Relation-5

Monday, October 19, 2020 11:06 AM

Equivalence Relations

Prob. $aRb = \{(a, b) \mid (a-b) \text{ is an integer}\}$

Is R an equivalence relation?

Because, $a - a = 0$ is an integer for all real numbers a, aRa holds. Hence, R is reflexive.

Now suppose that aRb. Then $(a-b)$ is an integer, so is $(b-a)$. Hence, aRb and bRa ~~are true~~. It follows ~~that~~ R is symmetric.

If aRb and bRc, then $(a-b)$ and $(b-c)$ are integers. therefore, $(a-c) = (a-b) + (b-c)$ is also an integers. Hence, aRc is also present. Thus, R is also transitive. Consequently, R is also an equivalent relation.

Prob-2

$a \equiv b \pmod{m}$ → Congruence modulo-m.

$\Rightarrow m \mid (a-b)$. $\Rightarrow$ If a and b are integers and m is a +ve integer, then a is congruent to b modulo m, if m divides $(a-b)$

* Determine whether 17 is congruent to 5 modulo-6.

$17 \equiv 5 \pmod 6 \Rightarrow 6 \mid (17-5) \Rightarrow 6 \mid 12$

* Determine whether 24 is congruent to 14 modulo-6

$24 - 14 = \underline{10} \Rightarrow \underline{6 \nmid 10}.$

$24 \not\equiv 14 \pmod 6.$

$a - a = 0$

$\frac{0}{m} = 0$

Note that $a - a = 0$ is divisible by m ($\because m \mid 0$). Hence, $a \equiv a \pmod m$. So congruence modulo-m is reflexive.

Suppose, $a \equiv b \pmod m$. Then $(a-b)$ is divisible by m. As, $m \mid (a-b)$ it implies $(a-b) = k.m$, where k is an integer. It follows that $(b-a) = (-k).\underline{m}$. which implies $m \mid (b-a)$. thus, $b \equiv a \pmod m$. Hence, Congruence modulo-m is symmetric.

Suppose, $a \equiv b \pmod m$ and $b \equiv c \pmod m$. Then, $m \mid (a-b)$ and $m \mid (b-c)$. Therefore, there are integers k & l s.t. $a - b = k.m$ and $b - c = l.m$. Adding these two we get, $a - c = (a-b) + (b-c) = (k+l).m$ which implies $m \mid (a-c)$. Therefore, $a \equiv c \pmod m$. Therefore, the congruence modulo-m is transitive. It follows that modulo-m is transitive. It follows that congruence modulo-m is an equivalence relation.

Prob-4

R is reflexive because $|x - x| = 0 < 1$, whenever $x \in R$.

R is symmetric for if xRy where x and y are real numbers, then $|x-y|$ implies. $|y-x| = |x-y| < 1$. So, yRx is also true. This implies that R is symmetric.

$\cancel{xRy \wedge yRz \rightarrow xRz}$.

for any arbitrary x, y & z.

If $x = 2.8$, $y = 1.9$, and $z = 1.1$. So $|x-y| = 0.9 < 1$, $|y-z| = |1.9 - 1.1| = 0.8 < 1$, but $|x-z| = |2.8 - 1.1| = 1.7 > 1$.

Relation R is not transitive and hence it is not an equivalence relation.

$\boxed{a \equiv b \pmod{m}} \Rightarrow m \mid (a-b)$

Prob-5

The equivalence class of 0 contains all integers 'a' s.t. $a \equiv 0 \pmod 4$. The integers in this class are those divisible by 4. Hence, the equivalence class of 0 for congruence modulo-4 is: $[0] = \{\ldots\ -8, -4, 0, 4, 8, \ldots\}$

This property used.

$m \mid (a-b)$

The equivalence class of 1 contains all integers 'a' s.t. $a \equiv 1 \pmod 4$. The integers in this equivalence class generates

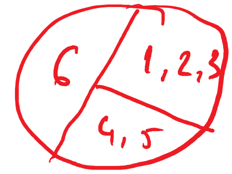
$m \mid (a-b)$
$\Updownarrow$
$a \bmod m = b$

integers a ...
integers in this equivalence class generates 1 as the remainder, when divided by 4.

$[1] : \{ \ldots -7, -3, 1, 5, 9, \ldots \}$.

Prob-6

$S = \{1, 2, 3, 4, 5, 6\}$



$A_1 = \{1, 2, 3\}$; $A_2 = \{4, 5\}$; $A_3 = \{6\}$

A_1, A_2, A_3
(subsets are equivalence classes of R)

The subsets in the partition are the equivalence classes of R. The pair $(a, b) \in R$ iff a and b are in the same subset of the partition. The pairs $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)$ belong to R because $A_1 = \{1, 2, 3\}$ is an equivalence class.

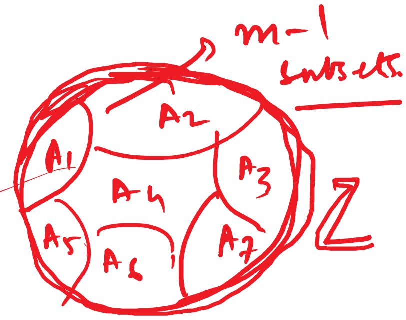
The pairs $(4, 4), (4, 5), (5, 5), (5, 4)$ belong to R because $A_2 = \{4, 5\}$ is an equivalence relation.

(reflexive, symmetric, transitive hold)

Finally, the pair $(6, 6) \in R$ because $A_3 = \{6\}$ is an equivalence class.

Prob.

Partition created by the equivalence relation congruence ... will have m ...



modulo-m will have 'm' subsets i.e. m equivalence classes.

R | Congruence modulo m

$[0]_m =$

$[1]_m =$

$\vdots$

$[m-1]_m =$

Congruence modulo-4 $\Rightarrow [0]_4$, $[1]_4$, $[2]_4$, and $[3]_4$.

$$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$$

$$[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$$

$$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$$

$$[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$$

Prob.

Because $a|a$ for every integer $a \in Z^+$. "$|$" is reflexive.

If $a|b$ and $b|a$, then $a = b$. Hence, "$|$" is anti-symmetric.

If $a|b$ and $b|c$, it implies $a|c$. Thus, "$|$" is transitive. It follows that "$|$" is a partial ordered reln. and the divisibility relation on the set of positive integers $(Z^+, |)$ is a poset.

$a|b \Rightarrow b = k.a$

$b|c \Rightarrow c = l.b$

$c = (kl).a$

$\Rightarrow a|c.$

Proof:

Because $A \subseteq A$ whenever A is a subset of S. Therefore $\subseteq$ is reflexive.

Because $A \subseteq B$ and $B \subseteq A$ imply $A = B$. Therefore, $\subseteq$ is anti-symmetric.

Because $A \subseteq B$ and $B \subseteq C$ imply $A \subseteq C$. Therefore, $\subseteq$ is transitive.

Hence, we say that $\subseteq$ relation is partially ordered on $P(S)$ and $(P(S), \subseteq)$ is a poset.