Indian Institute of Engineering Science and Technology, Shibpur

Dual Degree (B.Tech - M.Tech) 3rd Semester Mid-Term Examination, 2015

(For All Engineering Branches)

Subject: Mathematics-III(MA-301)

Time : 2 hours Full Marks : 100

Answer any FOUR questions

1. a) State axiomatic definition of probability. Using this definition show that for any two events A and B:

$$P(\overline{A}B) = P(B) - P(AB)$$

where $\overline{A}$ is the complementary event of A.
Hence prove that

$$P(A+B) = P(A) + P(B) - P(AB).$$

b) State and prove Baye's theorem. [25]

2. a) Let A and B are two events such that $P(A+B) = \frac{7}{8}$, $P(AB) = \frac{1}{4}$ and $P(\overline{A}) = \frac{5}{8}$. Find the conditional probability of A on the hypothesis that B does not occur.

b) In a bolt factory, machines A, B and C manufactures respectively 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% respectively are defective bolts. A bolt is drawn at random from their product and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C?

c) A man seeks advice regarding one of two possible courses of action from three advisers, who arrive at their recommendations independently. He follows the recommendations of the majority. The probabilities that the individual advisers are wrong are 0.1, 0.05 and 0.05 respectively. What is the probability that the man takes incorrect advice? [25]

3. a) Find the constant k so that the function

$$f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

is a probability density function. Find the distribution function and evaluate $P(1 < X < 2)$.

b) Prove that Poisson distribution is a limiting case of Binomial distribution.

c) Let X be a discrete random variable with probability mass function

$$P(X = r) = q^{r-1}p, \; r = 1, 2, 3, \cdots.$$

P.T.O.

(i) Show that it is a valid probability distribution.

(ii) Find the moment generating function of X and hence deduce mean and variance of X.

[25]

4. a) Find the Laplace transforms of

$$(i)\ \frac{1}{t}(1-e^t)\quad (ii)\ \frac{1}{t}(\cos at - \cos bt)$$

b) Find the Laplace transforms of $\frac{\sin at}{t}$. Hence show that $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$.

[25]

5. a) Find $L^{-1}\left\{\frac{4}{2s-3} - \left(\frac{3+4s}{9s^2-16}\right) + \frac{8-6s}{16s^2+9}\right\}$.

b) Find all the basic solutions to the following Problem:

$$\text{Maximize } Z = x_1 + 3x_2 + 3x_3$$

$$\text{Subject to } x_1 - 2x_2 + x_3 = 3,$$

$$2x_1 + x_2 + x_4 = 2.$$

Also find which of the basic solutions are (i) BFS, (ii) non-degenerate BFS and (iii) optimal BFS. [25]

6. a) A toy company manufactures two types of doll, a basic version - doll I and a delux version - doll II. Each doll of type II takes twice as long to produce as one of type I, and the company would have time to make a maximum of 2000 per day. The supply of plastic is sufficient to produce 1500 dolls per day (both I and II combined). The delux version requires a fancy dress of which there are only 600 per day available. If the company makes a profit of Rs.30 and Rs.50 per doll respectively on doll I and II, then how many of each doll should be produced per day in order to maximize the total profit. Formulate this problem and solve.

b) Define convex set and show that the point set given by

$$\{(x_1, x_2) : 2x_1 + x_2 \geq 20, x_1 + 2x_2 \leq 80, x_1 + x_2 \leq 50, x_1, x_2 \geq 0\}$$

is a convex set. [25]

Indian Institute of Engineering Science and Technology, Shibpur

Dual Degree (B. Tech.- M. Tech.) 3^{rd} Semester Final Examination, November 2015

Subject: Mathematics III (MA-301)

Time: 3 hours

Full Marks: 70

Use a separate answerscript for each half

1st Half (Full Marks - 40)

Answer Question No. 1 and ANY TWO of the rest

1. Answer ANY THREE questions: $3 \times 6 = 18$

   (a) A fair six-faced die is tossed until all the six faces are observed. If X is the number of tosses required, find $E(X)$.

   (b) X and Y are stochastically independent random variables each having $N(0,1)$ distribution. What is the distribution of the random variable $U = \frac{X+Y}{X-Y}$? Justify your answer.

   (c) Define Mean Square Error (MSE) of an estimator. Establish the bias-variance decomposition formula for MSE.

   (d) Prove that the correlation coefficient between two random variables always lies between -1 and 1.

   (e) A straight line of unit length is divided into three line segments by randomly choosing two points on it. Find the probability that the three line segments form a triangle.

2. (a) State and prove Tchebycheff's Inequality.

   (b) Show by Tchebycheff's Inequality that in 2000 throws with a coin, the probability that the number of heads lies between 900 and 1100 is at least 19/20.

   $5 + 6 = 11$

3. (a) X, Y are independent random variables having common density

$$f_{\alpha,\beta}(x) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, & \text{for } x > 0 \quad (\alpha, \beta > 0) \\ 0 & \text{otherwise.} \end{cases}$$

   Find the probability density function (p.d.f.) of $X + Y$.

   (b) Let T_1 and T_2 be two statistics with $E(T_1) = \theta_1 + \theta_2$ and $E(T_2) = \theta_1 - \theta_2$, find unbiased estimators of the parameters θ_1 and θ_2.

   $6 + 5 = 11$

4. (a) The following is a random sample of size 10 from a uniform distribution on $(0, \theta)$, $\theta > 0$: $5.6, 3.9, 3.2, 4.8, 7.8, 6.1, 10.2, 8.4, 7.1, 9.8$. Find the Maximum Likelihood Estimator (M.L.E.) of θ. Justify your answer.

   (b) $X_1, X_2, ..., X_n$ is a random sample of size n from a distribution having finite variance σ^2 . Prove that $T_n(X_1, X_2, ..., X_n) = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2$ is consistent but *not* unbiased for σ^2.

$5 + 6 = 11$

5. (a) Suppose $f(x)$ and $g(y)$ are two probability density functions. Prove that $h(x, y) = f(x)g(y)$ is a bivariate density with $f(x)$ and $g(y)$ as its marginal densities. Further, show that the two random variables are stochastically independent.

(b) Derive the normal equations for the simple linear regression model :

$$y_i = \alpha + \beta x_i + e_i, i = 1, 2, ..., n$$

$5 + 6 = 11$

2nd Half (Full Marks - 30)

Answer ANY THREE Questions

6. (a) Find the Laplace transforms of

$$(i) f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$$

$$(ii) f(t) = k\frac{t}{T} \text{ for } 0 < t < T \text{ and } f(t+T) = f(t).$$

(b) Using Laplace transform, evaluate $\displaystyle\int_0^{\infty} \frac{e^{-at} - e^{-bt}}{t} dt.$

$6 + 4 = 10$

7. (a) Find $L^{-1}\left\{\frac{4}{2s-3} - \left(\frac{3+4s}{9s^2-16}\right) + \frac{8-6s}{16s^2+9}\right\}$.

(b) Using Convolution theorem, find the inverse Laplace transform of $\frac{1}{(s-1)^2(s-2)^3}$.

$5 + 5 = 10$

8. (a) Solve the differential equation using Laplace transform:

$$(D^3 - 2D^2 + 5D)y = 0 \text{ with } y(0) = 0,\ y'(0) = 0,\ y''(0) = 1.$$

(b) $x_1 = 1,\ x_2 = 3,\ x_3 = 2$ is a feasible solution of the L.P.P.:

Maximize $Z = x_1 + 2x_2 + 4x_3$

Subject to $5x_1 + 4x_2 - x_3 = 15$,

$2x_1 + 7x_2 - 4x_3 = 15$,

$x_1, x_2, x_3 \geq 0$.

Find the basic feasible solutions from it.

optimal

$5 + 5 = 10$

$10x_1 + 8x_2 = 0$
$10x_1 + 35x_2 =$
$3\ 27x_2 =$
$15 - \frac{20}{3}$
$= \frac{45}{}$
$= 5$

$20x_1 - 4x_3 = 60$
$2x_1 - 4x_3 = 15$
$2\ 18x_1 = 45$

$\frac{15}{2}\ -15$

9. (a) Define convex set and show that the point set given by

$$X = \{(x_1, x_2) : 4x_1^2 + 9x_2^2 \le 36\}$$

is a convex set.

(b) Solve the following LPP by simplex method:

Maximize $Z = x_1 + x_2 + 3x_3$

Subject to $3x_1 + 2x_2 + x_3 \le 3$,

$2x_1 + x_2 + 2x_3 \le 2$,

$x_1, x_2, x_3 \ge 0$.

4 + 6 = 10

10. Solve the following LPP by Charnes' Big-M method:

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \ge 0$.

10

$1 - \frac{6}{5}$

$2 - \frac{3}{5}$

$1 - \frac{6}{5}$

$2 - \frac{3}{5}$

$10 - 4$

$15 - 12$