Graphs & Trees (Lecture – 5)

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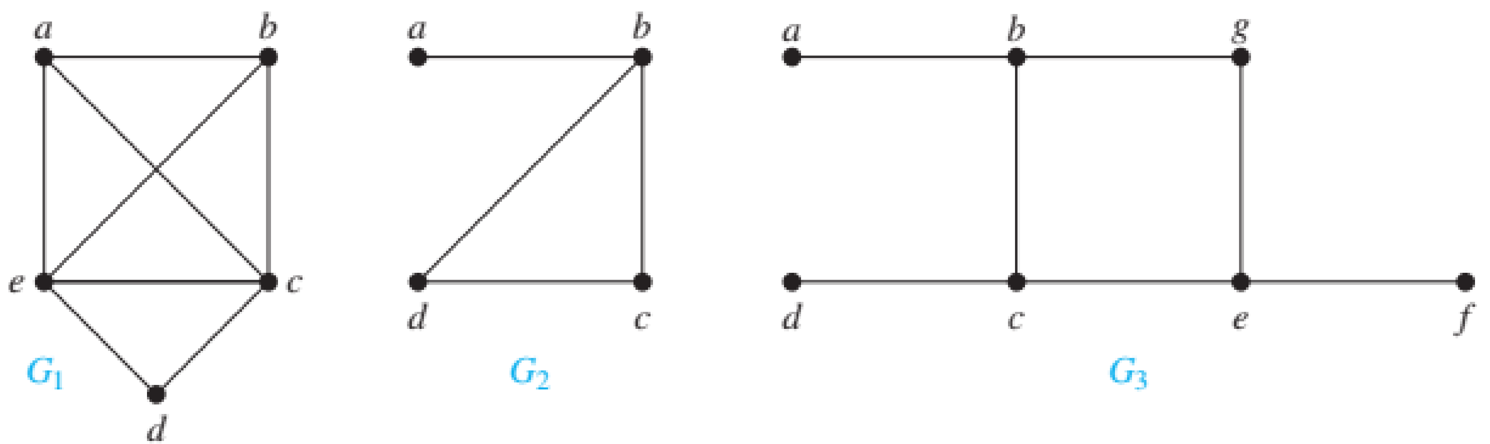
Hamilton Paths & Circuits

- Definition:

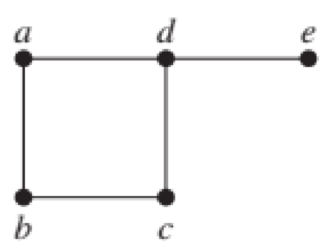
A simple path in a graph G that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph G that passes through every vertex exactly once is called a *Hamilton circuit*. That is, the simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path.

- Which of the following simple graphs have a Hamilton circuit or, if not, a Hamilton path?



Hamilton Paths & Circuits (*Contd…*)

- Conditions for the existence of Hamilton circuits
  - There are no known simple necessary and sufficient criteria for the existence of Hamilton circuits.
- Certain properties can be used to show that a graph has no Hamilton circuit.
  - A graph with a vertex of degree one cannot have a Hamilton circuit, because in a Hamilton circuit, each vertex is incident with two edges in the circuit.
  - If a vertex in the graph has degree two, then both edges that are incident with this vertex must be part of any Hamilton circuit.
  - When a Hamilton circuit is being constructed and this circuit has passed through a vertex, then all remaining edges incident with this vertex, other than the two used in the circuit, can be removed from consideration.
  - A Hamilton circuit cannot contain a smaller circuit within it.
- Show that neither graph displayed below has a Hamilton circuit.



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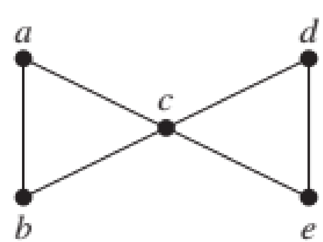

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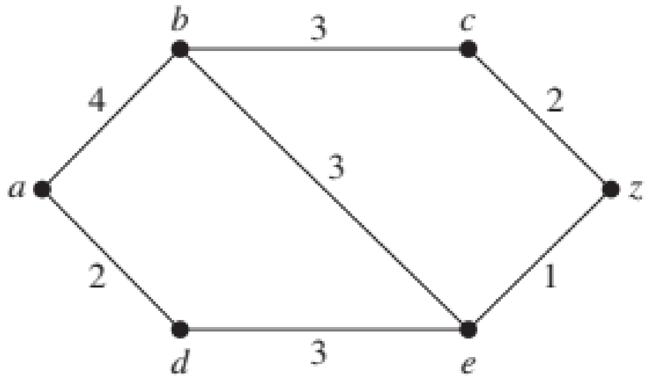
Shortest Path Problems

- **Weighted graphs**: Graphs that have a number assigned to each edge.
- Used to model airline systems, computer networks, etc.
- Airline system models – distance, fare, travel time
- Computer network models – distance, response time, lease rate.
- Two problems involving weighted graphs:
  - Determining a path of least *length* (different from the number of edges in a path in a graph without weights) between two vertices in a network (E.g., what is a shortest path in air distance between Boston and Los Angeles? What is the cheapest fare between these two cities? Which set of telephone lines gives a fastest response time for communications between San Francisco and New York?)
  - Determining a circuit of shortest total length that visits every vertex of a complete graph exactly once (E.g., traveling salesman problem)

Shortest Path Problems (*Contd…*)



- What is the length of a shortest path between *a* and *z* in this weighted graph?

- Finding a shortest path from *a* to *z* could be done by a brute force approach by examining the length of every path from *a* to *z*.
- However, this is infeasible for graphs with a large number of edges.
- **Dijkstra's Algorithm**: greedily computes shortest length path between a pair of vertices in a weighted graph
- The algorithm iteratively finds the length of a shortest path from *a* to a first vertex, the length of a shortest path from *a* to a second vertex, and so on, until the length of a shortest path from *a* to *z* is found.
- It can be easily extended to find the length of the shortest path from *a* to all other vertices of the graph, and not just to *z*.

Shortest Path Problems (*Contd…*)

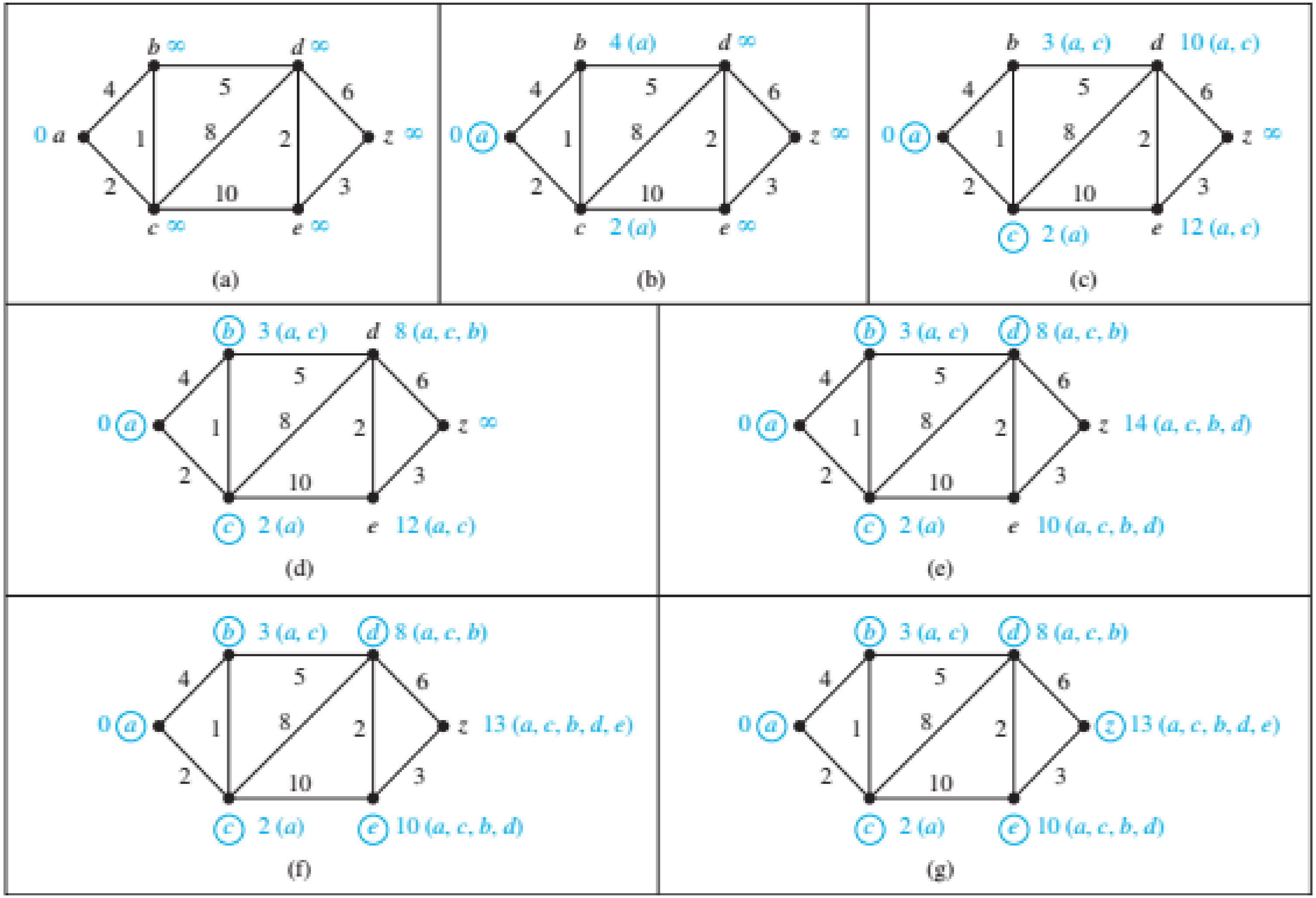
- A distinguished set of vertices is constructed by adding one vertex at each iteration.
- A labeling procedure is carried out at each iteration by which a vertex *w* is labeled with the length of a shortest path from *a* to *w*.
- This path contains only vertices in the distinguished set as the intermediate vertices.
- The vertex added to the distinguished set is one with a minimal label among those vertices not already in the set.

ALGORITHM 1 Dijkstra's Algorithm.

procedure *Dijkstra*(G: weighted connected simple graph, with
    all weights positive)
{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and lengths $w(v_i, v_j)$
    where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}
for $i := 1$ **to** n
    $L(v_i) := \infty$
$L(a) := 0$
$S := \emptyset$
{the labels are now initialized so that the label of a is 0 and all
    other labels are ∞, and S is the empty set}
while $z \notin S$
    $u :=$ a vertex not in S with $L(u)$ minimal
    $S := S \cup \{u\}$
    for all vertices v not in S
        if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$
        {this adds a vertex to S with minimal label and updates the
        labels of vertices not in S}
return $L(z)$ {$L(z) =$ length of a shortest path from a to z}

Shortest Path Problems (*Contd…*)

- Use Dijkstra's algorithm to find the length of a shortest path between the vertices *a* and *z* in the following weighted graph.



Shortest Path Problems (*Contd…*)

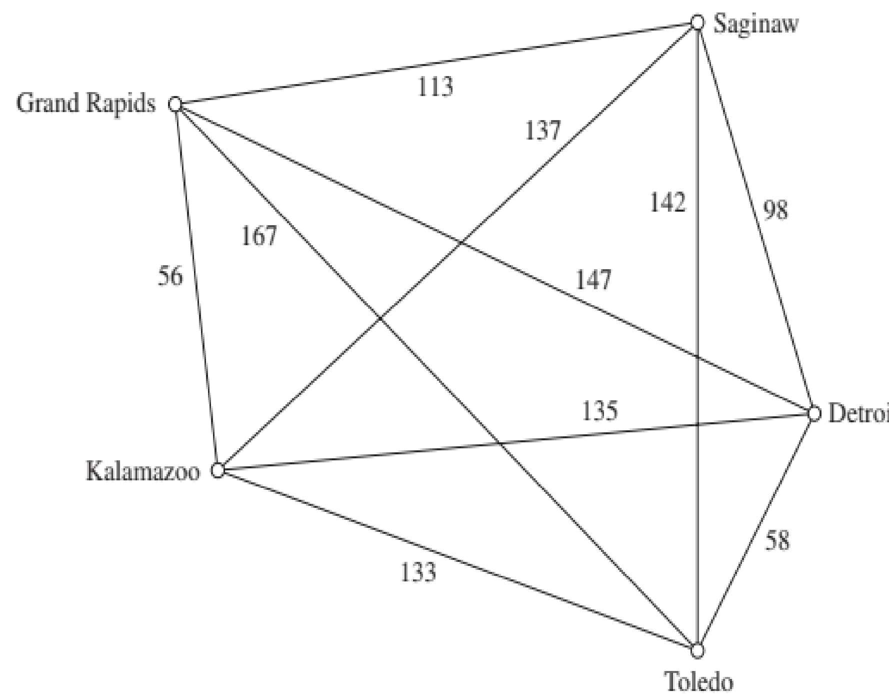
- A traveling salesperson wants to visit each of *n* cities exactly once and return to his starting point.

- In which order should he visit these cities to travel the minimum total distance?

- **<u>Traveling Salesman Problem</u>**: finding a circuit of minimum total weight in a weighted, complete, undirected graph that visits each vertex exactly once and returns to its starting point.

- Similar to *Hamilton circuit* with minimum total weight in the complete graph.

- Simplest strategy: examine all possible Hamilton circuits and select one of minimum total length.



Planar Graphs

- Can a graph can be drawn in the plane without edges crossing?
- Planar graph:

A graph is called *planar* if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint). Such a drawing is called a *planar representation* of the graph.

- A graph may be planar even if it is usually drawn with crossings, because it may be possible to draw it in a different way without crossings.

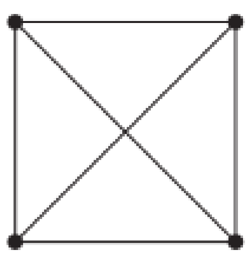


FIGURE 2 The Graph K_4.

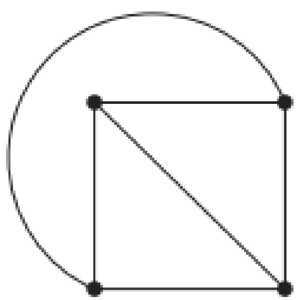


FIGURE 3 K_4 Drawn with No Crossings.

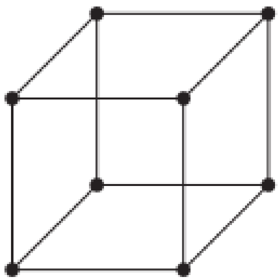


FIGURE 4 The Graph Q_3.

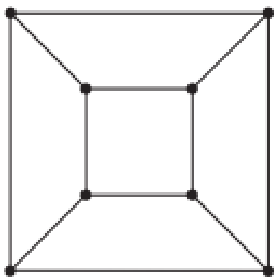
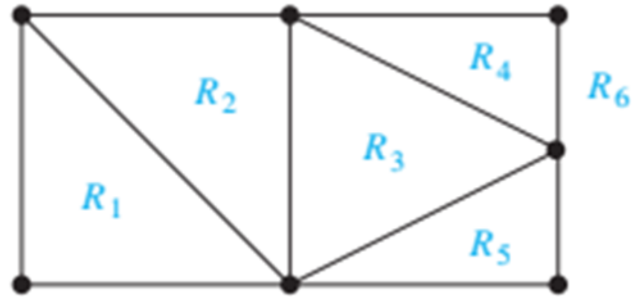


FIGURE 5 A Planar Representation of Q_3.

Planar Graphs (*Contd…*)

- We can show that a graph is planar by displaying a planar representation.
- A planar representation of a graph splits the plane into **regions**, including an unbounded region.
- Is $K_{3,3}$ planar?
- Applications of Planar graphs:
  - Design of electronic circuits: We can print a circuit on a single board with no connections crossing if the graph representing the circuit is planar.
  - Design of road network: Suppose we want to connect a group of cities by roads. We can built this road network without using underpasses or overpasses if the resulting graph is planar.



The regions of Planar Representation of a Graph

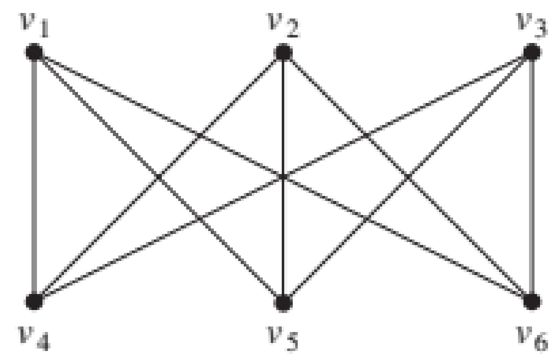
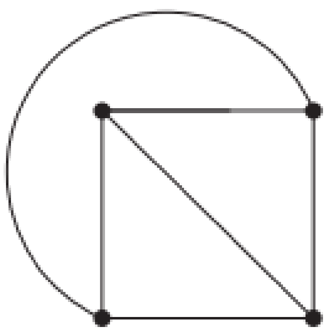
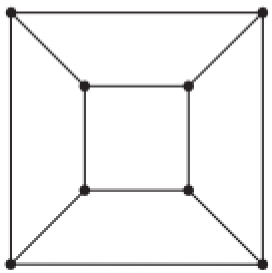


FIGURE 6 **The Graph $K_{3,3}$.**

Planar Graphs (*Contd...*)

- Degree of a region: Number of edges bordering a particular region.
- When an edge occurs twice on the boundary (so that it is traced out twice when the boundary is traced out), it contributes two to the degree.
- We denote the degree of a region *R* by *deg(R)*.
- Find the regions and degree of each region for planar graphs given below.





- Since the graphs discussed are simple graphs, no multiple edges that could produce regions of degree two, or loops that could produce regions of degree one, are permitted.

Planar Graphs (*Contd…*)

EULER'S FORMULA Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

- Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?
- Corollary-1:

If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

- Corollary-2:

If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

- Corollary-3:

If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.