B-TECH 3RD SEMESTER END-SEM EXAMINATION

JANUARY 2020

SUBJECT: MATHEMATICS - III [MA2101]

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No. of sheets uploaded: 11

Q) 1) Normal Distribution

→ A random variable X is said to have normal distribution with parameter $\mu \in (-\infty, \infty)$, and $\sigma > 0$, when it's probability density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

now $$A = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

let $z = \frac{x-\mu}{\sqrt{2}\sigma} \rightarrow dz = \frac{dx}{\sigma\sqrt{2}}$

$$\therefore A = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} dz = 2\int_{0}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} dz$$ (even fn.)

let $z^2 = t \rightarrow 2z\,dz = dt$

or $dz = \frac{1}{2} dt\; t^{-\frac{1}{2}}$

$$A = 2\int_{0}^{\infty} \frac{1}{\sqrt{\pi}} e^{-t} \times \frac{1}{2} dt \times t^{-\frac{1}{2}}$$

$$= 2\int_{0}^{\infty} \frac{1}{\sqrt{\pi}} \times \frac{1}{2}\; t^{-\frac{1}{2}} e^{-t} dt$$

$$= \frac{1}{\cancel{2}} \frac{1}{\sqrt{\pi}} \times \cancel{2} \times \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

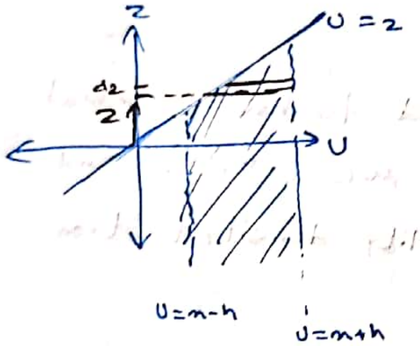
Q1) b) Let $F(x)$ be a distribution fn.

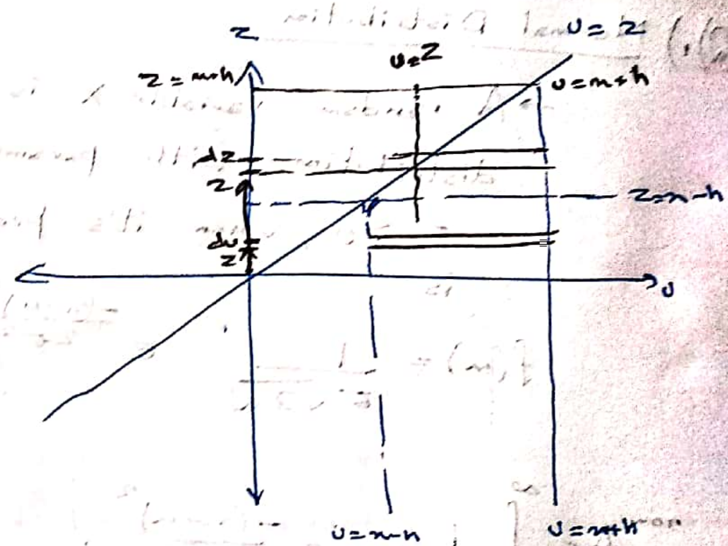
$$\therefore \quad F(x) = \int_{-\infty}^{x} f(x)\, dx .$$

where $f(x)$ is a probability density fn.

given $$G(x) = \frac{1}{2h} \int_{x-h}^{x+h} F(u)\, du$$

$$= \frac{1}{2h} \int_{x-h}^{x+h} \left[\int_{-\infty}^{u} f(z)\, dz \right] du$$





reversing integral

$$G(x) = \frac{1}{2h}\left[\int_{x-h}^{x+h} \left[\int_{z}^{x+h} f(z)\, du \right] dz + \int_{-\infty}^{x-h} \int_{x-h}^{x+h} f(z)\, du\, dz \right]$$

$$= \frac{1}{2h}\left[\int_{x-h}^{x+h} f(z)\,(x+h-z)\, dz + \int_{-\infty}^{x-h} f(z)\,[2h]\, dz \right]$$

let $z = x+h$
$dz = dx$.

$$= \frac{1}{2h}\left[\int_{x-2h}^{x} 0\, dx + 2h \int_{-\infty}^{x-h} f(z)\, dz \right]$$

$$= \frac{h}{h} \int_{-\infty}^{x-h} f(z)\, dz$$

$\therefore G(x)$ can also be a distribution fn.

Pg 2

Q2)a) <u>Binomial Distribution</u>

$$X \sim B(n,p)$$

$$P(X=r) = {}^nC_r \, p^r q^{n-r}$$

$$\text{mean} = E(X) = \sum_{r=0}^{n} r \, {}^nC_r \, p^r q^{n-r}$$

$$= \sum_{r=1}^{n} r \, {}^nC_r \, p^r q^{n-r}$$

$$= \sum_{r=1}^{n} \frac{r \times n!}{r!\,(n-r)!} p^r q^{n-r}$$

$$= np \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!\,(n-r)!} p^{r-1} q^{n-r}$$

$$= np\,(p+q)^{n-1} = np$$

$$\therefore \boxed{\text{mean} = E(X) = np}$$

now $\text{variance} = E[X(X-1)] - m(m-1)$

$$E(X(X-1)) = \sum_{r=0}^{n} r(r-1)\, {}^nC_r \, p^r q^{n-r}$$

$$= \sum_{r=2}^{n} \frac{r(r-1)\, n!}{r!\,(n-r)!} p^r q^{n-r}$$

$$= n(n-1)p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!\,(n-r)!} p^{r-2} q^{n-r}$$

$$= n(n-1)p^2 (p+q)^{n-2}$$

$$= n^2p^2 - np^2$$

$$\therefore Var(X) = n^2p^2 - np^2 - np(np-1)$$

$$= -np(p-1)$$

$$= npq$$

$$\therefore \boxed{Var(X) = npq}$$

Q2)b) $f(x) = \begin{cases} 1 - |1-x| & 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$

$= \begin{cases} 1 - 1 + x & 0 < x \leq 1 \\ 1 + 1 - x & 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} x & 0 < x \leq 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}$

$\therefore E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$= \int_0^1 x^2\, dx + \int_1^2 x(2-x)\, dx$

$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right]_1^2$

$= \frac{1}{3} + \left[4 - \frac{8}{3} - 1 + \frac{1}{3}\right]$

$= \cancel{\frac{1}{8}}\ 1$

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$= \int_0^1 x^3\, dx + \int_1^2 x^2(2-x)\, dx$

$= \int_0^1 x^3 dx + \int_1^2 2x^2 - x^3\, dx$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2}{3}x^3 - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{4} - 0 + \left[\frac{2}{3}\times 2^3 - \frac{2^4}{4} - \frac{2}{3} + \frac{1}{4}\right]$$

$$= \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}$$

$$= \frac{1}{2} + \frac{14}{3} - 4$$

$$= \frac{7}{6}$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{7}{6} - \cancel{\frac{1}{4}}\ 1$$

$$= \cancel{\frac{19}{18}}\ \frac{1}{6}$$

$\therefore$ mean = $\cancel{\frac{1}{3}}$ 1 variance = $\cancel{\frac{19}{18}}$ $\frac{1}{6}$

Q4) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-t}$, $y(0) = 1$
$y'(0) = 1$

taking Laplace,

$$\left[s^2 Y(s) - s\,y(0) - y'(0)\right] + 4\left[s\,Y(s) - y(0)\right] + 3\,Y(s) = \frac{1}{s+1}$$

or $\underline{s^2 Y(s)} - s - 1 + \underline{4sY(s)} - 4 + \underline{3Y(s)} = \frac{1}{s+1}$

or $Y(s)\left[s^2 + 4s + 3\right] = \frac{1}{s+1} + 5 + s$

$= \cancel{s+5}$

$$Y(s) = \frac{1}{(s+1)^2(s+3)} + \frac{s+5}{(s+3)(s+1)}$$

$$Y(s) = \frac{1}{(s+1)^2(s+3)} + \frac{s+5}{(s+1)(s+3)}$$

$$\cancel{Yt}\ y = L^{-1}\left\{\frac{1}{(s+1)^2(s+3)}\right\} + L^{-1}\left\{\frac{s+5}{(s+1)(s+3)}\right\} \quad (1)$$

I) $L^{-1}\left\{\frac{1}{(s+1)^2(s+3)}\right\}$

$$\frac{1}{(s+1)^2(s+3)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3}$$

$$\text{or} \quad 1 = A(s+1)(s+3) + B(s+3) + C(s+1)^2$$

$$s = -1 \Rightarrow 1 = 0 + 2B$$

$$\boxed{B = \frac{1}{2}}$$

$$s = -3 \Rightarrow 1 = 0 + 0 + C \times 4$$

$$\boxed{C = \frac{1}{4}}$$

$$s = 0 \Rightarrow 1 = 3A + 3B + C$$

$$3A = -\frac{3}{2} - \frac{1}{4} + 1$$

$$= \frac{3}{4}$$

$$\boxed{\cancel{A = \frac{1}{4}}} \quad \boxed{\cancel{A = -\frac{3}{4}}} \quad \boxed{A = -\frac{1}{4}}$$

$$\therefore L^{-1}\left\{\frac{1}{(s+1)^2(s+3)}\right\} = L^{-1}\left\{\frac{\overset{-1}{\cancel{-3}}}{4(s+1)}\right\} + L^{-1}\left\{\frac{1}{2} \times \frac{1}{(s+1)^2}\right\} + L^{-1}\left\{\frac{1}{4}\frac{1}{s+3}\right\}$$

now $L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

$$L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}\, t$$

$$L^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

$$\therefore L^{-1}\left\{\frac{1}{(s+1)^2(s+3)}\right\} = \frac{-1}{4}e^{-t} + \frac{e^{-t}t}{2} + \frac{e^{-3t}}{4} \quad (ii)$$

II) $L^{-1}\left\{\frac{s+5}{(s+1)(s+3)}\right\}$

$$\frac{s+5}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

or $s+5 = A(s+3) + B(s+1)$

$s=-3 \rightarrow 2 = B \times -2 \rightarrow \boxed{B=-1}$

$s=-1 \rightarrow 4 = 2A + 0 \rightarrow \boxed{A=2}$

$$\therefore L^{-1}\left\{\frac{s+5}{(s+1)(s+3)}\right\} = L^{-1}\left\{\frac{2}{s+1}\right\} + L^{-1}\left\{\frac{-1}{s+3}\right\}$$

$$= 2e^{-t} - e^{-3t} \quad (iii)$$

$\therefore$ using (ii) & (iii) in (i)

$$\rightarrow y = \left[\frac{-1}{4}e^{-t} + \frac{e^{-t}t}{2} + \frac{e^{-3t}}{4}\right] + \left[2e^{-t} - e^{-3t}\right]$$

$$\boxed{y = \frac{7}{4}e^{-t} + \frac{te^{-t}}{2} - \frac{11}{4}e^{-3t}}$$

5) a) let $L\{f(t)\} = \log\left(1+\frac{1}{s^2}\right)$

$$\frac{d}{ds} L\{f(t)\} = \frac{1}{1+\frac{1}{s^2}} \times \frac{-2}{s^3}$$

$$= \frac{s^2}{s^2+1} \times \frac{-2}{s^3}$$

$$= \frac{-2}{(s^2+1)(s)}$$

$$= -2 \times \frac{1}{s^2+1} \times \frac{1}{s}$$

now $L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$L^{-1}\left\{\frac{1}{s}\right\} = 1$

$\therefore L^{-1}\left\{\frac{1}{s^2+1} \times \frac{1}{s}\right\} = \int_0^t \sin u \, du = \left[-\cos u\right]_0^t = 1-\cos t$

$\therefore -\frac{d}{ds} L\{f(t)\} = L\{1+\cos t\}$

$\therefore$ as $L\{t\, g(t)\} = -\frac{d}{ds} L\{g(t)\}$

$t f(t) = (\cos t - 1)$

~~check: $-\frac{d}{ds}\left[\frac{s}{s^2+1} - \frac{1}{s}\right]$~~ or $f(t) = \frac{\cos t - 1}{t}$

~~check $= \int_0^t \frac{s}{s^2+1} - \frac{1}{s}\, ds = \log\frac{(s^2+1)}{s} \int_1^{t^2+1} \frac{1}{t}\, dt - [\log s]_0^t$~~

~~$s^2+1=t$~~
~~$2s\,ds = \frac{dt}{2}$~~

~~$= \log(t^2+1) - \log t$~~

5) b) A point $x \in X$ is said to be extreme point of the convex set X, if x cannot be expressed as convex combination of two point $x_1, x_2 \in X$

i.e $x \neq \lambda x_1 + (1-\lambda)x_2$

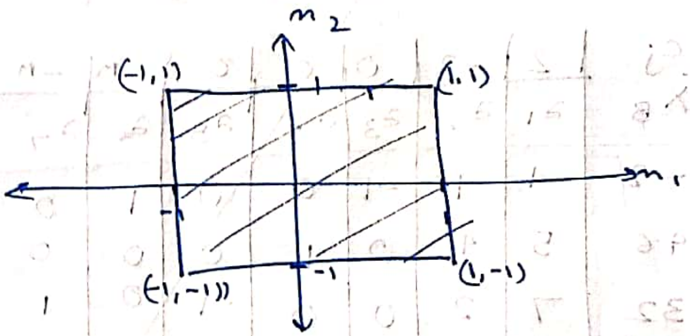
→ ~~These points~~

ii) → ~~The points are generally & corners or ends~~

Eg :→ corner point of rectangle.

→ Boundary of Circle, ~~e~~, etc.

i) $X_1 = \{(x_1, x_2) : |x_1| \leq 1, |x_2| \leq 1\}$

→ 

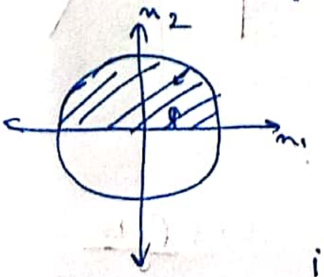
→ It's extreme points will be the corner points,

i.e $(1,1), (-1,1), (-1,-1), (1,-1)$

ii) $X_2 = \{(x_1, x_2) : x_1 + 2x_2 = 9\}$

→ as this is a straight line, there are no extreme points

iii) $X_3 = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1, x_1 \geq 0\}$



extreme point $= \{(x_1, x_2) : x_1^2 + x_2^2 = 1, x_1 \geq 0\}$

~~$\{(x_1, 0) : x_1 \in [-1, 1]\}$~~

i.e , the top arc ~~and the~~

c) max $z = 2x_1 - 3x_2$

st. $-x_1 + x_2 \geq -2$

$5x_1 + 4x_2 \leq 46$

$7x_1 + 2x_2 \geq 32$

$x_1, x_2 \geq 0$

→ adding slack, surplus, and artificial variables

max $z = 2x_1 - 3x_2 + 0x_3 + 0x_4 + 0x_5 - Mx_6$

st. $x_1 - x_2 + x_3 = 2$

$5x_1 + 4x_2 + x_4 = 46$

$7x_1 + 2x_2 - x_5 + x_6 = 32$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

C_B	B	c_j / X_B	2 / a_1	−3 / a_2	0 / a_3	0 / a_4	0 / a_5	−M / a_6	Remarks
0	a_3	2	$\boxed{1}$	−1	1	0	0	0	min $\left(\frac{2}{1}, \frac{46}{5}, \frac{32}{7}\right)$ ↑
0	a_4	46	5	4	0	1	0	0	
−M	a_6	32	7	2	0	0	−1	1	∴ index element = 1
		$z_j - c_j$	−7M −2 ↑	−2M +3	0	0	M	0	a_1 enter, a_3 exit
2	a_1	2	1	−1	1	0	0	0	min $\left(\frac{36}{9}, \frac{18}{9}\right)$ ↑
0	a_4	36	0	9	−5	1	0	0	key element 9
−M	a_6	18	0	$\boxed{9}$	−7	0	−1	1	a_6 exit
		$z_j - c_j$	0	−9M +1 ↑	7M +2	0	M	0	a_2 enter
2	a_1	4	1	0	$\frac{2}{9}$	0	$-\frac{1}{9}$	–	all $z_j - c_j \geq 0$
0	a_4	18	0	0	2	1	1	–	
−3	a_2	2	0	1	$-\frac{7}{9}$	0	$-\frac{1}{9}$	–	∴ optimal solution reached.
		$z_j - c_j$	0	0	$\frac{25}{9}$	0	$\frac{1}{9}$	–	

Hence $x_1 = 4$

$x_2 = 2$

and $\max z = 2 \times 4 - 3 \times 2$

$= 2$