Basic Discrete Structures

Sets, Functions, Sequences, Matrices, and Relations

(Lecture – 1)

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Sets

- It is the fundamental discrete structure on which all other discrete structures are built.
- Sets are used to group objects together. Often, but not always, the objects in a set have similar properties.
- Examples: all the students who are currently enrolled in your college make up a set; all the students currently taking a course in discrete mathematics at any college make up a set, etc.
- The language of sets is a means to study such collections in an organized fashion.

A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

Standard Sets & Intervals

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**
$\mathbf{R}$, the set of **real numbers**
$\mathbf{R}^+$, the set of **positive real numbers**
$\mathbf{C}$, the set of **complex numbers**.

- Note that some sources do not consider 0 a natural number
- According to them, $\{0, 1, 2, 3, \ldots..\}$ is a set of *whole numbers*
- Intervals of real numbers: when a and b are real numbers with $a < b$, we write
  - $[a, b] = \{x \mid a \leq x \leq b\}$
  - $[a, b) = \{x \mid a \leq x < b\}$
  - $(a, b] = \{x \mid a < x \leq b\}$
  - $(a, b) = \{x \mid a < x < b\}$
- **[a, b]** is called the *closed interval* from a to b and **(a, b)** is called the *open interval* from a to b.

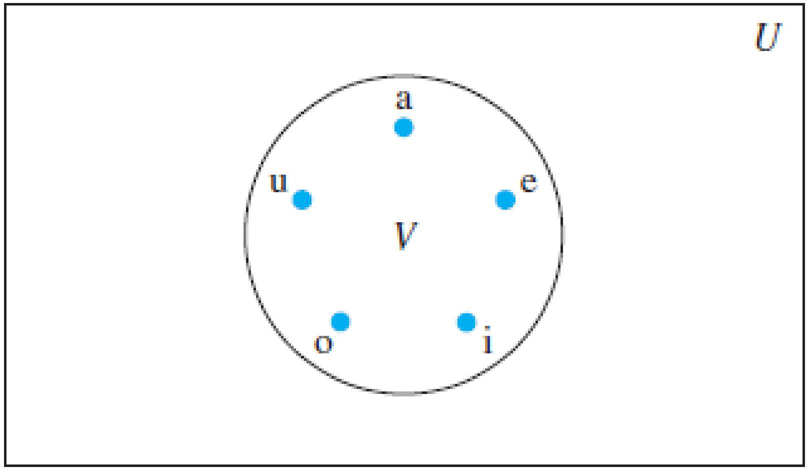
Definitions related to Sets

Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

- **EMPTY SET**: Special set that has no elements. It is also called null set, and is denoted by ∅. The empty set can also be denoted by { }.
  - Often, a set of elements with certain properties turns out to be the null set. For instance, the set of all positive integers that are greater than their squares is the null set.
- **SINGLETON SET**: A set with one element

Venn Diagrams

- Tool for representing sets in graphical form
- **Universal set** *U*: contains all the objects under consideration (represented by a rectangle).
  - The universal set varies depending on which objects are of interest.
- Inside this rectangle, circles or other geometrical figures are used to represent sets.
- Points are used to represent the
- particular elements of the set.
- Venn diagrams are often used to indicate the relationships between sets



Subsets

The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

Showing that A is a Subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a Subset of B To show that $A \not\subseteq B$, find a single $x \in A$ such that $x \notin B$.

- If set A is a subset of a set B but that $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B.
- For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A.

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

The Size of a Set, Power Set, Cartesian Product

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

A set is said to be *infinite* if it is not finite.

- Example: set of positive integers

Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

- Cartesian Products

The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, $\ldots$, and a_n as its nth element.

Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

Cartesian Product of More than Two Sets

The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words,

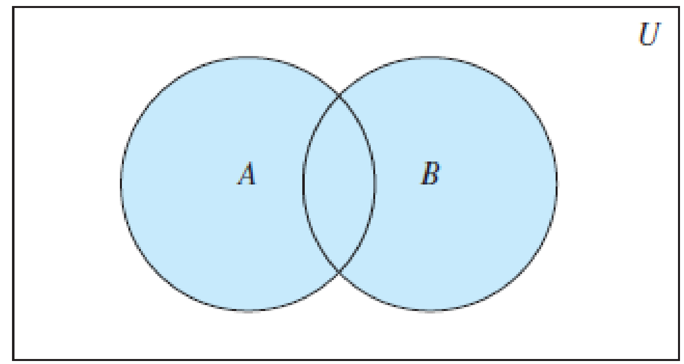
$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

- A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B.
- The elements of R are ordered pairs, where the first element belongs to A and the second to B.
  - Example: $R = \{(a, 0), (a, 1), (a, 3), (b, 1), (b, 2), (c, 0), (c, 3)\}$ is a relation from the set $\{a, b, c\}$ to the set $\{0, 1, 2, 3\}$.
- A relation from a set A to itself is called a relation on A.

Set Operations

Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

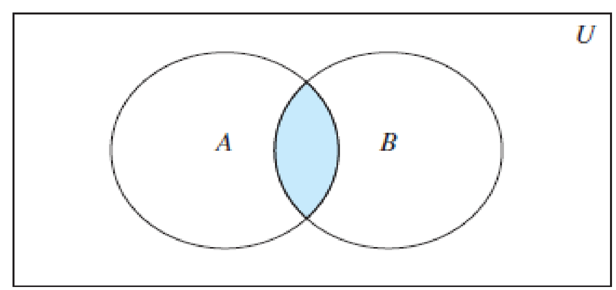
- $A \cup B = \{x \mid x \in A \lor x \in B\}$.



$A \cup B$ is shaded.

Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

- $A \cap B = \{x \mid x \in A \land x \in B\}$.

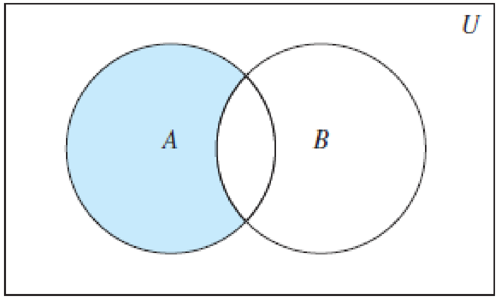


$A \cap B$ is shaded.

Set Operations

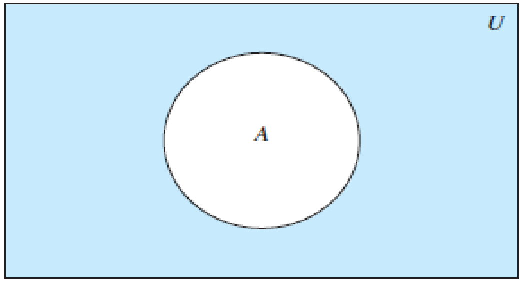
Two sets are called *disjoint* if their intersection is the empty set.

Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*.



$A - B$ is shaded.

Let U be the universal set. The *complement* of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.



$\overline{A}$ is shaded.

Set Identities

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws