Cardinality of sets

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Prob. $S = \{0, 2, 4, 6, \ldots\ldots\}$

We need to find a bijective function $f: Z^+ \to S$.

$Z^+ = \{1, 2, 3, 4, \ldots\}$

We define a function $f: x \to (2x-2)$

$1 \to 2.(1) - 2 = 0$

$2 \to 2.(2) - 2 = 4 - 2 = 2$

$3 \to 2.(3) - 2 = 6 - 2 = 4$

$4 \to 2.(4) - 2 = 8 - 2 = 6.$

One-to-one : $2x - 2 = 2y - 2 \Rightarrow 2x = 2y \Rightarrow x = y$

Onto : ~~$\forall s \in S$~~ $\forall s \in S, \left(\frac{s+2}{2}\right)$ is the preimage in Z^+.

Therefore, $|S| = |Z^+|$. Hence, the set S is Countable.

Prob. $Z = \{-\infty, \ldots\ldots 0, \ldots\ldots +\infty\}$

We define a bijection $f: Z^+ \to Z$.

... n is even.

We define a bijection f:

$f(n) = n/2$, when n is even.

$f(n) = \cancel{(n+1)} - \left(\frac{n-1}{2}\right)$ when n is odd.

$|Z| = |Z^+|$. $\Rightarrow$ the set of integers is countable.

Prob: We define a function $f: Z^+ \to Q^+$ by starting at $1/1$ and following the arrows as directed, skipping over any number that has already been counted.

Set $F(1) = 1/1$, $F(2) = 1/2$, $F(3) = 2/1$, $F(4) = 3/1$. Then we skip $2/2$ as $2/2 = 1$ is counted first.

After that set $F(5) = 1/3$, $F(6) = 1/4$, $F(7) = 2/3$, $F(8) = 3/2$, $F(9) = 4/1$ and $F(10) = 5/1$. We skip $4/2$, $3/3$, and $2/4$ and $F(11) = 1/5$.

Because all positive rational numbers are listed once, the set of positive rational numbers is countable.

Prob: $r_1 = 0.d_{11} d_{12} d_{13} \cdots$

Proof: $r_1 = 0.d_{11}d_{12}d_{13}\ldots\ldots$

$r_2 = 0.d_{21}d_{22}d_{23}\ldots\ldots$

$r_3 = 0.d_{31}d_{32}d_{33}\ldots\ldots$

$$\left.\begin{array}{l} 0.\,2\,0\,1\,4\,8\,8\,0\,2\ldots\ldots \\ 0.\,1\,1\,6\,6\,6\,0\,2\ldots\ldots \\ 0.\,0\,3\,3\,5\,3\,3\,2\ldots\ldots \\ 0.\,9\,6\,7\,7\,6\,8\,0\ldots\ldots \\ 0.\,0\,0\,0\,3\,1\,0\,0\ldots\ldots \end{array}\right\}$$

Construct a diagonal along all $d_{i,i}$'s.

We want to form a new number, say s

$s = 0.d_1d_2d_3d_4\ldots\ldots$ $\quad$ $s = 0.322\ldots\ldots$

$$d_i = \begin{cases} 2, & \text{if } d_{ii} \neq 2 \\ 3, & \text{if } d_{ii} = 2 \end{cases}$$

Since, $s \in \mathbb{R}^*$, $\exists t$ such that $s = r_t$.

Therefore,

$0.d_1d_2d_3d_4\ldots\ldots d_t = 0.d_{t1}d_{t2}d_{t3}d_{t4}\ldots d_{tt}\ldots\ldots$

$\Rightarrow \quad d_t = d_{tt}$

However, by construction d_t and d_{tt} cannot be equal.

$\therefore d_t \neq d_{tt} \Rightarrow r_t \neq s$.

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Every real number has a unique decimal expansion. therefore, the real number s is not equal to any of $r_1, r_2, r_3 \ldots$ because the decimal expansion of s is different from that of r_i in the i^{th} place to the right of the decimal point, for each i.

But $s \in \mathbb{R}^*$ and it is not in the list. Hence, this is a contradiction to our initial supposition that $\mathbb{R}^*$ is countable.

$\therefore \mathbb{R}^*$ is not countable (listable).

Likewise, we conclude $\mathbb{R}$ is also not countable

Theorem

If A and B are countable then $A \cup B$ is also countable.

→ Without loss of generality we assume that A and B are disjoint. There are three cases to consider: (i) A & B are both finite; (ii) A is infinite, B is finite; (iii) A & B are

(ii) A is infinite, B is finite, (iii) A and B are both infinite.

Case (i): Note when A and B are finite, $A \cup B$ is also finite and therefore countable.

Case (ii): Because A is countably infinite, so we can list the elements as $a_1, a_2, a_3 \ldots$ and B is finite, its terms can also be listed as $b_1, b_2, b_3, \ldots b_m$ for some positive integer m. We can list the elements of $A \cup B$ as $b_1, b_2, b_3, \ldots b_m, a_1, a_2, a_3, \ldots$. This implies that $A \cup B$ is countable.

Case (iii) Because both A and B are countably infinite, we can list their elements as $a_1, a_2, a_3 \ldots$ and $b_1, b_2, b_3 \ldots$, respectively. By alternating the terms of these two sets, we can list the elements of $A \cup B$ in the infinite sequence $a_1, b_1, a_2, b_2, \ldots a_n, b_n, \ldots$. This means $A \cup B$ is also countable.

Cantor's Theorem

Proof: Let $f: A \to P(A)$ be a map. For any set A, the map $f: A \to P(A) = \{a\}$, defined for all $a \in A$. This gives a well-defined injection from $A \to P(A)$.

Put $B = \{x \in A \mid x \notin f(x)\}$ i.e. B is a subset of elements from A whose images did not include themselves. Therefore, B is a well-defined subset of A. Any $x \in A$ is either $x \in f(x)$ or $x \notin f(x)$.
If $x \notin f(x)$ then it must be $x \in B$ as $f(x)$ cannot be equal to B by construction.

Therefore, we suppose there exists an element $y \in A$, such that $f(y) = B$.
This implies the following contradiction:

$$y \in f(y) \leftrightarrow y \in B \quad \text{(by assumption that } f(y) = B\text{)}$$

$$y \in B \leftrightarrow y \notin f(y). \quad \text{(by defn of } B\text{)}$$

Therefore, ~~it~~ this a contradiction to our supposition that there exists $y \in A$,

Therefore, this a contradiction
Supposition that there exists $y \in A$, such that $f(y) = B$. In other words, B is not in the image of f and f does not map to every element of $P(A)$ i.e. f is not surjective (onto).

$A = \{a, b, c\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$

One-to-one mapping

$|A| < |P(A)|$

Prob: Show that $|(0,1)| = |(0,1]|$.

→ Because $(0,1) \subset (0,1]$, $f(x) = x$ is a one-to-one function from $(0,1)$ to $(0,1]$.

Again, the function $g(x) = \frac{x}{2}$ is a one-to-one function that maps $(0,1]$ to $(0, 1/2] \subset (0,1)$. As

$(0,1]$ to $(0, 1/2] \subset (0,1)$. As we have found one-to-one functions from $(0,1)$ to $(0,1]$ and $(0,1]$ to $(0,1)$, the Schröder-Bernstein theorem states that $|(0,1)| = |(0,1]|$.