
$$E\{X(X-1)\} = \sum_{r=0}^{\infty} r(r-1)\,P(X = r) = \sum_{r=2}^{\infty} r(r-1) \frac{e^{-\lambda}\lambda^r}{r!}$$
$$= \lambda^2 e^{-\lambda} \sum_{r=2}^{\infty} \frac{\lambda^{r-2}}{(r-2)!} = \lambda^2 e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!}$$
[Replacing $(r-2)$ by r]
$$= \lambda^2 e^{-\lambda} \cdot e^{\lambda} = \lambda^2.$$

$$\therefore \quad Var(X) = E\{X(X-1)\} - m(m-1) \qquad \left[\begin{array}{l}\text{See art.8.6, property } (iv),\\ \text{Chapter 8. Here } m = E(X)\end{array}\right]$$
$$= \lambda^2 - \lambda(\lambda - 1) = \lambda.$$

Note: Standard deviation = $\sqrt{\lambda}$. (*W.B.U.T. 2006*)

Example: *Let X is a Poisson variate with $P(1) = P(2)$. Find (i) mean and s.d. of X (ii) $P(X = 4)$.* (*W.B.U.T. 2003, 2009*)

Solution: Given, X is a Poisson variate.

Let
$$P(r) = (X = r) = e^{-\lambda} \frac{\lambda^r}{r!},\ r = 0, 1, 2, \ldots$$

Given, $P(1) = P(2) \Rightarrow e^{-\lambda}\lambda = e^{-\lambda}\dfrac{\lambda^2}{2!} \Rightarrow \lambda = 2$

(*i*) $\therefore$ Mean $= E(X) = \lambda = 2$,

$$\text{Var}(X) = \lambda = 2 \quad \therefore \quad \text{S.D. of } X = \sqrt{2}.$$

(*ii*) $$P(X = 4) = e^{-\lambda}\frac{\lambda^4}{4!} = e^{-2}\frac{2^4}{24} = \frac{2}{3}e^{-2}.$$

3.7 POISSON DISTRIBUTION AS A LIMITING CASE OF BINOMIAL DISTRIBUTION (*W.B.U.T. 2002, 2003*)

A binomial distribution can be found if its parameters n and p are known. But in cases where n is very large and p is very small, application of binomial distribution becomes very labourious. Let us state a theorem regarding this.

Theorem: *Under the following assumptions:*

(*i*) *The number of trials is increased indefinitely* ($n \to \infty$)

(*ii*) *The probability of success in a single trial is very small* ($p \to 0$)

(*iii*) $\lambda = np$ *is a finite constant,*

the p.m.f. of Poisson variate can be obtained as a limiting case of the p.m.f. of Binomial variate, i.e.,

$P(X = r) = {}^nC_r\, p^r(1-p)^{n-r} \to \dfrac{e^{-\lambda}\lambda^r}{r!}$ *as* $n \to \infty$ *such that* $np = \lambda =$ *finite constant* > 0 ($r = 0$, *1, 2, 3, ...*).

Proof:

When $r \neq 0$: Now, for a Binomial distribution

$$P(X = r) = {}^nC_r\, p^r(1-p)^{n-r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\left(\frac{\lambda}{n}\right)^r\left(1-\frac{\lambda}{n}\right)^{n-r} \quad (\because \; np = \lambda)$$

$$= \frac{\lambda^r}{r!}\cdot\frac{n(n-1)(n-2)\ldots(n-r+1)}{n^r}\cdot\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!}\left(\frac{n}{n}\right)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left\{1-\frac{(r-1)}{n}\right\}\frac{\left[\left\{\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right\}\right]^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$\therefore \quad \lim_{n\to\infty} P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}, \left[\because \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e \text{ and } \frac{k}{n} \to 0 \text{ as } n \to \infty, k = 1, 2, \ldots, (n-1)\right] r = 1, 2, 3, 4, \ldots$$

When $r = 0$: $$P(X=0) = {}^nC_0\, p^0 (1-p)^{n-0} = \left(1-\frac{\lambda}{n}\right)^n$$

$$= \left\{\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right\}^{-\lambda} \to e^{-\lambda} \text{ as } n \to \infty.$$

$$\therefore \quad \lim_{n\to\infty} P(X=r) = \lim_{n\to\infty} {}^nC_r\, p^r (1-p)^{n-r} = \frac{e^{-\lambda}\lambda^r}{r!}, \quad r = 0, 1, 2, 3, \ldots$$

Example 1: *In a screw manufacturing factory, the probability that a screw is defective is known to be 0.02. If 100 screws are taken for inspection, then find the probability that (i) there is no defective screw, (ii) at most 3 defective screws and (iii) exactly 5 defective screws.*

Solution: Let the random variable X corresponds to the number of defective screws out of 100 screws.

The process of inspecting screws one after another follows Binomial distribution. Let the event 'a screw is defective' be called a success, then p = probability of success = 0.02 (given). Using Poisson approximation to Binomial distribution, we get

$$P(X=r) = e^{-\lambda}\frac{\lambda^r}{r!}, \quad \text{where } \lambda = np = 100 \times 0.02 = 2 \quad (r = 0, 1, 2, 3, \ldots)$$

(*i*) Required probability $= P(\text{zero defective screw})$

$$= P(X=0) = e^{-2}\frac{2^0}{0!} = e^{-2} = 0.135$$

(*ii*) Required probability $= P(\text{at most 3 defective screws})$

$$= P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= e^{-2}\frac{2^0}{0!} + e^{-2}\frac{2^1}{1!} + e^{-2}\frac{2^2}{2!} + e^{-2}\frac{2^3}{3!}$$

$$= \frac{19}{3}e^{-2} = 0.857$$

(*iii*) Required probability $= P(\text{exactly 5 defective screws})$

$$= P(X=5) = e^{-2}\frac{2^5}{5!} = \frac{4}{15}e^{-2} = 0.036.$$

Example 2: In *a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10. Calculate the number of packets containing no defective, one defective and two defective blades in a consignment of 10,000 packets. (Given: $e^{-0.02} = 0.9802$).* (*W.B.U.T. 2004*)

Solution: Given p = Probability of defective in a single trial = 0.002,

$$n = 10 \quad \therefore \quad \lambda = np = 10 \times 0.002 = 0.02.$$

Let the r.v. X corresponds to the no. of defective blades in a packet.

$$\therefore \quad P(r) = P(X=r) = e^{-\lambda}\frac{\lambda^r}{r!}, \quad r = 0, 1, 2, \ldots$$

N = Number of packets in the consignment = 10,000 (*i*) Probability for no defective $= P(0) = e^{-\lambda} = e^{-0.02} = 0.9802$

$\therefore$ Number of packets with no defective blades

$$= N\,P(0) = 10{,}000 \times 0.9802 = 9802$$

(*ii*) Probability for one defective $= P(1) = e^{-\lambda}\lambda = 0.9802 \times 0.02 = 0.019604$

$\therefore$ Number of packets with one defective blades

$$= N\,P(1)$$
$$= 10{,}000 \times 0.019604 = 196$$

(*iii*) Probability for two defectives $= P(2) = e^{-\lambda}\dfrac{\lambda^2}{2!} = 0.9802 \times \dfrac{(0.02)^2}{2}$

$$= \frac{1}{2} \times 0.9802 \times 0.0004$$
$$= 0.00019604$$

$\therefore$ Number of packets with two defective blades

$$= N\,P(2) = 10{,}000 \times 0.00019604 = 1.9604 = 2.$$

ILLUSTRATIVE EXAMPLES – II

Example 1: *A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with average number of demand per day 1.5. Calculate the proportion of days on which neither car is used and the proportion of days on which some demand is refused. (Given : $e^{-1.5} = 0.2231$).* *(W.B.U.T. 2003, 2006, 2007)*

Solution: Let the random variable X corresponds to the number of demands for a car on any day.

Given, X is a Poisson variate with parameter λ = average number of demand per day = 1.5.

$$\therefore \qquad P(X = r) = e^{-\lambda}\frac{\lambda^r}{r!} = e^{-1.5}\frac{(1.5)^r}{r!},\; r = 0, 1, 2, 3; \dots .$$

$\therefore$ Proportion of days on which neither car is used

$$= P(X = 0) = e^{-1.5} = 0.2231.$$

Proportion of days on which some demand is refused

$$= P(X > 2) = 1 - P(X \le 2) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$
$$= 1 - \left[e^{-1.5} + e^{-1.5} \times 1.5 + e^{-1.5} \times \frac{(1.5)^2}{2!}\right]$$
$$= 1 - e^{-1.5}\left\{1 + 1.5 + \frac{1}{2}(1.5)^2\right\} = 0.1912625.$$

Notes: (*i*) Number of days in a year on which neither car is used $= 365 \times 0.2231 = 81.4315 = 81$ days.

(*ii*) Number of days in a year when demand is refused

$$= 365 \times 0.1912625 = 69.81 = 70 \text{ days}.$$

Example 2: *The number of emergency admissions on each day to a hospital is found to have a Poisson distribution with mean 3.*

(i) Calculate the probability that on a particular day there will be no emergency admission.

(ii) At the beginning of one day, the hospital has four beds available for emergencies. Find the probability that there will be an insufficient number of beds for that day.

Solution: Let the random variable X corresponds to the number of emergency admissions per day. Given, X is a Poisson variate with parameter λ = average number of emergency admissions per day = 3.

$$\therefore \quad P(X = r) = e^{-\lambda}\frac{\lambda^r}{r!} = e^{-3}\frac{3^r}{r!},\ r = 0, 1, 2, 3, \dots .$$

(*i*) Required probability = P(no emergency admission)

$$= P(X = 0) = e^{-3}\frac{3^0}{0!} = e^{-3}$$

(*ii*) Required probability = P(insufficient number of beds)

$$= P(X > 4) = 1 - P(X \leq 4)$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)]$$

$$= 1 - e^{-3}\left\{1 + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!}\right\}.$$

Example 3: *The number of emergency calls received during a ten-minute interval on a hospital switchboard is found to have a Poisson distribution with mean 5. Find the probability that there are (i) at most 2 emergency calls in a ten-minute interval and (ii) exactly 3 emergency calls in a ten-minute interval.*

Solution: Let the random variable X corresponds to the number of emergency calls received in a ten-minute interval. Given, X is a Poisson variate with parameter λ = average number of emergency calls in a ten-minute interval = 5.

$$\therefore \quad P(X = r) = e^{-\lambda}\frac{\lambda^r}{r!} = e^{-5}\frac{5^r}{r!},\ r = 0, 1, 2, 3, \dots .$$

(*i*) Required probability = P(at most 2 emergency calls)

$$= P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= e^{-5}\frac{5^0}{0!} + e^{-5}\frac{5^1}{1!} + e^{-5}\frac{5^2}{2!} = \frac{37}{2}e^{-5}.$$

(*ii*) Required probability = P(exactly 3 emergency calls)

$$= P(X = 3) = e^{-5}\frac{5^3}{3!} = \frac{125}{6}e^{-5}.$$

Example 4: *If the probability that an individual suffers a bad reaction from a certain injection is 0.001. Find the probability that out of 2000 individuals (i) exactly 3 (ii) more than 2 will suffer a bad reaction.*

Solution: Let the random variable X corresponds to the number of individuals suffering a bad reaction out of 2000.

By Poisson distribution:

$$P(X = r) = e^{-\lambda} \frac{\lambda^r}{r!}, \ r = 0, 1, 2, 3, \dots .$$

where $\lambda = np = 2000 \times 0.001 = 2.$

(*i*) Required probability $= P(\text{exactly 3 will suffer}) = P(X = 3)$

$$= e^{-2} \frac{2^3}{3!} = \frac{4}{3} e^{-2} = 0.18045.$$

(*ii*) Required probability $= P(\text{more than 2 will suffer})$

$$= P(X > 2) = 1 - P(X \le 2)$$
$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$
$$= 1 - \left[e^{-2} \frac{2^0}{0!} + e^{-2} \frac{2^1}{1!} + e^{-2} \frac{2^2}{2!} \right]$$
$$= 1 - 5e^{-2} = 0.3233.$$

Example 5: *In a certain factory producing cycle tyres there is a small chance 1 in 500 for any tyre to be defective. The tyres are supplied in lots of 20. Using Poisson distribution, calculate the approximate number of lots containing no defective, one defective and two defective tyres respectively in a consignment of 20,000 tyres.*

Solution: Let the random variable X corresponds to the number of defective tyres in a lot of 20.

By Poisson distribution:

$$P(X = r) = e^{-\lambda} \frac{\lambda^r}{r!}, \ r = 0, 1, 2, 3, \dots .$$

where λ = Average number of defective tyres in a lot of 20

$$= np = 20 \times \frac{1}{500} = 0.04$$

$$\therefore \quad P(\text{no defective}) = P(X = 0) = e^{-0.04} \frac{(0.04)^0}{0!} = e^{-0.04} = 0.9608.$$

$$P(\text{one defective}) = P(X = 1) = e^{-0.04} \frac{(0.04)^1}{1!} = 0.0384$$

$$P(\text{two defectives}) = P(X = 2) = e^{-0.04} \frac{(0.04)^2}{2!} = 0.0008$$

Given, N = Number of lots in a consignment of 20,000 tyres

$$= \frac{20,000}{20} = 1000.$$

$\therefore$ Approximate number of lots containing

(*i*) no defective $= NP(X = 0) = 1000 \times 0.9608 = 961$

(*ii*) one defective $= NP(X = 1) = 1000 \times 0.0384 = 38$

(*iii*) two defectives $= NP(X = 2) = 1000 \times 0.0008 = 01.$