Counting
(Lecture – 2)

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PH Principle

- Problems:
  - Show that among any set of 5 integers, there are 2 with the same remainder when divided by 4.
  - How many students, each of whom comes from one of the 50 states, must be enrolled in a university to guaranteed that there are at least 100 who come from the same state?
  - During a month with 30 days, the number of games played by a baseball team increases by at least one each day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.
  - Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.
- **Theorem**:

Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Permutations

- A **permutation** of a set of distinct objects is an ordered arrangement of these objects.
- We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an ***r*-permutation**.
- The number of r-permutations of a set with n elements is denoted by $P(n, r)$.
- Examples:
  - Let $S = \{1, 2, 3\}$. The ordered arrangement 3, 1, 2 is a permutation of S. The ordered arrangement 3, 2 is a 2-permutation of S.
- <u>**Theorem**</u>: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are $P(n, r) = n(n-1)(n-2) \cdots (n-r+1)$ r-permutations of a set with n distinct elements.

Permutations

- **Corollary**:

If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \dfrac{n!}{(n-r)!}$.

- Examples:
  - In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?
  - How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?
  - Suppose that there are eight runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?
  - How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Combinations

- An ***r*-combination** of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.
- **Theorem**:

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals

$$C(n, r) = \frac{n!}{r!\,(n-r)!}.$$

- *C(n, r)*: also denoted as *choose (n, r)* or *binom(n, r)*
- Simplifying:

$$C(n, r) = \frac{n!}{r!\,(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

Combinations

- **Corollary**: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.
- Examples:
  - How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?
  - How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?
  - Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Permutations/Combinations with Repetition

- **Theorem #1**: The number of r-permutations of a set of n objects with repetition allowed is n^r
- Example:
  - How many strings of length r can be formed from the uppercase letters of the English alphabet?
- Example:
  - How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills? Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable, and that there are at least five bills of each type.
- **Theorem #2:** There are $C(n + r - 1, r) = C(n + r - 1, n - 1)$ r-combinations from a set with n elements (types) when repetition of elements is allowed.

Permutations/Combinations with Repetition

- Example:
  - Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.
  - How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, and x_3 are nonnegative integers?
  - How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where the variables are the integers with $x_1 >= 1$, $x_2 >= 2$ and $x_3 >= 3$?