Chapters 3 to 5

125

~~Module~~ I

CHAPTER 3

Some Important Probability Distributions

3.1 INTRODUCTION

In this chapter we shall discuss some important discrete and continuous probability distributions in which random variables are distributed according to some definite probability law which can be formulated mathematically.

3.2 BINOMIAL (OR BERNOULLI'S) PROBABILITY DISTRIBUTION

Binomial distribution was discovered by a Swiss mathematician **James Bernoulli** also known as **Jacques** or **Jakob** (1654–1705) and was published posthumously in 1713. It is also known as **Bernoulli's Distribution**.

Definition: Let (*i*) there are n independent trials in a random experiment, (*ii*) each trial has exactly two mutually exclusive outcomes namely success and failure (*i.e.*, happening or not happening of an event), (*iii*) the probability of success is p and the probability of failure is q in a single trial (and same for every trial) so that $p + q = 1$ and (*iv*) X denotes a random variable representing the number of successes in these n trials.

Then r successes can be obtained in n trials in ${}^nC_r$ ways and

$$P(X = r) = {}^nC_r\, P\left(\underbrace{SSS \ldots S}_{r \text{ times}}\ \underbrace{FF \ldots F}_{(n-r) \text{ times}}\right)$$

$$= {}^nC_r\, \underbrace{P(S)\, P(S) \ldots P(S)}_{r \text{ times}}\, \underbrace{P(F)P(F) \ldots P(F)}_{(n-r) \text{ times}} = {}^nC_r\, p^r q^{n-r}$$

[Since the trials are statistically independent, *i.e.*, the outcomes of any trial or sequence of trials do not affect the outcomes of subsequent trials.]

Hence $$P(X = r) = {}^nC_r p^r q^{n-r} \qquad \ldots(1)$$

where $0 \le p \le 1,\ p + q = 1$ and $r = 0, 1, 2, \ldots, n.$

126

The probability distribution (1) is called the **binomial probability distribution** and X is called the **binomial variate** denoted as $X \sim B(n, p)$, where n, p are called the **parameters** of the binomial distribution.

So, a discrete random variable X is said to have a **binomial distribution** with parameters $p(0 \le p \le 1)$ and n (a positive integer) if its distribution is given by

$$\begin{array}{rl} X = i: & 0 \quad 1 \quad 2 \quad \ldots.\, n \\ P(X = i) = f_i: & f_0 \quad f_1 \quad f_2 \quad \ldots.\, f_n, \end{array}$$

where the probability mass function (p.m.f.)

$$f_i = P(X = i) = {}^nC_i p^i (1-p)^{n-i} = \binom{n}{i} p^i (1-p)^{n-i},\ i = 0, 1, 2, \ldots, n$$

Observe that $f_i \ge 0$, $\forall\, i$ and $\displaystyle\sum_{i=0}^{n} f_i = \sum_{i-0}^{n} {}^nC_i (1-p)^{n-i} p^i = \{(1-p) + p\}^n$

(By Binomial expansion)

$$= 1.$$

So, this is a valid probability distribution.

Note: (*i*) The successive probabilities $P(X = r)$ in (1) for $r = 0, 1, 2, \ldots, n$ are ${}^nC_0 q^n, {}^nC_1 q^{n-1} p, {}^nC_2 q^{n-2} p^2, \ldots, {}^nC_n p^n$ which are the successive terms of the Binomial expansion of $(q + p)^n$. That is why this distribution is called **Binomial Distribution.**

(*ii*) A binomial random variable with parameters $(1, p)$ is also called a **Bernoulli random variable** and the underlying experiment is known as **Bernoulli trial.**

(*iii*) The distribution function is given by

$$F(x) = P(X \le x) = \sum_{\substack{i=0 \\ 0 \le r \le \min(n, x)}}^{r} P(X = i) = \sum_{\substack{i=0 \\ 0 \le r \le \min(n, x)}}^{r} {}^nC_i p^i (1-p)^{n-i}$$

(*iv*) **Uses:** The Binomial distribution can be used where there are two mutually exclusive and exhaustive outcomes and where the number of trials is finite. So, it is applicable in problems involving

(*a*) the tossing of a coin: heads or tails

(*b*) the result of a game: win or loss

(*c*) the result of an examination: success or failure

(*d*) the result of an inspection: acceptance or rejection.

(*v*) **$P(X = r)$ is usually written as $P(r)$.**

Example 1: *If 5% of bolts produced by a machine are defective, find the probability that out of 10 bolts (drawn at random)*

(*i*) *none* (*ii*) *1* (*iii*) *at most 2 bolts will be defective.*

Solution: Given, probability of defective bolts $= p = \dfrac{5}{100} = 0.05$.

So, probability of non-defective bolts $= q = 1 - p = 1 - 0.05 = 0.95$.

Total number of bolts $= n = 10$.

(*i*) Probability that none is defective

$$= P(0) = {}^{10}C_0 p^0 q^{10} = (0.95)^{10} = 0.599$$

(*ii*) Probability of 1 defective bolt

$$= P(1) = {}^{10}C_1 pq^9 = 10 \times 0.05 \times (0.95)^9 = 0.315$$

(*iii*) Probability of 2 defective bolts

$$= P(2) = {}^{10}C_2 p^2 q^8 = \frac{1}{2} \times 10 \times (10-1)\, p^2 q^8 = 45 \times (0.05)^2 (0.95)^8 = 0.0746$$

Probability of at most 2 defectives

$$= P(0 \text{ or } 1 \text{ or } 2) = P(0) + P(1) + P(2) = 0.599 + 0.315 + 0.0746 = 0.9886.$$

Example 2: *The overall percentage of failures in a certain examination is 40. What is the probability that out of a group of 6 candidates at least 4 passed the examination?* (*W.B.U.T. 2008*)

Solution: Let the random variable X corresponds to the number of candidates passed the examination out of 6 candidates.

Here $X \sim B(n, p)$, where $n = 6$,

p = probability of success in a single trial and

$q = 1 - p$ = probability of failure in a single trial.

Given: $$q = 1 - p = \frac{40}{100} = 0.4 \Rightarrow p = 0.6$$

$$\therefore \quad P(X = r) = {}^nC_r p^r q^{n-r} = {}^6C_r (0.6)^r (0.4)^{n-r}, r = 0, 1, 2, ..., 6.$$

$\therefore$ Required probability $= P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6)$

$$= {}^6C_4 (0.6)^4 (0.4)^2 + {}^6C_5 (0.6)^5\, 0.4 + {}^6C_6 (0.6)^6$$

$$= 0.54432 \qquad \left[\because\ {}^6C_4 = {}^6C_2 = \frac{1}{2} \times 6 \times (6-1) = 15,\ {}^6C_5 = 6 \text{ and } {}^6C_6 = 1\right]$$

3.3 RECURRENCE OR RECURSION FORMULA FOR THE BINOMIAL DISTRIBUTION

In a Binomial distribution,

$$P(X = r) = P(r) = {}^nC_r p^r q^{n-r} = \frac{n!}{r!(n-r)!} p^r q^{n-r}$$

$$P(X = r + 1) = P(r + 1) = {}^nC_{r+1} p^{r+1} q^{n-r-1} = \frac{n!}{(r+1)!(n-r-1)!} p^{r+1} q^{n-r-1}$$

$$\therefore \quad \frac{P(r+1)}{P(r)} = \frac{n!}{(r+1)!(n-r-1)!} \cdot \frac{r!(n-r)!}{n!} \cdot \frac{p}{q} = \left(\frac{n-r}{r+1}\right) \frac{p}{q}$$

$$\Rightarrow \quad P(r+1) = \left(\frac{n-r}{r+1}\right) \frac{p}{q} P(r)$$

This is the required **recurrence formula**. If $P(0)$ is known, then using this formula successively we can find $P(1)$, $P(2)$, ...

3.4 MEAN AND VARIANCE OF THE BINOMIAL DISTRIBUTION

Theorem: *If the random variable X has Binomial Distribution with parameters n and p then* (*i*) *mean* $= E(X) = np$ *and* (*ii*) $Var(X) = npq$ *where* $q = 1 - p$. (*W.B.U.T. 2005, 2006, 2009*)

Proof: Given, $X \sim B(n, p)$

$$\therefore \quad P(X = r) = P(r) = {}^nC_r p^r q^{n-r}, \text{ where } q = 1 - p \text{ and } r = 0, 1, 2, \ldots, n.$$

(*i*)

$$\text{Mean} = E(X) = \sum_{r=0}^{n} rP(r) = \sum_{r=1}^{n} r\,{}^nC_r p^r q^{n-r} = \sum_{r=1}^{n} r\frac{n!}{r!(n-r)!} p^r q^{n-r}$$

$$= np\sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!(n-r)!} p^{r-1} q^{n-r} = np\sum_{r=1}^{n} {}^{n-1}C_{r-1}\, p^{r-1} q^{n-r}$$

$$= np\sum_{r=0}^{n-1} {}^{n-1}C_r p^r q^{n-1-r} \qquad \text{[Replacing } (r-1) \text{ by } r]$$

$$= np(p+q)^{n-1} \qquad \text{[By Binomial expansion]}$$

$$= np \qquad [\because \; p + q = 1]$$

(*ii*) Now,

$$E\{X(X-1)\} = \sum_{r=0}^{n} r(r-1)\, P(r) = \sum_{r=2}^{n} r(r-1)\, {}^nC_r p^r q^{n-r}$$

$$= \sum_{r=2}^{n} r(r-1)\frac{n!}{r!(n-r)!} p^r q^{n-r}$$

$$= n(n-1)\, p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!(n-r)!} p^{r-2} q^{n-r}$$

$$= n(n-1)\, p^2 \sum_{r=2}^{n} {}^{n-2}C_{r-2}\, p^{r-2} q^{n-r}$$

$$= n(n-1)\, p^2 \sum_{r=0}^{n-2} {}^{n-2}C_r\, p^r q^{n-2-r} \qquad \text{[Replacing } (r-2) \text{ by } r]$$

$$= n(n-1)\, p^2 (p+q)^{n-2} \qquad \text{[By Binomial expansion]}$$

$$= n(n-1)\, p^2 \qquad [\because \; p + q = 1].$$

$$\therefore \quad Var(X) = E\{X(X-1)\} - m(m-1)$$

[See art. 2.6, property (*iv*), Chapter–2. Here $m = E(X)$]

$$= n(n-1)\, p^2 - np(np-1) = np(1-p) = npq \quad (\text{where } q = 1 - p)$$

Note: Standard deviation $= \sqrt{npq}$.

E-7 to E-11 E-7 C6

Place ~~C6 & C7~~ before Illustrative Examples-1 in p. ~~127~~ 129

3.5 Geometric Distribution

In the theory of probability, geometric distribution is the probability distribution of the number X of independent Bernoulli trials performed until a success occurs where the Bernoulli trials have a constant probability of success, say p. So, it is clear that X takes on the values 1, 2, 3, ... and

$P(X=x) = P$[x Bernoulli trials are performed until the first success occurs]

$\Rightarrow P(X=x) = P$[failures in the first $(x-1)$ Bernoulli trials and a success in the x^{th} trial]

$\Rightarrow P(X=x) = q^{x-1}p,\ x=1,2,3,\cdots$ (using the independence of events), where $q=1-p$, the constant probability of failure in each Bernouli trial.

Definition: A random variable X is said to have a geometric distribution with a parameter $p\,(0<p<1)$ if it takes on the values 1, 2, 3, ... and its probability mass function (p.m.f.) is given by

$$f(x;p) = P(X=x) = q^{x-1}p,\ x=1,2,3,\cdots,\ \text{where } q=1-p.$$

Notes: (i) If X is a geometric distributed random variable, i.e., $X \sim Geom(p)$, then the values of the p.m.f., viz. $p, qp, q^2p, q^3p, \ldots, q^{x-1}p, \ldots$ are the successive terms of a geometric progression series. This is the reason for calling geometric.

(ii) It is a valid probability distribution because $f(x,p)>0$, for $x=0,1,2,\cdots$ and also that

$$\sum_{x=1}^{\infty} f(x;p) = \sum_{x=1}^{\infty} q^{x-1}p = p(1+q+q^2+q^3+\cdots \text{to } \infty) = p\cdot\frac{1}{1-q} = 1.$$

Theorem 1: If the random variable X has geometric distribution with parameter p, then

(i) Mean $= E(X) = \frac{1}{p}$ and (ii) $Var(X) = \frac{q}{p^2}$, where $q = 1-p$.

Proof: (i) Mean $= E(X) = \sum_{x=1}^{\infty} x\, P(X=x) = \sum_{x=1}^{\infty} x q^{x-1} p$

$= p(1+2q+3q^2+4q^3+\cdots \text{ to } \infty) = p(1-q)^{-2} = \frac{1}{p}$.

(ii) $E\{X(X-1)\} = \sum_{x=1}^{\infty} x(x-1)\, q^{x-1} p = 2qp + 6q^2p + 12q^3p + 20q^4p + \cdots \text{ to } \infty$

$= 2qp\,(1+3q+6q^2+10q^3+\cdots \text{ to } \infty)$

$= 2qp\,(1-q)^{-3} = \frac{2q}{p^2}$.

$\therefore Var(X) = E\{X(X-1)\} - \frac{1}{p}\left(\frac{1}{p}-1\right) = \frac{2q}{p^2} - \frac{q}{p^2} = \frac{q}{p^2}$.

Example 1: A fair six-faced die is tossed until all the six faces are observed. If X is the number of tosses required, find $E(X)$. (IIESTS-2015)

Solution: One throw of the die shows one face. Let X_1 denotes the number of throws required to show a new face.

$\therefore X_1 \sim Geom(p),\ p = \frac{5}{6}$.

Let X_2 be the additional number of throws required to get yet another new face after two faces have been seen.

$\therefore X_2 \sim Geom(p),\ p = \frac{4}{6}$

Similarly, $X_3 \sim Geom\left(\frac{3}{6}\right),\ X_4 \sim Geom\left(\frac{2}{6}\right),\ X_5 \sim Geom\left(\frac{1}{6}\right)$

$\therefore X = 1 + X_1 + X_2 + X_3 + X_4 + X_5 \Rightarrow E(X) = 1 + \sum_{i=1}^{5} E(X_i)$

$\therefore E(X) = 1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1}$

$= 6\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right)$ $\left[\because E(Y) = \frac{1}{p}, \text{ when } Y \sim Geom(p)\right]$

3.6 MULTINOMIAL DISTRIBUTION

It can be regarded as a generalisation of Binomial distribution.

When there are more than two mutually exclusive outcomes of a random experiment (a trial), the observations lead to a multinomial distribution. Suppose $E_1, E_2, \cdots, E_K$ are K mutually exclusive and exhaustive outcomes of a random experiment with respective probabilities $p_1, p_2, \cdots, p_K$.

The probability that E_1 occurs x_1 times, E_2 occurs x_2 times, ... and E_K occurs x_K times in n independent trials is given by $p(x_1, x_2, \cdots, x_K) = c\, p_1^{x_1} p_2^{x_2} \cdots p_K^{x_K}$, where $\sum_{i=1}^{K} x_i = n$ and c is the number of permutation of the events $E_1, E_2, \cdots, E_K$.

To determine c, it is required to find the number of permutations of n objects of which x_1 are of one kind, x_2 of another kind, $\cdots$, x_K of the K^{th} kind, which is given by

$$c = \frac{n!}{x_1!\, x_2! \cdots x_K!}.$$

$$\therefore\ p(x_1, x_2, \cdots, x_K) = \frac{n!}{x_1!\, x_2! \cdots x_K!}\, p_1^{x_1} p_2^{x_2} \cdots p_K^{x_K}\ ;\ 0 \le x_i \le n,\ \sum_{i=1}^{K} x_i = n,$$

which is the required probability mass function of the multinomial distribution. It is so called since $p(x_1, x_2, \cdots, x_K)$ as stated is the general term in the multinomial expansion:

$$(p_1 + p_2 + \cdots + p_K)^n.$$

Since, the total probability is 1, we have

$$\sum_x p(x) = \sum_x \left\{ \frac{n!}{x_1! x_2! \cdots x_k!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} \right\} = (p_1+p_2+\cdots+p_k)^n = 1.$$

So, this is a valid probability distribution.

<u>Deduction of Binomial Distribution</u>

Putting $k=2$, $x_2 = n-x_1$, $p_1 = p$, $p_2 = 1-p$, since $p_1+p_2=1$, we get

$$p(x_1, x_2) = \frac{n!}{x_1!\,(n-x_1)!} p^{x_1} (1-p)^{n-x_1}, \text{ for } x_1 = 0, 1, 2, \ldots, n$$

Hence Binomial distribution is deduced from Multinomial distribution.

<u>Example</u> 1: Find the probability that in 8 throws of a die, the number 1, 4, 6 turn up 2, 3, 3 times respectively.

<u>Solution</u>: By Multinomial distribution, the required probability is

$$\frac{8!}{2!\,3!\,3!} \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^3 \left(\frac{1}{6}\right)^3,$$

since the probabilities of getting 1, 4, 6 from a single throw are all equal to $\frac{1}{6}$.

3.7. INFINITE SEQUENCE OF BERNOULLI TRIALS

Let $S = \{E_1, E_2\}$ be the event space connected to a random experiment E, where E_1 stands for the outcome 'success' and E_2 stands for 'failure'. Therefore, the possible events connected to E are E_1, E_2, O (impossible event) and S (certain event). An infinite sequence of trials of E is said to be independent if for any infinite sequence of events $A_1, A_2, \cdots, A_n, \cdots$ connected with E:

$$P(A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots) = P(A_1)P(A_2)\cdots P(A_n)\cdots$$

provided the infinite product in the right hand side is convergent. It is noted that each $A_i = E_1$ or E_2 or O or S.

An infinite sequence of independent trials of E is said to be an infinite sequence of Bernoulli trials if $P(E_1)$ and $P(E_2)$ remain constant in each trial of E.

Example: *The mean and standard deviation of a Binomial distribution are respectively 4 and* $\sqrt{\dfrac{8}{3}}$. *Find (i) n and p (ii) P(X = 0).*

Solution: We know that the mean and s.d. of a binomial variate are respectively np and $\sqrt{np(1-p)}$.

Given, $$np = 4 \text{ and } \sqrt{np(1-p)} = \sqrt{\frac{8}{3}}, \quad i.e., \quad np(1-p) = \frac{8}{3}.$$

Dividing: $$1 - p = \frac{2}{3} \quad \Rightarrow \quad p = 1 - \frac{2}{3} = \frac{1}{3}.$$

Now, $$np = 4. \quad \therefore \quad \frac{n}{3} = 4 \left(\because \ p = \frac{1}{3}\right) \quad \Rightarrow \quad n = 12.$$

$$\therefore \quad n = 12, p = \frac{1}{3}.$$

Now, $$P(X=0) = {}^nC_0 p^0 (1-p)^{n-0} = {}^{12}C_0 (1-p)^{12} = \left(1 - \frac{1}{3}\right)^{12} = \left(\frac{2}{3}\right)^{12}.$$

(*) Insert E-7 to E-11 →

ILLUSTRATIVE EXAMPLES – I

Example 1: *Comment on the statement 'a binomial variate has mean 3 and standard deviation 2'.*

Solution: Let $X \sim B(n, p)$.

$$\therefore \quad \text{Mean} = E(X) = np \quad \text{and} \quad \text{Var}(X) = np(1-p).$$

Given, $$np = 3 \quad \text{and} \quad \sqrt{np(1-p)} = 2, \ i.e., \ np(1-p) = 4$$

$$\therefore \quad \frac{np(1-p)}{np} = \frac{4}{3} \quad \Rightarrow \quad 1 - p = \frac{4}{3} \quad \Rightarrow \quad p = 1 - \frac{4}{3} = -\frac{1}{3} < 0,$$

which is not possible since $0 \le p \le 1$.

So the given statement is false.

Example 2: *If the mean of a Binomial distribution is 3 and the variance is* $\frac{3}{2}$, *find the probability of obtaining atmost 3 successes.* (*W.B.U.T. 2007*)

Solution: Let $X \sim B(n, p)$.

Here the r.v. X corresponds to the number of successes.

Now, $$P(X = r) = {}^nC_r p^r (1-p)^{n-r}, \ r = 0, 1, 2, ..., n$$

We know that $$\text{mean} = E(X) = np \quad \text{and} \quad \text{Var}(X) = np(1-p).$$

Given, $$np = 3 \quad \text{and} \quad np(1-p) = \frac{3}{2}.$$

$$\therefore \quad \frac{np(1-p)}{np} = \frac{3}{2} \times \frac{1}{3} \quad \Rightarrow \quad 1 - p = \frac{1}{2} \quad \Rightarrow \quad p = \frac{1}{2}.$$

130

Now, $np = 3$ and $p = \frac{1}{2}$. $\therefore$ $\frac{n}{2} = 3 \Rightarrow n = 6.$

Required probability $= P(X \le 3)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$$= {}^6C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^6 + {}^6C_1\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^5 + {}^6C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 + {}^6C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3$$

$$= \frac{1}{64}\left[1 + 6 + \frac{1}{2}\times 6\times(6-1) + \frac{6!}{3!3!}\right] = \frac{1}{64}[22 + 5\times 4] = \frac{21}{32}.$$

Example 3: *A random variable follows binomial distribution with mean 4 and standard deviation $\sqrt{2}$. Find the probability of assuming non-zero value of the variable.* (*W.B.U.T. 2009*)

Solution: Let $X \sim B(n, p)$.

Here the r.v. X corresponds to the number of successes.

Now, $P(X = r) = {}^nC_r p^r (1-p)^{n-r}$, $r = 0, 1, 2, \ldots, n$.

We know that mean $= E(X) = np$ and $\text{Var}(X) = np(1-p)$

Given, $np = 4$ and $\sqrt{np(1-p)} = \sqrt{2}$, *i.e.*, $np(1-p) = 2$.

$$\therefore \quad \frac{np(1-p)}{np} = \frac{2}{4} \Rightarrow 1 - p = \frac{1}{2} \Rightarrow p = \frac{1}{2}$$

Now, $np = 4$ and $p = \frac{1}{2}$. $\therefore$ $\frac{n}{2} = 4 \Rightarrow n = 8$

Required probability $= P(X > 0) = 1 - P(X = 0)$

$$= 1 - {}^8C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^8 = 1 - \frac{1}{256} = \frac{255}{256}.$$

Example 4: *It is observed that a cricket player becomes out within 10 runs in 3 out of 10 innings. If he played 4 innings, what is the probability that he will become (i) out twice (ii) out at least once within 10 runs?* (*W.B.U.T. 2007*)

Solution: Let the random variable X corresponds to the number of times of becoming out within 10 runs when the player plays 4 innings.

Here $X \sim B(n, p)$, where n = number of innings played = 4

and $p = P(\text{out within 10 in a single inning}) = \frac{3}{10}$.

$$\therefore \quad P(X = r) = {}^4C_r\left(\frac{3}{10}\right)^r\left(1 - \frac{3}{10}\right)^{4-r}, \; r = 0, 1, 2, 3, 4.$$

(*i*) Required probability $= P(X = 2) = {}^4C_2\left(\frac{3}{10}\right)^2\left(1 - \frac{3}{10}\right)^{4-2}$

$$= \frac{1}{2}\times 4\times(4-1)\times\frac{9\times 49}{10^4} = 0.2646$$

(*ii*) Required probability $= P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - {}^4C_0\left(\frac{3}{10}\right)^0\left(1-\frac{3}{10}\right)^4 = 1 - \left(\frac{7}{10}\right)^4$$

$$= 1 - \frac{2401}{10^4} = 1 - 0.2401 = 0.7599.$$

Example 5: *A and B play a game in which their chances of winning are in the ratio 3 : 2. Find A's chances of winning at least three games out of the five games played.* (*W.B.U.T. 2004*)

Solution: Let the random variable X corresponds to the number of times of winning (or success) of player A in 5 games.

Here $X \sim B(n, p)$,

where n = Number of games played = 5 and

p = Probability of winning (or success) of A in a single game (trial),

$q = 1 - p$ = Probability of failure of A in a single game

= Probability of winning of B in a single game.

Given, $$\frac{p}{q} = \frac{3}{2} \Rightarrow q = \frac{2}{3}p$$

$$\therefore \quad p + q = 1 \Rightarrow \left(1+\frac{2}{3}\right)p = 1 \Rightarrow p = \frac{3}{5} \text{ and } q = 1 - p = 1 - \frac{3}{5} = \frac{2}{5}.$$

$\therefore$ Required probability $= P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5)$

$$= {}^5C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2 + {}^5C_4\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right) + {}^5C_5\left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^0$$

$$= \frac{1}{5^5}\left[\frac{1}{2}\times 5\times(5-1)\times 108 + 5\times 162 + 243\right]$$

$$= \frac{2133}{3125} = 0.68256.$$

Example 6: *If 20% of the articles produced by a machine are defective, determine the probability that out of the 4 articles chosen at random less than 2 articles will be defective.* (*W.B.U.T. 2002*)

Solution: Let the random variable X corresponds to the number of defective articles out of the 4 articles chosen at random.

Here $X \sim B(n, p)$,

where n = Number of articles chosen = 4 and

p = Probability of drawing one defective article in a single trial

$$= \frac{20}{100} = 0.2$$

$$\therefore \quad P(X = r) = {}^4C_r(0.2)^r(1-0.2)^{4-r} = {}^4C_r(0.2)^r(0.8)^{4-r}, \; r = 0, 1, 2, 3, 4.$$

$\therefore$ Required probability $= P(X < 2) = P(X = 0) + P(X = 1)$

$$= {}^4C_0(0.2)^0(0.8)^4 + {}^4C_1(0.2)(0.8)^3$$

$$= (0.8)^3 [0.8 + 4 \times 0.2] = (0.8)^3 \times 1.6$$
$$= 0.8192.$$

Example 7: *A discrete random variable X has the mean 6 and variance 2. Assuming the distribution is binormal, find the probability that $5 \le X \le 7$.* (*W.B.U.T. 2002*)

Solution: Let $X \sim B(n, p)$.

$\therefore \quad P(X = r) = {}^nC_r p^r (1-p)^{n-r}, r = 0, 1, 2, ..., n$

We know that mean $= E(X) = np$ and $\text{Var}(X) = np(1-p)$

Given, $np = 6$ and $np(1-p) = 2 \quad \therefore \quad \dfrac{np(1-p)}{np} = \dfrac{2}{6}$

$$\Rightarrow \quad 1 - p = \frac{1}{3} \quad \Rightarrow \quad p = \frac{2}{3}$$

Now, $np = 6$ and $p = \dfrac{2}{3} \quad \therefore \quad n \cdot \dfrac{2}{3} = 6 \quad \Rightarrow \quad n = 9$

$$\therefore \quad P(5 \le X \le 7) = P(X = 5) + P(X = 6) + P(X = 7)$$

$$= {}^9C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^4 + {}^9C_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^3 + {}^9C_7 \left(\frac{2}{3}\right)^7 \left(\frac{1}{3}\right)^2$$

$$= \frac{1}{3^9}\left[\frac{9!}{5!4!} \times 32 + \frac{9!}{6!3!} \times 64 + \frac{1}{2} \times 9 \times (9-1) \times 128\right]$$

$$= \frac{1}{3^9} (4032 + 5376 + 4608) = \frac{14016}{19683} = 0.712.$$

Example 8: *A binomial variate X satisfies the relation $9P(X = 4) = P(X = 2)$ when $n = 6$. Find the value of the parameter p and $P(X = 1)$.*

Solution: Here $X \sim B(6, p)$

$\therefore \quad P(X = r) = {}^6C_r\, p^r q^{6-r}, \quad$ where $\quad q = 1 - p, r = 0, 1, ..., 6.$

Given, $\quad 9P(X = 4) = P(X = 2)$

$\Rightarrow \quad 9({}^6C_4\, p^4 q^2) = {}^6C_2\, p^2 q^4 \quad \Rightarrow \quad 9p^2 = q^2 \qquad [\because \; {}^nC_r = {}^nC_{n-r}]$

$\Rightarrow \quad 9p^2 = (1-p)^2 \qquad [\because \; q = 1 - p]$

$\Rightarrow \quad 8p^2 + 2p - 1 = 0$

$\Rightarrow \quad 8p^2 + 4p - 2p - 1 = 0$

$\Rightarrow \quad (2p + 1)(4p - 1) = 0$

$\Rightarrow \quad p = \dfrac{1}{4} \qquad (\because \; 0 \le p \le 1)$

$$\therefore \quad P(X = 1) = {}^6C_1 \left(\frac{1}{4}\right)\left(1 - \frac{1}{4}\right)^5 = \frac{6 \times 3^5}{4^6} = 0.356.$$

Example 9: *If the probability of hitting a target is 10% and 10 shots are fired independently. What is the probability that the target will be hit at least once?*

Solution: Let the random variable X corresponds to the number of hitting a target out of 10 shots.

Here $X \sim B(n, p)$, where n = Number of shots which are fired independently $= 10$

and $p = P(\text{hitting a target}) = \dfrac{10}{100} = 0.1$

$$\therefore \quad P(X = r) = {}^{10}C_r\,(0.1)^r(1-0.1)^{10-r} = {}^{10}C_r(0.1)^r(0.9)^{10-r}, \quad r = 0, 1, 2, \ldots, 10$$

$\therefore$ Required probability $= P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - {}^{10}C_0\,(0.1)^0(0.9)^{10}$$

$$= 1 - (0.9)^{10} = 0.6513.$$

Example 10: *In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts?*

Solution: Given, mean number of defectives $= 2 = np = 20p$.

$\therefore$ Probability of a defective part in a single trial is $p = \dfrac{2}{20} = 0.1$

So, $q = 1 - p = 0.9$ = probability of a non-defective part in a single trial.

$\therefore$ Probability of at least three defectives in a sample of 20

$$= P(X \geq 3) = 1 - P(X < 3)$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

[where the r.v. X corresponds to the no. of defective parts and $X \sim B\,(n, p) = B(20, 0.1)$]

$$= 1 - \left[{}^{20}C_0\,(0.9)^{20} + {}^{20}C_1\,(0.1)\,(0.9)^{19} + {}^{20}C_2\,(0.1)^2\,(0.9)^{18}\right]$$

$$= 1 - (0.9)^{18}\left[(0.9)^2 + 20 \times 0.1 \times 0.9 + \frac{1}{2} \times 20 \times (20 - 1) \times (0.1)^2\right]$$

$$= 0.323.$$

Therefore, the number of samples having at least three defective parts out of 1000 samples $= 1000 \times 0.323 = 323$.

Example 11: *The incidence of occupational disease in an industry is such that the workers have a 10 % chance of suffering from it. What is the probability that out of 6 workers, 3 or more will suffer from disease?*

Solution: Let the r.v. X corresponds to the number of workers suffering from disease out of 6 workers.

Here $X \sim B(n, p)$, where $n = 6$ and $p = \dfrac{10}{100} = 0.1$.

$$\therefore \quad P(X = r) = {}^{6}C_r\,(0.1)^r\,(1-0.1)^{6-r}$$

$$= {}^{6}C_r\,(0.1)^r(0.9)^{6-r}, \quad r = 0, 1, 2, \ldots, 6$$

$\therefore$ Required probability $= P(X \geq 3) = 1 - P(X < 3)$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - [{}^{6}C_0\,(0.9)^6 + {}^{6}C_1\,(0.1)\,(0.9)^5 + {}^{6}C_2\,(0.1)^2\,(0.9)^4]$$

$$= 1 - (0.9)^4 [(0.9)^2 + 6 \times 0.1 \times 0.9 + \frac{1}{2} \times 6 \times 5 \times (0.1)^2]$$

$$= 1 - (0.9)^4 \times 1.5 = 0.01585.$$

Example 12: *Six dice are thrown 729 times. How many times do you expect at least three dice to show a 5 or 6?*

Solution: Let the r.v. X corresponds to the number of times of occurrence of face 5 or 6 when six dice are thrown.

Here $X \sim B(n, p)$, where $n = 6$

and p = Probability of occurrence of face 5 or 6 when a single die is thrown

$$= \frac{2}{6} = \frac{1}{3}.$$

$$\therefore \quad P(X = r) = {}^6C_r \left(\frac{1}{3}\right)^r \left(1 - \frac{1}{3}\right)^{6-r} = {}^6C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{6-r}, \ r = 0, 1, 2, ..., 6.$$

$$\therefore \quad P(X \geq 3) = 1 - P(X < 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - \left[{}^6C_0 \left(\frac{2}{3}\right)^6 + {}^6C_1 \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^5 + .{}^6C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4\right]$$

$$= 1 - \left(\frac{2}{3}\right)^4 \left[\left(\frac{2}{3}\right)^2 + 6 \times \frac{1}{3} \times \frac{2}{3} + \frac{1}{2} \times 6 \times 5 \times \frac{1}{9}\right] = 1 - \frac{496}{729} = \frac{233}{729}$$

$$\therefore \quad \text{Expected number} = 729 \times \frac{233}{729} = 233.$$

Example 13: *The probability that a man aged 65 will live to be 70 is 0.7. Find the probability that out of 10 men, now 65, at least 7 will live to be 70.*

Solution: Let the r.v. X corresponds to the number of men, now 65, will live to be 70 out of 10 men.

Here $X \sim B\ (n, p)$, where $n = 10$

and p = Probability that a man aged 65 will live to be 70 = 0.7.

$$\therefore \quad P(X = r) = {}^{10}C_r\ (0.7)^r\ (1 - 0.7)^{10-r} = {}^{10}C_r\ (0.7)^r\ (0.3)^{10-r}, \ r = 0, 1, 2, ..., 10.$$

$\therefore$ Required probability $= P(X \geq 7)$

$$= P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

$$= {}^{10}C_7\ (0.7)^7\ (0.3)^3 + {}^{10}C_8\ (0.7)^8\ (0.3)^2 + {}^{10}C_9\ (0.7)^9\ (0.3) + {}^{10}C_{10}\ (0.7)^{10}$$

$$= (0.7)^7 \left[\frac{10!}{7!3!}(0.3)^3 + \frac{10!}{8!2!}(0.7)\ (0.3)^2 + 10(0.7)^2(0.3) + (0.7)^3\right]$$

$$= (0.7)^7\ [120\ (0.3)^3 + 45\ (0.7)\ (0.3)^2 + 3\ (0.7)^2 + (0.7)^3]$$

$$= (0.7)^7 \times 7.888 = 0.6496.$$

Example 14: *Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys and 2 girls (ii) at least one boy (iii) no girl (iv) at most two girls (v) children of both sexes? Assume equal probabilities for boys and girls.*

Solution: Let the random variable X corresponds to the number of boys among 4 children.

Here $X \sim B(n, p)$,

where $n = 4$

and p = Probability of having a boy in a single trial $= \frac{1}{2}$

$$\therefore \quad P(X = r) = {}^4C_r \left(\frac{1}{2}\right)^r \left(1 - \frac{1}{2}\right)^{4-r} = {}^4C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{4-r}, \; r = 0, 1, 2, 3, 4.$$

(*i*) The expected number of families having 2 boys and 2 girls

$$= 800\, P(X = 2) = 800\; {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2$$

$$= 800 \times \frac{1}{2} \times 4 \times 3 \times \frac{1}{16} = 300.$$

(*ii*) The expected number of families having at least one boy

$$= 800\, P(X \geq 1) = 800\,[1 - P(X = 0)]$$

$$= 800\left[1 - {}^4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4\right] = 800\left(1 - \frac{1}{16}\right) = 750.$$

(*iii*) The expected number of families having no girl (*i.e.*, having 4 boys)

$$= 800\, P(X = 4) = 800\; {}^4C_4 \left(\frac{1}{2}\right)^4 = 800 \times \frac{1}{16} = 50.$$

(*iv*) The expected number of families having at most two girls (*i.e.*, having at least two boys)

$$= 800\, P(X \geq 2) = 800\,[1 - P(X < 2)] = 800\left[1 - \{P(X = 0) + P(X = 1)\}\right]$$

$$= 800\left[1 - \left\{{}^4C_0 \left(\frac{1}{2}\right)^4 + {}^4C_1 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3\right\}\right] = 800\left[1 - \left(\frac{1}{16} + \frac{1}{4}\right)\right]$$

$$= 800 \times \frac{11}{16} = 550.$$

(*v*) The expected number of families having children of both sexes

$$= 800\, P(1 \leq X \leq 3) = 800\,\{P(X = 1) + P(X = 2) + P(X = 3)\}$$

$$= 800\left\{{}^4C_1 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 + {}^4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)\right\}$$

$$= 800 \times \frac{1}{16}(4 + 6 + 4) = 50 \times 14 = 700.$$

~~Place after Ex 14 in p.133~~

Example 15: Suppose that an airplane engine will fail, when in flight, with probability $(1-p)$ independently from engine to engine. Suppose that the airplane will make a successful flight if at least 50% of its engines remain operative. For what values of p is a four-engine plane preferable to a two-engine plane? (WBUT-2014)

Solution

Here, p = probability of an engine staying functional during flight

$$P(\text{four engined airplane can fly}) = {}^4C_4\, p^4 + {}^4C_3\, p^3(1-p) + {}^4C_2\, p^2(1-p)^2$$
$$= p^4 + 4p^3(1-p) + 6p^2(1-p)^2.$$

$$P(\text{two engined airplane can fly}) = {}^2C_2\, p^2 + {}^2C_1\, p(1-p) = p^2 + 2p(1-p).$$

Four engined plane is preferable if and only if

$$p^4 + 4p^3(1-p) + 6p^2(1-p)^2 > p^2 + 2p(1-p)$$
$$\Rightarrow p^4 + 4p^3 - 4p^4 + 6p^2(1-2p+p^2) > p^2 + 2p - 2p^2$$
$$\Rightarrow 3p^4 - 8p^3 + 6p^2 > 2p - p^2$$
$$\Rightarrow p(3p^3 - 8p^2 + 7p - 2) > 0$$
$$\Rightarrow 3p^3 - 8p^2 + 7p - 2 > 0$$
$$\Rightarrow 3p^3 - 3p^2 - 5p^2 + 5p + 2p - 2 > 0$$
$$\Rightarrow 3p^2(p-1) - 5p(p-1) + 2(p-1) > 0$$
$$\Rightarrow (p-1)(3p^2 - 5p + 2) > 0$$
$$\Rightarrow (p-1)(3p^2 - 3p - 2p + 2) > 0$$
$$\Rightarrow (p-1)^2(3p-2) > 0 \Rightarrow p > \frac{2}{3}.$$

$\therefore$ Four engined plane is preferable iff $\frac{2}{3} < p \le 1$. For $p = \frac{2}{3}$, the two and four engined planes are equally safe.

Example 16: Two persons A and B toss an ordinary die alternatively in succession which is to be won by the person whose first throw shows a 6. If A has the first throw, find the probability of winning for A and B.

Solution

Chance of throwing a 6 is $\frac{1}{6}$ and not throwing 6 is $1-\frac{1}{6}=\frac{5}{6}$.

A can win in 1^{st}, 3^{rd}, 5^{th}, ... throws

A's chance of getting 6 in the first throw $=\frac{1}{6}$.

A will get 3^{rd} chance if he fails in 1^{st} and B fails in 2^{nd}.

$\therefore$ A's chance of getting 6 in 3^{rd} throw $=\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}$.

Again A will get 5^{th} chance if he fails twice and B fails twice.

$\therefore$ A's chance of getting 6 in 5^{th} throw $=\left(\frac{5}{6}\right)^2\left(\frac{5}{6}\right)^2\frac{1}{6}$.

$\therefore$ A's probability of winning

$$=\frac{1}{6}+\left(\frac{5}{6}\right)^2\frac{1}{6}+\left(\frac{5}{6}\right)^4\frac{1}{6}+\cdots\cdots \text{ to } \infty$$

$$=\frac{\frac{1}{6}}{1-\left(\frac{5}{6}\right)^2}=\frac{6}{11}.$$

and B's probability of winning $=1-\frac{6}{11}=\frac{5}{11}$.