FOURIER HALF-RANGE SERIES

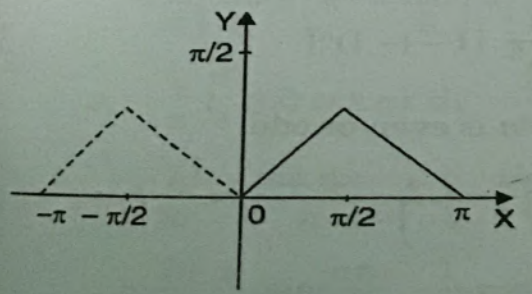
(Study material)

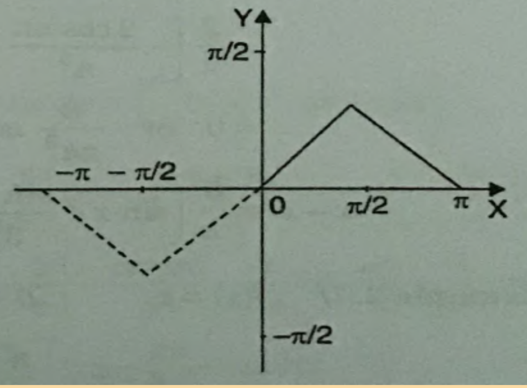
Reference book: Advanced Engineering Mathematics by Jain & Iyengar

Suppose that a function $f(x)$ is defined on some finite interval. It may also be the case that a periodic function $f(x)$ of period $2l$ is defined only on a half-interval $[0, l]$. It is possible to extend the definition of $f(x)$ to the other half $[-l, 0]$ of the interval $[-l, l]$ so that $f(x)$ is either an even or an odd function. In the first case, we call it an even periodic extension of $f(x)$ and in the second case, we call it an odd periodic extension of $f(x)$. If $f(x)$ is given and an *even periodic extension* is done then $f(x)$ is an even function in $[-l, l]$. Hence, $f(x)$ has a Fourier cosine series. If $f(x)$ is given and an *odd periodic extension* is done then $f(x)$ is an odd function in $[-l, l]$. Hence, $f(x)$ has now a Fourier sine series. Therefore, if a function $f(x)$ is defined only on a half interval $[0, l]$, then it is possible to obtain a Fourier cosine or a Fourier sine series expansion depending on the requirements of a particular problem, by suitable periodic extensions. We have the following results.

For example, consider the function

$$f(x) = \begin{cases} x, & 0 < x < \dfrac{\pi}{2} \\ \pi - x, & \dfrac{\pi}{2} < x < \pi \end{cases}$$





Theorem 9.4 (Fourier cosine series) Let $f(x)$ be piecewise continuous on $[0, l]$. Then, the Fourier cosine series expansion of $f(x)$ on the half-range interval $[0, l]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \tag{9.25}$$

where $$a_0 = \frac{2}{l}\int_0^l f(x)dx \quad \text{and} \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

The convergence Theorem 9.1 can be extended as follows.

If $x \in [0, l]$ and $f(x)$ has left and right hand derivatives at x, then at x, the Fourier cosine series converges to $[f(x+) + f(x-)]/2$. At a point of continuity, the Fourier cosine series converges to $f(x)$.

Theroem 9.5 (Fourier sine series) Let $f(x)$ be piecewise continuous on $[0, l]$. Then, the Fourier sine series expansion of $f(x)$ on $[0, l]$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \tag{9.26}$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

If $x \in [0, l]$ and $f(x)$ has left and right hand derivatives at x, then at x, the Fourier sine series converges to $[f(x+) + f(x-)]/2$. At both the end points $x = 0$ and l, the series converges to 0.

Example 2. *If* $f(x) = x, \qquad 0 < x < \dfrac{\pi}{2}$

$$= \pi - x, \qquad \frac{\pi}{2} < x < \pi$$

show that (i) $f(x) = \dfrac{4}{\pi}\left[\sin x - \dfrac{\sin 3x}{3^2} + \dfrac{\sin 5x}{5^2} - \ldots\ldots\right]$

(ii) $f(x) = \dfrac{\pi}{4} - \dfrac{2}{\pi}\left[\dfrac{\cos 2x}{1^2} + \dfrac{\cos 6x}{3^2} + \dfrac{\cos 10x}{5^2} + \ldots\ldots\right].$

Sol. (*i*) For the half-range sine series.

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Then $b_n = \dfrac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx = \dfrac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx\, dx\right]$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] + \frac{2}{\pi}\left[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{n\pi}{2}\right] = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

When n is even, $b_n = 0$.

$$\therefore \quad f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right].$$

(*ii*) **For the half-range cosine series.**

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right]$

$$= \frac{2}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi/2} + \left|\pi x - \frac{x^2}{2}\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right] = \frac{2}{\pi}\left[\frac{\pi^2}{4}\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[x\,.\,\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\,.\,\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\right] + \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1\right]$$

$\therefore$ $a_1 = 0,\ a_2 = \frac{2}{\pi\,.\,2^2}(2\cos\pi - \cos 2\pi - 1) = \frac{-2}{\pi\,.\,1^2}$,

$a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = \frac{2}{\pi\,.\,6^2}(2\cos 3\pi - \cos 6\pi - 1) = \frac{-2}{\pi\,.\,3^2}$,

$a_7 = 0,\ a_8 = 0,\ a_9 = 0,\ a_{10} = \frac{2}{\pi\,.\,10^2}(2\cos 5\pi - \cos 10\pi - 1) = \frac{-2}{\pi\,.\,5^2}$, ...

Hence $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\ldots\right]$.