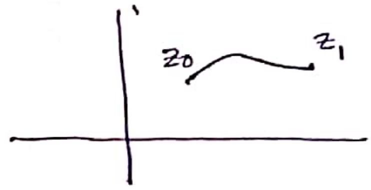
(1)

Complex Integration

Continuous arc: Let $x(t)$ and $y(t)$ be two continuous function of t in $[t_0, T]$, Then the set $\{z : z = x(t) + iy(t) ; t_0 \le t \le T\}$ is called a continuous arc.

[For $t = t_0$, $z = z_0$, for $t = T$, $z = z_1$
For all other t in (t_0, T), ~~the~~ z lies on the path]



A curve C is called simple if $t_1 \neq t_2$ implies $z(t_1) \neq z(t_2)$.

A point z_1 on the arc is called a multiple point if the equation $z_1 = x(t) + iy(t)$ is satisfied by more than one values of t in $[t_0, T]$.

Ex For $z = \cos t + i \sin t$, $0 \le t \le 2\pi$, $t = 0$ is the only multiple point.

An arc $z = x(t) + iy(t)$, $t_0 \le t \le T$ is called regular if it has continuously limiting tangent, that is $x(t)$ and $y(t)$ have continuous derivatives in $[t_0, T]$

A continuous chain of regular arc is said to be a Contour.

Contour

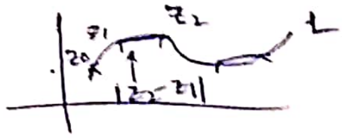
A contour is called closed contour if the two end points of the contour coincide.

closed contour.

Ex $z = a\cos t + ib \sin t$, $0 \le t \le 2\pi$ is a closed contour.

⑫

Rectifiable arc: Let L be a continuous arc and be defined by the set $\{z : z = x(t)+iy(t)\}$, where $x(t)$ and $y(t)$ are continuous functions in $[t_0, T]$. Let $D: t_0 < t_1 < t_2 < \cdots < t_n = T$ be a partition of $[t_0, T]$. Then this will correspond to the points $z_0, z_1, z_2, \ldots, z_n$ on L. We form the sum $\sum_{r=1}^{n} |z_r - z_{r-1}|$. If this sum tends to a definite limit as $\|D\| = \max|t_r - t_{r-1}| \to 0$, then the arc is called rectifiable and the limit is called its length. The length is $\int_{t_0}^{T} [x'^2(t) + y'^2(t)]^{\frac{1}{2}} dt$.



Integration along a regular arc.

Let $L: z = x(t)+iy(t)$, $t_0 \le t \le T$ be a regular arc and let $f(z)$ be defined on L. Let $D: t_0 < t_1 < t_2 < \cdots < t_n = T$ be a partition. Then this will correspond to the points $z_0, z_1, z_2, \ldots, z_n$ on L. Now form the sum $\sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})$, where ξ_r is any point on L lying between z_{r-1} and z_r. If this sum tends to a definite limit as $\|D\| = \max(t_r - t_{r-1}) \to 0$ for all choices of the point ξ_r, then $f(z)$ is called integrable on L and the limit is called its integral and is defined by $\int_L f(z)\,dz$.

$$\int_L f(z)\,dz = \int_{t_0}^{T} f(z(t))\{x'(t) + iy'(t)\}\,dt = \int_{t_0}^{T} F(t)\{x'(t) + iy'(t)\}\,dt.$$

(13)

Th: If L is a regular arc defined by $\{L: z = x(t) + iy(t)\}$ where $x'(t)$ and $y'(t)$ are continuous in $[t_0, T]$, then it is rectifiable and its length is

$$\int_{t_0}^{T} [x'^2(t) + y'^2(t)]^{\frac{1}{2}} dt.$$

Th: If C is a contour of length l and if $f(z)$ is continuous on C and $|f(z)| \le M$ for all z on C, then $\left|\int_C f(z)\,dz\right| \le Ml$.

Proof: Let $f(z) = f(z(t)) = F(t)$ say, then $|F(t)| \le M$ for t in $[t_0, T]$, where $z(t) = x(t) + iy(t)$, $t_0 \le t \le T$ is a regular arc.

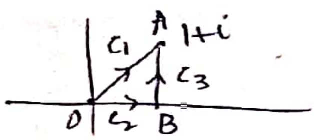
Hence
$$\left|\int_C f(z)\,dz\right| = \left|\int_{t_0}^{T} F(t)[x'(t) + iy'(t)]\,dt\right|$$
$$\le \int_{t_0}^{T} |F(t)|\,|x'(t) + iy'(t)|\,dt$$
$$\le \int_{t_0}^{T} M[x'^2(t) + y'^2(t)]^{\frac{1}{2}} dt = M\int_{t_0}^{T}[x'^2(t) + y'^2(t)]^{\frac{1}{2}} dt$$
$$= Ml$$

$$\therefore \left|\int_C f(z)\,dz\right| \le Ml.$$

Ex Find the value of $\displaystyle\int_0^{1+i} (x - y + ix^2)\,dz$

i) along the straight line from $z=0$ to $z=1+i$

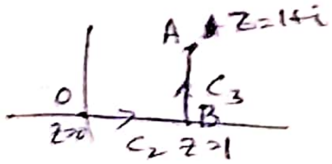
ii) along the real axis from $z=0$ to $z=1$ and then along a line parallel to the imaginary axis from $z=1$ to $z=1+i$



Soln i) Equation of the line C_1 is $x = t$, $y = t$, t varies from 0 to 1.

Then $\displaystyle\int_0^{1+i} (x - y + ix^2)\,dz = \int_0^1 [t - t + it^2][x'(t) + iy'(t)]\,dt$

(19) $= \int_0^1 [i t^2][1+i]\,dt = i(1+i)\frac{t^3}{3}\Big|_0^1 = \frac{(i-1)}{3}$



ii) Along C_2, $y=0$, $x=t$, t varies from 0 to 1. ∴ $\int_{C_2}(x-y+ix^2)\,dz = \int_0^1[t-0+it^2][1+i0]\,dt$

$= \int_0^1 (t+it^2)\,dt = \left(\frac{t^2}{2}+i\frac{t^3}{3}\right)_0^1 = \frac{1}{2}+\frac{i}{3}$

Along C_3, $x=1$, $y=t$, & ∴ $z=1+it$, t varies from 0 to 1. ∴ $\int_{C_3}(x-y+ix^2)\,dz = \int_0^1 (1-t+i)(x'(t)+iy'(t))\,dt$

$= \int_0^1 (1+i-t)(0+i)\,dt = \left[(-1+i)t - i\frac{t^2}{2}\right]_0^1$

$= (-1+i) - \frac{i}{2}$

Hence $\int_0^{1+i}(x-y+ix^2)\,dz = \frac{1}{2}+\frac{i}{3}-1+i-\frac{i}{2} = -\frac{1}{2}+\frac{5i}{6}$

(N.B. See that for part (i) & (ii) along two different paths, solution may not be same).

Ex. Evaluate $\int_C (2x+iy+1)\,dz$, where (i) c is the contour given by c: $z=(t+1)+i(2t^2-1)$, $0 \le t \le 1$

(ii) c is the st. line joining $1-i$ and $2+i$

Soln (i) Along c, $x(t)=t+1$, $y(t)=(2t^2-1)$, t varies from 0 to 1,

∴ $x'(t)+iy'(t) = 1+4it$

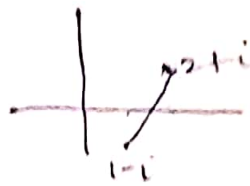
Hence $\int_C(2x+iy+1)\,dz = \int_0^1 [2(t+1)+1+i(2t^2-1)][1+4it]\,dt$

$= \int_0^1 [2t+2+1-4t(2t^2-1)+i\{2t^2-1+4t(2t+3)\}]\,dt$

$= \left(6\frac{t^2}{2}+3t-8\frac{t^4}{4}\right)_0^1 + i\left(10\frac{t^3}{3}-t+12\frac{t^2}{2}\right)_0^1$

$= 3+3-2+i\left(\frac{10}{3}-1+6\right) = 4+\frac{25}{3}i$

(ii) Equation of the st. line passing through $(1,-1)$ & $(2,1)$ is $\frac{y-1}{1+1} = \frac{x-2}{2-1} = t$ say



Thus parametric form of the line is

~~$y = 2t+1$~~ $x = 2+t,\ y = 2t+1$

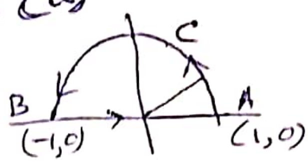
i.e. $z = 2+t+i(2t+1)$, t varies from -1 to 0.

$$\therefore \int_C f(z)\,dz = \int_{-1}^{0} [2+t+i(2t+1)][1+2i]\,dt$$

$$= (1+2i)\left[2t+\frac{t^2}{2}+i\left(\frac{2t^2}{2}+t\right)\right]_{-1}^{0} = (1+2i)\left[2-\frac{1}{2}+i(-1+1)\right]$$

$$= \frac{3}{2}(1+2i).$$

Ex Evaluate $\int_C (z-z^2)\,dz$, where c is the upper half of the positively oriented unit circle. (ii) C is the real line from B to A



Soln The contour C is

$c: z = \cos t + i\sin t$, t varies from 0 to π

so that $z'(t) = -\sin t + i\cos t$

Hence
$$\int_c (z-z^2)\,dz = \int_0^{\pi} [\cos t + i\sin t - (\cos t + i\sin t)^2][-\sin t + i\cos t]\,dt$$

$$= \int_0^{\pi} [e^{it} - e^{2it}]\, i e^{it}\,dt$$

$$= i\int_0^{\pi} [e^{2it} - e^{3it}]\,dt = i\left[\frac{e^{2it}}{2i} - \frac{e^{3it}}{3i}\right]_0^{\pi}$$

$$= \frac{1}{2}[e^{2\pi i} - 1] - \frac{1}{3}[e^{3\pi i} - 1]$$

$$= \frac{1}{2}(1-1) - \frac{1}{3}(-1-1) = \frac{2}{3}$$

(ii) Equation of the line is $c: z = t$, t varies from -1 to 1

$$\therefore \int_c (z-z^2)\,dz = \int_{-1}^{1} (t-t^2)\,1\,dt = \int_{-1}^{1} (t-t^2)\,dt$$

$$= 0 - 2\int_0^1 t^2\,dt = -2\frac{t^3}{3}\Big|_0^1 = -\frac{2}{3}$$