Graph - 4.

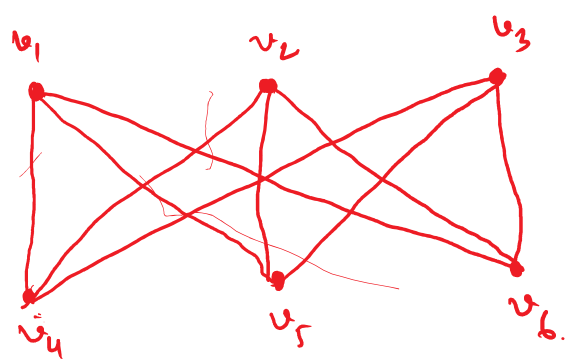
Monday, December 7, 2020 10:53 AM

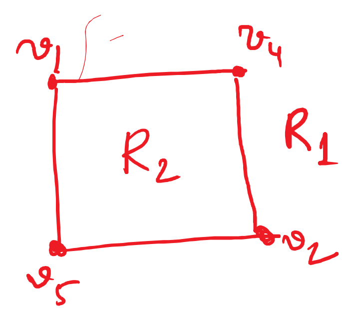
Best case
Avg. case
Worst case. → Lower bound.

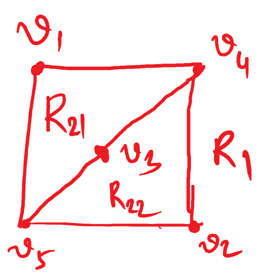
Travelling Salesman Problem (TSP)

- No algorithm with polynomial worst-case time complexity exists.
- Alternate approach — approximation algorithms.
- They do not necessarily produce the exact solution to the problem, but instead guarantees to produce a solution that is close to the exact solution.
- They may produce a HAM-CIRCUIT with a total length W' such that $W \leq W' \leq c \cdot W$, where W is the total length of an exact Solution and 'c' is a Constant.

$K_{3,3}$.







There is no way to place the final vertex v_6 without forcing a crossing,

Hence, $K_{3,3}$ is not a planar graph.

of

Prob. No. of vertices = 20.

Degree of each vertex = 3.

$\therefore$ Sum of the degrees of the vertices $= (20\times 3) = \underline{60}$

We know, $2e = \sum_{v\in V} \deg(v)$

$$\therefore \quad e = 60/2 = \underline{30}.$$

$\therefore$ No of regions in the planar representation $=$

$e - v + 2 = 30 - 20 + 2 = \underline{12}.$

Corollary-1

Proof: Let a connected planar simple graph divides a plane into r regions.

The degree of each region is at least three as the no. of vertices, $v \geq 3$.

As each edge occurs on the boundary of regions exactly twice, the sum of the degrees of the regions is exactly twice the no. of edges.

Because the degree of each region is at least three, it follows,

$$2e = \sum_{\text{all region } R} \deg(R) \geq 3r.$$

Hence, $(2/3)e \geq r$.

Usi Euler's formula we get,

$r = e - v + 2$

(3)

$r = e - v + 2$

Using Euler's formula we get,

$$e - v + 2 \leq \left(\frac{2}{3}\right)e$$

or, $\left(\frac{1}{3}\right)e \leq v - 2$

$\therefore$ $\boxed{e \leq 3v - 6}$

Corollary-2

Proof: If G has one or two vertices, the result is true. If G has at least three vertices, by Corollary-1, we have $e \leq 3v - 6$ so that

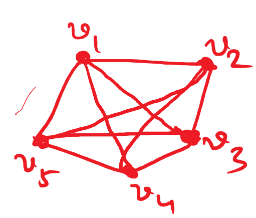
$\boxed{2e \leq 6v - 12}$

If the degree of every vertex were at least six then because $2e = \sum_{v \in V} \deg(v)$, we could have $\boxed{2e \geq 6v.}$

But this is a contradiction to the inequality $2e \leq 6v - 12$. It follows that there must be a vertex with degree no greater than five.

Prob. Show that K_5 is non planar using Corollary-1.

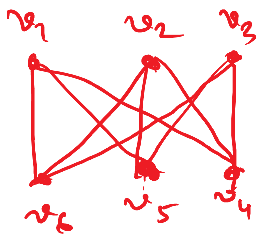
→ In graph K_5, there are 5 vertices and 10 edges.



H nality $e \leq 3v - 6$ is

However, the inequality $e \leq 3v - 6$ is not satisfied. Therefore, K_5 is not planar.

Prob. Show that $K_{3,3}$ is planar or non-planar using Corollary-1.



→ The graph $K_{3,3}$ has 6 vertices and 9 edges. the inequality $e \leq 3v - 6$

The ^in^equality satisfies. So according to Corollary-1 $K_{3,3}$ is planar. However, it can be shown that $K_{3,3}$ is nonplanar. Therefore, even if the inequality in Corollary-1 is satisfied, it does not imply that the graph is planar.

Corollary-3

If a connected planar ^simple^ graph has e edges and v vertices, with $v \geq 3$, and no circuit of length three exist, then $e \leq 2v - 4$.

Proof: As the connected planar simple graph does not have a circuit of length three, so the minimum degree of each region is four.

Therefore, as each region has degree greater than or equal to 4, it follows,

$$2e = \sum_{\text{for all region } R} \deg(R) \geq 4 \cdot r,$$

where r is the no. of regions into which h di 'd

of the graph divides the plane.
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or, $\left(\frac{1}{2}\right) e \geq r$

$\therefore$ Using $r = e - v + 2$ (Euler's formula) we have,

$$\left(\frac{1}{2}\right) e \geq e - v + 2$$

or, $e - v + 2 \leq \frac{1}{2} e$

or, $\boxed{e \leq 2v - 4.}$ (Proved)

Prob. Use Corollary-3 to show that $K_{3,3}$ is not planar.

No. of vertices = 6 and no. of edges = 9.

As no circuit of length three exists, so we can use the Corollary-3. Because, $e = 9$ and $2v - 4 = 8$, the inequality $e \leq 2v - 4$ is not satisfied. Consequently, $K_{3,3}$ is not a planar graph.

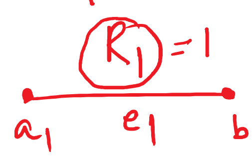
Euler's Formula.

Let G be a connected planar simple graph with e edges and v vertices. Let r be the no. of regions in a planar representation of G. Then $r = e - v + 2$.

Inductive defn: We will prove the formula by constructing a sequence of subgraphs $G_1, G_2, \ldots, G_e = G$, successively adding an edge at each stage.

We arbitrarily pick the edge of G to obtain G_1. Obtain G_n from G_{n-1} by arbitrarily adding an edge that is incident with a vertex already present in G_{n-1}, adding the other vertex incident with this edge if it is not in G_{n-1}. This construction is possible as G is connected. G is obtained after adding 'e' edges. Let r_n, e_n, and v_n represent the no. of regions, edges, and vertices of the planar representation of G_n, respectively.

Basis Step: The relationship $r_1 = e_1 - v_1 + 2$ is true for G_1 because $e_1 = 1$, $v_1 = 2$, and $r_1 = 1$.



Inductive hypothesis: Assume that $r_K = e_K - v_K + 2$. Let $\{a_{K+1}, b_{K+1}\}$ be the edge that is added to G_K to obtain G_{K+1}. There are two possibilities:

(i) Both a_{K+1} and b_{K+1} are already present in G_K:

These two vertices must be present on the boundary of a common region R, or else

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it would be impossible to draw the edge (a_{k+1}, b_{k+1}) to G_k without crossing.

The addition of this new edge splits R into two regions. Consequently, $r_{k+1} = r_k + 1$, $e_{k+1} = e_k + 1$, and $v_{k+1} = v_k$.

Therefore,

$$r_{k+1} = (e_k + 1) - (v_k) + 2$$
$$= e_{k+1} - v_{k+1} + 2$$

implying that the formula is still true.

(ii) One of the two vertices of the new edge is not already in G_k:

Suppose $a_{k+1} \in G_k$, but $b_{k+1} \notin G_k$. Adding a new edge does not produce a new region, because b_{k+1} must be in the same region as a_{k+1}. Consequently, $r_{k+1} = r_k$, $e_{k+1} = e_k + 1$, and $v_{k+1} = v_k + 1$.

Therefore,

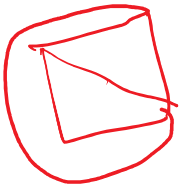
$$r_{k+1} = (e_k + 1) - (v_k + 1) + 2$$

$$\Rightarrow r_{k+1} = e_{k+1} - v_{k+1} + 2$$

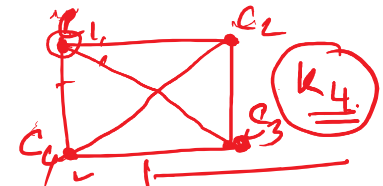
implying that the formula is still true.

Graph Coloring

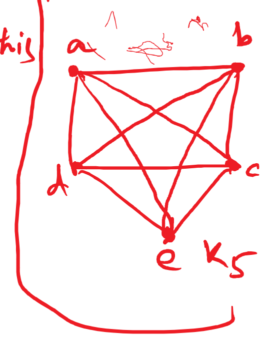
Prob: What is the chromatic number of K_n?



$\rightarrow$ The chromatic number of K_n is n i.e. $\chi(K_n) = n$.



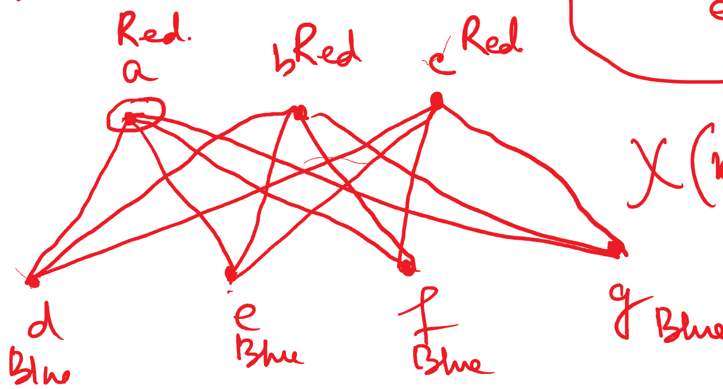
(K_n is not planar when $n \geq 5$, so this result does not contradict the four color theorem).



Prob:

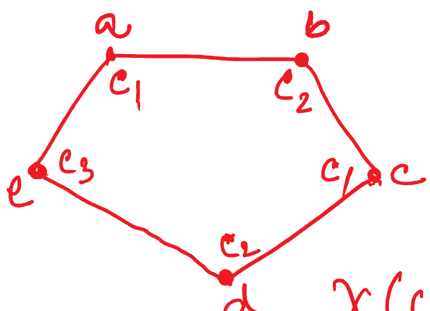
$m = 3$

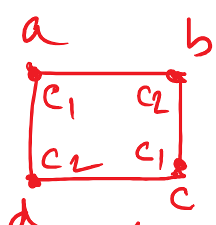
$n = 4$



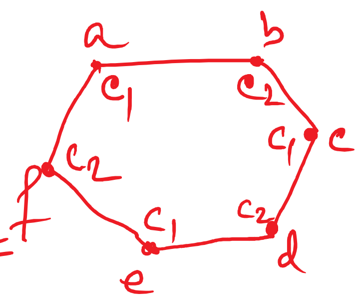
$\chi(K_{m,n}) = 2$

Ex: What is the chromatic number of the graph C_n, when $n \geq 3$



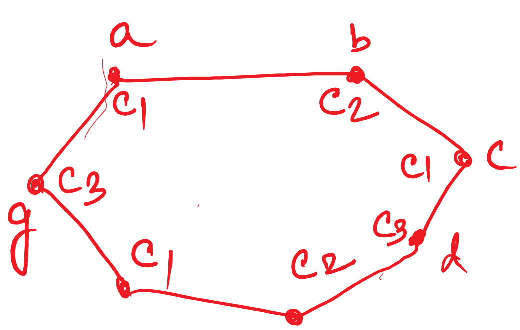


$\chi(C_4) = 2$

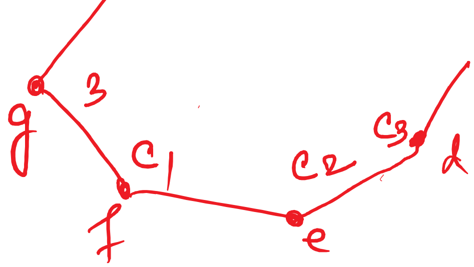


$\chi(C_5) = 3$.

$\chi(C_6) = 2$



$\chi(C_7) = 3$



$\chi(C_7) = 3$

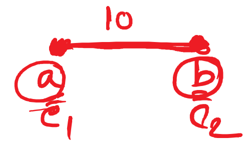
✓ $\chi(C_n) = 2$, when n is even for $n \geq 4$

✓ $\chi(C_n) = 3$, when n is odd. for $n \geq 3$

* The best algorithms known for finding the chromatic number of a graph have exponential worst-time complexity (in the no. of vertices of the graph).

Application of Graph Coloring

(1) Scheduling: How can the final exams at a school be scheduled so that no student has two exams at the same time?



Graph model
- Vertices: represent the courses.
- edges: represent if there is a common student b/w any two pair of courses.
- each time slot for the final exam is represented by a different color.

Scheduling of exam corresponds to coloring the associated graph model

Assignment Problem.

Prob: Television channels 2 through 13 are assigned to stations so that no two stations within 150 km can operate on the same channel. How can the of chan b modeled by h col ?

of
assignment of channels be modeled by graph coloring?

→ Graph model → Vertices: stations.
→ edges: represents that any pair of stations are at most 150 km apart.

→ an assignment of channels to stations corresponds to a coloring of the graph, where each color corresponds to different channels.