MATRIX THEORY ASSIGNMENT

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Q.) Find the values of λ for which the equation.

$$(\lambda-1)x + (3\lambda+1)y + 2\lambda z = 0$$
$$(\lambda-1)x + (4\lambda-2)y + (\lambda+3)z = 0$$
$$2x + (3\lambda+1)y + 3(\lambda-1)z = 0$$

are consistent and find the ratios $x:y:z$ when λ has the smallest of these values. What happens when λ has the greatest of these values?

Let $A = \begin{bmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3\lambda-3 \end{bmatrix}$

for homogeneous equation, the determinant of matrix is zero.

So, $|A| = 0$.

or, $(\lambda-1)\{(4\lambda-2)(3\lambda-3) - (\lambda+3)(3\lambda+1)\} - (3\lambda+1)\{(\lambda-1)(3\lambda-3) - 2(\lambda+3)\}$
$+ 2\lambda\{(\lambda-1)(3\lambda+1) - 2(4\lambda-2)\} = 0$.

or, $6\lambda(\lambda-3)^2 = 0$

or $\lambda = 3, 0$.

For $\lambda = 0$, smallest value.

$-x + y = 0 \longrightarrow x = y$
$-x + 2y + 3z = 0$
$2x + y - 3z = 0 \longrightarrow 3x = 3z \longrightarrow x = z$.

$\therefore x = y = z$, or $x:y:z = 1:1:1$

For $\lambda = 3$, greatest value.

$2x + 10y + 6z = 0$
$2x + 10y + 6z = 0$
$2x + 10y + 6z = 0$.

i.e, all equation comes out as same.

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Q2) Find the characteristic equation of

$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ and hence find it's Inverse.

Let $p(\lambda) = - \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix}$

$= (1-\lambda)\left[(3-\lambda)(-4-\lambda) - 12\right] - 1\left[(-4-\lambda) - 6\right] + 3\left[-4 + 2(3-\lambda)\right]$

$= (1-\lambda)\left[\lambda^2 + \lambda - 24\right] - 5\lambda + 16$

or, $p(\lambda) = \lambda^3 + 20\lambda + 8$

$\therefore$ now, we know that $p(A) = 0$.

$\therefore A^3 + 20A + 8 = 0$, this is the characteristic equation of A

multiplying both sides by A^{-1}, we get

$A^2 + 20I = -8A^{-1}$

or $A^{-1} = \dfrac{-(A^2 + 20I)}{8}$

now $A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$

$\Rightarrow A^{-1} = \dfrac{-1}{8} \begin{bmatrix} -4+20 & -8 & -12 \\ 10 & 22+20 & 6 \\ 2 & 2 & 22+20 \end{bmatrix}$

$= \dfrac{-1}{8} \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ \frac{-5}{4} & \frac{-1}{4} & \frac{-3}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$

3) If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that $A^n = A^{n-2} + A^2 - I$

Hence find A^{50}

Let $p(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 1)$

now ~~when~~ we know $p(A) = 0$.

$\therefore (I-A)(A^2-I) = 0$

or $A^2 - A^3 - I + A = 0$.

or $A^3 = A + A^2 - I$, $n = 3$. (i)

multiplying both sides by A

$A^4 = A^2 + A^3 - A$

or $A^4 = A^2 + \cancel{A} + A^2 - I - \cancel{A}$

or $A^4 = A^2 + A^2 - I$, $n = 4$ (ii)

multiplying both sides by A again we get.

$A^5 = A^3 + A^3 - A$

or $A^5 = A^3 + \cancel{A} + A^2 - I - \cancel{A}$

or $A^5 = A^3 + A^2 - I$, $n = 5$, (iii)

similarly we get

$A^6 = A^4 + A^2 - I$, $n = 6$

$A^7 = A^5 + A^2 - I$, $n = 7$

and so on, generalising, we get

$A^n = A^{n-2} + A^2 - I$

now $A^{50} = A^{48} + A^2 - I = A^{46} + 2(A^2 - I) = \ldots\ldots = A^2 + 24(A^2 - I)$

$= 25A^2 - 24I$

now $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$\therefore A^{50} = \begin{bmatrix} 25-24 & 0 & 0 \\ 25 & 25-24 & 0 \\ 25 & 0 & 25-24 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$

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Q4) Find matrix P which transforms the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$ to it's diagonal form. Hence find A^4

finding eigen values,

$|A - \lambda I| = 0$

or $$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

or $(1-\lambda)\left[(5-\lambda)(1-\lambda) - 1\right] - 1\left[1-\lambda-3\right] + 3\left[1 - 3(5-\lambda)\right]$

or $-\lambda^3 + 7\lambda^2 - 36 = 0$

solving this equation, we get $\lambda = 6, -2, 3$

now $$(A - \lambda I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

a) for $\lambda = 6$,

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, $-5x + y + 3z = 0 \rightarrow y = 2x$

$x - y + z = 0$

$3x + y - 5z = 0$ $\rightarrow 4x - 4z = 0 \rightarrow x = z$

$\therefore (x, y, z) = k(1, 2, 1)$

b) for $\lambda = -2$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, $3x + y + 3z = 0$

$x + 7y + z = 0$

$3x + y + 3z = 0$

or $y = 0$, $x = -z$

$\therefore (x, y, z) = m(1, 0, -1)$

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c) for $\lambda = 3$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$-2x + y + 3z = 0$$
$$x + 2y + z = 0$$
$$3x + y - 2z = 0$$

or $x = z, \quad y = -x$

$\therefore (x, y, z) = x(1, -1, 1)$

$\therefore$ let $P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\therefore \cancel{A = P^{-1}DP} \quad A = PDP^{-1}$

& now $A^4 = PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1}$

$= PD^4P^{-1}$

now $D^4 = \begin{bmatrix} 6^4 & 0 & 0 \\ 0 & -2^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 1296 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}$

$P^{-1} = \dfrac{\text{adj. } P}{|P|}$, $\quad |P| = 1(0-1) - 1(2+1) + 1(-2)$

$= -1 - 3 - 2 = -6$

~~adj P~~ $C_{11} = -1 \qquad C_{12} = -3 \qquad C_{13} = -2$

$C_{21} = -2 \qquad C_{22} = 0 \qquad C_{23} = 2$

$C_{31} = -1 \qquad C_{32} = 3 \qquad C_{33} = -2$

$\therefore P^{-1} = \begin{bmatrix} -1 & -3 & -2 \\ -2 & 0 & 2 \\ -1 & 3 & -2 \end{bmatrix}^T \times \dfrac{-1}{6} = \dfrac{-1}{6}\begin{bmatrix} -1 & -2 & -1 \\ -3 & 0 & 3 \\ -2 & 2 & -2 \end{bmatrix}$

$\cancel{= \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}}$

$A^4 = \dfrac{-1}{6}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1296 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} -1 & -2 & -1 \\ -3 & 0 & 3 \\ -2 & 2 & -2 \end{bmatrix}$

$= \dfrac{-1}{6}\begin{bmatrix} 1296 & 16 & 81 \\ 2592 & 0 & -81 \\ 1296 & -16 & 81 \end{bmatrix} \begin{bmatrix} -1 & -2 & -1 \\ -3 & 0 & 3 \\ -2 & 2 & -2 \end{bmatrix}$

$= \begin{bmatrix} 251 & 405 & 235 \\ 405 & 891 & 405 \\ 235 & 405 & 251 \end{bmatrix}$

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Q5) If λ is an eigen value of a non singular matrix A, show that $\frac{|A|}{\lambda}$ is an eigen value of (adj. A)

We know that

$$A\tilde{x} = \lambda \tilde{x}$$

multiplying both sides by A^{-1}

$$I\tilde{x} = \lambda A^{-1}\tilde{x}$$

or $$\frac{1}{\lambda}\tilde{x} = A^{-1}\tilde{x}$$

or $$\frac{(\text{adj } A)}{|A|}\tilde{x} = \frac{1}{\lambda}\tilde{x} \quad \left[\text{as } A^{-1} = \frac{\text{adj } A}{|A|}\right]$$

or $$(\text{adj } A)\tilde{x} = \frac{|A|}{\lambda}\tilde{x}$$

hence $\frac{|A|}{\lambda}$ is an eigen value of adj A

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