[It is assumed that the events E_1, E_2 and E_3 are mutually exclusive and exhaustive].

Also, by question: $P(A/E_1) = 1$, $P(A/E_2) = \frac{1}{4}$ and $P(A/E_3) = \frac{1}{8}$.

Required probability $= P(E_1/A)$

= P (an examinee knows the answer to a question given that he has correctly answered it)

$$= \frac{P(E_1)\, P(A/E_1)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + P(E_3)\, P(A/E_3)}$$

(By Baye's theorem)

$$= \frac{\frac{1}{2}\cdot 1}{\frac{1}{2}\cdot 1 + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{6}\cdot\frac{1}{8}} = \frac{1}{2}\cdot\frac{48}{24+4+1} = \frac{24}{29}.$$

1.7 INDEPENDENT EVENTS

Let A and B are two events connected to a given random experiment E. If $P(A/B) = P(A)$, where $P(B) \neq 0$ then we can say that the probability of A does not depend on the happening of B, *i.e.*, the events A and B are independent. Similarly, if $P(B/A) = P(B)$ where $P(A) \neq 0$ then we can say that the probability of B does not depend on the occurrence of A.

Therefore, we observe that if A, B are two independent events, then

$$\left.\begin{aligned} P(A/B) &= \frac{P(AB)}{P(B)} \\ P(B/A) &= \frac{P(AB)}{P(A)} \end{aligned}\right\} \Leftrightarrow P(AB) = P(A)\, P(B)$$

Hence, we come to the following theorem:

Theorem (Multiplication Theorem)

Two events A and B are independent if and only if $P(AB) = P(A)\, P(B)$.

So formally we can define independence of two events as follows:

Definition 1: Two events A and B are said to be **statistically independent** or simply **independent** if and only if $P(AB) = P(A)\, P(B)$.

Note: If $P(AB) \neq P(A)\, P(B)$ then A and B are said to be dependent.

Definition 2: Three events A, B, C are said to be **pairwise independent** if

$$P(AB) = P(A)\, P(B)$$
$$P(BC) = P(B)\, P(C)$$
$$P(CA) = P(C)\, P(A).$$

Definition 3: Three events A, B, C are said to be **mutually independent** if

$$P(AB) = P(A)\, P(B)$$
$$P(BC) = P(B)\, P(C)$$
$$P(CA) = P(C)\, P(A)$$
$$P(ABC) = P(A)\, P(B)\, P(C).$$

Example 1: *If A and B are two events such that* $P(A^c \cup B^c) = \frac{5}{6}$, $P(A) = \frac{1}{2}$ *and* $P(B^c) = \frac{2}{3}$. *Show that A and B are independent.* (*W.B.U.T. 2004*)

Solution: Given:

$$P(A^c \cup B^c) = \frac{5}{6},\ P(A) = \frac{1}{2},\ P(B^c) = \frac{2}{3} \qquad ...(1)$$

Now, $$P\{(A^c \cup B^c)^c\} = 1 - P(A^c \cup B^c) = 1 - \frac{5}{6} = \frac{1}{6} \qquad \text{[By (1)]}$$

$$\Rightarrow \quad P(A \cap B) = \frac{1}{6} \qquad \text{[Using De Morgan's law]} \quad ...(2)$$

Also, $$P(B) = 1 - P(B^c) = 1 - \frac{2}{3} = \frac{1}{3} \qquad \text{[By (1)]} \quad ...(3)$$

From (1), (2) and (3), we get

$$P(A \cap B) = P(A)\, P(B) = \frac{1}{6}.$$

Hence A and B are independent.

Example 2: *A man seeks advice regarding one of two possible courses of action from three advisers, who arrive at their recommendations independently. He follows the recommendations of the majority. The probabilities that the individual advisers are wrong are 0.1, 0.05 and 0.05 respectively. What is the probability that the man takes incorrect advice?*

Solution: *Let* A_i denotes the event 'i^{th} adviser gives wrong advice' ($i = 1, 2, 3$).

Given: $$P(A_1) = 0.1,\ P(A_2) = P(A_3) = 0.05.$$

Let A denotes the event 'the man takes incorrect advice'.

By question, the man follows the recommendations of the majority, so he will take the incorrect advice when any two or all of the three advisers give incorrect advice. Hence the event A can happen when any one of the following pairwise mutually exclusive events happens:

$$\bar{A}_1 A_2 A_3,\ A_1 \bar{A}_2 A_3,\ A_1 A_2 \bar{A}_3,\ A_1 A_2 A_3.$$

$\therefore$ Required probability

$$\begin{aligned}
&= P(A) = P(\bar{A}_1 A_2 A_3 + A_1 \bar{A}_2 A_3 + A_1 A_2 \bar{A}_3 + A_1 A_2 A_3) \\
&= P(\bar{A}_1 A_2 A_3) + P(A_1 \bar{A}_2 A_3) + P(A_1 A_2 \bar{A}_3) + P(A_1 A_2 A_3) \\
&= P(\bar{A}_1)\, P(A_2)\, P(A_3) + P(A_1)\, P(\bar{A}_2)\, P(A_3) + P(A_1) P(A_2) \\
&\qquad P(\bar{A}_3) + P(A_1)\, P(A_2)\, P(A_3)
\end{aligned}$$

(Since the events are all mutually independent)

$$\begin{aligned}
&= (0.9)\,(0.05)^2 + (0.1)\,(0.95)\,(0.05) + (0.1)\,(0.05)\,(0.95) \\
&\qquad + (0.1)\,(0.05)^2
\end{aligned}$$

$$[\because \ P(\bar{A}_i) = 1 - P(A_i);\ i = 1, 2, 3]$$

$$\begin{aligned}
&= (0.05)^2 + (0.2)\,(0.95)\,(0.05) \\
&= 0.012.
\end{aligned}$$

Note: P (the man takes correct advice) $= 1 - P(A) = 1 - 0.012 = 0.988$.

Example 17: *A problem in Mechanics is given to three students A, B and C whose chances of solving it are* $\frac{1}{2}$, $\frac{1}{3}$ *and* $\frac{1}{4}$ *respectively. What is the probability that the problem will be solved?*

Solution: Let A denotes the event 'student A will solve the problem'. Similarly for B and C.

By question, $$P(A) = \frac{1}{2},\ P(B) = \frac{1}{3},\ P(C) = \frac{1}{4}.$$

$\therefore$ Probability that A cannot solve the problem

$$= P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Similarly, $$P(\bar{B}) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$$

and $$P(\bar{C}) = 1 - P(C) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Now, $\bar{A}\bar{B}\bar{C}$ represents the event 'A, B, C cannot solve the problem'.

$$\therefore \quad P(\bar{A}\bar{B}\bar{C}) = P(\bar{A})\,P(\bar{B})\,P(\bar{C}) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4} = \frac{1}{4}$$

[Since A, B, C are mutually independent, so $\bar{A}, \bar{B}, \bar{C}$ are also mutually independent].

$\therefore$ P('the problem will be solved') $= 1 - \frac{1}{4} = \frac{3}{4}$.

Example 18: *There are three gentlemen aged 60, 65 and 70 years. The probability that they will live next 5 years are 0.8, 0.6 and 0.3 respectively. Find the probability that exactly two of them will remain alive 5 years hence.*

Solution: Let A, B and C denote the events 'gentleman aged 60 years will remain alive next 5 years', 'gentleman aged 65 years will remain alive next 5 years' and 'gentleman aged 70 years will remain alive next 5 years' respectively.

By question, $P(A) = 0.8$, $P(B) = 0.6$, $P(C) = 0.3$.

Let X denotes the event 'exactly two of them will remain alive next 5 years'.

$\therefore$ $X = \bar{A}BC + A\bar{B}C + AB\bar{C}$, where $\bar{A}$, $\bar{B}$, $\bar{C}$ are the complementary events of A, B, C respectively. Since $\bar{A}BC, A\bar{B}C, AB\bar{C}$, are pairwise m.e., we have by axiomatic definition of probability,

$$P(X) = P(\bar{A}BC + A\bar{B}C + AB\bar{C}) = P(\bar{A}BC) + P(A\bar{B}C) + P(AB\bar{C})$$
$$= P(\bar{A})\,P(B)\,P(C) + P(A)\,P(\bar{B})\,P(C) + P(A)\,P(B)\,P(\bar{C})$$

($\bar{A}$, B, C are mutually independent and so on)

$$= (1 - 0.8) \times 0.6 \times 0.3 + 0.8 \times (1 - 0.6) \times 0.3 + 0.8 \times 0.6 \times (1 - 0.3)$$
$$= 0.2 \times 0.6 \times 0.3 + 0.8 \times 0.4 \times 0.3 + 0.8 \times 0.6 \times 0.7 = 0.468,$$

this is the required probability.

Example 19: *An urn contains 3 white and 5 black balls. One ball is drawn and its colour is noted, kept aside and then another ball is drawn. What is the probability that it is (i) black (ii) white?*

(W.B.U.T. 2009).

Solution: Let W and B denote the events 'white ball' and 'black ball' respectively.

(*i*) Required probability

$$= P(WB + BB) = P(WB) + P(BB) \qquad (\because WB, BB \text{ are m.e.})$$

$$= P(W)\, P(B/W) + P(B)\, P(B/B)$$

$$= \frac{3}{8}\times\frac{5}{7}+\frac{5}{8}\times\frac{4}{7} = \frac{35}{56}=\frac{5}{8}.$$

(*ii*) Required probability

$$= P(BW + WW) = P(BW) + P(WW) \qquad (\because BW, WW \text{ are m.e.})$$

$$= P(B)\, P(W/B) + P(W)\, P(W/W)$$

$$= \frac{5}{8}\times\frac{3}{7}+\frac{3}{8}\times\frac{2}{7} = \frac{21}{56}=\frac{3}{8}.$$

Example 20: *A bag contains 3 black, 3 white and 2 red balls. One by one, three balls are drawn without replacement. Find the probability that the third ball is red.*

Solution: Let R and N denote the events 'Red ball' and 'Non-Red ball' respectively.

$\therefore$ Required probability

$$= P(RNR + NRR + NNR) \qquad (\because \text{ Number of red ball is 2})$$

$$= P(RNR) + P(NRR) + P(NNR) \qquad (\because RNR, NRR, NNR \text{ are m.e.})$$

$$= P(R)\, P(N/R)\, P(R/RN) + P(N)\, P(R/N)\, P(R/NR) + P(N)\, P(N/N)\, P(R/NN)$$

$$= \frac{2}{8}\times\frac{6}{7}\times\frac{1}{6}+\frac{6}{8}\times\frac{2}{7}\times\frac{1}{6}+\frac{6}{8}\times\frac{5}{7}\times\frac{2}{6}$$

$$= \frac{2}{56}+\frac{2}{56}+\frac{10}{56} = \frac{14}{56}=\frac{1}{4}.$$

Example 21: *Out of 15 persons 10 can speak Hindi and 8 can speak English. If two persons are chosen at random, find the probability that one person speaks Hindi only and the other speaks both Hindi and English.*

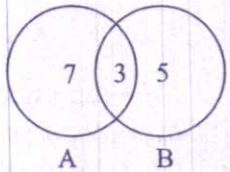
Solution: Let A and B represents the set of persons who can speak Hindi and English respectively.

$\therefore \qquad n(A) = 10,\ n(B) = 8 \quad \text{and} \quad n(A + B) = 15.$

$\therefore \qquad n(A) + n(B) - n(AB) = 15$

or $\qquad 10 + 8 - n(AB) = 15$

$\therefore \qquad n(AB) = 3.$



Now, two persons can be chosen out of 15 in

$${}^{15}C_2 = \frac{15!}{2!(15-2)!} = \frac{15\times14\times13!}{2\times1\times13!} = 105$$

Here 7 persons can speak Hindi only and 3 persons can speak Hindi and English.

$\therefore$ One person who can speak Hindi only can be chosen in ${}^7C_1$ ways and the other who can speak both Hindi and English can be chosen in ${}^3C_1$ ways.

If X denotes the event 'one person speaks Hindi only and the other speaks both Hindi and English', then total number of outcomes favourable to X is ${}^7C_1 \times {}^3C_1 = 7 \times 3 = 21$.

$\therefore \qquad P(X) = \frac{21}{105}=\frac{1}{5}$, this is the required probability.