Taylor's theorem:

Statement: Let $f(z)$ be analytic within the circle $C: |z-a| = R$ and let ξ be any point within C, then
$$f(\xi) = \sum_{n=0}^{\infty} \frac{(\xi-a)^n}{n!} f^n(a) \quad \text{—(1)}$$

The above series is called Taylor's series.

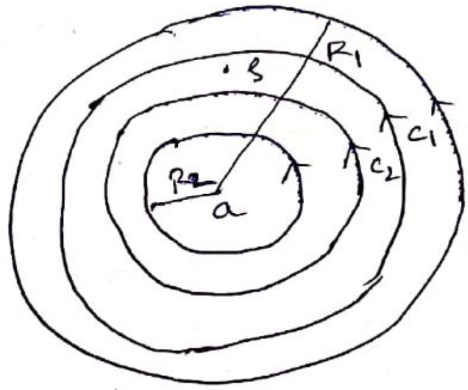
Laurent's theorem:

Statement: Let $f(z)$ be analytic within the ring shaped region $R_2 < |z-a| < R_1$, bounded by the circles $|z-a| = R_1$ and $|z-a| = R_2$. If ξ is any point within the region $R_2 < |z-a| < R_1$, then
$$f(\xi) = \sum_{n=0}^{\infty} a_n (\xi - a)^n + \sum_{n=1}^{\infty} b_n (\xi - a)^{-n}, \quad \text{—(2)}$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz$

and $b_n = \frac{1}{2\pi i} \int_{C_2} (z-a)^{n-1} f(z)\, dz$, and

C_1 & C_2 are any two circles of the form $C_1: |z-a| = \rho_1$, $C_2: |z-a| = \rho_2$, and $R_2 < \rho_2 < |\xi - a| < \rho_1 < R_1$.



The series (2) is called Laurent series.

(29) Isolated singularities of (analytic) function:

Let $f(z)$ be analytic in $0 < |z-a| < R$.

Then $f(z)$ can be expanded in a Laurent series of the form
$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \quad \text{—(1)}$$
where z is a point within the region $0 < |z-a| < R$ and the radius of the inner circle with centre a may be chosen as small as we please. Hence the expansion (1) is valid in $0 < |z-a| < R$.

The part $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ is called the principal part of the expansion of $f(z)$ about a.

Now three cases may arise:

(i) All the coefficients b_n are zero.

In this case the principal part vanishes and so we have $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$, $0 < |z-a| < R$.

In this case the point $z=a$ is called a removable singularity of $f(z)$. [That is the singularity can be removed by suitable transformation]

(ii) The principal part is terminating.

In this case all the coefficients b_n vanish after certain stage. In this case the point $z=a$ is called a pole of $f(z)$. If b_m is the last non vanishing coefficient in (1), then m is called the order of the pole. The number b_1 is called the residue of $f(z)$ at the pole $z=a$.

If $m=1$, the pole is called a simple pole.
(pole of order 1)

(iii) The principal part is not terminating.

In this case a is called an isolated essential singularity of $f(z)$. The number b_1 is called the residue of $f(z)$ at the isolated essential singularity $z=a$.

Illustration with example:

EX1 Let $f(z) = \frac{\sin z}{z}$. Then $f(z)$ is undefined at $z=0$. However for $0<|z|<\infty$,

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\cdots, \quad 0<|z|<\infty$$

Thus if we define $\frac{\sin z}{z} = 1$ at $z=0$, then the above series converges to 1 at $z=0$ and the function will become analytic at $z=0$.

Thus $z=0$ is a removable singularity.

(As it has no principal part).

EX2 Let $f(z) = \frac{z^2-3z+5}{z-1}$

Then $f(z) = \frac{z(z-1)-2(z-1)+3}{z-1} = z-3+\frac{3}{z-1}$

$$= (z-1)\,1 - 2 + \underbrace{\frac{3}{z-1}}_{\text{residue}}$$

$\therefore$ $z=1$ is a simple pole and 3 is the residue.
(pole of order 1)

EX3 Let $f(z) = \frac{z^3+5}{(z-1)^3} = \frac{(z-1)^3+3(z-1)^2+3(z-1)+6}{(z-1)^3}$

$$= 1 + \underbrace{\frac{3}{z-1}}_{\text{residue}} + \frac{3}{(z-1)^2} + \frac{6}{(z-1)^3} \longleftarrow \text{order of pole}$$

$\therefore$ $z=1$ is a pole of order 3 and 3 is the residue

EX4 consider $f(z) = e^{\frac{1}{z}}$.

Expanding we get ($e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$)

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\cdots, \quad 0 < |z| < \infty$$

Here the principal part $\frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\cdots$ is not terminating. Hence $z=0$ is an isolated essential singularity with residue 1.

EX4 $\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots\cdots, \quad 0 < |z| < \infty$

$$\left(\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \right)$$

$\therefore$ $z=0$ is an isolated essential singularity of $\sin\frac{1}{z}$.

Computation of residue for a pole.

Let $z=a$ be a pole of $f(z)$ of order m.

Then $f(z)$ can be expanded as

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$$

Multiplying both sides by $(z-a)^m$,

$$(z-a)^m f(z) = a_0(z-a)^m + a_1(z-a)^{m+1} + a_2(z-a)^{m+2} + \cdots\cdots$$
$$+ b_1(z-a)^{m-1} + b_2(z-a)^{m-2} + \cdots + b_m$$

$$\therefore \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right] = a_0[m(m-1)\cdots 2](z-a) + a_1[(m+1)m\cdots](z-a)^2$$
$$+ a_2[(m+2)(m+1)\cdots](z-a)^3 + \cdots\cdots$$
$$+ b_1(m-1)!$$

$$\therefore \left[\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]\right]_{z=a} = b_1(m-1)!$$

Hence $b_1 = \frac{1}{(m-1)!} \left.\frac{d^{m-1}}{dz^{m-1}}\left[f(z)(z-a)^m\right]\right|_{z=a}$ = Residue at $z=a$.

(32)

Ex 1 Expand $f(z) = \dfrac{z}{(z-1)(z-2)}$ in a Laurent series

for (i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$

(iv) $|z-1| > 1$ (v) $0 < |z-2| < 1$

Soln $f(z) = \dfrac{z}{(z-1)(z-2)} = \dfrac{A}{z-1} + \dfrac{B}{z-2}$ say

$$= \frac{A(z-2)+B(z-1)}{(z-1)(z-2)}$$

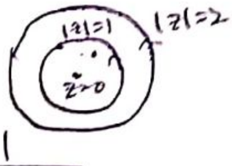
$\therefore z = A(z-2) + B(z-1)$ is an identity

for $z=1$, we get $1 = A(1-2) = -A \quad \therefore A = -1$

for $z=2$, we get $2 = B(2-1) = B \quad \therefore B = 2$

Hence $\dfrac{z}{(z-1)(z-2)} = -\dfrac{1}{z-1} + \dfrac{2}{z-2}$

(i) For $|z|<1$, we get $\left|\frac{z}{2}\right| < 1$

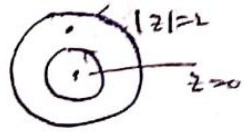


$$\therefore \frac{z}{(z-1)(z-2)} = \frac{1}{1-z} - \frac{\cancel{2}^{\,1}}{\cancel{2}\left(1-\frac{z}{2}\right)} = \frac{1}{-z+1} - \frac{1}{1-\frac{z}{2}}$$

$$= 1+z+z^2+z^3+\cdots - \left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

$$= z\left(1-\frac{1}{2}\right) + z^2\left(1-\frac{1}{2^2}\right) + z^3\left(1-\frac{1}{2^3}\right) + \cdots$$

(ii) $1 < |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$ & $\left|\frac{1}{z}\right| < 1$

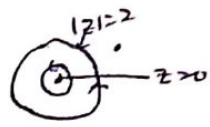


$$\therefore \frac{z}{(z-1)(z-2)} = -\frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{1-\frac{z}{2}}$$

$$= -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

$$= -1 - \frac{z}{2} - \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 - \cdots - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots$$

(iii) $|z| > 2$, $\left|\frac{2}{z}\right| < 1 \quad \therefore \left|\frac{1}{z}\right| < 1$

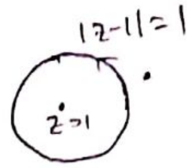


Hence $\dfrac{z}{(z-1)(z-2)} = \dfrac{1}{-z\left(1-\frac{1}{z}\right)} + \dfrac{2}{z\left(1-\frac{2}{z}\right)}$

$$= -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) + \frac{2}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right]$$

$$= \frac{1}{z}(2-1) + \frac{1}{z^2}(2^2-1) + \frac{1}{z^3}(2^3-1) + \cdots$$

(33) (iv) $|z-1|>1 \Rightarrow \left|\frac{1}{z-1}\right|<1$



$$\frac{z}{(z-1)(z-2)} = -\frac{1}{z-1} + \frac{2}{z-2}$$

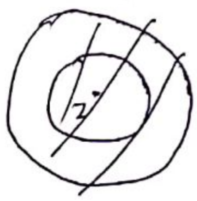
$$= -\frac{1}{z-1} + \frac{2}{z-1-1} = -\frac{1}{z-1} + \frac{2}{-1+(z-1)}$$

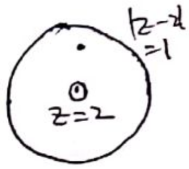
$$= -\frac{1}{z-1} + \frac{2}{(z-1)\left(1-\frac{1}{z-1}\right)}$$

$$= -\frac{1}{z-1} + \frac{2}{z-1}\left[1+\frac{1}{z-1}+\left(\frac{1}{z-1}\right)^2+\left(\frac{1}{z-1}\right)^3+\cdots\right]$$

$$= \frac{1}{z-1} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^4} + \cdots$$

(v) $0<|z-2|<1$,





$$\frac{z}{(z-1)(z-2)} = -\frac{1}{z-1} + \frac{2}{z-2}$$

$$= -\frac{1}{z-2+1} + \frac{2}{z-2}$$

$$= \frac{-1}{1+(z-2)} + \frac{2}{z-2}$$

$$= -\left[1-(z-2)+(z-2)^2-(z-2)^3+\cdots\right] + \frac{2}{z-2}$$

Ex2 Find the Laurent series expansion of $f(z) = \frac{e^z}{(z-2)^3}$ about the point $z=2$

$$f(z) = \frac{e^z}{(z-2)^3} = \frac{e^{z-2+2}}{(z-2)^3} = e^2 \cdot \frac{e^{z-2}}{(z-2)^3}$$

$$= \frac{e^2}{(z-2)^3}\left[1+z-2+\frac{(z-2)^2}{2!}+\frac{(z-2)^3}{3!}+\cdots\right]$$

$$= e^2\left[\frac{1}{(z-2)^3}+\frac{1}{(z-2)^2}+\frac{1}{2!\,(z-2)}+\frac{1}{3!}+\frac{z-2}{4!}+\frac{(z-2)^2}{5!}+\cdots\right]$$

(34)

EX3 : Compute the residue of the following functions

(i) $f(z) = \dfrac{z+1}{(z-1)(z-2)}$ at $z=1$ (ii) $f(z) = \dfrac{e^z}{z-3}$ at $z=3$

(iii) $f(z) = \dfrac{z-2}{(z-4)^2}$ at $z=4$, (iv) $f(z) = \dfrac{z}{(z-1)^3}$ at $z=1$

Soln (i) $z=1$ is a simple pole of $f(z)$

$$\text{Residue} = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{z+1}{z-2} = \frac{2}{-1} = -2$$

(ii) $z=3$ is a simple pole of $f(z)$.

$\therefore$ Residue at $z=3$ is $\lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3} e^z = e^3$

(iii) $z=4$ is a pole of order 2 of $f(z) = \dfrac{z-2}{(z-4)^2}$

$\therefore$ Residue of $f(z)$ at $z=4$ is

$$\lim_{z\to 4}\frac{d}{dz}\left[(z-4)^2 f(z)\right] = \lim_{z\to 4}\frac{d}{dz}[z-2] = 1$$

[Alternatively $\dfrac{z-2}{(z-4)^2} = \dfrac{z-4+2}{(z-4)^2} = \dfrac{1}{z-4} + \dfrac{2}{(z-4)^2}$

$\therefore$ coefficient of $\dfrac{1}{z-4}$ is 1 which is the residue]

(iv) $z=1$ is pole of order 3 of $f(z) = \dfrac{z}{(z-1)^3}$

$\therefore$ Residue of $f(z)$ at $z=1$ is

$$\lim_{z\to 1}\frac{1}{2!}\frac{d^2}{dz^2}\left[(z-1)^3 f(z)\right] = \lim_{z\to 1}\frac{1}{2}\frac{d^2}{dz^2}[z] = 0$$

[Alternatively, $f(z) = \dfrac{z}{(z-1)^3} = \dfrac{z-1+1}{(z-1)^3} = \dfrac{1}{(z-1)^2} + \dfrac{1}{(z-1)^3}$

coefficient of $\dfrac{1}{z-1}$ is zero. $\therefore$ Residue of $f(z)$ at $z=1$ is 0].

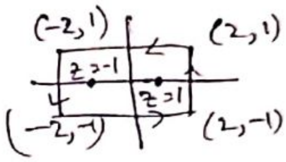
EX4 (i) Evaluate $\displaystyle\int_C \frac{(\cos\pi z)\,dz}{z^2-1}$ where C is a rectangle with vertices $(2,1)$, $(2,-1)$, $(-2,1)$, $(-2,-1)$ described positively.

(ii) Evaluate $\displaystyle\oint_C \frac{(z+7)\,dz}{z^2+2z+5}$ if C is the circle $|z-i| = \frac{3}{2}$

(iii) Evaluate $\displaystyle\oint_C \frac{\cos z}{(z+\pi i)^4}\,dz$ where Z is the closed square bounded by $x=\pm 4$, $y=\pm 4$.

(35)

Soln (i) $\oint_C \frac{\cos\pi z}{z^2-1}\,dz$



$= \oint_C \frac{\cos\pi z}{2}\left[\frac{-1}{z+1}+\frac{1}{z-1}\right]dz$

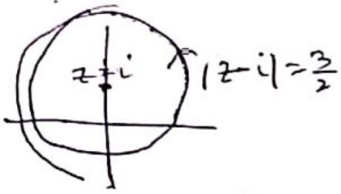
$= \frac{1}{2}\oint_C \frac{\cos\pi z}{z-1}\,dz - \frac{1}{2}\oint_C \frac{\cos\pi z}{z+1}\,dz$

$f(z)=\cos\pi z$ is analytic within and on C and both ~~the~~ $z=1$ & $z=-1$ lies within C. Hence by Cauchy's integral formula

$\oint_C \frac{\cos\pi z}{z-1}\,dz = 2\pi i\left[\cos\pi z\right]_{z=1} = 2\pi i[\cos\pi] = -2\pi i$

$\oint_C \frac{\cos\pi z}{z+1}\,dz = 2\pi i\left[\cos\pi z\right]_{z=-1} = 2\pi i[\cos(-\pi)] = -2\pi i$

Hence $\oint_C \frac{\cos\pi z}{z^2-1}\,dz = \frac{1}{2}[-2\pi i]-\frac{1}{2}[-2\pi i] = 0$



(ii) $\oint_C \frac{z+7}{z^2+2z+5}\,dz = \oint_C \frac{z+7}{[z-(-1+2i)][z-(-1-2i)]}\,dz$

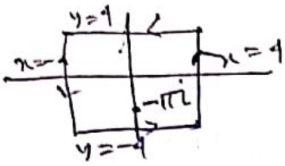
The function $f(z)=\frac{z+7}{z-(-1-2i)}$ is analytic within and on C and $z=-1+2i$ lies within C. Hence by Cauchy's integral formula

$\oint_C \frac{z+7}{z^2+2z+5}\,dz = \oint_C \frac{1}{z-(-1+2i)}\left[\frac{z+7}{z-(-1-2i)}\right]dz$

$= \cancel{2\pi \lim_{z\to -1+2i}[z-(-1+2i)]f(z)} = \cancel{2\pi\left[\frac{z+7}{\cdots}\right]_{z\to-1+2i}}$

$= 2\pi i\left[\frac{z+7}{z+1+2i}\right]_{z=-1+2i} = 2\pi i\left[\frac{-1+2i+7}{\cancel{-1}+2i+\cancel{1}+2i}\right]$

$= \frac{\pi}{2}[6+2i] = (3+i)\pi.$



(iii) $\cos z$ is analytic within and on C and $z=-\pi i$ which is a pole of order 4 lies within C. Hence by derivative of $f(z)=\frac{\cos z}{(z+\pi i)^4}$, formula

$\oint_C \frac{\cos z}{(z+\pi i)^4} = \frac{2\pi i}{3!}\frac{d^3}{dz^3}\cos z\Big|_{z=-\pi i} = \frac{2\pi i}{6}[\sin z]_{z=-\pi i}$

$= \frac{2\pi i}{6}\sin(-\pi i) = -\frac{\pi i}{3}\sin\pi i = -\frac{\pi i}{3}(i\sinh\pi) = \frac{\pi}{3}\sinh\pi.$

Scanned with CamScanner