Inverse Laplace transformation

Given the Laplace transform F(S), the operation of obtaining f(t) is termed the inverse Laplace transformation and is denoted by.

$$f(t) = \mathcal{L}^{-1}[F(S)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(S)\, e^{st}\, ds.$$

Properties of Laplace transform

1. Multiplication by a constant

Let K be a constant $\mathcal{L}[K f(t)] = K F(S)$

2. Sum and difference

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(S) \pm F_2(S)$$

3. Differentiation with respect to "t" (time-differentiation)

Let F(S) be the Laplace transform of f(t) and let $f(0^+)$ be the value of f(t) as t approaches 0.

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = SF(S) - \lim_{t\to 0} f(t) = SF(S) - f(0^+)$$

Laplace transform of the second derivative of f(t)

$$\mathcal{L}\left[\frac{d^2 f(t)}{dt^2}\right] = S^2 F(S) - S f(0^+) - f'(0^+)$$
$$= S^2 F(S) - S f(0^+) - \frac{df(0^+)}{dt}$$

$f'(0^+)$ is the value of the First derivative of f(t) as t approaches 0.

④ Integration by "t" (Time - Integration)

If $\mathcal{L}[f(t)] = F(s)$

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

$$\mathcal{L}\left[\int f(t)\,dt\right] = \mathcal{L}\left[\int_0^t f(t)\,dt + f^{-1}(0^+)\right]$$

$$= \frac{F(s)}{s} + \frac{f^{-1}(0^+)}{s}$$

where $f^{-1}(0^+)$ is the value of the integral $f(t)$ as t approaches zero.

$$\mathcal{L}\left[\int_0^{t_1}\int_0^{t_2}\int_0^{t_3}\cdots\int_0^{t_n} f(t)\,dt_1\,dt_2\cdots dt_n\right] = \frac{F(s)}{s^n}$$

⑤ [Differentiation with respect to "s" frequency differentiation]

$$\mathcal{L}[t.f(t)] = -\frac{dF(s)}{ds}$$

⑥ Integration by 's' frequency-Integration.

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s).\,ds$$

⑦ Shifting theorem :-

[a] Shifting in time (time-shifting)

$$\mathcal{L}[f(t-a).u(t-a)] = e^{-as}F(s)$$

[b] Shifting in frequency (frequency shifting)

$$\mathcal{L}\left[e^{-at} f(t)\right] = \int_0^\infty e^{-at} f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty f(t)\, e^{-(s+a)t}\, dt$$

$$= F(s+a)$$

$$\mathcal{L}\left[e^{at} f(t)\right] = F(s-a)$$

(8) Initial value Theorem :-

$$f(0^+) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} \left[s.F(s)\right]$$

If the function $f(t)$ and its first derivative $\frac{df(t)}{dt}$ are both Laplace transformable. then.

(9) Final value Theorem

The final value of a function $f(t)$ is given as.

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} \left[s.F(s)\right]$$

(10) Theorem for periodic Functions :-

The Laplace transform of a periodic function (wave) with period T is equal to $\frac{1}{1-e^{-Ts}}$ times the Laplace transform of the first cycle of that function (wave).

(11) convolution Theorem :- Given two functions $f_1(t)$ and $f_2(t)$ which are zero for $t<0$. If $\mathcal{L}[f_1(t)] = F_1(s)$ and $\mathcal{L}[f_2(t)] = F_2(s)$ then,

$$\mathcal{L}^{-1}\left[F_1(s).F_2(s)\right] = f_1(t) * f_2(t)$$ is called the convolution of $f_1(t)$ and $f_2(t)$ is equal to

$$\int_0^t f_1(t-\tau)\, f_2(\tau)\, d\tau \quad \text{or} \quad \int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau.$$

(12) Time scaling :- If Laplace transform of $f(t)$ is $F(S)$, then $\mathcal{L}[f(at)] = \frac{1}{|a|} F\left(\frac{s}{a}\right)$

Laplace transform pairs

$f(t)$	$F(S) = \int_0^{\infty} f(t) e^{-st} dt.$
1 or $u(t)$, K	$\frac{1}{s}, \frac{K}{s}$
t, t^n	$\frac{1}{s^2}, \frac{n!}{s^{n+1}}$
$\delta(t)$	1
$e^{\pm at}$	$\frac{1}{s \mp a}$
$t e^{\pm at}$	$\frac{1}{(s \mp a)^2}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$e^{\pm at} f(t)$	$F(s \mp a)$
$f(t \pm t_0)$	$e^{\pm t_0 s} F(s)$

Solution of Linear Differential Equations :-

A linear differential equation of the general form

$$a_0 \frac{d^n i}{dt^n} + a_1 \frac{d^{n-1} i}{dt^{n-1}} + \cdots\cdots + a_{n-1}\frac{di}{dt} + a_n i = v(t)$$

becomes, as a result of the Laplace transformation, an algebraic equation which may be solved for the unknown as.

$$I(S) = \frac{\mathcal{L}[v(t)] + \text{initial condition terms}}{a_0 S^n + a_1 S^{n-1} + \cdots + a_{n-1} S + a_n}$$

$$I(S) = \frac{P(S)}{Q(S)}$$

$$Q(S) = a_0 S^n + a_1 S^{n-1} + \cdots\cdots + a_{n-1} S + a_n$$

$$= a_0 (S+S_1) \cdots\cdots (S+S_n)$$

(i) Partial Fraction Expansion when all the Roots of Q(S) are simple.

If all roots of $Q(S)=0$, are simple then.

$$I(S) = \frac{P(S)}{(S+S_1)(S+S_2)\cdots(S+S_n)} = \frac{K_1}{S+S_1} + \frac{K_2}{S+S_2} + \cdots\cdots + \frac{K_n}{S+S_n}$$

where the K's are called Residues. Any of the residues $K_1, K_2, \text{---} K_n$ can be found by multiplying $I(S)$ by corresponding denominator factor and setting $(S+S_j)$ equal to zero.

i.e. $S = -S_j$, $K_j = \left[(S+S_j)\cdot\frac{P(S)}{Q(S)}\right]_{S=-S_j}$

Example

$$F(S) = \frac{2}{(S-1)(S-2)} = \frac{C_1}{S-1} + \frac{C_2}{S-2}$$

$$C_1 = (S-1) \cdot F(S) \Big|_{S=1} = (S-1) \cdot \frac{2}{(S-1)(S-2)} \Big|_{S=1} = -2$$

$$C_2 = (S-2)\, F(S) \Big|_{S=2} = (S-2) \cdot \frac{2}{(S-1)(S-2)} \Big|_{S=2} = 2$$

$$\therefore F(S) = \frac{-2}{S-1} + \frac{2}{S-2}$$

(II) Partial Fraction Expansion when some roots of $Q(S)$ are of multiple order.

If a root of $Q(S)=0$, is of multiplicity r, then

$$I(S) = \frac{P(S)}{(S+S_1)^r Q_1(S)} = \frac{K_{11}}{S+S_1} + \frac{K_{12}}{(S+S_1)^2} + \cdots + \frac{K_{1r}}{(S+S_1)^r} + \cdots$$

$$K_{1r} = (S+S_r)^r \cdot I(S) \Big|_{S=-S_1}$$

$$K_{1(r-1)} = \frac{d}{dS}\left[(S+S_1)^r \cdot I(S)\right] \Big|_{S=-S_1}$$

$$K_{1(r-2)} = \frac{1}{2!} \frac{d^2}{dS^2}\left[(S+S_1)^r \cdot I(S)\right]_{S=-S_1}$$

$$K_{11} = \frac{1}{(r-1)!} \frac{d^{r-1}}{dS^{r-1}}\left[(S+S_1)^r \cdot I(S)\right] \Big|_{S=-S_1}$$

example:-
$$F(s) = \frac{2s+1}{(s+2)^3}$$

$$= \frac{C_{11}}{(s+2)} + \frac{C_{21}}{(s+2)^2} + \frac{C_{31}}{(s+2)^3}$$

$$C_{31} = (s+2)^3 . F(s) \Big|_{s=-2}$$

$$= (s+2)^3 . \frac{(2s+1)}{(s+2)^3} \Big|_{s=-2} = (-2).2+1 = -3$$

$$C_{21} = \frac{d}{ds}\left[(s+2)^3 . F(s)\right]\Big|_{s=-2}$$

$$= \frac{d}{ds}\left[(s+2)^3 . \frac{2s+1}{(s+2)^3}\right]\Big|_{s=-2} = \frac{d}{ds}\left[2s+1\right]\Big|_{s=-2}$$

$$= 2$$

$$C_{11} = \frac{1}{(3-1)!} . \frac{d^2}{ds^2}\left[(s+2)^3 . F(s)\right]\Big|_{s=-2}$$

$$= \frac{1}{2!}\left[\frac{d^2}{ds^2}(2s+1)\right]_{s=-2} = 0$$

$$F(s) = \frac{2}{(s+2)^2} - \frac{3}{(s+2)^3}$$

(iii) partial expansion when two roots of $Q(S)$ are of complex conjugate pair.

If two ~~#~~ roots of $Q(S)=0$, which form a complex conjugate pair, then.

$$I(S) = \frac{P(S)}{(S+\alpha+j\omega)(S+\alpha-j\omega).Q(S)} = \frac{K_1}{(S+\alpha+j\omega)} + \frac{K_1^*}{(S+\alpha-j\omega)} + \cdots$$

$$K_1 = (S+\alpha+j\omega).I(S)\Big|_{S=-(\alpha+j\omega)}$$

K_1^* is a complex conjugate of K_1.

<u>example</u>

$$F(S) = \frac{S}{S^2+2S+2}$$

$$S^2+2S+2=0$$

$$S_{1,2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm j1$$

$$= \frac{C_1}{(S+1+j1)} + \frac{C_2}{(S+1-j1)}$$

$$C_1 = (S+1+j1)\,F(S)\Big|_{S=-1-j1}$$

$$= (S+1+j1)\frac{S}{(S+1+j1)(S+1-j1)}\Big|_{S=-1-j1} = \frac{-1-j1}{-j2} = 0.5-j0.5$$

$$C_2 = (S+1-j1)\frac{S}{(S+1+j1)(S+1-j1)}\Big|_{S=-1+j1}$$

$$= \frac{-1+j1}{j2} = 0.5+j0.5$$

~~$C_1 = C_2^*$~~ $\qquad C_2 = C_1^*$

Example

$$F(S) = \frac{2S+3}{(S+2)(S^2+4S+8)}$$

The roots of the denominator $D(S)$ are $-2, -2 \pm j2$

$$F(S) = \frac{A}{S+2} + \frac{B}{S+2+j2} + \frac{B^*}{S+2-j2}$$

$$= \frac{A}{S+2} + \frac{BS+C}{S^2+4S+8} \quad \text{[alternative method]}$$

$$= \frac{A(S^2+4S+8) + (S+2)(BS+C)}{(S+2)(S^2+4S+8)}$$

From which

$$2S+3 = A(S^2+4S+8) + (S+2)(BS+C)$$

$$= (A+B)S^2 + (4A+2B+C)S + (8A+2C)$$

Now equating the co-efficients of S^2, S and constant yields,

$$A+B=0, \quad 4A+2B+C=2$$
$$8A+2C=3$$

$$\therefore A = -\frac{1}{4}, \quad B = \frac{1}{4}, \quad C = 5/2$$

$$\therefore F(S) = \frac{-1}{4(S+2)} + \frac{\frac{1}{4}S + \frac{5}{2}}{S^2+4S+8}$$

Solve the Differential Equation.

$$x'' + 3x' + 2x = 0 \;,\quad x(0^+) = 2 \;,\quad x'(0^+) = -3$$

Soln

$$x'' + 3x' + 2x = 0$$

Taping Laplace transform.

$$s^2 X(s) - s X(0^+) - x'(0^+) + 3sX(s) - 3X(0^+) + 2X(s) = 0$$

$$\Rightarrow (s^2+3s+2) X(s) = s X(0^+) + x'(0^+) + 3x(0^+)$$

$$\Rightarrow (s^2+2s+2) X(s) = 2s+3$$

$$\Rightarrow \quad X(s) = \frac{2s+3}{s^2+3s+2} = \frac{2s+3}{(s+1)(s+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+2}.$$

$$\therefore K_1 = (s+1)\cdot X(s)\Big|_{s=-1} = \frac{\cancel{(s+1)}(2s+3)}{\cancel{(s+1)}(s+2)}\Big|_{s=-1} = \frac{-2+3}{-1+2} = 1$$

$$K_2 = (s+2) X(s)\Big|_{s=-2} = \frac{\cancel{(s+2)}(2s+3)}{\cancel{(s+2)}(s+1)}\Big|_{s=-2} = \frac{-4+3}{-2+1} = 1$$

$$\therefore X(s) = \frac{2s+3}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{1}{s+2}$$

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{1}{s+2}\right] = e^{-t} + e^{-2t}$$