(44) Evaluation of improper integrals of some particular form by contour integration

The integral $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R_1 \to \infty} \int_{-R_1}^{0} f(x)\,dx + \lim_{R_2 \to \infty} \int_{0}^{R_2} f(x)\,dx$,

where both the integrals in the right hand side exist, said to be exist and convergent.

However the value $\lim_{R_1 \to \infty} \left[\int_{-R_1}^{0} f(x)\,dx + \int_{0}^{R_1} f(x)\,dx \right]$

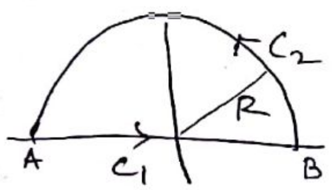
is called Cauchy's principal value.

If the function is even function and the Cauchy's principal value exist, then the integral is convergent and its value is Cauchy's principal value and in this case $\int_{0}^{\infty} f(x)\,dx = \frac{1}{2} \int_{-\infty}^{\infty} f(x)\,dx$.

Here our aim is to ~~evaluate~~ determine the Cauchy's principal value of the improper integrals of the form (i) $\int_{-\infty}^{\infty} f(x)\,dx$ and (ii) $\int_{-\infty}^{\infty} f(x) \cos mx \,(\text{or } \sin mx)\,dx$,

where $f(x)$ is of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials of degree m and n respectively.

To evaluate such type of integral we consider the positively oriented contour $C = C_1 + C_2$, where C_1 is $\{z : z = x,\ x \text{ varies from } -R \text{ to } R\}$ and C_2 is $\{z : z = Re^{i\theta},\ \theta \text{ varies from } 0 \text{ to } \pi\}$ as shown in the figure



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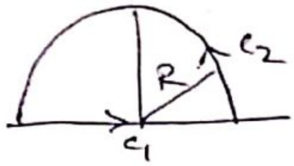
Here we shall consider result of two theorems:

(i) **Th I** If $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ & $q(x)$ are polinomials of degree m and n respectively, ~~then~~ and $n \geq m+2$, then $\underset{R\to\infty}{Lt} \int_{C_2} f(z)\,dz = 0$



(ii) **Th II** (Jordan's lemma)

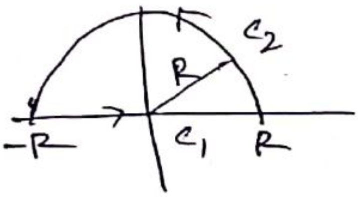
If $f(z)$ is analytic except a finite number of singularities and $f(z) \to 0$ uniformly as $z \to \infty$, then $\underset{R\to\infty}{Lt} \int_{C_2} e^{imz} f(z)\,dz = 0$, $m>0$.

EX1 Evaluate the integral

(i) $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+2)(x^2+6)}\,dx$ (ii) $\int_{0}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx$.

soln: (i) As $f(x) = \frac{x^2}{(x^2+2)(x^2+6)}$ is even function its value is equal to its principal value.

Let $f(z) = \frac{z^2}{(z^2+2)(z^2+6)}$ and let $C = C_1 + C_2$ be the positively oriented closed contour as shown in the figure, when R is large



Then $\int_C f(z)\,dz = \int_C \frac{z^2}{(z^2+2)(z^2+6)}\,dz$

Now $z = \pm\sqrt{2}i$ and $z = \pm\sqrt{6}i$ are the simple poles of $f(z)$ out of which $z = +\sqrt{2}i$ and $+\sqrt{6}i$ lies within C.

$$\operatorname{Res} f(z)\Big|_{z=\sqrt{2}i} = \frac{z^2}{(z+\sqrt{2}i)(z^2+6)}\Bigg|_{z=\sqrt{2}i} = \frac{-\cancel{2}}{\cancel{2}\sqrt{2}\,i(-2+6)} = \frac{-1}{4\sqrt{2}i}$$

(4b)

$$\operatorname{Res} f(z)\Big|_{z=\sqrt{6}i} = \frac{z^2}{(z^2+2)(z+\sqrt{6}i)}\Bigg|_{z=\sqrt{6}i} = \frac{-6}{(-6+2)2\sqrt{6}i} = \frac{3}{4\sqrt{6}i}$$

$$\therefore \int_C f(z)\,dz = 2\pi i\left[\operatorname{Res} f(z)\Big|_{z=\sqrt{2}i} + \operatorname{Res} f(z)\Big|_{z=\sqrt{6}i}\right]$$

$$= 2\pi i\left[-\frac{1}{4\sqrt{2}i} + \frac{3}{4\sqrt{6}i}\right] = \pi\left[-\frac{1}{2\sqrt{2}} + \frac{3}{2\sqrt{6}}\right]$$

$$= \frac{\pi}{2}\left[\frac{-\sqrt{3}+3}{\sqrt{6}}\right] \quad \text{---(1)}$$

Now $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

Now on C_1, $z=x$, x varies from $-R$ to R

$$\therefore \int_{C_1} f(z)\,dz = \int_{-R}^{R} \frac{x^2}{(x^2+2)(x^2+6)}\,dx \quad \text{---(2)}$$

As degree of numerator and denominator are 2 and 4 respectively $\lim_{R\to\infty} \int_{C_2} f(z)\,dz = 0$ —(3)

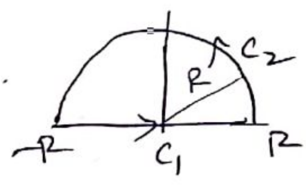
($n \geq m+2$, here $n=4$, $m=2$)

Hence approaching R to infinity, we get from (1), (2) and (3)

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+2)(x^2+6)}\,dx + 0 = \frac{\pi}{2}\left(\frac{3-\sqrt{3}}{\sqrt{6}}\right)$$

(ii) $\int_0^{\infty} \frac{x^2}{(x^2+1)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx$, as the integrant is even function

Let $f(z) = \frac{z^2}{(z^2+1)^2}$ and $C = C_1 + C_2$ be the positively oriented closed contour.



$z = \pm i$, are pole of order 2 of $f(z)$ out of which $z=+i$ lies within C. Hence by residue theorem

$$\int_C f(z)\,dz = 2\pi i\left[\operatorname{Res} f(z)\right]_{z=+i} = \cancel{2\pi i\left[\frac{(z-i)^2 \cdot z^2}{(z-i)^2(z+i)^2}\right]_{z=i}}$$

$$= \cancel{2\pi i \frac{z^2}{(z+i)^2}\Big|_{z=i} = 2\pi i \times \frac{-1}{-4}}$$

(47)

$$= 2\pi i\left[\frac{d}{dz}\left\{\frac{z^2}{(z+i)^2}\right\}\right]_{z=i} = 2\pi i\left[\frac{(z+i)^2 2z - z^2\cdot 2(z+i)}{(z+i)^4}\right]_{z=i}$$

$$= 2\pi i\left[\frac{-4\times 2i - (-1)\times 2\times 2i}{2^4}\right] = \frac{\pi}{8}[8-4] = \frac{\pi}{2} \quad \text{— (1)}$$

Now on C_1, $z = x$, x varies from $-R$ to R, Hence

$$\int_{C_1} f(z)\,dz = \int_{-R}^{R}\frac{x^2}{(x^2+1)^2}dx \quad \text{— (2)}$$

Also $\lim_{R\to\infty}\int_{C_2}\frac{z^2}{(z^2+1)^2}dz = 0$ — (3), as degree of z in the denominator $\geq$ degree of z in the numerator $+2$.

Hence approaching R to infinity we obtain from (1), (2) & (3),

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{(x^2+1)^2}dx + 0 = \frac{\pi}{2}$$

~~$$\text{or } \lim_{R\to\infty}\int_{0}^{R}\frac{x^2}{(x^2+1)^2}dx = 2\times\frac{\pi}{2}$$~~

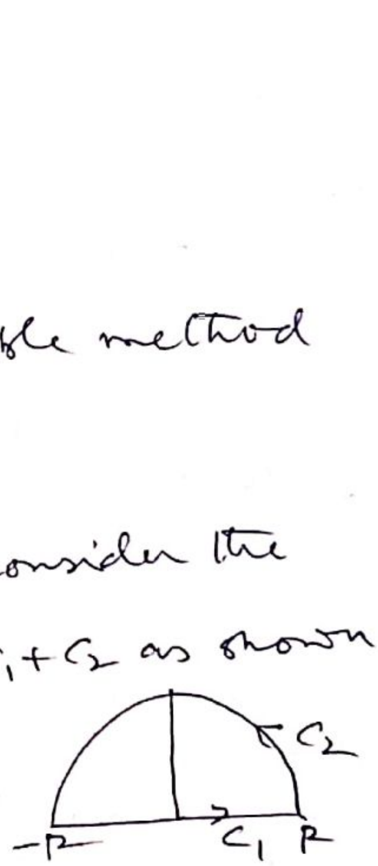
$$\text{or, } 2\times\int_0^\infty \frac{x^2}{(x^2+1)^2}dx = \frac{\pi}{2}$$

$$\text{or } \int_0^\infty \frac{x^2}{(x^2+1)^2}dx = \frac{\pi}{4}.$$

Ex 2 Evaluate by complex variable method

$$\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx, \quad a > b > 0.$$

Let $f(z) = \dfrac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$ and we consider the positively oriented contour $C = C_1 + C_2$ as shown in the figure. ~~$z = \pm\sqrt{a}\,i$ & $\pm\sqrt{b}\,i$~~ $z = \pm ai$ & $z = \pm bi$ are the simple poles of $f(z)$ out of which $z = ai$ & $z = bi$ lies within C.



Then by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i\left[\operatorname{Res} f(z)\Big|_{z=ai} + \operatorname{Res} f(z)\Big|_{z=bi}\right]$$

$$= 2\pi i\left[\frac{e^{iz}}{(z+ai)(z^2+b^2)}\Bigg|_{z=ai} + \frac{e^{iz}}{(z^2+a^2)(z+bi)}\Bigg|_{z=bi}\right]$$

$$= 2\pi i\left[\frac{e^{-a}}{2ai(b^2-a^2)} + \frac{e^{-b}}{2bi(a^2-b^2)}\right]$$

$$= \frac{\pi}{a^2-b^2}\left[-\frac{e^{-a}}{a} + \frac{e^{-b}}{b}\right] \quad \text{--- (1)}$$

On C_1, $z = x$, x varies from $-R$ to R

$$\therefore \int_{C_1} f(z)\,dz = \int_{-R}^{R} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx \quad \text{--- (2)}$$

~~Also as~~ $\cancel{\operatorname{Lt}_{R\to\infty} f(z) = 0}$~~, we have by Jordan's lemma~~ $\cancel{\operatorname{Lt}_{R\to\infty} \int_{C_2} e^{imz}}$

Also as $\dfrac{1}{(z^2+a^2)(z^2+b^2)} \to 0$ uniformly as $z \to \infty$, we have by Jordan's lemma

$$\operatorname{Lt}_{R\to\infty} \int_{C_2} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz = 0. \quad \text{--- (3)}$$

Thus approaching R to infinity, we obtain from (1), (2) & (3),

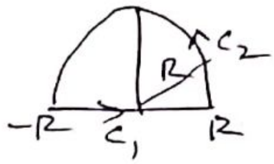
$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx + 0 = \frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

Taking real part, we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

Ex3 Using contour integration find the Cauchy's principal value of $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx$

(49) Let $f(z)=\frac{1}{z^2+4z+5}$ and let $C=C_1+C_2$ be the positively oriented closed contour as shown. We consider the integral $\int_C \frac{e^{iz}}{z^2+4z+5}\,dz$.



$z=\frac{-4\pm\sqrt{16-20}}{2} = -2\pm i$ are the simple poles out of which $-2+i$ lies within C. Hence by residue theorem

$$\int_C \frac{e^{iz}}{z^2+4z+5}\,dz = 2\pi i \times \operatorname{Res} f(z)\Big|_{z=-2+i}$$

$$=2\pi i\left[\frac{e^{iz}}{z-(-2-i)}\right]_{z=-2+i} = 2\pi i\left[\frac{e^{i(-2+i)}}{\cancel{-2}+i+\cancel{2}+i}\right]$$

$$=\pi\left[e^{-1}\cdot e^{-2i}\right] \quad \text{—— (1)}$$

On C_1, $z=x$, x varies from $-R$ to R

$$\therefore \int_{C_1}\frac{e^{iz}}{z^2+4z+5}\,dz = \int_{-R}^{R}\frac{e^{ix}}{x^2+4x+5}\,dx \quad \text{—— (2)}$$

Also as $\frac{1}{z^2+4z+5}\to 0$ as $z\to\infty$, we have

$$\lim_{R\to\infty}\int_{C_2}\frac{e^{iz}}{z^2+4z+5}\,dz = 0 \quad \text{—— (3)}$$

Using (2) & (3) we obtain from (1), approaches R to ∞

$$\text{Lt}\int_C \frac{e^{iz}}{z^2+4z+5}\,dz = \int_{C_1}\frac{e^{iz}}{z^2+4z+5}\,dz + \int_{C_2}\frac{e^{iz}}{z^2+4z+5}$$

$$\Rightarrow \frac{\pi}{e}e^{-2i} = \int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+4x+5}\,dx + 0$$

Taking imaginary part we get

$$\text{P.V. of} \int_{-\infty}^{\infty}\frac{\sin x}{x^2+4x+5}\,dx = -\sin 2\cdot\frac{\pi}{e}$$