

Counting - 1

Wednesday, November 11, 2020 10:09 AM

Pigeonhole principle

Theorem: If k is a positive integer and $(k+1)$ or more objects are to be placed in k boxes, then there is at least one box which contains two or more objects.

Proof: We will prove by contradiction. Suppose that one of the k boxes contain more than one object. Then, the total no. of objects would be at most k . This is a contradiction, because there are $(k+1)$ objects.

Generalized PH

Proof: We will be using the proof by contradiction. Suppose none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then the total no. of objects is at most

$$k \cdot (\lceil N/k \rceil - 1)$$

$$< k \left(\left(\frac{N}{k} + 1 \right) - 1 \right)$$

$$< N$$

$$\lceil N/k \rceil \equiv \left(\frac{N}{k} + 1 \right)$$

$$\lceil N \rceil / (N-1) \text{ has}$$

$< N$

Here, the inequality $\lceil \frac{N}{k} \rceil < \left(\frac{N}{k} + 1 \right)$ has been used. This is a contradiction because there are a total of N objects.

$$\lceil N/k \rceil \geq r.$$

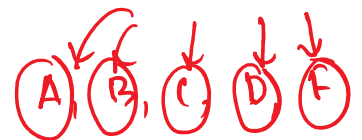
The smallest integer N with $\frac{N}{k} + 1 > r$, namely,

$N = k \cdot (r-1) + 1$

 satisfies the inequality $\lceil N/k \rceil \geq r$.

Is $\underline{N = k(r-1) + 1}$ the smallest integer?

→ Value smaller than N is not suffice, because if we had $k \cdot (r-1)$ objects, then we could put $(r-1)$ objects in each of k boxes and no box would have r objects.



Prob: Possible grades, $k = 5$

Minimum No. of students getting the same grade $r = 6$.

∴ Minimum no. of students in the class, $N = k(r-1) + 1$

$$= 5 \cdot (6-1) + 1$$

$$= \underline{26}.$$

Prob.

No. of suits, $k = 4$.

Min. no. of cards to be chosen from same suit, $r = 3$.

$$\begin{aligned} \therefore \text{Min. No. of cards to be chosen, } N &= k(r-1) + 1 \\ &= 4 \cdot (3-1) + 1 = \underline{9} \end{aligned}$$

In the worst case, we can select all the clubs, diamonds, and spades, 39 in all, before we select a single heart. The next three cards will be definitely all hearts, so we may need $(39+3)$ i.e. 42 cards to get three hearts.

Prob.

Possible remainder, $k = 4$ (0, 1, 2, 3)

No. of integers, $N = 5$.

\therefore If r is the least no. of integers which produces the same remainder then,

$$N = k(r-1) + 1$$

$$\text{or, } 5 = 4(r-1) + 1$$

$$\Rightarrow \boxed{r = 2.}$$