Basic Discrete Structures

Sets, Functions, Sequences, Matrices, and Relations
(Lecture – 5)

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Relations

- Relationships between elements of sets occur in many contexts
- They are represented using the structure called a **relation**, which is just a *subset of the Cartesian product of the sets.*
- Few applications: determining which pairs of cities are linked by airline flights in a network, finding a viable order for the different phases of a complicated project, or producing a useful way to store information in computer databases.
- The most direct way to express a relationship between elements of two sets is to use *ordered pairs* made up of two related elements.
- The sets of ordered pairs are called **binary relations**.

Binary Relation

Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

- A binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.
- The notation $a\ R\ b$ denotes that $(a, b) \in R$ and $a \not R\ b$ denotes that $(a, b) \notin R$.
- If $(a, b) \in R$, a is said to be **related to** b by R.
- **<u>Relations on a Set</u>**: A relation on a set A is a subset from AXA

Properties of Relations

- **Reflexive Property**

A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

- Consider the following relations on $\{1, 2, 3, 4\}$:

$$R1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$

$$R2 = \{(1, 1), (1, 2), (2, 1)\},$$

$$R3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$

$$R4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$

$$R5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$

$$R6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Properties of Relations

- **Symmetric Property**

A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

- A relation is *symmetric* if and only if a is related to b implies that b is related to a.
- A relation is *antisymmetric* if and only if there are no pairs of distinct elements a and b with a related to b and b related to a.
- The terms *symmetric* and *antisymmetric* are not opposites, because a relation can have both of these properties or may lack both of them.

Properties of Relations

- Consider the following relations on $\{1, 2, 3, 4\}$:

  $R1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$

  $R2 = \{(1, 1), (1, 2), (2, 1)\},$

  $R3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$

  $R4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$

  $R5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$

  $R6 = \{(3, 4)\}.$

Which of these relations are symmetric and antisymmetric?

- Consider these relations on the set of integers:

  $R1 = \{(a, b) \mid a \leq b\},$

  $R2 = \{(a, b) \mid a > b\},$

  $R3 = \{(a, b) \mid a = b \text{ or } a = -b\},$

  $R4 = \{(a, b) \mid a = b\},$

  $R5 = \{(a, b) \mid a = b + 1\},$

  $R6 = \{(a, b) \mid a + b \leq 3\}.$

  Which of these relations are symmetric and antisymmetric?

Properties of Relations

- **Transitive Property**

A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

- Consider the following relations on $\{1, 2, 3, 4\}$:

  $R1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

  $R2 = \{(1, 1), (1, 2), (2, 1)\}$,

  $R3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

  $R4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

  $R5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

  $R6 = \{(3, 4)\}$.

Which of these relations transitive?

- Consider these relations on the set of integers:

  $R1 = \{(a, b) \mid a \leq b\}$,

  $R2 = \{(a, b) \mid a > b\}$,

  $R3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,

  $R4 = \{(a, b) \mid a = b\}$,

  $R5 = \{(a, b) \mid a = b + 1\}$,

  $R6 = \{(a, b) \mid a + b \leq 3\}$.

  Which of these relations is transitive?

Combining Relations

- Combining Relations
  - Union, intersection, set difference, orthogonal sum

- Composite Relation

Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

- Recursive definition of Relation

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots,$ are defined recursively by

$$R^1 = R \qquad \text{and} \qquad R^{n+1} = R^n \circ R.$$

Representing Relations using Matrices

- A relation between finite sets can be represented using a zero–one matrix.
- Suppose that R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$. It can be represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

- In other words, the zero–one matrix representing R has a 1 as its (i, j) entry when a_i is related to b_j, and a 0 in this position if a_i is not related to b_j. (Such a representation depends on the orderings used for A and B.)