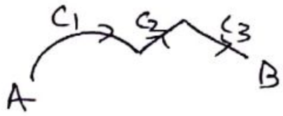
(22) **Note**: The following functions are analytic everywhere: e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, polynomial functions (like $a_0+a_1z+a_2z^2+a_3z^3+\cdots+a_nz^n$)

The function $\frac{1}{(z-z_1)^n}$ is analytic everywhere except at $z=z_1$.

In the figure $\int_{C_1} f(z)\,dz = -\int_{C_2} f(z)\,dz$,

where C_1 & C_2 are the curves joining A & B but orientation of C_1 is negative to that of C_2.



If $c = c_1+c_2+c_3$, Then

$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \int_{c_3} f(z)\,dz$$

and so on.

Singular point: If a function $f(z)$ fails to be analytic at a point z_0 but in every neighbourhood of z_0 there exist at least one point where the function is analytic, then z_0 is said to be a singular point or singularity of $f(z)$.

Analytic function sometime called regular function or holomorphic function.

Note: During problem solving, you must sketch the diagram properly to avoid any complication.

$\oint_c f(z)\,dz$ means the integration along the positively oriented closed contour. Sometimes it is written as $\int_c f(z)\,dz$ sometimes $\oint_c f(z)\,dz$.

but when it is written as $\int_c f(z)\,dz$, orientation of c must be given in the problem.

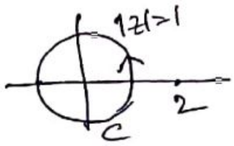
Ex 1 compute the following

(i) $\displaystyle\oint_{c:|z|=1} \frac{z\,dz}{z-2}$ (ii) $\displaystyle\oint_{c:|z|=1} \frac{z^2 dz}{z-1-i}$ (iii) $\displaystyle\int_{c:z-2=e^{i\theta}} \frac{(2z^2+1)\,dz}{z}$, where

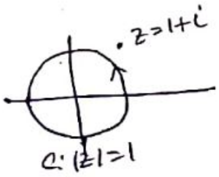
c is described in the positive sense,

(iv) $\displaystyle\int_{c:z-1=2e^{i\theta}} (e^z+z^2)\,dz$, c is described in the positive sense.

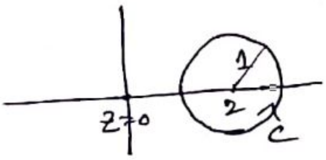
Soln: (i) As $z=2$ lies outside c, $f(z)=\dfrac{z}{z-2}$ is analytic within and on c. Hence $\displaystyle\oint_{c:|z|=1} \frac{z}{z-2}\,dz = 0$ (By Cauchy-Goursat Theorem)



(ii) $f(z)=\dfrac{z^2}{z-(1+i)}$ is analytic within and on c, as $z=1+i$ lies outside c. $\therefore \displaystyle\oint_{c:|z|=1} \frac{z^2}{z-1-i}\,dz = 0$.



(iii) $z-2=1e^{i\theta}$ is the circle with centre at $z=2$ and radius 1. and $z=0$ lies outside c.

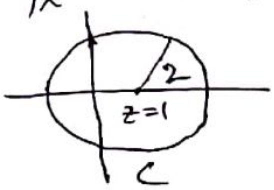


$\therefore \displaystyle\int_{c:|z-2|=1} \frac{(2z^2+1)}{z}\,dz = 0$, $[\,|z-2|=1$ or $z-2=e^{i\theta}$ are same.$]$

as $\dfrac{2z^2+1}{z}$ is analytic within and on c.

(iv) $f(z)=e^z+z^2$ is analytic everywhere,

Hence $\displaystyle\int_{c:|z-1|=2e^{i\theta}} (e^z+z^2)\,dz = 0$ (for any closed contour)

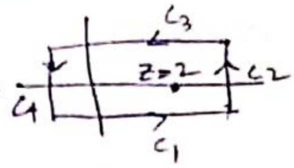


(21)

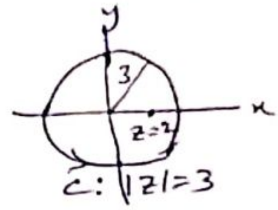
Ex2 Evaluate the following

(i) $\oint\limits_{C:|z|=3} \frac{z\,dz}{z-2}$ (ii) $\oint\limits_{C:|z|=2} \frac{z^2\,dz}{z-1-i}$ (iii) $\oint\limits_{C:|z|=1} \left(\frac{2z^2+1}{z}\right)dz$

(iv) $\oint\limits_{C} \frac{z}{(z-2)^2}\,dz$ where $C = C_1+C_2+C_3+C_4$ as shown in the figure.

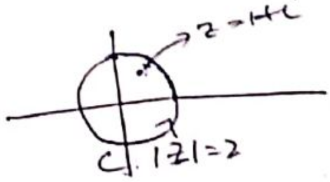


Soln (i) $f(z)=z$ is analytic within and on C and $z=2$ lies within C. Hence by Cauchy's integral formula $\left[f(z_0)=\frac{1}{2\pi i}\oint\limits_{C}\frac{f(z)}{z-z_0}\,dz\right]$



$$\oint\limits_{C:|z|=3} \frac{z}{z-2}\,dz = 2\pi i\, f(2) = 2\pi i \times 2 = 4\pi i$$

(ii) Here $f(z)=z^2$ is analytic within and on C and $z=1+i$ lies within C. Hence by Cauchy's integral formula

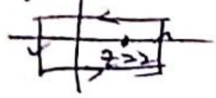


$$\int\limits_{C:|z|=2} \frac{z^2}{z-1-i}\,dz = 2\pi i\, f(1+i) = 2\pi i\,(1+i)^2 = 2\pi i(1-1+2i)$$
$$= -4\pi.$$

(iii) $f(z)=2z^2+1$ is analytic within and on C and $z=0$ lies within C,

$$\therefore \int\limits_{C}\left(\frac{2z^2+1}{z}\right)dz = 2\pi i f(0) = 2\pi i\,(0+1) = 2\pi i$$

(iv) $f(z)=z$ is analytic within and on C and $z=2$ lies within C. Hence by derivative formula $\left[f'(z_0)=\frac{1}{2\pi i}\int\limits_{C}\frac{f(z)}{(z-z_0)^2}\,dz\right]$



$$\int\limits_{C}\frac{z}{(z-2)^2}\,dz = 2\pi i \left.\frac{d}{dz}(z)\right|_{z=2} = 2\pi i$$

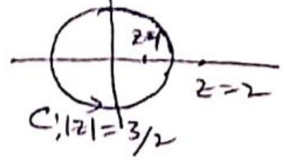
Scanned with CamScanner

EX3 Evaluate the following:

(i) $\displaystyle\oint_{C:|z|=\frac{3}{2}} \frac{z^3+3}{(z-1)(z-2)}\,dz$ (ii) $\displaystyle\oint_{|z|=4} \frac{z^3+3}{(z-1)(z-2)}\,dz$ (iii) $\displaystyle\oint_{C:|z|=2} \frac{z^3\,dz}{(z-1)^3}\,dz$

(iv) $\displaystyle\oint_{|z|=2} \frac{dz}{(z-1)^3}\,dz$

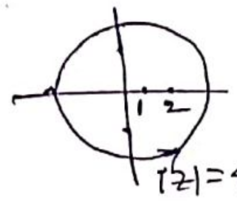
Soln (i) $f(z) = \dfrac{z^3+1}{z-2}$ is analytic within and on c and $z=1$ is within it



$$\therefore \oint_{C:|z|=\frac{3}{2}} \frac{z^3+3}{(z-2)}\cdot\frac{1}{(z-1)}\,dz$$

$$= 2\pi i\, f(1) = 2\pi i\left[\frac{1^3+3}{1-2}\right] = -8\pi i$$

(ii) Here both $z=2$ and $z=1$ are within C. Let us consider partial fraction.



$$\frac{z^3}{(z-1)(z-2)} = z+a+\frac{b}{z-1}+\frac{d}{z-2} \text{ say}$$

$$= \frac{(z+a)(z-1)(z-2)+b(z-2)+d(z-1)}{(z-1)(z-2)}$$

~~$a(z-1)(z-2)+b(z-2)+c(z-1)$ is an identity~~

$\therefore z^3 = (z+a)(z-1)(z-2)+b(z-2)+d(z-1)$ is an identity.

For $z=1$, $1=-b$ $\quad\therefore b=-1$

For $z=2$, $2^3=d$ $\quad\therefore d=8$

Hence $z^3 = (z+a)(z-1)(z-2)-(z-2)+8(z-1)$

(Equating constant terms)

$\Rightarrow 2a+2-8=0$

or $a=3$

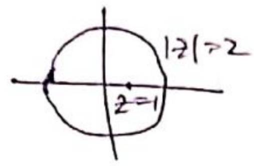
$$\therefore \oint_C \frac{z^3}{(z-1)(z-2)}\,dz = \oint_C\left[z+3+\frac{8}{z-2}-\frac{1}{z-1}\right]dz$$

$$= \oint_C (z+3)\,dz + \oint_C \frac{8}{z-2}\,dz - \oint_C \frac{dz}{z-1} = 0+2\pi i\times 8-2\pi i\times 1$$

$$= 14\pi i$$

(26)

(iii) $f(z)=z^3$ is analytic within and on the closed contour C and $z=1$ lies within it



$$\therefore \oint_C \frac{z^3}{(z-1)^3}\,dz = \frac{2\pi i}{2!}\frac{d^2}{dz^2}\left[z^3\right]_{z=1}$$

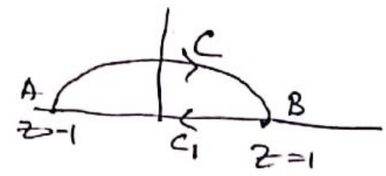
$$= \pi i\left[6z\right]_{z=1} = 6\pi i$$

$$\left[\text{Using } f^{n}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right]$$

(iv) $\displaystyle\oint_{|z|=2} \frac{dz}{(z-1)^3} = \frac{2\pi i}{2!}\frac{d^2}{dz^2}(1)\Big|_{z=1} = 0$, like before.

<u>EX4</u> Evaluate $\int_C (z^2+2z+1)\,dz$, where C is the upper half of the ellipse passing through $z=-1$ and $z=1$.

<u>Soln</u> Consider the closed contour $C+C_1$ (negatively oriented) where C_1 is the contour joining $z=1$ & -1 along the real line.



As $f(z)=z^2+2z+1$ is analytic within and on the closed contour $C+C_1$, $\displaystyle\int_{C+C_1}(z^2+2z+1)\,dz = 0$.

$$\therefore \int_C (z^2+2z+1)\,dz + \int_{C_1}(z^2+2z+1)\,dz = 0$$

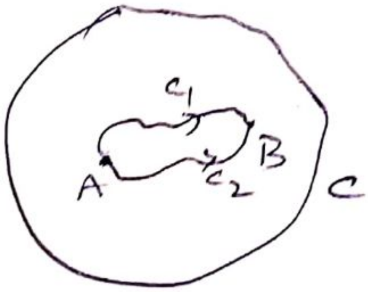
$$\text{or } \int_C (z^2+2z+1)\,dz = -\int_{C_1}(z^2+2z+1)\,dz$$

Now on C_1, $z=x$, x varies from 1 to -1, $\frac{dz}{dx}=1$

$$\therefore -\int_{C_1}(z^2+2z+1)\,dz = -\int_1^{-1}(x^2+2x+1)\,1\,dx = \int_{-1}^{1}(x^2+2x+1)\,dx = \frac{2}{3}+0+2$$

Hence $\displaystyle\int_C (z^2+2z+1)\,dz = \frac{8}{3}$

(27) **Ex 5** If $f(z)$ is analytic within and on the closed contour C and if A and B are any two points in C, then prove that $\int f(z)\,dz$ possesses the same value along any path joining A and B, that lies within C (ie $\int_{C_1} f(z)\,dz$ is independent of path)



Soln ~~As~~. Consider two paths C_1 and C_2 joining A and B that lies completely within C.

Then $C_1+(-C_2)$ forms a closed contour and as $f(z)$ is analytic within and on C, it is analytic within and on $C_1+(-C_2)$.

Hence by Cauchy-Goursat Theorem

$$\int_{C_1+(-C_2)} f(z)\,dz = 0 \quad \text{or} \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

$$\text{or,} \int_{C_1} f(z)\,dz = -\int_{-C_2} f(z)\,dz = \int_{C_2} f(z)\,dz$$

ie $\int f(z)\,dz$ is independent of path.