Proof by Induction (Lecture – 2)

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Proofs by Mathematical Induction (3)

Example: Prove $n < 2^n$ for all positive integers n.

- P(n): $n < 2^n$

Basis Step: $1 < 2^1$ (obvious)

Inductive Step: If P(n) is true then P(n+1) is true for each n.

- Suppose P(n): $n < 2^n$ is true
- Show P(n+1): $n+1 < 2^{n+1}$ is true.

$$
\begin{aligned}
n + 1 &< 2^n + 1 \\
&< 2^n + 2^n \\
&= 2^n(1 + 1) \\
&= 2^n(2) \\
&= 2^{n+1}
\end{aligned}
$$

Proofs by Mathematical Induction (4)

Example: Prove $n^3 - n$ is divisible by 3 for all positive integers.

- $P(n)$: $n^3 - n$ is divisible by 3

Basis Step: $P(1)$: $1^3 - 1 = 0$ is divisible by 3 (obvious)

Inductive Step: If $P(n)$ is true then $P(n+1)$ is true for each positive integer.

- Suppose $P(n)$: $n^3 - n$ is divisible by 3 is true.
- Show $P(n+1)$: $(n+1)^3 - (n+1)$ is divisible by 3.

$$
\begin{aligned}
(n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - n - 1 \\
&= (n^3 - n) + 3n^2 + 3n \\
&= (n^3 - n) + 3(n^2 + n)
\end{aligned}
$$

divisible by 3 divisible by 3

Proof by Mathematical Induction (5)

- Use mathematical induction to show that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.

Solution: Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer n.

BASIS STEP: $P(0)$ is true because $2^0 = 1 = 2^1 - 1$. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis, we assume that $P(k)$ is true for an arbitrary nonnegative integer k. That is, we assume that

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

To carry out the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k+1)$ is also true. That is, we must show that

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$$

assuming the inductive hypothesis $P(k)$. Under the assumption of $P(k)$, we see that

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} \\ &\overset{\text{IH}}{=} (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1. \end{aligned}$$

Note that we used the inductive hypothesis in the second equation in this string of equalities to replace $1 + 2 + 2^2 + \cdots + 2^k$ by $2^{k+1} - 1$. We have completed the inductive step.

Proof by Mathematical Induction (6)

- Use mathematical induction to prove that $2^n < n!$ for every integer n with $n \geq 4$. (Note that this inequality is false for $n = 1, 2$, and 3.)

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

BASIS STEP: To prove the inequality for $n \geq 4$ requires that the basis step be $P(4)$. Note that $P(4)$ is true, because $2^4 = 16 < 24 = 4!$

INDUCTIVE STEP: For the inductive step, we assume that $P(k)$ is true for an arbitrary integer k with $k \geq 4$. That is, we assume that $2^k < k!$ for the positive integer k with $k \geq 4$. We must show that under this hypothesis, $P(k+1)$ is also true. That is, we must show that if $2^k < k!$ for an arbitrary positive integer k where $k \geq 4$, then $2^{k+1} < (k+1)!$. We have

$$\begin{aligned}
2^{k+1} &= 2 \cdot 2^k && \text{by definition of exponent} \\
&< 2 \cdot k! && \text{by the inductive hypothesis} \\
&< (k+1)k! && \text{because } 2 < k+1 \\
&= (k+1)! && \text{by definition of factorial function.}
\end{aligned}$$

This shows that $P(k+1)$ is true when $P(k)$ is true. This completes the inductive step of the proof.

We have completed the basis step and the inductive step. Hence, by mathematical induction $P(n)$ is true for all integers n with $n \geq 4$. That is, we have proved that $2^n < n!$ is true for all integers n with $n \geq 4$. ◀

Proof by Mathematical Induction (7)

- Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

Solution: To construct the proof, let $P(n)$ denote the proposition: "$7^{n+2} + 8^{2n+1}$ is divisible by 57."

BASIS STEP: To complete the basis step, we must show that $P(0)$ is true, because we want to prove that $P(n)$ is true for every nonnegative integer. We see that $P(0)$ is true because $7^{0+2} + 8^{2\cdot 0+1} = 7^2 + 8^1 = 57$ is divisible by 57. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary nonnegative integer k; that is, we assume that $7^{k+2} + 8^{2k+1}$ is divisible by 57. To complete the inductive step, we must show that when we assume that the inductive hypothesis $P(k)$ is true, then $P(k+1)$, the statement that $7^{(k+1)+2} + 8^{2(k+1)+1}$ is divisible by 57, is also true.

The difficult part of the proof is to see how to use the inductive hypothesis. To take advantage of the inductive hypothesis, we use these steps:

$$\begin{aligned} 7^{(k+1)+2} + 8^{2(k+1)+1} &= 7^{k+3} + 8^{2k+3} \\ &= 7 \cdot 7^{k+2} + 8^2 \cdot 8^{2k+1} \\ &= 7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1} \\ &= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}. \end{aligned}$$

We can now use the inductive hypothesis, which states that $7^{k+2} + 8^{2k+1}$ is divisible by 57. We will use parts (i) and (ii) of Theorem 1 in Section 4.1. By part (ii) of this theorem, and the inductive hypothesis, we conclude that the first term in this last sum, $7(7^{k+2} + 8^{2k+1})$, is divisible by 57. By part (ii) of this theorem, the second term in this sum, $57 \cdot 8^{2k+1}$, is divisible by 57. Hence, by part (i) of this theorem, we conclude that $7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1} = 7^{k+3} + 8^{2k+3}$ is divisible by 57. This completes the inductive step.

Proof by Mathematical Induction (8)

- The harmonic numbers H_j, $j = 1, 2, 3, \ldots$, are defined by $H_j = 1 + {}^1/_2 + {}^1/_3 + \cdots + {}^1/_j$. For instance, $H_4 = 1 + {}^1/_2 + {}^1/_3 + {}^1/_4 = {}^{25}/_{12}$. Use mathematical induction to show that $H_{2^n} \geq 1 + {}^n/_2$, whenever n is a nonnegative integer.

Solution: To carry out the proof, let $P(n)$ be the proposition that $H_{2^n} \geq 1 + \frac{n}{2}$.

BASIS STEP: $P(0)$ is true, because $H_{2^0} = H_1 = 1 \geq 1 + \frac{0}{2}$.

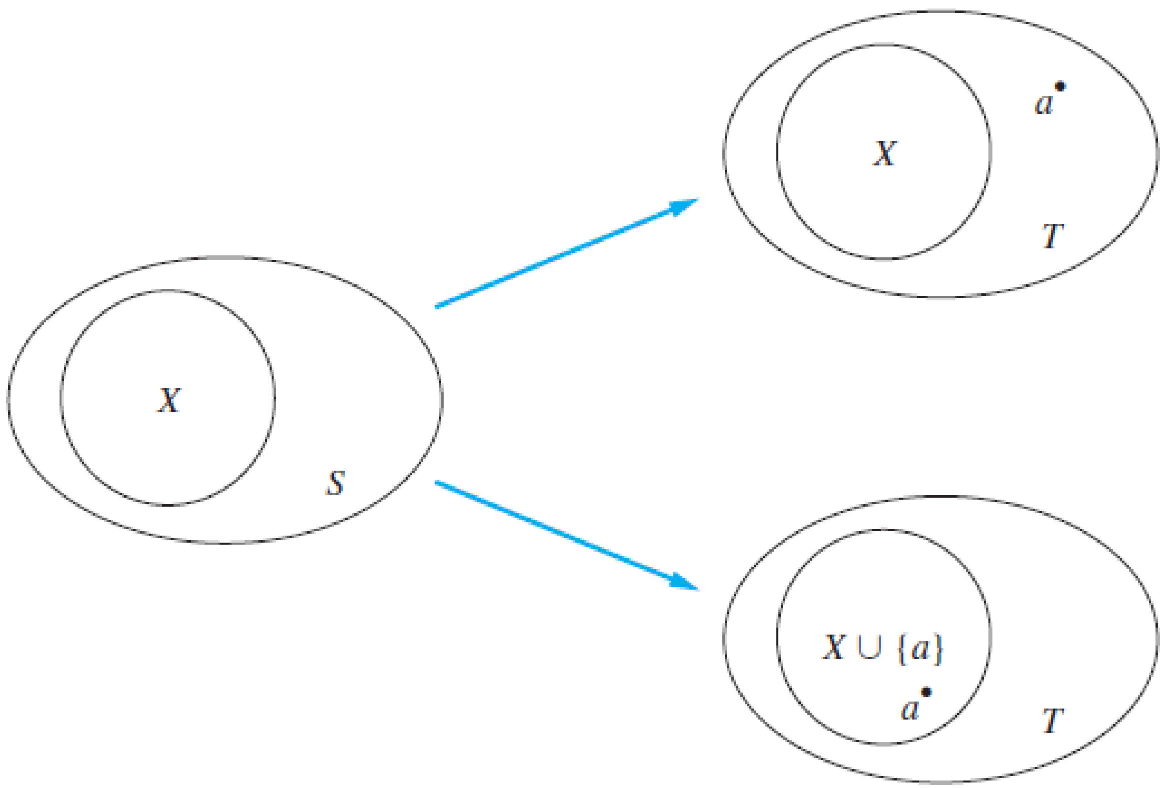
INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is, $H_{2^k} \geq 1 + \frac{k}{2}$, where k is an arbitrary nonnegative integer. We must show that if $P(k)$ is true, then $P(k+1)$, which states that $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$, is also true. So, assuming the inductive hypothesis, it follows that

$$
\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of harmonic number} \\
&= H_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of } 2^k\text{th harmonic number} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the inductive hypothesis} \\
&\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} && \text{because there are } 2^k \text{ terms each} \geq 1/2^{k+1} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2} && \text{canceling a common factor of } 2^k \text{ in second term} \\
&= 1 + \frac{k+1}{2}.
\end{aligned}
$$

This establishes the inductive step of the proof.

Proof by Mathematical Induction (9)

- **The Number of Subsets of a Finite Set** Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets..



Proof by Mathematical Induction (9)

- **The Number of Subsets of a Finite Set** Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets..

Solution: Let $P(n)$ be the proposition that a set with n elements has 2^n subsets.

BASIS STEP: $P(0)$ is true, because a set with zero elements, the empty set, has exactly $2^0 = 1$ subset, namely, itself.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary nonnegative integer k, that is, we assume that every set with k elements has 2^k subsets. It must be shown that under this assumption, $P(k+1)$, which is the statement that every set with $k+1$ elements has 2^{k+1} subsets, must also be true. To show this, let T be a set with $k+1$ elements. Then, it is possible to write $T = S \cup \{a\}$, where a is one of the elements of T and $S = T - \{a\}$ (and hence $|S| = k$). The subsets of T can be obtained in the following way. For each subset X of S there are exactly two subsets of T, namely, X and $X \cup \{a\}$. (This is illustrated in Figure 3.) These constitute all the subsets of T and are all distinct. We now use the inductive hypothesis to conclude that S has 2^k subsets, because it has k elements. We also know that there are two subsets of T for each subset of S. Therefore, there are $2 \cdot 2^k = 2^{k+1}$ subsets of T. This finishes the inductive argument.

Proof by Mathematical Induction (10)

Use mathematical induction to prove the following generalization of one of De Morgan's laws:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_n$ are subsets of a universal set U and $n \geq 2$.

Solution: Let $P(n)$ be the identity for n sets.

BASIS STEP: The statement $P(2)$ asserts that $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$. This is one of De Morgan's laws; it was proved in Example 11 of Section 2.2.

Proof by Mathematical Induction (10)

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, where k is an arbitrary integer with $k \geq 2$; that is, it is the statement that

$$\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_k$ are subsets of the universal set U. To carry out the inductive step, we need to show that this assumption implies that $P(k+1)$ is true. That is, we need to show that if this equality holds for every collection of k subsets of U, then it must also hold for every collection of $k+1$ subsets of U. Suppose that $A_1, A_2, \ldots, A_k, A_{k+1}$ are subsets of U. When the inductive hypothesis is assumed to hold, it follows that

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}} \quad \text{by the definition of intersection}$$

$$= \overline{\left(\bigcap_{j=1}^{k} A_j\right)} \cup \overline{A_{k+1}} \quad \text{by De Morgan's law (where the two sets are } \textstyle\bigcap_{j=1}^{k} A_j \text{ and } A_{k+1})$$

$$= \left(\bigcup_{j=1}^{k} \overline{A_j}\right) \cup \overline{A_{k+1}} \quad \text{by the inductive hypothesis}$$

$$= \bigcup_{j=1}^{k+1} \overline{A_j} \quad \text{by the definition of union.}$$

This completes the inductive step.

Strong Induction

- **<u>Strong induction</u>**: It is another form of mathematical induction which can often be used when we cannot easily prove a result using mathematical induction.
- Basic Principle:

STRONG INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that the proposition $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

- To prove that $P(n)$ is true for all positive integers n, inductive hypothesis for strong induction method is the assumption that $P(j)$ is true for $j = 1, 2, \ldots, k$.
- Here the inductive hypothesis includes all k statements $P(1), P(2), \ldots, P(k)$, unlike the mathematical induction which uses only the statement $P(k)$.

Strong Induction Vs. Mathematical Induction

- **Example**: Suppose we can reach the first and second rungs of an infinite ladder, and we know that if we can reach a rung, then we can reach two rungs higher. Can we prove that we can reach every rung using the principle of mathematical induction? Can we prove that we can reach every rung using strong induction?

Using Mathematical Induction

- *Basis Step:* The basis step of such a proof holds; here it simply verifies that we can reach the first rung.
- *Inductive Step:* The inductive hypothesis is the statement that we can reach the k^{th} rung of the ladder. To complete the inductive step, we need to show that if we assume the inductive hypothesis for the positive integer k, namely, if we assume that we can reach the k^{th} rung of the ladder, then we can show that we can reach the $(k + 1)^{th}$ rung of the ladder. However, there is no obvious way to complete this inductive step because we do not know from the given information that we can reach the $(k + 1)^{th}$ rung from the kth rung. After all, we only know that if we can reach a rung we can reach the rung two higher.

Strong Induction Vs. Mathematical Induction

Using Strong Induction

- *Basis Step:* The basis step of such a proof holds; here it simply verifies that we can reach the first rung.
- *Inductive Step:* The inductive hypothesis states that we can reach each of the first k rungs. To complete the inductive step, we need to show that if we assume that the inductive hypothesis is true, that is, if we can reach each of the first k rungs, then we can reach the $(k + 1)^{th}$ rung. We already know that we can reach the second rung. We can complete the inductive step by noting that as long as $k \geq 2$, we can reach the $(k + 1)^{th}$ rung from the $(k - 1)^{th}$ rung because we know we can climb two rungs from a rung we can already reach, and because $k - 1 \leq k$, by the inductive hypothesis we can reach the $(k - 1)^{th}$ rung. This completes the inductive step and finishes the proof by strong induction.