LAPLACE TRANSFORM

INTEGRAL TRANSFORM:

An **integral transform** maps a function into another function space through integration. The transformed function can be mapped back to the original function by performing another suitable integration which is called the *inverse transform*.

Use of integral transform is for mathematical convenience. It is a part of **Mathematical Methods** and has very important applications in different areas of mathematics.

There are several such transforms like Fourier Transform, Hankel transform, etc.

DEFINITION OF LAPLACE TRANSFORM

Let $F(t)$ be a function of $t > 0$. Then the Laplace transform of $F(t)$, denoted by L{ $F(t)$}, is defined by

$$\text{L}\{\,F(t)\} = f(s) = \int_0^\infty e^{-st}\,F(t)dt,$$

where the parameter s is assumed real or complex.

The Laplace transform of $F(t)$ is said to exist if the above integral is convergent.

NOTATION

If a function of t is indicated in terms of capital letter, such as $F(t)$, $G(t)$, $Y(t)$, etc, the Laplace transform of the function is denoted by the corresponding lower case letter, $f(s), g(s), y(s)$, etc.

EXAMPLES:

1) $F(t) = 1, t > 0$

$f(s)$ = L{1}

$$= \int_0^\infty e^{-st}(1)dt$$

$$= \lim_{P\to\infty} \int_0^P e^{-st}dt, \text{ if } s > 0$$

$$= \lim_{P\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^P$$

$$= \lim_{P\to\infty} \frac{1-e^{sP}}{s}$$

$$= \frac{1}{s}, \text{ if } s > 0.$$

2) $F(t) = t, t > 0$

$f(s)$ = L{t}

$$= \int_0^\infty e^{-st}(t)\,dt$$

$$= \lim_{P\to\infty} \int_0^P te^{-st}dt$$

$$= \lim_{P\to\infty} \left[(t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^P$$

$$= \lim_{P\to\infty} \left(\frac{1}{s^2} - \frac{e^{-sP}}{s^2} - \frac{Pe^{-sP}}{s}\right)$$

$$= \frac{1}{s^2} \text{ if } s > 0.$$

PIECEWISE CONTINUITY

A function is called *piecewise continuous* in an interval $\alpha \leq t \leq \beta$ if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits.

FUNCTIONS OF EXPONENTIAL ORDER

If real constants $k > 0$ and γ exist such that for all $t > N$

$$| e^{-\gamma t} F(t)| < k \text{ or, } |F(t)| < k\, e^{\gamma t}.$$

We say that $F(t)$ is a *function of exponential order* γ as $t \to \infty$, or, briefly , is of *exponential order*.

$F(t) = \ t^2$ is of exponential order 3 (for example), since

$$| t^2 | = t^2 < e^{3t} \text{ for all } t > 0.$$

$F(t) = e^{t^3}$ is not of exponential order, since $\left|e^{-\gamma t} e^{t^3}\right| = \left|e^{t^3 - \gamma t}\right|$ can be made larger than any given constant by increasing t.

CONDITION FOR EXISTENCE OF LAPLACE TRANSFORM

Theorem 1: If $F(t)$ is sectionally continuous in every finite interval $0 \leq t \leq N$ and of exponential order γ for $t > N$, then its Laplace transform $f(s)$ exists for all $s > \gamma$.

SOME PROPERTIES OF LAPLACE TRANSFORM

LINEARITY PROPERTY

$If c_1$ and c_2 are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transform $f_1(s)$ and $f_2(s)$ respectively, then

$$\mathrm{L}\{c_1 F_1(t) + c_2 F_2(t)\}$$

$$= c_1 \mathrm{L}\{F_1(t)\} + c_2 \mathrm{L}\{F_2(t)\}$$

$$= c_1 f_1(s) + c_2 f_2(s)$$

FIRST SHIFTING PROPERTY

If $\mathrm{L}\{F(t)\} = f(s)$, then

$$\mathrm{L}\{e^{at} F(t)\} = f(s-a).$$

SECOND SHIFTING PROPERTY

If L$\{F(t)\}$ = $f(s)$ and $G(t) = \begin{cases} F(t-a), & if\ t > a \\ 0, & if t < a \end{cases}$

then $$\text{L}\{G(t)\} = e^{-as} f(s).$$

CHANGE OF SCALE PROPERTY

If L$\{F(t)\}$ = $f(s)$, then $L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$.

Example: If $F(t) = e^{\alpha t}$, then

$$f(s) = \text{L}\{e^{\alpha t}\}$$

$$= \int_0^{\infty} e^{-st} e^{\alpha t}\, dt$$

$$= \lim_{P\to\infty} \int_0^{P} e^{-st} e^{\alpha t}\, dt$$

$$= \lim_{P\to\infty} \left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^P$$

$$= \lim_{P\to\infty} \frac{1-e^{-(s-a)P}}{s-a}$$

$$= \frac{1}{s-a}, \quad s > a.$$

Example: If $F(t) = coshat$, then

$$\mathrm{L}\{coshat\}$$

$$= \mathrm{L}\left\{\frac{1}{2}(e^{at} + e^{-at})\right\}$$

$$= \frac{1}{2}\mathrm{L}\{e^{at}\} + \frac{1}{2}\mathrm{L}\{e^{-at}\}$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$$= \frac{s}{s^2 - a^2}, \; s > |a|$$

Problem: Prove that

$$\mathrm{L}\{sinhat\} = \frac{a}{s^2 - a^2}, \; s > |a|$$

Example: If F(t) $= cosat$, then

$$f(s) = \mathrm{L}\{cosat\}$$

$$= \int_0^\infty e^{-st} \, cosat \, dt$$

$$= \lim_{P\to\infty} \int_0^P e^{-st} \, cosat \, dt$$

$$= \lim_{P\to\infty} \left[\left(\frac{e^{-st}}{s^2+a^2}\right)(-scosat + asinat)\right]_0^P$$

$$= \lim_{P\to\infty} \left\{\frac{s}{s^2+a^2} - \frac{e^{-sP}\ (scosaP\ - asinaP\ )}{s^2+a^2}\right\}$$

$$= \frac{s}{s^2+a^2}, \quad s > 0.$$

Problem: Prove that L$\{sinat\}$ $= \frac{a}{s^2+a^2}$

Example: Let $F(t) = t^2$

$f(s)$ = L $\{t^2\}$

$$=\int_0^\infty e^{-st}\ t^2 dt$$

$$= \lim_{P\to\infty} \int_0^P e^{-st} t^2 dt$$

Integrating by parts, we get

$$= \lim_{P\to\infty} \left\{\left[\frac{-t^2 e^{-st}}{s}\right]_0^P + \int_0^P 2t \frac{e^{-st}}{s} dt\right\}$$

$$= \lim_{P\to\infty} \left\{\frac{2}{s}\left[\frac{-e^{-st}}{s}\ t\right]_0^P + \frac{2}{s}\int_0^P \frac{e^{-st}}{s} dt\right\},$$

since $p^2 e^{-sp} \to 0$ as $p \to \infty$.

$$= \lim_{P \to \infty} \frac{2}{s^2}\left[\frac{e^{-st}}{-s}\right]_0^P = \frac{2}{s^3},\ s > 0.$$

Problem:

If n be positive, not necessarily an integer, then prove that

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}},\ s > 0.$$

Solution: L $\{t^n\}$

$$= \int_0^\infty e^{-st}\, t^n\, dt$$

$$= \int_0^\infty e^{-u}\, \frac{u^n}{s^{n+1}}\, du\ ,\ \ putting\ st = u$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}.$$

When n is a positive integer,

$$L\{t^n\} = \frac{n!}{s^{n+1}}\ \ [\because \Gamma(n+1) = n!]$$

Example: Let $F(t) = e^{-t}\, cos2t$

Since L$\{cos2t\}$ = $\frac{s}{s^2+4}$, we have ,

$$L\{e^{-t}\, cos2t\} = \frac{s+1}{(s+1)^2+4}$$

$$= \frac{s+1}{s^2+2s+5}.$$

Example: Let

$F(t)$= $4t^2 - 3cos2t + 5e^{-t}$,

L$\{4t^2 - 3cos2t + 5e^{-t}\}$

$$= 4L\{t^2\} - 3L\{cos2t\} + 5L\{e^{-t}\}$$

$$= 4\left(\frac{2!}{s^3}\right) - 3\left(\frac{s}{s^2+4}\right) + 5\left(\frac{1}{s+1}\right)$$

$$= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}.$$

Example: Let $F(t) = sin3t$.

Since L$\{sint\} = \frac{1}{s^2+1}$, we have

$$\text{L}\{sin3t\} = \frac{1}{3} \cdot \frac{1}{\left(\frac{s}{3}\right)^2 + 1}$$

$$= \frac{3}{s^2 + 9}.$$

LAPLACE TRANSFORM OF DERIVATIVES

Theorem: If L$\{F(t)\} = f(s)$, then

$$\text{L}\{F'(t)\} = sf(s) - F(0)$$

if $F(t)$is continuous for $0 \leq t \leq M$ and of exponential order for $t > M$, while $F'(t)$ is sectionally continuous for $0 \leq t \leq M$.

If in the above result $F(t)$ fails to be continuous at $t = a$, then

$$\text{L}\{F'(t)\} = sf(s) - F(0) - e^{-as}\{F(a+) - F(a-)\}$$

where $F(a+) - F(a-)$ is sometimes called the **jump** at the discontinuity at $t = a$.

Example: If $F(t) = cos3t$, then L$\{F(t)\}$= $\frac{s}{s^2+9}$

$$\text{L}\{F'(t)\} = \text{L}\{-3sin3t\}$$

$$= s\left(\frac{s}{s^2+9}\right) - 1$$

$$= \frac{-9}{s^2+9}\,.$$

General formula:

If L$\{F(t)\}$= $f(s)$, then

$$\text{L}\{F^{(n)}(t)\}$$

$$= s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - sF^{(n-2)}(0) - F^{(n-1)}(0)$$

if $F(t), F'(t), \ldots, F^{(n-1)}(t)$ are continuous for $0 \le t \le M$ and of exponential order for $t > M$, while $F^{(n)}(t)$ is sectionally continuous for $0 \le t \le M$.

LAPLACE TRANSFORM OF INTEGRALS

If L$\{F(t)\}$= $f(s)$, then

$$\text{L}\left\{\int_0^t F(u)du\right\} = \frac{f(s)}{s}$$

Example: Since L$\{sin2t\}$ = $\frac{2}{s^2+4}$, we have

$$\text{L}\left\{\int_0^t sin2udu\right\} = \frac{2}{s(s^2+4)}$$

Problem: Verify the above result by direct calculation.

MULTIPLICATION BY t^n

Theorem: If L$\{F(t)\}$= $f(s)$, then

$$\text{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

$$= (-1)^n f^{(n)}(s).$$

Example:

Since $\mathrm{L}\{e^{2t}\} = \frac{1}{s-2}$, we have

$$\mathrm{L}\{te^{2t}\} = -\frac{d}{ds}\left(\frac{1}{s-2}\right)$$

$$= \frac{1}{(s-2)^2}.$$

$$\mathrm{L}\{t^2e^{2t}\} = \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right)$$

$$= \frac{2}{(s-2)^3}$$

DIVISION BY $\boldsymbol{t}$

Theorem. If $\mathrm{L}\{F(t)\} = f(s)$, then

$$\mathrm{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)du$$

provided $\lim_{t\to 0} F(t)/t$ exists.

Example: Since L$\{sint\}$ = $\frac{1}{s^2+1}$

and $\lim_{t\to 0} \frac{sint}{t} = 1,$

we have,

$$\text{L}\left\{\frac{sint}{t}\right\} = \int_s^\infty \frac{du}{u^2+1}$$

$$= \tan^{-1}\left(\frac{1}{s}\right)$$

PERIODIC FUNCTIONS

Theorem: Let $F(t)$ have period $T > 0$ so that $F(t+T) = F(t)$

$$\text{Then L}\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\,dt}{1 - e^{-st}}$$

Prove that (a) L$\{sinhat\} = \frac{a}{s^2 - a^2}$

$$\text{L}\{sinhat\}$$

$$= \text{L}\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st} e^{at}\, dt \; - \; \frac{1}{2}\int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \frac{1}{2}\mathrm{L}\{e^{at}\} - \mathrm{L}\{e^{-at}\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\}$$

$$= \frac{a}{s^2-a^2} \text{ for } s > |a|$$

Problem:

Prove that $\mathrm{L}\{coshat\} = \frac{s}{s^2-a^2}$, if $s > |a|$.

THE INVERSE LAPLACE TRANSFORM

Definition: If the Laplace transform of $F(t)$ is $f(s)$, i.e. L$\{F(t)\}$= $f(s)$, then $F(t)$ is called *inverse Laplace transform* of $f(s)$ and we write symbolically $F(t)$ = $L^{-1}\{f(s)\}$where L^{-1} is called the inverse Laplace operator.

Example: Since L$\{e^{-3t}\}$= $\frac{1}{s+3}$ we can write

$$L^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}.$$

Two different functions with the same Laplace transform.

Example:

The two different function $F_1(t) = e^{-3t}$ and

$$F_2(t) = \begin{cases} 0, & t = 1 \\ e^{-3t}, & \text{otherwise} \end{cases}$$

have the same Laplace transform i.e. $\frac{1}{s+3}$.

We see that inverse Laplace transform is not unique.

Some Inverse Laplace transforms:

1. $L^{-1}\left\{\frac{1}{s}\right\} = 1$

2. $L^{-1}\left\{\frac{1}{s^2}\right\} = t$

3. $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}, n = 0,1, \ldots$

4. $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

5. $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{sinat}{a}$

6. $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = cosat$

7. $L^{-1}\left\{\frac{1}{s^2 - a^2}\right\} = \frac{sinhat}{a}$

8. $L^{-1}\left\{\frac{s}{s^2 - a^2}\right\} = coshat$

LINEARITY PROPERTY

$If c_1$ and c_2 are any constants while $f_1(s)$ and $f_2(s)$ are functions with Laplace transform $F_1(t)$ and $F_2(t)$ respectively, then

$L^{-1}\{c_1 f_1(s) + c_2 f_2(s)\}$

$$= c_1 L^{-1}\ \{f_1(s)\} + c_2 L^{-1}\ \{f_2(s)\}$$

$$= c_1 F_1(t) + c_2 F_2(t)$$

FIRST TRANSLATION OR SHIFTING PROPERTY

If $L^{-1}\{f(s)\} = F(t)$, then

$$L^{-1}\{f(s-a)\} = e^{at} F(t).$$

SECOND TRANSLATION OR SHIFTING PROPERTY

If $L^{-1}\{f(s)\} = F(t)$, then

$$L^{-1}\{e^{-as} f(s)\} = \begin{cases} F(t-a), & if\ t > a \\ 0, & if t < a \end{cases}.$$

CHANGE OF SCALE PROPERTY

If $L^{-1}\{f(s)\} = F(t)$, then

$$L^{-1}\{f(ks)\} = \frac{1}{k} f\left(\frac{t}{k}\right)$$

Example:

Find Inverse Laplace Transform of $\frac{1}{s^2 - 2s + 5}$

$$i.e.\ L^{-1}\left\{\frac{1}{s^2 - 2s + 5}\right\}$$

Since

$$L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{sin\,2t}{2}$$, we have

$$L^{-1}\left\{\frac{1}{s^2 - 2s + 5}\right\} = L^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = \frac{1}{2}\,e^t sin2t$$

Example:

Find Inverse Laplace Transform of $\frac{e^{-\frac{\pi}{3}s}}{s^2+1}$

$$i.e.\ L^{-1}\left\{\frac{e^{-\frac{\pi}{3}s}}{s^2+1}\right\}$$

Since

$L^{-1}\left\{\frac{1}{s^2+1}\right\} = \ sint$, we have

$$L^{-1}\left\{\frac{e^{-\frac{\pi}{3}s}}{s^2+1}\right\} = \begin{cases} sin\left(t-\frac{\pi}{3}\right) & ,if\ t > \frac{\pi}{3} \\ 0, & ,if t < \frac{\pi}{3} \end{cases}$$

Example:

Find the Inverse Laplace Transform of $\frac{2s}{(2s)^2+16}$

$$i.e.\ L^{-1}\left\{\frac{2s}{(2s)^2+16}\right\}$$

$L^{-1}\left\{\frac{s}{s^2+16}\right\}= cos4t$, we have

$$L^{-1}\left\{\frac{2s}{(2s)^2+16}\right\} = \ \frac{1}{2}cos\frac{4t}{2} = \ \frac{1}{2}cos2t.$$