Relation-7

Wednesday, October 28, 2020 8:48 AM

Lower & Upper Bounds

Prob-1

Subset: $\{a, b, c\}$

$UB(a) = \{a, b, c, d, e, f, g, h, j\}$

$UB(b) = \{b, d, e, f, g, h, j\}$

$UB(c) = \{c, e, f, h, j\}$

$\therefore UB(\{a, b, c\}) = \{e, f, h, j\}$ / $lub(\{a, b, c\}) = \{e\}$.

$LB(a) = \{a\}$

$LB(b) = \{a, b\}$

$LB(c) = \{a, c\}$

$\therefore LB(\{a, b, c\}) = \{a\}$. / $glb = \{a\}$.

Subset $\{j, h\}$

$UB(j) = \{j\}$

$UB(h) = \{h\}$

$UB(\{j, h\}) = \phi$ / $lub(\{j, h\}) = \phi$

$LB(j) = \{j, f, e, d, b, c, a\}$

$LB(h) = \{h, g, f, d, e, b, c, a\}$

$LB(h) = \{h, g, f, d, e, b, c, a\}$

$\therefore LB(\{j, h\}) = \{a, b, c, d, e, f\}$

$glb(\{j, h\}) = \{f\}$

Subset

$\{a, c, d, f\}$

$UB(a) = \{a, b, c, d, e, f, g, h, j\}$

$UB(c) = \{c, e, f, h, j\}$

$UB(d) = \{d, f, g, h, j\}$

$UB(f) = \{f, h, j\}$

$UB(\{a, c, d, f\}) = \{f, h, j\}$

$lub(\{a, c, d, f\}) = \{f\}$

$LB(a) = \{a\}$

$LB(c) = \{a, c\}$

$LB(d) = \{a, b, d\}$

$LB(f) = \{a, b, c, d, e, f\}$

$LB(\{a, c, d, f\}) = \{a\}$

$glb(\{a, c, d, f\}) = \{a\}$.

Prob. Given subset: $\{b, d, g\}$

$\{ \quad \}$

$LB(b) = \{\underline{a}, \underline{b}\}$

$LB(d) = \{\underline{a}, \underline{b}, d\}$

$LB(g) = \{\underline{a}, \underline{b}, d, g\}$

$LB(\{b, d, g\}) = \{a, b\}$ / $glb(\{b, d, g\}) = \{b\}$

$UB(b) = \{b, d, e, f, \underline{g}, \underline{h}, j\}$

$UB(d) = \{d, f, \underline{g}, \underline{h}, j\}$

$UB(g) = \{\underline{g}, \underline{h}\}$

$\therefore UB(\{b, d, g\}) = \{g, h\}$ / $lub(\{g, h\}) = \{g\}$

Prob. Given subsets are $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$. and the poset is $(Z^{+}, |)$.

glb here is same as the greatest common divisor (gcd)

An integer is a lower bound of $\{3, 9, 12\}$ if 3, 9, and 12 are divisible by this integer. The only such integers are 1 and 3. Because $1 | 3$, 3 is the $glb(\{3, 9, 12\})$.

lub is same as least common multiple (LCM)

An integer is an upper bound if it is divisible by 3, 9, and 12. i.e. the least common multiple (LCM) of 3, 9, and 12 which is 36. Hence, the $lub(\{3, 9, 12\})$ is $\underline{36}$.

least common multiple (LCM) (LCM) of 3, 9, and the lub($\{3, 9, 12\}$) is 36.

$\text{glb}(\{1, 2, 4, 5, 10\}) = 1$

$\text{lub}(\{1, 2, 4, 5, 10\}) = 20.$

Prob: Hasse diagram in ~~(a)~~ is a lattice because every pair of elements has both a lub and a glb.

Hasse diagram in (b) is not a lattice because for the pair $\{b, c\}$, there exists no lub.

Hasse diagram in (c) is a lattice as every pair has a lub and a glb.

Prob. Given poset: $(\mathbb{Z}^+, |)$

Let a and b be two positive integers. The lub and glb of these two integers are the LCM and GCD of these two integers, respectively. It follows that this poset is a lattice.

Prob. Given posets are $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$

→ Because 2 and 3 have no least common multiple in $(\{1, 2, 3, 4, 5\}, |)$, they do not

P 1

$(\{ \qquad , \}),$ they have a least upper bound. Hence, the first poset is not a lattice.

Every two elements of the second poset have both a lub (LCM) and a glb (GCD). Hence, the second poset is a lattice.

Prob. Let A and B be two subsets of S. Then lub and glb of A and B are $A \cup B$ and $A \cap B$, respectively. Hence, $(P(S), \subseteq)$ is a lattice.

Prob. The given Hasse diagram in (a) is a lattice, because every pair of elements has a glb and a lub.

The given Hasse diagram in (b) is not a lattice, because the pair $\{b, c\}$ does not have a lub.

The given Hasse diagram in (c) is a lattice as every pair of elements has a glb and a lub.

Topological Sorting

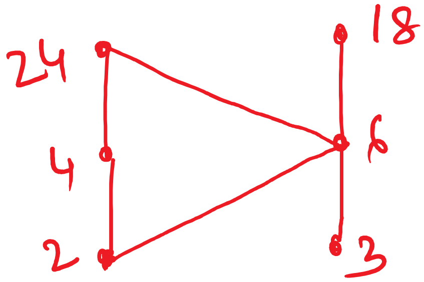
if $a | b$, then $a \leq b$. → total order if $a \mid b$, then $a \preceq b$.

but, if $a \leq b$, then a does not necessarily divides b.

Lemma: Every finite non-empty poset $(S, \preceq)$ has a minimal element.

Proof: Choose an element $a_0 \in S$. If a_0 is not minimal, then there is another element a_1 with $a_1 \prec a_0$. If a_1 is not minimal, then there is another element a_2, with $a_2 \prec a_1$. Continue this process, so that if a_n is not minimal, then there is an element a_{n+1}, such that $a_{n+1} \prec a_n$. Because there are only a finite no. of elements in S, this process must end with a minimal element a_n.

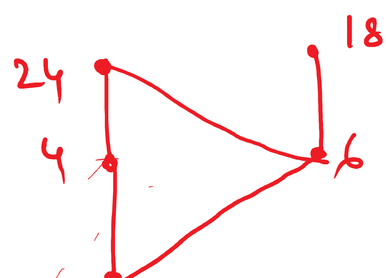
Prob.



Choose 3

$A' = A - \{3\}$

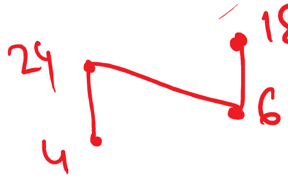
18



Choose 2

$A' = A - \{3, 2\}$

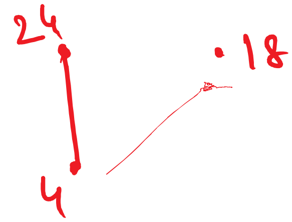
$A' - A - \{3\}$



Choose 6

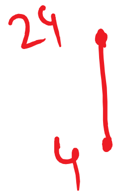
$A' = A - \{3, 2, 6\}$

$A - A - \{3, 2\}$



Choose 18

$A' = A - \{3, 2, 6, 18\}$



Choose 4

$A' = A - \{3, 2, 6, 18, 4\}$

Choose 24.

$A' = \phi$

Total order: $3 \lesssim 2 \lesssim 6 \lesssim 18 \lesssim 4 \lesssim 24$

This order is compatible with the "divides" partial order as for each pair of elements a and b in A, if $a \mid b$, then $a \lesssim b$.