Matrix I

1.1 INTRODUCTION

Matrix algebra provides us an integral part of the mathematical background necessary for various fields of engineering, sciences and technology besides diverse fields of mathematics, physics and others. Matrices originated as mere stores of information but now it is found very wide application. Matrix theory has an important relationship with systems of linear equations which occur in many engineering processes. The state-space representation of linear system models in applied systems engineering and control systems involve matrices. In this chapter, basic properties of matrices have been presented.

1.2 DEFINITION OF MATRIX

A matrix is defined as an array of mn numbers, real or complex, in m rows and n columns and is written in the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{m\times n}$$

or

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{m\times n}$$

It is called a matrix of order $m \times n$ or an $m \times n$ matrix and read as m by n matrix. The numbers a_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) are called the elements of the matrix, a_{ij} being the element lying at the junction of ith row and jth column.

Note: (*i*) If the elements a_{ij} are all real numbers then we say that this matrix A is over the real field. If the elements a_{ij} are all complex numbers then we say that this matrix A is over the complex field.

(*ii*) Matrix is an array of numbers, it is not a number.

(*iii*) $(a_{ij})_{m\times n}$ or (a_{ij}) or $[a_{ij}]_{m\times n}$ or $[a_{ij}]$ represents short form of matrix.

* This chapter is not included in the WBUT syllabus but is essential for the study.

Equality of matrices

Two matrices $(a_{ij})_{m \times n}$ and $(b_{ij})_{p \times q}$ are said to be equal if and only if they are of same order, *i.e.*, $m = p$, $n = q$ and $a_{ij} = b_{ij}$, for all i, j.

Example: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 5 & 6 \end{pmatrix}$, then $A \neq B$.

But if $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, then $A = B$.

1.3 DEFINITIONS OF VARIOUS TYPES OF MATRICES

1. Rectangular matrix

A matrix $A = (a_{ij})_{m \times n}$ is said to be rectangular if the number of rows is not equal to the number of columns, *i.e.*, if $m \neq n$.

Example: $\begin{pmatrix} 5 & 1 & 2 \\ 4 & 3 & 0 \end{pmatrix}$ is a rectangular matrix having two rows and three columns.

2. Square matrix

A matrix $A = (a_{ij})_{m \times n}$ is said to be a square matrix if the number of rows and columns are equal, *i.e.*, if $m = n$.

A square matrix $A = (a_{ij})_{n \times n}$ of order $n \times n$ is said to be a nth order square matrix and the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are said to form the principal diagonal of this matrix. These elements $a_{11}, a_{22}, \ldots, a_{nn}$ are known as the diagonal elements of A.

Example: $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 8 & 4 \\ 7 & 5 & 9 \end{pmatrix}$ is a 3rd order square matrix.

Here 1, 8, 9 are the diagonal elements which are lying in the principal diagonal of this matrix.

3. Row matrix or row vector

A matrix of n elements arranged in one row only is called a row matrix or a row vector, *i.e.*, it is an $1 \times n$ matrix.

Example: $(1\ 4\ 2)$ is a row matrix.

4. Column matrix or column vector

A matrix of m elements arranged in one column only is called a column matrix or a column vector, *i.e.*, it is an $m \times 1$ matrix.

Example: $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ is a column matrix.

5. Null matrix or zero matrix

A matrix $(a_{ij})_{m\times n}$ is said to be a null matrix or zero matrix if $a_{ij} = 0$, for all $i = 1, \ldots, m$ and $j = 1, 2, \ldots, n$. A null matrix of order $m \times n$ is denoted by $O_{m\times n}$.

Example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{3\times 2}$, $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = O_{2\times 4}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_{3\times 3}$.

6. Diagonal matrix

A square matrix in which all non-diagonal elements are zero is called a diagonal matrix, *i.e.*, $D = (d_{ij})_{n\times n}$ is said to be a diagonal matrix if $d_{ij} = 0$ for $i \neq j$; $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$. It is briefly written as diag. $(d_1, d_2, \ldots, d_n)$, where $d_1, d_2, \ldots, d_n$ are the diagonal elements.

Example: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$, $\begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are diagonal matrices.

7. Scalar matrix

A diagonal matrix, *i.e.*, all non-diagonal elements are zero, in which all the diagonal elements are equal to a scalar, say k, is called a scalar matrix.

Therefore, $A = (a_{ij})_{n\times n}$ is a scalar matrix if

$$a_{ij} = \begin{cases} 0, \text{when } i \neq j \\ k, \text{when } i = j \end{cases}$$

Example: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ are scalar matrices.

8. Unit matrix or identity matrix

A scalar matrix in which each diagonal element is unity (*i.e.*, 1) is said to be a unit matrix or identity matrix.

Therefore, a unit matrix is a square matrix in which all non-diagonal elements are zero and all diagonal elements are equal to 1, *i.e.*, $A = (d_{ij})_{n\times n}$ is a unit matrix if

$$a_{ij} = \begin{cases} 0, \text{when } i \neq j \\ 1, \text{when } i = j \end{cases}$$

A unit matrix of order n is denoted by I_n and if the order is evident, it may be simply denoted by I.

Example: $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

1.4. ALGEBRAIC OPERATIONS ON MATRICES

1. Multiplication by a scalar (or scalar multiplication)

The product (or multiplication) of a matrix $A = (a_{ij})_{m \times n}$ by a scalar c is a matrix defined by $cA = (ca_{ij})_{m \times n}$ (*i.e.*, a matrix whose elements are c times the corresponding elements of A).

Example: If $A = \begin{pmatrix} 1 & 3 & 2 \\ -5 & 0 & 8 \end{pmatrix}$, then $2A = \begin{pmatrix} 2 & 6 & 4 \\ -10 & 0 & 16 \end{pmatrix}$

Negative of a matrix $A = (a_{ij})$ is defined as $-A = (-a_{ij})_{m \times n}$.

Example: If $A = \begin{pmatrix} 1 & 0 \\ -2 & 3 \\ 5 & -4 \end{pmatrix}$, then $-A = \begin{pmatrix} -1 & 0 \\ 2 & -3 \\ -5 & 4 \end{pmatrix}$.

Properties

Let $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n}$ and c, d are scalars. Then the following results hold good.

(*i*) $c(dA) = (cd)A$

(*ii*) $oA = O_{m \times n}$

(*iii*) $c\,O_{m \times n} = O_{m \times n}$

(*iv*) $cI_n = \begin{pmatrix} c & 0 & 0 & \dots & 0 \\ 0 & c & 0 & \dots & 0 \\ 0 & 0 & c & \dots & 0 \\ 0 & 0 & 0 & \dots & c \end{pmatrix}$

(*v*) $1.A = A$

(*vi*) $cA = B$ if and only if $A = \frac{1}{c}B$, provided $c \neq 0$.

2. Addition of matrices

Two matrices A and B are said to be conformable (or meaningful) for addition if they have the same order.

If $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ then their sum $A + B$ is defined as $A + B = (a_{ij} + b_{ij})_{m \times n}$.

Example: $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix}$

3. Subtraction of matrices

Two matrices A and B are said to be conformable (or meaningful) for subtraction if they have the same order.

If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ then $A - B$ is defined as $A - B = (a_{ij} - b_{ij})_{m\times n}$.

Example: $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \end{pmatrix}$

Note: If A and B are matrices of different orders, then $A + B$, $A - B$ are not defined.

4. Multiplication of matrices

Two matrices A and B are said to be conformable (or meaningful) for the product AB if the number of columns of A is equal to the number of rows of B. If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$ then the product AB is a matrix C of order $m \times p$ defined by $AB = C = (c_{ij})_{m\times p}$, where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

The ijth element, *i.e.*, the element lying at the junction of ith row and jth column, of AB is obtained by adding the products of the corresponding elements of ith row of A and jth column of B. Therefore matrix multiplication is a row-by-column multiplication.

Note: If the number of columns of A is not equal to the number of rows of B, then AB is not defined.

Example: $\begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 6 & 5 \end{pmatrix}_{3\times 2} \times \begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 1\times1+4\times5 & 1\times2+4\times1 \\ 2\times1+3\times5 & 2\times2+3\times1 \\ 6\times1+5\times5 & 6\times2+5\times1 \end{pmatrix} = \begin{pmatrix} 21 & 6 \\ 17 & 7 \\ 31 & 17 \end{pmatrix}$

1.5 LAWS OF ALGEBRAIC OPERATIONS OF MATRICES

1. Commutative law of addition

If A and B are two matrices such that $A + B$ is defined then $A + B = B + A$.

2. Associative law of addition

If A, B, C are conformable for addition then $(A+B)+C = A+(B+C)$.

3. Distributive law for scalar multiplication on matrix addition

If A, B are conformable for addition then $k(A+B) = kA+kB$, where k is any number.

4. Additive law with null matrix (or zero matrix)

If A, O are matrices of same order, then $A + O = O + A = A$.

5. $A - A = O$, for any matrix A, where A, O are of same order.

6. Matrix multiplication is non-commutative

If A, B are two matrices, even if AB and BA are defined, $AB \neq BA$, in general.

(*W.B.U.T. 2005*)

Note: In some cases $AB = BA$.

Example: (*i*) Let us take

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \; B = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}$$

Then
$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 11 \\ 22 & 23 \end{pmatrix}$$

and
$$BA = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 19 & 28 \end{pmatrix}$$

$\therefore \quad AB \neq BA.$

(*ii*) If we take
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}, \text{ then}$$

$$AB = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}$$

and
$$BA = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}$$

$\therefore \quad AB = BA.$

7. Associative law for multiplication

If A, B, C are three matrices such that the products $BC, A(BC), AB, (AB)C$ are defined, then $A(BC) = (AB)C$.

8. Distributive law for multiplication with respect to matrix addition

If A, B, C are three matrices, then

(*i*) $A(B+C) = AB + AC$, provided both sides are defined

(*ii*) $(B+C)A = BA + CA$, provided both sides are defined.

9. Multiplication with identity matrix

(*i*) If A is a square matrix and I is an identity matrix such that AI is defined, then $AI = IA = A$.

(*ii*) If A is any matrix and I is an identity matrix such that AI is defined, then $AI = A$.

(*iii*) If A is any matrix and I is an identity matrix such that IA is defined, then $IA = A$.

1.6 SPECIAL MATRICES

1. Idempotent matrix

A square matrix A is said to be an idempotent matrix if $A^2 = A$.

Example: $A = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$ is an idempotent matrix since $A^2 = A \cdot A = A$.

Note: Identity matrix is an idempotent matrix.

2. Nilpotent matrix

A square matrix A is said to be a nilpotent matrix with index k, if k be the least positive integer for which $A^k = O$, a null (or zero) matrix.

Example: Let $$A = \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}$$

Here $$A^2 = A \cdot A = \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O.$$

Therefore, A is a nilpotent matrix with index 2.

Note: Null (or zero) matrix is a nilpotent matrix with index 1.

3. Involutary matrix

A square matrix A is said to be an involutary matrix if $A^2 = I$.

Example: $A = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$ is involutary since $A^2 = A \cdot A = I$.

Note: Unit matrix is an involutary matrix.

ILLUSTRATIVE EXAMPLES

Example 1: Determine the matrices A and B where

$$2A+B = \begin{pmatrix} 2 & 2 & 5 \\ 5 & 4 & 3 \\ 1 & 1 & 4 \end{pmatrix} \text{ and } A-2B = \begin{pmatrix} 1 & 6 & 5 \\ 5 & 2 & -1 \\ -2 & -2 & 2 \end{pmatrix}$$

Solution: Here $$2A+B = \begin{pmatrix} 2 & 2 & 5 \\ 5 & 4 & 3 \\ 1 & 1 & 4 \end{pmatrix} \quad \cdots(1)$$

and $$A-2B = \begin{pmatrix} 1 & 6 & 5 \\ 5 & 2 & -1 \\ -2 & -2 & 2 \end{pmatrix} \quad \cdots(2)$$

Multiplying (1) by 2 and then adding to (2), we get

$$2(2A+B)+A-2B = 2\begin{pmatrix} 2 & 2 & 5 \\ 5 & 4 & 3 \\ 1 & 1 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 6 & 5 \\ 5 & 2 & -1 \\ -2 & -2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 & 10 \\ 10 & 8 & 6 \\ 2 & 2 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 6 & 5 \\ 5 & 2 & -1 \\ -2 & -2 & 2 \end{pmatrix}$$

or $$5A = \begin{pmatrix} 5 & 10 & 15 \\ 15 & 10 & 5 \\ 0 & 0 & 10 \end{pmatrix}$$

or $$A = \frac{1}{5}\begin{pmatrix} 5 & 10 & 15 \\ 15 & 10 & 5 \\ 0 & 0 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

From (1), $$B = \begin{pmatrix} 2 & 2 & 5 \\ 5 & 4 & 3 \\ 1 & 1 & 4 \end{pmatrix} - 2A = \begin{pmatrix} 2 & 2 & 5 \\ 5 & 4 & 3 \\ 1 & 1 & 4 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 2 & 5 \\ 5 & 4 & 3 \\ 1 & 1 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 4 & 6 \\ 6 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Example 2: Find x, y, z and t if $3\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & 6 \\ -1 & 2t \end{pmatrix} + \begin{pmatrix} 4 & x+y \\ z+t & 3 \end{pmatrix}$.

Solution: The given equation is $\begin{pmatrix} 3x & 3y \\ 3z & 3t \end{pmatrix} = \begin{pmatrix} x+4 & 6+x+y \\ -1+z+t & 2t+3 \end{pmatrix}$

Equating corresponding elements from both sides, we get

$$3x = x + 4,\ 3y = 6 + x + y,\ 3z = -1 + z + t,\ 3t = 2t + 3$$

or $$2x = 4,\ 2y = 6 + x,\ 2z = -1 + t,\ t = 3$$

$\therefore$ $$x = 2,\ y = 4,\ z = 1,\ t = 3.$$

Example 3: Evaluate $(p\ q\ r)\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix}$.

Solution: We have

$$(p\ q\ r)\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = (ap + hq + gr \quad hp + bq + fr \quad gp + fq + cr)\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$= ap^2 + hpq + grp + hpq + bq^2 + fqr + grp + fqr + cr^2$$

$$= ap^2 + bq^2 + cr^2 + 2hpq + 2fqr + 2grp.$$

Example 4: Show with the help of an example that the matrix equation $AB = O$ does not necessarily mean that either $A = O$ or $B = O$, where 0 stands for the null matrix.

Solution: Let us take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$.

Here $A \neq 0$, $B \neq 0$,

but $AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1-1 & 0+0 \\ 1-1 & 0+0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$

Example 5: Prove by an example that $AB = AC$ does not imply $B = C$, in general.

Solution: Let us take $A = \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.

Therefore, $AB = \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1\times1+4\times2 & 1\times5+4\times1 \\ 0\times1+0\times2 & 0\times5+0\times1 \end{pmatrix} = \begin{pmatrix} 9 & 9 \\ 0 & 0 \end{pmatrix}$

and $AC = \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1\times1+4\times2 & 1\times1+4\times2 \\ 0\times1+0\times2 & 0\times1+0\times2 \end{pmatrix} = \begin{pmatrix} 9 & 9 \\ 0 & 0 \end{pmatrix}$

Obviously, $B \neq C$ though $AB = AC$.

Example 6: If $A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 1 & -1 \end{pmatrix}$, then prove that $BA = -AB$ and hence show that

$(A+B)^2 = A^2 + B^2$.

Solution: Here,

$$AB = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1+2 & 4-2 \\ -1-1 & -4+1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -2 & -3 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 1 & 4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1-4 & 2-4 \\ 1+1 & 2+1 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 2 & 3 \end{pmatrix}$$

$$= -\begin{pmatrix} 3 & 2 \\ -2 & -3 \end{pmatrix} = -AB.$$

Also, $(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B)$

$= (A^2 + AB) + (BA + B^2)$ [using distributive property]

$= A^2 + AB + BA + B^2$ [by associative property]

$= A^2 + AB - AB + B^2$ [$\because BA = -AB$]

$= A^2 + O + B^2 = A^2 + B^2$

Example 7: Show that the product of the matrices

$\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$ and $\begin{pmatrix} \cos^2\varphi & \cos\varphi\sin\varphi \\ \cos\varphi\sin\varphi & \sin^2\varphi \end{pmatrix}$ is a null matrix, where θ and φ differ by an odd multiple of $\frac{\pi}{2}$.

Solution: Here $\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}\begin{pmatrix} \cos^2\varphi & \cos\varphi\sin\varphi \\ \cos\varphi\sin\varphi & \sin^2\varphi \end{pmatrix}$

$$= \begin{pmatrix} \cos^2\theta\cos^2\varphi + \cos\theta\sin\theta\cos\varphi\sin\varphi & \cos^2\theta\cos\varphi\sin\varphi + \cos\theta\sin\theta\sin^2\varphi \\ \cos\theta\sin\theta\cos^2\varphi + \sin^2\theta\cos\varphi\sin\varphi & \cos\theta\sin\theta\cos\varphi\sin\varphi + \sin^2\theta\sin^2\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\varphi(\cos\theta\cos\varphi + \sin\theta\sin\varphi) & \cos\theta\sin\varphi(\cos\theta\cos\varphi + \sin\theta\sin\varphi) \\ \sin\theta\cos\varphi(\cos\theta\cos\varphi + \sin\theta\sin\varphi) & \sin\theta\sin\varphi(\cos\theta\cos\varphi + \sin\theta\sin\varphi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\varphi\cos(\theta-\varphi) & \cos\theta\sin\varphi\cos(\theta-\varphi) \\ \sin\theta\cos\varphi\cos(\theta-\varphi) & \sin\theta\sin\varphi\cos(\theta-\varphi) \end{pmatrix}$$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O_{2\times 2}$, a null matrix. $\left[\because \theta - \varphi = (2n+1)\frac{\pi}{2}, n = 0, \pm 1, \pm 2, \ldots \therefore \cos(\theta-\varphi) = 0\right]$

Example 8: By mathematical induction, prove that if

$A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, then $A^n = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$, where n is any positive integer.

Solution: For, $n = 1$, $A^1 = \begin{pmatrix} 1+2\cdot 1 & -4\cdot 1 \\ 1 & 1-2\cdot 1 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} = A$.

Therefore the result is true when $n = 1$.

Let us assume that the result is true for any positive integer m, *i.e.*,

$$A^m = \begin{pmatrix} 1+2m & -4m \\ m & 1-2m \end{pmatrix} \quad \ldots(1)$$

Now, $A^{m+1} = A^m \cdot A = \begin{pmatrix} 1+2m & -4m \\ m & 1-2m \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ [using (1)]

$$= \begin{pmatrix} 3(1+2m)-4m & -4(1+2m)+4m \\ 3m+1-2m & -4m-1+2m \end{pmatrix} = \begin{pmatrix} 3+2m & -4m-4 \\ m+1 & -2m-1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+2(m+1) & -4(m+1) \\ m+1 & 1-2(m+1) \end{pmatrix}.$$

Therefore, if the result is true for any positive integer m, then it is true for $(m+1)$.

But we have seen that the result is true for 1, therefore it is true for $1 + 1 = 2$, $2 + 1 = 3$, $3 + 1 = 4$, $4 + 1 = 5$ etc. *i.e.*, the result is true for any positive integer n.

Example 9: If $AB = B$ and $BA = A$, then prove that A, B are idempotent matrices.

Solution: Here $AB = B$ ...(1)

$BA = A$ ...(2)

Now, $A^2 = A\cdot A = A\cdot(BA)$ [using (2)]

$= (AB)\cdot A$ [by associative law]

$$= BA \qquad \text{[using (1)]}$$
$$= A \qquad \text{[using (2)]}$$

Therefore, A is an idempotent matrix.

Also, $B^2 = B \cdot B = B \cdot (AB)$ [using (1)]

$$= (BA) \cdot B \qquad \text{[by associative law]}$$
$$= AB \qquad \text{[using (2)]}$$
$$= B \qquad \text{[using (1)]}$$

Hence, B is an idempotent matrix.

1.7 TRANSPOSE OF A MATRIX

Let A be a matrix of order $m \times n$, then transpose of A, denoted by A^T or A', is defined as the matrix obtained by interchanging the rows and columns of A. Thus if $A = (a_{ij})_{m \times n}$, then $A^T = (a_{ji})_{n \times m}$.

Example: If $A = \begin{pmatrix} 1 & 5 & -2 \\ 6 & 9 & 4 \end{pmatrix}$ then its transpose $A^T = \begin{pmatrix} 1 & 6 \\ 5 & 9 \\ -2 & 4 \end{pmatrix}$.

Note: We observe that the element lying at the junction of ith row and jth column of A^T is same as the element lying at the junction of jth row and ith column of A.

Properties

If A and B are two matrices, then

(*i*) $(kA)^T = kA^T$, where k is a scalar (or number)

(*ii*) $(A^T)^T = A$

(*iii*) $(A+B)^T = A^T + B^T$, provided $A + B$ is defined

(*iv*) $(A-B)^T = A^T - B^T$, provided $A-B$ is defined

(*v*) $(cA+dB)^T = cA^T + dB^T$, provided $cA + dB$ is defined, where c, d are scalars (or numbers)

(*vi*) $(AB)^T = B^T A^T$, provided AB is defined.

(*vii*) $I_n^T = I_n$, I_n being identity matrix of order n.

ILLUSTRATIVE EXAMPLES

Example 1: For the matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, verify

(*i*) $(A^T)^T = A$, (*ii*) $(cA)^T = cA^T$, where c is a number.

Solution: Here

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

(*i*) $\therefore$
$$(A^T)^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A$$

Hence the verification.

(*ii*) Now,
$$cA = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \\ ca_{31} & ca_{32} & ca_{33} \end{pmatrix}$$

$\therefore$
$$(cA)^T = \begin{pmatrix} ca_{11} & ca_{21} & ca_{31} \\ ca_{12} & ca_{22} & ca_{32} \\ ca_{13} & ca_{23} & ca_{33} \end{pmatrix} = c\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = cA^T.$$

Hence the verification.

Example 2: If $A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & -5 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 4 & 3 \\ 1 & 9 & 5 \end{pmatrix}$, verify

(*i*) $(3A)^T = 3A^T$, (*ii*) $(A + B)^T = A^T + B^T$,

(*iii*) $(A - B)^T = A^T - B^T$, (*iv*) $(2A + 3B)^T = 2A^T + 3B^T$

Solution: (*i*) Here,

$$3A = 3\begin{pmatrix} 1 & 3 & -2 \\ 0 & -5 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 9 & -6 \\ 0 & -15 & 12 \end{pmatrix}.$$

$\therefore$
$$(3A)^T = \begin{pmatrix} 3 & 0 \\ 9 & -15 \\ -6 & 12 \end{pmatrix},\ 3A^T = 3\begin{pmatrix} 1 & 0 \\ 3 & -5 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 9 & -15 \\ -6 & 12 \end{pmatrix}.$$

$\therefore$ $(3A)^T = 3A^T$.

(*ii*) and (*iii*)

Now,
$$A+B = \begin{pmatrix} 1+6 & 3+4 & -2+3 \\ 0+1 & -5+9 & 4+5 \end{pmatrix} = \begin{pmatrix} 7 & 7 & 1 \\ 1 & 4 & 9 \end{pmatrix},$$

$$A-B = \begin{pmatrix} 1-6 & 3-4 & -2-3 \\ 0-1 & -5-9 & 4-5 \end{pmatrix} = \begin{pmatrix} -5 & -1 & -5 \\ -1 & -14 & -1 \end{pmatrix},$$

$$A^T = \begin{pmatrix} 1 & 0 \\ 3 & -5 \\ -2 & 4 \end{pmatrix},\ B^T = \begin{pmatrix} 6 & 1 \\ 4 & 9 \\ 3 & 5 \end{pmatrix}.$$

$$(A+B)^T = \begin{pmatrix} 7 & 1 \\ 7 & 4 \\ 1 & 9 \end{pmatrix},\ (A-B)^T = \begin{pmatrix} -5 & -1 \\ -1 & -14 \\ -5 & -1 \end{pmatrix}$$

$$A^T + B^T = \begin{pmatrix} 1+6 & 0+1 \\ 3+4 & -5+9 \\ -2+3 & 4+5 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 7 & 4 \\ 1 & 9 \end{pmatrix}.$$

$$A^T - B^T = \begin{pmatrix} 1-6 & 0-1 \\ 3-4 & -5-9 \\ -2-3 & 4-5 \end{pmatrix} = \begin{pmatrix} -5 & -1 \\ -1 & -14 \\ -5 & -1 \end{pmatrix}.$$

$\therefore$ $$(A+B)^T = A^T + B^T \text{ and } (A-B)^T = A^T - B^T$$

(*iv*) Here, $$2A+3B = 2\begin{pmatrix} 1 & 3 & -2 \\ 0 & -5 & 4 \end{pmatrix} + 3\begin{pmatrix} 6 & 4 & 3 \\ 1 & 9 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 6 & -4 \\ 0 & -10 & 8 \end{pmatrix} + \begin{pmatrix} 18 & 12 & 9 \\ 3 & 27 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} 2+18 & 6+12 & -4+9 \\ 0+3 & -10+27 & 8+15 \end{pmatrix} = \begin{pmatrix} 20 & 18 & 5 \\ 3 & 17 & 23 \end{pmatrix}$$

$\therefore$ $$(2A+3B)^T = \begin{pmatrix} 20 & 3 \\ 18 & 17 \\ 5 & 23 \end{pmatrix}$$

Again, $$2A^T + 3B^T = 2\begin{pmatrix} 1 & 0 \\ 3 & -5 \\ -2 & 4 \end{pmatrix} + 3\begin{pmatrix} 6 & 1 \\ 4 & 9 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 6 & -10 \\ -4 & 8 \end{pmatrix} + \begin{pmatrix} 18 & 3 \\ 12 & 27 \\ 9 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} 2+18 & 0+3 \\ 6+12 & -10+27 \\ -4+9 & 8+15 \end{pmatrix} = \begin{pmatrix} 20 & 3 \\ 18 & 17 \\ 5 & 23 \end{pmatrix}$$

Clearly $$(2A+3B)^T = 2A^T + 3B^T.$$

Example 3: If $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$, verify that $(AB)^T = B^T A^T$.

Solution: Now, $$AB = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1+4+0 & 0+2-1 & 0+0-3 \\ 3+0+0 & 0+0+2 & 0+0+6 \\ 4+10+0 & 0+5+0 & 0+0+0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 1 & -3 \\ 3 & 2 & 6 \\ 14 & 5 & 0 \end{pmatrix}$$

$$\therefore \qquad (AB)^T = \begin{pmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{pmatrix} \qquad \ldots(1)$$

Also,
$$A^T = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{pmatrix}, \quad B^T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\therefore \qquad B^T A^T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1+4+0 & 3+0+0 & 4+10+0 \\ 0+2-1 & 0+0+2 & 0+5+0 \\ 0+0-3 & 0+0+6 & 0+0+0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{pmatrix} \qquad \ldots(2)$$

From (1) and (2), we have $(AB)^T = B^T A^T$.

1.8 SYMMETRIC AND SKEW-SYMMETRIC MATRICES

1. A square matrix $A = (a_{ij})_{n \times n}$ is said to be symmetric if $A^T = A$, *i.e.*, if $a_{ij} = a_{ji}$ for $i, j = 1, 2, \ldots, n$.
2. A square matrix $A = (a_{ij})_{n \times n}$ is said to be skew-symmetric if $A^T = -A$, *i.e.*, if $a_{ij} = -a_{ji}$ for $i, j = 1, 2, \ldots, n$.

Example: The matrix $A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$ is symmetric since $A^T = A$.

The matrix $B = \begin{pmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{pmatrix}$ is skew-symmetric since

$$B^T = \begin{pmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{pmatrix} = -B.$$

Note: (*i*) In a skew-symmetric matrix $A = (a_{ij})_{n \times n}$, we have $a_{ij} = -a_{ji}$, therefore $a_{ii} = -a_{ii}$ (putting $j = i$), or $2a_{ii} = 0$, *i.e.*, $a_{ii} = 0$, $i = 1, 2, \ldots, n$. Hence, in a skew-symmetric matrix all the diagonal elements are zero.

(*ii*) We see that in a symmetric matrix, the elements equidistant from the principal diagonal are equal in magnitude and sign. In a skew-symmetric matrix, all the diagonal elements are zero and the elements equidistant from the principal diagonal are equal in magnitude and opposite in sign.

Properties

1. If A, B are symmetric matrices of same order, then $A + B$ is symmetric.

Proof: $(A+B)^T = A^T + B^T = A+B$, $\quad (\because A^T = A,\ B^T = B)$

2. The product AB of two symmetric matrices A, B of same order is symmetric if and only if $AB = BA$.

Proof: Let AB be symmetric, then $(AB)^T = AB$, or $B^TA^T = AB$, or $BA = AB$ $(\because A^T = A,\ B^T = B)$ conversely, let $AB = BA$, then $(AB)^T = B^TA^T = BA = AB$.

3. AA^T and A^TA are both symmetric for every matrix A of order $m \times n$.

Proof: AA^T and A^TA are square matrices of orders m and n respectively.

Now,
$$(AA^T)^T = (A^T)^T A^T = AA^T$$
and
$$(A^TA)^T = A^T(A^T)^T = A^TA.$$

4. If A be a square matrix, then $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric.

Proof:
$$(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T.$$
$$(A-A^T)^T = A^T - (A^T)^T = A^T - A = -(A-A^T).$$

5. Any square matrix can be expressed uniquely as a sum of a symmetric matrix and a skew-symmetric matrix. *(W.B.U.T. 2006, 2011, 2013)*

Proof: Let A be a square matrix of order n, then we can write
$$A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T) = B + C,$$
where $B = \frac{1}{2}(A+A^T)$ and $C = \frac{1}{2}(A-A^T)$.

Now,
$$B^T = \left(\frac{1}{2}(A+A^T)\right)^T = \frac{1}{2}(A+A^T)^T = \frac{1}{2}\{A^T + (A^T)^T\}$$
$$= \frac{1}{2}(A^T + A) = \frac{1}{2}(A+A^T) = B.$$

Also,
$$C^T = \left(\frac{1}{2}(A-A^T)\right)^T = \frac{1}{2}(A-A^T)^T = \frac{1}{2}\{A^T - (A^T)^T\}$$
$$= \frac{1}{2}(A^T - A) = -\frac{1}{2}(A-A^T) = -C.$$

Therefore, B is symmetric and C is skew-symmetric. Hence the result.

6. If A is a skew-symmetric matrix, then A^2 is symmetric.

Proof: $(A^2)^T = (A \cdot A)^T = A^T \cdot A^T = (-A)\cdot(-A) = A^2 \quad (\because A \text{ is skew-symmetric})$

ILLUSTRATIVE EXAMPLES

Example 1: Express the following as the sum of a symmetric and a skew-symmetric matrix:

$$\begin{pmatrix} 2 & 3 & 8 \\ 5 & -4 & 6 \\ 7 & 1 & 4 \end{pmatrix}.$$

Solution: Let
$$A = \begin{pmatrix} 2 & 3 & 8 \\ 5 & -4 & 6 \\ 7 & 1 & 4 \end{pmatrix}$$

$\therefore$
$$A^T = \begin{pmatrix} 2 & 5 & 7 \\ 3 & -4 & 1 \\ 8 & 6 & 4 \end{pmatrix}$$

Now,
$$B = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{pmatrix} 4 & 8 & 15 \\ 8 & -8 & 7 \\ 15 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 4 & \frac{15}{2} \\ 4 & -4 & \frac{7}{2} \\ \frac{15}{2} & \frac{7}{2} & 4 \end{pmatrix},$$

which is a symmetric matrix.

Also,
$$C = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & 5 \\ -1 & -5 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & \frac{1}{2} \\ 1 & 0 & \frac{5}{2} \\ -\frac{1}{2} & -\frac{5}{2} & 0 \end{pmatrix},$$

which is a skew-symmetric matrix.

Clearly
$$A = B + C$$

i.e.,
$$\begin{pmatrix} 2 & 3 & 8 \\ 5 & -4 & 6 \\ 7 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 & \frac{15}{2} \\ 4 & -4 & \frac{7}{2} \\ \frac{15}{2} & \frac{7}{2} & 4 \end{pmatrix} + \begin{pmatrix} 0 & -1 & \frac{1}{2} \\ 1 & 0 & \frac{5}{2} \\ -\frac{1}{2} & -\frac{5}{2} & 0 \end{pmatrix}.$$

Example 2: Prove that P^TAP is a symmetric or a skew-symmetric matrix according as A is symmetric or skew-symmetric. (*W.B.U.T. 2009*)

Solution: Now, $(P^TAP)^T = (P^T(AP))^T = (AP)^T(P^T)^T = (P^TA^T)P$ ...(1)

If A is a symmetric matrix, then $A^T = A$ and from (1),

$$(P^TAP)^T = P^TAP.$$

So, P^TAP is symmetric.

If A is a skew-symmetric matrix, then $A^T = -A$ and from (1),

$$(P^T AP)^T = (-P^T AP)$$

So, in this case $P^T AP$ is skew-symmetric.

MISCELLANEOUS EXAMPLES

Example 1: Find the matrices A and B when

(*i*) $A+B = 2B^T,\ 3A+2B = I_3$

(*ii*) $2A+3B^T = I_3,\ A^T + B = C$, where $C = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

Solution: (*i*) Here, $A + B = 2B^T$, therefore, $3A + 3B = 6B^T$ ...(1)

Also, by question, $3A + 2B = I_3$ ...(2)

Subtracting (2) from (1), we get $B = 6B^T - I_3$ ...(3)

$\therefore$

Thus, $B^T = (6B^T - I_3)^T = (6B^T)^T - I_3^T = 6(B^T)^T - I_3 = 6B - I_3$

$6B^T = 36B - 6I_3$ ...(4)

Adding (3) and (4), we get

$$B + 6B^T = 6B^T - I_3 + 36B - 6I_3$$

or $$-35B = -7I_3.$$

$$\therefore \quad B = \frac{1}{5}I_3 = \frac{1}{5}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$$

From (2), we get $$A = \frac{1}{3}(I_3 - 2B) = \frac{1}{3}\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2\begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}\right\}$$

$$= \frac{1}{3}\begin{pmatrix} 1-\frac{2}{5} & 0 & 0 \\ 0 & 1-\frac{2}{5} & 0 \\ 0 & 0 & 1-\frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}.$$

$= A(BC) + B(AC)$ [by associative law]

$= A(CB) + B(CA)$ [$\because$ BC and AC obey commutative law]

$= (AC)B + (BC)A$ [by associative law]

$= (CA)B + (CB)A$ [$\because$ AC and BC obey commutative law]

$= C(AB) + C(BA)$ [by associative law]

$= C(AB + BA).$ [by distributive law]

MULTIPLE CHOICE QUESTIONS

1. If $4A + \begin{pmatrix} 1 & 2 & 3 \\ -2 & 6 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 0 \end{pmatrix}$, then A is

   (*a*) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (*b*) $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (*c*) $\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ (*d*) none of these.

2. If $\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} A = \begin{pmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{pmatrix}$, then A is equal to

   (*a*) $(-1\ 2\ 1)$ (*b*) $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ (*c*) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (*d*) none of these.

3. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, the AB is defined

   (*a*) Yes (*b*) No.

4. If $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$, then

   (*a*) $AB \neq BA$ (*b*) AB is defined but BA is not defined

   (*c*) BA is defined but AB is not defined (*d*) $AB = BA$.

5. If $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then

   (*a*) $A(BC)$ is defined (*b*) $A(CB)$ is defined

   (*c*) $B(CA)$ is defined (*d*) $(CB)A$ is defined.

6. If $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, then it is

   (*a*) a nilpotent matrix
   (*b*) an idempotent matrix
   (*c*) an involutary matrix
   (*d*) none of these.

7. If $\begin{pmatrix} 4 & k \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 1 & 6 \end{pmatrix}$, then the value of k is

   (*a*) 6 (*b*) 5 (*c*) 1 (*d*) 4.

8. The matrix $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is

   (*a*) a symmetric matrix
   (*b*) a skew-symmetric matrix
   (*c*) a nilpotent matrix
   (*d*) none of these.

9. If $BA = A$ and $AB = B$, then $A^2 + B^2$ is equal to

   (*a*) $A + B$ (*b*) $2AB$ (*c*) $2BA$ (*d*) none of these.

10. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, then A^{100} is

    (*a*) $\begin{bmatrix} 1 & 0 \\ -150 & 1 \end{bmatrix}$
    (*b*) $\begin{bmatrix} 1 & 0 \\ -50 & 1 \end{bmatrix}$
    (*c*) $\begin{bmatrix} 1 & 0 \\ -100 & 1 \end{bmatrix}$
    (*d*) none of these. (*W.B.U.T. 2009*)

11. If A, B are two square matrices of same order such that $(A+B)^2 = A^2 + 2AB + B^2$, then

    (*a*) $A = B$ (*b*) $AB = BA$ (*c*) $A = -B$ (*d*) $A = -B^T$.

12. If A, B, C are three matrices such that B^TAB is defined, then

    (*a*) B^TAB is symmetric if A is symmetric
    (*b*) B^TAB is symmetric if A is skew-symmetric
    (*c*) B^TAB is skew-symmetric if A is symmetric
    (*d*) none of these.

13. If A, B, C are three matrices such that B^TAB is defined, then

    (*a*) B^TAB is symmetric if A is skew-symmetric
    (*b*) B^TAB is skew-symmetric if A is symmetric
    (*c*) B^TAB is skew-symmetric if A is skew-symmetric
    (*d*) none of these.

14. If A is a symmetric as well as a skew-symmetric matrix, then A must be

    (*a*) unit matrix
    (*b*) null matrix
    (*c*) involutary matrix
    (*d*) none of these.

15. If A is any square matrix, then $A + A^T$ is
   (*a*) symmetric (*b*) skew-symmetric
   (*c*) identity matrix (*d*) none of these.
16. If A is a square matrix, then $A - A^T$ is
   (*a*) symmetric (*b*) identity matrix
   (*c*) skew-symmetric (*d*) none of these.
17. If A is a skew-symmetric, then A^2 is
   (*a*) skew-symmetric (*b*) symmetric
   (*c*) idempotent (*d*) none of these.
18. If A is a matrix of order 3×4 and AB, BA are defined, then the order of B is
   (*a*) 4×4 (*b*) 3×3 (*c*) 3×4 (*d*) 4×3.
19. If $AB = B$ and $BA = A$, then A is
   (*a*) idempotent (*b*) symmetric
   (*c*) skew-symmetric (*d*) none of these.
20. If the order of A is 2×3, A^TB and B^TA are defined, then the order of B is
   (*a*) 3×3 (*b*) 3×2 (*c*) 2×3 (*d*) 2×2.
21. If the order of A is 2×3, A^TB and BA^T are defined, then the order of B is
   (*a*) 2×2 (*b*) 2×3 (*c*) 3×2 (*d*) 3×3.
22. If A, B are two matrices such that $AB = BA$ and the order of A is $n \times n$, then the order of B is
   (*a*) $n \times (n + 1)$ (*b*) $(n + 1) \times n$ (*c*) $n \times (n - 1)$ (*d*) $n \times n$.
23. If A, B, C are three square matrices such that $AB = BC = I$, then
   (*a*) $A = C$ (*b*) $A = B$ (*c*) $B = C$ (*d*) none of these.
24. The order of A, for which $(1\ 2\ 3\ 4\ 5)\,A = (3\ 1)$ holds, is
   (*a*) 5×5 (*b*) 2×5 (*c*) 5×2 (*d*) 2×2.
25. If $\begin{pmatrix} x & y \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 2 & x \\ 1 & y \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}$, then
   (*a*) $x = 1,\ y = -1$ (*b*) $x = 1, y = 1$ (*c*) $x = -1,\ y = 1$ (*d*) $x = -1,\ y = -1$.
26. If $A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, then AA^T is equal to
   (*a*) $\begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}$ (*b*) $\begin{pmatrix} 5 & -1 \\ 1 & 1 \end{pmatrix}$ (*c*) $\begin{pmatrix} 5 & 1 \\ -1 & 1 \end{pmatrix}$ (*d*) $\begin{pmatrix} 5 & -1 \\ -1 & 1 \end{pmatrix}$.
27. If $A^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then $A =$
   (*a*) $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ (*b*) $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$
   (*c*) $\frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ (*d*) $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ (*W.B.U.T. 2011*)

ANSWERS

1. (b) 2. (a) 3. (b) 4. (d)
5. (c) 6. (a) 7. (b) 8. (d)
9. (a) 10. (c) 11. (b) 12. (a)
13. (c) 14. (b) 15. (a) 16. (c)
17. (b) 18. (d) 19. (a) 20. (c)
21. (b) 22. (d) 23. (a) 24. (c)
25. (b) 26. (d). 27. (c)

PROBLEMS

1. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$, $C = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ -3 & -3 & 3 \end{bmatrix}$, verify the following laws:

(i) $A + C = C + A$ (ii) $A + (B + C) = (A + B) + C$
(iii) $A\ (B + C) = AB + AC$ (iv) $(B + C)\ A = BA + CA$
(iv) $A\ (BC) = (AB)\ C$.

Also show that $AB = O$, $BA \neq O$, $AC \neq O$, $CA = O$.

2. Given $A = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{pmatrix}$,

find AB and AC and show that they are equal.

3. If $A + 2B = \begin{pmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{pmatrix}$ and $2A - B = \begin{pmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{pmatrix}$, find A and B.

4. If $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{pmatrix}$, verify that $(A+B)^2 = A^2 + AB + BA + B^2$.

5. If $A = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 2 \\ -2 & -4 \end{pmatrix}$, show that $(A+B)^2 = A^2 + 2AB + B^2$.

6. Find the values of x, y, z and t which satisfy the matrix equation

$$\begin{pmatrix} x+3 & x+2y \\ z-1 & 4t-6 \end{pmatrix} = \begin{pmatrix} 0 & -7 \\ 3 & 2t \end{pmatrix}.$$