1. Give a direct proof of the theorem “If n is an odd integer, then n^2 is odd.”

2. Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square. (An integer a is a **perfect square** if there is an integer b such that $a = b^2$.)

3. Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

4. Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

5. Prove that the sum of two rational numbers is rational.

6. Prove that if n is an integer and n^2 is odd, then n is odd.

7. Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

8. Give a proof by contradiction of the theorem “If $3n + 2$ is odd, then n is odd.”

9. Proof by contradiction that there is no greatest integer.

10. Proof by contradiction that there is no integer that is both even and odd.

11. Prove that the sum of any rational number and any irrational number is irrational.

12. Prove that $1 + 3\sqrt{2}$ is irrational.

13. Prove the theorem “If n is an integer, then n is odd if and only if n^2 is odd.”

14. Show that these statements about the integer n are equivalent:

    p_1: n is even.
    p_2: $n - 1$ is odd.
    p_3: n^2 is even.

15. Prove that $(n + 1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

16. Prove that the only consecutive positive integers not exceeding 100 that are perfect powers are 8 and 9. (An integer is a **perfect power** if it equals n^a, where a is an integer greater than 1.)

17. Prove that if n is an integer, then $n^2 \geq n$.

18. Use a proof by cases to show that $|xy| = |x||y|$, where x and y are real numbers. (Recall that $|a|$, the absolute value of a, equals a when $a \geq 0$ and equals $-a$ when $a \leq 0$.)

19. Show that there are no solutions in integers x and y of $x^2 + 3y^2 = 8$.

20. Show that if x and y are integers and both xy and $x + y$ are even, then both x and y are even.

Given two positive real numbers x and y, their **arithmetic mean** is $(x + y)/2$ and their **geometric mean** is $\sqrt{xy}$. When we compare the arithmetic and geometric means of pairs of distinct positive real numbers, we find that the arithmetic mean is always greater than the geometric mean. [For example, when $x = 4$ and $y = 6$, we have $5 = (4 + 6)/2 > \sqrt{4 \cdot 6} = \sqrt{24}$.] Can we prove that this inequality is always true?

21. Suppose that two people play a game taking turns removing one, two, or three stones at a time from a pile that begins with 15 stones. The person who removes the last stone wins the game. Show that the first player can win the game no matter what the second player does.

22. "Every positive integer is a sum of the squares of (i) two integers (ii) three integers" – True or False?