CHAPTER 3

Matrix III

3.1 RANK OF A MATRIX

Submatrix

Submatrix of a matrix $A = (a_{ij})_{m \times n}$ is any matrix obtained from A by deleting some rows and columns in A.

$A = (a_{ij})_{m \times n}$ is a submatrix of itself (obtained by removing zero row and column).

Rank

Rank of a matrix $A = (a_{ij})_{m \times n}$ is the positive integer r such that

(*i*) there exists at least one rth order non-singular square submatrix of A.

(*ii*) all the square submatrices of A of order greater than r are singular.

Rank of matrix A is denoted by rank (A).

Notes: (*i*) Ranks of A and A^T are same.

(*ii*) Rank of a null matrix is zero.

(*iii*) For a matrix of order $m \times n$, rank of $A \leq \min(m, n)$.

(*iv*) For an nth order square matrix A, if rank $(A) = n$, then $|A| \neq 0$, *i.e.*, A is non-singular.

(*v*) For any square matrix A of order n, if rank $(A) < n$, then $|A| = 0$, *i.e.*, A is singular.

Example 1: Consider $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Since $|I_3| = 1 \neq 0$, therefore rank $(I_3) = 3$.

Now, $|I_n| = 1 \neq 0$, where I_n is the nth order identity matrix. Hence rank $(I_n) = n$.

Example 2: Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

Observe that all square submatrices of order ≥ 2 of A are singular and there is a submatrix (1) of order 1 which is non-singular. Therefore, rank $(A) = 1$.

Scanned with CamScanner

Example 3: Consider the matrix $A = \begin{pmatrix} 2 & 0 & 3 & 1 \\ 2 & 0 & 3 & 2 \\ 2 & 0 & 3 & 1 \end{pmatrix}$

Observe that all square submatrices of order 3 of A are singular and there is a submatrix $\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}$ of order 2 which is non-singular, since $\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = 6 - 3 = 3 \neq 0$. Therefore, rank $(A) = 2$.

3.2 ELEMENTARY ROW OPERATIONS ON A MATRIX

An elementary row operation on a matrix A is any one operation of the following three types:

(*i*) Interchange of any two rows of A. Interchange of ith and jth rows of A is denoted by R_{ij}.

(*ii*) Multiplication of a row by a non-zero scalar k. Multiplication of ith row of A by a non-zero scalar k is denoted by kR_i.

(*iii*) Addition of a scalar multiple of one row to another row. Addition of k times the elements of jth row to the corresponding elements of ith row is denoted by $R_i + kR_j$.

Example:

Type (*i*): $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 0 \\ 2 & 1 & 1 \\ 4 & 3 & 2 \end{pmatrix} \xrightarrow{R_{24}} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 2 & 1 & 1 \\ 4 & -2 & 0 \end{pmatrix}$

Type (*ii*): $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 2 \\ 4 & 3 & 1 \end{pmatrix} \xrightarrow{4R_1} \begin{pmatrix} 4 & 8 & -12 \\ 2 & 1 & 2 \\ 4 & 3 & 1 \end{pmatrix}$

Type (*iii*): $\begin{pmatrix} 1 & 2 & 3 & -4 \\ 2 & 3 & 0 & 1 \\ 1 & 4 & 3 & 2 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 3 & -4 \\ 0 & -1 & -6 & 9 \\ 1 & 4 & 3 & 2 \end{pmatrix}$.

3.3 ELEMENTARY COLUMN OPERATIONS ON A MATRIX

An elementary column operation on a matrix A is any one of the following three types:

(*i*) Interchange of any two columns of A. Interchange of ith and jth columns of A is denoted by C_{ij}.

(*ii*) Multiplication of a column by a non-zero scalar k. Multiplication of ith column of A by a non-zero scalar k is denoted by kC_i:

(*iii*) Addition of a scalar multiple of one column to another column. Addition of k times the elements of jth column to the corresponding elements of ith column is denoted by $C_i + kC_j$.

Scanned with CamScanner

Example:

Type (*i*): $\begin{pmatrix} 1 & 2 & 3 & -2 \\ 0 & 2 & 4 & 1 \\ 3 & -1 & 2 & 4 \end{pmatrix} \xrightarrow{C_{23}} \begin{pmatrix} 1 & 3 & 2 & -2 \\ 0 & 4 & 2 & 1 \\ 3 & 2 & -1 & 4 \end{pmatrix}$

Type (*ii*): $\begin{pmatrix} 1 & 2 & 3 & -2 \\ 0 & 2 & 4 & 1 \\ 3 & -1 & 2 & 4 \end{pmatrix} \xrightarrow{3C_2} \begin{pmatrix} 1 & 6 & 3 & -2 \\ 0 & 6 & 4 & 1 \\ 3 & -3 & 2 & 4 \end{pmatrix}$

Type (*iii*): $\begin{pmatrix} 1 & 2 & 3 & -2 \\ 0 & 2 & 4 & 1 \\ 3 & -1 & 2 & 4 \end{pmatrix} \xrightarrow{C_2-2C_4} \begin{pmatrix} 1 & 6 & 3 & -2 \\ 0 & 0 & 4 & 1 \\ 3 & -9 & 2 & 4 \end{pmatrix}$

Note: Elementary row and column operations are also known as elementary row and column transformations respectively.

3.4 ROW EQUIVALENT AND COLUMN EQUIVALENT MATRICES

A matrix B is said to be row equivalent of a matrix A if B is obtained by successive application of finite number of elementary row operations on A.

A matrix B is said to be column equivalent of a matrix A if B is obtained by successive application of finite number of elementary column operations on A.

Two matrices A and B are said to be equivalent, denoted by $A \sim B$, if one matrix can be obtained from another by a sequence of elementary (row and/or column) operations.

Minor of a matrix

The determinant of every square submatrix of a given matrix A is said to be a minor of the matrix A.

Example: (*i*) Let $A = \begin{pmatrix} 2 & 4 & 6 & -2 \\ 3 & 1 & 2 & 0 \\ -5 & 2 & 1 & 1 \end{pmatrix}$

Let us apply the following successive elementary row operations on A:

$$A = \begin{pmatrix} 2 & 4 & 6 & -2 \\ 3 & 1 & 2 & 0 \\ -5 & 2 & 1 & 1 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 2 & 3 & -1 \\ 3 & 1 & 2 & 0 \\ -5 & 2 & 1 & 1 \end{pmatrix} \xrightarrow{R_2-3R_1}$$

$$\begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -7 & 3 \\ -5 & 2 & 1 & 1 \end{pmatrix} \xrightarrow{R_3+5R_1} \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -7 & 3 \\ 0 & 12 & 16 & -4 \end{pmatrix} = B \text{ (say)}$$

Scanned with CamScanner

Therefore, B is row equivalent of A since B is obtained from A by three successive elementary row operations and we write $B \sim A$ or $A \sim B$. The matrices A and B are also said to be row equivalent.

(*ii*) Let $A = \begin{pmatrix} 3 & 6 & 6 \\ 2 & 1 & 3 \\ 4 & -2 & -6 \end{pmatrix}$

Let us apply the following successive elementary column operations on A:

$$A = \begin{pmatrix} 3 & 6 & 6 \\ 2 & 1 & 3 \\ 4 & -2 & -6 \end{pmatrix} \xrightarrow{\frac{1}{3}C_3} \begin{pmatrix} 3 & 6 & 2 \\ 2 & 1 & 1 \\ 4 & -2 & -2 \end{pmatrix} \xrightarrow{C_1 - 2C_3} \begin{pmatrix} -1 & 6 & 2 \\ 0 & 1 & 1 \\ 8 & -2 & -2 \end{pmatrix} = B \text{ (say)}$$

Here B is column equivalent of A since B is obtained from A by two elementary column operations and we write $B \sim A$ or $A \sim B$. A and B are also known as column equivalent matrices.

3.5 ZERO-ROW AND NON-ZERO ROW OF A MATRIX

If all elements of a row of a matrix are zero then this row is called a zero-row.

If at least one element of a row of a matrix is non-zero then this row is called a non-zero row.

Example: Let $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Here third row is a zero-row and first, second are non-zero rows.

3.6 ECHELON MATRIX

A matrix A is said to be an Echelon matrix or is said to be in echelon form if

(*i*) all zero-rows of A follow all non-zero rows,

(*ii*) the number of zeros preceding the first non-zero element of a row increases as we pass from row to row downwards.

Example: (*i*) The following are all Echelon matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 4 & 2 & 1 & 0 & 3 \\ 0 & 0 & 0 & 3 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Scanned with CamScanner

(*ii*) The following are not Echelon matrices:

$$\begin{pmatrix} 0 & 2 & 0 & 3 \\ 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

Let us state three important theorems without proofs.

Theorem 1. Every matrix can be made row equivalent to an Echelon matrix.

Theorem 2. Elementary operations (row and/or column) do not alter the rank of a matrix.

Theorem 3. If an Echelon matrix has r number of non-zero rows then the rank of this matrix is r.

ILLUSTRATIVE EXAMPLES

Example 1: Reduce the matrix $A = \begin{pmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{pmatrix}$ to an equivalent Echelon matrix and hence find its rank.

Solution: Let us apply the following successive elementary row operations on A:

$$A = \begin{pmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{pmatrix} \xrightarrow[R_4-2R_1]{R_2-2R_1,\ R_3+R_1,} \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3+R_2} \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix} \xrightarrow{R_{34}} \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = B \text{ (say)}$$

Therefore, the given matrix is equivalent to the Echelon matrix $B = \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ which has four non-zero rows. Therefore, Rank $(B) = 4$, by Theorem 3. Hence by Theorem 2, Rank (A) = Rank $(B) = 4$, since A and B are row equivalent matrices.

Scanned with CamScanner

Example 2: Find the rank of the rectangular matrix [W.B.U.T. 2006]

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$$

Solution: Let us apply the following elementary row operations on the given matrix A (say).

$$\therefore \quad A = \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix} \xrightarrow{R_3-2R_1,\ R_4-3R_1} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3+R_2} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix} \xrightarrow{R_{34}} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -5 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_3+\frac{5}{2}R_2}$$

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = B \text{ (say).}$$

Therefore, the given matrix A is equivalent to the Echelon matrix B which has three non-zero rows. Therefore, rank $(B) = 3$, by Theorem 3. Hence by Theorem 2, rank (A) = rank $(B) = 3$, since A and B are equivalent matrices. Thus the rank of the given matrix is 3.

Example 3: Find the rank of $A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{pmatrix}$ for different values of b.

Solution: Let us apply the following elementary row operations on the given matrix A.

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{pmatrix} \xrightarrow[R_4-9R_1]{R_2-4R_1,\ R_3-2R_1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ b-2 & 0 & 4 & 2 \\ 0 & 0 & b+9 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3-4R_2,\ R_4-3R_2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ b-2 & 0 & 0 & -2 \\ 0 & 0 & b+6 & 0 \end{pmatrix} = B \text{ (say)}$$

Case 1: Let $b = 2$. Here $|B| = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 8 & 0 \end{vmatrix} = 0$ (expanding along 1st column).

But there exists a minor of order 3, $\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{vmatrix} = -2 \neq 0$.

Therefore, rank $(B) = 3$ and hence rank $(A) = 3$, since A, B are equivalent matrices.

Case 2: Let $b = -6$. Here $|B| = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ -8 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$ (expanding along 4th row).

But there exists a minor of order 3, $\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{vmatrix} = -2 \neq 0$.

Therefore, rank $(B) = 3$ and hence rank $(A) = 3$, since A, B are equivalent matrices.

Case 3: Let b be any number different from 2 and -6, *i.e.*, $b \neq 2,\ b \neq -6$.

$$\therefore\ |B| = \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ b-2 & 0 & 0 & -2 \\ 0 & 0 & b+6 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & 1 & 1 \\ b-2 & 0 & -2 \\ 0 & b+6 & 0 \end{vmatrix}$$

$$= -(b-2)\begin{vmatrix} 1 & 1 \\ b+6 & 0 \end{vmatrix} = -(b-2)(b+6) \neq 0$$

$(\because\ b \neq 2,\ b \neq -6)$

Therefore, in this case rank (A) = rank $(B) = 4$, since A, B are equivalent matrices.

Example 4: Find the rank of the matrix $A = \begin{pmatrix} a & -1 & -1 \\ -1 & a & -1 \\ -1 & -1 & a \\ 1 & 1 & 1 \end{pmatrix}$ when (*i*) $a \neq -1$ and (*ii*) $a = -1$.

Solution: Let us apply the following elementary row operations on the given matrix A.

$$A = \begin{pmatrix} a & -1 & -1 \\ -1 & a & -1 \\ -1 & -1 & a \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_{14}} \begin{pmatrix} 1 & 1 & 1 \\ -1 & a & -1 \\ -1 & -1 & a \\ a & -1 & -1 \end{pmatrix} \xrightarrow[R_3+R_1,\ R_4-aR_1]{R_2+R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & a+1 & 0 \\ 0 & 0 & a+1 \\ 0 & -1-a & -1-a \end{pmatrix}$$

$$\xrightarrow{R_4+R_2+R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & a+1 & 0 \\ 0 & 0 & a+1 \\ 0 & 0 & 0 \end{pmatrix} = B \text{ (say)}$$

(i) $\boldsymbol{a \neq -1}$

In this case the given matrix A is equivalent to the Echelon matrix B which has three non-zero rows. Therefore, rank (B) = 3 and hence rank (A) = rank (B) = 3, since A and B are row equivalent matrices.

(ii) $\boldsymbol{a = -1}$

In this case the given matrix A is equivalent to the Echelon matrix B which has only one non-zero row. Therefore, rank (B) = 1 and hence rank (A) = rank (B) = 1, since A and B are row equivalent matrices.

Example 5: Find the rank of $\begin{pmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{pmatrix}$.

Solution: Let us apply the following elementary row operations on the given matrix A (say).

$$\therefore \quad A = \begin{pmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{pmatrix} \xrightarrow{R_{23}} \begin{pmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 1 & -2 & 1 & -4 & 2 \\ 4 & -7 & 4 & -4 & 5 \end{pmatrix}$$

$$\xrightarrow[R_4-2R_1]{2R_3-R_1} \begin{pmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & -1 & -9 & 4 \\ 0 & 1 & -2 & -6 & 5 \end{pmatrix} \xrightarrow{R_4-R_2} \begin{pmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & -1 & -9 & 4 \\ 0 & 0 & -1 & -9 & 4 \end{pmatrix}$$

Scanned with CamScanner

$$\xrightarrow{R_4-R_3}\begin{pmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & -1 & -9 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = B \text{ (say)}$$

Therefore, the given matrix A is equivalent to the Echelon matrix B which has three non-zero rows. Therefore, rank $(B) = 3$, by Theorem 3. Hence by Theorem 2, rank (A) = rank $(B) = 3$, since A and B are row equivalent matrices. Thus the rank of the given matrix is 3.

3.7 SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS BY MATRIX INVERSION METHOD

Let us consider a system of n linear equations in n variables.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}\right\} \qquad \ldots(3.7.1)$$

This system of equations can be written in the following matrix form:

$$AX = B \qquad \ldots(3.7.2)$$

where
$$A = \begin{pmatrix} a_{11}a_{12}\ldots a_{1n} \\ a_{21}a_{22}\ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots \\ a_{n1}a_{n2}\ldots a_{nn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

If $|A| \neq 0$, *i.e.*, if rank $(A) = n$ then A^{-1} exists. Premultiplying both sides of (3.7.2) by A^{-1} we get,

$$A^{-1}(AX) = A^{-1}B$$

or $\quad (A^{-1}A)X = A^{-1}B \quad$ (By associative law)

or $\quad IX = A^{-1}B \quad (\because A^{-1}A = I)$

$\therefore \quad X = A^{-1}B \quad (\because IX = X)$

This is the matrix solution of the system of equations (3.7.1).

Also this solution is unique. To prove this let us suppose that X_1 and X_2 are two solutions of equation (3.7.2).

Then $\quad AX_1 = B$ and $AX_2 = B$

$\therefore \quad AX_1 = AX_2$

or $\quad A^{-1}(AX_1) = A^{-1}(AX_2) \quad$ (Premultiplying both sides by A^{-1})

or $\quad (A^{-1}A)X_1 = (A^{-1}A)X_2 \quad$ (By associative law)

Scanned with CamScanner

or $$IX_1 = IX_2$$

$\therefore$ $$X_1 = X_2$$

Hence, the solution is unique.

Deduction of Cramer's Rule

We have $X = A^{-1}B$, provided $|A| \neq 0$.

$$\therefore \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$= \frac{1}{|A|}\begin{pmatrix} b_1A_{11} + b_2A_{21} + \dots + b_nA_{n1} \\ b_1A_{12} + b_2A_{22} + \dots + b_nA_{n2} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ b_1A_{1n} + b_2A_{2n} + \dots + b_nA_{nn} \end{pmatrix}$$

where A_{ij} is the cofactor of a_{ij} $(i = 1, 2, ..., n;\ j = 1, 2, ..., n)$ in $|A| = |a_{ij}|_{n\times n}$.

Equating the corresponding elements, we get

$$x_j = \frac{1}{|A|}(b_1A_{1j} + b_2A_{2j} + \dots + b_nA_{nj})$$

$$= \frac{|A_j|}{|A|}, \quad j = 1, 2, ..., n,$$

where $|A_j|$ is an nth order determinant obtained from $|A|$ by replacing the elements of its jth column by $b_1, b_2, ..., b_n$.

This is Cramer's rule.

ILLUSTRATIVE EXAMPLES

Example 1: Solve, by matrix method, the equations $x + y + z = 8$, $x - y + 2z = 6$, $3x + 5y - 7z = 14$.

Solution: The given equations are written in the matrix form as $AX = B$, ...(1)

where $$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 5 & -7 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ B = \begin{pmatrix} 8 \\ 6 \\ 14 \end{pmatrix}$$

Here $$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 5 & -7 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ 3 & 2 & -10 \end{vmatrix} = 20 - 2 = 18 \neq 0.$$

Scanned with CamScanner

Therefore, A^{-1} exists.

Now $$adj\ (A) = \begin{pmatrix} -3 & 13 & 8 \\ 12 & -10 & -2 \\ 3 & -1 & -2 \end{pmatrix}^T = \begin{pmatrix} -3 & 12 & 3 \\ 13 & -10 & -1 \\ 8 & -2 & -2 \end{pmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{18}\begin{pmatrix} -3 & 12 & 3 \\ 13 & -10 & -1 \\ 8 & -2 & -2 \end{pmatrix}$$

From (1), $$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B = \frac{1}{18}\begin{pmatrix} -3 & 12 & 3 \\ 13 & -10 & -1 \\ 8 & -2 & -2 \end{pmatrix}\begin{pmatrix} 8 \\ 6 \\ 14 \end{pmatrix}$$

$$= \frac{1}{18}\begin{pmatrix} 90 \\ 30 \\ 24 \end{pmatrix} = \begin{pmatrix} 5 \\ \frac{5}{3} \\ \frac{4}{3} \end{pmatrix}$$

Therefore, $x = 5,\ y = \frac{5}{3},\ z = \frac{4}{3}$.

Example 2: Find a matrix whose inverse is the matrix $A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 1 & -3 & 2 \end{pmatrix}$ and hence solve the equations

$$x - 2y + z = 1$$
$$2x + y - z = 2$$
$$x - 3y + 2z = 5$$

Solution: Since $(A^{-1})^{-1} = A$, the required matrix is A^{-1}.

Here $$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & -1 & 1 \end{vmatrix} (R_2 - 2R_1 \to R_2' \quad R_3 - R_1 \to R_3')$$

$$= 5 - 3 = 2 \neq 0.$$

Therefore, A^{-1} exists.

Now, $$adj\ (A) = \begin{pmatrix} -1 & -5 & -7 \\ 1 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix}^T = \begin{pmatrix} -1 & 1 & 1 \\ -5 & 1 & 3 \\ -7 & 1 & 5 \end{pmatrix}$$

Scanned with CamScanner

$$\therefore \quad A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ -5 & 1 & 3 \\ -7 & 1 & 5 \end{pmatrix}$$

The given system of equations can be written in the matrix form as

$AX = B$, where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\quad B = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$.

$$\therefore \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ -5 & 1 & 3 \\ -7 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 6 \\ 12 \\ 20 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix}$$

$$\therefore \quad x = 3,\ y = 6,\ z = 10.$$

Example 3: If $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$, show that $AB = 6I_3$.

Utilise this result to solve the following system of equations:

$$\begin{aligned} 2x + y + z &= 5 \\ x - y &= 0 \\ 2x + y - z &= 1 \end{aligned}$$

(*W.B.U.T. 2009*)

Solution: Here $AB = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 2+2+2 & 1-2+1 & 1+0-1 \\ 2-4+2 & 1+4+1 & 1+0-1 \\ 6+0-6 & 3+0-3 & 3+0+3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 6I_3$$

or $\quad \frac{1}{6}AB = I_3 \qquad \therefore \quad B^{-1} = \frac{1}{6}A = \frac{1}{6}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix}$

Scanned with CamScanner

The given system of equations can be written in the matrix form as

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \text{ or } B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5+0+1 \\ 5+0+1 \\ 15+0-3 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore \quad x = 1, y = 1, z = 2.$$

Example 4: Find the inverse of the matrix $\begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix}$. Hence solve the following system of equations:

$$2x - 3y + 4z = -4$$
$$x + z = 0$$
$$-y + 4z = 2$$

(W.B.U.T. 2005)

Solution: Let $A = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & 4 \end{vmatrix} \quad (C_3 - C_1 \to C_3^1)$$

$$= 10 \neq 0.$$

Therefore, A^{-1} exists.

Now,
$$adj\,(A) = \begin{pmatrix} 1 & -4 & -1 \\ 8 & 8 & 2 \\ -3 & 2 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 8 & -3 \\ -4 & 8 & 2 \\ -1 & 2 & 3 \end{pmatrix}$$

$$\therefore \quad A^{-1} = \frac{adj\,(A)}{|A|} = \frac{1}{10}\begin{pmatrix} 1 & 8 & -3 \\ -4 & 8 & 2 \\ -1 & 2 & 3 \end{pmatrix}$$

Scanned with CamScanner

The given system of equations can be written in the matrix form as $AX = B$,

where $$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ B = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix}$$

$\therefore$ $$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B = \frac{1}{10}\begin{pmatrix} 1 & 8 & -3 \\ -4 & 8 & 2 \\ -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -10 \\ 20 \\ 10 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$\therefore$ $x = -1,\ y = 2,\ z = 1.$

Example 5: Given the matrices $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$

Write down the linear equations given by $AX = B$ and solve for x, y, z by the matrix inversion method.

Solution: Here $AX = B$, ...(1)

where $$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ B = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

$\therefore$ $$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} \quad \text{or,} \quad \begin{pmatrix} x+y+z \\ x+2y+3z \\ x+4y+9z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

Using the definition of equality of two matrices, we have

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 3z &= 4 \\ x + 4y + 9z &= 6 \end{aligned}$$

These are the required linear equations.

Now, $$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 8 \end{vmatrix} = 8 - 6 = 2 \neq 0$$

Therefore, A^{-1} exists.

$$adj\ (A) = \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}^T = \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

Scanned with CamScanner

$$\therefore \qquad A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

From (1), we have

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B = \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 18-20+6 \\ -18+32-12 \\ 6-12+6 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore \qquad x = 2,\ y = 1,\ z = 0.$$

Example 6: Using the method of matrix inversion, solve the system

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} x & r \\ y & s \\ z & t \end{pmatrix} = \begin{pmatrix} 9 & 2 \\ 36 & 15 \\ 0 & -1 \end{pmatrix}.$$

Solution: Here $A\begin{pmatrix} x & r \\ y & s \\ z & t \end{pmatrix} = \begin{pmatrix} 9 & 2 \\ 36 & 15 \\ 0 & -1 \end{pmatrix}$, ...(1)

where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{pmatrix}$$

$$\therefore \qquad |A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 2 & -1 & -3 \end{vmatrix} = -9+5 = -4 \neq 0$$

Therefore, A^{-1} exists.

$$adj\ (A) = \begin{pmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & -5 & 3 \end{pmatrix}^T = \begin{pmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{pmatrix}$$

$$\therefore \qquad A^{-1} = \frac{adj(A)}{|A|} = -\frac{1}{4}\begin{pmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{pmatrix} \qquad ...(2)$$

Scanned with CamScanner

From (1), we have

$$\begin{pmatrix} x & r \\ y & s \\ z & t \end{pmatrix} = A^{-1}\begin{pmatrix} 9 & 2 \\ 36 & 15 \\ 0 & -1 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{pmatrix}\begin{pmatrix} 9 & 2 \\ 36 & 15 \\ 0 & -1 \end{pmatrix} \quad \text{[By (2)]}$$

$$= -\frac{1}{4}\begin{pmatrix} -36 & 4 \\ 36 & -8 \\ -36 & -4 \end{pmatrix} = \begin{pmatrix} 9 & -1 \\ -9 & 2 \\ 9 & 1 \end{pmatrix}$$

$\therefore$

$$x = 9, \quad r = -1$$
$$y = -9, \quad s = 2$$
$$z = 9, \quad t = 1.$$

3.8 CONSISTENCY AND INCONSISTENCY OF A SYSTEM OF HOMOGENEOUS AND INHOMOGENEOUS LINEAR SIMULTANEOUS EQUATIONS

Let us consider a system of m linear simultaneous equations in n number of variables $x_1, x_2, \ldots, x_n$ as stated below:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \ldots(3.8.1)$$

The above system of equations is said to be homogeneous or inhomogeneous (non-homogeneous) according as the constants $b_1, b_2, \ldots, b_m$ are all zero or at least one is non-zero.

The system of equation (3.8.1) is said to be consistent if it has at least one solution and it is said to be inconsistent if it has no solution.

Example: (*i*) The system $2x + 3y = 2$, $x + y = 1$ has an unique solution $x = 1, y = 0$. Therefore, this system is consistent. Geometrically, it represents a pair of straight lines in two dimensions, intersecting at the point (1, 0).

(*ii*) The system $2x + 3y = 2$, $4x + 6y = 4$ is consistent since (1, 0), *i.e.*, $x = 1, y = 0$, is a solution of this system. Observe that geometrically this system represents two coincident straight lines in two dimensions and therefore, it has infinitely many solutions, since all points on this straight line are solutions.

(*iii*) The system $2x + 3y = 2$, $2x + 3y = 3$ is inconsistent since it has no solution. Geometrically, this system represents a pair of parallel straight lines in two dimensions and therefore they have no intersecting point, *i.e.*, no solution.

Let us state two important theorems without proofs.

Scanned with CamScanner

Theorem 1 (Consistency of inhomogeneous system)

Let
$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \ldots(3.8.2)$$

be an inhomogeneous (*i.e.*, at least one of $b_1, b_2, \ldots, b_m$ is non-zero) system of m linear simultaneous equations in n variables $x_1, x_2, \ldots, x_n$.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ and $(A : B) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$

(*a*) The system (3.8.2) is consistent if and only if rank (A) = rank $(A : B)$

(*b*) If rank (A) = rank $(A : B) = n$ = number of unknowns, then the system (3.8.2) has unique solution.

(*c*) If rank (A) = rank $(A : B)$ < number of unknowns $(= n)$, then the sytem (3.8.2) has infinitely many solutions in terms of $(n - r)$ arbitrary constants.

Notes: (*i*) The matrices A and $(A : B)$ are known as coefficient matrix and augmented matrix respectively.

(*ii*) If rank $(A) \neq$ rank $(A : B)$, then system (3.8.2) is inconsistent, *i.e.*, it has no solution.

Theorem 2 (Existence of non-trivial solutions of homogeneous system)

Let
$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}\right\} \quad \ldots(3.8.3)$$

be a homogeneous system of m linear simultaneous equations in n variables $x_1, x_2, \ldots, x_n$.

Then this system has infinitely many non-trivial solutions [*i.e.*, other than (0, 0, ..., 0)] if and only if r = rank (A) < number of unknowns $(= n)$,

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, known as coefficient matrix. These non-trivial solutions can be expressed in terms of $(n - r)$ arbitrary constants.

Note: The homogeneous system (3.8.3) always has a trivial solution $x_1 = x_2 = \ldots = x_n = 0$.

Scanned with CamScanner

ILLUSTRATIVE EXAMPLES

Example 1: Examine whether the system of equations

$$3x_1 + 3x_2 + 2x_3 = 1$$
$$x_1 + 2x_2 = 4$$
$$10x_2 + 3x_3 = -2$$
$$2x_1 - 3x_2 - x_3 = 5$$

is consistent. Solve it, if it is consistent.

Solution: Here the coefficient matrix and augmented matrix are respectively

$$A = \begin{pmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{pmatrix} \text{ and } (A : B) = \begin{pmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{pmatrix}.$$

Now we see that A is a submatrix of $(A : B)$, so if we apply elementary row operations on $(A : B)$ then A will be automatically operated. Let us apply the following elementary row operations on $(A : B)$.

$$(A : B) = \begin{pmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{pmatrix} \xrightarrow[R_4-2R_1]{R_2-3R_1,} \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{pmatrix}$$

$$\xrightarrow[R_4-\frac{7}{3}R_2]{R_3+\frac{10}{3}R_2,} \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & \frac{29}{3} & \frac{-116}{3} \\ 0 & 0 & \frac{-17}{3} & \frac{68}{3} \end{pmatrix} \xrightarrow[\frac{3}{17}R_4]{\frac{3}{29}R_3,} \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -1 & 4 \end{pmatrix} \xrightarrow{R_4+R_3} \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} = C \text{ (say)}$$

Here C is an Echelon matrix with three non-zero rows and so, Rank $(A : B)$ = Rank (C) = 3 [as $(A : B)$ and C are row equivalent matrices]

Also A is row equivalent of $\begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, an Echelon matrix with three non-zero rows.

$\therefore$ rank $(A) = 3$.

Therefore, rank (A) = Rank $(A : B)$ = 3 = Number of unknowns.

So the given system of equations has unique solution.

Scanned with CamScanner

As augmented matrix is transformed to $C = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, the given system of equations is transformed to the following system:

$$x_1 + 2x_2 = 4$$
$$-3x_2 + 2x_3 = -11$$
$$x_3 = -4$$

From the last equation, we have $x_3 = -4$.

Putting this in 2nd, we get $x_2 = 1$.

Putting this in 1st, we get $x_1 = 2$.

So, the solution is $x_1 = 2$, $x_2 = 1$, $x_3 = -4$.

Example 2: Solve, if possible

$$x + y + z = 1$$
$$2x + y + 2z = 2$$
$$3x + 2y + 3z = 5$$

(*W.B.U.T. 2006, 2008*)

Solution: Here the coefficient matrix and augmented matrix are respectively

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \end{pmatrix} \text{ and } (A : B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 3 & 2 & 3 & 5 \end{pmatrix}$$

Let us apply the following elementary row operations on $(A : B)$ to get an Echelon matrix.

$$(A : B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 3 & 2 & 3 & 5 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1,} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = C \text{ (say)},$$

where C is an Echelon matrix with three non-zero rows. Therefore, Rank $(A : B)$ = Rank (C) = 3 [as $(A{:}B)$ and C are row equivalent].

Again the coefficient matrix A is row equivalent to $\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, an Echelon matrix having two non-zero rows, therefore rank $(A) = 2$.

Since rank $(A) \neq$ rank $(A : B)$, the given system of equations is inconsistent, *i.e.*, it cannot have any solution.

Example 3: Solve:

$$x_1 + x_2 - x_3 = 0$$
$$2x_1 - x_2 + x_3 = 3$$
$$4x_1 + 2x_2 - 2x_3 = 2$$

Solution: Here the coefficient matrix and augmented matrix are respectively

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 4 & 2 & -2 \end{pmatrix} \text{ and } (A : B) = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 3 \\ 4 & 2 & -2 & 2 \end{pmatrix}.$$

Let us apply the following elementary row operations on $(A : B)$ to get an Echelon matrix.

$$(A : B) = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 3 \\ 4 & 2 & -2 & 2 \end{pmatrix} \xrightarrow[R_3-4R_1]{R_2-2R_1,} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & -2 & 2 & 2 \end{pmatrix} \xrightarrow{R_3-\frac{2}{3}R_2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = C \text{ (say)},$$

where C is an Echelon matrix with two non-zero rows. Since $(A : B)$ and C are row equivalent, so rank $(A : B)$ = rank $(C) = 2$. Also the coefficient matrix A is row equivalent to $\begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, an Echelon matrix having two non-zero rows and hence rank $(A) = 2$.

Since rank (A) = rank $(A : B) = 2 <$ number of unknowns $(= 3)$, the given system is consistent but has infinitely many solutions in terms of $3 - 2 = 1$ arbitrary constant. Adding the first two equations, we have $3x_1 = 3$, or, $x_1 = 1$. Putting $x_1 = 1$ in the first equation, we get $x_2 = x_3 - 1$.

Therefore, the solutions are $x_1 = 1$, $x_2 = k - 1$, $x_3 = k$, where k is an arbitrary constant.

Example 4: Investigate for what values of λ and μ the following equations:

$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$
$$2x + 5y + \lambda z = \mu$$

has (*i*) no solution, (*ii*) unique solution, (*iii*) infinite number of solutions. Find the solutions in cases (*ii*) and (*iii*).

Solution: Here the coefficient matrix and augmented matrix are respectively

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & \lambda \end{pmatrix} \text{ and } (A : B) = \begin{pmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & \lambda & \mu \end{pmatrix}$$

Let us apply the following elementary row operations on $(A : B)$.

$$(A : B) = \begin{pmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & \lambda & \mu \end{pmatrix} \xrightarrow[R_3-2R_1]{R_2-R_1,} \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & \lambda-6 & \mu-12 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & \lambda-8 & \mu-15 \end{pmatrix}$$

Scanned with CamScanner

Case 1: $\lambda = 8$, $\mu \neq 15$. rank $(A) = 2$, rank $(A : B) = 3$. Therefore, rank $(A) \neq$ rank $(A : B)$ and hence the given system is inconsistent, *i.e.*, it has no solution.

Case 2: $\lambda \neq 8$, μ has any value. Here Rank $(A) =$ rank $(A : B) = 3 =$ number of unknowns. Therefore, the given system has unique solution.

The given system of equations is transformed to the following system:

$$x + 2y + 3z = 6$$
$$y + 2z = 3$$
$$(\lambda - 8)\ z = \mu - 15$$

$$\therefore \quad z = \frac{\mu - 15}{\lambda - 8},\ y = 3 - \frac{2(\mu - 15)}{\lambda - 8} = \frac{3\lambda - 2\mu + 6}{\lambda - 8},$$

$$x = 6 - \frac{2(3\lambda - 2\mu + 6)}{\lambda - 8} - \frac{3(\mu - 15)}{\lambda - 8} = \frac{\mu - 15}{\lambda - 8}.$$

Case 3: $\lambda = 8$, $\mu = 15$. In this case, rank $(A) =$ rank $(A : B) = 2 <$ number of unknowns $(= 3)$. Therefore, the given system has infinitely many solutions in terms of $3 - 2 = 1$ arbitrary constant.

The given system of equations is transformed to the following system:

$$x + 2y + 3z = 6$$
$$y + 2z = 3$$

$$\therefore \quad y = 3 - 2z,\ x = 6 - 2y - 3z = 6 - 2(3 - 2z) - 3z = z.$$

Therefore, the solutions are $x = k$, $y = 3 - 2k$, $z = k$, where k is an arbitrary constant.

Example 5: Solve the following homogeneous system of equations:

$$x_1 + x_2 - 2x_3 + 3x_4 = 0$$
$$x_1 - 2x_2 + x_3 - x_4 = 0$$
$$4x_1 + x_2 - 5x_3 + 8x_4 = 0$$
$$5x_1 - 7x_2 + 2x_3 - x_4 = 0$$

Solution: This system has a trivial solution $x_1 = x_2 = x_3 = x_4 = 0$.

Let us investigate whether it has infinitely many non-trivial solutions.

Here the coefficient matrix is $A = \begin{pmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{pmatrix}$

Let us apply the following elementary row operations on A to get an Echelon matrix.

$$A = \begin{pmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{pmatrix} \xrightarrow[R_4 - 5R_1]{R_2 - R_1,\ R_3 - 4R_1} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{pmatrix}$$

Scanned with CamScanner

$$\xrightarrow[R_4-4R_2]{R_3-R_2,}\begin{pmatrix}1 & 1 & -2 & 3\\0 & -3 & 3 & -4\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} = C \text{ (say)},$$

where C is an Echelon matrix having two non-zero rows, therefore rank $(C) = 2$. Since A and C are row equivalent, rank $(A) = 2 <$ number of unknowns ($= 4$). Therefore, the given system has infinitely many non-trivial solutions in terms of $4 - 2 = 2$ arbitrary constants.

The given system of equations is transformed to the following system:

$$x_1 + x_2 - 2x_3 + 3x_4 = 0$$
$$-3x_2 + 3x_3 - 4x_4 = 0$$

Choose $x_3 = k_1$ and $x_4 = k_2$, then $x_2 = \frac{1}{3}(3k_1 - 4k_2)$, $x_1 = 2k_1 - 3k_2 - \frac{1}{3}(3k_1 - 4k_2) = k_1 - \frac{5}{3}k_2$.

Therefore, the solutions are $x_1 = k_1 - \frac{5}{3}k_2$, $x_2 = k_1 - \frac{4}{3}k_2$, $x_3 = k_1$, $x_4 = k_2$, where k_1, k_2 are arbitrary constants.

Note: These solutions also include trivial solutions $x_1 = x_2 = x_3 = x_4 = 0$ for $k_1 = k_2 = 0$.

Example 6: Solve the following homogeneous system of equations

$$2x + y + 2z = 0$$
$$x + y + 3z = 0$$
$$4x + 3y + \lambda z = 0$$

for different values of λ.

Solution: Here the coefficient matrix is

$$A = \begin{pmatrix}2 & 1 & 2\\1 & 1 & 3\\4 & 3 & \lambda\end{pmatrix}.$$

Let us apply the following elementary row operations on A to get an echelon matrix.

$$A = \begin{pmatrix}2 & 1 & 2\\1 & 1 & 3\\4 & 3 & \lambda\end{pmatrix}\xrightarrow{R_{12}}\begin{pmatrix}1 & 1 & 3\\2 & 1 & 2\\4 & 3 & \lambda\end{pmatrix}\xrightarrow[R_3-4R_1]{R_2-2R_1,}\begin{pmatrix}1 & 1 & 3\\0 & -1 & -4\\0 & -1 & \lambda-12\end{pmatrix}\xrightarrow{R_3-R_2:}\begin{pmatrix}1 & 1 & 3\\0 & -1 & -4\\0 & 0 & \lambda-8\end{pmatrix}$$

Case 1: If $\lambda \neq 8$, then Rank $(A) = 3 =$ number of unknowns and hence $|A| \neq 0$, *i.e.*, A^{-1} exists. So the given system has trivial solution $x = y = z = 0$. It has no non-trivial solution.

Case 2: If $\lambda = 8$, then rank $(A) = 2 <$ number of unknowns ($= 3$). Therefore, in this case the given system has infinitely many solutions in terms of $3 - 2 = 1$ arbitrary constant.

Scanned with CamScanner

The given system of equations is transformed to the following system:

$$x + y + 3z = 0$$
$$-y - 4z = 0$$

$$\therefore \quad y = -4z, x = -y - 3z = 4z - 3z = z.$$

Therefore, the solutions are $x = k$, $y = -4k$, $z = k$, where k is an arbitrary constant.

Note that the trivial solution is obtained for $k = 0$ and infinite number of non-trivial solutions are obtained for different non-zero values of k.

Example 7: For what values of k, the following equations $x + y + z = 1$, $2x + y + 4z = k$, $4x + y + 10z = k^2$ have solutions and solve them completely in each case. [*W.B.U.T. 2003, B. Arch 2013*]

Solution: Here the coefficient matrix and augmented matrix are respectively

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{pmatrix} \text{ and } (A : B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & k \\ 4 & 1 & 10 & k^2 \end{pmatrix}$$

Let us apply the following elementary row operations on $(A : B)$ to get an Echelon matrix.

$$(A : B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & k \\ 4 & 1 & 10 & k^2 \end{pmatrix} \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & -3 & 6 & k^2-4 \end{pmatrix} \xrightarrow{R_3 - 3R_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & 0 & 0 & k^2-3k+2 \end{pmatrix}$$

Therefore, the coefficient matrix A is equivalent to Echelon matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ having two non-zero rows.

Hence rank $(A) = 2$ and also rank $(A : B) = 2$ if $k^2 - 3k + 2 = 0$, or, $(k - 1)(k - 2) = 0$, *i.e.*, $k = 1, 2$.

Therefore, the given system of equations has solutions only for $k = 1, 2$ and for these cases rank $(A) = $ Rank $(A : B) = 2 <$ number of unknowns (= 3), so it has infinitely many solutions in terms of $3 - 2 = 1$ arbitrary constant.

Case 1: $k = 1$. The given system of equations is transformed to the following system:

$$x + y + z = 1$$
$$-y + 2z = -1$$

$$\therefore \quad y = 1 + 2z, x = 1 - y - z = 1 - (1 + 2z) - z = -3z.$$

Hence the solutions are $x = -3a$, $y = 1 + 2a$, $z = a$, where a is an arbitrary constant.

Case 2: $k = 2$. The given system of equations is transformed to the following system:

$$x + y + z = 1$$
$$-y + 2z = 0$$

$$\therefore \quad y = 2z, \; x = 1 - y - z = 1 - 2z - z = 1 - 3z.$$

Scanned with CamScanner

Therefore, in this case the solutions are $x = 1 - 3b$, $y = 2b$, $z = b$, where b is an arbitrary constant.

Example 8: Determine the conditions under which the system of equations

$$x + y + z = 1$$
$$x + 2y - z = b$$
$$5x + 7y + az = b^2$$

admits of (*i*) only one solution, (*ii*) no solution (*iii*) many solutions (*W.B.U.T. 2009, 2010*)

Solution: Here the coefficient matrix and augmented matrix are respectively

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & a \end{pmatrix} \text{ and } (A : B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & b \\ 5 & 7 & a & b^2 \end{pmatrix}$$

Let us apply the following elementary row operations on $(A : B)$

$$(A : B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & b \\ 5 & 7 & a & b^2 \end{pmatrix} \xrightarrow[R_3 - 5R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & b-1 \\ 0 & 2 & a-5 & b^2-5 \end{pmatrix}$$

$$\xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & b-1 \\ 0 & 0 & a-1 & b^2-2b-3 \end{pmatrix} = C \text{ (say)}.$$

Case 1: $a \neq 1$, b has any value.

Here Rank (A) = rank $(A : B) = 3$ = number of unknowns, since C is an Echelon matrix having three non-zero rows. Therefore, the given system has only one solution.

Case 2: For having no solution, Rank $(A) \neq$ Rank $(A{:}B)$. This is possible if rank $(A) = 2$ and rank $(A : B) = 3$, *i.e.*, $a - 1 = 0$, or $a = 1$ and $b^2 - 2b - 3 \neq 0$, or $(b-3)(b+1) \neq 0$ or $b \neq 3$ and $b \neq -1$.

Therefore, the given system has no solution for $a = 1$, $b \neq 3$ or $b \neq -1$.

Case 3: $a = 1$, $b = 3$ or $b = -1$.

In this case $(A : B)$ is row equivalent to $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & b-1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore$ Rank (A) = rank $(A : B) = 2 <$ number of unknowns $(= 3)$. Therefore, in this case the given system has many solutions.

Scanned with CamScanner

Example 10: Investigate for what values of λ and μ the following equations:

$$x + 4y + 2z = 1$$
$$2x + 7y + 5z = 2\mu$$
$$4x + \lambda y + 10z = 2\mu + 1$$

has (*i*) unique solution, (*ii*) no solution, (*iii*) infinite number of solutions. Find the solutions in cases (*i*) and (*iii*).

Solution: Here the coefficient matrix and augmented matrix are respectively

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 7 & 5 \\ 4 & \lambda & 10 \end{pmatrix} \text{ and } (A : B) = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 2 & 7 & 5 & 2\mu \\ 4 & \lambda & 10 & 2\mu + 1 \end{pmatrix}$$

Let us apply the following elementary row operations on $(A : B)$ to get an Echelon matrix.

$$(A : B) = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 2 & 7 & 5 & 2\mu \\ 4 & \lambda & 10 & 2\mu + 1 \end{pmatrix} \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 2\mu - 2 \\ 0 & \lambda - 16 & 2 & 2\mu - 3 \end{pmatrix} \xrightarrow{R_3 + (\lambda - 16)R_2}$$

$$\begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 2\mu - 2 \\ 0 & 0 & \lambda - 14 & 2\mu - 3 + (\lambda - 16)(2\mu - 2) \end{pmatrix} = C \text{ (say)}.$$

Case 1: $\lambda \neq 14$, μ has any value. Here rank (A) = rank $(A : B) = 3$ = number of unknowns. Therefore, the given system has unique solution.

The given system of equations is transformed to the following system

$$x + 4y + 2z = 1$$
$$-y + z = 2\mu - 2$$
$$(\lambda - 14)z = 2\mu - 3 + (\lambda - 16)(2\mu - 2)$$

$$\therefore \quad z = \{2\mu - 3 + (\lambda - 16)(2\mu - 2)\}/(\lambda - 14),$$
$$y = z - (2\mu - 2) = (-2\mu + 1)/(\lambda - 14),$$
$$x = 1 - 4y - 2z = \{4\mu + 4 + (\lambda - 16)(5 - 4\mu)\}/(\lambda - 14).$$

Case 2: For having no solution, rank $(A) \neq$ rank $(A : B)$. This is possible if rank $(A) = 2$ and rank $(A : B) = 3$, *i.e.*, $\lambda - 14 = 0$, or, $\lambda = 14$ and $2\mu - 3 + (\lambda - 16)(2\mu - 2) \neq 0$, or $2\mu - 3 + (14 - 16)(2\mu - 2) \neq 0$, or, $\mu \neq \frac{1}{2}$.

Therefore, the given system has no solution for $\lambda = 14$ and $\mu \neq \frac{1}{2}$

Example 9: Choose λ that makes the following system of linear equations consistent and find the general solution of the system for that λ.

$x+y-z+t=2,\ 2y+4z+2t=3,\ x+2y+z+2t=\lambda.$

Solution: Here the coefficient matrix and augmented matrix are respectively

$$A=\begin{pmatrix}1&1&-1&1\\0&2&4&2\\1&2&1&2\end{pmatrix} \text{ and } (A:B)=\begin{pmatrix}1&1&-1&1&2\\0&2&4&2&3\\1&2&1&2&\lambda\end{pmatrix}$$

Let us apply the following elementary row operations on $(A : B)$ to get an Echelon matrix.

$$(A:B)=\begin{pmatrix}1&1&-1&1&2\\0&2&4&2&3\\1&2&1&2&\lambda\end{pmatrix}\xrightarrow{R_3-R_1}\begin{pmatrix}1&1&-1&1&2\\0&2&4&2&3\\0&1&2&1&\lambda-2\end{pmatrix}$$

$$\xrightarrow{R_3-\frac{1}{2}R_2}\begin{pmatrix}1&1&-1&1&2\\0&2&4&2&3\\0&0&0&0&\lambda-\frac{7}{2}\end{pmatrix}=C\text{ (say)}.$$

Therefore, the given system is consistent if $\lambda=\frac{7}{2}$ and in this case Rank (A) = rank $(A : B)$ = rank $(C) = 2 <$ number of unknowns (= 4). Hence for $\lambda=\frac{7}{2}$, the given system has infinitely many solutions in terms of $4-2=2$ arbitrary constants and the given system of equations is transformed to the following system:

$$x+y-z+t=2$$
$$2y+4z+2t=3$$

Choose $z=k_2$ and $t=k_1$, then $y=\frac{3}{2}-2k_2-k_1$, $x=2-y+z-t=2-\left(\frac{3}{2}-2k_2-k_1\right)+k_2-k_1$ $=\frac{1}{2}+3k_2$.

Therefore, for $\lambda=\frac{7}{2}$ the given system of linear equations is consistent and it has infinitely many solutions, namely

$x=\frac{1}{2}+3k_2$, $y=\frac{3}{2}-2k_2-k_1$, $z=k_2$, $t=k_1$, where k_1, k_2 are two arbitrary constants.

Scanned with CamScanner

Example 10: Investigate for what values of λ and μ the following equations:

$$x + 4y + 2z = 1$$
$$2x + 7y + 5z = 2\mu$$
$$4x + \lambda y + 10z = 2\mu + 1$$

has (*i*) unique solution, (*ii*) no solution, (*iii*) infinite number of solutions. Find the solutions in cases (*i*) and (*iii*).

Solution: Here the coefficient matrix and augmented matrix are respectively

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 7 & 5 \\ 4 & \lambda & 10 \end{pmatrix} \text{ and } (A : B) = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 2 & 7 & 5 & 2\mu \\ 4 & \lambda & 10 & 2\mu+1 \end{pmatrix}.$$

Let us apply the following elementary row operations on $(A : B)$ to get an Echelon matrix.

$$(A : B) = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 2 & 7 & 5 & 2\mu \\ 4 & \lambda & 10 & 2\mu+1 \end{pmatrix} \xrightarrow[R_3-4R_1]{R_2-2R_1} \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 2\mu-2 \\ 0 & \lambda-16 & 2 & 2\mu-3 \end{pmatrix} \xrightarrow{R_3+(\lambda-16)R_2}$$

$$\begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 2\mu-2 \\ 0 & 0 & \lambda-14 & 2\mu-3+(\lambda-16)(2\mu-2) \end{pmatrix} = C \text{ (say)}.$$

Case 1: $\lambda \neq 14$, μ has any value. Here rank (A) = rank $(A : B) = 3$ = number of unknowns. Therefore, the given system has unique solution.

The given system of equations is transformed to the following system:

$$x + 4y + 2z = 1$$
$$-y + z = 2\mu - 2$$
$$(\lambda-14)z = 2\mu - 3 + (\lambda-16)(2\mu-2)$$

$$\therefore \quad z = \{2\mu - 3 + (\lambda-16)(2\mu-2)\}/(\lambda-14).$$
$$y = z - (2\mu - 2) = (-2\mu+1)/(\lambda-14).$$
$$x = 1 - 4y - 2z = \{4\mu + 4 + (\lambda-16)(5-4\mu)\}/(\lambda-14).$$

Case 2: For having no solution, rank $(A) \neq$ rank $(A : B)$. This is possible if rank $(A) = 2$ and rank $(A : B) = 3$, *i.e.*, $\lambda - 14 = 0$, or, $\lambda = 14$ and $2\mu - 3 + (\lambda-16)(2\mu-2) \neq 0$, or $2\mu - 3 + (14-16)(2\mu-2) \neq 0$, or, $\mu \neq \dfrac{1}{2}$.

Therefore, the given system has no solution for $\lambda = 14$ and $\mu \neq \dfrac{1}{2}$.

Scanned with CamScanner

Case 3: $\lambda = 14$, $\mu = \frac{1}{2}$. In this case $(A : B)$ is row equivalent to $\begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Therefore, rank (A) = rank $(A : B) = 2 <$ number of unknowns (= 3). Therefore, in this case the given system has infinitely many solutions in terms of $3 - 2 = 1$ arbitrary constant.

The given system of equations is transformed to the following system:

$$x + 4y + 2z = 1$$
$$-y + z = -1$$
$$\therefore \quad y = 1 + z,\ x = 1 - 4y - 2z = 1 - 4(1 + z) - 2z = -6z - 3$$
$$\therefore \quad x = -6k - 3,\ y = k + 1,\ z = k,$$

where k is an arbitrary constant.

3.9 EIGEN VALUES AND EIGEN VECTORS

Definition: Let A be a square matrix of order n. A number (real or complex) λ is called an **eigen value** of A if there exists a non-zero (or non-null) column vector X such that $AX = \lambda X$ and then X is called an **eigen vector** corresponding to the eigen value λ.

Notes: (*i*) If X be an eigen vector corresponding to the eigen value λ of A, then αX, where α is any non-zero number, is also an eigen vector corresponding to eigen value λ of A.

(*ii*) Eigen value is also known as characteristic root (value) or latent root and eigen vector is also called characteristic vector or latent vector.

Example: Now, $\begin{pmatrix} 1 & 6 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $(AX = \lambda X)$.

This shows that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigen vector of $\begin{pmatrix} 1 & 6 \\ 2 & 5 \end{pmatrix}$ corresponding to the eigen value 7.

3.10 CHARACTERISTIC POLYNOMIAL AND CHARACTERISTIC EQUATION

Let $X = (x_1\ x_2\ \ldots\ x_n)^T \neq O$ be an eigen vector corresponding to the eigen value λ of $A = (a_{ij})_{n\times n}$. Then $AX = \lambda X$ so that $(A - \lambda I)X = O$, a null matrix. Hence $|A - \lambda I| = 0$, otherwise $|A - \lambda I| \neq 0$, *i.e.*, $(A - \lambda I)^{-1}$ exists and we get $X = O$, which is a contradiction.

Now, $$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{pmatrix}.$$

Scanned with CamScanner

Therefore, $|A-\lambda I| = 0$ gives

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0 \qquad \text{...(1)}$$

The expression in the left side of (1) is a polynomial in λ of degree n. This polynomial is called **characteristic polynomial** of A and the equation (1) is known as the **characteristic equation** of A. Since (1) is a polynomial equation in λ of degree n so it has n number of roots, say $\lambda_1, \lambda_2, \dots, \lambda_n$. These roots are known as characteristic roots (values) or eigen values or latent roots.

Theorem: The constant term, *i.e.*, the term independent of λ, of the characteristic polynomial of a square matrix A is equal to $|A|$.

Proof: The characteristic polynomial of a matrix $A = (a_{ij})_{n \times n}$ is

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}.$$

Since this is an nth order determinant, so after expansion the polynomial will be of the form $a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + a_n$, where a_n is the constant term.

$$\therefore \quad \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} \equiv a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n.$$

It is an identity and so it is satisfied for all values of λ.

Putting $\lambda = 0$, we get $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = a_n$,

i.e., $a_n = |A|$. This completes the proof.

Example: Find the eigen values and eigen vectors of the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}.$$

Scanned with CamScanner

Solution: The characteristic equation of A is $|A-\lambda I| = 0$

or $$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

or $$(1-\lambda)\{(5-\lambda)(1-\lambda)-1\}-(1-\lambda-3)+3\{1-3(5-\lambda)\} = 0$$

or $$(1-\lambda)(\lambda^2-6\lambda+4)+(\lambda+2)+3(3\lambda-14) = 0$$

or $$\lambda^2-6\lambda+4-\lambda^3+6\lambda^2-4\lambda+10\lambda-40 = 0$$

or $$-\lambda^3+7\lambda^2-36 = 0$$

or $$\lambda^3-7\lambda^2+36 = 0$$

or $$\lambda^3+2\lambda^2-9\lambda^2-18\lambda+18\lambda+36 = 0$$

or $$\lambda^2(\lambda+2)-9\lambda(\lambda+2)+18(\lambda+2) = 0$$

or $$(\lambda+2)(\lambda^2-9\lambda+18) = 0$$

or $$(\lambda+2)(\lambda^2-6\lambda-3\lambda+18) = 0$$

or $$(\lambda+2)\{(\lambda(\lambda-6)-3(\lambda-6)\} = 0$$

or $$(\lambda+2)(\lambda-6)(\lambda-3) = 0$$

$\therefore$ $\lambda = -2, 3, 6$ are the three eigen values of A.

Case 1: Eigen vectors corresponding to $\lambda = -2$

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to $\lambda = -2$, then $AX = -2X$

or $$(A+2I)X = 0$$

or $$\begin{pmatrix} 1+2 & 1 & 3 \\ 1 & 5+2 & 1 \\ 3 & 1 & 1+2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or
$$3x_1 + x_2 + 3x_3 = 0 \quad \ldots(1)$$
$$x_1 + 7x_2 + x_3 = 0 \quad \ldots(2)$$
$$3x_1 + x_2 + 3x_3 = 0 \quad \ldots(3)$$

Obviously, (1) and (3) are identical. From (1) and (2), we get $\dfrac{x_1}{-20} = \dfrac{x_2}{0} = \dfrac{x_3}{20}$, or $\dfrac{x_1}{-2} = \dfrac{x_2}{0}$

$= \dfrac{x_3}{2} = k$ (say).

Scanned with CamScanner

$\therefore\ x_1 = -2k,\ x_2 = 0,\ x_3 = 2k$, where k is any non-zero number. Thus the eigen vectors are $(-2k,\ 0,\ 2k)^T$, *e.g.*, $(-2,\ 0,\ 2)^T$ by putting $k = 1$.

Case 2: Eigen vectors corresponding to $\lambda = 3$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to $\lambda = 3$, then $AX = 3X$, or $(A - 3I)\,X = 0$

or $$\begin{pmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or $$\begin{aligned} -2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \\ 3x_1 + x_2 - 2x_3 &= 0 \end{aligned}$$

Here two equations are independent. Taking the first two equations, we get

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}, \text{ or } \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1} = k \text{ (say)}.$$

$\therefore\ x_1 = k,\ x_2 = -k,\ x_3 = k$, where k is any non-zero number. Thus the eigen vectors are $(k,\ -k,\ k)^T$, *e.g.*, $(1, -1, 1)^T$ by putting $k = 1$.

Case 3: Eigen vectors corresponding to $\lambda = 6$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to $\lambda = 6$, the $AX = 6X$,

or $$(A - 6I)\,X = 0$$

or $$\begin{pmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

or $$\begin{aligned} -5x_1 + x_2 + 3x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 - 5x_3 &= 0 \end{aligned}$$

Here two equations are independent. Taking the first two equations, we get

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}, \text{ or } \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1} = k \text{ (say)}.$$

$\therefore\ x_1 = k,\ x_2 = 2k,\ x_3 = k$, where k is any non-zero number. Thus the eigen vectors are $(k,\ 2k,\ k)^T$, *e.g.*, $(1,\ 2,\ 1)^T$ by putting $k = 1$.

Scanned with CamScanner

Observation: The characteristic polynomial of $A = (a_{ij})_{n \times n}$ is

$$f(\lambda) = |A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$$= (-1)^n (\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_n)$$

where the coefficients $p_1, p_2, ..., p_n$ are in terms of a_{ij}, $i = 1, 2, ..., n$; $j = 1, 2, ..., n$

Therefore, if $\alpha_1, \alpha_2, ..., \alpha_n$ are eigen values of $A = (a_{ij})_{n\times n}$, then $|A - \lambda I| = (\alpha_1 - \lambda)(\alpha_2 - \lambda)\ldots(\alpha_n - \lambda)$, since the coefficient of λ^n in left side is $(-1)^n$.

3.11 PROPERTIES OF EIGEN VALUES

Property 1: zero is an eigen value of a square matrix A if and only if A is singular.

Proof: The characteristic polynomial of A is $f(\lambda) = |A - \lambda I|$. So, characteristic equation of A is

$f(\lambda) = |A - \lambda I| = 0.$ ...(1)

Obviously, the constant term in this equation is $f(0) = |A|$.

Now, if A is a singular matrix then $f(0) = |A| = 0$ and hence 0 is a root of (1), *i.e.*, 0 is an eigen value of A.

Conversely, if 0 is an eigen value of A, *i.e.*, a root of (1), then $f(0) = |A| = 0$, *i.e.*, A is singular.

Ex. If the characteristic equation of a matrix A is $x^3 + 3x^2 + 5x + 9 = 0$, then $|A| = 9$.

(*W.B.U.T. 2012*)

Property 2: The eigen values of a diagonal matrix are its diagonal elements, *i.e.*, the elements of its principal diagonal.

Proof: Let us consider an nth order diagonal matrix

$$D_n = \begin{pmatrix} d_1 & 0 & 0 & \ldots & 0 \\ 0 & d_2 & 0 & \ldots & 0 \\ 0 & 0 & d_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & d_n \end{pmatrix}.$$

Its characteristic equation is $|D_n - \lambda I_n| = 0$,

or
$$\begin{vmatrix} d_1-\lambda & 0 & 0 & \ldots & 0 \\ 0 & d_2-\lambda & 0 & \ldots & 0 \\ 0 & 0 & d_3-\lambda & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & d_n-\lambda \end{vmatrix} = 0$$

Scanned with CamScanner

or $$(d_1 - \lambda)(d_2 - \lambda)(d_3 - \lambda)\ldots(d_n - \lambda) = 0$$

or $$\lambda = d_1, d_2, d_3, \ldots, d_n.$$

Hence, the eigen values of the diagonal matrix D_n are its diagonal elements.

Property 3: If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the eigen values of an nth order square matrix A and k is a number, then $(\alpha_1 - k), (\alpha_2 - k), \ldots, (\alpha_n - k)$ are the eigen values of the matrix $A - kI$, where I is an nth order identity matrix.

Proof: Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the eigen values of A, therefore

$$|A - \lambda I| = (\alpha_1 - \lambda)(\alpha_2 - \lambda)\ldots(\alpha_n - \lambda) \quad \ldots(1)$$

[$\because$ coefficient of λ^n in left side in $(-1)^n$]

Let $$B = A - kI$$

$\therefore$ $$|B - \lambda I| = |(A - kI) - \lambda I| = |B - \lambda' I| \quad (\text{where } \lambda' = k + \lambda) \quad \ldots(2)$$

Since (1) is an identity, therefore it is true for all values of λ and hence

$$|A - \lambda' I| = (\alpha_1 - \lambda')(\alpha_2 - \lambda')\ldots(\alpha_n - \lambda')$$

or $$|B - \lambda I| = \{\alpha_1 - (k + \lambda)\}\{\alpha_2 - (k + \lambda)\}\ldots\{\alpha_n - (k + \lambda)\}$$

[By (2) and since $\lambda' = k + \lambda$]

or $$|B - \lambda I| = \{(\alpha_1 - k) - \lambda\}\{(\alpha_2 - k) - \lambda\}\ldots\{(\alpha_n - k) - \lambda\}$$

This shows that $(\alpha_1 - k), (\alpha_2 - k), \ldots, (\alpha_n - k)$ are the eigen values of $B = A - kI$.

Property 4: The eigen values of a square matrix A and its transpose A^T are same.

Proof: Let A be an nth order square matrix, then the eigen values of A are the roots of

$$f(\lambda) = |A - \lambda I| = 0 \quad \ldots(1)$$

Since a determinant remains unaltered for interchange of rows and columns, therefore,

$$|A - \lambda I| = |(A - \lambda I)^T| = |A^T - \lambda I^T| = |A^T - \lambda I| \quad \ldots(2)$$

[$\because$ $(X + Y)^T = X^T + Y^T$ and $I^T = I$]

From (1) and (2), we have $f(\lambda) = |A - \lambda I| = |A^T - \lambda I| = 0$ which shows that the eigen values of a square matrix A and A^T are same.

Property 5: If A be a square matrix then the eigen values of kA are k times the eigen values of A, where k is a number.

Proof: Let X be an eigen vector of a square matrix A corresponding to the eigen value λ, then

$$AX = \lambda X \quad \ldots(1)$$

Now, $$(kA)X = k(AX) = k(\lambda X) \quad \text{[By (1)]}$$
$$= (k\lambda)X$$

Therefore, $k\lambda$ is an eigen value of kA and hence the eigen values of kA are k times the eigen values of A.

Scanned with CamScanner

Property 8: Let A be a square matrix, then

(*i*) sum of the eigen values of $A = Tr(A)$

(*ii*) product of the eigen values of $A = |A|$.

Proof: Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$.

The eigen values of A are obtained from the characteristic equation

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \ldots(1)$$

Let $\quad |A-\lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n \quad \ldots(2)$

Putting $\lambda = 0$, we get $|A| = a_n \quad \ldots(3)$

Also expanding $|A-\lambda I|$ with respect to first row we get

$$|A-\lambda I| = (-1)^n\lambda^n + (-1)^{n-1}(a_{11}+a_{22}+\ldots+a_{nn})\lambda^{n-1} + \ldots \quad \ldots(4)$$

Comparing (2) and (4), we get

$$a_0 = (-1)^n,\ a_1 = (-1)^{n-1}(a_{11}+a_{22}+\ldots+a_{nn}),\ \ldots \quad \ldots(5)$$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A, *i.e.*, the roots of (1), then sum of the eigen values = Sum of the roots

$$= \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

$$= -\frac{a_1}{a_0} = a_{11} + a_{22} + \ldots + a_{nn} \quad \text{[By (5)]}$$

$$= Tr(A).$$

Product of the eigen values = Product of the roots

$$= \lambda_1\lambda_2\ldots\lambda_n$$

$$= (-1)^n\frac{a_n}{a_0} = \frac{(-1)^n a_n}{(-1)^n}$$

$$= a_n = |A| \quad \text{[By (3)]}$$

Ex. If the two eigen values of the matrix $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ are 2 and -2, then the third eigen value is 2. (*W.B.U.T. 2012*)

Property 6: If λ is an eigen value of a non-singular matrix A then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

(*W.B.U.T. 2006*)

Proof: Let X be an eigen vector of a non-singular matrix A corresponding to the eigen value λ, then

$$AX = \lambda X \qquad ...(1)$$

Since A is non-singular, A^{-1} exists. Premultiplying both sides of (1) by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

or
$$(A^{-1}A)X = \lambda A^{-1}X$$

or
$$IX = \lambda A^{-1}X$$

$\therefore$
$$A^{-1}X = \frac{1}{\lambda}X.$$

Hence $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Property 7: If λ be an eigen value of A then λ^k is an eigen value of A^k where k is any positive integer.

Proof: Let X be an eigen vector of a square matrix A corresponding to the eigen value λ.

$\therefore$
$$AX = \lambda X \qquad ...(1)$$

$\therefore$
$$\begin{aligned} A^2X &= (AA)X = A(AX) = A(\lambda X) && \text{[by (1)]} \\ &= \lambda(AX) \\ &= \lambda(\lambda X) && \text{[by (1)]} \\ &= \lambda^2 X. \end{aligned}$$

Therefore, λ^2 is an eigen value of A^2.

Hence the result is true for $k = 1, 2$. Let us assume that the result is true for a positive integer m.

$\therefore$
$$A^mX = \lambda^m X \qquad \text{(By assumption)} \qquad ...(2)$$

$\therefore$
$$\begin{aligned} A^{m+1}X &= (AA^m)X = A(A^m X) \\ &= A(\lambda^m X) && \text{[By (2)]} \\ &= \lambda^m (AX) \\ &= \lambda^m (\lambda X) && \text{[By (1)]} \\ &= \lambda^{m+1} X. \end{aligned}$$

Therefore, if the result is true for a positive integer m, then it is also true for $m + 1$. But we have already seen that the result holds good for 1, 2, therefore it is true for $2 + 1 = 3$, $3 + 1 = 4$, $4 + 1 = 5$, etc., *i.e.*, the result is true for any positive integer k.

Scanned with CamScanner

Property 8: Let A be a square matrix, then

(*i*) sum of the eigen values of $A = Tr(A)$

(*ii*) product of the eigen values of $A = |A|$.

Proof: Let $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$.

The eigen values of A are obtained from the characteristic equation

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0 \qquad \dots(1)$$

Let $$|A-\lambda I| = a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_n \qquad \dots(2)$$

Putting $\lambda = 0$, we get $|A| = a_n$ ...(3)

Also expanding $|A-\lambda I|$ with respect to first row we get

$$|A-\lambda I| = (-1)^n \lambda^n + (-1)^{n-1}(a_{11}+a_{22}+\dots+a_{nn})\lambda^{n-1} + \dots \qquad \dots(4)$$

Comparing (2) and (4), we get

$$a_0 = (-1)^n,\ a_1 = (-1)^{n-1}(a_{11}+a_{22}+\dots+a_{nn}), \dots \qquad \dots(5)$$

If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigen values of A, *i.e.*, the roots of (1), then sum of the eigen values = Sum of the roots

$$= \lambda_1 + \lambda_2 + \dots + \lambda_n$$

$$= -\frac{a_1}{a_0} = a_{11} + a_{22} + \dots + a_{nn} \qquad \text{[By (5)]}$$

$$= Tr(A).$$

Product of the eigen values = Product of the roots

$$= \lambda_1 \lambda_2 \dots \lambda_n$$

$$= (-1)^n \frac{a_n}{a_0} = \frac{(-1)^n a_n}{(-1)^n}$$

$$= a_n = |A| \qquad \text{[By (3)]}$$

Ex. If the two eigen values of the matrix $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ are 2 and –2, then the third eigen value is 2. *(W.B.U.T. 2012)*

Property 9: If λ is an eigen value of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of $adj\,(A)$.

Proof: Since λ is an eigen value of a non-singular matrix, therefore $\lambda \neq 0$ (By Property 1). Also λ is an eigen value of A implies that there exists a non-null vector X such that $AX = \lambda X$.

$\therefore$ $$adj(A)(AX) = adj(A)(\lambda X)$$

or $$\{adj(A)A\}X = \lambda\, adj(A)X$$

or $$|A| IX = \lambda\, adj(A)X \qquad [\because\ adj\,(A)\,A = |A|\,I]$$

or $$|A|\,X = \lambda\, adj\,(A)\,X$$

$\therefore$ $$adj\,(A)\,X = \frac{|A|}{\lambda}X \qquad (\because\ \lambda \neq 0)$$

Since X is a non-zero vector, therefore $\frac{|A|}{\lambda}$ is an eigen value of the matrix $adj\,(A)$.

3.12 PROPERTIES OF EIGEN VECTORS

Condition of Orthogonality of Two Vectors

Two vectors X_1 and X_2 of same order, say $n \times 1$, are said to be **orthogonal** if $X_1^T X_2 = O$, *i.e.*, if

$$X_1 = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$ then condition of orthogonality is $a_1 b_1 + a_2 b_2 + \ldots + a_n b_n = 0$.

Property 1: If A be a square matrix, then there exists a unique eigen value corresponding to each eigen vector of A.

Proof: If possible let there are two distinct eigen values λ_1, λ_2, *i.e.*, $\lambda_1 \neq \lambda_2$, corresponding to an eigen vector X of a square matrix A. Then $AX = \lambda_1 X$ and $AX = \lambda_2 X$, subtracting we get $(\lambda_1 - \lambda_2)X = O$, a null matrix. This is a contradiction since X is a non-null vector and $\lambda_1 \neq \lambda_2$. Hence $\lambda_1 = \lambda_2$, therefore there exists a unique eigen value corresponding to each eigen vector of a square matrix A.

Property 2: The eigen vectors corresponding to distinct eigen values of a symmetric matrix are orthogonal.

Proof: Let A be a symmetric matrix and X_1, X_2 are two eigen vectors of A corresponding to two distinct eigen values λ_1, λ_2 respectively. Therefore, we have

$$A^T = A \qquad \ldots(1)$$

$$AX_1 = \lambda_1 X_1 \qquad \ldots(2)$$

and $$AX_2 = \lambda_2 X_2 \qquad \ldots(3)$$

Scanned with CamScanner

From (2), we have $(AX_1)^T = (\lambda_1 X_1)^T$

or $\quad X_1^T A^T = \lambda_1 X_1^T$

or $\quad X_1^T A = \lambda_1 X_1^T$ [By (1)]

Post-multiplying by X_2, we get

$$X_1^T AX_2 = \lambda_1 X_1^T X_2, \text{ or } X_1^T \lambda_2 X_2 = \lambda_1 X_1^T X_2 \quad \text{[by (3)]}$$

or $\lambda_2 X_1^T X_2 = \lambda_1 X_1^T X_2$, or $(\lambda_2 - \lambda_1) X_1^T X_2 = O$.

But $\lambda_1 \neq \lambda_2$, therefore $X_1^T X_2 = O$, *i.e.*, X_1 and X_2 are orthogonal.

Property 3: The modulus of each eigen value of an orthogonal matrix is unity.

Proof: Let A be an orthogonal matrix, then

$$AA^T = A^TA = I \quad ...(1)$$

Let λ be an eigen value of A corresponding to the eigen vector X.

$\therefore \quad AX = \lambda X \quad ...(2)$

Taking transpose of both sides of (2), we get

$$(AX)^T = (\lambda X)^T, \text{ or } X^T A^T = \lambda X^T \quad ...(3)$$

From (2) and (3), we get

$$X^T A^T AX = \lambda X^T \lambda X, \text{ or } X^T IX = \lambda^2 X^T X \quad \text{[by (1)]}$$

or $X^T X = \lambda^2 X^T X$, or $(1-\lambda^2) X^T X = O$

$\therefore \quad 1 - \lambda^2 = O \quad$ ($\because$ X is non-null because it is an eigen vector)

$\therefore \quad |\lambda| = 1.$

3.13 CAYLEY-HAMILTON THEOREM

Any square matrix A satisfies its own characteristic equation.

Proof: Let $|A - xI| = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n \quad ...(1)$

be the characteristic polynomial of an nth order square matrix A.

Now, $adj(A - xI)$ is a matrix polynomial of degree $(n - 1)$ since each element of the matrix $adj(A - xI)$ is a cofactor of $|A - xI|$ and hence each element of $adj(A - xI)$ is a polynomial of degree $\leq n - 1$.

Therefore, let $adj(A - xI) = B_0 + B_1 x + B_2 x^2 + \ldots + B_{n-2} x^{n-2} + B_{n-1} x^{n-1} \quad ...(2)$

Now, $(A - xI)\, adj\,(A - xI) = |A - xI|\, I \quad [\because\ C\, adj\,(C) = adj\,(C)\, C = |C|\, I]$

$$\therefore \quad (A - xI)(B_0 + B_1 x + B_2 x^2 + \ldots + B_{n-2} x^{n-2} + B_{n-1} x^{n-1})$$
$$= (a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n) I \quad \text{[by (1) and (2)]}$$

Scanned with CamScanner

This is an identity and so equating the coefficients of like powers of x from both sides, we get

$$AB_0 = a_0 I$$
$$AB_1 - B_0 = a_1 I$$
$$AB_2 - B_1 = a_2 I$$
$$\cdots\cdots\cdots\cdots$$
$$AB_{n-1} - B_{n-2} = a_{n-1} I$$
$$-B_{n-1} = a_n I$$

Pre-multiplying the above equations by $I, A, A^2, \ldots, A^n$ respectively, we get

$$\left.\begin{aligned} AB_0 &= a_0 I \\ A^2 B_1 - AB_0 &= a_1 A \\ A^3 B_2 - A^2 B_1 &= a_2 A^2 \\ &\cdots\cdots\cdots \\ A^n B_{n-1} - A^{n-1} B_{n-2} &= a_{n-1} A^{n-1} \\ -A^n B_{n-1} &= a_n A^n \end{aligned}\right| \quad \ldots(3)$$

Adding all the euations of (3), we get

$$a_0 I + a_1 A + a_2 A^2 + \ldots + a_n A^n = O$$

This completes the proof.

Note: The inverse of a non-singular matrix A can be calculated by using Cayley-Hamilton theorem as follows:

Let $|A - \lambda I| = a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n$ be the characteristic polynomial of A, then by Cayley-Hamilton theorem

$$a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = O. \quad \ldots(1)$$

Now, putting $\lambda = 0$ in $|A - \lambda I| = a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n$, we get $|A| = a_n \neq 0$ (since A is non-singular).

$$\therefore \quad I = -\frac{1}{a_n}(a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_{n-2} A^2 + a_{n-1} A) \quad \text{[by (1)]}$$

$$\therefore \quad A^{-1} = -\frac{1}{a_n}(a_0 A^{n-1} + a_1 A^{n-2} + \ldots + a_{n-2} A + a_{n-1} I) \quad \text{[multiplying both sides by } A^{-1}\text{]}$$

Example: Using Cayley-Hamilton theorem find the inverse of the matrix $\begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}$.

Solution: Let $A = \begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}$.

Scanned with CamScanner

The characteristic polynomial of A is

$$|A-\lambda I| = \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = (7-\lambda)\{(1+\lambda)^2-4\}-2\{6(1+\lambda)-12\}-2\{-12+6(1+\lambda)\}$$

$$= (7-\lambda)(\lambda^2+2\lambda-3)-24\lambda+24 = -\lambda^3+5\lambda^2-7\lambda+3.$$

By Cayley-Hamilton theorem

$$-A^3+5A^2-7A+3I = O, \text{ or } I = \frac{1}{3}(A^3-5A^2+7A).$$

Multiplying both sides by A^{-1}, we get

$$A^{-1} = \frac{1}{3}(A^2-5A+7I) \qquad \ldots(1)$$

Now, $$A^2 = AA = \begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}\begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{pmatrix}$$

Therefore, from (1),

$$A^{-1} = \frac{1}{3}\left[\begin{pmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{pmatrix} - 5\begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix} + 7\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{3}\begin{pmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{pmatrix}$$

ILLUSTRATIVE EXAMPLES

Example 1: If $X_1, X_2, \ldots, X_k$ are eigen vectors corresponding to an eigen value λ of a square matrix A, then $a_1X_1 + a_2X_2 + \ldots + a_kX_k$ ($a_1, a_2, \ldots, a_k$ are non-zero numbers) is also an eigen vector corresponding to λ of A.

Solution: Since X_i, $i = 1, 2, \ldots, k$, are eigen vectors corresponding to eigen value λ of A, we have $AX_1 = \lambda X_1$, $AX_2 = \lambda X_2$, ..., $AX_k = \lambda X_k$. Hence $A(a_1X_1) = \lambda(a_1X_1)$, $A(a_2X_2) = \lambda(a_2X_2)$, ..., $A(a_nX_n) = \lambda(a_nX_n)$.

$$\therefore \qquad A(a_1X_1+a_2X_2+\ldots+a_nX_n) = \lambda(a_1X_1+a_2X_2+\ldots+a_nX_n)$$

Therefore, $a_1X_1+a_2X_2+\ldots+a_nX_n$ is an eigen vector corresponding to the eigen value λ of A, where $a_1, a_2, \ldots, a_n$ are non-zero scalars.

Example 2: Find the eigen values of A^{-1} and A^5, where

$$A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}.$$

Scanned with CamScanner

Solution: The characteristic equation of A is $|A-\lambda I| = 0$

or $\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$, or $(3-\lambda)\{-(3+\lambda)(7-\lambda)+20\}-10\{-2(7-\lambda)+12\}$

$$+5\{-10+3(3+\lambda)\} = 0$$

or $$(3-\lambda)(\lambda^2-4\lambda-1)-20\lambda+20-5+15\lambda = 0$$

or $$-\lambda^3+7\lambda^2-16\lambda+12 = 0 \text{ or } \lambda^3-7\lambda^2+16\lambda-12 = 0$$

or $$\lambda^3-2\lambda^2-5\lambda^2+10\lambda+6\lambda-12 = 0$$

or $$\lambda^2(\lambda-2)-5\lambda(\lambda-2)+6(\lambda-2) = 0$$

or $$(\lambda-2)(\lambda^2-5\lambda+6) = 0$$

or $$(\lambda-2)(\lambda^2-2\lambda-3\lambda+6) = 0$$

or $$(\lambda-2)(\lambda-2)(\lambda-3) = 0$$

Therefore, eigen values of A are 2, 2, 3. Hence eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$ and eigen values of A^5 are $2^5, 2^5, 3^5$.

Example 3: Verify the statements that (*i*) the sum of the elements in the principal diagonal of a matrix is the sum of the eigen values of the matrix and (*ii*) the product of the eigen values of a matrix is equal to the determinant of that matrix, for $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

Solution: Here the characteristic polynomial of A is $f(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}$

$$= \begin{vmatrix} 5-\lambda & 7-\lambda & 5-\lambda \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} \quad [R_1+R_2+R_3 \to R_1']$$

$$= \begin{vmatrix} 5-\lambda & 7-\lambda & 0 \\ 1 & 5-\lambda & 0 \\ 3 & 1 & -2-\lambda \end{vmatrix} \quad [C_3-C_1 \to C_3']$$

$$= (-2-\lambda)\{(5-\lambda)^2-7+\lambda\} = -(\lambda+2)(\lambda^2-9\lambda+18)$$

$$= -(\lambda^3-7\lambda^2+36) = -\lambda^3+7\lambda^2-36$$

$\therefore\ |A| = f(0) = -36$ ...(1)

The characteristic equation of A is

$$\lambda^3-7\lambda^2+36 = 0 \qquad ...(2)$$

It is a cubic equation in λ and hence it has three roots and the three roots are the three eigen values of the matrix A.

The sum of the eigen values $= -\dfrac{(-7)}{1} = 7 = 1 + 5 + 1 =$ Sum of the diagonal elements.

Product of the eigen values $= -\dfrac{36}{1} = -36 = |A|$ [By (1)].

Hence the verification.

Example 4: Find the eigen values and the eigen vectors of the matrix $A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$.

Solution: The characteristic equation of A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 1-\lambda \end{vmatrix} = 0, \text{ or } (2-\lambda)(1-\lambda) - 12 = 0$$

or $$\lambda^2 - 3\lambda - 10 = 0, \text{ or } \lambda^2 - 5\lambda + 2\lambda - 10 = 0$$

or $$\lambda(\lambda - 5) + 2(\lambda - 5) = 0, \text{ or } (\lambda - 5)(\lambda + 2) = 0$$

$\therefore$ $\lambda = -2, 5$ are the eigen values of A.

Case 1: Eigen vectors corresponding to $\lambda = -2$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigen vector corresponding to $\lambda = -2$, then $AX = -2X$, or $(A + 2I)X = O$

or $$\begin{pmatrix} 2+2 & 4 \\ 3 & 1+2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ or } 4x_1 + 4x_2 = 0,\ 3x_1 + 3x_2 = 0$$

we get one independent equation $x_1 + x_2 = 0$, or $x_1 = -x_2$ $\therefore$ $x_1 = -k$, $x_2 = k$, where k is any non-zero number.

Thus the eigen vectors are $(-k, k)^T$, *e.g.*, $(-1, 1)$ by putting $k = 1$.

Case 2: Eigen vectors corresponding to $\lambda = 5$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigen vector corresponding to $\lambda = 5$, then $AX = 5X$, or $(A - 5I)X = O$

or $$\begin{pmatrix} 2-5 & 4 \\ 3 & 1-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } -3x_1 + 4x_2 = 0,\ 3x_1 - 4x_2 = 0$$

we get one independent equation $3x_1 - 4x_2 = 0$ or $\dfrac{x_1}{4} = \dfrac{x_2}{3} = k$ (say).

Scanned with CamScanner

$\therefore\ x_1 = 4k,\ x_2 = 3k,$ where k is any non-zero number. Hence the eigen vectors are $\begin{pmatrix} 4k \\ 3k \end{pmatrix}$, *e.g.*, $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ by putting $k = 1$.

Example 5: Determine the eigen values and eigen vectors of the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

(W.B.U.T. 2004)

Solution: The characteristic equation of A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0, \text{ or } \begin{vmatrix} 4-\lambda & 7-\lambda & 4-\lambda \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0 \quad (R_1 + R_2 + R_3 \to R_1')$$

or $$\begin{vmatrix} 4-\lambda & 7-\lambda & 0 \\ 1 & 3-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0 \quad (C_3 - C_1 \to C_3')$$

or $(1-\lambda)\{(4-\lambda)(3-\lambda)-(7-\lambda)\} = 0,$ or $(1-\lambda)(\lambda^2 - 7\lambda + 12 - 7 + \lambda) = 0$

or $(1-\lambda)(\lambda^2 - 6\lambda + 5) = 0,$ or $(1-\lambda)(\lambda^2 - 5\lambda - \lambda + 5) = 0,$ or $(\lambda-1)^2(\lambda-5) = 0$

$\therefore\ \lambda = 1, 1, 5$ are the three eigen values of A.

Case 1: Eigen vectors corresponding to $\lambda = 1$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to $\lambda = 1$, then $AX = X$ or $(A - I)\ X = O$,

or $$\begin{pmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or $$x_1 + 2x_2 + x_3 = 0, \text{ or } x_1 = -2x_2 - x_3$$

$\therefore\ x_1 = -2k_1 - k_2,\ x_2 = k_1,\ x_3 = k_2,$ where k_1, k_2 are any two non-zero numbers. Thus the eigen vectors are $(-2k_1 - k_2\ \ k_1\ \ k_2)^T$, where k_1, k_2 are arbitrary and k_1, k_2 are not simultaneously zero.

Case 2: Eigen vectors corresponding to $\lambda = 5$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to $\lambda = 5$, then $AX = 5X$, or $(A - 5I)X = O$,

or $$\begin{pmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or $$\begin{aligned} -3x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 - 3x_3 &= 0 \end{aligned}$$

Here two equations are independent. Taking the first-two equations,

we get $\dfrac{x_1}{2+2} = \dfrac{x_2}{1+3} = \dfrac{x_3}{6-2}$, or $x_1 = x_2 = x_3 = k$ (say), where k is any non-zero number.

Therefore, the eigen vectors are $\begin{pmatrix} k \\ k \\ k \end{pmatrix}$, *e.g.*, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ by putting $k = 1$.

Example 6: Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 4 & 6 \\ 2 & 9 \end{bmatrix}$

(*W.B.U.T. 2005, 2009*)

Solution: Let $A = \begin{bmatrix} 4 & 6 \\ 2 & 9 \end{bmatrix}$, therefore the characteristic equation of A is $|A - \lambda I| = 0$,

or $$\begin{vmatrix} 4-\lambda & 6 \\ 2 & 9-\lambda \end{vmatrix} = 0, \text{ or } (4-\lambda)(9-\lambda) - 12 = 0, \text{ or } \lambda^2 - 13\lambda + 24 = 0$$

$\therefore$ $$\lambda = \frac{1}{2}(13 \pm \sqrt{73}) \text{ are the eigen values of } A.$$

Case 1: Eigen vector corresponding to $\lambda = \frac{1}{2}(13+\sqrt{73})$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigen vector corresponding to $\lambda = \frac{1}{2}(13+\sqrt{73})$, then $AX = \frac{1}{2}(13+\sqrt{73})X$,

Scanned with CamScanner

or $\quad \left\{A-\frac{1}{2}(13+\sqrt{73})I\right\}X = O,$ or $\begin{bmatrix} 4-\frac{1}{2}(13+\sqrt{73}) & 6 \\ 2 & 9-\frac{1}{2}(13+\sqrt{73}) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or $\quad -\frac{1}{2}(5+\sqrt{73})x_1 + 6x_2 = 0$

$$2x_1 + \frac{1}{2}(5-\sqrt{73})x_2 = 0$$

These two are two identical equations and we get

$x_1 = \frac{1}{4}(\sqrt{73}-5)x_2$. $\quad\therefore\quad x_1 = \frac{1}{4}(\sqrt{73}-5)k,\ \ x_2 = k$, where k is any non-zero number.

Thus the eigen vectors are $\begin{bmatrix} \frac{1}{4}(\sqrt{73}-5)k \\ k \end{bmatrix}$, *e.g.* $\begin{bmatrix} \frac{1}{4}(\sqrt{73}-5) \\ 1 \end{bmatrix}$ by putting $k = 1$.

Case 2: Eigen vectors corresponding to $\lambda = \frac{1}{2}(13-\sqrt{73})$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigen vector corresponding to $\lambda = \frac{1}{2}(13-\sqrt{73})$, then $AX = \frac{1}{2}(13-\sqrt{73})X$,

or $\quad \left\{A-\frac{1}{2}(13-\sqrt{73})I\right\}X = O,$ or $\begin{bmatrix} 4-\frac{1}{2}(13-\sqrt{73}) & 6 \\ 2 & 9-\frac{1}{2}(13-\sqrt{73}) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

or $\quad \frac{1}{2}(\sqrt{73}-5)x_1 + 6x_2 = 0$

$$2x_1 + \frac{1}{2}(5+\sqrt{73})x_2 = 0$$

These are two identical equations and we get

$x_1 = -\frac{1}{4}(\sqrt{73}+5)x_2$. Therefore $x_1 = -\frac{1}{4}(\sqrt{73}+5)k,\ \ x_2 = k$, where k is any non-zero number.

Thus the eigen vectors are $\begin{bmatrix} -\frac{1}{4}(\sqrt{73}+5)k \\ k \end{bmatrix}$, *e.g.*, $\begin{bmatrix} -\frac{1}{4}(\sqrt{73}+5) \\ 1 \end{bmatrix}$ by putting $k = 1$.

Scanned with CamScanner

Example 7: Find the eigen values and corresponding eigen vectors of the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}$.

(*W.B.U.T. 2008*)

Solution: The characteristic equation of A is $|A - \lambda I| = 0$, or $\begin{vmatrix} 1-\lambda & -1 & 2 \\ 2 & -2-\lambda & 4 \\ 3 & -3 & 6-\lambda \end{vmatrix} = 0$.

or $\quad (1-\lambda)\{-(2+\lambda)(6-\lambda)+12\}+2(6-\lambda)-12+2\{-6+3(2+\lambda)\} = 0$

or $\quad (1-\lambda)(\lambda^2 - 4\lambda) - 2\lambda + 6\lambda = 0$

or $\quad -\lambda^3 + 5\lambda^2 - 4\lambda + 4\lambda = 0$

or $\quad \lambda^3 - 5\lambda^2 = 0$

or $\quad \lambda^2(\lambda - 5) = 0$

$\therefore$ $\lambda = 0, 0, 5$ are the three eigen values of A.

Case 1: Eigen vectors corresponding to $\lambda = 0$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector corresponding to $\lambda = 0$, then $AX = 0 \cdot X$, or $AX = O$,

or $\quad \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

or
$$x_1 - x_2 + 2x_3 = 0$$
$$2x_1 - 2x_2 + 4x_3 = 0$$
$$3x_1 - 3x_2 + 6x_3 = 0$$

These three equations are identical.

$\therefore$ $x_1 - x_2 + 2x_3 = 0$, or $x_2 = x_1 + 2x_3$

$\therefore$ $x_1 = k_1$, $x_2 = k_1 + 2k_2$, $x_3 = k_2$, where k_1, k_2 are arbitrary numbers and k_1, k_2 are not simultaneously zero.

Therefore, the eigen vectors are $\begin{bmatrix} k_1 \\ k_1 + 2k_2 \\ k_2 \end{bmatrix}$, *e.g.* $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ by putting $k_1 = k_2 = 1$.

Scanned with CamScanner

Case 2: Eigen vectors corresponding to $\lambda = 5$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector corresponding to $\lambda = 5$, then $AX = 5X$ or $(A - 5I)X = O$,

or $$\begin{bmatrix} 1-5 & -1 & 2 \\ 2 & -2-5 & 4 \\ 3 & -3 & 6-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

or $$-4x_1 - x_2 + 2x_3 = 0$$
$$2x_1 - 7x_2 + 4x_3 = 0$$
$$3x_1 - 3x_2 + x_3 = 0.$$

Here two equations are independent. Taking the first-two equations, we get

$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{30}$, or $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3} = k$ (say), where k is any non-zero number.

Therefore, the eigen vectors are $\begin{bmatrix} k \\ 2k \\ 3k \end{bmatrix}$, *e.g.*, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ by putting $k = 1$.

Example 8: If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, then verify that A satisfies its own characteristic equation. Hence find A^{-1}. *(W.B.U.T. 2007, 2008, 2012)*

Solution: The characteristic equation of A is $|A - \lambda I| = 0$, or $\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$,

or $$(1-\lambda)\{\lambda(1+\lambda)-1\} = 0, \text{ or } (\lambda-1)(\lambda^2+\lambda-1) = 0, \text{ or } \lambda^3 - 2\lambda + 1 = 0.$$

So, we have to verify whether $A^3 - 2A + I = O$ is true.

Now, $$A^2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$\therefore$ $$A^3 = A^2 \cdot A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{pmatrix}$$

Scanned with CamScanner

$$\therefore \quad A^3 - 2A + I = \begin{pmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O \qquad \text{...(1)}$$

Therefore, the given matrix A satisfies its own characteristic equation.

From (1), we have $2A - A^3 = I$, or $A(2I - A^2) = I$.

Using definition of inverse of a matrix, we have

$$A^{-1} = 2I - A^2 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Example 9: If $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$ then verify that A satisfies its own characteristic equation. Hence find A^{-1}. (*W.B.U.T. 2005*)

Solution: The characteristic equation of A is $|A - \lambda I| = 0$, or $\begin{vmatrix} 1-\lambda & 2 & 1 \\ 1 & -1-\lambda & 1 \\ 2 & 3 & -1-\lambda \end{vmatrix} = 0$

or $\quad (1-\lambda)\{(1+\lambda)^2 - 3\} - 2\{-(1+\lambda) - 2\} + 3 + 2(1+\lambda) = 0$

or $\quad (1-\lambda)(\lambda^2 + 2\lambda - 2) + 2\lambda + 6 + 5 + 2\lambda = 0$

or $\quad -\lambda^3 - \lambda^2 + 8\lambda + 9 = 0$, or $\lambda^3 + \lambda^2 - 8\lambda - 9 = 0$

So, we have to verify that $A^3 + A^2 - 8A - 9I = O$ ...(1)

Now, $$A^2 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 3 & 2 \\ 2 & 6 & -1 \\ 3 & -2 & 6 \end{pmatrix}$$

$$\therefore \quad A^3 = A^2 \cdot A = \begin{pmatrix} 5 & 3 & 2 \\ 2 & 6 & -1 \\ 3 & -2 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 12 & 13 & 6 \\ 6 & -5 & 9 \\ 13 & 26 & -5 \end{pmatrix}$$

Scanned with CamScanner

$$\therefore \quad A^3 + A^2 - 8A - 9I = \begin{pmatrix} 12 & 13 & 6 \\ 6 & -5 & 9 \\ 13 & 26 & -5 \end{pmatrix} + \begin{pmatrix} 5 & 3 & 2 \\ 2 & 6 & -1 \\ 3 & -2 & 6 \end{pmatrix} - 8\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} - 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus (1) is true, *i.e.*, A satisfies its own characteristic equation. From (1), we get

$$I = \frac{1}{9}(A^3 + A^2 - 8A). \quad \therefore \quad A^{-1}I = \frac{1}{9}(A^{-1}A^3 + A^{-1}A^2 - 8A^{-1}A)$$

or
$$A^{-1} = \frac{1}{9}[(A^{-1}A)A^2 + (A^{-1}A)A - 8I]$$

or
$$A^{-1} = \frac{1}{9}(I \cdot A^2 + I \cdot A - 8I)$$

$$\therefore \quad A^{-1} = \frac{1}{9}(A^2 + A - 8I)$$

$$= \frac{1}{9}\left\{\begin{pmatrix} 5 & 3 & 2 \\ 2 & 6 & -1 \\ 3 & -2 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} - 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$$= \frac{1}{9}\begin{pmatrix} -2 & 5 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{pmatrix}$$

Example 10: If A, B are two nth order square matrices and B is non-singular, prove that A and $B^{-1}AB$ have same eigen values.

Solution: Characteristic polynomial of $B^{-1}AB$

$$= |B^{-1}AB - \lambda I|$$

$$= |B^{-1}AB - B^{-1}(\lambda I)B| \qquad [\because B^{-1}(\lambda I)B = \lambda B^{-1}(IB) = \lambda B^{-1}B = \lambda I]$$

$$= |B^{-1}(A - \lambda I)B| = |B^{-1}||A - \lambda I||B|$$

$$= |B^{-1}||B||A - \lambda I| = |B^{-1}B||A - \lambda I|$$

$$= |I||A - \lambda I| = |A - \lambda I|$$

= Characteristic polynomial of A.

Therefore, the characteristic polynomials of A and $B^{-1}AB$ are same and hence A and $B^{-1}AB$ have same eigen values.

Scanned with CamScanner

Example 11: If A, B are two square matrices of same order and A is non-singular, then prove that $A^{-1}B$ and BA^{-1} have same eigen values.

Solution: Since A is non-singular, *i.e.*, $|A| \neq 0$, therefore A^{-1} exists. As A, B are square matrices of same order, therefore $A^{-1}B$ and BA^{-1} both exist.

Characteristic polynomial of $A^{-1}B$

$$= |A^{-1}B - \lambda I| = |AA^{-1}||A^{-1}B - \lambda I| \qquad (\because |AA^{-1}| = |I| = 1)$$

$$= |A||A^{-1}||A^{-1}B - \lambda I| = |A||A^{-1}B - \lambda I||A^{-1}|$$

$$= |A(A^{-1}B - \lambda I)A^{-1}|$$

$$= |(AA^{-1})BA^{-1} - A(\lambda I)A^{-1}|$$

$$= |IBA^{-1} - \lambda(AI)A^{-1}|$$

$$= |BA^{-1} - \lambda AA^{-1}|$$

$$= |BA^{-1} - \lambda I|$$

= Characteristic polynomial of BA^{-1}.

Therefore, the characteristic polynomials of $A^{-1}B$ and BA^{-1} are same and hence $A^{-1}B$ and BA^{-1} have same eigen values.

MULTIPLE CHOICE QUESTIONS

1. The rank of a sixth order identity matrix is
   (*a*) 1 (*b*) 0 (*c*) 36 (*d*) 6
2. The rank of a zero matrix (or null matrix) of any order is
   (*a*) 1 (*b*) 2 (*c*) 0 (*d*) 3
3. If the matrix $\begin{pmatrix} x+2 & 3 \\ 2 & x+3 \end{pmatrix}$ has rank 1, then
   (*a*) $x = 0$ (*b*) $x = -2$ (*c*) $x = -3$ (*d*) none of these
4. The rank of the matrix $\begin{pmatrix} 2 & 2 & 4 \\ 3 & 3 & 6 \end{pmatrix}$ is
   (*a*) 2 (*b*) 1 (*c*) 3 (*d*) none of these
5. The system of equations $3x + 2y - z = 2$, $6x + 4y - 2z = 4$ has
   (*a*) unique solution (*b*) no solution
   (*c*) infinitely many solutions (*d*) none of these

Scanned with CamScanner