CHAPTER 11

Infinite Series

11.1 SEQUENCE

A sequence is a function whose domain is the set N of positive integers and the range may be any set S. In other words, a sequence in a set S is a rule which assigns to each positive number a unique element of S.

11.2 REAL SEQUENCE

A real sequence is a function whose domain is the set N of positive integers and range is a subset of the set R of real numbers.

A real sequence is described by an ordered set of real numbers a_1, a_2, a_3, ..., a_n, ... which are formed according to some definite law so that corresponding to any definite positive integer n, there is a definite number a_n. It is denoted by $\{a_n\}$ where a_1, a_2, ... are called first, second, ... terms of this sequence. The mth and nth terms a_m and a_n for $m \neq n$ are treated as distinct terms even if $a_m = a_n$, *i.e.*, the terms occurring at different positions are treated as distinct terms even if they have the same value.

Examples:

(*i*) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots, a_n = \frac{1}{n}$

(*ii*) $-1, 2, -3, 5, \ldots, (-1)^n n, \ldots, a_n = (-1)^n n$

(*iii*) 3, 3, 3, 3, ..., 3, ... $a_n = 3$. It is known as a constant sequence.

11.3 BOUNDED SEQUENCE

A sequence $\{a_n\}$ of real numbers is said to be bounded above if there exists a real number M such that $a_n \leq M$, for $n = 1, 2, 3, \ldots$.

A sequence $\{a_n\}$ of real numbers is said to be bounded below if there exists a real number m such that $a_n \geq m$ for $n = 1, 2, 3, \ldots$.

A sequence $\{a_n\}$ of real numbers is said to be bounded if it is bounded both above and below. In other words, a sequence $\{a_n\}$ of real numbers is said to be bounded if there exists real numbers m and

M such that $m \le a_n \le M$, for $n = 1, 2, 3, \ldots$, otherwise it is said to be unbounded. The numbers m and M are known as the lower and upper bounds respectively of the sequence $\{a_n\}$ if it is bounded.

11.4 MONOTONIC SEQUENCE

Monotonic Increasing Sequence

A sequence $\{a_n\}$ of real numbers is said to be monotonically increasing if $a_n \le a_{n+1}$ for $n = 1, 2, \ldots$, *i.e.*, $a_1 \le a_2 \le a_3 \le a_4 \le \ldots$.

Monotonic Decreasing Sequence

A sequence $\{a_n\}$ of real numbers is said to be monotonically decreasing if $a_n \ge a_{n+1}$ for $n = 1, 2, \ldots$, *i.e.*, $a_1 \ge a_2 \ge a_3 \ge a_4 \ge \ldots$.

Monotonic Sequence

A sequence $\{a_n\}$ of real numbers is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

Examples:

(*i*) The sequence $\{a_n\}$ where $a_n = \frac{1}{n}$, *i.e.*, the sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ is bounded since $0 < a_n = \frac{1}{n} \le 1$ for $n = 1, 2, 3, \ldots$ and monotonically decreasing. It is known as harmonic sequence.

(*ii*) The sequence $\{a_n\}$ where $a_n = 2^n$, *i.e.*, the sequence $\{2, 2^2, 2^3, 2^4, \ldots\}$ is unbounded since 2^n becomes larger and larger as n is increasing and monotonically increasing.

11.5 LIMIT OF A SEQUENCE

A sequence $\{a_n\}$ of real numbers is said to tend to a limit L, if for every $\varepsilon > 0$, no matter however small, there exists a positive integer N depending on ε, in general, such that $|a_n - L| < \varepsilon$ for all $n \ge N$.

Then we write $\lim\limits_{n\to\infty} a_n = L$, or, simply $\lim a_n = L$, *i.e.*, $a_n \to L$ as $n \to \infty$.

Example: Consider a sequence $\{a_n\} = \left\{2 + \frac{1}{n}\right\}$. Plotting the values

n:	1	2	4	5	10	50	100	1000	10000	...
a_n:	3	2.5	2.25	2.2	2.1	2.02	2.01	2.001	2.0001	...

An n increases, $a_n = 2 + \frac{1}{n}$ becomes closer to 2. Thus the difference between $2 + \frac{1}{n}$ and 2 becomes smaller and smaller as n becomes larger and larger, *i.e.*, we can make $2+\frac{1}{n}$ as close as we please to 2, by choosing an appropriately large value for n, *i.e.*, the terms of this sequence cluster around this point.

Therefore, $\lim\limits_{n\to\infty} a_n = 2$

Note that $2+\frac{1}{n} \neq 2$, for $n = 1, 2, 3, \ldots$

Note: A monotonic sequence which is bounded has a limit.

11.6 CONVERGENCE, DIVERGENCE AND OSCILLATION OF A SEQUENCE

Convergent Sequence

A sequence $\{a_n\}$ is said to be convergent if it has a finite limit, *i.e.*, $\lim\limits_{n\to\infty} a_n = L$, where L is a unique finite number.

Divergent Sequence

A sequence $\{a_n\}$ is said to be divergent if $a_n \to +\infty$ or $-\infty$ as $n \to \infty$.

Oscillating Sequence

A sequence which is neither convergent nor divergent is said to be an oscillating sequence.

Examples:

(*i*) The sequence $\left\{\frac{1}{n^2}\right\}$ is convergent, since $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$, is a unique and finite number.

(*ii*) The sequence $\{n^2\}$ is divergent, since $n^2 \to \infty$ as $n \to \infty$.

(*iii*) The sequence $\{(-1)^n\}$ oscillates finitely, since

$$\lim_{n\to\infty} (-1)^n = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

(*iv*) The sequence $\{(-1)^n n\}$ oscillates infinitely, since $(-1)^n n \to +\infty$ or $-\infty$ as $n \to \infty$.

Note 1: If sequence $\{a_n\}$ converges to l_1 and $\{b_n\}$ converges to l_2, then

1. $\{a_n + b_n\}$ converges to $l_1 + l_2$
2. $\{c\, a_n\}$ converges to cl_1
3. $\{a_n\, b_n\}$ converges to $l_1\, l_2$
4. $\left\{\frac{a_n}{b_n}\right\}$ converges to $\frac{l_1}{l_2}$, provided $l_2 \neq 0$.

Note 2: Every convergent sequence is bounded.

Example: $\left\{a_n = \frac{1}{n}\right\}$ is convergent and is bounded, since $0 < a_n = \frac{1}{n} \leq 1$, for $n = 1, 2, 3, \ldots$.

But the converse is not true, *i.e.*, a bounded sequence may not be convergent.

Example: $\{a_n = (-1)^n\}$ is oscillatory but is bounded since $-1 \leq (-1)^n \leq 1$, for $n = 1, 2, 3, \ldots$.

Note 3: A bounded monotone sequence is convergent.

Example: $\left\{a_n = \frac{1}{n}\right\}$ is bounded since $0 < a_n \leq 1, n = 1, 2, 3, \ldots$ and monotonically decreasing since $\frac{1}{n} > \frac{1}{n+1}$, for $n = 1, 2, 3, \ldots$. Hence the sequence is convergent because $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$.

11.7 SOME STANDARD LIMITS

1. $\lim\limits_{n\to\infty} n^{1/n} = 1$
2. $\lim\limits_{n\to\infty} \frac{\log n}{n} = 0$
3. $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$, for any real x
4. $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1$, for $x > 0$
5. $\lim\limits_{n\to\infty} x^n = 0$, for $|x| < 1$, *i.e.*, $-1 < x < 1$
6. $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0$, for any real x.

ILLUSTRATIVE EXAMPLES

Determine the nature of the following sequences where a_n denotes the nth term.

Example 1: $a_n = \dfrac{n^2 + n}{2n^2 - n}$

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+n}{2n^2-n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2-\frac{1}{n}} = \frac{1}{2}. \quad \left(\because \frac{1}{n} \to 0 \text{ as } n \to \infty\right)$$

Therefore, the sequence $\{a_n\}$ is convergent since the limit of the sequence is unique and finite.

Example 2: $a_n = \cot hn$.

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \coth n = \lim_{n\to\infty} \frac{\cosh n}{\sinh n}$$

$$= \lim_{n\to\infty} \frac{e^n + e^{-n}}{e^n - e^{-n}} = \lim_{n\to\infty} \frac{1+\frac{1}{e^{2n}}}{1-\frac{1}{e^{2n}}} = 1$$

Hence, the sequence $\{a_n\}$ is convergent.

Example 3: $a_n = e^n$.

Solution: Here $a_n \to \infty$ as $n \to \infty$.

So, the sequence $\{a_n\}$ is divergent.

Example 4: $a_n = 1 + (-1)^n$.

Solution:

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \{1+(-1)^{2n}\} = 1 + 1 = 2$$

$$\lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} \{1+(-1)^{2n-1}\} = 1 - 1 = 0.$$

Therefore, the sequence $\{a_n\}$ is oscillating finitely since it has two finite limits.

Theorem: If for three convergent sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, there exists a positive integer N such that

(*i*) $x_n < y_n < z_n$, for all $n \geq N$ and

(*ii*) $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l$

then the sequence $\{y_n\}$ converges to the same limit l.

Example 5: Prove that $\lim_{n\to\infty} \left\{ \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}} \right\} = 1.$

Solution: Let $y_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}}$

Obviously,

$$\underbrace{\frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \ldots + \frac{1}{\sqrt{n^2+n}}}_{n \text{ terms}} < y_n < \underbrace{\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \ldots + \frac{1}{\sqrt{n^2+1}}}_{n \text{ terms}}, \quad \text{for } n \geq 2$$

or $$\frac{n}{\sqrt{n^2+n}} < y_n < \frac{n}{\sqrt{n^2+1}}, \quad \text{for } n \geq 2$$

or $$\frac{1}{\sqrt{1+\frac{1}{n}}} < y_n < \frac{1}{\sqrt{1+\frac{1}{n^2}}}, \quad \text{for } n \geq 2$$

$$\text{Since} \quad \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1,$$

$$\therefore \quad \lim_{n\to\infty} y_n = \lim_{n\to\infty} \left\{ \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}} \right\} = 1.$$

11.8 INFINITE SERIES

If $\{a_n\}$ be a sequence of real numbers, then the expression

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots \text{ to } \infty$$

is called an infinite series and is usually denoted by $\sum_{n=1}^{\infty} a_n$ or more briefly $\sum a_n$.

Many problems that cannot be solved in terms of elementary (algebraic and transcendental) functions may be solved by using infinite series. Our main aim is to study the nature (or behaviour) of convergence, divergence or oscillation of a given infinite series of real numbers.

To the given sequence $\{a_n\}$ which underlines the infinite series, we associate another sequence $\{\sigma_n\}$ defined as follows:

$$\sigma_n = a_1 + a_2 + \ldots + a_n$$

Thus σ_n denotes the sum of the first n terms of the infinite series $\sum_{n=1}^{\infty} a_n$. Here σ_n is known as the nth partial sum of this infinite series.

Convergence

An infinite series $\sum_{n=1}^{\infty} a_n$ is said to be convergent if

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \left(\sum_{k=1}^{n} a_k \right) = \lim_{n\to\infty} \sigma_n = \text{finite value} = S \text{ (say)}.$$

Here S is known as the sum (value) of he infinite series $\sum_{n=1}^{\infty} a_n$.

An infinite series which is not convergent may have any of the following four behaviours. It may

(i) diverge to plus infinity
(ii) diverge to minus infinity
(iii) oscillate finitely
(iv) oscillate infinitely

depending on the behaviour of the sequence of its partial sums.

Example: Consider the infinite series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots$ to ∞

Here $a_n = \frac{1}{2^{n-1}}$, $\sigma_n = 1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}} = \dfrac{1-\frac{1}{2^n}}{1-\frac{1}{2}}$

$$\therefore \quad \lim_{n\to\infty} \sigma_n = \frac{1}{1-\frac{1}{2}} = 2.$$

So the infinite series is convergent and converges to 2.

Example: Consider the infinite series $1+2+3+4+\ldots+n+\ldots$ to ∞

Here $a_n = n, \sigma_n = 1+2+\ldots+n = \frac{n(n+1)}{2}$.

$\therefore \quad \sigma_n \to \infty$ as $n \to \infty$.

So the infinite series is divergent and diverges to $+\infty$

Example: Consider the infinite series $1-1+1-1+\ldots$ to ∞.

$$\therefore \quad \sigma_n = 1-1+1-1+\ldots \text{ to } n \text{ terms} = \begin{cases} 0, & n \text{ is even} \\ 1, & n \text{ is odd} \end{cases}$$

Therefore, the infinite series oscillates finitely.

Example: Consider the infinite series $1-2+3-4+\ldots+(-1)^{n-1}n+\ldots$ to ∞.

$$\sigma_n = \begin{cases} -\frac{n}{2}, & n \text{ is even} \\ \frac{n+1}{2}, & n \text{ is odd} \end{cases}$$

So the infinite series oscillates infinitely.

11.9 SOME PROPERTIES OF AN INFINITE SERIES

1. If an infinite series $\sum a_n$ converges to a sum s then $c\sum a_n$ also converges to the sum cs, where c is a constant.
2. If an infinite series $\sum a_n$ is divergent, then $c\sum a_n$ is also divergent, where c is a constant.
3. If two infinite series $\sum a_n$ and $\sum b_n$ are convergent and converges to s_1 and s_2 respectively, then $\sum(c_1a_n + c_2b_n)$ is also convergent and converges to $c_1s_1 + c_2s_2$, where c_1, c_2 are constants.
4. The character of convergency or divergency of an infinite series remains unaltered by the addition or deletion of a finite number of terms.

11.10 TWO IMPORTANT INFINITE SERIES

Geometric Series

The series $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \ldots$ to ∞ is known as geometric series with common ratio r.

Theorem: A geometric series $\sum_{n=0}^{\infty} r^n$ is convergent if $-1 < r < 1$, divergent if $r \geq 1$ and it is oscillatory if $r \leq -1$.

Proof: Here the partial sum

$$\sigma_n = 1 + r + r^2 + \ldots + r^{n-1}$$

$$= \begin{cases} \dfrac{1-r^n}{1-r} & \text{if } r \neq 1 \\ n & \text{if } r = 1 \end{cases}$$

Case I: Let $-1 < r < 1$ we have

$$\lim_{n\to\infty} \sigma_n = \frac{1}{1-r}, \text{ a finite number} \qquad \left(\because \lim_{n\to\infty} r^n = 0, \text{ as } -1 < r < 1\right)$$

So the infinite series is convergent if $-1 < r < 1$.

Case II: Let $r \geq 1$.

When $r > 1$, $\quad \sigma_n = \dfrac{r^n - 1}{r-1} \to \infty$ as $n \to \infty$ $\qquad (\because r^n \to \infty$ as $n \to \infty$ because $r > 1)$

When $\quad r = 1$, $\quad \sigma_n = \underbrace{1+1+\ldots+1}_{n \text{ terms}} = n \to \infty$ as $n \to \infty$.

Thus the infinite series is divergent if $r \geq 1$.

Case III: Let $\quad r \leq -1$.

When $\quad r = -1$, $\quad \sigma_n = \underbrace{1-1+1-1+\ldots+(-1)^{n-1}}_{n \text{ terms}} = \begin{cases} 0, \text{ when } n \text{ is even} \\ 1, \text{ when } n \text{ is odd} \end{cases}$

When $\quad r < -1$, $\quad \sigma_n = \dfrac{r^n - 1}{r-1} \to +\infty$ or $-\infty$ as $n \to \infty$ according as n is odd or even respectively.

So the infinite series is oscillatory when $r \leq -1$.

Examples:

(*i*) The infinite series $1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \ldots$ to ∞ is convergent, as the series is a geometric series with common ratio $\dfrac{1}{2} < 1$.

(*ii*) The infinite series $1 + 2 + 2^2 + 2^3 + \dots$ to ∞ is divergent as the series is a geometric series with common ratio $2 > 1$.

(*iii*) The infinite series $1 - 2 + 2^2 - 2^3 + \dots +$ to ∞ is oscillatory (which is also divergent) as the series is a geometric series with common ratio $-2 < -1$.

(*iv*) The infinite series $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\dots$ to ∞ is convergent since $\frac{3}{5}+\frac{3}{5^3}+\dots$ to ∞ and $\frac{4}{5^2}+\frac{4}{5^4}+\dots$ to ∞ are two geometric series with common ratios $\frac{1}{5^2} < 1$.

Harmonic Series of order *p* or *p*-Harmonic Series or *p*-series

The infinite series $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\dots$ to ∞ is called harmonic series of order p or p-harmonic series or p-series.

Theorem: The p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

Proof: Case I: Let $p > 1$.

We have $\quad 2^n > n, \quad n = 1, 2, 3, \dots$

If $\sigma(n)$ denotes the sum of first n terms of the given infinite series, we have

$$\sigma(n) < \sigma(2^n)$$

(since the terms being all positive) ...(1)

Now, $$\sigma(2^n) = \frac{1}{1^p}+\frac{1}{2^p}+\dots+\frac{1}{(2^n)^p}$$

We have, $$\sigma(2^{n+1}-1) = \frac{1}{1^p}+\frac{1}{2^p}+\dots+\frac{1}{(2^{n+1}-1)^p}$$

$$= \frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\dots$$

$$+\left\{\frac{1}{(2^n)^p}+\frac{1}{(2^n+1)^p}+\dots+\frac{1}{(2^{n+1}-1)^p}\right\}$$

$$(\because\ 1+2+2^2+\dots+2^n = 2^{n+1}-1)$$

Now, $$\frac{1}{2^p}+\frac{1}{3^p}<\frac{2}{2^p} = \frac{1}{2^{p-1}}$$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{4}{4^p} = \frac{1}{2^{2(p-1)}}$$

....

....

$$\frac{1}{(2^n)^p}+\frac{1}{(2^n+1)^p}+\dots+\frac{1}{(2^{n+1}-1)^p}<\frac{2^{n+1}-2^n}{(2^n)^p}=\frac{2^n}{(2^n)^p}=\frac{1}{2^{n(p-1)}}$$

Thus we have, $\sigma(2^{n+1}-1)<\frac{1}{1^p}\frac{1}{2^{p-1}}+\frac{1}{2^{2(p-1)}}+\dots+\frac{1}{2^{n(p-1)}}$

$$=\frac{1-\left(\frac{1}{2^{p-1}}\right)^{n+1}}{1-\frac{1}{2^{p-1}}}<\frac{1}{1-\frac{1}{2^{p-1}}}=\frac{2^{p-1}}{2^{p-1}-1},\ n=1,2,3,\dots \qquad (\because p>1)$$

Now, $2^{n+1}-1>2^n$ which implies $\sigma(2^{n+1}-1)>\sigma(2^n)$ ...(2)

Thus from (1) and (2), we have

$\sigma(n)<\sigma(2^n)<\sigma(2^{n+1}-1)<\frac{2^{p-1}}{2^{p-1}-1}, n=1,2,3,\dots$ and as such the sequence $\sigma(n)$ is monotone and bounded

$$\therefore \qquad \sum_{n=1}^{\infty}\frac{1}{n^p}=\lim_{n\to\infty}\sigma(n)=\text{finite value.}$$

Thus the infinite series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent if $p>1$.

Case II: Let $p=1$.

We have the infinite series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots \text{ to } \infty$$

$$\therefore \qquad \sigma(2^n)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{2^n}$$

$$=\left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\dots$$

$$+\left(\frac{1}{2^{n-1}+1}+\frac{1}{2^{n-1}+2}+\dots+\frac{1}{2^n}\right)$$

$$(\because 2+2+2^2+2^3+\dots+2^{n-1}=2^n)$$

Now,

$$1+\frac{1}{2}>\frac{1}{2}$$

$$\frac{1}{3}+\frac{1}{4}>\frac{2}{4}=\frac{1}{2}$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{4}{8}=\frac{1}{2}$$

$$\frac{1}{2^{n-1}+1}+\frac{1}{2^{n-1}+2}+\ldots+\frac{1}{2^n}>\frac{1}{2^n}(2^n-2^{n-1})=\frac{2^{n-1}}{2^n}=\frac{1}{2}.$$

$$\therefore \qquad \sigma(2^n)>\underbrace{\frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2}}_{n\text{ terms}}=\frac{n}{2}\to\infty \text{ as } n\to\infty$$

As such the infinite series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

Case III: Let $p<1$.

Now, $\frac{1}{n^p}>\frac{1}{n}, n=2,3,\ldots$ (since $p<1$ implies $n^p>n=2,3,\ldots$)

Therefore, each term of the infinite series

$$\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots \text{ to } \infty$$

is greater than the corresponding term of the divergent series

$$\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots \text{ to } \infty$$

and as such the infinite series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent if $p<1$.

Examples:

(*i*) $\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$ to ∞ is a convergent series since $2>1$.

(*ii*) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots$ to ∞ is a divergent series since $\frac{1}{2}<1$.

(*iii*) The series $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\ldots$ to ∞ is convergent since the nth term $a_n=\frac{1}{n^{4/3}}$ with $p=\frac{4}{3}>1$.

11.11 NECESSARY CONDITION FOR CONVERGENCE

Theorem: If the infinite series $\sum_{n=1}^{\infty} a_n$ of real numbers is convergent, then $\lim_{n\to\infty} a_n = 0$.

Proof: Let $\sigma_n = a_1 + a_2 + \ldots + a_n$.

If the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \sigma_{n-1}$ = finite value = σ(say).

$$\therefore \quad a_n = \sigma_n - \sigma_{n-1} \text{ implies } \lim_{n\to\infty} a_n = \lim_{n\to\infty} (\sigma_n - \sigma_{n-1})$$

$$= \lim_{n\to\infty} \sigma_n - \lim_{n\to\infty} \sigma_{n-1} = \sigma - \sigma = 0$$

Hence the result.

Note 1: This is not a test for convergence. From the above theorem it follows that if a_n does not tend to zero as $n \to \infty$, then the infinite series $\sum_{n=1}^{\infty} a_n$ cannot converge.

Note 2: The converse of the above theorem is not true in general. For example, consider the p-series $\sum_{n=1}^{\infty} \frac{1}{n}$ with $p = 1$, which is divergent though $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example: The infinite series $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots + \frac{n}{n+1} + \ldots$ to ∞ is a divergent one since

$$a_n = n\text{th term} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1 \text{ as } n \to \infty.$$

Example: The infinite series $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$ is a divergent one since a_n = nth term = $\frac{n^2-1}{n^2+1}$

$$= \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}} \to 1 \text{ as } n \to \infty.$$

Example: Prove that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

Solution. Let $\quad a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, n = 1, 2, 3, \ldots$

$$\therefore \quad \sigma_n = a_1 + a_2 + a_3 + \ldots + a_n$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\therefore \quad \lim_{n\to\infty} \sigma_n = 1 \qquad \left(\because \frac{1}{n+1} \to 0 \text{ as } n \to \infty\right)$$

Hence the infinite series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

Example. Find the n^{th} partial sum of the infinite series:

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots \text{ to } \infty$$

Hence find the sum of the series if it is convergent. [*BESUS 2013*]

Solution: Here

$$a_n = n^{th}\text{term} = \frac{1}{(2n-1)(2n+1)}$$

$$= \frac{1}{2}\left\{\frac{1}{2n-1} - \frac{1}{2n+1}\right\}, \; n = 1, 2, 3, 4, \ldots .$$

The n^{th} partial sum of the given infinite series is

$$\sigma_n = a_1 + a_2 + a_3 + \ldots + a_n$$

$$= \frac{1}{2}\left\{\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right\}$$

$$= \frac{1}{2}\left\{1 - \frac{1}{2n+1}\right\} = \frac{n}{2n+1}$$

$$= \frac{1}{2 + \frac{1}{n}}$$

$$\therefore \quad \lim_{n\to\infty} \sigma_n = \frac{1}{2}.$$

Hence the given infinite series is convergent and converges to $\frac{1}{2}$ (*i.e.*, the sum is $\frac{1}{2}$).

Example: The infinite series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ cannot converge since a_n = nth term = $\frac{n^n}{n!}$ = $\left(\frac{n}{1}\right)\left(\frac{n}{2}\right)\left(\frac{n}{3}\right)\cdots\left(\frac{n}{n}\right) > 1$, when $n = 2, 3, 4, \ldots$.

Therefore, a_n cannot tend to zero as $n \to \infty$ and hence $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ cannot converge.

Theorem: If every term of an infinite series $\sum_{n=1}^{\infty} a_n$ is positive, then the series $\sum_{n=1}^{\infty} a_n$ either converges to a finite positive number or it diverges to $+\infty$.

Proof: Let $\sigma_n = a_1 + a_2 + \ldots + a_n$. Since each a_k ($k = 1, 2, 3, \ldots$) is positive, σ_n is also positive for every $n = 1, 2, 3, \ldots$ and $\sigma_{n+1} - \sigma_n = (a_1 + a_2 + \ldots + a_n + a_{n+1}) - (a_1 + a_2 \ldots + a_n) = a_{n+1} > 0$. So, $\sigma_{n+1} > \sigma_n$ for $n = 1, 2, 3, \ldots$.

Therefore, $\{\sigma_n\}$ is a strictly increasing sequence of positive numbers. If $\{\sigma_n\}$ is bounded, then by a theorem on monotonic sequence, $\{\sigma_n\}$ is convergent and because $\sigma_n (n = 1, 2, 3, \ldots)$ is positive, it converges to a positive number σ (say).

If the sequence $\{\sigma_n\}$ is unbounded, then since σ_n increases monotonically, by a property of monotonic sequence, we obtain that $\{\sigma_n\}$ is divergent and it diverges to $+\infty$.

11.12 COMPARISON TEST

Comparison test consists of 'comparison' between a given (unknown) infinite series with positive terms $\sum_{n=1}^{\infty} a_n$ and a (known) auxiliary infinite series with positive terms $\sum_{n=1}^{\infty} b_n$ whose nature of convergence or divergence is known.

We state below three theorems relating to comparison test for convergence and divergence of infinite series with positive terms.

Theorem 1: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series with positive terms and $\sum_{n=1}^{\infty} b_n$ is convergent. Then $\sum_{n=1}^{\infty} a_n$ is convergent if

(*i*) there exists a positive integer N such that $a_n \le kb_n$, for all $n \ge N$, where k is a positive constant,

(*ii*) $\lim_{n\to\infty} \frac{a_n}{b_n} = l$, where l is a finite number (may be 0).

Theorem 2: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series with positive terms and $\sum_{n=1}^{\infty} b_n$ is divergent. Then $\sum_{n=1}^{\infty} a_n$ is divergent if

(*i*) there exists a positive integer N such that $a_n \ge kb_n$, for all $n \ge N$, where k is a positive constant, (*ii*) $\lim_{n\to\infty} \frac{a_n}{b_n} = l$, where l is a non-zero number (may be $+\infty$).

Theorem 3: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series with positive terms

(*i*) if there exists a positive integer N such that $\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}$, for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

(*ii*) if there exists a positive integer N such that $\frac{a_{n+1}}{a_n} \ge \frac{b_{n+1}}{b_n}$, for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

Note 1: To apply comparison test we need another series whose convergence or divergence is known. The following series will be helpful.

(*i*) The geometric series $\sum_{n=0}^{\infty} r^n$ converges if $0 \le r < 1$ and diverges if $r \ge 1$.

(*ii*) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Note 2: Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with positive terms and suppose $a_n = \frac{x_n}{y_n}$ where x_n and y_n are expressions involving powers of n. Then for applying comparison test we choose the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where

p = (highest power of n in y_n) – (highest power of n in x_n).

ILLUSTRATIVE EXAMPLES

Example 1: Show that the series $\frac{1}{1^2+1} + \frac{1}{2^2+1} + \frac{1}{3^2+1} + \ldots$ to ∞ is convergent.

Solution: Let a_n be the nth term of the given series

$$\therefore \quad a_n = \frac{1}{n^2+1} < \frac{1}{n^2}, n = 1, 2, 3, \ldots$$

Also we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, since it is a p-series with $p = 2 > 1$.

Hence by comparison test the given series is convergent.

Alternative: We consider $\sum_{n=1}^{\infty} b_n$, where $b_n = \frac{1}{n^2}$.

Now,
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1, \text{ a finite number.}$$

Hence by comparison test, $\sum a_n$ is convergent or divergent if $\sum b_n$ is convergent or divergent respectively. But $\sum b_n = \sum \frac{1}{n^2}$ is a convergent series (since it is a p-series with $p = 2 > 1$).

Therefore the series $\sum a_n$, *i.e.*, the given series is convergent.

Example 2: Test the convergence of the series

$$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \ldots \text{ to } \infty.$$

Solution: Let a_n be the nth term of the given series

$$\therefore \quad a_n = \frac{n\text{th term of } 1, 3, 5, \ldots}{n(n+1)(n+2)} = \frac{2n-1}{n(n+1)(n+2)}$$

We observe that in a_n, highest power of n in denominator – highest power of n in numerator $= 3 - 1 = 2$.

So, let us compare $\sum a_n$ with $\sum b_n$, where $b_n = \frac{1}{n^2}$

$$\therefore \quad \frac{a_n}{b_n} = \frac{n^2(2n-1)}{n(n+1)(n+2)} = \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 2 \qquad \left(\because \frac{1}{n}, \frac{2}{n} \to 0 \text{ as } n \to \infty\right).$$

Therefore, $\sum a_n$ and $\sum b_n$ converge or diverge together. Since $\sum b_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2 > 1$, therefore $\sum \frac{1}{n^2}$ is convergent which implies the given series is also convergent.

Example 3: Test the convergence of the series

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots \text{ to } \infty.$$

Solution: Let a_n be the nth term of the given series.

$$\therefore \qquad a_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}.$$

Let us compare $\sum a_n$ with $\sum b_n$, where $b_n = \frac{1}{\sqrt{n}}$

$$\therefore \qquad \frac{a_n}{b_n} = \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$$

$$\therefore \qquad \lim_{n\to\infty}\frac{a_n}{b_n} = \frac{1}{2}\left(\because \frac{1}{n}\to 0 \text{ as } n\to\infty\right).$$

Since $\sum b_n = \sum \frac{1}{n^{1/2}}$ is of the form $\sum \frac{1}{n^p}$ with $p = \frac{1}{2} < 1$, therefore $\sum b_n$ is divergent which implies $\sum a_n$, *i.e.*, the given series is also divergent.

Example 4: Test the convergence of the series

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots \text{ to } \infty. \qquad (W.B.U.T.\ 2005)$$

Solution: Let us consider the infinite series (omitting the first term of the given series)

$$\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots \text{ to } \infty$$

Let a_n be the nth term of this series.

$$\therefore \qquad a_n = \frac{n^n}{(n+1)^{n+1}}$$

Let us compare $\sum a_n$ with $\sum b_n$, where $b_n = \frac{1}{n}$

$$\therefore \qquad \frac{a_n}{b_n} = \frac{n^{n+1}}{(n+1)^{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}$$

$$\therefore \qquad \lim_{n\to\infty}\frac{a_n}{b_n} = \frac{1}{e} \qquad \left(\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \text{ and } \frac{1}{n}\to 0 \text{ as } n\to\infty\right)$$

Therefore, $\sum a_n$ and $\sum b_n$ converge or diverge together.

Since $\sum b_n = \sum \frac{1}{n}$ is of the form $\sum \frac{1}{n^p}$ with $p = 1$, therefore $\sum b_n$ is divergent and hence

$\sum a_n$ is divergent and also the given series is divergent one (since addition or deletion of finite number of terms does not alter the nature of the series).

Example 5: Test the convergence of the series

$$\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\ldots \text{ to } \infty$$

Solution: The nth term $a_n = \frac{n+1}{n^p}$. Let $b_n = \frac{1}{n^{p-1}}$.

$$\therefore \qquad \frac{a_n}{b_n} = \frac{n+1}{n} = 1+\frac{1}{n}$$

$$\therefore \qquad \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \qquad \left(\because \frac{1}{n}\to 0 \text{ as } n\to\infty\right).$$

Also, $\sum b_n = \sum \frac{1}{n^{p-1}}$ is convergent if $p-1>1$ and divergent if $p-1\le 1$, *i.e.*, $\sum b_n$ is convergent if $p>2$ and divergent if $p\le 2$.

Hence by comparison test, $\sum a_n$, *i.e.*, the given infinite series is convergent if $p>2$ and divergent if $p\le 2$.

Example 6: Discuss the convergence or divergence of the infinite series $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$

Solution: The nth term $a_n = \sin\left(\frac{1}{n}\right)$. Let $b_n = \frac{1}{n}$.

$$\therefore \quad \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{m\to 0}\frac{\sin m}{m} \qquad \left(\text{let } m=\frac{1}{n},\ \therefore\ m\to 0 \text{ as } n\to\infty\right)$$

$$= 1.$$

Since $\sum b_n = \sum \frac{1}{n}$ is of the form $\sum \frac{1}{n^p}$ with $p=1$, therefore $\sum b_n$ is divergent and hence $\sum a_n$, *i.e.*, the given infinite series is also divergent.

Example 7: Examine the convergence of the infinite series

$$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n^p}.$$

Solution: The nth term $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^p} = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{n^p(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n^p(\sqrt{n+1}+\sqrt{n})}.$

In a_n, highest power of n in denominator – highest power of n in numerator $= n^{p+\frac{1}{2}}$.

So, let us compare $\sum a_n$ with $\sum b_n$, where $b_n = \dfrac{1}{n^{p+\frac{1}{2}}}$

$$\therefore \quad \frac{a_n}{b_n} = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$\therefore \quad \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2} \qquad \left(\because \frac{1}{n}\to 0 \text{ as } n\to\infty\right)$$

Therefore, $\sum a_n$ and $\sum b_n$ converge or diverge together.

Since $\sum b_n = \sum \dfrac{1}{n^{p+\frac{1}{2}}}$ is convergent if $p+\dfrac{1}{2}>1$ and divergent if $p+\dfrac{1}{2}\le 1$ *i.e.*, convergent if $p>\dfrac{1}{2}$ and divergent if $p\le\dfrac{1}{2}$, therefore $\sum a_n$, *i.e.*, the given infinite series is convergent if $p > 1/2$ and divergent if $p\le\dfrac{1}{2}$.

Example 8: Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\sin\frac{1}{n}$.

Solution: The nth term $a_n = \dfrac{1}{\sqrt{n}}\sin\dfrac{1}{n}$. Let $b_n = \dfrac{1}{n^{3/2}}$. *(W.B.U.T. 2001)*

$$\therefore \quad \frac{a_n}{b_n} = n\sin\frac{1}{n} = \frac{\sin\frac{1}{n}}{\frac{1}{n}}.$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{m\to 0}\frac{\sin m}{m} \qquad \left(\text{let } m=\frac{1}{n}, \therefore m\to 0 \text{ as } n\to\infty\right)$$

$$= 1.$$

Since $\sum b_n = \sum \frac{1}{n^{3/2}}$ is of the form $\sum \frac{1}{n^p}$ with $p = \frac{3}{2} > 1$, therefore $\sum b_n$ is convergent and hence $\sum a_n$, *i.e.*, the given infinite series is also convergent.

Example 9: Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$.

Solution: The nth term $a_n = \frac{1}{n \log n}$, $n = 2, 3, 4, \ldots$. Let $b_n = \frac{1}{n}$, $n = 2, 3, 4, \ldots$

$$\therefore \qquad \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1}{\log n} = 0.$$

Therefore, $\sum_{n=2}^{\infty} a_n$ and $\sum_{n=2}^{\infty} b_n$ converge or diverge together. Here $\sum_{n=2}^{\infty} b_n$ is divergent since $\sum_{n=2}^{\infty} \frac{1}{n}$ is a divergent series. Hence the given infinite series is a divergent one.

Example 10: Test the convergence of the series $\sum_{n=1}^{\infty} (\sqrt{n^4+1} - \sqrt{n^4-1})$. *(W.B.U.T. 2008)*

Solution: The nth term $a_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$$= \frac{(\sqrt{n^4+1} - \sqrt{n^4-1})(\sqrt{n^4+1} + \sqrt{n^4-1})}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$= \frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}}.$$

Let $$b_n = \frac{1}{n^2}, \; n = 1, 2, 3, 4, \ldots$$

$$\therefore \qquad \frac{a_n}{b_n} = \frac{2n^2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$= \frac{2}{\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}}.$$

$$\therefore \qquad \lim_{n \to \infty} \frac{a_n}{b_n} = 1.$$

Now, $\sum b_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2 > 1$. Therefore, $\sum b_n$ is convergent and hence by comparison test $\sum a_n$, *i.e.*, the given infinite series is also convergent.

Example 11: Test the convergence of the series

$$\frac{6}{1\cdot3\cdot5}+\frac{8}{3\cdot5\cdot7}+\frac{10}{5\cdot7\cdot9}+\ldots$$ (*W.B.U.T. 2008, 2013*)

Solution: The nth term $a_n = \dfrac{6+(n-1)2}{(2n-1)(2n+1)(2n+3)} = \dfrac{2n+4}{(2n-1)(2n+1)(2n+3)}$ and

let $b_n = \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$

$$\therefore \qquad \frac{a_n}{b_n} = \frac{2n^3+4n^2}{(2n-1)(2n+1)(2n+3)}$$

$$= \frac{2+\frac{4}{n}}{\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)}.$$

$$\therefore \qquad \lim_{n\to\infty}\frac{a_n}{b_n} = \frac{2}{8} = \frac{1}{4}, \quad \text{since} \quad \frac{1}{n}, \frac{3}{n}, \frac{4}{n} \to 0 \text{ as } n \to \infty.$$

Now, $\sum b_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2 > 1$. Therefore, $\sum b_n$ is convergent and hence by comparison test $\sum a_n$ *i.e.*, the given infinite series is also convergent.

Example 12: Test the convergence of the series $\sum a_n$, where $a_n = (n^3+1)^{1/3} - n$.

(*W.B.U.T. 2003, 2007*)

Solution: Here $a_n = (n^3+1)^{1/3} - n = \left\{n^3\left(1+\frac{1}{n^3}\right)\right\}^{1/3} - n$

$$= n\left(1+\frac{1}{n^3}\right)^{1/3} - n$$

$$= n\left\{1+\frac{1}{3}\cdot\frac{1}{n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2+\ldots\text{to }\infty\right\} - n$$

$$= \frac{1}{n^2}\left(\frac{1}{3} - \frac{1}{9}\cdot\frac{1}{n^3} + \ldots \text{ to } \infty\right)$$

Consider an infinite series $\sum b_n$, where $b_n = \dfrac{1}{n^2}$

$$\therefore \quad \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{9}\cdot\frac{1}{n^3} + \ldots \text{ to } \infty\right) = \frac{1}{3}.$$

Now, $\sum b_n = \sum \dfrac{1}{n^2}$ is of the form $\sum \dfrac{1}{n^p}$ with $p = 2 > 1$.

Therefore, $\sum b_n$ is convergent and hence by comparison test $\sum a_n$, *i.e.*, the given series is convergent.

11.13 D'ALEMBERT'S RATIO TEST

Let $\sum\limits_{n=1}^{\infty} a_n$ be an infinite series with positive terms such that $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}$ exists finitely and let

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = l.$$

Then the series is

(*i*) convergent if $l < 1$,

(*ii*) divergent if $l > 1$, and no conclusion can be drawn if $l = 1$, *i.e.*, the series may converge or diverge if $l = 1$. (*W.B.U.T. 2005*)

ILLUSTRATIVE EXAMPLES

Example 1: Test the convergence of the series

$$\frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \ldots \text{ to } \infty.$$

Solution: The nth term $a_n = \dfrac{1\cdot 2\cdot 3\ldots n}{3\cdot 5\cdot 7\ldots(2n+1)}$.

Also
$$a_{n+1} = \frac{1\cdot 2\cdot 3\ldots n(n+1)}{3\cdot 5\cdot 7\ldots(2n+1)(2n+3)}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{n+1}{2n+3} = \frac{1+\dfrac{1}{n}}{2+\dfrac{3}{n}}$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{3}{n}} = \frac{1}{2} < 1. \qquad \left(\because \frac{1}{n}, \frac{3}{n} \to 0 \text{ as } n \to \infty\right)$$

Therefore, by D'Alembert's ratio test the series $\sum a_n$, *i.e.*, the given infinite series is convergent.

Example 2: Test the convergence of the series

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 2}{3\cdot 5}\right)^2 + \left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2 + \ldots \text{ to } \infty.$$

(*W.B.U.T. 2002, 2007, 2010, 2012*)

Solution: The nth term $a_n = \left\{\frac{1\cdot 2\cdot 3\ldots n}{3\cdot 5\cdot 7\ldots(2n+1)}\right\}^2$

Also $\qquad a_{n+1} = \left\{\frac{1\cdot 2\cdot 3\ldots n(n+1)}{3\cdot 5\cdot 7\ldots(2n+1)(2n+3)}\right\}^2$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \left(\frac{n+1}{2n+3}\right)^2 = \left(\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\right)^2$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\right)^2 = \frac{1}{4} < 1 \qquad \left(\because \frac{1}{n}, \frac{3}{n} \to 0 \text{ as } n \to \infty\right)$$

Therefore, by D'Alembert's ratio test the given infinite series is convergent.

Example 3: Discuss the convergence of the series

$$1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots \text{ to } \infty, \ (p > 0).$$

[*W.B.U.T. 2008, W.B.U.T. (B. Arch.) 2013*]

Solution: The nth term $a_n = \frac{n^p}{n!}$, therefore, $a_{n+1} = \frac{(n+1)^p}{(n+1)!}$.

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(n+1)^p}{(n+1)!} \cdot \frac{n!}{n^p} = \frac{1}{(n+1)} \cdot \left(1+\frac{1}{n}\right)^p$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left\{\frac{1}{(n+1)} \cdot \left(1+\frac{1}{n}\right)^p\right\} = 0 < 1. \quad \left(\because \frac{1}{n}, \frac{1}{n+1} \to 0 \text{ as } n \to \infty\right)$$

Therefore, by D' Alembert's ratio test the given series is convergent.

Example 4: Test the convergence of the series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2^{n-1}+1}+\ldots \text{ to } \infty.$$

Solution: The nth term $a_n = \dfrac{1}{2^{n-1}+1}$ and $a_{n+1} = \dfrac{1}{2^n+1}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{2^{n-1}+1}{2^n+1} = \frac{2^{n-1}\left(1+\frac{1}{2^{n-1}}\right)}{2^n\left(1+\frac{1}{2^n}\right)} = \frac{1}{2}\cdot\frac{1+\frac{1}{2^{n-1}}}{1+\frac{1}{2^n}}$$

$$\therefore \quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1}{2}\cdot\frac{1+\frac{1}{2^{n-1}}}{1+\frac{1}{2^n}} = \frac{1}{2} < 1. \quad \left(\because \frac{1}{2^{n-1}}, \frac{1}{2^n} \to 0 \text{ as } n\to\infty\right)$$

Therefore, by D'Alembert's ratio test the given series is convergent.

Example 5: Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty}\frac{n!\,2^n}{n^n}$. *(W.B.U.T 2003)*

Solution: The nth term $a_n = \dfrac{n!\,2^n}{n^n}$ and $a_{n+1} = \dfrac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!\,2^n} = 2\cdot\frac{n^n}{(n+1)^n} = 2\frac{1}{\left(1+\frac{1}{n}\right)^n}.$$

$$\therefore \quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} \quad \left(\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right)$$

We know that $2 < e < 3$, therefore $\dfrac{2}{e} < 1$.

$$\therefore \quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} < 1.$$

Therefore, by D'Alembert's ratio test the given infinite series is convergent.

Example 6: Discuss the convergence of the series

$$\frac{x}{1\cdot 2}+\frac{x^2}{3\cdot 4}+\frac{x^3}{5\cdot 6}+\ldots \text{ to } \infty, (x>0).$$

Solution: The nth term $a_n = \dfrac{x^n}{(2n-1)2n}$ and $a_{n+1} = \dfrac{x^{n+1}}{(2n+1)\,2(n+1)}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(2n+1)\,2(n+1)} \cdot \frac{(2n-1)\,2n}{x^n} = x\frac{\left(2-\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} x\frac{\left(2-\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)} = x \qquad \left(\because\ \frac{1}{n}\to 0 \text{ as } n\to\infty\right)$$

Therefore, by D'Alembert's ratio test the given series is convergent if $x < 1$, divergent if $x > 1$ and this test fails if $x = 1$.

If $x = 1$, the given series becomes $\dfrac{1}{1\cdot 2}+\dfrac{1}{3\cdot 4}+\dfrac{1}{5\cdot 6}+\ldots$ to ∞.

The nth term $u_n = \dfrac{1}{(2n-1)2n}$. Let $v_n = \dfrac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{(2n-1)^{2n}} = \lim_{n\to\infty} \frac{1}{\left(2-\frac{1}{n}\right)2} = \frac{1}{4} \qquad \left(\because\ \frac{1}{n}\to 0 \text{ as } n\to\infty\right).$$

Therefore, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n = \sum \dfrac{1}{n^2}$ is of the form $\sum \dfrac{1}{n^p}$ with $p = 2$, therefore $\sum v_n$ is convergent and hence $\sum u_n$ is also convergent.

Therefore we conclude that the given infinite series is convergent if $x \le 1$ and divergent if $x > 1$.

Example 7: Test the convergence of the series

$$1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\ldots \text{ to } \infty, (x>0)$$

(*W.B.U.T. 2003, 2007*)

Solution: Let us consider the infinite series (omitting the first term of the given series)

$$\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\ldots \text{ to } \infty. \qquad \ldots(1)$$

The nth term $a_n = \dfrac{x^n}{n^2+1}$ and $a_{n+1} = \dfrac{x^{n+1}}{(n+1)^2+1}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)^2+1}\cdot\frac{n^2+1}{x^n} = x\cdot\frac{1+\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}$$

$$\therefore \quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = x \qquad \left(\because\ \frac{1}{n},\frac{1}{n^2}\to 0 \text{ as } n\to\infty\right).$$

Therefore, by D'Alembert's ratio test the series (1) is convergent when $x < 1$, divergent when $x > 1$ and no firm decision is possible when $x = 1$.

When $x = 1$, the series (1) becomes $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ ...(2)

Let $\qquad u_n = \frac{1}{n^2+1}$ and $v_n = \frac{1}{n^2}$.

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}} = 1 \qquad \left(\because\ \frac{1}{n^2}\to 0 \text{ as } n\to\infty\right).$$

Therefore, $\sum u_n$ and $\sum v_n$ converge or diverge together. Since $\sum v_n = \sum \frac{1}{n^2}$ is of the form $\sum\frac{1}{n^p}$ with $p = 2$, therefore $\sum v_n$ is convergent and hence $\sum u_n$ is also convergent.

Thus the given series is convergent when $x \le 1$ and divergent when $x > 1$ (since addition or deletion of finite number of terms does not alter the nature of the series).

Example 8: Examine the convergence or divergence of the series

$$1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\ldots+\frac{2^{n+1}-2}{2^{n+1}+1}x^n+\ldots \text{ to } \infty, (x>0).$$

Solution: Let us consider the infinite series (omitting the first term of the given series).

$$\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\ldots+\frac{2^{n+1}-2}{2^{n+1}+1}x^n+\ldots \text{ to } \infty \qquad \ldots(1)$$

The nth term $a_n = \frac{2^{n+1}-2}{2^{n+1}+1}x^n$ and $a_{n+1} = \frac{2^{n+2}-2}{2^{n+2}+1}x^{n+1}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{2^{n+2}-2}{2^{n+2}+1}\cdot\frac{2^{n+1}+1}{2^{n+1}-2}\cdot x = \frac{\left(1-\frac{2}{2^{n+2}}\right)\cdot\left(1+\frac{1}{2^{n+1}}\right)}{\left(1+\frac{1}{2^{n+2}}\right)\left(1-\frac{2}{2^{n+1}}\right)}\cdot x$$

$$\therefore \qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x \qquad \left(\because \frac{1}{2^{n+1}}, \frac{1}{2^{n+2}} \to 0 \text{ as } n \to \infty \right).$$

Therefore, by D'Alembert's ratio test the series (1) is convergent when $x < 1$, divergent when $x > 1$ and this test fails when $x = 1$.

When $x = 1$, the series (1) becomes $\sum_{n=1}^{\infty} \frac{2^{n+1}-2}{2^{n+1}+1}$ ...(2)

Therefore, nth term $\qquad u_n = \frac{2^{n+1}-2}{2^{n+1}+1} = \frac{1-\frac{2}{2^{n+1}}}{1+\frac{1}{2^{n+1}}}$

$\therefore \quad \lim_{n\to\infty} u_n = 1 \neq 0$ implies (2) is divergent.

Hence the given series is convergent when $x < 1$ and divergent when $x \geq 1$ (since addition or deletion of finite number of terms does not alter the nature of the series).

Example 9: Examine the convergence of the series

$$\frac{1}{1^p} + \frac{x}{3^p} + \frac{x^2}{5^p} + \ldots + \frac{x^{n-1}}{(2n-1)^p} + \ldots \text{ to } \infty, (x > 0).$$

Solution: The nth term $a_n = \frac{x^{n-1}}{(2n-1)^p}$ and $a_{n+1} = \frac{x^n}{(2n+1)^p}$

$$\therefore \qquad \frac{a_{n+1}}{a_n} = \left(\frac{2n-1}{2n+1}\right)^p x = \left(\frac{2-\frac{1}{n}}{2+\frac{1}{n}}\right)^p x$$

$$\therefore \qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x \qquad \left(\because \frac{1}{n} \to 0 \text{ as } n \to \infty \right).$$

Therefore, by D'Alembert's ratio test the given series is convergent when $x < 1$, divergent when $x > 1$ and no firm decision is possible when $x = 1$.

If $x = 1$, the given series becomes

$$\frac{1}{1^p} + \frac{1}{3^p} + \frac{1}{5^p} + \ldots + \frac{1}{(2n-1)^p} + \ldots \text{ to } \infty. \qquad \ldots(1)$$

The nth term $\qquad u_n = \frac{1}{(2n-1)^p}$. Let $v_n = \frac{1}{n^p}$

$$\therefore \quad \frac{u_n}{v_n} = \frac{n^p}{(2n-1)^p} = \frac{1}{\left(2-\frac{1}{n}\right)^p}$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2^p} \qquad \left(\because \frac{1}{n} \to 0 \text{ as } n \to \infty\right).$$

Also $\sum v_n = \sum \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$. Hence by comparison test, the series (1) is convergent if $p > 1$ and divergent if $p \le 1$.

Conclusions: If $x < 1$, the given series is convergent.
If $x > 1$, the given series is divergent.
If $x = 1$ and $p > 1$, the given series is convergent
If $x = 1$ and $p \le 1$, the given series is divergent.

Example 10: Test the convergence of the series

$$\frac{x}{1\cdot 3} + \frac{x^2}{3\cdot 5} + \frac{x^3}{5\cdot 7} + \dots \text{ to } \infty, (x > 0).$$ *(W.B.U.T. 2007)*

Solution: The nth term $a_n = \dfrac{x^n}{(2n-1)(2n+1)}$ and $a_{n+1} = \dfrac{x^{n+1}}{(2n+1)(2n+3)}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(2n-1)(2n+1)}{(2n+1)(2n+3)}\cdot x = \frac{2-\frac{1}{n}}{2+\frac{3}{n}}\cdot x$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x \qquad \left(\because \frac{1}{n}, \frac{3}{n} \to 0 \text{ as } n \to \infty\right)$$

Therefore, by D'Alembert's ratio test the given series is convergent when $x < 1$, divergent when $x > 1$ and this test fails when $x = 1$.

If $x = 1$, the given series becomes $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \dots$ to ∞.

The nth term $\quad u_n = \dfrac{1}{(2n-1)(2n+1)}$. Let $v_n = \dfrac{1}{n^2}$

$$\therefore \quad \frac{u_n}{v_n} = \frac{n^2}{(2n-1)(2n+1)} = \frac{1}{\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{4} \qquad \left(\because \frac{1}{n} \to 0 \text{ as } n \to \infty\right).$$

Therefore, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2$, therefore $\sum v_n$ is convergent and hence $\sum u_n$ is also convergent.

Therefore we conclude that the given infinite series is convergent if $0 < x \le 1$ and divergent if $x > 1$.

Example 11: Discuss the convergency of $\frac{1^2+2}{1^4}x + \frac{2^2+2}{2^4}x^2 + \frac{3^2+2}{3^4}x^3 + \ldots$ to ∞, $(x > 0)$.

(*W.B.U.T.* 2002)

Solution: The nth term $a_n = \frac{n^2+2}{n^4}x^n$ and $a_{n+1} = \frac{(n+1)^2+2}{(n+1)^4}x^{n+1}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(n+1)^2+2}{n^2+2} \cdot \frac{n^4}{(n+1)^4} \cdot x = \left\{ \frac{\left(1+\frac{1}{n}\right)^2 + \frac{2}{n^2}}{1+\frac{2}{n^2}} \right\} \frac{1}{\left(1+\frac{1}{n}\right)^4} \cdot x$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x \quad \left(\because \frac{1}{n}, \frac{2}{n^2} \to 0 \text{ as } n \to \infty \right).$$

Therefore, by D' Alembert's ratio test the given series is convergent when $x < 1$, divergent when $x > 1$ and this test fails when $x = 1$.

If $x = 1$, the given series becomes $\frac{1^2+2}{1^4} + \frac{2^2+2}{2^4} + \frac{3^2+2}{3^4} + \ldots$ to ∞.

The nth term $\quad u_n = \frac{n^2+2}{n^4}$. Let $v_n = \frac{1}{n^2}$.

$$\therefore \quad \frac{u_n}{v_n} = \frac{n^2+2}{n^2} = 1 + \frac{2}{n^2}$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = 1 \quad \left(\because \frac{2}{n^2} \to 0 \text{ as } n \to \infty \right).$$

Therefore, $\sum u_n$ and $\sum v_n$ converge or diverge together. Since $\sum v_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2$, therefore $\sum v_n$ is convergent and hence $\sum u_n$ is also convergent. Therefore, we conclude that the given infinite series is convergent if $x \le 1$ and divergent if $x > 1$.

Example 12: Test for convergence of $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1} x^n ; x > 0$ *(W.B.U.T. 2006)*

Solution: The nth term $a_n = \frac{n^2-1}{n^2+1} x^n, a_{n+1} = \frac{(n+1)^2-1}{(n+1)^2+1} x^{n+1}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \left\{\frac{(n+1)^2-1}{(n+1)^2+1}\right\}\left(\frac{n^2+1}{n^2-1}\right)x = \left\{\frac{\left(1+\frac{1}{n}\right)^2 - \frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2 + \frac{1}{n^2}}\right\}\left(\frac{1+\frac{1}{n^2}}{1-\frac{1}{n^2}}\right)x$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x \qquad \left(\because \frac{1}{n}, \frac{1}{n^2} \to 0 \text{ as } n \to \infty\right)$$

Therefore, by D'Alembert's ratio test the given series is convergent when $x < 1$, divergent when $x > 1$ and this test fails when $x = 1$.

If $x = 1$, the given series becomes $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$. It is a divergent one since

$$u_n = n\text{th term} = \frac{n^2-1}{n^2+1} = \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}} \to 1 \neq 0 \text{ as } n \to \infty.$$

Therefore we conclude that the given infinite series is convergent if $x < 1$ and divergent if $x \geq 1$.

Example 13: Show that the series $\frac{x^2}{2\log 2} + \frac{x^3}{3\log 3} + \frac{x^4}{4\log 4} + \ldots$ to ∞ is convergent when $0 < x < 1$ and divergent when $x > 1$.

Solution: The nth term $a_n = \frac{x^{n+1}}{(n+1)\log(n+1)}$ and $a_{n+1} = \frac{x^{n+2}}{(n+2)\log(n+2)}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = x\left(\frac{n+1}{n+2}\right)\frac{\log(n+1)}{\log(n+2)} = x\left(\frac{n+1}{n+2}\right)\frac{\log n + \log\left(1+\frac{1}{n}\right)}{\log n + \log\left(1+\frac{2}{n}\right)}$$

$$= x\left(\frac{n+1}{n+2}\right)\frac{\log n + \frac{1}{n} - \frac{1}{2}\cdot\frac{1}{n^2} + \ldots}{\log n + \frac{2}{n} - \frac{1}{2}\cdot\frac{4}{n^2} + \ldots} = x\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)\frac{1+\frac{1}{n\log n} - \frac{1}{2}\cdot\frac{1}{n^2\log n} + \ldots}{1+\frac{2}{n\log n} - \frac{1}{2}\cdot\frac{4}{n^2\log n} + \ldots}$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x \qquad \left(\because \ \frac{1}{n}, \frac{1}{n\log n} \to 0 \text{ as } n\to\infty\right)$$

Therefore, by D'Alembert's ratio test the results follow.

Example 14: Test the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$, $(x > 0)$.

Solution: The nth term $a_n = \frac{1}{x^n + x^{-n}}$ and $a_{n+1} = \frac{1}{x^{n+1} + x^{-(n+1)}}$.

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{x^n + x^{-n}}{x^{n+1} + x^{-(n+1)}} = \frac{x^{-n}(x^{2n}+1)}{x^{-(n+1)}(x^{2n+2}+1)} = \left(\frac{x^{2n}+1}{x^{2n+2}+1}\right)x$$

Case I: Let $x < 1$.

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x < 1 \qquad (\because \ x^{2n}, x^{2n+2} \to 0 \text{ as } n\to\infty \text{ if } 0 < x < 1)$$

Therefore, by D'Alembert's ratio test the given series is convergent when $x < 1$.

Case II: Let $x > 1$.

$$\frac{a_{n+1}}{a_n} = \left(\frac{x^{2n}+1}{x^{2n+2}+1}\right)x = \frac{x^{2n}\left(1+\frac{1}{x^{2n}}\right)}{x^{2n+2}\left(1+\frac{1}{x^{2n+2}}\right)}\cdot x$$

$$= \frac{\left(1+\frac{1}{x^{2n}}\right)}{\left(1+\frac{1}{x^{2n+2}}\right)}\cdot\frac{1}{x}$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{x} < 1 \qquad \left(\because \ \frac{1}{x^{2n}}, \frac{1}{x^{2n+2}} \to 0 \text{ as } n\to\infty \text{ if } x > 1\right)$$

Therefore, by D'Alembert's ratio test the given series is convergent when $x > 1$.

Case III: Let $x = 1$.

Here $\quad a_n = \frac{1}{1+1} = \frac{1}{2}$, for $n = 1, 2, 3, \ldots$

$$\therefore \quad \sigma_n = \text{sum of first } n \text{ terms} = \frac{n}{2} \to \infty \text{ as } n\to\infty.$$

Therefore, the given infinite series is divergent when $x = 1$.

Hence the given infinite series is convergent when $x < 1$ or $x > 1$ and divergent when $x = 1$.

Example 15: Test the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{a^n}{x^n + a^n}, (x > 0)$.

Solution: The nth term $u_n = \frac{a^n}{x^n + a^n}$ and $u_{n+1} = \frac{a^{n+1}}{x^{n+1} + a^{n+1}}$.

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{a(x^n + a^n)}{x^{n+1} + a^{n+1}}$$

Case I: Let $x < a$

$$\frac{u_{n+1}}{u_n} = \frac{a(x^n + a^n)}{x^{n+1} + a^{n+1}} = \frac{\left(\frac{x}{a}\right)^n + 1}{\left(\frac{x}{a}\right)^{n+1} + 1}$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1 \qquad \left(\because \left(\frac{x}{a}\right)^n, \left(\frac{x}{a}\right)^{n+1} \to 0 \text{ as } n \to \infty \text{ if } 0 < x < a\right)$$

Therefore D'Alembert's ratio test fails. But we observe that

$$u_n = \frac{a^n}{x^n + a^n} = \frac{1}{\left(\frac{x}{a}\right)^n + 1} \to 1 \text{ as } n \to \infty.$$

Hence the given series is divergent when $x < a$.

Case II: Let $x > a$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{a(x^n + a^n)}{x^{n+1} + a^{n+1}} = \left\{\frac{1 + \left(\frac{a}{x}\right)^n}{1 + \left(\frac{a}{x}\right)^{n+1}}\right\} \frac{a}{x}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{a}{x} < 1. \qquad \left(\because \left(\frac{a}{x}\right)^n, \left(\frac{a}{x}\right)^{n+1} \to 0 \text{ as } n \to \infty \text{ if } x > a\right)$$

Therefore, by D'Alembert's ratio test the given series is convergent when $x > a$.

Case III: Let $x = a$

Here

$$u_n = \frac{a^n}{a^n + a^n} = \frac{1}{2}, \; n = 1, 2, 3, \ldots$$

$$\therefore \quad \sigma_n = \text{sum of first } n \text{ terms} = \frac{n}{2} \to \infty \text{ as } n \to \infty.$$

Therefore, the given infinite series is divergent when $x = a$.

Hence the given infinite series is convergent when $x > a$ and divergent when $0 < x \leq a$.

Example 16: Test the convergence of the infinite series

$$1+\frac{\alpha+1}{\beta+1}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)}+\ldots \text{ to } \infty, (\alpha>0, \beta>0).$$

Solution: Let us consider the following infinite series (omitting the first term of the given infinite series)

$$\frac{\alpha+1}{\beta+1}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)}+\ldots \text{ to } \infty \qquad \ldots(1)$$

The nth term $$a_n = \frac{(\alpha+1)(2\alpha+1)\ldots(n\alpha+1)}{(\beta+1)(2\beta+1)\ldots(n\beta+1)}.$$

and $$a_{n+1} = \frac{(\alpha+1)(2\alpha+1)\ldots(n\alpha+1)\{(n+1)\alpha+1\}}{(\beta+1)(2\beta+1)\ldots(n\beta+1)\{(n+1)\beta+1\}}.$$

$$\therefore \quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)\alpha+1}{(n+1)\beta+1} = \lim_{n\to\infty}\frac{\alpha+\frac{1}{n+1}}{\beta+\frac{1}{n+1}} = \frac{\alpha}{\beta}.$$

Hence by D'Alembert's ratio test, the given infinite series is convergent when $\frac{\alpha}{\beta}<1$, *i.e.*, when $0<\alpha<\beta$ and divergent when $\frac{\alpha}{\beta}>1$, *i.e.*, when $\alpha>\beta>0$, while this test fails when $\frac{\alpha}{\beta}=1$, *i.e.*, $\alpha=\beta$.

when $\alpha=\beta$, $a_n = 1$, $n = 1, 2, 3, \ldots$

$\therefore$ σ_n = sum of first n term $= n \to \infty$ as $n\to\infty$.

Therefore, the given infinite series is divergent when $\alpha=\beta$.

Hence the given infinite series is convergent when $0<\alpha<\beta$ and divergent when $0<\beta\leq\alpha$.

Example 17: Discuss the convergency of $\sum_{n=1}^{\infty} n^4 e^{-n^2}$. (*W.B.U.T. 2005*)

Solution: The nth term $a_n = n^4 e^{-n^2}$ and $a_{n+1} = (n+1)^4 e^{-(n+1)^2}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^4 \frac{e^{-(n+1)^2}}{e^{-n^2}} = \left(1+\frac{1}{n}\right)^4 e^{n^2-(n+1)^2} = \left(1+\frac{1}{n}\right)^4 \frac{1}{e^{2n+1}}$$

$$\therefore \quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 0 < 1, \text{ since } \frac{1}{n}, \frac{1}{e^{2n+1}} \to 0 \text{ as } n\to\infty.$$

Therefore, by D'Alembert's ratio test the given series is convergent.

Example 18: Test the convergence of the series

$$x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\frac{5^5x^5}{5!}+\ldots \text{ to } \infty, \left(x \neq \frac{1}{e}\right).$$

Solution: The nth term $a_n = \dfrac{n^n x^n}{n!}$ and $a_{n+1} \dfrac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$, $n = 1, 2, 3, \ldots$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n x^n} = \frac{(n+1)^{n+1}}{(n+1)n^n}\cdot x = \left(1+\frac{1}{n}\right)^n x$$

$$\therefore \quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = ex \qquad \left(\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right).$$

Therefore, by D'Alembert's ratio test the given series is convergent if $ex < 1$, *i.e.*, $x < \dfrac{1}{e}$ and divergent if $ex > 1$, *i.e.*, $x > \dfrac{1}{e}$.

Example 19: Test the convergence of the series

$$1+\frac{2^2}{3^2}x+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}x^2+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}x^3+\ldots \text{ to } \infty, (x \neq 1)$$

[*W.B.U.T.* 2009, *W.B.U.T.* (*B. Arch.*) 2013]

Solution: Omitting the first term of the given infinite series, let a_n denotes the nth term of the resulting series.

$$\therefore \quad a_n = \frac{2^2\cdot 4^2\cdot 6^2\ldots(2n)^2}{3^2\cdot 5^2\cdot 7^2\ldots(2n+1)^2}x^n \text{ and } a_{n+1} = \frac{2^2\cdot 4^2\cdot 6^2\ldots(2n)^2(2n+2)^2}{3^2\cdot 5^2\cdot 7^2\ldots(2n+1)^2(2x+3)^2}x^{n+1}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \left(\frac{2n+2}{2n+3}\right)^2 x = \left(\frac{2+\frac{2}{n}}{2+\frac{3}{n}}\right)^2 x$$

$$\therefore \quad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = x, \text{ since } \frac{2}{n}, \frac{3}{n} \to 0 \text{ as } n \to \infty.$$

Hence by D'Alembert's ratio test the given infinite series is convergent when $x < 1$ and divergent when $x > 1$ (since addition or deletion of finite number of terms does not alter the nature of an infinite series).

11.14 RAABE'S TEST

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with positive positive terms and let

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = l$$

Then the infinite series $\sum_{n=1}^{\infty} a_n$ is

(*i*) convergent if $l > 1$,

(*ii*) divergent if $l < 1$,

(*iii*) no firm decision is possible if $l = 1$.

Notes: 1. If D'Alembert's ratio test fails, then we use Raabe's test.

2. It can be proved that if there exists $\delta > 0$ such that

$$\lim_{n\to\infty}\left[n^{\delta}\left\{n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right\}\right]$$

exists finitely, then the series $\sum_{n=1}^{\infty} a_n$ is divergent (by **Gauss's test**).

If Raabe's test fails, then we can use this test.

ILLUSTRATIVE EXAMPLES

Example 1: Test the convergence of the infinite series

$$1+\frac{2^2}{3^2}x+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}x^2+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}x^3+\ldots \text{ to } \infty, \quad (x>0)$$

Solution: Omitting the first term of the given infinite series, let an denotes the *n*th term of the resulting series,

$$\therefore \quad \frac{a_{n+1}}{a_n} = \left(\frac{2n+2}{2n+3}\right)^2 x = \left(\frac{2+\frac{2}{n}}{2+\frac{3}{n}}\right)^2 x \quad \ldots(1)$$

(see Examples 36, Art. 11.13)

$$\therefore \qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x, \quad \text{since} \frac{2}{n}, \frac{3}{n} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence by D' Alembert's ratio test the given infinite series is convergent when $x < 1$ and divergent when $x > 1$ (since addition or deletion of finite number of terms does not alter the nature of an infinite series). This test fails when $x = 1$. If $x = 1$, then by (1),

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{4n^2 + 5n}{(2n+2)^2} = \frac{4 + \frac{5}{n}}{\left(2 + \frac{2}{n}\right)^2} \qquad \text{...(2)}$$

$$\therefore \qquad \lim_{n\to\infty} n\left(\frac{a_{n+1}}{a_n} - 1\right) = 1.$$

and Raabe's test also fails.

From (2), we have

$$\lim_{n\to\infty} n\left\{n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right\} = \lim_{n\to\infty} \frac{-3n^2 - 4n}{(2n+2)^2}$$

$$= \lim_{n\to\infty} \frac{-3 - \frac{4}{n}}{\left(2 + \frac{2}{n}\right)^2} = -\frac{3}{4}.$$

Therefore by Gauss's test the given infinite series is found to be divergent for $x = 1$.

Hence, we conclude that the given infinite series is convergent when $0 < x < 1$ and divergent when $x \geq 1$.

Example 2: Test for convergence the infinite series

$$4x + \frac{4\cdot 7}{1\cdot 2}x^2 + \frac{4\cdot 7\cdot 10}{1\cdot 2\cdot 3}x^3 + \ldots \text{ to } \infty, (x > 0). \qquad [\textit{BESUS (B. Arch.) 2013}]$$

Solution: Let a_n be the nth term of the given infinite series.

$$\therefore \qquad a_n = \frac{4\cdot 7\cdot 10\ldots(3n+1)}{1\cdot 2\cdot 3\ldots n}x^n \text{ and } a_{n+1} = \frac{4\cdot 7\cdot 10\ldots(3n+1)(3n+4)}{1\cdot 2\cdot 3\ldots n\cdot(n+1)}x^{n+1},$$

$n = 1, 2, 3, \ldots$

$$\therefore \qquad \frac{a_{n+1}}{a_n} = \left(\frac{3n+4}{n+1}\right)x = \left(\frac{3 + \frac{4}{n}}{1 + \frac{1}{n}}\right)x \qquad \text{...(1)}$$

$$\therefore \qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3x$$

Therefore, by D'Alembert's ratio test the given series is convergent if $3x < 1$, *i.e.*, $x < \frac{1}{3}$ and divergent if $3x > 1$, *i.e.*, $x > \frac{1}{3}$. This test fails if $3x = 1$, *i.e.*, $x = \frac{1}{3}$.

Let us apply the Raabe's test when $3x = 1$, *i.e.*, $x = \frac{1}{3}$.

From (1), we have

$$\frac{a_n}{a_{n+1}} = \frac{3(n+1)}{3n+4} \qquad \left(\because x = \frac{1}{3}\right)$$

$$\therefore \qquad n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{-n}{3n+4} = -\frac{1}{3+\frac{4}{n}}$$

$$\therefore \qquad \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = -\frac{1}{3} < 1.$$

Therefore, by Raabe's test the given series is divergent for $x = \frac{1}{3}$.

Therefore, we conclude that the given infinite series is convergent for $0 < x < \frac{1}{3}$ and divergent for $x \geq \frac{1}{3}$.

Example 3: Discuss the convergence of the hypergeometric series

$$1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2\cdot\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \ldots \text{ to } \infty, (x > 0).$$

Solution: Omitting the first term of the given infinite series, let a_n denotes the nth term of the resulting series.

$$\therefore \qquad a_{n+1} = a_n \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}x, \; n = 1, 2, 3, \ldots \qquad \ldots(1)$$

$$\therefore \qquad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}x = \lim_{n\to\infty}\frac{\left(\frac{\alpha}{n}+1\right)\left(\frac{\beta}{n}+1\right)}{\left(1+\frac{1}{n}\right)\left(\frac{\gamma}{n}+1\right)}$$

$$= x.$$

Hence by D'Alembert's ratio test the given infinite series is convergent when $x < 1$ and divergent when $x > 1$ (since addition or deletion of finite number of terms does not alter the nature of an infinite series). This test fails when $x = 1$.

Let us apply Raabe's test when $x = 1$.

From (1), we have

$$n\left(\frac{a_n}{a_{n+1}}-1\right)=\frac{(1+\gamma-\alpha-\beta)n^2+(\gamma-\alpha\beta)n}{(\alpha+n)(\beta+n)}=\frac{1+\gamma+\alpha-\beta+\frac{1}{n}(\gamma-\alpha\beta)}{\left(\frac{\alpha}{n}+1\right)\left(\frac{\beta}{n}+1\right)}$$

$$(\because x=1) \qquad \text{...(2)}$$

$$\therefore \qquad \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = 1+\gamma-\alpha-\beta.$$

Thus the given infinite series is convergent for $1+\gamma-\alpha-\beta > 1$, *i.e.*, for $\gamma > \alpha+\beta$ and divergent for $1+\gamma-\alpha-\beta < 1$, *i.e.*, for $\gamma < \alpha+\beta$. But it fails when $\gamma = \alpha+\beta$.

From (2), we have

$$\lim_{n\to\infty} n\left\{n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right\}=\lim_{n\to\infty}\frac{-\alpha\beta n(n+1)}{(\alpha+n)(\beta+n)}$$

$$=\lim_{n\to\infty}\frac{-\alpha\beta}{\left(\frac{\alpha}{n}+1\right)\left(\frac{\beta}{n}+1\right)}=-\alpha\beta.$$

Therefore by Gauss's test the given infinite series is found to be divergent for $x = 1$ and $\gamma = \alpha + \beta$.

Hence we conclude that the given infinite series is convergent for $0 < x < 1$ and divergent for $x > 1$. When $x = 1$, this infinite series converges for $\gamma > \alpha + \beta$ and diverges for $\gamma < \alpha + \beta$.

Example 4: Test the convergence of the following infinite series:

$$\frac{1}{3}x+\frac{1\cdot 2}{3\cdot 5}x^2+\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}x^3+\ldots \text{ to } \infty, (x>0).$$

Solution: Let a_n be the nth term of the given infinite series.

$$\therefore \qquad a_n=\frac{1\cdot 2\cdot 3\ldots n}{3\cdot 5\cdot 7\ldots(2n+1)}x^n \text{ and } a_{n+1}=\frac{1\cdot 2\cdot 3\ldots n(n+1)}{3\cdot 5\cdot 7\ldots(2n+1)(2n+3)}x^{n+1}$$

$$\therefore \qquad \frac{a_{n+1}}{a_n}=\left(\frac{n+1}{2n+3}\right)x=\left(\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\right)x \qquad \text{...(1)}$$

$$\therefore \qquad \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{x}{2}.$$

Therefore, by D' Alembert's ratio test the given infinite series is convergent if $\frac{x}{2} < 1$, *i.e.*, $x < 2$ and divergent if $\frac{x}{2} > 1$, *i.e.*, $x > 2$. This test fails if $\frac{x}{2} = 1$, *i.e.*, $x = 2$.

Let us apply the Raabe's test when $x = 2$.

From (1), we have

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{n}{2(n+1)} = \frac{1}{2\left(1+\frac{1}{n}\right)} \qquad (\because x = 2)$$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{1}{2} < 1.$$

Therefore, by Raabe's test the given series is divergent for $x = 2$.

Hence we conclude that the given infinite series is convergent for $0 < x < 2$ and divergent for $x \geq 2$.

Example 5: Discuss the convergence of the following infinite series:

$$1 + \frac{(1!)^2}{2!}x + \frac{(2!)^2}{4!}x^2 + \frac{(3!)^2}{6!}x^3 + \dots \text{ to } \infty, (x > 0).$$

Solution: Let a_n be the nth terms of the given infinite series (omitting the first term).

$$\therefore \qquad a_n = \frac{(n!)^2}{(2n)!}x^n \text{ and } a_{n+1} = \frac{\{(n+1)!\}^2}{\{2(n+1)\}!}x^{n+1}$$

$$\therefore \qquad \frac{a_{n+1}}{a_n} = \frac{\{(n+1)!\}^2}{\{2(n+1)\}!}\cdot\frac{(2n)!}{(n!)^2}x = \frac{(x+1)^2(n!)^2(2n)!}{(2n)!(2n+1)(2n+2)(n!)^2}x$$

$$= \frac{x+1}{2(2n+1)}x = \frac{1+\frac{1}{n}}{2\left(2+\frac{1}{n}\right)}x \qquad \text{...(1)}$$

$$\therefore \qquad \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{x}{4}.$$

Therefore, by D'Alembert's ratio test the given infinite series is convergent if $\frac{x}{4} < 1$, *i.e.*, $x < 4$ and divergent if $\frac{x}{4} > 1$, *i.e.*, $x > 4$. The test fails if $\frac{x}{4} = 1$, *i.e.*, $x = 4$.

Let us now apply the Raabe's test when $x = 4$.

From (1), we have

$$n\left(\frac{a_n}{a_{n+1}}-1\right) = -\frac{n}{2(n+1)} = -\frac{1}{2\left(1+\frac{1}{n}\right)} \qquad (\because x=4)$$

$$\therefore \qquad \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = -\frac{1}{2}<1.$$

Therefore, by Raabe's test the given series is divergent for $x = 4$.

Hence we conclude that the given infinite series is convergent for $0 < x < 4$ and divergent for $x \geq 4$ (since addition or deletion of finite number of terms does not alter the nature of an infinite series).

Example 6: Test the convergence of the following infinite series:

$$\frac{x}{y}+\frac{x(x+1)}{y(y+1)}+\frac{x(x+1)(x+2)}{y(y+1)(y+2)}+\ldots \text{ to } \infty, \quad (x>0, y>0)$$

Solution: The nth term $a_n = \dfrac{x(x+1)\ldots(x+n-1)}{y(y+1)\ldots(y+n-1)}$.

$$\therefore \qquad a_{n+1} = \frac{x(x+1)\ldots(x+n-1)(x+n)}{y(y+1)\ldots(y+n-1)(y+n)}.$$

$$\therefore \qquad \frac{a_{n+1}}{a_n} = \frac{x+n}{y+n} = \frac{\frac{x}{n}+1}{\frac{y}{n}+1} \qquad \ldots(1)$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1$$

Therefore D' Alembert's ratio test fails. Let us not apply Raabe's test.

From (1), we have

$$n\left(\frac{a_n}{a_{n+1}}-1\right) = \frac{n(y-x)}{x+n} = \frac{y-x}{\frac{x}{n}+1}$$

$$\therefore \qquad \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty}\frac{y-x}{\frac{x}{n}+1} = y-x.$$

Hence by Raabe's test the given infinite series is convergent if $y - x > 1$ and divergent if $y-x<1$. The test fails if $y - x = 1$. If $y - x = 1$, *i.e.*, $y = x + 1$, then nth term

$$a_n = \frac{x(x+1)\ldots(x+n-2)(x+n-1)}{y(y+1)\ldots(y+n-2)(y+n-1)} = \frac{x(x+1)\ldots(x+n-2)(x+n-1)}{(x+1)(x+2)\ldots(x+n-1)(x+n)}$$

$$= \frac{x}{x+n}, \text{ and let } b_n = \frac{1}{n}, \quad n = 1, 2, 3, \ldots$$

$$\therefore \quad \frac{a_n}{b_n} = \frac{nx}{x+n} = \frac{x}{\frac{x}{n}+1}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{x}{\frac{x}{n}+1} = x \neq 0.$$

Now, $\sum b_n = \sum \frac{1}{n}$ is of the form $\sum \frac{1}{n^p}$ with $p = 1$.

Therefore, $\sum b_n$ is divergent and hence by comparison test $\sum a_n$, *i.e.*, the given infinite series is also divergent for $y - x = 1$.

Hence we conclude that the given infinite series is convergent when $y - x > 1$ and divergent when $y - x \leq 1$.

11.15 CAUCHY'S ROOT TEST

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with positive terms such that $\lim_{n\to\infty} [a_n]^{1/n}$ exists finitely and let

$$\lim_{n\to\infty} [a_n]^{1/n} = l$$

Then the series is

(*i*) convergent if $l < 1$,

(*ii*) divergent if $l > 1$, and no conclusion can be drawn if $l = 1$, *i.e.*, the series may converge or diverge if $l = 1$.

(*W.B.U.T. 2004*)

ILLUSTRATIVE EXAMPLES

Example 1: Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$.

Solution: The nth term $a_n = \frac{1}{(\log n)^n}$, $n = 2, 3, \ldots$

$$\therefore \quad \lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{1}{\log n} = 0 < 1.$$

Hence by Cauchy's root test the given series is convergent.

Example 2: Show that the infinite series $\sum_{n=1}^{\infty} e^{-\sqrt{n}} x^n$ converges if $0 < x < 1$ and diverges if $x > 1$.

Solution: The nth term $a_n = e^{-\sqrt{n}} x^n$.

$$\therefore \quad \lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} e^{-\frac{1}{\sqrt{n}}} x = x.$$

Therefore by Cauchy's root test the given series converges if $0 < x < 1$ and diverges if $x > 1$.

Example 3: Test the convergence of $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$.

Solution: The nth term $a_n = \left(1+\frac{1}{n}\right)^{-n^2}$.

$$\therefore \quad \lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \frac{1}{e} < 1 \quad \left(\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \text{ and } 2 < e < 3\right)$$

Hence by Cauchy's root test the given infinite series is convergent.

Example 4: Discuss the convergence of the series $\sum_{n=1}^{\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$. *(W.B.U.T. 2004)*

Solution: The nth term $a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

$$\therefore \quad \lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}} = \lim_{m\to\infty} \left(1+\frac{1}{m}\right)^{-m}$$

(put $m = \sqrt{n}$, $\therefore m \to \infty$ as $n \to \infty$)

$$= \frac{1}{\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m} = \frac{1}{e} < 1 \quad \left(\because \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e \text{ and } 2 < e < 3\right).$$

Hence by Cauchy's root test the given infinite series is convergent.

Example 5: Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^3 + a}{2^n + a}$.

Solution: The nth term $a_n = \dfrac{n^3 + a}{2^n + a}$ and let $b_n = \dfrac{n^3}{2^n}$

$$\therefore \quad \frac{a_n}{b_n} = \left(\frac{n^3+a}{2^n+a}\right)\frac{2^n}{n^3} = \left(\frac{n^3+a}{n^3}\right)\left(\frac{2^n}{2^n+a}\right) = \left(1+\frac{a}{n^3}\right)\cdot\frac{1}{\left(1+\frac{a}{2^n}\right)}$$

$$\therefore \quad \lim_{n\to\infty}\frac{a_n}{b_n} = 1 \qquad \left(\because \ \frac{1}{n^3}, \frac{1}{2^n} \to 0 \text{ as } n\to\infty\right).$$

Therefore, by comparison test, the given infinite series is convergent or divergent according as $\sum_{n=1}^{\infty}\frac{n^3}{2^n}$ is convergent or divergent respectively.

$$\text{Now,} \qquad \lim_{n\to\infty} b_n^{1/n} = \lim_{n\to\infty}\frac{n^{3/n}}{2} = \frac{1}{2} < 1 \qquad (\because \ \lim_{n\to\infty} n^{1/n} = 1).$$

Therefore, by Cauchy's root test $\sum b_n$ is convergent and hence $\sum a_n$, *i.e.*,the given infinite series is also convergent.

Example 6: Examine the convergence of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \ldots \text{ to } \infty.$$

(W.B.U.T. 2001, 2003, 2005, 2010)

Solution: The nth term $a_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n}$

$$\therefore \quad a_n^{1/n} = \left[\left(1+\frac{1}{n}\right)\left\{\left(1+\frac{1}{n}\right)^n - 1\right\}\right]^{-1}$$

$$\therefore \quad \lim_{n\to\infty} a_n^{1/n} = \frac{1}{e-1} < 1 \quad \left(\because \ \frac{1}{n}\to 0 \text{ as } n\to\infty, \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \text{ and } 2<e<3\right)$$

Hence by Cauchy's root test the given series is convergent.

Example 7: Test the convergence of the infinite series

$$\left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots \text{ to } \infty, x>0.$$

Solution: The nth term $a_n = \left(\frac{n+1}{n+2}\right)^n x^n$.

$$\therefore \quad a_n^{1/n} = \left(\frac{n+1}{n+2}\right)x = \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)x.$$

$$\therefore \quad \lim_{n\to\infty} a_n^{1/n} = x \qquad \left(\because \frac{1}{n}, \frac{2}{n} \to 0 \text{ as } n \to \infty\right)$$

Therefore by Cauchy's root test $\sum a_n$ is convergent if $x < 1$ and divergent if $x > 1$. If $x = 1$, this test fails.

$$\text{When} \quad x = 1, \quad a_n = \left(\frac{n+1}{n+2}\right)^n = \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^n$$

$$\therefore \quad \lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}{\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n} = \frac{e}{e^2} \qquad \left(\because \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x\right)$$

$$= \frac{1}{e} > 0.$$

Therefore $\sum a_n$ is divergent when $x = 1$.

Hence the given infinite series is convergent when $0 < x < 1$ and divergent when $x \geq 1$.

Example 8: Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{(n+1)^n x^n}{n^{n+1}}$.

Solution: The nth term $a_n = \frac{(n+1)^n x^n}{n^{n+1}}$, $n = 1, 2, 3, \ldots$

$$\therefore \quad a_n^{1/n} = \left\{\frac{(n+1)^n x^n}{n^n} \cdot \frac{1}{n}\right\}^{1/n} = \left(\frac{n+1}{n}\right) x \cdot \frac{1}{n^{1/n}} = \left(1+\frac{1}{n}\right) x \cdot \frac{1}{n^{1/n}}$$

$$\therefore \quad \lim_{n\to\infty} a_n^{1/n} = x \qquad \left(\because \frac{1}{n} \to 0 \text{ as } n \to \infty \text{ and } \lim_{n\to\infty} n^{1/n} = 1\right)$$

Therefore by Cauchy's root test $\sum a_n$ is convergent if $x < 1$ and divergent if $x > 1$. If $x = 1$, this test fails. When $x = 1$, $a_n = \frac{(n+1)^n}{n^{n+1}}$. Let $b_n = \frac{1}{n}$.

$$\therefore \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Therefore, $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge together.

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = 1$, therefore it is divergent and hence $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}$ is also divergent.

Hence we conclude that the given infinite series is convergent if $x < 1$ and divergent if $x \geq 1$.

11.16 ALTERNATING SERIES

So far we have considered positive term series, we proceed to consider series whose terms are alternatively positive and negative. Such series which often occur in many applications are known as Alternating series. The convergence of an alliterating series can be decided by applying the following theorem.

Leibnitz's Theorem

Let $\{a_n\}$ be a sequence such that for $n = 1, 2, 3, \ldots$

(i) $a_n \geq 0$

(ii) $a_{n+1} \leq a_n$

(iii) $\lim_{n\to\infty} a_n = 0.$

Then the alternating series

$$a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n-1} a_n + \ldots$$

is convergent.

ILLUSTRATIVE EXAMPLES

Example 1: Show that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ to ∞ is convergent.

Solution: The given series is $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, where $a_n = \frac{1}{n}$ and it is an alternating series.

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{n}{n+1} < 1, \text{ i.e., } a_{n+1} < a_n, \text{ for } n = 1, 2, 3, \ldots$$

Hence $\{a_n\}$ is a monotonic decreasing sequence.

Also $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, by Leibnitz's theorem the given series is convergent.

Example 2: Test the convergence of the series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \text{ to } \infty.$$ *(W.B.U.T. 2006, 2009, 2011, 2013)*

Solution: The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where $a_n = \frac{1}{n^2}$ and it is an alternating series.

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} = \left(\frac{n}{n+1}\right)^2 < 1, \text{ i.e., } a_{n+1} < a_n, \text{ for } n = 1, 2, 3, \ldots$$

Also
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n^2} = 0.$$

Therefore, by Leibnitz's theorem the given series is convergent.

Example 3: Examine the convergence of the series

$$\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \ldots \text{ to } \infty.$$

Solution: The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where $a_n = \frac{1}{(2n-1)2n}$ and it is an alternating series.

Therefore,
$$a_{n+1} = \frac{1}{(2n+1)(2n+2)}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(2n-1)2n}{(2n+1)2(n+1)} = \frac{(2n-1)}{(2n+1)}\left(\frac{n}{n+1}\right) < 1. \text{ i.e.,}$$

$$a_{n+1} < a_n, n = 1, 2, 3, \ldots$$

Also
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{(2n-1)2n} = 0.$$

Therefore, by Leibnitz's theorem the given series is convergent.

Example 4: Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2+1}$. (*W.B.U.T. 2002*)

Solution: The given series can be written as

$$-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+1} \quad (\because \cos n\pi = (-1)^n) \qquad \ldots(1)$$

Let us consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where $a_n = \frac{1}{n^2+1}$ ...(2)

It is an alternating series.

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{n^2+1}{(n+1)^2+1} < 1, \text{ i.e., } a_{n+1} < a_n, \text{ for } n = 1, 2, 3, \ldots$$

Also $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0.$$

Therefore, by Leibnitz's theorem the series (2) is convergent and hence the series (1), *i.e.*, the given series is also convergent.

Example 5: Examine the convergence of the series

$$\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \ldots \text{ to } \infty.$$

Solution: The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where

$$a_n = \frac{1}{(n+1)^3}(1+2+3+\ldots+n) = \frac{1}{(n+1)^3}\cdot\frac{n(n+1)}{2} = \frac{1}{2}\frac{n}{(n+1)^2}.$$

It is an alternating series.

$$a_{n+1} = \frac{1}{2}\frac{n+1}{(n+2)^2}. \qquad \therefore\ a_{n+1} - a_n = \frac{1}{2}\left\{\frac{n+1}{(n+2)^2} - \frac{n}{(n+1)^2}\right\}$$

$$= \frac{(n+1)^3 - n(n+2)^2}{2(n+2)^2(n+1)^2} = \frac{-(n^2+n-1)}{2(n+2)^2(n+1)^2} < 0, \text{ i.e., } a_{n+1} < a_n, \text{ for } n = 1, 2, 3, \ldots.$$

Also, $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{2}\frac{n}{(n+1)^2} = \lim_{n\to\infty}\frac{1}{2}\frac{\frac{1}{n}}{\left(1+\frac{1}{n}\right)^2} = 0$$

Therefore, by Leibnitz's theorem the given series is convergent.

Example 6: Discuss the convergence of the series

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \ldots \text{ to } \infty, (0 < x < 1).$$

Solution: The given series is an alternating series and it can be written as $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where $a_n = \frac{x^n}{1+x^n}$.

$$\therefore \quad a_{n+1} - a_n = \frac{x^{n+1}}{1+x^{n+1}} - \frac{x^n}{1+x^n} = x^n\left(\frac{x}{1+x^{n+1}} - \frac{1}{1+x^n}\right)$$

$$= -\frac{(1-x)x^n}{(1+x^{n+1})(1+x^n)} < 0, \text{ since } 0 < x < 1.$$

$\therefore\ a_{n+1} < a_n$, for $n = 1, 2, 3, \ldots$

Now, $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{x^n}{1+x^n} = 0 \qquad (\because\ 0<x<1,\ \therefore\ x^n \to 0 \text{ as } n\to\infty)$$

Therefore, by Leibnitz's theorem the given series is convergent.

Example 7: Examine the convergence of the series

$$\frac{1}{1\cdot 2^3} - \frac{1}{2\cdot 3^3}(1^2+2^2) + \frac{1}{3\cdot 4^3}(1^2+2^2+3^2) - \frac{1}{4\cdot 5^3}(1^2+2^2+3^2+4^2) + \ldots \text{ to } \infty.$$

Solution: The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where

$$a_n = \frac{1^2+2^2+3^3+\ldots+n^2}{n(n+1)^3} = \frac{n(n+1)(2n+1)}{6n(n+1)^3} = \frac{2n+1}{6(n+1)^2}.$$

If is an alternating series.

$$a_{n+1} - a_n = \frac{2(n+1)+1}{6(n+2)^2} - \frac{(2n+1)}{6(n+1)^2} = \frac{-(2n^2+4n+1)}{6(n+1)^2(n+2)^2} < 0,$$

i.e., $a_{n+1} < a_n$, for $n = 1, 2, 3, \ldots$

Also, $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{6(n+1)^2} = \lim_{n\to\infty} \frac{\frac{2}{n}+\frac{1}{n^2}}{6\left(1+\frac{1}{n}\right)^2} = 0.$$

Therefore, by Leibnitz's theorem the given series is convergent.

Example 8: Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\sin(2n-1)\frac{\pi}{2}}{n+1}$.

Solution: We know that $\sin(2n-1)\frac{\pi}{2} = (-1)^{n-1}$, $n = 1, 2, 3, \ldots$

Therefore the given series is $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where $a_n = \frac{1}{n+1}$.

It is an alternating series.

$$a_{n+1} - a_n = \frac{1}{n+2} - \frac{1}{n+1} = -\frac{1}{(n+1)(n+2)} < 0, \textit{ i.e.,}$$

$a_{n+1} < a_n$, for $n = 1, 2, 3, \ldots$

Also, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+1} = 0.$

Hence by Leibnitz's theorem the given series is convergent.

Example 9: Test the convergence of the series $2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\ldots$ to ∞

Solution: The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where $a_n = \frac{n+1}{n} = 1+\frac{1}{n}$, which is an alternating series.

$$a_{n+1}-a_n = \left(1+\frac{1}{n+1}\right)-\left(1+\frac{1}{n}\right) = \frac{-1}{(n+1)n} < 0,$$

i.e., $a_{n+1} < a_n$, for $n = 1, 2, 3, \ldots$

Also, $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \neq 0.$$

Therefore, the given series does not converge.

Example 10: Discuss the convergence of the series

$$1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots\text{to }\infty$$

Solution: The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where $a_n = \frac{1}{\sqrt{n}}$, it is an alternating series.

Now, $$\frac{a_{n+1}}{a_n} = \sqrt{\frac{n}{n+1}} < 1, \text{ i.e., } a_{n+1} < a_n, \text{ for } n = 1, 2, 3, \ldots$$

Also, $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0.$$

Therefore, by Leibnitz's theorem the given series is convergent.

11.17 ABSOLUTE CONVERGENCE AND CONDITIONAL CONVERGENCE

Absolutely Convergent Series

Definition: As infinite series $\sum_{n=1}^{\infty} a_n$ of real numbers is said to be absolutely convergent if $\sum_{n=1}^{\infty}|a_n|$ is convergent.

Note: When $\sum a_n$ is an infinite series with positive terms, $\sum a_n$ and $\sum |a_n|$ are the same series and so if $\sum a_n$ is convergent, then it is also absolutely convergent. Therefore for an infinite series with positive terms, the concept of convergence and absolute convergence is same.

Conditionally Convergent Series

Definition: An infinite series $\sum_{n=1}^{\infty} a_n$ of real numbers is said to be conditionally convergent if $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

A conditionally convergent series is also known as non-absolutely convergent or semi convergent series.

Theorem: If the infinite series $\sum_{n=1}^{\infty} a_n$ of real numbers is absolutely convergent, then it is convergent.

Proof: Let $\sum a_n$ be absolutely convergent, so that $\sum |a_n|$ is convergent.

Let
$$u_n = \begin{cases} a_n, & \text{if } a_n \ge 0 \\ 0, & \text{if } a_n < 0 \end{cases} \quad \text{and} \quad v_n = \begin{cases} -a_n, & \text{if } a_n \le 0 \\ 0, & \text{if } a_n > 0 \end{cases}$$

Therefore, $u_n \ge 0, v_n \ge 0$ and $a_n = u_n - v_n$, $n = 1, 2, 3, \ldots$

Now, $u_n \le |a_n|, v_n \le |a_n|$, $n = 1, 2, 3, \ldots$

Since $\sum |a_n|$ is convergent, therefore, by comparison test, both $\sum u_n$ and $\sum v_n$ are convergent and let convergent to l_1 and l_2 respectively.

$$\therefore \quad \sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} (u_n - v_n)$$

$$= \lim_{k\to\infty} \sum_{n=1}^{k} u_n - \lim_{k\to\infty} \sum_{n=1}^{k} v_n = l_1 - l_2$$

Hence $\sum_{n=1}^{\infty} a_n$ is convergent.

ILLUSTRATIVE EXAMPLES

Example 1: Prove that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ to ∞ is a conditionally convergent series.

Solution: The given infinite series is convergent (see Example 1 Art. 11.16).

But $\sum_{n=1}^{\infty} |a_n|$, where $a_n = (-1)^{n-1} \cdot \frac{1}{n}$, $n = 1, 2, 3, \ldots$ is divergent since it is a p-series with $p = 1$.

Hence the given infinite series converges conditionally.

Example 2: Show that for any fixed value of x, the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is absolutely convergent.

Solution: Let the given series be $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{\sin nx}{n^2}$, $n = 1, 2, 3, \ldots$

Now, $$|a_n| = \frac{|\sin nx|}{n^2} \le \frac{1}{n^2}, \quad (\because\ |\sin nx| \le 1, \text{ for all } n \text{ and } x)\ n = 1, 2, 3, \ldots$$

Let us consider $\sum_{n=1}^{\infty} b_n$, where $b_n = \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$

$\therefore$ $$|a_n| \le b_n,\ n = 1, 2, 3, \ldots$$

But $\sum b_n = \sum \frac{1}{n^2}$ is convergent since it is a p-series with $p = 2$.

Hence by comparison test the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Thus the infinite series $\sum_{n=1}^{\infty} a_n$, *i.e.*, the given series is absolutely convergent.

Example 3: Show that the series $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ to ∞ converges absolutely for all values of x and hence deduce that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

Solution: For $x = 0$, the given series evidently converges absolutely.

Let $x \ne 0$. The given series can be written as $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{x^n}{n!}$, $n = 1, 2, 3, \ldots$

$\therefore$ $$a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$\therefore$ $$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1.$$

Therefore, by D'Alembert's ratio test $\sum_{n=1}^{\infty} |a_n|$ is convergent for all x ($\ne 0$).

Hence the given series converges absolutely for all values of x.

Since for a convergent series $\sum_{n=1}^{\infty} u_n$, $\lim_{n\to\infty} u_n = 0$.

Therefore $\lim_{n\to\infty} \frac{|x^n|}{n!} = 0$, *i.e.*, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, for all x.

Example 4: Prove that the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n+1}\frac{x^n}{n} + \ldots$ to ∞ is absolutely convergent when $|x| < 1$ and conditionally convergent when $|x| = 1$. (*W.B.U.T. 2007*)

Solution: The given series can be written as

$$\sum_{n=1}^{\infty} a_n, \text{ where } \quad a_n = (-1)^{n+1}\frac{x^n}{n}.$$

For $x = 0$, the given series evidently converges absolutely.

Let $x \neq 0$. Here $\quad |a_n| = \frac{|x|^n}{n}, |a_{n+1}| = \frac{|x|^{n+1}}{n+1}$, $n = 1, 2, 3, \ldots$

$$\therefore \quad \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n}{n+1}|x| = \lim_{n\to\infty} \frac{|x|}{1+\frac{1}{n}} = |x|.$$

By D'Alembert's ratio test the series $\sum_{n=1}^{\infty} |a_n|$ is convergent if $|x| < 1$, divergent if $|x| > 1$ and the test fails when $|x| = 1$. Hence the given series is absolutely convergent when $|x| < 1$.

If $x = 1$, then the give series becomes $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ which is convergent by Leibnitz's theorem (see Example 1, Art. 11.16). If $x = -1$, then the given series reduces to $-\sum_{n=1}^{\infty}\frac{1}{n}$ which is divergent one since $\sum_{n=1}^{\infty}\frac{1}{n}$ is a p-series with $p = 1$.

Note: The given infinite series is known as the logarithmic series.

Example 5: Show that the binomial series

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \text{ to } \infty$$

is absolutely convergent when $|x| < 1$.

Solution: For $x = 0$, the given series evidently converges absolutely. For $x \neq 0$, omitting the first term of the given infinite series, let a_r denotes the r^{th} term of the resulting series.

Then $$|a_r| = \frac{n(n-1)\ldots(n-r+1)}{r!}|x|^r$$

and $$|a_{r+1}| = \frac{n(n-1)\ldots(n-r+1)(n-r)}{(r+1)!}|x|^{r+1}, \; r = 1, 2, 3, \ldots$$

$$\therefore \quad \frac{|a_{r+1}|}{|a_r|} = \left|\frac{n-r}{r+1}\right||x| = \frac{\left|\frac{n}{r}-1\right|}{1+\frac{1}{r}}|x|$$

$$\therefore \quad \lim_{r\to\infty}\frac{|a_{r+1}|}{|a_r|} = |x|$$

By D'Alembert's ratio test the series $\sum_{r=1}^{\infty}|a_n|$ is convergent when $|x|<1$.

Hence the given series is absolutely convergent when $|x|<1$ (since addition or deletion of finite number of terms does not alter the nature of the series).

Example 6: Prove that the series $2\sin\frac{x}{3}+4\sin\frac{x}{9}+8\sin\frac{x}{27}+\ldots$ to ∞ converges absolutely for all values of x.

Solution: For $x = 0$, the given series evidently converges absolutely. For $x \neq 0$, let the given series is denoted by $\sum_{n=1}^{\infty}a_n$, then we have $a_n = 2^n\sin(x/3^n)$ and $a_{n+1} = 2^{n+1}\sin(x/3^{n+1})$, $n = 1, 2, 3, \ldots$

$$\therefore \quad \frac{|a_{n+1}|}{|a_n|} = 2\frac{|\sin(x/3^{n+1})|}{|\sin(x/3^n)|} = \frac{2}{3}\left|\frac{\sin(x/3^{n+1})}{x/3^{n+1}}\right|\left|\frac{x/3^n}{\sin(x/3^n)}\right|.$$

$$\therefore \quad \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \frac{2}{3}\left(\because \lim_{n\to\infty}\frac{\sin(x/3^n)}{x/3^n} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1, \theta = \frac{x}{3^n}, \text{ for all values of } x(\neq 0)\right)$$
$$< 1.$$

Therefore, by D'Alembert's ratio test the series $\sum_{n=1}^{\infty}|a_n|$ is convergent.

Hence the given series converges absolutely for all values of x.

MULTIPLE CHOICE QUESTIONS

1. Which of the following sequence is monotonic increasing ?
   (a) $\{2, 5, 7, 1, 8, 9, 3, \ldots\}$ (b) $\{2, -2, 2, -2, \ldots\}$
   (c) $\{2, 4, 8, 16, 32, 64, 128, \ldots\}$ (d) $\{\sin k\}_{k=1}^{\infty}$.

2. The sequence $\left\{\frac{1}{2^n}\right\}$ is
   (a) monotonic increasing (b) divergent
   (c) monotonic decreasing (d) oscillatory.

3. Which of the following is not true for the sequence $\left\{\frac{1}{n^2}\right\}$?
   (a) Oscillatory (b) monotonic decreasing
   (c) bounded (d) convergent.

4. The sequence $\left\{1+\left(-\frac{1}{3}\right)^n\right\}$ is
   (a) divergent (b) convergent and converges to $\frac{1}{3}$
   (c) convergent and converges to 1 (d) oscillatory.

5. Bounds of the sequence $\left\{\left(-\frac{1}{2}\right)^n\right\}$ are
   (a) $-\frac{1}{2}$ and $\frac{1}{4}$ (b) $-\frac{1}{2}$ and 0
   (c) 0 and $\frac{1}{4}$ (d) none of these.

6. The sequence $\{(-1)^n\}$ is
   (a) convergent (b) oscillatory
   (c) divergent (d) none of these. (*W.B.U.T. 2006*)

7. Which of the following sequence is convergent?
   (a) $\{2^n\}$ (b) $\{(-2^n)\}$
   (c) $\left\{\left(-\frac{1}{2}\right)^n\right\}$ (d) $\{1, -1, 1, -1, \ldots\}$

8. Which of the following sequence is divergent?

(a) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \dots\right\}$ (b) $\left\{\left(1+\frac{1}{n}\right)^n\right\}$

(c) $\left\{\frac{1}{n}\sin n\pi\right\}$ (d) $\{2^n\}$

9. Indicate the correct one from the following statements:

(a) A bounded monotonic sequence is convergent

(b) A bounded monotonic sequence is divergent

(c) A convergent sequence may not be bounded

(d) The monotonic sequence is convergent.

10. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if

(a) $p = 1$ (b) $p < 1$ (c) $p \le 1$ (d) $p > 1$.

(W.B.U.T. 2012)

11. For $x = -1$, the geometric series $1 + x + x^2 + x^3 + \dots$ to ∞

(a) converges (b) oscillates finitely

(c) oscillates infinitely (d) diverges to $+\infty$.

12. The series $\sin x + \sin^2 x + \sin^3 x + \sin^4 x + \dots$ to ∞, $0 < \pi < \frac{\pi}{2}$

(a) converges (b) diverges to $+\infty$

(c) oscillates finitely (d) oscillates infinitely.

13. The series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \dots$ to ∞ is

(a) convergent (b) divergent and diverges to $+\infty$

(c) oscillatory (d) none of these.

14. The series $\frac{1}{\sqrt[3]{1}} + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \dots$ to ∞ is

(a) convergent (b) divergent and diverges to $+\infty$

(c) oscillatory (d) none of these.

15. The series $\frac{1}{\sqrt[5]{1^3}} + \frac{1}{\sqrt[5]{2^3}} + \frac{1}{\sqrt[5]{3^3}} + \frac{1}{\sqrt[5]{4^3}} + \dots$ to ∞ is

(a) convergent (b) divergent and diverges to $+\infty$

(c) oscillatory (d) none of these.

16. If for the series $\sum_{n=1}^{\infty} a_n, a_n > 0$, $n = 1, 2, 3, \dots$ and $\lim_{n\to\infty} a_n = 0$, then the series is

(a) convergent (b) divergent and diverges to $+\infty$

(c) oscillatory (d) nothing can be said.

17. If for the series $\sum_{n=1}^{\infty} a_n$ and $\lim_{n\to\infty} a_n \neq 0$, then the series is

(a) convergent (b) divergent
(c) nothing can be said (d) none of these.

18. Which of the following series is convergent?

(a) $\sum_{n=1}^{\infty} 2^n$ (b) $\sum_{n=1}^{\infty} (-2)^n$
(c) $\sum_{n=1}^{\infty} 2$ (d) $\sum_{n=1}^{\infty} 2^{-n}$.

19. The series $\sum_{n=1}^{\infty} (-2)^n$

(a) converges (b) oscillates finitely
(c) oscillates infinitely (d) none of these.

20. The series $1 + r + r^2 + r^3 + r^4 + \ldots$ to ∞ is convergent if

(a) $-1 < r < 1$ (b) $r \geq 1$
(c) $r \leq -1$ (d) $r = 1$.

21. If $\sum_{n=1}^{\infty} u_n$ is convergent then $\sum_{n=1}^{k} a_n + \sum_{n=1}^{\infty} u_n$ is

(a) divergent (b) oscillatory
(c) convergent (d) nothing can be said.

22. The series $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ is *(W.B.U.T. 2007)*

(a) convergent (b) divergent
(c) neither convergent nor divergent (d) none of these.

23. Let $S = \sum_{k=1}^{\infty} a_k$ and $S_n = \sum_{k=1}^{n} a_k$, then S will be convergent if

(a) $\{S_n\}$ is convergent (b) $\{S_n\}$ is monotone increasing but not bounded above
(c) $\{S_n\}$ is divergent (d) none of these.

24. The series $\sum_{n=1}^{\infty} \frac{1}{n+1}$ is

(a) convergent (b) divergent
(c) oscillatory (d) none of these.

25. If $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is also convergent, then $\sum_{n=1}^{\infty} (a_n + b_n)$ is

(a) convergent (b) divergent
(c) conditionally convergent (d) nothing can be said.

26. If $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is also convergent, then $\sum_{n=1}^{\infty} (a_n - b_n)$ is

(*a*) convergent (*b*) divergent
(*c*) conditionally convergent (*d*) nothing can be said.

27. The series $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ is

(*a*) convergent (*b*) divergent
(*c*) oscillatory (*d*) none of these.

28. $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of positive terms and $\sum_{n=1}^{\infty} b_n$ is divergent. Also $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, then $\sum_{n=1}^{\infty} a_n$ is

(*a*) convergent (*b*) oscillatory
(*c*) divergent (*d*) none of these.

29. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ to ∞ is

(*a*) oscillatory (*b*) absolutely convergent
(*c*) conditionally convergent (*d*) none of these.

30. The series $1 - 1/2 + 1/4 - 1/8 + \dots$ to ∞ is

(*a*) oscillatory (*b*) absolutely convergent
(*c*) conditionally convergent (*d*) none of these.

31. On alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$

(*a*) Cauchy's root test is applied (*b*) D'Alembert's ratio test is applied
(*c*) Comparison test is applied (*d*) Leibnitz's test is applied.

32. The sequence $\left\{\frac{1}{3^n}\right\}$ is

(*a*) monotonic increasing (*b*) oscillatory
(*c*) divergent (*d*) monotonic decreasing. (*W.B.U.T. 2008*)

33. The series $\sum \frac{1}{n^p}$ is convergent if

(*a*) $p \geq 1$ (*b*) $p > 1$
(*c*) $p < 1$ (*d*) $p \leq 1$. (*W.B.U.T. 2008, 2010*)

34. The series $\sum_{n=1}^{\infty} \frac{2}{e^n}$ is

(a) convergent (b) divergent

(c) oscillatory (d) none of these. (W.B.U.T. 2009)

ANSWERS

1. (c)	2. (c)	3. (a)	4. (c)	5. (a)
6. (b)	7. (c)	8. (d)	9. (a)	10. (d)
11. (b)	12. (a)	13. (b)	14. (b)	15. (b)
16. (d)	17. (b)	18. (d)	19. (c)	20. (a)
21. (c)	22. (b)	23. (a)	24. (b)	25. (a)
26. (a)	27. (b)	28. (c)	29. (c)	30. (c)
31. (d)	32. (d)	33. (b)	34. (a).	

PROBLEMS

1. Define convergent series. If $\sum_{n=1}^{\infty} a_n$ is a convergent series then prove that $\lim_{n\to\infty} a_n = 0$. Is the converse true? Justify your answer with an example. (BESUS 2013)

2. Prove that the series $\sum_{n=1}^{\infty} x^n$ is convergent if $-1 < x < 1$.

3. State comparison test for the convergence of an infinite series and use it to show that $\sum_{n=1}^{\infty} \frac{2n+1}{n^3}$ is convergent and $\sum_{n=1}^{\infty} \frac{2n+1}{n^2}$ is divergent.

4. State Cauchy's root test for the convergence of an infinite series. Use it to show that the series $\sum_{n=1}^{\infty} \frac{(1+nx)^n}{n^n}$ is convergent if $x < 1$ and divergent if $x \geq 1$.

5. State D'Alembert's ratio test for the convergence of an infinite series. Use it to show that the series $1+3x+5x^2+7x^3+\ldots$ to ∞ is convergent if $x < 1$ and divergent if $x \geq 1$.

6. State Leibnitz's test for the convergence of an alternating series. Use it to show that the series $\sum_{n=2}^{\infty} \frac{(-1)^{n-1} x^n}{n(n-1)}$, $0 < x < 1$, is convergent.

7. Define absolute convergent series and conditionally convergent series. Give an example of a convergent series which is not absolutely convergent with justification. Show that the infinite

series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2+1}$ is conditionally convergent.

8. Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos nx}{n^3}$ is absolutely convergent.

Test the following series for convergence or divergence:

9. $\frac{1^2}{2^2} + \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} x + \frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \frac{5^2}{6^2} x^2 + \ldots$ to ∞, $x > 1$ (W.B.U.T. 2001)

10. $1 + \frac{\sqrt{2}-1}{1!} + \frac{(\sqrt{2}-1)^2}{2!} + \frac{(\sqrt{2}-1)^3}{3!} + \ldots$ to ∞ (W.B.U.T. 2001)

11. $\frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 7} + \frac{1 \cdot 3 \cdot 5}{4 \cdot 7 \cdot 10} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 7 \cdot 10 \cdot 13} + \ldots$ to ∞

12. $\sum_{n=1}^{\infty} \frac{1}{n^p (n+1)^p}$

13. $\frac{4}{5} + \frac{4 \cdot 7}{5 \cdot 8} + \frac{4 \cdot 7 \cdot 10}{5 \cdot 8 \cdot 11} + \ldots$ to ∞.

14. $\sum \frac{1}{a^n + a^{-n}}, (a > 0)$.

15. $x + \frac{2^2 \cdot x^2}{2!} + \frac{3^3 \cdot x^3}{3!} + \frac{4^4 x^4}{4!} + \ldots$ to ∞, $x > 0$.

16. $\sum_{n=1}^{\infty} \frac{1}{a+nb}, a > 0, b > 0$

17. $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$

18. $\sum_{n=1}^{\infty} \frac{1^2 + 2^2 + \ldots + n^2}{n^4 + 1}$

19. $\sum_{n=1}^{\infty} \frac{n^p}{\sqrt{n+1} + \sqrt{n}}$

20. $\frac{2^p}{1^q} + \frac{3^p}{2^q} + \frac{4^p}{3^q} + \ldots$ to ∞, where p and q are positive numbers.

21. $\frac{2^3}{1^k + 3^k} + \frac{3^3}{2^k + 5^k} + \frac{4^3}{3^k + 7^k} + \ldots$ to ∞.

22. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

23. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

24. $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$

25. $\sum_{n=1}^{\infty} \frac{x^3 + 1}{2^n + 1}$

26. $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}} x^n, x > 0$.

27. $1+\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\ldots$ to ∞, $x>0$

28. $\sum_{n=1}^{\infty}\frac{x^2+1}{5^n}$

29. $\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\ldots$ to ∞.

30. $\sum_{n=1}^{\infty}\frac{x^n}{n}$

31. $\frac{1}{1+x}+\frac{1}{1+2x^2}+\frac{1}{1+3x^3}+\ldots$ to ∞, $x>0$.

32. $\frac{1}{3}x+\frac{1\cdot 2}{3\cdot 5}x^2+\frac{1\cdot 2.3}{3.5.7}x^3+\ldots$ to ∞, $0<x<2$.

33. $\sum_{n=1}^{\infty}\frac{x^n}{n^n}, x>0$

34. $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\ldots$ to ∞, $x\geq 0$.

35. $\sum_{n=1}^{\infty}3^{-n-(-1)^n}$

36. $\sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}\right)^n x^n, x>0$

37. $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$

38. $\sum_{n=1}^{\infty}\frac{(1+nx)^n}{n^n}, x>0.$

39. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\log(n+1)}.$

40. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{3n-2}.$

41. $\left(1+\frac{1}{2}\right)-\left(1+\frac{1}{4}\right)+\left(1+\frac{1}{8}\right)-\left(1+\frac{1}{16}\right)+\ldots$ to ∞

42. $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+1}-\sqrt{n})$

43. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}, p>0.$

44. $1+\frac{2}{1^2+1}x+\frac{4}{2^2+1}x^2+\frac{6}{3^2+1}x^3+\ldots$ to ∞, $(x>0)$.

45. $\frac{x}{1}+\frac{1\cdot x^2}{2\cdot 3}+\frac{1\cdot 3\cdot x^3}{2\cdot 4\cdot 5}+\ldots$ to ∞, $(x>0)$.

46. $\frac{1^2}{4^2}+\frac{1^2\cdot 5^2}{4^2\cdot 8^2}+\frac{1^2\cdot 5^2\cdot 9^2}{4^2\cdot 8^2\cdot 12^2}+\ldots$ to ∞.

47. $1+\frac{2^2}{3^2}+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}+\ldots$ to ∞.

48. $\sum_{n=1}^{\infty}\frac{2^2\cdot 4^2\cdot 6^2\ldots(2n-2)^2}{3\cdot 4\cdot 5\ldots(2n-1)}x^{2n}, (x>0).$

49. $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\ldots+\frac{x^n}{x^2+1}+\ldots$ to ∞, $(x>0)$.

50. Examine the convergence and absolute convergence of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^2+1}$.

51. Show that the infinite series $\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$ to ∞ is conditionally convergent.

52. Show the given exponential series $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ to ∞ converges absolutely for all values of x.

ANSWERS TO PROBLEMS

9. divergent
10. convergent
11. convergent
12. convergent if $p>\frac{1}{2}$.
13. divergent
14. convergent if $a>1$ or $0<a<1$ and divergent if $a=1$.
15. convergent if $0<x<\frac{1}{e}$; divergent if $x\geq\frac{1}{e}$.
16. divergent
17. divergent
18. divergent
19. convergent if $p<-\frac{1}{2}$.
20. convergent if $q>p+1$ and divergent if $q\leq p+1$.
21. convergent if $k>4$ and divergent if $k\leq 4$.
22. divergent
23. convergent
24. divergent
25. convergent
26. convergent if $x<1$ and divergent if $x\geq 1$.
27. convergent if $x<1$ and divergent if $x\geq 1$.
28. convergent
29. convergent
30. convergent if $x<1$ and divergent if $x\geq 1$.
31. convergent if $x>1$ and divergent if $x\leq 1$.
32. convergent if $x<2$ and divergent if $x\geq 2$.
33. convergent

34. convergent
35. convergent
36. convergent if $x < 1$ and divergent if $x \geq 1$.
37. convergent
38. convergent if $x < 1$ and divergent if $x \geq 1$.
39. convergent
40. oscillatory
41. oscillatory
42. convergent
43. convergent
44. convergent if $x < 1$ and divergent if $x \geq 1$.
45. convergent if $x^2 \leq 1$
46. convergent
47. divergent
48. convergent if $x \leq 1$
49. convergent if $x \leq 1$.
50. conditionally convergent.