Example ~~17~~ 18: *A batch of 50 fuses is known to have 10 defective fuses. If 10 fuses are selected at random, find the probability that all will be good.*

Solution: 10 fuses can be selected from 50 fuses in ${}^{50}C_{10}$ ways, so the total number of outcomes is

$$n = {}^{50}C_{10} = \frac{50\,!}{10\,!\,(50-10)\,!}$$

Let A denotes the event 'all 10 fuses will be good'.

$$\therefore \quad m(A) = {}^{50-10}C_{10} = {}^{40}C_{10} = \frac{40\,!}{10\,!\,(40-10)\,!}.$$

$$\therefore \quad P(A) = \frac{m(A)}{n} = \frac{40\,!}{10!\,30\,!}\cdot\frac{10\,!\,40\,!}{50\,!} = \frac{(40\,!)^2}{30\,!\,50\,!},$$ this is the required probability.

1.5 AXIOMATIC DEFINITION OF PROBABILITY *(W.B.U.T. 2005)*

Let E be a given random experiment and S be the corresponding event space. Let A be an event associated with the random experiment E, then probability of A, denoted by $P(A)$, is a real number which satisfies the following axioms:

(A_1) $P(A) \geq 0$ for every $A \in S$

(A_2) $P(S) = 1$

(A_3) if $A_1, A_2, \ldots, A_n, \ldots$ be a finite or infinite number of pairwise mutually exclusive events, *i.e.*, $A_i A_j = \varphi$ for $i \neq j$ and $A_i, A_j \in S$, then $P(A_1 + A_2 + \ldots + A_n + \ldots) = P(A_1) + P(A_2) + \ldots + P(A_n) + \ldots$

The entire mathematical structure of the theory of probability can be logically built up on the basis of these three axioms.

Deductions from Axiomatic Definition

Theorem 1: *Probability of an impossible event is zero, i.e., $P(\varphi) = 0$.*

Proof: Now, $S + \varphi = S.$

Both S and φ are disjoint, *i.e.*,

$$S\,\varphi = \varphi.$$

$$\therefore \quad P(S + \varphi) = P(S)$$

$$\Rightarrow \quad P(S) + P(\varphi) = P(S) \qquad [\text{By } (A_3)]$$

$$\Rightarrow \quad P(\varphi) = 0.$$

Theorem 2: *Probability of the complementary event $\bar{A}$ of A is $P(\bar{A}) = 1 - P(A)$. Also, $0 \leq P(A) \leq 1$.*

Proof: A and $\bar{A}$ are disjoint events.

Moreover, $A + \bar{A} = S$

$$\therefore \quad P(A + \bar{A}) = P(S)$$

$$\Rightarrow \quad P(A) + P(\bar{A}) = P(S) \qquad [\text{By } A_3]$$

$$\Rightarrow \quad P(A) + P(\bar{A}) = 1 \qquad [\text{By } A_2]$$

$$\Rightarrow \quad P(\bar{A}) = 1 - P(A)$$

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Again since $P(A) \geq 0$ [By A_1]

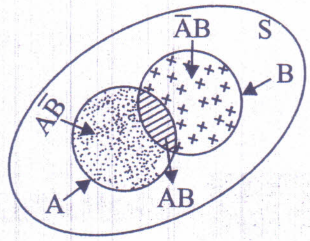
$\therefore \quad 0 \leq P(\bar{A}) \leq 1$

$\Rightarrow \quad 0 \leq P(A) \leq 1.$

Theorem 3: *For any two events A and B,*

$$P(\bar{A}B) = P(B) - P(AB).$$

Proof: Now, AB and $\bar{A}B$ are disjoint.



Also, $\quad AB + \bar{A}B = B$

$\Rightarrow \quad P(AB + \bar{A}B) = P(B)$

$\Rightarrow \quad P(AB) + P(\bar{A}B) = P(B)$ [By A_3]

$\Rightarrow \quad P(\bar{A}B) = P(B) - P(AB).$

Note: $\bar{A}\ \bar{B} + (A + B) = S.$

Theorem 4 *(Addition Theorem)*: *For any two events A and B (may not be mutually exclusive),*

$$P(A + B) = P(A) + P(B) - P(AB).$$ *(W.B.U.T. 2004, 2007)*

Proof: From the figure of Theorem 3, we have $A + B = A + \bar{A}B$, also A and $\bar{A}B$ are disjoint.

$\therefore \quad P(A + B) = P(A + \bar{A}B)$

$= P(A) + P(\bar{A}B)$ [By A_3]

$= P(A) + P(B) - P(AB)$ [By Theorem 3]

Theorem 5 *(Extension of Theorem 4)*: *For any three events A, B, C*

$$P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC).$$

Proof: $\quad \text{LHS} = P(A + B + C) = P\{(A + B) + C\}$

$= P(A + B) + P(C) - P\{(A + B)C\}$ [By Theorem 4]

$= P(A) + P(B) - P(AB) + P(C) - P(AC + BC)$ [By Theorem 4]

$= P(A) + P(B) + P(C) - P(AB) - \{P(AC) + P(BC) - P(ABC)\}$

[By Theorem 4 and since $(AC)(BC) = ABC$]

$= \text{RHS}.$

Deduction of Classical Definition

Let the event space (of a random experiment E) contains n distinct event points (*i.e.*, simple events) $u_1, u_2, \ldots, u_n$.

$\therefore \quad u_1 + u_2 + \ldots + u_n = S$

$\Rightarrow \quad P(u_1 + u_2 + \ldots + u_n) = P(S)$

$\Rightarrow \quad P(u_1) + P(u_2) + \ldots + P(u_n) = 1.$

Let the simple events have equal probability,

i.e., $$P(u_1) = P(u_2) = \ldots = P(u_n) = \frac{1}{n}.$$

Now suppose A be an event connected to the given random experiment E. If A contains m ($\leq n$) event points of S, then without loss of generality we can write

$$A = u_1 + u_2 + \dots + u_m$$

$$\begin{aligned}\therefore \quad P(A) &= P(u_1 + u_2 + \dots + u_m)\\ &= P(u_1) + P(u_2) + \dots + P(u_m)\\ &= \underbrace{\frac{1}{n} + \frac{1}{n} + \dots\dots + \frac{1}{n}}_{\rightarrow m \text{ terms} \leftarrow}\\ &= \frac{m}{n} = \frac{m(A)}{n}\end{aligned}$$

which is the classical definition of probability deduced from axiomatic definition of probability.

Theorem 6: *For any two events A and B, $P(AB) \leq P(A) \leq P(A + B) \leq P(A) + P(B)$.*

(*W.B.U.T. 2004, 2008*)

Proof: Now, AB, $A\bar{B}$ are disjoint and $AB + A\bar{B} = A$ (See the Fig. of Th. 3)

$$\therefore \quad P(AB + A\bar{B}) = P(A)$$

$$\Rightarrow \quad P(AB) + P(A\bar{B}) = P(A)$$

Since $0 \leq P(A\bar{B}) \leq 1$, so $P(AB)$ can never exceed $P(A)$

i.e., $$P(AB) \leq P(A) \qquad \dots(1)$$

Again, $$P(A + B) = P(A) + P(B) - P(AB) \qquad \text{(By Theorem 4)}$$
$$= P(A) + P(\bar{A}B) \qquad \text{(By Theorem 3)}$$

Since $$0 \leq P(\bar{A}B) \leq 1, \text{ so } P(A) \leq P(A + B) \qquad \dots(2)$$

Now, $$P(A + B) = P(A) + P(B) - P(AB) \leq P(A) + P(B) \quad [\because \ 0 \leq P(AB) \leq 1] \qquad \dots(3)$$

Combining (1), (2) and (3), we get

$$P(AB) \leq P(A) \leq P(A + B) \leq P(A) + P(B).$$

Note: Similarly, $P(AB) \leq P(B) \leq P(A + B) \leq P(A) + P(B)$

Theorem 7 (*Boole's Inequality*): *For any n events $A_1, A_2, \dots, A_n$ connected to a random experiment E,*

$$P(A_1 + A_2 + \dots + A_n) \leq P(A_1) + P(A_2) + \dots + P(A_n)$$

Proof: We have for any two events A_1, A_2,

$$P(A_1 + A_2) = P(A_1) + P(A_2) - P(A_1 A_2) \leq P(A_1) + P(A_2)$$
$$[\because \ P(A_1 A_2) \geq 0] \qquad \dots(1)$$

Therefore the given inequality is true for $n = 1, 2$.

Let us assume that the given inequality is true for any positive integer $m \geq 2$,

i.e., $$P(A_1 + A_2 + \dots + A_m) \leq P(A_1) + P(A_2) + \dots + P(A_m) \qquad \dots(2)$$

$$\begin{aligned}\therefore \quad P(A_1 + A_2 + \dots + A_m + A_{m+1}) &= P[(A_1 + A_2 + \dots + A_m) + A_{m+1}]\\ &\leq P(A_1 + A_2 + \dots + A_m) + P(A_{m+1}) \qquad \text{[By (1)]}\\ &\leq P(A_1) + P(A_2) + \dots + P(A_m) + P(A_{m+1}) \qquad \text{[By (2)]}\end{aligned}$$

Thus the given inequality is true for $m + 1$ whenever it is true for m. But we have already seen that this inequality is true for 2, hence it is true for $2 + 1 = 3$, $3 + 1 = 4$, $4 + 1 = 5$, etc. Hence the given inequality is true for any positive integer n.

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Theorem 8 ***(Bonferroni's Inequalities):*** *For any n events $A_1, A_2, \ldots, A_n$ connected to a random experiment E,*

$$(i)\ P(A_1 A_2 \ldots A_n) \geq 1 - \sum_{i=1}^{n} P(\bar{A}_i)$$

$$(ii)\ P(A_1 A_2 \ldots A_n) \geq \sum_{i=1}^{n} P(A_i) - (n-1).$$ *(W.B.U.T. 2003)*

Proof: (*i*) We have by Boole's inequality,

$$P(\bar{A}_1 + \bar{A}_2 + \ldots + \bar{A}_n) \leq P(\bar{A}_1) + P(\bar{A}_2) + \ldots + P(\bar{A}_n)$$

$$\Rightarrow \quad P(\overline{A_1 A_2 \ldots A_n}) \leq P(\bar{A}_1) + P(\bar{A}_2) + \ldots + P(\bar{A}_n) \quad \text{[By De Morgan's law]}$$

$$\Rightarrow \quad 1 - P(A_1 A_2 \ldots\ldots A_n) \leq \sum_{i=1}^{n} P(\bar{A}_i)$$

$$\Rightarrow \quad 1 - \sum_{i=1}^{n} P(\bar{A}_i) \leq P(A_1 A_2 \ldots A_n)$$

$$\Rightarrow \quad P(A_1 A_2 \ldots A_n) \geq 1 - \sum_{i=1}^{n} P(\bar{A}_i)$$

(*ii*) We know that $\quad P(\bar{A}_i) = 1 - P(A_i)$

$$\therefore \quad 1 - \sum_{i=1}^{n} P(\bar{A}_i) = 1 - \sum_{i=1}^{n} [1 - P(A_i)] = \sum_{i=1}^{n} P(A_i) - (n-1)$$

Using (*i*): $\quad P(A_1 A_2 \ldots A_n) \geq \sum_{i=1}^{n} P(A_i) - (n-1).$

Example: *In an examination 60%, 70% and 75% students have passed in the subjects Physics, Chemistry and Mathematics respectively. Find the minimum percentage of students who have passed in all the subjects.*

Solution: Let *A*, *B* and *C* be the events defined as follows:

$A \equiv$ student chosen passes the subject Physics,

$B \equiv$ student chosen passes the subject Chemistry,

$C \equiv$ student chosen passes the subject Mathematics.

Given: $\quad P(A) = \dfrac{60}{100} = \dfrac{3}{5},\ P(B) = \dfrac{70}{100} = \dfrac{7}{10},\ P(C) = \dfrac{75}{100} = \dfrac{15}{20}.$

$$\therefore \quad P(A) + P(B) + P(C) = \frac{3}{5} + \frac{7}{10} + \frac{15}{20} = \frac{12 + 14 + 15}{20} = \frac{41}{20}.$$

Now, applying Bonferroni's inequality, we have *P*(student chosen passes all subjects)

$$= P(ABC) \geq P(A) + P(B) + P(C) - (3 - 1)$$

$$= \frac{41}{20} - 2 = \frac{1}{20} = 0.05$$

Hence, the minimum value of $P(ABC)$ is 0.05. Therefore, at least 5% students may pass in all the subjects.

1.6 CONDITIONAL PROBABILITY

Let A and B are any two events connected to a given random experiment E. The **conditional probability** of the event A on the hypothesis that the event B has occurred is denoted and defined by

$$P(A/B) = \frac{P(AB)}{P(B)}, \text{ provided } P(B) \neq 0.$$

Similarly, we can define

$$P(B/A) = \frac{P(AB)}{P(A)}, \text{ provided } P(A) \neq 0.$$

Note: The existence of conditional probabilities are guaranteed by Theorem 6, Art. 1.5.

Example 1: Let E be the random experiment of throwing of a die with event space S. Let A denotes the event 'even face' and B denotes the event 'multiple of 3'.

$$\therefore \quad S = \{1, 2, 3, 4, 5, 6\}, A = \{2, 4, 6\}, B = \{3, 6\}, AB = \{6\}$$

$$\therefore \quad P(A) = \frac{3}{6} = \frac{1}{2}, P(B) = \frac{2}{6} = \frac{1}{3}, P(AB) = \frac{1}{6}.$$

$$\therefore \quad P(A/B) = \frac{P(AB)}{P(B)} = \frac{1/6}{1/3} = \frac{1}{2}.$$

$$P(B/A) = \frac{P(AB)}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3}.$$

Example 2: *Let A and B are two events such that $P(A \cup B) = 7/8$, $P(A \cap B) = 1/4$ and $P(A^c) = 5/8$. Find the conditional probability of A on the hypothesis that B does not occur.* (W.B.U.T. 2003)

Solution: Given: $P(A \cup B) = 7/8$, $P(A \cap B) = 1/4$, $P(A^c) = 5/8$.

We have to find $$P(A/B^c) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{P(A) - P(A \cap B)}{1 - P(B)} \qquad ...(1)$$

Now, $$P(A) = 1 - P(A^c) = 1 - \frac{5}{8} = \frac{3}{8}.$$

We know that, $$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

$$\Rightarrow \quad \frac{7}{8} = \frac{3}{8} + P(B) - \frac{1}{4}$$

$$\Rightarrow \quad P(B) = \frac{7}{8} - \frac{3}{8} + \frac{1}{4} = \frac{7 - 3 + 2}{8} = \frac{3}{4}.$$

$$\therefore \text{ Required probability } = P(A/B^c) = \frac{P(A) - P(A \cap B)}{1 - P(B)} \qquad \text{[By (1)]}$$

$$= \frac{\frac{3}{8} - \frac{1}{4}}{1 - \frac{3}{4}} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}.$$

Theorem 1: *If $A_1, A_2, ..., A_n$ are pairwise mutually exclusive events one of which certainly occurs, then*

$$P(B) = \sum_{i=1}^{n} P(A_i)\, P(B/A_i)$$

where B is any event connected to the same random experiment provided the conditional probabilities are defined.

Proof: Here $A_i A_j = \varphi$, for all $i \neq j$; $i, j = 1, 2, ..., n$.

$A_1 + A_2 + ... + A_n$ = certain event = S, since one of $A_1, A_2, ..., A_n$ certainly occurs.

$\therefore$ $$B = SB = (A_1 + A_2 + ... + A_n)\, B = A_1 B + A_2 B + ... + A_n B.$$

Also, $$(A_i B)(A_j B) = (A_i A_j) B = \varphi B = \varphi \quad (i \neq j;\ i, j = 1, 2, ..., n)$$

$\therefore$
$$\begin{aligned} P(B) &= P(A_1 B + A_2 B + ... + A_n B) \\ &= P(A_1 B) + P(A_2 B) + ... + P(A_n B) \\ &= \sum_{i=1}^{n} P(A_i B) \\ &= \sum_{i=1}^{n} P(A_i)\, P(B/A_i) \qquad \left[\because\ P(B/A_i) = \frac{P(A_i B)}{P(A_i)},\ i = 1, 2, ..., n\right] \end{aligned}$$

$\therefore$ $$P(B) = \sum_{i=1}^{n} P(A_i)\, P(B/A_i) \qquad ...(1)$$

Notes: (*i*) This theorem is known as the **theorem on total probability.**

(*ii*) By definition: $P(A_i/B) = \dfrac{P(A_i B)}{P(B)} = \dfrac{P(A_i)\, P(B/A_i)}{P(B)}$

Using (1): $P(A_i/B) = \dfrac{P(A_i)\, P(B/A_i)}{\sum_{r=1}^{n} P(A_r)\, P(B/A_r)}$, provided $P(B) \neq 0$.

This is known as **Baye's theorem.** (*W.B.U.T. 2012*)

Example 3: *A bag contains 4 white and 2 black balls and a second bag contains 3 of each colours. A bag is selected at random and a ball is then taken out at random from the bag chosen. What is the probability that the ball selected is a white?*

Solution: Let A_i $(i = 1, 2)$ be the event of selecting i^{th} bag and B be the event of selecting a white ball.

$\therefore$ $$P(A_1) = P(A_2) = \frac{1}{2};\ P(B/A_1) = \frac{4}{6} = \frac{2}{3};\ P(B/A_2) = \frac{3}{6} = \frac{1}{2}.$$

$\therefore$
$$\begin{aligned} P(B) &= P(A_1)\, P(B/A_1) + P(A_2)\, P(B/A_2) \\ &= \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}. \end{aligned}$$

Example 4: *Suppose that there is a chance for newly constructed house to collapse whether the design is faulty or not. The chance that the design is faulty is 10%. The chance that the house collapse if the design is faulty is 95% and otherwise it is 45%. It is seen that the house collapsed. What is the probability that it is due to faulty design?*

Solution: Let A, $\overline{A}$ and B denote the following events:

$A \equiv$ Design is faulty, $\bar{A} \equiv$ design is not faulty and $B \equiv$ the house collapses.

Given, $P(A) = 0.1$, $P(B/A) = 0.95$, $P(B/\bar{A}) = 0.45$.

$\therefore$ $P(\bar{A}) = 1 - P(A) = 0.9$.

$\therefore$ Required probability $= P(A/B)$

$$= \frac{P(A)\,P(B/A)}{P(A)\,P(B/A) + P(\bar{A})\,P(B/\bar{A})} \quad \text{[By Baye's theorem]}$$

$$= \frac{0.1 \times 0.95}{0.1 \times 0.95 + 0.9 \times 0.45} = 0.19.$$

Example 5: *The chance that a doctor will diagnose a certain disease correctly is 60%. The chance that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of the doctor, who had the disease, dies. What is the probability that the disease was diagnosed correctly?*

Solution: Let A_1 denotes the event 'the disease was diagnosed correctly by the doctor' and A_2 denotes the event 'a patient who has the disease dies'.

Given, $P(A_1) = \frac{60}{100} = \frac{3}{5}$, $P(\bar{A}_1) = 1 - P(A_1) = 1 - \frac{3}{5} = \frac{2}{5}$,

$P(A_2/A_1) = \frac{40}{100} = \frac{2}{5}$ and $P(A_2/\bar{A}_1) = \frac{70}{100} = \frac{7}{10}$.

Required probability

= P(Disease was diagnosed correctly given that the patient dies)

$$= P(A_1/A_2) = \frac{P(A_1)\,P(A_2/A_1)}{P(A_1)\,P(A_2/A_1) + P(\bar{A}_1)\,P(A_2/\bar{A}_1)}$$

[By Baye's theorem]

$$= \frac{\frac{3}{5} \times \frac{2}{5}}{\frac{3}{5} \times \frac{2}{5} + \frac{2}{5} \times \frac{7}{10}} = \frac{12}{12 + 14} = \frac{6}{13}.$$

Example 6: *There were three candidates A, B and C for the position of a manager whose chances of getting the appointment are in the proportion 4 : 2 : 3 respectively. The probability that A, if selected, would launch a new product in the market is 0.3. The probabilities of B and C doing the same are 0.5 and 0.8 respectively. What is the probability that the new product was launched in the market by C?*

(W.B.U.T. 2006)

Solution: Let X, Y, Z respectively denote the events that A, B, C are selected as a manager and N denotes the event that a new product is launched in the market.

Given, $P(X) = \frac{4}{9}$, $P(Y) = \frac{2}{9}$, $P(Z) = \frac{3}{9}$,

$P(N/X)$ = Pr. that a new product is launched in market by $A = 0.3$

$P(N/Y)$ = Pr. that a new product is launched in market by $B = 0.5$

$P(N/Z)$ = Pr. that a new product is launched in market by $C = 0.8$

Using Baye's theorem, the required probability is

$$P(Z/N) = \frac{P(Z)\,P(N/Z)}{P(X)\,P(N/X) + P(Y)\,P(N/Y) + P(Z)\,P(N/Z)}$$

$$= \frac{\frac{3}{9}\times 0.8}{\frac{4}{9}\times 0.3 + \frac{2}{9}\times 0.5 + \frac{3}{9}\times 0.8} = \frac{24}{12+10+24} = \frac{24}{46} = \frac{12}{23}.$$

Example 7: *The probabilities of X, Y and Z becoming the Principal of a college are respectively 0.3, 0.5 and 0.2. The probabilities that 'Student Aid-Fund' will be introduced in the college if X, Y and Z become Principal, are 0.4, 0.6 and 0.1 respectively. Given that 'Student Aid-Fund' has been introduced, find the probability that Y has been appointed as the Principal.* (*W.B.U.T. 2004*)

Solution: Let A, B and C respectively denote the events that X, Y and Z are becoming the Principal and F denotes the event that 'Student Aid-Fund' is introduced.

Given: $P(A) = 0.3$, $P(B) = 0.5$, $P(C) = 0.2$

$P(F/A) = P$(Student Aid-Fund will be introduced if X becomes Principal) = 0.4.

Similarly, $P(F/B) = 0.6$ and $P(F/C) = 0.1$.

Required probability

$= P(B/F)$

$= P(Y$ becomes Principal given Student Aid-Fund has been introduced).

Using Baye's theorem,

$$P(B/F) = \frac{P(B)\,P(F/B)}{P(A)\,P(F/A) + P(B)\,P(F/B) + P(C)\,P(F/C)}$$

$$= \frac{0.5\times 0.6}{0.3\times 0.4 + 0.5\times 0.6 + 0.2\times 0.1} = \frac{30}{12+30+2} = \frac{30}{44} = \frac{15}{22}.$$

Example 8: *Three identical boxes, I, II and III contain respectively 4 white and 3 red balls, 3 white and 7 red balls, 2 white and 3 red balls. A box is chosen at random and a ball is drawn out of it. If the ball is found to be white, what is the probability that box II is selected?* (*W.B.U.T. 2007*)

Solution: Let A_1, A_2 and A_3 denote the events that the ball is drawn from boxes I, II and III respectively. Clearly the events A_1, A_2 and A_3 are mutually exclusive and exhaustive.

$$\therefore \quad P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}.$$

Also let A denote the event that the drawn ball is white.

$$\therefore \quad P(A/A_1) = \frac{4}{7},\ P(A/A_2) = \frac{3}{10} \text{ and } P(A/A_3) = \frac{2}{5}.$$

Using Baye's theorem, the required probability is

$$P(A_2/A) = \frac{P(A_2)\,P(A/A_2)}{P(A_1)\,P(A/A_1) + P(A_2)\,P(A/A_2) + P(A_3)\,P(A/A_3)}$$