# CURVES AND SURFACES

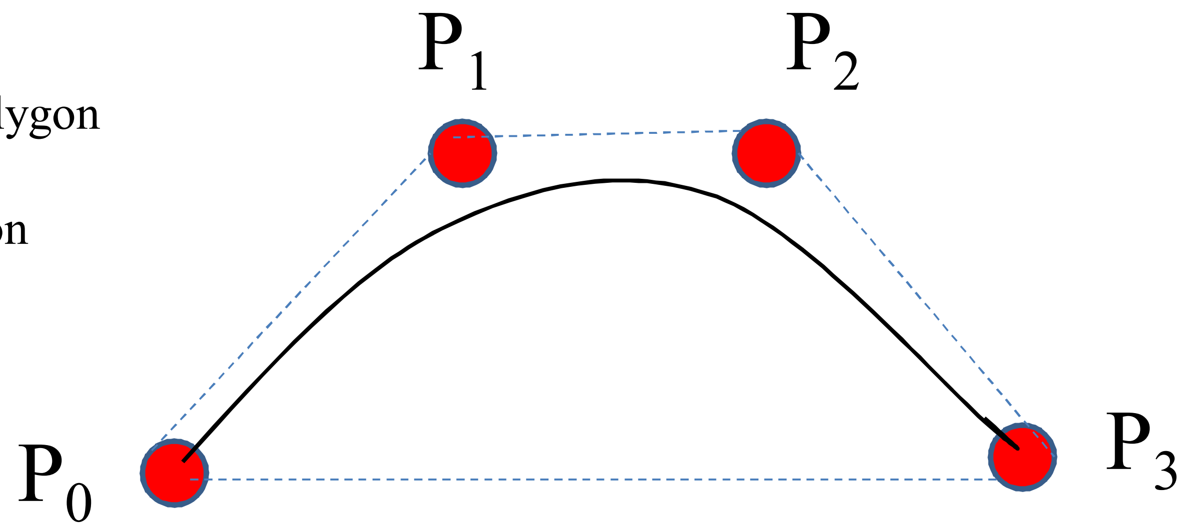
# BEZIER CURVE

Specified by set of coordinates known as control points

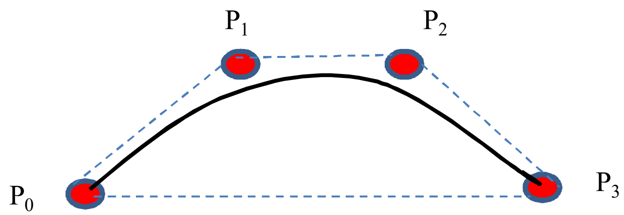
Control points indicates the shape of the curve

A Bezier curve is an approximation curve

Characteristic polygon
or
Control polygon




| Degree | Control points | Name |
|---|---|---|
| 2 | 3 | Quadratic Bezier |
| 3 | 4 | Cubic Bezier |
| 4 | 5 | Quartic Bezier |
| 5 | 6 | Quintic Bezier |




$$P_0 = (x_0, y_0, z_0)$$
$$P_1 = (x_1, y_1, z_1)$$
$$P_2 = (x_2, y_2, z_2)$$
$$P_3 = (x_3, y_3, z_3)$$

$$x(u) = (1-u)^3 x_0 + 3(1-u)^2 u x_1 + 3(1-u)u^2 x_2 + u^3 x_3 \qquad 0 \le u \le 1$$
$$= {}^3C_0(1-u)^3 x_0 + {}^3C_1(1-u)^2 u x_1 + {}^3C_2(1-u)u^2 x_2 + {}^3C_3 u^3 x_3$$
$$= \boxed{\sum_{i=0}^{3} {}^3C_i (1-u)^{3-i}\ u^i}\ x_i$$

Bezier basis function
or
Bernstein polynomial

$$y(u) = \sum_{i=0}^{3} {}^3C_i (1-u)^{3-i}\ u^i\ y_i$$

$$z(u) = \sum_{i=0}^{3} {}^3C_i (1-u)^{3-i}\ u^i\ z_i$$

**Quartic Bezier Curve**

$$x(u) = \sum_{i=0}^{4} {}^{4}C_i (1-u)^{4-i} u^i \, x_i$$

**Quintic Bezier Curve**

$$x(u) = \sum_{i=0}^{5} {}^{5}C_i (1-u)^{5-i} u^i \, x_i$$

**N$^{th}$ degree Bezier Curve**

$$x(u) = \sum_{i=0}^{N} {}^{N}C_i (1-u)^{N-i} u^i \, x_i$$

# Properties of BEZIER CURVE

**Property-I.**

a)Tangent (f’) at each endpoint is defined by the endpoint and the adjacent point

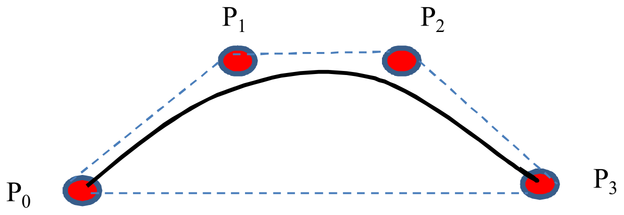
b) Curvature(f’’) at each endpoint is defined by the endpoint and its two adjacent points

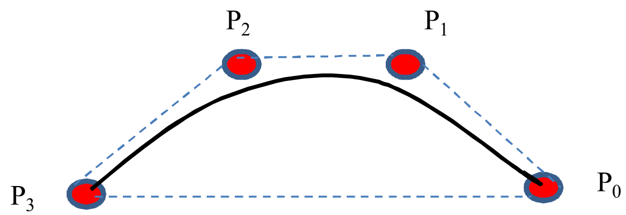
c) f’’’ at each endpoint is defined by the endpoint and its three adjacent points

# Properties of BEZIER CURVE

**Property-II.**

Reversing the sequence of control points does not change the shape of the curve

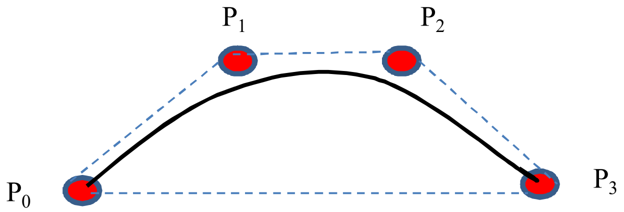


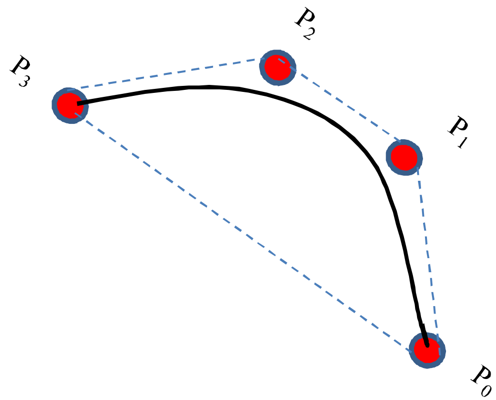



# Properties of BEZIER CURVE

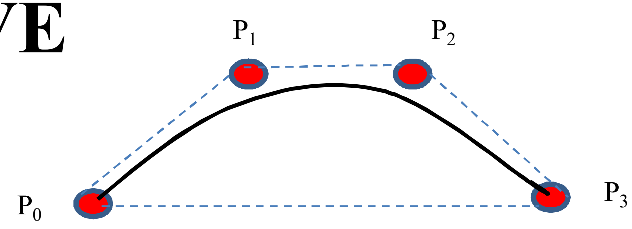
**Property-III.**

The curve is invariant under translation, rotation, scaling and sheering



# Properties of BEZIER CURVE



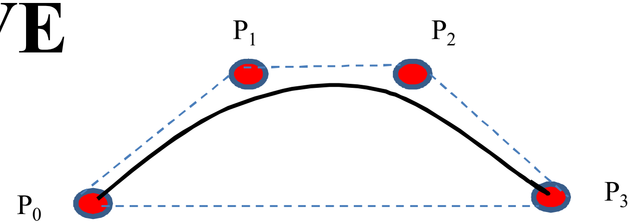

**Property-IV.***(Property of convex hull)*

The curve lies entirely inside characteristic polygon

•Bound of a curve

•Two Bezier curves are intersecting or not

# Properties of BEZIER CURVE

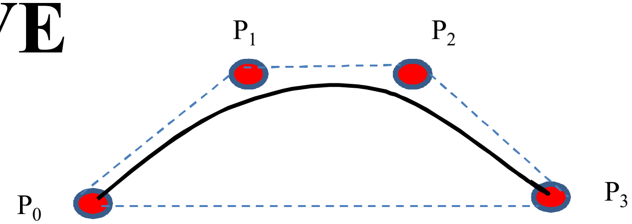



**Property-V.**

The curve exhibits partition of unity

$$\boxed{\sum_{i=0}^{3} {}^{3}C_i (1-u)^{3-i}\ u^i = 1}$$

# Properties of  BEZIER CURVE




**Property-VI.**

The curve can be represented in matrix form

$$x(u) = (1-u)^3 x_0 + 3(1-u)^2 u x_1 + 3(1-u)u^2 x_2 + u^3 x_3 \qquad 0 \le u \le 1$$

$$= (1-3u+3u^2-u^3)x_0 + (3u-6u^2+3u^3)x_1 + (3u^2-3u^3)x_2 + u^3 x_3$$

$$= u^3(-x_0+3x_1-3x_2+x_3) + u^2(3x_0-6x_1+3x_2) + u(-3x_0+3x_1) + x_0$$

$$x(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

**Global propagation of change**

Changing one control point changes the globally

**Local propagation of change**

Changing one control point changes the locally

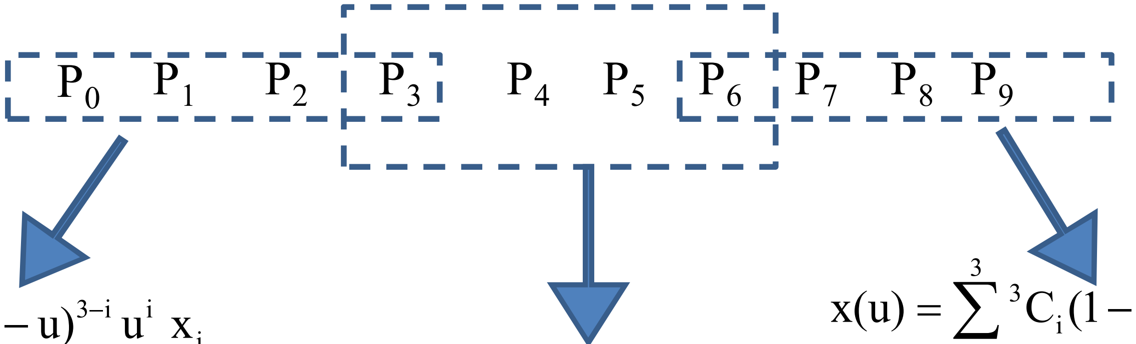
$$x(u) = {}^3C_0(1-u)^3x_0 + {}^3C_1(1-u)^2ux_1 + {}^3C_2(1-u)u^2x_2 + {}^3C_3u^3x_3$$

# Piecewise BEZIER CURVE

N=9

Control points:10

$$x(u) = \sum_{i=0}^{9} {}^{9}C_i (1-u)^{9-i} u^i \ x_i$$




$$x(u) = \sum_{i=0}^{3} {}^{3}C_i (1-u)^{3-i} u^i \ x_i$$

$$x(u) = \sum_{i=0}^{3} {}^{3}C_i (1-u)^{3-i} u^i \ x_i$$

$$x(u) = \sum_{i=0}^{3} {}^{3}C_i (1-u)^{3-i} u^i \ x_i$$

# Implementation issue

```
Void Bezier(int x[],int y[])
{
       for (u=0;u<1;u=u+0.0005)
              {
              float xu=f(x,u);
               float yu=f(y,u);
              }
          plot_point(round(x(u)),round(y(u)),COLOR);

}
```

# BEZIER SURFACE

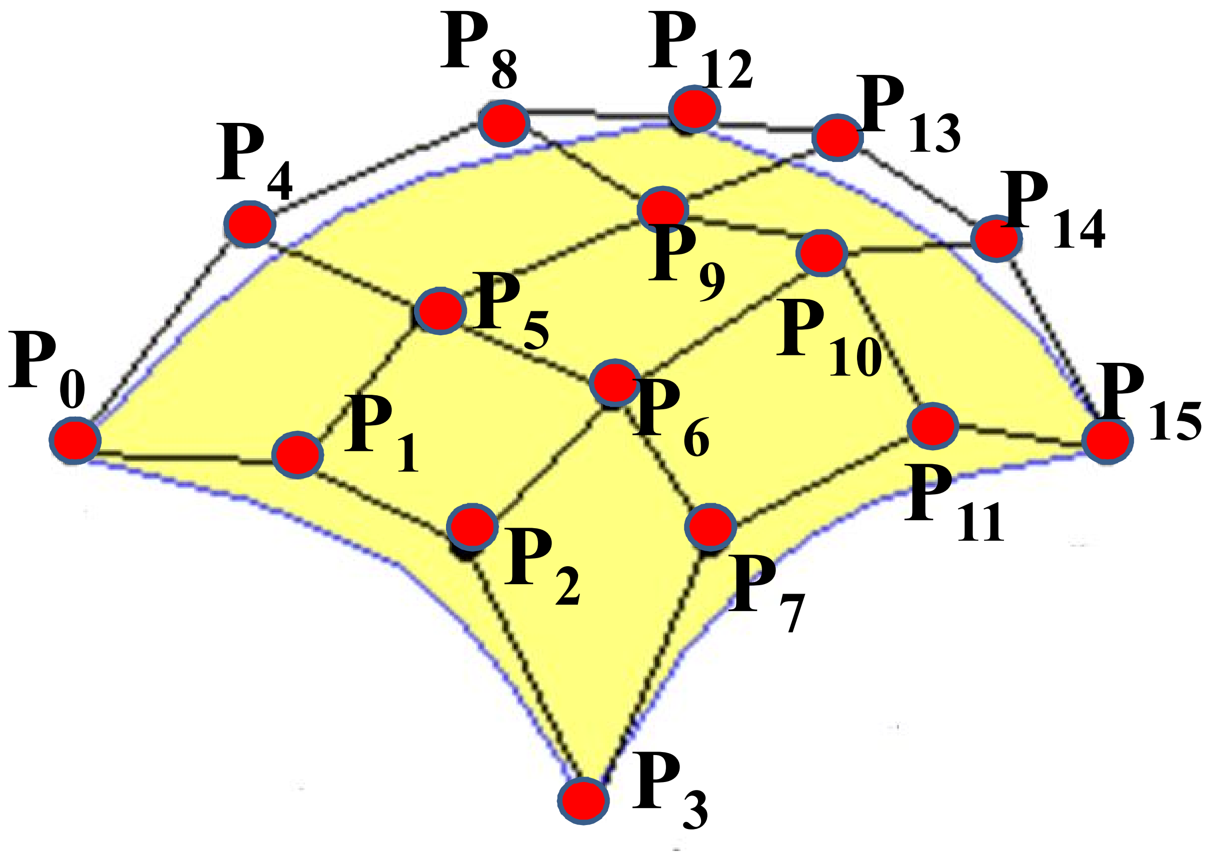
**Cubic Bezier curve**

$$x(u) = \sum_{i=0}^{3} {}^{3}C_{i}(1-u)^{3-i}\, u^{i}\; x_{i}$$

**Cubic-Cubic Bezier surface / Bi-cubic Bezier surface**

$$x(u,w) = \sum_{i=0}^{3}\sum_{j=0}^{3}({}^{3}C_{i}(1-u)^{3-i}\, u^{i})({}^{3}C_{j}(1-u)^{3-j} w^{j}\;)\, x_{ij}$$

**Bi-cubic Bezier surface**




https://www.google.co.in/url?sa=i&rct=j&q=&esrc=s&source=images&cd=&cad=rja&uact=8&ved=&url=http%3A%2F%2Ffuzzyphoton.tripod.com%2Frtref%2Frtref_b.htm&psig=AFQjCNE1pIh6AKewLh2bHw6umQ49XOtdRQ&ust=1467879043590516

# Properties of BEZIER SURFACE

1. Passes through four corner points

2. Tangent vectors are defined by adjacent points

3. Convex hull

4. Partition of unity

# Non uniform B-SPLINE CURVE

**Order of curve: K**

**No. of control points: n+1**

**No of segments of a curve: n-K+2**

**B-Spline basis function**

$$\boxed{x(u) = \sum_{i=0}^{n} N_{i,k}(u)\ x_i \qquad 0 \le u \le n - K + 2}$$

$$\boxed{x(u)=\sum_{i=0}^{n} N_{i,k}(u)\ x_i \qquad 0\le u\le n-K+2}$$

$$N_{i,k}(u)=\frac{(u-t_i)\,N_{i,K-1}(u)}{t_{i+K-1}-t_i}+\frac{(t_{i+K}-u)\,N_{i+1,K-1}(u)}{t_{i+K}-t_{i+1}}$$

$$\begin{aligned} N_{i,1}(u) &= 1 \qquad \text{if} \quad t_i\le u\le t_{i+1} \\ &= 0 \end{aligned}$$

$$\text{Knot values}: \ t_i \qquad (0\le i\le n+K)$$

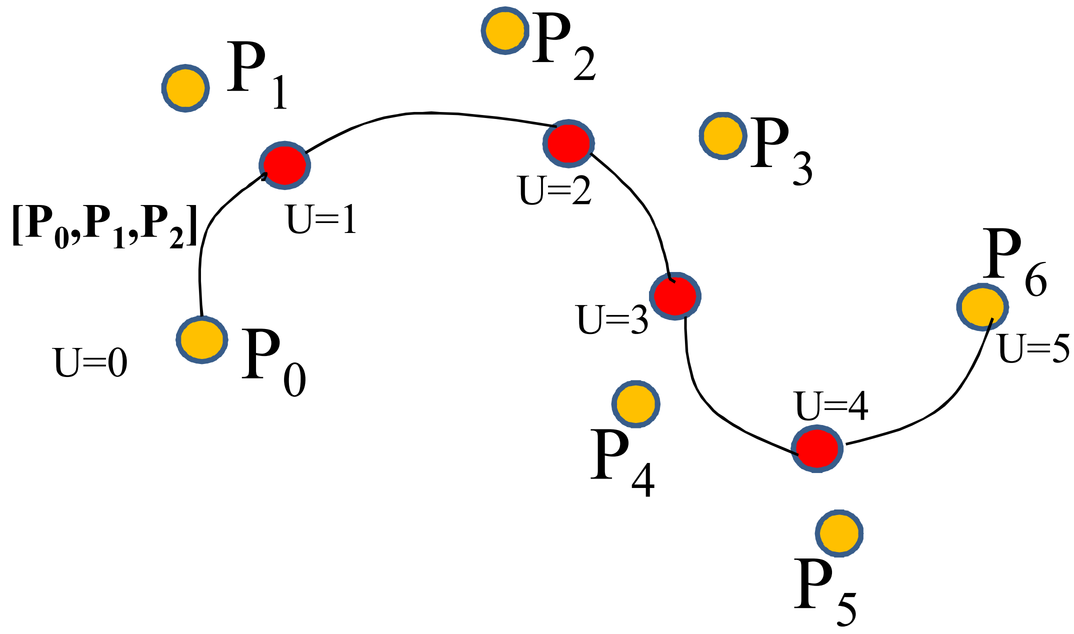
$$\begin{aligned} t_i &= 0 && \text{if} \quad i<K \\ &= i-K+1 && \text{if} \quad K\le i\le n \\ &= n-K+2 && \text{if} \quad i>n \end{aligned}$$

# Non uniform B-SPLINE CURVE example

**Order of curve:**
**K =3**

**No. of control points:**
**N+1=7**

**No of segments of a curve:**
**N-K+2=5**




| SEGMENT | $U_1$ | $U_2$ | CONTROL POINT |
|---|---|---|---|
| Segment-1 | 0 | 1 | $[P_0,P_1,P_2]$ |
| Segment-2 | 1 | 2 | $[P_1,P_2,P_3]$ |
| Segment-3 | 2 | 3 | $[P_2,P_3,P_4]$ |
| Segment-4 | 3 | 4 | $[P_3,P_4,P_5]$ |
| Segment-5 | 4 | 5 | $[P_4,P_5,P_6]$ |

| SEGMENT | $U_1$ | $U_2$ | CONTROL POINT |
|---|---|---|---|
| Segment-1 | 0 | 1 | $[P_0,P_1,P_2]$ |
| Segment-2 | 1 | 2 | $[P_1,P_2,P_3]$ |
| Segment-3 | 2 | 3 | $[P_2,P_3,P_4]$ |
| Segment-4 | 3 | 4 | $[P_3,P_4,P_5]$ |
| Segment-5 | 4 | 5 | $[P_4,P_5,P_6]$ |

| CONTROL POINT | SEGMENT | NUMBER OF SEGMENT(S) |
|---|---|---|
| $P_0$ | Segment-1 | 1 |
| $P_1$ | Segment-1<br>Segment-2 | 2 |
| $P_2$ | Segment-1<br>Segment-2<br>Segment-3 | 3 |
| $P_3$ | Segment-2<br>Segment-3<br>Segment-4 | 3 |
| $P_4$ | Segment-3<br>Segment-4<br>Segment-5 | 3 |
| $P_5$ | Segment-4<br>Segment-5 | 2 |
| $P_6$ | Segment-5 | 1 |

# CONTINUITY

**$C_0$ continuity/Point Continuity**

All points are connected

**$C_1$ continuity/Slope Continuity**

All points have unique slope(f')

**$C_2$ continuity/Curvature Continuity**

All points have unique curvature(f'')

# Properties of Non uniform B-SPLINE CURVE

1. The curve is $C_{K-2}$ continuous
2. Made up of N-K+2 segments
3. Only K control points affect any segment
4. A control point affects at most K curve segments

# Non uniform B-SPLINE SURFACE

## Non uniform B-SPLINE curve

$$\boxed{x(u) = \sum_{i=0}^{N} N_{i,k}(u)\ x_i}$$

## Non uniform B-SPLINE surface

$$\boxed{x(u,w) = \sum_{i=0}^{N} \sum_{j=0}^{M} N_{i,k}(u)\ N_{j,l}(w)\ x_{ij}}$$

# Non uniform rational B-SPLINE (NURBS)

**NURBS curve**

$$x(u) = \frac{\sum_{i=0}^{N} h_i N_{i,k}(u)\ x_i}{\sum_{i=0}^{N} h_i N_{i,k}(u)}$$

$h_i$ is the weightage of $x_i$

Family of NURBS

$h_i$=1 for non uniform B-Spline

**NURBS surface**

$$x(u,w) = \frac{\sum_{i=0}^{N} \sum_{j=0}^{M} h_i h_j N_{i,k}(u)\ N_{j,l}(w)\ x_{ij}}{\sum_{i=0}^{N} \sum_{j=0}^{M} h_i h_j N_{i,k}(u)\ N_{j,l}(w)}$$

END OF CHAPTER