Proof by Induction (Lecture – 1)

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Proof Techniques

- **Basic proof methods:**
  - Direct, Indirect (Contraposition, Contradiction), By cases, etc.
- **Proof of quantified statements:**
  - There exists x with some property P(x).
    - It is sufficient to find one element for which the property holds.
  - For all x some property P(x) holds**.**
    - Proofs of 'For all x some property P(x) holds' must cover all x and can be harder.
- **Mathematical induction** is a technique that can be applied to prove the universal statements for sets of positive integers or their associated sequences.

Mathematical Induction

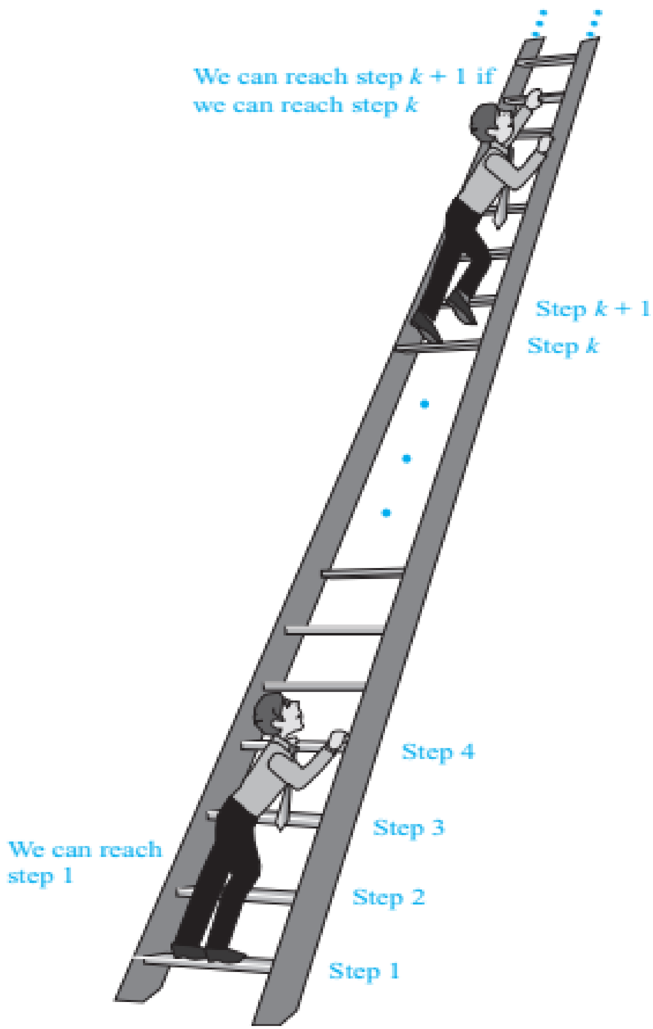

FIGURE 1 **Climbing an Infinite Ladder.**

- We know two things:
  - We can reach the first rung of the ladder.
  - If we can reach a particular rung of the ladder, then we can reach the next rung.
- But can we conclude that we are able to reach every rung of this infinite ladder?
  - Yes, by **mathematical induction** and its underlying *well-ordering property*

Mathematical Induction

- In general, mathematical induction can be used to prove statements that assert that $P(n)$ is true for all positive integers n i.e., $\forall n\, P(n)$, where $P(n)$ is a propositional function.

PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

- In the inductive step, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k+1)$ must also be true. The assumption that $P(k)$ is true is called the **inductive hypothesis**.
- Validity of mathematical induction:
  - **Based on the well-ordering property:** Every nonempty subset of nonnegative integers has a **least element**.

The Good and the Bad of Mathematical Induction

- Pros:
  - Mathematical induction is an extremely important proof technique that can be used to prove a conjecture once it is has been made (and is true).
  - It is used extensively to prove results about a large variety of discrete objects – proving results about the complexity of algorithms, correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities.
- Cons
  - It cannot be used to find new theorems.
  - Mathematicians sometimes find proofs by mathematical induction unsatisfying because they do not provide insights as to why theorems are true.

Mathematical Induction: General Steps

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

Proofs by Mathematical Induction (1)

- Show that if n is a positive integer, then $1 + 2 + \cdots + n = {}^{n(n+1)}/_{2}$

Solution: Let $P(n)$ be the proposition that the sum of the first n positive integers, $1 + 2 + \cdots n = \frac{n(n+1)}{2}$, is $n(n+1)/2$. We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \ldots$. Namely, we must show that $P(1)$ is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3, \ldots$.

BASIS STEP: $P(1)$ is true, because $1 = \frac{1(1+1)}{2}$. (The left-hand side of this equation is 1 because 1 is the sum of the first positive integer. The right-hand side is found by substituting 1 for n in $n(n+1)/2$.)

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

is also true. When we add $k + 1$ to both sides of the equation in $P(k)$, we obtain

$$\begin{aligned}1 + 2 + \cdots + k + (k+1) &\overset{\text{IH}}{=} \frac{k(k+1)}{2} + (k+1)\\ &= \frac{k(k+1) + 2(k+1)}{2}\\ &= \frac{(k+1)(k+2)}{2}.\end{aligned}$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

Proofs by Mathematical Induction (2)

Example: Prove the sum of first n odd integers is n^2 .

i.e. $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$ for all positive integers.

Proof:

- What is P(n)? P(n): $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$

Basis Step Show P(1) is true

- Trivial: $1 = 1^2$

Inductive Step Show if P(n) is true then P(n+1) is true for all n.

- Suppose P(n) is true, that is $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$
- Show P(n+1): $1 + 3 + 5 + 7 + \ldots + (2n - 1) + (2n + 1) = (n+1)^2$ follows:
- $$\underbrace{1 + 3 + 5 + 7 + \ldots + (2n - 1)}_{n^2} + (2n + 1) = n^2 + (2n+1) = (n+1)^2$$