4 VECTOR SPACE

4.1. Vector space. Let V be a non-empty set and $\oplus$ be a composition in V i.e. $\oplus$ composes two elements of V ; F be a field with respect to the compositions + and $\bullet$. Let $\odot$ be a composition which composes the elements of F with the elements of V (i.e. $\odot$ is a mapping whose domain is $F \times V$).

V is said to be a vector space over the field F, if the following axioms are satisfied :

I. (i) $\alpha \oplus \beta \in V$ for all α and β in V [closure property under $\oplus$]

(ii) $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$ for all α, β, γ in V [Associative property under $\oplus$]

(iii) V contains an element, say θ, such that $\alpha \oplus \theta = \alpha$ for all α in V.

(iv) Corresponding to each element α in V, there exists an element, say $-\alpha$ in V such that $\alpha \oplus (-\alpha) = \theta$

(v) $\alpha \oplus \beta = \beta \oplus \alpha$ for all α, β in V [commutative property under $\oplus$]

II. (i) $a \odot \alpha, \in V$ for all a in F and for all α in V.

(ii) $e \odot \alpha = \alpha$ for all α in V, where e is multiplicative identity of the field F.

(iii) $(a.b) \odot \alpha = a \odot (b \odot \alpha)$ for all a, b in F and for all α in V.

(iv) $a \odot (\alpha \oplus \beta) = a \odot \alpha \oplus a \odot \beta$ for all a in F and for all α, β in V.

(v) $(a+b) \odot \alpha = a \odot \alpha \oplus b \odot \alpha$ for all a, b in F and for all α in V.

Remark. For convenience we shall use the symbol + to denote both the compositions $\oplus$ and + in V and F.

Thus if $\alpha, \beta \in V$ then, hence forth, $\alpha + \beta$ stands for $\alpha \oplus \beta$; if $a, b \in F$ then $a+b$ stands for $a+b$. Similarly justaposition of elements will be used to denote both the compositions $\odot$ and $\odot$. Thus if $a \in F, \alpha \in V$ then $a\alpha$ stands for $a \odot \alpha$, if $a, b \in F$ then ab stands for $a.b$

Vectors and Scalars.

The elements of a vector space are called vectors and the elements of the related field are called scalars.

Null vector and Additive inverse of a vector.

In a vector space V, we get a vector θ having the property $\alpha+\theta=\alpha$ for all α in V. This vector θ is called the null vector of V.

Illustrations.

1. Consider the set R^n (or V_n) $= R\times R\times R\times.....\times R$ (n times) where R is set of real numbers. So $R^n=\{(x_1,x_2,...x_n): x_i\in R\}$.

Let $\alpha=(x_1,x_2,...x_n)$ and $\beta=(y_1,y_2,\cdots y_n)$ be two elements of R^n. The composition $+$ between them is defined as $\alpha+\beta=(x_1+y_1,x_2+y_2,......,x_n+y_n)$ and the composition between an element a of the field R and an element α of R^n is defined as

$$a\alpha=a(x_1,x_2.....x_n)=(ax_1,ax_2,.....ax_n).$$

We shall show R^n is a vector space of the field R.

Let $\alpha,\beta,\gamma\in R^n$. So $\alpha=(x_1,x_2,...x_n)$, $\beta=(y_1,y_2\cdots y_n)$ and $\gamma=(z_1,z_2,....z_n)$ where x_i,y_i and z_i are all real numbers.

I (i) Then $\alpha+\beta=(x_1+y_1,x_2+y_2,...,x_n+y_n)$

$\therefore \alpha+\beta\in R^n$ since x_i+y_i is also real.

(ii) Now $\alpha+(\beta+\gamma)=(x_1,x_2,...x_n)+\{(y_1,y_2\cdots y_n)+(z_1,z_2\cdots z_n)\}$

$$=(x_1,x_2,....x_n)+(y_1+z_1,y_2+z_2,.....y_n+z_n)$$
$$=(x_1+y_1+z_1,x_2+y_2+z_2,....,x_n+y_n+z_n)$$
$$=\{(x_1,x_2,....,x_n)+(y_1,y_2,....y_n)\}+(z_1,z_2,....z_n)$$
$$=(\alpha+\beta)+\gamma$$

(iii) Since 0 is a real number so $\theta=(0,0,.....0)\in R^n$

Moreover, we see $\alpha+\theta=(x_1,x_2,....x_n)+(0,0,....0)$

$$=(x_1,x_2,....x_n)=\alpha$$

$\therefore R^n$ contains null vector $\theta=(0,0,....0)$

(iv) Since $-x_1, -x_2, \ldots\ldots x_n$ are also real numbers

so $-\alpha = (-x_1, -x_2, \ldots -x_n) \in R^n$.

Moreover, we see

$$\alpha + (-\alpha) = (x_1, x_2, \ldots x_n) + (-x_1, -x_2, \ldots -x_n)$$
$$= (x_1 - x_1, x_2 - x_2, \ldots, x_n - x_n) = (0,0,\ldots 0) = \theta$$

$\therefore R^n$ contains additive inverse of each element.

(v) $\alpha + \beta = (x_1, x_2, \ldots\ldots x_n) + (y_1, y_2, \ldots y_n)$

$$= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$
$$= (y_1 + x_1, y_2 + x_2 \ldots\ldots, y_n + x_n)$$
$$= (y_1, y_2, \ldots\ldots y_n) + (x_1, x_2, \ldots\ldots x_n) = \beta + \alpha$$

II. Let $a, b \in R$

(i) Then $a\alpha = a(x_1, x_2, \ldots x_n) = (ax_1, ax_2, \ldots ax_n) \in R^n$ since ax_i are all reals.

(ii) $1\alpha = 1(x_1, x_2, \ldots x_n) = (x_1, x_2, \ldots x_n) = \alpha$

(iii) $(ab)\alpha = (ab)(x_1, x_2, \ldots x_n) = (abx_1, abx_2, \ldots abx_n)$

$$= a(bx_1, bx_2, \ldots bx_n) = a\{b(x_1, x_2, \ldots x_n)\}$$
$$= a(b\alpha)$$

(iv) $a(\alpha + \beta) = a\{(x_1, x_2, \ldots x_n) + (y_1, y_2, \ldots y_n)\}$

$$= a(x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n)$$
$$= (a(x_1 + y_1), a(x_2 + y_2), \ldots\ldots a(x_n + y_n))$$
$$= (ax_1 + ay_1, ax_2 + ay_2, \ldots, ax_n + ay_n)$$
$$= (ax_1, ax_2, \ldots\ldots, ax_n) + (ay_1, ay_2, \ldots\ldots ay_n)$$
$$= a(x_1, x_2, \ldots x_n) + a(y_1, y_2 \ldots y_n) = a\alpha + a\beta$$

(v) $(a+b)\alpha = (a+b)(x_1, x_2, \ldots x_n)$

$$= ((a+b)x_1, (a+b)x_2, \ldots (a+b)x_n)$$
$$= (ax_1 + bx_1, ax_2 + bx_2, \cdots, ax_n + bx_n)$$

$$= (ax_1, ax_2, ..., ax_n) + (bx_1, bx_2, ... bx_n)$$

$$= a(x_1, x_2, ..., x_n) + b(x_1, x_2, ... x_n)$$

$$= a\alpha + b\alpha$$

Thus all the axioms of vector space are satisfied. Therefore R^n is a vector space over the field of real numbers.

Note. (1) Thus an n-tuple is a vector e.g. $\alpha = (1, -2, 0, 8)$ is a vector belonging to the vector space R^4 or V_4.

(2) $\theta = (0,0,0,0,0)$ is the null vector belonging to R^5 or V_5.

Theorems: In a vector space V over the field F

(1) $0\alpha = \theta$ (null vector) for all α in V (where 0 is the zero element of F)

(2) $a\theta = \theta$ for all a in F

(3) $-e\alpha = -\alpha$ for all α in V (where e is the identity element of F).

(4) $a\alpha = \theta \Rightarrow a = 0$ or $\alpha = \theta$

Proof. (1) Let $a \in F$. We know $a + 0 = a$

$\therefore (a + 0)\alpha = a\alpha$ or, $a\alpha + 0\alpha = a\alpha$

or, $-(a\alpha) + a\alpha + 0\alpha = -(a.\alpha) + a\alpha$

or, $\theta + 0\alpha = \theta$ $\quad [\because -(a\alpha)$ is additive inverse of $a\alpha]$

or, $0\alpha = \theta$

(2) Let $\alpha \in V$ and $a \in F$.

We know, $\alpha + \theta = \alpha$ $\therefore a(\alpha + \theta) = a.\alpha$

or, $a.\alpha + a.\theta = a.\alpha$ by an axiom of vector space.

or, $-(a.\alpha) + (a.\alpha + a.\theta) = -(a.\alpha) + a.\alpha$

or, $\{-(a\alpha) + (a\alpha)\} + a\theta = \theta$

or, $\theta + a\theta = \theta$ or, $a\theta = \theta$ $\quad [\because \theta$ is null vector]

(3) we see, $\alpha + (-e\alpha) = (e\alpha) + (-e\alpha)$ $\quad [e\alpha = \alpha$ by an axiom of vector space]

$= (e + (-e))\alpha = 0\alpha = \theta$ by Th 1.

Thus $\alpha+(-e\alpha)=\theta$

So $-e\alpha$ is additive inverse of α, i.e. $-e\alpha=-\alpha$

(4) Let $a\alpha=\theta$ holds but $a\neq 0$. (1)

Then the multiplicative inverse of a, say a' exists in the field F. So from (1) we get

$a'(a\alpha)=a'\theta$ or, $(a'a)\alpha=\theta$ by Th.2

or, $e\alpha=\theta$ or, $\alpha=\theta$ using an axiom of vector space.

Thus if $a\neq 0$ then $\alpha=\theta$.

Again let (1) holds but $\alpha\neq\theta$. We have to show $a=0$.

Let, if possible, $a\neq 0$. Then by the above process we can show $\alpha=0$ which contradicts our hypothesis $\alpha\neq\theta$.

So $a=0$.

4.2. Subspace. Let V a vector space of the field F. A subset S of V is called subspace of V if S itself is a vector space over the same field F under the same composition of V.

Theorem.(A criterion for subspace)

A subset S of a vector space V is subspace, if and only if

(i) $\alpha+\beta\in S$ for all α,β in S

(ii) $c\alpha\in S$ for all c in R, α in S

Proof. Beyond the scope of this book.

Illustration. The set $S=\{(x,y,0): x,y \text{ are real}\}$ is a subspace of the vector space R^3.

Clearly $S\subset R^3$.

Let $\alpha,\beta\in S$. So $\alpha=(a_1,b_1,0)$ and $\beta=(a_2,b_2,0)$ where a_i,b_i are reals. Then $\alpha+\beta=(a_1+a_2,b_1+b_2,0)\in S$ because a_1+a_2, b_1+b_2 are also reals.

Again for any real number c, $c\alpha=(ca_1,cb_1,0)\in S$ because ca_1,cb_1 are reals.

Thus S satisfies the two critieria of subspace.

Hence S is a subspace of R^3.

Theorem. Intersection of two subspaces is a subspace.

Proof. Let V be a vector space of the field F. S and T be two subspaces of V.

We shall show that $S \cap T$ is a subspace of V.

Since S and T are subsets of V, so their intersection $S \cap T \subset V$.

Let $\alpha, \beta \in S \cap T$. So $\alpha, \beta \in S$ and $\alpha, \beta \in T$.

$\therefore \alpha + \beta \in S$ and $\alpha + \beta \in T$, since S and T are subspaces.

So, $\alpha + \beta \in S \cap T$.

Let $\alpha \in S \cap T$ and $\alpha \in F$. $\quad \therefore \alpha \in S$ and $\alpha \in T$

Since S and T are subspaces so $a\alpha \in S$ and $a\alpha \in T$.

Hence $a\alpha \in S \cap T$

So, $S \cap T$ is a subspace of V.

Remark. *Union of two subspaces may not be a subspace.*

For example $S = \{(0, x) : x \text{ is real }\}$ and $T = \{(x, 0) : x \text{ is real }\}$ are two subspaces of the vector space R^2.

We see $\alpha = (0,3) \in S \cup T (\because \alpha \in S)$ and $\beta = (3,0) \in S \cup T (\because \beta \in T)$.

But $\alpha + \beta = (0,3) + (3,0) = (3,3) \notin S \cup T$, because it neither belongs to S nor to T. Thus closure property does not hold in $S \cup T$.

So, $S \cup T$ is not a subspace.

4.3. Linear Combination of vectors.

Let $\alpha_1, \alpha_2, \ldots.. \alpha_r$ be r number of vectors in a vector space V over a field F. Then $c_1\alpha_1 + c_2\alpha_2 + \ldots + c_r\alpha_r = \sum_{i=1}^{r} c_i\alpha_i$ is called a linear combination of the vectors $\alpha_1, \alpha_2 \ldots . \alpha_r$, where $c_1, c_2, \ldots . c_r$ belong to F.

Example: Let $\alpha_1 = (3,0), \alpha_2 = (1,4), \alpha_3 = (4,-2)$ be three vectors of $R^2 = R \times R$

Then $3\alpha_1 - 7\alpha_2 + 0\alpha_3 = 3(3,0) - 7(1,4) + 0(4,-2) = (9,0) - (7,28) + (0,0)$

$= (9 - 7 + 0, 0 - 28 + 0) = (2, -28)$ is a linear combination of α_1, α_2 and α_3.

Note. By the axioms of vector space, a linear combination of some vectors in a vector space V again belongs to the same vector space.

4.4. Linear dependence and independence of vectors.

Let V be a vector space over a field F. The vectors $\alpha_1,\alpha_2,...\alpha_r$ are called linearly dependent (or simply dependent), if it is possible to get r scalars $c_1,c_2,....c_r$ in F, at least one non-zero, such that

$$c_1\alpha_1+c_2\alpha_2+...+c_r\alpha_r=\theta$$

If $\alpha_1,\alpha_2,.....\alpha_r$ are not linearly dependent then they are called linearly independent (or simply independent); that is $\alpha_1,\alpha_2,.....\alpha_r$ are linearly independent if $c_1\alpha_1+c_2\alpha_2+....+c_r\alpha_r=\theta$ implies $c_1=c_2=c_3=.....=c_r=0$ only.

Illustrations.

(i) The vectors $\alpha_1=(1,2,3),\alpha_2=(3,-1,4)$ and $\alpha_3=(4,1,7)$ of R^3 are linearly dependent because we get the numbers 1,1 and -1 in R such that $1\alpha_1+1\alpha_2+(-1)\alpha_3=(0,0,0)=\theta$

(ii) The vector $(3,2),(6,4),(9,15)$ are linearly dependent because we get three reals 1, $-\frac{1}{2}$ and 0 for which

$$1(3,2)+\left(-\frac{1}{2}\right)(6,4)+0(9,15)=(0,0)=\theta$$

(iii) *The vector* $\alpha_1=(1,0,0)$, $\alpha_2=(0,1,0)$ *and* $\alpha_3=(0,0,1)$ *are linearly independent,* because

$$c_1\alpha_1+c_2\alpha_2+c_3\alpha_3=\theta \Rightarrow c_1(1,0,0)+c_2(0,1,0)+c_3(0,0,1)=(0,0,0)$$

$$\Rightarrow (c_1,c_2,c_3)=(0,0,0)$$

$$\Rightarrow c_1=0,c_2=0 \text{ and } c_3=0$$

(iv) *The vectors in* R^n, $\alpha_1=(1,0,....0),\alpha_2=(0,1,0....0).....\alpha_n$ $=(0,0,...0,1)$ *are linearly independent,* because $c_1\alpha_1+c_2\alpha_2+.......+c_r\alpha_r=\theta$

$$\Rightarrow c_1(1,0,....0)+c_2(0,1,...0)+...+c_r(0,0,...1)=(0,0,....0)$$

$$(c_1,c_2,....c_r)=(0,0,...0)$$

$$\Rightarrow c_1=0,\ c_2=0,\ .....c_r=0$$

So $\alpha_1, \alpha_2, \ldots\ldots \alpha_r$ are independent.

Theorem 1. A collection of vectors containing null vector is linearly dependent.

Proof. Let $\alpha_1, \alpha_2, \ldots\ldots \alpha_r$, θ be a collection of vectors in a vector space V over the field F.

We can write $0\alpha_1 + 0\alpha_2 + \ldots + 0\alpha_r + e\theta = \theta + \theta + \ldots + \theta = \theta$

($\because 0\alpha = \theta$ and $\alpha + \theta = \alpha$ for all α in V).

We see among the scalars $0, 0, \ldots 0, e$, at least one, namely e is non-zero.

So, $\alpha_1, \alpha_2, \ldots \alpha_r, \theta$ are linearly dependent.

Illustartion. The vectors $\alpha = (4,1,3,4), \beta = (-1,2,-2,4)$ and $\gamma = (0,0,0,0)$ of R^4 are linearly dependent.

Theorem2. A collection of vectors which contains a collection of linearly dependent vectors is linearly dependent.

Proof. Let $\alpha_1, \alpha_2, \ldots \alpha_p, \alpha_{p+1}, \ldots\ldots \alpha_r$ be a collection of vectors, of which, the collection $\alpha_1, \alpha_2, \ldots \alpha_p$ are linearly dependent. So, we get the scalars $c_1, c_2, \ldots\ldots c_p$, at least one non-zero, such that

$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_p\alpha_p = \theta$. This implies

$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_p\alpha_p + 0\alpha_{p+1} + \ldots + 0\alpha_r = \theta$

Since at least one of $c_1, c_2, \ldots c_p$ is non-zero so at least one of $c_1, c_2, c_p, 0, 0, \ldots 0$ is non-zero.

So, $\alpha_1, \alpha_2, \ldots\ldots \alpha_p, \alpha_{p+1}, \ldots \alpha_r$ are linearly dependent.

Illustrations. Consider the five vectors $\alpha_1 = (1,2,3)$, $\alpha_2 = (2,4,6)$ $\alpha_3 = (5,9,1)$, $\alpha_4 = (-6,7,8)$, and $\alpha_5 = (11,2,5)$ of the vector space R^3 over the real field R.

We see $2(1,2,3) + (-1)(2,4,6) = (0,0,0)$. So, $(1,2,3)$ and $(2,4,6)$ are linearly dependent. So, by above theorem, the given five vectors are also linearly dependent.

D **Theorem 3.** Any part of a collection of linearly independent vectors is linearly independent.

Proof. Follows from the above theorem.

Illustration. Since the four vectors $(1,0,0,0), (0,1,0,0), (0,0,1,0)$ and $(0,1,0,0)$ are linearly independent so the vectors $(1,0,0,0)$, $(0,0,1,0)$ and $(0,1,0,0)$ are also linearly independent.

An important theorem is stated without proof.

Theorem 4. The n number of vectors

$$\alpha_1 = (a_{11}, a_{12}, \ldots a_{1n}), \alpha_2 = (a_{21}, a_{22}, \ldots a_{2n}), \ldots, \alpha_n = (a_{n1}, a_{n2}, \ldots\ldots a_{nn})$$

will be independent, if and only if, the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

Illustration. The three vectors $(1,2,3)$, $(-1,1,0)$ and $(0,3,3)$ are dependent, because the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 0 & 3 & 3 \end{vmatrix} = 0$$

4.5. Generator or Spanning vectors. Let V be a vector space over the field F. Also let $\alpha_1, \alpha_2, \ldots \alpha_r$ be r number of vectors of V and S be a subspace of V (S may be equal to V). If every element of S can be expressed as a linear combination of $\alpha_1, \alpha_2, \ldots \alpha_r$ then we say $\alpha_1, \alpha_2, \ldots \alpha_r$ generate or span the subspace S.

Illustration.

(i) Consider $S = \{(x, y, 0) : x, y$ are any real numbers $\}$ which is a subspace of V_3.

We see $(x, y, 0) = x(1,0,0) + y(0,1,0)$ i.e. every element of S can be expressed as linear combination of the two vectors $\alpha_1 = (1,0,0)$ and $\alpha_2 = (0,1,0)$. Hence the two vectors $(1,0,0)$ and $(0,1,0)$ generate S.

(ii) $\alpha_1 = (1,0,0,0), \alpha_2 = (0,1,0,0), \alpha_3 = (0,0,1,0), \alpha_4 = (0,0,0,1)$, *are generators of the entire vector space* R^4.

For any element of R^4, say (x_1,x_2,x_3,x_4) can be expressed as $(x_1,x_2,x_3,x_4)=x_1(1,0,0,0)+x_2(0,1,0,0)+x_3(0,0,1,0)+x_4(0,0,0,1)$

(iii) *The n number of vectors* $\alpha_1=(1,0,0,.....0)$, $\alpha_2=(0,1,0,...0)$ .....$\alpha_n=(0,0,......1)$ *generate the vector space* R^n because for any arbitrary element $\alpha=(x_1,x_2,...,x_n)$ of R^n, we can express

$$\alpha=x_1(1,0,...0)+x_2(0,1,0,....0)+....+x_n(0,0,....0,1)$$
$$=x_1\alpha_1+x_2\alpha_2+....+x_n\alpha_n$$

which shows that every element of R^n can be expressed as linear combination of the vectors $\alpha_1,\alpha_2,...,\alpha_n$.

4.6.Basis. Let V be a vector space over the field F. A set of vectors $\{\alpha_1,\alpha_2,...,\alpha_r\}$ is said to be a basis of V if $\alpha_1,\alpha_2,...\alpha_r$ are linearly independent and if they generate V.

Since a subspace is itself a vector space, so a basis of a subspace can be defined likewise.

Example. The three vectors $\alpha_1=(1,0,0),\alpha_2=(0,1,0)$ and $\alpha_3=(0,0,1)$ are independent and they generate the vector space R^3. So $\{\alpha_1,\alpha_2,\alpha_3\}$ is a basis of R^3.

Again we see the three vectors $\beta_1=(1,1,1)$, $\beta_2=(1,1,0)$ and $\beta_3=(1,0,0)$ are independent as $\begin{vmatrix}1&1&1\\1&1&0\\1&0&0\end{vmatrix}\neq 0$ and generate the vector space R^3 as any vector (x_1,x_2,x_3) can be expressed as $(x_1,x_2,x_3)=x_3(1,1,1)+(x_2-x_3)(1,1,0)+(x_1-x_2)(1,0,0)$. So the three vectors β_1,β_2,β_3 also from a basis of R^3.

Note. In the above example we see though a vector space may have more than one basis, the number of vectors in each of the basis is same.

In fact we have the following theorem.

Theorem. Any two basis of a vector space have same number of vectors.

Proof. Beyond the scope of the book.

4.7. Dimension or Rank of a vector space.

The number of vectors present in a basis of a vector space V is called the dimension of V. It is denoted by dim (V).

Example: (1) Dimension of the vector space R^4 is 4. Since the four vectors $(1,0,0,0),(0,1,0,0),(0,0,1,0)$ and $(0,0,0,1)$ from a basis of R^4.

(2) dim $(R^n)=n$ because $\alpha_1=(1,0,....0),\alpha_2=(0,1,....0)$ ......$\alpha_n=(0,0,...1)$ form a basis of R^n.

Note. (1) The dimension of a vector space may be infinite. Here we are concerned with finite dimensional vector space.

(2) In light of the above theorem, we see the dimension of a space is unique.

Theorem 1. If a vector space has dimension r, then the vector space may have at most r number of linearly independent vectors.

Proof. Beyond the scope of the book.

Illustration. The vectors $\alpha=(1,5)$, $\beta=(22,10)$, $\gamma=(3,14)$ and $\delta=(16,2)$ are linearly dependent in the vector space R^2, because dim (R^2) $=2$

Theorem 2. If the dimension of a vector space is r and if a collection of r number of vectors of V are linearly independent, then the collection is a basis of V.

Proof. Beyond the scope of this book.

Illustration. (i) The four vectors $\alpha_1=(1,2,3,0)$, $\alpha_2=(2,1,0,3)$, $\alpha_3=(1,1,1,1)$ and $\alpha_4=(2,3,4,1)$ of R^4 are independent because

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 \end{vmatrix} \neq 0$$

So by the above theorem we can say they would form a basis of R^4.

Illustrative Examples.

Example 1. Prove that the set $R^3 = \{(x,y,z): x,y,z \in R\}$ is a vector space over real field where the two compositions + and • are defined as $(x_1,y_1,z_1)+(x_2+y_2+z_2)=(x_1+x_2,y_1+y_2,z_1+z_2)$ and $a.(x_1,y_1,z_1)=(ax_1,ay_1,az_1)$, where a is a real number.

Solution : Let us verify the axioms of vector space

Let $\alpha,\beta,\gamma \in R^3$. So $\alpha=(x_1,x_2,x_3)$, $\beta=(y_1,y_2,y_3)$ and $\gamma=(z_1,z_2,z_3)$ where x_i, y_i and z_i are all reals.

I.. Then (i) $\alpha+\beta=(x_1+y_1,x_2+y_2,x_3+y_3)$

$\therefore \alpha+\beta \in R^3$ since x_1+y_1, x_2+y_2 and x_3+y_3 are also real.

(ii) $\alpha+(\beta+\gamma)=(x_1,x_2,x_3)+\{(y_1,y_2,y_3)+(z_1,z_2,z_3)\}$

$$=(x_1,x_2,x_3)+(y_1+z_1,y_2+z_2,y_3+z_3)$$
$$=(x_1+y_1+z_1,x_2+y_2+z_2,x_3+y_3+z_3)$$
$$=(x_1+y_1,x_2+y_2,x_3+y_3)+(z_1,z_2,z_3)$$
$$=\{(x_1,x_2,x_3)+(y_1,y_2,y_3)\}+(z_1,z_2,z_3)$$
$$=(\alpha+\beta)+\gamma$$

(iii) Since 0 is a real number so $\theta=(0,0,0)\in R^3$.

Moreover we see $\alpha+\theta=(x_1,x_2,x_3)+(0,0,0)=(x_1,x_2,x_3)=\alpha$

$\therefore R^3$ contains will null vector $\theta=(0,0,0)$

(iv) Since $-x_1,-x_2,-x_3$ are also reals, so $-\alpha=(-x_1,-x_2,-x_3)\in R^3$.

Moreover we see

$$\alpha+(-\alpha)=(x_1,x_2,x_3)+(-x_1,-x_2,-x_3)=(0,0,0)=\theta$$

$\therefore R^3$ contains additive inverse of each element.

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(v) $\alpha+\beta=(x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3)$

$=(y_1+x_1,y_2+x_2,y_3+x_3)=(y_1,y_2,y_3)+(x_1,x_2,x_3)$

$=\beta+\alpha$

II. Let $a,b,c\in R$

Then (i) $c.\alpha=c.(x_1,x_2,x_3)=(cx_1,cx_2,cx_3)\in R^3$ since cx_1,cx_2 and cx_3 are all reals.

(ii) 1 is identity element of the field of all reals R. Then we see

$1.\alpha=1.(x_1,x_2,x_3)=(a_1,a_2,a_3)=\alpha$

(iii) $(ab).\alpha=(ab).(x_1,x_2,x_3)=(abx_1,abx_2,abx_3)$

$=a.(bx_1,bx_2,bx_3)$

$=a.\{b.(x_1,x_2,x_3)\}=a.(b.\alpha)$

(iv) $a.(\alpha+\beta)=a.\{(x_1,x_2,x_3)+(y_1,y_2,y_3)\}$

$=a.(x_1+y_1,x_2+y_2,x_3+y_3)$

$=(a(x_1+y_1),a(x_2+y_2),a(x_3+y_3))$

$=(ax_1+ay_1,ax_2+ay_2,ax_3+ay_3)$

$=(ax_1,ax_2,ax_3)+(ay_1,ay_2,ay_3)$

$=a.(x_1,x_2,x_3)+a.(y_1,y_2,y_3)=a.\alpha+a.\beta$

(v) $(a.b).\alpha=(a+b).(x_1,x_2,x_3)$

$=((a+b)x_1,(a+b)x_2,(a+b)x_3)$

$=(ax_1+bx_1,ax_2+bx_2,ax_3+bx_3)$

$=(ax_1,ax_2,ax_3)+(bx_1,bx_2,bx_3)$

$=a.(x_1,x_2,x_3)+b.(x_1,x_2,x_3)$

$=a.\alpha+b.\alpha$

Thus all the axioms of vector space are satisfied.

So R^3 is a vector space over real field.

Example 2. Find whether the set $R^2 = \{(x,y): x,y$ are reals $\}$ is a vector space over real field where the compositions + and • are defined as $(x_1,y_1)+(x_2,y_2)=(3y_1+3y_2,-x_1-x_2)$ and $c.(x,y)=3cy-cx$

Solution : Let us verify the axioms of vector space.

Let $\alpha,\beta,\gamma \in R^2$. So $\alpha=(x_1,y_1)$, $\beta=(x_2,y_2)$, $\gamma=(x_3,y_3)$ where x_i, y_i and z_i are all real numbers.

I. Then (i) $\alpha+\beta=(x_1,y_1)+(x_2,y_2)$

$$=(3y_1+3y_2,-x_1-x_2).$$

$\therefore \alpha+\beta \in R^2$ since $3y_1+3y_2$ and $-x_1-x_2$ are also reals.

(ii) $\alpha+(\beta+\gamma)=(x_1,y_1)+\{(x_2,y_2)+(x_3+y_3)\}$

$$=(x_1,y_1)+(3y_2+3y_3,-x_2-x_3)$$

$$=(3y_1+3(-x_2-x_3),-x_1-(3y_2+3y_3))$$

$$=(3y_1-3x_2-3x_3,-x_1-3y_2-3y_3)$$

and $(\alpha+\beta)+\gamma=\{(x_1,y_1)+(x_2,y_2)\}+(x_3,y_3)$

$$=(3y_1+3y_2,-x_1-x_2)+(x_3,y_3)$$

$$=(3(-x_1-x_2)+3y_3,-(3y_1+3y_2)-x_3)$$

$$=(-3x_1-3x_2+3y_3,-3y_1-3y_2-x_3)$$

$$\therefore \alpha+(\beta+\gamma)\neq(\alpha+\beta)+\gamma$$

$\therefore$ the associative law does not hold.

So R^3 is not a vector space with respect to these compositions.

Example 3. Prove that the set of all second order real square matrices is a vector space over real field with respect to matrix addition and multiplication of a matrix by a real number.

Solution : Let M be the set of all second order square matrices $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ where x_i, y_i are real numbers.

Let us verify the axioms of vector space. Let $\alpha,\beta,\gamma \in M$

$\therefore \alpha = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $\beta = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix}$, $\gamma = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ where a_i, b_i, x_i etc. are all real numbers.

I. Then (i) $\alpha+\beta = \begin{pmatrix} a_1+c_1 & b_1+d_1 \\ a_2+c_2 & b_2+d_2 \end{pmatrix} \in M \because a_1+c_1, b_1+d_1$ etc. are all real.

(ii) The associative law $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ is well known for matrix addition.

(iii) We see $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M$ as 0 is real and $\alpha + O = \alpha$ $\therefore M$ contains null vector which is O.

(iv) Now, $-\alpha = \begin{pmatrix} -a_1 & -b_1 \\ -a_2 & -b_2 \end{pmatrix} \in M$ as $-a_i, -b_i$ are all reals. We see

$\alpha+(-\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$. So M contains additive inverse of every element.

(v) $\alpha+\beta = \beta+\alpha$ is also well known for matrix addition.

II. Let $a,b,c \in R$

Then (i) $a.\alpha = a.\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} aa_1 & ab_1 \\ aa_2 & ab_2 \end{pmatrix} \in M$

since aa_i, ab_i are also reals.

(ii) 1 is the identity element of the real field.

Now, $1.\alpha = 1.\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \alpha$

(iii) $(ab).\alpha = (ab).\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} aba_1 & abb_1 \\ aba_2 & abb_2 \end{pmatrix}$

$$= a.\begin{pmatrix} ba_1 & bb_1 \\ ba_2 & bb_2 \end{pmatrix} = a.\left\{ b.\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \right\} = a.(b.\alpha)$$

(iv) $a.(\alpha+\beta)=a.\alpha+a.\beta$ is well known for matrix.

(v) $(a+b).\alpha=a.\alpha+b.\alpha$ is well known for matrix.

Thus all the axioms of vector space are satisfied. Hence M is a vector space over real field.

Example 4. Show that $S=\left\{(x_1,x_2,x_3,x_4)\in R^4 : x_1-x_2+x_3=x_4\right\}$ is a subspace of R^4.

Solution : Obviously $S\subset R^4$

Let $\alpha,\beta\in S$

So, if $\alpha=(x_1,x_2,x_3,x_4)$ and $\beta=(y_1,y_2,y_3,y_4)$ then

$x_1-x_2+x_3=x_4$ and $y_1-y_2+y_3=y_4$ (1)

Now $\alpha+\beta=(x_1+y_1,x_2+y_2,x_3+y_3,x_4+y_4)$

Here $(x_1+y_1)-(x_2+y_2)+(x_3+y_3)$

$$=(x_1-x_2+x_3)+(y_1-y_2+y_3)=x_4+y_4$$

$\therefore \alpha+\beta\in S$

Let c be any real number. Then $c.\alpha=(cx_1,cx_2,cx_3,cx_4)$. Here we see $cx_1-cx_2+cx_3=c(x_1-x_2+x_3)=cx_4$ by (1)

$\therefore c.\alpha\in S$

Thus S is a subspace of R^4.

Example 5. Show that the set $\{(0,x,y):x,y$ are reals $\}$ is a subspace of R^3.

Solution : Let $S=\{(0,x,y):x,y$ are reals $\}$ obviously $S\subset R^3$.

Let $\alpha,\beta\in S$ $\therefore \alpha=(0,x_1,y_1)$ and $\beta=(0,x_2,y_2)$ where x_i,y_i are reals.

$\therefore \alpha+\beta=(0,x_1,y_1)+(0,x_2,y_2)=(0,x_1+x_2,y_1+y_2)\in S$

since x_1+x_2 and y_1+y_2 are real numbers.

Let c be any real number. Then $c.\alpha=(0,cx_1,cy_1)\in S$ since cx_1,cy_1 are real numbers.

Thus S is a subspace of R^3.

Example 6. Show that the set $S=\{(x,y,z):x+y-z=0,2x-y+z=0\}$ is a subspace of R^3.

Solution : Obviously $S\subset R^3$

Let $\alpha,\beta\in R^3$. $\therefore \alpha=(x_1,y_1,z_1)$ and $\beta=(x_2,y_2,z_2)$

where $x_1+y_1-z_1=0$ (1)

$2x_1-y_1+z_2=0$ (2)

and $x_2+y_2-z_2=0$ (3)

$2x_2-y_2+z_2=0$ (4)

Adding (1) and (3) we get $(x_1+x_2)+(y_1+y_2)-(z_1+z_2)=0$

Adding (2) and (4) we get $2(x_1+x_2)-(y_1+y_2)+(z_1+z_2)=0$

These show that $(x_1+x_2,y_1+y_2,z_1+z_2)\in S$

i.e. $\alpha+\beta\in S$

Let c be any real number. Multiplying (1) and (2) by c we get

$$cx_1+cy_1-cz_1=0$$

$$2cx_1-cy_1+cz_1=0$$

These show that $(cx_1,cy_1,cz_1)\in S$ i.e. $c\alpha\in S$.

Hence S is a subspace of R^3.

Example 7. Is $W=\{(x,2y,3z):x,y,z\in R\}$ a subspace of R^3? [C.P. 2007]

Solution : Obviously $W\subset R^3$

Let $\alpha,\beta\in W$. $\therefore \alpha=(x_1,2y_1,3z_1)$ and $\beta=(x_2,2y_2,3z_2)$

$\therefore \alpha+\beta=(x_1+x_2,2(y_1+y_2),3(z_1+z_2))\in W$ since x_1+x_2,y_1+y_2 etc. are real.

Let $c\in R$. Then $c.\alpha=c.(x_1,2y_1,3z_1)$

$=(cx_1,2cy_1,3cz_1)\in W$ since cx_1,cy_1,cz_1 are real.

Example 8. Show that the set $S=\{(x,y,z): x^2+y^2=z^2\}$ is not a subspace of V^3.

Solution : Let $\alpha,\beta\in S$ $\quad\therefore \alpha=(x_1,y_1,z_1)$ and $\beta=(x_2,y_2,z_2)$

$$\therefore x_1^2+y_1^2=z_1^2,\ x_2^2+y_2^2=z_2^2$$

Now $\alpha+\beta=(x_1+x_2,y_1+y_2,z_1+z_2)$

Here $(x_1+x_2)^2+(y_1+y_2)^2=x_1^2+x_2^2+2x_1x_2+y_1^2+y_2^2+2y_1y_2$

$=z_1^2+z_2^2+2x_1x_2+2y_1y_2\neq(z_1+z_2)^2$ because z_1z_2 may not be equal to $x_1x_2+y_1y_2$.

So, S is not a subspace of V_3.

Example 9. Express $(-1,2,4)$ as linear combination of $(-1,2,0)$, $(0,-1,1)$ and $(3,-4,2)$ in the vector space R^3 over real field.

Solution : Let $(-1,2,4)=c_1(-1,2,0)+c_2(0,-1,1)+c_3(3,-4,2)$

or, $(-1,2,4)=(-c_1+3c_3,2c_1-c_2-4c_3,c_2+2c_3)$

$$\therefore -c_1+3c_3=-1$$

$$2c_1-c_2-4c_3=2$$

$$c_1+2c_3=4$$

Solving these three equations we get $c_1=4$, $c_2=2$, $c_3=1$

$$\therefore (-1,2,4)=4(-1,2,0)+2(0,-1,1)+1(3,-4,2).$$

Ex.10. Express the vector $(1,2,5)$ as a linear combination of the vectors $(1,1,1),(2,1,2)$ and $(3,2,3)$.

Solution: Let $\alpha=(1,2,5);\alpha_1=(1,1,1)$, $\alpha_2=(2,1,2)$, $\alpha_3=(3,2,3)$ we express $\alpha=c_1\alpha_1+c_2\alpha_2+c_3\alpha_3$

or, $(1,2,5)=c_1(1,1,1)+c_2(2,1,2)+c_3(3,2,3)$

or, $(1,2,5)=(c_1+2c_2+3c_3, c_1+c_2+2c_3, c_1+2c_2+3c_3)$

This gives $c_1+2c_2+3c_3=1$

$$c_1+c_2+2c_3=2$$

$$c_1+2c_2+3c_3=5$$

These three equations are not solvable, since the coefficient determinant

$$\begin{vmatrix}1 & 2 & 3\\ 1 & 1 & 2\\ 1 & 2 & 3\end{vmatrix}=0$$

$\therefore$ the given vector α can not be expressed as linear combination of α_1, α_2 and α_3.

Example 11. Find whether the vectors $(2,4,0),(0,1,0)$ and $(2,6,2)$ are linearly independent in the real vector space R^3.

Solution : *1st process (using definition) :*

$$c_1(2,4,0)+c_2(0,1,0)+c_3(2,6,2)=(0,0,0)$$

$$\Rightarrow (2c_1+2c_3, 4c_1+c_2+6c_3, 2c_3)=(0,0,0)$$

$$\Rightarrow 2c_1+2c_3=0,\ 4c_1+c_2+6c_3=0 \text{ and } 2c_3=0$$

$$\Rightarrow c_3=0.$$

So from the first $c_1=0$. Then from the second $c_2=0$. Hence the three vectors are linearly independent.

2nd process (Using a Theorem):

We see the determinant $\begin{vmatrix}2 & 4 & 0\\ 0 & 1 & 0\\ 2 & 6 & 2\end{vmatrix}=2(2-0)-4(0-0)+0=4\neq 0$

$\therefore$ the vectors $(2,4,0)$, $(0,1,0)$ and $(2,6,2)$ are independent.

Example 12. Find whether the set $\{(1,0,0),(0,1,0),(8,-1,0)\}$ is linearly independent in the vector space R^3 [C.P. 2005]

Solution :

1st process : *(Using definition)*

$$c_1(1,0,0)+c_2(0,1,0)+c_3(8,-1,0)=(0,0,0)$$

$$\Rightarrow (c_1 + 8c_3, c_2 - c_3, 0) = (0,0,0)$$
$$\Rightarrow c_1 + 8c_3 = 0$$
$$c_2 - c_3 = 0$$

These give $c_2 = c_3$ and $c_1 = -8c_3$

So if c_3 take any value other than 0 say 1 then $c_2 = 1$, $c_1 = -8$. Then we see $-8(1,0,0) + 1(0,1,0) + 1(8,-1,0) = (0,0,0)$

i.e. the vectors $(1,0,0),(0,1,0)$ and $(8,-1,0)$ are linearly dependent.

2nd process (using a Theorem):

We see the determinant $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 8 & -1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} - 0 + 0$

$= 0$

$\therefore$ the vectors $(1,0,0),(0,1,0)$ and $(8,-1,0)$ are linearly dependent.

Example 13. Find k so that the vectors $(1,-1,2),(0,k,3)$ and $(-1,2,3)$ are linearly dependent. [*C.P. 1998,2004*]

Solution : Since the given vectors are linearly dependent

so $\begin{vmatrix} 1 & -1 & 2 \\ 0 & k & 3 \\ -1 & 2 & 3 \end{vmatrix} = 0$

or, $\begin{vmatrix} k & 3 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 0 & k \\ -1 & 2 \end{vmatrix} = 0$

or, $3k - 6 + 3 + 2(0 + k) = 0$

or, $5k - 3 = 0$ $\quad\quad \therefore k = \frac{3}{5}$

Example 14. For what values of x the three vectors $(1,1,2),(x,1,1)$ and $(1,2,1)$ are linearly independent.

Solution : Since the given three vectors are linearly independent, therefore $\begin{vmatrix} 1 & 1 & 2 \\ x & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \neq 0$

or, $1(1-2)-(x-1)+2(2x-1) \neq 0$

or, $-1-x+1+4x-2 \neq 0$

or, $3x-2 \neq 0$ or, $x \neq \frac{2}{3}$

$\therefore$ the value of x is any real number other than $\frac{2}{3}$.

Example 15. Let V be the vector space of all 2×2 matrices over the field of real numbers. Show that the set

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$ is linearly independent.

[B.P. 2003]

Solution : $c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow c_1 = 0,\ c_2 = 0, c_3 = 0$ and $c_4 = 0$, only.

Hence the given set is linearly independent.

Example.16. Verify whether the set of vectors $S = \{(4,3,2),(2,1,4),(2,3,-8)\}$ is a basis of R^3. [C.P. 2008]

Solution : We see the determinant

$$\begin{vmatrix} 4 & 3 & 2 \\ 2 & 1 & 4 \\ 2 & 3 & -8 \end{vmatrix} = 4\begin{vmatrix} 1 & 4 \\ 3 & -8 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 2 & -8 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 4(-8-12)-3(-16-8)+2(6-2)$$

$$= -80+72+8 = 0$$

So the given three vectors in S are linearly dependent. Hence S can not be a basis.

Example 17. Do the vectors $(1,1,0),(1,0,1)$ and $(0,1,1)$ form a basis of the vector space R^3 over the field of real numbers ? [C.P. 2000]

Solution : We see that the determinant $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$

$$= 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 0 = (0-1)-(1-0) = -1-1 = -2 \neq 0$$

$\therefore$ the three vectors $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ are linearly independent in the vector space R^3. Since the dimension of R^3 is 3, so these three vector will form a basis of R^3.

Example 18. Show that $B = \{(1,2,1),(0,1,0),(0,0,1)\}$ is a basis of R^3. Express the vector $(1,2,3) \in R^3$ as a linear combination of the basis B. [C.P. 1999]

Solution : We see that the determinant $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= 1(1-0) - 2(0-0) + 1(0-0) = 1 \neq 0$

So the three vectors in B are linear by independent. Since $\dim(R^3) = 3$ and number of vectors in B is also 3, so B is a basis of the vector space R^3.

Let $(1,2,3) = c_1(1,2,1) + c_2(0,1,0) + c_3(0,0,1)$

or, $(1,2,3) = (c_1, 2c_1, c_1) + (0, c_2, 0) + (0,0,c_3) = (c_1, 2c_1 + c_2, c_1 + c_3)$

$\therefore c_1 = 1$, $2c_1 + c_2 = 2$ and $c_1 + c_3 = 3$

These give $c_2 = 2 - 2\times 1 = 0$, $c_3 = 3 - 1 = 2$.

Thus the vector $(1,2,3)$ is expressed as

$(1,2,3) = 1(1,2,1) + 0(0,1,0) + 2(0,0,1)$ which is a linear combination of B.

Example 19. Find a basis of the real vector space R^3 containing the vectors $(1,2,1)$ and $(2,1,1)$

Solution : We know $\{(1,0,0),(0,1,0),(0,0,1)\}$ is basis of R^3. We pick up the first vector $(1,0,0)$ and then find whether the three vectors $(1,2,1),(2,1,1)$ and $(1,0,0)$ are independent.

We see $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1(0-0)-2(0-1)+1(0-1) = 2-1 = 1 \neq 0$.

$\therefore$ the set $S = \{(1,2,1),(2,1,1),(1,0,0)\}$ is linearly independent.

Since $\dim(R^3) = 3$ and S contains 3 independent vectors, so S is a basis of R^3.

Therefore the required basis is $\{(1,2,1),(2,1,1),(1,0,0)\}$

Example 20. Extend the set $\{(2,1,1),(1,1,1)\}$ to a basis of R^3.

Solution : We know $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of R^3. We select the first vector $(1,0,0)$ from this and then find whether the three vectors $(2,1,1),(1,1,1),(1,0,0)$ are independent.

We see $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 2(0-0)-(0-1)+(0-1) = 1-1 = 0$

$\therefore (2,1,1),(1,1,1),(1,0,0)$ are not independent.

So we replace the selected vector $(1,0,0)$ by $(0,1,0)$ and find whether the vectors $(2,1,1),(1,1,1),(0,1,0)$ are independent.

We see $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 2(0-1)-(0-0)+(1-0) = -2+1 = -1 \neq 0$

$\therefore$ the set $\{(2,1,1),(1,1,1),(0,1,0)\}$ are independent.

Since $\dim(R^3) = 3$ and there are 3 vectors in this set, so this set $\{(2,1,1),(1,1,1),(0,1,0)\}$ is a basis of R^3.

Example 21. Find a basis and the dimension of the subspace W of R^3 where $W = \{(x,y,z) \in R^3 : x+y+z=0\}$ [C.P. 2006]

Solution : Let $\alpha = (x,y,z)$ be arbitrary element of W.

$\therefore x+y+z=0$ or, $z=-x-y$

$\therefore \alpha = (x,y,-x-y)$

$= (x,o,-x)+(o,y,-y)$

$= x(1,0,-1)+y(0,1,-1)$

$\therefore$ every element of W can be expressed as linear combination of the two vectors $\alpha_1 = (1,0-1)$ and $\alpha_2 = (0,1-1)$.

Now $c_1\alpha_1 + c_2\alpha_2 = \theta$

$\Rightarrow c_1(1,0,-1)+c_2(0,1,-1)=(0,0,0)$

$\Rightarrow c_1 = 0$, $c_2 = 0$, $-c_1 - c_2 = 0$, 0,0)

$\Rightarrow (c_1, c_2, -c_1 - c_2) = (0, 0, 0)$

$\Rightarrow c_1 = 0$ and $c_2 = 0$ only.

Hence α_1 and α_2 are independent.

$\therefore \{\alpha_1, \alpha_2\}$ is a basis of W.

Since a basis of W contains two vectors, so $\dim(w) = 2$

Example22. Show that $S = \{(x,2y,3x) : x,y$ are reals $\}$ is a subspace of R^3. Find two basis of S. What is your conclusion about dimension of S?

Solution : Obviously $S \subset R^3$

Let $\alpha, \beta \in S$ $\therefore \alpha(x_1,2y_1,3x_1)$ and $\beta = (x_2,2y_2,3x_2)$

$\therefore \alpha + \beta = (x_1,2y_1,3x_1)+(x_2,2y_2,3x_2)$

$= (x_1+x_2),2(y_1+y_2),3(x_1+x_2) \in S$

and also $c.\alpha = c.(x_1,2y_1,3x_1) = (cx_1,2cy_1,3cx_1) \in S$ for all $c \in R$. So S is a subspace of R^3.

Any vector of S, $(x,2y,3x)$ can be expressed as

$(x,2y,3x) = (x,0,3x)+(0,2y,0) = x(1,0,3)+y(0,2,0)$ (1)

$\therefore$ Any vector of S can be expressed as linear combination of $\alpha_1 = (1,0,3)$ and $\alpha_2 = (0,2,0)$.

$\therefore \alpha_1, \alpha_2$ generate S.

Now, $c_1(1,0,3) + c_2(0,2,0) = (0,0,0)$

$\Rightarrow (c_1, 2c_2, 3c_1) = (0,0,0)$

$\Rightarrow c_1 = 0,\ 2c_2 = 0,\ 3c_1 = 0$

$\Rightarrow c_1 = 0,\ c_2 = 0$ only.

$\therefore \alpha_1, \alpha_2$ are linearly independent.

$\therefore \{\alpha_1, \alpha_2\}$ is a basis of S.

Again (1) can be written as

$(x, 2y, 3x) = -x(-1,0,-3) + (-y)(0,-2,0)$

$\therefore \beta_1 = (-1,0,-3),\ \beta_2 = (0,-2,0)$ generate S.

As above we can show that β_1, β_2 are independent.

$\therefore \{\beta_1, \beta_2\}$ is also a basis of S.

Since a basis of S contain two vectors, therefore $\dim(S) = 2$

Example 23. Prove that $S = \{(x,0,0,0) \in R^4\}$ is a subspace of R^4. Find a basis of this subspace. Hence determine its dimension.

Solution : Obviously $S \subset R^4$.

Let $\alpha, \beta \in S$ $\quad \therefore \alpha(x_1,0,0,0)$ and $\beta = (x_2,0,0,0)$

$\therefore \alpha + \beta = (x_1 + x_2, 0,0,0)$. So $\alpha + \beta \in S$

Let $c \in R$ $\therefore c.\alpha = c(x_1,0,0,0) = (cx_1,0,0,0)$

$\therefore c.\alpha \in S$

So S is a subspace of R^4.

For any $\alpha = (x,0,0,0)$ in S we can express $\alpha = x(1,0,0,0)$

$\therefore S$ is generated by the vector $(1,0,0,0)$

Now, $c_1(1,0,0,0) = \theta \Rightarrow (c_1,0,0,0) = (0,0,0,0) \Rightarrow c_1 = 0$

$\therefore (1,0,0,0)$ is independent. Hence $\{(1,0,0,0)\}$ is a basis of S.

Since the basis contains only one vector, $\therefore \dim(s) = 1$

Exercise-IV

1. Explain the following concept with an example of each :

(i) Vector space;

(ii) Generators of a finite dimensional vector space,

(iii) Basis of a finite dimensional vector space. [C.P. 2005]

2. Define a subspace of a finite dimensional vector space. [C.P. 2006]

3. Answer whether the basis of the vector space R^3 over R is unique. [C.P. 2005]

4. Can the vectors $(1,2,5)$, $(3,1,9)$ form a basis of R^3? Give reasons.

5. Find whether the set $S = \{(x,-x) \in R^2\}$ is a subspace of R^2.

6. Obtain the vectors which generate the subspace $\{(x,0,0,y) : x, y$ are real $\}$.

7. Can the five vectors $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and θ be linearly independent? Give reasons.

8. Is the set $\{(1,0,0,),(0,1,0),(0,0,1),(1,1,1)\}$ a basis of R^3? Give reasons.

9. The vectors $(1,2,3,4)$ and $(7,1,9,1)$ may form a basis of R^4 — comment.

10. The set of vectors $S = \{(1,0,0),(1,2,3),(10,10,12),(5,6,7)\}$ is a basis of R^3 —— comment.

11. Find whether the set $W = \{(x_1, x_2) : x_1, x_2$ are real $\}$ is a vector space where the compositions are defined as

(i) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $c.(x_1, y_1) = (0, cy_1)$

(ii) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $c.(x_1, y_1) = (c^2 x_1, c^2 y_1)$

12. Show that the set of all complex numbers is a vector space over real field w.r.t the usual addition between two complex numbers and usual multiplication between a real number and a complex number.

13. Prove that the set $\left\{\begin{pmatrix} x & y \\ y & z \end{pmatrix} : x, y, z \in R\right\}$ is a vector space under usual matrix addition and usual multiplication between a real number and a matrix.

14. Show that the set $S = \{(x, 2x, 3x) : x \text{ is real}\}$ is a subspace of R^3.

15. Show that the set $S = \{(x+2y, y, -x+3y) : x, y \text{ are real}\}$ is a subspace of R^3.

16. Prove that the set $S = \{(x, y, 0) : x, y \in R\}$ is a subspace of R^3.

[C.P. 2008]

17. Find whether the set $S = \{(1, x, y) \in R^3\}$ is a subspace of R^3.

18. Prove that the set $\{(x, y, z) \in R^3 : 2x + y - z = 0\}$ is a subspace of R^3.

19. Prove that the set $\{(x, y, z) \in R^3 : x + y + z = 0\}$ is a subspace of R^3.

20. Show if $W = \{(x, y, z) : x - 3y + 4z = 0\}$ is a subspace of R^3.

[B.P. 1998]

21. Show that the set $S = \{(x, y, z) : x - y + z = 2\}$ is a subset of R^3 but not a subspace.

22. Show that the set $S = \{(x, y, z) : x + y + z = 1\}$ is not a subspace of R^3.

23. Let $S = \{(x, y, z) : x + y + z = 0, 2x - 3y + z = 0\}$. Show that S is a subspace of R^3.

24. Show that $W = \{(x, y, z) \in R^3 : x + 2y - z = 0, 2x - y + 3z = 0\}$ is a subspace of R^3.

25. Express the vector $(7, 11)$ as a linear combination of the vectors $(2, 3)$ and $(3, 5)$.

[C.P. 1997]

26. Express $(5,2,1)$ as a linear combination of $(1,4,0),(2,2,1)$ and $(3,0,1)$. [C.P. 2008]

27. Show that the vector $(2,-5,3)$ can not be expressed as linear combination of $(1,-3,2),(2,-4,-1)$ and $(1,-5,7)$.

28. Show that the vectors $(2,3,1),(2,1,3)$ and $(1,1,1)$ are linearly dependent.

29. Show that the vectors $(1,5,2),(1,1,0)$ and $(0,0,1)$ are linearly independent in the real vector space R^3. [C.P. 2003]

30. Show that the vectors $(2,4,0),(0,1,0)$ and $(2,6,2)$ are linearly independent in the real vector space R^3. [C.P. 2001]

31. Show that the vectors $(2,1,0),(1,1,0),(4,2,0)$ of R^3 are not linearly independent. [C.P. 1999]

32. Do the vectors $(1,1,2),(3,5,3)$ and $(1,0,0)$ form a linearly independent subset of R^3 ? Justify. [C.P. 1997]

33. Are the vectors $(1,-2,0)$, $(3,0,-2)$ and $(0,-1,-5)$ linearly independent in the real vector space R^3? [B.P. 1996]

34. Are the vectors $(0,2,-4),(1,-2,-1),(1,-4,3)$ linearly independent?

35. Determine x such that the vectors $(1,2,1),(x,3,1)$ and $(2,x,0)$ are linearly dependent.

36. Determine k so that the vectors $(1,3,1),(2,k,0)$ and $(0,4,1)$ are linearly dependent in R^3. [C.P. 2000]

37. $u=(1,-3,2)$ and $v=(2,-1,1)$ in R^3 are two vectors. Is the vector $w=(1,7,-4)$ linearly dependent on u and v ? [B.P. 2002]

38. Show that the set of vectors $\{(1,2,1),(2,1,0),(1,-1,2)\}$ is a basis of R^3. [C.P. 2007, 2002, 1998]

39. Examine whether the set $S=\{(1,0,1),(0,1,1),(1,1,0)\}$ forms a basis of R^3 over R. [C.P. 2005, 2004]

40. Show that the subset $S=\{(2,4,0),(0,1,0),(2,6,2)\}$ of the real vector space R^3 is a basis of R^3. [C.P. 2001]

41. Prove that the vectors $(1,-2,3)$, $(2,3,1)$ and $(-1,3,2)$ form a basis of R^3.

42. Find whether the set of vectors $S=\{(2,1,4),(1,-1,2),(3,1,-2)\}$ is a basis of R^3.

43. Find whether the set $S=\{(1,1,2),(1,2,5),(5,3,4)\}$ is a basis of the vector space R^3.

44. Find a basis of the real vector space R^3 containing the vectors $(1,1,2)$ and $(3,5,2)$. [C.P. 2003]

45. Find a basis of the real vector space R^3 containing the vectors $(1,2,1)$ and $(2,1,1)$.

46. Find a basis for the vector space V_3 that contains the vector $(1,2,1)$ and $(3,6,2)$.

47. Find a basis of R^3 containing the vectors $(1,2,0),(1,3,1)$.

48. Show that the set $S=\{(x,2x,3x):x \text{ is real }\}$ is a subspace of R^3. Find a basis of S. Hence determine its dimension.

49. Find a basis and hence the dimension of the subspace

$$S=\{(x,y,z)\in R^3: 2x+y-z=0\}.$$

50. Show that $W=\{(x,y,z)\in R^3: 2x-y+3z=0\}$ is a subspace of R^3. Find a basis of W. What is its dimension ?

51. Prove that the set $W=\{(x,y,z,w): x=y=0 \text{ and } x,y,z,w\in R\}$ is a subspace of R^4. Find a basis of W. Hence determine $\dim(W)$.

Answers

4. no

5. yes

3. no

7. no

8. no

6. $(1,0,0,0)$ and $(0,0,0,1)$

9. Two vectors can not form a basis of R^4

10. Any basis of R^3 must contain three vectors

11. (i) not (ii) not

17. no

20. yes

25. $(7,11) = 2(2,3) + 1(3,5)$

26. $(5,2,1) = 1(1,4,0) - 1(2,2,1) + 2(3,0,1)$

32. yes

33. yes

34. no

35. $x = 2,1$

36. --2

37. yes

39. is basis

42. yes

43. no

44. $\{(1,1,2),(3,5,2),(1,0,0)\}$ is a basis

45. $\{(1,2,1),(2,1,1),(1,0,0)\}$ is a basis

46. $\{(1,2,1),(3,6,2),(1,0,0)\}$ is a basis

47. $\{(1,2,0),(1,3,1),(0,0,1)\}$ is a basis

48. $\{(1,2,3)\}$ is a basis, $\dim(S) = 1$

49. $\{(1,0,2),(0,1,1)\}; \dim(S) = 2$

50. $\{(1,2,0),(0,3,1)\}$ is basis; $\dim(W) = 2$

51. $\{(0,0,1,0),(0,0,0,1)\}$ is basis, $\dim(W) = 2$