Q3) show that $X = \{(x_1, x_2) \mid x_2 - 3 \geq -x_1^2, x_1, x_2 \geq 0\}$ is not a convex set.

→ Let (x_{11}, x_{12}) and (x_{21}, x_{22}) be ~~part of~~ a point in X

∴ we can say that

$$x_{12} - 3 \geq -x_{11}^2 \quad , x_{11}, x_{12} \geq 0 \quad (i)$$
$$x_{22} - 3 \geq -x_{21}^2 \quad , x_{21}, x_{22} \geq 0 \quad (ii)$$

→ now if we can show that convex combination of these two points also lies in X, then X is a convex set

→ let $(x_{31}, x_{32}) = \lambda (x_{11}, x_{12}) + (1-\lambda)(x_{21}, x_{22})$

$= \cancel{\lambda x_{11}+} \quad (\lambda \in [0,1])$

∴ $x_{31} = \lambda x_{11} + (1-\lambda) x_{21}$

$x_{32} = \lambda x_{12} + (1-\lambda) x_{22}$

→ now $x_{31}, x_{32} \geq 0$ as both λ and $(1-\lambda)$ are greater then zero and $x_{11}, x_{21}, x_{12}, x_{22} \geq 0$

→ now checking if $x_{32} - 3 \geq -x_{31}^2$ is true or not

$$\lambda x_{12} + (1-\lambda) x_{22} - 3 \geq -(\lambda x_{11} + (1-\lambda) x_{21})^2$$

$$\lambda x_{12} + (1-\lambda) x_{22} - 3 \geq -\lambda^2 x_{11}^2 - (1-\lambda)^2 x_{21}^2 - 2\lambda(1-\lambda) x_{11} x_{21}$$

$$\geq -\lambda^2 x_{11}^2 - (1-\lambda)^2 x_{21}^2 - 2\lambda(1-\lambda)\left[\frac{x_{11}^2 + x_{21}^2}{2}\right]$$

$$\cancel{\geq -\lambda}$$

$$\geq -\lambda x_{11}^2 (\lambda + 1 - \lambda) - (1-\lambda) x_{21}^2 [1 - \lambda + \lambda]$$

$$\geq -\lambda^2$$

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→ now we have to see that if

$$x_{32} - 3 \geq -x_{31}^2$$

or $x_{32} + x_{31}^2 \geq 3$

is true or not.

consider ~~$x_{32} + x_{31}^2$ x_{32}~~

consider

$$x_{32} + x_{31}^2 = \lambda x_{12} + (1-\lambda)x_{22} + \left[\lambda x_{11} + (1-\lambda)x_{21}\right]^2$$

$$= \lambda x_{12} + (1-\lambda)x_{22} + \left[\lambda^2 x_{11}^2 + (1-\lambda)^2 x_{21}^2 + 2\lambda(1-\lambda)x_{11}x_{21}\right]$$

as AM $\geq$ GM

we can say that $\frac{x_{11}^2+x_{21}^2}{2} \geq x_{11}x_{21}$

$$\therefore x_{32}+x_{31}^2 \leq \lambda x_{12} + (1-\lambda)x_{22} + \left[\lambda^2 x_{11}^2 + (1-\lambda)^2 x_{21}^2 + \lambda(1-\lambda)\left[x_{11}^2+x_{21}^2\right]\right]$$

$$\leq \lambda x_{12} + (1-\lambda)x_{22} + \lambda x_{11}^2\left[\lambda+1-\lambda\right] + x_{21}^2(1-\lambda)\left[1-\lambda+\lambda\right]$$

$$\leq \lambda x_{12} + (1-\lambda)x_{22} + \lambda x_{11}^2 + (1-\lambda)x_{21}^2$$

$$\leq \lambda(x_{12}+x_{11}^2) + (1-\lambda)(x_{22}+x_{21}^2) \quad \text{(iii)}$$

~~$\leq 3\lambda + 3(1-\lambda)$~~

~~≤ 3~~

∴ ~~we can see that $(x_{31}, x_{32}) \notin X$~~

∴ ~~X is not a convex set~~

→ now if we include (i) and (ii) in (iii), we see that in some point inequality may reverse

→ hence ~~whole of~~ we can not say that ~~set~~ convex combination of two point inside X, will stay inside X

→ hence X is not a convex set