# Sets & Functions-1

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## ~~Ros~~ Roster method of representing set

1. $V = \{a, e, i, o, u\} \rightarrow$ set of vowels.
2. $O = \{1, 3, 5, 7, 9\} \rightarrow$ odd positive integers less than 10.
3. $T = \{1, 2, 3, \ldots, 99\} \rightarrow$ set of positive integers $< 100$.

## Set builder notation for describing sets.

1. $O = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$
2. $Q^+ = \{x \in R \mid x = p/q, \text{ for some positive integers } p \,\&\, q\}$

**Theorem:** For every set $S$, (i) $\phi \subseteq S$ and (ii) $S \subseteq S$.

$A \subseteq B$.

$\boxed{\forall x (x \in A \rightarrow x \in B)}$

**Proof:** To show that ~~$\phi \in$~~ $\phi \subseteq S$, we must show $\forall x (x \in \phi \rightarrow x \in S)$ is true. Because empty set do not contain any element it follows that $x \in \phi$ is always false. It follows that the conditional statement ~~$x \Rightarrow$~~ $x \in \phi \rightarrow x \in S$ is always true, because the hypothesis of the conditional statement is false. Therefore, $\phi \subseteq S$ is always true.

... vacuous proof.

false. Therefore, $\phi \subseteq$ [illegible]

This is an example vacuous proof.

To show that $S \subseteq S$, we must show $\forall x (x \in S \rightarrow x \in S)$. This can be shown by trivial proof.

**Powerset**

If $|S| = n$, then $|P(S)| = 2^n$. If we discard the empty set then $|P(S)| = 2^n - 1$.

* What is the power set of the empty set?

→ The set $\{\phi\}$ has exactly two subsets: $\phi$ and $\{\phi\}$.

Consequently, $P(\{\phi\}) = \{\phi, \{\phi\}\}$

* Empty set and the set itself are the members of this power set.

General formula: if a set has $n$ elements, then its power set has $2^n$ elements.

**Cartesian Product**

$A = \{1, 2\}$, $B = \{a, b, c\}$

Find (i) $A \times B$ and (ii) $B \times A$.

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), \ldots\}$

Thus, $\boxed{A \times B \neq B \times A}$

$\boxed{R \subseteq A \times B}$

$S = \{0, 1, 2, 3\}$

What are the ordered pairs in the '$\leq$' relation?

$R \subseteq S \times S$.

$\hookrightarrow (a \leq b)$

$\underline{R} = \{(0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

$A = \{0, 1\}$, $B = \{1, 2\}$, $C = \{0, 1, 2\}$

The Cartesian product $A \times B \times C$ will contain all ordered triplets $(a, b, c)$ where $a \in A$, $b \in B$, and $c \in C$.

$A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2),$
$(0,2,0), (0,2,1), (0,2,2),$
$(1,1,0), (1,1,1), (1,1,2),$
$(1,2,0), (1,2,1), (1,2,2)\}$

$(1, 2, 0), (1, 2, 1), (\ldots)$

## Using set notations with Quantifiers:

$\forall x \in R \ (x^2 \geq 0)$ : The square of every real number is nonnegative (TRUE)

$\exists x \in Z \ (x^2 = 1)$ : There is an integer whose square is 1. (TRUE)

## Truth sets and Quantifiers

Given a predicate P, and a domain D, we define the truth set of P to be the set of elements $x \in D$ for which $P(x)$ is true. The truth set of $P(x)$ is denoted as: $\{x \in D \mid P(x)\}$.

Ex: $P(x)$ : $|x| = 1$.
$Q(x)$ : $x^2 = 2$
$R(x)$ : $|x| = x$

domain : set of integers. ($Z$)

Truth set of $P(x)$: $\{x \in Z \mid |x| = 1\}$

↳ set contains integers 1 & −1

✓ Truth set of $Q(n)$: $\{x \in Z \mid x^2 = 2\}$ integers 1 & -1.

↳ Empty set because no integer $x$ ~~satif~~ satisfies $x^2 = 2$.

✓ Truth set of $R(n)$: $\{x \in Z \mid |x| = x\}$

↳ Set contains all nonnegative integers.

Prob-1. $\overline{A \cap B} = \{x \mid x \notin A \cap B\}$ by defn. of complement

$= \{x \mid \neg(x \in A \cap B)\}$ by defn. of does not belong symbol.

$= \{x \mid \neg((x \in A) \wedge (x \in B))\}$ → by defn of intersection

$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$, by the First De Morgan's law for logical equivalence

$= \{x \mid (x \notin A) \vee (x \notin B)\}$, by definition of does not contain

$= \{x \mid (x \in \bar{A}) \vee (x \in \bar{B})\}$, by defn of

$= \{x \mid (x \in \overline{A}) \vee (x \in \overline{B})\}$, by defn. of complement

$= \{x \mid x \in \overline{A} \cup \overline{B}\}$, by defn. of union.

$= \overline{A} \cup \overline{B}$, by meaning of set builder notation

**Proof-2**

$A \cap (B \cup C) = \{x \mid (x \in A) \wedge x \in (B \cup C)\}$

$= \{x \mid (x \in A) \wedge ((x \in B) \vee (x \in C))\}$

$= \{x \mid ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))\}$

$= \{x \mid (x \in A \cap B) \vee (x \in A \cap C)\}$

$= (A \cap B) \cup (A \cap C)$

**Proof-3**

$\overline{A \cup (B \cap C)} = \overline{A} \cap (\overline{B \cap C})$, by first

$$A \cup (B \cap C) = A \cap (B \cap C),$$ by first De Morgan's law.

$$= \bar{A} \cap (\bar{B} \cup \bar{C}),$$ by second De Morgan's law.

$$= (\bar{B} \cup \bar{C}) \cap \bar{A},$$ by the Commutative property of intersection

$$= (\bar{C} \cup \bar{B}) \cap \bar{A},$$ by the Commutative property of union.

## Union of a collection of sets.

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^{n} A_i$$

## Intersection

$$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{i=1}^{n} A_i$$

$$\boxed{\begin{aligned} A_1 &= (1,2,3\dots) \\ A_2 &= (2,3,4,\dots) \\ A_3 &= (3,4,5\dots) \\ A_n &= (n, n+1, n+2\dots \end{aligned}}$$

**Ex:** For $i = 1, 2, \dots,$ let $A_i = \{i, i+1, i+2\dots\}$

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2\dots\} = \{1, 2, 3\dots\}$$

$$\bigcap^{n} A_i = \bigcap^{n} \{i, i+1, i+2, \dots\} = A_n.$$

$$\bigcap_{i=1} A_i = \bigcap_{i=1} \{i, i+1, i+2, \ldots\} = A_n.$$

$$A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n \cup \ldots = \bigcup_{i=1}^{\infty} A_i$$

$$A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n \cap \ldots = \bigcap_{i=1}^{\infty} A_i$$

$I$ : Set containing integers $1, 2, 3, \ldots$

$$\Rightarrow \bigcap_{i \in I} A_i = \{x \mid \forall_{i \in I} (x \in A_i)\}$$

$$\Rightarrow \bigcup_{i \in I} A_i = \{x \mid \exists_{i \in I} (x \in A_i)\}$$