Counting - 2

Wednesday, November 18, 2020 8:55 AM

PH Principle

Prob-3 Let a_j be the no. of games played on the j^{th} day of the month.

Then $a_1, a_2, \ldots a_{30}$ is an increasing sequence of distinct positive integers, with $1 \leq a_j \leq 45$.

Moreover, $a_1+14, a_2+14, \ldots, a_{30}+14$ is also an increasing sequence of distinct positive integers, with $15 \leq a_j+14 \leq 59$

The 60 positive integers $a_1, a_2, \ldots, a_{30}$ and $a_1+14, a_2+14, \ldots a_{30}+14$ are less than or equal to 59. Hence, by PH principle, two of these integers are equal. Because, the integers a_j, $j = 1, 2, \ldots 30$ and a_j+14, for $j = 1, 2, \ldots 30$ are all distinct, there must be indices i and j for which $a_i = a_j + 14$. This means that exactly 14 games were played from day $(j+1)$ to i.

Prob-4. We write each of $(n+1)$ integers as $a_1, a_2, \ldots a_{n+1}$ as the power of 2 times an odd integer i.e. $a_j = 2^{k_j} \cdot q_j$ for $j = 1, 2, \ldots (n+1)$, where k_j is a nonnegative integer and q_j is odd.

The integers $q_1, q_2, \ldots q_{n+1}$ are all odd positive

- integers which are less than $2n$.

Because there are only n odd positive integers less than $2n$, it follows from the PH principle that there are two of the integers in the sequence $q_1, q_2, \ldots q_{n+1}$, must be equal. Therefore, there are distinct integers i & j such that $q_i = q_j = q$ (say).

Therefore, $a_i = 2^{k_i} \cdot q$ and $a_j = 2^{k_j} \cdot q$.

It follows that if $k_i < k_j$, then $a_i \mid a_j$

Otherwise, if $k_i > k_j$, then $a_j \mid a_i$

Theorem. Every sequence of n^2+1 distinct real numbers contains a subsequence of length $(n+1)$ that is either strictly increasing or strictly decreasing.

→ Let $a_1, a_2, \ldots a_{n^2+1}$ be a sequence of (n^2+1) distinct real numbers. Associate an ordered pair with each term of the sequence, ~~namely~~ (i_k, d_k), where i_k is the length of longest increasing subsequence starting at a_k and d_k is the length of the longest decreasing subsequence which starts at a_k.

Suppose that there are no such increasing or decreasing subsequences of length $(n+1)$. Then i_k and d_k they are both positive integers less than or equal to n, for $k = 1, 2, \ldots n^2+1$. Hence, by the product rule, there are n^2 possible ordered pairs for (i_k, d_k)

But as we have (n^2+1) integers, so at most we should have (n^2+1) ordered pairs of (i_k, d_k).

By the PH principle, two of these n^2+1 ordered pairs are equal. In other words, there exists terms a_s and a_t with $s < t$ such that $i_s = i_t$ and $d_s = d_t$.

(i_t, d_t) a_t

(i_s, d_s) a_s $1 + i_t$

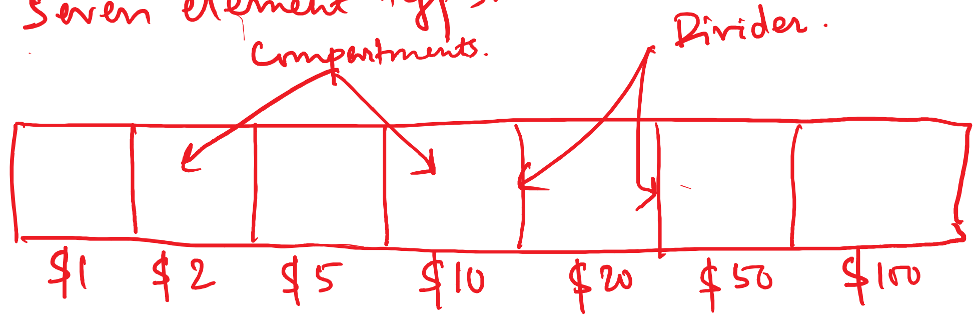
$i_s \neq i_t$

Because the terms of the sequence are distinct, either $a_s < a_t$ or $a_s > a_t$. If $a_s < a_t$, then because, $i_s = i_t$ an increasing subsequence of length $i_t + 1$ can be built starting at a_s followed by an increasing subsequence of length i_t beginning at a_t. This is a contradiction as $i_s = 1 + i_t$

Similarly, if $a_s > a_t$, the same reasoning shows that d_s must be greater than d_t, which is a contradiction.

Combination with repetition.

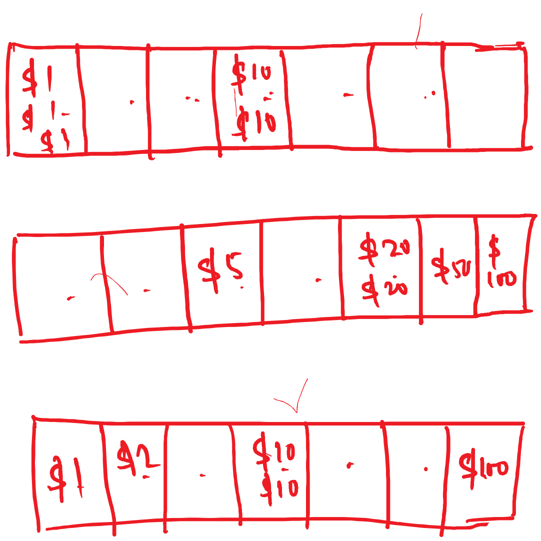
Prob. Because the order in which the bills are selected does not matter and seven different types of bills can be selected as many as five times, this problem involves counting of 5-combinations with repetitions allowed from a set with seven element types.



7 Compartments

6 Dividers.

The choice of five bills corresponds to placing five markers in the compartments holding different types of bills.



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6 dividers.

5 markers.
(bills to be selected)

11 positions.

The no. of ways to select five bills corresponds to the no. of ways to arrange six bars and five markers/stars in a row with 11 positions.

Consequently, the no. of ways to select five bills is the no. of ways to select the positions of 5 stars from 11 positions.

$7+5-1=11$

$\therefore$ Consequently there are $C(11,5) = \frac{11!}{5!\,6!} =$ 462 ways to choose five bills from a cashbox with seven types of bills.

$C(n+r-1, r)$

$= C(n+r-1, n-1)$