Basic Discrete Structures

Sets, Functions, Sequences, Matrices, and Relations

(Lecture – 4)

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Cardinality of Sets

- **Theorem:** The set of real numbers is an uncountable set.
- **Proof**: We will be using proof by contradiction. Suppose that the real numbers are countable. Then every subset of the reals is countable, in particular, the interval [0,1] is countable. This implies the elements of this set can be listed say r_1, r_2, r_3, … where
  - $r_1 = 0.d_{11}d_{12}d_{13}d_{14} \ldots$
  - $r_2 = 0.d_{21}d_{22}d_{23}d_{24} \ldots$
  - $r_3 = 0.d_{31}d_{32}d_{33}d_{34} \ldots.$

  Where, the $d_{ij}\, \varepsilon\, \{0,1,2,3,4,5,6,7,8,9\}$.
  - Use *Cantor's diagonalization argument* to contradict the supposition!

Cardinality of Sets

- **<u>Theorem</u>:**

Show that if A and B are sets, A is uncountable, and $A \subseteq B$, then B is uncountable.

If A and B are countable sets, then $A \cup B$ is also countable.

- **<u>Cantor's Theorem (The Power Set Theorem)</u>:**

If A is any set, then there is an injunction from A to $P(A)$, but no surjection, so $|A| < |P(A)|$

- **<u>Theorem</u>:**

SCHRÖDER-BERNSTEIN THEOREM If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

Sequences

- A sequence is a discrete structure used to represent an ordered list.

• Definition

A **sequence** is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.

- An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of a_k depend on k.

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term a* and the *common ratio r* are real numbers.

An *arithmetic progression* is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

where the *initial term a* and the *common difference d* are real numbers.

Summations

- Summation notations: the sum of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$ is expressed as:

$$\sum_{j=m}^{n} a_j, \qquad \sum\nolimits_{j=m}^{n} a_j, \qquad \text{or} \qquad \sum\nolimits_{m \le j \le n} a_j$$

- Index of summation: j; Upper limit: n; Lower limit: m
- Double summations arise in many contexts (as in the analysis of nested loops in computer programs) $\displaystyle\sum_{i=1}^{4}\sum_{j=1}^{3} ij.$
- To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation
- We can also use summation notation to add all values of a function, or terms of an indexed set, where the index of summation runs over all values in a set $\displaystyle\sum_{s \in S} f(s)$

Summations (Contd...)

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Summations (Contd…)

- Recursive definition of summation: separating off a final term, adding on a final term

$$\sum_{k=m}^{m} a_k = a_m \qquad \text{and} \qquad \sum_{k=m}^{n} a_k = \sum_{k=m}^{n-1} a_k + a_n \quad \text{for all integers } n > m.$$

- Product Notation

• Definition

If m and n are integers and $m \le n$, the symbol $\prod_{k=m}^{n} a_k$, read the **product from k equals m to n of a-sub-k**, is the product of all the terms $a_m,\ a_{m+1},\ a_{m+2},\ \ldots,\ a_n$. We write

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

- Recursive definition of product: If m is any integer then,

$$\prod_{k=m}^{m} a_k = a_m \qquad \text{and} \qquad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_n \quad \text{for all integers } n > m.$$

Summations (Contd...)

- Properties of Summation and Product

Theorem 5.1.1

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$$

2. $$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$$ generalized distributive law

3. $$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k).$$

Sequences

- **Recurrence relations**: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
  - A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
  - The **initial conditions** for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect.

The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, is defined by the initial conditions $f_0 = 0$, $f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \ldots$.

Matrices

A *matrix* is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix. The plural of matrix is *matrices*. A matrix with the same number of rows as columns is called *square*. Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

- Matrix addition:

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ matrices. The *sum* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} + \mathbf{B}$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i, j)th element. In other words, $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$.

- The sum of two matrices of the same size is obtained by adding elements in the corresponding positions.
- Matrices of different sizes cannot be added, because the sum of two matrices is defined only when both matrices have the same number of rows and the same number of columns.

Matrices

- Matrix multiplication:

Let $\mathbf{A}$ be an $m \times k$ matrix and $\mathbf{B}$ be a $k \times n$ matrix. The *product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{AB}$, is the $m \times n$ matrix with its (i, j)th entry equal to the sum of the products of the corresponding elements from the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$. In other words, if $\mathbf{AB} = [c_{ij}]$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}.$$

- The product of two matrices is not defined when the *number of columns in the first matrix and the number of rows in the second matrix are not the same*.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{kn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

Identity and Transposes Matrices

- Identity matrix

The *identity matrix of order n* is the $n \times n$ matrix $\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Hence

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

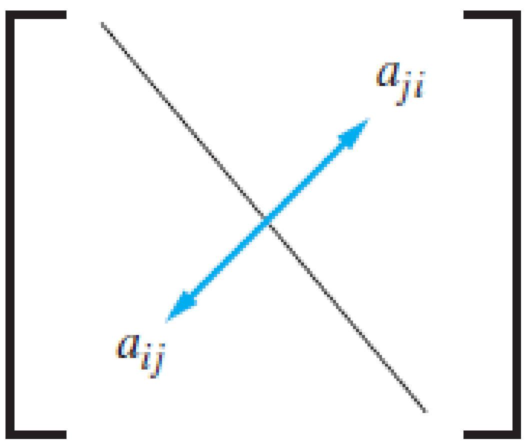
- Transpose of a matrix

Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix. The *transpose* of $\mathbf{A}$, denoted by $\mathbf{A}^t$, is the $n \times m$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$. In other words, if $\mathbf{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Square and Symmetric Matrices

A square matrix $\mathbf{A}$ is called *symmetric* if $\mathbf{A} = \mathbf{A}^t$. Thus $\mathbf{A} = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all i and j with $1 \le i \le n$ and $1 \le j \le n$.

- A matrix is symmetric if and only if it is square and it is symmetric with respect to its main diagonal (which consists of entries that are in the i^{th} row and i^{th} column for some i).



Relations

- Relationships between elements of sets occur in many contexts
- They are represented using the structure called a **relation**, which is just a *subset of the Cartesian product of the sets.*
- Few applications: determining which pairs of cities are linked by airline flights in a network, finding a viable order for the different phases of a complicated project, or producing a useful way to store information in computer databases.
- The most direct way to express a relationship between elements of two sets is to use *ordered pairs* made up of two related elements.
- The sets of ordered pairs are called **binary relations**.

Binary Relation

Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

- A binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.
- The notation $a \, R \, b$ denotes that $(a, b) \in R$ and $a \not{R} \, b$ denotes that $(a, b) \notin R$.
- If $(a, b) \in R$, a is said to be **related to** b by R.
- **<u>Relations on a Set</u>**: A relation on a set A is a subset from AXA