CHAPTER 12

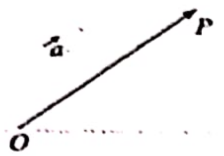
Vector Algebra

12.1 VECTORS

The physical quantities are primarily divided into two distinct classes: (*i*) **Scalars** (*ii*) **Vectors**.

A **scalar** is a quantity specified by magnitude only and does not involve the notion of direction, *e.g.*, mass, volume, time, temperature, density etc.

A **vector** is a quantity which requires both magnitude and direction, such as force, velocity, acceleration, displacement, electric current etc. A vector is represented by a directed line segment, say, $\overline{OP}$. The **magnitude** (or modulus) of this vector is indicated by the length OP, O is called the origin or initial point and P is called the terminal point or terminus.



A vector is also represented by a letter with an arrow over it such as $\vec{a}$ and its magnitude is denoted by $|\vec{a}|$.

Unit vector: A vector whose magnitude is unity is called a **unit vector**. An unit vector in the direction of a vector $\vec{a}$ is usually denoted by $\hat{a}$.

Zero vector or Null vector: A vector having magnitude zero is called a **zero vector** or **null vector** and is denoted by $\vec{0}$.

Equal vectors: Two vectors are said to be equal if they have the same magnitude and direction irrespective of the position of their initial points.

Like vectors: The vectors having the same direction are called **like vectors,** otherwise they are called **unlike vectors.**

Negative of a vector: It is a vector whose magnitude is equal to that of the given vector and is parallel but opposite in direction of the given vector.

12.2 ALGEBRA OF VECTORS

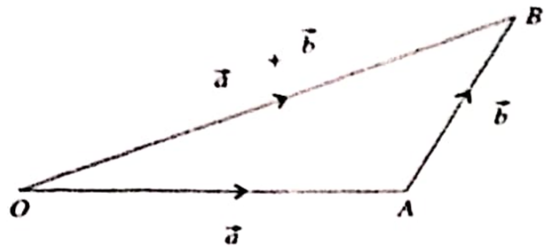
Multiplication of a Vector by a Real Number

Let λ is a non-zero real number, then $\lambda\vec{a}$ (or $\vec{a}\,\lambda$) is a vector whose magnitude is $|\lambda|$ times that of $\vec{a}$ and direction is the same or opposite as that of $\vec{a}$ according as λ is positive or negative.

For any vector $\vec{a}$, $0\vec{a}$ is defined as a zero vector having zero magnitude and indeterminate direction, denoted by $\vec{0}$.

It is observed that $\vec{a}$ and $-\vec{a}$ are vectors having same magnitude but opposite in direction, that is, one is negative to the other.

Addition of Vectors



A vector whose effect is the same as that of two given vectors, is called the sum or the resultant of the given vectors.

Let $\vec{a}$ and $\vec{b}$ are two given vectors. If $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$, then the vector $\overrightarrow{OB}$ is known as the sum of $\vec{a}$ and $\vec{b}$ and is symbolically written as

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

or

$$\vec{a} + \vec{b} = \overrightarrow{OB}$$

This is known as the **triangle law of addition.**

Properties

For any three vectors $\vec{a}, \vec{b}, \vec{c}$

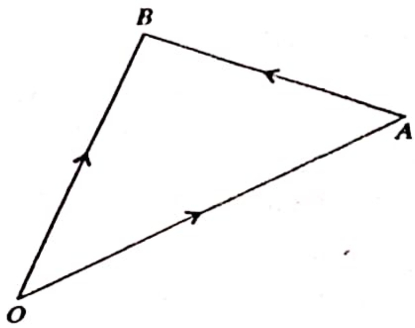
(*i*) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ (commutative law)

(*ii*) $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$ (associative law)

(*iii*) $\vec{a}-\vec{b} = \vec{a}+(-\vec{b}), \vec{a}-\vec{a} = \vec{a}+(-\vec{a}) = \vec{o}$

(*iv*) $\lambda(\vec{a}+\vec{b}) = \lambda\vec{a}+\lambda\vec{b}$, where λ is any real number.

Position Vector of a Point



In order to specify the position of a point P in space we require to choose an arbitrary point O in space as origin, then $\overrightarrow{OP}$ determines the position of P relative to O. The vector $\overrightarrow{OP}$ is called the **position vector** (p.v.) of P relative to a fixed origin O (O is selected in advance).

Using triangle law of addition, we have

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

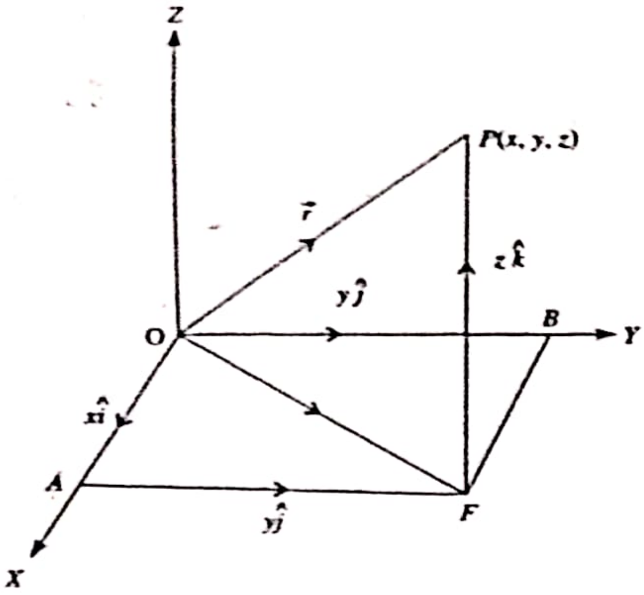
or

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$\therefore$ $\overrightarrow{AB}$ = Position vector of B – Position vector of A.

Let OX, OY, OZ be the three rectangular axes in space and let $\hat{i}, \hat{j}, \hat{k}$ be three unit vectors which are respectively parallel to X, Y, Z-axes having the positive directions of X, Y, Z-axes such that $\hat{i}, \hat{j}, \hat{k}$ form a right handed system .

Let $\overrightarrow{OP} = \vec{r}$ and the co-ordinates of P are (x, y, z). Using the definition of equality of two vectors and triangle law of vector addition, we get



$$\vec{r} = \overrightarrow{OP} = \overrightarrow{OF} + \overrightarrow{FP} = (\overrightarrow{OA} + \overrightarrow{AF}) + \overrightarrow{FP} = x\hat{i} + y\hat{j} + z\hat{k}.$$

Therefore, the position vector of the point P with respect to O is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Obviously $\quad OP = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

Therefore, if P, Q are two points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, then

$$\begin{aligned} \overrightarrow{PQ} &= \text{Position vector of } Q - \text{Position vector of } P \\ &= x_2\hat{i} + y_2\hat{j} + z_2\hat{k} - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) \\ &= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \end{aligned}$$

and $\quad PQ = |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Ratio Formula

The position vector of the point R which divides the line segment joining the points P and Q, with position vectors $\vec{a}$ and $\vec{b}$ respectively, in the ratio $m : n$ is $\dfrac{m\vec{b} + n\vec{a}}{m+n}$ or $\dfrac{m\vec{b} - n\vec{a}}{m-n}$ according as R divides PQ internally or externally respectively.

Note: Here $\hat{i}, \hat{j}, \hat{k}$ are independent unit vectors in the sense that there is no component of one along others since they are mutually perpendicular.

Collinearity of Three Points

Theorem

Three distinct points A, B, C will lie on a straight line (*i.e.*, collinear) if there exist three numbers x, y, z, not all zero, such that

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0},\ x + y + z = 0$$

where $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of A, B, C respectively with respect to a chosen origin O and conversely.

Proof. Let A, B, C are three distinct points lying on a straight line.

Therefore C must divide AB either internally or externally in some ratio $m : n$, where $\frac{m}{n} \neq 0, \infty$ or -1 and so $m, n, m + n \neq 0$.

$$\therefore \qquad \overrightarrow{AC} = \frac{m}{n}\overrightarrow{CB} \qquad (\because AC : CB = m : n)$$

or $$n(\vec{c} - \vec{a}) = m(\vec{b} - \vec{c}) \qquad [\because \overrightarrow{AC} = \text{p.v. of } C - \text{p.v. of } A = \vec{c} - \vec{a}]$$

or $$-n\vec{a} - m\vec{b} + (n + m)\vec{c} = \vec{0},$$

i.e., $$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0},$$

where $x = -n$, $y = -m$, $z = n + m$, so $x + y + z = 0$.

Hence, the conditions (1) follow from the fact of collinearity of A, B, C.

Conversely, let the conditions (1) hold.

Since x, y, z are not all zero, let $z \neq 0$, then $z = -(x + y)$.

Now, $$z\vec{c} = -(x\vec{a} + y\vec{b})$$

$$\therefore \qquad \vec{c} = \frac{-(x\vec{a} + y\vec{b})}{z} = \frac{-(x\vec{a} + y\vec{b})}{-(x + y)} = \frac{x\vec{a} + y\vec{b}}{x + y}.$$

This proves that $\vec{c}$ lies on the line of join of $\vec{a}$ and $\vec{b}$ and also divides the segment in the ratio $y : x$. Hence, A, B, C are collinear.

Coplanarity of Four Points

Theorem

Four points A, B, C, D (no three of them are collinear) will lie on the same plane (*i.e.*, coplanar) if there exist four numbers x, y, z, t, not all zero, such that

$$x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = \vec{0},\ x + y + z + t = 0 \qquad \ldots(2)$$

where $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are the position vectors of the points A, B, C, D respectively with respect to an origin O and conversely.

Proof: Let A, B, C, D are four points, no three collinear, lying on the same plane.

Therefore AB divides CD at some point P (not A, B, C, D). Suppose P divides AB and CD in the ratios $m : n$ and $p : q$ respectively.

$$\therefore \qquad \vec{p} = \frac{n\vec{a} + m\vec{b}}{n + m} = \frac{q\vec{c} + p\vec{d}}{q + p}, \qquad \ldots(3)$$

where $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{p}$ are the position vectors of A, B, C, D, P respectively with respect to an origin O and $\frac{m}{n}, \frac{p}{q} \neq 0, -1$.

From (3) we have

$$\frac{n}{m+n}\bar{a} + \frac{m}{m+n}\bar{b} - \frac{q}{p+q}\bar{c} - \frac{p}{p+q}\bar{d} = \bar{0}$$

or
$$x\bar{a} + y\bar{b} + z\bar{c} + t\bar{d} = \bar{0},$$

where $x = \frac{n}{m+n}$, $y = \frac{m}{m+n}$, $z = -\frac{q}{p+q}$, $t = -\frac{p}{p+q}$, so $x + y + z + t = 0$.

Hence, the conditions (2) follow from the fact of coplanarity of A, B, C, D.

Conversely let the conditions (2) hold

Observe that at least one of $x + y, y + z, z + t, t + x, x + z, y + t$ is not zero, for if they are all zero then $x = y = z = t = 0$, which is against the given condition.

Let us assume that $z + t \neq 0$ and hence, $x + y = -(z + t) \neq 0$. Dividing $x\bar{a} + y\bar{b} = -(z\bar{c} + t\bar{d})$ by $x + y$ and $-(z + t)$, we get

$$\frac{x\bar{a} + y\bar{b}}{x+y} = \frac{-(z\bar{c} + t\bar{d})}{-(z+t)} = \frac{z\bar{c} + t\bar{d}}{z+t} = \bar{p} \text{ (say)}$$

This proves that there exists a point P (with position vector $\bar{p}$) which divides AB in the ratio $y : x$ and CD in the ratio $t : z$, *i.e.*, AB and CD intersect at P. Hence, A, B, C, D must be coplanar.

Note: When $z + t = 0$, hence, $x + y = 0$, we get $x(\bar{a} - \bar{b}) = -z(\bar{c} - \bar{d}) = z(\bar{d} - \bar{c})$, *i.e.*, $x\overrightarrow{BA} = z\overrightarrow{CD}$, *i.e.*, AB and CD are parallel. Hence, A, B, C, D are coplanar.

Collinear and Coplanar Vectors

Collinear vectors: Two vectors $\bar{a}$ and $\bar{b}$ are said to be collinear (or parallel or like) if $\bar{a} = \lambda\bar{b}$ where λ is an arbitrary constant and conversely. A system of vectors is said to be collinear (or parallel or like) if they are parallel to the same straight line.

Coplanar vectors: A system of vectors is called coplanar if they are parallel to the same plane. For example, $\alpha\bar{a} + \beta\bar{b}$ is coplanar with the vectors $\bar{a}$ and $\bar{b}$, whatever the numbers α and β may be.

Theorems

1. If $\bar{a}$ and $\bar{b}$ are two non-zero collinear vectors then there exist non-zero numbers α, β such that $\alpha\bar{a} + \beta\bar{b} = \bar{0}$ and conversely.

(Note: If $\bar{a}, \bar{b}$ are non-collinear vectors then $\alpha\bar{a} + \beta\bar{b} = \bar{0}$ implies $\alpha = \beta = 0$.)

2. If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors then every vector $\vec{r}$ coplanar with $\vec{a}$ and $\vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}$ and $\vec{b}$, *i.e.*, $\vec{r} = \alpha\vec{a} + \beta\vec{b}$, where α, β are numbers.

Linearly Dependent and Independent Vectors

The vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n$ are said to be **linearly dependent** if there exist numbers $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$, not all zero, such that

$$\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \lambda_3\vec{a}_3 + \ldots + \lambda_n\vec{a}_n = \vec{0} \qquad \ldots(1)$$

Otherwise if (1) is true only for $\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_n = 0$, then the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n$ are said to be **linearly independent**.

12.3 PRODUCT OF TWO VECTORS

Scalar or Dot Product

The scalar or dot product of two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\cdot\vec{b}$ and is defined as

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

where $|\vec{a}|, |\vec{b}|$ are the magnitudes of the vectors $\vec{a}, \vec{b}$ respectively and θ is the angle between the vectors $\vec{a}$ and $\vec{b}$.

Hence, the angle between $\vec{a}$ and $\vec{b}$ is given by

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

Note: Scalar or dot product is a number, *i.e.*, it is a scalar.

Properties

(*i*) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

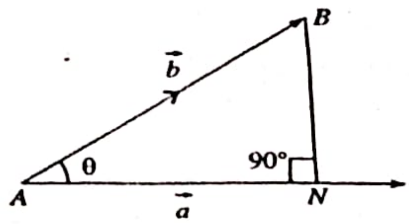
(*ii*) $\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos\theta = |\vec{a}|^2$ consequently $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$. ($\because \theta = 0$)

(*iii*) $\vec{a}\cdot\vec{b} = 0$ implies either $|\vec{a}| = 0$ (*i.e.*, $\vec{a}$ is a zero vector) or, $|\vec{b}| = 0$ (*i.e.*, $\vec{b}$ is a zero vector) or, $\vec{a}$ and $\vec{b}$ are mutually perpendicular, when neither $\vec{a} = \vec{0}$ nor $\vec{b} = \vec{0}$ and conversely.

Consequently, $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$, since $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along positive directions of *X*, *Y*, *Z* axes respectively and so they are mutually perpendicular.

(*iv*) Projection of $\vec{b}$ on $\vec{a} = AN = |\vec{b}|\cos\theta$

$$= |\vec{b}|\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$



Similarly projection of $\vec{a}$ on $\boxed{\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}}$.

(v) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c},\ (\vec{b}+\vec{c})\cdot\vec{a} = \vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}$

(vi) Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$.

Applying (v), (iii), (ii) respectively, we get

$$\vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3,$$

$$\vec{a}\cdot\vec{a} = |\vec{a}|^2 = a_1^2+a_2^2+a_3^2,$$

$$\vec{b}\cdot\vec{b} = |\vec{b}|^2 = b_1^2+b_2^2+b_3^2.$$

Therefore, the angle between $\vec{a}$ and $\vec{b}$ is given by

$$\cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\ \sqrt{b_1^2+b_2^2+b_3^2}}\right).$$

Observations

1. As $(\vec{a}\cdot\vec{b})\cdot\vec{c}$ is not defined (since scalar product is only defined between two vectors, not between a vector and a scalar), the associative law between three vectors need not be considered for scalar product. It is also noted that, in general, $(\vec{a}\cdot\vec{b})\vec{c} \neq \vec{a}(\vec{b}\cdot\vec{c})$, though both sides are defined.

2. $(\vec{a}+\vec{b})^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$ [using (v)]

$$= |\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2 \qquad (\because\ \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a})$$

Similarly, $(\vec{a}-\vec{b})^2 = |\vec{a}|^2-2\vec{a}\cdot\vec{b}+|\vec{b}|^2$

and $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2-|\vec{b}|^2$

3. Let $\vec{c}\neq\vec{0}$ and $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$

$\therefore$ $(\vec{a}-\vec{b})\cdot\vec{c} = 0$

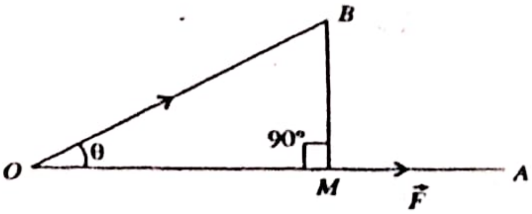
Therefore, either $\vec{a}-\vec{b} = \vec{0}$, *i.e.*, $\vec{a} = \vec{b}$ or $\vec{a}-\vec{b}$ is perpendicular to $\vec{c}$.

Hence, we conclude that we cannot cancel $\vec{c}$ in $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$ and obtain $\vec{a} = \vec{b}$ unless we know that $\vec{a}-\vec{b}$ and $\vec{c}$ are not perpendicular.

4. If the angle between $\vec{a}$ and $\vec{b}$ be θ, then the angle between $\alpha\vec{a}$ and $\vec{b}$ is either θ or $\pi-\theta$ according as α is positive or negative. The angle between $\vec{a}$ and $-\vec{b}$ (or $-\vec{a}$ and $\vec{b}$) is $\pi-\theta$, whereas the angle between $-\vec{a}$ and $-\vec{b}$ is θ.

Work Done

I. A force $\vec{F}$ is said to do work when its point of application moves. Suppose the force $\vec{F}$ is acting at a point O in the direction OA and suppose it displaces the point of application from O to B. Then the displacement in the direction of force $= OM = OB\cos\theta$ as shown in the adjacent figure. If W be the work done, then $W = |\vec{F}|OM = |\vec{F}|OB\cos\theta = \vec{F}\cdot\overline{OB}$.



II. If $\vec{F}$ be the resultant (or sum) of two forces $\vec{F}_1$ and $\vec{F}_2$ acting at the same point O, then $\vec{F} = \vec{F}_1 + \vec{F}_2$.

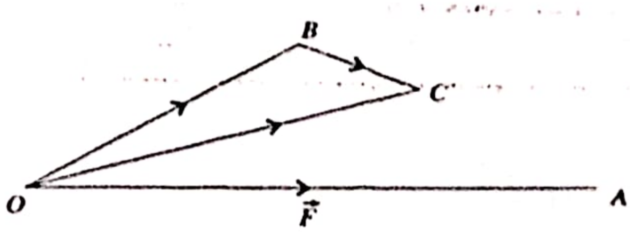
So, the work done by $\vec{F}$ is given by

$$W = \vec{F} \cdot \overline{OB} = (\vec{F}_1 + \vec{F}_2) \cdot \overline{OB}$$
$$= \vec{F}_1 \cdot \overline{OB} + \vec{F}_2 \cdot \overline{OB}$$

It shows that the work done by the resultant is equal to the sum of the works done by the component forces.

III. If the point of application of a force $\vec{F}$ moves from O to B and then from B to C as shown in the adjacent figure, the work done is

$$W = \vec{F} \cdot \overline{OB} + \vec{F} \cdot \overline{BC}$$
$$= \vec{F} \cdot (\overline{OB} + \overline{BC}) = \vec{F} \cdot \overline{OC}.$$



It shows that the sum of the works done by a force $\vec{F}$ in two consecutive displacements $\overline{OB}$ and $\overline{BC}$ is equal to the work done by $\vec{F}$ in the resultant displacements $\overline{OC}$.

Illustration: A particle is acted on by forces 9 and 10 units at direction whose direction cosines are $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$ and $0, \frac{4}{5}, \frac{3}{5}$ respectively displaces a particle from the point $2\hat{i} - \hat{j} - 3\hat{k}$ to the point $4\hat{i} - 3\hat{j} + 7\hat{k}$. Calculate the total work done by the forces.

Solution: Let the position vectors of the points A and B are respectively $2\hat{i} - \hat{j} - 3\hat{k}$ and $4\hat{i} - 3\hat{j} + 7\hat{k}$.

$\therefore$
$$\overline{AB} = \text{p.v. of B} - \text{p.v. of A}$$
$$= 4\hat{i} - 3\hat{j} + 7\hat{k} - (2\hat{i} - \hat{j} - 3\hat{k})$$
$$= 2\hat{i} - 2\hat{j} + 10\hat{k}$$

Given forces are $\qquad \overline{F}_1 = 9\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

and $\qquad \overline{F}_2 = 10\left(\frac{4}{5}\hat{j} + \frac{3}{5}\hat{k}\right)$,

where $\hat{i}, \hat{j}, \hat{k}$ are three unit vectors along three rectangular axes (in the positive directions) such that they form a right-handed system.

$\therefore$ Resultant force $\qquad \overline{F} = \overline{F}_1 + \overline{F}_2$

$$= 9\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) + 10\left(\frac{4}{5}\hat{j} + \frac{3}{5}\hat{k}\right)$$

$$= 6\hat{i} + 11\hat{j} + 12\hat{k}$$

$\therefore$ Required work done $\qquad W = \overline{F} \cdot \overline{AB}$

$$= (6\hat{i} + 11\hat{j} + 12\hat{k}) \cdot (2\hat{i} - 2\hat{j} + 10\hat{k})$$

$$= 12 - 22 + 120$$

$$= 110 \text{ unit.}$$

Vector or Cross Product

The vector or cross product of two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ and is defined as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin \theta \, \hat{n}$$

where $|\vec{a}|$ and $|\vec{b}|$ are the magnitudes of the vectors $\vec{a}$ and $\vec{b}$ respectively, θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If a right-handed screw be rotated from $\vec{a}$ to $\vec{b}$ through an angle $< 180°$, then it undergoes translation in the direction of $\hat{n}$.

It is obvious that $\vec{a} \times \vec{b}$ is normal to the plane containing $\vec{a}$ and $\vec{b}$.

Note: Vector product is a vector whereas the scalar product is a scalar.

Properties

(*i*) Vector product is not commutative, *i.e.*, $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ since $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

(*ii*) For any scalar (or number) α, $(\alpha\vec{a}) \times \vec{b} = \alpha(\vec{a} \times \vec{b}) = \vec{a} \times (\alpha\vec{b})$

(*iii*) $\vec{a} \times \vec{b} = \vec{0}$ implies either $|\vec{a}| = 0$ (*i.e.*, $\vec{a}$ is a null vector) or, $|\vec{b}| = 0$ (*i.e.*, $\vec{b}$ is a null vector) or, $\vec{a}$ and $\vec{b}$ are parallel, when none of them is a null vector and conversely.

Consequently, $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$.

(*iv*) $\vec{a} \times \vec{b}$ is a vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ form a right handed triad (*i.e.*, obeys right-handed screw law).

Hence, $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$, provided $\hat{i}, \hat{j}, \hat{k}$ form a right handed triad. So,

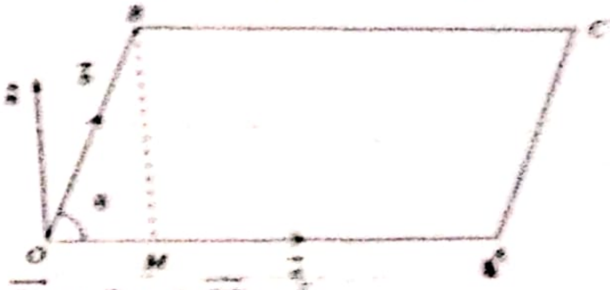
$\hat{j}\times\hat{i} = -\hat{k}$, $\hat{k}\times\hat{j} = -\hat{i}$, $\hat{i}\times\hat{k} = -\hat{j}$.

(v) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$, $(\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a}+\vec{c}\times\vec{a}$

(vi) Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$. Using (v), (iv), (iii) respectively, we get

$$\vec{a}\times\vec{b} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$$
$$= (a_2b_3-a_3b_2)\hat{i}+(a_3b_1-a_1b_3)\hat{j}+(a_1b_2-a_2b_1)\hat{k}$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(vii) Geometrically, $\vec{a}\times\vec{b}$ represents the vector area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$.



Proof.

$$\vec{a}\times\vec{b} = (OA\cdot OB\ \sin\theta)\,\hat{n}$$
$$= (OA\cdot BM)\,\hat{n}$$
$$= (\text{Base}\times\text{Height})\,\hat{n}$$

$\therefore\ \vec{a}\times\vec{b}$ = vector area of parallelogram $OACB$.

(viii) $\vec{a}\times\vec{b} = 2\times$ (vector area of $\Delta\,OAB$), where $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

(ix) The vector area $(\vec{\Delta})$ of the ΔPQR whose vertices have position vectors $\vec{a}, \vec{b}, \vec{c}$ is given by

$$\vec{\Delta} = \frac{1}{2}(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$$

Proof

$$\vec{\Delta} = \frac{1}{2}(\overrightarrow{PQ}\times\overrightarrow{PR})$$
$$= \frac{1}{2}\{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\}$$
$$= \frac{1}{2}(\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a})$$
$$= \frac{1}{2}(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$$

$[\because\ \vec{a}\times\vec{a} = \vec{0},\ -\vec{b}\times\vec{a} = \vec{a}\times\vec{b},\ -\vec{a}\times\vec{c} = \vec{c}\times\vec{a}]$

(x) Let $\vec{c}\neq\vec{0}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{c}$

$\therefore\ (\vec{a}-\vec{b})\times\vec{c} = \vec{0}$

Therefore, either $\vec{a}-\vec{b} = \vec{0}$, *i.e.*, $\vec{a} = \vec{b}$ or $\vec{a}-\vec{b}$ is parallel to $\vec{c}$.

Hence, we conclude that we cannot cancel $\vec{c}$ from $\vec{a}\times\vec{c} = \vec{b}\times\vec{c}$ to obtain $\vec{a} = \vec{b}$, it may happen that $\vec{a}-\vec{b}$ and $\vec{c}$ are parallel.

Conversely if $\quad \vec{a} = \vec{b}+\alpha\vec{c}$ (α is a scalar *i.e.*, a number)

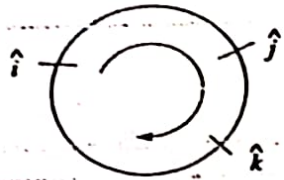
then
$$\vec{a}\times\vec{c} = (\vec{b}+\alpha\vec{c})\times\vec{c} = \vec{b}\times\vec{c}+\alpha(\vec{c}\times\vec{c})$$
$$= \vec{b}\times\vec{c} \quad (\because \vec{c}\times\vec{c}=\vec{0}).$$

$$\begin{vmatrix} |a|^2 & a^2 \\ b^2 & |b|^2 \end{vmatrix}$$

(xi) $(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2$

Proof.
$$(\vec{a}\times\vec{b})^2 = (\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$$
$$= (|\vec{a}||\vec{b}|\sin\theta)^2 \qquad [\because \vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n} \text{ and } \hat{n}\cdot\hat{n}=1]$$
$$= |\vec{a}|^2|\vec{b}|^2\sin^2\theta$$
$$= |\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)$$
$$= |\vec{a}|^2|\vec{b}|^2-(|\vec{a}||\vec{b}|\cos\theta)^2$$
$$= |\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2 \qquad [\because \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta]$$

Note: We always assume that $\hat{i},\hat{j},\hat{k}$ form a right-handed system of three mutually perpendicular unit vectors, so that $\hat{i}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$ and $\hat{k}\times\hat{i}=\hat{j}$



Vector moment of a force about a point (physical interpretation of vector cross product)

The vector moment or torque $\overline{M}$ of a force $\overline{F}$ about any point O is in magnitude equal to $|\overline{F}|$ times the perpendicular distance of O from the line of action of the force $\overline{F}$.

$$\therefore \qquad |\overline{M}| = |\overline{F}|\cdot ON = |\overline{F}|\cdot OP\cdot\frac{ON}{OP}$$
$$= |\overline{F}|\,OP\sin\theta \qquad ...(1)$$

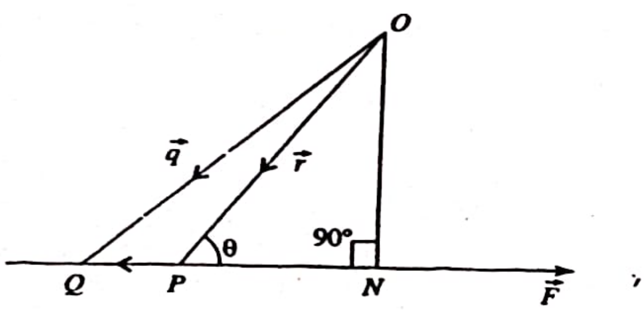


Fig.

If $\vec{r}$ be the position vector of any point P on the line of action of force $\vec{F}$, then

$$|\vec{r}\times\vec{F}| = |\vec{F}|OP\sin\theta \quad ...(2)$$

From (1) and (2), we have $|\vec{M}| = |\vec{r}\times\vec{F}|$.

The direction of vector moment is to be determined by the right-handed screw law. Thus in the adjacent figure the direction of the vector moment is along the normal drawn through O on the plane of the paper (*i.e.*, the plane containing $\vec{F}$ and $\vec{r}$) and pointing towards the reader.

$\therefore$ $\vec{M} = \vec{r}\times\vec{F}$ and its magnitude is $|\vec{F}|ON = |\vec{F}|\cdot OP\sin\theta$.

Properties:

1. Vector moment is independent of the position of the point P on the line of action of $\vec{F}$.

Let us choose any point Q (other then P) on the line of action of $\vec{F}$ and let the position vector of Q be $\vec{q}$ relative to O. Then the moment $\vec{M'}$ of $\vec{F}$ about O is

$$\vec{M'} = \vec{q}\times\vec{F} = (\vec{r}+\vec{PQ})\times\vec{F}$$

$$= \vec{r}\times\vec{F} = \vec{M}$$

($\because \vec{PQ}\times\vec{F} = \vec{O}$ as both $\vec{PQ}$ and $\vec{F}$ are in the same line).

2. The algebraic sum of the moments of a system of forces about any point is equal to the moment of their resultant about the same point.

If $\vec{F}$ be the resultant of a system of forces $\vec{F_1}, \vec{F_2}, ...$ acting through a point, then

$$\vec{F} = \vec{F_1}+\vec{F_2}+...$$

$\therefore$

$$\vec{M} = \vec{r}\times\vec{F} = \vec{r}\times(\vec{F_1}+\vec{F_2}+...)$$

$$= \vec{r}\times\vec{F_1}+\vec{r}\times\vec{F_2}+...$$

Illustration: A force $\vec{F} = 3\hat{i}+2\hat{j}-4\hat{k}$ is acting at the point (1, −1, 2). Find the moment of $\vec{F}$ about the point (2, −1, 3).

Solution: Let $A \equiv (2, -1, 3)$ and $P \equiv (1, -1, 2)$.

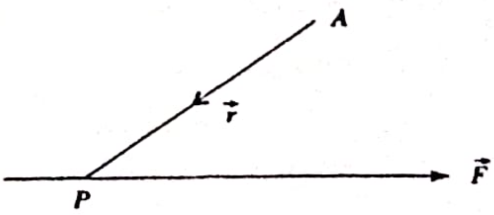
$\therefore$ $\vec{r} = \vec{AP}$ = position vector of P − position vector of A

$$= \hat{i}-\hat{j}+2\hat{k}-(2\hat{i}-\hat{j}+3\hat{k}) = -\hat{i}-\hat{k}$$

$\therefore$ Required moment $= \vec{r}\times\vec{F}$

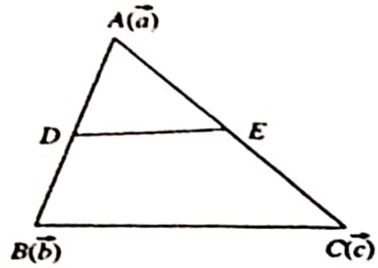
$$= (-\hat{i}-\hat{k})\times(3\hat{i}+2\hat{j}-4\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix} = 2\hat{i}-7\hat{j}-2\hat{k}.$$



ILLUSTRATIVE EXAMPLES

Example 1: Prove that the line joining the mid-points of two sides of a triangle is parallel and half of the third side.



Solution: Let ABC be a triangle and $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of A, B, C respectively. Let D and E be the mid-points of the sides AB and AC respectively.

Position vector of $D = \frac{1}{2}(\vec{a}+\vec{b})$

Position vector of $E = \frac{1}{2}(\vec{a}+\vec{c})$

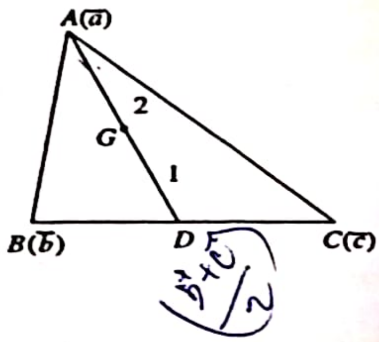
$$\therefore \quad \overrightarrow{DE} = \text{Position vector of } E - \text{Position vector of } D$$

$$= \frac{1}{2}(\vec{a}+\vec{c}) - \frac{1}{2}(\vec{a}+\vec{b}) = \frac{1}{2}(\vec{c}-\vec{b})$$

$$= \frac{1}{2}\overrightarrow{BC}$$

This proves that DE is parallel to BC and $DE = \frac{1}{2}BC$.

Example 2: Show by vector method that the medians of a triangle are concurrent.



Solution: Let ABC be a triangle and $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of A, B, C respectively.

If D is the mid-point of BC, then the position vector of

$$D = \frac{1}{2}(\vec{b}+\vec{c})$$

If the point G divides AD in the ratio 2 : 1 (as shown in fig.), then the position vector of

$$G = \frac{2\cdot\frac{1}{2}(\vec{b}+\vec{c})+\vec{a}}{2+1} = \frac{1}{3}(\vec{a}+\vec{b}+\vec{c}).$$

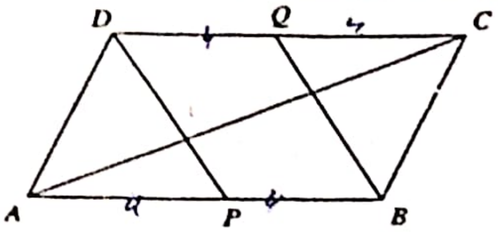
The symmetry of this expression proves that the other two medians will be divided in the ratio 2 : 1 at the same point G.

Hence, the medians are concurrent.

Example 3: $ABCD$ is a parallelogram. P, Q are the mid-points of the sides AB and CD respectively. Show that DP and BQ trisect AC and are trisected by AC.

Solution: Let the position vectors of A, B, C, D, P, Q are $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{p}, \vec{q}$ respectively.

Since P and Q are the mid-points of AB and CD respectively,



$$\bar{p} = \frac{1}{2}(\bar{a}+\bar{b}),$$

$$\bar{q} = \frac{1}{2}(\bar{c}+\bar{d}).$$

Here $ABCD$ is a parallelogram.

$\therefore$ $$\overrightarrow{AB} = \overrightarrow{DC}, \text{ or, } \bar{b}-\bar{a} = \bar{c}-\bar{d} \quad \text{...(1)}$$

$\therefore$ $$\bar{p} = \frac{1}{2}(\bar{a}+\bar{b}) = \frac{1}{2}(\bar{a}+\bar{a}+\bar{c}-\bar{d}) \quad \text{[by (1)]}$$

or $$2\bar{a}+\bar{c} = 2\bar{p}+\bar{d},$$

or $$\frac{2\bar{a}+\bar{c}}{3} = \frac{2\bar{p}+\bar{d}}{3}$$

This result shows that the common point of AC and PD intersects them in the ratio 1 : 2.

Also $$\bar{q} = \frac{1}{2}(\bar{c}+\bar{d}) = \frac{1}{2}(\bar{c}+\bar{c}+\bar{a}-\bar{b}) \quad \text{[by (1)]}$$

or $$2\bar{c}+\bar{a} = 2\bar{q}+\bar{b},$$

or $$\frac{2\bar{c}+\bar{a}}{3} = \frac{2q+\bar{b}}{3}.$$

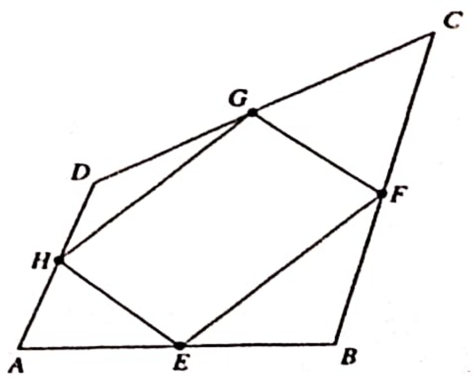
Hence, CA and QB are intersected by their common point in the ratio 1 : 2.

Therefore, DP and BQ trisect AC and are trisected by AC.

Example 4: If the mid-points of the consecutive sides of any quadrilateral are connected by straight lines then prove by vector method that the resulting quadrilateral is a parallelogram.

Solution: Let $ABCD$ be a quadrilateral and E, F, G, H are the mid-points of the consecutive sides AB, BC, CD and DA respectively. Let the position vectors of A, B, C, D are $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ respectively.

Then the position vectors of E, F, G, H are respectively $\frac{1}{2}(\bar{a}+\bar{b}), \frac{1}{2}(\bar{b}+\bar{c}), \frac{1}{2}(\bar{c}+\bar{d}), \frac{1}{2}(\bar{d}+\bar{a})$.



$\therefore$ $$\overrightarrow{EF} = \text{p.v. of } F - \text{p.v. of } E$$

$$= \frac{1}{2}(\bar{b}+\bar{c}) - \frac{1}{2}(\bar{a}+\bar{b})$$

$$= \frac{1}{2}(\bar{c}-\bar{a}).$$

and $$\overrightarrow{HG} = \text{p.v. of } G - \text{p.v. of } H$$

$$= \frac{1}{2}(\bar{c}+\bar{d}) - \frac{1}{2}(\bar{d}+\bar{a})$$

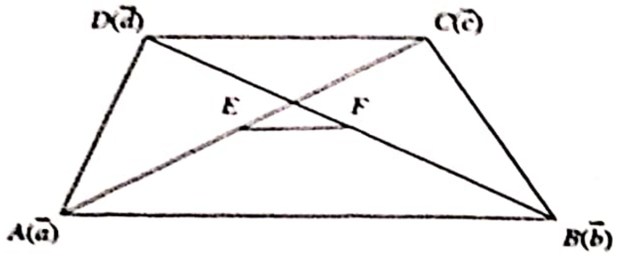
$$= \frac{1}{2}(\vec{c} - \vec{a}).$$

$$\therefore \qquad \overrightarrow{EF} = \overrightarrow{HG}$$

Hence, $EF = HG$ and $EF \parallel HG$.

Therefore, $EFGH$ is a parallelogram.

Example 5: In a trapezium, prove that the straight line joining the mid-points of the diagonals is parallel to the parallel sides and half their difference.



Solution: Here $ABCD$ is a trapezium with parallel sides AB and DC. Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are the position vectors of A, B, C, and D respectively.

Since DC is parallel to AB,

$$\overrightarrow{DC} = \lambda \overrightarrow{AB}, \text{ for some number } \lambda. \qquad \text{...(1)}$$

or
$$\vec{c} - \vec{d} = \lambda(\vec{b} - \vec{a}) \qquad \text{...(2)}$$

The position vector of the mid-point E of AC is $\frac{1}{2}(\vec{a} + \vec{c})$

The position vector of the mid-point F of BD is $\frac{1}{2}(\vec{b} + \vec{d})$

$$\therefore \qquad \overrightarrow{EF} = \text{p.v. of } F - \text{p.v. of } E$$

$$= \frac{1}{2}(\vec{b} + \vec{d}) - \frac{1}{2}(\vec{a} + \vec{c})$$

$$= \frac{1}{2}\{(\vec{b} - \vec{a}) - (\vec{c} - \vec{d})\}$$

$$= \frac{1}{2}\{(\vec{b} - \vec{a}) - \lambda(\vec{b} - \vec{a})\} \qquad \text{[by (2)]}$$

$$= \frac{1}{2}(1-\lambda)(\vec{b} - \vec{a}) = \frac{1}{2}(1-\lambda)\overrightarrow{AB}$$

$$\therefore \qquad \overrightarrow{EF} = \frac{1}{2}(1-\lambda)\overrightarrow{AB} \qquad \text{...(3)}$$

Hence, EF is parallel to AB.

Also from (1) and (3), we have

$$\overrightarrow{EF} = \frac{1}{2}(1-\lambda)\overrightarrow{AB} = \frac{1}{2}(\overrightarrow{AB} - \lambda\overrightarrow{AB}) = \frac{1}{2}(\overrightarrow{AB} - \overrightarrow{DC})$$

$$\therefore \qquad \overrightarrow{EF} \cdot \overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AB} - \overrightarrow{DC}) \cdot \frac{1}{2}(\overrightarrow{AB} - \overrightarrow{DC})$$

$$\therefore \quad EF^2 = \frac{1}{4}(\overrightarrow{AB} - \overrightarrow{DC})\cdot(\overrightarrow{AB} - \overrightarrow{DC})$$

$$= \frac{1}{4}(AB^2 - 2\overrightarrow{AB}\cdot\overrightarrow{DC} + DC^2) \qquad [\because \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta]$$

$$= \frac{1}{4}(AB^2 - 2AB\cdot DC\cos 0 + DC^2)$$

$$= \frac{1}{4}(AB - DC)^2$$

$$\therefore \quad EF = \frac{1}{2}|AB - DC|$$

Hence, the result.

Example 6: If the position vectors of A, B, C are $2\hat{i}+4\hat{j}-\hat{k}$, $4\hat{i}+5\hat{j}+\hat{k}$ and $3\hat{i}+6\hat{j}-3\hat{k}$ respectively, prove that ΔABC is right-angled.

Solution: Here $\overrightarrow{AB}$ = p.v. of B – p.v. of A

$$= 4\hat{i}+5\hat{j}+\hat{k}-(2\hat{i}+4\hat{j}-\hat{k})$$

$$= 2\hat{i}+\hat{j}+2\hat{k}$$

$\overrightarrow{BC}$ = p.v. of C – p.v. of B

$$= 3\hat{i}+6\hat{j}-3\hat{k}-(4\hat{i}+5\hat{j}+\hat{k})$$

$$= -\hat{i}+\hat{j}-4\hat{k}$$

$\overrightarrow{CA}$ = p.v. of A – p.v. of C

$$2\hat{i}+4\hat{j}-\hat{k}-(3\hat{i}+6\hat{j}-3\hat{k}) =$$

$$= -\hat{i}-2\hat{j}+2\hat{k}$$

Now, $\overrightarrow{AB}\cdot\overrightarrow{AB} = (2\hat{i}+\hat{j}+2\hat{k})\cdot(2\hat{i}+\hat{j}+2\hat{k})$

or $AB\cdot AB\cos 0 = 2^2+1^2+2^2 \qquad [\because \hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1,\ \hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0]$

or $AB^2 = 9$

Similarly, $\overrightarrow{BC}\cdot\overrightarrow{BC} = (-\hat{i}+\hat{j}-4\hat{k})\cdot(-\hat{i}+\hat{j}-4\hat{k})$

$\therefore \quad BC^2 = (-1)^2+1^2+(-4)^2 = 18,$

$\overrightarrow{CA}\cdot\overrightarrow{CA} = (-\hat{i}-2\hat{j}+2\hat{k})\cdot(-\hat{i}-2\hat{j}+2\hat{k})$

$\therefore \quad CA^2 = (-1)^2+(-2)^2+2^2 = 9.$

$\therefore \quad AB^2+CA^2 = 9+9 = 18 = BC^2.$

Hence, $\Delta\ ABC$ is a right-angled triangle.

Example 7: Show that the points $A(\hat{i}-2\hat{j}+3\hat{k})$, $B(2\hat{i}-3\hat{j}+4\hat{k})$ and $C(-2\hat{i}+\hat{j})$ are collinear.

Solution: Let $(\hat{i}-2\hat{j}+3\hat{k})+y(2\hat{i}-3\hat{j}+4\hat{k})+z(-2\hat{i}+\hat{j})=\vec{0}$

or $\quad (1+2y-2z)\hat{i}+(-2-3y+z)\hat{j}+(3+4y)\hat{k}=\vec{0}$

$\therefore 1+2y-2z=0,\ -2-3y+z=0,\ 3+4y=0$

($\because \hat{i},\hat{j},\hat{k}$ are independent unit vectors as $\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0$)

$\therefore y=-\dfrac{3}{4},\ z=2+3y=2-\dfrac{9}{4}=-\dfrac{1}{4}.$

Hence, $1+y+z=0$

Therefore, the given points A, B, C are collinear.

Example 8: Show by vector method that the points $P(1, 5, -1)$, $Q(0, 4, 5)$, $R(-1, 5, 1)$ and $S(2, 4, 3)$ are coplanar.

Solution: Let O be the origin and $\hat{i},\hat{j},\hat{k}$ are unit vectors along the positive directions of x, y, z-axes respectively.

Therefore, the position vectors of P, Q, R, S are $\overline{OP}=\vec{r}_1=\hat{i}+5\hat{j}-\hat{k}$, $\overline{OQ}=\vec{r}_2=4\hat{j}+5\hat{k}$, $\overline{OR}=\vec{r}_3=-\hat{i}+5\hat{j}+\hat{k}$ and $\overline{OS}=\vec{r}_4=2\hat{i}+4\hat{j}+3\hat{k}$ with respect to O.

If the given points are coplanar then there exist three numbers x, y, z such that

$$\vec{r}_1+x\vec{r}_2+y\vec{r}_3+z\vec{r}_4=\vec{0} \qquad ...(1)$$

and
$$1+x+y+z=0 \qquad ...(2)$$

From (1), we get $1-y+2z=0$, $5+4x+5y+4z=0$, $-1+5x+y+3z=0$ ($\because \hat{i},\hat{j},\hat{k}$ are independent vectors).

Solving, we get $x=1, y=-1, z=-1$.

Therefore, (2) is satisfied. Hence, the given four points are coplanar.

Example 9: If $\vec{\alpha},\vec{\beta},\vec{\gamma}$ be three vectors defined by $\vec{\alpha}=\lambda(2\hat{i}+3\hat{j}+6\hat{k})$, $\vec{\beta}=\lambda(3\hat{i}-6\hat{j}+2\hat{k})$ and $\vec{\gamma}=\lambda(6\hat{i}+2\hat{j}-3\hat{k})$ where λ is a number and $\hat{i},\hat{j},\hat{k}$ are the three orthogonal unit vectors, determine λ for which $\vec{\alpha},\vec{\beta}$ and $\vec{\gamma}$ may be each of unit length. Also prove that $\vec{\alpha},\vec{\beta}$ and $\vec{\gamma}$ are mutually perpendicular.

Solution: Now, $\quad \vec{\alpha}\cdot\vec{\alpha}=\lambda^2(2\hat{i}+3\hat{j}+6\hat{k})\cdot(2\hat{i}+3\hat{j}+6\hat{k})$

$\therefore \quad |\vec{\alpha}|^2=\lambda^2(2^2+3^2+6^2)=49\lambda^2$

[$\because \vec{\alpha}\cdot\vec{\alpha}=|\vec{\alpha}||\vec{\alpha}|\cos 0$

$\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$,

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0 \text{ etc.})$$

$\therefore$ $$|\vec{\alpha}| = 7\lambda.$$

Similarly, $$\vec{\beta}\cdot\vec{\beta} = \lambda^2(3\hat{i} - 6\hat{j} + 2\hat{k})\cdot(3\hat{i} - 6\hat{j} + 2\hat{k})$$

$\therefore$ $$|\vec{\beta}|^2 = \lambda^2\{3^2 + (-6)^2 + 2^2\} = 49\lambda^2.$$

$\therefore$ $$|\vec{\beta}| = 7\lambda.$$

$$\vec{\gamma}\cdot\vec{\gamma} = \lambda^2(6\hat{i} + 2\hat{j} - 3\hat{k})\cdot(6\hat{i} + 2\hat{j} - 3\hat{k})$$

$\therefore$ $$|\vec{\gamma}|^2 = \lambda^2\{6^2 + 2^2 + (-3)^2\} = 49\lambda^2.$$

$\therefore$ $$|\vec{\gamma}| = 7\lambda.$$

Therefore, $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ will be of unit length if $\lambda = \frac{1}{7}$

Now, $$\vec{\alpha}\cdot\vec{\beta} = \lambda^2(2\hat{i} + 3\hat{j} + 6\hat{k})\cdot(3\hat{i} - 6\hat{j} + 2\hat{k})$$
$$= \lambda^2(6 - 18 + 12) = 0.$$
$$\vec{\beta}\cdot\vec{\gamma} = \lambda^2(3\hat{i} - 6\hat{j} + 2\hat{k})\cdot(6\hat{i} + 2\hat{j} - 3\hat{k})$$
$$= \lambda^2(18 - 12 - 6) = 0.$$
$$\vec{\gamma}\cdot\vec{\alpha} = \lambda^2(6\hat{i} + 2\hat{j} - 3\hat{k})\cdot(2\hat{i} + 3\hat{j} + 6\hat{k})$$
$$= \lambda^2(12 + 6 - 18) = 0.$$

Hence, $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are mutually perpendicular.

Example 14: The position vectors of two points P and Q are $3\hat{i} + 7\hat{j} - 4\hat{k}$ and $6\hat{i} - 2\hat{j} + 12\hat{k}$ respectively, where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the positive directions of x, y, z-axes respectively. Calculate the angle between OP and OQ where O is the origin.

Solution: Here $\overline{OP} = 3\hat{i} + 7\hat{j} - 4\hat{k}$ and $\overline{OQ} = 6\hat{i} - 2\hat{j} + 12\hat{k}$. If θ be the angle between them, then

$$\overline{OP}\cdot\overline{OQ} = OP\cdot OQ\cos\theta \qquad \text{...(1)}$$

Now, $$\overline{OP}\cdot\overline{OQ} = (3\hat{i} + 7\hat{j} - 4\hat{k})\cdot(6\hat{i} - 2\hat{j} + 12\hat{k})$$
$$= 18 - 14 - 48 = -44.$$
$$OP^2 = \overline{OP}\cdot\overline{OP} = (3\hat{i} + 7\hat{j} - 4\hat{k})\cdot(3\hat{i} + 7\hat{j} - 4\hat{k})$$
$$= 3^2 + 7^2 + (-4)^2 = 74.$$

$\therefore$ $$OP = \sqrt{74}.$$

$$OQ^2 = \overline{OQ}\cdot\overline{OQ} = (6\hat{i} - 2\hat{j} + 12\hat{k})\cdot(6\hat{i} - 2\hat{j} + 12\hat{k})$$
$$= 6^2 + (-2)^2 + (12)^2 = 184.$$

$$\therefore \qquad OQ = \sqrt{184}.$$

Therefore, using (1), we get

$$\cos\theta = \frac{\overrightarrow{OP}\cdot\overrightarrow{OQ}}{OP\cdot OQ} = \frac{-44}{\sqrt{74}\sqrt{184}} = \frac{-11}{\sqrt{37}\sqrt{23}} = \frac{-11}{\sqrt{851}}$$

$$\therefore \qquad \theta = \cos^{-1}\left(\frac{-11}{\sqrt{851}}\right)$$

Example 11: If $|\vec{\alpha}| = 3$ and $|\vec{\beta}| = 4$, then find the values of the scalar c for which the vectors $\vec{\alpha} + c\vec{\beta}$ and $\vec{\alpha} - c\vec{\beta}$ will be perpendicular to one another . *(W.B.U.T. 2005)*

Solution: The vectors $\vec{\alpha} + c\vec{\beta}$ and $\vec{\alpha} - c\vec{\beta}$ will be perpendicular to one another if

$$(\vec{\alpha} + c\vec{\beta})\cdot(\vec{\alpha} - c\vec{\beta}) = 0, \text{ or, } \vec{\alpha}\cdot\vec{\alpha} - c\vec{\alpha}\cdot\vec{\beta} + c\vec{\beta}\cdot\vec{\alpha} - c^2\vec{\beta}\cdot\vec{\beta} = 0$$

$$\therefore \qquad |\vec{\alpha}|^2 - c^2|\vec{\beta}|^2 = 0, \text{ or, } 3^2 - c^2 4^2 = 0$$

$$\therefore \qquad c = \pm\frac{3}{4}.$$

Example 12: Given two vectors $\vec{\alpha} = 3\hat{i} - \hat{j}$ and $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$, express $\vec{\beta}$ in the form $\vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$. *(W.B.U.T. 2005)*

Solution: The vector $c\vec{\alpha}$ is parallel to $\vec{\alpha}$ for all values of the scalar c, because $(c\vec{\alpha})\times\vec{\alpha} = c(\vec{\alpha}\times\vec{\alpha}) = c\vec{0} = \vec{0}$. Let us take $\vec{\beta}_1 = c\vec{\alpha} = c(3\hat{i} - \hat{j})$.

Let $\vec{\beta}_2 = x\hat{i} + y\hat{j} + z\hat{k}$.

Since $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$, therefore,

$$\vec{\alpha}\cdot\vec{\beta}_2 = 0, \text{ or, } (3\hat{i} - \hat{j})\cdot(x\hat{i} + y\hat{j} + z\hat{k}) = 0$$

or $$3x - y = 0, \text{ or, } y = 3x \qquad \ldots(1)$$

$$\therefore \qquad \vec{\beta}_2 = x\hat{i} + 3x\hat{j} + z\hat{k}$$

Here $$\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$$

$$\therefore \qquad 2\hat{i} + \hat{j} - 3\hat{k} = c(3\hat{i} - \hat{j}) + x\hat{i} + 3x\hat{j} + z\hat{k}$$

$$= (3c + x)\hat{i} + (3x - c)\hat{j} + z\hat{k}$$

Since $\hat{i}, \hat{j}, \hat{k}$ are independent, equating coefficients of $\hat{i}, \hat{j}, \hat{k}$ from both sides, we get

$$3c + x = 2 \qquad \ldots(2)$$

$$3x - c = 1 \qquad \ldots(3)$$

$$z = -3 \qquad \ldots(4)$$

Solving (2) and (3), we get $x = \frac{1}{2}$, $c = \frac{1}{2}$.

From (1), $$y = 3x = \frac{3}{2}.$$

$$\therefore \quad \vec{\beta}_1 = \frac{1}{2}(3\hat{i} - \hat{j}),\ \vec{\beta}_2 = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}.$$

Example 13: Find the unit vector perpendicular to each of $\vec{a} = 6\hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 6\hat{j} - 2\hat{k}$.

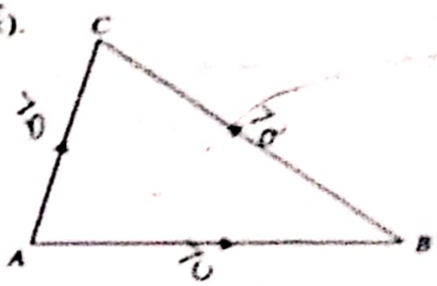
Solution: Here $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 2 & 3 \\ 3 & -6 & -2 \end{vmatrix} = 14\hat{i} + 21\hat{j} - 42\hat{k}.$$

It is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

Also, $$|\vec{a} \times \vec{b}|^2 = (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b})$$
$$= (14\hat{i} + 21\hat{j} - 42\hat{k}) \cdot (14\hat{i} + 21\hat{j} - 42\hat{k})$$
$$= (14)^2 + (21)^2 + (-42)^2$$

Therefore, the required unit vector is

$$\frac{14\hat{i} + 21\hat{j} - 42\hat{k}}{\sqrt{(14)^2 + (21)^2 + (42)^2}} = \frac{2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{1}{7}(2\hat{i} + 3\hat{j} - 6\hat{k}).$$



Example 14: Prove, by vector method, the trigonometrical formula $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Solution: Let ABC be a triangle.

$$\therefore \quad \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \quad \text{(By triangle law of addition of vectors)}$$

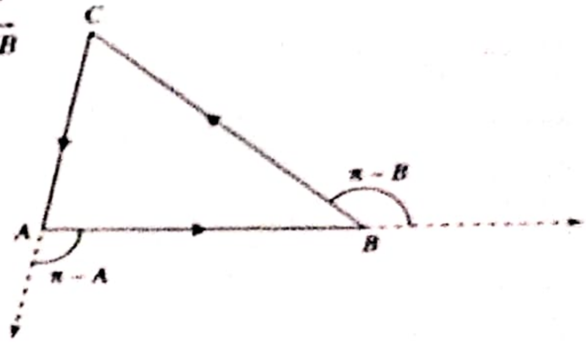
$$\therefore \quad \overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{BC} = \overrightarrow{CB} - \overrightarrow{CA} \quad (\because -\overrightarrow{BC} = \overrightarrow{CB},\ \overrightarrow{AC} = -\overrightarrow{CA})$$

$$\therefore \quad \overrightarrow{AB} \cdot \overrightarrow{AB} = (\overrightarrow{CB} - \overrightarrow{CA}) \cdot (\overrightarrow{CB} - \overrightarrow{CA})$$

or $$AB^2 = CB^2 + CA^2 - 2\overrightarrow{CA} \cdot \overrightarrow{CB}$$

or $$c^2 = a^2 + b^2 - 2ba\cos C$$

$$\therefore \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$



Example 15: Prove, by vector method, the trigonometrical formula
$$c = a\cos B + b\cos A,$$
where the symbols have their usual meanings.

Solution: In $\Delta\, ABC$, $\overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AC}$

$\therefore \quad \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} = \vec{0}$

$\therefore \quad \overrightarrow{AB}\cdot(\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}) = \overrightarrow{AB}\cdot\vec{0} = 0$

or $\quad \overrightarrow{AB}\cdot\overrightarrow{AB}+\overrightarrow{AB}\cdot\overrightarrow{BC}+\overrightarrow{AB}\cdot\overrightarrow{CA} = 0$

or $\quad AB^2 + AB\cdot BC\cos(\pi-B) + AB\cdot CA\cos(\pi-A) = 0$

or $\quad c^2 + ca\cos(\pi-B) + cb\cos(\pi-A) = 0$

or $\quad c - a\cos B - b\cos A = 0$

$\therefore \quad c = a\cos B + b\cos A.$

Example 16: Prove by vector method $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ in a triangle ABC.

Solution: In $\Delta\, ABC$, $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} = \vec{0}$

$\therefore \quad \overrightarrow{AB}\times(\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}) = \overrightarrow{AB}\times\vec{0} = \vec{0}$

or $\quad \overrightarrow{AB}\times\overrightarrow{BC}+\overrightarrow{AB}\times\overrightarrow{CA} = \vec{0}$ $\quad [\overrightarrow{AB}\times\overrightarrow{AB} = AB\cdot AB\sin 0\,\hat{n} = \vec{0}]$.

or $\quad AB\cdot BC\sin(\pi-B)\,\hat{n} - AB\cdot CA\sin(\pi-A)\hat{n} = \vec{0}$

[See Fig. of Ex. 15 and since the movements of a right-handed screw are opposite]

or $\quad (ca\sin B - cb\sin A)\,\hat{n} = \vec{0}$

$\therefore \quad a\sin B - b\sin A = 0$ $\quad (\because\ \hat{n}$ is a unit vector)

or $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B}.$

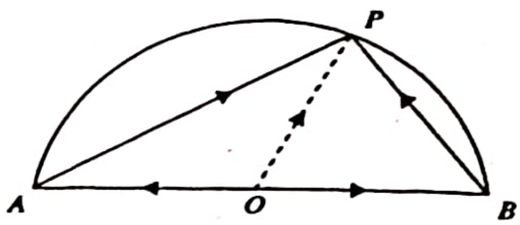
Similarly considering $\overrightarrow{BC}\times(\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}) = \vec{0}$, we get

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Example 17: Show, by vector method, that an angle inscribed in a semicircle is a right angle.

Solution: Let P be any point on the semicircle APB with centre at O and AOB as diameter. Considering O as vector origin, we have



$$\overrightarrow{AP}\cdot\overrightarrow{BP} = (\overrightarrow{OP}-\overrightarrow{OA})\cdot(\overrightarrow{OP}-\overrightarrow{OB})$$

$$= (\overrightarrow{OP}-\overrightarrow{OA})\cdot(\overrightarrow{OP}+\overrightarrow{OA}) \quad [\because \overrightarrow{OB} = -\overrightarrow{OA}]$$

$$= \overrightarrow{OP}\cdot\overrightarrow{OP}-\overrightarrow{OA}\cdot\overrightarrow{OA} = OP^2 - OA^2 = 0 \quad [\because\ OP = OA = \text{radius}]$$

Therefore, AP and BP are at right angle. Hence, APB is a right angle.

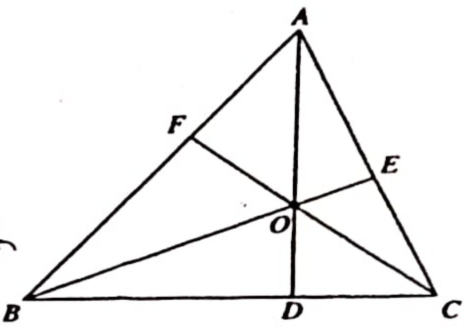
Example 18: Show, by vector method, that the perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.

Solution: Let ABC be a triangle and AD, BE are perpendiculars to BC and CA drawn from A and B respectively.

Let they intersect at O. CO is joined and extended to meet AB at F.

With O as vector origin, we have $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$ and $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$.

Since AD is perpendicular to BC,



$$\overrightarrow{BC} \cdot \overrightarrow{OA} = 0,$$

or $$(\overrightarrow{OC} - \overrightarrow{OB}) \cdot \overrightarrow{OA} = 0,$$

or $$\overrightarrow{OC} \cdot \overrightarrow{OA} - \overrightarrow{OB} \cdot \overrightarrow{OA} = 0 \quad ...(1)$$

Similarly, $$\overrightarrow{CA} \cdot \overrightarrow{OB} = 0,$$

or $$(\overrightarrow{OA} - \overrightarrow{OC}) \cdot \overrightarrow{OB} = 0$$

or $$\overrightarrow{OA} \cdot \overrightarrow{OB} - \overrightarrow{OC} \cdot \overrightarrow{OB} = 0 \quad ...(2)$$

Adding (1) and (2), we get

$$\overrightarrow{OC} \cdot \overrightarrow{OA} - \overrightarrow{OC} \cdot \overrightarrow{OB} = 0,$$

or $$(\overrightarrow{OA} - \overrightarrow{OB}) \cdot \overrightarrow{OC} = 0$$

or $$\overrightarrow{BA} \cdot \overrightarrow{OC} = 0$$

Therefore, CF is perpendicular to AB.

Hence, the result follows.

Example 19: Prove, by vector method, that in a plane

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \text{ for all values of } A \text{ and } B.$$

Solution: Let $\hat{i}$ and $\hat{j}$ be two unit vectors along positive directions of rectangular axes OX and OY respectively in the XY-plane.

Let $\hat{a}$ and $\hat{b}$ be two unit vectors such that the angle between $\hat{i}$ and $\hat{a}$ be A and the angle between $\hat{i}$ and $\hat{b}$ be $A + B$.

$$\therefore \left.\begin{aligned} \hat{a} &= \hat{i}\cos A + \hat{j}\sin A \\ \hat{b} &= \hat{i}\cos(A+B) + \hat{j}\sin(A+B) \end{aligned}\right\} \quad ...(1)$$

If $\hat{b}$ is referred to a system of rectangular axes along

$$\hat{i}_1 = \hat{a} \text{ and } \hat{j}_1 = \hat{i}\cos\left(A+\frac{\pi}{2}\right) + \hat{j}\sin\left(A+\frac{\pi}{2}\right)$$

[it is possible since $\hat{i}_1 \cdot \hat{j}_1 = \hat{a} \cdot \hat{j}_1 = (\hat{i}\cos A + \hat{j}\sin A) \cdot \hat{j}_1 = \cos A \cos\left(A+\frac{\pi}{2}\right) + \sin A \sin\left(A+\frac{\pi}{2}\right)$ $= -\cos A \sin A + \sin A \cos A = 0$], then $\quad \hat{b} = \hat{i}_1 \cos B + \hat{j}_1 \sin B \qquad$ [$\because$ angle between $\hat{a}$ and $\hat{b}$ be B]

$$= \hat{a}\cos B + \left\{\hat{i}\cos\left(A+\frac{\pi}{2}\right) + \hat{j}\sin\left(A+\frac{\pi}{2}\right)\right\}\sin B$$

$$= (\hat{i}\cos A + \hat{j}\sin A)\cos B + \left\{\hat{i}\cos\left(A+\frac{\pi}{2}\right) + \hat{j}\sin\left(A+\frac{\pi}{2}\right)\right\}\sin B$$

[by (1)]

$$= (\cos A\cos B - \sin A\sin B)\,\hat{i} + (\sin A\cos B + \cos A\sin B)\,\hat{j} \qquad ...(2)$$

Since $\hat{i}$ and $\hat{j}$ are independent, equating coefficients of $\hat{i}$ and $\hat{j}$ in $\hat{b}$ from (1) and (2), we get

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B.$$

Example 20: If $|\vec{a}+\vec{b}| = 60$, $|\vec{a}-\vec{b}| = 40$ and $|\vec{b}| = 46$, find $|\vec{a}|$.

Solution: $\quad (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}+\vec{b}|^2 = 3600$

or $\quad |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}\cdot\vec{b} = 3600 \qquad ...(1)$

Also, $\quad (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}-\vec{b}|^2 = 1600$

or $\quad |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b} = 1600 \qquad ...(2)$

Adding (1) and (2), we get

$$2|\vec{a}|^2 + 2|\vec{b}|^2 = 5200$$

or $\quad |\vec{a}|^2 + (46)^2 = 2600$

or $\quad |\vec{a}|^2 = 484$

$\therefore \quad |\vec{a}| = 22.$

Example 21: If for two vectors $\vec{A}$ and $\vec{B}$,

$$|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$$

find the angle between $\vec{A}$ and $\vec{B}$.

Solution: Here $\quad |\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$

or $\quad |\vec{A}+\vec{B}|^2 = |\vec{A}-\vec{B}|^2$

or $\quad (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) = (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})$

or $\quad |\vec{A}|^2 + |\vec{B}|^2 + 2\vec{A}\cdot\vec{B} = |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A}\cdot\vec{B}$

or $\quad 4\,\vec{A}\cdot\vec{B} = 0$

or $\quad \vec{A}\cdot\vec{B} = 0$

Hence, $\vec{A}$ and $\vec{B}$ are perpendicular to each other, *i.e.*, the angle between $\vec{A}$ and $\vec{B}$ is 90°.

Example 22: If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, $|\vec{a}|=3$, $|\vec{b}|=5$, $|\vec{c}|=7$, find the angle between $\vec{a}$ and $\vec{b}$.

Solution: Here $\quad \vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\therefore \quad \vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{a}\cdot\vec{0}=0$

or $\quad |\vec{a}|^2+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0$

$\therefore \quad \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=-9 \qquad (\because |\vec{a}|=3) \qquad \ldots(1)$

Again, $\quad \vec{b}\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{b}\cdot\vec{0}=0$

or $\quad \vec{b}\cdot\vec{a}+|\vec{b}|^2+\vec{b}\cdot\vec{c}=0$

$\therefore \quad \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}=-25 \qquad (\because |\vec{b}|=5) \qquad \ldots(2)$

Also, $\quad \vec{c}\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{c}\cdot\vec{0}=0$

or $\quad \vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+|\vec{c}|^2=0$

$\therefore \quad \vec{c}\cdot\vec{a}+\vec{b}\cdot\vec{c}=-49 \qquad (a+b+c)^2=0 \qquad (\because |\vec{c}|=7) \qquad \ldots(3)$

Adding (1), (2) and (3), we get

$$2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=-83$$

or $\quad \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\dfrac{83}{2}$

$\therefore \quad \vec{a}\cdot\vec{b}=-\dfrac{83}{2}+49=\dfrac{15}{2} \qquad$ [using (3)]

or $\quad |\vec{a}||\vec{b}|\cos\theta=\dfrac{15}{2}$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

$\therefore \quad \cos\theta=\dfrac{1}{2} \qquad (\because |\vec{a}|=3, |\vec{b}|=5)$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{3}=60°$.

Example 23: If $\vec{\alpha},\vec{\beta},\vec{\gamma}$ are three vectors satisfying the condition $\vec{\alpha}+\vec{\beta}+\vec{\gamma}=\vec{0}$, where $|\vec{\alpha}|=3$, $|\vec{\beta}|=4$ and $|\vec{\gamma}|=5$, show that $\vec{\alpha}\cdot\vec{\beta}+\vec{\beta}\cdot\vec{\gamma}+\vec{\gamma}\cdot\vec{\alpha}=-25$.

Solution: Here $\quad \vec{\alpha}+\vec{\beta}+\vec{\gamma}=\vec{0}$.

$\therefore \quad (\vec{\alpha}+\vec{\beta}+\vec{\gamma})\cdot(\vec{\alpha}+\vec{\beta}+\vec{\gamma})=\vec{0}\cdot\vec{0}$

or $\quad \vec{\alpha}\cdot\vec{\alpha}+\vec{\beta}\cdot\vec{\beta}+\vec{\gamma}\cdot\vec{\gamma}+2(\vec{\alpha}\cdot\vec{\beta}+\vec{\beta}\cdot\vec{\gamma}+\vec{\gamma}\cdot\vec{\alpha})=0$

or $\quad |\vec{\alpha}|^2+|\vec{\beta}|^2+|\vec{\gamma}|^2+2(\vec{\alpha}\cdot\vec{\beta}+\vec{\beta}\cdot\vec{\gamma}+\vec{\gamma}\cdot\vec{\alpha})=0$

or $\quad 3^2+4^2+5^2+2(\vec{\alpha}\cdot\vec{\beta}+\vec{\beta}\cdot\vec{\gamma}+\vec{\gamma}\cdot\vec{\alpha})=0 \qquad (\because |\vec{\alpha}|=3, |\vec{\beta}|=4, |\vec{\gamma}|=5)$.

$\therefore \quad \vec{\alpha}\cdot\vec{\beta}+\vec{\beta}\cdot\vec{\gamma}+\vec{\gamma}\cdot\vec{\alpha}=-25$.

Example 24: Determine y and z, by using vectors, such that the points $(-1, 3, 2)$, $(-4, 2, -2)$ and $(5, y, z)$ lie on a straight line.

Solution: Let A, B, C be three points whose coordinates are $(-1, 3, 2)$, $(-4, 2, -2)$, $(5, y, z)$ respectively.

$\therefore$ Position vector of $A = -\hat{i}+3\hat{j}+2\hat{k}$

Position vector of $B = -4\hat{i}+2\hat{j}-2\hat{k}$

Position vector of $C = 5\hat{i}+y\hat{j}+z\hat{k}.$

$$\therefore \quad \overrightarrow{AB} = \text{p.v. of } B - \text{p.v. of } A$$
$$= -4\hat{i}+2\hat{j}-2\hat{k}-(-\hat{i}+3\hat{j}+2\hat{k})$$
$$= -3\hat{i}-\hat{j}-4\hat{k}$$
$$\overrightarrow{BC} = \text{p.v. of } C - \text{p.v. of } B$$
$$= 5\hat{i}+y\hat{j}+z\hat{k}-(-4\hat{i}+2\hat{j}-2\hat{k})$$
$$= 9\hat{i}+(y-2)\hat{j}+(z+2)\hat{k}$$

If A, B, C lie on a straight line, *i.e.*, $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are in the same straight line, then

$$\overrightarrow{AB}\times\overrightarrow{BC} = \bar{0}, \text{ i.e., } \begin{vmatrix} \hat{i} & \hat{j} & -\hat{k} \\ -3 & -1 & -4 \\ 9 & y-2 & z+2 \end{vmatrix} = \bar{0}$$

or $(-z+4y-10)\hat{i}+(3z-30)\hat{j}+(-3y+15)\hat{k} = \bar{0}$

Since $\hat{i}, \hat{j}, \hat{k}$ are independent vectors, Therefore,

$$-z+4y-10 = 0$$
$$3z-30 = 0, \text{ or, } z = 10$$
$$-3y+15 = 0, \text{ or, } y = 5.$$

Also $y = 5$, $z = 10$ satisfy the first equation.

$$\therefore \quad y = 5, z = 10.$$

Example 25: If $\bar{a}\times\bar{r} = \bar{b}+\lambda\bar{a}$ and $\bar{a}\cdot\bar{r} = 3$, where $\bar{a} = 2\hat{i}+\hat{j}-\hat{k}$ and $\bar{b} = -\hat{i}-2\hat{j}+\hat{k}$, then find $\bar{r}$ and λ.

Solution: Let $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$$\therefore \quad \bar{a}\times\bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ x & y & z \end{vmatrix} = (z+y)\hat{i}-(x+2z)\hat{j}+(2y-x)\hat{k} \quad \ldots(1)$$

Also, by question,

$$\bar{a}\times\bar{r} = \bar{b}+\lambda\bar{a} = -\hat{i}-2\hat{j}+\hat{k}+\lambda(2\hat{i}+\hat{j}-\hat{k})$$

$$= (2\lambda-1)\hat{i}-(2-\lambda)\hat{j}+(1-\lambda)\hat{k} \qquad ...(2)$$

From (1) and (2), equating the coefficients of i, j, k we get

$$z+y = 2\lambda-1 \qquad ...(3)$$

$$x+2z = 2-\lambda \qquad ...(4)$$

$$2y-x = 1-\lambda \qquad ...(5)$$

Adding (4) and (5), we get $2(y+z) = 3-2\lambda$, or, $y+z = \frac{3}{2}-\lambda$ ...(6)

From (3) and (6), we have $2\lambda-1 = \frac{3}{2}-\lambda$, or, $\lambda = \frac{5}{6}$

Also, $\bar{a}\cdot\bar{r} = 3$, therefore, $2x+y-z = 3$ ...(7)

Putting $\lambda = \frac{5}{6}$ in (3), (4) and (5), we get

$$y+z = \frac{2}{3},\ x+2z = \frac{7}{6},\ 2y-x = \frac{1}{6}.$$

Solving, we get $x = \frac{7}{6},\ y = \frac{2}{3},\ z = 0.$

$\therefore$ $\bar{r} = \frac{7}{6}\hat{i}+\frac{2}{3}\hat{j}$ and $l = \frac{5}{6}.$

Example 26: Find unit vectors in the plane of $\bar{\alpha} = \hat{i}+2\hat{j}-\hat{k}$ and $\bar{\beta} = \hat{i}+\hat{j}-2\hat{k}$ which are perpendicular to the vector $\bar{\gamma} = 2\hat{i}-\hat{j}+\hat{k}.$

Solution: A vector in the plane of $\bar{\alpha}$ and $\bar{\beta}$ is of the form $x\bar{\alpha}+y\bar{\beta}, x, y$ are scalars. If it is perpendicular to $\bar{\gamma}$, then $(x\bar{\alpha}+y\bar{\beta})\cdot\bar{\gamma} = 0$

or $\{x(\hat{i}+2\hat{j}-\hat{k})+y(\hat{i}+\hat{j}-2\hat{k})\}\cdot(2\hat{i}-\hat{j}+\hat{k}) = 0$

or $\{(x+y)\hat{i}+(2x+y)\hat{j}-(x+2y)\hat{k}\}\cdot(2\hat{i}-\hat{j}+\hat{k}) = 0$

or $2(x+y)-(2x+y)-(x+2y) = 0$

or $y = -x.$

$\therefore$ $x\bar{\alpha}+y\bar{\beta} = x(\bar{\alpha}-\bar{\beta}) = x\{(\hat{i}+2\hat{j}-\hat{k})-(\hat{i}+\hat{j}-2\hat{k})\} = x(\hat{j}+\hat{k})$

Therefore, unit vector in this direction is

$$\frac{x(\hat{j}+\hat{k})}{\sqrt{x^2+x^2}} = \frac{1}{\sqrt{2}}(\hat{j}+\hat{k}).$$

Another unit vector in the opposite direction $= -\frac{1}{\sqrt{2}}(\hat{j}+\hat{k})$

Therefore, the required unit vectors are $\pm\frac{1}{\sqrt{2}}(\hat{j}+\hat{k}).$

or $$\vec{a}\times\vec{a}+\vec{a}\times\vec{b} = \vec{c}\times\vec{a}$$

$\therefore$ $$\vec{a}\times\vec{b} = \vec{c}\times\vec{a} \qquad (\because \vec{a}\times\vec{a}=\vec{0}) \qquad ...(1)$$

Also, $$\vec{b}+\vec{c} = -\vec{a}$$

$\therefore$ $$\vec{b}\times(\vec{b}+\vec{c}) = \vec{b}\times(-\vec{a}) = -\vec{b}\times\vec{a}$$

or $$\vec{b}\times\vec{b}+\vec{b}\times\vec{c} = \vec{a}\times\vec{b}$$

$\therefore$ $$\vec{b}\times\vec{c} = \vec{a}\times\vec{b} \qquad (\because \vec{b}\times\vec{b}=\vec{0}) \qquad ...(2)$$

From (1) and (2), we have $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$.

Example 30: If the position vectors of the vertices of a triangle are $\hat{i}+\hat{j}+2\hat{k}$, $2\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-\hat{j}-\hat{k}$, find its vector and scalar areas.

Solution: Let the position vectors of the vertices A, B, C of ΔABC are respectively $\hat{i}+\hat{j}+2\hat{k}$, $2\hat{i}+2\hat{j}+3\hat{k}$, $3\hat{i}-\hat{j}-\hat{k}$.

$\therefore$
$$\begin{aligned}\overrightarrow{AB} &= \text{p.v. of } B - \text{p.v. of } A\\ &= 2\hat{i}+2\hat{j}+3\hat{k}-(\hat{i}+\hat{j}+2\hat{k})\\ &= \hat{i}+\hat{j}+\hat{k}.\end{aligned}$$

$$\begin{aligned}\overrightarrow{AC} &= \text{p.v. of } C - \text{p.v. of } A\\ &= 3\hat{i}-\hat{j}-\hat{k}-(\hat{i}+\hat{j}+2\hat{k})\\ &= 2\hat{i}-2\hat{j}-3\hat{k}.\end{aligned}$$

The required vector area of ΔABC

$$= \frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC} = \frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 2 & -2 & -3\end{vmatrix}$$

$$= \frac{1}{2}(-\hat{i}+5\hat{j}-4\hat{k})$$

The scalar area of ΔABC is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$$= \frac{1}{2}\sqrt{(-1)^2+5^2+(-4)^2} = \frac{1}{2}\sqrt{42} \text{ square units.}$$

Example 31: Find the vector and scalar areas of a parallelogram whose adjacent sides are $\hat{i}-2\hat{j}+3\hat{k}$ and $2\hat{i}+\hat{j}-4\hat{k}$.

Solution: Vector area of the given parallelogram

$$= (\hat{i}-2\hat{j}+3\hat{k})\times(2\hat{i}+\hat{j}-4\hat{k})$$

Example 27: Two non-null vectors $\bar{r}_1$ and $\bar{r}_2$ are given by $\bar{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\bar{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, prove that the necessary and sufficient condition that $\bar{r}_1$ and $\bar{r}_2$ may be parallel is that $x_1 : y_1 : z_1 = x_2 : y_2 : z_2$.

Solution: If $\bar{r}_1$ and $\bar{r}_2$ are non-null vectors then the condition that $\bar{r}_1$ and $\bar{r}_2$ are parallel is $\bar{r}_1 \times \bar{r}_2 = \bar{0}$. Hence, for parallelism of $\bar{r}_1$ and $\bar{r}_2$ we have

$$\bar{r}_1 \times \bar{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \bar{0},$$

or
$$(y_1z_2 - y_2z_1)\hat{i} + (z_1x_2 - z_2x_1)\hat{j} + (x_1y_2 - x_2y_1)\hat{k} = \bar{0} \qquad \ldots(1)$$

Since $\hat{i}, \hat{j}, \hat{k}$ are non-coplanar, three scalar coefficients must be separately zero, *i.e.*,

$y_1z_2 - y_2z_1 = 0$, $z_1x_2 - z_2x_1 = 0$, $x_1y_2 - x_2y_1 = 0$ which gives $x_1 : y_1 : z_1 = x_2 : y_2 : z_2$.

Conversely, if $x_1 : y_1 : z_1 = x_2 : y_2 : z_2$, then we may take $x_1 = \lambda x_2$, $y_1 = \lambda y_2$, $z_1 = \lambda z_2$, where λ is a constant. These values of x_1, y_1, z_1 will make left side of (1) zero, *i.e.*, $\bar{r}_1 \times \bar{r}_2 = \bar{0}$.

Example 28: Using vector method, prove Schwarz's Inequality: $(a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$, the equality sign holds only when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

Solution: Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

We know that $\qquad \bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta$

$\therefore \qquad (\bar{a}\cdot\bar{b})^2 = |\bar{a}|^2|\bar{b}|^2\cos^2\theta \le |\bar{a}|^2|\bar{b}|^2$

i.e., $\qquad (\bar{a}\cdot\bar{b})^2 \le |\bar{a}|^2|\bar{b}|^2$

or
$$\{(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})\}^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

$$\therefore \quad (a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

The equality sign holds only when $\theta = 0$ or π, *i.e.*, only when $\bar{a}$ and $\bar{b}$ are collinear, *i.e.*, only when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

Example 29: If $\bar{a} + \bar{b} + \bar{c} = \bar{0}$, prove that $\bar{a}\times\bar{b} = \bar{b}\times\bar{c} = \bar{c}\times\bar{a}$.

Solution: Here $\bar{a} + \bar{b} = -\bar{c}$.

$$\therefore \qquad \bar{a}\times(\bar{a}+\bar{b}) = \bar{a}\times(-\bar{c}) = -\bar{a}\times\bar{c}$$

or $$\bar{a}\times\bar{a}+\bar{a}\times\bar{b} = \bar{c}\times\bar{a}$$

$\therefore$ $$\bar{a}\times\bar{b} = \bar{c}\times\bar{a} \qquad (\because\ \bar{a}\times\bar{a}=\bar{0}) \qquad ...(1)$$

Also, $$\bar{b}+\bar{c} = -\bar{a}$$

$\therefore$ $$\bar{b}\times(\bar{b}+\bar{c}) = \bar{b}\times(-\bar{a}) = -\bar{b}\times\bar{a}$$

or $$\bar{b}\times\bar{b}+\bar{b}\times\bar{c} = \bar{a}\times\bar{b}$$

$\therefore$ $$\bar{b}\times\bar{c} = \bar{a}\times\bar{b} \qquad (\because\ \bar{b}\times\bar{b}=\bar{0}) \qquad ...(2)$$

From (1) and (2), we have $\bar{a}\times\bar{b} = \bar{b}\times\bar{c} = \bar{c}\times\bar{a}$.

Example 30: If the position vectors of the vertices of a triangle are $\hat{i}+\hat{j}+2\hat{k}$, $2\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-\hat{j}-\hat{k}$, find its vector and scalar areas.

Solution: Let the position vectors of the vertices A, B, C of $\Delta\, ABC$ are respectively $\hat{i}+\hat{j}+2\hat{k}$, $2\hat{i}+2\hat{j}+3\hat{k}$, $3\hat{i}-\hat{j}-\hat{k}$.

$\therefore$
$$\begin{aligned}\overrightarrow{AB} &= \text{p.v. of } B - \text{p.v. of } A\\ &= 2\hat{i}+2\hat{j}+3\hat{k}-(\hat{i}+j+2\hat{k})\\ &= \hat{i}+j+\hat{k}.\end{aligned}$$

$$\begin{aligned}\overrightarrow{AC} &= \text{p.v. of } C - \text{p.v. of } A\\ &= 3\hat{i}-\hat{j}-\hat{k}-(\hat{i}+j+2\hat{k})\\ &= 2\hat{i}-2j-3\hat{k}.\end{aligned}$$

The required vector area of $\Delta\, ABC$

$$= \frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC} = \frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 2 & -2 & -3\end{vmatrix}$$

$$= \frac{1}{2}(-\hat{i}+5\hat{j}-4\hat{k})$$

The scalar area of $\Delta\, ABC$ is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

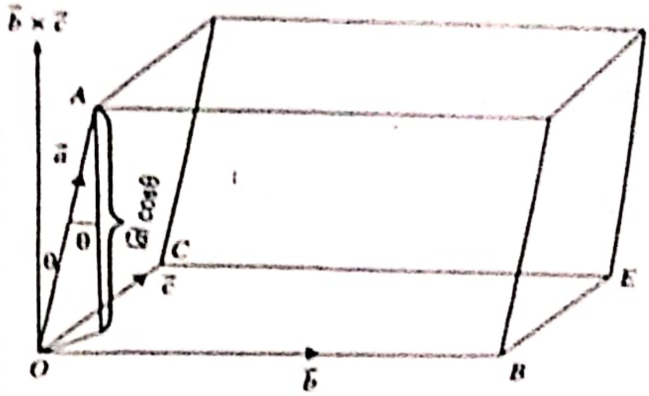
$$= \frac{1}{2}\sqrt{(-1)^2+5^2+(-4)^2} = \frac{1}{2}\sqrt{42} \text{ square units.}$$

Example 31: Find the vector and scalar areas of a parallelogram whose adjacent sides are $\hat{i}-2\hat{j}+3\hat{k}$ and $2\hat{i}+\hat{j}-4\hat{k}$.

Solution: Vector area of the given parallelogram

$$= (\hat{i}-2\hat{j}+3\hat{k})\times(2\hat{i}+\hat{j}-4\hat{k})$$

Geometrical Significance



Let $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ represent the vectors $\vec{a}, \vec{b}, \vec{c}$ respectively. Here $\vec{b}\times\vec{c}$ is a vector perpendicular to both $\vec{b}$ and $\vec{c}$ and hence perpendicular to the plane $OBEC$ of the parallelopiped whose three co-terminus edges are OB, OC and OA as shown in the adjacent figure.

Let θ be the angle between $\vec{a}$ and $\vec{b}\times\vec{c}$. Then $|\vec{a}|\cos\theta$ is the altitude of the parallelopiped and $|\vec{b}\times\vec{c}|$ is the area of the base $OBEC$.

Now,
$$\begin{aligned}\vec{a}\cdot(\vec{b}\times\vec{c}) &= |\vec{a}||\vec{b}\times\vec{c}|\cos\theta\\ &= |\vec{b}\times\vec{c}||\vec{a}|\cos\theta\\ &= \text{area of the base } OBEC \times \text{altitude of the parallelopiped}\\ &= \text{volume of the parallelopiped.}\end{aligned}$$

Thus the scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})$ is numerically equal to the volume V of a parallelopiped having $\vec{a}, \vec{b}, \vec{c}$ as co-terminus edges. Its sign is positive or negative according as $\vec{a}, \vec{b}, \vec{c}$ form a right-handed triad or left-handed triad.

Because of this geometric meaning of $\vec{a}\cdot(\vec{b}\times\vec{c})$, it is also known as box product and is denoted by $[\vec{a}\;\vec{b}\;\vec{c}]$.

Properties

(*i*) $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} = [\vec{a}\;\vec{b}\;\vec{c}]$

(*ii*) $[\vec{a}\;\vec{b}\;\vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = -\vec{a}\cdot(\vec{c}\times\vec{b}) = -[\vec{a}\;\vec{c}\;\vec{b}]$

(*iii*) $[\vec{a}\;\vec{b}\;\vec{c}] = [\vec{b}\;\vec{c}\;\vec{a}] = [\vec{c}\;\vec{a}\;\vec{b}] = -[\vec{a}\;\vec{c}\;\vec{b}] = -[\vec{b}\;\vec{a}\;\vec{c}] = -[\vec{c}\;\vec{b}\;\vec{a}]$

(*iv*) $[\vec{i}\;\vec{j}\;\vec{k}] = [\vec{j}\;\vec{k}\;\vec{i}] = [\vec{k}\;\vec{i}\;\vec{j}] = 1$

and $[\vec{i}\;\vec{k}\;\vec{j}] = [\vec{k}\;\vec{j}\;\vec{i}] = [\vec{j}\;\vec{i}\;\vec{k}] = -1.$

(*v*) $[\vec{a}\;\vec{b}\;\vec{c}] = 0$ implies

either one of $\vec{a}, \vec{b}, \vec{c}$ is a zero vector or, two of them are parallel (or equal) or, the volume of the parallelopiped formed by them is zero, *i.e.*, they are coplanar.

Consequently, $[\vec{a}\;\vec{a}\;\vec{c}] = \vec{a}\cdot(\vec{a}\times\vec{c}) = 0$ as, $\vec{a}, \vec{a}, \vec{c}$ are coplanar. Also note that $\vec{a}\times\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{c}$ and hence, $\vec{a}\cdot(\vec{a}\times\vec{c}) = 0$.

Note: 1. From (*i*), (*ii*) and (*iii*), we observe that the value of the scalar triple product depends on the cyclic order of the vectors as they appear in the product but not on the position of dot and cross.

2. $[\vec{a}\;\vec{b}\;\vec{c}]$ is also written as $[\vec{a}, \vec{b}, \vec{c}]$.

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 1 & -4 \end{vmatrix} = 5\hat{i} + 10\hat{j} + 5\hat{k}$$

Scalar area = $|5\hat{i} + 10\hat{j} + 5\hat{k}| = \sqrt{5^2 + 10^2 + 5^2} = 5\sqrt{6}$ square units.

Example 32: If AC and BD are two diagonals of a quadrilateral, show that its vector area is $\frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$.

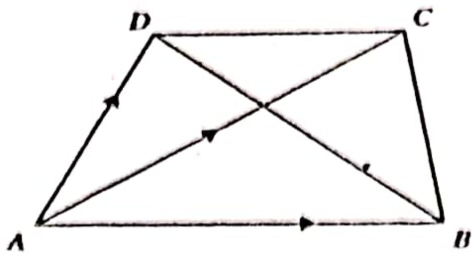
Solution: Vector area of the quadrilateral $ABCD$ = vector area of ΔABC + vector area of ΔACD

$$= \frac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC} + \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{AD}$$

$$= \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{AD} - \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{AB}$$

$$= \frac{1}{2}\overrightarrow{AC} \times (\overrightarrow{AD} - \overrightarrow{AB})$$

$$= \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$$



12.4 PRODUCT OF THREE VECTORS

Scalar Triple Product

If $\vec{a}, \vec{b}, \vec{c}$ are three vectors then $\vec{a} \cdot (\vec{b} \times \vec{c})$, which is a scalar quantity, is known as scalar triple product and is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\hat{i} + (b_3c_1 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}$$

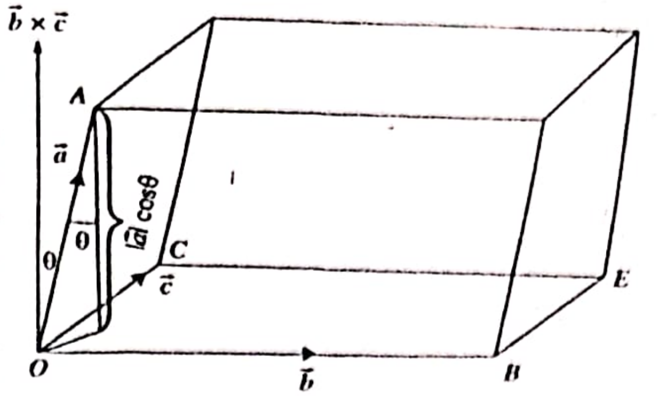
$$\therefore \quad \vec{a} \cdot (\vec{b} \times \vec{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Note: Scalar triple product is also known as box product.

Geometrical Significance

Let $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ represent the vectors $\vec{a}, \vec{b}, \vec{c}$ respectively. Here $\vec{b}\times\vec{c}$ is a vector perpendicular to both $\vec{b}$ and $\vec{c}$ and hence perpendicular to the plane $OBEC$ of the parallelopiped whose three co-terminus edges are OB, OC and OA as shown in the adjacent figure.



Let θ be the angle between $\vec{a}$ and $\vec{b}\times\vec{c}$. Then $|\vec{a}|\cos\theta$ is the altitude of the parallelopiped and $|\vec{b}\times\vec{c}|$ is the area of the base $OBEC$.

Now, $\vec{a}\cdot(\vec{b}\times\vec{c}) = |\vec{a}||\vec{b}\times\vec{c}|\cos\theta$

$= |\vec{b}\times\vec{c}||\vec{a}|\cos\theta$

= area of the base $OBEC$ × altitude of the parallelopiped

= volume of the parallelopiped.

Thus the scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})$ is numerically equal to the volume V of a parallelopiped having $\vec{a}, \vec{b}, \vec{c}$ as co-terminus edges. Its sign is positive or negative according as $\vec{a}, \vec{b}, \vec{c}$ form a right-handed triad or left-handed triad.

Because of this geometric meaning of $\vec{a}\cdot(\vec{b}\times\vec{c})$, it is also known as box product and is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$.

Properties

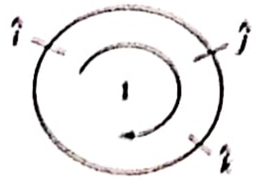
(i) $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]$

(ii) $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = -\vec{a}\cdot(\vec{c}\times\vec{b}) = -[\vec{a}\ \vec{c}\ \vec{b}]$

(iii) $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}] = -[\vec{a}\ \vec{c}\ \vec{b}] = -[\vec{b}\ \vec{a}\ \vec{c}] = -[\vec{c}\ \vec{b}\ \vec{a}]$

(iv) $[\hat{i}\ \hat{j}\ \hat{k}] = [\hat{j}\ \hat{k}\ \hat{i}] = [\hat{k}\ \hat{i}\ \hat{j}] = 1$

and $[\hat{i}\ \hat{k}\ \hat{j}] = [\hat{k}\ \hat{j}\ \hat{i}] = [\hat{j}\ \hat{i}\ \hat{k}] = -1.$



(v) $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ implies

either one of $\vec{a}, \vec{b}, \vec{c}$ is a zero vector or, two of them are parallel (or equal) or, the volume of the parallelopiped formed by them is zero, *i.e.*, they are coplanar.

Consequently, $[\vec{a}\ \vec{a}\ \vec{c}] = \vec{a}\cdot(\vec{a}\times\vec{c}) = 0$ as, $\vec{a}, \vec{a}, \vec{c}$ are coplanar. Also note that $\vec{a}\times\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{c}$ and hence, $\vec{a}\cdot(\vec{a}\times\vec{c}) = 0$.

Note: 1. From (i), (ii) and (iii), we observe that the value of the scalar triple product depends on the cyclic order of the vectors as they appear in the product but not on the position of dot and cross.

2. $[\vec{a}\ \vec{b}\ \vec{c}]$ is also written as $[\vec{a}, \vec{b}, \vec{c}]$.

Vector Triple Product

For three vectors $\vec{a}, \vec{b}, \vec{c}$, the expressions $\vec{a}\times(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b})\times\vec{c}$, which are vectors, are called vector triple products of $\vec{a}, \vec{b}, \vec{c}$. Obviously $\vec{a}\times(\vec{b}\times\vec{c})$ lies in the plane of $\vec{b}$ and $\vec{c}$ where as $(\vec{a}\times\vec{b})\times\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$.

Let $\vec{i}, \vec{j}, \vec{k}$ be the unit vectors along the positive directions of x, y, z-axes respectively. Without loss of generality, we may choose $\vec{a}$ along $\vec{i}$, $\vec{b}$ in the plane of $\vec{i}$ and $\vec{j}$. Therefore, we can write $\vec{a}, \vec{b}, \vec{c}$ in the following form: $\vec{a} = a\hat{i}, \vec{b} = b_1\hat{i}+b_2\hat{j}, \vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$.

$$\therefore \quad \vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = b_2c_3\,\hat{i} - b_1c_3\,\hat{j} + (b_1c_2 - b_2c_1)\,\hat{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & 0 & 0 \\ b_2c_3 & -b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$$= -a(b_1c_2 - b_2c_1)\,\hat{j} - ab_1c_3\,\hat{k}$$

$$= a\,c_1b_1\,\hat{i} + a\,c_1b_2\,\hat{j} - a\,c_1b_1\,\hat{i} - a\,b_1c_2\,\hat{j} - a\,b_1c_3\,\hat{k}$$

$$= a\,c_1(b_1\,\hat{i} + b_2\,\hat{j}) - a\,b_1(c_1\,\hat{i} + c_2\,\hat{j} + c_3\hat{k})$$

$$= (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$$

$$\therefore \quad (\vec{a}\times\vec{b})\times\vec{c} = -\vec{c}\times(\vec{a}\times\vec{b}) = -\{(\vec{c}\cdot\vec{b})\,\vec{a} - (\vec{c}\cdot\vec{a})\,\vec{b}\}$$

$$= (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{b}\cdot\vec{c})\,\vec{a}$$

Thus $\boxed{\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}}$ *(W.B.U.T. 2005)*

and $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{b}\cdot\vec{c})\,\vec{a}$

Note: (*i*) It is evident from the above results that $\vec{a}\times(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b})\times\vec{c}$ are not equal in general.

(*ii*) If $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$ for three non-zero vectors $\vec{a}, \vec{b}, \vec{c}$, then

$$(\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c} = -\vec{c}\times(\vec{a}\times\vec{b}) = \vec{c}\times(\vec{b}\times\vec{a})$$

or $$(\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c} = (\vec{c}\cdot\vec{a})\,\vec{b} - (\vec{c}\cdot\vec{b})\,\vec{a}$$

or $$(\vec{c}\cdot\vec{b})\,\vec{a} - (\vec{a}\cdot\vec{b})\,\vec{c} = \vec{0}$$

or $$(\vec{b}\cdot\vec{c})\,\vec{a} - (\vec{b}\cdot\vec{a})\,\vec{c} = \vec{0}$$

or $$\vec{b}\times(\vec{a}\times\vec{c}) = \vec{0}$$

i.e., either $\vec{b}$ is parallel to $\vec{a}\times\vec{c}$ or $\vec{a}\times\vec{c} = \vec{0}$ $(\because \vec{b} \neq \vec{0})$

i.e., either $\vec{b}$ is parallel to $\vec{a}\times\vec{c}$
or, $\vec{a}$ and $\vec{c}$ are collinear.

Reciprocal System of Vectors

Definition: If $\vec{a},\vec{b},\vec{c}$ be a system of three non-coplanar vectors, *i.e.*, $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$, then the system of vectors $\vec{a}',\vec{b}',\vec{c}'$ which satisfy the relations

$$\vec{a}\cdot\vec{a}' = \vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1$$

and

$$\vec{a}\cdot\vec{b}' = \vec{a}\cdot\vec{c}' = \vec{b}\cdot\vec{a}' = \vec{b}\cdot\vec{c}' = \vec{c}\cdot\vec{a}' = \vec{c}\cdot\vec{b}' = 0 \quad \ldots(1)$$

are said to be the reciprocal system to the vectors $\vec{a},\vec{b},\vec{c}$.

Expressions for $\vec{a}',\vec{b}',\vec{c}'$ in terms of $\vec{a},\vec{b},\vec{c}$

The relations $\vec{b}\cdot\vec{a}' = \vec{c}\cdot\vec{a}' = 0$ imply that $\vec{a}'$ is perpendicular to both $\vec{b}$ and $\vec{c}$ and hence, parallel to $\vec{b}\times\vec{c}$. In other words, $a' = n(\vec{b}\times\vec{c})$, where n is a number to be determined. Further from $\vec{a}\cdot\vec{a}' = 1$, we get $\vec{a}\cdot n(\vec{b}\times\vec{c}) = 1$, *i.e.*,

$$n = \frac{1}{\vec{a}\cdot(\vec{b}\times\vec{c})} = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}.$$

We thus obtain

$$\vec{a}' = n(\vec{b}\times\vec{c}) = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Similarly,

$$\vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

and

$$\vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]} \quad \ldots(2)$$

Symmetry of the relations in (1) shows that if $\vec{a}',\vec{b}',\vec{c}'$ be the reciprocal system to $\vec{a},\vec{b},\vec{c}$, then the system $\vec{a},\vec{b},\vec{c}$ is reciprocal to $\vec{a}',\vec{b}',\vec{c}'$.

Therefore, we have

$$\vec{a} = \frac{\vec{b}'\times\vec{c}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']},\ \vec{b} = \frac{\vec{c}'\times\vec{a}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']},\ \vec{c} = \frac{\vec{a}'\times\vec{b}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']} \quad \ldots(3)$$

Derivation of (1) from (2)

$$\vec{a}\cdot\vec{a}' = \vec{a}\cdot\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} = 1$$

and

$$\vec{a}\cdot\vec{b}' = \vec{a}\cdot\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{[\vec{a}\ \vec{c}\ \vec{a}]}{[\vec{a}\ \vec{b}\ \vec{c}]} = 0 \qquad (\because [\vec{a}\ \vec{c}\ \vec{a}] = 0).$$

Similarly other relations of (1) can be obtained.

Observation

From (2), we have

$$\bar{b}'\times\bar{c}' = \frac{(\bar{c}\times\bar{a})\times(\bar{a}\times\bar{b})}{[\bar{a}\,\bar{b}\,\bar{c}]^2} = \frac{[\bar{c}\,\bar{a}\,\bar{b}]}{[\bar{a}\,\bar{b}\,\bar{c}]^2}\bar{a} = \frac{\bar{a}}{[\bar{a}\,\bar{b}\,\bar{c}]} \qquad ...(4)$$

$$[\because (\bar{c}\times\bar{a})\times(\bar{a}\times\bar{b}) = \{(\bar{c}\times\bar{a})\cdot\bar{b}\}\,\bar{a} - \{(\bar{c}\times\bar{a})\cdot\bar{a}\}\,\bar{b}$$

$$= \{(\bar{c}\times\bar{a})\cdot\bar{b}\}\,\bar{a} = [\bar{c}\,\bar{a}\,\bar{b}]\,\bar{a},\ \text{since}\ (\bar{c}\times\bar{a})\cdot\bar{a} = 0]$$

$$\therefore \qquad \bar{a}'\cdot(\bar{b}'\times\bar{c}') = \frac{\bar{b}\times\bar{c}}{[\bar{a}\,\bar{b}\,\bar{c}]}\cdot\frac{\bar{a}}{[\bar{a}\,\bar{b}\,\bar{c}]} = \frac{[\bar{b}\,\bar{c}\,\bar{a}]}{[\bar{a}\,\bar{b}\,\bar{c}]^2}$$

or

$$[\bar{a}'\,\bar{b}'\,\bar{c}'] = \frac{1}{[\bar{a}\,\bar{b}\,\bar{c}]}$$

$$\therefore \qquad [\bar{a}\,\bar{b}\,\bar{c}]\,[\bar{a}'\,\bar{b}'\,\bar{c}'] = 1 \qquad ...(5)$$

From (4) and (5), we have

$$\bar{a} = \frac{\bar{b}'\times\bar{c}'}{[\bar{a}'\,\bar{b}'\,\bar{c}']}.$$

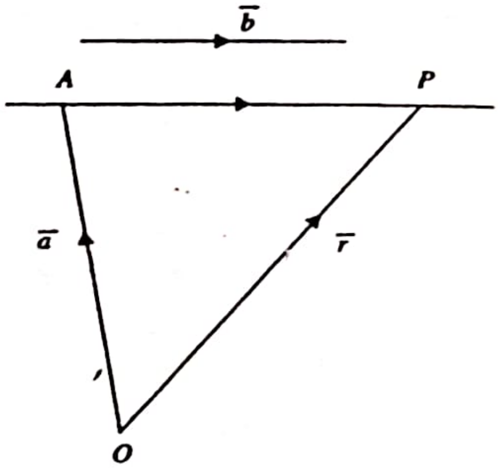
Similarly other relations of (3) can be derived. From (5), we see that $[\bar{a}\,\bar{b}\,\bar{c}]$ and $[\bar{a}'\,\bar{b}'\,\bar{c}']$ are reciprocal to each other, this gives justification for the name reciprocal. From (5), we also observe that $[\bar{a}\,\bar{b}\,\bar{c}]$ and $[\bar{a}'\,\bar{b}'\,\bar{c}']$ must have the same sign.

12.5 VECTOR EQUATIONS OF STRAIGHT LINE, PLANE AND SPHERE

Vector equation of a straight line passing through a given point and parallel to a given vector

Given: A be the given point whose position vector is $\bar{a}$ with respect to O and $\bar{b}$ is the given vector.

Let P be any point on the required line AP, which is parallel to $\bar{b}$ and the position vector of P be $\bar{r}$ with respect to O.



$$\therefore \qquad \overline{AP} = t\,\bar{b}$$

Now, $$\overline{OP} = \overline{OA} + \overline{AP}$$

or $$\bar{r} = \bar{a} + t\bar{b},\ \text{where } t \text{ is a scalar.} \quad ...(1)$$

This is the required vector equation of the straight line.

Note: (*i*) The vector equation of a straight line passing through the origin O and parallel to $\bar{b}$ is clearly $\bar{r} = t\bar{b}$, t is a scalar.

(*ii*) The equation (1) can also be written as $\bar{r}\times\bar{b} = \bar{a}\times\bar{b}$.

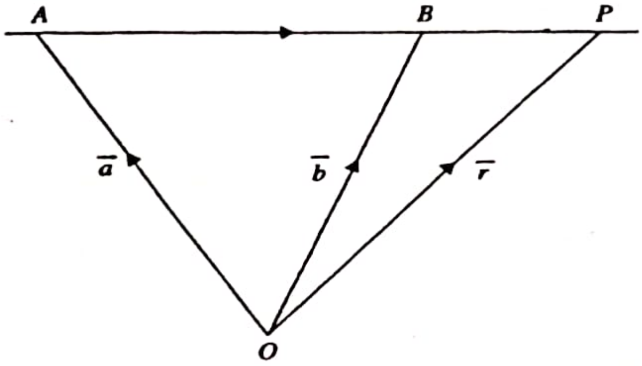
(*iii*) With reference to a system of three rectangular axes through O, let the co-ordinates of A and P be (x_1, y_1, z_1) and (x, y, z) respectively and let $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then (1) gives

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + t(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

or $(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k} = t(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

or $$\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3} = t$$

Vector equation of a straight line passing through two given points



Given: A and B are two given points whose position vectors are $\bar{a}$ and $\bar{b}$ respectively with respect to O.

Let P be any point on the required line AP and the position vector of P be $\bar{r}$ with respect to O.

Now, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

or $\boxed{\bar{r} = \bar{a} + t\,\overline{AB}}$

or $\bar{r} = \bar{a} + t(\bar{b} - \bar{a})$

$\therefore$ $\bar{r} = (1-t)\,\bar{a} + t\,\bar{b}$, where t is a scalar ...(2)

This is the required vector equation of the straight line.

Note: (*i*) The equation (2) can be written as $\bar{r} - (1-t)\,\bar{a} - t\,\bar{b} = \bar{O}$. The sum of scalar coefficients $= 1 - (1-t) - t = 0$, therefore, the condition of collinearity of the three points A, B, P is satisfied.

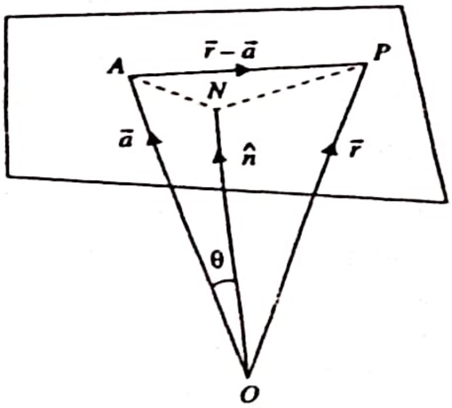
(*ii*) With reference to a system of three rectangular axes through O, let the co-ordinates of A, B and P are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x, y, z) respectively. Then (2) gives

$$x\hat{i} + y\hat{j} + z\hat{k} = (1-t)\;(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + t\;(x_2\hat{i} + y_2\hat{j} + z_2\hat{k})$$

or $(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k} = t(x_2 - x_1)\hat{i} + t(y_2 - y_1)\hat{j} + t(z_2 - z_1)\hat{k}$.

$$\therefore \quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t.$$

Vector equation of a plane passing through a given point and perpendicular to a given vector



Given: A is a point on the required plane and $\hat{n}$ is the unit vector perpendicular to the plane. The position vector of A is $\bar{a}$ with respect to O.

Let P be any point on the required plane and the position vector of P be $\bar{r}$ w.r.t. O.

$\bar{r} = \bar{a} + x\bar{b}$

Now, $$\overline{AP} = \bar{r} - \bar{a}$$

$\therefore$ $$(\bar{r} - \bar{a}) \cdot \hat{n} = 0 \qquad (\because \overrightarrow{AP} \perp \hat{n})$$

or $$\bar{r} \cdot \hat{n} = \bar{a} \cdot \hat{n} = p,$$

where $\bar{a} \cdot \hat{n} = a \cos\theta = ON$ = length of the perpendicular from O to the plane $= p$ (say).

This is the required equation of the plane and is known as the normal form of the equation of plane.

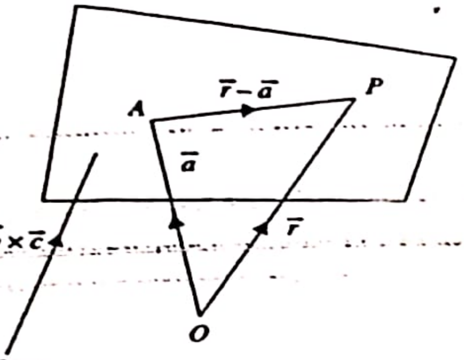
If the plane passes through the origin then this equation becomes $\bar{r} \cdot \hat{n} = 0$.

Note: The equation of a plane passing through a point A with position vector $\bar{a}$ and perpendicular to the vector $\bar{m}$ is

$$(\bar{r} - \bar{a}) \cdot \bar{m} = 0, \text{ or, } \bar{r} \cdot \bar{m} = \bar{a} \cdot \bar{m} = k \text{ (a scalar constant)},$$

where $\bar{r}$ is the position vector of any point on the plane. If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}, \bar{m} = m_1\hat{i} + m_2\hat{j} + m_3\hat{k}$, then the Cartesian form of the equation of this plane is $m_1x + m_2y + m_3z$ = constant.

Vector equation of a plane passing through a given point and parallel to two given vectors



Given: A be the given point whose position vector is $\bar{a}$ w.r.t. O and $\bar{b}, \bar{c}$ are given vectors.

Let P be any point on the required plane and the position vector of P be $\bar{r}$ w.r.t. O. Since $\bar{b}, \bar{c}$ are parallel to this plane, $\bar{b} \times \bar{c}$ is perpendicular to this plane and hence $\bar{b} \times \bar{c}$ is perpendicular to $\overrightarrow{AP}$.

$\therefore$ $$\overline{AP}.(\bar{b} \times \bar{c}) = 0, \text{ or, } (\bar{r} - \bar{a}) \cdot (\bar{b} \times \bar{c}) = 0$$

or $$\bar{r} \cdot (\bar{b} \times \bar{c}) = \bar{a} \cdot (\bar{b} \times \bar{c}), \text{ or, } [\bar{r}\ \bar{b}\ \bar{c}] = [\bar{a}\ \bar{b}\ \bar{c}]$$

This is the required vector equation of the plane.

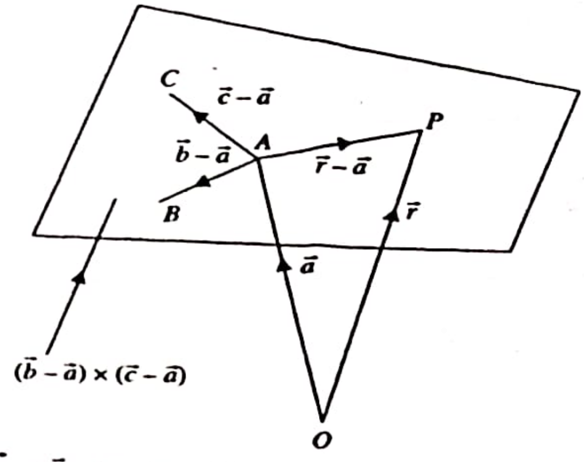
Note: (*i*) The equation of a plane through the origin and parallel to two given vectors $\bar{a}$ and $\bar{b}$ is $[\bar{r}\ \bar{a}\ \bar{b}] = 0$.

(*ii*) The plane passing through two points with position vectors $\bar{a}$ and $\bar{b}$ and parallel to $\bar{c}$ is identical with the plane passing through $\bar{a}$ and parallel to $\bar{b} - \bar{a}, \bar{c}$. Hence, the equation of this plane is $[\bar{r}\ (\bar{b} - \bar{a})\ \bar{c}] = [\bar{a}\ \bar{b}\ \bar{c}]$.

Vector equation of a plane passing through three given points

Given : A, B, C are three given points whose position vectors are $\bar{a}, \bar{b}, \bar{c}$ respectively w.r.t. O.

Let P be any point on the required plane and the position vector of P be $\bar{r}$ w.r.t. O. Here $(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})$ is perpendicular to this plane and hence, perpendicular to $\overline{AP}$.



$\therefore$ $\overrightarrow{AP}\cdot\{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\}=0$

or $(\vec{r}-\vec{a})\cdot\{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\}=0$

or $\vec{r}\cdot(\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}) = \vec{a}\cdot(\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a})$

or $\vec{r}\cdot\vec{m} = [\vec{a}\,\vec{b}\,\vec{c}]$, where $m = \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$.

This is the required equation of the plane.

Vector equation of a plane through two given lines

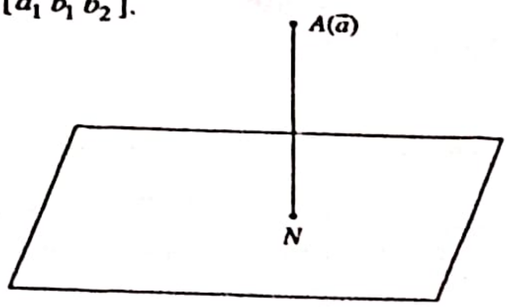
Given: $\vec{r} = \vec{a}_1+t\vec{b}_1$ and $\vec{r} = \vec{a}_2+s\vec{b}_2$, where t, s are scalars, are two given lines.

Clearly, the plane through these lines passes through the point $\vec{a}_1$ and is parallel to $\vec{b}_1$ and $\vec{b}_2$. Therefore, the equation of this plane is $[\vec{r}\,\vec{b}_1\,\vec{b}_2] = [\vec{a}_1\,\vec{b}_1\,\vec{b}_2]$, where $\vec{r}$ is the position vector of any point on the plane.

Condition of coplanarity of two straight lines

Since the plane passes through the point $\vec{a}_2$. Therefore, the condition of coplanarity of two straight lines $\vec{r} = \vec{a}_1+t\vec{b}_1$ and $\vec{r} = \vec{a}_2+s\vec{b}_2$ is $[\vec{a}_2\,\vec{b}_1\,\vec{b}_2] = [\vec{a}_1\,\vec{b}_1\,\vec{b}_2]$.

Distance of a point from a given plane



Given: A be the given point whose position vector is $\vec{a}$ and the equation of the given plane be $\vec{r}\cdot\vec{m} = k$, where $\vec{m}$ is normal to the plane.

Let AN is perpendicular to the plane where N lies on the plane. The equation of AN is $\vec{r} = \vec{a}+t\vec{m}$, t is a scalar. At the point of intersection N of this line and the plane,

$$(\vec{a}+t\vec{m})\cdot\vec{m} = k,\text{ or, } \vec{a}\cdot\vec{m}+t|\vec{m}|^2 = k,\text{ or, } t = \frac{k-\vec{a}\cdot\vec{m}}{|\vec{m}|^2}$$

Therefore, the position vector of $N = \vec{a}+t\vec{m} = \vec{a}+\dfrac{(k-\vec{a}\cdot\vec{m})\vec{m}}{|\vec{m}|^2}$

Equation of the line of shortest distance

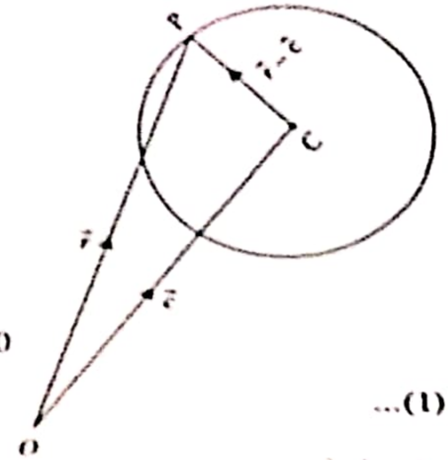
The line of shortest distance is the line of intersection of the two planes drawn through the given skew lines and the line of shortest distance (LM).

The equation of the plane drawn through $\vec{r} = \vec{a} + t\vec{b}$ and LM is $(\vec{r} - \vec{a}) \cdot [\vec{b} \times (\vec{b} \times \vec{d})] = 0$ ...(2)

The equation of the plane drawn through $\vec{r} = \vec{c} + s\vec{d}$ and LM is $(\vec{r} - \vec{c}) \cdot [\vec{d} \times (\vec{b} \times \vec{d})] = 0$ ...(3)

The line of intersection of the planes (2) and (3) is the required line of shortest distance.

Vector equation of a sphere



Given: C be the centre of the sphere and a is the radius of the sphere. The position vector of C is $\vec{c}$ w.r.t. O.

Let P be any point on the surface of the required sphere and the position vector of P be $\vec{r}$ w.r.t. O.

$\therefore$ $\overrightarrow{CP} = \vec{r} - \vec{c}$ and $CP = |\overrightarrow{CP}| = |\vec{r} - \vec{c}| = a =$ radius of the sphere.

$\therefore$ $$CP^2 = (\vec{r} - \vec{c})^2 = a^2 \text{ or } |\vec{r}|^2 - 2\vec{r} \cdot \vec{c} + |\vec{c}|^2 - a^2 = 0$$

$\therefore$ $$|\vec{r}|^2 - 2\vec{r} \cdot \vec{c} + k = 0 \quad ...(1)$$

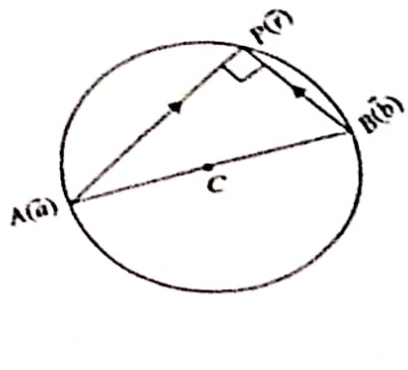
where $k = |\vec{c}|^2 - a^2$

Since the relation (1) is true for any point P on the surface of the sphere and by no others, therefore, it represents the vector equation of the sphere.

Particular cases

1. When the origin O lies on the surface of the sphere $OC = a$, *i.e.*, $|\vec{c}| = a$ and hence, $k = 0$. The equation of the sphere becomes $|\vec{r}|^2 - 2\vec{r} \cdot \vec{c} = 0$.
2. When the origin O coincides with the centre C, we have simply $\overrightarrow{CP} = \vec{r}$. Therefore, $\overrightarrow{CP} \cdot \overrightarrow{CP} = \vec{r} \cdot \vec{r}$, or $|\vec{r}|^2 = CP^2$, *i.e.*, $|\vec{r}|^2 = a^2$ is the required equation.

Vector equation of a sphere with two given points as the extremities of a diameter



Given: A and B are the two given points which are the extremities of a diameter of the required sphere. The position vectors of A and B are $\vec{a}$ and $\vec{b}$ respectively w.r.t. O.

Let P be any point on the surface of the required sphere and the position vector of P be $\vec{r}$ w.r.t. O.

$\therefore$ $$\overrightarrow{AP} = \vec{r} - \vec{a} \text{ and } \overrightarrow{BP} = \vec{r} - \vec{b}.$$

Since $$\angle APB = 90^\circ, \text{ we have } (\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0.$$

This is satisfied by the position vector of any point on the surface of the sphere and by no others and hence, it is the required equation of the sphere.

Hence, $\overline{AN}$ = p.v. of N – p.v. of A = $\bar{a}+\dfrac{(k-\bar{a}\cdot\bar{m})\bar{m}}{|\bar{m}|^2}-\bar{a} = \dfrac{(k-\bar{a}\cdot\bar{m})\bar{m}}{|\bar{m}|^2}$

Therefore, the required distance

$$= AN = \frac{|k-\bar{a}\cdot\bar{m}||\bar{m}|}{|\bar{m}|^2} = \frac{|k-\bar{a}\cdot\bar{m}|}{|\bar{m}|}$$

Distance of a point from a given line

N

Given: A be the given point whose position vector is $\bar{a}$. The equation of the given line BN be $\bar{r} = \bar{b}+t\hat{n}$ where t is a scalar, $\bar{b}$ is the position vector of the point B and $\hat{n}$ is the unit vector parallel to BN.

AN is perpendicular to BN.

Now $\quad \overline{BA} = \bar{a}-\bar{b}.$

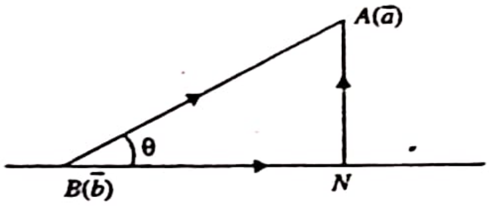
$\therefore \quad BA^2 = \overline{BA}\cdot\overline{BA} = (\bar{a}-\bar{b})^2$

$BN = BA\cos\theta = (\bar{a}-\bar{b})\cdot\hat{n}$

$\therefore \quad AN^2 = BA^2 - BN^2$

$= (\bar{a}-\bar{b})^2 - \{(\bar{a}-\bar{b})\cdot\hat{n}\}^2,$

m 3)



which gives the required distance AN.

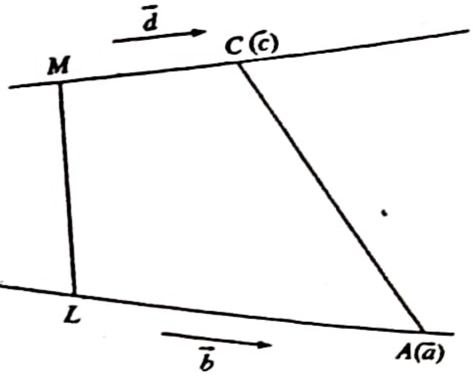
Note: (*i*) $\overline{NA} = \overline{BA}-\overline{BN} = (\bar{a}-\bar{b})-\{(\bar{a}-\bar{b})\cdot\hat{n}\}\hat{n}$

(*ii*) If $\hat{n}$ is not the unit vector then replace $\hat{n}$ by $\dfrac{\hat{n}}{|\hat{n}|}$, where $|\hat{n}|$ is the magnitude of $\hat{n}$.

n

Shortest distance between two skew lines

Given: LA and MC be two skew (*i.e.*, not coplanar) lines and their equations are $\bar{r} = \bar{a}+t\bar{b}$ and $\bar{r} = \bar{c}+s\bar{d}$ respectively. Here t, s are scalars, $\bar{a}, \bar{c}$ are the position vectors of A, C respectively and LA, MC are parallel to $\bar{b}, \bar{d}$ respectively.



Let LM be the shortest distance between the given skew lines. Then LM is perpendicular to both the lines, so it is parallel to $\bar{b}\times\bar{d}$. The shortest distance is the projection of $\overline{AC}$ upon $\overline{LM}$, *i.e.*, the projection of $\bar{c}-\bar{a}$ on $\bar{b}\times\bar{d}$ (since LM is parallel to $\bar{b}\times\bar{d}$).

$$\therefore \quad LM = \frac{|(\bar{c}-\bar{a})\cdot(\bar{b}\times\bar{d})|}{|\bar{b}\times\bar{d}|} = \frac{|[\bar{c}\,\bar{b}\,\bar{d}]-[\bar{a}\,\bar{b}\,\bar{d}]|}{|\bar{b}\times\bar{d}|} \quad ...(1)$$

Note: If $\bar{r} = \bar{a}+t\bar{b}$, $\bar{r} = \bar{c}+s\bar{d}$ are two non-parallel intersecting straight lines, then they are coplanar. Therefore, $[\bar{a}\,\bar{b}\,\bar{d}] = [\bar{c}\,\bar{b}\,\bar{d}]$, *i.e.*, the shortest distance LM is zero.

Equation of the line of shortest distance

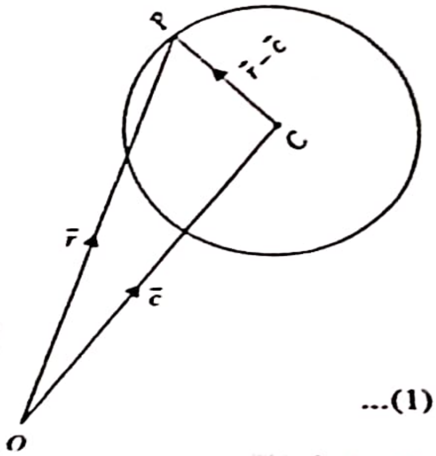
The line of shortest distance is the line of intersection of the two planes drawn through the given skew lines and the line of shortest distance (LM).

The equation of the plane drawn through $\bar{r} = \bar{a}+t\bar{b}$ and LM is $(\bar{r}-\bar{a})\cdot\{\bar{b}\times(\bar{b}\times\bar{d})\} = 0$ ...(2)

The equation of the plane drawn through $\bar{r} = \bar{c}+s\bar{d}$ and LM is $(\bar{r}-\bar{c})\cdot\{\bar{d}\times(\bar{b}\times\bar{d})\} = 0$ ...(3)

The line of intersection of the planes (2) and (3) is the required line of shortest distance.

Vector equation of a sphere



Given: C be the centre of the sphere and a is the radius of the sphere. The position vector of C is $\bar{c}$ w.r.t. O.

Let P be any point on the surface of the required sphere and the position vector of P be $\bar{r}$ w.r.t. O.

$\therefore$ $\overrightarrow{CP} = \bar{r}-\bar{c}$ and $CP = |\overrightarrow{CP}| = |\bar{r}-\bar{c}| = a =$ radius of the sphere.

$$\therefore \quad CP^2 = (\bar{r}-\bar{c})^2 = a^2 \text{ or } |\bar{r}|^2 - 2\bar{r}\cdot\bar{c} + |\bar{c}|^2 - a^2 = 0$$

$$\therefore \quad |\bar{r}|^2 - 2\bar{r}\cdot\bar{c} + k = 0 \qquad ...(1)$$

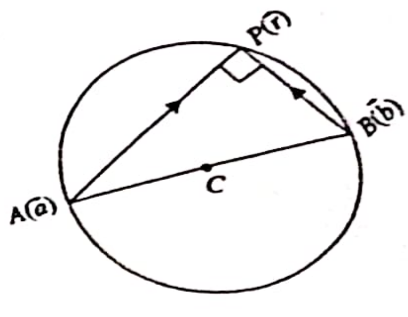
where $k = |\bar{c}|^2 - a^2$.

Since the relation (1) is true for any point P on the surface of the sphere and by no others, therefore, it represents the vector equation of the sphere.

Particular cases

1. When the origin O lies on the surface of the sphere $OC = a$, *i.e.*, $|\bar{c}| = a$ and hence, $k = 0$. The equation of the sphere becomes $|\bar{r}|^2 - 2\bar{r}\cdot\bar{c} = 0$.
2. When the origin O coincides with the centre C, we have simply $\overrightarrow{CP} = \bar{r}$. Therefore, $\overrightarrow{CP}\cdot\overrightarrow{CP} = \bar{r}\cdot\bar{r}$, or $|\bar{r}|^2 = CP^2$, *i.e.*, $|\bar{r}|^2 = a^2$ is the required equation.

Vector equation of a sphere with two given points as the extremities of a diameter



Given: A and B are the two given points which are the extremities of a diameter of the required sphere. The position vectors of A and B are $\bar{a}$ and $\bar{b}$ respectively w.r.t. O.

Let P be any point on the surface of the required sphere and the position vector of P be $\bar{r}$ w.r.t. O.

$\therefore$ $\overrightarrow{AP} = \bar{r}-\bar{a}$ and $\overrightarrow{BP} = \bar{r}-\bar{b}$.

Since $\angle APB = 90°$, we have $(\bar{r}-\bar{a})\cdot(\bar{r}-\bar{b}) = 0$.

This is satisfied by the position vector of any point on the surface of the sphere and by no others and hence, it is the required equation of the sphere.

ILLUSTRATIVE EXAMPLES

Example 1: Prove that the vectors $\vec{A} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{B} = \hat{i} - 3\hat{j} - 5\hat{k}$, $\vec{C} = 3\hat{i} - 4\hat{j} - 4\hat{k}$ form the sides of a right-angled triangle.

Solution: If $\vec{A}, \vec{B}, \vec{C}$ form a triangle, then they must be coplanar. To prove the coplanarity we have to show $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$.

Now,
$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & -5 \\ 3 & -4 & -4 \end{vmatrix} = -8\hat{i} - 11\hat{j} + 5\hat{k}$$

$\therefore$
$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (2\hat{i} - \hat{j} + \hat{k}) \cdot (-8\hat{i} - 11\hat{j} + 5\hat{k})$$
$$= -16 + 11 + 5 = 0.$$

Hence, $\vec{A}, \vec{B}, \vec{C}$ are coplanar.

Now,
$$|\vec{A}|^2 = 2^2 + (-1)^2 + 1^2 = 6$$
$$|\vec{B}|^2 = 1^2 + (-3)^2 + (-5)^2 = 35$$
$$|\vec{C}|^2 = 3^2 + (-4)^2 + (-4)^2 = 41$$

$\therefore$
$$|\vec{A}|^2 + |\vec{B}|^2 = |\vec{C}|^2.$$

Hence, $\vec{A}, \vec{B}, \vec{C}$ form the sides of a right-angled triangle.

Example 2: If $\vec{a} = 2\hat{i} - 10\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{c} = 2\hat{i} + \hat{j} + 3\hat{k}$, find the vector $\vec{a} \times (\vec{b} \times \vec{c})$ and interpret the result geometrically.

Solution: We know that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Here
$$\vec{a} \cdot \vec{c} = (2\hat{i} - 10\hat{j} + 2\hat{k}) \cdot (2\hat{i} + \hat{j} + 3\hat{k})$$
$$= 4 - 10 + 6 = 0$$
$$\vec{a} \cdot \vec{b} = (2\hat{i} - 10\hat{j} + 2\hat{k}) \cdot (3\hat{i} + \hat{j} + 2\hat{k})$$
$$= 6 - 10 + 4 = 0$$

$\therefore$ $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{0}$ and hence, $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$.

Example 3: Solve $t\vec{r} + \vec{r} \times \vec{a} = \vec{b}$, where t is a non-zero number and $\vec{a}, \vec{b}$ are two given vectors.

Solution: Here $t\vec{r} + \vec{r} \times \vec{a} = \vec{b}$ ...(1)

Taking dot product with $\vec{a}$,

$t(\vec{a} \cdot \vec{r}) = \vec{a} \cdot \vec{b}$ $[\because \vec{a} \cdot (\vec{r} \times \vec{a}) = 0]$ ...(2)

From (1), taking cross-product with $\vec{a}$,
$$t(\vec{a} \times \vec{r}) + \vec{a} \times (\vec{r} \times \vec{a}) = \vec{a} \times \vec{b}$$

or $\quad t(t\vec{r}-\vec{b})+(\vec{a}\cdot\vec{a})\,\vec{r}-(\vec{a}\cdot\vec{r})\,\vec{a} \;=\; \vec{a}\times\vec{b}$ [by (1)]

or $\quad t(t\vec{r}-\vec{b})+|\vec{a}|^2\,\vec{r}-\dfrac{(\vec{a}\cdot\vec{b})}{t}\vec{a} \;=\; \vec{a}\times\vec{b}$ [by (2)]

or $$(t^2+|\vec{a}|^2)\,\vec{r} \;=\; t\vec{b}+\frac{(\vec{a}\cdot\vec{b})}{t}\vec{a}+\vec{a}\times\vec{b}$$

$\therefore$ $$\vec{r} \;=\; \frac{1}{(t^2+|\vec{a}|^2)}\left\{t\vec{b}+\frac{(\vec{a}\cdot\vec{b})}{t}\vec{a}+\vec{a}\times\vec{b}\right\}$$

This is the required solution.

Example 4: Show that $[\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}] \;=\; 2[\vec{a}\;\vec{b}\;\vec{c}]$, where $\vec{a}\;\vec{b}\;\vec{c}$ are any three vectors.

Solution: **L.H.S.**
$$\begin{aligned}
&= [\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}]\\
&= (\vec{a}+\vec{b})\cdot\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}\\
&= (\vec{a}+\vec{b})\cdot\{\vec{b}\times\vec{c}-\vec{a}\times\vec{b}+\vec{c}\times\vec{a}\}\\
&= \vec{a}\cdot(\vec{b}\times\vec{c})-\vec{a}\cdot(\vec{a}\times\vec{b})+\vec{a}\cdot(\vec{c}\times\vec{a})\\
&\quad +\vec{b}\cdot(\vec{b}\times\vec{c})-\vec{b}\cdot(\vec{a}\times\vec{b})+\vec{b}\cdot(\vec{c}\times\vec{a})\\
&= [\vec{a}\;\vec{b}\;\vec{c}]-0+0+0-0+[\vec{b}\;\vec{c}\;a]\\
&= [\vec{a}\;\vec{b}\;\vec{c}]+[\vec{b}\;\vec{c}\;a] \;=\; 2[\vec{a}\;\vec{b}\;\vec{c}]\\
&= \text{R.H.S.}
\end{aligned}$$

Note: If $\vec{a},\vec{b},\vec{c}$ are coplanar, then $\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}$ are also so.

Example 5: A vector $\vec{r}$ satisfies the equations $\vec{r}\times\vec{b} \;=\; \vec{c}\times\vec{b}$ and $\vec{r}\cdot\vec{a} \;=0$. Show that
$$\vec{r} \;=\; \vec{c}-\frac{(\vec{a}\cdot\vec{c})\,\vec{b}}{\vec{a}\cdot\vec{b}}$$

Solution: Here $\vec{r}\times\vec{b} \;=\; \vec{c}\times\vec{b}$

Considering the vector product with $\vec{a}$, we get
$$\vec{a}\times(\vec{r}\times\vec{b}) \;=\; \vec{a}\times(\vec{c}\times\vec{b})$$

or $$(\vec{a}\cdot\vec{b})\,\vec{r}-(\vec{a}\cdot\vec{r})\,\vec{b} \;=\; (\vec{a}\cdot\vec{b})\,\vec{c}-(\vec{a}\cdot\vec{c})\,\vec{b}$$

or $$(\vec{a}\cdot\vec{b})\,\vec{r} \;=\; (\vec{a}\cdot\vec{b})\,\vec{c}-(\vec{a}\cdot\vec{c})\,\vec{b} \qquad [\because \vec{r}\cdot\vec{a}=0]$$

$\therefore$ $$\vec{r} \;=\; \vec{c}-\frac{(\vec{a}\cdot\vec{c})\,\vec{b}}{\vec{a}\cdot\vec{b}}$$

Example 6: Prove that the four points with position vectors $3\hat{i}-2\hat{j}+4\hat{k}$, $6\hat{i}+3\hat{j}+\hat{k}$, $5\hat{i}+7\hat{j}+3\hat{k}$ and $2\hat{i}+2\hat{j}+6\hat{k}$ are coplanar.

Solution: Let the position vectors of the points A, B, C, D are respectively $3\hat{i}-2\hat{j}+4\hat{k}$, $6\hat{i}+3\hat{j}+\hat{k}$, $5\hat{i}+7\hat{j}+3\hat{k}$, $2\hat{i}+2\hat{j}+6\hat{k}$.

$\therefore \quad \overrightarrow{AB}$ = p.v. of B − p.v. of A = $6\hat{i}+3\hat{j}+\hat{k}-(3\hat{i}-2\hat{j}+4\hat{k}) = 3\hat{i}+5\hat{j}-3\hat{k}$

$\overrightarrow{AC}$ = p.v. of C − p.v. of A = $5\hat{i}+7\hat{j}+3\hat{k}-(3\hat{i}-2\hat{j}+4\hat{k}) = 2\hat{i}+9\hat{j}-\hat{k}$

$\overrightarrow{AD}$ = p.v. of D − p.v. of A = $2\hat{i}+2\hat{j}+6\hat{k}-(3\hat{i}-2\hat{j}+4\hat{k}) = -\hat{i}+4\hat{j}+2\hat{k}$

$$\therefore \quad \overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD}) = \begin{vmatrix} 3 & 5 & -3 \\ 2 & 9 & -1 \\ -1 & 4 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 5 & -3 \\ -1 & 4 & 2 \\ -1 & 4 & 2 \end{vmatrix} \quad (R_2 - R_1 \to R_2^1) = 0.$$

Therefore, $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar, *i.e.*, the four given points A, B, C, D are coplanar.

Example 7: Find the constant m such that the vectors $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-3\hat{k}$, $\vec{c} = 3\hat{i}+m\hat{j}+5\hat{k}$ are coplanar. *(W.B.U.T. 2004)*

Solution: If $\vec{a},\vec{b},\vec{c}$ are coplanar, then we have

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0, \text{ i.e., } \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & m & 5 \end{vmatrix} = 0,$$

or $\quad 2(10+3m)+(5+9)+(m-6) = 0,$

or $\quad 7m + 28 = 0. \quad \therefore m = -4.$

Example 8: If four points with position vectors $\vec{a},\vec{b},\vec{c},\vec{d}$ are coplanar, prove that

$$[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{a}\,\vec{b}\,\vec{d}]+[\vec{a}\,\vec{d}\,\vec{c}]+[\vec{d}\,\vec{b}\,\vec{c}].$$

Solution: Let $\vec{a},\vec{b},\vec{c},\vec{d}$ be the position vectors of the points A, B, C, D respectively.

$$\therefore \quad \overrightarrow{AD} = \vec{d}-\vec{a},\ \overrightarrow{BD} = \vec{d}-\vec{b},\ \overrightarrow{CD} = \vec{d}-\vec{c}.$$

If A, B, C, D are coplanar, *i.e.*, $\overrightarrow{AD}, \overrightarrow{BD}, \overrightarrow{CD}$ are coplanar, then $\overrightarrow{AD}\cdot(\overrightarrow{BD}\times\overrightarrow{CD}) = 0$,

or $\quad (\vec{d}-\vec{a})\cdot\{(\vec{d}-\vec{b})\times(\vec{d}-\vec{c})\} = 0$

or $\quad (\vec{d}-\vec{a})\cdot(\vec{d}\times\vec{d}-\vec{d}\times\vec{c}-\vec{b}\times\vec{d}+\vec{b}\times\vec{c}) = 0$

or $\quad (\vec{d}-\vec{a})\cdot(-\vec{d}\times\vec{c}-\vec{b}\times\vec{d}+\vec{b}\times\vec{c}) = 0$

or $\quad -\vec{d}\cdot(\vec{d}\times\vec{c})-\vec{d}\cdot(\vec{b}\times\vec{d})+\vec{d}\cdot(\vec{b}\times\vec{c})$

$\quad +\vec{a}\cdot(\vec{d}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{d})-\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$

or $\quad -0-0+[\vec{d}\,\vec{b}\,\vec{c}]+[\vec{a}\,\vec{d}\,\vec{c}]+[\vec{a}\,\vec{b}\,\vec{d}]-[\vec{a}\,\vec{b}\,\vec{c}] = 0$

$$\therefore \quad [\vec{a}\,\vec{b}\,\vec{c}] = [\vec{a}\,\vec{b}\,\vec{d}]+[\vec{a}\,\vec{d}\,\vec{c}]+[\vec{d}\,\vec{b}\,\vec{c}]$$

Example 9: Show that

(i) $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$

(ii) $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k}) = 2\vec{a}$

Solution: (i) Now,

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c} \quad ...(1)$$

$$\vec{b}\times(\vec{c}\times\vec{a}) = (\vec{b}\cdot\vec{a})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a} \quad ...(2)$$

$$\vec{c}\times(\vec{a}\times\vec{b}) = (\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b} \quad ...(3)$$

Adding (1), (2) and (3), we have

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$$

(ii) Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

Now,
$$\hat{i}\times(a\times\hat{i}) = (\hat{i}\cdot\hat{i})\vec{a}-(\hat{i}\cdot\vec{a})\hat{i}$$
$$= \vec{a}-a_1\hat{i}$$
$$\hat{j}\times(\vec{a}\times\hat{j}) = (\hat{j}\cdot\hat{j})\vec{a}-(\hat{j}\cdot\vec{a})\hat{j}$$
$$= \vec{a}-a_2\hat{j}$$
$$k\times(\vec{a}\times k) = (\hat{k}\cdot\hat{k})\vec{a}-(\hat{k}\cdot\vec{a})\hat{k}$$
$$= \vec{a}-a_3\hat{k}.$$

Adding, we get $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})$

$$= 3\vec{a}-(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}) = 3\vec{a}-\vec{a} = 2\vec{a}$$

Example 10: Show that for any three vectors $\vec{a},\vec{b}$ and $\vec{c}$, *(W.B.U.T.2006)*

$$[\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$$

Solution: L.H.S.
$$= [\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}]$$
$$= (\vec{a}\times\vec{b})\cdot\{\vec{\alpha}\times(\vec{c}\times\vec{a})\} \quad [\text{where } \vec{\alpha}=\vec{b}\times\vec{c}]$$
$$= (\vec{a}\times\vec{b})\cdot\{(\vec{\alpha}\cdot\vec{a})\vec{c}-(\vec{\alpha}\cdot\vec{c})\vec{a}\}$$
$$= (\vec{a}\times\vec{b})\cdot\{[\vec{b}\,\vec{c}\,\vec{a}]\vec{c}-0\} \quad [\because \vec{\alpha}=\vec{b}\times\vec{c}]$$
$$= [\vec{b}\,\vec{c}\,\vec{a}](\vec{a}\times\vec{b})\cdot\vec{c}$$
$$= [\vec{b}\,\vec{c}\,\vec{a}][\vec{a}\,\vec{b}\,\vec{c}] = [\vec{a}\,\vec{b}\,\vec{c}]^2 = \text{R.H.S.}$$

Note: If $\vec{a},\vec{b},\vec{c}$ are coplanar, then $\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}$ are also coplanar.

Example 11: Show that $[\vec{a}\times\vec{b}\ \ \vec{c}\times\vec{d}\ \ \vec{e}\times\vec{f}]$

$$= [\vec{a}\,\vec{b}\,\vec{e}][\vec{c}\,\vec{d}\,\vec{f}]-[\vec{a}\,\vec{b}\,\vec{f}][\vec{c}\,\vec{d}\,\vec{e}].$$

Solution: Now,

$$\begin{aligned}(\vec{c}\times\vec{d})\times(\vec{e}\times\vec{f}) &= \vec{\alpha}\times(\vec{e}\times\vec{f}) && [\text{where } \vec{\alpha}=\vec{c}\times\vec{d}\,]\\ &= (\vec{\alpha}\cdot\vec{f})\vec{e}-(\vec{\alpha}\cdot\vec{e})\vec{f}\\ &= [\vec{c}\,\vec{d}\,\vec{f}]\vec{e}-[\vec{c}\,\vec{d}\,\vec{e}]\vec{f} && [\because \alpha=\vec{c}\times\vec{d}\,] \quad \ldots(1)\end{aligned}$$

$$\begin{aligned}\therefore \text{ L.H.S.} &= [\vec{a}\times\vec{b}\ \ \vec{c}\times\vec{d}\ \ \vec{e}\times\vec{f}]\\ &= (\vec{a}\times\vec{b})\cdot\{(\vec{c}\times\vec{d})\times(\vec{e}\times\vec{f})\}\\ &= (\vec{a}\times\vec{b})\cdot\{[\vec{c}\,\vec{d}\,\vec{f}]\vec{e}-[\vec{c}\,\vec{d}\,\vec{e}]\vec{f}\} && [\text{using (1)}]\\ &= [\vec{a}\,\vec{b}\,\vec{e}][\vec{c}\,\vec{d}\,\vec{f}]-[\vec{a}\,\vec{b}\,\vec{f}][\vec{c}\,\vec{d}\,\vec{e}]. = \text{R.H.S.}\end{aligned}$$

Example 12: Prove that

$$[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} & \vec{c}\cdot\vec{c}\end{vmatrix}$$

Solution: Let $a = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}, \vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}.$

$$\text{L.H.S.} = [\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}^2 = \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$= \begin{vmatrix}a_1^2+a_2^2+a_3^2 & a_1b_1+a_2b_2+a_3b_3 & a_1c_1+a_2c_2+a_3c_3\\ b_1a_1+b_2a_2+b_3a_3 & b_1^2+b_2^2+b_3^2 & b_1c_1+b_2c_2+b_3c_3\\ c_1a_1+c_2a_2+c_3a_3 & c_1b_1+c_2b_2+c_3b_3 & c_1^2+c_2^2+c_3^2\end{vmatrix}$$

$$= \begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c}\end{vmatrix} = \begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} & \vec{c}\cdot\vec{c}\end{vmatrix} = \text{R.H.S.}$$

Example 13: If the vectors $\vec{a}$ and $\vec{c}$ are perpendicular to each other, then prove that the vectors $\vec{a}\times(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b})\times\vec{c}$ are also perpendicular to each other.

Solution: If $\vec{a}$ and $\vec{c}$ are perpendicular to each other, then $\vec{a}\cdot\vec{c} = 0$ ...(1)

Now, $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c} = -(\vec{a}\cdot\vec{b})\vec{c}$ [by (1)]

$$(\vec{a}\times\vec{b})\times\vec{c} = -\vec{c}\times(\vec{a}\times\vec{b}) = \vec{c}\times(\vec{b}\times\vec{a})$$

$$= (\vec{c}\cdot\vec{a})\vec{b}-(\vec{c}\cdot\vec{b})\,\vec{a} = -(\vec{c}\cdot\vec{b})\,\vec{a} \qquad \text{[by (1)]}$$

$\therefore$ $[\vec{a}\times(\vec{b}\times\vec{c})]\cdot[(\vec{a}\times\vec{b})\times\vec{c}] = (\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{b})\,\vec{c}\cdot\vec{a} = 0$ [by (1)]

Hence the result.

Example 14: If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors such that $\vec{a} = \vec{c}+\vec{d}$, where $\vec{c}$ is a vector parallel to $\vec{b}$ and $\vec{d}$ is a vector perpendicular to $\vec{b}$, then find $\vec{c}$ and $\vec{d}$ in terms of $\vec{a}$ and $\vec{b}$.

Solution: Here $\vec{c}$ is parallel to $\vec{b}$, therefore, $\vec{c}\times\vec{b} = \vec{0}$ ...(1)

Also, $\vec{d}$ is perpendicular to $\vec{b}$, therefore, $\vec{d}\cdot\vec{b} = 0$ ...(2)

$$\vec{a} = \vec{c}+\vec{d} \qquad \text{...(3)}$$

Considering the vector product with $\vec{b}$, we get

$$\vec{a}\times\vec{b} = \vec{c}\times\vec{b}+\vec{d}\times\vec{b} = \vec{d}\times\vec{b} \qquad \text{[by (1)]}$$

Again considering vector product with $\vec{b}$, we get

$$\vec{b}\times(\vec{a}\times\vec{b}) = \vec{b}\times(\vec{d}\times\vec{b})$$

or $$(\vec{b}\cdot\vec{b})\,\vec{a}-(\vec{b}\cdot\vec{a})\,\vec{b} = (\vec{b}\cdot\vec{b})\,\vec{d}-(\vec{b}\cdot\vec{d})\,\vec{b}$$

or $$|\vec{b}|^2\,\vec{a}-(\vec{a}\cdot\vec{b})\,\vec{b} = |\vec{b}|^2\,\vec{d} \qquad \text{[by (2)]}$$

$\therefore$ $$\vec{d} = \vec{a}-\frac{(\vec{a}\cdot\vec{b})}{|\vec{b}|^2}\,\vec{b}$$

From (3), $$\vec{c} = \vec{a}-\vec{d} = \frac{(\vec{a}\cdot\vec{b})}{|\vec{b}|^2}\,\vec{b}.$$

Example 15: Find the vector $\vec{c}$ and the scalar k which satisfy the conditions:

$$\vec{a}\times\vec{c} = \vec{b}+k\vec{a} \text{ and } \vec{a}\cdot\vec{c} = 2$$

where $\vec{a} = 3\hat{i}-2\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+3\hat{j}-4\hat{k}$

Solution: Here $$\vec{a}\times\vec{c} = \vec{b}+k\vec{a}$$

$\therefore$ $$\vec{a}\cdot(\vec{a}\times\vec{c}) = \vec{a}\cdot\vec{b}+k\vec{a}\cdot\vec{a}$$

or $$0 = \vec{a}\cdot\vec{b}+k|\vec{a}|^2$$

$\therefore$ $$k = -\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} = -\frac{(3-6-4)}{14} = \frac{1}{2} \qquad \text{...(1)}$$

Now, $$\vec{a}\times\vec{c} = \vec{b}+k\vec{a}$$

$\therefore$ $$(\vec{a}\times\vec{c})\cdot\vec{c} = \vec{b}\cdot\vec{c}+k\vec{a}\cdot\vec{c}$$

or $$0 = \vec{b}\cdot\vec{c}+\frac{1}{2}\cdot 2$$

$\therefore$ $$\vec{b}\cdot\vec{c} = -1 \qquad \text{...(2)}$$

Again, $\quad \bar{b}\times(\bar{a}\times\bar{c}) = \bar{b}\times\bar{b}+k(\bar{b}\times\bar{a})$

or $\quad (\bar{b}\cdot\bar{c})\,\bar{a}-(\bar{b}\cdot\bar{a})\,\bar{c} = k(\bar{b}\times\bar{a})$

$\therefore$
$$(\bar{b}\cdot\bar{a})\,\bar{c} = (\bar{b}\cdot\bar{c})\,\bar{a}-k(\bar{b}\times\bar{a})$$
$$= -\bar{a}-\frac{1}{2}(\bar{b}\times\bar{a}) \qquad \text{[by (1) and (2)]}$$

or
$$(3-6-4)\,\bar{c} = -\bar{a}-\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 3 & -4\\ 3 & -2 & 1\end{vmatrix}$$

$\therefore$
$$\bar{c} = \frac{1}{7}\bar{a}+\frac{1}{14}\{(3-8)\,\hat{i}+(-12-1)\,\hat{j}+(-2-9)\,\hat{k}\}$$
$$= \frac{1}{7}(3\hat{i}-2\hat{j}+\hat{k})+\frac{1}{14}(-5\hat{i}-13\hat{j}-11\hat{k})$$
$$= \frac{1}{14}(\hat{i}-17\hat{j}-9\hat{k})$$

Example 16: Find a vector $\bar{\alpha}$ satisfying the condition $\bar{\alpha}\cdot\bar{a} = 3$ and $\bar{\alpha}\times\bar{b} = \bar{c}$, where $\bar{a} = \hat{i}+\hat{j}-\hat{k}$, $\bar{b} = 2\hat{i}+\hat{j}+\hat{k}$, $\bar{c} = \hat{i}+2\hat{j}+\hat{k}$

Solution: Here $\quad \bar{\alpha}\times\bar{b} = \bar{c}$

$\therefore$
$$\bar{a}\times(\bar{\alpha}\times\bar{b}) = \bar{a}\times\bar{c}$$

or
$$(\bar{a}\cdot\bar{b})\,\bar{\alpha}-(\bar{a}\cdot\bar{\alpha})\,\bar{b} = \bar{a}\times\bar{c}$$

$\therefore$
$$\bar{\alpha} = \frac{3\bar{b}+\bar{a}\times\bar{c}}{\bar{a}\cdot\bar{b}} \qquad [\because \bar{\alpha}\cdot\bar{a}=3]\ldots(1)$$

Now,
$$\bar{a}\times\bar{c} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & -1\\ 1 & 2 & 1\end{vmatrix} = 3\hat{i}-2\hat{j}+\hat{k}$$

and
$$\bar{a}\cdot\bar{b} = (\hat{i}+\hat{j}-\hat{k})\cdot(2\hat{i}+\hat{j}+\hat{k}) = 2+1-1 = 2.$$

Therefore, from (1),
$$\bar{\alpha} = \frac{3}{2}\bar{b}+\frac{1}{2}\bar{a}\times\bar{c}$$
$$= \frac{3}{2}(2\hat{i}+\hat{j}+\hat{k})+\frac{1}{2}(3\hat{i}-2\hat{j}+\hat{k})$$
$$= \frac{9}{2}\hat{i}+\frac{1}{2}\hat{j}+2\hat{k}$$

Example 17: If $\vec{a} \neq \vec{0}$ and $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}, \vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, then show that $\vec{b} = \vec{c}$.

Solution: Here $\qquad \vec{a}\times\vec{b} = \vec{a}\times\vec{c}$

$\therefore \qquad \vec{a}\times(\vec{a}\times\vec{b}) = \vec{a}\times(\vec{a}\times\vec{c})$

or $\qquad (\vec{a}\cdot\vec{b})\,\vec{a}-(\vec{a}\cdot\vec{a})\,\vec{b} = (\vec{a}\cdot\vec{c})\,\vec{a}-(\vec{a}\cdot\vec{a})\,\vec{c}$

or $\qquad |\vec{a}|^2\,\vec{b} = |\vec{a}|^2\,\vec{c} \qquad (\because \vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c})$

or $\qquad |\vec{a}|^2\,(\vec{b}-\vec{c}) = \vec{0}$

$\therefore \qquad \vec{b}-\vec{c} = 0 \qquad (\because \vec{a}\neq\vec{0}, \text{i.e.}, |\vec{a}|^2\neq 0)$

or $\qquad \vec{b} = \vec{c}$.

Example 18: Find the equation of the plane through the points $A\,(-1, 1, 2)$, $B\,(1, -2, 1)$ and $C\,(2, 2, 5)$.

Solution: The position vectors of A, B, C are respectively $-\hat{i}+\hat{j}+2\hat{k}, \hat{i}-2\hat{j}+\hat{k}, 2\hat{i}+2\hat{j}+5\hat{k}$ w.r.t. origin.

Let P be any point on the required plane whose position vector is $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ w.r.t. origin.

Here

$$\overrightarrow{AB} = \text{p.v. of } B - \text{p.v. of } A = \hat{i}-2\hat{j}+\hat{k}-(-\hat{i}+\hat{j}+2\hat{k}) = 2\hat{i}-3\hat{j}-\hat{k}$$

$$\overrightarrow{AC} = \text{p.v. of } C - \text{p.v. of } A = 2\hat{i}+2\hat{j}+5\hat{k}-(-\hat{i}+\hat{j}+2\hat{k}) = 3\hat{i}+\hat{j}+3\hat{k}$$

$$\overrightarrow{AP} = \text{p.v. of } P - \text{p.v. of } A = x\hat{i}+y\hat{j}+z\hat{k}-(-\hat{i}+\hat{j}+2\hat{k})$$

$$= (x+1)\,\hat{i}+(y-1)\,\hat{j}+(z-2)\,\hat{k}$$

Eq of plane passing through 3 points.

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$

Since $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AP}$ are coplanar, therefore we have

$$\overrightarrow{AP}\cdot(\overrightarrow{AB}\times\overrightarrow{AC}) = \begin{vmatrix} x+1 & y-1 & z-2 \\ 2 & -3 & -1 \\ 3 & 1 & 3 \end{vmatrix} = 0$$

or $\qquad (x+1)(-9+1)+(y-1)(-3-6)+(z-2)(2+9) = 0$

or $\qquad -8x-9y+11z-8+9-22 = 0$

or $\qquad 8x+9y-11z+21 = 0$

It is the required equation of the plane.

Example 19: If the vector equations of two lines are $\vec{r} = \vec{r}_1+t\vec{\alpha}, \vec{r} = \vec{r}_2+s\vec{\beta}$, where t, s are scalars and $\vec{r}_1, \vec{\alpha}, \vec{r}_2, \vec{\beta}$ are vectors with co-ordinates (1, 4, 5), (2, 1, 2), (2, 8, 11) and (–1, 3, 4) respectively, prove that the lines are coplanar.

Solution: Here $\vec{r}_1 = \hat{i}+4\hat{j}+5\hat{k}, \vec{\alpha} = 2\hat{i}+\hat{j}+2\hat{k}, \ \vec{r}_2 = 2\hat{i}+8\hat{j}+11\hat{k}, \ \vec{\beta} = -\hat{i}+3\hat{j}+4\hat{k}$.

The given lines will be coplanar if

$$[\bar{r}_1\ \bar{\alpha}\ \bar{\beta}] = [\bar{r}_2\ \bar{\alpha}\ \bar{\beta}].$$

Here
$$[\bar{r}_1\ \bar{\alpha}\ \bar{\beta}] = \begin{vmatrix} 1 & 4 & 5 \\ 2 & 1 & 2 \\ -1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 1 & 1 \\ -1 & 3 & 1 \end{vmatrix} \quad (C_3 - C_2 \to C_3)$$

$$= \begin{vmatrix} 1 & 4 & 1 \\ 1 & -3 & 0 \\ -2 & -1 & 0 \end{vmatrix} = -1 - 6 = -7$$

$$[\bar{r}_2\ \bar{\alpha}\ \bar{\beta}] = \begin{vmatrix} 2 & 8 & 11 \\ 2 & 1 & 2 \\ -1 & 3 & 4 \end{vmatrix} \begin{aligned} &= 2(4-6) + 8(-2-8) + 11(6+1) \\ &= -4 - 80 + 77 = -7. \end{aligned}$$

$\therefore$ $[\bar{r}_1\ \bar{\alpha}\ \bar{\beta}] = [\bar{r}_2\ \bar{\alpha}\ \bar{\beta}]$ and hence the given lines are coplanar.

Example 20: Show that the lines $\bar{r}\times\bar{a} = \bar{b}\times\bar{a}$ and $\bar{r}\times\bar{b} = \bar{a}\times\bar{b}$ intersect and find their point of intersection.

Solution: The equations of the given lines are

$$\bar{r}\times\bar{a} = \bar{b}\times\bar{a}, \text{ or } (\bar{r}-\bar{b})\times\bar{a} = \bar{0} \qquad ...(1)$$

$$\bar{r}\times\bar{b} = \bar{a}\times\bar{b}, \text{ or } (\bar{r}-\bar{a})\times\bar{b} = \bar{0} \qquad ...(2)$$

The line (1) passes through the point whose position vector is $\bar{b}$ and it is parallel to $\bar{a}$. Hence the equation of this line can be written as

$$\bar{r} = \bar{b} + t\bar{a}, \text{ where } t \text{ is a scalar} \qquad ...(3)$$

Similarly, the equation (2) can be written as

$$\bar{r} = \bar{a} + s\bar{b}, \text{ where } s \text{ is a scalar} \qquad ...(4)$$

If these lines intersect, they lie in the plane which is parallel to $\bar{b}-\bar{a}, \bar{a}$ and $\bar{b}$ and hence these vectors must be coplanar.

Since $(\bar{b}-\bar{a})\cdot(\bar{a}\times\bar{b}) = \bar{b}\cdot(\bar{a}\times\bar{b}) - \bar{a}\cdot(\bar{a}\times\bar{b}) = 0 - 0 = 0,$

therefore, $(\bar{b}-\bar{a}), \bar{a}$ and $\bar{b}$ are coplanar. Hence the given lines intersect.

For the point of intersection we should have same value of $\bar{r}$ for the two lines and obviously it happens for $t = s = 1$.

Therefore, the position vector of the required point of intersection is $\bar{a}+\bar{b}$.

Example 21: Find the shortest distance between two skew lines $\bar{r} = \bar{r}_1 + t\bar{\alpha}, \bar{r} = \bar{r}_2 + s\bar{\beta}$, where t, s are scalars and $\bar{r}_1, \bar{\alpha}, \bar{r}_2, \bar{\beta}$ are $\hat{i}-2\hat{j}+3\hat{k},\ 2\hat{i}+\hat{j}+\hat{k},\ -2\hat{i}+2\hat{j}-\hat{k},\ -3\hat{i}+\hat{j}+2\hat{k}$ respectively.

Solution: Here shortest distance $= \dfrac{|(\bar{r}_2-\bar{r}_1)\cdot(\bar{\alpha}\times\bar{\beta})|}{|\bar{\alpha}\times\bar{\beta}|}$

Now,
$$\bar{r}_2-\bar{r}_1 = -2\hat{i}+2\hat{j}-\hat{k}-(\hat{i}-2\hat{j}+3\hat{k})$$
$$= -3\hat{i}+4\hat{j}-4\hat{k}$$

$$\bar{\alpha}\times\bar{\beta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ -3 & 1 & 2 \end{vmatrix} = \hat{i}-7\hat{j}+5\hat{k}$$

Therefore, the required shortest distance

$$= \frac{|(-3\hat{i}+4\hat{j}-4\hat{k})\cdot(\hat{i}-7\hat{j}+5\hat{k})|}{\sqrt{1^2+(-7)^2+5^2}} = \frac{|-3-28-20|}{5\sqrt{3}} = \frac{51}{5\sqrt{3}} = \frac{17\sqrt{3}}{5}.$$

Example 22: Find the equation of the plane passing through the point (2, 3, –1) and perpendicular to the vector (3, – 4, 7). Find the length of the perpendicular from the origin to the plane.

Solution: Let P be any point on the required plane whose position vector is $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$ w.r.t. origin. By question this plane passes through a point whose position vector is $\bar{a} = 2\hat{i}+3\hat{j}-\hat{k}$ w.r.t. origin and perpendicular to $\bar{m} = 3\hat{i}-4\hat{j}+7\hat{k}$. Therefore, the vector equation of the required plane is

$$(\bar{r}-\bar{a})\cdot\bar{m} = \bar{0}$$

or $$\{x\hat{i}+y\hat{j}+z\hat{k}-(2\hat{i}+3\hat{j}-\hat{k})\}\cdot(3\hat{i}-4\hat{j}+7\hat{k}) = 0$$

or $$\{(x-2)\hat{i}+(y-3)\hat{j}+(z+1)\hat{k}\}\cdot(3\hat{i}-4\hat{j}+7\hat{k}) = 0$$

or $$3(x-2)-4(y-3)+7(z+1) = 0$$

or $$3x-4y+7z+13 = 0.$$

Length of the perpendicular from the origin to the plane

$$= \frac{|k-\bar{0}\cdot\bar{m}|}{|\bar{m}|}$$

[where $k = \bar{a}\cdot\bar{m} = (2\hat{i}+3\hat{j}-\hat{k})\cdot(3\hat{i}-4\hat{j}+7\hat{k}) = 6-12-7 = -13$]

$$= \frac{|-13|}{\sqrt{3^2+(-4)^2+7^2}} = \frac{13}{\sqrt{74}}$$

Example 23: Find the equation of the plane passing through the point (3, –2, –1) and parallel to the vectors $\hat{i}-2\hat{j}+4\hat{k}$ and $3\hat{i}+2\hat{j}-5\hat{k}$.

Solution: Let P be any point on the required plane whose position vector is $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$ w.r.t. origin. By question this plane passes through a point A with position vector $3\hat{i}-2\hat{j}-\hat{k}$ w.r.t. origin and parallel to the vectors $\hat{i}-2\hat{j}+4\hat{k}$ and $3\hat{i}+2\hat{j}-5\hat{k}$. So, $(\hat{i}-2\hat{j}+4\hat{k})\times(3\hat{i}+2\hat{j}-5\hat{k})$ is perpendicular to this plane and hence perpendicular to $\overrightarrow{AP}$.

$\therefore$

$$\overrightarrow{AP}\cdot\{(\hat{i}-2\hat{j}+4\hat{k})\times(3\hat{i}+2\hat{j}-5\hat{k})\}=0 \quad ...(1)$$

Here

$$\begin{aligned}\overrightarrow{AP} &= \text{p.v. of } P - \text{p.v. of } A\\ &= x\hat{i}+y\hat{j}+z\hat{k}-(3\hat{i}-2\hat{j}-\hat{k})\\ &= (x-3)\hat{i}+(y+2)\hat{j}+(z+1)\hat{k} \quad ...(2)\end{aligned}$$

From (1) and (2), we have

$$\begin{vmatrix} x-3 & y+2 & z+1\\ 1 & -2 & 4\\ 3 & 2 & -5\end{vmatrix}=0$$

or

$$(x-3)(10-8)+(y+2)(12+5)+(z+1)(2+6)=0$$

or

$$2x+17y+8z+36=0$$

This is the equation of the required plane.

Example 24: Find the position vector of the point of intersection of the straight line joining the points $\hat{i}+\hat{j}+\hat{k}$ and $3\hat{i}+2\hat{j}-\hat{k}$ with the plane $\vec{r}\cdot(\hat{k}-\hat{j})=5$.

Solution: The given straight line passes through the point $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=3\hat{i}+2\hat{j}-\hat{k}$, so its equation is

$$\vec{r}=\vec{a}+t(\vec{b}-\vec{a})=\hat{i}+\hat{j}+\hat{k}+t\{3\hat{i}+2\hat{j}-\hat{k}-(\hat{i}+\hat{j}+\hat{k})\}$$

or

$$\vec{r}=\hat{i}+\hat{j}+\hat{k}+t(2\hat{i}+\hat{j}-2\hat{k}) \quad ...(1)$$

where $\vec{r}$ is the current vector and t is a scalar. At the point of intersection of this line and the plane $\vec{r}\cdot(\hat{k}-\hat{j})=5$, $\{\hat{i}+\hat{j}+\hat{k}+t(2\hat{i}+\hat{j}-2\hat{k}\}\cdot(\hat{k}-\hat{j})=5$

or $\{(2t+1)\hat{i}+(t+1)\hat{j}+(1-2t)\hat{k}\}\cdot(0\hat{i}-\hat{j}+\hat{k})=5$

or $(2t+1)\cdot 0+(t+1)\cdot(-1)+(1-2t)\cdot 1=5$

or $-3t=5$, *i.e.*, $t=-\dfrac{5}{3}$

Putting $t=-\dfrac{5}{3}$ in (1), we obtain the position vector of the required point of intersection which is

$$\hat{i}+\hat{j}+\hat{k}-\frac{5}{3}(2\hat{i}+\hat{j}-2\hat{k})=-\frac{7}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{13}{3}\hat{k}$$

Example 25: Find the position vector of the point of intersection of the line joining the points (1, –2, 1) and (0, –2, 3) with the plane through the points (0, 0, 0), (2, 4, 1) and (4, 0, 2). Use vector method.

Solution: The vector equation of line joining two given points is

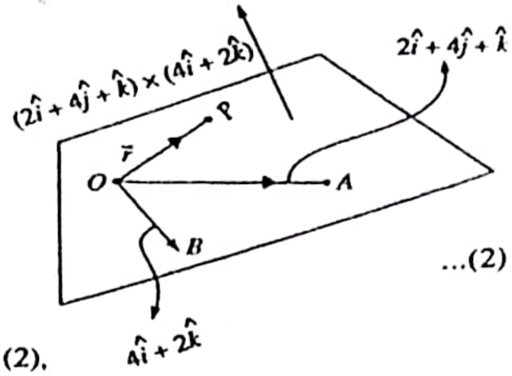
$$\vec{r}=\hat{i}-2\hat{j}+\hat{k}+t\{-2\hat{j}+3\hat{k}-(\hat{i}-2\hat{j}+\hat{k})\}$$

or $\quad \bar{r} = \hat{i} - 2\hat{j} + \hat{k} + t(-\hat{i} + 2\hat{k})$

or $\quad \bar{r} = (1-t)\hat{i} - 2\hat{j} + (1+2t)\hat{k} \quad \ldots(1)$

where $\bar{r}$ is the current vector and t is a scalar.

The vector equation of the plane through the three given points is



$$\bar{r}\cdot\{(2\hat{i}+4\hat{j}+\hat{k})\times(4\hat{i}+2\hat{k})\} = 0 \quad \ldots(2)$$

where $\bar{r}$ is the current vector.

At the point of intersection of the line (1) and the plane (2),

$$\{(1-t)\hat{i} - 2\hat{j} + (1+2t)\hat{k}\}\cdot\{(2\hat{i}+4\hat{j}+\hat{k})\times(4\hat{i}+2\hat{k})\} = 0$$

or
$$\begin{vmatrix} 1-t & -2 & 1+2t \\ 2 & 4 & 1 \\ 4 & 0 & 2 \end{vmatrix} = 0$$

or $\quad 8(1-t) - 16(1+2t) = 0$, or $-8-40t = 0$,

i.e., $\quad t = -\frac{1}{5}$.

Therefore, the position vector of the required point of intersection, obtained by putting

$$t = -\frac{1}{5} \text{ in (1), is}$$

$$\left(1+\frac{1}{5}\right)\hat{i} - 2\hat{j} + \left(1-\frac{2}{5}\right)\hat{k} = \frac{1}{5}(6\hat{i} - 10\hat{j} + 3\hat{k}).$$

Example 26: Find in terms of K, the shortest distance between the lines $\bar{r} = \bar{\alpha} + t\bar{\beta}$ and $\bar{r} = \bar{\gamma} + s\bar{\delta}$, t, s are scalars, where

$$\bar{\alpha} = \hat{i} + 2\hat{j} + 3\hat{k}$$
$$\bar{\beta} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$
$$\bar{\gamma} = K\hat{i} + 3\hat{j} + 4\hat{k}$$

and
$$\bar{\delta} = 3\hat{i} + 4\hat{j} + 5\hat{k}$$

For what value of K are the lines coplanar?

Solution: The shortest distance (S.D.) between the two lines $\bar{r} = \bar{\alpha} + t\bar{\beta}$, $\bar{r} = \bar{\gamma} + s\bar{\delta}$ is given by

$$p = \frac{|(\bar{\alpha}-\bar{\gamma})\cdot(\bar{\beta}\times\bar{\delta})|}{|\bar{\beta}\times\bar{\delta}|}$$

Now, $\quad \bar{\alpha} - \bar{\gamma} = \hat{i} + 2\hat{j} + 3\hat{k} - (K\hat{i} + 3\hat{j} + 4\hat{k})$

$$= (1-K)\hat{i} - \hat{j} - \hat{k}.$$

$$\bar{\beta}\times\bar{\delta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -\hat{i} + 2\hat{j} - \hat{k}$$

$\therefore$ $$|\bar{\beta}\times\bar{\delta}| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}$$

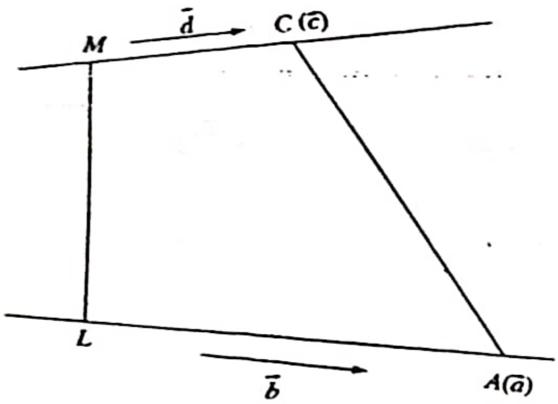
$\therefore$ $$p = \frac{|\{(1-K)\hat{i} - \hat{j} - \hat{k}\}\cdot(-\hat{i} + 2\hat{j} - \hat{k})|}{\sqrt{6}}$$

$$= \frac{|-1+K-2+1|}{\sqrt{6}} = \frac{|K-2|}{\sqrt{6}}$$

Therefore, the given two lines are coplanar if S.D. $= 0$, *i.e.*, $K = 2$.

Example 27: Find the shortest distance between the two lines through A with position vector $\bar{a} = 6\hat{i} + 2\hat{j} + 2\hat{k}$ and C with position vector $\bar{c} = -4\hat{i} - \hat{k}$ and parallel to the vectors $\bar{b} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\bar{d} = 3\hat{i} - 2\hat{j} - 2\hat{k}$ respectively. Find where the lines meet the common perpendicular (line of S.D.).

Solution:



Here $\overrightarrow{AC}$ = p.v. of C − p.v. of A

$$= \bar{c} - \bar{a}$$
$$= -4\hat{i} - \hat{k} - (6\hat{i} + 2\hat{j} + 2\hat{k})$$
$$= -10\hat{i} - 2\hat{j} - 3\hat{k}$$
$$\bar{b} = \hat{i} - 2\hat{j} + 2\hat{k}$$
$$\bar{d} = 3\hat{i} - 2\hat{j} - 2\hat{k}$$

$\therefore$ $$\bar{b}\times\bar{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = 8\hat{i} + 8\hat{j} + 4\hat{k}.$$

$$|\bar{b}\times\bar{d}| = \sqrt{8^2 + 8^2 + 4^2} = 12.$$

$\therefore$ $$\text{S.D.} = p = \frac{|\overrightarrow{AC}\cdot(\bar{b}\times\bar{d})|}{|\bar{b}\times\bar{d}|} = \frac{|(-10\hat{i} - 2\hat{j} - 3\hat{k})\cdot(8\hat{i} + 8\hat{j} + 4\hat{k})|}{12}$$

$$= \frac{|-80-16-12|}{12} = \frac{108}{12} = 9.$$

The plane ALM is given by $(\bar{r} - \bar{a})\cdot\{\bar{b}\times(\bar{b}\times\bar{d})\} = 0$

or $\{(x-6)\hat{i}+(y-2)\hat{j}+(z-2)\hat{k}\}\cdot\{(\hat{i}-2\hat{j}+2\hat{k})\times(8\hat{i}+8\hat{j}+4\hat{k})\}$

or $\begin{vmatrix} x-6 & y-2 & z-2 \\ 1 & -2 & 2 \\ 8 & 8 & 4 \end{vmatrix} = 0$, or $-24(x-6)+12(y-2)+24(z-2) = 0$

or $$2x-y-2z = 6. \qquad \ldots(1)$$

The line CM has the equation $\vec{r} = \vec{c}+t\vec{d}$

or $$x\hat{i}+y\hat{j}+z\hat{k} = (-4+3t)\hat{i}-2t\hat{j}-(1+2t)\hat{k}$$

$\therefore$ $$x = -4+3t,\ y = -2t,\ z = -(1+2t)$$

Putting these values of x, y, z in (1), we get $-8+6t+2t+2+4t = 6$, or $t = 1$.

Therefore, the line CM meets the plane ALM at the point M whose co-ordinates are $(-1, -2, -3)$.

The plane CML is given by $(\vec{r}-\vec{c})\cdot\{\vec{d}\times(\vec{b}\times\vec{d})\} = 0$

or $\{(x+4)\hat{i}+y\hat{j}+(z+1)\hat{k}\}\cdot\{(3\hat{i}-2\hat{j}-2\hat{k})\times(8\hat{i}+8\hat{j}+4\hat{k})\} = 0$

or $\begin{vmatrix} x+4 & y & z+1 \\ 3 & -2 & -2 \\ 8 & 8 & 4 \end{vmatrix} = 0$, or $8(x+4)-28y+40(z+1) = 0$

or $$2x-7y+10z+18 = 0 \qquad \ldots(2)$$

The line AL has the equation $\vec{r} = \vec{a}+s\vec{b}$, or $x\hat{i}+y\hat{j}+z\hat{k} = (6+s)\hat{i}+(2-2s)\hat{j}+(2+2s)\hat{k}$

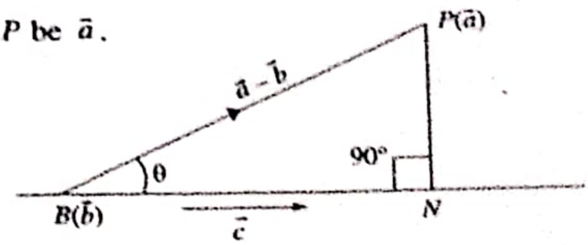
$\therefore x = 6+s,\ y = 2-2s,\ z = 2+2s$

Putting these values of x, y, z in (2), we get $12+2s-14+14s+20+20s+18 = 0$, or $s = -1$.

Therefore, the line AL meets the plane CML at the point L whose co-ordinates are $(5, 4, 0)$.

Example 28: If $\vec{a}$ be the position vector of a point P, find the distance of P from the straight line $\vec{r} = \vec{b}+t\vec{c}$ where the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ have the co-ordinates $(5, -6, 2)$, $(1, -1, 2)$ and $(0, -4, -3)$ respectively.

Solution: The equation of the given line BN be $\vec{r} = \vec{b}+t\vec{c}$, where t is a scalar, $\vec{b}$ is the position vector of the point B and $\vec{c}$ is a vector parallel to BN.



PN is perpendicular to BN. The position vector of P be $\vec{a}$.

Now $\overrightarrow{BP}$ = p.v. of P – p.v. of B = $\vec{a}-\vec{b}$.

$\therefore$ $BP^2 = \overrightarrow{BP}\cdot\overrightarrow{BP} = (\vec{a}-\vec{b})^2$

$$BN = BP\cos\theta = (\vec{a}-\vec{b})\cdot\frac{\vec{c}}{|\vec{c}|}$$

$$PN^2 = BP^2 - BN^2 = (\vec{a}-\vec{b})^2 - \left\{(\vec{a}-\vec{b})\cdot\frac{\vec{c}}{|\vec{c}|}\right\}^2 \qquad \ldots(1)$$

Here $\vec{a} = 5\hat{i} - 6\hat{j} + 2\hat{k}$

$\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$

$\vec{c} = -4\hat{j} - 3\hat{k}$

$\therefore \quad \vec{a} - \vec{b} = 5\hat{i} - 6\hat{j} + 2\hat{k} - (\hat{i} - \hat{j} + 2\hat{k}) = 4\hat{i} - 5\hat{j}$

$\therefore \quad (\vec{a} - \vec{b})^2 = (4\hat{i} - 5\hat{j}) \cdot (4\hat{i} - 5\hat{j}) = 16 + 25 = 41$

$$(\vec{a} - \vec{b}) \cdot \vec{c} = (4\hat{i} - 5\hat{j}) \cdot (-4\hat{j} - 3\hat{k})$$
$$= (4\hat{i} - 5\hat{j} + 0\hat{k}) \cdot (0\hat{i} - 4\hat{j} - 3\hat{k})$$
$$= 0 + 20 - 0 = 20$$
$$|\vec{c}| = \sqrt{0^2 + (-4)^2 + (-3)^2} = 5$$

Therefore, from (1),

$$PN^2 = 41 - \left\{\frac{20}{5}\right\}^2 = 41 - 16 = 25.$$

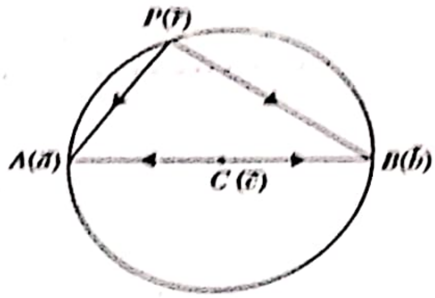
$\therefore \quad PN = 5$ units, which is the required distance.

Example 29: *Prove that any diameter of a sphere subtends a right angle at a point on the surface.*

Solution: Let C be the centre of the sphere and a is the radius. Let ACB is a diameter and A, B are points on the sphere.

Let the position vectors of A, B, C are $\vec{a}, \vec{b}, \vec{c}$ respectively.

Let P be any point on the surface of the sphere with position vector $\vec{r}$. Therefore, the vector equation of this sphere is



$$|\vec{r}|^2 - 2\vec{r} \cdot \vec{c} + K = 0 \quad ...(1)$$

where $K = |\vec{c}|^2 - a^2$

Here $\vec{CA} =$ p.v. of A − p.v. of $C = \vec{a} - \vec{c}$

$\vec{CB} =$ p.v. of B − p.v. of $C = \vec{b} - \vec{c}$.

Since ACB is a diameter, C is the centre and A, B are points on the sphere (1), we have

$$\vec{CA} = -\vec{CB}, \text{ or } (\vec{a} - \vec{c}) = -(\vec{b} - \vec{c})$$

$\therefore \quad \vec{a} + \vec{b} = 2\vec{c} \quad ...(2)$

Since $A(\vec{a})$ lies on the sphere (1), therefore, $|\vec{a}|^2 - 2\vec{a} \cdot \vec{c} + K = 0$

or $\quad |\vec{a}|^2 - \vec{a} \cdot (\vec{a} + \vec{b}) + K = 0 \quad$ [by (2)]

or $\quad |\vec{a}|^2 - |\vec{a}|^2 - \vec{a} \cdot \vec{b} + K = 0$

$\therefore \quad \vec{a} \cdot \vec{b} = K \quad ...(3)$

Let P be any point on the sphere (1) and the position vector of P be $\vec{r}$.

Now, $\quad \vec{PA} \cdot \vec{PB} = (\vec{a} - \vec{r}) \cdot (\vec{b} - \vec{r})$

$$= \vec{a}\cdot\vec{b} - \vec{r}\cdot(\vec{a}+\vec{b}) + |\vec{r}|^2$$

$$= K - 2\vec{r}\cdot\vec{c} + |\vec{r}|^2 \qquad \text{[by (2) and (3)]}$$

$$= 0 \qquad [\because P(\vec{r}) \text{ lies on (1)}]$$

$\therefore$ $\angle APB = 90°$. Hence the result.

MULTIPLE CHOICE QUESTIONS

1. The magnitude of the vector $\vec{a}+\vec{b}$ where $\vec{a} = 2\hat{i}+\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-\hat{k}$ is

   (*a*) $2\sqrt{6}$ (*b*) 0 (*c*) $2\sqrt{3}$ (*d*) $3\sqrt{2}$.

2. The unit vector along the vector $2\hat{i}+\hat{j}-2\hat{k}$ is

   (*a*) $2\hat{i}+\hat{j}-2\hat{k}$ (*b*) $\frac{1}{9}(2\hat{i}+\hat{j}-2\hat{k})$ (*c*) $\frac{1}{3}(2\hat{i}+\hat{j}-2\hat{k})$ (*d*) none of these.

3. The angle between the vectors $\hat{i}-2\hat{j}+2\hat{k}$ and $4\hat{i}+3\hat{k}$ is

   (*a*) $\cos^{-1}\left(\frac{2}{3}\right)$ (*b*) $\cos^{-1}\left(\frac{1}{15}\right)$ (*c*) $\cos^{-1}\left(\frac{3}{2}\right)$ (*d*) none of these.

4. If $\vec{a} = \hat{i}+2\hat{j}-\hat{k}$, $\vec{b} = 2\hat{i}+4\hat{j}-2\hat{k}$ then $\vec{a}$ and $\vec{b}$ are

   (*a*) independent (*b*) collinear
   (*c*) mutually perpendicular (*d*) none of these.

5. If $A = (2, 4, -5)$, $B = (3, 5, -6)$, then $\overrightarrow{AB}$ is

   (*a*) $5\hat{i}+9\hat{j}-11\hat{k}$ (*b*) $-\hat{i}-\hat{j}+\hat{k}$
   (*c*) $\hat{i}+\hat{j}-\hat{k}$ (*d*) none of these.

6. The value of λ for which the vectors $\vec{a} = 2\hat{i}+3\hat{j}-4\hat{k}$ and $\vec{b} = \hat{i}+\lambda\hat{j}+2\hat{k}$ are mutually perpendicular is

   (*a*) -4 (*b*) 0 (*c*) -2 (*d*) 2.

7. If θ be an angle between the vectors $\vec{a} = 6\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b} = 2\hat{i}-9\hat{j}+6\hat{k}$, then

   (*a*) $\theta = \cos^{-1}\left(\frac{12}{77}\right)$ (*b*) $\theta = \sin^{-1}\left(\frac{12}{77}\right)$
   (*c*) $\theta = \tan^{-1}\left(\frac{12}{77}\right)$ (*d*) none of these. (*W.B.U.T. 2009*)

8. If $\vec{a}$ and $\vec{b}$ are perpendicular to each other, then $|\vec{a}\times\vec{b}|$ is equal to

   (*a*) 0 (*b*) $|\vec{a}||\vec{b}|$ (*c*) $|\vec{a}\cdot\vec{b}|$ (*d*) none of these.

9. If $(x-2)\hat{i}+(x^2y-4)\hat{j}+(yz-3)\hat{k} = \vec{0}$, then