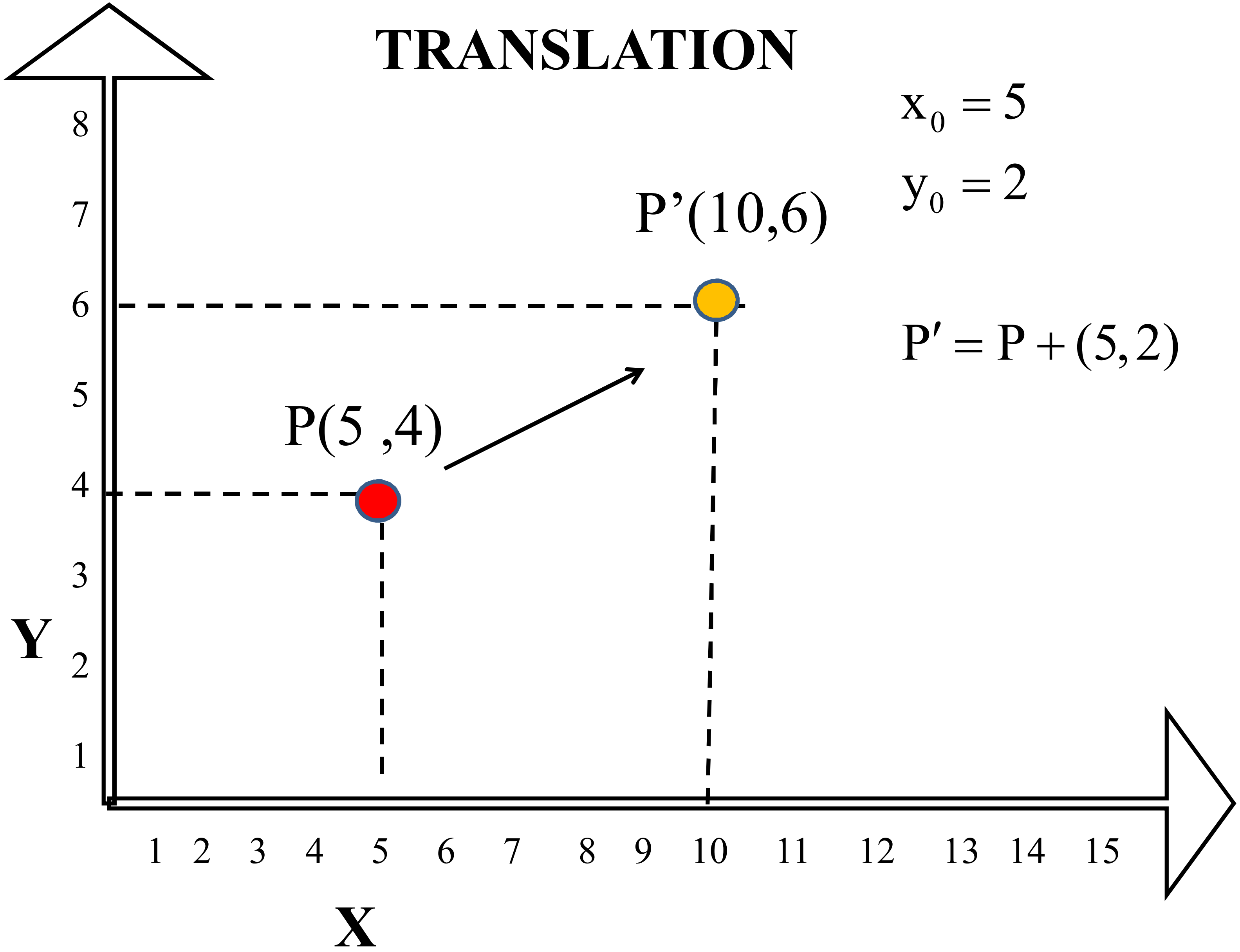
SPATIAL TRANSFORMATION

TRANSLATION

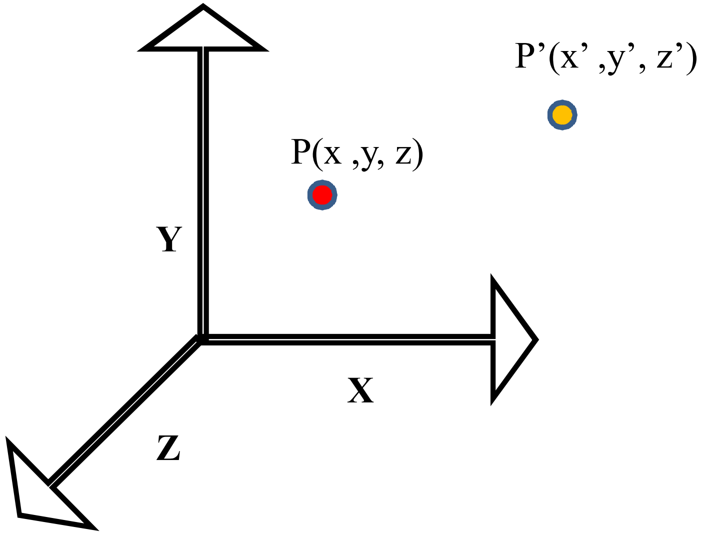
$x_0 = 5$

$y_0 = 2$

$P' = P + (5, 2)$



TRANSLATION



Operation

$$x' = x + x_0$$

$$y' = y + y_0$$

$$z' = z + z_0$$

Matrix representation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Symmetric matrix representation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

SCALING

Matrix representation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

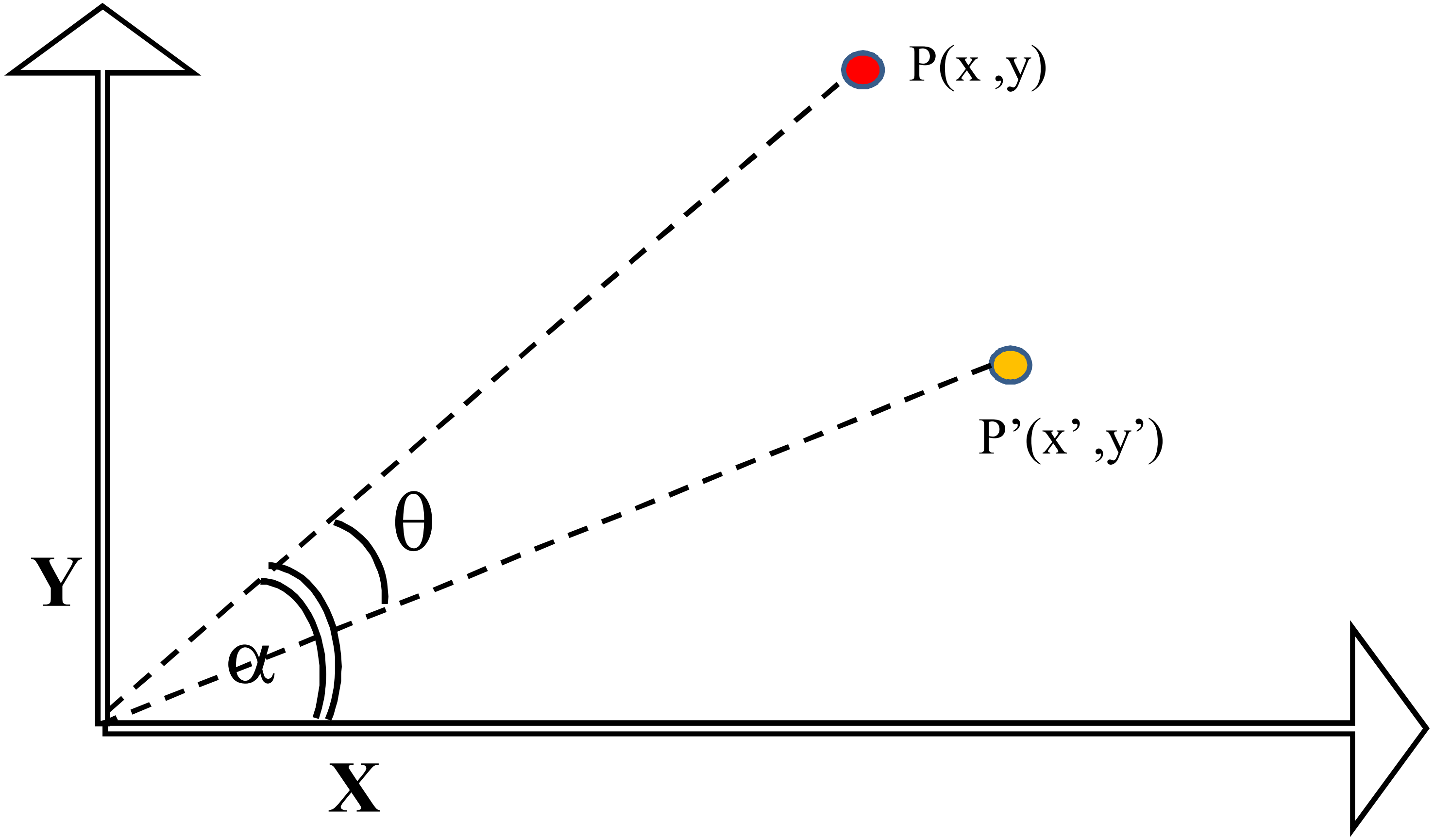
Operation

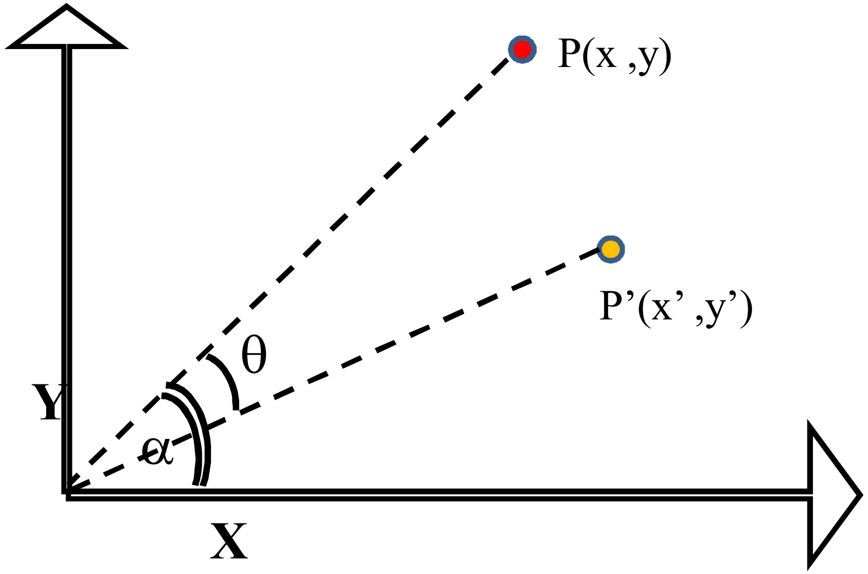
$$x' = x . s_x$$
$$y' = y . s_y$$
$$z' = z . s_z$$

Symmetric matrix representation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 1 \\ 0 & s_y & 0 & 1 \\ 0 & 0 & s_z & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

ROTATION about origin





$$x = r\cos\alpha$$

$$y = r\sin\alpha$$

$$x' = r\cos(\alpha - \theta)$$

$$= r\cos\alpha\cos\theta + r\sin\alpha\sin\theta$$

$$\boxed{x' = x\cos\theta + y\sin\theta}$$

$$y' = r\sin(\alpha - \theta)$$

$$= r\sin\alpha\cos\theta - r\cos\alpha\sin\theta$$

$$\boxed{y' = y\cos\theta - x\sin\theta}$$

Operation

$$x' = x\cos\theta + y\sin\theta$$
$$y' = y\cos\theta - x\sin\theta$$

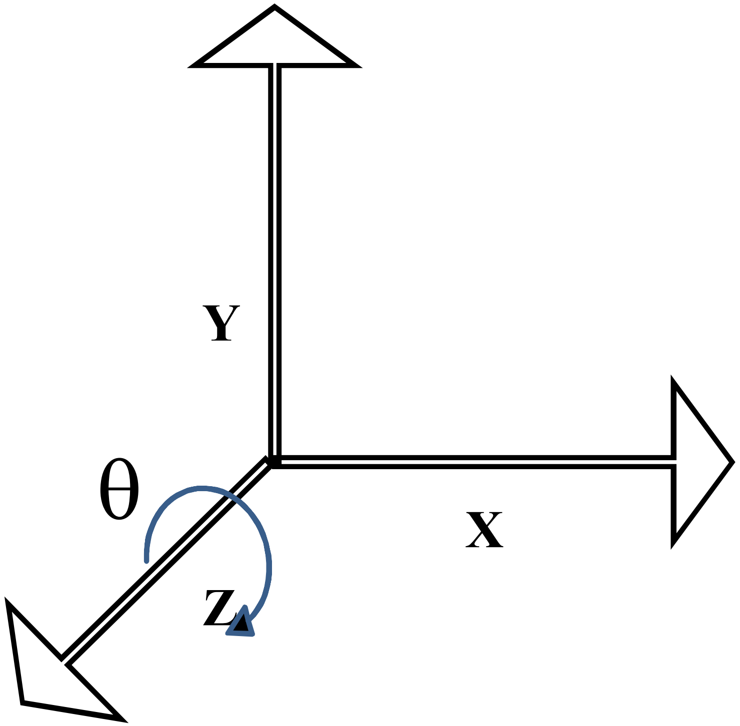
Matrix representation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Symmetric matrix representation

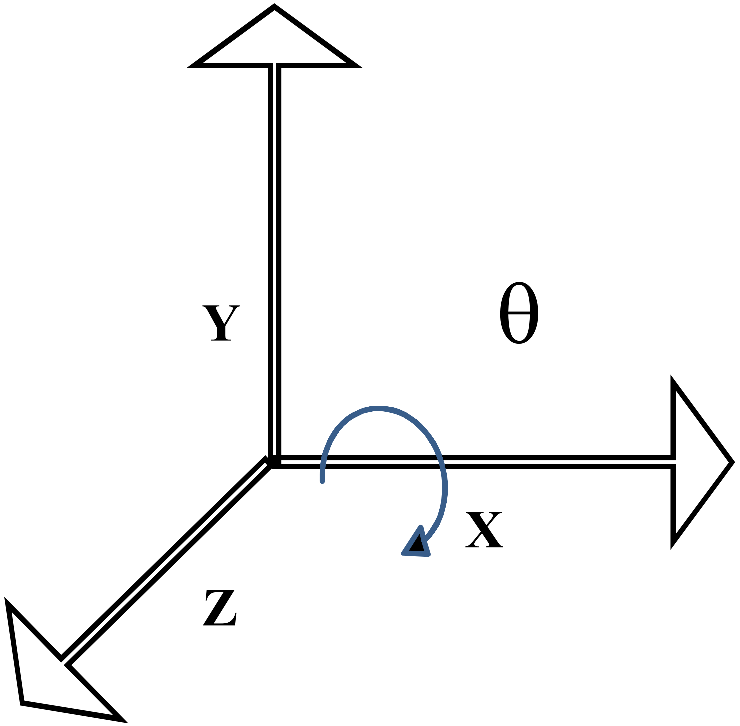
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

ROTATION about Z-axis



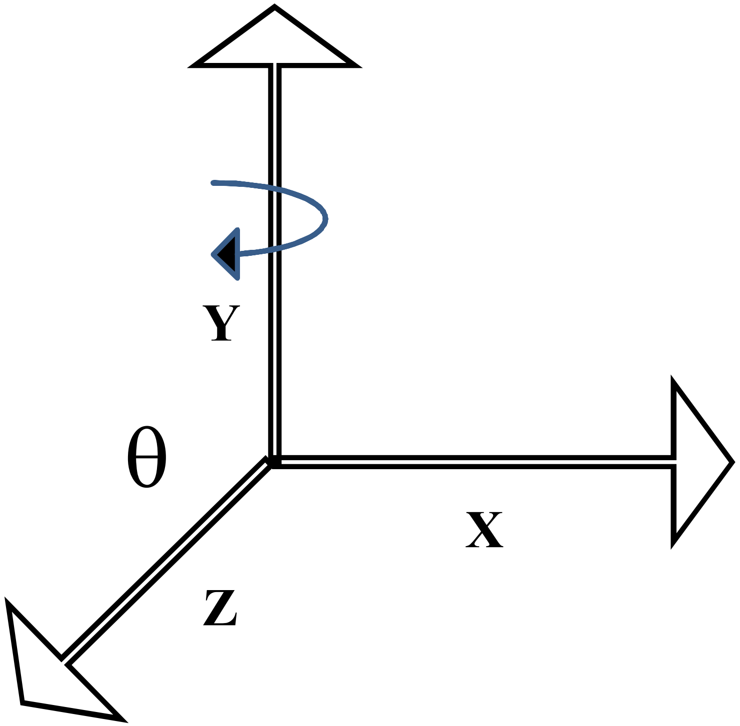
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

ROTATION about X-axis



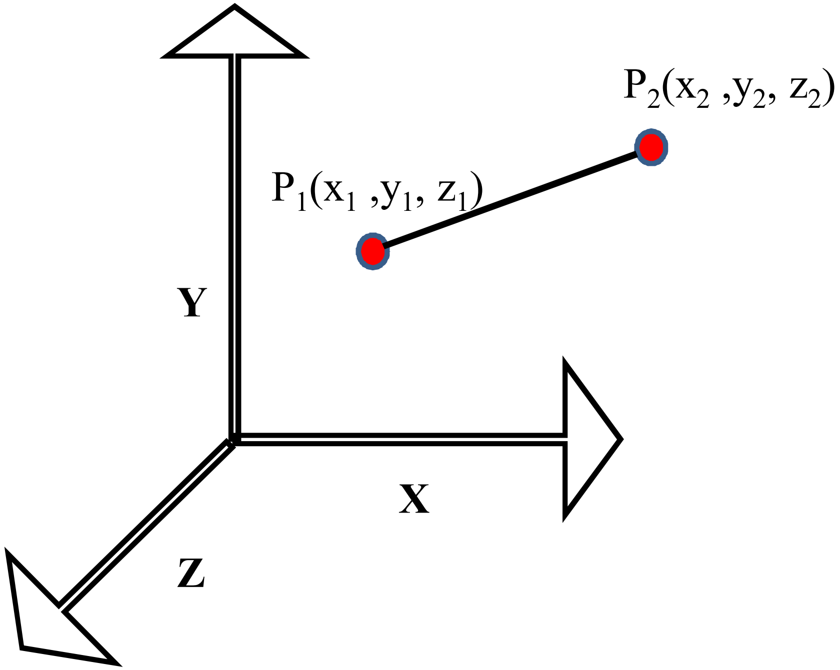
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

ROTATION about Y-axis



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

ROTATION about an arbitrary axis



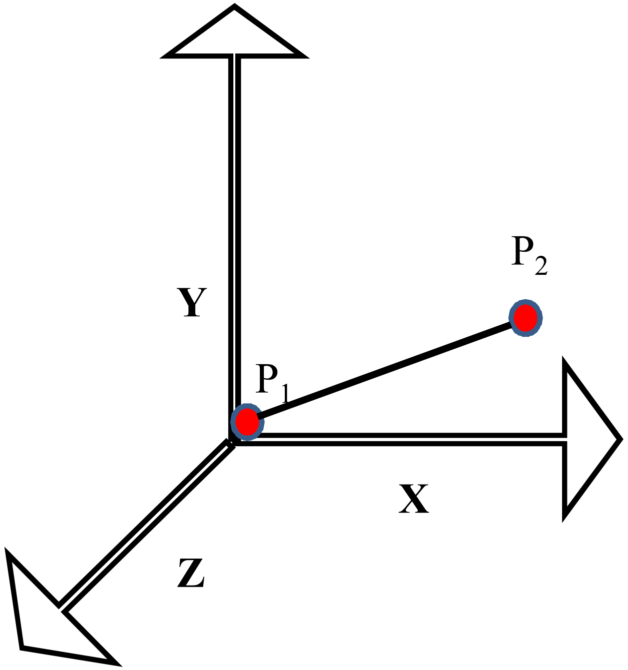
$$u = (a, b, c)$$

$$v = P_2 - P_1 = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

$$u = \frac{v}{|v|}$$

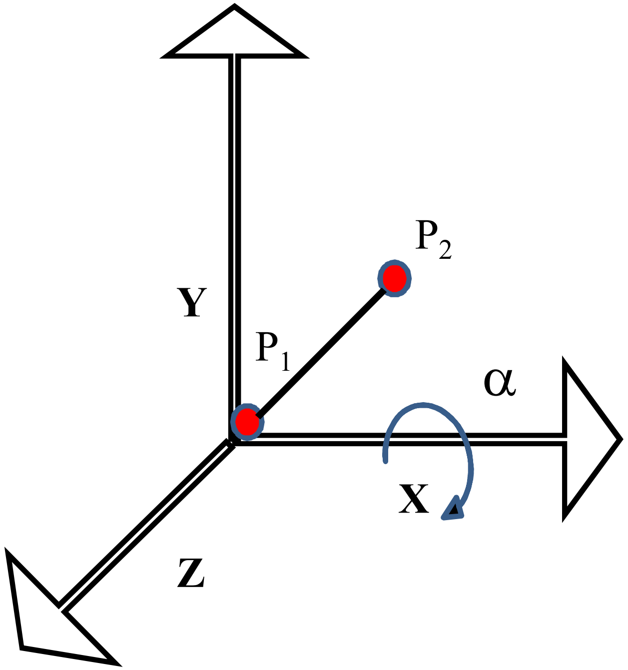
$$a = \frac{x_2 - x_1}{|v|} \qquad b = \frac{y_2 - y_1}{|v|} \qquad c = \frac{z_2 - z_1}{|v|}$$

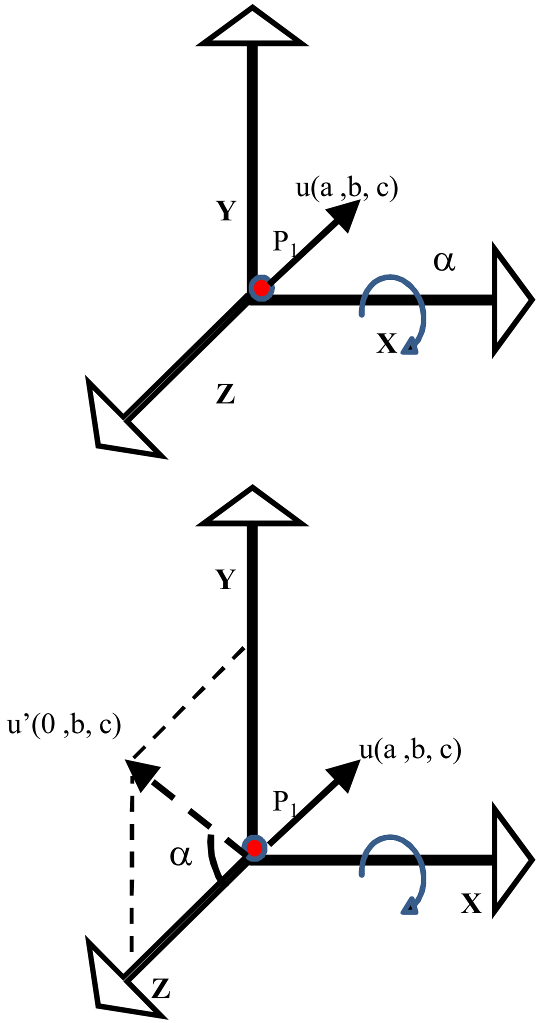
Step1: Translate $P_1(x_1, y_1, z_1)$ to origin

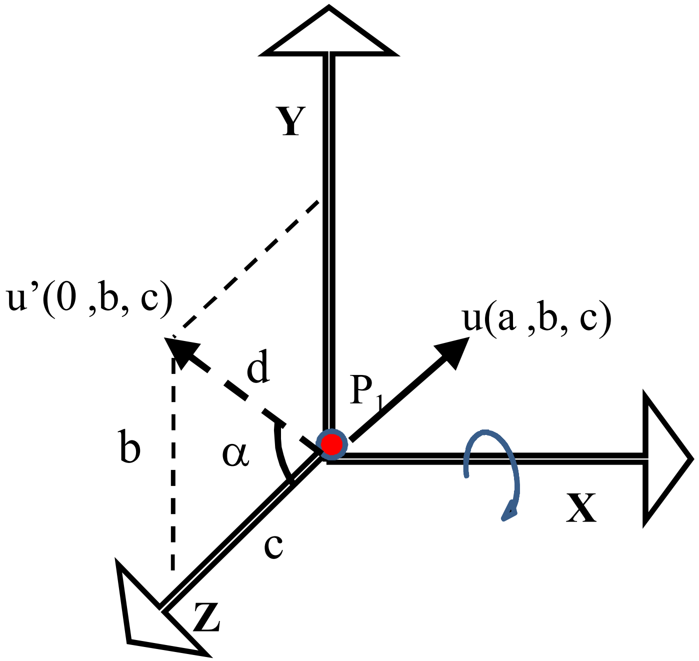


$$T = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step2: Rotate X axis at an angle α to origin





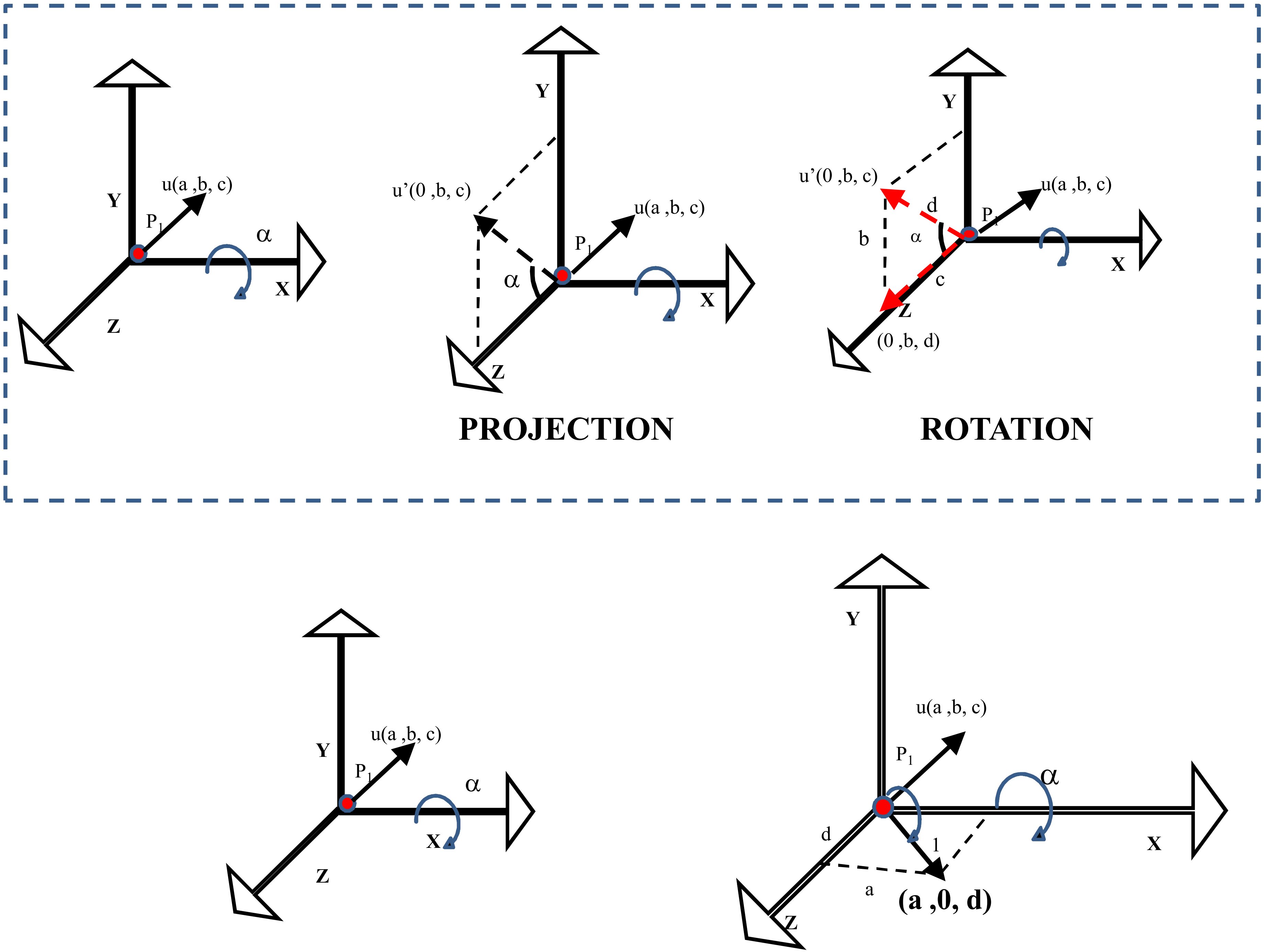


$$\sin\alpha = \frac{b}{d} \qquad \cos\alpha = \frac{c}{d} \qquad d = \sqrt{b^2 + c^2}$$

$$R_X(-\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\alpha) & \sin(-\alpha) & 0 \\ 0 & -\sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

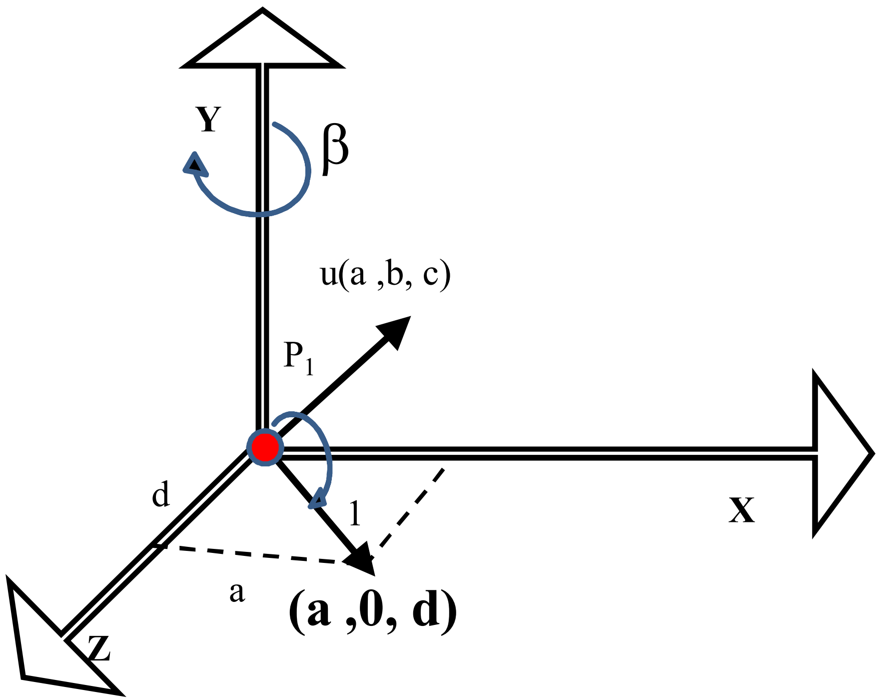
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{d} & \frac{-b}{d} & 0 \\ 0 & \frac{b}{d} & \frac{c}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Y
u(a ,b, c)
P1
α
X
Z
Y
u'(0 ,b, c)
u(a ,b, c)
P1
α
X
Z
PROJECTION
Y
u'(0 ,b, c)
u(a ,b, c)
d
P1
b
α
c
X
Z
(0 ,b, d)
ROTATION
Y
u(a ,b, c)
P1
α
X
Z
Y
u(a ,b, c)
P1
α
d
1
X
a
(a ,0, d)
Z

Step3: Rotate Y axis at an angle β to origin

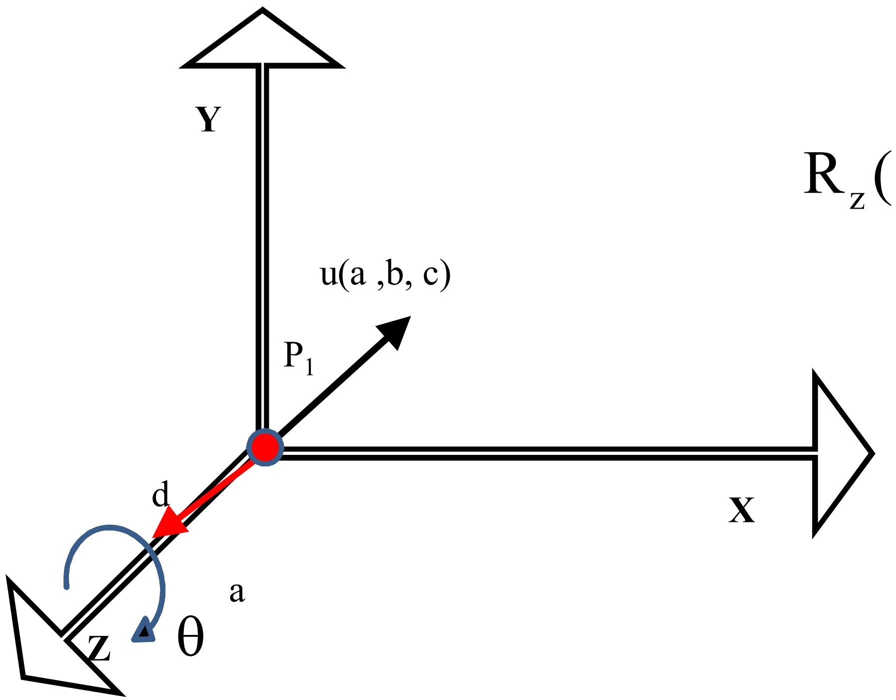


$\sin\beta = a \qquad \cos\beta = d$

$$R_Y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} d & 0 & a & 0 \\ 0 & 1 & 0 & 0 \\ -a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step4: Rotate Z axis at an angle θ to origin



$$R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step1: Translate $P_1(x_1, y_1, z_1)$ to origin

Step2: Rotate X axis at an angle α to origin

Step3: Rotate Y axis at an angle β to origin

Step4: Rotate Z axis at an angle θ to origin

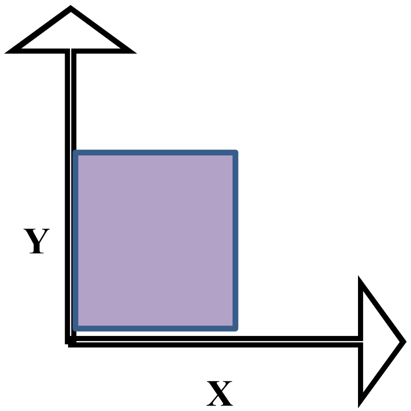
Step5:Reverse of ‘3’

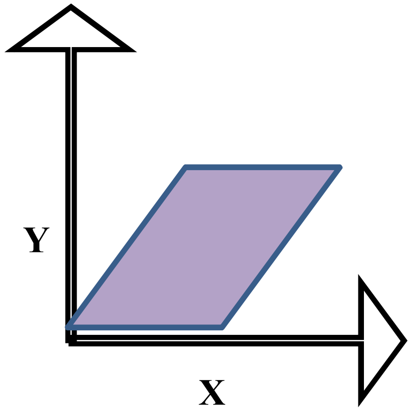
Step6: Reverse of ‘2’

Step7:Reverse of ‘1’

2D SHEER

Sheer relative to X axis





Operation

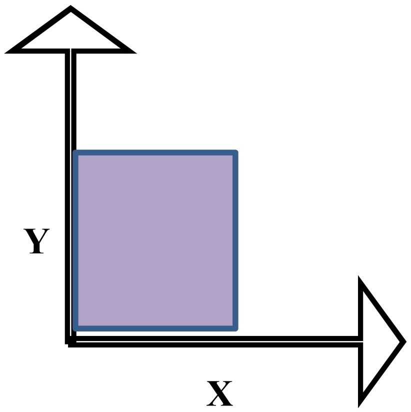
$$x' = x + sh_x y$$
$$y' = y$$

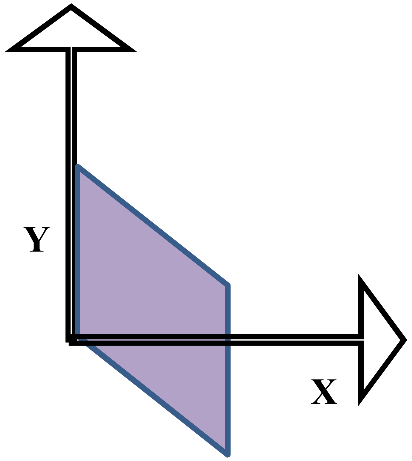
Symmetric matrix representation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D SHEER

Sheer relative to Y axis





Operation

$$x' = x$$
$$y' = y + sh_y x$$

Symmetric matrix representation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3D SHEER

Sheer relative to Z axis

Operation

$$x' = x + az$$

$$y' = y + bz$$

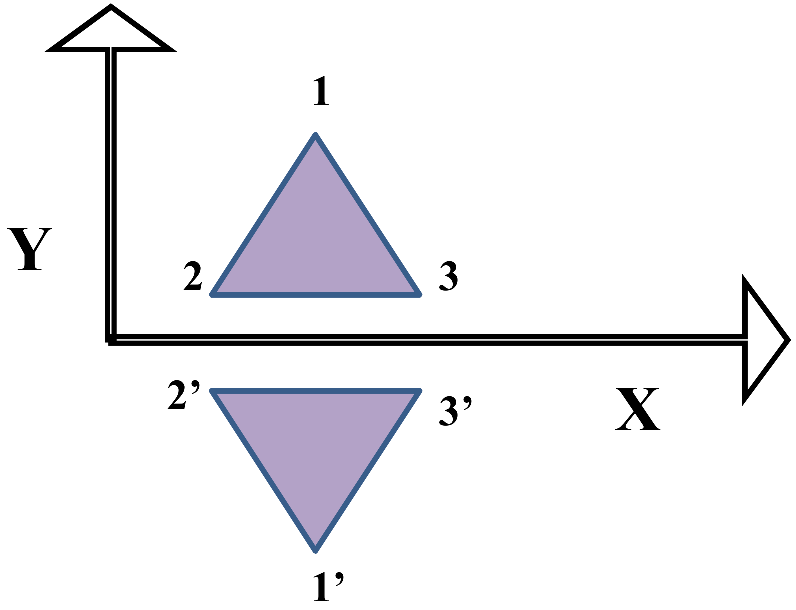
$$z' = z$$

Symmetric matrix representation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

2D REFLECTION

Refection about to X axis



Operation

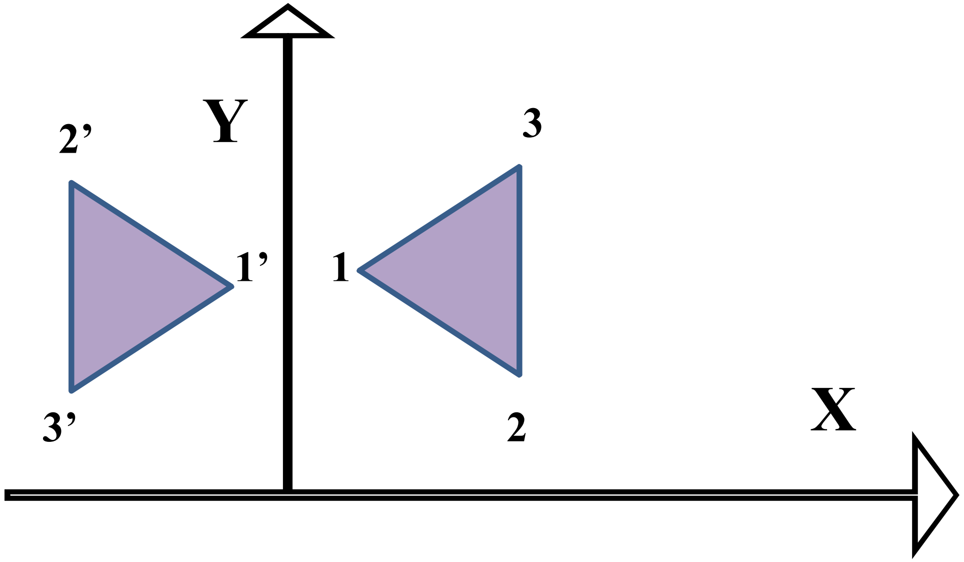
$$x' = x$$
$$y' = -y$$

Symmetric matrix representation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D REFLECTION

Refection about to Y axis



Operation

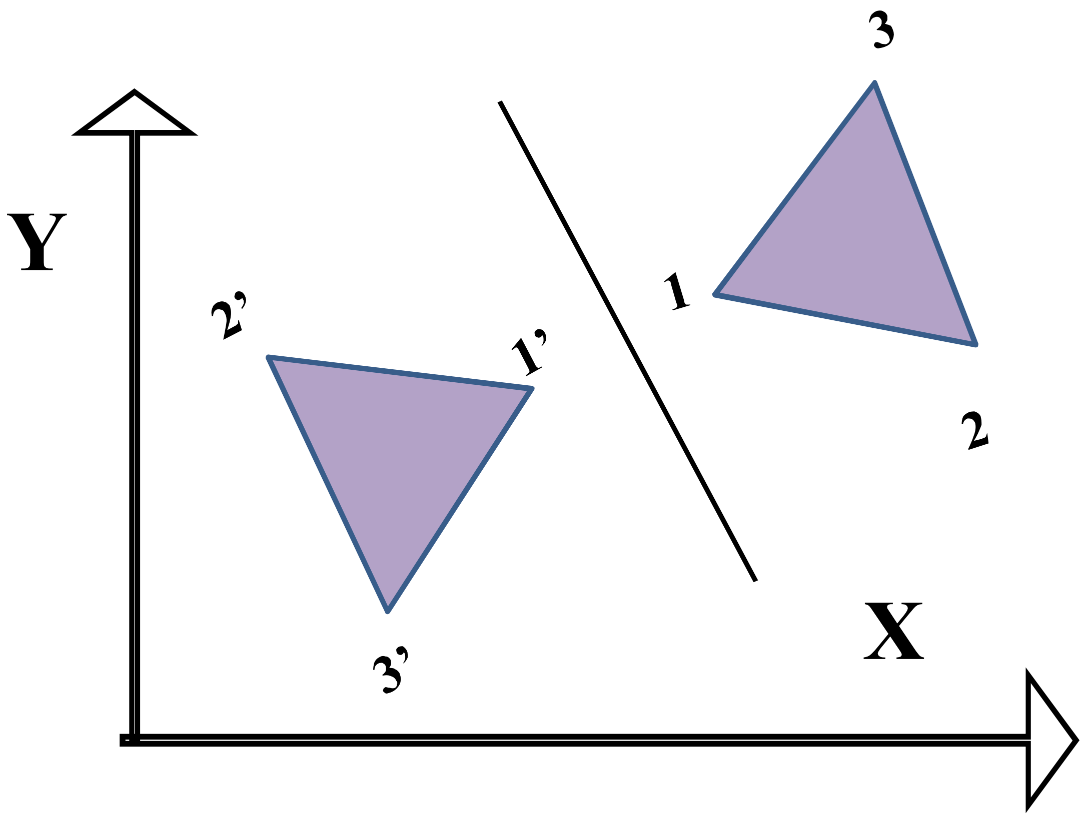
$$x' = -x$$
$$y' = y$$

Symmetric matrix representation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D REFLECTION

Refection about to any axis



3D REFLECTION

Operation

$x' = x$

$y' = y$

$z' = z$

Symmetric matrix representation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

COMPOSITE TRANSFORMATION

TRANSLATION

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 0 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \hat{V} = T(V)$$

SCALING

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 1 \\ 0 & s_y & 0 & 1 \\ 0 & 0 & s_z & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \hat{V} = S(V)$$

ROTATION about Z-axis at angle θ

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \hat{V} = R_Z(\theta)(V)$$

COMPOSITE TRANSFORMATION

$$\begin{aligned} \hat{V} &= (R_Z(\theta)(S(T(V)))) \\ &= (R_Z(\theta) S.T)(V) \\ &= A(V) \end{aligned}$$

END OF CHAPTER