Basic Discrete Structures

Sets, Functions, Sequences, Matrices, and Relations

(Lecture – 9)

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Equivalence Relations

A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

- **Reflexive property**: Every element should be equivalent to itself.
- **Symmetric property**: It makes sense to say that a and b are related (not just that a is related to b) by an equivalence relation, because when a is related to b, b is also related to a.
- **Transitive property**: If a and b are equivalent and b and c are equivalent, it follows that a and c are equivalent.

Equivalence Classes

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

- In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is $[a]_R = \{s \mid (a, s) \in R\}$.
- If $b \in [a]_R$, then b is called a **representative** of this equivalence class.
  - Any element of a class can be used as a representative of this class.

Equivalence Class and Partition

Theorem 1:

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

$$(i)\ aRb \qquad (ii)\ [a]=[b] \qquad (iii)\ [a]\cap[b]\neq\emptyset$$

- Let A be the set of students who are majoring in exactly one subject, and let R be the relation on A consisting of pairs (x, y), where x and y are students with the same major.
  - R is an equivalence relation.
- R splits all students in A into a collection of disjoint subsets, where each subset contains students with a specified major.
  - For instance, one subset contains all students majoring (just) in computer science, and a second subset contains all students majoring in history.
  - These subsets are equivalence classes of R.
- The equivalence classes of an equivalence relation partition a set into disjoint, nonempty subsets.

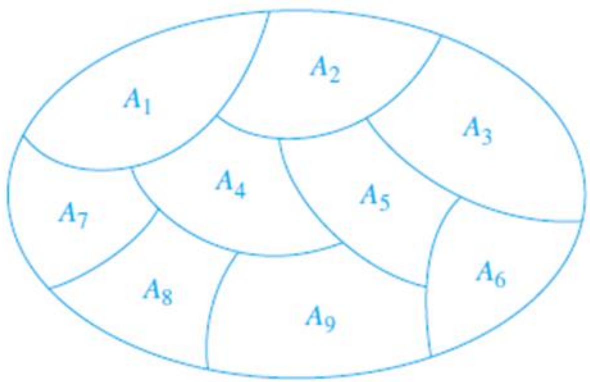

FIGURE 1 **A Partition of a Set.**

Partitioning of a Set by Equivalence Relation

- Let R be an equivalence relation on a set A. The union of the equivalence classes of R is all of A, because an element a of A is in its own equivalence class, namely, $[a]_R$.

$$\bigcup_{a \in A} [a]_R = A.$$

- From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$, when $[a]_R \neq [b]_R$.
  - Equivalence classes form a partition of A, because they split A into disjoint subsets.
- More precisely, a **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union.
- Collection of subsets A_i, $i \in I$ (where I is an index set) forms a partition of S if and only if
  - $A_i \neq \emptyset$ for $i \in I$,
  - $A_i \cap A_j = \emptyset$ when $i = j$,
  - *and* $\bigcup_{i \in I} A_i = S.$

Theorem 2:

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Partial Orderings

Definition: A relation R on a set S is called a *partial ordering*, or *partial order*, if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

Example: Assume R denotes the "greater than or equal" relation ($\geq$) on the set S={1,2,3,4,5}.

- Is the relation reflexive? Yes
- Is it antisymmetric? Yes
- Is it transitive? Yes
- **Conclusion:** R is a partial ordering.