Place C10 – C15 before 'Multiple Choice Questions' in p.176

3.16. Moment Generating Function (m.g.f.) (about origin)

Definition: The moment generating function (about origin) of a random variable (r.v.) X is denoted and defined as

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_r e^{tx_r} p_r, & \text{if X is a d.r.v.} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, & \text{if X is a c.r.v.,} \end{cases}$$

where t is a real variable.

Deductions

(i) Let X be discrete

Assuming term by term differentiation w.r.t. t is permissible, we have

$$\frac{d}{dt} M_X(t) = \sum_r x_r e^{tx_r} p_r$$

$$\frac{d^2}{dt^2} M_X(t) = \sum_r x_r^2 e^{tx_r} p_r$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{d^k}{dt^k} M_X(t) = \sum_r x_r^k e^{tx_r} p_r$$

$$\Rightarrow \boxed{\left[\frac{d^k}{dt^k} M_X(t)\right]_{t=0} = \sum_r x_r^k p_r = E(X^k) = \alpha_k;\ k = 1, 2, 3, \ldots}$$

(ii) Let X be continuous

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx.$$

Assuming the differentiation under the sign of integration, we have

$$\frac{d}{dt} M_x(t) = \int_{-\infty}^{\infty} x e^{tx} f(x)\, dx$$

$$\frac{d^2}{dt^2} M_X(t) = \int_{-\infty}^{\infty} x^2 e^{tx} f(x)\, dx$$

. . . .

$$\frac{d^K}{dt^K} M_X(t) = \int_{-\infty}^{\infty} x^K e^{tx} f(x)\, dx.$$

$$\Rightarrow \boxed{\left[\frac{d^K}{dt^K} M_X(t)\right]_{t=0} = \int_{-\infty}^{\infty} x^K f(x)\, dx = E(X^K) = \alpha_K ;\ K = 1, 2, 3, \cdots}$$

Example 1. Find the MGF of Poisson variate $X(\lambda)$ and hence find its mean and variance.

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Solution. For Poisson variate $X(\lambda)$:

$$P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!} ;\ x = 0, 1, 2, \cdots$$

$$\therefore M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!}$$

$$= e^{-\lambda} . e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

$$\therefore \text{Mean} = E(X) = \left.\frac{d}{dt} M_X(t)\right|_{t=0} = \left.\lambda e^t e^{\lambda(e^t - 1)}\right|_{t=0} = \lambda.$$

$$E(X^2) = \left.\frac{d^2}{dt^2} M_X(t)\right|_{t=0} = \left.\lambda e^t (1 + \lambda e^t) e^{\lambda(e^t - 1)}\right|_{t=0} = \lambda(1 + \lambda).$$

$$\therefore \text{Var}(X) = E(X^2) - \{E(X)\}^2 = \lambda(1 + \lambda) - \lambda^2 = \lambda.$$

Example 2. Find the moment generating function of $X \sim N(\mu, \sigma)$ and hence find its mean and variance.

Solution The p.d.f. of $X \sim N(\mu, \sigma)$ is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty.$$

$$\therefore M_X(t) = E(e^{tX}) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-z^2/2} dz \quad \left(\text{setting } \frac{x-\mu}{\sigma} = z\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu t} e^{-\frac{1}{2}(z^2 - 2t\sigma z)} dz = \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\{(z-\sigma t)^2 - \sigma^2 t^2\}} dz$$

$$= \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du \quad (\text{setting } z - \sigma t = u)$$

Now, $\displaystyle\int_{-\infty}^{\infty} e^{-u^2/2} du = 2\int_{0}^{\infty} e^{-u^2/2} du$ $\left[\because e^{-u^2/2} \text{ is an even function}\right]$

$$= \frac{2}{\sqrt{2}} \int_{0}^{\infty} e^{-y} y^{-\frac{1}{2}} dy \quad \left(\text{setting } \frac{u^2}{2} = y \Rightarrow u\,du = dy \Rightarrow du = \frac{dy}{\sqrt{2y}}\right)$$

$$= \sqrt{2} \int_{0}^{\infty} e^{-y} y^{\frac{1}{2}-1} dy = \sqrt{2}\,\Gamma\left(\tfrac{1}{2}\right) = \sqrt{2\pi}.$$

$$\therefore M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$\therefore \text{Mean} = E(X) = \frac{d}{dt} M_X(t)\Big|_{t=0} = (\mu + \sigma^2 t) e^{\mu t + \frac{1}{2}\sigma^2 t^2}\Big|_{t=0} = \mu.$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = \left\{\sigma^2 + (\mu + \sigma^2 t)^2\right\} e^{\mu t + \frac{1}{2}\sigma^2 t^2}\Big|_{t=0} = \sigma^2 + \mu^2.$$

$$\therefore \text{Var}(X) = E(X^2) - \{E(X)\}^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2.$$

Example 3 Find the moment generating function of geometric distribution and hence find its mean and variance.

Solution If X is a geometric random variable with parameter p, then X has the following probability mass function:

$$f(x;p) = P(X=x) = q^{x-1}p, \quad x=1,2,3,\cdots; \; p+q=1, \; 0 \le p \le 1$$

Hence the m.g.f. of X is given by

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} P(X=x) = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1} = pe^t \{1 + (qe^t) + (qe^t)^2 + (qe^t)^3 + \cdots\}$$

$$= pe^t (1-qe^t)^{-1} = \frac{pe^t}{1-qe^t}.$$

Now, $$\frac{d}{dt} M_X(t) = \frac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2} = \frac{pe^t}{(1-qe^t)^2},$$

$$\frac{d^2}{dt^2} M_X(t) = \frac{(1-qe^t)^2 pe^t - pe^t\{2(1-qe^t)(-qe^t)\}}{(1-qe^t)^4}$$

$$= \frac{pe^t(1-q^2e^{2t})}{(1-qe^t)^4} = \frac{pe^t(1+qe^t)}{(1-qe^t)^3}.$$

$$\therefore \text{Mean} = E(X) = \left.\frac{d}{dt} M_X(t)\right|_{t=0} = \frac{p}{(1-q)^2} = \frac{1}{p} \quad [\because p+q=1]$$

$$E(X^2) = \left.\frac{d^2}{dt^2} M_X(t)\right|_{t=0} = \frac{p(1+q)}{(1-q)^3} = \frac{1+q}{p^2}$$

$$\therefore Var(X) = E(X^2) - \{E(X)\}^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}.$$

Example 4. Determine the moment generating function of an exponential random variable and hence find its mean and variance.

Solution Let X be an exponential random variable with parameter $\lambda (>0)$. Its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{elsewhere}. \end{cases}$$

By definition, the m.g.f. of X is given by

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x}\,dx = \lambda \lim_{B \to \infty} \int_0^B e^{-(\lambda - t)x}\,dx$$

$$= \lambda \lim_{B \to \infty} \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^B,$$ which converges provided $\lambda - |t| > 0$.

In this case, we have: $M_X(t) = \dfrac{\lambda}{\lambda - t}$, for $|t| < \lambda$.

$$\therefore M_X(t) = E(e^{tX}) = \frac{1}{1 - \left(\frac{t}{\lambda}\right)} = \left\{1 - \left(\frac{t}{\lambda}\right)\right\}^{-1} = 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \frac{t^3}{\lambda^3} + \cdots$$

$$\Rightarrow E\left\{1 + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + \cdots\right\} = 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \frac{t^3}{\lambda^3} + \cdots$$

$$\Rightarrow 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots = 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \frac{t^3}{\lambda^3} + \cdots$$

$$\Rightarrow E(X) = \frac{1}{\lambda}, \quad E(X^2) = \frac{2}{\lambda^2}.$$

$\therefore$ Mean $= E(X) = \dfrac{1}{\lambda}$. $\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = \dfrac{2}{\lambda^2} - \dfrac{1}{\lambda^2} = \dfrac{1}{\lambda^2}$

Example 5: Find the moment generating function of the binomial variate X with parameters n, p and hence find its mean and variance.

Solution. Here $X \sim B(n,p)$

$\therefore P(X=x) = {}^nC_x p^x q^{n-x}$, where $0 \le p \le 1$, $p+q=1$,

$x = 0, 1, 2, 3, \ldots, n$

$$\therefore M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} P(X=x) = \sum_{x=0}^{n} e^{tx}\, {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x} = (q+pe^t)^n.$$

$$\therefore \text{Mean} = E(X) = \frac{d}{dt} M_X(t)\Big|_{t=0} = n(q+pe^t)^{n-1} pe^t\Big|_{t=0} = np.$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = n(q+pe^t)^{n-1} pe^t + n(n-1)(q+pe^t)^{n-2} p^2e^{2t}$$

$$= n(q+pe^t)^{n-1} pe^t + n(n-1)(q+pe^t)^{n-2} p^2 e^{2t}\Big|_{t=0}$$

$$= np + n(n-1)p^2$$

$$\therefore Var(X) = E(X^2) - \{E(X)\}^2 = np + n(n-1)p^2 - n^2p^2$$

$$= np - np^2 = np(1-p) = npq.$$