Basic Discrete Structures

Sets, Functions, Sequences, Matrices, and Relations

(Lecture-6)

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Representing Properties of Relation using Square Matrices

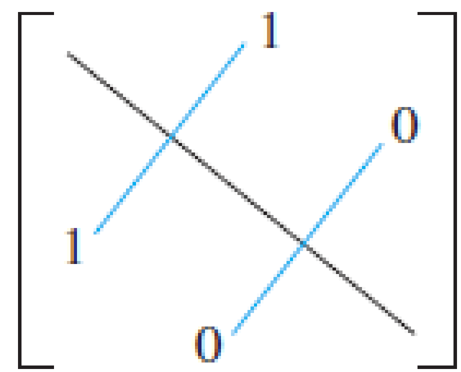
- Relation R on A is **reflexive** if $(a, a) \in R$ whenever $a \in A$.
  - R is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots, n$.
  - R is reflexive if and only if $m_{i,i} = 1$, for $i = 1, 2, \ldots, n$.
  - R is reflexive if all the elements on the main diagonal of M_R are equal to 1 (note that the elements off the main diagonal can be either 0 or 1

$$\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

FIGURE 1 The Zero–One Matrix for a Reflexive Relation. (Off

Representing Properties of Relation using Square Matrices

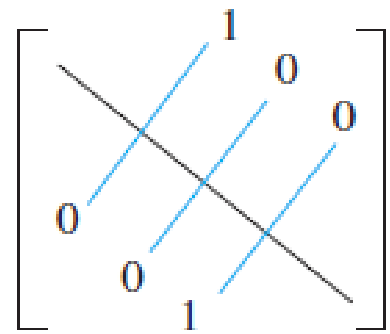
- Relation R on A is **symmetric** if $(a, b) \in R$ implies that $(b, a) \in R$, for all $a, b \in A$.
  - R is symmetric if and only if $(a_j, a_i) \in R$ whenever $(a_i, a_j) \in R$ for $i = 1, 2, \ldots, n$.
  - R is symmetric if and only if $m_{j,i} = 1$ whenever $m_{j,i} = 1$, for $i = 1, 2, \ldots, n$. This also means $m_{j,i} = 0$ whenever $m_{i,j} = 0$.
  - R is symmetric if and only if $m_{i,j} = m_{j,i}$ **for all pair of integers** i and j, with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.
  - R is symmetric if and only if $\boldsymbol{M_R = (M_R)^t}$
    - M_R is a symmetric matrix and



(a) Symmetric

Representing Properties of Relation using Square Matrices

- Relation R on A is **antisymmetric** if and only if $(a, b) \in R$ and $(b, a) \in R$, implies $a = b$, for some $a, b \in A$.
  - R is antisymmetric if it has the property that **<u>if $m_{i,j}=1$, then $m_{j,i}=0$ (for $i \neq j$)</u>**
  - In other words, either $m_{i,j} = 0$ or $m_{j,i} = 0$ when $i \neq j$.



(b) Antisymmetric

- Relation R on A is **transitive** if and only if **<u>$(a, b) \in R^2$ implies $(a, b) \in R$</u>** for some $a, b \in A$.
  - Take composition of the given relation with itself
    - Find Boolean product of the relation matrix by multiplying with itself.
  - $M_{R \circ R} = M_R^2 = M_R \odot M_R = M_R^{[2]}$
  - R is transitive **<u>if $m^{(2)}_{i,j}=1$, then $m_{i,j}=1$ (for all i,j)</u>**

Operations on Relational Matrices

- Boolean operations join and meet can be used to find the matrices representing the union and the intersection of two relations.
- Suppose that R_1 and R_2 are relations on a set A represented by the matrices $\boldsymbol{M}_{R1}$ and $\boldsymbol{M}_{R2}$, respectively.
- The matrix representing the union of these relations has a 1 in the positions where either $\boldsymbol{M}_{R1}$ or $\boldsymbol{M}_{R2}$ has a 1.
- The matrix representing the intersection of these relations has a 1 in the positions where both $\boldsymbol{M}_{R1}$ and $\boldsymbol{M}_{R2}$ have a 1.

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \qquad \text{and} \qquad \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}.$$

Representing Relations Using Digraphs

A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the *terminal vertex* of this edge.

- An edge of the form *(a, a)* is represented using an arc from the vertex a back to itself.
  - Such an edge is called a loop.

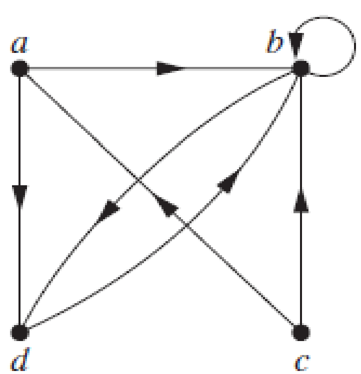


FIGURE 3
A Directed Graph.

- The relation R on a set A is represented by the directed graph that has the elements of A as its vertices and the ordered pairs (a, b), where $(a, b) \in R$, as edges.
- This assignment sets up a one-to-one correspondence between the relations on a set A and the directed graphs with A as their set of vertices.
- Thus, every statement about relations corresponds to a statement about directed graphs, and vice versa.
- Directed graphs give a visual display of information about relations.

Representing Relations Using Digraphs

- Directed graphs representing a relation can be used to determine whether the relation has various properties:
  - **Reflexive**: A relation is reflexive if and only if there is a loop at every vertex of the directed graph, so that every ordered pair of the form (x, x) occurs in the relation.
  - **Symmetric**: A relation is symmetric if and only if for every edge between distinct vertices in its digraph, there is an edge in the opposite direction, so that (y, x) is in the relation whenever (x, y) is in the relation.
  - **Antisymmetric**: A relation is antisymmetric if and only if there are never two edges in opposite directions between distinct vertices.
  - **Transitive**: A relation is transitive if and only if whenever there is an edge from a vertex x to a vertex y and an edge from a vertex y to a vertex z, there is an edge from x to z

Practice Set

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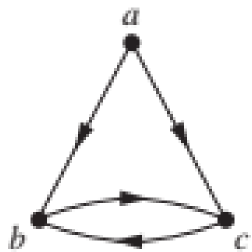

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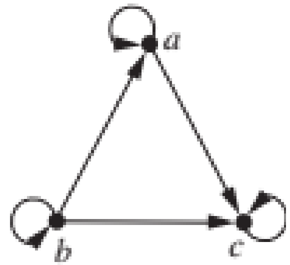

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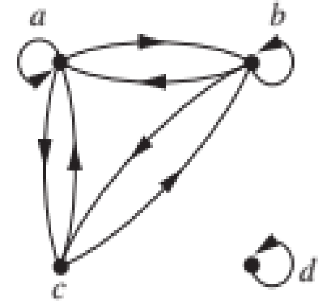

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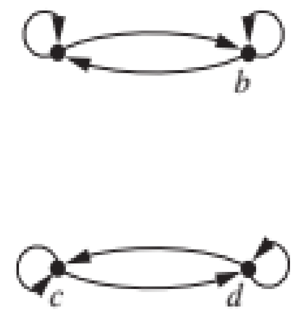

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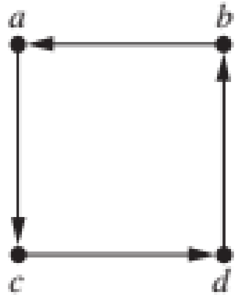

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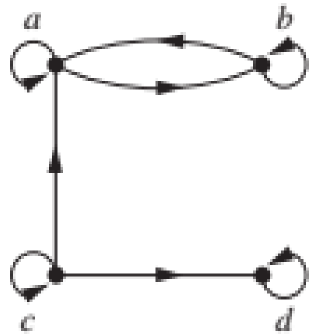

Determine whether the relations represented by the directed graphs are reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive.

a) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Determine whether the relations represented by the matrices in are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.