By Poisson distribution:

$$\text{Required probability} = P(X = m) = e^{-\lambda}\,\frac{\lambda^m}{m!} = e^{-100}\,\frac{(100)^m}{m!}.$$

3.8 UNIFORM (OR RECTANGULAR) DISTRIBUTION

Definition: A continuous random variable X is said to have a **uniform** (or **rectangular**) **distribution** over the interval $[a, b]$, $-\infty < a < b < \infty$, if its probability density function (p.d.f.) is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0 & , \text{elsewhere} \end{cases}$$

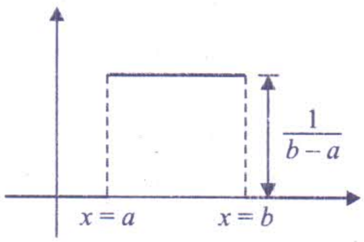
where a, b are two parameters of the distribution.

Obviously:

(*i*) $f(x) \ge 0$, $\forall\, x$.

(*ii*) $\int_{-\infty}^{\infty} f(x)\,dx = \int_a^b \dfrac{dx}{b-a} = 1.$

So, this is a valid probability distribution. The density curve is shown adjacent.



Distribution Function

The distribution function $F(x)$ of the r.v. X is given by

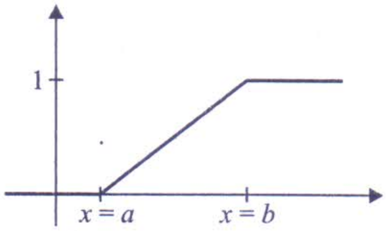
$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

when $\quad x < a : F(x) = 0$

when $\quad a \le x < b : F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_a^x \dfrac{dt}{b-a} = \dfrac{x-a}{b-a}$

when $\quad x \ge b : F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_a^b \dfrac{dt}{b-a} = 1.$

The distribution curve is shown below:



Example 1: *Electric trains on a certain line run every half hour between mid-night and 5 in the morning. Find the probability that a man entering the station at a random time during this period will have to wait at least fifteen minutes.*

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Solution: Let the random variable X corresponds to the waiting time for the next train between mid-night and 5 in the morning. So, X is distributed uniformly over the interval [0, 30] with p.d.f.:

$$f(x) = \begin{cases} \frac{1}{30}, & 0 \le x \le 30 \\ 0, & \text{elsewhere} \end{cases}$$

$\therefore$ P(the man has to wait at least 15 minutes)

$$= P(X \ge 15) = \int_{15}^{30} \frac{dx}{30} = \frac{1}{2}.$$

Example 2: *The random variable X corresponds to the position of a point chosen at random in the interval [a, b] in such a way that the probability that it lies in any sub-interval of [a, b] is proportional to the length of the sub-interval. Find the distribution function of X.*

Solution: Let $F(x)$ be the distribution function of the random variable X. From the conditions of the problem, we have

$$F(x) = P(X \le x) = \begin{cases} 0 & , -\infty < x < a \\ \lambda(x-a), & a \le x \le b \ (\lambda \text{ is a constant}) \\ 1 & , b < x < \infty \end{cases}$$

Since $F(b+0) = F(b)$, we have $1 = \lambda(b-a) \Rightarrow \lambda = 1/(b-a)$.

Hence X is uniformly distributed over $[a, b]$.

Remark: Usually the phrase 'a point is chosen at random in a given interval' means that the probability of its occurrence in any sub-interval of the given interval is proportional to the length of the sub-interval, *i.e.*, the random point has a uniform distribution over the given interval.

3.9 MEAN AND VARIANCE OF THE UNIFORM DISTRIBUTION

Theorem: *If a continuous random variable X has uniform distribution with parameters a and b, then*

(*i*) *mean* $= E(X) = \frac{1}{2}(a+b)$ *and* (*ii*) $Var(X) = \frac{(a-b)^2}{12}$.

Proof: The probability density function of the r.v. X is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0 & , \text{elsewhere} \end{cases}$$

(*i*) Mean $= E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{a} x \cdot 0\,dx + \int_{a}^{b} x \cdot \frac{1}{b-a}\,dx + \int_{b}^{\infty} x \cdot 0\,dx$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a} \cdot \frac{1}{2}(b^2 - a^2) = \frac{1}{2}(a+b).$$

(*ii*) Now, $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{a}^{b} x^2 \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$

$$= \frac{1}{3}(b^2 + ba + a^2)$$

$$\therefore \quad \text{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{1}{3}(a^2 + ab + b^2) - \frac{1}{4}(a+b)^2$$

$$= \frac{1}{12}(a-b)^2.$$

Example: *If X is a uniformly distributed random variable with mean 1 and variance 4/3, determine P(X < 0).*

Solution: Given, X is a uniformly distributed r.v. with mean 1 and variance 4/3. Let the parameters of the distribution are a, b ($> a$).

$$\therefore \quad E(X) = \frac{1}{2}(a+b) = 1 \text{ and } \text{Var}(X) = \frac{1}{12}(a-b)^2 = \frac{4}{3}$$

$$\Rightarrow \quad a + b = 2 \text{ and } b - a = 4 \quad (\because \; b > a). \quad \therefore \quad a = -1, b = 3.$$

Therefore, the probability density function of the random variable X is

$$f(x) = \begin{cases} \dfrac{1}{b-a} = \dfrac{1}{4}, & -1 \le x \le 3 \\ 0 & \text{, elsewhere} \end{cases}$$

$$\therefore \quad P(X < 0) = \int_{-\infty}^{0} f(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} \frac{1}{4}\,dx = \frac{1}{4}.$$

ILLUSTRATIVE EXAMPLES – III

Example 1: *If X is uniformly distributed over [1, 2], find u such that* $P(X > u + \bar{X}) = \frac{1}{6}$, *where* $\bar{X} = E(X)$.

Solution: Given, X is uniformly distributed over [1, 2], so its p.d.f. is

$$f(x) = \begin{cases} \dfrac{1}{2-1} = 1, & 1 \le x \le 2 \\ 0 & \text{, elsewhere} \end{cases}$$

$$\therefore \quad \bar{X} = E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx = \int_{1}^{2} x \,.\, 1\,dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{2}(2^2 - 1^2) = \frac{3}{2}$$

$$\therefore \quad P(X > u + \bar{X}) = \frac{1}{6} \Rightarrow P\left(X > u + \frac{3}{2}\right) = \frac{1}{6} \Rightarrow \int_{u+\frac{3}{2}}^{\infty} f(x)\,dx = \frac{1}{6}$$

$$\therefore \quad \int_{u+\frac{3}{2}}^{2} dx = \frac{1}{6} \Rightarrow [x]_{u+\frac{3}{2}}^{2} = \frac{1}{6} \Rightarrow 2 - u - \frac{3}{2} = \frac{1}{6} \Rightarrow u = \frac{1}{3}.$$

Example 2: *A random variable X has uniform distribution over (– 4, 4). Find*

(i) $P(X = 2)$, $P(X < 3)$, $P(|X| \le 2)$, $P(|X - 2| < 3)$,

(ii) λ *for which* $P(X > \lambda) = \dfrac{1}{3}$.

Solution: Given, the r.v. X is uniformly distributed over (–4, 4), so its p.d.f. is

$$f(x) = \begin{cases} \dfrac{1}{4-(-4)} = \dfrac{1}{8} &, -4 < x < 4 \\ 0 &, \text{elsewhere} \end{cases}$$

(i) $P(X = 2) = 0$, since the probability of a continuous random variable at a particular point is zero.

$$P(X < 3) = \int_{-\infty}^{3} f(x)\,dx = \int_{-\infty}^{-4} 0\,dx + \int_{-4}^{3} \frac{1}{8}\,dx = \frac{7}{8}.$$

$$P(|X| \le 2) = P(-2 \le X \le 2) = \int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} \frac{1}{8}\,dx = \frac{4}{8} = \frac{1}{2}.$$

$$P(|X-2| < 3) = P(-3 < X-2 < 3) = P(-1 < X < 5)$$

$$= \int_{-1}^{5} f(x)\,dx = \int_{-1}^{4} \frac{1}{8}\,dx + \int_{4}^{5} 0\,dx = \frac{5}{8}.$$

(ii)
$$P(X > \lambda) = \frac{1}{3} \Rightarrow \int_{\lambda}^{\infty} f(x)\,dx = \frac{1}{3} \Rightarrow \int_{\lambda}^{4} \frac{1}{8}\,dx = \frac{1}{3}$$

$\therefore$
$$\frac{1}{8}(4-\lambda) = \frac{1}{3}, \text{ or } 12 - 3\lambda = 8 \quad \therefore \quad \lambda = \frac{4}{3}.$$

Example 3: *A passenger arrives at a bus stop at 9 am knowing that the bus will arrive at some time uniformly distributed between 9 am and 9.30 am.*

(i) Find the probability that he will have to wait longer than 10 min.

(ii) If at 9.15 am the bus has not yet arrived, find the probability that he will have to wait at least 10 additional minutes.

Solution: Let the random variable X corresponds to the waiting time. According to the question X is uniformly distributed with p.d.f.:

$$f(x) = \begin{cases} \dfrac{1}{30} &, 0 \le x \le 30 \\ 0 &, \text{elsewhere} \end{cases}$$

(i) P(he will have to wait longer than 10 min.)

$$= P(X > 10)$$

$$= \int_{10}^{\infty} f(x)\,dx = \int_{10}^{30} \frac{1}{30}\,dx + \int_{30}^{\infty} 0\,dx = \frac{20}{30} = \frac{2}{3}.$$

(ii) Required probability

$$= P(X \ge 25 / X > 15)$$

$$= \frac{P\{(X \ge 25) \cap (X > 15)\}}{P(X > 15)}$$

$$= \frac{P(X \geq 25)}{P(X > 15)}.$$

Now, $$P(X \geq 15) = \int_{25}^{\infty} f(x)\,dx = \int_{25}^{30} \frac{1}{30}\,dx = \frac{5}{30}$$

and $$P(X > 15) = \int_{15}^{\infty} f(x)\,dx = \int_{15}^{30} \frac{1}{30}\,dx = \frac{15}{30}.$$

$\therefore$ Required probability $= \frac{5}{15} = \frac{1}{3}.$

3.10 EXPONENTIAL DISTRIBUTION

Definition: A continuous random variable X assuming non-negative values is said to have an **exponential distribution** with parameter λ (> 0) if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{elsewhere} \end{cases}$$

Observe that:

(*i*) $$f(x) \geq 0, \forall x$$

(*ii*) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx = \lim_{B \to \infty} \int_{0}^{B} \lambda e^{-\lambda x}\,dx$$

$$= \lim_{B \to \infty} \left[\frac{\lambda}{-\lambda} e^{-\lambda x} \right]_0^B = \lim_{B \to \infty} (1 - e^{-\lambda B}) = 1 \quad (\because \quad \lambda > 0)$$

So, this is a valid probability distribution.

Distribution Function

The distribution function $F(x)$ of the r.v. X is given by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

when $x < 0$: $$F(x) = 0.$$

when $x \geq 0$: $$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} \lambda e^{-\lambda t}\,dt = \lambda \left[\frac{e^{-\lambda t}}{-\lambda} \right]_0^x$$

$$= 1 - e^{-\lambda x}.$$

Mean and Variance

Theorem 1: *If a continuous random variable X has an exponential distribution with parameter* λ *(> 0), then*

(*i*) *mean* $= E(X) = \frac{1}{\lambda}$ (*ii*) *Var* $(X) = \frac{1}{\lambda^2}.$

Proof: The probability density function of the r.v. X is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

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(*i*) $$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^{\infty} x\,\lambda e^{-\lambda x}\, dx$$

$$= \lim_{B \to \infty} \int_0^B \lambda\, x e^{-\lambda x}\, dx = \lim_{B \to \infty} \int_0^{\lambda B} \frac{u}{\lambda} e^{-u}\, du \qquad \left[\begin{array}{l}\text{Put } \lambda x = u \\ \Rightarrow \lambda\, dx = du,\ x = \dfrac{u}{\lambda}\end{array}\right]$$

$$= \frac{1}{\lambda} \int_0^{\infty} u e^{-u}\, du \qquad (\because\ \lambda > 0)$$

$$= \frac{1}{\lambda} \int_0^{\infty} e^{-u}\, u^{2-1}\, du = \frac{\Gamma(2)}{\lambda} = \frac{1!}{\lambda} = \frac{1}{\lambda}.$$

(*ii*) Now, $$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\, dx = \int_0^{x} x^2\, \lambda e^{-\lambda x}\, dx$$

$$= \lim_{B \to \infty} \int_0^B \lambda x^2 e^{-\lambda x}\, dx = \lim_{B \to \infty} \int_0^{\lambda B} \frac{u^2}{\lambda^2} e^{-u}\, du \qquad [\text{Put } \lambda x = u]$$

$$= \frac{1}{\lambda^2} \int_0^{\infty} e^{-u} u^{3-1}\, du = \frac{\Gamma(3)}{\lambda^2} = \frac{2!}{\lambda^2} = \frac{2}{\lambda^2}.$$

$$\therefore \qquad \text{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Memoryless Property

Theorem 2: *If X is an exponentially distributed random variable then*

$$P(X \ge s + t/X \ge s) = P(X \ge t),\ \forall\ s,\ t > 0.$$

Proof: Given, X is an exponentially distributed random variable. So, its p.d.f. is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0\ (\lambda > 0) \\ 0 & , x < 0 \end{cases}$$

$$\therefore \qquad P(X \ge s) = \int_s^{\infty} f(x)\, dx = \lim_{B \to \infty} \int_s^B \lambda e^{-\lambda x} dx = \lim_{B \to \infty} \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_s^B$$

$$= \lim_{B \to \infty} (e^{-\lambda s} - e^{-\lambda B}) = e^{-\lambda s} \quad [\because\ e^{-\lambda B} \to 0 \text{ as } B \to \infty, \text{ because } \lambda > 0]$$

$$\therefore \qquad P(X \ge s + t/X \ge s) = \frac{P\{(X \ge s + t) \cap (X \ge s)\}}{P(X \ge s)} = \frac{P(X \ge s + t)}{P(X \ge s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X \ge t).$$

Note: It can be proved that the converse of this theorem is also true, *i.e.*, if X is a continuous random variable assuming non-negative values and possessing memoryless property $P(X \ge s + t/X \ge s) = P(X \ge t)$, $\forall s$, $t > 0$, then X follows an exponential distribution.

(*) Place E-12 →

Place after 'Note' in p. ~~148~~ 152

E-12

Remark: It is also common in the literature to define the exponential distribution using an alternate parametrization given by

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & \text{for } x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

Here θ is known as the survival parameter. In this alternate specification, the r.v. X is the duration of time that a biological or mechanical system manages to survive at the rate of θ. Here $E(X) = \theta$ and $Var(X) = \theta^2$, that is the expected rate of survival is θ and the expected variance of survival is θ^2.