⑥

Ex Show that the function $f(z)=\sqrt{|xy|}$ satisfies C-R equations at (0,0) but not differentiable at (0,0).

Soln $f(z)=\sqrt{|xy|} \Rightarrow u(x,y)=\sqrt{|xy|}$, $v(x,y)=0$

Now $\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{h\to 0}\frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt{h\times 0}-0}{h}=0$

$\left.\frac{\partial u}{\partial y}\right|_{(0,0)} = \lim_{k\to 0}\frac{u(0,k)-u(0,0)}{k} = \lim_{k\to 0}\frac{0-0}{k}=0$

$\left.\frac{\partial v}{\partial x}\right|_{(0,0)} = \lim_{h\to 0}\frac{v(h,0)-v(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h}=0$

$\left.\frac{\partial v}{\partial y}\right|_{(0,0)} = \lim_{k\to 0}\frac{v(0,k)-v(0,0)}{k} = \lim_{k\to 0}\frac{0-0}{k}=0$

$\therefore$ At (0,0), $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$, that is C-R equations are satisfied at (0,0)

Now $f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{(x,y)\to(0,0)}\frac{\sqrt{|xy|}-0}{x+iy}$

Now along the path $y=mx$,

$f'(0)=\lim_{x\to 0}\frac{\sqrt{|m|x^2}}{x(1+im)} = \frac{\sqrt{|m|}}{1+im}$ which is different for different values of m. Hence $f'(0)$ doesn't exist.

Sufficient Condition.

We have seen that the necessary condition for a function to be derivable (or differentiable) is that it will satisfy C-R equation. The condition is not sufficient, that is if a function satisfy C-R equation then it may not be differentiable at some point (which is evident from the examples given). Now a function $f(z)=u(x,y)+iv(x,y)$ is derivable at some point if (i) the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous at that point and (ii) satisfy C-R equations there.

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Harmonic function

A function $u(x,y)$ which possesses continuous partial derivatives of first and second orders and satisfies Laplace equation i.e the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, is called a harmonic function.

Let $f(z)=u(x,y)+iv(x,y)$ is an analytic function, then C-R equations will be satisfied, i.e $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$. If $u(x,y)$ and $v(x,y)$ possess continuous second order derivatives,

then $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y}$ & $\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x}=-\frac{\partial^2 v}{\partial x\partial y}$,

so that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

Similarly $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$.

Thus $u(x,y)$ is a harmonic function and $v(x,y)$ is called a harmonic function conjugate to $u(x,y)$.

Ex Show that $u=x+xy$ is a harmonic function. Find the harmonic conjugate to u.

Soln $\frac{\partial u}{\partial x}=1+y$, $\frac{\partial u}{\partial y}=x$, $\frac{\partial^2 u}{\partial x^2}=0$, $\frac{\partial^2 u}{\partial y^2}=0$

$\therefore$ u_x, u_y, u_{xx}, u_{yy} are continuous and $u_{xx}+u_{yy}=0$.

$\therefore u(x,y)$ is a harmonic function.

Let $v(x,y)$ be the conjugate harmonic function.

Then $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\Rightarrow\frac{\partial v}{\partial y}=1+y$ or, $v=y+\frac{y^2}{2}+g(x)$, where $g(x)$ is a function of x only.

Now $-\frac{\partial v}{\partial x}=-g'(x)$ and $-\frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}$ gives

$-g'(x)=x$ or $g(x)=-\frac{x^2}{2}+c$, where c is a constant

$\therefore v(x,y)=y+\frac{y^2}{2}-\frac{x^2}{2}+c$ is the conjugate harmonic function

(8)

Ex Find the analytic function $f(z) = u + iv$, whose real part is $u = -\sin x \sinh y$.

Soln $u = -\sin x \sinh y$, $u_x = -\cos x \sinh y$, $u_y = -\sin x \cosh y$.
C-R equation gives, $u_x = v_y$ & $u_y = -v_x$, where v is the conjugate harmonic function.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = -\cos x \sinh y$ or, $v = -\cos x \cosh y + \alpha(x)$,
$\alpha(x)$ is a fn. of x only

Hence $\frac{\partial v}{\partial x} = \sin x \cosh y + \alpha'(x)$

Also $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ gives $\sin x \cosh y + \alpha'(x) = \sin x \cosh y$

or, $\alpha'(x) = 0$ $\therefore \alpha(x) = \text{const} = C$

Hence $v(x,y) = -\cos x \cosh y + C$

Hence $f(z) = u + iv = -\sin x \sinh y + i(-\cos x \cosh y + C)$

An alternative method for the construction of Analytic function (Milne-Thomson method)

Let $f(z) = u + iv$ be an analytic function.
$z = x + iy$ so that $\bar{z} = x - iy$ $\therefore x = \frac{1}{2}(z + \bar{z})$, $y = \frac{1}{2i}(z - \bar{z})$

Now $f(z) = u(x,y) + iv(x,y) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$

When $z = \bar{z}$, $y = 0$ and $x = z$, so that

$f(z) = u(z,0) + iv(z,0)$

$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (by C-R equations)

$\therefore f'(z) = \phi_1(x,y) - i\phi_2(x,y) = \phi_1(z,0) - i\phi_2(z,0)$ at $x = z, y = 0$

Integrating we get

$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + C$, where C is a constant

(9)

Ex Prove that $u(x,y) = x^3-3xy^2+3x^2-3y^2+1$ is a harmonic function and determine the corresponding analytic function.

Soln
$$\frac{\partial u}{\partial x} = 3x^2-3y^2+6x, \quad \frac{\partial^2 u}{\partial x^2} = 6x+6$$

$$\frac{\partial u}{\partial y} = -6xy-6y, \quad \frac{\partial^2 u}{\partial y^2} = -6x-6, \quad \frac{\partial^2 u}{\partial x \partial y} = -6y$$

$\therefore$ $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}$ are continuous and

further $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x+6-6x-6 = 0$.

$\therefore$ $u(x,y)$ is a harmonic function.

Let $\phi_1(x,y) = \frac{\partial u}{\partial x} = 3x^2-3y^2+6x$

& $\phi_2(x,y) = \frac{\partial u}{\partial y} = -6xy-6y$

By Milne Thomson method,

$f'(z) = \phi_1(z,0) - i\phi_2(z,0) = 3z^2+6z-i(0)$

Hence $f(z) = \frac{3z^3}{3} + \frac{6z^2}{2} + c = z^3+3z^2+c$, c is a constant.

Ex Find the analytic function $f(z) = u+iv$ whose real part is $u = e^x(x\cos y - y\sin y)$.

Soln $\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y)$
$= e^x(x\cos y - y\sin y + \cos y) = \phi_1(x,y)$ say

$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y) = \phi_2(x,y)$ say

By Milne Thomson method

$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$ [when $x=z, y=0$]
$= e^z(z+1) - ie^z(0-0-0) = e^z(z+1)$

$\therefore f(z) = \int e^z(z+1)\,dz = (z+1)e^z - \int e^z\,dz + c = (z+1)e^z - e^z + c$
$= ze^z + c$.

Ex Show that the function $f(z)$ defined by

$$f(z) = \frac{(\bar{z})^2}{z}, \quad z \neq 0$$
$$= 0, \quad z = 0$$

is not differentiable at $z=0$, though the C-R equations are satisfied at $(0,0)$

Ex Show that the following functions are not differentiable at origin though continuous everywhere:

(i) $f(z) = |z|$, (ii) $f(z) = z\,\mathrm{Re}(z)$ (iii) $f(z) = \mathrm{Im}(z)$

Ex Prove that the following functions are harmonic. Find the conjugate harmonic function and hence determine the analytic function whose real part is the given function:

(i) $u = x^3 - 3xy^2$

(ii) $u = x^2 - y^2$

(iii) $u = e^x \cos y$

(iv) $u = e^x \sin y - x$.