Logic & Proofs
(Lecture – 3)

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Application of Satisfiability

- Many problems are modeled in terms of propositional satisfiability: *robotics, software testing, computer-aided design, machine vision, integrated circuit design, computer networking, genetics*, etc.
- We will discuss about modeling **Sudoku puzzles** using propositional satisfiability

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

- For each puzzle, some of the 81 cells, called **givens**, are assigned one of the numbers 1, 2, . . . , 9, and the other cells are blank.
- Assign a number to each blank cell so that *every row, every column, and every one of the nine 3 × 3 blocks contains each of the nine possible numbers*.
- Example: where to place 4?
  - One possibility: 2nd row, 6th column

Application of Satisfiability

- Let *p (i, j, n)* denote the proposition that is true when the number *n* is in the cell in the *i*-th row and *j*-th column
- We need to find truth assignments to 729 propositions *p (i, j, n)* with *i, j , and n* each ranging from 1 to 9 that makes the conjunction of all these compound propositions true
- Asserting every row contains every number: $\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, n)$
- Asserting every column contains every number $\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i, j, n)$
- Asserting each of the nine 3X3 blocks contains every number:

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r + i, 3s + j, n)$$

Application of Satisfiability

- Asserting each of the nine 3X3 blocks contains every number:

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r+i, 3s+j, n)$$

- To assert that no cell contains more than one number, we take the conjunction over all values of n, m, i, and j where each variable ranges from 1 to 9 and $n \neq m$ of $p(i, j, n) \rightarrow \neg p(i, j, m)$.

- Take conjunctions of all the listed assertions to find a solution to a given Sudoko puzzle.

Predicates

- **Predicate logic**: used to express the meaning of a wide range of statements in mathematics and computer science and permits us to reason and explore relationships between objects.
- The statement: "x is greater than 3" has two parts:
  - First part – variable x (subject)
  - Second part – greater than 3, refers a property of the subject (predicate)
- The predicate can be denoted by a **propositional function *P(x)***
- Once a value is assigned to x, the statement $P(x)$ becomes a proposition and has a truth value
- Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?
- In general, a statement involving the n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.

Quantifiers

- Quantification – enables us to create a proposition from a propositional function
- Expresses whether a predicate is true over a range of elements.
- Common English terms: *all*, *some*, *many*, *none*, and *few*
- We will study two types of quantifiers:
  - *Universal quantification*: states that the predicate is true for every element under consideration
  - *Existential quantification:* states that for at least one element under consideration, the predicate is true
- **Predicate calculus**: the area of logic which deals with predicates and quantifiers

Universal Quantifier

The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier.** We read $\forall x P(x)$ as "for all $x P(x)$" or "for every $x P(x)$." An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$.

- Asserts if a property is true for all values of a variable in a particular domain, called the **domain of discourse** (or the **universe of discourse**).

Existential Quantifier

The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the *existential quantifier*.

- A domain must always be specified when a statement $\exists x\ P(x)$ is used
- Meaning of $\exists x\ P(x)$ changes when the domain changes.
- Uniqueness quantifier: denoted by $\exists!$ or $\exists_1$
  - There exists a unique x such that $P(x)$ is true
  - Example: $\exists! x(x - 1 = 0)$, where the domain is the set of real numbers

TABLE 1 **Quantifiers.**

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Negating Quantified Expressions

- Consider the statement: "Every student in your class has taken a course in calculus."
- Representation in terms of universal quantification: $\forall x\, P(x)$
  - $P(x)$ is the statement: "Student x has taken a course in calculus"
- The negation of the statement is: "There is a student in your class who has not taken a course in calculus."
- Same as the existential quantification of the negation of the original propositional function $\exists x\ \neg P(x)$.

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x).$$

Negating Quantified Expressions

- Consider the statement: "There is a student in this class who has taken a course in calculus."
- Representation in terms of existential quantification: $\exists x\, Q(x)$
  - $Q(x)$ is the statement: "Student x has taken a course in calculus"
- The negation of the statement is: "Every student in this class has not taken calculus."
- Same as the universal quantification of the negation of the original propositional function $\forall x\ \neg Q(x)$.

$$\neg \exists x\, Q(x) \equiv \forall x\, \neg Q(x).$$

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg \exists x\, P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x\, P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Nested Quantifier

- One quantifier is within the scope of another quantifier
  - Example: $\forall x\ \exists y\ (x + y = 0)$; domain of x, y consists of all real numbers
    - For every real number x there is a real number y such that x + y = 0. This states that every real number has an additive inverse
  - Example: $\forall x\ \forall y((x > 0) \land (y < 0) \rightarrow (xy < 0))$: domain of x, y consists of all real numbers
    - This statement says that for every real number x and for every real number y, if $x > 0$ and $y < 0$, then $xy < 0$.
    - This can be stated more succinctly as "The product of a positive real number and a negative real number is always a negative real number".
- Nested quantifiers can be looked into as loops:
  - $\forall x\ \forall y\ P(x, y)$: loop through the values for x, and for each x we loop through the values for y
  - $\forall x\ \exists y\ P(x, y)$: for each x we loop through the values for y until we find a y for which $P(x, y)$ is true
  - $\exists x\ \forall y\ P(x, y)$: we loop through the values for x until we find an x for which $P(x, y)$ is always true when we loop through all values for y
  - $\exists x\ \exists y\ P(x, y)$: we loop through the values for x, where for each x we loop through the values for y until we hit an x for which we hit a y for which $P(x, y)$ is true.

Order of Quantifiers

- The order of nested universal quantifiers in a statement without other quantifiers can be changed without changing the meaning of the quantified statement.

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Translating English Sentences into Logical Expressions

- Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.
  - *We introduce the propositional functions $F(x)$ to represent "x is female," $P(x)$ to represent "x is a parent," and $M(x, y)$ to represent "x is the mother of y." The original statement can be represented as* $\forall x\, ((F(x) \wedge P(x)) \rightarrow \exists y\, M(x, y))$.
- Express the statement "Everyone has exactly one best friend" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.
  - *When we introduce the predicate $B(x, y)$ to be the statement "y is the best friend of x," the statement that x has exactly one best friend can be represented as* $\exists y\, (B(x, y) \wedge \forall z\, ((z \neq y) \rightarrow \neg B(x, z)))$. *Consequently, our original statement can be expressed as* $\forall x\, \exists y\, (B(x, y) \wedge \forall z\, ((z \neq y) \rightarrow \neg B(x, z)))$.

Translating English Sentences into Logical Expressions

- Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."
  - *Let $P(w, f)$ be "w has taken f" and $Q(f, a)$ be "f is a flight on a." We can express the statement as* $\exists w \, \forall a \, \exists f \, (P(w, f) \land Q(f, a))$*, where the domains of discourse for w, f, and a consist of all the women in the world, all airplane flights, and all airlines, respectively.*