Solution of Ordinary Differential Equation by use of Laplace Transform

Ex: Solve: $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4e^{2t}$,

given $y(0) = -3,\ y'(0) = 5$.

Solution:

We take Laplace Transform on both sides. By linearity property,

$$L[y''] - 3L[y'] + 2L[y] = 4L[e^{2t}].$$

Now we have the general formula:

$$L[F^n(x)] = p^n L[F(x)] - p^{n-1}\, F(0) - \cdots - pF^{n-2}(0) - F^{n-1}(0).$$

Applying the above formula

$$\{p^2 L[y] - py(0) - y'(0)\} - 3\{pL[y] - y(0)\} + 2L[y] = \frac{4}{p-2}$$

or, $\quad (p^2 - 3p + 2)L[y] = \frac{4}{p-2} - 3p + 14$

or, $\quad L[y] = \frac{-3p^2+20p-24}{(p-2)^2\,(p-1)}$

or, $\quad L[y] = -\frac{7}{p-1} + \frac{4}{p-2} + \frac{4}{(p-2)^2}.$

Therefore,

$$y(t) = -7L^{-1}\left[\frac{1}{p-1}\right] + 4L^{-1}\left[\frac{1}{p-2}\right] + 4L^{-1}\left[\frac{1}{(p-2)^2}\right]$$

$$= -7e^{-t} + 4e^{2t} + 4te^{2t}.$$

Thus, the required solution is $y(t) = -7e^{-t} + 4e^{2t} + 4te^{2t}.$

Ex: Solve $\{tD^2 + (1-2t)D - 2\}\, y = 0,\ D \equiv \frac{d}{dt}$, given $y(0) = 1$ and $y'(0) = 2$.

Solution: The equation is $ty'' + y' - 2ty' - 2y = 0$,

where $y'' \equiv D^2 y$ and $y' \equiv Dy$.

Taking Laplace Transform on both sides, it becomes

$$L[ty''] + L[y'] \ - 2L[ty'] - 2L[y] = 0$$

or, $$-\frac{d}{dp}L[y''] + L[y'] + 2\frac{d}{dp}L[y'] - 2L[y] = 0$$

or, $$-\frac{d}{dp}\{p^2L[y] - py(0) - y'(0)\} \ + \{pL[y] - y(0)\} +2\frac{d}{dp}\{pL[y] - y(0)\} - 2L[y] = 0$$

or, $$-\frac{d}{dp}\{p^2L[y] - p - 2\} + \{pL[y] - 1\} +2\frac{d}{dp}\{pL[y] - 1\} - 2L[y] = 0$$

or, $-(p^2 - 2p)\frac{dL[y]}{dp} - pL[y] = 0$

or, $\frac{dL[y]}{L[y]} + \frac{1}{p-2}dp = 0.$

Integrating, we get

$$\ln L[y] + \ln(p - 2) = \ln C_1$$

Or, $L[y] = \frac{C_1}{p-2}.$

Therefore, $$y(t) = C_1 L^{-1}\left\{\frac{1}{p-2}\right\} = C_1 e^{2t}.$$

From the given condition $y(0) = 1$, we get $y(t) = e^{2t}$.