CHAPTER 8

Calculus of Functions of Several Variables

8.1 INTRODUCTION

Real world phenomena can be successfully explained in mathematical language using parametric equations and functions such as trigonometric functions which describe cyclic, repetitive activities; exponential, logarithmic functions which describe growth and decay and polynomial functions which approximate these and most other functions. In physical and social problems, most variables that we need to know depend not on one but several independent variables. The problems in engineering physics, computer science, operations research, probability, statistics, economics etc. deal with functions of two or more independent variables. Due to these reasons the calculus of functions of several variables becomes more richer. In this chapter, we shall study limit, continuity and partial derivative of real valued functions of two or more real variables.

8.2 FUNCTIONS OF TWO OR MORE VARIABLES

A variable z is said to be a function of two variables x and y if it takes a value corresponding to a pair of values (x, y). This is expressed by $z = f(x, y)$, where z is a dependent variable and x, y are independent variables. For examples, the area of an ellipse is πab, *i.e.*, it depends on two independent variables a and b; the total surface area of a rectangular parallelopiped is $2(xy + yz + zx)$ and it depends on three independent variables; the velocity of a fluid particle moving in space depends on four independent variables x, y, z, t.

The calculus of functions of several variables can be developed as functions of single variable. The concepts of limit, continuity and differentiability can also be introduced for functions of two or more variables.

Geometrically, $z = f(x, y)$ represents a surface in three dimensional xyz-coordinate space. The set of values (or points) (x, y) for which the function $z = f(x, y)$ is defined is known as the domain of definition or simply domain of this function. For example, if $z = \sqrt{1 - x^2 - y^2}$, the domain for which z is real consists of the set of points (x, y) such that $x^2 + y^2 \leq 1$, *i.e.*, the set of points inside and on a circle in the xy-plane having its centre at $(0, 0)$ and radius 1.

Note: If z is a function of n independent variables $x_1, x_2, \ldots, x_n$, then we write $z = f(x_1, x_2, \ldots, x_n)$.

For example, if $u = x_1^2 + x_2^2 + x_3^2 + x_4^2$, then u is a function of four variables x_1, x_2, x_3 and x_4, *i.e.*, $u = f(x_1, x_2, x_3, x_4)$.

8.3 LIMIT AND CONTINUITY OF A FUNCTION OF TWO VARIABLES

Neighbourhood of a Point

We begin with the concept of neighbourhood of a point in 2-dimensional space.

Definition 1: The set of all values of (x, y) in the xy-plane other than (a, b) that lie in a square with centre at (a, b) bounded by the four lines $x = a - \delta$, $x = a + \delta$, $y = b - \delta$, $y = b + \delta$, *i.e.*, $a - \delta < x < a + \delta$ and $b - \delta < y < b + \delta$, where δ is any arbitrarily small positive number is said to form a square neighbourhood of (a, b).

In other words, the set of all values of (x, y) in the xy-plane other than (a, b) that satisfy the inequalities $0 < |x - a| < \delta$ and $0 < |y - b| < \delta$, where δ is any arbitrarily small positive number is said to form a square neighbourhood of (a, b).

2. The set of all values of (x, y) in the xy-plane other than (a, b) that satisfy the inequalities

$$0 < (x-a)^2 + (x-b)^2 < \delta^2$$

where δ is any arbitrarily small positive number is said to form a circular neighbourhood of (a, b).

Definition: Limit: Let $f(x, y)$ be a function defined in a region R of xy-plane and (a, b) be a point in R. The function $f(x, y)$ is said to have a limit l (say) as (x, y) tends to (a, b) in any manner if for any preassigned positive number ε, no matter however small, there exists a positive number δ such that

$$|f(x,y) - l| < \varepsilon$$

for $$0 < |x - a| < \delta \text{ and } 0 < |y - b| < \delta$$

or for $$0 < (x-a)^2 + (y-b)^2 < \delta^2$$

In other words, the function $f(x, y)$ is said to have a limit l as (x, y) tends to (a, b) if the value of $f(x, y)$ can be made as close as we please to l for all those (x, y) in an appropriately chosen δ-neighbourhood of (a, b).

This situation is expressed by writing

$$\lim_{(x,y)\to(a,b)} f(x,y) = l, \quad \text{or} \quad \lim_{\substack{x\to a\\ y\to b}} f(x,y) = l$$

Here l is sometimes called the simultaneous limit or the double limit.

Definition: Repeated limits: Let $f(x, y)$ be a function defined in a region R of xy-plane and (a, b) be a point in R, then $\lim\limits_{x\to a} f(x, y)$, if it exists, is a function of y, say $g(y)$. If $\lim\limits_{y\to b} g(y)$ exists and equals to l (say), then l is called the repeated or iterated limit of $f(x, y)$ as $x \to a$ and then $y \to b$, and we write

$$\boxed{\lim_{y \to b} \lim_{x \to a} f(x, y) = l}$$

By changing the order of limits, *i.e.*,

$$\lim_{x \to a} \lim_{y \to b} f(x, y)$$

may give (not necessarily always) a different value (or the limit may not exist).

Definition: Continuity: Let a function $f(x, y)$ be defined in a region R of xy-plane and (a, b) lies in R. The function $f(x, y)$ is said to be continuous at (a, b) if

$$\boxed{\lim_{\substack{x \to a \\ y \to b}} f(x, y) = f(a, b).}$$

If $f(x, y)$ is continuous at each point (a, b) in R, then $f(x, y)$ is said to be continuous in R.

Analytically, $f(x, y)$ is said to be continuous at (a, b) if given $\varepsilon > 0$, no matter however small, there exists a $\delta > 0$ such that

$$\boxed{|f(x, y) - f(a, b)| < \varepsilon}$$

for $$\boxed{|x - a| < \delta \text{ and } |y - b| < \delta}$$

or for $$(x - a)^2 + (y - b)^2 < \delta^2.$$

Theorems on limit and continuity

Theorem 1: If $f(x, y)$ and $g(x, y)$ be two functions defined in the same region R of xy-plane such that

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l \text{ and } \lim_{\substack{x \to a \\ y \to b}} g(x, y) = m, \text{ then}$$

(i) $\lim\limits_{\substack{x \to a \\ y \to b}} \{c_1 f(x, y) \pm c_2 g(x, y)\} = c_1 l \pm c_2 m$, where c_1 and c_2 are constants,

(ii) $\lim\limits_{\substack{x \to a \\ y \to b}} \{f(x, y) \cdot g(x, y)\} = l \cdot m$ and

(iii) $\lim\limits_{\substack{x \to a \\ y \to b}} \dfrac{f(x, y)}{g(x, y)} = \dfrac{l}{m}$, provided $m \neq 0$.

Theorem 2: Let $f(x, y)$ and $g(x, y)$ are continuous at (a, b), then

(i) $f(x, y) \pm g(x, y)$ are continuous at (a, b),

(ii) $f(x, y) \cdot g(x, y)$ is continuous at (a, b)

and *(iii)* $\dfrac{f(x, y)}{g(x, y)}$ is continuous at (a, b) provided $g(a, b) \neq 0$.

ILLUSTRATIVE EXAMPLES

Example 1: Show that $\lim\limits_{\substack{x\to 0\\ y\to 0}} xy\dfrac{x^2-y^2}{x^2+y^2}=0.$

Solution: To prove this, we have to find a $\boxed{\delta>0}$ for given $\boxed{\varepsilon>0}$, such that

$$\left|xy\frac{x^2-y^2}{x^2+y^2}-0\right|<\varepsilon, \text{ for } \boxed{0<|x-0|<\delta \text{ and } 0<|y-0|<\delta}$$

Now, $$\left|xy\frac{x^2-y^2}{x^2+y^2}-0\right| = |x||y|\frac{|x^2-y^2|}{x^2+y^2}<|x||y| \quad (\because |x^2-y^2|<x^2+y^2, x\neq 0, y\neq 0)$$

Therefore, if we take $\delta=\sqrt{\varepsilon}$, then $\left|xy\dfrac{x^2-y^2}{x^2+y^2}-0\right|<|x||y|<\varepsilon$

for $$0<|x-0|<\delta \text{ and } 0<|y-0|<\delta.$$

Hence by definition the given double limit exists and equals to zero, *i.e.*,

$$\lim_{\substack{x\to 0\\ y\to 0}} xy\frac{x^2-y^2}{x^2+y^2}=0.$$

Note: Let (a, b) be a point within the region R of xy-plane and (x, y) be any point in R. The point (x, y) moves in the region R and tends to the point (a, b) along any specified curve in R but for the existence of the double (or simultaneous) limit, the limiting value must be unique along whatever path (x, y) tends to (a, b). If the limiting values are different for different approaches to the point (a, b) along different curves within the region R, then the limit does not exist.

Example 2: Show that the repeated limits exist and

$$\lim_{y\to 0}\lim_{x\to 0}\frac{xy}{x^2+y^2}=\lim_{x\to 0}\lim_{y\to 0}\frac{xy}{x^2+y^2}$$

but the double limit $\boxed{\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{xy}{x^2+y^2} \text{ does not exist.}}$

Solution: 1st part:

Since $$\lim_{x\to 0}\frac{xy}{x^2+y^2}=\frac{0.y}{0^2+y^2}=0 \qquad (\because y\neq 0)$$

We have $$\lim_{y\to 0}\lim_{x\to 0}\frac{xy}{x^2+y^2}=\lim_{y\to 0}0=0$$

$$(\because x\neq 0)$$

Similarly, $$\lim_{x\to 0}\lim_{y\to 0}\frac{xy}{x^2+y^2}=\lim_{x\to 0}0=0$$

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and hence the result of the first part follows.

2nd part: Put $y = mx$, then $x \to 0$ implies $(x, y) \to (0, 0)$.

Therefore, $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{xmx}{x^2+m^2x^2} \quad (\because y = mx)$$

$$= \lim_{x\to 0} \frac{m}{1+m^2} \, (\because x \neq 0) = \frac{m}{1+m^2},$$

which depends on m, *i.e.*, has different values for different m. Hence the double (or simultaneous) limit does not exist.

Example 3: Show that $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^4}{(x^2+y^4)^2}$ does not exist.

Solution: Let $y^2 = mx$, then $x \to 0$ implies $(x, y) \to (0, 0)$

$$\therefore \quad \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^4}{(x^2+y^4)^2} = \lim_{x\to 0} \frac{x^2 \cdot m^2x^2}{(x^2+m^2x^2)^2} \quad (\because y^2 = mx)$$

$$= \lim_{x\to 0} \frac{m^2}{(1+m^2)^2} \, (\because x \neq 0) = \frac{m^2}{(1+m^2)^2},$$

which has different values for different m.

Hence the double limit does not exist.

Example 4: Show that the repeated limits exist and

$$\lim_{y\to 0}\lim_{x\to 0} \frac{x+y}{x-y} = -1 \text{ and } \lim_{x\to 0}\lim_{y\to 0} \frac{x+y}{x-y} = 1$$

also the double limit $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x+y}{x-y}$ does not exist.

Solution: 1st part:

Since $$\lim_{x\to 0} \frac{x+y}{x-y} = \lim_{x\to 0} \frac{0+y}{0-y}$$

$$= \lim_{x\to 0}(-1) \, (\because y \neq 0) = -1,$$

we have $$\lim_{y\to 0}\lim_{x\to 0} \frac{x+y}{x-y} = \lim_{y\to 0}(-1) = -1$$

Similarly, $$\lim_{x\to 0}\lim_{y\to 0} \frac{x+y}{x-y} = \lim_{x\to 0} 1 = 1 \quad (\because x \neq 0),$$

and hence the results of the first part follow.

2nd part: Put $y = mx$, then $x \to 0$ implies $(x, y) \to (0, 0)$

Therefore, $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x+y}{x-y} = \lim_{x\to 0} \frac{x+mx}{x-mx} \qquad (\because y = mx)$$

$$= \lim_{x\to 0} \frac{1+m}{1-m} (\because x \neq 0) = \frac{1+m}{1-m},$$

which has different values for different m ($\neq 1$) and hence the given double limit does not exist.

Example 5: Show that the limit of $f(x, y)$ exists at the origin but the repeated limits do not, where

$$f(x,y) = \begin{cases} x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right), & xy \neq 0 \\ 0, & xy = 0 \end{cases}.$$

Solution: Here

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0}\left\{x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right)\right\}$$

$$= y\lim_{x\to 0}\sin\left(\frac{1}{x}\right) \qquad \left(\because -1 \le \sin\left(\frac{1}{y}\right) \le 1, \text{ for } y \neq 0\right)$$

but it is oscillating and hence $\lim_{x\to 0} f(x,y)$ does not exist. Similarly $\lim_{y\to 0} f(x,y)$ does not exist. Therefore, $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ and $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ do not exist.

To prove that the double (or simultaneous) limit exists, we have to find a $\delta > 0$ for given $\varepsilon > 0$, such that

$$\left| x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right) - 0\right| < \varepsilon, \text{ for } 0 < |x-0| < \delta \text{ and } 0 < |y-0| < \delta$$

Now, $$\left| x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right) - 0\right| \le |x|\left|\sin\left(\frac{1}{y}\right)\right| + |y|\left|\sin\left(\frac{1}{x}\right)\right| \le |x| + |y|$$

$$\left(\because 0 \le \left|\sin\left(\frac{1}{x}\right)\right|, \left|\sin\left(\frac{1}{y}\right)\right| \le 1\right)$$

Therefore, if we take $\delta = \frac{\varepsilon}{2}$, then

$$|f(x,y) - 0| \le |x| + |y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for } 0 < |x-0| < \delta \text{ and } 0 < |y-0| < \delta$$

Hence by definition

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0.$$

Note: Here $f(x, y)$ is continuous at $(0, 0)$ since $f(0, 0) = 0$ and $|f(x, y) - f(0, 0)| < \varepsilon$ for $|x-0| < \delta$ and $|y-0| < \delta$, where $\delta = \dfrac{\varepsilon}{2}$.

Example 6: Prove that the repeated limits of $f(x, y)$ exist at the origin and are equal but the simultaneous limit does not, where

$$f(x,y) = \begin{cases} 1, & \text{if } xy \neq 0 \\ 0, & \text{if } xy = 0 \end{cases}.$$

Solution: Here

$$\lim_{x\to 0} f(x,y) = \begin{cases} 1, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}$$

and hence $\quad \displaystyle\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 1 \qquad (\because y \to 0 \Rightarrow y \neq 0)$

Similarly, $\quad \displaystyle\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 1$

Therefore the repeated limits exist and are equal.

Again, since there are points arbitrarily near to $(0, 0)$ at which f is equal to 0 and also there are points arbitrarily, near to $(0, 0)$ at which f is equal to 1, therefore, there exists $\varepsilon > 0$, such that

$$|f(x,y) - l| \not< \varepsilon,$$

for all points in any neighbourhood of $(0, 0)$ where l is any fixed real number.

Example 7: Show that

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$

is not continuous at $(0, 0)$. (*W.B.U.T. 2009*)

Solution: Put $y = mx$, then $x \to 0$ implies $(x, y) \to (0, 0)$

Therefore, $\quad \displaystyle\lim_{\substack{x\to 0\\ y\to 0}} \frac{2xy}{x^2+y^2} = \lim_{x\to 0}\frac{2x\cdot mx}{x^2+m^2x^2} = \lim_{x\to 0}\frac{2m}{1+m^2} \qquad (\because x \neq 0)$

$$= \frac{2m}{1+m^2},$$

which has different values for different m. Thus, $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist and hence $f(x, y)$ is not continuous at $(0, 0)$.

Example 8: Show that

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & \text{for } (x, y) \neq (0, 0) \\ 0, & \text{for } (x, y) = (0, 0) \end{cases}$$

is continuous at (0, 0).

Solution: Now

$$\left| \frac{xy}{\sqrt{x^2 + y^2}} \right| = |r \cos\theta \sin\theta| \qquad \text{(putting } x = r\cos\theta,\ y = r\sin\theta,\ r > 0\text{)}$$

$$\leq r = \sqrt{x^2 + y^2}$$

Therefore,

$$|f(x,y) - f(0, 0)| = \left| \frac{xy}{\sqrt{x^2 + y^2}} - 0 \right|$$

$$\leq \sqrt{x^2 + y^2} < \varepsilon, \text{ whenever } \sqrt{x^2 + y^2} < \varepsilon$$

Therefore, given $\varepsilon > 0$, no matter however small, there exists a $\delta = \varepsilon^2 > 0$, such that

$$|f(x, y) - f(0, 0)| < \varepsilon \text{ for } (x - 0)^2 + (y - 0)^2 < \delta.$$

Hence by definition $f(x, y)$ is continuous at (0, 0).

Example 9: Prove, using definition, that the function $f(x, y) = x^2 + y$ is continuous at (1, 2).

Solution: Here $f(x, y) = x^2 + y$, therefore $f(1, 2) = 3$.

Now, we have to prove that given $\varepsilon > 0$, no matter however small, there exists a $\delta > 0$, such that

$$|f(x, y) - f(1, 2)| = |x^2 + y - 3| < \varepsilon \text{ when } |x - 1| < \delta, |y - 2| < \delta$$

If $|x - 1| < \delta$ and $|y - 2| < \delta$, then $1 - \delta < x < 1 + \delta$ and $2 - \delta < y < 2 + \delta$.

Assuming $0 < \delta < 1$, we have $1 - 2\delta + \delta^2 < x^2 < 1 + 2\delta + \delta^2$, hence

$$3 - 3\delta + \delta^2 < x^2 + y < 3 + 3\delta + \delta^2,$$

or $$-3\delta + \delta^2 < x^2 + y - 3 < 3\delta + \delta^2, \text{ or } -4\delta < x^2 + y - 3 < 4\delta$$

Thus choosing $\delta = \dfrac{\varepsilon}{4}$, we get $|x^2 + y - 3| < \varepsilon$,

i.e., $|f(x, y) - f(1, 2)| < \varepsilon$, whenever $|x - 1 < \delta, 1y - 2| < \delta$

Therefore, $f(x, y) = x^2 + y$ is continuous at (1, 2).

Note: Here $\lim\limits_{\substack{x \to 1 \\ y \to 2}} f(x, y) = f(1, 2) = 3$.

8.4 PARTIAL DERIVATIVES

Let $z = f(x_1, x_2, \ldots, x_n)$ be a function of n independent variables $x_1, x_2, \ldots, x_n$. The derivative of f with respect to one of the independent variables, say x_i, keeping all other independent variables constant is called the partial derivative of f with respect to x_i and is generally denoted by $\dfrac{\partial f}{\partial x_i}$ or f_{x_i} or $f_{x_i}(x_1, x_2, \ldots, x_n)$.

Therefore, $\dfrac{\partial f}{\partial x_i} = \lim\limits_{h_i \to 0} \dfrac{f(x_1, x_2, \ldots, x_i + h_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots, x_n)}{h_i}$ provided the limit exists.

Thus if $z = f(x, y)$, then

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \text{ provided the limit exists.}$$

Similarly,

$$\frac{\partial f}{\partial y} = f_y = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}, \text{ provided the limit exists.}$$

The partial derivatives at a particular point (a, b) are often denoted by

$$\left[\frac{\partial f}{\partial x}\right]_{(a,\, b)} \text{ or } \frac{\partial f(a, b)}{\partial x} \text{ or } f_x(a, b)$$

and

$$\left[\frac{\partial f}{\partial y}\right]_{(a,\, b)} \text{ or } \frac{\partial f(a, b)}{\partial y} \text{ or } f_y(a, b)$$

Therefore,

$$\boxed{f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}}$$

and

$$\boxed{f_y(a, b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}}, \text{ provided the limits exist.}$$

Now, f_x, f_y may be either constants or function of x, y and so each may again possess partial derivatives. The partial derivatives of f_x and f_y are denoted and defined as follows:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial f_x}{\partial x} = \lim_{h \to 0} \frac{f_x(x+h, y) - f_x(x, y)}{h},$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f_x}{\partial y} = \lim_{k \to 0} \frac{f_x(x, y+k) - f_x(x, y)}{k},$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f_y}{\partial x} = \lim_{h \to 0} \frac{f_y(x+h, y) - f_y(x, y)}{h},$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial f_y}{\partial y} = \lim_{k \to 0} \frac{f_y(x, y+k) - f_y(x, y)}{k},$$

provided each of the above limits exists.

The four partial derivatives $f_{xx}, f_{yx}, f_{xy}, f_{yy}$ are known as the second order partial derivatives of $f(x, y)$. Third and higher order partial derivatives of $f(x, y)$ are defined in a similar manner and in general,

$$\frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right) = \frac{\partial^n f}{\partial x^n}; \quad \frac{\partial}{\partial y}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right) = \frac{\partial^n f}{\partial y\, \partial x^{n-1}} \text{ etc.}$$

Notes:

(*i*) The above definitions of partial derivatives can be extended for functions of three or more variables.

(*ii*) If any of the above limits does not exist, then we say the corresponding partial derivative does not exist.

(*iii*) In general $f_{xy} \neq f_{yx}$, sufficient conditions for $f_{xy} = f_{yx}$, are stated in the following theorem.

Schwarz's theorem: Let $f(x, y)$ be a function defined in a region R of the xy-plane and (a, b) is in R such that

1. f_x exists in some neighbourhood of (a, b),
2. f_{xy} is continuous at (a, b), then f_{yx} exists at (a, b) and $f_{xy}(a, b) = f_{yx}(a, b)$.

(*iv*) We can also state that if the mixed partial derivatives f_{xy} and f_{yx} of a function $f(x, y)$ are continuous in a region R, then $f_{xy} = f_{yx}$ holds everywhere in R.

ILLUSTRATIVE EXAMPLES

Example 1: Find, from first principle, $f_x(1, 2)$ and $f_y(1, 2)$ for $f(x, y) = \dfrac{x+2y-1}{x+2y+1}$.

Solution: By definition

$$f_x(1,2) = \lim_{h\to 0} \frac{f(1+h,2)-f(1,2)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{4+h}{6+h} - \frac{2}{3}}{h} = \lim_{h\to 0} \frac{1}{3(6+h)} \quad (\because h \neq 0) = \frac{1}{18}$$

$$f_y(1,2) = \lim_{k\to 0} \frac{f(1,2+k)-f(1,2)}{k} = \lim_{k\to 0} \frac{\frac{2+k}{3+k} - \frac{2}{3}}{k}$$

$$= \lim_{k\to 0} \frac{1}{3(3+k)} \quad (\because k \neq 0) = \frac{1}{9}.$$

Example 2: Find, from definition, $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ where $z = x^2 + y^2$.

Solution: By definition, we get

$$\frac{\partial z}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \qquad (\text{where } z = f(x,y) = x^2 + y^2)$$

$$= \lim_{h\to 0} \frac{(x+h)^2 + y^2 - x^2 - y^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h}$$

$$= \lim_{h\to 0} (2x+h) \quad (\because h \to 0 \Rightarrow h \neq 0) = 2x$$

Similarly, $$\frac{\partial z}{\partial y} = 2y.$$

Observation: Evaluation of partial derivatives involve no new difficulty and one can always treat the given function to be a function of a single variable, the other variable or variables if any, are kept as constants. Therefore using the previous knowledge of derivatives of functions of a single variable, the result will come very easily.

Example 3: Find $\left(\frac{\partial f}{\partial x}\right)_{(0,2)}$ where $f(x, y) = x^2 + xy + y^2 + e^{xy^2}$.

Solution: By definition of partial derivative we keep y constant at the time of finding $\frac{\partial f}{\partial x}$.

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial x}(e^{xy^2})$$

$(\because y \text{ is constant})$

$$= 2x + y + y^2 e^{xy^2}$$

$$\therefore \quad \left(\frac{\partial f}{\partial x}\right)_{(0,2)} = 2\times 0 + 2 + 2^2 e^{0\times 2^2} = 6.$$

Example 4: If $u = \log(\tan x + \tan y)$, prove that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = 2.$$

(*W.B.U.T. 2005*)

Solution: Here $$u = \log(\tan x + \tan y)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y} \text{ and } \frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y}$$

$$\text{L.H.S.} = \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y}(\sin 2x \sec^2 x + \sin 2y \sec^2 y)$$

$$= \frac{1}{\tan x + \tan y}\left(\frac{2\sin x \cos x}{\cos^2 x} + \frac{2\sin y \cos y}{\cos^2 y}\right) = \frac{2(\tan x + \tan y)}{\tan x + \tan y}$$

$$= 2 = \text{R.H.S.}$$

Example 5: Find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ if $u = e^{r\cos\theta}\cos(r\sin\theta)$.

Solution: Here

$$u = e^{r\cos\theta}\cos(r\sin\theta)$$

$$\therefore \quad \frac{\partial u}{\partial r} = e^{r\cos\theta}\{-\sin(r\sin\theta)\sin\theta\} + e^{r\cos\theta}\cos\theta\cos(r\sin\theta)$$

$$= e^{r\cos\theta}\{\cos(r\sin\theta)\cos\theta - \sin(r\sin\theta)\sin\theta\}$$

$$= e^{r\cos\theta}\cos(r\sin\theta+\theta)$$

$$\frac{\partial u}{\partial \theta} = e^{r\cos\theta}\{-\sin(r\sin\theta)\}\, r\cos\theta + e^{r\cos\theta}(-r\sin\theta)\cos(r\sin\theta)$$

$$= -re^{r\cos\theta}\{\sin(r\sin\theta)\cos\theta + \cos(r\sin\theta)\sin\theta\}$$

$$= -re^{r\cos\theta}\sin(r\sin\theta+\theta).$$

Example 6: If $z = e^{ax+by} f(ax-by)$, show that

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2\,abz$$

Solution: Here $z = e^{ax+by} f(ax-by)$

$$\therefore \quad \frac{\partial z}{\partial x} = a e^{ax+by} f(ax-by) + e^{ax+by}\cdot a f'(ax-by)$$

$$\left[\text{Here } f'(ax-by) = \frac{d\,f(u)}{du}, u = ax-by\right]$$

$$\therefore \quad b\frac{\partial z}{\partial x} = ab\, e^{ax+by} f(ax-by) + ab\, e^{ax+by} f'(ax-by) \quad ...(1)$$

$$\frac{\partial z}{\partial y} = b\, e^{ax+by} f(ax-by) + e^{ax+by}(-b)\, f'(ax-by)$$

$$\therefore \quad a\frac{\partial z}{\partial y} = ab\, e^{ax+by} f(ax-by) - ab\, e^{ax+by} f'(ax-by) \quad ...(2)$$

Adding (1) and (2), we get

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab\, e^{ax+by} f(ax-by) = 2abz.$$

Example 7: If $u = e^{xyz}$, find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$.

Solution: Here $u = e^{xyz}$

$$\therefore \quad \frac{\partial u}{\partial z} = e^{xyz}\cdot xy$$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}(e^{xyz}\cdot xy) = e^{xyz}\cdot xz\cdot xy + e^{xyz}\cdot x$$

$$= e^{xyz}(x + x^2yz)$$

$$\frac{\partial^2 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = e^{xyz} \cdot yz(x + x^2 yz) + e^{xyz}(1 + 2xyz)$$

$$= e^{xyz}(xyz + x^2y^2z^2 + 1 + 2xyz)$$

$$= e^{xyz}(1 + 3xyz + x^2y^2z^2).$$

Example 8: Prove that

$$y = f(x+ct) + g(x-ct) \text{ satisfies}$$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

where f and g are assumed to be at least twice differentiable and c is any constant.

Solution:

Here $y = f(x+ct) + g(x-ct)$ ...(1)

Differentiating both sides of (1) partially with respect to x, we get

$$\frac{\partial y}{\partial x} = f'(x+ct) + g'(x-ct)$$

$$\frac{\partial^2 y}{\partial x^2} = f''(x+ct) + g''(x-ct) \quad ...(2)$$

Differentiating both sides of (1) partially with respect to t, we get

$$\frac{\partial y}{\partial t} = cf'(x+ct) - cg'(x-ct)$$

$$\therefore \quad \frac{\partial^2 y}{\partial t^2} = c^2 f''(x+ct) + c^2 g''(x-ct)$$

$$= c^2 \frac{\partial^2 y}{\partial x^2}. \quad \text{[By (2)]}$$

Example 9: If

$$x^x y^y z^z = c, \text{ show that at } x = y = z,$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\{x \log_e(ex)\}^{-1}.$$

Solution: Here $x^x y^y z^z = c$, where z is a function of x and y.

Now taking logarithm of both sides, we get

$$x \log_e x + y \log_e y + z \log_e z = \log_e c \quad ...(1)$$

Differentiating both sides of (1) partially with respect to x, we get

$$\left(x \cdot \frac{1}{x} + \log_e x\right) + \left(z \cdot \frac{1}{z} + \log_e z\right)\frac{\partial z}{\partial x} = 0$$

or $$\frac{\partial z}{\partial x} = -\left(\frac{1+\log_e x}{1+\log_e z}\right) \qquad ...(2)$$

Similarly, by symmetry, we have

$$\frac{\partial z}{\partial y} = -\left(\frac{1+\log_e y}{1+\log_e z}\right)$$

$$\therefore \quad \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left\{-\left(\frac{1+\log_e y}{1+\log_e z}\right)\right\}$$

$$= -(1+\log_e y)\frac{\partial}{\partial x}(1+\log_e z)^{-1} = \frac{(1+\log_e y)}{z(1+\log_e z)^2}\cdot\frac{\partial z}{\partial x}$$

$$= -\frac{(1+\log_e y)\,(1+\log_e x)}{z(1+\log_e z)^3} \qquad \text{[By (2)]}$$

$$\therefore \quad \left(\frac{\partial^2 z}{\partial x \partial y}\right)_{x=y=z} = -\frac{1}{x(1+\log_e x)} = -\{x\log_e(ex)\}^{-1}.$$

Example 10: If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that

(*i*) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x+y+z}$

(*ii*) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = -\dfrac{3}{(x+y+z)^2}$ (*W.B.U.T. 2003*)

(*iii*) $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = -\dfrac{9}{(x+y+z)^2}$.

Solution: (*i*) We know that

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega),$$

where ω, ω^2 are the imaginary cube roots of unity.

Therefore $\qquad \omega^3 = 1$ and $1 + \omega + \omega^2 = 0 \qquad ...(1)$

$$\therefore \quad u = \log(x+y+z) + \log(x+y\omega+z\omega^2) + \log(x+y\omega^2+z\omega)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{x+y+z} + \frac{1}{x+y\omega+z\omega^2} + \frac{1}{x+y\omega^2+z\omega}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x+y+z} + \frac{\omega}{x+y\omega+z\omega^2} + \frac{\omega^2}{x+y\omega^2+z\omega}$$

$$\frac{\partial u}{\partial z} = \frac{1}{x+y+z} + \frac{\omega^2}{x+y\omega+z\omega^2} + \frac{\omega}{x+y\omega^2+z\omega}$$

Adding and using (1), we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}.$$

(*ii*) Now,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = -\frac{1}{(x+y+z)^2} - \frac{1}{(x+y\omega+z\omega^2)^2} - \frac{1}{(x+y\omega^2+z\omega)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{1}{(x+y+z)^2} - \frac{\omega^2}{(x+y\omega+z\omega^2)^2} - \frac{\omega^4}{(x+y\omega^2+z\omega)^2}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = -\frac{1}{(x+y+z)^2} - \frac{\omega^4}{(x+y\omega+z\omega^2)^2} - \frac{\omega^2}{(x+y\omega^2+z\omega)^2}$$

Adding and using (1), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^2}. \qquad (\because \omega^4 = \omega^3 \cdot \omega = \omega)$$

(*iii*) Here,

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

[From (*i*)]

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{9}{(x+y+z)^2}.$$

Example 11: (*i*) If $U = \sqrt{xy}$, find the value of $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$. (*W.B.U.T. 2001*)

(*ii*) If $U(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, then show that $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$.

Solution: (*i*) Here $U(x, y) = \sqrt{xy} = x^{\frac{1}{2}} y^{\frac{1}{2}}$

$$\therefore \qquad \frac{\partial U}{\partial x} = \frac{1}{2} x^{-\frac{1}{2}} y^{\frac{1}{2}},$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = -\frac{1}{4} x^{-\frac{3}{2}} y^{\frac{1}{2}}$$

Since U is symmetric in x, y, therefore,

$$\frac{\partial^2 U}{\partial y^2} = -\frac{1}{4} x^{\frac{1}{2}} y^{-\frac{3}{2}}$$

$$\therefore \qquad \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -\frac{1}{4}\left(x^{-\frac{3}{2}} y^{\frac{1}{2}} + x^{\frac{1}{2}} y^{-\frac{3}{2}}\right).$$

(*ii*) Here $\qquad \dfrac{\partial U}{\partial x} = -\dfrac{1}{2} \cdot \dfrac{2x}{(x^2+y^2+z^2)^{3/2}} = -\dfrac{x}{(x^2+y^2+z^2)^{3/2}}$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = -\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\therefore \qquad \frac{\partial^2 U}{\partial x^2} = \frac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{5/2}} \qquad \text{...(1)}$$

Since $U(x, y, z)$ is symmetric in x, y, z,

therefore, $\qquad \dfrac{\partial^2 U}{\partial y^2} = \dfrac{2y^2 - z^2 - x^2}{(x^2+y^2+z^2)^{5/2}} \qquad$ ...(2)

and $\qquad \dfrac{\partial^2 U}{\partial z^2} = \dfrac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}} \qquad$ ...(3)

Adding (1), (2) and (3), we get

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0.$$

Example 12: If $x = r\cos\theta$, $y = r\sin\theta$, then show that

(*i*) $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$, where $u = f(r)$,

(*ii*) $\quad \dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right]$,

(iii) $\dfrac{\partial^2\theta}{\partial x^2}+\dfrac{\partial^2\theta}{\partial y^2}=0$, provided $x \neq 0$, $y \neq 0$.

Solution: (i) Here $u = f(r)$, $x = r\cos\theta$, $y = r\sin\theta$, therefore, $u = f(\sqrt{x^2+y^2})$ which is symmetric in x, y.

$$\therefore \quad \frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x}, \quad \frac{\partial^2 u}{\partial x^2} = f''(r)\cdot\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2}$$

Similarly, since u is symmetric in x, y

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f''(r)\left\{\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right\}+f'(r)\left\{\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}\right\} \quad ...(1)$$

Since $\quad r = \sqrt{x^2+y^2}$

therefore, $\quad \dfrac{\partial r}{\partial x} = \dfrac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = \dfrac{x}{r} \quad ...(2)$

and $\quad \dfrac{\partial^2 r}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial r}{\partial x}\right) = \dfrac{r\cdot 1 - x\cdot\dfrac{\partial r}{\partial x}}{r^2} = \dfrac{r^2-x^2}{r^3} = \dfrac{y^2}{r^3} \quad (\because r^2 = x^2+y^2) \quad ...(3)$

Similarly, since r is symmetric in x, y

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad ...(4)$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3} \quad ...(5)$$

Therefore, from (1) – (5), we have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f''(r)\left\{\frac{x^2+y^2}{r^2}\right\}+f'(r)\left\{\frac{y^2+x^2}{r^3}\right\}$$

$$= f''(r)+\frac{1}{r}f'(r)$$

(ii) From (2) – (5), we get

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left\{\left(\frac{y}{r}\right)^2+\left(\frac{x}{r}\right)^2\right\} = \frac{1}{r}\left\{\left(\frac{\partial r}{\partial y}\right)^2+\left(\frac{\partial r}{\partial x}\right)^2\right\}$$

$$= \frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\}$$

(iii) Since $x = r\cos\theta$, $y = r\sin\theta$, therefore

$$\theta = \tan^{-1}\left(\frac{y}{x}\right),$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

and
$$\frac{\partial^2\theta}{\partial x^2} = -y\left\{\frac{-1}{(x^2+y^2)^2}\cdot 2x\right\} = \frac{2xy}{(x^2+y^2)^2} \qquad \text{...(6)}$$

Again
$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

and
$$\frac{\partial^2\theta}{\partial y^2} = \frac{-x}{(x^2+y^2)^2}\cdot 2y = \frac{-2xy}{(x^2+y^2)^2} \qquad \text{...(7)}$$

Adding (6) and (7), we get

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$$

Example 13: If $f(x, y) = \frac{1}{\sqrt{y}}e^{-\frac{(x-a)^2}{4y}}$, verify that

$$f_{xy}(x, y) = f_{yx}(x, y).$$

Solution: Here

$$f(x, y) = y^{-\frac{1}{2}}e^{-\frac{(x-a)^2}{4y}} \qquad \text{...(1)}$$

Differentiating $f(x, y)$ partially with respect to x, we get

$$f_x(x, y) = \frac{\partial f}{\partial x} = y^{-\frac{1}{2}}\cdot\frac{1}{4y}\{-2(x-a)\}\,e-(x-a)^2/4y$$

$$= -\frac{1}{2}y^{-3/2}(x-a)\,e-(x-a)^2/4y$$

Differentiating again partially with respect to y, we have

$$f_{yx}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{3}{4}y^{-5/2}(x-a)e-(x-a)^2/4y$$

$$-\frac{1}{8}y^{-7/2}(x-a)^3e-(x-a)^2/4y$$

$$= \frac{1}{8}(x-a)y^{-7/2}e-(x-a)^2/4y\,\{6y-(x-a)^2\} \quad ...(2)$$

Differentiating (1) partially with respect to y, we get

$$f_y(x,y) = -\frac{1}{2}y^{-3/2}e-(x-a)^2/4y + \frac{1}{4}y^{-5/2}(x-a)^2e-(x-a)^2/4y$$

$$\therefore \quad f_{xy}(x,y) = \frac{\partial}{\partial x}(f_y) = \frac{1}{4}y^{-5/2}(x-a)e-(x-a)^2/4y$$

$$+\frac{1}{2}y^{-5/2}(x-a)e-(x-a)^2/4y - \frac{1}{8}y^{-7/2}(x-a)^3e-(x-a)^2/4y$$

$$= \frac{1}{8}(x-a)y^{-7/2}e-(x-a)^2/4y\{2y+4y-(x-a)^2\}$$

$$= \frac{1}{8}(x-a)y^{-7/2}e-(x-a)^2/4y\{6y-(x-a)^2\} \quad ...(3)$$

From (2) and (3), we get

$$f_{xy}(x,y) = f_{yx}(x,y).$$

Example 14: If $\quad f(x,y) = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right),$ (W.B.U.T. 2011, 2013)

show that $\quad f_{xy} = f_{yx}.$

Solution: Here $\quad f(x,y) = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right) \quad ...(1)$

Differentiating (1) partially with respect to y, we get

$$f_y = \frac{\partial f}{\partial y} = \frac{x^2}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} - \left\{2y\tan^{-1}\left(\frac{x}{y}\right) + y^2\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right)\right\}$$

$$= \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right) + \frac{xy^2}{x^2+y^2} = x - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\therefore \quad f_{xy} = \frac{\partial}{\partial x}(f_y) = 1 - 2y \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} \qquad ...(2)$$

Differentiating (1) partially with respect to x, we have

$$f_x = \frac{\partial f}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) + x^2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) - y^2 \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2y}{x^2+y^2} - \frac{y^3}{x^2+y^2} = 2x\tan^{-1}\left(\frac{y}{x}\right) - y$$

$$\therefore \quad f_{yx} = \frac{\partial}{\partial y}(f_x) = 2x \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2} \qquad ...(3)$$

From (2) and (3), we get $f_{xy} = f_{yx}$.

A **Example 15:** Given the function

$$f(x, y) = \begin{cases} xy(x^2-y^2)/(x^2+y^2), & \text{when } (x, y) \neq (0, 0) \\ 0, & \text{when } (x, y) = (0, 0) \end{cases}$$

find, from first principle, $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.

Verify whether $f_{xy}(0, 0) = f_{yx}(0, 0)$. (*W.B.U.T. 2003, 2010*)

Solution: First step:

$$f_{xy}(0, 0) = \lim_{h\to 0} \frac{f_y(0+h, 0) - f_y(0, 0)}{h}$$

Also,
$$f_y(h, 0) = \lim_{k\to 0} \frac{f(h, 0+k) - f(h, 0)}{k}$$

$$= \lim_{k\to 0} \frac{hk\dfrac{h^2-k^2}{h^2+k^2} - h\times 0\dfrac{h^2-0^2}{h^2+0^2}}{k} = \lim_{k\to 0} \frac{h(h^2-k^2)}{h^2+k^2} \qquad (\because k \neq 0)$$

$$= \frac{h^3}{h^2} = h$$

and $$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

$\therefore$ $$f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1 \qquad (\because h \neq 0)$$

Second step: $$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k)-f_x(0,0)}{k}$$

Also, $$f_x(0,k) = \lim_{h\to 0} \frac{f(0+h,k)-f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{hk\dfrac{h^2-k^2}{h^2+k^2} - 0\times k\dfrac{0^2-k^2}{0^2+k^2}}{h} = \lim_{h\to 0} \frac{k(h^2-k^2)}{h^2+k^2} \qquad (\because h \neq 0)$$

$$= \frac{-k^3}{k^2} = -k$$

and $$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\therefore$ $$f_{yx}(0,0) = \lim_{k\to 0} \frac{-k-0}{k} = -1 \qquad (\because k \neq 0)$$

Third step: Since $f_{xy}(0,0) = 1$ and $f_{yx}(0,0) = -1$,
we conclude that $$f_{xy}(0,0) \neq f_{yx}(0,0).$$

Example 16. If $\theta = t^n e^{\frac{-r^2}{4t}}$, find what value of n will make

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t} \qquad \textit{(W.B.U.T. 2012)}$$

Solution: Given,

$$\theta = t^n e^{\frac{-r^2}{4t}}, \text{ a function of } r \text{ and } t \qquad ...(1)$$

$\therefore$ $$\frac{\partial \theta}{\partial r} = t^n e^{\frac{-r^2}{4t}}\left(-\frac{2r}{4t}\right) = -\frac{r\theta}{2t} \qquad [\text{by } (1)]$$

$\therefore$ $$\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{r^3\theta}{2t}\right)$$

$$= -\frac{1}{2t}\left\{3r^2\theta + r^3\frac{\partial \theta}{\partial r}\right\}$$

$$= -\frac{1}{2t}\left\{3r^2\theta + r^3\left(-\frac{r\theta}{2t}\right)\right\}$$

$$= -\frac{1}{2t}\left\{3r^2\theta - \frac{r^4\theta}{2t}\right\}$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{1}{2t}\left\{3\theta - \frac{r^2\theta}{2t}\right\} \quad \text{...(2)}$$

Again,
$$\frac{\partial\theta}{\partial t} = nt^{n-1}e^{\frac{-r^2}{4t}} + t^n e^{\frac{-r^2}{4t}}\left(\frac{r^2}{4t^2}\right)$$

$$= \frac{\theta}{t}\left(n + \frac{r^2}{4t}\right) \quad \text{...(3)}$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$$

$$\Rightarrow \quad \frac{\theta}{t}\left\{-\frac{3}{2} + \frac{r^2}{4t}\right\} = \frac{\theta}{t}\left\{n + \frac{r^2}{4t}\right\} \quad \text{[using (2) and (3)]}$$

$$\Rightarrow \quad n = -\frac{3}{2}$$

8.5 DIFFERENTIABILITY AND TOTAL DIFFERENTIAL

Definition 1: Let $f(x, y)$ be a function of two independent variables x and y and let $\Delta x = dx$ and $\Delta y = dy$ be the increments of x and y respectively. Then

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x, y)$$

is called total increment or simply increment of the function $f(x, y)$.

Definition 2: The function $f(x, y)$ is said to be differentiable at (x, y) when its total increment Δf can be expressed in the form

$$\Delta f = (A\Delta x + B\Delta y) + \varepsilon_1 \cdot \Delta x + \varepsilon_2 \cdot \Delta y \quad \text{...(1)}$$

where A and B are independent of Δx and Δy, ε_1 and ε_2 are functions of Δx and Δy such that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$ in any manner.

Theorem 1: If a function $f(x, y)$ is differentiable at (x, y), then the first order partial derivatives

$$f_x = \frac{\partial f}{\partial x} \text{ and } f_y = \frac{\partial f}{\partial y} \text{ exist at } (x, y).$$

Proof: Put $\Delta y = 0$ in (1) and then divide by Δx and make $\Delta x \to 0$, we get

$$A = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} = \frac{\partial f}{\partial x}. \qquad (\because \varepsilon_1 \to 0 \text{ as } \Delta x \to 0)$$

Similarly, $B = \dfrac{\partial f}{\partial y}$. Therefore the differentiability of $f(x, y)$ at (x, y) implies the existence of f_x and f_y at (x, y).

Theorem 2: If a function $f(x, y)$ is differentiable at (x, y), then it must be continuous at (x, y).

Proof: Since $f(x, y)$ is differentiable at (x, y), therefore there exist ε_1, ε_2 such that

$$f(x+\Delta x, y+\Delta y) - f(x, y) = (A\Delta x + B\Delta y) + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where A, B are independent of Δx and Δy and $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $\Delta x \to 0$, $\Delta y \to 0$ in any manner.

$$\therefore \quad \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f(x+\Delta x, y+\Delta y) = f(x, y)$$

which proves that $f(x, y)$ is continuous at (x, y).

Total differential: Using the above theorem, the total increment Δf of a differentiable function $f(x, y)$ can be written in the form

$$\Delta f = (f_x \Delta x + f_y \Delta y) + \varepsilon_1 \cdot \Delta x + \varepsilon_2 \cdot \Delta y$$

where $\varepsilon_1 \to 0, \varepsilon_2 \to 0$ as $\Delta x \to 0$, $\Delta y \to 0$.

When f_x and f_y exist at (x, y), the part $f_x \Delta x + f_y \Delta y$ of Δf, written as df, is called total differential of f.

$$\therefore \quad \boxed{df = f_x \Delta x + f_y \Delta y = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy} \qquad (\because \Delta x = dx, \Delta y = dy)$$

Note: *(i)* The above concept of total differential can be extended to the functions of three or more variables. Thus for a differentiable function $f(x, y, z)$, we have

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

(ii) If $f(x, y) =$ constant, then $df = 0$. This result is also true for differentiable functions of more variables.

Second order differential

Let $z = f(x, y)$ be a function of two independent variables x and y, then the second order differential of $z = f(x, y)$ is denoted by $d^2 z$ and it is defined by $d^2 z = d(dz)$.

Theorem 3: If a function $z = f(x, y)$ possesses continuous second order partial derivatives, then

$$d^2 z = \frac{\partial^2 z}{\partial x^2}(dx)^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx\, dy + \frac{\partial^2 z}{\partial y^2}(dy)^2$$

$$= \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right)^2 z.$$

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Proof: Now,
$$d^2z = d(dz) = d\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)$$
$$= \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)\right\}dx + \left\{\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)\right\}dy$$
$$= \left(\frac{\partial^2 z}{\partial x^2}dx + \frac{\partial^2 z}{\partial x \partial y}dy\right)dx + \left(\frac{\partial^2 z}{\partial y \partial x}dx + \frac{\partial^2 z}{\partial y^2}dy\right)dy$$

[since dx, dy are independent of x, y, we have $\frac{\partial}{\partial x}(dx) = \frac{\partial}{\partial y}(dx) = \frac{\partial}{\partial x}(dy) = \frac{\partial}{\partial y}(dy) = 0$].

Since $z = f(x, y)$ is assumed to have continuous second order derivatives, therefore $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ and so we get

$$d^2z = \frac{\partial^2 z}{\partial x^2}(dx)^2 + 2\frac{\partial^2 z}{\partial x \partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}(dy)^2.$$

Note: 1. The above theorem can be extended for functions of more than two variables.

If $u = f(x, y, z)$, then $d^2u = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)^2 u$.

2. Total differential of a function of several variables has many applications of which the problems of error and approximation are very important.

ILLUSTRATIVE EXAMPLES

Example 1: If $z = f(x, y) = xy^2 - x$, find Δz. Hence find whether z is differentiable. If so find dz.

Solution: Here
$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y)$$
$$= \{(x+\Delta x)(y+\Delta y)^2 - (x+\Delta x)\} - (xy^2 - x)$$
$$= y^2\Delta x - \Delta x + 2xy\Delta y + 2y\Delta x\Delta y + x(\Delta y)^2 + \Delta x(\Delta y)^2$$
$$= (y^2-1)\Delta x + 2xy\Delta y + (\Delta y)^2\Delta x + (2y\Delta x + x\Delta y)\Delta y$$
$$= (y^2-1)\Delta x + 2xy\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

We observe that $\varepsilon_1 = (\Delta y)^2 \to 0$, $\varepsilon_2 = 2y\Delta x + x\Delta y \to 0$ as $\Delta x \to 0$, $\Delta y \to 0$ in any manner

Therefore, by definition, the function $f(x, y)$ is differentiable.

The total differential, $dz = (y^2-1)\Delta x + 2xy\Delta y$
$$= (y^2-1)dx + 2xy\,dy.$$

Example 2: Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{when } (x, y) \neq (0, 0) \\ 0, & \text{when } (x, y) = (0, 0) \end{cases}$$

is not differentiable at (0, 0), although it is continuous and has equal partial derivatives at (0, 0).

Solution: In view of Example 8, Art. 8.3, $f(x, y)$ is continuous at (0, 0).

Next to find the partial derivatives we see that,

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

and

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, 0+k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

Therefore the partial derivatives at (0, 0) exist and each equals zero.

Now, let, if possible, $f(x, y)$ be differentiable at (0, 0) therefore for increments h of x and k of y, we can write

$$f(0+h, 0+k) - f(0, 0) = hf_x(0, 0) + kf_y(0, 0) + \varepsilon_1 h + \varepsilon_2 k$$

where $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $h \to 0$, $k \to 0$ in any manner.

$$\therefore \quad \frac{hk}{\sqrt{h^2+k^2}} - 0 = 0 \cdot h + 0 \cdot k + \varepsilon_1 h + \varepsilon_2 k$$

or

$$\lim_{\substack{h \to 0 \\ k \to 0}} \frac{hk}{\sqrt{h^2+k^2}} = \lim_{\substack{h \to 0 \\ k \to 0}} (\varepsilon_1 h + \varepsilon_2 k)$$

$$\therefore \quad \lim_{\substack{h \to 0 \\ k \to 0}} \frac{hk}{\sqrt{h^2+k^2}} = 0 \qquad (\because \varepsilon_1 \to 0, \varepsilon_2 \to 0 \text{ as } h \to 0, k \to 0)$$

But this is not true since the limit on the left side does not exist as we see that the limit takes different values for different choice of paths. In fact, it is seen that the limiting value is $\dfrac{mh}{\sqrt{1+m^2}}$ as $h \to 0$, $k \to 0$ along the path $k = mh$. Therefore $f(x, y)$ is not differentiable at (0, 0).

Example 3: Using the concept of total differential compute the approximate value of $(4.16)^2 \times 6.88 + (6.88)^2 \times 4.16 + 2 \times 4.16$.

Solution: Consider the function $f(x, y) = x^2y + y^2x + 2x$. We choose $x = 4$, $\Delta x = 0.16$, $y = 7$ and $\Delta y = -0.12$. Since Δx, Δy are small we take $df \simeq \Delta f$.

Now,

$$f(x+\Delta x, y+\Delta y) - f(x, y) = \Delta f \simeq df$$

Also,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = (2xy + y^2 + 2)dx + (x^2 + 2yx)dy$$

$$= (2\times4\times7+7^2+2)\times0.16+(4^2+2\times7\times4)\times(-0.12) = 8.48$$

$(\because dx = \Delta x, dy = \Delta y)$

Therefore, from above $f(4 + 0.16, 7 - 0.12) - f(4,7) \simeq 8.48$

or $f(4.16, 6.88) \simeq f(4,7) + 8.48$

$$\therefore\ (4.16)^2\times6.88+(6.88)^2\times4.16+2\times4.16$$

$$\simeq 4^2\times7+7^2\times4+2\times4+8.48 = 324.48.$$

Example 4: If $z = x^2y + y^2x$, find d^2z.

Solution: We know that

$$d^2z = \frac{\partial^2 z}{\partial x^2}(dx)^2 + 2\frac{\partial^2 z}{\partial x\partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}(dy)^2$$

Here
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(2xy+y^2) = 2y,$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(x^2+2yx) = 2x$$

and
$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(x^2+2yx) = 2x+2y$$

$$\therefore\quad d^2z = 2y(dx)^2 + 4(x+y)\,dx\,dy + 2x(dy)^2.$$

Example 5: If $u = xyz$, find d^2u.

Solution: Here $u = xyz$, therefore

$$d^2u = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)^2 u$$

$$= (dx)^2\frac{\partial^2 u}{\partial x^2} + (dy)^2\frac{\partial^2 u}{\partial y^2} + (dz)^2\frac{\partial^2 u}{\partial z^2}$$

$$+ 2\,dx\,dy\frac{\partial^2 u}{\partial x\partial y} + 2dy\,dz\frac{\partial^2 u}{\partial y\partial z} + 2dz\,dx\frac{\partial^2 u}{\partial z\partial x} \qquad \text{...(1)}$$

Now,
$$\frac{\partial u}{\partial x} = yz,\ \frac{\partial u}{\partial y} = xz,\ \frac{\partial u}{\partial z} = xy,$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial z^2} = 0,$$

$$\frac{\partial^2 u}{\partial x\partial y} = z,\ \frac{\partial^2 u}{\partial y\partial z} = x,\ \frac{\partial^2 u}{\partial z\partial x} = y$$

Therefore, from (1), we have

$$d^2u = 2z\,dx\,dy + 2x\,dy\,dz + 2y\,dz\,dx.$$

Example 6: The indicated horse-power of an engine is calculated from the formula

$$I = \frac{PLAN}{33{,}000}, \text{ where } A = \frac{\pi}{4}D^2$$

Assuming that errors of r percent may have been made in measuring P, L, N and D, find the approximate percentage error in I.

Solution: Here $\quad I = \dfrac{PLAN}{33{,}000} = \dfrac{PL\pi D^2 N}{1{,}32{,}000} \qquad \left(\because A = \dfrac{\pi}{4}D^2\right)$

$$\therefore \quad \log I = \log P + \log L + \log \pi + 2\log D + \log N - \log 132000$$

$$\therefore \quad \frac{dI}{I} = \frac{dP}{P} + \frac{dL}{L} + 2\frac{dD}{D} + \frac{dN}{N} \qquad \left[\because d(\log x) = \frac{d}{dx}(\log x)dx = \frac{dx}{x}\right]$$

$$\therefore \quad 100\frac{dI}{I} = 100\frac{dP}{P} + 100\frac{dL}{L} + 200\frac{dD}{D} + 100\frac{dN}{N}$$

By question, $\quad 100\dfrac{dP}{P} = 100\dfrac{dL}{L} = 100\dfrac{dD}{D} = 100\dfrac{dN}{N} = r,$

so the above result gives

$$100\frac{dI}{I} = r + r + 2r + r = 5r.$$

Therefore, approximate percentage error in I is $5r\%$.

Example 7: The diameter and altitude of a can in the shape of a right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximate errors in the values computed for the volume and lateral surface.

Solution: Diameter of the cylindrical can, $D = 4$ cm. Height of the cylindrical can, $h = 6$ cm. It is given that $dD = dh = 0.1$ cm.

Volume of the can, $\quad V = \dfrac{\pi}{4}D^2h$

$$\therefore \quad dV = \frac{\partial V}{\partial D}\cdot dD + \frac{\partial V}{\partial h}\cdot dh = \frac{\pi}{h}\,[2Dh\,dD + D^2\,dh]$$

$$= \frac{\pi}{4}[2\times4\times6\times0.1 + 16\times0.1] = \pi(1.2 + 0.4)$$

$$= 1.6\pi = 5.0265\text{ cm}^3.$$

Lateral surface of the can, $S = \pi Dh$.

$$\therefore \quad dS = \frac{\partial S}{\partial D}dD + \frac{\partial S}{\partial h}dh = \pi\,[h\,dD + D\,dh]$$

$$= \pi(6\times0.1 + 4\times0.1) = \pi = 3.142\text{ cm}^2.$$

Therefore approximate errors in the values computed for the volume and lateral surface are 5.0265 cm^3 and 3.142 cm^2 respectively.

Example 8: The deflection at the centre of a rod of length l and diameter D supported at its ends and loaded at the centre with a weight W varies as Wl^3D^{-4}. What is approximate percentage increase in the deflection corresponding to the percentage increase in W, l and D are 3, 2 and 1 respectively.

Solution: Let the deflection of the rod at the centre be E.

$$\therefore \quad E = k\frac{Wl^3}{D^4}, \quad k \text{ is a constant}$$

$$\therefore \quad \log E = \log k + \log W + 3\log l - 4\log D$$

$$\therefore \quad \frac{dE}{E} = \frac{dW}{W} + 3\frac{dl}{l} - 4\frac{dD}{D} \qquad \left[\because d(\log x) = \frac{d}{dx}(\log x)dx = \frac{dx}{x}\right]$$

$$\therefore \quad 100\frac{dE}{E} = 100\frac{dW}{W} + 3\times 100\frac{dl}{l} - 4\times 100\frac{dD}{D}$$

$$= 3 + 3\times 2 - 4\times 1 = 5$$

Hence, the required approximate percentage increase in the deflection is 5%.

8.6 HOMOGENEOUS FUNCTIONS

Definition: A function $f(x, y)$ of two independent variables x, y is said to be homogeneous of degree n if one of the followings holds:

(*i*) $f(x, y) = x^n\varphi\left(\frac{y}{x}\right)$

or (*ii*) $f(x, y) = y^n\psi\left(\frac{x}{y}\right)$

or (*iii*) $f(tx, ty) = t^n f(x, y)$, for every positive t.

Example: The function $f(x, y) = \frac{x^2}{y} + \frac{2y^2}{x}$ is homogeneous of degree one since we could write

$$f(x, y) = x\left\{\frac{1}{\frac{y}{x}} + 2\left(\frac{y}{x}\right)^2\right\} = x\varphi\left(\frac{y}{x}\right),$$

or $$f(x, y) = y\left\{\left(\frac{x}{y}\right)^2 + \frac{2}{\frac{x}{y}}\right\} = y\psi\left(\frac{x}{y}\right),$$

or $$f(tx, ty) = \frac{t^2x^2}{ty} + 2\frac{t^2y^2}{tx} = t\,f(x, y).$$

Example: The function $f(x, y) = \dfrac{x-y}{x+y}$ is homogeneous of degree zero since

$$f(tx, ty) = \frac{tx-ty}{tx+ty} = t^{0} f(x,y).$$

Example: The function $f(x, y) = x^2 + y + y^2$ is not a homogeneous function.

Note: Similar definition applies to functions of three and more variables.

Example: The function $f(x, y, z) = 1/(\sqrt{x}+\sqrt{y}+\sqrt{z})$ is homogeneous of degree $-\dfrac{1}{2}$ since

$$f(tx, ty, tz) = 1/(\sqrt{tx}+\sqrt{ty}+\sqrt{tz}) = t^{-\frac{1}{2}} f(x,y,z).$$

Theorem: Euler's theorem for two variables

If $f(x, y)$ be a homogeneous function of x and y of degree n having continuous partial derivatives, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n f(x,y)$$

Proof: Since $f(x, y)$ is a homogeneous function of degree n, therefore, we may express

$$f(x,y) = x^n \varphi\left(\frac{y}{x}\right)$$

$$\therefore \quad \frac{\partial f}{\partial x} = n x^{n-1}\varphi\left(\frac{y}{x}\right) + x^n \varphi'\left(\frac{y}{x}\right)\times\left(-\frac{y}{x^2}\right) \quad \left[\text{where } \varphi'\left(\frac{y}{x}\right) = \frac{d\varphi}{du}, u \equiv \frac{y}{x}\right]$$

$$\therefore \quad x\frac{\partial f}{\partial x} = nx^n\varphi\left(\frac{y}{x}\right) - x^{n-1} y\, \varphi'\left(\frac{y}{x}\right)$$

Again,
$$\frac{\partial f}{\partial y} = x^n \varphi'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$

$$\therefore \quad y\frac{\partial f}{\partial y} = x^{n-1} y\, \varphi'\left(\frac{y}{x}\right)$$

Hence,
$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n x^n \varphi\left(\frac{y}{x}\right) = n f(x,y).$$

Extension of Euler's theorem for two variables

If $f(x, y)$ be a homogeneous function of x and y of degree n having second order continuous partial derivatives, then

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1) f(x,y)$$

or $$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(x,y) = n(n-1)f(x,y).$$ [BESUS (B.Arch) 2013]

Proof: Since $f(x, y)$ is a homogeneous function of degree n, we have by Euler's theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = nf \qquad ...(1)$$

Differentiating both sides of (1) partially with respect to x, we get

$$\frac{\partial f}{\partial x}+x\frac{\partial^2 f}{\partial x^2}+y\frac{\partial^2 f}{\partial x\partial y} = n\frac{\partial f}{\partial x} \qquad ...(2)$$

Again differentiating both sides of (1) partially with respect to y, we get

$$x\frac{\partial^2 f}{\partial y\partial x}+\frac{\partial f}{\partial y}+y\frac{\partial^2 f}{\partial y^2} = n\frac{\partial f}{\partial y} \qquad ...(3)$$

Multiplying (2) by x and (3) by y and adding, we obtain

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}+\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)=n\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)$$

$\left(\because \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}\right.$ as second order partial derivatives are continuous)

$$\therefore \quad x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f(x,y). \qquad \text{[using (1)]}$$

Theorem: Euler's theorem for three variables

If $f(x, y, z)$ be a homogeneous function of x, y and z of degree n having continuous partial derivatives, then

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z} = nf(x,y,z).$$

Proof: Since $f(x, y, z)$ is a homogeneous function of degree n, therefore, we may write

$$f(tx,ty,tz) = t^n f(x,y,z) \qquad ...(1)$$

where t is independent of x, y, z.

Let us put $u = tx$, $v = ty$, $w = tz$ and then differentiating both sides of (1) partially with respect to t, we get

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial t}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial t}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial t} = nt^{n-1}f(x,y,z) \qquad \text{[using chain rule, see art. 8.7]}$$

or $$\frac{\partial f}{\partial u}\cdot x+\frac{\partial f}{\partial v}\cdot y+\frac{\partial f}{\partial w}\cdot z = nt^{n-1}f(x,y,z) \qquad (\because u=tx, v=ty, w=tz)$$

Example 1: Verify Euler's theorem for the function

$$f(x, y) = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}.$$

Solution: Here $\quad f(tx, ty) = t^{o} f(x, y)$

Therefore $f(x, y)$ is a homogeneous function in x, y of degree zero.

So, we have to verify the relation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0 \cdot f(x,y) = 0 \qquad ...(1)$$

Now, $$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$\therefore$ $$x\frac{\partial f}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \qquad ...(2)$$

Also, $$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\left(-\frac{x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$\therefore$ $$y\frac{\partial f}{\partial y} = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \qquad ...(3)$$

Adding (2) and (3), we get $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0.$

Hence relation (1) is verified, *i.e.*, Euler's theorem is verified.

Example 2: If $\quad f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2},$

Prove that $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}+2f = 0.$

Solution: Here $\quad f(x,y) = \dfrac{1}{x^2}+\dfrac{1}{xy}+\dfrac{\log\left(\dfrac{x}{y}\right)}{x^2+y^2}$

$$\therefore \qquad f(tx,ty) = t^{-2}f(x,y)$$

Therefore, $f(x, y)$ is a homogeneous function in x, y of degree -2 and hence by Euler's theorem, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = -2f, \text{ or } x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+2f = 0.$$

Example 3: If $\quad u = \sin^{-1}\dfrac{x^2+y^2}{x+y},$

show that $\quad x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y} = \tan u.$

Solution: Here $\quad u = \sin^{-1}\dfrac{x^2+y^2}{x+y},$

which is not a homogeneous function but if we put

$$f(x,y) \equiv \sin u = \frac{x^2+y^2}{x+y},$$

then $f(x, y)$ is a homogeneous function in x, y of degree 1 since $f(tx, ty) = t\, f(x, y)$.

Therefore by applying Euler's theorem, we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = f \qquad \text{...(1)}$$

Since f is a function of u, u is a function of x and y, so by Chain Rule

$$\frac{\partial f}{\partial x} = \frac{df}{du}\cdot\frac{\partial u}{\partial x} = (\cos u)\frac{\partial u}{\partial x}$$

and $$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} = (\cos u)\frac{\partial u}{\partial y} \qquad [\because f = \sin u]$$

Therefore, from (1), we get

$$x(\cos u)\frac{\partial u}{\partial x}+y(\cos u)\frac{\partial u}{\partial y} = \sin u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \tan u.$$

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Example 4: If $u = \cos^{-1}\left\{\frac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$, then prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u = 0.$$ (W.B.U.T. 2009, BESUS 2013)

Solution: Here $u = \cos^{-1}\left\{\frac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$

$\therefore$ $$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = f(x,y) \text{ (say)}$$

Now $$f(tx,ty) = t^{1/2}f(x,y)$$

and therefore $f(x, y)$ is homogeneous in x, y of degree $\frac{1}{2}$.

Applying Euler's theorem, we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = \frac{1}{2}f, \text{ or } x\frac{df}{du}\cdot\frac{\partial u}{\partial x}+y\frac{df}{du}\cdot\frac{\partial u}{\partial y} = \frac{1}{2}f$$

$$\therefore\ x(-\sin u)\frac{\partial u}{\partial x}+y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u \qquad [\because f=\cos u]$$

or $$-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y} = \frac{1}{2}\cot u$$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u = 0.$$

Example 5: If $u = \sin^{-1}\sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$ then verify whether the following identity is true:

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right).$$ (W.B.U.T. 2008)

Solution: Here $u = \sin^{-1}\sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$

$\therefore$ $$\sin u = \sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}} = f(x,y) \text{ (say)}$$

Now $$f(tx,ty) = \sqrt{\frac{(tx)^{1/3}+(ty)^{1/3}}{(tx)^{1/2}+(ty)^{1/2}}} = t^{-1/12}f(x,y)$$

and therefore $f(x, y)$ is homogeneous in x, y of degree $-\frac{1}{12}$.

Applying Euler's theorem, we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = -\frac{1}{12}f,$$

or
$$x\frac{df}{du}\frac{\partial u}{\partial x}+y\frac{df}{\partial u}\frac{\partial u}{\partial y} = -\frac{1}{12}f$$

$$\therefore\ x(\cos u)\frac{\partial u}{\partial x}+y(\cos u)\frac{\partial u}{\partial y} = -\frac{1}{12}\sin u \qquad (\because f=\sin u)$$

or
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u \qquad \text{...(1)}$$

Differentiating both sides of (1) partially with respect to x, we get

$$\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y} = -\frac{1}{12}\sec^2 u\frac{\partial u}{\partial x} \qquad \text{...(2)}$$

Again, differentiating both sides of (1) partially with respect to y, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2} = -\frac{1}{12}\sec^2 u\frac{\partial u}{\partial y} \qquad \text{...(3)}$$

Multiplying (2) by x and (3) by y and adding, assuming $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$, we obtain

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) = \left(-\frac{1}{12}\sec^2 u\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$\therefore\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = \left(-\frac{1}{12}\sec^2 u-1\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$= -\left(\frac{1}{12}\sec^2 u+1\right)\left(-\frac{1}{12}\tan u\right) \qquad \text{[using (1)]}$$

$$= \frac{\tan u}{12}\left\{\frac{1}{12}(1+\tan^2 u)+1\right\} = \frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right).$$

Example 6: If $u = xf\left(\frac{y}{x}\right)+g\left(\frac{y}{x}\right)$, then show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = xf\left(\frac{y}{x}\right)$$

and $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2} = 0.$ *(W.B.U.T. 2004, 2007, 2011, BESUS 2013)*

Solution: Let

$$\left.\begin{aligned} u &= v + w, \\ \text{where} \quad v &= x f\left(\frac{y}{x}\right) \\ \text{and} \quad w &= g\left(\frac{y}{x}\right) = x^{o} g\left(\frac{y}{x}\right) \end{aligned}\right\} \quad \ldots(1)$$

Therefore, v and w are homogeneous functions in x, y of degree 1 and 0 respectively.

Using Euler's theorem, we get

$$\left.\begin{aligned} x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} &= 1.v = v \\ \text{and} \quad x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} &= 0.w = 0 \end{aligned}\right\} \quad \ldots(2)$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial}{\partial x}(v+w) + y\frac{\partial}{\partial y}(v+w)$$

$$= \left\{x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right\} + \left\{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}\right\}$$

$$= v + 0 \qquad \text{[by (2)]}$$

$$= x f\left(\frac{y}{x}\right) \qquad \text{[by (1)]}$$

i.e.,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x f\left(\frac{y}{x}\right) \quad \ldots(3)$$

Differentiating both sides of (3) partially with respect to x, we get

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = f\left(\frac{y}{x}\right) + x f'\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right)$$

$$= f\left(\frac{y}{x}\right) - \frac{y}{x} f'\left(\frac{y}{x}\right) \quad \ldots(4)$$

Again differentiating both sides of (3) partially with respect to y, we get

$$x\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = x f'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right) = f'\left(\frac{y}{x}\right) \quad \ldots(5)$$

Multiplying (4) by x and (5) by y and adding, assuming

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, \text{ we obtain}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = x f\left(\frac{y}{x}\right)$$

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$$\therefore \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0. \qquad \text{[using (3)]}$$

Example 7: If z is a homogeneous function in x, y of degree n having continuous second order partial derivatives and $z = f(u)$, then prove that

$$(i) \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)},$$

$$(ii) \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\{g'(u) - 1\},$$

where $g(u) = n \dfrac{f(u)}{f'(u)}$.

Solution: (*i*) Since z is homogeneous in x, y of degree n, we have, by Euler's theorem

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \qquad \ldots(1)$$

Now, $z = f(u)$, therefore,

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \frac{\partial u}{\partial x} = f'(u) \frac{\partial u}{\partial x}$$

and

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \frac{\partial u}{\partial y} = f'(u) \frac{\partial u}{\partial y}$$

Putting these in (1), we obtain

$$x f'(u) \frac{\partial u}{\partial x} + y f'(u) \frac{\partial u}{\partial y} = n f(u)$$

or

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)} \qquad \ldots(2)$$

(*ii*) From (2), we have

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = g(u), \qquad \ldots(3)$$

where $g(u) = n \dfrac{f(u)}{f'(u)}$.

Differentiating both sides of (3) partially with respect to x, we get

$$\frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = g'(u) \frac{\partial u}{\partial x} \qquad \ldots(4)$$

Again differentiating both sides of (3) partially with respect to y, we get

$$x \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = g'(u) \frac{\partial u}{\partial y} \qquad \ldots(5)$$

Multiplying (4) by x and (5) by y and adding, we obtain

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)g'(u)$$

$$\left[\because \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}\text{, as second order partial derivatives are continuous}\right]$$

$$\therefore\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = g(u)\{g'(u)-1\}. \qquad \text{[using (3)]}$$

Example 8. If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then show that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u$. [W.B.U.T. 2012]

Solution: Given,

$$u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$$

$$\Rightarrow \quad \tan u=\frac{x^3+y^3}{x-y}$$

$$= f(x, y), \text{ say.}$$

Now $f(tx,ty)=t^2 f(x,y)$ and so $f(x, y)$ is homogeneous in x, y of degree 2.

Applying Euler's theorem, we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = 2f,$$

or
$$x\frac{df}{du}\frac{\partial u}{\partial x}+y\frac{df}{du}\frac{\partial u}{\partial y} = 2f$$

$$\therefore \quad x(\sec^2 u)\frac{\partial u}{\partial x}+y(\sec^2 u)\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u \qquad ...(1)$$

Differentiating both sides of (1) partially with respect to x, we get

$$\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y} = (2\cos 2u)\frac{\partial u}{\partial x} \qquad ...(2)$$

Again, differentiating both sides of (1) partially with respect to y, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2} = (2\cos 2u)\frac{\partial u}{\partial y} \qquad ...(3)$$

Multiplying (2) by x and (3) by y and adding, assuming $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, we obtain

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right) = (2\cos 2u)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right)$$

Using (1), we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (2\cos 2u - 1)\sin 2u$$
$$= (1 - 4\sin^2 u)\sin 2u$$

8.7 CHANGE OF VARIABLES : CHAIN RULES

Total Derivative

(*i*) Let $f = f(u, v)$, where $u = u(t)$, $v = v(t)$ and f, u, v are differentiable functions of their arguments, then f is a function of t only and the total derivative of f with respect to t is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial t}$$

(*ii*) Let $f = f(u,v,w)$, where $u = u\ (t)$, $v = v\ (t)$, $w = w\ (t)$ and f, u, v, w are differentiable functions of their arguments, then f is a function of t only and the total derivative of f with respect to t is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial u} \cdot \frac{du}{dt} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dt} + \frac{\partial f}{\partial w} \cdot \frac{dw}{dt}.$$

Chain Rules

(*i*) Let $f = f(u, v)$, where $u = u(x, y)$, $v = v(x, y)$ and f, u, v are differentiable functions of their arguments, then f is a function of x, y and

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

(*i*) Let $f = f(u, v, w)$, w[illegible] $= v\ (x, y, z)$, $w = w\ (x, y, z)$ and f, u, v, w are differentiable f[illegible] f is a function of x, y, z and

Example 1: If $u = f(x-y, y-z, z-x)$, prove that $u_x + u_y + u_z = 0$.

Solution: Let $h_1 = x - y$, $h_2 = y - z$, $h_3 = z - x$, then $u = f(h_1, h_2, h_3)$.

Applying chain rules, we get

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial h_1}\cdot\frac{\partial h_1}{\partial x} + \frac{\partial u}{\partial h_2}\cdot\frac{\partial h_2}{\partial x} + \frac{\partial u}{\partial h_3}\cdot\frac{\partial h_3}{\partial x} = \frac{\partial u}{\partial h_1}\cdot 1 + \frac{\partial u}{\partial h_2}\cdot 0 + \frac{\partial u}{\partial h_3}\cdot(-1)$$

$$= \frac{\partial u}{\partial h_1} - \frac{\partial u}{\partial h_3}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial h_1}\cdot\frac{\partial h_1}{\partial y} + \frac{\partial u}{\partial h_2}\cdot\frac{\partial h_2}{\partial y} + \frac{\partial u}{\partial h_3}\cdot\frac{\partial h_3}{\partial y} = \frac{\partial u}{\partial h_1}\cdot(-1) + \frac{\partial u}{\partial h_2}\cdot 1 + \frac{\partial u}{\partial h_3}\cdot 0$$

$$= -\frac{\partial u}{\partial h_1} + \frac{\partial u}{\partial h_2}$$

$$u_z = \frac{\partial u}{\partial z} = \frac{\partial u}{\partial h_1}\cdot\frac{\partial h_1}{\partial z} + \frac{\partial u}{\partial h_2}\cdot\frac{\partial h_2}{\partial z} + \frac{\partial u}{\partial h_3}\cdot\frac{\partial h_3}{\partial z} = \frac{\partial u}{\partial h_1}\cdot 0 + \frac{\partial u}{\partial h_2}\cdot(-1) + \frac{\partial u}{\partial h_3}\cdot 1$$

$$= -\frac{\partial u}{\partial h_2} + \frac{\partial u}{\partial h_3}$$

$$\therefore\quad u_x + u_y + u_z = \left(\frac{\partial u}{\partial h_1} - \frac{\partial u}{\partial h_3}\right) + \left(-\frac{\partial u}{\partial h_1} + \frac{\partial u}{\partial h_2}\right) + \left(-\frac{\partial u}{\partial h_2} + \frac{\partial u}{\partial h_3}\right) = 0.$$

Example 2: If $z = f(x, y)$ where $x = e^u \cos v$ and $y = e^u \sin v$ then show that

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}.$$

(W.B.U.T. 2006, 2009, 2013)

Solution: Here $z = f(x, y)$, $x = e^u \cos v$, $y = e^u \sin v$.

Applying chain rules, we get

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \left(\frac{\partial z}{\partial x}\right)e^u \cos v + \left(\frac{\partial z}{\partial y}\right)e^u \sin v$$

nd

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \left(\frac{\partial z}{\partial x}\right)(-e^u \sin v) + \left(\frac{\partial z}{\partial y}\right)e^u \cos v$$

$$\therefore\quad y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = (e^u \sin v)\left\{\left(\frac{\partial z}{\partial x}\right) \text{[illegible]} \left(\frac{\partial z}{\partial y}\right)e^u \sin v\right\}$$

$$+(e^u \cos v)\left\{\left(\frac{\partial z}{\partial x}\right)(-e^u \sin v)+\left(\frac{\partial z}{\partial y}\right)e^u \cos v\right\}$$

$$= e^{2u}\left(\frac{\partial z}{\partial x}\right)(\sin v\cos v - \sin v\cos v) + e^{2u}\left(\frac{\partial z}{\partial y}\right)(\sin^2 v + \cos^2 v)$$

$$= e^{2u}\frac{\partial z}{\partial y}.$$

Example 3: If $z = \sin uv$ where $u = 3x^2$ and $v = \log x$, find $\frac{dz}{dx}$. (W.B.U.T. 2004)

Solution: Here $z = \sin uv$, $u = 3x^2$, $v = \log x$

By using chain rule,

$$\frac{dz}{dx} = \frac{\partial z}{\partial u}\cdot\frac{du}{dx} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dx}$$

$$\therefore \quad \frac{dz}{dx} = (v\cos uv)6x + (u\cos uv)\cdot\frac{1}{x}.$$

Example 4: If $f(v^2-x^2, v^2-y^2, v^2-z^2) = 0$, where v is a function of x, y, z, show that

$$\frac{1}{x}\frac{\partial v}{\partial x} + \frac{1}{y}\frac{\partial v}{\partial y} + \frac{1}{z}\frac{\partial v}{\partial z} = \frac{1}{v}. \quad \text{(W.B.U.T. 2005)}$$

Solution: Let $r = v^2 - x^2$, $s = v^2 - y^2$, $t = v^2 - z^2$, then $f(r, s, t) = 0$. Differentiating both sides of $f(r, s, t) = 0$ *w.r.t.* x, we get

$$\frac{\partial f}{\partial x} = 0. \text{ Using chain rules,}$$

$$\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x} = 0$$

or
$$\frac{\partial f}{\partial r}\left(2v\frac{\partial v}{\partial x} - 2x\right) + \frac{\partial f}{\partial s}\left(2v\frac{\partial v}{\partial x} - 0\right) + \frac{\partial f}{\partial t}\left(2v\frac{\partial v}{\partial x} - 0\right) = 0$$

or
$$v\frac{\partial f}{\partial r}\frac{\partial v}{\partial x} - x\frac{\partial f}{\partial r} + v\frac{\partial f}{\partial s}\cdot\frac{\partial v}{\partial x} + v\frac{\partial f}{\partial t}\cdot\frac{\partial v}{\partial x} = 0$$

or
$$v\frac{\partial v}{\partial x}\left(\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\right) = x\frac{\partial f}{\partial r}$$

$$\frac{v}{x}\frac{\partial v}{\partial x} = \frac{\frac{\partial f}{\partial r}}{\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}} \quad \text{...(1)}$$

Similarly, since f is symmetric in x, y, z, we get

$$\frac{v}{y}\frac{\partial v}{\partial y} = \frac{\dfrac{\partial f}{\partial s}}{\dfrac{\partial f}{\partial r}+\dfrac{\partial f}{\partial s}+\dfrac{\partial f}{\partial t}} \qquad \ldots(2)$$

$$\frac{v}{z}\frac{\partial v}{\partial z} = \frac{\dfrac{\partial f}{\partial t}}{\dfrac{\partial f}{\partial r}+\dfrac{\partial f}{\partial s}+\dfrac{\partial f}{\partial t}} \qquad \ldots(3)$$

Adding (1), (2) and (3), we obtain

$$\frac{v}{x}\frac{\partial v}{\partial x}+\frac{v}{y}\frac{\partial v}{\partial y}+\frac{v}{z}\frac{\partial v}{\partial z} = 1, \quad \text{or} \quad \frac{1}{x}\frac{\partial v}{\partial x}+\frac{1}{y}\frac{\partial v}{\partial y}+\frac{1}{z}\frac{\partial v}{\partial z} = \frac{1}{v}.$$

Example 5: If z is a function of x and y and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$;

prove that
$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$$

Solution: Here $z = z(x, y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, therefore using chain rules, we get

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cdot e^u+\frac{\partial z}{\partial y}(-e^{-u}),$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}(-e^v)$$

$$\therefore \quad \frac{\partial z}{\partial u}-\frac{\partial z}{\partial v} = (e^u+e^{-v})\frac{\partial z}{\partial x}-(e^{-u}-e^v)\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$$

Example 6: If $u = f(x^2+2yz, y^2+2zx)$, show that

$$(y^2-zx)\frac{\partial u}{\partial x}+(x^2-yz)\frac{\partial u}{\partial y}+(z^2-xy)\frac{\partial u}{\partial z} = 0. \qquad \textit{(W.B.U.T. 2001, 2012)}$$

Solution: Let $r = x^2+2yz$, $s = y^2+2zx$, therefore, $u = f(r, s)$.

Using chain rules, we get

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} = 2x\frac{\partial u}{\partial r}+2z\frac{\partial u}{\partial s},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} = 2z\frac{\partial u}{\partial r}+2y\frac{\partial u}{\partial s}$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} = 2y\frac{\partial u}{\partial r}+2x\frac{\partial u}{\partial s}$$

$$\therefore \quad (y^2-zx)\frac{\partial u}{\partial x}+(x^2-yz)\frac{\partial u}{\partial y}+(z^2-xy)\frac{\partial u}{\partial z}$$

$$= (y^2-zx)\left\{2x\frac{\partial u}{\partial r}+2z\frac{\partial u}{\partial s}\right\}+(x^2-yz)\left\{2z\frac{\partial u}{\partial r}+2y\frac{\partial u}{\partial s}\right\}$$

$$+(z^2-xy)\left\{2y\frac{\partial u}{\partial r}+2x\frac{\partial u}{\partial s}\right\}$$

$$= \{2x(y^2-zx)+2z(x^2-yz)+2y(z^2-xy)\}\frac{\partial u}{\partial r}$$

$$+\{2z(y^2-zx)+2y(x^2-yz)+2x(z^2-xy)\}\frac{\partial u}{\partial s}$$

$$= 0\cdot\frac{\partial u}{\partial r}+0\cdot\frac{\partial u}{\partial s} = 0.$$

Example 7: If $u = f(r, s, t)$, where $r = \dfrac{x}{y}$, $s = \dfrac{y}{z}$, $t = \dfrac{z}{x}$,

show that $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = 0.$$

Solution: Here $u = f(r, s, t)$,

$$r = \frac{x}{y},\ s = \frac{y}{z},\ t = \frac{z}{x}.$$

Therefore, applying chain rules, we get

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{1}{y}+\frac{\partial u}{\partial s}\cdot 0+\frac{\partial u}{\partial t}\cdot\left(-\frac{z}{x^2}\right)$$

$$\therefore \quad x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r}-\frac{z}{x}\frac{\partial u}{\partial t} \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot\left(-\frac{x}{y^2}\right)+\frac{\partial u}{\partial s}\cdot\frac{1}{z}+\frac{\partial u}{\partial t}\cdot 0$$

$$\therefore \quad y\frac{\partial u}{\partial y} = -\frac{x}{y}\frac{\partial u}{\partial r}+\frac{y}{z}\frac{\partial u}{\partial s} \qquad ...(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}\cdot 0+\frac{\partial u}{\partial s}\cdot\left(-\frac{y}{z^2}\right)+\frac{\partial u}{\partial t}\cdot\frac{1}{x}$$

$$\therefore \quad z\frac{\partial u}{\partial z} = -\frac{y}{z}\frac{\partial u}{\partial s}+\frac{z}{x}\frac{\partial u}{\partial t} \qquad ...(3)$$

Adding (1), (2) and (3), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = \left(\frac{x}{y}\frac{\partial u}{\partial r}-\frac{z}{x}\frac{\partial u}{\partial t}\right)+\left(-\frac{x}{y}\frac{\partial u}{\partial r}+\frac{y}{z}\frac{\partial u}{\partial s}\right)+\left(-\frac{y}{z}\frac{\partial u}{\partial s}+\frac{z}{x}\frac{\partial u}{\partial t}\right) = 0.$$

Note: The above result can also be verified easily by applying Euler's theorem.

Example 8: If $u = u\left(\dfrac{y-x}{xy}, \dfrac{z-x}{zx}\right)$

prove that $x^2\dfrac{\partial u}{\partial x}+y^2\dfrac{\partial u}{\partial y}+z^2\dfrac{\partial u}{\partial z} = 0.$

Solution: Here $u = u\,(r, s)$,

where
$$r = \frac{y-x}{xy} = \frac{1}{x}-\frac{1}{y},$$

$$s = \frac{z-x}{zx} = \frac{1}{x}-\frac{1}{z}$$

By chain rules,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} = \frac{\partial u}{\partial r}\cdot\left(-\frac{1}{x^2}\right)+\frac{\partial u}{\partial s}\cdot\left(-\frac{1}{x^2}\right)$$

$$\therefore \quad x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial r}-\frac{\partial u}{\partial s} \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{1}{y^2}+\frac{\partial u}{\partial s}\cdot 0$$

$$\therefore \quad y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \qquad ...(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} = \frac{\partial u}{\partial r}\cdot 0+\frac{\partial u}{\partial s}\cdot\frac{1}{z^2}$$

$$\therefore \qquad z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s} \qquad \text{...(3)}$$

Adding (1), (2) and (3), we get

$$x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial r}-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s} = 0.$$

Example 9: If $\varphi(cx-az, cy-bz)=0$, show that

$$ap+bq = c,$$

where $$p = \frac{\partial z}{\partial x}$$

and $$q = \frac{\partial z}{\partial y}.$$

Solution: Here $\varphi(r, s) = 0$, where $r = cx-az, s = cy-bz$.

Now, $$\frac{\partial \varphi}{\partial x} = 0,$$

therefore, using chain rules,

$$\frac{\partial \varphi}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial \varphi}{\partial s}\cdot\frac{\partial s}{\partial x} = 0, \text{ or } \frac{\partial \varphi}{\partial r}\left(c-a\frac{\partial z}{\partial x}\right)+\frac{\partial \varphi}{\partial s}\cdot\left(-b\frac{\partial z}{\partial x}\right)=0$$

or $$c\frac{\partial \varphi}{\partial r}-\left(a\frac{\partial \varphi}{\partial r}+b\frac{\partial \varphi}{\partial s}\right)\frac{\partial z}{\partial x} = 0$$

$$\therefore \qquad a\frac{\partial z}{\partial x} = \frac{ac\frac{\partial \varphi}{\partial r}}{a\frac{\partial \varphi}{\partial r}+b\frac{\partial \varphi}{\partial s}} \qquad \text{...(1)}$$

Also $\frac{\partial \varphi}{\partial y} = 0$, therefore, using chain rules,

$$\frac{\partial \varphi}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial \varphi}{\partial s}\cdot\frac{\partial s}{\partial y} = 0, \text{ or } \frac{\partial \varphi}{\partial r}\cdot\left(-a\frac{\partial z}{\partial y}\right)+\frac{\partial \varphi}{\partial s}\left(c-b\frac{\partial z}{\partial y}\right)=0$$

or $$c\frac{\partial \varphi}{\partial s}-\left(a\frac{\partial \varphi}{\partial r}+b\frac{\partial \varphi}{\partial s}\right)\frac{\partial z}{\partial y} = 0$$

$$\therefore \qquad b\frac{\partial z}{\partial y} = \frac{bc\frac{\partial \varphi}{\partial s}}{a\frac{\partial \varphi}{\partial r}+b\frac{\partial \varphi}{\partial s}} \qquad \text{...(2)}$$

Adding (1) and (2), we obtain

$$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y} = \frac{ac\dfrac{\partial \varphi}{\partial r}+bc\dfrac{\partial \varphi}{\partial s}}{a\dfrac{\partial \varphi}{\partial r}+b\dfrac{\partial \varphi}{\partial s}} = c$$

$\therefore\ ap+bq=c$, where $p=\dfrac{\partial z}{\partial x}$ and $q=\dfrac{\partial z}{\partial y}$.

Example 10: If $z=z(u,v)$, where $u=lx+my$ and $v=ly-mx$,

prove that $$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = (l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right).$$

Solution: Here $z=z(u,v)$, $u=lx+my$ and $v=ly-mx$, therefore by using chain rules, we get

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\cdot l+\frac{\partial z}{\partial v}\cdot(-m) = l\frac{\partial z}{\partial u}-m\frac{\partial z}{\partial v}$$

$$\therefore\quad \frac{\partial}{\partial x} \equiv l\frac{\partial}{\partial u}-m\frac{\partial}{\partial v} \quad \text{...(1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\cdot m+\frac{\partial z}{\partial v}\cdot l = m\frac{\partial z}{\partial u}+l\frac{\partial z}{\partial v}$$

$$\therefore\quad \frac{\partial}{\partial y} \equiv m\frac{\partial}{\partial u}+l\frac{\partial}{\partial v} \quad \text{...(2)}$$

$$\therefore\quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(l\frac{\partial}{\partial u}-m\frac{\partial}{\partial v}\right)\left(l\frac{\partial z}{\partial u}-m\frac{\partial z}{\partial v}\right) \quad \text{[using (1)]}$$

$$= l^2\frac{\partial^2 z}{\partial u^2}-2lm\frac{\partial^2 z}{\partial u\partial v}+m^2\frac{\partial^2 z}{\partial v^2} \quad \text{...(3)}$$

$$\left(\text{assuming } \frac{\partial^2 z}{\partial u\partial v}=\frac{\partial^2 z}{\partial v\partial u}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(m\frac{\partial}{\partial u}+l\frac{\partial}{\partial v}\right)\left(m\frac{\partial z}{\partial u}+l\frac{\partial z}{\partial v}\right) \quad \text{[using (2)]}$$

$$= m^2\frac{\partial^2 z}{\partial u^2}+2lm\frac{\partial^2 z}{\partial u\partial v}+l^2\frac{\partial^2 z}{\partial v^2} \quad \text{...(4)}$$

$$\left(\text{assuming } \frac{\partial^2 z}{\partial u\partial v}=\frac{\partial^2 z}{\partial v\partial u}\right)$$

Adding (3) and (4), we get

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=(l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right).$$

Example 11: If $f=f(u, v)$, where $u=e^x\cos y$ and $v=e^x\sin y$, show that

(*i*) $\frac{\partial f}{\partial x}=u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}$ (*ii*) $\frac{\partial f}{\partial y}=-v\frac{\partial f}{\partial u}+u\frac{\partial f}{\partial v}$

and (*iii*) $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=(u^2+v^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right)$.

Solution: Here $f=f(u, v)$, $u=e^x\cos y$ and $v=e^x\sin y$, therefore by using chain rules we get the following:

(*i*) $$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}$$

$$\therefore \quad \frac{\partial}{\partial x}\equiv u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v} \qquad \text{...(1)}$$

(*ii*) $$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=-v\frac{\partial f}{\partial u}+u\frac{\partial f}{\partial v}$$

$$\therefore \quad \frac{\partial}{\partial y}\equiv -v\frac{\partial}{\partial u}+u\frac{\partial}{\partial v} \qquad \text{...(2)}$$

Now, $$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\left(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}\right) \qquad \text{[by (1)]}$$

$$=u\frac{\partial}{\partial u}\left(u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}\right)+v\frac{\partial}{\partial v}\left(u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}\right)$$

$$=u\frac{\partial f}{\partial u}+u^2\frac{\partial^2 f}{\partial u^2}+uv\frac{\partial^2 f}{\partial u\partial v}+uv\frac{\partial^2 f}{\partial v\partial u}+v\frac{\partial f}{\partial v}+v^2\frac{\partial^2 f}{\ldots}$$

$$=\left(u^2\frac{\partial^2 f}{\partial u^2}+uv\frac{\partial^2}{\ldots}\right. \ldots \qquad \text{...(3)}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\ldots} \qquad \text{[by (2)]}$$

$$= v^2\frac{\partial^2 f}{\partial u^2} - v\frac{\partial f}{\partial v} - uv\frac{\partial^2 f}{\partial u \partial v} - u\frac{\partial f}{\partial u} - uv\frac{\partial^2 f}{\partial v \partial u} + u^2\frac{\partial^2 f}{\partial v^2}$$

$$= \left(v^2\frac{\partial^2 f}{\partial u^2} - uv\frac{\partial^2 f}{\partial u \partial v} - uv\frac{\partial^2 f}{\partial v \partial u} + u^2\frac{\partial^2 f}{\partial v^2}\right) - \left(u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}\right) \qquad ...(4)$$

Adding (3) and (4), we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right).$$

Note: $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = (u^2+v^2)\left\{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right\}$ [*BESUS (B. Arch) 2013*]

Example 12: A twice differentiable function $f(x, y)$ when expressed in terms of the new variables

$$x = \frac{1}{2}(u+v),\ y = \sqrt{uv} \text{ becomes } g(u, v)$$

prove that $$\frac{\partial^2 g}{\partial u \partial v} = \frac{1}{4}\left[\frac{\partial^2 f}{\partial x^2} + 2\frac{x}{y}\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} + \frac{1}{y}\frac{\partial f}{\partial y}\right].$$

Solution: Here $g(u, v) = f(x, y)$, where

$$x = \frac{1}{2}(u+v),\ y = \sqrt{uv}$$

$$\therefore \frac{\partial x}{\partial u} = \frac{\partial x}{\partial v} = \frac{1}{2},\ \frac{\partial y}{\partial u} = \frac{1}{2}\sqrt{\frac{v}{u}},\ \frac{\partial y}{\partial v} = \frac{1}{2}\sqrt{\frac{u}{v}} \qquad ...(1)$$

By chain rules we get,

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2}\sqrt{\frac{v}{u}}\frac{\partial f}{\partial y} \qquad \text{[by (1)]}$$

$$\therefore \frac{\partial}{\partial u} \equiv \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2}\sqrt{\frac{v}{u}}\frac{\partial}{\partial y} \qquad [\because g(u,v) = f(x,y)] \qquad ...(2)$$

Also, $$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2}\sqrt{\frac{u}{v}}\frac{\partial f}{\partial y} \qquad \text{[by (1)]}$$

$$\therefore \frac{\partial^2 g}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial v}\right) = \frac{\partial}{\partial u}\left[\frac{1}{2}\frac{\partial f}{\partial x} \ldots \frac{\partial f}{\ldots}\right]$$

$$= \frac{1}{2}\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) + \frac{1}{4\sqrt{uv}}\frac{\partial f}{\partial y} \ldots$$

$$= \frac{1}{2}\left(\frac{1}{2}\frac{\partial}{\partial x}+\frac{1}{2}\sqrt{\frac{v}{u}}\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x}\right)+\frac{1}{4\sqrt{uv}}\frac{\partial f}{\partial y}+\frac{1}{2}\sqrt{\frac{u}{v}}\left(\frac{1}{2}\frac{\partial}{\partial x}+\frac{1}{2}\sqrt{\frac{v}{u}}\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial y}\right) \quad \text{[by (2)]}$$

$$= \frac{1}{4}\frac{\partial^2 f}{\partial x^2}+\frac{1}{4}\sqrt{\frac{v}{u}}\frac{\partial^2 f}{\partial y\partial x}+\frac{1}{4\sqrt{uv}}\frac{\partial f}{\partial y}+\frac{1}{4}\sqrt{\frac{u}{v}}\frac{\partial^2 f}{\partial x\partial y}+\frac{1}{4}\frac{\partial^2 f}{\partial y^2}$$

$$= \frac{1}{4}\left[\frac{\partial^2 f}{\partial x^2}+\left(\sqrt{\frac{v}{u}}+\sqrt{\frac{u}{v}}\right)\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y^2}+\frac{1}{\sqrt{uv}}\frac{\partial f}{\partial y}\right] \quad \left(\text{assuming } \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}\right)$$

$$= \frac{1}{4}\left[\frac{\partial^2 f}{\partial x^2}+2\frac{x}{y}\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y^2}+\frac{1}{y}\frac{\partial f}{\partial y}\right] \quad \left[\because \sqrt{\frac{v}{u}}+\sqrt{\frac{u}{v}}=\frac{v+u}{\sqrt{uv}}=2\frac{x}{y}\right]$$

Example 13: Show that the transformation $u = x - ct$, $v = x + ct$ reduces the equation $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$ to the equation $\frac{\partial^2 z}{\partial u\partial v} = 0$.

Solution: Here $u = x - ct$, $v = x + ct$

$$\therefore \quad \frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial t} = -c, \quad \frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial t} = c \qquad \text{...(1)}$$

Using chain rules we get,

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial t}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial t} = -c\frac{\partial z}{\partial u}+c\frac{\partial z}{\partial v}$$

$$= -c\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial v}\right)z \qquad \text{[by (1)]}$$

$$\therefore \quad \frac{\partial}{\partial t} \equiv -c\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial v}\right) \qquad \text{...(2)}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$$

$$= \left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)z \qquad \text{[by (1)]}$$

$$\therefore \quad \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u}+\frac{\partial}{\partial v} \qquad \text{...(3)}$$

$$\therefore \quad \frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right) = c^2\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right) \qquad \text{[by (2)]}$$

$$= c^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right) \qquad \text{...(4)}$$

$$\left(\text{assuming } \frac{\partial^2 z}{\partial u\partial v}=\frac{\partial^2 z}{\partial v\partial u}\right)$$

and $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2} \qquad ...(5)$$

Putting these values of $\frac{\partial^2 z}{\partial t^2}, \frac{\partial^2 z}{\partial x^2}$, from (4) and (5), in the equation $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$, we get

$$c^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}\right) = c^2\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}\right)$$

$$\therefore \qquad \frac{\partial^2 z}{\partial u\,\partial v} = 0.$$

Observation: Let $x = r\cos\theta$, $y = r\sin\theta$

Therefore, $$r = x\sec\theta \qquad ...(1)$$

Also, $$r^2 = x^2 + y^2 \qquad ...(2)$$

Differentiating both sides of (1) partially with respect to x, keeping θ as constant, we get

$$\frac{\partial r}{\partial x} = \sec\theta \qquad ...(3)$$

Differentiating both sides of (2) partially with respect to x, keeping y as constant, we get

$$2r\frac{\partial r}{\partial x} = 2x, \text{ or } \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \qquad ...(4)$$

From (3), $\frac{\partial r}{\partial x} = \sec\theta$ and from (4), $\frac{\partial r}{\partial x} = \cos\theta$, which make confusion. To avoid confusion we use the following conventions.

(i) Take y as constant for $\frac{\partial}{\partial x}$ (ii) Take x as constant for $\frac{\partial}{\partial y}$

(iii) Take θ as constant for $\frac{\partial}{\partial r}$ (iv) Take r as constant for $\frac{\partial}{\partial \theta}$

(v) $\left(\frac{\partial r}{\partial x}\right)_\theta$ means the partial derivatives of r with respect to x, keeping θ as constant. Similar meaning for $\left(\frac{\partial r}{\partial x}\right)_y$, *i.e.*, partial derivative of r with respect to x, keeping y as constant.

Example 14: If $u = f(x, y)$, where $x = r\cos\theta$, $y = r\sin\theta$, then prove that

(i) $$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

CALCULUS OF FUNC

(ii) $\frac{\partial^2 u}{\partial x^2}+$

Solution: (i

and

or

Squarin

(ii) F

$\therefore$

(ii) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$ (*W.B.U.T. 2002, 2008*)

Solution: (*i*) Here $x = r\cos\theta$ and $y = r\sin\theta$. By chain rule, we get

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}=\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y} \quad ...(1)$$

and

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}=-r\sin\theta\frac{\partial u}{\partial x}+r\cos\theta\frac{\partial u}{\partial y} \quad ...(2)$$

or

$$\frac{1}{r}\frac{\partial u}{\partial \theta}=-\sin\theta\frac{\partial u}{\partial x}+\cos\theta\frac{\partial u}{\partial y} \quad ...(3)$$

Squaring and adding (1) and (3), we get

$$\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=(\cos^2\theta+\sin^2\theta)\left(\frac{\partial u}{\partial x}\right)^2+(\sin^2\theta+\cos^2\theta)\left(\frac{\partial u}{\partial y}\right)^2$$

$$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$$

(*ii*) From (1) and (2), we get

$$\frac{\partial}{\partial r}\equiv\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y} \quad ...(4)$$

$$\frac{\partial}{\partial \theta}\equiv-r\sin\theta\frac{\partial}{\partial x}+r\cos\theta\frac{\partial}{\partial y} \quad ...(5)$$

$$\therefore \quad \frac{\partial^2 u}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)=\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}\right) \quad \text{[by (1)]}$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)+\sin\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)$$

$$=\cos\theta\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right)$$

$$+\sin\theta\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right) \quad \text{[by (4)]}$$

$$=\cos^2\theta\frac{\partial^2 u}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}+\sin^2\theta\frac{\partial^2 u}{\partial y^2} \quad ...(6)$$

$$\left(\text{assuming } \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}\right)$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(-r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}\right)$$ [by (2

$$= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right) - r\sin\theta\frac{\partial u}{\partial y} + r\cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right)$$

$$= -r\left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right) - r\sin\theta\left(-r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}\right)\frac{\partial u}{\partial x}$$

$$+ r\cos\theta\left(-r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}\right)\frac{\partial u}{\partial y}$$ [by (5)]

$$= -r\frac{\partial u}{\partial r} + r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2}$$

$$\left(\text{assuming } \frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}\right)$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2} \quad ...(7)$$

Adding (6) and (7), we have

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

CHAPTER 9

Extrema for Functions of Several Variables

9.1 INTRODUCTION

To optimize something means either to maximize or minimize in some aspects of it. An important application of calculus of functions of several variables is to find the maximum and minimum values of functions and where they occur. Determination of extrema, *i.e.*, maxima or minima, is very important in the study of stability of the equilibrium states of mechanical and physical systems. Lagrange's multiplier method developed by mathematician Lagrange in 1755 is a powerful technique for evaluating extreme values of constrained functions in designing multi-stage rockets in engineering, in economics etc. In this chapter we study the extrema of functions of two and three variables.

9.2 MAXIMA AND MINIMA FOR FUNCTIONS OF TWO VARIABLES

Let $z = f(x, y)$ be a function of independent variables x and y.

Relative maximum: A point (a, b) is called a local or relative maximum point (or simply maximum point) of $f(x, y)$ if there exists a region surrounding the point (a, b) in which $f(x, y) < f(a, b)$ for all points (x, y), except (a, b), in this region.

In otherwords, $f(x, y)$ is said to have a local or relative maximum (or simply maximum) at a point (a, b) if there exist $h > 0$, $k > 0$, however small h, k may be, such that $f(a \pm h_1, b \pm k_1) < f(a, b)$, where $0 < h_1 \le h, 0 < k_1 \le k$.

Relative minimum: A point (a, b) is called a local or relative minimum point (or simply minimum point) of $f(x, y)$ if there exists a region surrounding the point (a, b), in which $f(x, y) > f(a, b)$ for all points (x, y), except (a, b) in this region.

In otherwords, $f(x, y)$ has a local or relative minimum (or simply minimum) at a point (a, b) if there exist $h > 0$, $k > 0$, however small h, k may be, such that

$$f(a \pm h_1, b \pm k_1) > f(a, b), \text{ where } 0 < h_1 \le h, 0 < k_1 \le k.$$

Note: The value of a function f at an extremum (maximum or minimum) point is known as the extremum (maximum or minimum) value of the function f.

Theorem 1: (Necessary Conditions of Extrema):

If a function $z = f(x, y)$ has a maximum or minimum point at (a, b), then

$$f_x(a,b) = f_y(a,b) = 0, \text{ provided these partial derivatives exist.}$$

Notes: (*i*) The converse of the above theorem is not true in general. Let us consider the function $f(x, y) = x^2y^3$ in support of this statement. Here $f_x = 2xy^3$, $f_y = 3x^2y^2$ and so $f_x(0,0) = f_y(0, 0) = 0$. But $f(0, 0)$ is not an extreme value, since there exist no $h > 0$, $k > 0$, such that $f(0 \pm h_1, 0 \pm k_1) - f(0,0) = f(\pm h_1, \pm k_1)$ keeps the same sign for $0 < h_1 \le h, 0 < k_1 \le k$. For example, $f(\varepsilon, \varepsilon) = \varepsilon^5 > 0$, while $f(\varepsilon, -\varepsilon) = -\varepsilon^5 < 0$ for any small $\varepsilon > 0$.

(*ii*) A function $f(x, y)$ may have extreme value at (a, b) though $f_x(a, b)$ and $f_y(a, b)$ do not exist.

For example let us take $f(x,y) = |x|+|y|$ [*BESUS* (*B. Arch.*) *2013*]

$$\therefore \quad \lim_{h\to 0+} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0+} \frac{h-0}{h} \quad (\because h>0)$$

$$= 1 \quad (\because h \neq 0)$$

and
$$\lim_{h\to 0-} \frac{f(0+h)-f(0,0)}{h} = \lim_{h\to 0-} \frac{-h-0}{h} \quad (\because h<0)$$

$$= -1. \quad (\because h \neq 0)$$

Hence $f_x(0, 0)$ does not exist and also, by symmetry, $f_y(0, 0)$ does not exist. But $f(0, 0) = 0$ is a minimum value of $f(x, y)$. Therefore $f(x, y)$ has minimum at $(0, 0)$ though $f_x(0, 0)$, $f_y(0, 0)$ do not exist.

(*iii*) For extrema $f_x = f_y = 0$ and hence $df = f_x dy + f_y dy = 0$. Therefore, $df = 0$ may be considered as the necessary condition for the existence of extrema of a function $f(x, y)$.

Theorem 2: (Conditions of Extrema):

Let $z = f(x, y)$ be a continuous function having second order partial derivatives and (a, b) be a point satisfying the equations $f_x = f_y = 0$, *i.e.*, $f_x(a, b) = f_y(a, b) = 0$. If H is defined by $H(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - \{f_{xy}(x, y)\}^2$, then

(*i*) $f(a, b)$ is a maximum value of $f(x, y)$ at (a, b) if $H(a, b) > 0$ and $f_{xx}(a, b) < 0$ (or $f_{yy}(a, b) < 0$),

(*ii*) $f(a, b)$ is a minimum value of $f(x, y)$ at (a, b) if $H(a, b) > 0$ and $f_{xx}(a, b) > 0$ (or $f_{yy}(a, b) > 0$),

(*iii*) $f(a, b)$ is neither a maximum nor a minimum value of $f(x, y)$ at (a, b) if $H(a, b) < 0$ and

(*iv*) the case is doubtful and need further investigation if $H(a, b) = 0$.

Saddle Point:

A point (a, b) is said to be a saddle point of a function $f(x, y)$ if it has neither a maximum nor a minimum at (a, b) though $f_x(a, b) = f_y(a, b) = 0$.

Note: Clearly a point (a, b) will be a saddle point of $f(x, y)$ if condition (*iii*) of Theorem 2, *i.e.*, $f_{xx}(a, b) f_{yy}(a, b) - \{f_{xy}(a, b)\}^2 < 0$ is satisfied.

Critical (or Stationary) Point

A point (a, b) is said to be a critical (or stationary) point of a function $f(x, y)$ if $f_x(a, b) = f_y(a, b) = 0$.

Working Rules

Given a function $f(x, y)$, we follow the following steps to find its extrema (*i.e.*, maxima and minima).

Step 1: Solve: $f_x = 0$, $f_y = 0$, to find critical (or stationary) points. Let (a, b) be a critical point.

Step 2: Calculate $H(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \{f_{xy}(a,b)\}^2$.

Step 3: (A) If $H(a, b) > 0$, then $f(x, y)$ has

(*i*) maximum at (a, b) if $f_{xx}(a, b) < 0$ or $f_{yy}(a, b) < 0$,

(*ii*) minimum at (a, b) if $f_{xx}(a, b) > 0$ or $f_{yy}(a, b) > 0$.

(B) If $H(a, b) < 0$, then $f(x, y)$ has neither a maximum nor a minimum at (a, b). The point (a, b) is called saddle point of $f(x, y)$.

(C) If $H(a, b) = 0$, no information is obtained about maximum or minimum of $f(x, y)$ at (a, b) and it needs further investigation.

ILLUSTRATIVE EXAMPLES

Example 1: Find the maxima and minima of the function $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

[*BESUS* (*B. Arch.*) 2013]

Solution: Let $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$, so,

$$f_x = \frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72 \text{ and } f_y = \frac{\partial f}{\partial y} = 6xy - 30y$$

The critical points are given by

$$f_x = 3x^2 + 3y^2 - 30x + 72 = 0 \quad \text{...(1)}$$

$$f_y = 6y(x-5) = 0 \quad \text{...(2)}$$

From (2), $y = 0$ or $x = 5$

Putting $y = 0$ in (1), we get $3x^2 - 30x + 72 = 0$

or $x^2 - 10x + 24 = 0$

or $x^2 - 6x - 4x + 24 = 0$

or $(x - 6)(x - 4)$. Therefore $x = 6, 4$.

Putting $x = 5$ in (1), we get $75 + 3y^2 - 150 + 72 = 0$

or $y^2 = 1$, therefore $y = \pm 1$.

So, the critical points are (6, 0), (4, 0), (5, 1), (5, –1).

Here $\qquad f_{xx} = 6x-30,\ f_{yy} = 6x-30,\ f_{xy} = 6y$

So, $\qquad H(x,y) = f_{xx}f_{yy} - \{f_{xy}\}^2 = 36(x-5)^2 - 36y^2$

$$= 36\{(x-5)^2 - y^2\}.$$

(*i*) At the critical point (6, 0), we have $H(6, 0) = 36 > 0$, $f_{xx}(6, 0) = 6 > 0$. Therefore, (6, 0) is a minimum point of the given function and the minimum value is

$$f(6, 0) = 6^3 - 0 - 15 \times 6^2 - 0 + 72 \times 6 = 108.$$

(*ii*) At (4, 0), we have $H(4, 0) = 36 > 0$, $f_{xx}(4, 0) = -6 < 0$. So a maximum value of the given function occurs at (4, 0) and the maximum value is $f(4, 0) = 4^3 + 0 - 15 \times 4^2 - 0 + 72 \times 4 = 112$.

(*iii*) At (5, 1), we have $H(5, 1) = -36 < 0$, so (5, 1) is a saddle point, *i.e.*, the function has neither maximum nor minimum at (5, 1).

(*iv*) At (5, –1), we have $H(5, -1) = -36 < 0$, so the function has neither maximum nor minimum at (5, –1), *i.e.*, it is a saddle point.

Example 2: Find the maximum and minimum values of the function

$$f(x,y) = x^3 + y^3 - 3axy.$$

(*W.B.U.T. 2002, 2008*)

Solution: Here $f(x,y) = x^3 + y^3 - 3axy$, so,

$$f_x = \frac{\partial f}{\partial x} = 3x^2 - 3ay = 3(x^2 - ay),\ f_y = \frac{\partial f}{\partial y} = 3y^2 - 3ax = 3(y^2 - ax).$$

The critical points are given by

$$f_x = 0,\quad i.e.,\quad x^2 - ay = 0 \qquad \ldots(1)$$

$$f_y = 0,\quad i.e.,\quad y^2 - ax = 0 \qquad \ldots(2)$$

Subtracting (2) from (1), we get

$$x^2 - y^2 + a(x-y) = 0, \text{ or } (x-y)(x+y+a) = 0$$

Therefore $\qquad x = y$, or $x + y + a = 0$

Putting $y = x$ in (1), we get $x = 0, a$ and hence in this case solutions are (0, 0), (*a*, *a*).

Putting $y = -x - a$ in (1), we get $x^2 + ax + a^2 = 0$, which has no real solution. Thus (0, 0), (*a*, *a*) are the only critical points of $f(x, y)$.

Also, $\qquad f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = -3a.$

So, $\qquad H(x, y) = f_{xx}f_{yy} - \{f_{xy}\}^2 = 36xy - 9a^2$

(*i*) At (0, 0), we have $H(0, 0) = -9a^2 < 0$, so $f(x, y)$ has neither maximum nor minimum at (0, 0), *i.e.*, it is a saddle point.

(*ii*) At (*a*, *a*), we have $H(a, a) = 36a^2 - 9a^2 = 27a^2 > 0$ and $f_{xx}(a,a) = 6a \gtrless 0$ according as $a \gtrless 0$. Therefore $f(x, y)$ has a maximum when $a < 0$ and the maximum value is $f(a, a) = a^3 + a^3 - 3a^3 = -a^3$.

The function $f(x, y)$ has a minimum when $a > 0$ and the minimum value is $f(a, a) = -a^3$.

Example 3: *In a plane triangle, find the maximum value of* $\cos A \cos B \cos C$.

Solution: Here $A + B + C = \pi$, so, $C = \pi - (A + B)$.

$\therefore$

$$\cos A \cos B \cos C = \cos A \cos B \cos\{\pi - (A+B)\}$$
$$= -\cos A \cos B \cos(A+B) = f(A, B) \text{ say}$$

Therefore,

$$f_A = \frac{\partial f}{\partial A} = \cos B\{\sin A \cos(A+B) + \cos A \sin(A+B)\}$$
$$= \cos B \sin(2A+B)$$

and, by symmetry,

$$f_B = \frac{\partial f}{\partial B} = \cos A \sin(A+2B).$$

The critical points are given by

$$f_A = \cos B \sin(2A+B) = 0 \qquad ...(1)$$
$$f_B = \cos A \sin(A+2B) = 0 \qquad ...(2)$$

If $\cos B = 0$, then $B = \dfrac{\pi}{2}$ and from (2), $\cos A \sin(A+\pi) = 0$, or $\cos A(-\sin A) = 0$ which gives $\cos A = 0$, *i.e.*, $A = \dfrac{\pi}{2}$, which is not possible, since $A + B + C = \pi$ gives $C = 0$. Otherwise $\sin A = 0$, *i.e.*, $A = 0$ or π which is also not possible. Hence $\cos B \neq 0$ and similarly $\cos A \neq 0$. Thus from (1) and (2), we have

$$\sin(2A+B) = 0, \quad \text{or} \quad 2A + B = \pi$$

and

$$\sin(A+2B) = 0, \quad \text{or} \quad A + 2B = \pi.$$

Solving these equations, we get $A = B = \dfrac{\pi}{3}$. ...(3)

Now,

$$f_{AA} = 2\cos B \cos(2A+B), \quad f_{BB} = 2\cos A \cos(A+2B),$$
$$f_{AB} = -\sin A \sin(A+2B) + \cos A \cos(A+2B) = \cos(2A+2B)$$

So,

$$H(A, B) = f_{AA}\, f_{BB} - \{f_{AB}\}^2 = 4\cos A \cos B \cos(A+2B)\cos(2A+B) - \cos^2(2A+2B).$$

Now,

$$H\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = 4\cos^2\frac{\pi}{3}\cos^2\pi - \cos^2\left(\frac{4\pi}{3}\right)$$
$$= 4\cdot\frac{1}{4}\cdot 1 - \cos^2\left(\pi + \frac{\pi}{3}\right) = 1 - \frac{1}{4} = \frac{3}{4} > 0.$$

Also,

$$f_{AA}\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = 2\cos\frac{\pi}{3}\cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right) = -1 < 0.$$

Therefore, $f(A, B)$ is maximum at $A = B = \dfrac{\pi}{3}$

and the maximum value is
$$f\left(\frac{\pi}{3},\frac{\pi}{3}\right) = -\cos\frac{\pi}{3}\cos\frac{\pi}{3}\cos\frac{2\pi}{3}$$
$$= -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{1}{8}$$

Hence $\cos A \cos B \cos C$ is maximum when each of the angle is $\frac{\pi}{3}$, *i.e.*, the triangle is equilateral, and its maximum value is $\frac{1}{8}$.

Example 4: A rectangular box, open at the top, is to have a given capacity V. Find the dimensions of the box requiring least material for its construction.

Solution: Let x, y and z be the length, breadth and height respectively of the given box.

By equation, $xyz = V$, a constant ...(1)

Therefore, total surface area of the box is
$$S = xy + 2yz + 2zx = xy + \frac{2V}{x} + \frac{2V}{y} \quad \text{[using (1)]}$$
$$= f(x, y), \text{ say.}$$

Now,
$$f_x = y - \frac{2V}{x^2}, \quad f_y = x - \frac{2V}{y^2}.$$

The critical points are given by
$$f_x = y - \frac{2V}{x^2} = 0 \quad \text{...(2)}$$
$$f_y = x - \frac{2V}{y^2} = 0 \quad \text{...(3)}$$

From (2), $y = \frac{2V}{x^2}$ and putting this value in (3), we get
$$x - 2V\cdot\frac{x^4}{4V^2} = 0, \text{ or } x\left(1 - \frac{x^3}{2V}\right) = 0, \text{ or } x = (2V)^{1/3} \quad (\because x \neq 0)$$

and
$$y = \frac{2V}{x^2} = \frac{2V}{(2V)^{2/3}} = (2V)^{1/3}$$

Hence $((2V)^{1/3}, (2V)^{1/3})$ is a critical point.

Now,
$$f_{xx} = \frac{4V}{x^3}, \quad f_{yy} = \frac{4V}{y^3}, \quad f_{xy} = 1$$

$$\therefore \quad H(x, y) = f_{xx}f_{yy} - \{f_{xy}\}^2 = \left(\frac{4V}{x^3}\right)\left(\frac{4V}{y^3}\right) - 1$$

$\therefore \quad H((2V)^{1/3}, (2V)^{1/3}) = 2 \cdot 2 - 1 = 4 - 1 = 3 > 0$

Also, $f_{xx}((2V)^{1/3}, (2V)^{1/3}) = \frac{4V}{2V} = 2 > 0$

Hence S is minimum when $x = y = (2V)^{1/3}$

also $$z = \frac{V}{xy} \quad \text{[by (1)]}$$

$$= \frac{V}{(2V)^{2/3}} = \frac{V^{1/3}}{2^{2/3}} = \frac{(2V)^{1/3}}{2}.$$

Therefore, the length, breadth and height of the given box should be $(2V)^{1/3}$, $(2V)^{1/3}$, $\frac{1}{2}(2V)^{1/3}$ respectively for least materials for its construction.

Example 5: Find the shortest distance from origin to the surface $xyz^2 = 2$.

Solution: Let $P(x, y, z)$ be any point on the surface $xyz^2 = 2$. The distance between $P(x, y, z)$ and origin is given by $\{(x-0)^2 + (y-0)^2 + (z-0)^2\}^{1/2}$. If the distance is maximum or minimum, so will be the square of the distance.

Let $$f(x, y, z) = x^2 + y^2 + z^2 = x^2 + y^2 + \frac{2}{xy} \quad [\because (x, y, z) \text{ lies on } xyz^2 = 2]$$

$$= g(x, y) \text{ (say)}.$$

So, $$g_x = 2x - \frac{2}{x^2 y} \text{ and } g_y = 2y - \frac{2}{xy^2}.$$

The critical points are given by

$$g_x = 0, \text{ or } \frac{x^3 y - 1}{x^2 y} = 0, \text{ or } x^3 y = 1 \quad ...(1)$$

$$g_y = 0, \frac{xy^3 - 1}{xy^2} = 0, \text{ or } xy^3 = 1 \quad ...(2)$$

From (1) and (2), $x^3 y - xy^3 = 0$, or $xy(x^2 - y^2) = 0$,

since $x \neq 0$, $y \neq 0$, so $x = \pm y$ $\quad [x \neq 0, y \neq 0$ because (x, y, z) lies on $xyz^2 = 2]$ ...(3)

From (1), (2) and (3), two possible critical points are (1, 1) and (–1, –1).

Now, $$g_{xx} = 2 + \frac{4}{x^3 y}, \; g_{yy} = 2 + \frac{4}{xy^3} \text{ and } g_{xy} = \frac{2}{x^2 y^2}$$

So, $$H(x, y) = g_{xx} g_{yy} - \{g_{xy}\}^2 = \left(2 + \frac{4}{x^3 y}\right)\left(2 + \frac{4}{xy^3}\right) - \frac{4}{x^4 y^4}$$

$$= \frac{12}{x^4 y^4} + \frac{8}{xy^3} + \frac{8}{x^3 y} + 4.$$

(*i*) At (1, 1), we have $H(1, 1) = 32 > 0$, $g_{xx}(1, 1) = 6 > 0$.

(*ii*) At (–1, –1), we have $H(-1, -1) = 32 > 0$, $g_{xx}(-1, -1) = 6 > 0$.

So minimum occurs at $(1, 1, \sqrt{2})$, $(1, 1, -\sqrt{2})$, $(-1, -1, \sqrt{2})$, $(-1, -1, -\sqrt{2})$, since $z^2 = \frac{2}{xy}$.

...refore the required shortest distance is

$$\sqrt{1^2 + 1^2 + (\sqrt{2})^2} = \sqrt{4} = 2.$$

...A OF CONSTRAINED FUNCTIONS: LAGRANGE'S ...IER METHOD

Let us consider a function $f(x, y, z)$ subject to the constraint $\varphi(x, y, z) = 0$. ...(1)

The necessary condition for the existence of extremum for $f(x, y, z)$ is

$$df = 0, \ i.e., \ \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \qquad ...(2)$$

From (1), we have

$$d\varphi = 0, \ i.e., \ \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz = 0 \qquad ...(3)$$

...iplying equation (3) by λ and adding to (2), we get

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \lambda\left(\frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz\right) = 0$$

or

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \varphi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \varphi}{\partial z}\right)dz = 0 \qquad ...(4)$$

Since dx, dy and dz are independent, therefore

$$\frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x} = 0, \ \frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} + \lambda\frac{\partial \varphi}{\partial y} = 0, \ \frac{\partial L}{\partial z} = \frac{\partial f}{\partial z} + \lambda\frac{\partial \varphi}{\partial z} = 0, \qquad ...(5)$$

where $\quad L(x, y, z, \lambda) = f(x, y, z) + \lambda\varphi(x, y, z)$

is known as Lagrangian function and the parameter λ is called Lagrangian multiplier which is independent of x, y, z.

The equations (5) are called Lagrange's equations and solving these equations along with equation (1), we get λ and the critical (or stationary) points.

To test whether these critical points are maximum or minimum, we apply the method described in Art. 9.2.

Note: This method can be extended to functions of more than three variables.

ILLUSTRATIVE EXAMPLES

Example 1: Find the point upon the plane $ax + by + cz = p$ at which the function $f(x, y, z) = x^2 + y^2 + z^2$ has a minimum value and find this minimum. Use Lagrange's multiplier method.

Solution: Here $f(x, y, z) = x^2 + y^2 + z^2$ and the constraint is $ax + by + cz = p$.

First part:

The Lagrangian function is

$$L(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda(ax + by + cz - p),$$

where λ is a Lagrangian multiplier.

Now, for the critical points, we have

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 0, \text{ i.e., } 2x + \lambda a = 0, \text{ or } x = -\frac{\lambda a}{2} \\ \frac{\partial L}{\partial y} &= 0, \text{ i.e., } 2y + \lambda b = 0, \text{ or } y = -\frac{\lambda b}{2} \\ \frac{\partial L}{\partial z} &= 0, \text{ i.e., } 2z + \lambda c = 0, \text{ or } z = -\frac{\lambda c}{2} \end{aligned}\right\} \quad \text{...(1)}$$

$$\frac{\partial L}{\partial \lambda} = 0, \text{ i.e., } ax + by + cz - p = 0 \quad \text{...(2)}$$

Putting the values of x, y, z from (1) in (2), we get

$$a\left(-\frac{\lambda a}{2}\right) + b\left(-\frac{\lambda b}{2}\right) + c\left(-\frac{\lambda c}{2}\right) = p$$

or $$\lambda(a^2 + b^2 + c^2) = -2p, \text{ or } \lambda = \frac{-2p}{a^2 + b^2 + c^2}.$$

$$\therefore \quad x = \frac{ap}{a^2 + b^2 + c^2}, \quad y = \frac{bp}{a^2 + b^2 + c^2}, \quad z = \frac{cp}{a^2 + b^2 + c^2}$$

Therefore, the critical point is $\left(\frac{ap}{a^2 + b^2 + c^2}, \frac{bp}{a^2 + b^2 + c^2}, \frac{cp}{a^2 + b^2 + c^2}\right)$

Second part:

Now, from (2), $$z = \frac{1}{c}(p - ax - by)$$

$$\therefore \quad \frac{\partial z}{\partial x} = -\frac{a}{c}, \quad \frac{\partial z}{\partial y} = -\frac{b}{c}$$

From $$f(x, y, z) = x^2 + y^2 + z^2, \text{ we get}$$

$$\frac{\partial f}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} = 2\left(x - \frac{az}{c}\right) \text{ and } \frac{\partial f}{\partial y} = 2y + 2z\frac{\partial z}{\partial y} = 2\left(y - \frac{bz}{c}\right)$$

$$\therefore \quad \frac{\partial^2 f}{\partial x^2} = 2\left(1 - \frac{a}{c}\frac{\partial z}{\partial x}\right) = 2\left(1 + \frac{a^2}{c^2}\right), \frac{\partial^2 f}{\partial y^2} = 2\left(1 - \frac{b}{c}\frac{\partial z}{\partial y}\right) = 2\left(1 + \frac{b^2}{c^2}\right)$$

and

$$\frac{\partial^2 f}{\partial x \partial y} = -2\frac{b}{c}\frac{\partial z}{\partial x} = \frac{2ab}{c^2}.$$

Now,

$$H = f_{xx}f_{yy} - \{f_{xy}\}^2 = 4\left(1 + \frac{a^2}{c^2}\right)\left(1 + \frac{b^2}{c^2}\right) - \frac{4a^2b^2}{c^4}$$

$$= 4\left(1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = 2\left(1 + \frac{a^2}{c^2}\right)$$

Therefore, at the point $\left(\frac{ap}{a^2+b^2+c^2}, \frac{bp}{a^2+b^2+c^2}, \frac{cp}{a^2+b^2+c^2}\right)$,

$$H > 0, \ \frac{\partial^2 f}{\partial x^2} > 0.$$

Therefore, the function $f(x, y, z)$ is minimum at $\left(\frac{ap}{a^2+b^2+c^2}, \frac{bp}{a^2+b^2+c^2}, \frac{cp}{a^2+b^2+c^2}\right)$ and the minimum value of this function is

$$\frac{a^2p^2}{(a^2+b^2+c^2)^2} + \frac{b^2p^2}{(a^2+b^2+c^2)^2} + \frac{c^2p^2}{(a^2+b^2+c^2)^2} = \frac{p^2}{a^2+b^2+c^2}.$$

Example 2: Find a point in the plane $x + 2y + 3z = 13$ nearest to the point $(1, 1, 1)$ using the method of Lagrange's multipliers. [*W.B.U.T. 2001, 2002*]

Solution: Let $P(x, y, z)$ be any point on the plane $x + 2y + 3z = 13$. The distance between the point $P(x, y, z)$ and $A(1, 1, 1)$ is given by $\{(x-1)^2 + (y-1)^2 + (z-1)^2\}^{1/2}$. If the distance is maximum or minimum, so will be the square of the distance.

Let $$f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$$

subject to the constraint $x + 2y + 3z = 13$.

First part:

The Lagrangian function is

$$L(x, y, z, \lambda) = (x-1)^2 + (y-1)^2 + (z-1)^2 + \lambda(x + 2y + 3z - 13),$$

λ is a Lagrangian multiplier.

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For the critical points, we have

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 0, \text{ i.e., } 2(x-1)+\lambda = 0, \text{ or } x = -\frac{\lambda}{2}+1 \\ \frac{\partial L}{\partial y} &= 0, \text{ i.e., } 2(y-1)+2\lambda = 0, \text{ or } y = -\lambda+1 \\ \frac{\partial L}{\partial z} &= 0, \text{ i.e., } 2(z-1)+3\lambda = 0, \text{ or } z = -\frac{3\lambda}{2}+1 \end{aligned}\right\} \quad \ldots(1)$$

$$\frac{\partial L}{\partial \lambda} = 0, \text{ i.e., } x+2y+3z-13 = 0 \quad \ldots(2)$$

Putting the values of x, y, z from (1) in (2), we get

$$\left(-\frac{\lambda}{2}+1\right)+(-2\lambda+2)+\left(-\frac{9\lambda}{2}+3\right)-13 = 0, \text{ or } \lambda = -1.$$

Therefore, using (1), the critical point is $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$.

Second part:

Now, from (2), $z = \frac{1}{3}(13-x-2y)$, therefore $\frac{\partial z}{\partial x} = -\frac{1}{3}$ and $\frac{\partial z}{\partial y} = -\frac{2}{3}$.

From $f(x, y, z) = (x-1)^2+(y-1)^2+(z-1)^2$, we get

$$\frac{\partial f}{\partial x} = 2(x-1)+2(z-1)\frac{\partial z}{\partial x} = 2(x-1)-\frac{2}{3}(z-1),$$

$$\frac{\partial f}{\partial y} = 2(y-1)+2(z-1)\frac{\partial z}{\partial y} = 2(y-1)-\frac{4}{3}(z-1),$$

$$\frac{\partial^2 f}{\partial x^2} = 2-\frac{2}{3}\frac{\partial z}{\partial x} = 2+\frac{2}{9} = \frac{20}{9}, \quad \frac{\partial^2 f}{\partial y^2} = 2-\frac{4}{3}\frac{\partial z}{\partial y} = 2+\frac{8}{9} = \frac{26}{9},$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{4}{3}\frac{\partial z}{\partial x} = \frac{4}{9}.$$

$$\therefore \quad H = f_{xx}f_{yy}-\{f_{xy}\}^2 = \frac{20}{9}\times\frac{26}{9}-\frac{16}{81} = \frac{1}{81}(20\times 26-16) > 0$$

Therefore, at the point $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$,

$$H > 0, \quad \frac{\partial^2 f}{\partial x^2} > 0.$$

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