Relation - 3

Monday, October 12, 2020 10:35 AM

Closure of a relation

Prob-1) $R = \{(a,b) \mid a < b\} \cup \{(a,a) \mid a \in Z\}$

$\Rightarrow R = \{(a,b) \mid a \leq b\}$.

Inverse of a relation

✓ If R is a relation from A to B ($R \subseteq A \times B$), then the relation R^{-1} from B to A can be defined by interchanging the elements of all the ordered pairs of R.

$$R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}.$$

Prob: $R = \{(a,b) \mid a > b\}$.

$R \cup R^{-1} = \{(a,b) \mid a > b\} \cup \{(b,a) \mid a < b\}$

$\Rightarrow R \cup R^{-1} = \{(a,b) \mid a \neq b\}$.

Theorem-1:

(i) $(a,b) \in R^n \rightarrow$ path of length n from a to b.

(ii) Path of length n from a to b $\longrightarrow (a,b) \in R^n$

(i) [illegible]

(ii) Path of length n from a to b $\longrightarrow$ $(a,b) \in R^n$

<u>Proof (i)</u>: Let $(a,b) \in R^n$ is true ~~from~~ for $n = 1, 2, 3 \ldots$

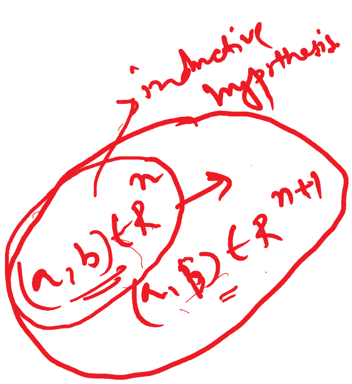
It follows that $(a,b) \in R^n$ implies $(a,c) \in R$ and $(c,b) \in R^{n-1}$ for some $c \in A$ by the definion of composition of $(a,b) \in R^n$.

Because $(a,c) \in R$ means there is a path of length 1 from a to c, and $(c,b) \in R^{n-1}$ means there is a path of length $(n-1)$ from c to b. It follows that there is a path of length n from a to b.

<u>Proof (ii)</u>: We will use mathematical induction to prove this part.

$\therefore$ By definition, for $n=1$, a path exists b/w a and b.

Therefore, $(a,b) \in R$ (basis step)

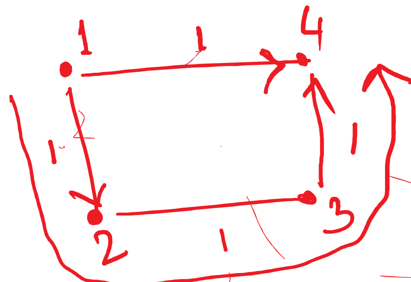


Assume that the theorem is true for the positive integer n i.e. $(a,b) \in R^n$ (inductive hypothesis)

$\Leftarrow$ To complete the proof we have to show $(a,b) \in R^{n+1}$

There is a path of length $(n+1)$ from a to b iff there is an element $c \in A$ s.t.

there is a path of length one from a to c i.e. $(a, c) \in R$ (basis step), and a path of length n from c to b i.e. $(c, b) \in R^n$ (inductive hypothesis).
Consequently, by using basis step and inductive hypothesis, there is a path of length $(n+1)$ from a to b iff $(a, b) \in R^{n+1}$.
This completes our proof.



$A = \{1, 2, 3, 4\}$

$R = \{(1,2), (2,3), (3,4), (1,4)\}$

$R^2 = \{(1,3), (2,4)\}$

$R^3 = \{1,4\}$

$R^4 = \{\phi$

$R^* = R^1 \cup R^2 \cup R^3 \cup R^4 \cup \ldots\ldots \cup R^n \cup \ldots$

$R^* = R^1 \cup R^2 \cup R^3$

$R^* = \{(1,2), (1,3), (1,4), (2,3), (3,4)\}$

Theorem - 2

(i) $R^n \subseteq R \rightarrow R$ is transitive

(ii) R is transitive $\rightarrow R^n \subseteq R$.

Proof(i): Let $R^n \subseteq R$ is true for $n = 1, 2, 3, \ldots$

$\therefore$ It follows that $R^2 \subseteq R$ which implies that if $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition $(a, c) \in R^2$. Because, $R^2 \subseteq R$, this means $(a, c) \in R$. Hence, R is transitive.

Proof(ii): We will use mathematical induction to prove if R is transitive, then $R^n \subseteq R$.

By ~~definition~~ defn., for $n = 1$, $R \subseteq R$ (basis step)

Assume that this theorem is true for the positive integer n i.e. $R^n \subseteq R$ (inductive hypothesis)

To complete the inductive step, we must show that $R^{n+1} \subseteq R$.

Let $(a, b) \in R^{n+1}$. Then as $R^{n+1} = R^n \circ R$, there is an element c with $c \in A$, s.t. $(a, c) \in R$ and $(c, b) \in R^n$. Consequently, using inductive hypothesis, with $R^n \subseteq R$ then $(c, b) \in R$. Furthermore, as R is transitive, and $(a, c) \in R$ and $(c, b) \in R$, it follows that $(a, b) \in R$. This shows that $R^{n+1} \subseteq R$, completing the proof.

Proof-3. To prove that R^* is the transitive closure of a relation R we need to prove:

(i) R^* contains R $(R^* \supseteq R)$

(ii) R^* is transitive

(iii) R^* is contained in every relation S that contains R.

$$\boxed{R^* = \bigcup_{n=1}^{\infty} R^n}$$

(i) By definition of connectivity relation, $R^* \supseteq R$ (R^* contains R)

(ii) If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c. We obtain a path from a to c by starting from a and reach b, then ~~from~~ starting from b to reach c. Hence, $(a, c) \in R^*$. Therefore, R^* is transitive.

(iii) Suppose S is a transitive relation containing R. Because, S is transitive, S^n is also transitive (to be verified) and $S^n \subseteq S$ (from theorem-2). Furthermore, because

$$S^* = \bigcup_{k=1}^{\infty} S^k \text{ and } S^k \subseteq S,$$

it follows that $S^* \subseteq S$.

Now, it is to be noted that if $R \subseteq S$, then $R^* \subseteq S^*$, because any path in R is also a path in S.

Consequently, $R^* \subseteq S^* \subseteq S$.

Thus, any transitive relation that contains R must also contain R^*. Therefore, R^* is the transitive closure of R.

Problem

$R = \{(a, b) \mid a \text{ has met } b\}$

$R^n \rightarrow$ consists of all pairs (a, b) such that there are people $x_1, x_2, \ldots x_{n-1}$ such that a has met x_1, x_1 has met x_2, ..., and x_{n-1} has met b.

$R^* \rightarrow$ Contains (a, b) if there is a sequence of people, starting with a and ending with b, s.t. each person in the sequence has met the next person in the sequence.

Problem.

$R = \{(a, b) \mid a \text{ shares border with } b\}$.

$R^n \rightarrow$ Consists of the pairs (a, b) where it is possible to go from state a to state b by crossing exactly n state borders.

$R^* \rightarrow$ Consists of pairs (a, b) where it is possible to go from state a to b by crossing as many state borders as necessary.