Distributive Laws

6a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ 6b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Complement Laws

7a. $A \cup A' = S$ 7b. $A \cap A' = \varphi$

8a. $S' = \varphi$ 8b. $\varphi' = S$

9. $(A')' = A$.

De Morgan's Laws

10a. $(A \cup B)' = A' \cap B'$ 10b. $(A \cap B)' = A' \cup B'$.

Note: These laws can be verified with the help of Venn Diagrams.

1.3 BASIC TERMINOLOGY

1. Die: It is a small cube. Dots 1, 2, 3, 4, 5, 6 are marked on its six faces. Plural of the die is dice. The outcome of throwing a die is the number of dots on its upper face.

2. Cards: A pack of cards consists of four suits, namely, Spades, Hearts, Diamonds and Clubs. Each suit contains 13 cards, among these nine cards numbered 2, 3, 4, ..., 10, an Ace, a King, a Queen and a Jack or Knave. Colour of Spades and Clubs is black and that of Hearts and Diamonds is red. Aces, Kings, Queens and Jacks are called **face** cards.

3. Random experiment: An experiment E is called a **random experiment,** if (*i*) all possible outcomes of E are known in advance, (*ii*) it is impossible to predict which outcome will occur at a particular performance of E, and (*iii*) E can be repeated under identical conditions infinite number of times, *i.e.,* a large number of times.

Example: (*i*) The experiment of tossing a coin is an example of a random experiment. Here the possible outcomes are 'head' or 'tail'. It is impossible to predict which outcome will occur at a particular toss of the coin under given condition.

(*ii*) Other examples of random experiment are 'throwing a die', 'drawing a card from a pack of 52 cards at random,' etc.

Note: Performing a random experiment is called a **trial.**

4. Event space: The set of all possible outcomes of a given random experiment E is called the **event space** or **sample space** of E and is denoted by S.

Example: (*i*) The event space S of the random experiment of tossing a coin is $\{H, T\}$, where H and T correspond to the outcomes 'head' and 'tail' respectively. This is an example of a finite event space since it contains a finite number of elements.

(*ii*) The event space S corresponding to the experiment of choosing a number at random from the interval (2, 4) is the set (2, 4) itself which is an infinite event space.

5. Event: Let us consider the random experiment of throwing a die, its event space is $S = \{1, 2, 3, 4, 5, 6\}$. Here $A = \{2, 4, 6\}$ is an event which can be described as 'Even number appears in throwing a die'. The event A happens in a specific trial of the given random experiment if and only if exactly one of the outcomes 2, 4 or 6 occurs in the trial.

Therefore an **event** A of a given random experiment can be defined as a subset of the corresponding event space S. So, outcome of a random experiment is termed as **event.**

6. Impossible event: An event of a given random experiment is called an **impossible event** if it can never happen in any performance (trial) of the random experiment under identical conditions. Such an event is described by the empty subset φ of the corresponding event space S.

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Example: In the random experiment 'throwing a die' the event 'face marked 7' is an impossible event.

7. Certain event: An event of a given random experiment is called 'certain' if it happens in every performance of the corresponding random experiment under identical conditions.

Example: In connection with the random experiment of tossing a coin, the event H or T is a certain event. It is described by the event space $S = \{H, T\}$.

Event space is an event since it is a subset of itself. It is a certain event.

Simple events: An event A is called a **simple event** or an **elementary event** if A contains exactly one element. In other words, events which cannot be further decomposed are called **simple events** or **elementary events.**

Note: Let E be a random experiment, then the simple events connected with E will be called **event points** or **simply points.** The set of all possible event points is known as the **event space** S of E.

The experiment E must be such that its event space S is completely known.

Any subset A of S will be called an **event** connected with E, *i.e.*, an **event** is an aggregate of some of the event points.

The entire space is the certain event and the empty subset φ is the **impossible event.**

8. Compound events: An event A is called a **compound event** or **composite event** if A contains more than one element. In other words, events which can be decomposed into simple events are known as **compound events** or **composite events.**

Example: $E \equiv$ throwing of an ordinary die

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{1\} \rightarrow$ only 1,

$B = \{3, 6\} \rightarrow$ multiple of 3,

$C = \{2, 4, 6\} \rightarrow$ even faces.

$A \rightarrow$ simple event; $B, C \rightarrow$ compound events.

9. Complementary event: Let A be an event. An event which is the negative of the event A is called the **complementary event** of A and is denoted by $\overline{A}$ or A' or A^c. Therefore, $\overline{A}$ contains all the elements of the event space which are not in A.

Example: Let us consider the random experiment of throwing a die. The complementary event of 'multiple of 3' is obviously 'not multiple of 3' which can be decomposed into the simple events '1', '2', '4' and '5'.

Note: Complementary event of a certain event is an impossible event and vice-versa.

10. Simultaneous occurrence of two or more events: Let A and B are two events, then the set $A \cap B$ represents the simultaneous occurrence of the two events A and B. This event is also denoted by AB. Similarly, the simultaneous occurrence of n events $A_1, A_2, ..., A_n$ is the set $A_1 A_2 ... A_n$.

Example: Let us consider the random experiment of throwing a die. Let A = 'even face' = $\{2, 4, 6\}$ and B = 'multiple of 3' = $\{3, 6\}$. Then $A \cap B = AB = \{6\}$ is the event whose occurrence shows the simultaneous occurrence of A = 'even face' and B = 'multiple of 3'.

11. At least one of two or more events: Let A and B are two events, then the set $A \cup B$ represents the occurrence of at least one of two events A and B. This event is also denoted by $A + B$. Similarly, the occurrence of at least one of n events $A_1, A_2, ..., A_n$ is the set $A_1 \cup A_2 \cup ... \cup A_n = A_1 + A_2 + ... + A_n$.

Example: Let us consider the random experiment of throwing a die. Let A = 'odd face' = $\{1, 3, 5\}$ and B = 'multiple of 3' = $\{3, 6\}$. Then $A \cup B = A + B = \{1, 3, 5, 6\}$ is the event whose occurrence shows the occurrence of at least one of A = 'odd face' and B = 'multiple of 3'.

12. Mutually exclusive events (m.e.): Two events A and B connected to a given random experiment E are said to be **mutually exclusive (m.e.)** if A and B can never happen simultaneously in any performance of E, *i.e.*, if $A \cap B = AB = \varphi$ (*i.e.*, A, B are disjoint).

Example: $E \equiv$ throwing of a die, $S = \{1, 2, 3, 4, 5, 6\}$.

A = 'even face' = $\{2, 4, 6\}$, B = 'odd face' = $\{1, 3, 5\}$,

C = 'multiple of 3' = $\{3, 6\}$.

A, C are not mutually exclusive since 6 is an even number as well as a multiple of 3, *i.e.*, $AC \neq \varphi$. Similarly B, C are not mutually exclusive events. But A, B are mutually exclusive events since $AB = \varphi$.

13. Exhaustive events: A collection of events is said to be **exhaustive** if at least one event belonging to the collection is sure to occur in every performance of the underlying random experiment. Therefore, two or more events, say, $A_1, A_2, A_3, \ldots$ are said to be exhaustive if $A_1 \cup A_2 \cup A_3 \cup \ldots$ = certain event, *i.e.*, if at least one of them must occur.

Example: $E \equiv$ throwing of a die, $S \equiv \{1, 2, 3, 4, 5, 6\}$.

$A_1 \equiv \{2, 4, 6\} \rightarrow$ 'even face', $A_2 \equiv \{1\} \rightarrow$ only 1 appear,

$A_3 \equiv \{3, 5\} \rightarrow$ odd faces except 1.

The collection of events $\{A_1, A_2, A_3\}$ is exhaustive since $A_1 \cup A_2 \cup A_3 = A_1 + A_2 + A_3 = S$ = certain event.

14. Principle of counting: If an event can occur in n_1 ways and thereafter for each of these events a second event can occur in n_2 ways and thereafter for each of these events a third event can occur in n_3 ways, then the number of ways these three events can happen is given by $n_1 n_2 n_3$. This result can be extended for any number of events.

15. Equally likely events: If one of the events cannot be expected to occur in preference to another then such events are called **equally likely,** or **equally probable.** For example, in tossing a coin, the occurrence of the head or the tail is equally likely.

Let us consider the random experiment of throwing a die. The turning up of the six different faces of the die are mutually exclusive, exhaustive and equally likely.

16. Favourable events: The outcomes which ensure the happening of a desired event are called **favourable events** (or **cases**). Thus in throwing a die, the favourable events of getting an even number are 2, 4 and 6.

17. Factorial: If n be a positive integer, then $n\,!$ or $\underline{|n}$ (read as factorial n) stands for the product of first n natural numbers (*i.e.*, positive integers).

$$\begin{aligned}\therefore \quad n\,! &= \underline{|n} \\ &= 1 \times 2 \times 3 \times \ldots \times (n-2) \times (n-1) \times n \\ &= n \times \{(n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1\} \\ &= n \times (n-1) \times \{(n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1\} \\ \therefore \quad n\,! &= n\,\{(n-1)\,!\} = n\,(n-1)\,\{(n-2)\,!\} = n\,(n-1)\,(n-2)\,\{(n-3)\,!\}.\end{aligned}$$

Notes: (*i*) $\underline{|0} = 0\,! = 1$ (by definition)

(*ii*) $\underline{|1} = 1\,! = 1$, $\underline{|2} = 2\,! = 2 \times 1 = 2$, $\underline{|3} = 3\,! = 3 \times 2 \times 1 = 6$.

18. Permutation: If n and r ($\leq n$) are positive integers, then ${}^nP_r$ stands for the number of all possible arrangements (or permutations) of n distinct objects taken r at a time in some definite order.

Thus, ab, ba are two different arrangements of the two letters a, b.

$$\therefore \quad {}^2P_2 = 2.$$

The arrangements (or permutations) of three letters a, b, c taken two at a time are ab, ba, ac, ca, bc, cb.

$$\therefore \quad {}^3P_2 = 6.$$

$$\therefore \quad {}^nP_r = n(n-1)(n-2)\ldots\{n-(r-2)\}\{n-(r-1)\}$$

$$= \frac{n!}{(n-r)!}.$$

$$\therefore \quad {}^nP_n = n!, {}^nP_0 = 1, {}^nP_1 = n, {}^nP_2 = n(n-1), \text{etc.}$$

19. Combination: If n and r ($\le n$) are positive integers, then ${}^nC_r$ stands for the number of all possible selections or combinations (without regard to the order of their arrangements) of n distinct objects taken r at a time. If we take any one of these combinations, its r objects can be arranged in $r!$ ways. So the total number of arrangements (or permutations) which can be obtained from all these combinations is

$${}^nP_r = {}^nC_r \cdot r!. \quad \therefore \quad {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!}.$$

Thus, the combinations (selections) of three letters a, b, c taken two at a time are ab, bc, ca.

$$\therefore \quad {}^3C_2 = 3 = \frac{3!}{2!(3-2)!}$$

It can be proved that

(*i*) ${}^nC_r = {}^nC_{n-r}$ (*ii*) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

Note: $${}^nC_0 = {}^nC_n = 1; {}^nC_1 = {}^nC_{n-1} = n;$$

$${}^nC_2 = {}^nC_{n-2} = \frac{1}{2}n(n-1).$$

Example: *From six engineers and four architects, a committee is to be formed having three engineers and two architects. How many different committees can be formed if (i) there is no restriction, (ii) two particular engineers must be included, (iii) one particular architect must be excluded?*

Solution:

(*i*) Number of committees $= {}^6C_3 \times {}^4C_2 = \dfrac{6!}{3!(6-3)!} \times \dfrac{4!}{2!(4-2)!}$

$$= \frac{6\times5\times4\times3!}{3\times2\times1\times3!} \times \frac{4\times3\times2!}{2\times1\times2!} = 20 \times 6 = 120.$$

(*ii*) Here we have to select one engineer from the remaining four engineers.

$\therefore$ Number of committees $= {}^4C_1 \times {}^4C_2 = 4 \times 6 = 24.$

(*iii*) Here we have to choose two architects from the remaining three architects.

$\therefore$ Number of committees $= {}^6C_3 \times {}^3C_2 = 20 \times 3 = 60.$

1.4 CLASSICAL DEFINITION OF PROBABILITY

Let E be a random experiment such that its event space S contains a finite number, say n, of event points, all of which are known to be equally likely. If any event A connected with E contains

$m(A)$ number of these event points, then the probability of A, denoted by $P(A)$, will be defined by

$$P(A) = \frac{m(A)}{n}.$$

Deductions:

(*i*) $0 \le P(A) \le 1$ (*ii*) $P(S) = 1$

(*iii*) $P(\varphi) = 0$ (*iv*) $P(\bar{A}) = 1 - P(A)$.

Proof: (*i*) We have, $P(A) = \frac{m}{n}$.

Obviously, $0 \le m \le n \;\Rightarrow\; 0 \le \frac{m}{n} \le 1 \;\Rightarrow\; 0 \le P(A) \le 1$.

(*ii*) When $A = S$, then $m = n$.

$\therefore$ $P(A) = P(S) = \frac{n}{n} = 1$.

(*iii*) $P(\varphi) = \frac{0}{n} = 0$

(*iv*) $P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$.

ILLUSTRATIVE EXAMPLES—I

Example 1: Let E denotes the random experiment of tossing a coin three times in succession. A typical event point is, say, 'head, head, tail' which may be denoted by the symbol (H, H, T). The event space S consists of 8 $(= 2^3)$ points $U_1, U_2, ..., U_8$ given by

$$U_1 = (H, H, H),\ U_2 = (H, H, T),\ U_3 = (H, T, H),\ U_4 = (T, H, H),$$
$$U_5 = (T, T, H),\ U_6 = (T, H, T),\ U_7 = (H, T, T),\ U_8 = (T, T, T).$$

and we write $S = U_1 + U_2 + ... + U_8$.

Let A denotes the event 'two heads', then A contains the 3 points U_2, U_3, U_4, *i.e.*, $A = U_2 + U_3 + U_4$.

If B be the event 'head in the first trial', then

$$B = U_1 + U_2 + U_3 + U_7$$
$$A + B = U_1 + U_2 + U_3 + U_4 + U_7$$
$$AB = U_2 + U_3;\ A - AB = U_4,\ B - AB = U_1 + U_7.$$

We note that the events AB, $A - AB$ and $B - AB$ are pairwise mutually exclusive.

The event 'no head or all tails' is obviously the event point U_8, so that the event 'at least one head' is the complementary event

$$\bar{U}_8 = S - U_8 = U_1 + U_2 + ... + U_7.$$

In the following examples we assume that any event space in question is classical in nature, *i.e.*, it contains a finite number of event points, all of which are equally probable, unless stated otherwise.

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Example 2: *A coin is tossed 3 times in succession. Find the probability of (a) 2 heads, (b) 2 consecutive heads.*

Solution: Here the total number of points in the event space is $n = 2^3 = 8$.

(*a*) Let A denotes the event '2 heads'. Then A contains 3 event points, *viz.*;

$$U_2, U_3, U_4, \text{ i.e., } m(A) = 3.$$

$$\therefore \quad P(A) = \frac{m(A)}{n} = \frac{3}{8}.$$

(*b*) Let B denotes the event '2 consecutive heads'. Then B consists of 2 event points, *viz.*; U_2, U_4 so that $m(B) = 2$.

$$\therefore \quad P(B) = \frac{m(B)}{n} = \frac{2}{8} = \frac{1}{4}.$$

Example 3: *A room has 3 lamps. From a collection of 10 light bulbs of which 5 are defective, a person selects 3 at random and puts them in the sockets. What is the probability that he will have light?*

Solution: There are 10 bulbs in all.

$\therefore$ Total number of ways of taking 3 out of 10 is

$$n = {}^{10}C_3 = \frac{10!}{3!(10-3)!} = \frac{10\times 9\times 8\times 7!}{3\times 2\times 1\times 7!}$$

$$= 10 \times 3 \times 4 = 120.$$

Now 5 bulbs are defective.

$\therefore$ Total number of ways of getting 3 defective bulbs is

$$m = {}^5C_3 = \frac{5!}{3!(5-3)!} = \frac{5\times 4\times 3!}{3!2!} = 10$$

Hence the probability of getting all the defective bulbs (*i.e.*, probability of not getting light)

$$= \frac{10}{120} = \frac{1}{12}.$$

$$\therefore \quad \text{Probability of light} = 1 - \frac{1}{12} = \frac{11}{12}$$ [See deduction (*iv*) of Art. 1.4]

Example 4: *A traffic census shows that out of 1000 vehicles passing a junction point on a highway 550 turned to the right. Find the probability of a car turning to the right at this junction.*

Solution: By question,

Total number of cars $= 1000.$

Total number of cars turning to the right $= 550.$

Hence the required probability $= \dfrac{550}{1000} = \dfrac{11}{20}.$

Example 5: *Among the digits 1, 2, 3, 4, 5, at first one is chosen and then a second selection is made among the remaining four digits. Assuming that all possible outcomes are equally likely, find the probability that an odd digit will be selected at (i) the first time, (ii) second time, (iii) both times.*

Solution: Total number of outcomes is $n = {}^5C_1 \times {}^4C_1 = 5 \times 4 = 20.$

(*i*) There are 12 outcomes in which the first digit chosen is odd:

$$(1, 2), (1, 3), (1, 4), (1, 5)$$

(3, 1), (3, 2), (3, 4), (3, 5)

(5, 1), (5, 2), (5, 3), (5, 4)

$\therefore \quad P$ (first digit chosen is odd) $= \frac{12}{20} = \frac{3}{5}$.

(*ii*) Similarly, there are 12 outcomes in which the second digit chosen is odd:

(2, 1), (3, 1), (4, 1), (5, 1)

(1, 3), (2, 3), (4, 3), (5, 3)

(1, 5), (2, 5), (3, 5), (4, 5)

$\therefore$ P(second digit chosen is odd) $= \frac{12}{20} = \frac{3}{5}$.

(*iii*) There are 6 outcomes in which both digits chosen are odd:

(1, 3), (3, 1)

(1, 5), (5, 1)

(3, 5), (5, 3)

$\therefore \quad P$(both digits chosen are odd) $= \frac{6}{20} = \frac{3}{10}$.

Example 6: *What is the probability that a leap year, selected at random, will contain 53 Sundays?* (*W.B.U.T. 2009*)

Solution: A leap year consists of 366 days that is 52 full weeks (= 364 days) and two days extra. These extra two days may be either (Sunday, Monday), or (Monday, Tuesday), or (Tuesday, Wednesday), or (Wednesday, Thursday), or (Thursday, Friday), or (Friday, Saturday), or (Saturday, Sunday).

Therefore, a leap year will contain 53 Sundays if one of the two extra days is Sunday. Now out of the above seven cases two are favourable (*i.e.*, contains Sunday).

$\therefore \quad$ Required probability $= \frac{2}{7}$.

Example 7: *Two dice are thrown simultaneously. What is the probability that the sum total of the points on the dice will be 6?*

Solution: Now the first die may show 6 possible outcomes and the second die may also show 6 possible outcomes. Hence the total number of outcomes is

$$n = 6 \times 6 = 36.$$

Let A denotes the event 'sum total of the points on the dice is 6'.

$$\therefore \quad A \equiv \{(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)\}.$$

$$\therefore \quad m(A) = 5.$$

$$\therefore \quad P(A) = \frac{m(A)}{n} = \frac{5}{36}.$$

Example 8: *A pair of dice is thrown. Find the probability of getting a sum of 7, when it is known that the digit in the first die is greater than that of the second.* (*W.B.U.T. 2009*)

Solution: There are six numbers (1, 2, 3, 4, 5, 6) written on the six faces of each die. So, the first die may show 6 possible outcomes and the second die may also show 6 possible outcomes. Hence the total number of outcomes is $n = 6 \times 6 = 36$.

Example 17: Each coefficient in the equation $ax^2+bx+c=0$ is determined by throwing a regular shape die. Find the probability that the equation will have real roots. [IIESTS: MA-301/1]

Solution: It is noted that each of the coefficients a, b and c of the given equation can take the values from 1 to 6 since its value is determined by throwing an ordinary die.

$\therefore$ Total number of possible outcomes $= n(S) = 6^3 = 216$.

Let A denotes the event that the given quadratic equation will have real roots and so the number of outcomes favourable to A are obtained if the following condition is satisfied:

$$b^2 \geqslant 4ac \quad \cdots\cdots (1)$$

Since the maximum value that b can take is 6, the maximum value that ac can take to satisfy (1) is clearly equal to $\frac{b^2}{4} = \frac{6^2}{4} = 9$.

Let us construct the following table:

ac	a	c	4ac	b ($b^2 \geq 4ac$)	No. of cases
1	1	1	4	2, 3, 4, 5, 6	5
2	1 2	2 1	8	3, 4, 5, 6	$2 \times 4 = 8$
3	1 3	3 1	12	4, 5, 6	$2 \times 3 = 6$
4	1 2 4	4 2 1	16	4, 5, 6	$3 \times 3 = 9$
5	1 5	5 1	20	5, 6	$2 \times 2 = 4$
6	1 2 3 6	6 3 2 1	24	5, 6	$4 \times 2 = 8$
8	2 4	4 2	32	6	$2 \times 1 = 2$
9	3	3	36	6	$1 \times 1 = 1$

$\therefore n(A) = 5 + 8 + 6 + 9 + 4 + 8 + 2 + 1 = 43.$

$\therefore$ Required probability $= \dfrac{n(A)}{n(S)} = \dfrac{43}{216}.$