Some problems on vector calculus that includes, Green's theorem, Stokes theorem etc.

Gauss' Divergence theorem:

Let $\bar{F}$ be a continuously differentiable vector point function and S be a closed surface enclosing volume V. Then
$$\iiint_V \operatorname{div} \bar{F}\, dV = \iint_S \bar{F}\cdot\hat{n}\, ds$$
where $\hat{n}$ is the unit normal vector drawn ~~outward~~ outward to S.

Stokes' theorem

Let $\bar{F}$ be a continuously differentiable vector point function and S be an open two-sided surface bounded by a simple closed curve Γ.

Then $$\iint_S \operatorname{curl} \bar{F}\cdot\hat{n}\, ds = \oint_\Gamma \bar{F}\cdot d\bar{r},$$

where the ~~F~~ curve Γ is traversed in the positive direction and $\hat{n}$ is the unit normal vector at any point on S drawn in the sense in which a right handed screw would move when rotated in a sense of description of Γ.

Cartesian representation of Gauss' divergence theorem: is

$$\iiint_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dx\,dy\,dz = \iint_S \left[F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy\right]$$
where $\bar{F} = F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$.

Stokes' Theorem:

$$\iint_S \left[\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)dy\,dz+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)dx\,dz + \left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dx\,dy\right] = \int_\Gamma (F_1\,dx+F_2\,dy+F_3\,dz)$$

where $\bar{F} = F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$.

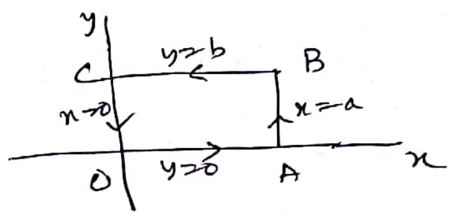
Green's Theorem in the plane:

If R be a closed region of the xy-plane bounded by a simple closed curve Γ and if P and Q are continuous functions of x and y having continuous derivatives in R, then

$$\oint_\Gamma (P\,dx+Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

Examples

EX1 verify Stokes' theorem for the vector function $\bar{F} = (x^2-y^2)\hat{i}+2x\hat{j}$ round the rectangle bounded by the straight lines $x=0$, $x=a$, $y=0$, $y=b$



Soln

$$\text{curl}\,\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2x & 0 \end{vmatrix}$$

$$= (2+2y)\hat{k}$$

$$\therefore \iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds = \iint_R (2+2y)\hat{k}\cdot\hat{k}\,dx\,dy = \int_{y=0}^{b}\int_{x=0}^{a}(2+2y)\,dx\,dy$$

[Here $\hat{n}=\hat{k}$ and $ds = dx\,dy$ is the elementary area]

$$= \int_{y=0}^{b} \left[2x+2xy\right]_{x=0}^{a} dy = \int_{y=0}^{b} (2a+2ay)\,dy = 2ab + \frac{2ab^2}{2}$$

$$= 2ab + ab^2 \quad \text{—— (1)}$$

Now $\int_{\Gamma} \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CO} \bar{F}\cdot d\bar{r}$ —— (2)

Now On OA, $y=0$, $\therefore$ ~~$d\bar{r}=dx$~~ $d\bar{r} = dx\,\hat{i}$ ($d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$)
(Here $z=0$ for all the line integrals)
and x varies from 0 to a

$$\therefore \int_{OA} \bar{F}\cdot d\bar{r} = \int_0^a \left[(x^2-0)\,\hat{i} + 2x\,\hat{j}\right]\cdot dx\,\hat{i} = \int_0^a x^2\,dx = \frac{a^3}{3} \quad \text{—— (3)}$$

Similarly, On AB, $x=a$ $\therefore dx=0$,

$$\bar{F}\cdot d\bar{r} = \left[(a^2-y^2)\,\hat{i} + 2a\,\hat{j}\right]\cdot dy\,\hat{j} = 2a\,dy,$$

y varies from 0 to b.

$$\therefore \int_{AB} \bar{F}\cdot d\bar{r} = \int_0^b 2a\,dy = 2ab \quad \text{—— (4)}$$

On, BC, $y=b$ $\therefore dy=0$, $\bar{F}\cdot d\bar{r} = \left[(x^2-b^2)\,\hat{i} + 2x\,\hat{j}\right]\cdot dx\,\hat{i}$,
$= (x^2-b^2)\,dx$,

x varies from a to 0.

$$\therefore \int_{BC} \bar{F}\cdot d\bar{r} = \int_a^0 (x^2-b^2)\,dx = \left(\frac{x^3}{3} - b^2x\right)_a^0 = -\frac{a^3}{3} + b^2a \quad \text{—— (5)}$$

On CO, $x=0$, $\therefore \bar{F}\cdot d\bar{r} = \left[-y^2\,\hat{i}\right]\cdot dy\,\hat{j} = 0$. —— (6)

$$\therefore \int_{CO} \bar{F}\cdot d\bar{r} = 0.$$

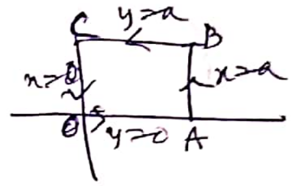
Thus Using (3), (4), (5) & (6) we obtain from (2)

$$\int_{\Gamma} \bar{F}\cdot d\bar{r} = \cancel{\frac{a^3}{3}} + 2ab - \cancel{\frac{a^3}{3}} + b^2a + 0 = 2ab + ab^2 \quad \text{—— (7)}$$

Thus Stokes' theorem is verified (From (1) & (7)).

(4)

EX2 Verify Stokes' theorem for the function $\bar{F} = x^2\hat{i} + xy\hat{j}$ integrated round the square in the plane $z=0$ whose sides are along the straight lines $x=0, y=0, x=a, y=a$.



Soln. Stokes' theorem is

$$\iint_R \text{curl}\,\bar{F}\cdot\hat{n}\,ds = \oint_\Gamma \bar{F}\cdot d\bar{r} \quad \text{--- (1)}$$

Here
$$\text{curl}\,\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = (y-0)\hat{k} = y\hat{k}$$

and $\hat{n} = \hat{k}$, $ds = dx\,dy$.

$$\therefore \iint_R \text{curl}\,\bar{F}\cdot\hat{n}\,ds = \int_{y=0}^{a}\int_{x=0}^{a} y\hat{k}\cdot\hat{k}\,dx\,dy = \int_{y=0}^{a}\int_{x=0}^{a} y\,dx\,dy$$

$$= \int_{y=0}^{a} yx\Big]_0^a dy = \int_{y=0}^{a} ay\,dy = \frac{ay^2}{2}\Big]_0^a = \frac{a^3}{2} \quad \text{--- (2)}$$

$$\oint_\Gamma \bar{F}\cdot d\bar{r} = \int_{OA}\bar{F}\cdot d\bar{r} + \int_{AB}\bar{F}\cdot d\bar{r} + \int_{BC}\bar{F}\cdot d\bar{r} + \int_{CO}\bar{F}\cdot d\bar{r} \quad \text{--- (3)}$$

On OA, $y=0$, $d\bar{r} = dx\,\hat{i}$, x varies from 0 to a

$$\bar{F}\cdot d\bar{r} = x^2\hat{i}\cdot dx\,\hat{i} = x^2 dx$$

$$\therefore \int_{OA}\bar{F}\cdot d\bar{r} = \int_0^a x^2 dx = \frac{a^3}{3} \quad \text{--- (4)}$$

On BA, $x=a$, $d\bar{r} = dy\,\hat{j}$, $\bar{F}\cdot d\bar{r} = (a^2\hat{i} + ay\hat{j})\cdot dy\,\hat{j}$
$= ay\,dy$

$$\therefore \int_{BA}\bar{F}\cdot d\bar{r} = \int_{y=0}^{a} ay\,dy = a\cdot\frac{a^2}{2} = \frac{a^3}{2} \quad \text{--- (5)}$$

On BC, $y=a$, $d\bar{r} = dx\,\hat{i}$, $\bar{F}\cdot d\bar{r} = (x^2\hat{i} + ax\hat{j})\cdot dx\,\hat{i}$
$= x^2 dx$

$$\therefore \int_{BC}\bar{F}\cdot d\bar{r} = \int_{x=a}^{0} x^2 dx = -\frac{a^3}{3} \quad \text{--- (6)}$$

(5) On CO, $x=0$, $d\bar{r}=dy\,\hat{j}$, $\bar{F}\cdot d\bar{r}=(0\hat{i}+0\hat{j})\cdot dy\,\hat{j}=0$

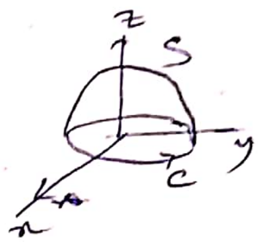
$$\therefore \int_{CO}\bar{F}\cdot d\bar{r}=0 \quad \text{——} \quad (7)$$

Thus using (4), (5), (6) & (7) we obtain from (3),

$$\oint_{\Gamma}\bar{F}\cdot d\bar{r}=\cancel{\frac{a^3}{3}}+\frac{a^3}{2}-\cancel{\frac{a^3}{3}}+0=\frac{a^3}{2}=\iint_{R}\text{curl}\,\bar{F}\cdot\hat{n}\,ds \quad (\text{see }(2))$$

Thus Stokes' Theorem is verified.

<u>Ex3</u> Verify Stokes' Theorem for $\bar{F}=(2x-y)\hat{i}-yz^2\hat{j}-y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary



<u>Soln</u> Stokes' Theorem is

$$\iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds=\oint_C \bar{F}\cdot d\bar{r} \quad \text{——} \quad (1)$$

$$\text{curl}\,\bar{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x-y & -yz^2 & -y^2z\end{vmatrix}=\hat{i}(-2yz+2yz)-\hat{j}(0-0)+\hat{k}(0+1)=\hat{k}$$

On the surface of the sphere $x^2+y^2+z^2=1$,

$$\hat{n}=\frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}=x\hat{i}+y\hat{j}+z\hat{k}$$

as on the sphere $x^2+y^2+z^2=1$ & unit normal is $\frac{\nabla f}{|\nabla f|}$.

$$\therefore \iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds=\iint_S \hat{k}\cdot(x\hat{i}+y\hat{j}+z\hat{k})\,ds=\iint_S z\,ds$$

$$=\iint_R z\,\frac{dx\,dy}{\hat{n}\cdot\hat{k}}=\iint_R dx\,dy=\text{area of the circle } C=\pi(1^2)=\pi. \quad \text{——} \quad (2)$$

[~~Ele~~ R is the projection of S on the xy-plane and elementary area in xy-plane ~~ds~~ $dx\,dy=ds\,(\hat{n}\cdot\hat{k})$.
$\therefore dS=\frac{dx\,dy}{\hat{n}\cdot\hat{k}}$]

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On the ~~boundary C, z=0~~ xy plane $z=0$ and the boundary C is given by the circle $x^2+y^2=1$

On C, $x=\cos\theta$, $y=\sin\theta$

$\therefore d\bar{r} = dx\,\hat{i} + dy\,\hat{j} = -\sin\theta\, d\theta\,\hat{i} + \cos\theta\, d\theta\,\hat{j}$

$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_C (2x-y)\,dx$, as on C, $z=0$

$$= \int_0^{2\pi}(2\cos\theta-\sin\theta)(-\sin\theta)\,d\theta = -\int_0^{2\pi}\sin 2\theta\, d\theta + \int_0^{2\pi}\sin^2\theta\, d\theta$$

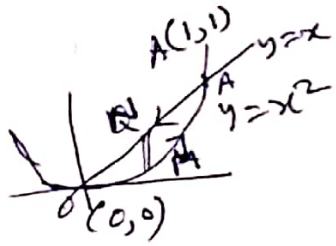
$$= +\frac{\cos 2\theta}{2}\Big|_0^{2\pi} + \frac{1}{2}\int_0^{2\pi}(1-\cos 2\theta)\,d\theta = 0+\frac{1}{2}(2\pi)-\frac{1}{2}(0)$$

$$= \pi \quad \text{—— (3)}$$

Thus we see from (2) and (3) $\iint_S \operatorname{curl}\bar{F}\cdot\hat{n}\, ds = \oint_C \bar{F}\cdot d\bar{r} = \pi$

Thus, Stokes' theorem is verified.

Ex 4 Verify Green's theorem in a plane for $\oint_C[(x^2+xy)dx + x\,dy]$ where c is the curve enclosing the region bounded by $y=x^2$ and $y=x$.



Soln

Green's Theorem is

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Intersection of the curves $y=x$ & $y=x^2$ is given by $y=x^2=y^2$ or $y(y-1)=0$

$\therefore y=0,1 \Rightarrow x=0$ & $x=1$. Therefore $(0,0)$ and $(1,1)$ are the points of intersection of the curves.

Now $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_R [1-x]\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x} (1-x)\,dy\,dx$

(7)

$$\iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dx\,dy = \int_{x=0}^{1} (y-xy)_{y=x^2}^{x} dx = \int_{x=0}^{1}\left[x-x^2-x^2+x\cdot x^2\right] dx$$

$$= \left[\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}\right]_0^1 = \frac{1}{2}-\frac{2}{3}+\frac{1}{4} = \frac{6-8+3}{12} = \frac{1}{12}$$

Also Along the curve $y=x^2$ from O to A, $y=x^2$ $\therefore dy=2x\,dx$

Hence $\int_{OMA} \left[(x^2+xy)\,dx+x\,dy\right] = \int_{x=0}^{1}\left[(x^2+x\cdot x^2)\,dx+x\cdot 2x\,dx\right]$

$$= \left[\frac{x^3}{3}+\frac{x^4}{4}+2\frac{x^3}{3}\right]_0^1 = \frac{1}{3}+\frac{1}{4}+\frac{2}{3}$$

Along the curve $y=x$ from A to O, $y=x$ $\therefore dy=dx$, x varies from 1 to 0.

$$\therefore \int_{ANO}\left[(x^2+xy)\,dx+x\,dy\right] = \int_{x=1}^{0}\left[(x^2+x\cdot x)\,dx+x\,dx\right]$$

$$= \left[\frac{2x^3}{3}+\frac{x^2}{2}\right]_{x=1}^{0} = -\frac{2}{3}-\frac{1}{2}$$

Hence $\oint_C \left[P\,dx+Q\,dy\right] = \int_{OMA}(P\,dx+Q\,dy)+\int_{ANO}(P\,dx+Q\,dy)$

$$= \frac{1}{3}+\frac{1}{4}+\cancel{\frac{2}{3}}-\cancel{\frac{2}{3}}-\frac{1}{2} = \frac{4+3-6}{12} = \frac{1}{12}$$

Hence $\iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy = \frac{1}{12} = \oint_C (P\,dx+Q\,dy)$

Thus Green's theorem is verified.

EX5 Use Stokes' theorem to show that $\int_C (yz\,dx+zx\,dy+xy\,dz)=0$ where C is the curve $x^2+y^2=1$, $z=y^2$.

Soln: Intersection of two surfaces is a curve, represents a curve.

Thus $x^2+y^2=1$, $z=y^2$

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⑧ We consider a surface S on the curve. Then by Stokes' theorem $\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,dS$.

Let us take $\bar{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$.

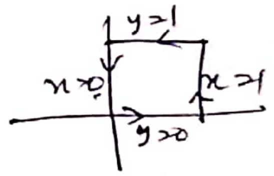
Then $\int_C \bar{F}\cdot d\bar{r} = \int_C [yz\,dx + xz\,dy + xy\,dz]$

Now
$$\text{curl}\,\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z) = \bar{0}$$

Then $\iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,dS = 0$.

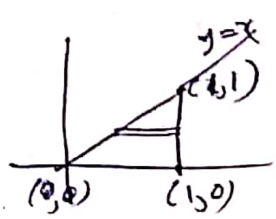
Hence $\int_C [yz\,dx + xz\,dy + xy\,dz] = 0$.

EX6 Use Green's theorem to prove that $\int_C (y\,dx + 2x\,dy) = 1$, where C is the boundary of the square $0 \le x \le 1, 0 \le y \le 1$, taken in the positive sense.



Soln By Green's Theorem
$$\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$
$$\therefore \int_C (y\,dx + 2x\,dy) = \iint_R (2-1)\,dx\,dy = \iint_R dx\,dy = \text{Area of the square} = 1.$$

EX7 Using Green's theorem show that $\int_C [2xy\,dx + (e^x + x^2)\,dy] = 1$, where C is the boundary of the triangle with vertices $(0,0), (1,0), (1,1)$ taken in the positive sense.



Soln
$$\int_C [2xy\,dx + (e^x + x^2)\,dy] = \iint_R [e^x + 2x - 2x]\,dx\,dy \text{ by Green's Theorem}$$
$$= \iint_R e^x\,dx\,dy$$

Now
$$\iint_R e^x\,dx\,dy = \int_0^1\int_{x=y}^{1} e^x\,dx\,dy = \int_{y=0}^1 [e^x]_{x=y}^1\,dy = \int_{y=0}^1 [e^1 - e^y]\,dy$$
$$= [ye - e^y]_{y=0}^1 = [e - e^1 - (0-1)] = 1$$