INDIAN INSTITUTE OF ENGINEERING SCIENCE
AND TECHNOLOGY, SHIBPUR

B. TECH 5^{TH} SEMESTER EXAMINATION, DECEMBER 2021

Graph Algorithms [CS 3104]

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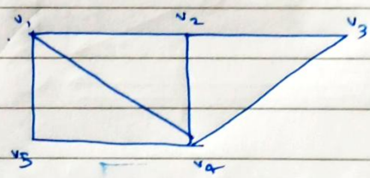
Q1) a) Hamiltonian Path.

→ Hamiltonian Path of a graph G is a path that travels through every vertex exactly once.

Eg



an hamiltonian path in above graph

$v_1 \to v_2 \to v_3 \to v_4 \to v_5$

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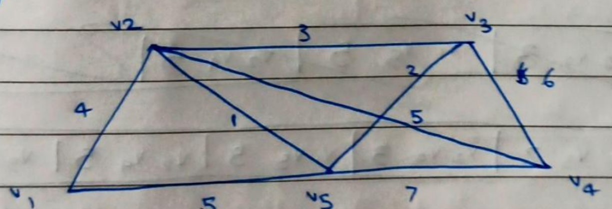
Q1) b) Graph Density

→ Graph Density of a graph G is the ratio of the ~~no. of~~ size of it's existing edges to the maximum possible size of edges.

→ If $G = (V, E)$ and $|V| = n$ and $|E| = m$, then maximum edges possible $= {}^nC_2 = \frac{n(n-1)}{2 \times 1}$

$\therefore \rho(G) = \frac{2m}{n(n-1)}$

Q2) a) Prim Algorithm



(I) → we start with vertex v_1, push (v_1, v_2) & (v_1, v_5) into a min heap.

priority queue = | $(v_1 v_2\ 4)$ | $v_1\ v_5\ 5$ |

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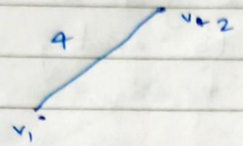
(II) select (v_1 v_2 4) from ~~min heap~~ priority queue. ,add all it's edges to priority queue.

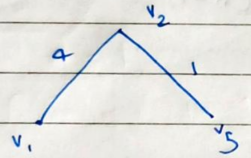
MST



Priority queue = | v_2 v_5 1 | v_2 v_3 3 | v_1 v_5 5 | v_2 v_4 5 |

III → select (v_2 v_5 1) from Priority queue & add it to MST
→ add all edges of v_5 to Priority queue (except v_2 v_5 1)
[also exclude v_1 v_5 5 as v_1 is already visited]

MST

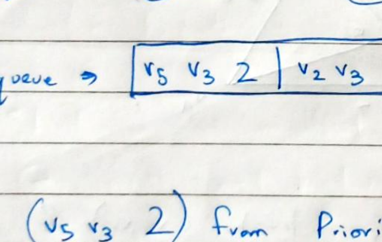


~~Priority queue ⇒ | v_2 v_3 3 | v_1 v_5 5 | v_2 v_4 5 |~~

Priority queue ⇒ | v_5 v_3 2 | v_2 v_3 3 | v_1 v_5 5 | v_2 v_4 5 | v_5 v_4 7 |

IV) → select (v_5 v_3 2) from Priority queue & add it to MST
→ add all edges of v_3 except already visited destination one.

MST



PQ → | v_2 v_3 3 | v_1 v_5 5 | v_2 v_4 5 | v_3 v_4 6 | v_5 v_4 7 |

(V) → select (v_2 v_3 3) from PQ

→ we discard it as v_3 already visited

PQ → | v_1 v_5 5 | v_2 v_4 5 | v_3 v_4 6 | v_5 v_4 7 |

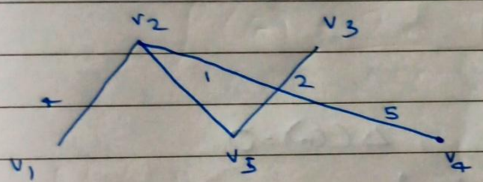
(VI) → select (v_1 v_5 5) from PQ and discard it as v_5 already in MST

PQ → | v_2 v_4 5 | v_3 v_4 6 | v_5 v_4 7 |

(VII) → select (v_2 v_4 5) from PQ

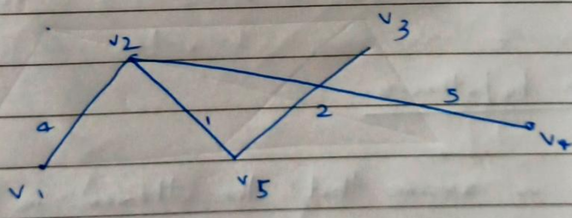
→ and ~~add~~ add all edges of v_4 already not visited.

MST



VIII All ~~e~~ edges in PQ will be discarded now as all vertex included.

End MST



MST weight = 4 + 1 + 2 + 5 = 12

Q2) b) PT Chromatic number of a graph will not exceed by more than one the maximum degree of vertices in a graph

Proof

Let $G = (V, E)$ ~~with~~ be a connected graph

with → ~~that~~ $|V| = n$ → no. of vertices in G

$|E| = m$ → no. of edges in G

$\Delta(G)$ → max (degree of vertex) for G

$\chi(G)$ → chromatic number of G

Let us prove this theorem by ~~pro~~ Induction :

(I) Base case

$n = 1$, $\Delta(G) = 0$

$\chi(G)$ will be 1

∴ The theorem is true.

(II) Inductive hypothesis.

→ "Let us assume that the theorem is true for every Graph G with $n-1$ vertices $(n \geq 2)$

→ we will try to ~~prove~~ prove that theorem will ~~to~~ be true for ~~G~~ a ~~pa~~ graph G_n with n vertices.

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Let G be a graph with n vertices.

let a vertex $v \in V$ be a vertex, $(G - v) = G'$ will be a graph with $n-1$ vertex and there we have assumed that theorem is true.

Two cases arrives

(i) $\Delta(G) == \Delta(G - v)$

→ There is at least one colour of $\Delta(G-v) + 1 = \Delta(G)+1$ color not used by the neighbours of v. So v can be coloured with the extra color.

This gives $\chi(G-v) = \chi(G)+1$

so $\chi(G) \leq \Delta(G)+1$

(ii) $\Delta(G-v) < \Delta(G)$

→ using a new color of v, we will have a $\Delta(G-v)+2$ colors of G

→ Since $\Delta(G-v) + 2 \leq \Delta(G) + 1$, we have

$\chi(G) \leq \Delta(G)+1$

→ In both cases $\chi(G) \leq \Delta(G)+1$ & it holds for $n=1$

So by induction hypothesis, the theorem is proved

Q3) a) Vertex ~~Co~~ Coloring

→ Vertex coloring of a graph G is function which colors vertices of G in such a way that no two adjacent vertices of G will have same color.

$$C : V \rightarrow \{C_1, C_2, \ldots C_k\} \quad s.t \quad \forall (u,v) \in E$$
$$c(u) \neq c(v)$$

5 - Color Theorem

→ Every Planar graph with n vertices can be colored using at most 5 colors.

Proof by Induction

(I) Base Case :

→ for $1 \leq n \leq 5$ this Theorem is trivially true.

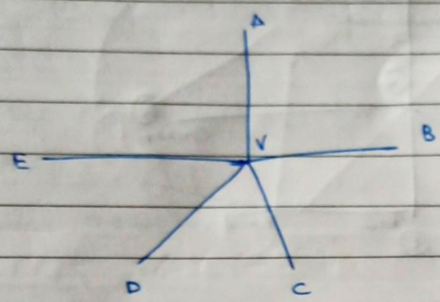
→ as we can give each vertex different color.

(II) Inductive Step

→ Let us assume that $P(n)$ is true, i.e we can color a planar graph of n vertices with at most 5 colors

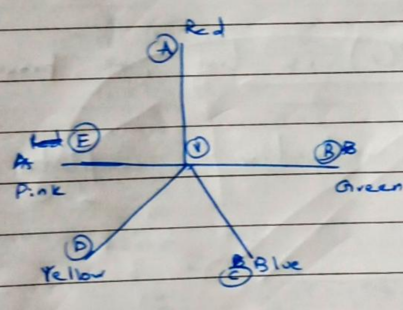
→ we will show that $P(n+1)$ is true.

→ we know that every planar graph has at least one vertex of degree ≤ 5, let this vertex be v



→ for the case $\deg(v) \leq 4$, we can color all vertices of v using 4 colors and color v using fifth color.

→ now for $\deg(v) = 5$, ~~the~~ let coloring be as follows.

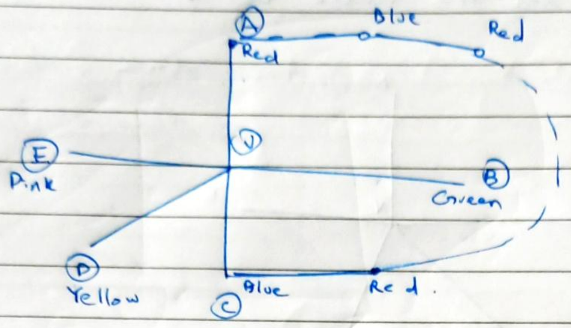


→ Case*: Assume no path from A to C

→ we can change color of A to Blue and color V red

→ we also have change all vertices adjacent to red having color blue to red.

Case: There is a path from A to C



→ As this graph is planar, we can't have a path from B to D

→ so we can chage B to yellow and all neighbour of B having color yellow to green

→ and then we can ~~ch~~ define V's ~~to~~ color to green

→ hence assuming $P(n)$ is true, we proved $P(n+1)$ is true

→ hence proof by induction is complete.

Q3) b) given graph $G = (V, E)$, algorithm to find connected components.

we will use Disjoint Set operations to achive this.

→ Disjoint set operations that will be used are as follows.

i) MAKE_SET (x)

→ creates new set whose only member is x

ii) ~~Union~~ UNION (x, y)

→ ~~unites~~

→ first we check if the representative elements of sets in which x & y are present are same or not [using FIND_SET ()]

→ if same we don't need to Union

→ if different, we merge those two sets and update representative of the union set accordingly.

iii) FIND_SET (u)

→ returns the representative element of the set in which u exists.

Algorithm to get connected components

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connected Components (G):
    for each vertex v ∈ G.V :
        MAKE_SET (v)

    for each edge (u, v) ∈ G.E :
        if FIND-SET (u) ≠ FIND-SET (v) :
            UNION (u, v)
return the disjoint set
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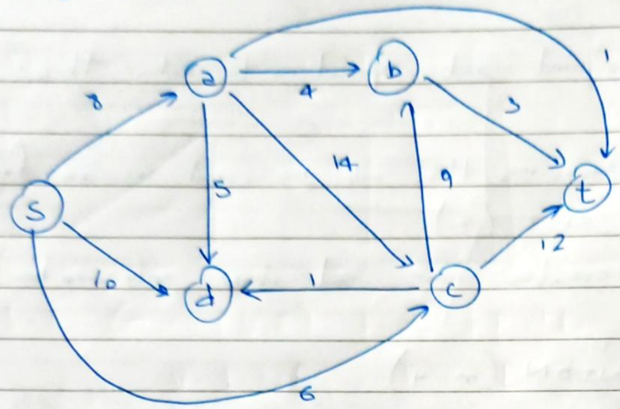
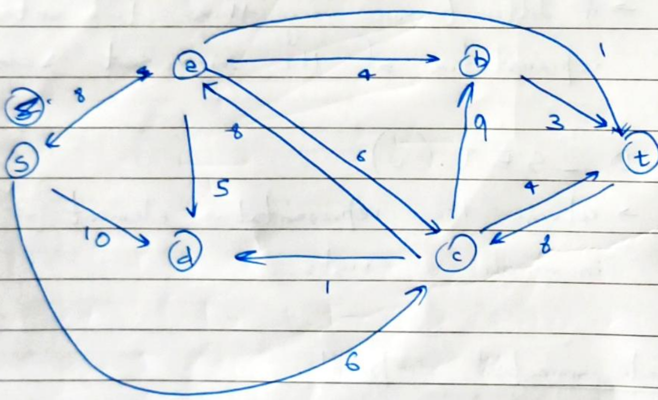
Q5) given graph, find maximum flow



I) path → $s \xrightarrow{8} a \xrightarrow{14} c \xrightarrow{12} t$

min $c_f(p) = 8$ → negate 8 from path p

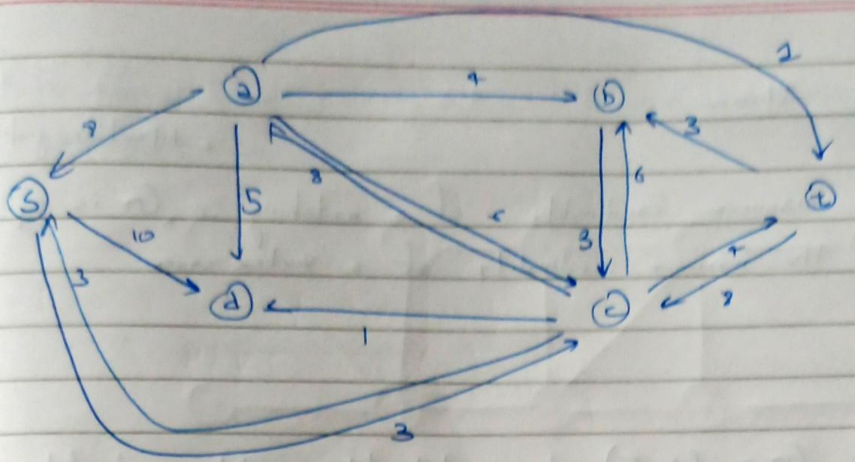


II) path

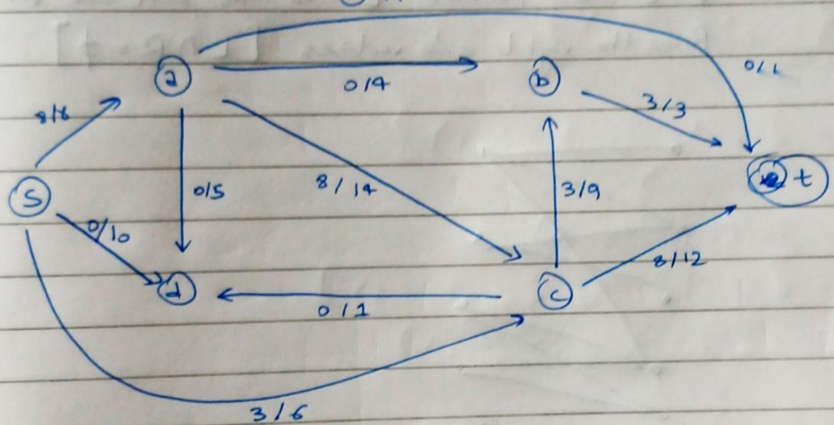
~~s → c → a → b → t~~

$p \Rightarrow s \xrightarrow{6} c \xrightarrow{9} b \xrightarrow{3} t$

min $c_f(p) = 3$

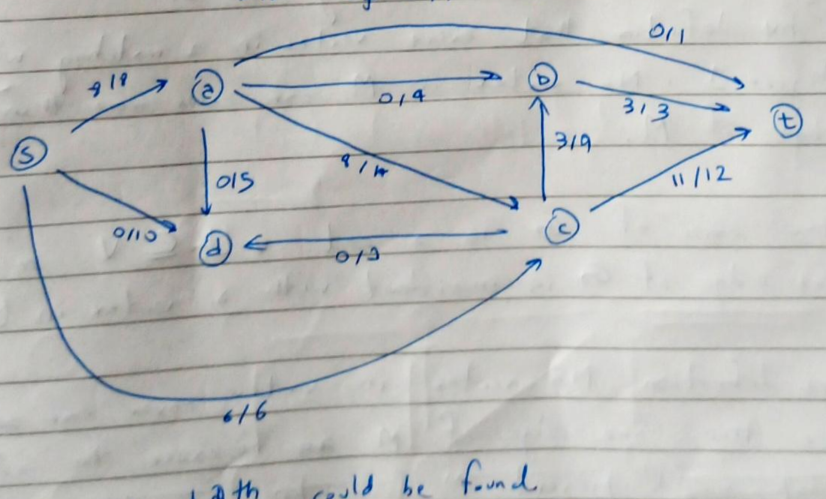


OR



III path $s \xrightarrow{3} c \xrightarrow{4} t$

min $c_f(p) = 3$



→ no more path could be found

maximum flow = 11+ 3+0 = 14

Q(a)(2) Konig's Theorem

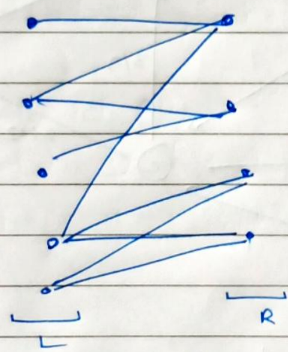
→ The maximum cardinality of a matching in a bipartite graph G is equal to the maximum cardinality of a vertex cover

~~Proof~~

Proof

Let G = (V, E) be a bipartite graph and let $V = L \cup R$, where L is the left side vertices and R is the right side vertices $[L \cap R = \phi]$

Eg



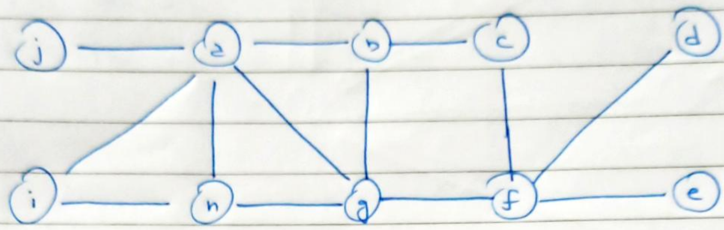
→ suppose M is a maximum matching for G
so by definition that every vertex in a matching will have degree 1. No vertex in M can cover more than one edge of M.

→ Vertex cover of G is a set $U \subseteq V$ if every edge of G is incident with a vertex in U

→ So by definition No vertex in vertex cover can cover more than one edge of M because the edge overlap would prevent M from being a matching.

→ So if a vertex cover with |M| vertices can be constructed, it must be a minimum cover.

Q a) b) given Graph and initial matching , we need to expand it



$M = \{(i,a), (f,e)\}$

I let $P = \{(j,a), (a,i), (i,h)\}$

so $M = (M-P) \cup (P-M)$

$= \{(f,e)\} \cup \{(j,a), (i,h)\}$

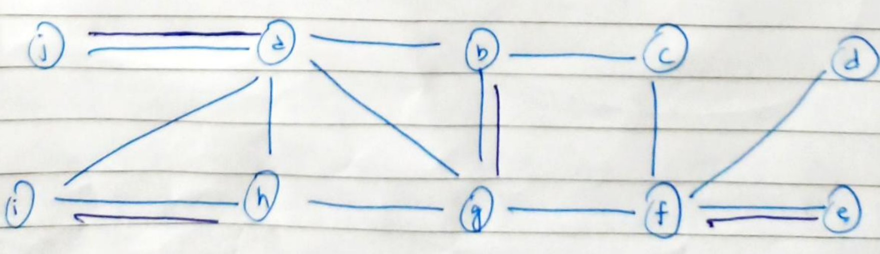
$= \{(f,e), (j,a), (i,h)\}$

II let $P = \{(b,g)\}$

$M = \{(f,e), (j,a), (i,h), (b,g)\}$

no more alternating path can be found

so ~~end~~ matching



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