Relation-4.

Wednesday, October 14, 2020 8:48 AM

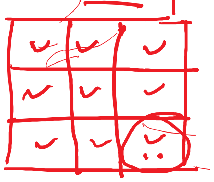
Lemma-1

Proof: Suppose there is a ~~pa~~ path from a to b in R. Let m be the length of the shortest such path. Suppose that $x_0, x_1, x_2, \dots x_{m-1}, x_m$ where $x_0 = a$ and $x_m = b$.

Case(i): When a and b are the terminal vertices of a circuit/cycle.

Suppose that $a = b$ and $m > n$, so then $m \geq n+1$. Because there are n vertices, among the m vertices of $x_0, x_1, x_2 \dots x_m$, at least two are equal. This implies $m \leq n$.

PH Principle.



n boxes.
k objects.

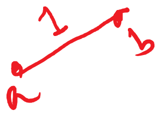
$k > n$

At least one box will have more than one obj.

Case(ii): When $a \neq b$, ~~but~~ a circuit/cycle exists in the path.

Suppose $x_i = x_j$, with $0 \leq i < j \leq m-1$. Then the path contains a cycle from x_i to itself. This circuit can be deleted from the path from a to b, ~~lea~~ leaving a path $x_0, x_1, x_2 \dots x_{m-1}, x_m$ from a to b of shorter length. Hence, the length of the shortest path must be less than or equal to n.

Case(iii): When $a \neq b$ and there is no circuit / cycle.



basis step ~~case~~

By definition, if there is a single vertex, $l(1) = 0$.

If there are two vertices, $l(2) = 1$.

Assume that the lemma is true for a positive integer n. i.e. $l(n) = n-1$ (inductive hypothesis).

To complete the proof we have to show $l(n+1) = n$.

Let A contains $(n+1)$ vertices. Consequently, the length of the shortest path will be $l(n+1)$.

Using inductive hypothesis, get $l(n+1) = l(n) + 1$.

i.e. $l(n+1) = n - \cancel{1} + \cancel{1} = n$.

Therefore, the length of the shortest path in a graph with n vertices is $(n-1)$.

Prob

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_R^{[2]} = M_R \odot M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_R^{[3]} = M_R^{[2]} \odot M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Algorithm to compute transitive closure.

Compute boolean powers of $M_R, M_R^{[2]}, \ldots, M_R^{[n]}$ requires $(n-1)$ boolean product of $n \times n$ zero-one matrices.

To generate a single element in a Boolean power $M_R^{[i]}$, n products and $(n-1)$ joins are performed.

Therefore, the no. of bit operations reqd. to generate a single element $= n + n - 1 = 2n - 1$. This has to be repeated for n^2 positions in $M_R^{[i]}$. The no. of bit operations to generate $M_R^{[i]} = n^2(2n-1)$

This has to be repeated for $(n-1)$ M_R's yielding $n^2(2n-1)(n-1)$ bit operations.

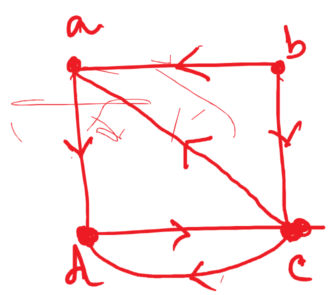
$n^2(2n-1)(n-1)$ bit operations.

To find M_{R}^{*} from n boolean powers of M_R, $(n-1)$ joins (disjunction) of zero-one matrices have to performed. Computing each of these joins uses n^2 bit operations, yielding a total of $n^2(n-1)$ bit operations.

Total no. of bit operations $= n^2(2n-1)(n-1) + n^2(n-1)$

$$= 2n^3(n-1) \approx O(n^4).$$

Prob.



Let $v_1 = a$, $v_2 = b$, $v_3 = c$ & $v_4 = d$.

W_0 is the matrix of the relation. Hence,

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

W_1 has 1 in the $(i,j)^{th}$ position if there is a path from v_i to v_j that has ~~ed~~ only $v_1 = a$

as an interior vertex. We get a new path from b to d namely, b, a, d. Hence,

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

✓ W_2 will have 1 in the $(i,j)^{th}$ position if there is a path from v_i to v_j that has only $v_1 = a$ and/or $v_2 = b$ as its interior vertices. Because there are no new path so $W_2 = W_1$.

✓ W_3 will have 1 in the $(i,j)^{th}$ position if there is a path from v_i to v_j that has only $v_1 = a$, $v_2 = b$, and/or $v_3 = c$ as the interior vertices. We now have two paths – (i) d to a, namely, d, c, a and (ii) d to d, namely d, c, d. Hence,

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

✓ W_4 will have 1 in $(i,j)^{th}$ position if there is a path from v_i to v_j that has

$v_1 = a$, $v_2 = b$, $v_3 = c$, and/or $v_4 = d$ as the interior vertices.

$$W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$\therefore$ W_4 is the required transitive closure.