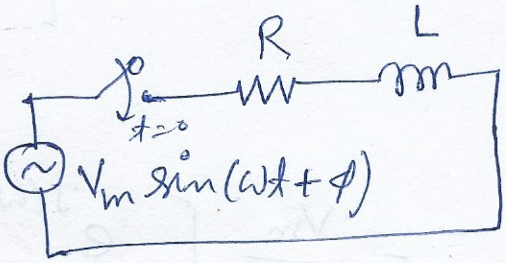
Transient Response of series R-L circuit having sinusoidal Excitation.



Applying KVL

$$L\frac{di(t)}{dt} + R\,i(t) = V_m \sin(\omega t+\phi)$$

$$\Rightarrow \frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{V_m}{L}\sin(\omega t+\phi)$$

This is a non-homogeneous equation, the current $i(t)$ consist of the sum of complementary function $i_c(t)$ and particular integral $i_p(t)$

$$i = i_c + i_p$$

The complementary function of equation is

$$i_c = K e^{-R/L\, t}$$

and the ~~integral~~ particular integral of equation

$$\left[\frac{dy(t)}{dt} + P\,y(t) = Q \Rightarrow y(t) = e^{-Pt}\int Q e^{Pt}\,dt + Ke^{-Pt}\right]$$

$$i_p(t) = e^{-\frac{R}{L}t}\int \frac{V_m}{L}\sin(\omega t+\phi)\, e^{+\frac{R}{L}t}\,dt$$

$$\boxed{\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j} \qquad \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}}$$

$$= e^{-R/L\,t}\int \frac{V_m}{L}\left[\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j}\right] e^{+R/L\,t}\,dt$$

$$= \frac{e^{-R/L\,t}\cdot V_m}{2jL}\int\left(e^{j(\omega t+\phi)+\frac{R}{L}t} - e^{-j(\omega t+\phi)+\frac{R}{L}t}\right)dt$$

$$= \frac{e^{-\frac{R}{L}t}\cdot V_m}{2jL}\left[\frac{e^{j(\omega t+\phi)+\frac{R}{L}t}}{j\omega+\frac{R}{L}} - \frac{e^{-j(\omega t+\phi)+\frac{R}{L}t}}{-j\omega+\frac{R}{L}}\right]$$

$$= \frac{V_m}{2jL}\left[\frac{e^{j(\omega t+\phi)}}{\frac{R}{L}+j\omega} - \frac{e^{-j(\omega t+\phi)}}{\frac{R}{L}-j\omega}\right]$$

$$= \frac{V_m}{2jL}\left[\frac{e^{j(\omega t+\phi)}\left(\frac{R}{L}-j\omega\right) - e^{-j(\omega t+\phi)}\cdot\left(\frac{R}{L}+j\omega\right)}{\left(\frac{R}{L}+j\omega\right)\left(\frac{R}{L}-j\omega\right)}\right]$$

$$= \frac{V_m}{2jL}\left[\frac{\frac{R}{L}\left\{e^{+j(\omega t+\phi)} - e^{-j(\omega t+\phi)}\right\} - j\omega\left\{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right\}}{\left(\frac{R}{L}\right)^2+(\omega)^2}\right]$$

$$\cancel{= \frac{V_m}{2jL}\left[\frac{R}{L}\right.}$$

$$= \frac{V_m}{L\left\{\frac{R^2}{L^2}+\omega^2\right\}}\left[\frac{R}{L}\left\{\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j}\right\} - j\omega\left\{\frac{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}}{2j}\right\}\right]$$

$$= \frac{V_m}{R^2+\omega^2L^2}\left[R\cdot\sin(\omega t+\phi) - \cancel{j}\,\omega L\cos(\omega t+\phi)\right]$$

Let $R = C\cos\theta$, $-\omega L = C\sin\theta$.

$C = \sqrt{R^2+(\omega L)^2}$ $\qquad \theta = -\tan^{-1}\left(\frac{\omega L}{R}\right)$

$$i_p(t) = \cancel{\frac{V_m}{\sqrt{R^2+\omega^2L^2}} \sin(\omega t + \phi - \tan^{-1}}$$

$$= \frac{V_m}{R^2+\omega^2L^2} \cdot \left[C \sin(\omega t + \phi + \theta)\right]$$

$$= \frac{V_m}{R^2+\omega^2L^2} \cdot \sqrt{R^2+\omega^2L^2} \cdot \sin\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right)$$

$$= \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cdot \sin\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right)$$

$$\therefore i(t) = K e^{-R/L\, t} + \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cdot \sin\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right)$$

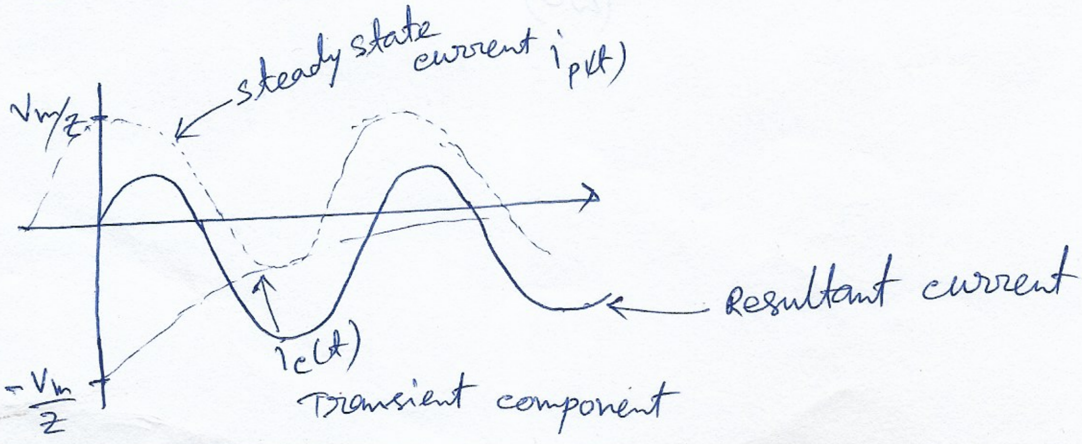
$i(0+) = 0$, (inductor behaves as a open cut at switching)

$$0 = \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cdot \sin\left(\phi - \tan^{-1}\frac{\omega L}{R}\right) + K$$

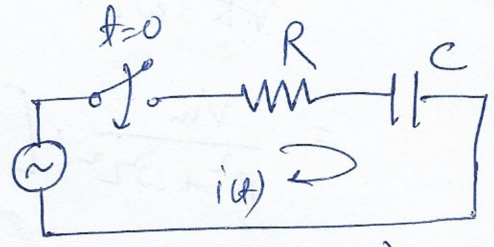
$$\therefore K = -\frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cdot \sin\left(\phi - \tan^{-1}\frac{\omega L}{R}\right)$$

$$\therefore i(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cdot \left[\sin(\omega t + \phi + \theta) - \sin(\phi + \theta) \cdot e^{-R/L\, t}\right]$$

$$Z = \sqrt{R^2+\omega^2L^2}. \quad \theta = -\tan^{-1}\frac{\omega L}{R}.$$



Transient Response of series R-C circuit Having Sinusoidal excitation.



$v = V_m \sin(\omega t+\phi)$

Applying KVL

$$R\,i(t) + \frac{1}{C}\int_0^t i(t)\,dt + V_c(0^+) = V_m \sin(\omega t+\phi)$$

on differentiating, we get

$$\frac{di(t)}{dt} + \frac{1}{RC}\,i(t) = \frac{V_m\omega}{R}\cos(\omega t+\phi)$$

General solution of this differential equation is

$$i(t) = i_c(t) + i_p(t)$$

$$= K e^{-t/RC} + \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}} \sin\left(\omega t+\phi+\tan^{-1}\frac{1}{\omega CR}\right)$$

Since capacitor behaves as a short circuit at switching

$$i(0^+) = \frac{V_m \sin\phi}{R}$$

or, $$\frac{V_m \sin\phi}{R} = K + \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}} \sin\left(\phi+\tan^{-1}\frac{1}{\omega CR}\right)$$

or, $$K = \frac{V_m \sin\phi}{R} - \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}} \sin\left(\phi+\tan^{-1}\frac{1}{\omega CR}\right)$$

$$\therefore\ i(t) = \left[\frac{V_m \sin\phi}{R} - \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}} \sin\left(\phi + \tan^{-1}\frac{1}{\omega CR}\right)\right] e^{-t/RC}$$

$$+ \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \phi + \tan^{-1}\frac{1}{\omega CR}\right)$$

or, $$i(t) = \left[\frac{V_m \sin\phi}{R} - \frac{V_m}{Z}\sin(\phi+\theta)\right] e^{-t/RC} + \frac{V_m}{Z}\sin(\omega t + \phi + \theta)$$

$$\theta = \tan^{-1}\left(\frac{1}{\omega CR}\right), \quad Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

Example 1:- A 50HZ 300V (peak value) sinusoidal voltage is applied at $t=0$ to a series R-L circuit having resistance 2.5Ω and inductance 0.1H. Find an expression of current at any instant t. Also, calculate the value of the transient current, steady state current and resultant current 0.01 sec. after switching on.

Solution:- Impedance of R-L circuit.

$$Z = \sqrt{(2.5)^2 + (2\pi \times 50 \times 0.1)^2} = 31.5\Omega$$

Transient current $i_c(t) = K e^{-\frac{R}{L}t} = K e^{-25t}$

steady state current $i_p(t) = \frac{V_m}{Z}\sin\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right)$

$$= \frac{300}{31.5}\sin(314t - 85.45^\circ)$$

Since voltage is applied at $t=0$, i.e. $\phi = 0$ and

$$\tan^{-1}\left(\frac{314 \times 0.1}{2.5}\right) = 85.45^\circ$$

$= 9.52 \sin(314t - 1.49)$ [since $85.45° = 1.49$ rad]

$\therefore i(t) = i_c(t) + i_p(t)$

$= K e^{-25t} + 9.52 \sin(314t - 1.49)$

Since $i(0^+) = 0$

$0 = K + 9.52 \sin(-1.49)$

or, $K = 9.49$

$\therefore i(t) = 9.49 e^{-25t} + 9.52 \sin(314t - 1.49)$

at $t = 0.01$ sec $i_c(t) = 9.49\, e^{-25 \times 0.01} = 7.39$ A

$i_p(t) = 9.52 \sin(314 \times 0.01 - 1.49) = 9.49$ A

$i(t) = i_c(t) + i_p(t) = 16.88$ A.

Problem 2 A voltage $v = 300 \sin 314t$ is applied at $t = 2.14$ msec to a series R-C circuit having resistance 10Ω and capacitance $200\mu F$. Find an expression for current. Also, find the value of current 1 msec. after switching on.

Soln:- It may be noted that the voltage is not applied at $t=0$, but at ϕ where $\phi = 2.14$ msec

$= 2.14 \times 10^{-3} \times 314 = 0.672$ rad.

$\therefore$

Impedance of R-C circuit

$$Z = \sqrt{(10)^2 + \left(\frac{1}{314 \times 200 \times 10^{-6}}\right)^2} = 18.8\Omega$$

Transient current

$$i_c(t) = Ke^{-t/RC} = Ke^{-\frac{t}{10\times 200\times 10^{-6}}} = Ke^{-500t}$$

Steady state current

$$i_p(t) = \frac{V_m}{Z}\sin\left(\omega t + \phi + \tan^{-1}\frac{1}{\omega CR}\right)$$

$$= \frac{300}{18.8}\sin\left[314t + 0.672 + \tan^{-1}\left(\frac{1}{314\times 200\times 10^{-6}\times 10}\right)\right]$$

$$= 15.96\sin(314t + 0.672 + \overset{1.01}{\cancel{1.59}})$$

Therefore $i(t) = i_c(t) + i_p(t)$

$$= Ke^{-500t} + 15.96\sin(314t + 0.672 + \overset{1.01}{\cancel{1.59}})$$

Since capacitor behaves as a short-circuit at switching

$$\therefore i(2.14\,\text{msec}) = \frac{300\times\sin(314\times 2.14\times 10^{-3})}{10} = 18.67\text{ A}$$

Hence we get

$$18.67 = K(1) + 15.96\sin(0.672 + \overset{1.01}{\cancel{1.59}})$$

or, $K = 18.67 - \overset{15.86}{\cancel{12.29}} = \cancel{6.38}\ \cancel{2.808}\ 2.81$

Therefore, the required expression of current is given by

$$i(t) = \overset{2.81}{\cancel{6.38}}e^{-500t} + 15.96\sin(314t + \overset{1.682}{\cancel{2.262}})$$

after 1 msec. the current becomes.

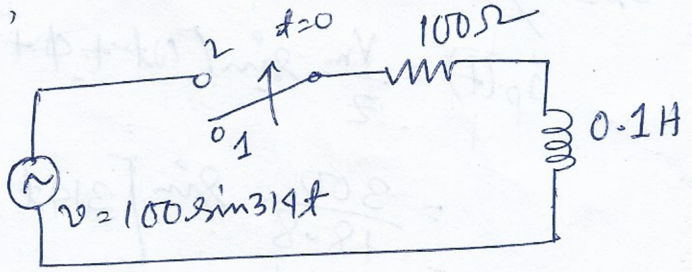
$$i = \overset{2.81}{\cancel{6.38}}e^{-500\times 10^{-3}} + 15.96\sin(314\times 10^{-3} + \overset{1.682}{\cancel{2.262}})$$

$$= \overset{1.70}{\cancel{3.87}} + \overset{14.53}{\cancel{8.55}} = \cancel{12.42\text{ A}}$$

$$= 16.23\text{ A}$$

Problem 3 :- obtain the current at $t>0$ if a.c. voltage v is applied when the switch S is moved to 2 from 1 at $t=0$. Assume a steady state current of 1 A in the LR circuit when the switch was at position 1.

Ans:- At position 1 of the switch 'S' the steady state current in the circuit is 1 A, $i(0^-) = 1\,A = i(0^+)$



$$Z = R + jX_L = 100 + j\,2\pi \times 50 \times 0.1$$
$$= 104.8 \angle 17.47^\circ$$

Applying KVL in the RL circuit

$$R.i + L\frac{di}{dt} = v$$

$$\Rightarrow 100\,i + 0.1\frac{di}{dt} = 100 \sin 314t$$

$$\Rightarrow \frac{di}{dt} + 10^3 i = 10^3 \sin 314t$$

$$\Rightarrow i_c = C e^{-R/L\,t} = C e^{-\frac{100}{0.1}t} = C e^{-1000t}$$

$$i_p = \frac{V_m}{Z} \sin\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right)$$

$$= \frac{100}{104.8} \sin(314t - 17.47^\circ) \quad [\phi = 0]$$

$$= 0.954 \sin(314t - 0.304)$$

$$\therefore i = i_c + i_p = C e^{-1000t} + 0.954 \sin(314t - 0.304)$$

$$C = 1.28$$

problem 4 :- In a series R-L-C circuit, $R = 5\Omega$, $L = 1H$ and $C = 1F$. A d.c. voltage of 20V is applied at $t=0$, obtain $i(t)$.

Solution :- Applying of KVL yields, in the series RLC circuits.

$$iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = V \qquad (1)$$

$$\Rightarrow L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0$$

$$\Rightarrow Lp^2 + Rp + \frac{1}{C} = 0 \Rightarrow p^2 + 5p + 1 = 0$$

$$P_1, P_2 = \frac{-5 \pm \sqrt{25-4}}{2} = \frac{-5 \pm 4.58}{2} = -0.21, -4.79$$

The solution becomes, $i(t) = K_1 e^{-0.21t} + K_2 e^{-4.79t}$ — (2)

As current in inductance cannot change instantaneouly and also the voltage across the capacitor does not change instantaneouly. $i(0^+) = 0$ and $\frac{1}{C}\int i\,dt = 0$

Thus from (1) $L\frac{di}{dt} = V$ (at $t = 0^+$)

$$\frac{di}{dt}(0^+) = \frac{V}{L} = \frac{20}{1} = 20\ A/sec.$$

Again substituting $i=0$ at $t=0^+$ in (2)

$$0 = K_1 e^{-0.21\times 0} + K_2 e^{-4.79\times 0}$$

$$0 = K_1 + K_2$$

Again differentiating equation (2)

$$\frac{di}{dt} = -0.21 K_1 e^{-0.21t} - 4.79\, e^{-4.79t}$$

at $t = 0^+$, $20 = -0.21 K_1 e^{-0.21t} - 4.79\, e^{-4.79t}$

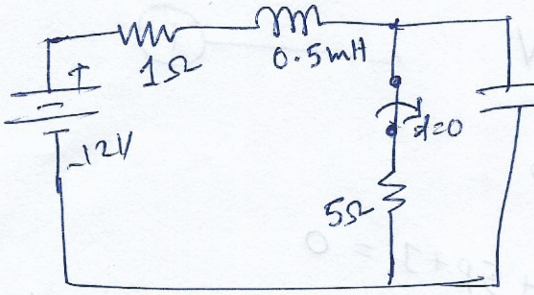
$$20 = -0.21K_1 - 4.79K_2$$

$$20 = -4.58K_2 \qquad [K_2 = -K_1]$$

$K_2 = -4.37$ Hence $K_1 = 4.37$

$$i(t) = (4.37\, e^{-0.21t} - 4.37\, e^{-4.79t})\ A$$

problem 5



The switch in circuit of figure has been closed for a very long time. It opens at $t=0$, Find $V_c(t)$ for $t>0$ using differential equation approach.

Ans:- At steady state (with switch is closed) inductor behaves as a short circuit while capacitor behaves as a open circuit.

$$i_L(0^-) = \frac{12}{1+5} = 2A, \quad V_C(0^-) = 2\times 5 = 10V.$$

at $t=0$, switch is open, then applying KVL

$$12 = 1.\, i(t) + 0.5 \frac{di(t)}{dt} + \frac{1}{2}\int_0^t i(t)dt + 10$$

$$\Rightarrow 2 = i(t) + 0.5 \frac{di(t)}{dt} + \frac{1}{2}\int_0^t i(t)\, dt$$

on differentiating we get

$$\Rightarrow 0 = \frac{di(t)}{dt} + 0.5 \frac{d^2 i(t)}{dt^2} + \frac{1}{2} i(t)$$

$$\Rightarrow \frac{d^2 i(t)}{dt^2} + 2\frac{di(t)}{dt} + i(t) = 0 \Rightarrow p^2 + 2p + 1 = 0$$

$$p_{1,2} = -1$$

$$\Rightarrow i(t) = (K_1 + K_2 t)e^{-t}$$

at $t=0^+$ $i(0^+) = i_L(0^-) = 2 = K_1 e^0 \Rightarrow K_1 = 2$

$$V_C(0^+) = V_C(0^-) = 10 = \frac{1}{2}\int_0^t i(t)\, dt\Big|_{t=0} + 10 \Rightarrow K_2 = 2$$

$$i(t) = (2+2t)e^{-t}$$

$$V_C(t) = \frac{1}{2}\int_0^t i(t)dt + 10 = \int_0^t (1+t)e^{-t}dt + 10 = 12 - 2e^{-t} - te^{-t}$$