Graphs & Trees (Lecture – 4)

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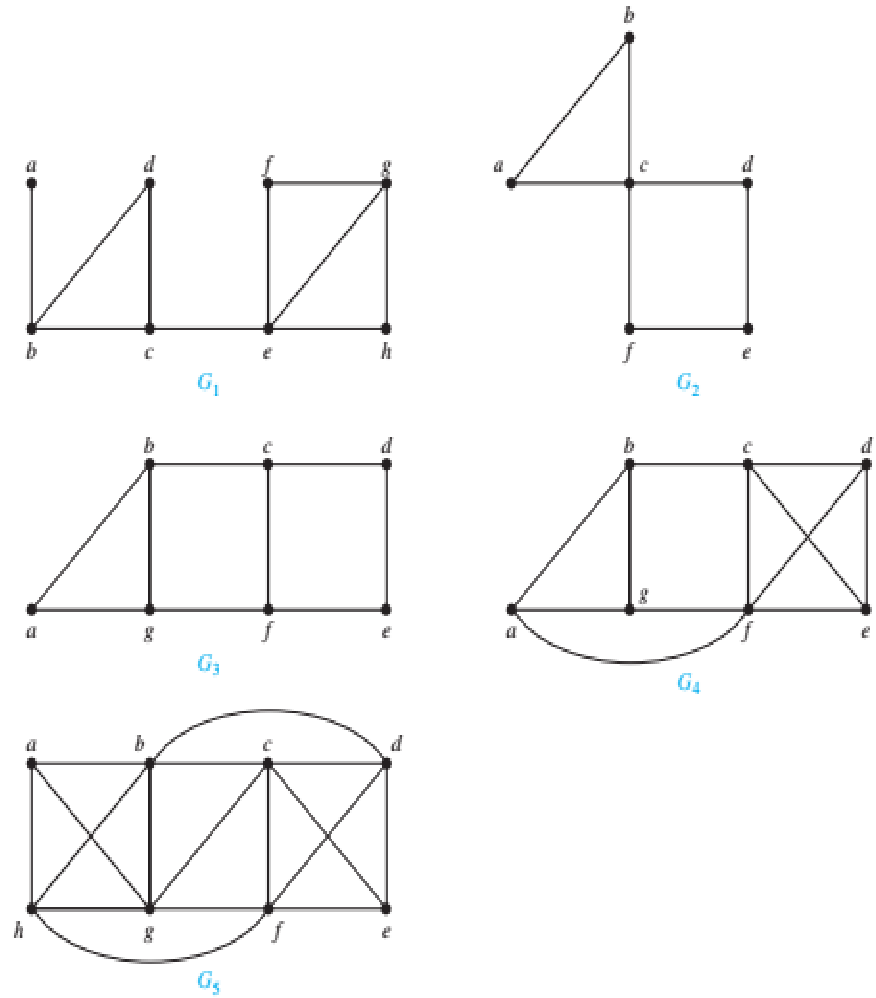
Connectivity (*Contd…*)

- How connected in a graph?
  - **Cut vertex**: removal of a cut vertex from a connected graph produces a subgraph that is not connected.
  - **Cut edge or bridge**: an edge whose removal produces a graph with more connected components than in the original graph.
    - Example: In a computer network, a cut vertex is an essential router and cut edge is an essential link that cannot fail for a computer network to be operational.
- **Vertex cut**: A subset *V'* of the vertex set *V* of $G = (V, E)$ is a *vertex cut, or separating set*, if $G - V'$ is disconnected.
  - For complete graph, no minimum vertex cut exist.
  - ***Vertex connectivity*** of a noncomplete graph *G*, denoted by $\kappa(G)$, is the minimum number of vertices in a ***vertex cut***

Connectivity (*Contd…*)

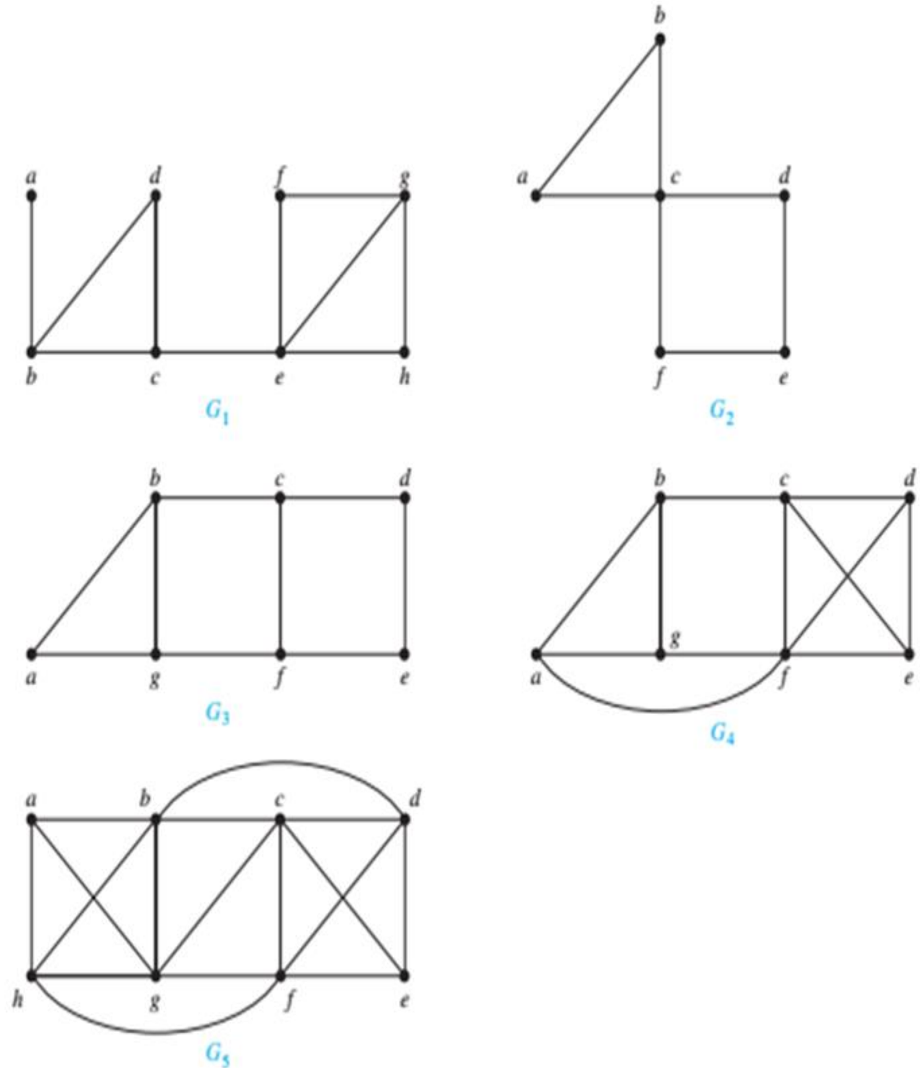
- In simple term, *vertex connectivity* of a noncomplete graph G, denoted by $\kappa(G)$, is the minimum number of vertices that can be removed from G to either disconnect G or produce a graph with a single vertex
- The larger $\kappa(G)$ is, the more connected G to be
- <u>k-connected</u>: a graph is k-connected (or k-vertex-connected), if $\kappa(G) \geq k$.
- If G is a k-connected graph, then G is also j-connected graph for all j with $0 \leq j \leq k$.
- Find the vertex connectivity of each of the following graphs.



Connectivity (*Contd...*)

- Edge Connectivity:
  - If a graph has a *cut edge*, then we need to remove it only to disconnect *G*.
  - If *G* does not have a *cut edge*, we look for the *smallest* set of edges that can be removed to disconnect it.
  - **<u>Edge cut</u>**: A set of edges *E'* is called an *edge cut* of *G* if the subgraph *G - E'* is disconnected.
  - **<u>Edge connectivity</u>**: The edge connectivity of a graph *G*, denoted by $\lambda(G)$, is the minimum number of edges in an edge cut of *G*.

- Find the edge connectivity of each of the following graphs.

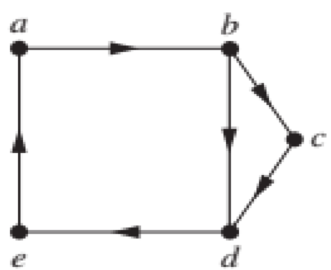


Connectedness in Directed Graphs

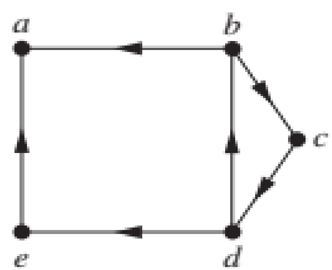
A directed graph is *strongly connected* if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graph.

- A directed graph is weakly connected if and only if there is always a path between two vertices when the directions of the edges are disregarded.
- Clearly, any strongly connected directed graph is also weakly connected.
- Are the directed graphs G and H shown below are strongly connected? Are they weakly connected?



G

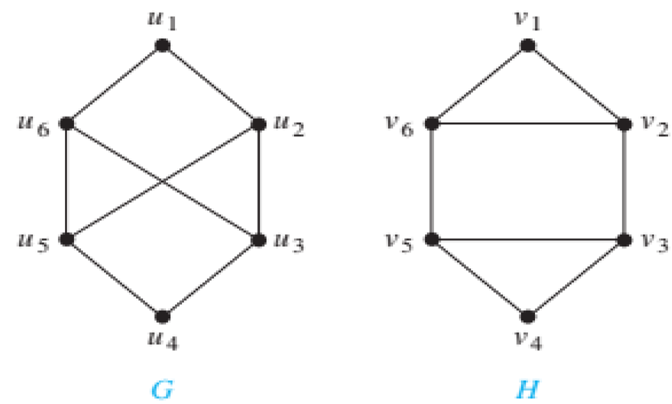


H

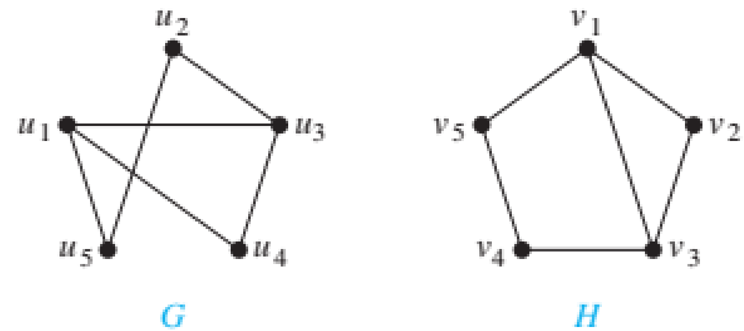
- Strongly Connected Components (SCC)
  - The maximal strongly connected subgraphs, are called the *strongly connected components* or *strong components* of G.
  - **Definition**: A strongly connected component of a directed graph $G = (V, E)$ is a maximal set $\mathcal{V} \subseteq V$ such that for each pair of vertices u and v in $\mathcal{V}$, we have $u \rightarrow v$ and $v \rightarrow u$, i.e. vertices u and v are both reachable from each other.

Paths and Isomorphism

- Paths and circuits can help determine whether two graphs are isomorphic.
- The existence of a simple circuit of a particular length can be used to show that two graphs are not isomorphic.
- A useful isomorphic invariant for simple graphs is the existence of a simple circuit of length k, where k is a positive integer greater than 2.



Determine if the graphs *G* and *H* are isomorphic.



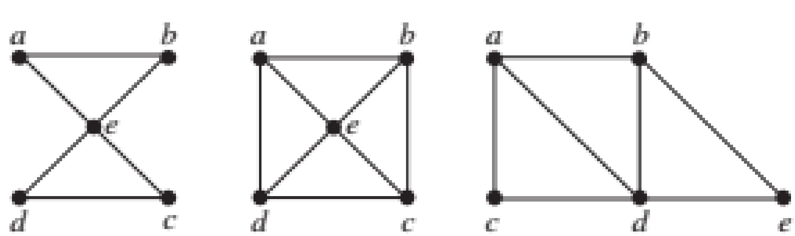
Determine if the graphs *G* and *H* are isomorphic.

Euler Circuits & Paths

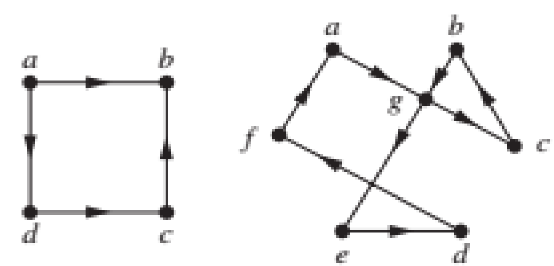
- <u>Euler Circuit and Path</u>:
  - Travel along the edges of a graph starting at a vertex and returning to it by traversing each edge of the graph exactly once.

An *Euler circuit* in a graph G is a simple circuit containing every edge of G. An *Euler path* in G is a simple path containing every edge of G.

- Which of the undirected graphs have an Euler circuit? Of those that do not, which have an Euler path?
- Which of the directed graphs have an Euler circuit? Of those that do not, which have an Euler path?



G_1 G_2 G_3



H_1 H_2 H_3

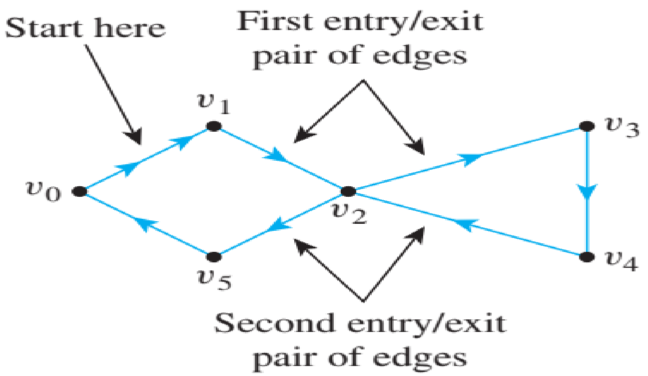
Euler Circuits & Paths (*Contd…*)

Theorem 10.2.2

If a graph has an Euler circuit, then every vertex of the graph has positive even degree.

Proof:

Suppose G is a graph that has an Euler circuit. *[We must show that given any vertex v of G, the degree of v is even.]* Let v be any particular but arbitrarily chosen vertex of G. Since the Euler circuit contains every edge of G, it contains all edges incident on v. Now imagine taking a journey that begins in the middle of one of the edges adjacent to the start of the Euler circuit and continues around the Euler circuit to end in the middle of the starting edge. (See Figure 10.2.3. There is such a starting edge because the Euler circuit has at least one edge.) Each time v is entered by traveling along one edge, it is immediately exited by traveling along another edge (since the journey ends in the *middle* of an edge).



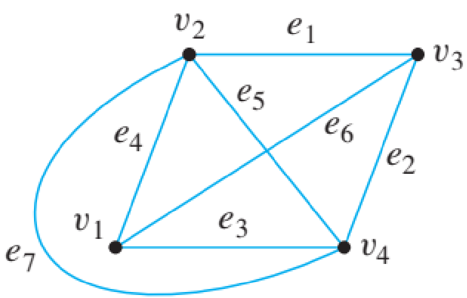
In this example, the Euler circuit is $v_0 v_1 v_2 v_3 v_4 v_5 v_0$, and v is v_2. Each time v_2 is entered by one edge, it is exited by another edge.

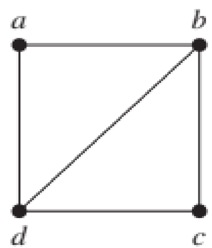
Figure 10.2.3 **Example for the Proof of Theorem 10.2.2**

Euler Circuits & Paths (*Contd…*)

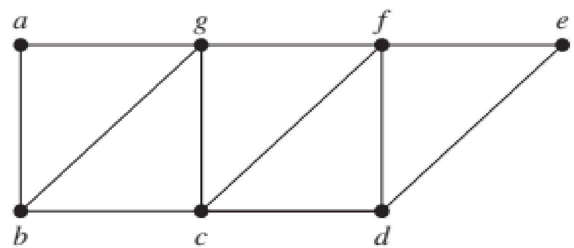
Contrapositive Version of Theorem 10.2.2

If some vertex of a graph has odd degree, then the graph does not have an Euler circuit.

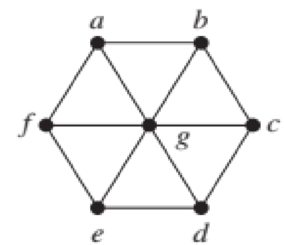




G_1



G_2



G_3

- Theorem:

A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Euler Circuits & Paths (*Contd…*)

- Applications of Euler paths and circuits:
  - Many applications ask for a path or circuit that traverses each street in a neighborhood, each road in a transportation network, each connection in a utility grid, or each link in a communications network exactly once.
  - Finding an Euler path or circuit in the appropriate graph model can solve such problems.
  - Example: if a postman can find an Euler path in the graph that represents the streets he needs to cover, it produces a route that traverses each street of the route exactly once.
  - If no Euler path exists, some streets will have to be traversed more than once.

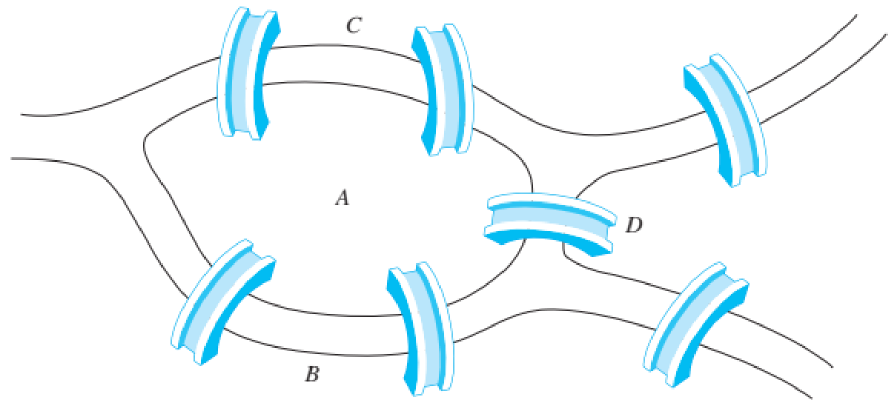


FIGURE 1 **The Seven Bridges of Königsberg.**

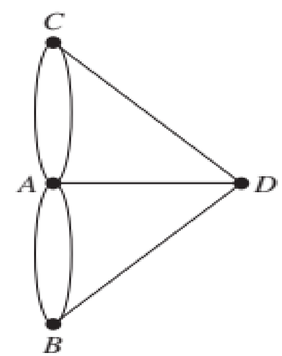


FIGURE 2 **Multigraph Model of the Town of Königsberg.**

Does this multigraph contain Euler circuits and paths?

Hamilton Paths & Circuits

- Definition:

A simple path in a graph G that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph G that passes through every vertex exactly once is called a *Hamilton circuit*. That is, the simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path.

- Which of the following simple graphs have a Hamilton circuit or, if not, a Hamilton path?