B-Tech 3rd Semester Mid Term Examination

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1) Baye's Theorem

→ when Sample Space S is defined as union of n differect disjoint sets $A_1, A_2, A_3, \ldots A_n$, and B is a event set in S, the the Baye's Theorem States that,

$$P(A_i/B) = \frac{P(B/A_i)\,P(A_i)}{\sum_{j=1}^{n} P(B/A_j)\,P(A_j)}, \quad i = 1, 2, \ldots n$$

Proof

given $S = A_1 + A_2 + \ldots\ldots A_n$

$$= \sum_{j=1}^{n} A_j$$

$$B = BS = B(A_1 + A_2 + \ldots\ldots A_n)$$

$$= BA_1 + BA_2 + \ldots BA_n$$

$$P(B) = P(BA_1) + P(BA_2) + \ldots\ldots P(BA_n)$$

(as $A_1, A_2 \ldots A_n$ disjoint)

→ now $P(B/A_j) = \frac{P(BA_j)}{P(A_j)}$

or $P(BA_j) = P(B/A_j)\,P(A_j)$, $j = 1, 2, \ldots n$

$$\therefore P(B) = \sum_{j=1}^{n} P(B/A_j)\,P(A_j)$$

→ now $P(A_i/B) = \frac{P(A_i B)}{P(B)}$

now by following previos steps

$$P(A_i B) = P(B/A_i)\, P(A_i)$$

$$\therefore P(A_i/B) = \frac{P(B/A_i)\, P(A_i)}{\sum_{j=1}^{n} P(B/A_j)\, P(A_j)}$$

$\therefore$ Hence proved.

2) given $P(\text{correctly Diagnose}) = P(C) = 0.6$

$P(\text{die / correctly Diagnose}) = P(D/C) = 40\% = 0.4$

$P(\text{die / Incorrect Diagnose}) = P(D/I) = 0.7\%$.

given patient dies, to find $P(\text{diagnose correct / Die})$

$\therefore$ From the question, $P(C) = 0.6$, $P(D/C) = 0.4$

$P(I) = 0.4$ $\quad P(D/I) = 0.7$

$$\therefore P(C/D) = \frac{P(D/C)\, P(C)}{P(D/C).\, P(C) + P(D/I)\, P(I)}$$

(from Baye's Theorem)

$\therefore P(C/D) = \dfrac{0.4 \times 0.6}{0.4 \times 0.6 + 0.7 \times 0.4}$

$= \dfrac{0.6}{0.6 + 0.7} = \dfrac{6}{13}$

3) i) given $f(t) = \frac{1}{t}(1 - e^{t})$, to find $L\{f(t)\}$

→ first lets find $L\{1 - e^{t}\}$

→ we know that $L\{1\} = \frac{1}{s}$

$L\{e^{t}\} = \frac{1}{s-1}$

$\therefore L\{1 - e^{t}\} = L\{1\} - L\{e^{t}\}$

$= \frac{1}{s} - \frac{1}{s-1}$

now $L\{\frac{1}{t}(1 - e^{t})\} = \int_s^\infty L\{1 - e^{t}\}\, ds$

$= \int_s^\infty \frac{1}{s} - \frac{1}{s-1}\, ds$

$= \lim_{D \to \infty} \int_s^D \frac{1}{s} - \frac{1}{s-1}\, ds$

$= \lim_{D \to \infty} [\ln s - \ln s - 1]_s^D$

$= \lim_{D \to \infty} \left[\ln \frac{s}{s-1}\right]_s^D$

$= \lim_{D \to \infty} \left[\ln \frac{1}{1 - \frac{1}{s}}\right]_s^D$

$= \lim_{D \to \infty} \ln \frac{1}{1 - \frac{1}{D}} - \ln \frac{1}{1 - \frac{1}{s}}$

$= \ln 1 - \ln \frac{s}{s-1}$

$$\boxed{L\{\tfrac{1}{t}(1 - e^{t})\} = -\ln \frac{s}{s-1}}$$

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3) ii) $f(t) = \frac{1}{t}(\cos at - \cos bt)$, to find $L\{f(t)\}$

→ we know that $L\{\cos at\} = \frac{s}{s^2+a^2}$

$\therefore L\{\cos at - \cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$$L\{\frac{1}{t}(\cos at - \cos bt)\} = \int_s^\infty \frac{s}{s^2+a^2}\, ds - \int_s^\infty \frac{s}{s^2+b^2}\, ds$$

$$= I_1 - I_2$$

→ now $I_1 = \int_s^\infty \frac{s}{s^2+a^2}\, ds$

let $s^2+a^2 = k$

$2s\,ds = dk$

$s\,ds = \frac{dk}{2}$

$\therefore I_1 = \int_{s^2+a^2}^\infty \frac{1}{2}\frac{dk}{k}$

→ now ~~Similarly~~ $I_2 = \cancel{\int_{s^2+b^2}^\infty} \int_s^\infty \frac{s}{s^2+b^2}\, ds$

let $\alpha = s^2+b^2$

$\frac{d\alpha}{2} = s\,ds$

$\therefore I_2 = \int_{s^2+b^2}^\infty \frac{1}{2}\frac{d\alpha}{\alpha}$

$$\therefore L\left\{\frac{1}{t}(\cos at - \cos bt)\right\} = \lim_{D\to\infty} \int_{s^2+a^2}^{D} \frac{1}{2} \frac{dk}{k} - \int_{s^2+b^2}^{D} \frac{1}{2} \frac{dk}{k}$$

$$= \lim_{D\to\infty} \frac{1}{2}\left[\ln D - \ln s^2+a^2 - \ln D + \ln s^2+b^2\right]$$

$$= \frac{1}{2}\left[\ln s^2+b^2 - \ln s^2+a^2\right]$$

$$\boxed{L\left\{\frac{1}{t}(\cos at - \cos bt)\right\} = \frac{1}{2} \ln\left[\frac{s^2+b^2}{s^2+a^2}\right]}$$

4) given $f(t)$ is periodic with time period T,

To prove $L\{f(t)\} = \dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T e^{-st} f(t)\,dt$

→ as $f(t)$ is periodic with Time period T

$f(t+kT) = f(t)$ for integer values of k

now $L\{f(t)\} = \displaystyle\int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)\,e^{-st}dt + \int_{2T}^{3T} f(t)e^{-st}dt$

$+ \ldots \infty$

→ put $t = t+T$ in second integral,
$t = t+2T$ in third integral,
$t = t+3T$ in forth integral and so on..

→ we get $L\{f(t)\} = \displaystyle\int_0^T f(t)e^{-st}dt + e^{-sT}\int_0^T f(t)e^{-st}dt$

$+ e^{-2sT}\displaystyle\int_0^T f(t)e^{-st}dt + \ldots \infty$

$\therefore L\{f(t)\} = \int_0^T f(t)e^{-st}\,dt\left[1+e^{-sT}+e^{-2sT}+e^{-3sT}+e^{-4sT}+\dots\infty\right]$

→ now $1+e^{-sT}+e^{-2sT}+\dots\infty$

is an infinite G.P sum

$\therefore 1+e^{-sT}+e^{-2sT}+\dots\infty = \frac{1}{1-e^{-sT}}$

$\therefore L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}\,dt$

$\therefore$ Hence Proved.

5) <u>Convex Set</u>

a set X in said to be a convex set in $\mathbb{R}^n$, when for $x_1, x_2, \dots, x_n \in X$, convex combination of these n points resides inside X, i.e

$x_k \in X$,

where $x_k = \lambda_1 x_1 + \lambda_2 x_2 + \dots \lambda_n x_n$

where $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$

To show that $X = \{(x_1, x_2) : 4x_1^2 + 9x_2^2 \le 36\}$ Pg 7

is a convex set in $\mathbb{R}^2$

→ for a point $(x_1, y_1) \in X$

$$4x_1^2 + 9y_1^2 \le 36$$

→ for another point $(x_2, y_2) \in X$

$$4x_2^2 + 9y_2^2 \le 36$$

→ now let (x_3, y_3) be convex combination of (x_1, y_1) and (x_2, y_2)

$$\therefore (x_3, y_3) = \lambda (x_1, y_1) + (1-\lambda)(x_2, y_2)$$

or $x_3 = \lambda x_1 + (1-\lambda) x_2$
$y_3 = \lambda y_1 + (1-\lambda) y_2$

then $4x_3^2 + 9y_3^2$

$$= 4[\lambda x_1 + (1-\lambda)x_2]^2 + 9[\lambda y_1 + (1-\lambda)y_2]^2$$

$$= 4\lambda^2 x_1^2 + 4(1-\lambda)^2 x_2^2 + 8\lambda(1-\lambda)x_1 x_2 + 9\lambda^2 y_1^2 + 9(1-\lambda)^2 y_2^2 + 18\lambda(1-\lambda)y_1 y_2$$

$$= \lambda^2[4x_1^2 + 9y_1^2] + (1-\lambda)^2[4x_2^2 + 9y_2^2] + 2\lambda(1-\lambda)[4x_1 x_2 + 9y_1 y_2]$$

→ ~~no~~ now as AM $\geq$ GM

$$\frac{4x_1^2 + 4x_2^2}{2} \ge \sqrt{4^2 x_1^2 x_2^2} \ge 4x_1 x_2$$

and $$\frac{9y_1^2 + 9y_2^2}{2} \ge 9y_1 y_2$$

so, $4x_3^2+9y_3^2 \leq \lambda^2[4x_1^2+9y_1^2] + (1-\lambda)^2[4x_2^2+9y_2^2]$

$$+ 2\lambda(1-\lambda)\left[\frac{4x_1^2+9y_1^2+4x_2^2+9y_2^2}{2}\right]$$

$$\leq \lambda^2 \times 36 + (1+\lambda^2-2\lambda)\times 36$$

$$+ 2(\lambda-\lambda^2)\times 36$$

$$\leq \cancel{2\lambda^2 36} + 36 - \cancel{2\lambda\times 36} + \cancel{2\lambda\times 36} - \cancel{2\lambda^2 36}$$

$$\leq 36$$

$\therefore$ $4x_3^2+9y_3^2 \leq 36$

$\therefore$ (x_3,y_3) also resides in X, i.e $(x_3,y_3) \in X$

$\therefore$ as convex combination of two points in X, also resides in X, X is a convex set.

Extreme Points of $X = \{(x_1,x_2) : 4x_1^2+9x_2^2 \leq 36\}$

→ As the set X is interior part of an ellipse, it's Extreme point is an infinite set in $\mathbb{R}^2$, which satisfies $4x_1^2+9x_2^2 = 36$, ie

→ The extreme points set $B = \{(x,y): 4x^2+9y^2=36\}$

2) ~~get~~ given $X_i = \begin{cases} 1 & i^{th} \text{ student agrees} \\ 0 & \text{otherwise} \end{cases}$

$X_i \rightarrow$ Bernoulli's random variable with parameter p

$\therefore E(X_i) = np$

~~Var~~ $\sigma^2 = np(1-p)$

Chebyshev's Inequaily

$$P(|n-m| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

or $P(|n-m| \leq \varepsilon) > 1 - \frac{\sigma^2}{\varepsilon^2}$

or $P(-\varepsilon \leq n-m < \varepsilon) > 1 - \frac{\sigma^2}{\varepsilon^2}$

or $P(-\varepsilon - m < n < \varepsilon + m) > 1 - \frac{\sigma^2}{\varepsilon^2}$

now to find $(\cancel{0.05}\, p - 0.05 < n < p + 0.05)$

$\therefore -\varepsilon - m = p - 0.05$

$\cancel{+\varepsilon m = p + 0.05}$

$\therefore -\varepsilon - np = p - 0.05 \rightarrow \varepsilon = p - 0.05 - np$

$\cancel{+\varepsilon + np = p + 0.05}$

$\therefore P(p - 0.05 < n < p + 0.05) > 1 - \cancel{n^2 p}\, \frac{npq}{(p - 0.05 - np)^2}$

$\therefore$ ~~np~~