Boolean Algebra

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Basic Concepts

Boolean Algebra:

- An algebraic system defined on the set $\{0, 1\}$, with two binary operators (AND and OR) and one unary operator NOT. AND operation is also called Logical Product. OR operation and NOT operation are called Logical sum and complement respectively.
- Boolean (switching) variables are two valued variables that can take two distinct values 0 and 1.
- Boolean expression is an expression consisting of Boolean variables, constants and operators.
- Given a Boolean expression, how to prove it? Suppose $x + xy = x$, how do we prove it?
  (i) By verifying the expression for all possible values of the variable. Called truth table verification or perfect induction.
  (ii) By using algebraic manipulation using some rules.

Basic Laws of Boolean Algebra

Basic identities

$$x + 1 = 1$$
$$x + 0 = x$$
$$x.1 = x$$
$$x.0 = 0$$

Idempotent Law

$$x + x = x$$
$$x.x = x$$

- Commutative law

  $x + y = y + x \qquad x.y = y.x$

  Order of variable does not matter.
- Complementation law

  $x + \overline{x} = 1 \qquad x.\overline{x} = 0$
- Associative law

  $(x + y + z = x + (y + z) \qquad (x.y).z = x.(y.z)$
- Distributive law

  $x.(y + z) = x.y + x.z$

  $x + (y.z) = (x + y).(x + z)$

Basic Laws of Boolean Algebra

- Absorption law
  (i) $x + (x.y) = x$
  $x + x.y$
  $=x.1 + x.y$
  $=x.(1 + y) \Rightarrow$ distribution law
  $=x.1$
  $=x$
  (ii) $x.(x + y) = x$ $x.(x + y)$
  $=(x + 0).(x + y)$
  $=x + 0.y \Rightarrow$ distribution law
  $=x + 0$
  $=x$

Basic Laws of Boolean Algebra

- Useful law

  (i) $x + (\overline{x}.y) = x + y$

  $x + (\overline{x}.y)$

  $=x.1 + \overline{x}.y$

  $=x.(y + \overline{y}) + \overline{x}.y$

  $=x.y + x.\overline{y} + \overline{x}.y$

  $=x.y + x.y + x.\overline{y} + \overline{x}.y$

  $=x.y + x.\overline{y} + x.y + \overline{x}.y$

  $=x.(y + \overline{y}) + y.(x + \overline{x})$

  $=x.1 + y.1$

  $=x + y$

  (ii) $x.(\overline{x} + y) = x.y$

Basic Laws of Boolean Algebra

- Consensus Theorem

  (i) $x.y + \overline{x}.z + y.z = x.y + \overline{x}.z$

  (ii) $(x + y).(\overline{x} + z).(y + z) = (x + y).(\overline{x} + z)$
- Involution

  $\overline{(\overline{x})} = x$

These rules may be used for algebraic manipulation of boolean expression.

Boolean Function Minimization

Given a Boolean expression, we can simplify it by using basic laws of Boolean algebra. We can reduce the number of terms in the expression. We can also reduce the number of literals (variables).

Example:

$x.y \ + \ x.y.z \ + \ \overline{y}.z$

$= x.y.1 \ + \ x.y.z \ + \ \overline{y}.z$

$= x.y(1 \ + \ z) \ + \ \overline{y}.z$

$= x.y.1 \ + \ \overline{y}.z$

$= x.y \ + \ \overline{y}.z$

Algebraic Manipulation

Principle of Duality

- Most of the rules discussed so far appear in pairs.
- Principle of duality states that a Boolean expression E_2 can be obtained from a given Boolean expression E_1 by interchanging the operations AND and OR and constant 0 and 1. E_1 and E_2 are said to be dual of each other.

Examples:
$x + \overline{x}.y = x + y \Rightarrow$ if this true then,
$x.(\overline{x} + y) = x.y$ is also true (from duality principle)
$x + 1 = 1$ is the dual of $x.0 = 0$

Simplification Examples

Example- 1

$F = x\overline{y} + xy + yz$

$F = x(\overline{y} + y) + yz$

$F = x.1 + yz$

$F = x + yz$

Example - 2

$F = \overline{x}yz + x\overline{y}z + x.y\overline{z} + xyz$

We know that $x + x + x = x$

$F = \overline{x}yz + xyz + x\overline{y}z + xyz + x.y\overline{z} + xyz$

$F = yz(\overline{x} + x) + xz(\overline{y} + y) + xy(\overline{z} + z)$

$F = yz.1 + xz.1 + xy.1$

$F = xy + yz + xz$

De Morgan's Theorem

This is very important law that you can apply for transforming expressions in variety of ways.

For two variables x and y, De Morgan's theorems state that

(i) $\overline{(x \; + \; y)} \; = \; \overline{x}.\overline{y}$

(ii) $\overline{(x.y)} \; = \; \overline{x} \; + \; \overline{y}$

Simplification using De Morgan's Theorems

$$F = \overline{(x+y)}.(\overline{x}+\overline{y}) = \overline{x}.\overline{y}.(\overline{x}+\overline{y}) = \overline{x}.\overline{y}+\overline{x}.\overline{y} = \overline{x}.\overline{y} \tag{1}$$

$$\begin{aligned} F &= \overline{(x.\overline{y}+\overline{x}.y)} \\ &= \overline{(x.\overline{y})}.\overline{(\overline{x}.y)} \\ &= (\overline{x}+y).(x+\overline{y}) \\ &= x\overline{x}+\overline{x}.\overline{y}+x.y+y.\overline{y} \\ &= x.y+\overline{x}.\overline{y} \end{aligned}$$