Convolution:

The convolution of two functions is given by

$$(F * G)(t) = \int_0^t F(u)G(t-u)du.$$

Convolution Properties:

i) $F * G = G * F,$
ii) $(F * G) * H = F * (G * H),$
iii) $F * (G + H) = F * G + F * H.$

Proof of $F * G = G * F$

We know

$$(F * G)(t) = \int_0^t F(u)G(t-u)du.$$

Letting $t - u = v$ i.e $u = t - v.$

Then

$$(F * G)(t) = \int_0^t F(u)G(t-u)du.$$

$$= \int_0^t F(t-v)G(v)dv.$$

$$= (\mathrm{G} * \mathrm{F})(\mathrm{t}).$$

$\therefore \qquad \mathrm{F} * \mathrm{G} = \mathrm{G} * \mathrm{F}.$

The other properties are proved in a similar way.

Convolution theorem:

If $\boldsymbol{L^{-}\{f(s)\} = F(t)}$ and $\boldsymbol{L^{-}\{g(s)\} = G(t)}$
then

$$\boldsymbol{L^{-}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)du = (F * G)(t).}$$

Example: $L^{-}\left\{\frac{1}{(s-1)(s-2)}\right\}$

Since $\quad L^{-}\{\frac{1}{s-1}\} = e^{t} = F(t)$

and $\quad L^{-}\{\frac{1}{s-2}\} = e^{2t} = G(t).$

We have

$$L^{-}\left\{\frac{1}{(s-1)(s-2)}\right\} = (F * G)(t)$$

$$= \int_0^t e^{u} e^{2(t-u)} du$$

$$= e^{2t} - e^{t}.$$

Example: $\mathrm{L}^{-}\left\{\frac{\mathrm{s}}{(\mathrm{s}^2+\mathrm{a}^2)^2}\right\}$

$$\frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2}$$

We have

$$\mathrm{L}^{-}\left\{\frac{\mathrm{s}}{s^2+a^2}\right\} = cos\ at,$$

$$\mathrm{L}^{-}\left\{\frac{1}{s^2+a^2}\right\} = \frac{sin\ at}{a}$$

$$\therefore \mathrm{L}^{-}\left\{\frac{\mathrm{s}}{(\mathrm{s}^2+\mathrm{a}^2)^2}\right\} = \int_0^t cos\ au\ \frac{sin\ a(t-u)}{a} du$$

$$= \frac{1}{\mathrm{a}} \int_0^{\mathrm{t}} \cos \mathrm{au}(\sin \mathrm{at} \cos \mathrm{au} - \cos \mathrm{at} \sin \mathrm{au}) \mathrm{du}$$

$$= \frac{sin\ at}{\mathrm{a}} \int_0^{\mathrm{t}} \cos^2 \mathrm{au\ du} - \frac{cos\ at}{\mathrm{a}} \int_0^{\mathrm{t}} \sin \mathrm{au} \cos \mathrm{au\ du}$$

$$= \frac{sin\ at}{\mathrm{a}} \int_0^{\mathrm{t}} \frac{(1+\cos 2\mathrm{au})}{2}\ \mathrm{du} - \frac{cos\ at}{\mathrm{a}} \int_0^{\mathrm{t}} \frac{\sin 2\mathrm{au}}{2}\ \mathrm{du}$$

$$= \frac{\sin \mathrm{at}}{2\mathrm{a}}\left(\mathrm{t} + \frac{\sin 2\mathrm{at}}{2\mathrm{a}}\right) - \frac{\cos \mathrm{at}}{2\mathrm{a}}\left(\frac{1-\cos 2\mathrm{at}}{2\mathrm{a}}\right)$$

$$= \frac{\sin \mathrm{at}}{2\mathrm{a}}\left(\mathrm{t} + \frac{2\sin at \ \cdot cos\ at}{2\mathrm{a}}\right) - \frac{\cos \mathrm{at}}{2\mathrm{a}}\left(\frac{2\sin^2 \mathrm{at}}{2\mathrm{a}}\right)$$

$$= \frac{\mathrm{t} \sin \mathrm{at}}{2\mathrm{a}}.$$