Logic & Proofs
(Lecture – 7 & 8)

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Methods of Proving Theorems: Proof by Contradiction

- Suppose we want to prove that a statement p is true. Furthermore, suppose that we can find a contradiction q such that $\neg \boldsymbol{p} \rightarrow \boldsymbol{q}$ is true. Because q is false, but $\neg \boldsymbol{p} \rightarrow \boldsymbol{q}$ is true, we can conclude that $\neg p$ is false, which means that p is true.
  - How can we find a contradiction q that might help us prove that p is true in this way?
- The statement $\boldsymbol{r} \wedge \neg \boldsymbol{r}$ is a contradiction whenever r is a proposition.
- We can prove that p is true if we can show that $\neg \boldsymbol{p} \rightarrow (\boldsymbol{r} \wedge \neg \boldsymbol{r})$ is true for some proposition r.
- Proof of this type are called **proof by contradiction**.
  - Because a proof by contradiction does not prove a result directly, it is another type of indirect proof.

Proof by Cases & Exhaustive Proof

- Sometimes we cannot prove a theorem using a single argument that holds for all possible cases.
- We need to consider different cases separately.
- To prove: conditional statement of the form $(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q$
  - Tautology $[(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q] \boxed{\leftrightarrow} [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \wedge (p_n \rightarrow q)]$ can be used as a rule of inference.
- The original conditional statement with a hypothesis made up of a disjunction of the propositions $p_1, p_2, \ldots, p_n$ can be proved by proving each of the n conditional statements $\boldsymbol{p_i \rightarrow q}$, $i = 1, 2, \ldots, n$, individually.
- Such an argument is called a **proof by cases**.
- <u>**Proof by exhaustion**</u>:
  - Some theorems can be proved by examining a relatively **small number of examples**. Such proofs are called **exhaustive proofs**, or **proofs by exhaustion**
  - These proofs proceed by exhausting all possibilities.
  - It is a special case of proof by cases, where each case involves checking a single example

Without Loss of Generality (WLOG)

- When we use the phrase "without loss of generality" in a proof (often abbreviated as WLOG), we assert that by proving one case of a theorem, no additional argument is required to prove other specified cases.
  - Other cases follow by making straightforward changes to the argument, or by filling in some straightforward initial step.
- Proofs by cases can often be made much more efficient when the notion of without loss of generality is employed.
- Incorrect use of this principle can lead to errors
  - Errors occur mostly while making certain assumptions that lead to a loss in generality.
  - Such assumptions usually skip a case that may be substantially different from others.

Proof Strategies

- **Forward reasoning**
  - Most common type of reasoning, used as a starting point for both direct and indirect proof methods
  - <u>For direct proofs</u>: usually starts with the premises and constructs proof by applying axioms and known theorems as a sequence of steps that leads to the conclusion.
  - <u>For indirect proofs</u>: starts with the negation of the conclusion and, using a sequence of steps, obtain the negation of the premises.
  - Often difficult to use to prove more complicated results, because the reasoning needed to reach the desired conclusion may be far from obvious.
- **Backward reasoning**
  - To reason backward to prove a statement q, we find a statement p that we can prove with the property that $p \rightarrow q$.

Proof Strategies

- **Looking for counterexamples**
  - When confronted with a conjecture, first try to prove this conjecture
  - If your attempts are unsuccessful, try to find a counterexample, first by looking at the simplest, smallest examples.
  - Looking for counterexamples is an extremely important pursuit, which often provides insights into problems
  - <u>Strategy</u>
    - *Formulating conjectures* – examination of special cases, identification of possible patterns, altering the hypotheses and conclusions of known theorems, based on intuition or a belief that a result holds.
    - If the conjecture is believed to be true, then one needs to find a proof.
    - If it cannot be proven, counterexamples are explored.
    - If counterexamples are not available, try for other proof methods/strategies.

Tiling of Checkerboards

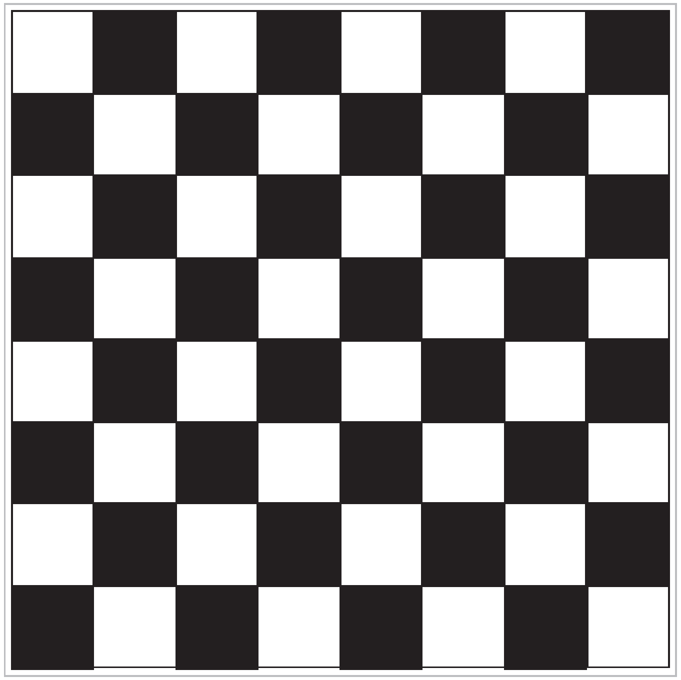

FIGURE 2 **The Standard Checkerboard.**

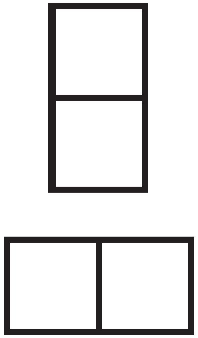

FIGURE 3
Two Dominoes.

- Checkerboard: A checkerboard is a rectangle divided into squares of the same size by horizontal and vertical lines.
  - Standard checkerboard: played on a board with 8 rows and 8 columns
- Domino: A domino is a rectangular piece that is one square by two squares
  - We say that a board is tiled by dominoes when all its squares are covered with no overlapping dominoes and no dominoes overhanging the board.

Tiling of Checkerboards

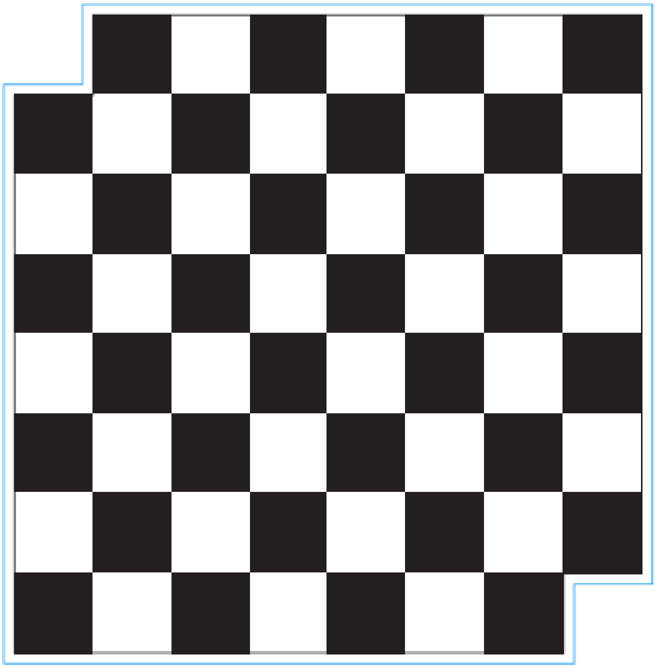

FIGURE 5 The Standard Checkerboard with the Upper Left and Lower Right Squares Removed.

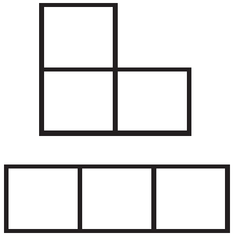

FIGURE 6 A Right Triomino and a Straight Triomino.

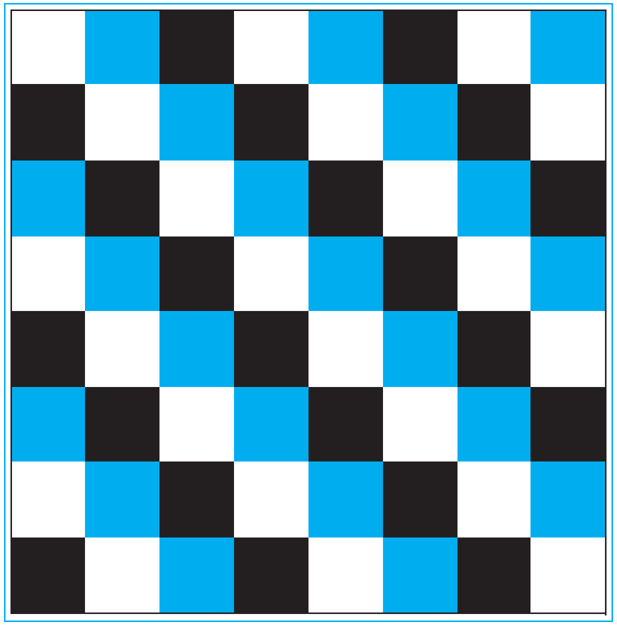

FIGURE 7 Coloring the Squares of the Standard Checkerboard with Three Colors.