Counting (Lecture – 3)

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Distributing Objects into Boxes

- Many counting problems can be solved by enumerating the ways objects can be placed into boxes (where the order these objects are placed into the boxes does not matter).
- The objects can be either distinguishable, or indistinguishable
  - Distinguishable objects are sometimes said to be labeled, whereas indistinguishable objects are said to be unlabeled.
- Similarly, boxes can be distinguishable, or indistinguishable.
  - Distinguishable boxes are often said to be labeled, while indistinguishable boxes are said to be unlabeled.
- Need to determine whether the objects are distinguishable and whether the boxes are distinguishable.
- Although the context of the counting problem makes these two decisions clear, counting problems are sometimes ambiguous.

Distinguishable Objects, Distinguishable Boxes

- Model: counting the number of ways of placing *n* distinguishable objects into *k* distinguishable boxes
- Example:
  - How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?
- **Theorem**:

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$, equals

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Indistinguishable Objects, Distinguishable Boxes

- Model: counting the number of ways of placing *n* indistinguishable objects into *k* distinguishable boxes
- Same as counting the number of *n*-combinations for a set with *k* element types when repetitions are allowed.
- One-to-one correspondence between *n*-combinations from a set with *k* element types when repetition is allowed and the ways to place *n* indistinguishable balls into *k* distinguishable boxes.
- Example:
  - How many ways are there to place 10 indistinguishable balls into eight distinguishable bins?

Distinguishable Objects, Indistinguishable Boxes

- Model: counting the number of ways of placing *n* distinguishable objects into *k* indistinguishable boxes
- Difficult than counting the number of ways to place distinguishable or indistinguishable objects into distinguishable boxes.
- There is no simple closed formula for the number of ways to distribute *n* distinguishable objects into *k* indistinguishable boxes.
- Example:
  - How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

Distinguishable Objects, Indistinguishable Boxes

- Let $S(n, j)$ denote the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no box is empty. The numbers $S(n, j)$ are called **Stirling numbers of the second kind.**

$$S(n, j) = \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j - i)^n.$$

- Consequently, the number of ways to distribute n distinguishable objects into k indistinguishable boxes equals

$$\sum_{j=1}^{k} S(n, j) = \sum_{j=1}^{k} \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j - i)^n.$$

Indistinguishable Objects, Indistinguishable Boxes

- Model: counting the number of ways of placing n indistinguishable objects into k indistinguishable boxes
- Example:
  - How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?
- No simple closed formula exists for this number.
- Distributing n indistinguishable objects into k indistinguishable boxes is the same as writing n as the sum of at most k positive integers in non-increasing order.
- If $a_1+a_2+\cdots+a_k = n$, where $a_1, a_2, \ldots, a_k$ are positive integers with $a_1 \geq a_2 \geq \cdots \geq a_k$, we say that $a_1, a_2, \ldots, a_k$ is a partition of the positive integer n into k positive integers.
- If $p_k(n)$ is the number of partitions of n into at most k positive integers
  - Implies $p_k(n)$ ways to distribute n indistinguishable objects into k indistinguishable boxes.