Proof by induction - 1

Monday, November 9, 2020 10:54 AM

Prob-3

$P(n)$: $n < 2^n$, for all positive integers n.

Basis step: for $n = 1$, $1 < 2^1$ i.e. $P(1)$ is true.

Inductive step: If $P(k)$ is true then $P(k+1)$ is true for any arbitrary positive integer k.

Suppose, $P(k)$: $k < 2^k$ is true (inductive hypothesis)

Show $P(k+1)$: $k+1 < 2^{k+1}$

$k < 2^k$ (by inductive hypothesis $P(k)$)

$\Rightarrow k+1 < 2^k + 1$

$\Rightarrow k+1 < 2^k + 2^k$

$= k+1 < 2^k \cdot 2$

$= k+1 < 2^{k+1} \Rightarrow P(k+1)$

Prob-4.

$P(n)$: $n^3 - n$ is divisible by 3 for all positive integers

Basic step: $P(1)$: $1^3 - 1 = 0$ is divisible by 3.

Inductive step: If $P(k)$ is true then $P(k+1)$ is also true for any arbitrary positive integer k.

true for any arbitrary positive integer k.

Suppose $P(k)$: $k^3 - k$ is divisible by 3 is true.
(inductive hypothesis)

Show $P(k+1) = (k+1)^3 - (k+1)$ is also divisible by 3.

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + \cancel{1} - k - \cancel{1}$$
$$= \underbrace{\underbrace{(k^3 - k)}_{\substack{\text{divisible by} \\ 3.}} + \underbrace{3(k^2 + k)}_{\substack{\text{divisible by 3} \\ (\text{If } a|b, \text{ then} \\ a|bc \\ \text{for all integer} \\ c)}}}_{\text{divisible by 3}}$$

(If $a|b$ and $a|c$, then $a|b+c$)

Prob-5

Use mathematical induction to show that $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$, for all non-negative integer n.

$P(n)$: $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$ for the integer n

Basis step: $P(0)$: $2^0 = 1 = 2^{0+1} - 1$ is true.

Inductive step: If $P(k)$ is true for any arbitrary non-negative integer k, then $P(k+1)$ is also true.

$P(k): 1+2+2^2+\dots+2^k = 2^{k+1}-1.$

(inductive hypothesis)

Show $P(k+1) = 1+2+2^2+\dots+2^{k+1} = 2^{k+2}-1$

$$1+2+2^2+\dots+2^k+2^{k+1} = (1+2+2^2+\dots+2^k)+2^{k+1}$$
$$\overset{IH}{=} (2^{k+1}-1)+2^{k+1}$$
$$= 2.2^{k+1}-1$$
$$= 2^{k+2}-1$$

This completes the proof by induction.

Prob-6. Use mathematical induction to prove that $2^n < n!$ for every integer $n \geq 4$.

positive

non-negative

($P(n): 2^n < n!$ is false for $n = 1, 2, 3$)

Basis step: $P(4): 2^4 < 4!$ is true as

$$16 < 24.$$

Inductive step: If $P(k)$ is true for any arbitrary integer $k \geq 4$, then $P(k+1)$ is also true.

Suppose $P(k): 2^k < k!$

Show $P(k+1): 2^{k+1} < (k+1)!$

We have:

$$2^{k+1} = 2.2^{k} \text{ (by the definition of exponent)}$$

$$< 2.k! \text{ (by inductive hypothesis } 2^k < k!\text{)}$$

$$< (k+1).k! \text{ (as, } 2 < k+1\text{)}$$

$$< (k+1)! \text{ (by the defn. of factorial)}$$

This shows that $P(k+1)$ is true when $P(k)$ is true. this completes the inductive step of the proof.

Prob-7. Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

$P(n): 7^{n+2} + 8^{2n+1}$ is divisible by 57.

Basis step: $P(0): 7^{0+2} + 8^{0+1} = 7^2 + 8 = 49+8 = 57$

$P(0)$ is true because 57 is divisible by 57.

Inductive Step: If $P(k)$ is true for any arbitrary non-negative integer k, then $P(k+1)$ is also true.

Suppose $P(k)$: $7^{k+2} + 8^{2k+1}$ is divisible by 57.
(inductive hypothesis)

Show $P(k+1)$: $7^{(k+1)+2} + 8^{2(k+1)+1}$ is divisible by 57.

We have,

$$7^{(k+1)+2} + 8^{2(k+1)+1}$$
$$= 7^{k+3} + 8^{2k+3}$$
$$= 7.7^{k+2} + 8^2.8^{2k+1}$$
$$= 7.7^{k+2} + 64.8^{2k+1}$$
$$= 7.7^{k+2} + 7.8^{2k+1} + 57.8^{2k+1}$$
$$= \underbrace{\underbrace{7(7^{k+2} + 8^{2k+1})}_{\substack{\text{divisible by 57} \\ \text{(inductive hypothesis)}}} + \underbrace{57.8^{2k+1}}_{\substack{\text{divisible by 57} \\ \text{(if } a|b, \text{ then} \\ a|bc \text{ for all} \\ \text{integer } c)}}}_{\substack{\text{divisible by 57} \\ \text{(if } a|b \text{ and } a|c, \text{ then } a|b+c)}}$$

We conclude that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

Prob-8

The harmonic numbers H_j, for $j = 1, 2, 3 \ldots$ are defined

Prob-8

The harmonic numbers H_j, for $j = 1, 2, 3 \ldots$ are defined by $H_j = 1 + 1/2 + 1/3 + \ldots + 1/j$.

Use mathematical induction to show that

$H_{2^n} \geq 1 + n/2$, whenever n is a nonnegative integer.

$\rightarrow$ $P(n) : H_{2^n} \geq 1 + n/2$ for any non-negative integer n.

Basis step: $P(0) : H_{2^0} \geq 1 + 0/2$ i.e. $H_1 = 1 \geq 1$.

Inductive step: If $P(k)$ is true for any arbitrary non-negative integer k, then $P(k+1)$ is also true.

Suppose, $P(k) : H_{2^k} \geq 1 + k/2$ (inductive hypothesis)

Show, $P(k+1) : H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$

We have,

$$H_{2^{k+1}} = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^k}\right) + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \ldots + \frac{1}{2^{k+1}}$$

$$\Rightarrow H_{2^{k+1}} = H_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \ldots + \frac{1}{2^{k+1}}$$ (by inductive hypothesis)

2^{k+1} (hypothesis)

$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$

$2^2 = 4$ terms.

each $\geq 1/8$

$\frac{1}{2^k+1}+\frac{1}{2^k+2}+\frac{1}{2^k+3}+\dots+\frac{1}{2^{k+1}}$

$2^k \Rightarrow$ terms.

$$\Rightarrow H_{2^{k+1}} \geq (1+k/2) + \left[\frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^{k+1}}\right]$$

$$\geq (1+k/2) + 2^k \cdot \frac{1}{2^{k+1}}$$ (because there are 2^k terms and each term $\geq \frac{1}{2^{k+1}}$)

$$\geq (1+k/2) + \frac{1}{2}$$

$$\geq \left(1+\frac{k+1}{2}\right)$$

This completes the inductive step of the proof.

Prob-9

Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.

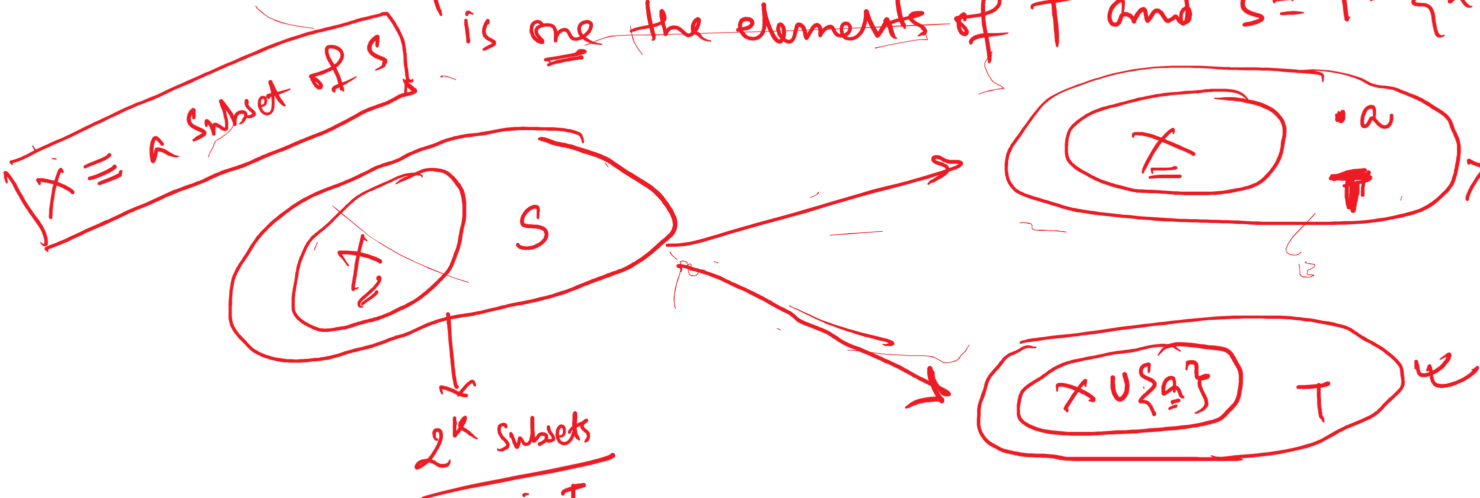
$P(n)$: a set with n elements has 2^n subsets.

Basis step: $P(0)$ is true, because a set with zero elements, the empty set, has exactly $2^0 = 1$ subsets, i.e. itself.

Inductive step: If $P(k)$ is true for any arbitrary nonnegative integer k, then $P(k+1)$ is also true.

Suppose $P(k)$: a set with k elements has 2^k subsets.
Show $P(k+1)$: a set with $(k+1)$ elements has 2^{k+1} subsets.

Let T is a set with $(k+1)$ elements. It is possible to write $T = S \cup \{a\}$, where a is one the elements of T and $S = T - \{a\}$.



Two subsets in T for each subset in S:
$= 2 \cdot 2^k$ subsets possible subsets in T
$\Rightarrow 2^{k+1}$ subsets.

For each subset X of S, there are exactly two subsets in T, namely X and $X \cup \{a\}$. This constitutes that all subsets of T are distinct. By inductive hypothesis, we have 2^k subsets for S which has k elements.

$\Rightarrow$ Therefore, there are $2 \cdot 2^k = 2^{k+1}$ subsets of T which has $(k+1)$ elements. This completes the inductive step.

Prob-10, Use mathematical induction to prove one

De Morgan's law
w, $\overline{A_1 \cap A_2} = \bar{A_1} \cup$

Prob-10 Use mathematical induction to prove one of De Morgan's laws:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever $A_1, A_2, \ldots A_n$ are subsets of a universal set U and $n \geq 2$.

$\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$

$\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$

Basis step: $P(2)$ asserts $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$. This is one of De Morgan's laws. Thus, $P(2)$ is true.

Inductive step: If $P(k)$ is true for any arbitrary integer $k \geq 2$, then $P(k+1)$ is also true.

Suppose $P(k)$: $\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j}$ (inductive hypothesis)

Show $P(k+1)$: $\overline{\bigcap_{j=1}^{k+1} A_j} = \bigcup_{j=1}^{k+1} \overline{A_j}$

We have,

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}}$$ (by defn. of intersection)

$j=1$ intersection)

$$= \left(\bigcap_{j=1}^{k} \overline{A_j}\right) \cup \overline{A_{k+1}}$$ (by applying De Morgan's law)

$$= \left(\overline{\bigcup_{j=1}^{k} A_j}\right) \cup \overline{A_{k+1}}$$ (by using inductive hypothesis)

$$= \overline{\bigcup_{j=1}^{k+1} A_j}$$ (by the defn. of union)

This completes the inductive step.