Basic Discrete Structures

Sets, Functions, Sequences, Matrices, and Relations (Lecture – 7)

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Reflexive Closure

- The relation $R = \{(1, 1), (1, 2), (2, 1), (3, 2)\}$ on the set $A = \{1, 2, 3\}$ is not reflexive.
- How can we produce a reflexive relation containing R that is as small as possible?
- This can be done by adding $(2, 2)$ and $(3, 3)$ to R, because these are the only pairs of the form (a, a) that are not in R.
- Then $S= \{(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)\}$

  $R \subseteq S$

  The minimal set $S \supseteq R$ is called **the reflexive closure of R**

- Set S is called ***the reflexive closure of R*** if it:
  - Contains R
  - Has reflexive property
  - Is contained in every reflexive relation Q that contains R
- Definition of Closure

Definition: Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

Symmetric Closure

- The relation $R = \{(1, 2), (1, 3), (2, 2)\}$ on the set $A = \{1, 2, 3\}$ is not symmetric.
- How can we produce a symmetric relation containing R that is as small as possible?
- This can be done by adding $(2, 1)$ and $(3, 1)$ to R.
- Then $S = \{(1,2),(1,3),(2,2),(2,1),(3,1)\}$

$$R \subseteq S$$

- The minimal set S is the ***Symmetric closure of R***:
  - Contains R
  - Has symmetric property
  - Is contained in every symmetric relation Q that contains R

Transitive Closure

- Let relation $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on the set $\{1, 2, 3, 4\}$.
- This relation is not transitive because it does not contain all pairs of the form (a, c) where (a, b) and (b, c) are in R.
- Add pairs to make R transitive: $(1, 2)$, $(2, 3)$, $(2, 4)$, and $(3, 1)$.
- Adding these pairs does *not* produce a transitive relation, because the resulting relation contains $(3, 1)$ and $(1, 4)$ but does not contain $(3, 4)$.
- This shows that constructing the transitive closure of a relation is more complicated than constructing either the reflexive or symmetric closure.

Transitive Closure: Paths in Directed Graph

A *path* from a to b in the directed graph G is a sequence of edges (x_0, x_1), (x_1, x_2), $(x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a nonnegative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has *length n*. We view the empty set of edges as a path of length zero from a to a. A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

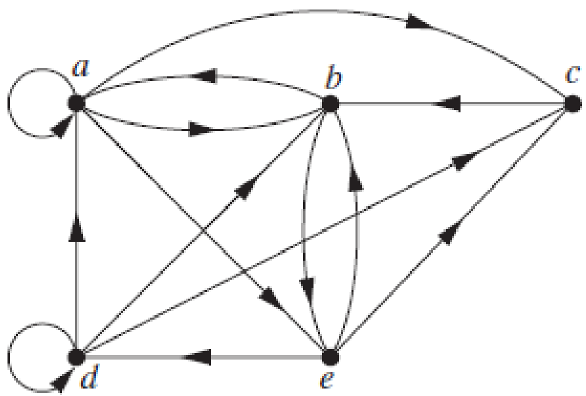
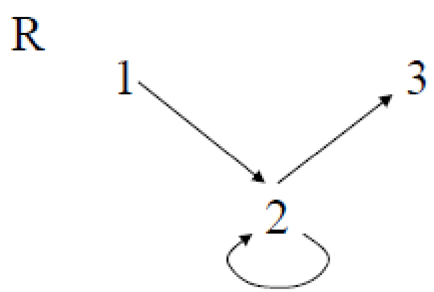


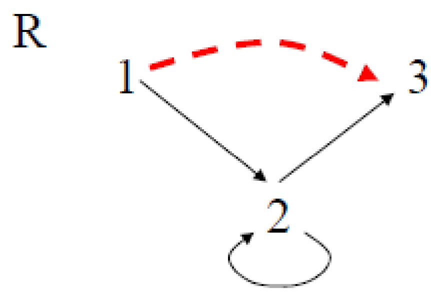
FIGURE 1 **A Directed Graph.**

- Which of the following are paths in the directed graph shown in Figure 1: *a, b, e, d*; *a, e, c, d, b*; *b, a, c, b, a, a, b*; *d, c*; *c, b, a*; *e, b, a, b, a, b, e*? What are the lengths of those that are paths? Which of the paths in this list are circuits?

Transitive Closure: Paths in Directed Graph

- Assume $R=\{(1,2),(2,2),(2,3)\}$ on $A=\{1,2,3\}$.
  - Is R transitive?
    - No
- How to make it transitive?
  - $S = \{(1,2),(2,2),(2,3)\} \cup \{(1,3)\} = \{(1,2),(2,2),(2,3),(1,3)\}$
  - S is the transitive closure of R
- We can represent the relation on the graph. Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path (or digraph).





Transitive Closure: Paths in Directed Graph

- Theorem 1:

Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

- Definition:

Let R be a relation on a set A. The *connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

- Because R^n consists of the pairs (a, b) such that there is a path of length n from a to b, it follows that R^* is the union of all the sets R^n. In other words,

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

Transitive Closure: Paths in Directed Graph

- Theorem 2:

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

- Theorem 3:

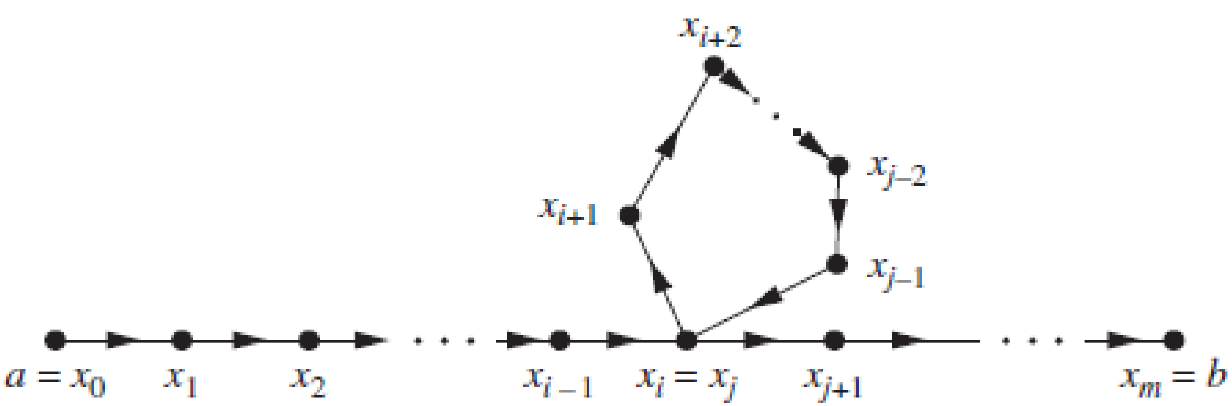
The transitive closure of a relation R equals the connectivity relation R^*.

- Computing the transitive closure:
  - We do not need to examine arbitrarily long paths to determine whether there is a path between two vertices in a finite directed graph.
  - It is sufficient to examine paths containing no more than n edges, where n is the number of elements in the set.

Transitive Closure: Paths in Directed Graph

- Lemma 1:

Let A be a set with n elements, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Moreover, when $a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $n - 1$.



Producing a Path with Length Not Exceeding *n*