Practice problems

1. (a) Prove that the union of two subspaces of a vector space $\mathbf{V}$ over a field $\mathbb{F}$ is a subspace of $\mathbf{V}$.

   (b) If $\mathbf{U_1}$, $\mathbf{U_2}$ be two subspaces of a vector space $\mathbf{V}$ over a field $\mathbb{F}$, then the union $\mathbf{U_1} \cup \mathbf{U_2}$ is a subspace of $\mathbf{V}$ if and only if either $\mathbf{U_1} \subset \mathbf{U_2}$ or $\mathbf{U_2} \subset \mathbf{U_1}$.

2. Examine if the set $\mathbf{S}$ is a subspace of $\mathbf{V}$:

   (a) $\mathbf{S} = \{(x, y, z) \in \mathbb{R}^3 : x - y = 1 \ , \ 2z + y = 0\} \ , \ \mathbf{V} = \mathbb{R}^3$.

   (b) $\mathbf{S} = \{(x, y, z, w) \in \mathbb{R}^4 : x = 2y \ , \ x + y + z + w = 0\} \ , \ \mathbf{V} = \mathbb{R}^4$.

   (c) $\mathbf{S} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2} : det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \right\} \ , \ \mathbf{V} = \mathbb{R}^{2\times 2}$

   (d) $\mathbf{S}$ = Set of all $n \times n$ upper triangular matrices , $\mathbf{V} = \mathbb{R}^{n\times n}$.

3. (a) For what conditions in x, y the vectors $\{(x, y, y), (y, x, y), (y, y, x)\}$ is linearly dependent in $\mathbb{R}^3$.

   (b) Let $\{\alpha, \beta, \gamma\}$ be a basis of a real vector space $\mathbf{V}$ and c be a non-zero real number. Prove that $\{\alpha + c\beta \ , \ \beta + c\gamma \ , \ \gamma + c\alpha\}$ may not be a basis of $\mathbf{V}$.

4. Find basis and dimension of the subspace $\mathbf{S}$ of $\mathbf{V}$ defined by :

   (a) $\mathbf{S} = \{(x, y, z, w) \in \mathbb{R}^4 : \ x + y + z = 0\}$, $\mathbf{V} = \mathbb{R}^4$.

   (b) $\mathbf{S} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2} : a = 2b \right\}$, $\mathbf{V} = \mathbb{R}^{2\times 2}$.

   (c) $\mathbf{S}$ = Set of all $n \times n$ real symmetric matrices , $\mathbf{V} = \mathbb{R}^{n\times n}$.

5. Find the coordinate vector of $\alpha = (0, 3, 1) \in \mathbb{R}^3$ relative to the ordered basis $\{(1, 1, 0), \ (1, 0, 1), \ (0, 1, 1)\}$.

   Coordinate Vector :

   Let $\mathbf{B} = \{\alpha_1, \ \alpha_2 \ , \cdots , \ \alpha_n\}$ be an ordered basis of a vector space $\mathbf{V}$ over a field $\mathbb{F}$. Then for each vector $\alpha \in \mathbf{V}$, $\exists \ n$ scalars $c_1 \ , \ c_2 \ , \cdots , \ c_n \in \mathbb{F}$ s.t. $\alpha \ = c_1\alpha_1 \ + \ c_2\alpha_2 + \cdots + \ c_n\alpha_n$.
   The ordered n-tuple $(c_1, \ c_2 \ , \ \cdots \ , \ c_n)$ is said to be the coordinate vector of α relative to the ordered basis $\mathbf{B}$.

6. (a) Examine whether $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (yz\ ,\ zx\ , xy)\ ,\ (x, y, z) \in \mathbb{R}^3$ is a linear map.

   (b) Examine whether $T : \mathbb{R}^{2\times2} \longrightarrow \mathbb{R}^{2\times2}$ defined by $T(A) = \frac{1}{2}(A + A^t)\ ,\ A \in \mathbb{R}^{2\times2}$ is a linear map.

7. Determine the linear mapping $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that maps the basis vectors $(2, 1, 1),\ (1, 2, 1),\ (1, 1, 2)$ of $\mathbb{R}^3$ to the vectors $(1, 1, 1),\ (1, 1, 1),\ (1, 1, 1)$ respectively. Find $Ker(T)$ and $Im(T)$. Verify that $\boldsymbol{dim(Ker(T)) + dim(Im(T))} = 3$.