18. Prove that the perpendicular distance of the point P with position vector $\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$ from the plane $\vec{r}\cdot(4\hat{i}-3\hat{j}+\hat{k}) = 18$ is $\frac{20}{\sqrt{26}}$ units.

19. The position vectors of four points A, B, C, P are $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\rho}$ respectively w.r.t. origin O, where A, B, C are fixed points and P is a variable point. Interpret the following vector equation $(\bar{\rho}-\bar{\alpha})\times(\bar{\beta}\times\bar{\gamma}) = \bar{0}$.

20. Find the shortest distance between two lines, one joining the points A and B with position vectors $\vec{a} = -\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b} = -16\hat{i}+6\hat{j}+4\hat{k}$ respectively and the other joining the points C and D with position vectors $\vec{c} = \hat{i}-\hat{j}+3\hat{k}$ and $\vec{d} = 4\hat{i}+9\hat{j}+7\hat{k}$ respectively.

21. If $\bar{\alpha}$ be the position vector of a given point and $\bar{\rho}$, the position vector of any variable point P, find the locus of P if $(\bar{\rho}-\bar{\alpha})\cdot\bar{\rho} = 0$.

ANSWERS TO PROBLEMS

1. $\lambda = 3, \mu = -6$
2. $\hat{i}+4\hat{j}+2\hat{k},\ \hat{i}-8\hat{k},\ 2\hat{i}+4\hat{j}-6\hat{k}$
7. $d = 0$
8. $c = -1, d = 1$
9. $\pm\frac{11\sqrt{3}}{3}(\hat{i}+\hat{j}-\hat{k})$
10. $\pm\frac{1}{5}(3\hat{i}+4\hat{j})$
13. 0, coplanar.
19. Equation of a straight line passing through A and perpendicular to the plane of OB and OC.
20. S.D. = 7 units.
21. A sphere. Its radius is $\frac{1}{2}|\bar{\alpha}|$ and the position vector of the centre be $\frac{1}{2}\bar{\alpha}$.

CHAPTER 13

Vector Calculus

13.1 DERIVATIVE OF A VECTOR FUNCTION

Vector Function

If corresponding to each value of a scalar variable t we associate a unique vector $\bar{F}$, determined by any law whatsoever, then $\bar{F}$ is called a vector function of t denoted by $\bar{F}(t)$.

Example: Let a particle moves in space in such a way that at any time t, its position vector $\bar{r}(t)$ w.r.t. origin O is given by

$$\bar{r}(t) = (t^2+4t+1)\hat{i}+(t-2)\hat{j}+(t^3+t^2+3)\hat{k}.$$

Then to each value of the scalar variable we associate a unique vector $\bar{r}(t)$. Therefore, $\bar{r}(t)$ is a vector function of the scalar variable t.

Note: (*i*) We always assume that $\hat{i}, \hat{j}, \hat{k}$ be three unit vectors along three mutually perpendicular fixed directions so that $\hat{i}\times\hat{j} = \hat{k}$. Therefore, $\hat{j}\times\hat{k} = \hat{i}$ and $\hat{k}\times\hat{i} = \hat{j}$.

(*ii*) Any vector function $\bar{F}(t)$ can be decomposed as $\bar{F}(t) = F_1(t)\hat{i}+F_2(t)\hat{j}+F_3(t)\hat{k}$, where $F_1(t)$, $F_2(t)$ and $F_3(t)$ are scalar functions of t.

Such a relation may also be expressed by

$$\bar{F} = (F_1, F_2, F_3).$$

Derivative

1. Let $\bar{F}(t)$ be a vector function of a scalar variable t, then

$$\frac{d\bar{F}}{dt} = \lim_{h\to 0}\frac{\bar{F}(t+h)-\bar{F}(t)}{h}, \text{ provided it exists.}$$

Example: Let O be the origin and P, Q are the positions of a moving particle at time t and $t+\Delta t$ respectively. Let $\overline{OP} = \bar{r}$ and $\overline{OQ} = \bar{r}+\Delta\bar{r}$.

$$= \frac{dF_1}{dt}\hat{i} + \frac{dF_2}{dt}\hat{j} + \frac{dF_3}{dt}\hat{k}$$

Similarly, $$\frac{d^2\bar{F}}{dt^2} = \frac{d^2F_1}{dt^2}\hat{i} + \frac{d^2F_2}{dt^2}\hat{j} + \frac{d^2F_3}{dt^2}\hat{k}.$$

13.2 FORMULAE OF DIFFERENTIATION

If $\bar{F}$, $\bar{G}$, $\bar{H}$ are derivable vector functions of a scalar variable t and φ is a derivable scalar function of t, then

(i) $$\frac{d}{dt}(\bar{F} \pm \bar{G}) = \frac{d\bar{F}}{dt} \pm \frac{d\bar{G}}{dt}$$

(ii) $$\frac{d}{dt}(\bar{F} \cdot \bar{G}) = \frac{d\bar{F}}{dt} \cdot \bar{G} + \bar{F} \cdot \frac{d\bar{G}}{dt}$$

(iii) $$\frac{d}{dt}(\bar{F} \times \bar{G}) = \frac{d\bar{F}}{dt} \times \bar{G} + \bar{F} \times \frac{d\bar{G}}{dt}$$

(iv) $$\frac{d}{dt}(\varphi\bar{F}) = \varphi\frac{d\bar{F}}{dt} + \bar{F}\frac{d\varphi}{dt}$$

(v) $$\frac{d}{dt}\{\bar{F} \cdot (\bar{G} \times \bar{H})\} = \frac{d\bar{F}}{dt} \cdot (\bar{G} \times \bar{H}) + \bar{F} \cdot \left(\frac{d\bar{G}}{dt} \times \bar{H}\right) + \bar{F} \cdot \left(\bar{G} \times \frac{d\bar{H}}{dt}\right)$$

(vi) $$\frac{d}{dt}\{\bar{F} \times (\bar{G} \times \bar{H})\} = \frac{d\bar{F}}{dt} \times (\bar{G} \times \bar{H}) + \bar{F} \times \left(\frac{d\bar{G}}{dt} \times \bar{H}\right) + \bar{F} \times \left(\bar{G} \times \frac{d\bar{H}}{dt}\right).$$

Note: In all the terms on the right sides of the formulae (*iii*) and (*vi*), $\bar{F}$, $\bar{G}$ and $\bar{H}$ occupy the same positions as they do on the left sides. It would not be correct to interchange the positions unless the sign is changed at the same time.

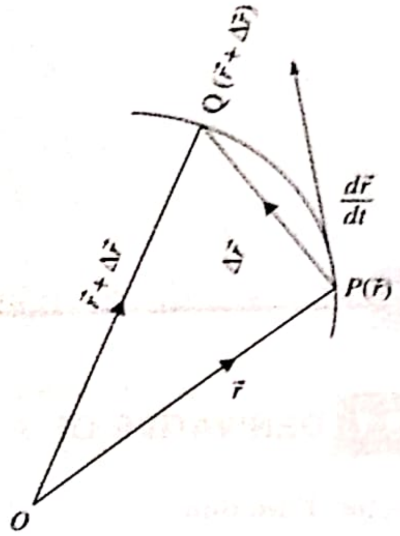
13.3 TWO IMPORTANT THEOREMS

Let $\bar{u}$ be a proper vector, *i.e.*, a non-zero vector. Therefore, $\bar{u}$ has a magnitude as well as a direction. When $\bar{u}$ changes, there may be change in its magnitude or in direction or in both. In this connection we prove the following two important theorems:

Theorem 1: Vectors with Constant Magnitude

The necessary and sufficient condition for a proper vector (*i.e.* non-zero vector) $\bar{u}$ to have constant magnitude is that

$$\therefore \qquad \overline{PQ} = \overline{OQ} - \overline{OP}$$
$$= (\bar{r} + \Delta\bar{r}) - \bar{r} = \Delta\bar{r}.$$



Here $\frac{\Delta\bar{r}}{\Delta t}$ is a vector. As $\Delta t \to 0$, Q tends to P and the chord $\overline{PQ} = \Delta\bar{r}$ tends to the tangent at P.

Therefore, $\frac{d\bar{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\bar{r}}{\Delta t}$ is a vector in the direction of the tangent at P. $\frac{d\bar{r}}{dt}$ gives the velocity of the particle at P, which is along the tangent to its path, and $\frac{d^2\bar{r}}{dt^2}$ gives the acceleration of the particle at P.

2. Let $\bar{a}$ be a constant vector, *i.e.*, its magnitude is constant and direction remains parallel to a fixed line. Therefore, an increment Δt in the scalar variable t produces no change in $\bar{a}$, *i.e.*, $\Delta\bar{a} = \bar{0}$, so $\frac{\Delta\bar{a}}{\Delta t} = \bar{0}$ for every increment Δt and hence $\frac{d\bar{a}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\bar{a}}{\Delta t} = \bar{0}$.

The converse is also true.

Therefore, $\frac{d\bar{a}}{dt} = \bar{0}$, where $\bar{a}$ is a constant vector and conversely if $\frac{d\bar{a}}{dt} = \bar{0}$ then $\bar{a}$ is a constant vector.

3. Let $\bar{u}$ be a derivable function of a scalar variable s and s be a derivable function of another scalar variable t, then

$$\frac{d\bar{u}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\bar{u}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\bar{u}}{\Delta s} \cdot \frac{\Delta s}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{\Delta\bar{u}}{\Delta s} \cdot \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \qquad (\because \ \Delta s \to 0 \text{ as } \Delta t \to 0)$$
$$= \frac{d\bar{u}}{ds} \cdot \frac{ds}{dt}$$

Let $\bar{F}(t) = F_1(t)\hat{i} + F_2(t)\hat{j} + F_3(t)\hat{k}$, then

$$\frac{d\bar{F}}{dt} = \lim_{h \to 0} \frac{\bar{F}(t+h) - \bar{F}(t)}{h}$$
$$= \left\{\lim_{h \to 0} \frac{F_1(t+h) - F_1(t)}{h}\right\} \hat{i} + \left\{\lim_{h \to 0} \frac{F_2(t+h) - F_2(t)}{h}\right\} \hat{j}$$
$$+ \left\{\lim_{h \to 0} \frac{F_3(t+h) - F_3(t)}{h}\right\} \hat{k}$$

$$= \frac{dF_1}{dt}\hat{i} + \frac{dF_2}{dt}\hat{j} + \frac{dF_3}{dt}\hat{k}$$

Similarly, $$\frac{d^2\vec{F}}{dt^2} = \frac{d^2F_1}{dt^2}\hat{i} + \frac{d^2F_2}{dt^2}\hat{j} + \frac{d^2F_3}{dt^2}\hat{k}.$$

13.2 FORMULAE OF DIFFERENTIATION

If $\vec{F}$, $\vec{G}$, $\vec{H}$ are derivable vector functions of a scalar variable t and φ is a derivable scalar function of t, then

(i) $$\frac{d}{dt}(\vec{F} \pm \vec{G}) = \frac{d\vec{F}}{dt} \pm \frac{d\vec{G}}{dt}$$

(ii) $$\frac{d}{dt}(\vec{F}\cdot\vec{G}) = \frac{d\vec{F}}{dt}\cdot\vec{G} + \vec{F}\cdot\frac{d\vec{G}}{dt}$$

(iii) $$\frac{d}{dt}(\vec{F}\times\vec{G}) = \frac{d\vec{F}}{dt}\times\vec{G} + \vec{F}\times\frac{d\vec{G}}{dt}$$

(iv) $$\frac{d}{dt}(\varphi\vec{F}) = \varphi\frac{d\vec{F}}{dt} + \vec{F}\frac{d\varphi}{dt}$$

(v) $$\frac{d}{dt}\{\vec{F}\cdot(\vec{G}\times\vec{H})\} = \frac{d\vec{F}}{dt}\cdot(\vec{G}\times\vec{H}) + \vec{F}\cdot\left(\frac{d\vec{G}}{dt}\times\vec{H}\right) + \vec{F}\cdot\left(\vec{G}\times\frac{d\vec{H}}{dt}\right)$$

(vi) $$\frac{d}{dt}\{\vec{F}\times(\vec{G}\times\vec{H})\} = \frac{d\vec{F}}{dt}\times(\vec{G}\times\vec{H}) + \vec{F}\times\left(\frac{d\vec{G}}{dt}\times\vec{H}\right) + \vec{F}\times\left(\vec{G}\times\frac{d\vec{H}}{dt}\right).$$

Note: In all the terms on the right sides of the formulae (*iii*) and (*vi*), $\vec{F}$, $\vec{G}$ and $\vec{H}$ occupy the same positions as they do on the left sides. It would not be correct to interchange the positions unless the sign is changed at the same time.

13.3 TWO IMPORTANT THEOREMS

Let $\vec{u}$ be a proper vector, *i.e.*, a non-zero vector. Therefore, $\vec{u}$ has a magnitude as well as a direction. When $\vec{u}$ changes, there may be change in its magnitude or in direction or in both. In this connection we prove the following two important theorems:

Theorem 1: Vectors with Constant Magnitude

The necessary and sufficient condition for a proper vector (*i.e.* non-zero vector) $\vec{u}$ to have constant magnitude is that

Scanned with CamScanner

$$\boxed{\bar{u}\cdot\frac{d\bar{u}}{dt}=0.}$$

Proof: Now, $$\frac{d}{dt}|\bar{u}|^2 = \frac{d}{dt}(\bar{u}\cdot\bar{u}) = \frac{d\bar{u}}{dt}\cdot\bar{u}+\bar{u}\cdot\frac{d\bar{u}}{dt}$$

$$= 2\bar{u}\cdot\frac{d\bar{u}}{dt}.$$

Therefore, $|\bar{u}|$ = constant implies $\bar{u}\cdot\frac{d\bar{u}}{dt}=0$ and conversely $\bar{u}\cdot\frac{d\bar{u}}{dt}=0$ implies $\frac{d}{dt}|\bar{u}|^2=0$, *i.e.*, $|\bar{u}|$ = constant. Hence the theorem.

Note: This theorem expresses the fact that the derivative of a vector of constant magnitude is perpendicular to that vector.

Theorem 2: Vectors with Constant Direction

The necessary and sufficient condition for a proper vector $\bar{u}$ to have constant direction (*i.e.*, it remains parallel to a fixed line) is that

$$\boxed{\bar{u}\times\frac{d\bar{u}}{dt}=\bar{0}.}$$

Proof: When $\bar{u}$ has a constant direction

Let $\bar{u} = u\hat{u}$, where $\hat{u}$ is the unit vector in the direction of $\bar{u}$ and u be its magnitude.

$$\therefore \quad \bar{u}\times\frac{d\bar{u}}{dt} = u\hat{u}\times\frac{d}{dt}(u\hat{u}) = u\hat{u}\times\left(\frac{du}{dt}\hat{u}+u\frac{d\hat{u}}{dt}\right)$$

$$= (\hat{u}\times\hat{u})\,u\frac{du}{dt}+u^2\,\hat{u}\times\frac{d\hat{u}}{dt}$$

$$= u^2\,\hat{u}\times\frac{d\hat{u}}{dt} \qquad (\because\ \hat{u}\times\hat{u}=\bar{0}) \qquad \ldots(1)$$

When the direction of $\bar{u}$ is constant, *i.e.*, it remains parallel to a fixed line, $\hat{u}$ is a constant vector and hence $\frac{d\hat{u}}{dt}=\bar{0}$.

Therefore, from (1), $\bar{u}\times\frac{d\bar{u}}{dt}=\bar{0}$

Conversely, let $\quad \bar{u}\times\frac{d\bar{u}}{dt}=\bar{0}$

Here $\quad \bar{u}\times\frac{d\bar{u}}{dt}=\bar{0}$, or $u\hat{u}\times\frac{d}{dt}(u\hat{u})=\bar{0}$

or $$u^2\,\hat{u}\times\frac{d\hat{u}}{dt} = \bar{0}$$ [by (1)]

$$\therefore\quad \hat{u}\times\frac{d\hat{u}}{dt} = \bar{0} \qquad (\because\ \hat{u}\neq\bar{0}) \qquad \text{...(2)}$$

But $\hat{u}$ has a constant magnitude unity.

Therefore, by Theorem 1, we get

$$\hat{u}\cdot\frac{d\hat{u}}{dt} = 0. \qquad \text{...(3)}$$

The equations (2) and (3) are contradictory unless

$$\frac{d\hat{u}}{dt} = \bar{0},$$

therefore the unit vector $\hat{u}$ is constant, which means that $\bar{u}$ has a constant direction, *i.e.*, it remains parallel to a fixed line.

ILLUSTRATIVE EXAMPLES

Example 1: If $\bar{\alpha} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ and $\bar{\beta} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$ where $\hat{i}, \hat{j}, \hat{k}$ have their usual meanings, find $\frac{d}{dt}\left(\bar{\alpha}\times\frac{d\bar{\beta}}{dt}\right)$ at $t = 2$.

Solution: Now, $$\frac{d}{dt}\left(\bar{\alpha}\times\frac{d\bar{\beta}}{dt}\right) = \bar{\alpha}\times\frac{d^2\bar{\beta}}{dt^2} + \frac{d\bar{\alpha}}{dt}\times\frac{d\bar{\beta}}{dt}$$

Here $\bar{\alpha} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$, $\bar{\beta} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$

$$\therefore\quad \frac{d\bar{\alpha}}{dt} = 2t\hat{i} - \hat{j} + 2\hat{k},\quad \frac{d\bar{\beta}}{dt} = 2\hat{i} - \hat{k},\quad \frac{d^2\bar{\beta}}{dt^2} = \frac{d}{dt}\left(\frac{d\bar{\beta}}{dt}\right) = \bar{0}.$$

At $$t = 2,\quad \frac{d\bar{\alpha}}{dt}\times\frac{d\bar{\beta}}{dt} = (4\hat{i} - \hat{j} + 2\hat{k})\times(2\hat{i} - \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix} = \hat{i} + 8\hat{j} + 2\hat{k}$$

$$\therefore\quad \frac{d}{dt}\left(\bar{\alpha}\times\frac{d\bar{\beta}}{dt}\right) = \hat{i} + 8\hat{j} + 2\hat{k} \quad \text{at } t = 2, \text{ since } \frac{d^2\bar{\beta}}{dt^2} = \bar{0}.$$

Example 2: Evaluate $[\dot{\bar{r}}\,\ddot{\bar{r}}\,\dddot{\bar{r}}]$ where $\bar{r} = a\cos u\hat{i} + a\sin u\hat{j} + bu\hat{k}$. (*W.B.U.T. 2008*)

Solution: Here $\bar{r} = a\cos u\hat{i} + a\sin u\hat{j} + b\bar{u}\hat{k}$

$$\therefore \quad \dot{\bar{r}} = \frac{d\bar{r}}{d\bar{u}} = -(a\sin u)\hat{i} + (a\cos u)\hat{j} + b\hat{k}$$

$$\ddot{\bar{r}} = \frac{d^2\bar{r}}{du^2} = -(a\cos u)\hat{i} - (a\sin u)\hat{j}$$

$$\dddot{\bar{r}} = \frac{d^3\bar{r}}{du^3} = (a\sin u)\hat{i} - (a\cos u)\hat{j}$$

$$\therefore \quad \ddot{\bar{r}}\times\dddot{\bar{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\cos u & -a\sin u & 0 \\ a\sin u & -a\cos u & 0 \end{vmatrix} = \begin{vmatrix} -a\cos u & -a\sin u \\ a\sin u & -a\cos u \end{vmatrix}\hat{k}$$

$$= (a^2\cos^2 u + a^2\sin^2 u)\hat{k} = a^2\hat{k}$$

$$\therefore \quad [\dot{\bar{r}}\,\ddot{\bar{r}}\,\dddot{\bar{r}}] = \dot{\bar{r}}\cdot(\ddot{\bar{r}}\times\dddot{\bar{r}})$$

$$= \{-(a\sin u)\hat{i} + (a\cos u)\hat{j} + b\hat{k}\}\cdot(a^2\hat{k})$$

$$= a^2 b.$$

Example 3: If $\bar{r} = 3t\hat{i} + 3t^2\hat{j} + 2t^3\hat{k}$, find the value of

$$\left[\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}, \frac{d^3\bar{r}}{dt^3}\right].$$

Solution: Here $\bar{r} = 3t\hat{i} + 3t^2\hat{j} + 2t^3\hat{k}$

$$\therefore \quad \frac{d\bar{r}}{dt} = 3\hat{i} + 6t\hat{j} + 6t^2\hat{k}$$

$$\frac{d^2\bar{r}}{dt^2} = 6\hat{j} + 12t\hat{k} \text{ and } \frac{d^3\bar{r}}{dt^3} = 12\hat{k}$$

$$\therefore \quad \left[\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}, \frac{d^3\bar{r}}{dt^3}\right] = \frac{d\bar{r}}{dt}\cdot\left(\frac{d^2\bar{r}}{dt^2}\times\frac{d^3\bar{r}}{dt^3}\right)$$

$$= \frac{d\bar{r}}{dt}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 6 & 12t \\ 0 & 0 & 12 \end{vmatrix} = (3\hat{i} + 6t\hat{j} + 6t^2\hat{k})\cdot(72\hat{i}) = 216$$

Example 4: If $\bar{\omega}$ is a constant vector, $\bar{r}$ and $\bar{s}$ are vector functions of a scalar variable t and if $\frac{d\bar{r}}{dt} = \bar{\omega}\times\bar{r}$, $\frac{d\bar{s}}{dt} = \bar{\omega}\times\bar{s}$, then prove that

$$\frac{d}{dt}(\bar{r}\times\bar{s}) = \bar{\omega}\times(\bar{r}\times\bar{s}).$$

Solution: L.H.S. $= \frac{d}{dt}(\bar{r}\times\bar{s}) = \bar{r}\times\frac{d\bar{s}}{dt}+\frac{d\bar{r}}{dt}\times\bar{s}$

$$= \bar{r}\times(\bar{\omega}\times\bar{s})+(\bar{\omega}\times\bar{r})\times\bar{s} \qquad \left(\because \frac{d\bar{r}}{dt}=\bar{\omega}\times\bar{r} \text{ and } \frac{d\bar{s}}{dt}=\bar{\omega}\times\bar{s}\right)$$

$$= \bar{r}\times(\bar{\omega}\times\bar{s})-\bar{s}\times(\bar{\omega}\times\bar{r})$$

$$= (\bar{r}\cdot\bar{s})\bar{\omega}-(\bar{r}\cdot\bar{\omega})\bar{s}-(\bar{s}\cdot\bar{r})\bar{\omega}+(\bar{s}\cdot\bar{\omega})\bar{r}$$

$$= (\bar{s}\cdot\bar{\omega})\bar{r}-(\bar{r}\cdot\bar{\omega})\bar{s} \qquad [\because \bar{s}\cdot\bar{r}=\bar{r}\cdot\bar{s}]$$

$$= (\bar{\omega}\cdot\bar{s})\bar{r}-(\bar{\omega}\cdot\bar{r})\bar{s} = \bar{\omega}\times(\bar{r}\times\bar{s}) = \text{R.H.S.}$$

Example 5: A particle moves on the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of velocity and acceleration at time $t = 1$ in the direction $\hat{i}-3\hat{j}+2\hat{k}$.

(*W.B.U.T. 2002, 2009*)

Solution: If $\bar{r}$ be the position vector of any point (x, y, z) on the given curve, then

$$\bar{r} = x\hat{i}+y\hat{j}+z\hat{k} = 2t^2\hat{i}+(t^2-4t)\hat{j}+(3t-5)\hat{k}$$

Therefore, velocity $\bar{v} = \frac{d\bar{r}}{dt} = 4t\hat{i}+(2t-4)\hat{j}+3\hat{k}$

and acceleration $\bar{a} = \frac{d^2\bar{r}}{dt^2} = 4\hat{i}+2\hat{j}$

Therefore, at $t = 1$, $\bar{v} = 4\hat{i}-2\hat{j}+3\hat{k}$ and $\bar{a} = 4\hat{i}+2\hat{j}$.

The component of $\bar{v}$ along $\bar{b} = \hat{i}-3\hat{j}+2\hat{k}$ is

$$\frac{\bar{v}\cdot\bar{b}}{|\bar{b}|} = \frac{(4\hat{i}-2\hat{j}+3\hat{k})\cdot(\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{1^2+(-3)^2+2^2}} = \frac{4+6+6}{\sqrt{14}} = \frac{16}{\sqrt{14}} = \frac{8\sqrt{14}}{7}$$

and the component of $\bar{a}$ along $\bar{b} = \hat{i}-3\hat{j}+2\hat{k}$ is

$$\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|} = \frac{(4\hat{i}+2\hat{j}+0\hat{k})\cdot(\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{14}} = \frac{4-6+0}{\sqrt{14}} = -\frac{2}{\sqrt{14}} = -\frac{\sqrt{14}}{7}.$$

Example 6: Establish the following formula for the derivative of the cross product of two vectors $\vec{\alpha}$ and $\vec{\beta}$ which depend on a scalar variable t.

$$\frac{d}{dt}(\vec{\alpha}\times\vec{\beta}) = \vec{\alpha}\times\frac{d\vec{\beta}}{dt}+\frac{d\vec{\alpha}}{dt}\times\vec{\beta}$$

Hence, prove that $\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = \vec{r}\times\frac{d^2\vec{r}}{dt^2}$.

Solution: If $\vec{\alpha}$ and $\vec{\beta}$ produce increments $\Delta\vec{\alpha}$ and $\Delta\vec{\beta}$ respectively for an increment Δt in t, then the increment in $\vec{\alpha}\times\vec{\beta}$ is given by

$$\begin{aligned}\Delta(\vec{\alpha}\times\vec{\beta}) &= (\vec{\alpha}+\Delta\vec{\alpha})\times(\vec{\beta}+\Delta\vec{\beta})-\vec{\alpha}\times\vec{\beta}\\ &= \vec{\alpha}\times\Delta\vec{\beta}+\Delta\vec{\alpha}\times\vec{\beta}+\Delta\vec{\alpha}\times\Delta\vec{\beta}\end{aligned}$$

$$\therefore \quad \lim_{\Delta t\to 0}\frac{\Delta(\vec{\alpha}\times\vec{\beta})}{\Delta t} = \vec{\alpha}\times\lim_{\Delta t\to 0}\frac{\Delta\vec{\beta}}{\Delta t}+\left(\lim_{\Delta t\to 0}\frac{\Delta\vec{\alpha}}{\Delta t}\right)\times\vec{\beta}+\left(\lim_{\Delta t\to 0}\frac{\Delta\vec{\alpha}}{\Delta t}\right)\times\left(\lim_{\Delta t\to 0}\Delta\vec{\beta}\right)$$

$$\therefore \quad \frac{d}{dt}(\vec{\alpha}\times\vec{\beta}) = \vec{\alpha}\times\frac{d\vec{\beta}}{dt}+\frac{d\vec{\alpha}}{dt}\times\vec{\beta} \qquad [\because \Delta\vec{\beta}\to\vec{0} \text{ as } \Delta t\to 0]$$

Putting $\vec{\alpha} = \vec{r}$ and $\vec{\beta} = \frac{d\vec{r}}{dt}$, we get

$$\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = \vec{r}\times\frac{d^2\vec{r}}{dt^2}+\frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt} = \vec{r}\times\frac{d^2\vec{r}}{dt^2} \qquad \left(\because \frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt}=\vec{0}\right)$$

Example 7: Prove that

$$\frac{d}{dt}[\vec{p}\;\vec{q}\;\vec{r}] = \left[\frac{d\vec{p}}{dt}\;\vec{q}\;\vec{r}\right]+\left[\vec{p}\;\frac{d\vec{q}}{dt}\;\vec{r}\right]+\left[\vec{p}\;\vec{q}\;\frac{d\vec{r}}{dt}\right]$$

where $\vec{p}$, $\vec{q}$, $\vec{r}$ are functions of t. Hence find

$$\frac{d}{dt}\left[\vec{r}\;\frac{d\vec{r}}{dt}\;\frac{d^2\vec{r}}{dt^2}\right]$$

where $\vec{r}$ is a function of t.

Solution: $[\vec{p}\;\vec{q}\;\vec{r}] = \vec{p}\cdot(\vec{q}\times\vec{r})$

$$\begin{aligned}\therefore \quad \frac{d}{dt}[\vec{p}\;\vec{q}\;\vec{r}] &= \left(\frac{d\vec{p}}{dt}\right)\cdot(\vec{q}\times\vec{r})+\vec{p}\cdot\frac{d}{dt}(\vec{q}\times\vec{r})\\ &= \left(\frac{d\vec{p}}{dt}\right)\cdot(\vec{q}\times\vec{r})+\vec{p}\cdot\left\{\frac{d\vec{q}}{dt}\times\vec{r}+\vec{q}\times\frac{d\vec{r}}{dt}\right\}\\ &= \left[\frac{d\vec{p}}{dt}\;\vec{q}\;\vec{r}\right]+\left[\vec{p}\;\frac{d\vec{q}}{dt}\;\vec{r}\right]+\left[\vec{p}\;\vec{q}\;\frac{d\vec{r}}{dt}\right]\end{aligned}$$

Replacing $\vec{p}$, $\vec{q}$ and $\vec{r}$ by $\vec{r}$, $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$ respectively, we get

$$\frac{d}{dt}\left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\right] = \left[\frac{d\vec{r}}{dt}\ \frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\right] + \left[\vec{r}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^2\vec{r}}{dt^2}\right] + \left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^3\vec{r}}{dt^3}\right]$$

$$= \left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^3\vec{r}}{dt^3}\right]$$

($\because$ $[\vec{a}\ \vec{b}\ \vec{c}\,]$ vanishes if any two of $\vec{a}$, $\vec{b}$, $\vec{c}$, are equal).

Example 8: If $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + at\tan\theta\,\hat{k}$, then find the values of

$$\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right| \text{ and } \left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right].$$

Solution: Here $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + at\tan\theta\,\hat{k}$

$\therefore$
$$\frac{d\vec{r}}{dt} = -(a\sin t)\hat{i} + (a\cos t)\hat{j} + (a\tan\theta)\hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} = -(a\cos t)\hat{i} - (a\sin t)\hat{j}$$

$\therefore$
$$\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & a\tan\theta \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= \hat{i}\,a^2\sin t\tan\theta - \hat{j}\,a^2\cos t\tan\theta + a^2\hat{k}$$

$\therefore$
$$\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{a^4\sin^2 t\tan^2\theta + a^4\cos^2 t\tan^2\theta + a^4}$$

$$= \sqrt{a^4\tan^2\theta + a^4} = a^2\sec\theta$$

Now,
$$\frac{d^3\vec{r}}{dt^3} = \frac{d}{dt}\left(\frac{d^2\vec{r}}{dt^2}\right) = (a\sin t)\hat{i} - (a\cos t)\hat{j}$$

$\therefore$
$$\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = \begin{vmatrix} -a\sin t & a\cos t & a\tan\theta \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix} = (a^2\cos^2 t + a^2\sin^2 t)a\tan\theta$$

$$= a^3\tan\theta.$$

Example 9: Establish the following formula for the derivative of the scalar product of two vectors $\vec{\alpha}$ and $\vec{\beta}$ which depend on a scalar variable t.

$$\frac{d}{dt}(\vec{\alpha}\cdot\vec{\beta}) = \vec{\alpha}\cdot\frac{d\vec{\beta}}{dt}+\frac{d\vec{\alpha}}{dt}\cdot\vec{\beta}$$

Hence, prove that $\frac{d}{dt}\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)=\left(\frac{d\vec{r}}{dt}\right)^2+\vec{r}\cdot\frac{d^2\vec{r}}{dt^2}$.

Solution: If $\vec{\alpha}$ and $\vec{\beta}$ produce increments $\Delta\vec{\alpha}$ and $\Delta\vec{\beta}$ respectively for an increment Δt in t, then the increment in $\vec{\alpha}\cdot\vec{\beta}$ is given by

$$\Delta(\vec{\alpha}\cdot\vec{\beta}) = (\vec{\alpha}+\Delta\vec{\alpha})\cdot(\vec{\beta}+\Delta\vec{\beta})-\vec{\alpha}\cdot\vec{\beta}$$

$$= \vec{\alpha}\cdot\Delta\vec{\beta}+\Delta\vec{\alpha}\cdot\vec{\beta}+\Delta\vec{\alpha}\cdot\Delta\vec{\beta}$$

$$\therefore \quad \lim_{\Delta t\to 0}\frac{\Delta(\vec{\alpha}\cdot\vec{\beta})}{\Delta t} = \vec{\alpha}\cdot\lim_{\Delta t\to 0}\frac{\Delta\vec{\beta}}{\Delta t}+\left(\lim_{\Delta t\to 0}\frac{\Delta\vec{\alpha}}{\Delta t}\right)\cdot\vec{\beta}+\left(\lim_{\Delta t\to 0}\frac{\Delta\vec{\alpha}}{\Delta t}\right)\cdot\lim_{\Delta t\to 0}\Delta\vec{\beta}$$

$$\therefore \quad \frac{d}{dt}(\vec{\alpha}\cdot\vec{\beta}) = \vec{\alpha}\cdot\frac{d\vec{\beta}}{dt}+\frac{d\vec{\alpha}}{dt}\cdot\vec{\beta} \qquad [\because \Delta\vec{\beta}\to\vec{0} \text{ as } \Delta t\to 0]$$

Replacing $\vec{\alpha}$ and $\vec{\beta}$ by $\vec{r}$ and $\frac{d\vec{r}}{dt}$ respectively, we get

$$\frac{d}{dt}\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right) = \vec{r}\cdot\frac{d^2\vec{r}}{dt^2}+\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt} = \left(\frac{d\vec{r}}{dt}\right)^2+\vec{r}\cdot\frac{d^2\vec{r}}{dt^2}.$$

Example 10: Find the unit vector in the direction of the tangent at any point on the curve $\vec{r} = (a\cos t)\hat{i}+(a\sin t)\hat{j}+bt\hat{k}$.

Solution: Here $\frac{d\vec{r}}{dt} = -(a\sin t)\hat{i}+(a\cos t)\hat{j}+b\hat{k}$

Therefore, the unit vector in the direction of the tangent

$$= \left(\frac{d\vec{r}}{dt}\right)\bigg/\left|\frac{d\vec{r}}{dt}\right| = \frac{-(a\sin t)\hat{i}+(a\cos t)\hat{j}+b\hat{k}}{\sqrt{a^2\sin^2 t+a^2\cos^2 t+b^2}} = \frac{-(a\sin t)\hat{i}+(a\cos t)\hat{j}+b\hat{k}}{\sqrt{a^2+b^2}}.$$

Example 11: Find the angle between the tangents to the curve

$$x = t-4,\ y = t^3-4t,\ z = t^2 \text{ at } t = 1, 2.$$

Solution: Let $\vec{r}$ be the position vector of any point $P(x, y, z)$ on the given curve. Then

$$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k} = (t-4)\hat{i}+(t^3-4t)\hat{j}+t^2\hat{k}.$$

Now, $\frac{d\vec{r}}{dt} = \hat{i}+(3t^2-4)\hat{j}+2t\hat{k}$ is a vector along the tangent at any point t.

Let $\vec{T}_1$ and $\vec{T}_2$ are the vectors along the tangents at $t = 1$ and $t = 2$ respectively.

$\therefore \quad \vec{T}_1 = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{T}_2 = \hat{i} + 8\hat{j} + 4\hat{k}$

$\therefore \quad \vec{T}_1 \cdot \vec{T}_2 = 1 - 8 + 8 = 1,\ |\vec{T}_1| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$

and $\quad |\vec{T}_2| = \sqrt{1^2 + 8^2 + 4^2} = 9$

Hence the angle between the tangents at $t = 1$ and $t = 2$ is

$$\cos^{-1}\left(\frac{\vec{T}_1 \cdot \vec{T}_2}{|\vec{T}_1||\vec{T}_2|}\right) = \cos^{-1}\left(\frac{1}{9\sqrt{6}}\right).$$

Example 12: A particle moves along the curve $\vec{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$ where t denotes time. Find the magnitude of acceleration along the tangent and normal at time $t = 2$.

Solution: Here velocity $\vec{v} = \dfrac{d\vec{r}}{dt} = (3t^2 - 4)\hat{i} + (2t + 4)\hat{j} + (16t - 9t^2)\hat{k}$ and acceleration $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = 6t\hat{i} + 2\hat{j} + (16 - 18t)\hat{k}$.

Therefore, at $t = 2$, velocity $\vec{v} = 8\hat{i} + 8\hat{j} - 4\hat{k}$ and acceleration $\vec{a} = 12\hat{i} + 2\hat{j} - 20\hat{k}$.

We know that the velocity is along the tangent to the curve along which the particle is moving.

Therefore, the magnitude of the comsponent of $\vec{a}$ along the tangent

$$= \vec{a} \cdot \frac{\vec{v}}{|\vec{v}|} = (12\hat{i} + 2\hat{j} - 20\hat{k}) \cdot \frac{(8\hat{i} + 8\hat{j} - 4\hat{k})}{\sqrt{64 + 64 + 16}}$$

$$= \frac{96 + 16 + 80}{12} = \frac{192}{12} = 16.$$

Now, the magnitude of the component of $\vec{a}$ along the normal

$= |\vec{a} -$ resolved part of $\vec{a}$ along the tangent $|$

[By triangle law of vector addition]

$$= \left| 12\hat{i} + 2\hat{j} - 20\hat{k} - 16\,\frac{(8\hat{i} + 8\hat{j} - 4\hat{k})}{12} \right|$$

$$= \left| \frac{1}{3}(4\hat{i} - 26\hat{j} - 44\hat{k}) \right| = 2\sqrt{73}.$$

13.4 SCALAR AND VECTOR FIELDS

Scalar Point Functions: Scalar Fields

If at each point P of a certain region R, there corresponds (by any law whatsoever) a definite scalar quantity $\varphi(P)$, then φ is called a **scalar point function** over the region R. The points of the region R will form a **scalar field** defined by φ.

Examples of scalar point functions are (*i*) density of a body, (*ii*) temperature distribution in a medium, (*iii*) potential of body due to gravity etc.

Vector Point Functions: Vector Fields

If at each point P of a certain region R, there is associated a definite vector quantity $\bar{F}(P)$, then $\bar{F}$ is called a **vector point function** over the region R. The points of R constitute a **vector field** defined by the vector point function $\bar{F}$.

Examples of vector point functions are (*i*) velocity of a moving fluid, (*ii*) gravitational force etc.

Level Surfaces

Let $\varphi(P)$ define a scalar field over a region R. The points P satisfying $\varphi(P) = c$, where c is an arbitrary constant, form a **family of curved lines** in two-dimensional space and a **family of surfaces** in three-dimensional space. The curved lines given by $\varphi(P) = c$, in two-dimensional space, are called **contour lines** and the surfaces $\varphi(P) = c$, in three-dimensional space, are called **level surfaces.**

Example: For the two-dimensional scalar field $\varphi(P) = x^2 + y^2$, the contour lines are circles $x^2 + y^2 = c^2$, c is an arbitrary constant.

For the three-dimensional scalar field $\varphi(P) = x^2 + y^2 + z^2$, the **level surfaces** are spheres $x^2 + y^2 + z^2 = c^2$, c is an arbitrary constant.

13.5 GRADIENT OF A SCALAR POINT FUNCTION

Let us first introduce a symbol del (or nabla) $\bar{\nabla}$, in rectangular cartesian system, as a vector operator defined by

$$\bar{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \equiv \Sigma\hat{i}\frac{\partial}{\partial x}.$$

Definition: Let $\varphi(x, y, z)$ be a continuously differentiable scalar point function defined in a certain region R of three-dimensional space. The gradient of φ, written as grad φ or $\bar{\nabla}\varphi$ is defined by

$$\text{grad } \varphi = \bar{\nabla}\varphi = \hat{i}\frac{\partial \varphi}{\partial x} + \hat{j}\frac{\partial \varphi}{\partial y} + \hat{k}\frac{\partial \varphi}{\partial z} = \Sigma\hat{i}\frac{\partial \varphi}{\partial x}.$$

Note: (*i*) Here $\bar{\nabla}\varphi$ is a vector point function, where φ is a scalar point function. Therefore, the gradient of a scalar field defines a vector field.

(*ii*) Now,

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz$$

$$= \left(\hat{i}\frac{\partial \varphi}{\partial x} + \hat{j}\frac{\partial \varphi}{\partial y} + \hat{k}\frac{\partial \varphi}{\partial z}\right)\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= (\bar{\nabla}\varphi)\cdot d\bar{r} \qquad [\text{where } \bar{r} = \hat{i}x + \hat{j}y + \hat{k}z. \quad \therefore d\bar{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz]$$

$$= |\vec{\nabla}\varphi|\,|d\bar{r}|\cos\theta \text{ [where } \theta \text{ is the angle between the directions of } \vec{\nabla}\varphi \text{ and } d\bar{r}\text{]}$$

$$\leq |\vec{\nabla}\varphi|\,|d\bar{r}|.$$

The value of $d\varphi$ is maximum when $\theta = 0$, *i.e.*, when $d\bar{r}$ and $\nabla\bar{\varphi}$ are in the same direction. Therefore at any point, the direction of the vector $\vec{\nabla}\varphi$ indicates the direction in which one must move from the point to get the most rapid increase in the function φ. For this reason the name, gradient of φ.

Geometrical Interpretation of $\vec{\nabla}\varphi$

Theorem:

If $\varphi(x, y, z)$ be a scalar point function defined in a certain region R of three-dimensional space, then $\vec{\nabla}\varphi$ is a vector normal to the level surface $\varphi(x, y, z) = c$, where c is a constant.

Proof: Suppose $\bar{r}$ and $\bar{r} + \delta\bar{r}$ are the position vectors of two neighbouring points $P(x, y, z)$ and $Q(x + \delta x, y + \delta y, z + \delta z)$ respectively on the level surface $\varphi(x, y, z) = c$.

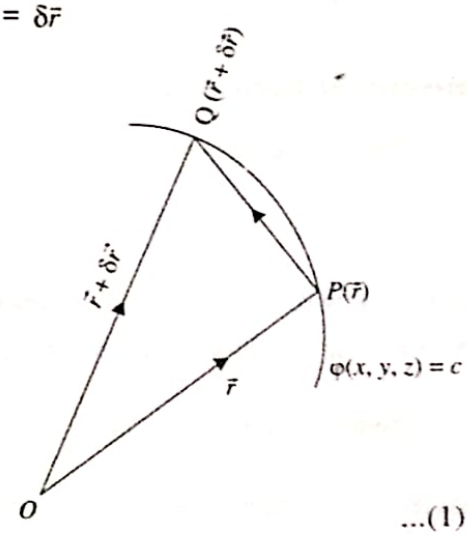
$$\therefore \quad \bar{r} = \hat{i}x + \hat{j}y + z\hat{k}, \quad \bar{r} + \delta\bar{r} = \hat{i}(x + \delta x) + \hat{j}(y + \delta y) + \hat{k}(z + \delta z),$$

$$\overline{PQ} = p.v. \text{ of } Q - p.v. \text{ of } P = (\bar{r} + \delta\bar{r}) - \bar{r} = \delta\bar{r}$$

$$= \hat{i}\,\delta x + \hat{j}\,\delta y + \hat{k}\,\delta z$$

As $Q \to P$, the line PQ tends to lie on the tangent plane at P to the level surface. Hence $d\bar{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$ lies on the tangent plane to the surface $\varphi(x, y, z) = c$ at P.

Now the given level surface be $\varphi(x, y, z) = c$.



$$\therefore \quad 0 = d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$

$$= \left(\hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z}\right)\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \vec{\nabla}\varphi \cdot d\bar{r} \qquad \text{...(1)}$$

Now Q is an arbitrary neighbouring point of P on the level surface and hence $d\bar{r}$ is an arbitrary vector lying on the tangent plane at P of that level surface. Therefore, from (1) we conclude that $\vec{\nabla}\varphi$ is perpendicular to the tangent plane at P of the given level surface $\varphi(x, y, z) = c$.

Tangent Plane

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point $P(x, y, z)$ of the required tangent plane at $P_0(x_0, y_0, z_0)$ of the given level surface $\varphi(x, y, z) = c$. The position vector of P_0 be $\bar{r}_0 = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$ and $(\vec{\nabla}\varphi)_0$ denotes the value of $\vec{\nabla}\varphi$ at $P_0(x_0, y_0, z_0)$.

Then $(\vec{r}-\vec{r}_0)$ is a vector lying on the tangent plane and $(\vec{\nabla}\varphi)_0$ is normal to the tangent plane at P_0 to the given level surface $\varphi(x, y, z) = c$.

Therefore, $(\vec{r}-\vec{r}_0)\cdot(\vec{\nabla}\varphi)_0 = 0$ ...(2)

is satisfied by any point with position vector $\vec{r}$ lying on the required tangent plane and by no others. Hence (2) represents the vector equation of the tangent plane at P_0 of the given level surface $\varphi(x, y, z) = c$.

Normal

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point $P(x, y, z)$ on the required normal and let $\vec{r}_0 = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$ be the position vector of the given point $P_0(x_0, y_0, z_0)$ on the level surface $\varphi(x, y, z) = c$ at which the normal is drawn. Therefore $(\vec{r}-\vec{r}_0)$ lies along the normal and $(\vec{\nabla}\varphi)_0$ is also normal to the tangent plane at P_0 and hence these two vectors are parallel.

$\therefore$ $(\vec{r}-\vec{r}_0)\times(\vec{\nabla}\varphi)_0 = \vec{0}$ ...(3)

Now (3) is satisfied by any point with position vector $\vec{r}$ lying on the normal at P_0 and by no others. Hence (3) represents the vector equation of the normal at P_0 of the given level surface $\varphi(x, y, z) = c$.

Directional Derivative

The component of $\vec{\nabla}\varphi$ in th e direction of a vector $\vec{a}$ is equal to

$$\vec{\nabla}\varphi\cdot\frac{\vec{a}}{|\vec{a}|} = \vec{\nabla}\varphi\cdot\hat{a}$$ (where $\hat{a}$ is the unit vector in the direction of $\vec{a}$)

and is known as the directional derivative of φ in the direction of $\vec{a}$. Physically, the directional derivative in the direction of $\vec{a}$ means the rate of change of φ at (x, y, z) in the direction of $\vec{a}$.

Note: *(i)* $\vec{\nabla}\varphi\cdot\frac{\vec{a}}{|\vec{a}|} = \vec{\nabla}\varphi\cdot\hat{a} = |\vec{\nabla}\varphi|\cos\theta$, where θ is the angle between $\vec{\nabla}\varphi$ and $\vec{a}$. Therefore, the directional derivative of φ is maximum in the direction of $\vec{\nabla}\varphi$ and the maximum value is $|\vec{\nabla}\varphi|$.

(ii) Remember that $\vec{\nabla}\varphi$ is a vector in the direction in which the maximum value of the directional derivative of φ occurs and the length of $\vec{\nabla}\varphi = |\vec{\nabla}\varphi|$ = maximum rate of change of φ.

ILLUSTRATIVE EXAMPLES

Example 1: If $\varphi = x^2y - y^3z^2$, find $\vec{\nabla}\varphi$ at $(1, 2, -1)$.

Solution: $$\vec{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y - y^3z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(x^2y - y^3z^2)$$

$$= 2xy\hat{i} + (x^2 - 3y^2z^2)\hat{j} - 2y^3z\hat{k}$$

$$\therefore \qquad \vec{\nabla}\varphi\Big|_{(1,2,-1)} = 4\hat{i} - 11\hat{j} + 16\hat{k}.$$

Example 2: Find a unit vector normal to the surface $x^2 + 3y^2 + 2z^2 = 6$ at P (2, 0, 1).

Solution: Here $\varphi = x^2 + 3y^2 + 2z^2 = 6$. Now $\vec{\nabla}\varphi$ is a normal vector.

$$\vec{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + 3y^2 + 2z^2)$$

$$= 2x\hat{i} + 6y\hat{j} + 4z\hat{k}$$

Therefore, normal vector at (2, 0, 1) is $4\hat{i} + 4\hat{k}$. Hence unit normal vector at (2, 0, 1) is

$$\frac{4\hat{i} + 4\hat{k}}{\sqrt{4^2 + 4^2}} = \frac{1}{\sqrt{2}}(\hat{i} + \hat{k}).$$

Example 3: If $\varphi \equiv \varphi(x, y, z, t)$, prove that

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + \vec{\nabla}\varphi\cdot\frac{d\vec{r}}{dt}, \text{ where } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

and x, y, z are differentiable functions of t.

Solution:
$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial\varphi}{\partial t} + \left(\hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z}\right)\cdot\left(\hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt}\right)$$

$$= \frac{\partial\varphi}{\partial t} + \vec{\nabla}\varphi\cdot\frac{d\vec{r}}{dt} \qquad [\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$$

Example 4: Find the equations of the tangent plane and normal line to the surface $2x^2 + y^2 + 2z = 3$ at the point (2, 1, −3). (*W.B.U.T. 2002*)

Solution: Here the level surface is

$$\varphi = 2x^2 + y^2 + 2z = 3 \qquad \ldots(1)$$

The position vectors of P (x, y, z) and P_0 (2, 1, −3) are respectively $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and

$$\vec{r}_0 = 2\hat{i} + \hat{j} - 3\hat{k}$$

$$\vec{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x^2 + y^2 + 2z) = 4x\hat{i} + 2y\hat{j} + 2\hat{k}$$

$$(\vec{\nabla}\varphi)_0 = \vec{\nabla}\varphi/_{P_0} = 8\hat{i}+2\hat{j}+2\hat{k}$$

$$\vec{r}-\vec{r}_0 = (x-2)\hat{i}+(y-1)\hat{j}+(z+3)\hat{k}$$

Therefore, the equation of the tangent plane to the surface (1) at the point (2,1, –3) is

$$(\vec{r}-\vec{r}_0)\cdot(\vec{\nabla}\varphi)_0 = 0, \text{ or } 8(x-2)+2(y-1)+2(z+3) = 0$$

or $$4x+y+z = 6.$$

The equation of the normal to the surface (1) at the point (2, 1, –3) is

$$(\vec{r}-\vec{r}_0)\times(\vec{\nabla}\varphi)_0 = \vec{0}$$

or $$\vec{r}-\vec{r}_0 = \lambda(\vec{\nabla}\varphi)_0 \qquad \text{(since } \vec{r}-\vec{r}_0 \text{ and } (\vec{\nabla}\varphi)_0 \text{ are parallel)}$$

where λ is a scalar.

or $$(x-2)\hat{i}+(y-1)\hat{j}+(z+3)\hat{k} = \lambda(8\hat{i}+2\hat{j}+2\hat{k}).$$

Equating the coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ from both sides, since they are independent, we get

$$\frac{x-2}{8} = \frac{y-1}{2} = \frac{z+3}{2}, \text{ or } \frac{x-2}{4} = \frac{y-1}{1} = \frac{z+3}{1}.$$

Note: The normal line is parallel to $(\vec{\nabla}\varphi)_0$, *i.e.*, parallel to the vector $4\hat{i}+\hat{j}+\hat{k}$.

Example 5: Show that the equation of the tangent plane to the surface $z = xy$ at the point (2, 3, 6) is $3x + 2y = z + 6$.

Solution: Here the level surface is

$$\varphi = z - xy = 0 \qquad ...(1)$$

The position vectors of $P(x, y, z)$ and P_0 (2, 3, 6) are $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and $\vec{r}_0 = 2\hat{i}+3\hat{j}+6\hat{k}$ respectively.

Now, $$\vec{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(z-xy) = -y\hat{i}-x\hat{j}+\hat{k}$$

$\therefore$ $$(\vec{\nabla}\varphi)_0 = \vec{\nabla}\varphi/_{P_0} = -3\hat{i}-2\hat{j}+\hat{k}$$

$$\vec{r}-\vec{r}_0 = (x-2)\hat{i}+(y-3)\hat{j}+(z-6)\hat{k}$$

Therefore, the equation of the tangent plane to the surface (1) at the point (2, 3, 6) is

$$(\vec{r}-\vec{r}_0)\cdot(\vec{\nabla}\varphi)_0 = 0, \text{ or } -3(x-2)-2(y-3)+z-6 = 0$$

or $$3x+2y = z+6.$$

Example 6: Find the angle of intersection of the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, –1, 2).

Solution: The given surfaces are

$$\varphi = x^2 + y^2 + z^2 = 9 \quad \text{...(1)}$$

$$\psi = x^2 + y^2 - z = 3 \quad \text{...(2)}$$

$$\therefore \quad \vec{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\vec{\nabla}\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 - z) = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\therefore \quad (\vec{\nabla}\varphi)_0 = \vec{\nabla}\varphi/_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

$$(\vec{\nabla}\psi)_0 = \vec{\nabla}\psi/_{(2,-1,2)} = 4\hat{i} - 2\hat{j} - \hat{k}$$

The angle of intersection at a common point of the surfaces (1) and (2), by definition, is the angle between their tangent planes at that point, which is also the angle between their normals at that point.

Now $(\vec{\nabla}\varphi)_0$ and $(\vec{\nabla}\psi)_0$ are along the normals to the surfaces (1) and (2) respectively at (2, –1, 2). Thus the required angle between the surfaces (1) and (2) at (2, –1, 2), *i.e.*, the angle between $(\vec{\nabla}\varphi)_0$ and $(\vec{\nabla}\psi)_0$, is given by

$$\theta = \cos^{-1}\left\{\frac{(\vec{\nabla}\varphi)_0 \cdot (\vec{\nabla}\psi)_0}{|(\vec{\nabla}\varphi)_0||(\vec{\nabla}\psi)_0|}\right\}$$

$$= \cos^{-1}\left\{\frac{(4\hat{i} - 2\hat{j} + 4\hat{k})\cdot(4\hat{i} - 2\hat{j} - \hat{k})}{\sqrt{16+4+16}\ \sqrt{16+4+1}}\right\} = \cos^{-1}\left(\frac{16}{6\sqrt{21}}\right)$$

$$= \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

Example 7: Find the constants a and b so that the surface $ax^2 - byz + z^2 = (a-2)x$ will be orthogonal to the surface $4x^2y + z^3 = 12$ at the point (1, 1, 2).

Solution: Here (1, 1, 2) lies on the surface

$$ax^2 - byz + z^2 = (a-2)x$$

$$\therefore \quad a - 2b + 4 = a - 2, \text{ or } b = 3.$$

The given surfaces are

$$\varphi = ax^2 - 3yz + z^2 - (a-2)x = 0 \quad ...(1)$$

$$\psi = 4x^2y + z^3 = 12 \quad ...(2)$$

$$\therefore \quad \vec{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\{ax^2 - 3yz + z^2 - (a-2)x\}$$

$$= (2ax - a + 2)\hat{i} - 3z\hat{j} + (2z - 3y)\hat{k}$$

$$\vec{\nabla}\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^2y + z^3) = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$$

$$(\vec{\nabla}\varphi)_0 = \vec{\nabla}\varphi\big|_{(1,1,2)} = (a+2)\hat{i} - 6\hat{j} + \hat{k}$$

and

$$(\vec{\nabla}\psi)_0 = \vec{\nabla}\psi\big|_{(1,1,2)} = 8\hat{i} + 4\hat{j} + 12\hat{k}$$

If the surfaces (1) and (2) intersect orthogonally, the angle between their normals at the common point (1, 1, 2) must be 90°. Here $(\vec{\nabla}\varphi)_0$ and $(\vec{\nabla}\psi)_0$ are along the normals to the surfaces (1) and (2) respectively at (1, 1, 2). Hence the given surfaces will be orthogonal at the point (1, 1, 2) if

$$(\vec{\nabla}\varphi)_0 \cdot (\vec{\nabla}\psi)_0 = 0,$$

or

$$\{(a+2)\hat{i} - 6\hat{j} + \hat{k}\} \cdot (8\hat{i} + 4\hat{j} + 12\hat{k}) = 0$$

or

$$8a + 16 - 24 + 12 = 0, \text{ or } a = -\frac{1}{2}.$$

Therefore, the required values are $a = -\frac{1}{2}$, $b = 3$.

Example 8: If r = $|\bar{r}|$, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

(*i*) $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$ (*W.B.U.T. 2013*)

(*ii*) $\vec{\nabla}(r^n) = nr^{n-2}\bar{r}$ (*W.B.U.T. 2004, 2012*)

Solution: Here $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$\therefore \quad r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2} \text{ and } r^2 = x^2 + y^2 + z^2$$

$$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x, \text{ or } \frac{\partial r}{\partial x} = \frac{x}{r}$$

...(1)

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \quad \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

(*i*)
$$\vec{\nabla}\left(\frac{1}{r}\right) = \hat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\,\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\,\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$
$$= \hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$
$$= -\hat{i}\,\frac{1}{r^2}\cdot\frac{x}{r} - \hat{j}\,\frac{1}{r^2}\cdot\frac{y}{r} - \hat{k}\,\frac{1}{r^2}\cdot\frac{z}{r} \qquad \text{[by (1)]}$$
$$= -\frac{1}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) = -\frac{\bar{r}}{r^3}.$$

(*ii*)
$$\vec{\nabla}(r^n) = \hat{i}\,\frac{\partial}{\partial x}(r^n) + \hat{j}\,\frac{\partial}{\partial y}(r^n) + \hat{k}\,\frac{\partial}{\partial z}(r^n)$$
$$= \hat{i}\,n r^{n-1}\frac{\partial r}{\partial x} + \hat{j}\,n r^{n-1}\frac{\partial r}{\partial y} + \hat{k}\,n r^{n-1}\frac{\partial r}{\partial z}$$
$$= \hat{i}\,n r^{n-1}\frac{x}{r} + \hat{j}\,n r^{n-1}\frac{y}{r} + \hat{k}\,n r^{n-1}\frac{z}{r} \qquad \text{[by (1)]}$$
$$= n r^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= n r^{n-2}\,\bar{r}.$$

Example 9: Find $\varphi(r)$ such that $\vec{\nabla}\varphi = \dfrac{\bar{r}}{r^4}$ and $\varphi(1) = 0$, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\bar{r}|$.

Solution: Here $\vec{\nabla}\varphi = \dfrac{\bar{r}}{r^4}$

$$\therefore \quad \hat{i}\,\frac{\partial\varphi}{\partial x} + \hat{j}\,\frac{\partial\varphi}{\partial y} + \hat{k}\,\frac{\partial\varphi}{\partial z} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^4}$$

Since $\hat{i}, \hat{j}, \hat{k}$ are independent, equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ from both sides, we get

$$\frac{\partial\varphi}{\partial x} = \frac{x}{r^4}, \quad \frac{\partial\varphi}{\partial y} = \frac{y}{r^4}, \quad \frac{\partial\varphi}{\partial z} = \frac{z}{r^4}$$

$$\therefore \quad d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$
$$= \frac{x}{r^4}dx + \frac{y}{r^4}dy + \frac{z}{r^4}dz$$

$$= \frac{x\,dx + y\,dy + z\,dz}{r^4}$$

$$= \frac{r\,dr}{r^4} \qquad [\because r^2 = x^2 + y^2 + z^2 \;\therefore 2r\,dr = 2x\,dx + 2y\,dy + 2z\,dz]$$

$$= \frac{dr}{r^3}.$$

$$\therefore \quad d\varphi = \frac{dr}{r^3}$$

$$\therefore \quad \int d\varphi = \int \frac{dr}{r^3}$$

or $$\varphi(r) = -\frac{1}{2r^2} + c$$

But $\varphi(1) = 0$, therefore $0 = -\frac{1}{2} + c$, or $c = \frac{1}{2}$

$$\therefore \quad \varphi(r) = -\frac{1}{2r^2} + \frac{1}{2} = \frac{1}{2}\left(1 - \frac{1}{r^2}\right).$$

Example 10: Find the directional derivative of the scalar point function $f(x, y, z) = x^2 + xy + z^2$ at the point A (1, −1, −1) in the direction of $\overline{AB}$ where B has coordinates (3, 2, 1).

Solution: Here $f(x, y, z) = x^2 + xy + z^2$

$$\therefore \quad \vec{\nabla} f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + xy + z^2)$$

$$= (2x + y)\hat{i} + x\hat{j} + 2z\hat{k}$$

$$\therefore \quad [\vec{\nabla} f]_A = [\vec{\nabla} f]_{(1,-1,-1)} = \hat{i} + \hat{j} - 2\hat{k} \qquad \text{...(1)}$$

$$\overline{AB} = \text{p.v. of } B - \text{p.v. of } A = 3\hat{i} + 2\hat{j} + \hat{k} - (\hat{i} - \hat{j} - \hat{k})$$

$$= 2\hat{i} + 3\hat{j} + 2\hat{k}$$

Unit vector in the direction of $\overline{AB}$ is

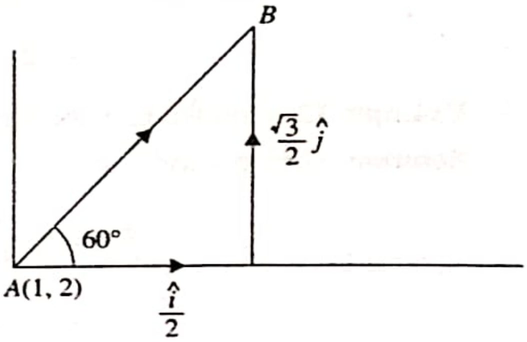
$$\hat{a} = \frac{2\hat{i} + 3\hat{j} + 2\hat{k}}{\sqrt{2^2 + 3^2 + 2^2}} = \frac{1}{\sqrt{17}}(2\hat{i} + 3\hat{j} + 2\hat{k}) \qquad \text{...(2)}$$

Therefore, the required directional derivative is

$$[\vec{\nabla} f]_A \cdot \hat{a} = (\hat{i} + \hat{j} - 2\hat{k}) \cdot \frac{1}{\sqrt{17}}(2\hat{i} + 3\hat{j} + 2\hat{k}) \qquad \text{[by (1) and (2)]}$$

$$= \frac{1}{\sqrt{17}}(2 + 3 - 4) = \frac{1}{\sqrt{17}}.$$

Example 11: For the function $\varphi(x, y) = \dfrac{xy}{x^2+y^2}$, find the magnitude of the directional derivative along a line making an angle 60° with the positive x-axis at (1, 2).



Solution: Here $\bar{\nabla}\varphi = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)\dfrac{xy}{x^2+y^2}$

$$= \left\{\frac{y}{x^2+y^2} - \frac{2x^2y}{(x^2+y^2)^2}\right\}\hat{i}$$

$$+\left\{\frac{x}{x^2+y^2} - \frac{2xy^2}{(x^2+y^2)^2}\right\}\hat{j}$$

$$= \frac{(y^2-x^2)y}{(x^2+y^2)^2}\hat{i} + \frac{(x^2-y^2)x}{(x^2+y^2)^2}\hat{j}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}(-y\hat{i}+x\hat{j})$$

$$\therefore \quad [\bar{\nabla}\varphi]_{(1,2)} = \frac{3}{25}(2\hat{i}-\hat{j})$$

Unit vector in the direction of $\overline{AB}$ is

$$\hat{a} = \frac{\hat{i}}{2} + \frac{\sqrt{3}}{2}\hat{j}$$

$\therefore$ Directional derivative at (1, 2) along $\overrightarrow{AB}$

$$= [\bar{\nabla}\varphi]_{(1,2)} \cdot \hat{a} = \frac{3}{25}(2\hat{i}-\hat{j})\cdot\frac{1}{2}(\hat{i}+\sqrt{3}\hat{j})$$

$$= \frac{3}{25} - \frac{3\sqrt{3}}{50} = \frac{3(2-\sqrt{3})}{50}$$

Therefore, the required magnitude of the directional derivative is $\dfrac{3(2-\sqrt{3})}{50}$.

Example 12: Find the maximum value of the directional derivative of $\varphi = x^2 + y^2 + z^2$ at the point (1, 2, 3). Find also the direction in which it occurs.

Solution: Here $\varphi(x, y, z) = x^2 + y^2 + z^2$

$$\therefore \quad \bar{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+y^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\therefore \quad [\bar{\nabla}\varphi]_{(1,2,3)} = 2\hat{i} + 4\hat{j} + 6\hat{k}$$

Since the directional derivative is maximum in the direction of $\vec{\nabla}\varphi$, therefore the required direction is $2\hat{i}+4\hat{j}+6\hat{k}$.

The required maximum value of the directional derivative is the magnitude of $[\vec{\nabla}\varphi]_{(1,2,3)}$

$$= \left|2\hat{i}+4\hat{j}+6\hat{k}\right| = \sqrt{2^2+4^2+6^2} = \sqrt{56} = 2\sqrt{14}.$$

Example 13: Find the maximum rate of change of $\varphi = xy^2 + x^2yz^2$ at $(1, 1, -2)$.

Solution: Here $\varphi = xy^2 + x^2yz^2$

$$\vec{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(xy^2+x^2yz^2)$$

$$= (y^2+2xyz^2)\hat{i}+(2xy+x^2z^2)\hat{j}+2x^2yz\hat{k}$$

$$\therefore \quad [\vec{\nabla}\varphi]_{(1,1,-2)} = 9\hat{i}+6\hat{j}-4\hat{k}$$

Therefore, the maximum rate of change of φ at $(1, 1, -2)$ is $|\vec{\nabla}\varphi|$ at $(1, 1, -2) = \sqrt{9^2+6^2+(-4)^2}$ $= \sqrt{133}$.

Example 14: Find the directional derivative of $\vec{v}^2$, where $\vec{v} = xy^2\hat{i}+zy^2\hat{j}+xz^2\hat{k}$ at the point $(1, 2, 1)$ in the direction of the normal to the sphere $x^2+y^2+z^2 = 6$ at the point $(2, 1, 1)$.

Solution: Here $v^2 = \vec{v}^2 = \vec{v}\cdot\vec{v}$

$$= (xy^2\hat{i}+zy^2\hat{j}+xz^2\hat{k})\cdot(xy^2\hat{i}+zy^2\hat{j}+xz^2\hat{k})$$

$$= x^2y^4+z^2y^4+x^2z^4$$

$$\vec{\nabla}v^2 = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2y^4+z^2y^4+x^2z^4)$$

$$= (2xy^4+2xz^4)\hat{i}+(4x^2y^3+4z^2y^3)\hat{j}+(2zy^4+4x^2z^3)\hat{k}$$

$$\therefore \quad [\vec{\nabla}v^2]_{(1,2,1)} = 34\hat{i}+64\hat{j}+36\hat{k}$$

Normal to $f = x^2+y^2+z^2 = 6$ is

$$\vec{\nabla}f = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$$

$$= 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

Unit normal vector at $(2, 1, 1)$

$$= \frac{4\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{4^2+2^2+2^2}} = \frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}$$

Therefore, the required directional derivative is

$$[\vec{\nabla}v^2]_{(1,2,1)} \cdot \frac{(2\hat{i}+\hat{j}+\hat{k})}{\sqrt{6}} = \frac{1}{\sqrt{6}}(34\hat{i}+64\hat{j}+36\hat{k})\cdot(2\hat{i}+\hat{j}+\hat{k})$$

$$= \frac{1}{\sqrt{6}}(68+64+36) = \frac{168}{\sqrt{6}}.$$

Example 15: Find the directional derivative of $\varphi(x, y, z) = x^2y^2z^2$ at the point $(1, 1, -1)$ in the direction of the tangent to the curve $x = e^t$, $y = 2\sin t + 1$, $z = t - \cos t$, at $t = 0$.

Solution: Here $\varphi(x, y, z) = x^2y^2z^2$

$$\therefore \quad \vec{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)x^2y^2z^2 = 2xy^2z^2\,\hat{i}+2yz^2x^2\,\hat{j}+2zx^2y^2\,\hat{k}$$

$$\therefore \quad [\vec{\nabla}\varphi]_{(1,1,-1)} = 2\hat{i}+2\hat{j}-2\hat{k}$$

If $\bar{r}$ be the position vector of a point $P(x, y, z)$ on the curve $x = e^t$, $y = 2\sin t + 1$, $z = t - \cos t$ with respect to origin, then

$$\bar{r} = x\hat{i}+y\hat{j}+z\hat{k} = e^t\,\hat{i}+(2\sin t+1)\,\hat{j}+(t-\cos t)\hat{k}$$

$$\therefore \quad \frac{d\bar{r}}{dt} = e^t\,\hat{i}+(2\cos t)\,\hat{j}+(1+\sin t)\hat{k}$$

is a vector in the direction of the tangent at P on the given curve.

Therefore, unit vector in the direction of the tangent at $t = 0$ is

$$\hat{a} = \frac{\hat{i}+2\hat{j}+\hat{k}}{\sqrt{1^2+2^2+1^2}} = \frac{1}{\sqrt{6}}(\hat{i}+2\hat{j}+\hat{k})$$

Hence the directional derivative of φ in the direction of the tangent at $t = 0$ to the given curve is

$$[\vec{\nabla}\varphi]_{(1,1,-1)}\cdot\hat{a} = (2\hat{i}+2\hat{j}-2\hat{k})\cdot\frac{1}{\sqrt{6}}(\hat{i}+2\hat{j}+\hat{k})$$

$$= \frac{1}{\sqrt{6}}(2+4-2) = \frac{4}{\sqrt{6}}.$$

Example 16: Find the directional derivative of $\varphi = e^{2x}\cos yz$ at the origin in the direction of the tangent to the curve $x = a\sin t$, $y = a\cos t$, $z = at$ at $t = \frac{\pi}{4}$.

Solution: Here $\varphi = e^{2x}\cos yz$

$$\therefore \quad \vec{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)e^{2x}\cos yz$$

$$= \hat{i}\,2e^{2x}\cos yz - \hat{j}\,ze^{2x}\sin yz - \hat{k}\,ye^{2x}\sin yz$$

$$\therefore \quad [\vec{\nabla}\varphi]_{(0,0,0)} = 2\hat{i}$$

If $\bar{r}$ be the position vector of a point $P(x, y, z)$ on the curve $x = a\sin t$, $y = a\cos t$, $z = at$ with respect to origin, then $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k} = \hat{i}\,a\sin t+\hat{j}\,a\cos t+at\hat{k}$. Therefore, $\therefore \quad \frac{d\bar{r}}{dt} = \hat{i}\,a\cos t - \hat{j}\,a\sin t+a\hat{k}$ is a vector in the direction of the tangent at P on the given curve. Therefore, unit vector in the direction of the tangent at $t = \frac{\pi}{4}$ is

$$\hat{a} = \left(\frac{a}{\sqrt{2}}\hat{i}-\frac{a}{\sqrt{2}}\hat{j}+a\hat{k}\right)\Big/\sqrt{\frac{a^2}{2}+\frac{a^2}{2}+a^2}$$

$$= \frac{\hat{i}}{2}-\frac{\hat{j}}{2}+\frac{\hat{k}}{\sqrt{2}}$$

$\therefore$ Directional derivative of φ at origin in the direction of the tangent at $t = \frac{\pi}{4}$ to the given curve is

$$[\bar{\nabla}\varphi]_{(0,0,0)}\cdot\bar{a} = 2\hat{i}\cdot\left(\frac{\hat{i}}{2}-\frac{\hat{j}}{2}+\frac{\hat{k}}{\sqrt{2}}\right) = 1.$$

Example 17: Find the directional derivatives of $\varphi = x^2y^2z^2$ at $(1, -1, 1)$ in the direction making equal angles with the positive X, Y and Z-axes.

Solution: Here $\varphi = x^2y^2z^2$

$$\therefore \quad \bar{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)x^2y^2z^2 = 2xy^2z^2\,\hat{i}+2yx^2z^2\,\hat{j}+2zx^2y^2\,\hat{k}$$

$$\therefore \quad [\bar{\nabla}\varphi]_{(1,-1,1)} = 2\hat{i}-2\hat{j}+2\hat{k}$$

Let $\hat{a}$ be the unit vector in the direction making equal angles θ with the positive X, Y and Z-axes.

$\therefore \quad \hat{a} = \hat{i}\cos\theta+\hat{j}\cos\theta+\hat{k}\cos\theta$, where $\cos^2\theta+\cos^2\theta+\cos^2\theta = 1$, or $3\cos^2\theta = 1$

$$\therefore \quad \cos\theta = \pm\frac{1}{\sqrt{3}}$$

$$\therefore \quad \hat{a} = \pm\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$$

Therefore, the required directional derivatives are

$$[\bar{\nabla}\varphi]_{(1,-1,1)}\cdot\hat{a} = \pm\frac{1}{\sqrt{3}}(2\hat{i}-2\hat{j}+2\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})$$

$$= \pm\frac{(2-2+2)}{\sqrt{3}} = \pm\frac{2}{\sqrt{3}}.$$

Example 18: In what direction from the point (1, 1, –1) is the directional derivative of $\varphi(x, y, z) = x^2 - 2y^2 + 4z^2$ a maximum? What is the magnitude of this directional derivative?

(W.B.U.T. 2003, 2007)

Solution: Here $\varphi(x, y, z) = x^2 - 2y^2 + 4z^2$

$$\therefore \quad \bar{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - 2y^2 + 4z^2) = 2x\hat{i} - 4y\hat{j} + 8z\hat{k}$$

$$\therefore \quad [\bar{\nabla}\varphi]_{(1,1,-1)} = 2\hat{i} - 4\hat{j} - 8\hat{k}$$

Since the directional derivative is maximum in the direction of $\bar{\nabla}\varphi$, therefore the required direction is $2\hat{i} - 4\hat{j} - 8\hat{k}$.

The required maximum value of the directional derivative is the magnitude of

$$[\bar{\nabla}\varphi]_{(1,1,-1)} = |2\hat{i} - 4\hat{j} - 8\hat{k}| = \sqrt{2^2 + (-4)^2 + (-8)^2}$$

$$= \sqrt{84} = 2\sqrt{21}.$$

Example 19: Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1, –2, 1). *(W.B.U.T. 2002)*

Solution: The given surfaces are

$$\varphi = xy^2z - 3x - z^2 = 0 \quad \text{...(1)}$$

$$\psi = 3x^2 - y^2 + 2z = 1 \quad \text{...(2)}$$

$$\therefore \quad \bar{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2z - 3x - z^2)$$

$$= (y^2z - 3)\hat{i} + 2xyz\hat{j} + (xy^2 - 2z)\hat{k}$$

$$\bar{\nabla}\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2 - y^2 + 2z)$$

$$= 6x\hat{i} - 2y\hat{j} + 2\hat{k}$$

$$\therefore \quad (\bar{\nabla}\varphi)_0 = \bar{\nabla}\varphi/_{(1,-2,1)} = \hat{i} - 4\hat{j} + 2\hat{k}$$

$$(\bar{\nabla}\psi)_0 = \bar{\nabla}\psi/_{(1,-2,1)} = 6\hat{i} + 4\hat{j} + 2\hat{k}$$

The angle of intersection at a common point of the surfaces (1) and (2), by definition, is the angle between their tangent planes at that point, which is also the angle between their normals at that point. Now $(\bar{\nabla}\varphi)_0$ and $(\bar{\nabla}\psi)_0$ are along the normals to the surfaces (1) and (2) respectively at (1, –2, 1).

Thus the required angle between the surfaces (1) and (2) at (1, –2, 1), *i.e.*, the angle between $(\bar{\nabla}\varphi)_0$ and $(\bar{\nabla}\psi)_0$, is given by

$$\theta = \cos^{-1}\left\{\frac{(\bar{\nabla}\varphi)_0 \cdot (\bar{\nabla}\psi)_0}{|(\bar{\nabla}\varphi)_0|\,|(\bar{\nabla}\psi)_0|}\right\}$$

$$= \cos^{-1}\left\{\frac{(\hat{i}-4\hat{j}+2\hat{k})\cdot(6\hat{i}+4\hat{j}+2\hat{k})}{\sqrt{1^2+(-4)^2+2^2}\,\sqrt{6^2+4^2+2^2}}\right\}$$

$$= \cos^{-1}\left(\frac{6-16+4}{\sqrt{21}\,\sqrt{56}}\right) = \cos^{-1}\left(-\frac{3}{7\sqrt{6}}\right).$$

Example 20: Find the directional derivative of $f(x, y, z) = x^2yz + 4xz^2$ at the point (1, 2, –1) in the direction of the vector $2\hat{i}-\hat{j}-2\hat{k}$. (*W.B.U.T. 2008*)

Solution: Here $f(x, y, z) = x^2yz + 4xz^2$

$$\therefore \qquad \bar{\nabla}f = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2yz+4xz^2)$$

$$= (2xyz+4z^2)\hat{i}+x^2z\hat{j}+(x^2y+8xz)\hat{k}$$

$$\therefore \qquad [\bar{\nabla}f]_{(1,2,-1)} = -\hat{j}-6\hat{k}$$

Therefore, the required directional derivative is

$$[\bar{\nabla}f]_{(1,2,-1)}\cdot\frac{(2\hat{i}-\hat{j}-2\hat{k})}{\sqrt{2^2+(-1)^2+(-2)^2}} = \frac{1}{3}(0\hat{i}-\hat{j}-6\hat{k})\cdot(2\hat{i}-\hat{j}-2\hat{k})$$

$$= \frac{1}{3}(1+12) = \frac{13}{3}.$$

Example 21: If $u = x + y + z$, $v = x^2 + y^2 + z^2$ and $\omega = xy + yz + zx$, prove that grad u, grad v and grad ω are coplanar.

Solution: Here

$$\text{grad } u = \bar{\nabla}u = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x+y+z)$$

$$= \hat{i}+\hat{j}+\hat{k}.$$

$$\text{grad } v = \bar{\nabla}v = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$$

$$= 2x\hat{i}+2y\hat{j}+2z\hat{k}.$$

$$\text{grad } \omega = \bar{\nabla}\omega = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(xy+yz+zx)$$

$$= (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}.$$

We know that three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

Here, [grad u, grad v, grad ω]

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & z+x & x+y \end{vmatrix} (R_2' \to R_2 + R_3)$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0$$

Therefore, grad u, grad v and grad ω are coplanar.

Example 22: Find the directional derivative of $\varphi = 5x^2y - 5y^2z + 2z^2x$ at the point P (1, 1, 1) in the direction of the line $\dfrac{x-1}{2} = \dfrac{y-3}{-2} = z$.

Solution: Here

$$\bar{\nabla}\varphi = \hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z}$$

$$= (10xy + 2z^2)\hat{i} + (5x^2 - 10yz)\hat{j} + (-5y^2 + 4zx)\hat{k}$$

$$= 12\hat{i} - 5\hat{j} - \hat{k} \text{ at } P\ (1, 1, 1).$$

Also, $\hat{n} = \dfrac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$ is a unit vector parallel to the given line (see Example 4).

Hence the required directional derivative is

$$(\bar{\nabla}\varphi)\cdot\hat{n} = (12\hat{i} - 5\hat{j} - \hat{k})\cdot\frac{1}{3}(2\hat{i} - 2\hat{j} + \hat{k})$$

$$= \frac{1}{3}(24 + 10 - 1) = \frac{33}{3} = 11.$$

Example 23: Find the values of the constants a, b, c so that the directional derivative of $\varphi = axy^2 + byz + cz^2x^3$ at (1, 2, –1) has a maximum magnitude 64 in the direction of positive z-axis.

Solution: Here $\varphi = axy^2 + byz + cz^2x^3$

$$\therefore \quad \bar{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(axy^2 + byz + cz^2x^3)$$

$$= (ay^2 + 3cz^2x^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2czx^3)\hat{k}$$

$$\therefore \quad (\bar{\nabla}\varphi)_0 = \bar{\nabla}\varphi/_{(1,\,2,\,-1)}$$

$$= (4a + 3c)\hat{i} + (4a - b)\hat{j} + (2b - 2c)\hat{k} \qquad ...(1)$$

By question directional derivative of φ at (1, 2, –1) has a maximum magnitude 64 in the direction of positive z-axis, *i.e.*, in the direction of unit vector $\hat{k}$.

$$\therefore \quad \left|(\bar{\nabla}\varphi)_0 \cdot \hat{k}\right| = 64, \text{ or } \left|\{(4a + 3c)\hat{i} + (4a - b)\hat{j} + (2b - 2c)\hat{k}\} \cdot \hat{k}\right| = 64$$

$$\therefore \quad |2b - 2c| = 64, \text{ or } b - c = \pm 32 \qquad ...(2)$$

Also $(\bar{\nabla}\varphi)_0$ is parallel to $\hat{k}$, *i.e.*, z-axis [since the directional derivative of φ at (1, 2, –1) is maximum in the direction of $(\bar{\nabla}\varphi)_0$], therefore $(\bar{\nabla}\varphi)_0$ is perpendicular to the x and y-axes.

Thus $(\bar{\nabla}\varphi)_0 \cdot \hat{i} = 0$, or $4a + 3c = 0$ [by (1)] ...(3)

and $(\bar{\nabla}\varphi)_0 \cdot \hat{j} = 0$, or $4a - b = 0$ [by (1)] ...(4)

Solving (2), (3) and (4), we get

$$a = 6,\ b = 24,\ c = -8, \text{ or } a = -6,\ b = -24,\ c = 8.$$

Example 24: If the directional derivative of $\varphi = ax^2y + by^2z + cz^2x$ at the point (1, 1, 1) has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$, find the values of a, b and c.

Solution: Here $\varphi = ax^2y + by^2z + cz^2x$.

$$\therefore \quad \bar{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(ax^2y + by^2z + cz^2x)$$

$$= (2axy + cz^2)\hat{i} + (ax^2 + 2byz)\hat{j} + (by^2 + 2czx)\hat{k}$$

$$\therefore \quad (\bar{\nabla}\varphi)_0 = \bar{\nabla}\varphi\big|_{(1,\,1,\,1)}$$

$$= (2a + c)\hat{i} + (a + 2b)\hat{j} + (b + 2c)\hat{k} \qquad ...(1)$$

By question directional derivative of φ at (1, 1, 1) has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$, *i.e.*, in the direction parallel to the vector $2\hat{i} - 2\hat{j} + \hat{k}$ (see Example 4).

$$\therefore \quad \left|(\bar{\nabla}\varphi)_0 \cdot \frac{(2\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{2^2 + (-2)^2 + 1^2}}\right| = 15$$

or $\frac{1}{3}\left|2(2a+c)-2(a+2b)+(b+2c)\right| = 15$ [using (1)]

$$\therefore \quad 2a-3b+4c = \pm 45 \quad \text{...(2)}$$

Also we know that the directional derivative of φ at (1, 1, 1) is maximum in the direction of $(\vec{\nabla}\varphi)_0$, therefore, $(\vec{\nabla}\varphi)_0$ is parallel to $2\hat{i}-2\hat{j}+\hat{k}$, *i.e.*, $(\vec{\nabla}\varphi)_0 = \lambda(2\hat{i}-2\hat{j}+\hat{k})$, λ is a scalar.

Using (1), we get

$$2a+c = 2\lambda, \quad a+2b = -2\lambda, \quad b+2c = \lambda$$

Solving, we get

$$a = \frac{4}{9}\lambda, \quad b = -\frac{11}{9}\lambda, \quad c = \frac{10}{9}\lambda$$

Putting these values of a, b, c in (2), we get

$$\lambda = \pm 5.$$

Therefore, the required values of a, b, c are

$$a = \pm\frac{20}{9}, \quad b = \mp\frac{55}{9} \quad \text{and} \quad c = \pm\frac{50}{9}.$$

13.6 DIVERGENCE OF A VECTOR POINT FUNCTION

Definition: Let $\vec{F}(x, y, z)$ be a continuously differentiable vector point function defined in a certain region R of three-dimensional space, then the divergence of $\vec{F}$, denoted by div $\vec{F}$ or $\vec{\nabla}\cdot\vec{F}$, is defined by

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \hat{i}\cdot\frac{\partial\vec{F}}{\partial x}+\hat{j}\cdot\frac{\partial\vec{F}}{\partial y}+\hat{k}\cdot\frac{\partial\vec{F}}{\partial z} = \Sigma\hat{i}\cdot\frac{\partial\vec{F}}{\partial x}.$$

Notes: (*i*) Divergence is defined on a vector point function and the result is a scalar quantity.

(*ii*) If c is a scalar constant, then $\vec{\nabla}\cdot(c\vec{F}) = c\vec{\nabla}\cdot\vec{F}$

(*iii*) If $\vec{F}(x, y, z) = \hat{i}F_1(x, y, z)+\hat{j}F_2(x, y, z)+\hat{k}F_3(x, y, z)$, then

$$\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}F_1+\hat{j}F_2+\hat{k}F_3)$$

$$= \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

Now,

$$\vec{F}\cdot\vec{\nabla} = (F_1\hat{i}+F_2\hat{j}+F_3\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)$$

$$= F_1\frac{\partial}{\partial x}+F_2\frac{\partial}{\partial y}+F_3\frac{\partial}{\partial z}$$

$$\therefore \qquad \bar{F}\cdot\bar{\nabla} \neq \bar{\nabla}\cdot\bar{F}$$

Laplacian Operator $\bar{\nabla}^2$

The Laplacian operator $\bar{\nabla}^2$ is defined by

$$\bar{\nabla}^2 = \bar{\nabla}\cdot\bar{\nabla} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} = \Sigma\frac{\partial^2}{\partial x^2}$$

If $\varphi(x, y, z)$ be a scalar point function and $\bar{F}(x, y, z)$ be a vector point function then

$$\bar{\nabla}^2\varphi = \frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}$$

and

$$\bar{\nabla}^2\bar{F} = \frac{\partial^2\bar{F}}{\partial x^2}+\frac{\partial^2\bar{F}}{\partial y^2}+\frac{\partial^2\bar{F}}{\partial z^2}$$

The equation $\bar{\nabla}^2\varphi = 0$ is known as **Laplace's equation** and any of its solution is called an **Harmonic function.**

Note: div grad $\varphi = \bar{\nabla}\cdot\bar{\nabla}\ \varphi = \bar{\nabla}^2\varphi$ and hence div grad $\equiv \bar{\nabla}^2$.

Solenoidal Vector Function

A vector point function $\bar{F}(x, y, z)$ is said to be solenoidal if div $\bar{F} \equiv \bar{\nabla}\cdot\bar{F} = 0$

Let $\bar{F}(x, y, z) = \hat{i}\,F_1(x, y, z)+\hat{j}\,F_2(x, y, z)+\hat{k}\,F_3(x, y, z)$.

Therefore, $\bar{F}$ is solenoidal if div $\bar{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z} = 0.$

13.7 CURL OR ROT OF A VECTOR POINT FUNCTION

Definition: Let $\bar{F}(x, y, z)$ be a continuously differentiable vector point function defined in a certain region R of three-dimensional space, then the curl of $\bar{F}$ (also known as rot of $\bar{F}$), denoted by curl $\bar{F}$ or rot $\bar{F}$ or $\bar{\nabla}\times\bar{F}$, is defined by curl $\bar{F}$ = rot $\bar{F} = \bar{\nabla}\times\bar{F} = \hat{i}\times\frac{\partial\bar{F}}{\partial x}+\hat{j}\times\frac{\partial\bar{F}}{\partial y}+\hat{k}\times\frac{\partial\bar{F}}{\partial z} = \Sigma\hat{i}\times\frac{\partial\bar{F}}{\partial x}$.

Notes: (*i*) Curl is defined on a vector point function and the result is a vector quantity.

(*ii*) If c is a scalar constant, then $\bar{\nabla}\times(c\bar{F}) = c\bar{\nabla}\times\bar{F}$

(*iii*) $\bar{F}\times\bar{\nabla} \neq \bar{\nabla}\times\bar{F}$

(*iv*) If $\bar{F}(x, y, z) = \hat{i}\,F_1(x, y, z)+\hat{j}\,F_2(x, y, z)+\hat{k}\,F_3(x, y, z)$, then

$$\text{Curl } \bar{F} = \bar{\nabla}\times\bar{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(\hat{i}\,F_1+\hat{j}\,F_2+\hat{k}\,F_3)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\hat{i}+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\hat{j}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\hat{k}$$

(*v*) If $\bar{a}$ is a constant vector, then $\bar{\nabla}\times\bar{a} = \bar{0}$.

Irrotational Vector Function

A vector point function $\bar{F}$ is said to be irrotational if curl $\bar{F} = \bar{\nabla}\times\bar{F} = \bar{0}$.

For example, the vector point function $\bar{F} = (x+2y+4z)\hat{i}+(2x-3y-z)\hat{j}+(4x-y+2z)\hat{k}$ is irrotational since

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+4z & 2x-3y-z & 4x-y+2z \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(4x-y+2z)-\frac{\partial}{\partial z}(2x-3y-z)\right\}\hat{i}$$

$$+\left\{\frac{\partial}{\partial z}(x+2y+4z)-\frac{\partial}{\partial x}(4x-y+2z)\right\}\hat{j}$$

$$+\left\{\frac{\partial}{\partial x}(2x-3y-z)-\frac{\partial}{\partial y}(x+2y+4z)\right\}\hat{k}$$

$$= (-1+1)\hat{i}+(4-4)\hat{j}+(2-2)\hat{k} = \bar{0}.$$

13.8 IMPORTANT IDENTITIES

Let $f(x, y, z)$ and $g(x, y, z)$ are continuously differentiable scalar point functions and let $\bar{F}(x, y, z)$ and $\bar{G}(x, y, z)$ are continuously differentiable vector point functions. Then we have the following identities:

Sums

I. $\text{grad } (f \pm g) = \text{grad } f \pm \text{grad } g$

II. $\text{div } (\vec{F} \pm \vec{G}) = \text{div } \vec{F} \pm \text{div } \vec{G}$

III. $\text{curl } (\vec{F} \pm \vec{G}) = \text{curl } \vec{F} \pm \text{curl } \vec{G}$

Products

IV. $\text{grad } (fg) = f \text{ grad } g + g \text{ grad } f$

V. $\text{grad } (\vec{F} \cdot \vec{G}) = \vec{F} \times \text{curl } \vec{G} + \vec{G} \times \text{curl } \vec{F} + (\vec{F} \cdot \vec{\nabla})\vec{G} + (\vec{G} \cdot \vec{\nabla})\vec{F}$

VI. $\text{div } (\vec{F} \times \vec{G}) = \vec{G} \cdot \text{curl } \vec{F} - \vec{F} \cdot \text{curl } \vec{G}$

VII. $\text{curl } (\vec{F} \times \vec{G}) = \vec{F} \text{ div } \vec{G} - \vec{G} \text{ div } \vec{F} + (\vec{G} \cdot \vec{\nabla})\vec{F} - (\vec{F} \cdot \vec{\nabla})\vec{G}$

Divergence and Curl of One Vector Function Multiplied by One Scalar Function

VIII. $\text{div } (f\vec{F}) = f \text{ div } \vec{F} + \vec{F} \cdot \text{grade } f$

IX. $\text{curl } (f\vec{F}) = \text{grad } f \times \vec{F} + f \text{ curl } \vec{F}$

Second Order Differential Operators

X. $\text{curl (grad } f) = \vec{0}$ *(BESUS 2013)*

XI. $\text{div (curl } \vec{F}) = 0$ *(BESUS 2013)*

XII. $\text{curl (curl } \vec{F}) = \text{grad (div } \vec{F}) - \text{Laplacian } \vec{F}$

Proof: Proofs of I, II and III follow directly from the definitions.

IV. L.H.S.

$$= \vec{\nabla}(fg) = \hat{i}\frac{\partial}{\partial x}(fg) + \hat{j}\frac{\partial}{\partial y}(fg) + \hat{k}\frac{\partial}{\partial z}(fg)$$

$$= \hat{i}\left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right) + \hat{j}\left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right) + \hat{k}\left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)$$

$$= f\left(\hat{i}\frac{\partial g}{\partial x} + \hat{j}\frac{\partial g}{\partial y} + \hat{k}\frac{\partial g}{\partial z}\right) + g\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

$$= f\vec{\nabla}g + g\vec{\nabla}f = \text{R.H.S.}$$

V.

$$\vec{\nabla}(\vec{F} \cdot \vec{G}) = \sum \hat{i}\frac{\partial}{\partial x}(\vec{F} \cdot \vec{G}) = \sum \hat{i}\left(\vec{F} \cdot \frac{\partial \vec{G}}{\partial x} + \vec{G} \cdot \frac{\partial \vec{F}}{\partial x}\right)$$

$$= \sum \hat{i}\left(\vec{F} \cdot \frac{\partial \vec{G}}{\partial x}\right) + \sum \hat{i}\left(\vec{G} \cdot \frac{\partial \vec{F}}{\partial x}\right) \qquad ...(1)$$

Now, $\qquad \vec{F}\times\left(\hat{i}\times\frac{\partial\vec{G}}{\partial x}\right) = \left(\vec{F}\cdot\frac{\partial\vec{G}}{\partial x}\right)\hat{i} - (\vec{F}\cdot\hat{i})\,\frac{\partial\vec{G}}{\partial x} \qquad [\because\ \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}]$

$\therefore \qquad \hat{i}\left(\vec{F}\cdot\frac{\partial\vec{G}}{\partial x}\right) = \vec{F}\times\left(\hat{i}\times\frac{\partial\vec{G}}{\partial x}\right) + (\vec{F}\cdot\hat{i})\,\frac{\partial\vec{G}}{\partial x}$

Similarly, $\qquad \hat{i}\left(\vec{G}\cdot\frac{\partial\vec{F}}{\partial x}\right) = \vec{G}\times\left(\hat{i}\times\frac{\partial\vec{F}}{\partial x}\right) + (\vec{G}\cdot\hat{i})\,\frac{\partial\vec{F}}{\partial x}$

Therefore, from (1), we get

$$\begin{aligned}
\vec{\nabla}(\vec{F}\cdot\vec{G}) &= \sum\vec{F}\times\left(\hat{i}\times\frac{\partial\vec{G}}{\partial x}\right) + \sum(\vec{F}\cdot\hat{i})\frac{\partial\vec{G}}{\partial x} \\
&\quad + \sum\vec{G}\times\left(\hat{i}\times\frac{\partial\vec{F}}{\partial x}\right) + \sum(\vec{G}\cdot\hat{i})\frac{\partial\vec{F}}{\partial x} \\
&= \vec{F}\times\sum\hat{i}\times\frac{\partial\vec{G}}{\partial x} + \vec{G}\times\sum\hat{i}\times\frac{\partial\vec{F}}{\partial x} \\
&\quad + \sum(\vec{F}\cdot\hat{i})\,\frac{\partial\vec{G}}{\partial x} + \sum(\vec{G}\cdot\hat{i})\,\frac{\partial\vec{F}}{\partial x} \\
&= \vec{F}\times(\vec{\nabla}\times\vec{G}) + \vec{G}\times(\vec{\nabla}\times\vec{F}) + (\vec{F}\cdot\vec{\nabla})\vec{G} + (\vec{G}\cdot\vec{\nabla})\vec{F}
\end{aligned}$$

VI. L.H.S.

$$\begin{aligned}
&= \vec{\nabla}\cdot(\vec{F}\times\vec{G}) = \sum\hat{i}\cdot\frac{\partial}{\partial x}\,(\vec{F}\times\vec{G}) \\
&= \sum\hat{i}\cdot\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G} + \vec{F}\times\frac{\partial\vec{G}}{\partial x}\right) = \sum\hat{i}\cdot\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) + \sum\hat{i}\cdot\left(\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right) \\
&= \sum\hat{i}\cdot\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) - \sum\hat{i}\cdot\left(\frac{\partial\vec{G}}{\partial x}\times\vec{F}\right) \\
&= \sum\left(\hat{i}\times\frac{\partial\vec{F}}{\partial x}\right)\cdot\vec{G} - \sum\left(\hat{i}\times\frac{\partial\vec{G}}{\partial x}\right)\cdot\vec{F} \qquad [\because\ \vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}] \\
&= \vec{G}\cdot\sum\left(\hat{i}\times\frac{\partial\vec{F}}{\partial x}\right) - \vec{F}\cdot\sum\left(\hat{i}\times\frac{\partial\vec{G}}{\partial x}\right) \\
&= \vec{G}\cdot(\vec{\nabla}\times\vec{F}) - \vec{F}\cdot(\vec{\nabla}\times\vec{G}) = \text{R.H.S.}
\end{aligned}$$

VII. L.H.S.

$$\begin{aligned}
&= \vec{\nabla}\times(\vec{F}\times\vec{G}) = \sum\hat{i}\times\frac{\partial}{\partial x}\,(\vec{F}\times\vec{G}) \\
&= \sum\hat{i}\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G} + \vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)
\end{aligned}$$

$$= \sum \hat{i} \times \left(\frac{\partial \vec{F}}{\partial x} \times \vec{G} \right) + \sum \hat{i} \times \left(\vec{F} \times \frac{\partial \vec{G}}{\partial x} \right)$$

$$= \sum \left\{ (\hat{i} \cdot \vec{G}) \frac{\partial \vec{F}}{\partial x} - \left(\hat{i} \cdot \frac{\partial \vec{F}}{\partial x} \right) \vec{G} \right\} + \sum \left\{ \left(\hat{i} \cdot \frac{\partial \vec{G}}{\partial x} \right) \vec{F} - (\hat{i} \cdot \vec{F}) \frac{\partial \vec{G}}{\partial x} \right\}$$

$$= \vec{F} \sum \hat{i} \cdot \frac{\partial \vec{G}}{\partial x} - \vec{G} \sum \hat{i} \cdot \frac{\partial \vec{F}}{\partial x} + \sum (\vec{G} \cdot \hat{i}) \frac{\partial \vec{F}}{\partial x} - \sum (\vec{F} \cdot \hat{i}) \frac{\partial \vec{G}}{\partial x}$$

$$= \vec{F}(\vec{\nabla} \cdot \vec{G}) - \vec{G}(\vec{\nabla} \cdot \vec{F}) + (\vec{G} \cdot \vec{\nabla}) \vec{F} - (\vec{F} \cdot \vec{\nabla}) \vec{G}$$

= R.H.S.

VIII. L.H.S.

$$= \vec{\nabla} \cdot (f\vec{F}) = \sum \hat{i} \cdot \frac{\partial}{\partial x}(f\vec{F}) = \sum \hat{i} \cdot \left(\frac{\partial f}{\partial x} \vec{F} + f \frac{\partial \vec{F}}{\partial x} \right)$$

$$= \sum \hat{i} \cdot \left(\frac{\partial f}{\partial x} \vec{F} \right) + \sum \hat{i} \cdot \left(f \frac{\partial \vec{F}}{\partial x} \right)$$

$$= \left(\sum \hat{i} \frac{\partial f}{\partial x} \right) \cdot \vec{F} + f \sum \left(\hat{i} \cdot \frac{\partial \vec{F}}{\partial x} \right)$$

$$= \vec{F} \cdot \vec{\nabla} f + f(\vec{\nabla} \cdot \vec{F})$$

= R.H.S.

IX. L.H.S.

$$= \vec{\nabla} \times (f\vec{F}) = \sum \hat{i} \times \frac{\partial}{\partial x}(f\vec{F}) = \sum \hat{i} \times \left(\frac{\partial f}{\partial x} \vec{F} + f \frac{\partial \vec{F}}{\partial x} \right)$$

$$= \sum \hat{i} \times \frac{\partial f}{\partial x} \vec{F} + \sum \hat{i} \times f \frac{\partial \vec{F}}{\partial x}$$

$$= \left(\sum \hat{i} \frac{\partial f}{\partial x} \right) \times \vec{F} + f \sum \left(\hat{i} \times \frac{\partial \vec{F}}{\partial x} \right)$$

$$= \vec{\nabla} f \times \vec{F} + f \vec{\nabla} \times \vec{F} = \text{R.H.S.}$$

X. L.H.S.

$$= \vec{\nabla} \times \vec{\nabla} f = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times \left(\hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right) + \hat{j}\left(\frac{\partial^2 f}{\partial z\partial x} - \frac{\partial^2 f}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\right)$$

$$= \vec{0} \qquad \left[\text{assuming } \frac{\partial^2 f}{\partial y\partial z} = \frac{\partial^2 f}{\partial z\partial y} \text{ etc.}\right]$$

$$= \text{R.H.S.}$$

XI. Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\therefore \quad \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\therefore \quad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left\{\hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right\}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} + \frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y}$$

$$= 0 \qquad \left[\text{assuming } \frac{\partial^2 F_1}{\partial y\partial z} = \frac{\partial^2 F_1}{\partial z\partial y} \text{ etc.}\right]$$

XII. Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\therefore \quad \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\therefore \text{ L.H.S.} = \vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial F_3}{\partial y}-\dfrac{\partial F_2}{\partial z} & \dfrac{\partial F_1}{\partial z}-\dfrac{\partial F_3}{\partial x} & \dfrac{\partial F_2}{\partial x}-\dfrac{\partial F_1}{\partial y} \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\right\}\hat{i}$$

$$+\left\{\frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)-\frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\right\}\hat{j}$$

$$+\left\{\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\right\}\hat{k}$$

$$= -\left(\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\right)\hat{i}-\left(\frac{\partial^2 F_2}{\partial x^2}+\frac{\partial^2 F_2}{\partial z^2}\right)\hat{j}-\left(\frac{\partial^2 F_3}{\partial x^2}+\frac{\partial^2 F_3}{\partial y^2}\right)\hat{k}$$

$$+\left(\frac{\partial^2 F_2}{\partial y\partial x}+\frac{\partial^2 F_3}{\partial z\partial x}\right)\hat{i}+\left(\frac{\partial^2 F_3}{\partial z\partial y}+\frac{\partial^2 F_1}{\partial x\partial y}\right)\hat{j}+\left(\frac{\partial^2 F_1}{\partial x\partial z}+\frac{\partial^2 F_2}{\partial y\partial z}\right)\hat{k}$$

$$= -\left(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\right)\hat{i}-\left(\frac{\partial^2 F_2}{\partial x^2}+\frac{\partial^2 F_2}{\partial y^2}+\frac{\partial^2 F_2}{\partial z^2}\right)\hat{j}$$

$$-\left(\frac{\partial^2 F_3}{\partial x^2}+\frac{\partial^2 F_3}{\partial y^2}+\frac{\partial^2 F_3}{\partial z^2}\right)\hat{k}+\left(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_2}{\partial y\partial x}+\frac{\partial^2 F_3}{\partial z\partial x}\right)\hat{i}$$

$$+\left(\frac{\partial^2 F_2}{\partial y^2}+\frac{\partial^2 F_3}{\partial z\partial y}+\frac{\partial^2 F_1}{\partial x\partial y}\right)\hat{j}+\left(\frac{\partial^2 F_3}{\partial z^2}+\frac{\partial^2 F_1}{\partial x\partial z}+\frac{\partial^2 F_2}{\partial y\partial z}\right)\hat{k}$$

$$= -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(F_1\hat{i}+F_2\hat{j}+F_3\hat{k})$$

$$+\hat{i}\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)+\hat{j}\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)$$

$$+\hat{k}\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)$$

$$\left[\text{assuming } \frac{\partial^2 F_2}{\partial x\partial y}=\frac{\partial^2 F_2}{\partial y\partial x} \text{ etc.}\right]$$

$$= -\vec{\nabla}^2\vec{F}+\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\vec{\nabla}\cdot\vec{F}$$

$$= -\vec{\nabla}^2\vec{F}+\vec{\nabla}(\vec{\nabla}\cdot\vec{F}) = \text{R.H.S.}$$

ILLUSTRATIVE EXAMPLES

Example 1: Find div $\bar{F}$ and curl $\bar{F}$ where

$$\bar{F} = \text{grad } (x^3+y^3+z^3-3xyz)$$ [W.B.U.T. 2001, 2009, 2012]

Solution: Here $\bar{F} = \text{grad } (x^3+y^3+z^3-3xyz)$

$$= \hat{i}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)+\hat{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)$$

$$+\hat{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$= (3x^2-3yz)\hat{i}+(3y^2-3xz)\hat{j}+(3z^2-3xy)\hat{k}$$

$$\therefore \quad \text{div } \bar{F} = \bar{\nabla}\cdot\bar{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\{(3x^2-3yz)\hat{i}$$

$$+(3y^2-3xz)\hat{j}+(3z^2-3xy)\hat{k}\}$$

$$= \frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3xz)+\frac{\partial}{\partial z}(3z^2-3xy)$$

$$= 6x+6y+6z = 6(x+y+z)$$

$$\text{curl } \bar{F} = \bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= (-3x+3x)\hat{i}+(-3y+3y)\hat{j}+(-3z+3z)\hat{k} = \bar{0}.$$

Example 2: Show that $\bar{A} = (6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}$ is irrotational. Find a scalar function φ such that $\bar{A} = \bar{\nabla}\varphi$. (W.B.U.T. 2002, 2004, 2013)

Solution: First part:

Here $\bar{A} = (6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}$

$$\therefore \text{ curl } \bar{A} = \bar{\nabla}\times\bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right\}\hat{i} + \left\{\frac{\partial}{\partial z}(6xy + z^3) - \frac{\partial}{\partial x}(3xz^2 - y)\right\}\hat{j}$$

$$+\left\{\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3)\right\}\hat{k}$$

$$= (-1+1)\hat{i} + (3z^2 - 3z^2)\hat{j} + (6x - 6x)\hat{k} = \vec{0}$$

Therefore, $\vec{A}$ is irrotational.

Second part:

$$\because \qquad \vec{A} = \vec{\nabla}\varphi$$

$$\therefore \quad (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k} = \hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z}$$

Equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ from both sides (since $\hat{i}, \hat{j}, \hat{k}$ are independent), we get

$$\frac{\partial\varphi}{\partial x} = 6xy + z^3, \quad \frac{\partial\varphi}{\partial y} = 3x^2 - z, \quad \frac{\partial\varphi}{\partial z} = 3xz^2 - y$$

$$\therefore \qquad d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$

$$= (6xy + z^3)\,dx + (3x^2 - z)\,dy + (3xz^2 - y)\,dz$$

$$= (6xy\;dx + 3x^2\;dy) + (z^3\;dx + 3xz^2)\,dz - (z\;dy + y\;dz)$$

$$= 3\{yd(x^2) + x^2\;dy\} + \{z^3\;dx + x\,d(z^3)\} - d(yz)$$

$$\therefore \qquad d\varphi = 3d(x^2y) + d(xz^3) - d(yz)$$

Integrating, we get

$$\varphi = 3x^2y + xz^3 - yz + c, \text{ } c \text{ is an arbitrary constant.}$$

This is the required scalar function.

Note: Here $\vec{A}$ is called **conservative (irrotational)** vector field and φ is known as the **scalar potential.**

Example 3: If the vectors $\vec{F}$ and $\vec{G}$ are irrotational then show that the vector $\vec{F} \times \vec{G}$ is solenoidal.

(*W.B.U.T. 2004, 2006, 2013*)

Solution: Since $\vec{F}$ and $\vec{G}$ are irrotational, therefore curl $\vec{F} = \vec{0}$ and curl $\vec{G} = \vec{0}$.

We know that $\quad \vec{\nabla}\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot\text{curl}\,\vec{F} - \vec{F}\cdot\text{curl}\,\vec{G}$

$$\therefore \qquad \vec{\nabla}\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot\vec{0} - \vec{F}\cdot\vec{0} = 0.$$

Hence $\vec{F}\times\vec{G}$ is solenoidal.

Example 4: Show that curl grad $f = \vec{0}$ where $f = x^2y + 2xy + z^2$. (*W.B.U.T. 2003*)

Solution: Here

$$\text{grad } f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y + 2xy + z^2)$$

$$= (2xy + 2y)\hat{i} + (x^2 + 2x)\hat{j} + 2z\hat{k}$$

$$\therefore \qquad \text{curl grad } f = \vec{\nabla} \times \vec{\nabla} f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + 2y & x^2 + 2x & 2z \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(2z) - \frac{\partial}{\partial z}(x^2 + 2x)\right\}\hat{i} + \left\{\frac{\partial}{\partial z}(2xy + 2y) - \frac{\partial}{\partial x}(2z)\right\}\hat{j}$$

$$+ \left\{\frac{\partial}{\partial x}(x^2 + 2x) - \frac{\partial}{\partial y}(2xy + 2y)\right\}\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + \{(2x + 2) - (2x + 2)\}\hat{k} = \vec{0}.$$

Example 5: Prove that $\vec{\nabla} \cdot \left[\frac{f(r)}{r}\vec{r}\right] = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$.

Solution:

$$\vec{\nabla} \cdot \left[\frac{f(r)}{r}\vec{r}\right] = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[\frac{f(r)}{r}\vec{r}\right]$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[\frac{f(r)}{r}(x\hat{i} + y\hat{j} + z\hat{k})\right]$$

$$= \frac{\partial}{\partial x}\left\{x\frac{f(r)}{r}\right\} + \frac{\partial}{\partial y}\left\{y\frac{f(r)}{r}\right\} + \frac{\partial}{\partial z}\left\{z\frac{f(r)}{r}\right\} \qquad \text{...(1)}$$

Now,

$$\frac{\partial}{\partial x}\left\{x\frac{f(r)}{r}\right\} = \frac{f(r)}{r} + x\frac{\partial}{\partial x}\left\{\frac{f(r)}{r}\right\} = \frac{f(r)}{r} + x\frac{d}{dr}\left\{\frac{f(r)}{r}\right\}\frac{\partial r}{\partial x}$$

$$= \frac{f(r)}{r} + x\left\{\frac{f'(r)}{r} - \frac{1}{r^2}f(r)\right\}\frac{x}{r}$$

$$\left[\because\ r^2 = x^2 + y^2 + z^2,\ \therefore\ 2r\frac{\partial r}{\partial x} = 2x,\ i.e.,\ \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$= \frac{f(r)}{r} + \frac{x^2}{r^2}f'(r) - \frac{x^2}{r^3}f(r) \qquad \text{...(2)}$$

Similarly, $$\frac{\partial}{\partial y}\left\{y\frac{f(r)}{r}\right\} = \frac{f(r)}{r}+\frac{y^2}{r^2}f'(r)-\frac{y^2}{r^3}f(r) \quad ...(3)$$

$$\frac{\partial}{\partial z}\left\{z\frac{f(r)}{r}\right\} = \frac{f(r)}{r}+\frac{z^2}{r^2}f'(r)-\frac{z^2}{r^3}f(r) \quad ...(4)$$

From (1) – (4), we get

$$\bar{\nabla}\cdot\left[\frac{f(r)}{r}\bar{r}\right] = \frac{3f(r)}{r}+\frac{1}{r^2}(x^2+y^2+z^2)f'(r)-\frac{1}{r^3}(x^2+y^2+z^2)f(r)$$

$$= \frac{3f(r)}{r}+\frac{r^2}{r^2}f'(r)-\frac{r^2}{r^3}f(r)$$

$$= \frac{3f(r)}{r}+f'(r)-\frac{f(r)}{r} = \frac{2f(r)}{r}+f'(r)$$

$$= \frac{1}{r^2}\{2r\,f(r)+r^2 f'(r)\} = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)].$$

Example 6: Show that $\bar{\nabla}\times[f(r)\bar{r}] = \bar{0}$, *i.e.*, $f(r)\bar{r}$ is irrotational, where $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and $r = |\bar{r}|$.

Solution: L.H.S. $= \bar{\nabla}\times[f(r)\bar{r}]$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times[f(r)(x\hat{i}+y\hat{j}+z\hat{k})]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x f(r) & y f(r) & z f(r) \end{vmatrix}$$

$$= \left\{z\frac{\partial}{\partial y}f(r)-y\frac{\partial}{\partial z}f(r)\right\}\hat{i}+\left\{x\frac{\partial}{\partial z}f(r)-z\frac{\partial}{\partial x}f(r)\right\}\hat{j}$$
$$+\left\{y\frac{\partial}{\partial x}f(r)-x\frac{\partial}{\partial y}f(r)\right\}\hat{k}$$

$$= \left\{z\frac{d}{dr}f(r)\frac{\partial r}{\partial y}-y\frac{d}{dr}f(r)\frac{\partial r}{\partial z}\right\}\hat{i}+\left\{x\frac{d}{dr}f(r)\frac{\partial r}{\partial z}-z\frac{d}{dr}f(r)\frac{\partial r}{\partial x}\right\}\hat{j}$$
$$+\left\{y\frac{d}{dr}f(r)\frac{\partial r}{\partial x}-x\frac{d}{dr}f(r)\frac{\partial r}{\partial y}\right\}\hat{k}$$

$$= \left\{z f'(r)\frac{y}{r} - y f'(r)\frac{z}{r}\right\}\hat{i} + \left\{x f'(r)\frac{z}{r} - z f'(r)\frac{x}{r}\right\}\hat{j} + \left\{y f'(r)\frac{x}{r} - x f'(r)\frac{y}{r}\right\}\hat{k}$$

$$\left[\because r^2 = x^2 + y^2 + z^2, \therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}\right]$$

$$= \frac{f'(r)}{r}\{(yz - yz)\hat{i} + (zx - zx)\hat{j} + (xy - xy)\hat{k}\}$$

$$= \bar{0} = \text{R.H.S.}$$

Example 7: Show that the vector $\bar{F} = (2x - yz)\hat{i} + (2y - zx)\hat{j} + (2z - xy)\hat{k}$ is irrotational. For this $\bar{F}$, find a scalar function φ such that $\bar{F} = \text{grad}\ \varphi$.

Solution: First part:

Here $\bar{F} = (2x - yz)\hat{i} + (2y - zx)\hat{j} + (2z - xy)\hat{k}$

$$\therefore \quad \text{curl}\ \bar{F} = \bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - yz & 2y - zx & 2z - xy \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(2z - xy) - \frac{\partial}{\partial z}(2y - zx)\right\}\hat{i} + \left\{\frac{\partial}{\partial z}(2x - yz) - \frac{\partial}{\partial x}(2z - xy)\right\}\hat{j}$$

$$+ \left\{\frac{\partial}{\partial x}(2y - zx) - \frac{\partial}{\partial y}(2x - yz)\right\}\hat{k}$$

$$= (-x + x)\hat{i} + (-y + y)\hat{j} + (-z + z)\hat{k} = \bar{0}$$

Therefore, $\bar{F}$ is irrotational.

Second part:

$$\because \quad \bar{F} = \text{grad}\ \varphi$$

$$\therefore \quad (2x - yz)\hat{i} + (2y - zx)\hat{j} + (2z - xy)\hat{k} = \hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z}.$$

Equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ from both sides (since $\hat{i}, \hat{j}, \hat{k}$ are independent), we get

$$\frac{\partial\varphi}{\partial x} = 2x - yz, \quad \frac{\partial\varphi}{\partial y} = 2y - zx, \quad \frac{\partial\varphi}{\partial z} = 2z - xy$$

$$\therefore \quad d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = (2x - yz)\,dx + (2y - zx)\,dy + (2z - xy)\,dz$$

$$= (2x\,dx + 2y\,dy + 2z\,dz) - (yz\,dx + zx\,dy + xy\,dz)$$

$$= d(x^2 + y^2 + z^2) - d(xyz)$$

(*ii*) L.H.S. $= \bar{\nabla}\cdot\left(\frac{\bar{r}}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left\{\frac{1}{r}(x\hat{i}+y\hat{j}+z\hat{k})\right\}$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{1}{r}+x\frac{d}{dr}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x}+\frac{1}{r}+y\frac{d}{dr}\left(\frac{1}{r}\right)\frac{\partial r}{\partial y}+\frac{1}{r}+z\frac{d}{dr}\left(\frac{1}{r}\right)\frac{\partial r}{\partial z}$$

$$= \frac{1}{r}-\frac{x}{r^2}\cdot\frac{x}{r}+\frac{1}{r}-\frac{y}{r^2}\cdot\frac{y}{r}+\frac{1}{r}-\frac{z}{r^2}\cdot\frac{z}{r} \quad \text{[by (1)]}$$

$$= \frac{3}{r}-\frac{1}{r^3}(x^2+y^2+z^2)$$

$$= \frac{3}{r}-\frac{r^2}{r^3} = \frac{3}{r}-\frac{1}{r} = \frac{2}{r} = \text{R.H.S.}$$

Example 9: Prove that, if $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and $r = |\bar{r}|$, then

(*i*) div grad $r^m = m(m+1)\,r^{m-2}$

(*ii*) curl grad $r^m = \bar{0}$.

Solution: Here $r^2 = \bar{r}\cdot\bar{r} = x^2+y^2+z^2$

$\therefore$ $\quad 2r\frac{\partial r}{\partial x} = 2x$, or $\frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\quad \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$ ...(1)

Now, grad $r^m = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)r^m$

$$= \left(mr^{m-1}\frac{\partial r}{\partial x}\right)\hat{i}+\left(mr^{m-1}\frac{\partial r}{\partial y}\right)\hat{j}+\left(mr^{m-1}\frac{\partial r}{\partial z}\right)\hat{k}$$

$$= mr^{m-1}\left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right) \quad \text{[by (1)]}$$

$$= mr^{m-2}(x\hat{i}+y\hat{j}+z\hat{k})$$

(*i*) L.H.S. $= \text{div grad } r^m = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\{mr^{m-2}(x\hat{i}+y\hat{j}+z\hat{k})\}$

$$= \frac{\partial}{\partial x}(mxr^{m-2})+\frac{\partial}{\partial y}(myr^{m-2})+\frac{\partial}{\partial z}(mzr^{m-2})$$

$\therefore$
$$d\varphi = d(x^2 + y^2 + z^2) - d(xyz)$$

Integrating, we get

$$\varphi = x^2 + y^2 + z^2 - xyz + c, \ c \text{ is an arbitrary constant.}$$

This is the required scalar function.

Example 8: Prove that, if $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\bar{r}|$, then

(*i*) curl $\dfrac{\bar{r}}{r} = \bar{0}$

(*ii*) $\bar{\nabla} \cdot \left(\dfrac{\bar{r}}{r}\right) = \dfrac{2}{r}$.

Solution: Here $r^2 = \bar{r} \cdot \bar{r} = x^2 + y^2 + z^2$.

$\therefore$

Similarly,
$$\left.\begin{aligned} 2r\frac{\partial r}{\partial x} = 2x, \text{ or } \frac{\partial r}{\partial x} = \frac{x}{r} \\ \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r} \end{aligned}\right\} \quad \ldots(1)$$

(*i*)
$$\text{L.H.S.} = \bar{\nabla} \times \left(\frac{\bar{r}}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left\{\frac{1}{r}(x\hat{i} + y\hat{j} + z\hat{k})\right\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{r} & \dfrac{y}{r} & \dfrac{z}{r} \end{vmatrix} = \left\{\frac{\partial}{\partial y}\left(\frac{z}{r}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r}\right)\right\}\hat{i}$$

$$+ \left\{\frac{\partial}{\partial z}\left(\frac{x}{r}\right) - \frac{\partial}{\partial x}\left(\frac{z}{r}\right)\right\}\hat{j} + \left\{\frac{\partial}{\partial x}\left(\frac{y}{r}\right) - \frac{\partial}{\partial y}\left(\frac{x}{r}\right)\right\}\hat{k}$$

$$= \left\{-\frac{z}{r^2}\frac{\partial r}{\partial y} + \frac{y}{r^2}\frac{\partial r}{\partial z}\right\}\hat{i} + \left\{-\frac{x}{r^2}\frac{\partial r}{\partial z} + \frac{z}{r^2}\frac{\partial r}{\partial x}\right\}\hat{j} + \left\{-\frac{y}{r^2}\frac{\partial r}{\partial x} + \frac{x}{r^2}\frac{\partial r}{\partial y}\right\}\hat{k}$$

$$= \left(-\frac{z}{r^2}\cdot\frac{y}{r} + \frac{y}{r^2}\cdot\frac{z}{r}\right)\hat{i} + \left(-\frac{x}{r^2}\cdot\frac{z}{r} + \frac{z}{r^2}\cdot\frac{x}{r}\right)\hat{j} + \left(-\frac{y}{r^2}\cdot\frac{x}{r} + \frac{x}{r^2}\cdot\frac{y}{r}\right)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \bar{0} = \text{R.H.S.} \qquad \text{[By (1)]}$$

(*ii*) L.H.S. $= \vec{\nabla}\cdot\left(\dfrac{\vec{r}}{r}\right) = \left(\hat{i}\dfrac{\partial}{\partial x}+\hat{j}\dfrac{\partial}{\partial y}+\hat{k}\dfrac{\partial}{\partial z}\right)\cdot\left\{\dfrac{1}{r}(x\hat{i}+y\hat{j}+z\hat{k})\right\}$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{1}{r}+x\frac{d}{dr}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x}+\frac{1}{r}+y\frac{d}{dr}\left(\frac{1}{r}\right)\frac{\partial r}{\partial y}+\frac{1}{r}+z\frac{d}{dr}\left(\frac{1}{r}\right)\frac{\partial r}{\partial z}$$

$$= \frac{1}{r}-\frac{x}{r^2}\cdot\frac{x}{r}+\frac{1}{r}-\frac{y}{r^2}\cdot\frac{y}{r}+\frac{1}{r}-\frac{z}{r^2}\cdot\frac{z}{r} \qquad \text{[by (1)]}$$

$$= \frac{3}{r}-\frac{1}{r^3}(x^2+y^2+z^2)$$

$$= \frac{3}{r}-\frac{r^2}{r^3} = \frac{3}{r}-\frac{1}{r} = \frac{2}{r} = \text{R.H.S.}$$

Example 9: Prove that, if $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and $r = |\vec{r}|$, then

(*i*) div grad $r^m = m(m+1)\,r^{m-2}$

(*ii*) curl grad $r^m = \vec{0}$.

Solution: Here $r^2 = \vec{r}\cdot\vec{r} = x^2+y^2+z^2$

$$\therefore \quad \left.\begin{aligned} 2r\frac{\partial r}{\partial x} &= 2x, \text{ or } \frac{\partial r}{\partial x} = \frac{x}{r} \\ \text{Similarly,}\quad \frac{\partial r}{\partial y} &= \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r} \end{aligned}\right\} \qquad \ldots(1)$$

Now, grad $r^m = \left(\hat{i}\dfrac{\partial}{\partial x}+\hat{j}\dfrac{\partial}{\partial y}+\hat{k}\dfrac{\partial}{\partial z}\right)r^m$

$$= \left(m r^{m-1}\frac{\partial r}{\partial x}\right)\hat{i}+\left(m r^{m-1}\frac{\partial r}{\partial y}\right)\hat{j}+\left(m r^{m-1}\frac{\partial r}{\partial z}\right)\hat{k}$$

$$= m r^{m-1}\left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right) \qquad \text{[by (1)]}$$

$$= m r^{m-2}(x\hat{i}+y\hat{j}+z\hat{k}) \qquad \ldots(2)$$

(*i*) L.H.S. $=$ div grad $r^m = \left(\hat{i}\dfrac{\partial}{\partial x}+\hat{j}\dfrac{\partial}{\partial y}+\hat{k}\dfrac{\partial}{\partial z}\right)\cdot\{m r^{m-2}(x\hat{i}+y\hat{j}+z\hat{k})\}$ [by (2)]

$$= \frac{\partial}{\partial x}(mx r^{m-2})+\frac{\partial}{\partial y}(my r^{m-2})+\frac{\partial}{\partial z}(mz r^{m-2})$$

$$= m\left\{r^{m-2} + x(m-2)r^{m-3}\frac{\partial r}{\partial x} + r^{m-2} + y(m-2)r^{m-3}\frac{\partial r}{\partial y}\right.$$

$$\left. + r^{m-2} + z(m-2)r^{m-3}\frac{\partial r}{\partial z}\right\}$$

$$= m\left\{r^{m-2} + x(m-2)r^{m-3}\cdot\frac{x}{r} + r^{m-2} + y(m-2)r^{m-3}\cdot\frac{y}{r}\right.$$

$$\left. + r^{m-2} + z(m-2)r^{m-3}\cdot\frac{z}{r}\right\} \qquad \text{[by (1)]}$$

$$= m\{3r^{m-2} + (m-2)r^{m-4}(x^2+y^2+z^2)\}$$

$$= m\{3r^{m-2} + (m-2)r^{m-2}\}$$

$$= m(m+1)r^{m-2} = \text{R.H.S.}$$

(*ii*) L.H.S.

$$= \text{curl grad } r^m = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \{m r^{m-2}(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})\}$$

[by (2)]

$$= m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x r^{m-2} & y r^{m-2} & z r^{m-2} \end{vmatrix}$$

$$= m\left\{\left(z\frac{\partial}{\partial y}r^{m-2} - y\frac{\partial}{\partial z}r^{m-2}\right)\hat{i} + \left(x\frac{\partial}{\partial z}r^{m-2} - z\frac{\partial}{\partial x}r^{m-2}\right)\hat{j}\right.$$

$$\left. + \left(y\frac{\partial}{\partial x}r^{m-2} - x\frac{\partial}{\partial y}r^{m-2}\right)\hat{k}\right\}$$

$$= m(m-2)r^{m-3}\left\{\left(z\frac{\partial r}{\partial y} - y\frac{\partial r}{\partial z}\right)\hat{i}\right.$$

$$\left. + \left(x\frac{\partial r}{\partial z} - z\frac{\partial r}{\partial x}\right)\hat{j} + \left(y\frac{\partial r}{\partial x} - x\frac{\partial r}{\partial y}\right)\hat{k}\right\}$$

$$= m(m-2)r^{m-3}\left\{\left(z\cdot\frac{y}{r}-y\cdot\frac{z}{r}\right)\hat{i}\right.$$

$$\left.+\left(x\cdot\frac{z}{r}-z\cdot\frac{x}{r}\right)\hat{j}+\left(y\cdot\frac{x}{r}-x\cdot\frac{y}{r}\right)\hat{k}\right\} \qquad \text{[by (1)]}$$

$$= m(m-2)r^{m-3}\,(0\hat{i}+0\hat{j}+0\hat{k}) = \vec{0} = \text{R.H.S.}$$

Example 10: A function $\varphi(x, y, z)$ is such that

$$\vec{\nabla}\varphi = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}.$$

If $\varphi(1, -2, 2) = 4$, find $\varphi(x, y, z)$.

Solution: Here $\vec{\nabla}\varphi = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$.

$$\therefore \quad \hat{i}\frac{\partial\varphi}{\partial x}+\hat{j}\frac{\partial\varphi}{\partial y}+\hat{k}\frac{\partial\varphi}{\partial z} = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$$

Equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ from both sides (since $\hat{i}, \hat{j}, \hat{k}$ are independent), we get

$$\frac{\partial\varphi}{\partial x} = 2xyz^3, \quad \frac{\partial\varphi}{\partial y} = x^2z^3, \quad \frac{\partial\varphi}{\partial z} = 3x^2yz^2.$$

$$\therefore \quad d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$

$$= 2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^2\,dz$$

$$= d(x^2yz^3)$$

$$\therefore \quad d\varphi = d(x^2yz^3).$$

Integrating we get

$$\varphi(x, y, z) = x^2yz^3 + c.$$

By question, $\varphi(1, -2, 2) = 4$

$$\therefore \quad 4 = 1^2\cdot(-2)\cdot 2^3 + c$$

$$\therefore \quad c = 16 + 4 = 20$$

$$\therefore \quad \varphi(x, y, z) = x^2yz^3 + 20.$$

Example 11: If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\bar{r}|$, prove that grad $f(r)\times\bar{r} = \bar{0}$. (*W.B.U.T. 2005*)

Solution: Here $r^2 = \bar{r}\cdot\bar{r} = (x\hat{i}+y\hat{j}+z\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k}) = x^2+y^2+z^2$

$$\left.\begin{aligned} &\therefore \quad 2r\frac{\partial r}{\partial x} = 2x \quad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \\ &\text{Similarly,} \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r} \end{aligned}\right\} \qquad \ldots(1)$$

Now $\quad \text{grad } f(r) = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right) f(r)$

$$= \hat{i}\,\frac{d}{dr}f(r)\frac{\partial r}{\partial x} + \hat{j}\,\frac{d}{dr}f(r)\frac{\partial r}{\partial y} + \hat{k}\,\frac{d}{dr}f(r)\frac{\partial r}{\partial z}$$

$$= \hat{i}\,f'(r)\frac{x}{r} + \hat{j}\,f'(r)\frac{y}{r} + \hat{k}\,f'(r)\frac{z}{r} \qquad \text{[by (1)]}$$

$$= \frac{f'(r)}{r}(x\,\hat{i} + y\,\hat{j} + z\,\hat{k}) = \frac{f'(r)}{r}\,\bar{r}$$

$\therefore \quad \text{grad } f(r) \times \bar{r} = \dfrac{f'(r)}{r}\,\bar{r}\times\bar{r} = \bar{0}.$

Example 12: If $\bar{\nabla}\cdot\bar{E} = 0$, $\bar{\nabla}\cdot\bar{H} = 0$, $\bar{\nabla}\times\bar{E} = -\dfrac{\partial \bar{H}}{\partial t}$ and $\bar{\nabla}\times\bar{H} = \dfrac{\partial \bar{E}}{\partial t}$, then show that

$$\bar{\nabla}^2\bar{H} = \frac{\partial^2\bar{H}}{\partial t^2} \text{ and } \bar{\nabla}^2\bar{E} = \frac{\partial^2\bar{E}}{\partial t^2}.$$

Solution: We know that

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{F}) = -\bar{\nabla}^2\bar{F} + \bar{\nabla}(\bar{\nabla}\cdot\bar{F}) \qquad ...(1)$$

Here $\quad \bar{\nabla}\times(\bar{\nabla}\times\bar{H}) = \bar{\nabla}\times\dfrac{\partial\bar{E}}{\partial t} \qquad \left[\because \bar{\nabla}\times\bar{H} = \dfrac{\partial \bar{E}}{\partial t}\right]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\frac{\partial\bar{E}}{\partial t}$$

$$= \hat{i}\times\frac{\partial^2\bar{E}}{\partial x\partial t} + \hat{j}\times\frac{\partial^2\bar{E}}{\partial y\partial t} + \hat{k}\times\frac{\partial^2\bar{E}}{\partial z\partial t}$$

$$= \hat{i}\times\frac{\partial^2\bar{E}}{\partial t\partial x} + \hat{j}\times\frac{\partial^2\bar{E}}{\partial t\partial y} + \hat{k}\times\frac{\partial^2\bar{E}}{\partial t\partial z}$$

$$= \frac{\partial}{\partial t}\left(\hat{i}\times\frac{\partial\bar{E}}{\partial x} + \hat{j}\times\frac{\partial\bar{E}}{\partial y} + \hat{k}\times\frac{\partial\bar{E}}{\partial z}\right)$$

$$= \frac{\partial}{\partial t}\left\{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\bar{E}\right\}$$

$$= \frac{\partial}{\partial t}\{\bar{\nabla}\times\bar{E}\} = -\frac{\partial^2\bar{H}}{\partial t^2} \qquad \left[\because \bar{\nabla}\times\bar{E} = -\frac{\partial\bar{H}}{\partial t}\right]$$

$$\therefore \quad \bar{\nabla}\times(\bar{\nabla}\times\bar{H}) = -\frac{\partial^2\bar{H}}{\partial t^2} \qquad ...(2)$$

From (1), replacing $\vec{F}$ by $\vec{H}$, we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = -\vec{\nabla}^2\vec{H}+\vec{\nabla}(\vec{\nabla}\cdot\vec{H})$$

$$= -\vec{\nabla}^2\vec{H} \qquad [\because \vec{\nabla}\cdot\vec{H}=0]$$

$$\therefore \qquad \vec{\nabla}^2\vec{H} = \frac{\partial^2\vec{H}}{\partial t^2} \qquad \text{[using (2)]}$$

Also,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\times\frac{\partial\vec{H}}{\partial t} \qquad \left[\because \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{H}}{\partial t}\right]$$

$$= -\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\frac{\partial\vec{H}}{\partial t}$$

$$= -\left(\hat{i}\times\frac{\partial^2\vec{H}}{\partial x\partial t}+\hat{j}\times\frac{\partial^2\vec{H}}{\partial y\partial t}+\hat{k}\times\frac{\partial^2\vec{H}}{\partial z\partial t}\right)$$

$$= -\left(\hat{i}\times\frac{\partial^2\vec{H}}{\partial t\partial x}+\hat{j}\times\frac{\partial^2\vec{H}}{\partial t\partial y}+\hat{k}\times\frac{\partial^2\vec{H}}{\partial t\partial z}\right)$$

$$= -\frac{\partial}{\partial t}\left(\hat{i}\times\frac{\partial\vec{H}}{\partial x}+\hat{j}\times\frac{\partial\vec{H}}{\partial y}+\hat{k}\times\frac{\partial\vec{H}}{\partial z}\right)$$

$$= -\frac{\partial}{\partial t}\left\{\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\vec{H}\right\}$$

$$= -\frac{\partial}{\partial t}\{\vec{\nabla}\times\vec{H}\} = -\frac{\partial^2\vec{E}}{\partial t^2} \qquad \left[\because \vec{\nabla}\times\vec{H}=\frac{\partial\vec{E}}{\partial t}\right]$$

$$\therefore \qquad \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial^2\vec{E}}{\partial t^2} \qquad \text{...(3)}$$

From (1), replacing $\vec{F}$ by $\vec{E}$, we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}^2\vec{E}+\vec{\nabla}(\vec{\nabla}\cdot\vec{E})$$

$$= -\vec{\nabla}^2\vec{E} \qquad [\because \vec{\nabla}\cdot\vec{E}=0]$$

$$\therefore \qquad \vec{\nabla}^2\vec{E} = \frac{\partial^2\vec{E}}{\partial t^2}. \qquad \text{[using (3)]}$$

Example 13: If $\vec{F}$ is solenoidal prove that

$$\vec{\nabla}\times\vec{\nabla}\times\vec{\nabla}\times\vec{\nabla}\times\vec{F} = \vec{\nabla}^4\vec{F}.$$

Solution: Since $\vec{F}$ is solenoidal, therefore

$$\vec{\nabla}\cdot\vec{F} = 0 \qquad \text{...(1)}$$

Also, $\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{F})-\vec{\nabla}^2\vec{F}$

$= -\vec{\nabla}^2\vec{F}$ [by (1)] ...(2)

$\therefore$ L.H.S. $= \vec{\nabla}\times\vec{\nabla}\times(-\vec{\nabla}^2\vec{F}) = \vec{\nabla}\times\vec{\nabla}\times\vec{G}$ (where $\vec{G}=-\vec{\nabla}^2\vec{F}$)

$= \vec{\nabla}(\vec{\nabla}\cdot\vec{G})-\vec{\nabla}^2\vec{G}$

$= -\vec{\nabla}^2\vec{G}$ $[\because \vec{\nabla}\cdot\vec{G}=\vec{\nabla}\cdot(-\vec{\nabla}^2\vec{F})=-\vec{\nabla}^2(\vec{\nabla}\cdot\vec{F})=0$, by (1)]

$= -\vec{\nabla}^2(-\vec{\nabla}^2\vec{F}) = \vec{\nabla}^4\vec{F}$ = R.H.S.

Example 14: If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and $\vec{a}$ is a constant vector, then prove that $\vec{\nabla}\cdot(\vec{a}\times\vec{r}) = 0$.

Solution: Now, $\vec{\nabla}\cdot(\vec{a}\times\vec{r}) = \vec{r}\cdot(\vec{\nabla}\times\vec{a})-\vec{a}\cdot(\vec{\nabla}\times\vec{r})$.

$$\vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0},$$

$\vec{\nabla}\times\vec{a} = \vec{0}$, since $\vec{a}$ is a constant vector.

$\therefore$ $\vec{\nabla}\cdot(\vec{a}\times\vec{r}) = 0$.

Example 15: If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and $\vec{a}$ is a constant vector, then show that $\vec{\nabla}\times(\vec{a}\times\vec{r}) = 2\vec{a}$.

Solution: We know that

$$\vec{\nabla}\times(\vec{a}\times\vec{r}) = \vec{a}(\vec{\nabla}\cdot\vec{r})-\vec{r}(\vec{\nabla}\cdot\vec{a}) + (\vec{r}\cdot\vec{\nabla})\vec{a}-(\vec{a}\cdot\vec{\nabla})\vec{r} \quad ...(1)$$

Here $\vec{a}$ is a constant vector and let it can be expressed as $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, where a_1, a_2, a_3 are constants.

Now, $$\vec{\nabla}\cdot\vec{r} = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z) = 3 \quad ...(2)$$

$$\vec{\nabla}\cdot\vec{a} = \frac{\partial}{\partial x}(a_1)+\frac{\partial}{\partial y}(a_2)+\frac{\partial}{\partial z}(a_3) = 0 \quad ...(3)$$

$$(\vec{r}\cdot\vec{\nabla})\vec{a} = \left\{(x\hat{i}+y\hat{j}+z\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\right\}\vec{a}$$

$$= \left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})$$

$$= \vec{0} \quad ...(4)$$

$$(\vec{a}\cdot\vec{\nabla})\vec{r} = \left\{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\right\}\vec{r}$$

$$= \left(a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = \bar{a} \qquad \text{...(5)}$$

Therefore, from (1), using (2) – (5), we get

$$\bar{\nabla}\times(\bar{a}\times\bar{r}) = 3\bar{a} - \bar{a} = 2\bar{a}.$$

Example 16: If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\bar{a}$ is a constant vector, prove that

$$\bar{\nabla}\times\left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = \frac{1}{r^n}(2-n)\bar{a} + \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$$

where $r = |\bar{r}|$.

Solution: Here

$$r^2 = \bar{r}\cdot\bar{r} = (x\hat{i} + y\hat{j} + z\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= x^2 + y^2 + z^2.$$

$\therefore$ $$2r\frac{\partial r}{\partial x} = 2x, \text{ or } \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $$\left.\begin{aligned}\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}\end{aligned}\right\} \qquad \text{...(1)}$$

$\therefore$ $$\bar{\nabla}(r^{-n}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)r^{-n}$$

$$= -n r^{-(n+1)}\left(\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right)$$

$$= -n r^{-(n+2)}\bar{r} \qquad \text{[by (1)]} \qquad \text{...(2)}$$

Now, $$\bar{\nabla}\times\left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = \bar{\nabla}\times[r^{-n}(\bar{a}\times\bar{r})]$$

$$= r^{-n}\,\bar{\nabla}\times(\bar{a}\times\bar{r}) + (\bar{\nabla}r^{-n})\times(\bar{a}\times\bar{r})$$

$$= r^{-n}\{(\bar{\nabla}\cdot\bar{r})\bar{a} - (\bar{a}\cdot\bar{\nabla})\bar{r}\}$$

$$-n r^{-(n+2)}\,\bar{r}\times(\bar{a}\times\bar{r}) \qquad \text{...(3)}$$

[by (2) and since $\bar{\nabla}\cdot\bar{a} = 0$, $(\bar{r}\cdot\bar{\nabla})\bar{a} = \bar{0}$, $\bar{a}$ is a constant vector]

Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; a_1, a_2, a_3 are constants.

$\therefore$ $$\bar{\nabla}\cdot\bar{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$(\bar{a} \cdot \bar{\nabla})\bar{r} = \left(a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} = \bar{a}$$

Therefore, from (3),

$$\bar{\nabla} \times \left(\frac{\bar{a} \times \bar{r}}{r^n}\right) = r^{-n}(3\bar{a} - \bar{a}) - n r^{-(n+2)}\{(\bar{r} \cdot \bar{r})\bar{a} - (\bar{r} \cdot \bar{a})\bar{r}\}$$

$$= \frac{2\bar{a}}{r^n} - \frac{n}{r^{n+2}}\{r^2 \bar{a} - (\bar{a} \cdot \bar{r})\bar{r}\}$$

$$= \frac{1}{r^n}(2-n)\bar{a} + \frac{n(\bar{a} \cdot \bar{r})}{r^{n+2}}\bar{r}.$$

13.9 GREEN'S THEOREM IN THE PLANE

Statement: If $P(x, y)$, $Q(x, y)$, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous functions over a region R of the xy-plane bounded by a simple closed curve Γ, then

$$\oint_{\Gamma}(P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

where Γ is traversed in counter clockwise direction.

In vector notation: Green's theorem in vector form is

$$\oint_{\Gamma} \bar{F} \cdot d\bar{r} = \iint_R (\bar{\nabla} \times \bar{F}) \cdot \hat{k}\,ds$$

where $\bar{F} = P\hat{i} + Q\hat{j}$, $\bar{r} = x\hat{i} + y\hat{j}$, $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the positive directions of x, y, z-axes respectively with $\hat{i} \times \hat{j} = \hat{k}$ and $ds = dx\,dy$.

ILLUSTRATIVE EXAMPLES

Example 1: A vector field $\bar{F}$ is given by $\bar{F} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$. Evaluate the line integral $\int_{\Gamma} \bar{F} \cdot d\bar{r}$ where Γ is the circular path given by $x^2 + y^2 = a^2$.

Solution: Here $\bar{F} = (\sin y)\hat{i} + x(1+\cos y)\hat{j}$.

$$\therefore \quad \int_\Gamma \bar{F}\cdot d\bar{r} = \int_\Gamma \{(\sin y)\hat{i} + x(1+\cos y)\hat{j}\}\cdot(\hat{i}\,dx + \hat{j}\,dy)$$

$$= \int_\Gamma \{\sin y\,dx + x(1+\cos y)\,dy\}$$

$$= \iint_R \left[\frac{\partial}{\partial x}\{x(1+\cos y)\} - \frac{\partial}{\partial y}\sin y\right]dxdy$$

[using Green's theorem. Here the region R is bounded by $x^2 + y^2 = a^2$]

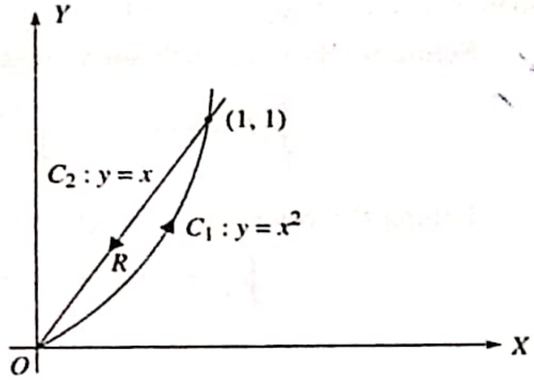
$$= \iint_R (1+\cos y - \cos y)\,dxdy$$

$$= \iint_R dxdy = \text{Area of the circle} = \pi a^2.$$

Example 2: Verify Green's theorem in the plane for $\oint_C [(xy+y^2)\,dx + x^2 dy]$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$. (*W.B.U.T. 2001, 2003*)

Solution: The Green's theorem in the plane is

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$



Here $\qquad P = xy + y^2,\ Q = x^2$

$$\therefore \quad \oint_C (Pdx + Qdy) = \int_{C_1} (Pdx + Qdy) + \int_{C_2} (Pdx + Qdy)$$

$$= \int_0^1 [\{x\cdot x^2 + (x^2)^2\}dx + x^2\cdot 2xdx] + \int_1^0 \{(x\cdot x + x^2)dx + x^2 dx\}$$

[$\because$ on C_1, $y = x^2$, $dy = 2xdx$ and on C_2, $y = x$, $dy = dx$]

$$= \int_0^1 (3x^3 + x^4)\,dx + \int_1^0 3x^2\,dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 + \left[3\cdot\frac{x^3}{3}\right]_1^0$$

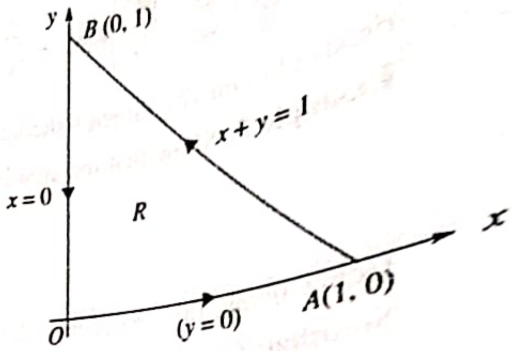
$$= \frac{3}{4} + \frac{1}{5} - 1 = -\frac{1}{20}$$

$$\therefore \quad \oint_C (Pdx + Qdy) = -\frac{1}{20} \qquad \ldots(1)$$

Example 4: Verify Green's theorem for

$$\oint_C \{(3x-8y^2)\,dx+(4y-6xy)\,dy\}$$

where C is the boundary of the region bounded by $x = 0$, $y = 0$ and $z + y = 1$. *(W.B.U.T. 2002, 2013)*



Solution: The Green's theorem in the plane is

$$\oint_C (Pdx+Qdy) = \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

Here $P = 3x - 8y^2$ and $Q = 4y - 6xy$

$$\therefore \quad \oint_C (Pdx+Qdy) = \int_{OA}(Pdx+Qdy)+\int_{AB}(Pdx+Qdy)+\int_{BO}(Pdx+Qdy)$$

$$= \int_0^1 3x\,dx+\int_1^0 [\{3x-8(1-x)^2\}dx+\{4(1-x)-6x(1-x)\}(-dx)]$$

$$+\int_1^0 4y\,dy$$

[$\because$ on OA, $y = 0$, $dy = 0$, on AB, $x + y = 1$, $dy = -dx$, on BO, $x = 0$, $dx = 0$]

$$= \left[3\frac{x^2}{2}\right]_0^1+\int_1^0(-14x^2+29x-12)\,dx+\left[4\frac{y^2}{2}\right]_1^0$$

$$= \frac{3}{2}+\left[-14\cdot\frac{x^3}{3}+29\cdot\frac{x^2}{2}-12x\right]_1^0-2$$

$$= \frac{3}{2}+\frac{14}{3}-\frac{29}{2}+12-2 = \frac{5}{3}$$

$$\therefore \quad \oint_C (Pdx+Qdy) = \frac{5}{3}$$

Also, $$\iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy = \iint_R\left\{\frac{\partial}{\partial x}(4y-6xy)-\frac{\partial}{\partial y}(3x-8y^2)\right\}dxdy \quad \ldots(1)$$

$$= \int_0^1\int_{y=0}^{1-x}(-6y+16y)\,dydx = \int_0^1\left[10\cdot\frac{y^2}{2}\right]_{y=0}^{1-x}dx$$

$$= \int_0^1 5(1-x)^2\,dx = 5\left[-\frac{(1-x)^3}{3}\right]_0^1 = \frac{5}{3}$$

Also, $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_0^1 \int_{y=x^2}^{x} \left\{\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right\} dydx$

$$= \int_0^1 \int_{y=x^2}^{x} (2x - x - 2y)\, dydx = \int_0^1 [xy - y^2]_{y=x^2}^{x}\, dx$$

$$= \int_0^1 (x^2 - x^2 - x^3 + x^4)\, dx = \int_0^1 (x^4 - x^3)\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

$\therefore \qquad \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = -\frac{1}{20}$ ...(2)

From (1) and (2), Green's theorem is verified.

Example 3: Evaluate by Green's theorem

$$\oint_C \{(\cos x \sin y - xy)\, dx + \sin x \cos y\, dy\}$$

where C is the circle $x^2 + y^2 = 1$. (*W.B.U.T. 2004, BESUS 2013*)

Solution: By Green's theorem in the plane, we have

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Putting $P = \cos x \sin y - xy$, $Q = \sin x \cos y$, we get

$$\oint_C \{(\cos x \sin y - xy)\, dx + \sin x \cos y\, dy\}$$

$$= \iint_R \left\{\frac{\partial}{\partial x}(\sin x \cos y) - \frac{\partial}{\partial y}(\cos x \sin y - xy)\right\} dxdy$$

$$= \iint_R (\cos x \cos y - \cos x \cos y + x)\, dxdy$$

$$= \iint_R x\, dxdy$$

$$= \int_{-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\, dydx = \int_{-1}^{1} [y]_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\, x\, dx$$

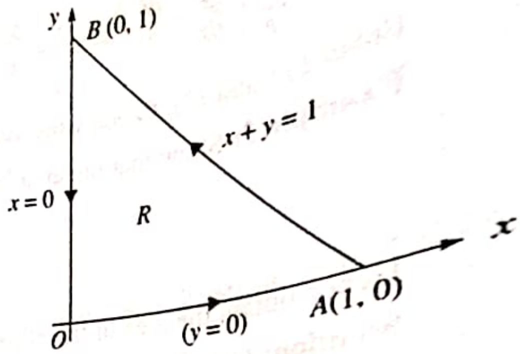
[Since the region R is bounded by the circle $x^2 + y^2 = 1$]

$$= 2\int_{-1}^{1} x\sqrt{1-x^2}\, dx = 0. \qquad \text{[Since } x\sqrt{1-x^2} \text{ is an odd function]}$$

Example 4: Verify Green's theorem for

$$\oint_C \{(3x - 8y^2)\,dx + (4y - 6xy)\,dy\}$$

where C is the boundary of the region bounded by $x = 0$, $y = 0$ and $z + y = 1$. *(W.B.U.T. 2002, 2013)*



Solution: The Green's theorem in the plane is

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Here $P = 3x - 8y^2$ and $Q = 4y - 6xy$

$$\therefore \qquad \oint_C (Pdx + Qdy) = \int_{OA} (Pdx + Qdy) + \int_{AB} (Pdx + Qdy) + \int_{BO} (Pdx + Qdy)$$

$$= \int_0^1 3x\,dx + \int_1^0 [\{3x - 8(1-x)^2\}dx + \{4(1-x) - 6x(1-x)\}(-dx)]$$

$$+ \int_1^0 4y\,dy$$

[$\because$ on OA, $y = 0$, $dy = 0$, on AB, $x + y = 1$, $dy = -dx$, on BO, $x = 0$, $dx = 0$]

$$= \left[3\frac{x^2}{2}\right]_0^1 + \int_1^0 (-14x^2 + 29x - 12)\,dx + \left[4\frac{y^2}{2}\right]_1^0$$

$$= \frac{3}{2} + \left[-14\cdot\frac{x^3}{3} + 29\cdot\frac{x^2}{2} - 12x\right]_1^0 - 2$$

$$= \frac{3}{2} + \frac{14}{3} - \frac{29}{2} + 12 - 2 = \frac{5}{3}$$

$$\therefore \qquad \oint_C (Pdx + Qdy) = \frac{5}{3}$$

Also, $$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_R \left\{\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x - 8y^2)\right\} dxdy \qquad \cdots(1)$$

$$= \int_0^1 \int_{y=0}^{1-x} (-6y + 16y)\,dydx = \int_0^1 \left[10\cdot\frac{y^2}{2}\right]_{y=0}^{1-x} dx$$

$$= \int_0^1 5(1-x)^2\,dx = 5\left[-\frac{(1-x)^3}{3}\right]_0^1 = \frac{5}{3}$$

$$\therefore \quad \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \frac{5}{3} \qquad \ldots(2)$$

From (1) and (2), Green's theorem is verified.

Example 5: Show that the area bounded by a simple closed curve C is given by

$$\frac{1}{2}\oint_C (x\,dy - y\,dx).$$

Hence obtain the area of the ellipse $x = a\cos t$, $y = b\sin t$. (*W.B.U.T. 2011, 2013*)

Solution: Green's theorem in the plane is

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

where R is the region bounded by a simple closed curve C.

Putting $P = -y$, $Q = x$, we get

$$\oint_C (-y\,dx + x\,dy) = \iint_R \left\{\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right\} dxdy$$

$$= 2\iint_R dxdy = 2.\text{ (Area bounded by } C)$$

$$\therefore \quad \text{Area bounded by } C = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$$\text{The area of the ellipse} = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\int_{t=0}^{2\pi} \{a\cos t\, d(b\sin t) - b\sin t\, d(a\cos t)\}$$

$$= \frac{1}{2}\int_{t=0}^{2\pi} (ab\cos^2 t + ab\sin^2 t)\,dt = \frac{1}{2}ab\int_{t=0}^{2\pi} dt$$

$$= \frac{1}{2}ab\,[t]_0^{2\pi} = \frac{1}{2}ab\cdot 2\pi = \pi ab.$$

Example 6: If C be a simple closed curve in the xy-plane not enclosing the origin, prove that

$$\int_C \bar{F}\cdot d\bar{r} = 0, \text{ where } \bar{F} = \frac{y\hat{i} - x\hat{j}}{x^2+y^2},\ \bar{r} = x\hat{i} + y\hat{j}.$$

Solution: $$\int_C \bar{F}\cdot d\bar{r} = \int_C \frac{(y\hat{i} - x\hat{j})}{x^2+y^2}\cdot(\hat{i}\,dx + \hat{j}\,dy)$$

$$= \int_C \frac{y\,dx - x\,dy}{x^2+y^2} = \int_C (P\,dx + Q\,dy)$$

$$\left[\text{where } P = \frac{y}{x^2+y^2},\ Q = -\frac{x}{x^2+y^2}\right]$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

[by Green's theorem, where R is the region bounded by C]

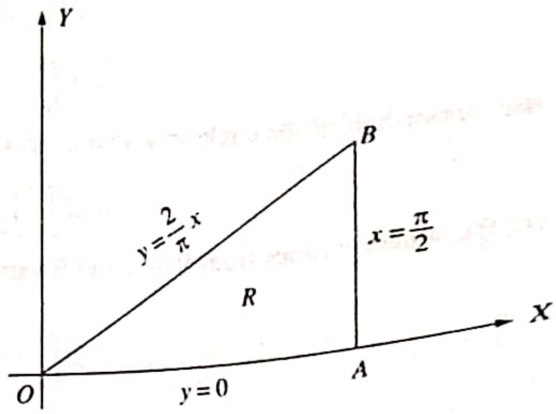
$$= \iint_R \left[\frac{-(x^2+y^2)+x(2x)}{(x^2+y^2)^2} - \frac{\{(x^2+y^2)-y(2y)\}}{(x^2+y^2)^2}\right] dxdy$$

$$= \iint_R \left[\frac{x^2-y^2}{(x^2+y^2)^2} - \frac{(x^2-y^2)}{(x^2+y^2)^2}\right] dxdy = 0.$$

Example 7: Using Green's theorem, evaluate

$$\int_C \{(y - \sin x)dx + \cos x\,dy\}$$

where C is the boundary of the region bounded by $y = 0$, $x = \frac{\pi}{2}$ and $y = \frac{2}{\pi}x$.



Solution: By Green's theorem in the plane, we have

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Putting $P = y - \sin x$, $Q = \cos x$, we get

$$\int_C \{(y-\sin x)dx + \cos x\,dy\} = \int_0^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1)\,dydx$$

$$= -\int_0^{\pi/2} (1+\sin x)\,[y]_0^{\frac{2x}{\pi}}\,dx$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} x(1+\sin x)\,dx$$

$$= -\frac{2}{\pi}\left[\frac{x^2}{2} - x\cos x + \sin x\right]_0^{\frac{\pi}{2}}$$

$$\left[\because \int x\sin x\,dx = x\int \sin x\,dx - \int\left\{\left(\frac{dx}{dx}\right)\Big/\int \sin x\right\} dx = -x\cos x + \sin x\right]$$

$$\therefore \quad \oint_{\Gamma}(Pdx+Qdy) = \int_{OA}(Pdx+Qdy)+\int_{AB}(Pdx+Qdy)+\int_{BC}(Pdx+Qdy)$$

$$+\int_{CO}(Pdx+Qdy) = \int_0^a x^2\,dx+\int_0^a ay\,dy+\int_a^0 x^2\,dx+\int_a^0 0\,dy$$

[$\because$ on OA : $y=0$, $dy=0$, x varies from 0 to a
on AB : $x=a$, $dx=0$, y varies from 0 to a
on BC : $y=a$, $dy=0$, x varies from a to 0
on CO : $x=0$, $dx=0$, y varies from a to 0]

$$\therefore \quad \oint_{\Gamma}(Pdx+Qdy) = \left[\frac{x^3}{3}\right]_0^a + a\left[\frac{y^2}{2}\right]_0^a + \left[\frac{x^3}{3}\right]_a^0 = \frac{a^3}{2} \qquad \text{...(1)}$$

Also, $$\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy = \iint_R\left\{\frac{\partial}{\partial x}(xy)-\frac{\partial}{\partial y}(x^2)\right\}dx\,dy$$

$$= \int_{y=0}^a\int_{x=0}^a y\,dx\,dy = \int_0^a y\,dy\int_0^a dx$$

$$= \left[\frac{y^2}{2}\right]_0^a [x]_0^a = \frac{a^3}{2} \qquad \text{...(2)}$$

From (1) and (2), Green's theorem is verified.

13.10 GAUSS' DIVERGENCE THEOREM

Statement: Let S be a closed surface enclosing a volume V and $\bar{F}$ be a continuously differentiable vector point function in V, then

$$\oint_S \bar{F}\cdot\hat{n}\,dS = \int_V \text{div}\,\bar{F}\,dV$$

where $\hat{n}$ is the unit outward drawn normal vector to the surface S.

If $\bar{F} = F_1(x,y,z)\hat{i}+F_2(x,y,z)\hat{j}+F_3(x,y,z)\hat{k}$, then the Cartesian form is

$$\iint_S(F_1\,dy\,dz+F_2\,dz\,dx+F_3\,dx\,dy) = \iiint_V\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dx\,dy\,dz.$$

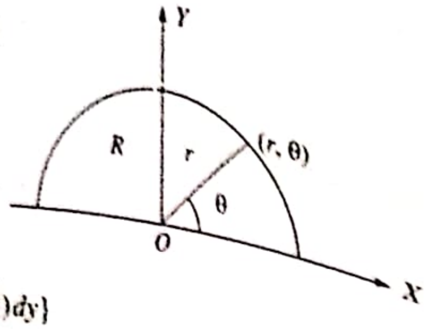
$$= -\frac{2}{\pi}\left[\frac{\pi^2}{8}+1-0\right] = -\frac{2}{\pi}\left(\frac{\pi^2}{8}+1\right)$$

$$= -\left(\frac{\pi}{4}+\frac{2}{\pi}\right).$$

Example 8: Apply Green's theorem to evaluate

$$\int_C \{(2x^2 - y^2)dx + (x^2 + y^2)dy\}$$

where C is the boundary of the area enclosed by the x-axis and the upper-half of the circle $x^2 + y^2 = a^2$.



Solution: By Green's theorem

$$\int_C \{(2x^2 - y^2)dx + (x^2 + y^2)dy\}$$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(2x^2 - y^2)\right] dxdy$$

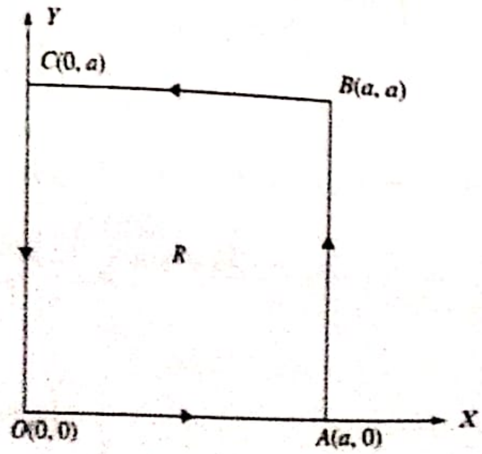
$= 2\iint_R (x + y)\,dxdy$ [here R is the region bounded by the x-axis and the upper-half of the circle $x^2 + y^2 = a^2$ as shown in the figure]

$= 2\int_0^a \int_0^\pi r(\cos\theta + \sin\theta) r\,d\theta\,dr$ [Changing to polar coordinates (r, θ), where r varies from 0 to a and θ varies from 0 to π]

$$= 2\int_0^a r^2\,dr \int_0^\pi (\cos\theta + \sin\theta)\,d\theta = 2\left[\frac{r^3}{3}\right]_0^a [\sin\theta - \cos\theta]_0^\pi$$

$$= 2\cdot\frac{a^3}{3}\cdot(1+1) = \frac{4a^3}{3}.$$

Example 9: Verify Green's theorem in the plane for $\oint_\Gamma (x^2\,dx + xy\,dy)$, where Γ is the square in the xy-plane given by $x = 0$, $y = 0$, $x = a$, $y = a$ $(a > 0)$ described in the positive sense. *(W.B.U.T. 2005)*



Solution: The Green's theorem in the plane is

$$\oint_\Gamma (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Here $P = x^2$ and $Q = xy$

Scanned with CamScanner

$$\therefore \quad \oint_\Gamma (Pdx + Qdy) = \int_{OA} (Pdx + Qdy) + \int_{AB} (Pdx + Qdy) + \int_{BC} (Pdx + Qdy)$$

$$+ \int_{CO} (Pdx + Qdy) = \int_0^a x^2\, dx + \int_0^a ay\, dy + \int_a^0 x^2\, dx + \int_a^0 0\, dy$$

[$\because$ on OA : $y = 0$, $dy = 0$, x varies from 0 to a
on AB : $x = a$, $dx = 0$, y varies from 0 to a
on BC : $y = a$, $dy = 0$, x varies from a to 0
on CO : $x = 0$, $dx = 0$, y varies from a to 0]

$$\therefore \quad \oint_\Gamma (Pdx + Qdy) = \left[\frac{x^3}{3}\right]_0^a + a\left[\frac{y^2}{2}\right]_0^a + \left[\frac{x^3}{3}\right]_a^0 = \frac{a^3}{2} \qquad \ldots(1)$$

Also, $$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_R \left\{\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2)\right\} dx\, dy$$

$$= \int_{y=0}^a \int_{x=0}^a y\, dx\, dy = \int_0^a y\, dy \int_0^a dx$$

$$= \left[\frac{y^2}{2}\right]_0^a [x]_0^a = \frac{a^3}{2} \qquad \ldots(2)$$

From (1) and (2), Green's theorem is verified.

13.10 GAUSS' DIVERGENCE THEOREM

Statement: Let S be a closed surface enclosing a volume V and $\bar{F}$ be a continuously differentiable vector point function in V, then

$$\oint_S \bar{F} \cdot \hat{n}\, dS = \int_V \text{div}\, \bar{F}\, dV$$

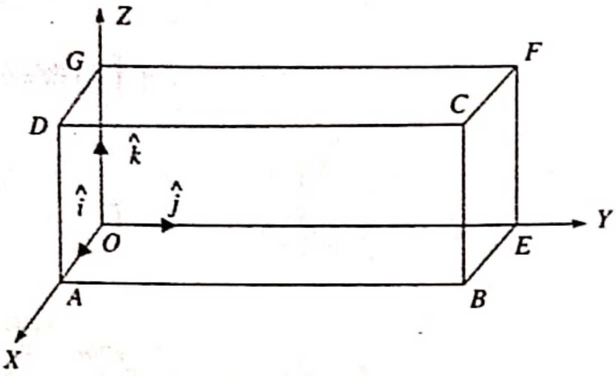
where $\hat{n}$ is the unit outward drawn normal vector to the surface S.

If $\bar{F} = F_1(x, y, z)\hat{i} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$, then the Cartesian form is

$$\iint_S (F_1\, dydz + F_2\, dzdx + F_3\, dx\, dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz.$$

ILLUSTRATIVE EXAMPLES

Example 1: Verify divergence theorem for $\bar{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.



Solution: Divergence theorem states that

$$\oint_S \bar{F}\cdot\hat{n}\,dS = \int_V \text{div}\,\bar{F}\,dV$$

Here
$$\text{div}\,\bar{F} = \bar{\nabla}\cdot\bar{F}$$
$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$
$$= 2(x + y + z)$$

$\therefore$
$$\int_V \text{div}\,\bar{F}\,dV = 2\int_V (x+y+z)\,dx\,dy\,dz$$
$$= 2\int_{z=0}^{c}\left[\int_{y=0}^{b}\left\{\int_{x=0}^{a}(x+y+z)\,dx\right\}dy\right]dz$$
$$= 2\int_0^c\int_0^b\left[\frac{1}{2}x^2 + xy + xz\right]_2^a dy\,dz$$
$$= 2\int_0^c\int_0^b\left[\frac{a^2}{2} + ay + az\right]dy\,dz = 2\int_0^c\left[\frac{a^2}{2}y + \frac{ay^2}{2} + azy\right]_{y=0}^{b} dz$$
$$= 2\int_0^c\left[\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right]dz = 2\left[\left(\frac{a^2b}{2} + \frac{ab^2}{2}\right)z + ab\,\frac{z^2}{2}\right]_{z=0}^{c}$$
$$= (a^2b + ab^2)c + abc^2 = abc(a+b+c)$$

$\therefore$
$$\int_V \text{div}\,\bar{F}\,dV = abc(a+b+c) \qquad \text{...(1)}$$

Now,
$$\int_S \bar{F}\cdot\hat{n}\,dS = \int_{S_1}\bar{F}\cdot\hat{n}\,dS + \int_{S_2}\bar{F}\cdot\hat{n}\,dS + \int_{S_3}\bar{F}\cdot\hat{n}\,dS$$
$$+ \int_{S_4}\bar{F}\cdot\hat{n}\,dS + \int_{S_5}\bar{F}\cdot\hat{n}\,dS + \int_{S_6}\bar{F}\cdot\hat{n}\,dS \qquad \text{...(2)}$$

where
$$\int_{S_1}\bar{F}\cdot\hat{n}\,dS = \int_{ABCD}\{(a^2 - yz)\hat{i} + (y^2 - za)\hat{j} + (z^2 - ay)\hat{k}\}\cdot\hat{i}\,dS$$

[$\because$ On $ABCD$: $\hat{n} = \hat{i}$, $x = a$]

$$= \int_{z=0}^{c} \int_{y=0}^{b} (a^2 - yz)\, dy\, dz \qquad [\because \text{ On } ABCD,\ dS = dy\, dz]$$

$$= \int_0^c \left[a^2 y - \frac{y^2}{2} z \right]_{y=0}^{b} dz$$

$$= \int_0^c \left(a^2 b - \frac{b^2}{2} z \right) dz = \left[a^2 bz - \frac{b^2}{2} \cdot \frac{z^2}{2} \right]_{z=0}^{c}$$

$$= a^2 bc - \frac{b^2 c^2}{4} \qquad \ldots(3)$$

$$\int_{S_2} \bar{F} \cdot \hat{n}\, dS = \int_{BEFC} \{(x^2 - bz)\hat{i} + (b^2 - zx)\hat{j} + (z^2 - xb)\hat{k}\} \cdot \hat{j}\, dS$$

$$[\because \text{ On } BEFC,\ \hat{n} = \hat{j},\ y = b]$$

$$= \int_{z=0}^{c} \int_{x=0}^{a} (b^2 - zx)\, dx\, dz = \int_0^c \left[b^2 x - \frac{zx^2}{2} \right]_{x=0}^{a} dz$$

$$[\because \text{ On } BEFC,\ dS = dx\, dz]$$

$$= \int_0^c \left(b^2 a - \frac{a^2}{2} z \right) dz = \left[b^2 az - \frac{a^2}{2} \cdot \frac{z^2}{2} \right]_{z=0}^{c}$$

$$= b^2 ac - \frac{a^2 c^2}{4} \qquad \ldots(4)$$

$$\int_{S_3} \bar{F} \cdot \hat{n}\, dS = \int_{OEFG} (-yz\hat{i} + y^2 \hat{j} + z^2 \hat{k}) \cdot (-\hat{i})\, dS \qquad [\because \text{ On } OEFG,\ \hat{n} = -\hat{i},\ x = 0]$$

$$= \int_{z=0}^{c} \int_{y=0}^{b} yz\, dy\, dz = \left[\frac{y^2}{2} \right]_0^b \left[\frac{z^2}{2} \right]_0^c = \frac{b^2 c^2}{4}$$

$$[\because \text{ On } OEFG,\ dS = dy\, dz] \qquad \ldots(5)$$

$$\int_{S_4} \bar{F} \cdot \hat{n}\, dS = \int_{AOGD} (x^2 \hat{i} - zx\hat{j} + z^2 \hat{k}) \cdot (-\hat{j})\, dS \qquad [\because \text{ On } AOGD,\ \hat{n} = -\hat{j},\ y = 0]$$

$$= \int_{z=0}^{c} \int_{x=0}^{a} zx\, dx\, dz = \left[\frac{z^2}{2} \right]_0^c \left[\frac{x^2}{2} \right]_0^a$$

$$= \frac{c^2 a^2}{4} \qquad [\because \text{ On } AOGD,\ dS = dx\, dz] \qquad \ldots(6)$$

$$\int_{S_5} \bar{F} \cdot \hat{n}\, dS = \int_{ABEO} (x^2 \hat{i} + y^2 \hat{j} - xy\hat{k}) \cdot (-\hat{k})\, dS \qquad [\because \text{ On } ABEO,\ \hat{n} = -\hat{k},\ z = 0]$$

$$\int_{y=0}^{b}\int_{x=0}^{a} xy\,dx\,dy = \left[\frac{x^2}{2}\right]_0^a \left[\frac{y^2}{2}\right]_0^b \qquad [\because \text{ On } ABEO,\ dS = dx\,dy]$$

$$= \frac{a^2b^2}{4} \qquad \ldots(7)$$

$$\int_{S_6} \bar{F}\cdot\hat{n}\,dS = \int_{DCFG} \{(x^2-cy)\hat{i}+(y^2-cx)\hat{j}+(c^2-xy)\hat{k}\}\cdot\hat{k}\,dS$$

$$[\because \text{ On } DCFG,\ \hat{n}=\hat{k},\ z=c]$$

$$= \int_{y=0}^{b}\int_{x=0}^{a}(c^2-xy)\,dx\,dy \qquad [\because \text{ On } DCFG,\ dS = dx\,dy]$$

$$= \int_0^b \left[c^2x - \frac{x^2}{2}y\right]_{x=0}^{a} dy$$

$$= \int_0^b \left(c^2a - \frac{a^2}{2}y\right)dy = \left[c^2ay - \frac{a^2}{2}\cdot\frac{y^2}{2}\right]_0^b$$

$$= c^2ab - \frac{a^2b^2}{4} \qquad \ldots(8)$$

From (2) – (8), we have

$$\int_S \bar{F}\cdot\hat{n}\,dS = a^2bc + b^2ca + c^2ab = abc(a+b+c) \qquad \ldots(9)$$

From (1) and (9) the divergence theorem is verified.

Example 2: Use the divergence theorem to evaluate

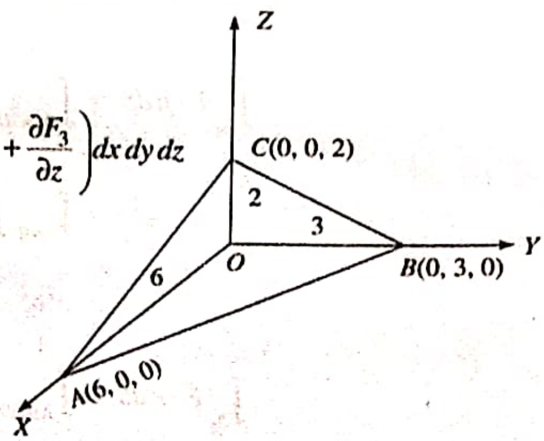
$$\iint_{S_1} (x\,dydz + y\,dzdx + z\,dxdy)$$

where S_1 is the portion of the plane $x + 2y + 3z = 6$ which lies in the first octant.

Solution: Divergence theorem states that

$$\iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) = \iiint_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dx\,dy\,dz$$

where S is a closed surface bounding a volume V.



Here $$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

$$= \iint_{ABC} (x\,dydz + y\,dzdx + z\,dxdy)$$

$$+\iint_{AOC}(0+0\,dzdx+0)+\iint_{OBC}(0\,dydz+0+0)$$

$$+\iint_{ABO}(0+0+0\,dxdy)$$

[$\because$ On AOC : $y = 0$, $dy = 0$. On OBC : $x = 0$, $dx = 0$. On ABO : $z = 0$, $dz = 0$]

$$= \iint_{ABC}(x\,dydz+y\,dzdx+z\,dxdy)$$

$$= \iiint_V\left(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\right)dxdydz \qquad \text{[by divergence theorem]}$$

$$= 3\iiint_V dx\,dy\,dz$$

= 3 (volume of the tetrahedron $OABC$)

= 3 [($\frac{1}{3}$ Area of the base ΔAOB) × height OC]

$$= 3\left[\frac{1}{3}\left(\frac{1}{2}\times 6\times 3\right)\times 2\right] = 18$$

$\therefore$

$$\iint_{S_1}(x\,dydz+y\,dzdx+z\,dxdy) = 18$$

where S_1 is the portion of the plane $x + 2y + 3z = 6$ which lies in the first octant.

Example 3: Evaluate $\iint_S \{xz^2\,dydz+(x^2y-z^3)\,dzdx+(2xy+y^2z)\,dxdy\}$ where S is the surface of the hemispherical region bounded by $z=\sqrt{a^2-x^2-y^2}$ and $z = 0$.

Solution: By divergence theorem $\iint_S (F_1\,dydz+F_2\,dzdx+F_3\,dxdy)$

$$= \iiint_V\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dxdydz$$

where S is a closed surface bounding a volume V.

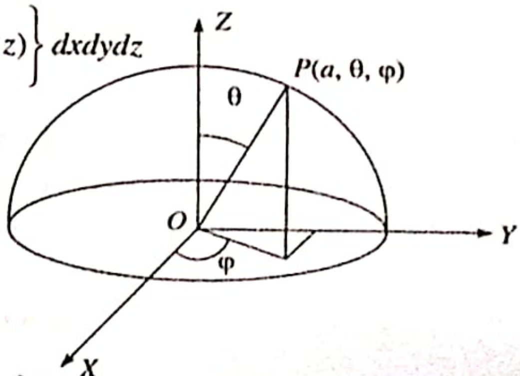
$\therefore$

$$\iint_S \{xz^2\,dydz+(x^2y-z^3)\,dzdx+(2xy+y^2z)\,dxdy\}$$

$$= \iiint_V\left\{\frac{\partial}{\partial x}(xz^2)+\frac{\partial}{\partial y}(x^2y-z^3)+\frac{\partial}{\partial z}(2xy+y^2z)\right\}dxdydz$$

$$= \iiint_V (z^2+x^2+y^2)\,dxdydz$$

$$= \iiint_V r^2\,(r^2\sin\theta\,dr\,d\theta\,d\varphi)$$



[where $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$]

$$= \int_0^a r^4\,dr \int_0^{\pi/2} \sin\theta\,d\theta \int_0^{2\pi} d\varphi$$

$$= \left[\frac{r^5}{5}\right]_0^a [-\cos\theta]_0^{\pi/2}\,[\varphi]_0^{2\pi}$$

$$= \frac{2\pi a^5}{5}.$$

Example 4: Prove that $\int_S (x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k})\cdot\hat{n}\,dS = 0$, where S is the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $\hat{n}$ is the unit outward drawn normal vector to the surface S.

Solution: By divergence theorem

$$\int_S (x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k})\cdot\hat{n}\,dS = \int_V \vec{\nabla}\cdot(x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k})\,dV$$

$$= \int_V \left\{\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)\right\} dV$$

$$= 2\int_V (x + y + z)\,dV$$

$$= 2\int_{z=-c}^{c} \int_{y=-b\sqrt{1-\frac{z^2}{c^2}}}^{b\sqrt{1-\frac{z^2}{c^2}}} \int_{x=-a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}}^{a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}} (x + y + z)\,dx\,dy\,dz$$

$$= 4a\int_{z=-c}^{c} \left\{\int_{y=-b\sqrt{1-\frac{z^2}{c^2}}}^{b\sqrt{1-\frac{z^2}{c^2}}} (y+z)\sqrt{1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\,dy\right\} dz$$

[$\because x$ is an odd function]

$$= 8a\int_{-c}^{c} \left\{\int_0^{b\sqrt{1-\frac{z^2}{c^2}}} \sqrt{1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\,dy\right\} z\,dz$$

$\left[\because y\sqrt{1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\right.$ is an odd function of y and $\sqrt{1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}}$ is an even function of y]

$$= 8ab\int_{-c}^{c} \left\{\int_0^{\sqrt{1-\frac{z^2}{c^2}}} \sqrt{\left(1 - \frac{z^2}{c^2}\right) - u^2}\,du\right\} z\,dz \qquad \text{(where } bu = y\text{)}$$

$$= 8ab\int_{-c}^{c}\left[\frac{u}{2}\sqrt{\left(1-\frac{z^2}{c^2}\right)-u^2}+\frac{1}{2}\left(1-\frac{z^2}{c^2}\right)\sin^{-1}\frac{u}{\sqrt{1-\frac{z^2}{c^2}}}\right]_{u=0}^{\sqrt{1-\frac{z^2}{c^2}}} zdz$$

$$= 2\pi ab\int_{-c}^{c}\left(1-\frac{z^2}{c^2}\right)zdz = 0 \quad \left[\because \left(1-\frac{z^2}{c^2}\right)z \text{ is an odd function of } z\right]$$

Example 5: Show that $\iint_S \vec{r}\cdot d\vec{S} = 3V$, where V is the volume enclosed by the closed surface S and $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$.

Solution: We have, by divergence theorem,

$$\iint_S \vec{r}\cdot d\vec{S} = \iint_S \vec{r}\cdot\hat{n}\,dS = \iiint_V \vec{\nabla}\cdot\vec{r}\,dV$$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i}+y\hat{j}+z\hat{k})\,dV$$

$$= \iiint_V \left(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\right)dV = \iiint_V 3dV = 3V.$$

Example 6: Use divergence theorem to evaluate $\iint_S \vec{F}\cdot\hat{n}\,dS$, where $\vec{F} = xz\hat{i}+y^2\hat{j}+2yz\hat{k}$ and S is the surface of the cube bounded by $x=0, y=0, z=0, x=1, y=1, z=1$.

Solution: By divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV$$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(xz\hat{i}+y^2\hat{j}+2yz\hat{k})\,dV$$

$$= \iiint_V \left\{\frac{\partial}{\partial x}(xz)+\frac{\partial}{\partial y}(y^2)+\frac{\partial}{\partial z}(2yz)\right\}dV$$

$$= \iiint_V (z+2y+2y)\,dV$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1}(z+4y)\,dx\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}[zx+4xy]_{x=0}^{1}\,dy\,dz$$

Scanned with CamScanner

Example 8: Let φ and ψ are scalar point functions which together with their derivatives in any direction are continuous within the region V bounded by a closed surface S, then prove that

$$\int_V (\varphi\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\varphi)\,dV = \int_S (\varphi\vec{\nabla}\psi - \psi\vec{\nabla}\varphi)\cdot d\vec{S},$$

where $d\vec{S} = \hat{n}\,dS$, $\hat{n}$ is the unit outward drawn normal vector to the surface S.

Solution: Divergence theorem states that

$$\int_V \vec{\nabla}\cdot\vec{F}\,dV = \int_S \vec{F}\cdot\hat{n}\,dS \qquad \ldots(1)$$

Let $\vec{F} = \varphi\vec{\nabla}\psi$, therefore

$$\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot(\varphi\vec{\nabla}\psi) = \varphi(\vec{\nabla}\cdot\vec{\nabla}\psi) + \vec{\nabla}\varphi\cdot\vec{\nabla}\psi$$

$$= \varphi\vec{\nabla}^2\psi + \vec{\nabla}\varphi\cdot\vec{\nabla}\psi \qquad [\because \vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot\vec{\nabla}f]$$

Therefore, from (1)

$$\int_V (\varphi\vec{\nabla}^2\psi + \vec{\nabla}\varphi\cdot\vec{\nabla}\psi)\,dV = \int_S (\varphi\vec{\nabla}\psi)\cdot\hat{n}\,dS \qquad \ldots(2)$$

This is known as **Green's first identity theorem**. Interchanging φ and ψ in (2), we get

$$\int_V (\psi\vec{\nabla}^2\varphi + \vec{\nabla}\psi\cdot\vec{\nabla}\varphi)\,dV = \int_S (\psi\vec{\nabla}\varphi)\cdot\hat{n}\,dS \qquad \ldots(3)$$

Subtracting (3) from (2), we get

$$\int_V (\varphi\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\varphi)\,dV = \int_S (\varphi\vec{\nabla}\psi - \psi\vec{\nabla}\varphi)\cdot\hat{n}\,dS$$

This is known as **Green's second identity theorem** or **Green's theorem in symmetrical form.**

Example 9: Prove that $\int_V \vec{\nabla}\varphi\,dV = \int_S \varphi\hat{n}\,dS$ and hence deduce that $\int_S \hat{n}\,dS = \vec{0}$, where S is a closed surface bounding a volume V and $\hat{n}$ is the unit outward drawn normal vector to the surface S.

Solution: Divergence theorem states that

$$\int_V \vec{\nabla}\cdot\vec{F}\,dV = \int_S \vec{F}\cdot\hat{n}\,dS \qquad \ldots(1)$$

Let $\vec{F} = \varphi\vec{C}$, where $\vec{C}$ is a constant vector and $\vec{C} \neq \vec{0}$.

Also,
$$\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot(\varphi\vec{C}) = (\vec{\nabla}\varphi)\cdot\vec{C} + \varphi\vec{\nabla}\cdot\vec{C}$$

$$= (\vec{\nabla}\varphi)\cdot\vec{C} \qquad [\text{Since } \vec{C} \text{ is a constant vector, } \vec{\nabla}\cdot\vec{C} = 0]$$

and
$$\vec{F}\cdot\hat{n} = (\varphi\vec{C})\cdot\hat{n} = \vec{C}\cdot(\varphi\hat{n})$$

Therefore, (1) gives

$$\int_V (\vec{\nabla}\varphi)\cdot\vec{C}\,dV = \int_S \vec{C}\cdot(\varphi\hat{n})\,dS$$

or
$$\vec{C}\cdot\left[\int_V \vec{\nabla}\varphi\,dV - \int_S \varphi\hat{n}\,dS\right] = 0$$

$$= \int_{z=0}^{1} \int_{y=0}^{1} [z+4y]\,dy\,dz = \int_0^1 [zy+2y^2]_{y=0}^{1}\,dz$$

$$= \int_0^1 (z+2)\,dz$$

$$= \left[\frac{z^2}{2}+2z\right]_0^1 = \frac{1}{2}+2 = \frac{5}{2}.$$

Example 7: Use divergence theorem to evaluate

$$\iint_S (x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy)$$

where S is the closed surface consisting of the cylinder $x^2+y^2=4$ $(0 \le z \le 5)$ and the circular disc $z=0$ $(x^2+y^2 \le 4)$ and $z=5$ $(x^2+y^2 \le 4)$.

Solution: Using divergence theorem, we have

$$\iint_S (x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy)$$

$$= \iiint_V \left\{\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial z}(x^2z)\right\} dx\,dy\,dz$$

$$= \int_{z=0}^{5} \int_{y=-2}^{2} \int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 5x^2\,dx\,dy\,dz$$

$$= 5\int_{z=0}^{5} \int_{y=-2}^{2} \left[\frac{x^3}{3}\right]_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy\,dz$$

$$= \frac{2.5}{3} \int_{z=0}^{5} \int_{y=-2}^{2} (4-y^2)^{3/2}\,dy\,dz$$

$$= \frac{10}{3} \int_{z=0}^{5} dz \int_{y=-2}^{2} (4-y^2)^{3/2}\,dy$$

$$= \frac{20}{3} [z]_{z=0}^{5} \int_0^2 (4-y^2)^{3/2}\,dy$$

$$= \frac{100}{3} \int_{\frac{\pi}{2}}^{0} (2^3 \sin^3\theta)(-2\sin\theta)\,d\theta \qquad [\text{put } y = 2\cos\theta, \therefore\ dy = -2\sin\theta\,d\theta]$$

$$= \frac{100}{3} \cdot 2^4 \int_0^{\pi/2} \sin^4\theta\,d\theta = \frac{100}{3} \cdot 2^4 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\left[\because \int_0^{\pi/2} \sin^n\theta\,d\theta = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \text{ if } n \text{ is even positive integer}\right]$$

$= 100\pi.$

Example 8: Let φ and ψ are scalar point functions which together with their derivatives in any direction are continuous within the region V bounded by a closed surface S, then prove that

$$\int_V (\varphi\bar{\nabla}^2\psi - \psi\bar{\nabla}^2\varphi)\,dV = \int_S (\varphi\bar{\nabla}\psi - \psi\bar{\nabla}\varphi)\cdot d\bar{S},$$

where $d\bar{S} = \hat{n}\,dS$, $\hat{n}$ is the unit outward drawn normal vector to the surface S.

Solution: Divergence theorem states that

$$\int_V \bar{\nabla}\cdot\bar{F}\,dV = \int_S \bar{F}\cdot\hat{n}\,dS \qquad ...(1)$$

Let $\bar{F} = \varphi\bar{\nabla}\psi$, therefore

$$\bar{\nabla}\cdot\bar{F} = \bar{\nabla}\cdot(\varphi\bar{\nabla}\psi) = \varphi(\bar{\nabla}\cdot\bar{\nabla}\psi) + \bar{\nabla}\varphi\cdot\bar{\nabla}\psi$$

$$= \varphi\bar{\nabla}^2\psi + \bar{\nabla}\varphi\cdot\bar{\nabla}\psi \qquad [\because\ \bar{\nabla}\cdot(f\bar{A}) = f(\bar{\nabla}\cdot\bar{A}) + \bar{A}\cdot\bar{\nabla}f]$$

Therefore, from (1)

$$\int_V (\varphi\bar{\nabla}^2\psi + \bar{\nabla}\varphi\cdot\bar{\nabla}\psi)\,dV = \int_S (\varphi\bar{\nabla}\psi)\cdot\hat{n}\,dS \qquad ...(2)$$

This is known as **Green's first identity theorem**. Interchanging φ and ψ in (2), we get

$$\int_V (\psi\bar{\nabla}^2\varphi + \bar{\nabla}\psi\cdot\bar{\nabla}\varphi)\,dV = \int_S (\psi\bar{\nabla}\varphi)\cdot\hat{n}\,dS \qquad ...(3)$$

Subtracting (3) from (2), we get

$$\int_V (\varphi\bar{\nabla}^2\psi - \psi\bar{\nabla}^2\varphi)\,dV = \int_S (\varphi\bar{\nabla}\psi - \psi\bar{\nabla}\varphi)\cdot\hat{n}\,dS$$

This is known as **Green's second identity theorem** or **Green's theorem in symmetrical form.**

Example 9: Prove that $\int_V \bar{\nabla}\varphi\,dV = \int_S \varphi\hat{n}\,dS$ and hence deduce that $\int_S \hat{n}\,dS = \bar{0}$, where S is a closed surface bounding a volume V and $\hat{n}$ is the unit outward drawn normal vector to the surface S.

Solution: Divergence theorem states that

$$\int_V \bar{\nabla}\cdot\bar{F}\,dV = \int_S \bar{F}\cdot\hat{n}\,dS \qquad ...(1)$$

Let $\bar{F} = \varphi\bar{C}$, where $\bar{C}$ is a constant vector and $\bar{C}\neq\bar{0}$.

Also, $$\bar{\nabla}\cdot\bar{F} = \bar{\nabla}\cdot(\varphi\bar{C}) = (\bar{\nabla}\varphi)\cdot\bar{C} + \varphi\bar{\nabla}\cdot\bar{C}$$

$$= (\bar{\nabla}\varphi)\cdot\bar{C} \qquad [\text{Since } \bar{C} \text{ is a constant vector, } \bar{\nabla}\cdot\bar{C} = 0]$$

and $$\bar{F}\cdot\hat{n} = (\varphi\bar{C})\cdot\hat{n} = \bar{C}\cdot(\varphi\hat{n})$$

Therefore, (1) gives

$$\int_V (\bar{\nabla}\varphi)\cdot\bar{C}\,dV = \int_S \bar{C}\cdot(\varphi\bar{n})\,dS$$

or $$\bar{C}\cdot\left[\int_V \bar{\nabla}\varphi\,dV - \int_S \varphi\hat{n}\,dS\right] = 0$$

Since $\bar{C}$ is any arbitrary constant vector, we have

$$\int_V \bar{\nabla}\varphi\, dV - \int_S \varphi\hat{n}\, dS = \bar{0},$$

or
$$\int_V \bar{\nabla}\varphi\, dV = \int_S \varphi\hat{n}\, dS.$$

In particular, if $\varphi(\neq 0)$ be a constant, then $\bar{\nabla}\varphi = \bar{0}$ and hence

$$\int_S \varphi\hat{n}\, dS = \bar{0},$$

or
$$\int_S \hat{n}\, dS = \bar{0}.$$

Example 10: Show that $\int_V \bar{\nabla}\times\bar{B}\, dV = \int_S \hat{n}\times\bar{B}\, dS.$

Solution: By the divergence theorem, we have

$$\int_V \bar{\nabla}\cdot\bar{F}\, dV = \int_S \bar{F}\cdot\hat{n}\, dS \qquad \text{...(1)}$$

Let $\bar{F} = \bar{B}\times\bar{C}$, where $\bar{C}(\neq\bar{0})$ is a constant vector.

Now,
$$\bar{\nabla}\cdot\bar{F} = \bar{\nabla}\cdot(\bar{B}\times\bar{C}) = \bar{C}\cdot\bar{\nabla}\times\bar{B} - \bar{B}\cdot\bar{\nabla}\times\bar{C}$$
$$= \bar{C}\cdot\bar{\nabla}\times\bar{B} \qquad [\because\ \bar{\nabla}\times\bar{C} = \bar{0},\ \bar{C} \text{ is a constant vector}]$$

and
$$\bar{F}\cdot\hat{n} = (\bar{B}\times\bar{C})\cdot\hat{n} = [\bar{B}\,\bar{C}\,\hat{n}] = [\bar{C}\,\hat{n}\,\bar{B}] = \bar{C}\cdot(\hat{n}\times\bar{B})$$

Therefore, (1) gives

$$\bar{C}\cdot\int_V (\bar{\nabla}\times\bar{B})dV = \bar{C}\cdot\int_S (\hat{n}\times\bar{B})\,dS$$

or
$$\bar{C}\cdot\left[\int_V \bar{\nabla}\times\bar{B}\, dV - \int_S \hat{n}\times\bar{B}\, dS\right] = 0$$

Since $\bar{C}$ is any arbitrary constant vector, it follows that

$$\int_V \bar{\nabla}\times\bar{B}\, dV = \int_S \hat{n}\times\bar{B}\, dS.$$

Example 11: Verify Gauss' divergence theorem for $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the cylindrical region $x^2 + y^2 \le a^2$, $z = 0$ and $z = h > 0$.

Solution: Divergence theorem states that

$$\int_V \bar{\nabla}\cdot\bar{F}\, dV = \int_S \bar{F}\cdot\hat{n}\, dS$$

Here
$$\bar{\nabla}\cdot\bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})$$
$$= \frac{\partial}{\partial x}(4x) - \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$$

$$\therefore \quad \int_V \bar{\nabla} \cdot \bar{F}\, dV = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{h} (4-4y+2z)\,dz\,dy\,dx$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} [(4-4y)z+z^2]_{z=0}^{h}\,dy\,dx$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (4h+h^2-4hy)\,dx\,dy$$

$$= 2(4h+h^2)\int_{x=-a}^{a} \sqrt{a^2-x^2}\,dx$$

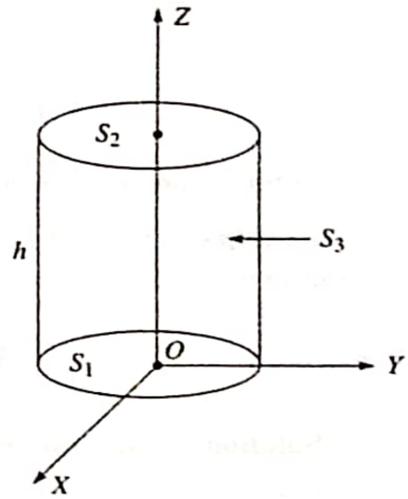
$$= 4h(h+4)\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= 4h(h+4)\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{x=0}^{a}$$

$$= \pi a^2 h(h+4)$$

$$\therefore \quad \int_V \bar{\nabla}\cdot\bar{F}\,dV = \pi a^2 h(h+4) \qquad \text{...(1)}$$

The surface S of the cylindrical region has a base S_1, the top S_2 and the curved surface S_3.



On S_1, $z = 0$, $\hat{n} = -\hat{k}$ and $\bar{F}\cdot\hat{n} = \bar{F}\cdot(-\hat{k}) = -z^2 = 0$

$$\therefore \quad \int_{S_1} \bar{F}\cdot\hat{n}\,dS = 0 \qquad \text{...(2)}$$

On S_2, $z = h$, $\hat{n} = \hat{k}$ and $\bar{F}\cdot\hat{n} = \bar{F}\cdot\hat{k} = z^2 = h^2$, $dS = dx\,dy$.

$$\therefore \quad \int_{S_2} \bar{F}\cdot\hat{n}\,dS = \iint_R h^2\,dx\,dy$$

(R is the region bounded by the circle $x^2+y^2=a^2$, $z=h$)

$$= h^2 \iint_R dx\,dy = h^2\pi a^2 \qquad \text{...(3)}$$

Let $\qquad \varphi = x^2 + y^2$

$$\therefore \quad \bar{\nabla}\varphi = \hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} = 2x\hat{i} + 2y\hat{j}$$

On S_3, $\qquad \hat{n} = \dfrac{\bar{\nabla}\varphi}{|\bar{\nabla}\varphi|} = \dfrac{2x\hat{i}+2y\hat{j}}{2\sqrt{x^2+y^2}} = \dfrac{x\hat{i}+y\hat{j}}{a} \qquad (\because x^2+y^2=a^2)$

$$\therefore \quad \bar{F}\cdot\hat{n} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\cdot\frac{1}{a}(x\hat{i}+y\hat{j}+0\hat{k})$$

Now, $$\vec{\nabla}\cdot\frac{\vec{r}}{r^3} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\frac{1}{r^3}(\hat{i}\,x+\hat{j}\,y+\hat{z}\,k)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1}{r^3}+x\frac{d}{dr}\left(\frac{1}{r^3}\right)\frac{\partial r}{\partial x}$$

$$= \frac{1}{r^3}-x\frac{3}{r^4}\cdot\frac{x}{r} \qquad \text{[by (1)]}$$

$$= \frac{1}{r^3}-\frac{3x^2}{r^5}$$

Similarly, $$\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = \frac{1}{r^3}-\frac{3y^2}{r^4},\quad \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{1}{r^3}-\frac{3z^2}{r^5}$$

$\therefore$ $$\vec{\nabla}\cdot\frac{\vec{r}}{r^3} = \frac{3}{r^3}-\frac{3}{r^5}(x^2+y^2+z^2)$$

$$= \frac{3}{r^3}-\frac{3r^2}{r^5} = \frac{3}{r^3}-\frac{3}{r^3} = 0 \qquad \text{...(2)}$$

Therefore, by the divergence theorem

$$\int_S \frac{\vec{r}}{r^3}\cdot\hat{n}\,dS = \int_V \vec{\nabla}\cdot\frac{\vec{r}}{r^3}\,dV = 0 \qquad \text{[by (2)]}$$

if the origin O lies outside S.

(*ii*) Let the origin O lies inside S

Let us consider a small sphere S_1 of radius a enclosing the origin O, where S_1 is within S. Now apply the divergence theorem within the region V bounded by S and S_1.

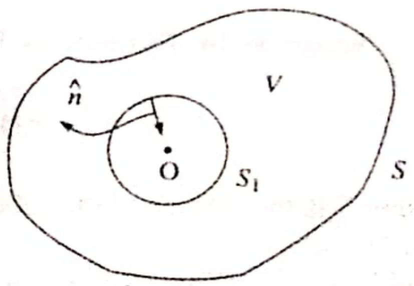
$$\int_{S+S_1}\frac{\vec{r}}{r^3}\cdot\hat{n}\,dS = \int_V \vec{\nabla}\cdot\frac{\vec{r}}{r^3}\,dV = 0 \qquad \text{[by (2), since } r\neq 0 \text{ in } V]$$

or $$\int_S\frac{\vec{r}}{r^3}\cdot\hat{n}\,dS+\int_{S_1}\frac{\vec{r}}{r^3}\cdot\hat{n}\,dS = 0$$

$\therefore$ $$\int_S\frac{\vec{r}}{r^3}\cdot\hat{n}\,dS = -\int_{S_1}\frac{\vec{r}}{r^3}\cdot\hat{n}\,dS \qquad \text{...(3)}$$

On S_1, $r = a$, $\hat{n} = -\dfrac{\vec{r}}{a}$, therefore, $$\frac{\vec{r}}{r^3}\cdot\hat{n} = \frac{\vec{r}}{a^3}\cdot\left(-\frac{\vec{r}}{a}\right)$$

$$= -\frac{a^2}{a^4} = -\frac{1}{a^2}$$



$$= \frac{1}{a}(4x^2 - 2y^3)$$

$x^2 + y^2 = a^2$, $\therefore$ $x = a\cos\theta$, $y = a\sin\theta$, $0 \le \theta \le 2\pi$, $dS = a\,d\theta\,dz$

$$\therefore \quad \int_{S_3} \bar{F}\cdot\hat{n}\,dS = \int_{z=0}^{h}\int_{\theta=0}^{2\pi} \frac{1}{a}(4a^2\cos^2\theta - 2a^3\sin^3\theta)\,a\,d\theta\,dz$$

$$= \int_0^h dz \int_0^{2\pi} (4a^2\cos^2\theta - 2a^3\sin^3\theta)\,d\theta$$

$$= h\int_0^{2\pi} (4a^2\cos^2\theta - 2a^3\sin^3\theta)\,d\theta$$

Now $2\cos^2\theta = 1 + \cos 2\theta$ and $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$

$$\therefore \quad \int_0^{2\pi} 2\cos^2\theta\,d\theta = \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 2\pi$$

and
$$\int_0^{2\pi} \sin^3\theta\,d\theta = \frac{1}{4}\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_0^{2\pi} = 0$$

$$\therefore \quad \int_{S_3} \bar{F}\cdot\hat{n}\,dS = 4\pi a^2 h \qquad ...(4)$$

$$\therefore \quad \int_S \bar{F}\cdot\hat{n}\,dS = \int_{S_1} \bar{F}\cdot\hat{n}\,dS + \int_{S_2} \bar{F}\cdot\hat{n}\,dS + \int_{S_3} \bar{F}\cdot\hat{n}\,dS$$

$$= 0 + \pi a^2 h^2 + 4\pi a^2 h \qquad \text{[by (2), (3) and (4)]}$$

$$\therefore \quad \int_S \bar{F}\cdot\hat{n}\,dS = \pi a^2 h(h+4) \qquad ...(5)$$

From (1) and (5) the divergence theorem is verified.

Example 12: Let S be a closed surface and let $\bar{r}$ be the position vector of any point (x, y, z) with respect to origin O and $r = |\bar{r}|$. Show that

$$\int_S \frac{\bar{r}}{r^3}\cdot\hat{n}\,dS = \begin{cases} 0, & \text{if } O \text{ lies outside } S \\ 4\pi, & \text{if } O \text{ lies inside } S. \end{cases}$$

Solution: (*i*) **Let the origin** O **lies outside** S.

Here
$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore \quad r^2 = \bar{r}\cdot\bar{r} = x^2 + y^2 + z^2 \neq 0$$

in V, since O lies outside S.

$$\left.\begin{aligned} &\therefore \quad 2r\frac{\partial r}{\partial x} = 2x, \text{ or } \frac{\partial r}{\partial x} = \frac{x}{r} \\ &\text{Similarly,} \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial r}{\partial z} = \frac{z}{r} \end{aligned}\right\} \qquad ...(1)$$

Now,
$$\bar{\nabla}\cdot\frac{\bar{r}}{r^3} = \left(\hat{i}\,\frac{\partial}{\partial x}+\hat{j}\,\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}\right)\cdot\frac{1}{r^3}\,(\hat{i}\,x+\hat{j}\,y+\hat{z}\,k)$$
$$= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)$$
$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1}{r^3}+x\,\frac{d}{dr}\left(\frac{1}{r^3}\right)\frac{\partial r}{\partial x}$$
$$= \frac{1}{r^3}-x\,\frac{3}{r^4}\cdot\frac{x}{r} \qquad \text{[by (1)]}$$
$$= \frac{1}{r^3}-\frac{3x^2}{r^5}$$

Similarly,
$$\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = \frac{1}{r^3}-\frac{3y^2}{r^4},\quad \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{1}{r^3}-\frac{3z^2}{r^5}$$

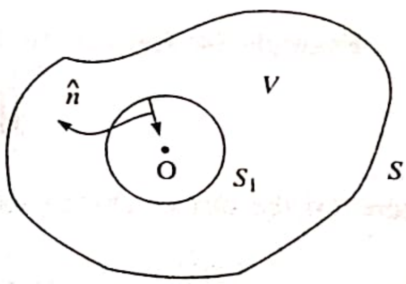
∴
$$\bar{\nabla}\cdot\frac{\bar{r}}{r^3} = \frac{3}{r^3}-\frac{3}{r^5}\,(x^2+y^2+z^2)$$
$$= \frac{3}{r^3}-\frac{3r^2}{r^5} = \frac{3}{r^3}-\frac{3}{r^3} = 0 \qquad \text{...(2)}$$

Therefore, by the divergence theorem
$$\int_S \frac{\bar{r}}{r^3}\cdot\hat{n}\,dS = \int_V \bar{\nabla}\cdot\frac{\bar{r}}{r^3}\,dV = 0 \qquad \text{[by (2)]}$$
if the origin O lies outside S.

(*ii*) Let the origin O lies inside S

Let us consider a small sphere S_1 of radius a enclosing the origin O, where S_1 is within S. Now apply the divergence theorem within the region V bounded by S and S_1.
$$\int_{S+S_1} \frac{\bar{r}}{r^3}\cdot\hat{n}\,dS = \int_V \bar{\nabla}\cdot\frac{\bar{r}}{r^3}\,dV = 0 \qquad \text{[by (2), since } r\neq 0 \text{ in } V]$$
or
$$\int_S \frac{\bar{r}}{r^3}\cdot\hat{n}\,dS+\int_{S_1} \frac{\bar{r}}{r^3}\cdot\hat{n}\,dS = 0$$
∴
$$\int_S \frac{\bar{r}}{r^3}\cdot\hat{n}\,dS = -\int_{S_1} \frac{\bar{r}}{r^3}\cdot\hat{n}\,dS \qquad \text{...(3)}$$

On S_1, $r = a$, $\hat{n} = -\dfrac{\bar{r}}{a}$, therefore, $\dfrac{\bar{r}}{r^3}\cdot\hat{n} = \dfrac{\bar{r}}{a^3}\cdot\left(-\dfrac{\bar{r}}{a}\right)$
$$= -\frac{a^2}{a^4} = -\frac{1}{a^2}$$

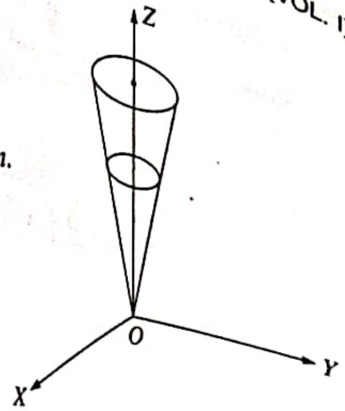


So, by (3),

$$\int_S \frac{\vec{r}}{r^3}\cdot\hat{n}\,dS = \frac{1}{a^2}\int_{S_1} dS = \frac{1}{a^2}(4\pi a^2) = 4\pi,$$

if the origin O lies inside S.

Note: The statement of Example 12 is known as *Gauss' theorem.*



Example 13: Evaluate $\int_S \vec{F}\cdot\hat{n}\,dS$ over the entire surface of the region above the xy-plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = h > 0$, where $\vec{F} = xz^2\,\hat{i} + yz^2\,\hat{j} + xz\,\hat{k}$.

Solution: By divergence theorem,

$$\int_S \vec{F}\cdot\hat{n}\,dS = \int_V \vec{\nabla}\cdot\vec{F}\,dV$$

Here

$$\vec{\nabla}\cdot\vec{F} = \left(i\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(xz^2\,\hat{i}+yz^2\,\hat{j}+xz\hat{k})$$

$$= \frac{\partial}{\partial x}(xz^2)+\frac{\partial}{\partial y}(yz^2)+\frac{\partial}{\partial z}(xz)$$

$$= z^2+z^2+x = 2z^2+x.$$

In V, for a particular z, $x^2 + y^2 = z^2$ is a circle of radius z. So we can assume y varies from $-z$ to z and $-\sqrt{z^2-y^2} \le x \le \sqrt{z^2-y^2}$

$$\therefore \quad \int_S \vec{F}\cdot\hat{n}\,dS = \int_{z=0}^{h}\int_{y=-z}^{z}\int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} (2z^2+x)\,dx\,dy\,dz$$

$$= 4\int_{z=0}^{h}\int_{y=-z}^{z} z^2\sqrt{z^2-y^2}\,dy\,dz \qquad \left[\because \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} x\,dx = 0\right]$$

$$= 4\int_0^h z^2\left[\frac{y\sqrt{z^2-y^2}}{2}+\frac{z^2}{2}\sin^{-1}\frac{y}{z}\right]_{y=-z}^{z} dz$$

$$= 2\pi\int_0^h z^4\,dz = 2\pi\left[\frac{z^5}{5}\right]_0^h = \frac{2}{5}\pi h^5.$$

Example 14: Evaluate by Divergence theorem

$$\iint_S \{x^2 dy\,dz + y^2\,dz\,dx + 2z(xy-x-y)\,dx\,dy\},$$

(W.B.U.T. 2005)

where S is the surface of the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

Solution: By divergence theorem

$$\iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

where S is a closed surface bounding a volume V.

$$\therefore \quad \iint_S \{x^2 dy\,dz + y^2\,dz\,dx + 2z(xy - x - y)\,dx\,dy\}$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}\{2z(xy - x - y)\}\right] dx\,dy\,dz$$

$$= \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} \{2x + 2y + 2(xy - x - y)\}\,dx\,dy\,dz$$

$$= \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} 2xy\,dx\,dy\,dz$$

$$= 2\int_0^1 dz \int_0^1 y\,dy \int_0^1 x\,dx$$

$$= 2[z]_0^1 \left[\frac{y^2}{2}\right]_0^1 \left[\frac{x^2}{2}\right]_0^1$$

$$= 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}.$$

13.11 STOKES' THEOREM

Statement: Let $\bar{F}$ be a continuously differentiable vector point function and S be an open surface bounded by a simple closed curve Γ, then

$$\oint_\Gamma \bar{F} \cdot d\bar{r} = \int_S \text{curl}\, \bar{F} \cdot \hat{n}\, dS$$

where $\hat{n}$ is a unit normal vector at any point of S drawn in the sense in which a right-handed screw would move when rotated in the sense of description of the curve Γ.

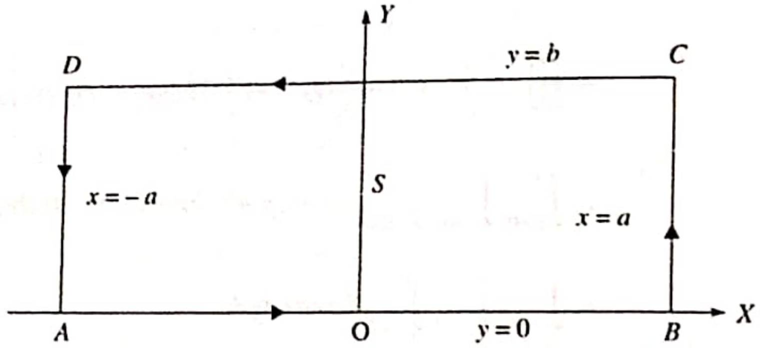
If $\bar{F} = F_1(x, y, z)\hat{i} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$, then the Cartesian form is

$$\int_\Gamma (F_1\,dx + F_2\,dy + F_3\,dz) = \iint_S \left\{\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy\,dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz\,dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy\right\}.$$

ILLUSTRATIVE EXAMPLES

Example 1: Verify Stokes' theorem for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$. (W.B.U.T. 2003)

Solution: Let $ABCD$ be the given rectangle.



$$\therefore \quad \oint_{ABCDA} \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r} \quad \ldots(1)$$

Here $\vec{F}\cdot d\vec{r} = \{(x^2+y^2)\hat{i} - 2xy\hat{j}\}\cdot(\hat{i}\,dx + \hat{j}\,dy)$

$[\because \vec{r} = x\hat{i} + y\hat{j},\ \because d\vec{r} = i\,dx + \hat{j}\,dy]$

$$= (x^2+y^2)\,dx - 2xy\,dy$$

Along AB, $y = 0$, $dy = 0$ and x varies from $-a$ to a.

$$\therefore \quad \int_{AB} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{2a^3}{3}$$

Along BC, $x = a$, $dx = 0$ and y varies from 0 to b.

$$\therefore \quad \int_{BC} \vec{F}\cdot d\vec{r} = -2a\int_0^b y\,dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2.$$

Along CD, $y = b$, $dy = 0$ and x varies from a to $-a$.

$$\therefore \quad \int_{CD} \vec{F}\cdot d\vec{r} = \int_a^{-a} (x^2+b^2)\,dx = \left[\frac{x^3}{3} + b^2x\right]_a^{-a} = -\frac{2a^3}{3} - 2ab^2$$

Along DA, $x = -a$, $dx = 0$ and y varies from b to 0.

$$\therefore \quad \int_{DA} \vec{F}\cdot d\vec{r} = \int_b^0 2ay\,dy = 2a\left[\frac{y^2}{2}\right]_b^0 = -ab^2$$

$$\therefore \quad \oint_\Gamma \vec{F}\cdot d\vec{r} = \oint_{ABCA} \vec{F}\cdot d\vec{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2$$

$$= -4ab^2 \qquad \ldots(2)$$

For the surface $ABCD$, $\hat{n} = \hat{k}$ and

$$\text{curl } \vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = (-2y-2y)\hat{k} = -4y\hat{k}$$

$$\therefore \quad \int_S \text{curl } \vec{F}\cdot\hat{n}\,dS = \int_S -4y\hat{k}\cdot\hat{k}\,dS = -4\int_{y=0}^{b}\int_{x=-a}^{a} y\,dx\,dy$$

$$\therefore \quad \int_S \text{curl } \vec{F}\cdot\hat{n}\,dS = -4\int_0^b y\,dy\int_{-a}^{a} dx = -4\left[\frac{y^2}{2}\right]_0^b [x]_{-a}^{a}$$

$$= -4\cdot\frac{b^2}{2}\cdot 2a = -4ab^2 \qquad \ldots(3)$$

From (2) and (3), we have

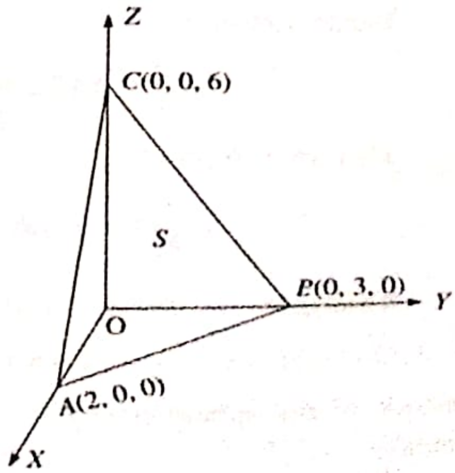
$$\oint_\Gamma \vec{F}\cdot d\vec{r} = \int_S \text{curl } \vec{F}\cdot\hat{n}\,dS$$

Hence Stokes' theorem is verified.

Example 2: Using Stokes' theorem evaluate

$$\int_\Gamma \{(x+y)\,dx + (2x-z)\,dy + (y+z)\,dz\}$$

where Γ is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).



Solution: Here $\quad \vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$

$$\therefore \quad \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\hat{i} + \hat{k}$$

Equation of the plane through A (2, 0, 0), B (0, 3, 0) and C (0, 0, 6) is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1, \text{ or } 3x + 2y + z = 6.$$

Let $\quad \varphi = 3x + 2y + z$

$$\therefore \quad \vec{\nabla}\varphi = \hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z} = 3\hat{i} + 2\hat{j} + \hat{k}.$$

Unit normal to the plane ABC is

$$\hat{n} = \frac{\vec{\nabla}\varphi}{|\vec{\nabla}\varphi|} = \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{3^2+2^2+1^2}} = \frac{1}{\sqrt{14}}(3\hat{i}+2\hat{j}+\hat{k})$$

$$\therefore \quad \int_\Gamma \{(x+y)dx+(2x-z)dy+(y+z)dz\} = \int_\Gamma \vec{F}\cdot d\vec{r}$$

$$= \int_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS$$

(by Stokes' theorem, where S is the surface bounded by the triangle ABC)

$$= \int_S (2\hat{i}+0\hat{j}+\hat{k})\cdot\frac{1}{\sqrt{14}}(3\hat{i}+2\hat{j}+\hat{k})dS = \frac{7}{\sqrt{14}}\int_S dS$$

$$= \frac{7}{\sqrt{14}}\text{ (Area of } \Delta ABC) \qquad \ldots(1)$$

Here $\overline{AB}$ = p.v. of B – p.v. of A = $-2\hat{i}+3\hat{j}$

and $\overline{AC}$ = p.v. of C – p.v. of A = $-2\hat{i}+6\hat{k}$

$$\therefore \quad \overline{AB}\times\overline{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & 0 \\ -2 & 0 & 6 \end{vmatrix} = 18\hat{i}+12\hat{j}+6\hat{k}$$

$$\therefore \quad |\overline{AB}\times\overline{AC}| = \sqrt{(18)^2+(12)^2+6^2} = 6\sqrt{9+4+1} = 6\sqrt{14}$$

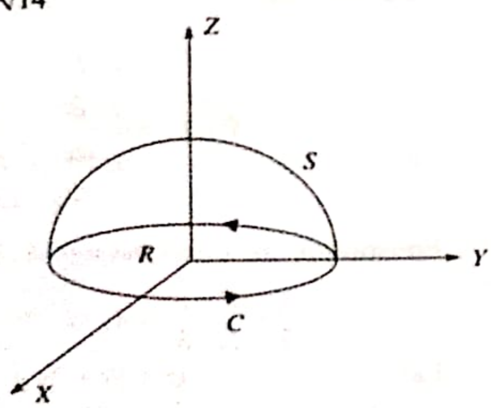
Hence area of

$$\Delta ABC = \frac{1}{2}\left|\overline{AB}\times\overline{AC}\right| = \frac{1}{2}\cdot 6\sqrt{14} = 3\sqrt{14}$$

Therefore, from (1),

$$\int_\Gamma \{(x+y)dx+(2x-z)dy+(y+z)dz\} = \frac{7}{\sqrt{14}}\cdot 3\sqrt{14} = 21.$$

Example 3: Verify Stokes' theorem for $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2 = 1$ and C is its boundary. *(W.B.U.T. 2006, 2012)*



Solution: The boundary C of the upper half surface S of the sphere $x^2+y^2+z^2 = 1$ on the xy-plane (*i.e.*, $z = 0$) is $x^2+y^2 = 1$.

$$\therefore \quad \oint_C \vec{F}\cdot d\vec{r} = \oint_C (2x-y)\hat{i}\cdot(\hat{i}\,dx+\hat{j}\,dy) \quad (\because z = 0,\ dz = 0 \text{ in the } xy\text{-plane})$$

$$= \oint_C (2x-y)dx = \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\, d\theta$$

[Putting $x = \cos\theta$, $y = \sin\theta$. $\therefore$ $dx = -\sin\theta\, d\theta$]

$$= \int_0^{2\pi} (-2\sin\theta\cos\theta + \sin^2\theta)\, d\theta$$

$$= \int_0^{2\pi} \left\{-\sin 2\theta + \frac{1}{2}(1-\cos 2\theta)\right\} d\theta$$

$$= \left[\frac{1}{2}\cos 2\theta + \frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \pi$$

$\therefore$ $$\oint_C \vec{F}\cdot d\vec{r} = \pi \qquad \ldots(1)$$

Now, $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = (-2yz + 2yz)\hat{i} + 0\,\hat{j} + \hat{k} = \hat{k}$$

$\therefore$ $$\int_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, dS = \int_S \hat{k}\cdot\hat{n}\, dS$$

$$= \iint_R dx\, dy \quad [\because \hat{k}\cdot\hat{n}\, dS = \text{Projection of } dS \text{ on the } xy\text{-plane} = dxdy]$$

where R is the projection of S on the xy-plane and hence R is the region enclosed by the circle $x^2 + y^2 = 1$.

$\therefore$ $$\int_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, dS = \iint_R dx\, dy = (\text{Area of the circle } x^2 + y^2 = 1)$$

$$= \pi \qquad \ldots(2)$$

From (1) and (2), we have

$$\oint_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\, \vec{F}\cdot\hat{n}\, dS.$$

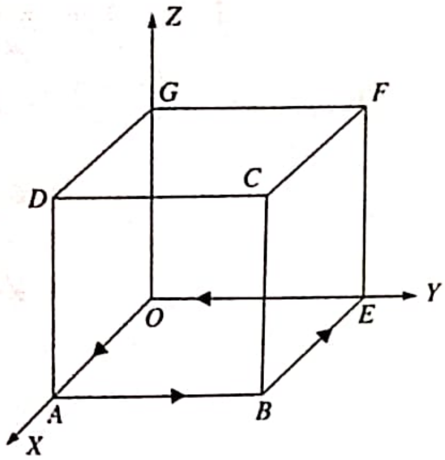
Hence Stokes' theorem is verified.

Example 4: Verify Stokes' theorem for $\vec{A} = (y-z+2)\hat{i} + (yz+4)\hat{j} - xz\hat{k}$ over the surface of the cube $x = y = z = 0$ and $x = y = z = 2$ above xy-plane. *(W.B.U.T. 2007)*

Solution: Let us denote five faces $ABCD$, $BEFC$, $EOGF$, $OADG$ and $DCFG$ of the given cube above xy-plane by S_1, S_2, S_3, S_4 and S_5 respectively.

Now, $$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= -y\hat{i} + (z-1)\hat{j} - \hat{k}$$



On S_1, $\hat{n} = \hat{i}$, $x = 2$, $dx = 0$, $0 \le y,\ z \le 2$, $dS = dydz$.

$$\therefore \quad \int_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS = \int_{S_1} \{-y\hat{i} + (z-1)\hat{j} - \hat{k}\} \cdot \hat{i}\, dS$$

$$= -\int_0^2 \int_0^2 y\, dy\, dz$$

$$= -\left[\frac{y^2}{2}\right]_0^2 [z]_0^2 = -2 \cdot 2 = -4.$$

On S_2, $\hat{n} = \hat{j}$, $y = 2$, $dy = 0$, $0 \le x,\ z \le 2$, $dS = dxdz$.

$$\therefore \quad \int_{S_2} (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS = \int_{S_2} \{-2\hat{i} + (z-1)\hat{j} - \hat{k}\} \cdot \hat{j}\, dS$$

$$= \int_0^2 \int_0^2 (z-1)\, dx\, dz = [x]_0^2 \left[\frac{z^2}{2} - z\right]_0^2 = 0.$$

On S_3, $\hat{n} = -\hat{i}$, $x = 0$, $dx = 0$, $0 \le y,\ z \le 2$, $dS = dydz$.

$$\therefore \quad \int_{S_3} (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS = \int_{S_3} \{-y\hat{i} + (z-1)\hat{j} - \hat{k}\} \cdot (-\hat{i})\, dS = 4$$

On S_4, $\hat{n} = -\hat{j}$, $y = 0$, $dy = 0$, $0 \le x,\ z \le 2$, $dS = dxdz$.

$$\therefore \quad \int_{S_4} (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS = \int_{S_4} \{(z-1)\hat{j} - \hat{k}\} \cdot (-\hat{j})\, dS = 0$$

On S_5, $\hat{n} = \hat{k}$, $z = 2$, $dz = 0$, $0 \le x,\ y \le 2$, $dS = dxdy$

$$\therefore \quad \int_{S_5} (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS = \int_{S_5} \{-y\hat{i} + \hat{j} - \hat{k}\} \cdot \hat{k}\, dx\, dy$$

$$= \int_0^2 \int_0^2 (-1)\, dx\, dy = -[x]_0^2 [y]_0^2 = -4.$$

$$\therefore \quad \int_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS = \int_{S_1} (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS + \int_{S_2} (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS + \int_{S_3} (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS$$

$$+ \int_{S_4} (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS + \int_{S_5} (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS$$

$$= -4 + 0 + 4 + 0 - 4 = -4$$

$$\therefore \quad \int_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS = -4 \qquad \ldots(1)$$

Now, on the xy-plane, $z = 0$, $dz = 0$

$$\therefore \quad \oint_{ABEOA} \bar{A}\cdot d\bar{r} = \oint_{ABEOA} \{(y+2)\hat{i} + 4\hat{j}\}\cdot(\hat{i}\,dx + \hat{j}\,dy)$$

$$\oint_{ABEOA} \{(y+2)dx + 4dy\} = \oint_{AB} \{(y+2)dx + 4dy\} + \oint_{BE} \{(y+2)dx + 4dy\}$$

$$+\oint_{EO} \{(y+2)dx + 4dy\} + \oint_{OA} \{(y+2)dx + 4dy\}$$

$$= \int_0^2 4\,dy + \int_2^0 4\,dx + \int_2^0 4\,dy + \int_0^2 2\,dx$$

$$= 8 - 8 - 8 + 4 = -4$$

[$\because$ On AB : $x = 2$, $dx = 0$, y varies from 0 to 2,
On BE : $y = 2$, $dy = 0$, x varies from 2 to 0,
On EO : $x = 0$, $dx = 0$, y varies from 2 to 0,
On OA : $y = 0$, $dy = 0$, x varies from 0 to 2.]

$$\therefore \quad \oint_{ABEOA} \bar{A}\cdot d\bar{r} = -4 \qquad \text{...(2)}$$

From (1) and (2), we have

$$\oint_{ABEOA} \bar{A}\cdot d\bar{r} = \int_S (\bar{\nabla}\times\bar{A})\cdot\hat{n}\,dS$$

Hence Stokes' theorem is verified.

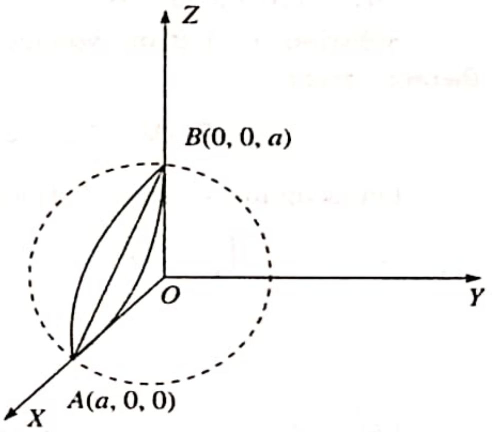
Example 5: Apply Stokes' theorem to evaluate

$$\oint_C (y\,dx + z\,dy + x\,dz)$$

where C is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$. *(W.B.U.T. 2001)*

Solution: We know that the intersection of a sphere and a plane is a circle. Therefore, the given curve C is a circle lying on the plane $x + z = a$ with a diameter AB where $A \equiv (a, 0, 0)$ and $B \equiv (0, 0, a)$.

So its radius is $\dfrac{a}{\sqrt{2}}$.



The circle lies on the plane $x + z = a$.

Let $\qquad \varphi = x + z$

$$\therefore \quad \bar{\nabla}\varphi = \hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z} = \hat{i} + \hat{k}$$

$$\therefore \quad \hat{n} = \frac{\bar{\nabla}\varphi}{|\bar{\nabla}\varphi|} = \frac{\hat{i}+\hat{k}}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}(\hat{i}+\hat{k})$$

Now, $\qquad \bar{r} = x\hat{i} + y\hat{j} + z\hat{k},$

so $\qquad d\bar{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$$\therefore \quad y\,dx + z\,dy + x\,dz = (y\hat{i} + z\hat{j} + x\hat{k})\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \vec{F} \cdot d\vec{r}, \text{ where } \vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$$

$$\therefore \quad \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k} = -(\hat{i} + \hat{j} + \hat{k})$$

$$\therefore \quad (\vec{\nabla} \times \vec{F}) \cdot \hat{n} = -(\hat{i} + \hat{j} + \hat{k}) \cdot \frac{1}{\sqrt{2}} (\hat{i} + 0\hat{j} + \hat{k})$$

$$= -\frac{1}{\sqrt{2}}(1+1) = -\frac{2}{\sqrt{2}} = -\sqrt{2} \qquad \ldots(1)$$

$$\therefore \quad \oint_C (y\,dx + z\,dy + x\,dz) = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \int_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS$$

[by Stokes' theorem, where S is the region bounded by the circle C with radius $\frac{a}{\sqrt{2}}$]

$$= -\sqrt{2} \int_S dS \qquad \text{[by (1)]}$$

$$= -\sqrt{2} \text{ (Area of the circle } C \text{ with radius } \frac{a}{\sqrt{2}})$$

$$= -\sqrt{2} \cdot \frac{\pi a^2}{2} = -\frac{\pi a^2}{\sqrt{2}}.$$

Example 6: Use Stokes' theorem, prove that

(*i*) div curl $\vec{F} = 0$ (*W.B.U.T. 2002*)

(*ii*) curl grad $\varphi = \vec{0}$

Solution: (*i*) Let the volume V be enclosed by a closed surface S. Then Gauss' divergence theorem gives

$$\int_V \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) dV = \int_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS \qquad \ldots(1)$$

Let us divide the closed surface S into two open surfaces S_1 and S_2 by means of a closed curve C.

$$\therefore \quad \int_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS = \int_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS + \int_{S_2} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS$$

$$= \oint_C \vec{F} \cdot d\vec{r} - \oint_C \vec{F} \cdot d\vec{r} \qquad \text{(by Stokes' theorem)}$$

$$= 0$$

[Here the integrals are in opposite signs since the senses of enclosing C for the two surfaces are opposite in character with respect to these surfaces.]

Therefore, from (1),

$$\int_V \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})\,dV = 0.$$

This is true for all volume elements V and hence

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0, \text{ i.e., div curl } \vec{F} = 0.$$

(ii) Let S be a surface enclosed by a simple closed curve Γ. By Stokes' theorem, we have

$$\int_S (\vec{\nabla}\times\vec{\nabla}\varphi)\cdot\hat{n}\,dS = \oint_\Gamma \vec{\nabla}\varphi\cdot d\vec{r} \quad \text{...(2)}$$

Now,
$$\vec{\nabla}\varphi\cdot d\vec{r} = \left(\hat{i}\frac{\partial\varphi}{\partial x}+\hat{j}\frac{\partial\varphi}{\partial y}+\hat{k}\frac{\partial\varphi}{\partial z}\right)\cdot(\hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz)$$

$$= \frac{\partial\varphi}{\partial x}dx+\frac{\partial\varphi}{\partial y}dy+\frac{\partial\varphi}{\partial z}dz = d\varphi$$

Therefore, from (2),

$$\int_S (\vec{\nabla}\times\vec{\nabla}\varphi)\cdot\hat{n}\,dS = \oint_\Gamma d\varphi = 0 \qquad (\because \Gamma \text{ is a closed curve})$$

This is true for all surface elements S and consequently

$$\vec{\nabla}\times(\vec{\nabla}\varphi) = \vec{0}, \text{ i.e., curl grad } \varphi = \vec{0}.$$

Example 7: Prove that $\oint_\Gamma \vec{r}\times d\vec{r} = 2\int_S \hat{n}\,dS$ where S be the surface enclosed by a simple closed curve Γ.

Solution: Let us apply Stokes' theorem to the function $\vec{A}\times\vec{r}$, where $\vec{A}$ is any arbitrary constant vector and $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$.

$$\therefore \quad \oint_\Gamma (\vec{A}\times\vec{r})\cdot d\vec{r} = \int_S \{\vec{\nabla}\times(\vec{A}\times\vec{r})\}\cdot\hat{n}\,dS \quad \text{...(1)}$$

We may express $\vec{A} = A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$, where A_1, A_2, A_3 are constants.

$$\therefore \quad \vec{A}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

$$= (A_2 z - A_3 y)\hat{i}+(A_3 x - A_1 z)\hat{j}+(A_1 y - A_2 x)\hat{k}$$

Hence
$$\vec{\nabla}\times(\vec{A}\times\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_2 z - A_3 y & A_3 x - A_1 z & A_1 y - A_2 x \end{vmatrix}$$

$$= (A_1+A_1)\hat{i}+(A_2+A_2)\hat{j}+(A_3+A_3)\hat{k} = 2\vec{A}$$

Therefore, from (1),

$$\oint_\Gamma (\vec{A}\times\vec{r})\cdot d\vec{r} = 2\int_S \vec{A}\cdot\hat{n}\,dS$$

or
$$\vec{A}\cdot\oint_\Gamma \vec{r}\times d\vec{r} - 2\vec{A}\cdot\int_S \hat{n}\,S = 0$$

[since $\vec{A}$ is a constant vector and $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$]

or
$$\vec{A}\cdot\left[\oint_\Gamma \vec{r}\times d\vec{r} - 2\int_S \hat{n}\,dS\right] = 0$$

$\therefore$
$$\oint_\Gamma \vec{r}\times d\vec{r} - 2\int_S \hat{n}\,dS = \vec{0}$$
(since $\vec{A}$ is any arbitrary constant vector)

or
$$\oint_\Gamma \vec{r}\times d\vec{r} = 2\int_S \hat{n}\,dS.$$

Example 8: Evaluate $\oint_\Gamma (e^x\,dx + 2y\,dy - dz)$ by using Stokes' theorem, where Γ is the curve $x^2 + y^2 = a^2$, $z = h$.

Solution: Here

$$e^x\,dx + 2y\,dy - dz = (e^x\,\hat{i} + 2y\,\hat{j} - \hat{k})\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$
$$= \vec{F}\cdot d\vec{r}, \text{ where } \vec{F} = e^x\,\hat{i} + 2y\,\hat{j} - \hat{k} \qquad \ldots(1)$$

$\therefore$
$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0} \qquad \ldots(2)$$

Therefore, by Stokes' theorem

$$\oint_\Gamma \vec{F}\cdot d\vec{r} = \int_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS = 0 \qquad \text{[by (2)]}$$

or
$$\oint_\Gamma (e^x\,dx + 2y\,dy - dz) = 0. \qquad \text{[by (1)]}$$

Example 9: If $\vec{F} = (y^2 + z^2 - x^2)\hat{i} + (z^2 + x^2 - y^2)\hat{j} + (x^2 + y^2 - z^2)\hat{k}$ and S be the portion of the surface $x^2 + y^2 - 2ax + az = 0$ for which $z \geq 0$, then applying Stokes' theorem, show that

$$\int_S \text{curl } \vec{F}\cdot\hat{n}\,dS = 2\pi a^3.$$

Solution: In the xy-plane (*i.e.*, $z = 0$), the boundary C of the given surface S is the circle $x^2 + y^2 - 2ax = 0$, or $(x-a)^2 + y^2 = a^2$. The parametric form of this circle is $x = a + a\cos\theta$ and $y = a\sin\theta$, where $0 \leq \theta \leq 2\pi$. On xy-plane; $z = 0$, $dz = 0$

$\therefore$
$$\vec{F}\cdot d\vec{r} = \{(y^2 - x^2)\hat{i} + (x^2 - y^2)\hat{j} + (x^2 + y^2)\hat{k}\}\cdot(\hat{i}\,dx + \hat{j}\,dy + 0\,\hat{k})$$

$$= (y^2 - x^2)\,dx + (x^2 - y^2)\,dy = (x^2 - y^2)(dy - dx)$$

$$\therefore \quad \oint_C \vec{F}\cdot d\vec{r} = \oint_C (x^2 - y^2)(dy - dx)$$

$$= \int_0^{2\pi} \{(a + a\cos\theta)^2 - a^2\sin^2\theta\}\{d(a\sin\theta) - d(a + a\cos\theta)\}$$

$$= \int_0^{2\pi} a^2(1 + 2\cos\theta + \cos^2\theta - \sin^2\theta)\,a(\cos\theta + \sin\theta)\,d\theta$$

$$= a^3\int_0^{2\pi}(1 + 2\cos\theta + \cos 2\theta)(\cos\theta + \sin\theta)\,d\theta$$

$$= a^3\int_0^{2\pi}(\cos\theta + 2\cos^2\theta + \cos 2\theta\cos\theta + \sin\theta$$
$$+ 2\cos\theta\sin\theta + \cos 2\theta\sin\theta)\,d\theta$$

$$= a^3\int_0^{2\pi}\left\{\cos\theta + (1 + \cos 2\theta) + \frac{1}{2}(\cos 3\theta + \cos\theta)\right.$$
$$\left. + \sin\theta + \sin 2\theta + \frac{1}{2}(\sin 3\theta - \sin\theta)\right\}d\theta$$

$$= a^3\left[\sin\theta + \theta + \frac{1}{2}\sin 2\theta + \frac{1}{2}\left(\frac{\sin 3\theta}{3} + \sin\theta\right)\right.$$
$$\left. - \cos\theta - \frac{\cos 2\theta}{2} + \frac{1}{2}\left(-\frac{\cos 3\theta}{3} + \cos\theta\right)\right]_0^{2\pi}$$

$$= a^3[0 + 2\pi + 0 + 0 - 0 - 0 + 0] = 2\pi a^3$$

Therefore, by Stokes' theorem

$$\int_S \text{curl } \vec{F}\cdot\hat{n}\,dS = \oint_C \vec{F}\cdot d\vec{r} = 2\pi a^3 \quad \textbf{(Proved).}$$

MULTIPLE CHOICE QUESTIONS

1. The unit tangent vector at $t = 1$ on the curve $x = t^2 + 4$, $y = 4t - 6$, $z = t^2 + 2t$ is

(*a*) $\frac{1}{3}(\hat{i} + \hat{j} + \hat{k})$ (*b*) $\frac{2}{3}(\hat{i} + \hat{j} + \hat{k})$

(*c*) $\frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$ (*d*) none of these.

2. The angle between the tangents to the curve $\vec{r} = t\hat{i} + 2t\hat{j} - t^2\hat{k}$ at the points $t = \pm 1$ is

(*a*) $\cos^{-1}\left(\frac{1}{3}\right)$ (*b*) $\cos^{-1}\left(\frac{1}{9}\right)$ (*c*) $\cos^{-1}\left(\frac{2}{9}\right)$ (*d*) none of these.