Basic Discrete Structures

Sets, Functions, Sequences, Matrices, and Relations
(Lecture – 8)

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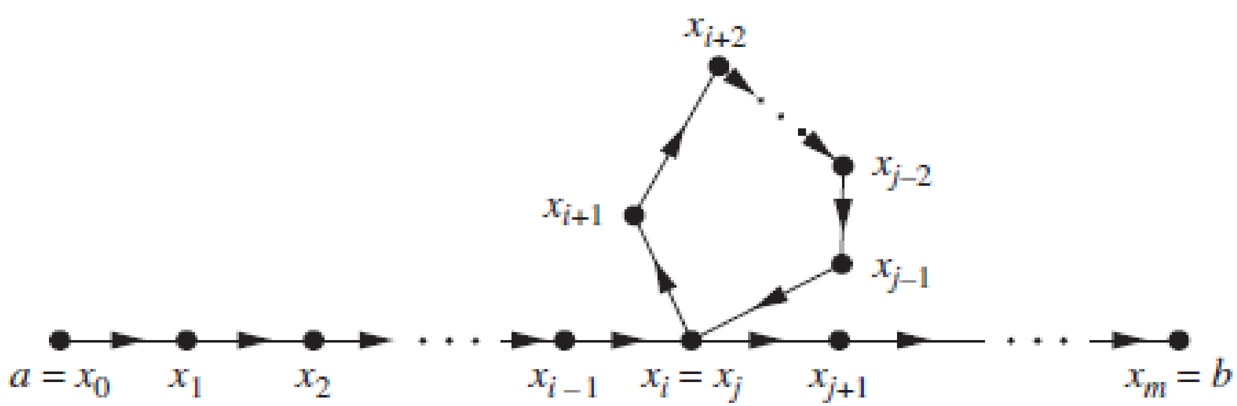
Transitive Closure: Paths in Directed Graph

- Lemma 1:

Let A be a set with n elements, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Moreover, when $a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $n - 1$.



Producing a Path with Length Not Exceeding *n*

Transitive Closure: Zero-One Matrix

- The zero–one matrix representing a union of relations is the join of the zero–one matrices of these relations
- The zero–one matrix for the transitive closure is the join of the zero–one matrices of the first n powers of the zero–one matrix of R.

Let $\mathbf{M}_R$ be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$$

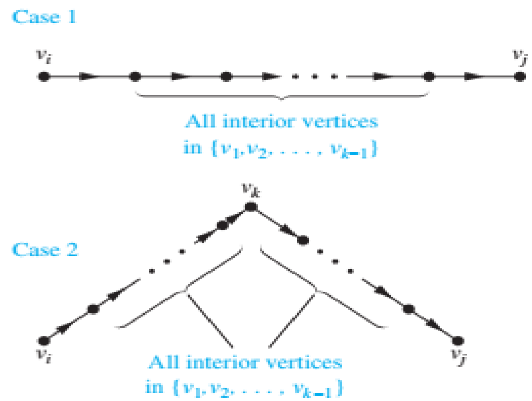
ALGORITHM 1 A Procedure for Computing the Transitive Closure.

procedure *transitive closure* ($\mathbf{M}_R$: zero–one $n \times n$ matrix)
$\mathbf{A} := \mathbf{M}_R$
$\mathbf{B} := \mathbf{A}$
for $i := 2$ **to** n
    $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$
    $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$
return $\mathbf{B}$ {$\mathbf{B}$ is the zero–one matrix for R^*}

Transitive Closure: Warshall's Algorithm

- More efficient in terms of number of bit operations in computing the transitive closure.
- Uses the concept of ***internal vertices*** of a path.
  - If $a, x_1, x_2, \ldots, x_{m-1}, b$ is a path, its interior vertices are $x_1, x_2, \ldots, x_{m-1}$, that is, all the vertices of the path that occur somewhere other than as the first and last vertices in the path.
- Warshall's algorithm is also based on the construction of a sequence of zero-one matrices
  - These matrices are $\boldsymbol{W}_0, \boldsymbol{W}_1, \ldots, \boldsymbol{W}_n$, where $\boldsymbol{W}_0 = \boldsymbol{M}_R$ is the zero–one matrix of this relation
- $\boldsymbol{W}_k = [w_{ij}^{(k)}]$, where $w_{ij}^{(k)} = 1$ if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, \ldots, v_k\}$ (the first k vertices in the list) and is 0 otherwise.
  - Note $\boldsymbol{W}_n = \boldsymbol{M}_{R*}$, because the $(i, j)^{th}$ entry of $\boldsymbol{M}_{R*}$ is 1 if and only if there is a path from v_i to v_j, with all interior vertices in the set $\{v_1, v_2, .., v_n\}$.

Transitive Closure: Warshall's Algorithm



- Lemma 2

Let $\mathbf{W}_k = [w_{ij}^{[k]}]$ be the zero–one matrix that has a 1 in its (i, j)th position if and only if there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \ldots, v_k\}$. Then

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]}),$$

whenever i, j, and k are positive integers not exceeding n.

ALGORITHM 2 Warshall Algorithm.

procedure *Warshall* ($\mathbf{M}_R : n \times n$ zero–one matrix)
$\mathbf{W} := \mathbf{M}_R$
for $k := 1$ **to** n
  for $i := 1$ **to** n
    for $j := 1$ **to** n
      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return $\mathbf{W}\{\mathbf{W} = [w_{ij}]$ is $\mathbf{M}_{R^*}\}$