$$\therefore \quad E(C_1) = 40\,P(X < 200) + 10\,P(X \geq 200)$$
$$= 40\,(1 - e^{-4/3}) + 10\,e^{-4/3} = 40 - 30e^{-4/3} = ₹\,32.09$$

Similarly,
$$C_2 = \begin{cases} ₹\,50, & \text{for } X < 200 \\ ₹\,20, & \text{for } X \geq 200 \end{cases}$$

$$\therefore \quad E(C_2) = 50\,P(X < 200) + 20\,P(X \geq 200)$$
$$= 50\,(1 - e^{-1}) + 20\,e^{-1} = 50 - 30\,e^{-1} = ₹\,38.96.$$

Since $E(C_1) < E(C_2)$, so process I should be adopted.

Example 6: *The length of the shower in a tropical island in a rainy season follows an exponential distribution with parameter λ (> 0), time being measured in minutes. Find the probability that it will last for at least one more minute.*

Solution: Let the random variable X corresponds to the length of the shower. According to the problem X follows an exponential distribution with parameter $\lambda > 0$. So, the p.d.f. of X is

$$f(x) = \begin{cases} \lambda\, e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$\therefore$ Required probability $= P$(shower will last for at least one more minute)

$$= P(X \geq s + 1 / X \geq s) = P(X \geq 1) \quad \text{[By memoryless property]}$$
$$= \int_1^\infty f(x)\,dx = \lim_{B \to \infty} \int_1^B \lambda\, e^{-\lambda x} dx = \lim_{B \to \infty} [-e^{-\lambda x}]_1^B$$
$$= \lim_{B \to \infty} (e^{-\lambda} - e^{-\lambda B}) = e^{-\lambda}.$$

$[\because \quad e^{-\lambda B} \to 0$ as $B \to \infty$, because $\lambda > 0]$

3.11 NORMAL DISTRIBUTION

Definition: A continuous random variable X is said to have a **normal distribution** with parameters $\mu(-\infty < \mu < \infty)$ and σ (> 0) if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty.$$

In this case, we say X is a **normal variate** with parameters μ and σ and is denoted by $X \sim N(\mu, \sigma)$. *(W.B.U.T. 2002)*

Observe that (*i*) $f(x) \geq 0, \forall\, x$.

(*ii*) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{\infty} f(x)\,dx$

$$= \lim_{B_1 \to \infty} \int_{-B_1}^{\mu} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2} dx$$
$$+ \lim_{B_2 \to \infty} \int_{\mu}^{B_2} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \lim_{B_1 \to \infty} -\int_{\frac{B_1+\mu}{\sigma\sqrt{2}}}^{0} \frac{1}{\sqrt{\pi}} e^{-u^2} du + \lim_{B_2 \to \infty} \int_0^{\frac{B_2-\mu}{\sigma\sqrt{2}}} \frac{1}{\sqrt{\pi}} e^{-v^2} dv$$

$$\left[\text{Putting } u = -\frac{(x-\mu)}{\sigma\sqrt{2}} \text{ and } v = \frac{x-\mu}{\sigma\sqrt{2}}\right]$$

$$= \lim_{B_1 \to \infty} \frac{1}{\sqrt{\pi}} \int_0^{\frac{B_1+\mu}{\sigma\sqrt{2}}} e^{-u^2} du + \int_0^{\infty} \frac{1}{\sqrt{\pi}} e^{-v^2} dv = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du$$

$$= \frac{2}{\sqrt{\pi}} \lim_{B \to \infty} \int_0^{B} e^{-u^2} du = \frac{2}{\sqrt{\pi}} \lim_{B \to \infty} \int_0^{B^2} \frac{1}{2} e^{-z} z^{-\frac{1}{2}} dz$$

$$\left[\begin{array}{lll} \text{Putting } u^2 = z & \Rightarrow & 2u\, du = dz \\ & \Rightarrow & du = \frac{1}{2} z^{-\frac{1}{2}} dz \end{array}\right]$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-z} z^{\frac{1}{2}-1} dz = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1.$$

So, this is a valid probability distribution.

Distribution Function

The **distribution function** $F(x)$ of the r.v. X is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\, dt = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(t-\mu)^2/2\sigma^2} dt.$$

Theorem 1: *If X is a normal variate with parameters μ and σ then (i) mean = $E(X) = \mu$ (ii) Var $(X) = \sigma^2$.*

Proof: The probability density function of the r.v. X is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty.$$

(*i*)
$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{\infty} (x-\mu)\, f(x)\, dx + \mu \int_{-\infty}^{\infty} f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)\, e^{-(x-\mu)^2/2\sigma^2} dx + \mu$$

$$\left[\because \int_{-\infty}^{\infty} f(x)\, dx = 1\right]$$

Now,
$$\int_{-\infty}^{\infty} (x-\mu)\, e^{-(x-\mu)^2/2\sigma^2} dx = \lim_{\substack{B_1 \to \infty \\ B_2 \to \infty}} \int_{-B_1}^{B_2} (x-\mu)\, e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \lim_{\substack{B_1 \to \infty \\ B_2 \to \infty}} \frac{1}{2} \int_{(B_1+\mu)^2}^{(B_2-\mu)^2} e^{-z/2\sigma^2} dz = \lim_{\substack{B_1 \to \infty \\ B_2 \to \infty}} \frac{1}{2} \left[-2\sigma^2 e^{-z/2\sigma^2}\right]_{(B_1+\mu)^2}^{(B_2-\mu)^2}$$

$$\left[\text{Put } z = (x-\mu)^2 \Rightarrow (x-\mu)\, dx = \frac{1}{2} dz\right]$$

$$= \lim_{\substack{B_1 \to \infty \\ B_2 \to \infty}} \sigma^2 \left[e^{-(B_1+\mu)^2/2\sigma^2} - e^{-(B_2-\mu)^2/2\sigma^2} \right] = 0.$$

$\therefore$ Mean $= E(X) = \mu$.

(*ii*) $$\text{Var}(X) = E\left\{(X-\mu)^2\right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{1}{\sqrt{\pi}} \lim_{B_1 \to \infty} \int_{-B_1}^{\mu} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2} \frac{dx}{\sigma\sqrt{2}}$$

$$+ \frac{1}{\sqrt{\pi}} \lim_{B_2 \to \infty} \int_{\mu}^{B_2} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2} \frac{dx}{\sigma\sqrt{2}}$$

$$= \frac{1}{\sqrt{\pi}} \lim_{B_1 \to \infty} \int_{\frac{B_1+\mu}{\sigma\sqrt{2}}}^{0} -2\sigma^2 u^2 e^{-u^2} du + \frac{1}{\sqrt{\pi}} \lim_{B_2 \to \infty} \int_{0}^{\frac{B_2-\mu}{\sigma\sqrt{2}}} 2\sigma^2 v^2 e^{-v^2} dv$$

$$\left[\text{Putting } u = -\frac{(x-\mu)}{\sigma\sqrt{2}} \text{ and } v = \frac{x-\mu}{\sigma\sqrt{2}}\right]$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \lim_{B_1 \to \infty} \int_{0}^{\frac{B_1+\mu}{\sigma\sqrt{2}}} u^2 e^{-u^2} du + \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} v^2 e^{-v^2} dv$$

$$= \frac{4\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} u^2 e^{-u^2} du = \frac{2\sigma^2}{\sqrt{\pi}} \lim_{B \to \infty} \int_{0}^{B} 2u^2 e^{-u^2} du$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \lim_{B \to \infty} \int_{0}^{B^2} e^{-z} z^{1/2} dz \quad [\text{Put } u^2 = z \Rightarrow 2udu = dz,\ u = z^{1/2}]$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z} z^{\frac{3}{2}-1} dz = \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \quad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= \frac{\sigma^2}{\sqrt{\pi}} \cdot \sqrt{\pi} = \sigma^2.$$

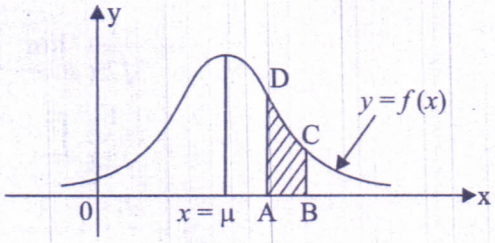
Note: S.D. of $X = \sigma$.

Normal Probability Curve (or Normal Curve)

The graph of the probability density function of a normal variate is called **normal probability curve** or **normal curve.** So, the normal probability curve with mean μ and variance σ^2 is given by the equation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

and the graph of $y = f(x)$ is given adjacent:



(*i*) The curve is bell shaped and symmetrical about the highest ordinate, *i.e.*, about the line $x = \mu$.

(*ii*) The maximum probability occurs around the point $x = \mu$.

(*iii*) Since $f(x) \geq 0$, it can never be negative and hence no portion of the curve lies below the x-axis.

(*iv*) x-axis is an asymptote to this curve.

(*v*) **Area property:** If A and B are any two points on the x-axis, the shaded area, $ABCD$, bounded by the curve, the ordinates at A and at B and the x-axis is equal to the probability that x lies between A and B. **The total area bounded by the curve and x-axis is equal to 1.** The probability distribution defined in this manner by means of a normal probability curve is called **normal probability distribution.** This probability distribution is completely known if the values of the parameters μ and σ are known.

Standard Normal Distribution

The normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$ is called **standard normal distribution.** A random variable X having standard normal distribution is called **standard normal variate** and is denoted as $X \sim N(0, 1)$.

The **probability density function** and **distribution function** of a **standard normal variate** X are denoted by φ and ϕ respectively.

$$\therefore \qquad \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \; -\infty < x < \infty$$

and

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

Note: We have already seen that for a normal distribution the probability that the r.v. X will take a value between $x = a$ and $x = b$ is equal to the area enclosed by the normal probability curve, x-axis and the ordinates at $x = a$ and $x = b$. As such the problem of finding the probability reduces to that of finding the area between the two ordinates of the normal curve. But for different values of μ and σ we get different normal curves. The problem of finding area between two given ordinates will multiply into too many problems if we are to find the area between the given ordinates for different normal curves with different values of μ and σ. All such problems can be reduced to a single one by reducing all normal distributions to a single normal distribution called **standard normal distribution** by using the following theorem:

Theorem 2: *If a continuous random variable X has normal distribution with parameters μ and σ, then the random variable $Z = \dfrac{X - \mu}{\sigma}$ has standard normal distribution.*

Proof: Given, $X \sim N(\mu, \sigma)$ and $Z = \dfrac{X - \mu}{\sigma} \Rightarrow X = \mu + \sigma Z.$

Now,
$$F(z) = P(Z \leq z) = P\left(\frac{X - \mu}{\sigma} \leq z\right) = P(X \leq \mu + \sigma z)$$

$$\therefore \qquad F(z) = \int_{-\infty}^{\mu + \sigma z} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx = \frac{1}{\sigma\sqrt{2\pi}} \lim_{B \to \infty} \int_{-B}^{\mu + \sigma z} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{B \to \infty} \int_{-(B+\mu)/\sigma}^{z} e^{-t^2/2} dt \qquad \left(\text{Putting } t = \frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$$

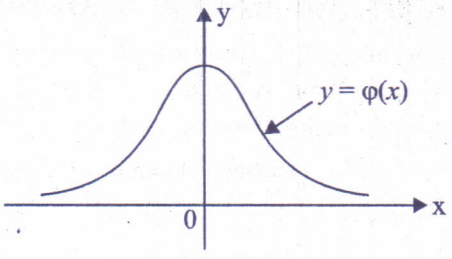
$$\therefore \qquad f(z) = \frac{dF}{dz} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, -\infty < z < \infty. \text{ Hence } Z \sim N(0, 1).$$

Standard Normal Curve

The graph of the probability density function of a standard normal variate is called **standard normal curve.** So, the standard normal curve is given by the equation

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, -\infty < x < \infty$$

and the graph of $y = \varphi(x)$ is given adjacent:



(*i*) The curve is bell shaped and symmetrical about the y-axis, *i.e.*, about the line $x = 0$.

(*ii*) The maximum probability occurs around the point $x = 0$.

(*iii*) Since $\varphi(x) \geq 0$, it can never be negative and hence no portion of the curve lies below the x-axis.

(*iv*) x-axis is an asymptote to this curve.

(*v*) Since $\int_{-\infty}^{\infty} \varphi(x)\, dx = 1$, so the total area bounded by the curve and x-axis is equal to 1.

Area Property

If $X \sim N(\mu, \sigma)$ then the probability that the random variable X will lie between μ and x_1 is given by

$$P(\mu < X < x_1) = \int_{\mu}^{x_1} f(x)\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{x_1} e^{-(x-\mu)^2/2\sigma^2} dx$$

Put $\qquad \dfrac{X-\mu}{\sigma} = Z, \quad i.e., \quad X - \mu = \sigma Z$

When $\qquad X = \mu, Z = 0$ and when $X = x_1, Z = \dfrac{x_1 - \mu}{\sigma} = z_1$ (say)

$$\therefore \qquad P(\mu < X < x_1) = P(0 < Z < z_1) = \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2} dz = \int_0^{z_1} \varphi(z)\, dz$$

where $\varphi(z) = \dfrac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the p.d.f. of the standard normal variate Z (see Theorem 2).

The definite integral: $\int_0^{z_1} \varphi(z)\, dz$ is known as **normal probability integral** and gives the area under standard normal curve between the ordinates at $z = 0$ and $z = z_1$. The area under the standard normal curve for different values of z_1 are found from **Standard Normal Table.**

Important Observations

(*i*) $$P(\mu - \sigma < X < \mu + \sigma) = \int_{\mu-\sigma}^{\mu+\sigma} f(x)\,dx$$

$$\Rightarrow \quad P(-1 < Z < 1) = \int_{-1}^{1} \varphi(z)\,dz \left[\text{where } Z = \frac{X-\mu}{\sigma}\right]$$

$$= 2\int_0^1 \varphi(z)\,dz \qquad \text{[Since } \varphi(z) \text{ is an even function]}$$

$$= 2 \times 0.3413 = 0.6826 \qquad \text{(From table)}$$

Similarly,

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(-2 < Z < 2) = \int_{-2}^{2} \varphi(z)\,dz$$

$$= 2\int_0^2 \varphi(z)dz = 2 \times 0.4772 = 0.9544$$

(*ii*) The total area under normal probability curve is unity,

i.e., $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \varphi(z)\,dz = 1.$$

(*iii*) Since in the tables, the area under standard normal curve is given, in numerical problems we shall deal with standard normal variate Z rather than the r.v. X itself.

Hence, if we want to find area under normal curve, we shall convert the given area to the form $P(0 < Z < z_1)$ since the areas have been given in this form in the tables.

Example: *X is a normal variate with mean 30 and variance 25. Find*

(*i*) $P(26 \le X \le 40)$ (*ii*) $P(X < 35)$

(*iii*) $P(X \ge 45)$ *and* (*iv*) $P(|X - 30| > 5)$.

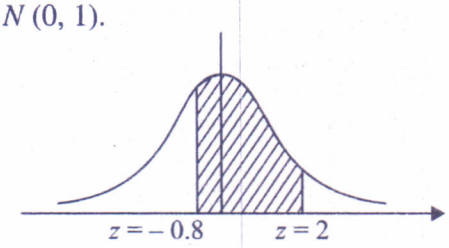
Solution: Here $X \sim N(\mu, \sigma)$.

Given, mean $= \mu = 30$ and Var $(X) = \sigma^2 = 25$. $\therefore$ $\sigma = 5$.

Let $$Z = \frac{X-\mu}{\sigma} = \frac{X-30}{5}. \quad \therefore \quad Z \sim N(0, 1).$$

(*i*) When $$X = 26, Z = \frac{26-30}{5} = -0.8$$

When $$X = 40, Z = \frac{40-30}{5} = 2$$



$$\therefore \quad P(26 \le X \le 40) = P(-0.8 \le Z \le 2)$$

$$= P(-0.8 \le Z \le 0) + P(0 < Z \le 2)$$

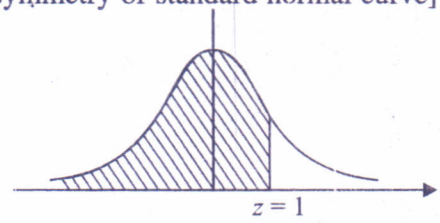
$$= P(0 \le Z \le 0.8) + P(0 < Z \le 2)$$

[From symmetry of standard normal curve]

$$= 0.2881 + 0.4772 = 0.7653.$$

(*ii*) When $$X = 35, Z = \frac{35-30}{5} = 1.$$

$$\therefore \quad P(X < 35) = P(Z < 1)$$

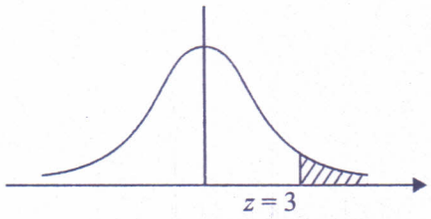


$$= P(-\infty < Z \le 0) + P(0 < Z < 1)$$
$$= 0.5 + 0.3413 = 0.8413.$$

(*iii*) When $X = 45, Z = \dfrac{45-30}{5} = 3.$

$\therefore$ $P(X \ge 45) = P(Z \ge 3)$

$$= 0.5 - P(0 \le Z \le 3)$$
$$= 0.5 - 0.49865$$
$$= 0.00135.$$



(*iv*) $P(|X - 30| > 5) = 1 - P(|X - 30| \le 5)$

$$= 1 - P(-5 \le X - 30 \le 5)$$
$$= 1 - P(25 \le X \le 35)$$
$$= 1 - P(-1 \le Z \le 1) \quad \left[\text{Use } Z = \frac{X-30}{5}\right]$$
$$= 1 - 2P(0 \le Z \le 1)$$
$$= 1 - 2 \times 0.3413 = 1 - 0.6826 = 0.3174.$$

3.12 NORMAL APPROXIMATION TO BINOMIAL DISTRIBUTION

Theorem: Let the random variable X follows Binomial distribution with probability mass function

$$P(X = i) = {}^nC_i\, p^i (1-p)^{n-i}, \; i = 0, 1, 2, \ldots, n$$

where n and p are parameters. If $n \to \infty$ and p is not very small but of moderate magnitude, then the distribution of the random variable

(*i*) X approaches to $N\left(np, \sqrt{np(1-p)}\right)$,

(*ii*) $Z = \dfrac{X - np}{\sqrt{np(1-p)}}$ approaches to $N(0, 1)$.

Proof is omitted

Notes: (*i*) The random variable $Z = \dfrac{X - np}{\sqrt{np(1-p)}}$ is called standardized Binomial variate.

(*ii*) We apply this theorem when $np \ge 5$ and $n(1-p) \ge 5$.

Example: *A fair coin is tossed 400 times. Using normal approximation to binomial distribution find the probability of obtaining (i) exactly 200 heads (ii) between 190 and 210 heads, both inclusive. Given that the area under standard normal curve between $z = 0$ and $z = 0.05$ is 0.0199 and between $z = 0$ and $z = 1.05$ is 0.3531.* (*W.B.U.T. 2007, 2010*)

Solution: Let the random variable X corresponds to the number of heads in 400 tosses. Then clearly X follows binomial distribution with parameters: n = number of trials = 400 and p = probability of success in a single trial = 1/2.

Since n is large, we suppose that X is an approximately normal variate with parameters:

$$\mu = np = 400 \times \frac{1}{2} = 200 \text{ and } \sigma = \sqrt{np(1-p)} = \sqrt{400 \times \frac{1}{2} \times \frac{1}{2}} = 10.$$

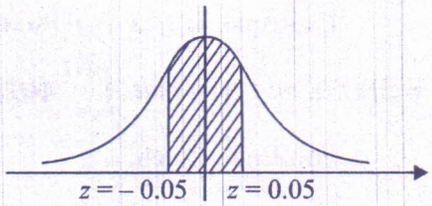
So, $Z = \dfrac{X - \mu}{\sigma} = \dfrac{X - 200}{10}$ is approximately standard normal variate.

(*i*) When $X = 199.5, Z = \dfrac{199.5 - 200}{10} = -0.05.$

When $X = 200.5, Z = 0.05.$

Using normal approximation:

Required probability



$= P(X = 200) = P(199.5 \le X \le 200.5)$

[In terms of normal approximation]

$= P(-0.05 \le Z \le 0.05)$

$= 2P(0 \le Z \le 0.05)$

$= 2 \times 0.0199 = 0.0398.$

(*ii*) When $X = 189.5, Z = \dfrac{189.5 - 200}{10} = -1.05$

When $X = 210.5, Z = 1.05$

Using normal approximation:

Required probability $= P(190 \le X \le 210) = P(189.5 \le X \le 210.5)$

[In terms of normal approximation]

$= P(-1.05 \le Z \le 1.05) = 2P(0 \le Z \le 1.05)$

$= 2 \times 0.3531 = 0.7062.$

ILLUSTRATIVE EXAMPLES – V

Example 1: *If X follows a normal distribution with mean 12 and variance 16, find $P(X \ge 20)$.*

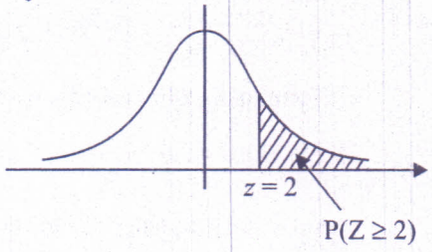
$$\left[\text{Given}: \int_{-\infty}^{2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt = 0.977725\right]$$ (*W.B.U.T. 2009*)

Solution: Here $X \sim N(\mu, \sigma)$

Given, mean $= \mu = 12$ and Var $(X) = \sigma^2 = 16.$ $\therefore$ $\sigma = 4.$

Let $Z = \dfrac{X - \mu}{\sigma} = \dfrac{X - 12}{4}$ $\therefore$ $Z \sim N(0, 1).$

When $X = 20, Z = \dfrac{20 - 12}{4} = 2.$



$\therefore$ $P(X \ge 20) = P(Z \ge 2)$

$= 1 - P(Z < 2)$

$= 1 - \displaystyle\int_{-\infty}^{2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = 1 - 0.977725 = 0.022275.$

Example 2: *If X is normally distributed with zero mean and unit variance, find the expectation of X^2.* (*W.B.U.T. 2002*)

Solution: Given, $E(X) = 0$ ...(1)

and Var $(X) = 1 \Rightarrow E(X^2) - \{E(X)\}^2 = 1$

$\Rightarrow$ $E(X^2) = 1$ [By (1)]

Example 3: *If X is normally distributed with mean 3 and s.d. 2, find c such that P(X > c) = 2P(X ≤ c). Given that* $\int_{-\infty}^{0.43} \varphi(t)\,dt = 0.6666.$ *(W.B.U.T. 2007)*

Solution: Given, $X \sim N(\mu, \sigma)$, where $\mu = 3$ and $\sigma = 2$.

Let $$Z = \frac{X-\mu}{\sigma} = \frac{X-3}{2} \quad \therefore \quad Z \sim N(0, 1).$$

$$\therefore \quad P(X > c) = P\left(\frac{X-3}{2} > \frac{c-3}{2}\right) = P\left(Z > \frac{c-3}{2}\right) = 1 - P\left(Z \le \frac{c-3}{2}\right)$$

$$= 1 - \int_{-\infty}^{(c-3)/2} \varphi(t)\,dt,$$

where $\varphi(t)$ is the p.d.f. of $Z \sim N(0,1)$.

Also, $$P(X \le c) = P\left(\frac{X-3}{2} \le \frac{c-3}{2}\right) = P\left(Z \le \frac{c-3}{2}\right) = \int_{-\infty}^{(c-3)/2} \varphi(t)\,dt$$

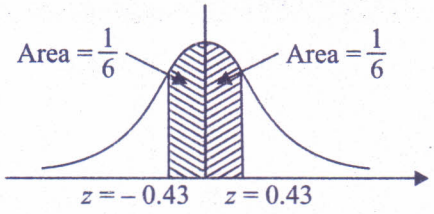
Given: $$P(X > c) = 2P(X \le c)$$

$$\Rightarrow \quad 1 - \int_{-\infty}^{(c-3)/2} \varphi(t)\,dt = 2\int_{-\infty}^{(c-3)/2} \varphi(t)\,dt$$

$$\Rightarrow \quad \int_{-\infty}^{(c-3)/2} \varphi(t)\,dt = \frac{1}{3} \qquad \text{...(1)}$$

Given, $$P(Z \le 0.43) = \int_{-\infty}^{0.43} \varphi(t)\,dt$$

$$= 0.6666 = \frac{2}{3}$$



$$\therefore \quad P(-0.43 \le Z \le 0) = P(0 \le Z \le 0.43)$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$\therefore \quad P(Z \le -0.43) = \int_{-\infty}^{-0.43} \varphi(t)\,dt = \frac{1}{2} - \frac{1}{6} = \frac{1}{3} \qquad \text{...(2)}$$

Comparing (1) and (2), we get

$$\frac{c-3}{2} = -0.43 \quad \Rightarrow \quad c = 3 - 0.86 = 2.14.$$

Note: By symmetry of the standard normal curve about y-axis, we have

$$P(Z \le -0.43) = P(Z \ge 0.43) = 1 - P(Z \le 0.43) = 1 - \frac{2}{3} = \frac{1}{3}.$$

Example 4: *If a random variable X follows normal distribution such that*

$$P(9.6 \le X \le 13.8) = 0.7008 \text{ and } P(X \ge 9.6) = 0.8159,$$

where $$\int_{-\infty}^{0.9} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt = 0.8159 \text{ and } \int_{-\infty}^{1.2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt = 0.8849,$$

find the mean and variance of X. *(W.B.U.T. 2008)*

Solution: Given, $X \sim N(\mu, \sigma)$, where $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$

Let $$Z = \frac{X-\mu}{\sigma} \quad \therefore \quad Z \sim N(0, 1)$$

Let φ is the distribution function of Z.

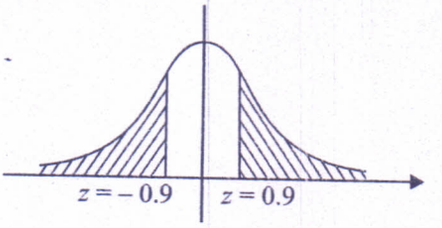
$$\therefore \quad P\left(\frac{X-\mu}{\sigma} \le x\right) = P(Z \le x) = \varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

Given: $\varphi(0.9) = 0.8159$ and $\varphi(1.2) = 0.8849$.

By symmetry of the standard normal curve about y-axis, we have

$$\varphi(-0.9) = P(Z \le -0.9) = P(Z \ge 0.9) = 1 - P(Z \le 0.9)$$
$$= 1 - \varphi(0.9) = 1 - 0.8159 = 0.1841.$$

$$\therefore \quad P(-0.9 \le Z \le 1.2) = \varphi(1.2) - \varphi(-0.9)$$
$$= 0.8849 - 0.1841$$
$$= 0.7008$$



$$\Rightarrow \quad P\left(-0.9 \le \frac{X-\mu}{\sigma} \le 1.2\right) = 0.7008 \qquad ...(1)$$

Given, $P(9.6 \le X \le 13.8) = 0.7008$

$$\Rightarrow \quad P\left(\frac{9.6-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{13.8-\mu}{\sigma}\right) = 0.7008 \qquad ...(2)$$

Comparing (1) and (2), we get

$$\frac{9.6-\mu}{\sigma} = -0.9 \text{ and } \frac{13.8-\mu}{\sigma} = 1.2 \quad \Rightarrow \quad \mu = 9.6 + 0.9\sigma = 13.8 - 1.2\sigma$$

$$\therefore \quad \sigma = \frac{4.2}{2.1} = 2 \text{ and } \mu = 9.6 + 0.9 \times 2 = 11.4.$$

$\therefore$ Mean of $X = 11.4$ and $\text{Var}(X) = \sigma^2 = 4$.

Example 5: *Prove that for normal distribution, the mean deviation from the mean equals 4/5 of the standard deviation nearly.*

Solution: Let X be a normal variate with mean μ and standard deviation σ. By definition:

Mean deviation from the mean

$$= \int_{-\infty}^{\infty} |x-\mu| f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} |x-\mu| e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{dx}{\sigma} + \int_{0}^{\infty} |x-\mu| e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{dx}{\sigma} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\lim_{B_1 \to \infty} \int_{-B_1}^{0} |x-\mu| e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{dx}{\sigma} + \lim_{B_2 \to \infty} \int_{0}^{B_2} |x-\mu| e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{dx}{\sigma} \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\lim_{B_1 \to \infty} - \int_{\frac{B_1+\mu}{\sigma}}^{0} \sigma |u| e^{-u^2/2} du + \lim_{B_2 \to \infty} \int_0^{\frac{B_2-\mu}{\sigma}} \sigma |v| e^{-v^2/2} dv\right]$$

$$\left(\text{Putting } u = \frac{-x+\mu}{\sigma} \text{ and } v = \frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left[\lim_{B_1 \to \infty} \int_0^{(B_1+\mu)/\sigma} \sigma u e^{-u^2/2} du + \int_0^{\infty} \sigma v e^{-v^2/2} dv\right]$$

$$= \sigma\sqrt{\frac{2}{\pi}} \int_0^{\infty} u e^{-u^2/2} du$$

$$= \sigma\sqrt{\frac{2}{\pi}} \lim_{B \to \infty} \int_0^{B} u e^{-u^2/2} du = \sigma\sqrt{\frac{2}{\pi}} \lim_{B \to \infty} \int_0^{B^2/2} e^{-t} dt$$

$$= \sigma\sqrt{\frac{2}{\pi}} \lim_{B \to \infty} \left[-e^{-t}\right]_0^{B^2/2} = \sigma\sqrt{\frac{2}{\pi}} \lim_{B \to \infty} \left[1 - e^{-B^2/2}\right]$$

$$= \sigma\sqrt{\frac{2}{\pi}} = 0.7979\,\sigma = 0.8\,\sigma = \frac{4}{5}\sigma \text{ (nearly).} \quad \textbf{(Proved)}$$

Example 6: *The mean height of 1,000 soldiers in a regiment is 68.22 inches and the variance is 10.8 inches square. Assuming that the heights are normally distributed, find how many soldiers to be over 6 feet tall?*

[Given: Area under the standard normal curve between $z = 0$ and $z = 1.15$ is 0.3746]

Solution: Let the random variable X corresponds to the height of a soldier.

Given, $X \sim N(\mu, \sigma)$, where μ = mean of $X = 68.22$

σ = S.D. of $X = \sqrt{10.8}$.

Let $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X - 68.22}{\sqrt{10.8}}$. $\therefore$ $Z \sim N(0, 1)$.

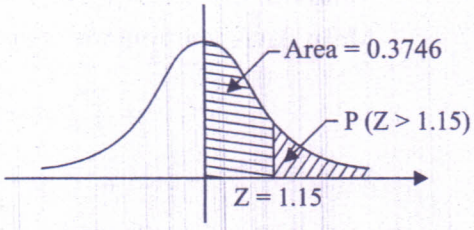
When $X = 6$ feet $= 72$ inches, $Z = \dfrac{72 - 68.22}{\sqrt{10.8}} = 1.15$

$\therefore$ $P(X > 72) = P(Z > 1.15)$

$= 0.5 - P(0 \le Z \le 1.15)$

$= 0.5 - 0.3746$

$= 0.1254$



$\therefore$ Expected number of soldiers

$= 1000 \times 0.1254 = 125$ (nearly).

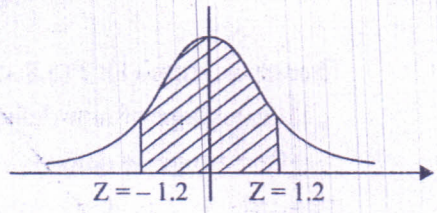
Example 7: *The mean inside diameter of a sample of 200 washers produced by a machine is 0.502 cm and the standard deviation is 0.005 cm. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 0.496 to 0.508 cms. Determine the percentage of defective washers produced by the machine assuming the diameters are normally distributed. (Given: area under the standard normal curve from $Z = 0$ to $Z = 1.2$ is 0.3849).*

Solution: Let the random variable X corresponds to the inside diameter of a washer.
Given, $X \sim N(\mu, \sigma)$, where μ = mean of $X = 0.502$ and σ = s.d. of $X = 0.005$.

Let $$Z = \frac{X-\mu}{\sigma} = \frac{X-0.502}{0.005} \quad \therefore \quad Z \sim N(0, 1)$$

When $$X = 0.496, Z = \frac{0.496-0.502}{0.005} = -1.2.$$

When $$X = 0.508, Z = \frac{0.508-0.502}{0.005} = 1.2.$$



$\therefore \quad P(0.496 \le X \le 0.508) = P(-1.2 \le Z \le 1.2)$

= Shaded area between, $Z = -1.2$ to $Z = 1.2$.

$= 2P(0 \le Z \le 1.2)$

[$\because$ Standard normal curve is symmetrical about y-axis]

$= 2 \times 0.3849$

$= 0.7698$

$\therefore$ Percentage of washers between tolerance limits

= 76.98 = 77% (nearly)

$\therefore$ Percentage of defective washers

= 100 – 77 = 23% (nearly).

Example 8: *The length of bolts produced by a machine is normally distributed with mean 4 and s.d. 0.5. A bolt is defective if its length does not lie in the interval (3.8, 4.3). Find the percentage of defective bolts produced by the machine.*

$$\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.6} e^{-t^2/2}dt = 0.7257, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.4} e^{-t^2/2}dt = 0.6554\right] \qquad \text{(W.B.U.T. 2004)}$$

Solution: Let the random variable X corresponds to the length of a bolt.

Given, $X \sim N(\mu, \sigma)$, where μ = mean of $X = 4$

and σ = s.d. of $X = 0.5$

Let $$Z = \frac{X-\mu}{\sigma} = \frac{X-4}{0.5} \quad \therefore \quad Z \sim N(0,1).$$

Let ϕ is the distribution function of Z.

When $$X = 3.8, Z = \frac{3.8-4}{0.5} = -0.4.$$

When $$X = 4.3, Z = \frac{4.3-4}{0.5} = 0.6.$$

$\therefore \quad P(3.8 < X < 4.3) = P(-0.4 < Z < 0.6) = \phi(0.6) - \phi(-0.4) \qquad ...(1)$

Now, $\phi(-0.4) = P(Z \le -0.4) = P(Z \ge 0.4)$

[By symmetry of the standard normal curve about y-axis]

$$= 1 - P(Z < 0.4) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.4} e^{-t^2/2} dt$$

$$= 1 - 0.6554 = 0.3446$$

and $$\phi(0.6) = P(Z < 0.6) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.6} e^{-t^2/2} dt = 0.7257.$$

Therefore, from (1): $P(3.8 < X < 4.3) = 0.7257 - 0.3446 = 0.3811$.

$\therefore$ Percentage of non-defective bolts produced by the machine = 38.11 = 38% (nearly)

$\therefore$ Percentage of defective bolts produced by the machine = 100 – 38 = 62% (nearly).

Example 9: *The marks obtained by 1000 students to a final examination are found to be approximately normally distributed with mean 70 and standard deviation 5. Estimate the number of students whose marks will be between 60 and 75 both inclusive. Given that area under the normal curve* $\varphi(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$ *between Z = 0 and Z = 2 is 0.4772 and between Z = 0 and Z = 1 is 0.3413.*

(W.B.U.T. 2002)

Solution: Let the random variable X corresponds to the marks obtained by a student.

Given, $X \sim N(\mu, \sigma)$, where μ = mean of $X = 70$ and σ = s.d. of $X = 5$.

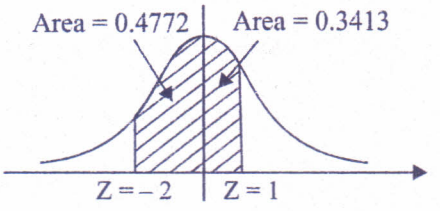
Let $$Z = \frac{X - \mu}{\sigma} = \frac{X - 70}{5}. \quad \therefore \quad Z \sim N(0, 1).$$

When $X = 60$,

$$Z = \frac{60 - 70}{5} = -2.$$

When $X = 75$,

$$Z = \frac{75 - 70}{5} = 1.$$



$\therefore$ $P(60 \le X \le 75)$

$$= P(-2 \le Z \le 1) = P(-2 \le Z \le 0) + P(0 < Z \le 1)$$

$$= P(0 \le Z \le 2) + P(0 < Z \le 1)$$

[$\because$ Standard normal curve is symmetrical about y-axis]

$$= 0.4772 + 0.3413 = 0.8185.$$

$\therefore$ Number of students whose marks will be between 60 and 75 both inclusive = 1000 × 0.8185 = 818 (nearly).

Example 10: *The weight of a student in a college is normally distributed with m = 40 kg, σ = 5 kg. Find the percentage of the students that have weight*

(i) greater than 40 kg

(ii) greater than 50 kg

(iii) between 38 kg and 52 kg.

$$\left[\text{Given}: \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{2} e^{-t^2/2} dt = 0.9772, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.4} e^{-t^2/2} dt = 0.6554, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{2.4} e^{-t^2/2} dt = 0.9918\right]$$

(W.B.U.T. 2003)