Signals and Systems
Part - B

Abhik M

CST Dept, IIEST Shibpur

November 15, 2020

Overview

1. Linear time invariant systems
2. Filter design
3. Sampling of signals
4. Basics of Communication systems
   - Amplitude modulation systems
   - Frequency modulation systems
5. Basics of Feedback control systems
   - Classical control in frequency domain
   - Modern time domain control
   - State space methods in Discrete domain

Linear time invariant systems- significance

- Superposition property allows arbitrary signals to be expressed as linear combination of basic signals and compute the output in terms of response to the basic signals.
- General signals can be represented as linear combination of delayed impulses which enables usage of the time invariance property.
- Studying the response to unit impulse is enough to completely characterize the system, this representation is convolution sum or integral.

Discrete time signals as impulse sum

- Sifting property

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

- When $k = n$, $\delta[n-k] = \delta[0] = 1$ the only position in the summation where the impulse is present.
- Hence the signal can be represented as sequence of individual shifted impulses $\delta[n-k]$ weighted by the sequence $x[k]$.
- Now signals like unit step can be written as

$$u[n] = \sum_{k=0}^{\infty} 1.\delta[n-k]$$

noting that the unit step signal exists with value unity only for $n > 0$ and $\delta[n-k]$ exists only for $n = k$.

Continuous time signals using impulse

- Basic idea is to express the continuous signal as sum of approximated impulse responses.
- Taking $\delta_\Delta = \frac{1}{\Delta}$ in the interval $0 \leq t \leq \Delta$ so that the product $\Delta\delta_\Delta(t) = 1$ to resemble impulse function.
- The approximated signal is

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

  which approaches an integral as $\Delta \to 0$ and $\hat{x}(t) \to x(t)$.
- The convolution integral is

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

Response of discrete systems to unit impulse

- Introduce $h_k[n]$ as response of the system to the shifted unit impulse $\delta[n-k]$
- Then output is superposition sum

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

- Due to time invariance, $h_k[n] = h_0[n-k]$ and unit impulse response is denoted $h[n] = h_0[n]$
- Hence the output expression becomes

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- This is convolution sum or superposition sum, symbolically $y[n] = x[n] * h[n]$

Response of discrete systems - Example

- Consider $x[n] = \alpha^n u[n], h[n] = u[n]$ with $0 < \alpha < 1$.
- Then $x[k]h[n-k] = \alpha^k$ for the interval $0 \leq k \leq n$ since $h[k]$ is zero for negative index.
- Hence for $n \geq 0$,

$$y[n] = \sum_{k=0}^{n} \alpha^k$$

- which gives $y[n] = (\frac{1-\alpha^{n+1}}{1-\alpha})u[n]$ for all n.

Response of continuous systems to unit impulse

- Response is denoted by $h(t)$ for the impulse $\delta(t)$
- Then

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

  or $y(t) = x(t) * h(t)$
- Consider $x(t) = e^{-at}u(t)$ and $h(t) = u(t)$ so that $x(\tau)h(t-\tau) = e^{-a\tau}$ in the zone $0 < \tau < t$ and 0 elsewhere.
- Hence

$$y(t) = \int_0^t e^{-a\tau}d\tau = \frac{1}{a}(1 - e^{-at})u(t)$$

  for all t.

Properties of linear time invariant systems

- Commutative– $x[n] * h[n] = h[n] * x[n]$ and $x(t) * h(t) = h(t) * x(t)$ can be proved taking $r = n - k$ or equivalently $k = n - r$ to interchange the roles of $h[n]$ and $x[n]$.
- Distributive – $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$ and $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$
- Also from the above two, $(x_1[n] + x_2[n]) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$ and $[x_1(t) + x_2(t)] * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$
- Associative– $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$ and $x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$
- These properties help resolve or simplify complex interconnected systems. Such cascading is typical of LTI systems only, change of order in nonlinear system results in different system altogether.

Memory of LTI systems

- Only way to have memoryless discrete LTI systems is $h[n] = 0, n \neq 0$ whereby $h[n] = K\delta[n]$ with $K = h[0]$ a constant.
- The convolution sum reduces to $y[n] = Kx[n]$. Similarly for continuous time, $y(t) = Kx(t)$
- For $K = 1$ this represents the identity system.

Invertibility of LTI systems

- The inverse system response $h_1(t)$ together with $h(t)$ can reproduce the original signal.
- Hence $h(t) * h_1(t) = \delta(t)$. Similarly, $h[n] * h_1[n] = \delta[n]$.
- Consider a system $y(t) = x(t - t_0)$ which is a delay. Then the system response is $h(t) = \delta(t - t_0)$ so that output looks like $x(t - t_0) = x(t) * \delta(t - t_0)$. To recover back the original signal, take $h_1(t) = \delta(t + t_0)$ to shift the delayed signal back. This is the inverse of the system response. Hence $h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$.

Causality of LTI systems

- For discrete system to be causal, $y[n]$ must not depend on $x[k]$ for $k > n$. For that $h[n-k] = 0$ for all $k > n$. i.e. $h[n] = 0$ for negative n.
- Hence the expression for convolution sum becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

- For continuous time,

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Accumulator, time delay are causal. But time shift in advance in non-causal.

Stability of LTI systems

- For bounded input, output has to be bounded. Taking $x[n] \leq B$,

$$|y[n]| \leq B \sum_{k=-\infty}^{+\infty} |h[k]|$$

which implies that if the impulse sum is bounded, system is stable.

- For continuous time,

$$\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$$

is the condition for stability.

- The accumulator, integrator are unstable systems. The time delay is stable system.

Response to unit step input

- The response is denoted by

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{n} h[k]$$

  so that $h[n] = s[n] - s[n-1]$.
- Similarly,

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

  so that unit impulse response is the derivative of the unit step response $h(t) = \frac{ds(t)}{dt}$.
- Thus unit step response can be used to characterize the LTI system, since unit impulse response can be computed from it.

Differential equation based description

- General expression of n^{th} order system is

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

.

- When $N = 0$ this reduces to

$$y(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

.

- Solution of this system is the natural response of the system and it requires initial conditions to be specified. Various methods would be introduced later.

Difference equation- finite impulse response (FIR) system

- The discrete time equation is

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

which is recursive in nature.

- For $N = 0$ this reduces to

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$$

.

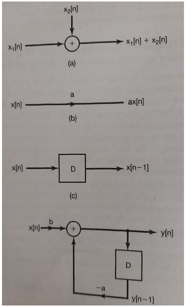
- This is called the finite impulse response (FIR) system, non-recursive since previous outputs not used, only convolution sum of inputs are present.
- Solution of this system is the natural response of the system and it requires initial conditions to be specified. Various methods would be introduced later.

Infinite impulse response (IIR) systems

- Consider $y[n] - \frac{1}{2}y[n-1] = x[n]$ or $y[n] = x[n] + \frac{1}{2}y[n-1]$.
- Consider an input $x[n] = K\delta[n]$.
- Then $y[0] = x[0] + \frac{1}{2}y[-1] = K$,
- $y[1] = x[1] + \frac{1}{2}y[0] = \frac{1}{2}K$,
- $y[2] = x[2] + \frac{1}{2}y[1] = (\frac{1}{2})^2K, \ldots$
- $y[n] = x[n] + \frac{1}{2}y[n-1] = (\frac{1}{2})^nK$
- These systems are called infinite impulse response(IIR) systems.

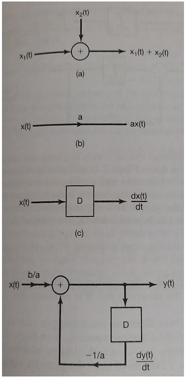
Cascading through block diagrams – delay difference

- First order difference equation: $y[n] + ay[n-1] = bx[n]$
- rewritten as $y[n] = -ay[n-1] + bx[n]$.
- This can be realized using a gain multiplier, a delay feedback element and a summer.



Cascading through block diagrams – Differentiation

- First order differential equation: $\frac{dy(t)}{dt} + ay(t) = bx(t)$
- rewritten as $y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$.
- This can be realized using the multiplier gains, a differentiator and an adder.

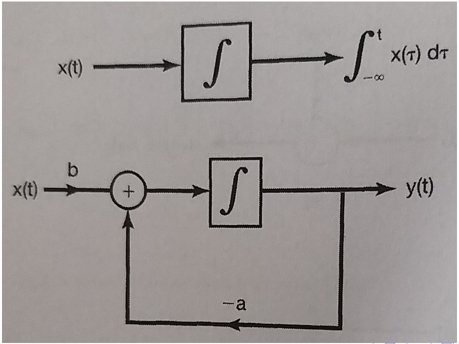


Cascading through block diagrams – integration

- Again rewriting above system in the form

$$y(t) = \int_{-\infty}^{t} (bx(\tau) - ay(\tau))d\tau$$

- The system can be implemented using multiplier, adder and an integrator.



Implementation of the blocks

- The integrator or differentiator can be realized using opamp circuit.
- Integrator is better as differentiation is error prone.
- This is called analog computer and may be extended to cover more complex systems, useful to simulate dynamical systems.

Unit impulse singularity and convolution

- Take the identity system impulse response $x(t) = x(t) * \delta(t)$ and together with $x(t) = \delta(t)$ it implies $\delta(t) = \delta(t) * \delta(t)$.
- This property is very basic of the idealized rectangular pulse with unity area for all limiting cases.
- Unity area can be shown by taking $x(t) = 1$ since

$$1 = x(t) = x(t) * \delta(t) = \delta(t) * x(t) = \int_{-\infty}^{+\infty} \delta(\tau) d\tau$$

- Now consider a time reversed signal $g(-t) = g(-t) * \delta(t) = \int g(\tau - t)\delta(\tau)d\tau$ and for $t = 0$, $g(0) = \int g(\tau)\delta(\tau)d\tau$. This means that unit impulse when multiplied with a signal $g(t)$ and then integrated on both sides to infinity, produces the value $g(0)$. This can be extended to product of two signals as well.

Derivative of the impulse

- Output is derivative of input $y(t) = \frac{dx(t)}{dt}$. Its unit impulse response is called dervative of impulse, unit doublet $u_1(t)$.
- Hence $\frac{dx(t)}{dt} = x(t) * u_1(t)$,
$\frac{d^2x(t)}{dt^2} = x(t) * u_2(t) = \frac{d}{dt}(\frac{dx(t)}{dt}) = x(t) * u_1(t) * u_1(t)$ so that $u_2(t) = u_1(t) * u_1(t)$. This way, $u_k(t) = u_1(t) \ldots u_1(t)$ a cascade of k derivatives.
- Unit doublet has zero area since for $x(t) = 1$, $\frac{dx(t)}{dt} = 0$ so that

$$\int_{-\infty}^{+\infty} u_1(\tau) d\tau = 0$$

.

Integration of the impulse

- Similar approach can be used for integrator of impulse.

$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} u(\tau) d\tau$$

  gives $u_{-2}(t) = tu(t)$ which is unit ramp function. In general, $u_{-k}(t) = \frac{t^{k-1}}{(k-1)!}u(t)$. To fit to this, $\delta(t) = u_0(t)$ and $u(t) = u_{-1}(t)$.
- Thus $u_k(t)$ represent cascade of k differentiators, $u_0(t)$ represents impulse response of identity system and when k is negative, it represents response of cascaded integrators.
- Again, $u_{-1}(t) * u_1(t) = u_0(t)$ which emphasizes the fact that differentiator is inverse system of integrator.
- Cascading of integrators and differentiators may be realized with manipulation of index in $u_{k+r}(t) = u_k(t) * u_r(t)$.

System response to complex exponential

- For continuous domain, consider the signal $x(t) = e^{st}$
- Then

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$

- The above can be split to obtain

$$y(t) = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{(-\tau)}d\tau = e^{st}H(s)$$

System response to complex exponential

- Thus the response $y[z] = z^n H(z)$ implies the response to the complex exponential is the same signal multiplied by an amplitude factor.
- Here z^n is the eigenfunction and the complex constant $H(z)$ is the corresponding eigenvalue.
- For signal

$$x(n) = \sum_k a_k z_k^n$$

  the output will be

$$y(n) = \sum_k a_k H(z_k) z_k^n$$

  due to superposition.

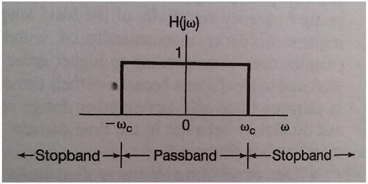
Filters for frequency shaping

- Change the relative amplitudes of the frequency components or even eliminate them
- Magnitude is in the units of $20log_{10}|H(j\omega)|$ decibels or dB. Frequency is given in Hz $(\omega/2\pi)$.
- Differentiating filter - Filter output is derivative of filter input $y(t) = dx(t)/dt$. With $x(t) = e^{j\omega t}$ the output will be $y(t) = j\omega e^{j\omega t}$ so that the frequency response becomes $H(j\omega) = j\omega$.Larger the frequency , larger the amplification.
- This can enhance rapid variations or transitions in signal e.g. used to detect edges in image processing.

Frequency selective filters

- Low pass filter: $H(j\omega) = 1$ for $|\omega| \leq \omega_c$ 0 otherwise.
- Simple RC circuit can be used to realize it. The system equation is $RC\frac{dV_c}{dt} + v_c(t) = v_s(t)$ which yields for applied voltage $v_s(t) = e^{j\omega t}$, and output $v_c(t) = H(j\omega)e^{j\omega t}$ which yields $H(j\omega) = \frac{1}{1+RCj\omega}$. The amplitude of this response is $|H(j\omega) \approx 1$ for $\omega \approx 0$ and the amplitude steadily decreases as the frequency increases.
- When voltage across the resistor is chosen as output, the filter becomes high pass filter. $RC\frac{dV_r(t)}{dt} + v_r(t) = RC\frac{dV_s(t)}{dt}$ yields $G(j\omega) = \frac{j\omega RC}{1+j\omega RC}$
- Band pass filter: $H(j\omega) = 1$ for $\omega_{c_1} \leq |\omega| \leq \omega_{c_2}$ 0 otherwise.

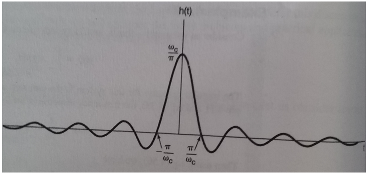
Low pass filter bandwidth



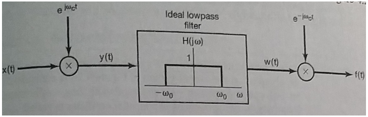
Time domain equations for filters

- The time domain response of lowpass filter is $h(t) = \frac{1}{RC}e^{-t/RC}u(t)$ and the step response is $s(t) = (1 - e^{-t/RC})u(t)$.
- For highpass filter it is $v_r(t) = e^{-t/RC}u(t)$.
- Trade-off needed between time domain response and frequency domain response.
- For large RC, the attenuation at higher frequency is better but the time domain response to step input is sluggish.
- On the contrary, smaller RC results in faster step response but the frequency response suffers.
- Such design is based on first order differential equations. More tuning is available for higher order differential equation based filters.

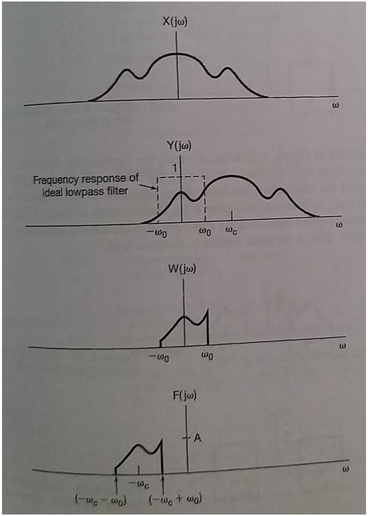
Impulse response of Low pass filters



Band pass filter bandwidth



Band pass filter spectra



Recursive discrete time filters

- First order difference equation $y[n] - ay[n-1] = x[n]$. If $x[n] = e^{j\omega n}$, then $y[n] = H(e^{j\omega})e^{j\omega n}$.
- Manipulating the index, we obtain $H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$.
- For positive value of a, the response behaves like lowpass filter. For negative a, the filter is highpass. The magnitude of a controls the size of passband, broader with reduction of the magnitude.
- Impulse response is $h[n] = a^n u[n]$ and Step response is given by $s[n] = u[n] * h[n] = \frac{1-a^{n+1}}{1-a} u[n]$.
- Trade-off is clear from these responses. Also system becomes unstable for $|a| \geq 1$.

Non Recursive discrete time filters

- Nonrecursive difference equation is given by

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k]$$

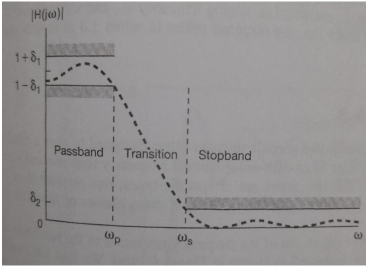
. Coefficients b_k provide the weights for the sequence terms. This has a moving average like tendency.

- The frequency response is given by $H(e^{j\omega}) = \frac{1}{N+M+1} \Sigma_{k=-N}^{M} e^{-j\omega k}$.
- The summation evaluates to $H(e^{j\omega}) = \frac{1}{N+M+1} e^{j\omega((N-M)/2)} \frac{sin(\omega(M+N+1)/2)}{sin(\omega/2)}$.
- The cutoff frequency can be varied by adjusting the moving window $N + M + 1$.

Fourier transform analysis of filters

Consider frequency response $H(j\omega) = 1$ below the cutoff frequency $|\omega| < \omega_c$.
Using duality, $h(t) = \frac{sin\omega_c t}{\pi t}$.
Again, for $h(t) = e^{-t}u(t)$ frequency response is $H(j\omega) = \frac{1}{1+j\omega}$.
This can be implemented using simple RC circuit.

Frequency domain response - basic idea



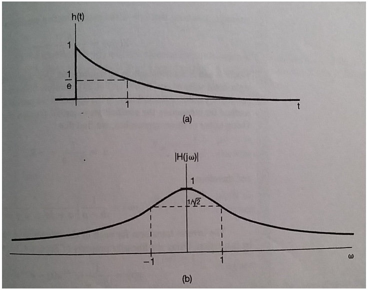
Time domain properties of filters

- Ideal filters have sharp cutoff frequencies with response of 0 and 1 respectively, which corresponds to the impulse response of the form $sinx/x$ in the time domain.
- The step response contains a rise time followed by frequent overshoots and undershoots before settling to the desired final value in the time domain. The rise time and other time domain features depend on the bandwidth of the cutoff frequency.
- Nonideal filters have a passband followed by a transition to a stopband.
- This behaviour corresponds to rise time, overshoot, undershoot and settling time in the time domain for a step input.

First order system response - continuous systems

Consider $\tau \frac{dy(t)}{dt} + y(t) = x(t)$.
Here τ is called time constant of the system.
Then frequency response would be $H(j\omega) = \frac{1}{j\omega\tau+1}$.
So impulse response is $h(t) = \frac{1}{\tau}e^{-t/\tau}u(t)$.
Step response is $s(t) = [1 - e^{-t/\tau}]u(t)$.

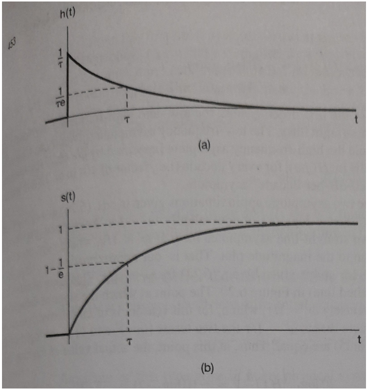
Impulse response of LTI systems



Asymptotic Significance of the time constant - Magnitude

When $t = \tau$ the impulse response reaches $1/e$ times its value at $t = 0$ so that with decrease in τ the decay is sharper while for step response the rise to reach unity is sharper without any oscillations.
Magnitude is $20log_{10}|H(j\omega)| = -10log_{10}[(\omega\tau)^2 + 1]$.
This implies that for $\omega << 1/\tau$ the log magnitude is approximately zero.
For high frequency the plot approaches asymptotically a straight line decreasing at $20dB$ per decade.
In fact the break frequency point $\omega = 1/\tau$ is called the $3dB$ point since $20log_{10}|H((j\frac{1}{\tau})| = 20log_{10}(|\frac{1}{1+j1}|) = -10log_{10}(2) \approx -3dB$.

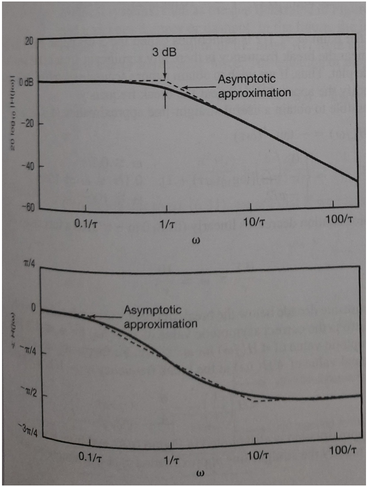
Impulse and step response - magnitude



Asymptotic Significance of the time constant - Phase

Phase is given by $\langle H(j\omega) = -tan^{-1}(\omega\tau)$.
The phase is $\approx 0, \omega \leq 0.1\tau$;
$\approx \pi/2 for \omega \geq 10/\tau$.
In the intermediate region, phase is linear in log scale
$\approx -(\pi/4)[log_{10}(\omega\tau) + 1]$.
It is easy to check that the phase angle is 45^o at the break frequency point.

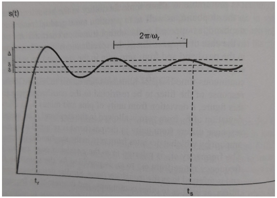
Impulse and step response - phase

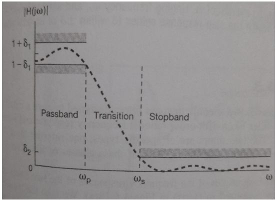


Second order differential equation

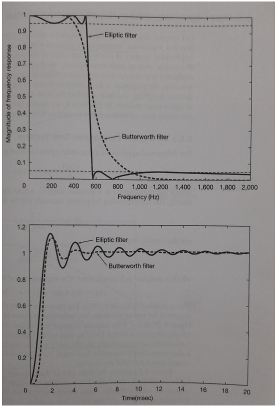
These are of the form $\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$
The frequency response is $H(j\omega) = \frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n^2}$.
Expanding by partial fractions, $H(j\omega) = \frac{M}{j\omega-c_1} - \frac{M}{j\omega-c_2}$
Here, $c_1 = -\zeta\omega_n + \omega_n\sqrt{(\zeta^2-1)}$;
$c_2 = -\zeta\omega_n - \omega_n\sqrt{(\zeta^2-1)}$;
$M = \frac{\omega_n}{2\sqrt{(\zeta^2-1)}}$
When $\zeta = 1$ then $c_1 = c_2 = -\omega_n$, $H(j\omega) = \frac{\omega_n^2}{(j\omega+\omega_n)^2}$ and
The response is actually a function of ω/ω_n. Here ζ is the damping ratio and ω_n the natural frequency.

Time and frequency domain response of second order system





Time and frequency domain response for different order filters



Second order system response in time domain

Hence impulse response is $h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$ or $h(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{2j\sqrt{1-\zeta^2}}$.

The impulse response for $\zeta = 1$ becomes $h(t) = \omega_n^2 t e^{-\omega_n t} u(t)$.

Thus for the range $0 < \zeta < 1$ the response is underdamped oscillatory. For $\zeta > 1$ system is overdamped with difference of two exponential signals.

For the case $\zeta = 1$ the system is critically damped.

Step response for $\zeta \neq 1$ is $s(t) = \{1 + M[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}]\}u(t)$.

Step response for $\zeta = 1$ is $s(t) = [1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}]u(t)$

Magnitude and phase plots

Magnitude plot obtained as
$20log_{10}|H(j\omega)| = -10log_{10}\{[1-(\frac{\omega}{\omega_n})^2]^2 + 4\zeta^2(\frac{\omega}{\omega_n})^2\}$.
Hence the log magnitude is zero for $\omega << \omega_n$ and $-40log\omega + 40log\omega_n$ for $\omega >> \omega_n$ implying a slope of $40dB$ per decade.
Magnitude maximizes at $\omega_{max} = \omega_n\sqrt{1-2\zeta^2}$ with value of magnitude being $\frac{1}{2\zeta\sqrt{1-\zeta^2}}$.
Quality factor indicating sharpness of the peak is $Q = \frac{1}{2\zeta}$.
Phase plot yields $\langle H(j\omega) = -tan^{-1}(\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2})$.
Here approximation yields zero for frequencies below $0.1\omega_n$, $-\pi$ for frequencies above $10\omega_n$ and in between the phase angle is approximately $-\frac{\pi}{2}[log_{10}(\frac{\omega}{\omega_n}) + 1]$.

First order system response - discrete systems

Magnitude is given by $|H(e^{j\omega})| = \frac{1}{(1+a^2-2acos\omega)^{1/2}}$.
For $a > 0$ the system attenuates high frequencies (magnitude is smaller near $\omega \approx \pm\pi$).
For $a < 0$ the system attenuates low frequencies. The maximum and minimum values are $\frac{1}{1+a}$ and $\frac{1}{1-a}$ respectively and for small $|a|$ the peaks are flattened.
For $|a|$ closer to unity, the peaks are sharp and hence the filtering over narrow band is achieved.
Phase is given by $\langle H(e^{j\omega}) = -tan^{-1}[\frac{asin\omega}{1-acos\omega}]$.

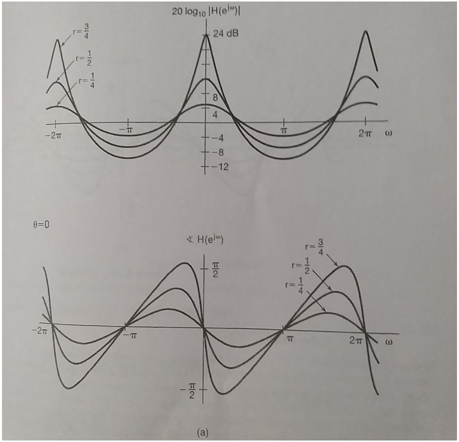
Second order system response - discrete systems

Consider $y[n] - 2rcos\theta y[n-1] + r^2 y[n-2] = x[n]$ with $0 < r < 1$ and $0 \leq \theta \leq \pi$.
This yields $H(e^{j\omega}) = \frac{1}{[1-re^{j\theta}e^{-j\omega}][1-re^{-j\theta}e^{-j\omega}]}$.
Partial fraction yields the coefficients as $A = \frac{e^{j\theta}}{2jsin\theta}$ and $B = \frac{e^{-j\theta}}{2jsin\theta}$.
The impulse response becomes $h[n] = r^n \frac{sin[(n+1)\theta]}{sin\theta} u[n]$.
For $\theta = 0$, $h[n] = (n+1)r^n u[n]$
For $\theta = \pi$, $h[n] = (n+1)(-r)^n u[n]$.

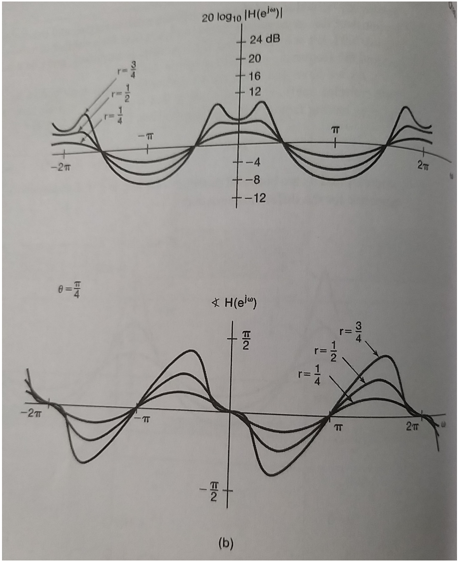
Real coefficient scenario

For real coefficients, $H(e^{j\omega}) = \frac{1}{(1-d_1 e^{-j\omega})(1-d_2 e^{-j\omega})}$
for the system $y[n] - (d_1 + d_2)y[n-1] + d_1 d_2 y[n-2] = x[n]$.
Then it simplifies to $H(e^{j\omega}) = \frac{\frac{d_1}{d_1-d_2}}{1-d_1 e^{-j\omega}} + \frac{\frac{d_2}{d_2-d_1}}{1-d_2 e^{-j\omega}}$.
Hence $h[n] = [\frac{d_1}{d_1-d_2} d_1^n + \frac{d_2}{d_2-d_1} d_2^n]u[n]$
- sum of two decaying exponentials or cascading of two first order systems.

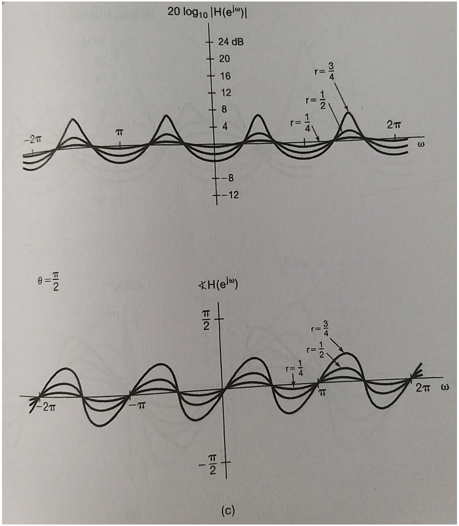
Magnitude phase plot showing response for $\theta = 0$



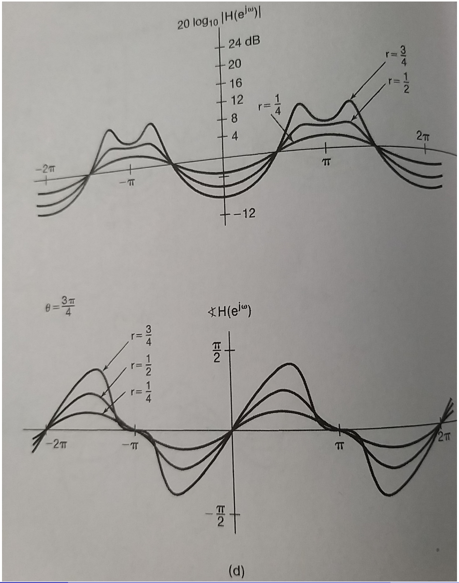
Magnitude phase plot showing response for $\theta = \pi/4$



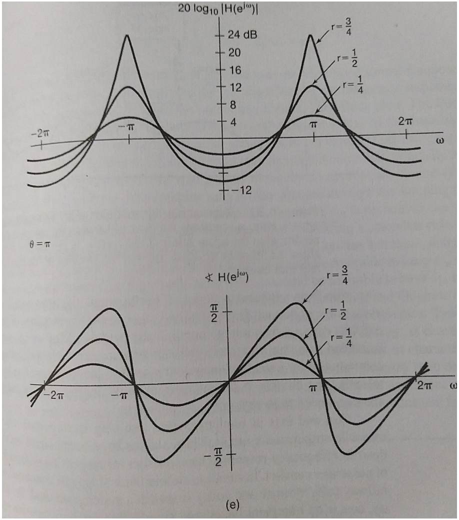
Magnitude phase plot showing response for $\theta = \pi/2$



Magnitude phase plot showing response for $\theta = 3\pi/4$



Magnitude phase plot showing response for $\theta = \pi$



Discrete domain filters

Butterworth FIlters

Sampling with impulse train

Let $x(t)$ be a bandlimited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$.
Pulse train

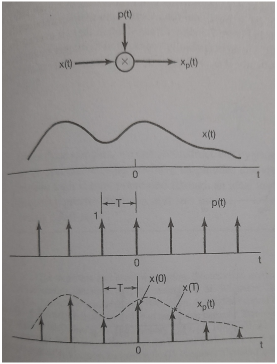
$$p(t) = \sum_{-\infty}^{+\infty} \delta(t - nT)$$

is used to modulate the signal.
Fundamental frequency of $p(t)$ is $\omega_s = \frac{2\pi}{T}$.
This gives pulse train modulated signal $x_p(t) = x(t)p(t)$
Multiplying $x(t)$ with unit impulse samples the signal value at the point where the signal is located $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$.
Hence

$$x_p(t) = \sum_{-\infty}^{+\infty} x(nT)\delta(t - nT)$$

.

Representation of impulse train sampled signals in time domain



Sample recovery with impulse train

Since

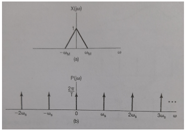
$$P(j\omega) = \frac{2\pi}{T} \sum_{-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

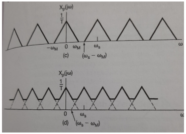
, and since convolution with signal shifts the frequency, It follows that

$$X_p(j\omega) = \frac{1}{T} \sum_{-\infty}^{+\infty} X(j(\omega - k\omega_s))$$

Here $X_p(j\omega)$ is a periodic function of ω with superposition of shifted replicas of $X(j\omega)$ scaled by $\frac{1}{T}$.
When $\omega_M < (\omega_S - \omega_M)$, i.e. $\omega_S > 2\omega_M$, there is no overlap between the shifted replicas. But with $\omega_S < 2\omega_M$, there is overlap.
Hence with $\omega_S > 2\omega_M$, $x(t)$ can be recovered from $x_p(t)$ by using a lowpass filter with cutoff frequency between ω_M and $\omega_S - \omega_M$.

Sampled signal recovery in frequency domain

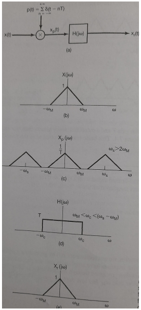




Sampling theorem

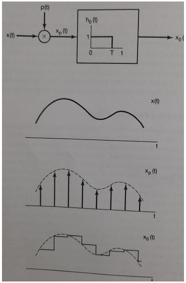
Let $x(t)$ be a bandlimited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$.
Then $x(t)$ is uniquely determined from its samples $x(nT)$,
$n = 0, \pm 1, \pm 2, \dots$ if $\omega_S > 2\omega_M$ where $\omega_S = \frac{2\pi}{T}$.
Given these samples, the signal can be reconstructed by generating a periodic impulse train with successive impulse amplitudes being the successive sample values.
This impulse train is passed through an ideal lowpass filter with gain T and cutoff frequecny greater than ω_M and less than $\omega_S - \omega_M$.
The resulting output would exactly match with input signal.

Samples retrieved from impulse trains



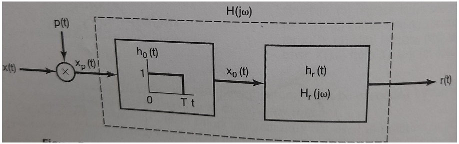
(a) System for sampling and reconstruction
(b) Representative spectrum for signal
(c) Corresponding spectrum for sampled (pulse train modulated) signal
(d) Ideal lowpass filter to recover signal from sampled signal
(e) Spectrum of recovered signal

Zero order hold based signal and system

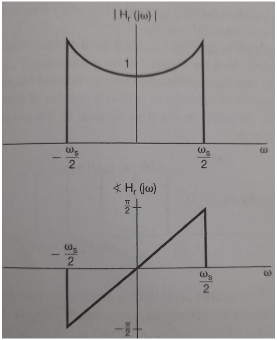


Impulse train sampling followed by an LTI system with rectangular impulse response

Zero order hold followed by Reconstruction



Signal reconstruction filter – magnitude and phase



Effect of aliasing

Conversions from continuous to discrete systems

Conversions from discrete to continuous systems

Band limited filters

Sampling of discrete time signals

Interpolation of Discrete time signals

Amplitude modulation (AM) with Exponential carrier

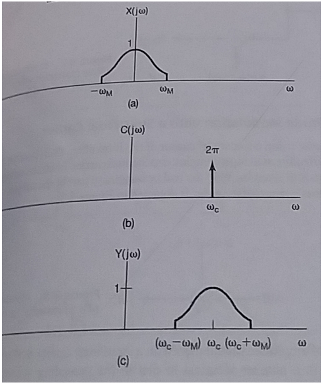
Signal $x(t)$ Carrier signal $c(t)$
Modulated signal $y(t) = x(t)c(t)$.
Exponential Carrier signal $c(t) = e^{j(\omega_c t + \theta_c)}$
Keeping $\theta_c = 0$,

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)C(j\omega - \theta)d\theta$$

With $C(j\omega) = 2\pi\delta(\omega - \omega_c)$ for complex exponential,
$Y(j\omega) = X(j\omega - j\omega_c)$.

Frequency Spectrum with Exponential carrier

For bandlimited signal with bandwidth $2\omega_M$, the spectrum of modulated signal is centered around ω_c with width $\omega_c \pm \omega_M$. See spectrum plots below (taken from book).



Demodulation with exponential carrier

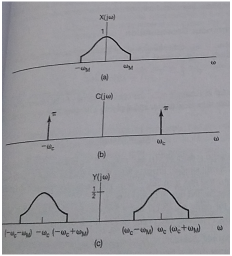
Demodulation can be achieved by shifting back in frequency. But when the phases are not synchronized, the process needs to be revisited.
Let $y(t) = e^{j(\omega_c t+\theta_c)}x(t)$ and $w(t) = e^{-j(\omega_c t+\phi_c)}y(t) = e^{j}(\theta_c - \phi_c)x(t)$ with $\theta_c \neq \phi_c$.
Magnitude of the demodulated signal corresponds with the original signal since $|w(t)| = x(t)$.

Amplitude modulation with Sinusoidal carrier

Sinusoidal Carrier signal $c(t) = cos(\omega_c t + \theta_c)$.
Keeping $\theta_c = 0$,

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) C(j\omega - \theta) d\theta$$

With $C(j\omega) = \pi(\delta(\omega - \omega_c) + \delta(\omega + \omega_c)$ for sinusoidal.
Hence $Y(j\omega) = X(j\omega - j\omega_c) + X(j\omega + j\omega_c)$.
Frequency Spectrum with Sinusoidal carrier plots below (taken from book).



It may be seen that Demodulation cannot be achieved by simple shift of

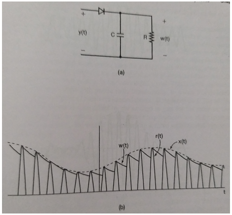
Demodulation- synchronous

Assuming $\omega_c > \omega_M$, signal can be recovered by performing modulation again followed by low pass filtering.
With $y(t) = x(t)cos\omega_c t$, modulation gives
$w(t) = y(t)cos\omega_c t = x(t)cos^2\omega_c t$.
Using trigonometry, $w(t) = \frac{1}{2}x(t) + \frac{1}{2}x(t)cos2\omega_c t$.
Applying lowpass filtering on the signal $w(t)$ the part $\frac{1}{2}x(t)$ can be recovered by eliminating the other part having higher frequency.
If phases are not synchronized, expression becomes
$w(t) = \frac{1}{2}cos(\theta_c - \phi_c)x(t) + \frac{1}{2}x(t)cos(2\omega_c t + \theta_c + \phi_c)$.
Oscillators used at either end should match in both phase and frequency to maximize the output upon demodulation.

Demodulation- asynchronous

Assuming $\omega_c >> \omega_M$, signal can be recovered by performing envelope detection using diode half wave rectifier circuit which may be followed by low pass filtering to remove the ripples.

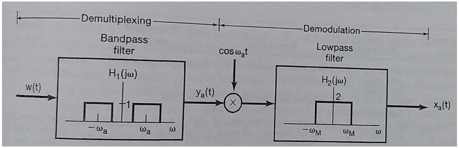
For this system to work, $x(t)$ must be positive and vary much slowly compared to the carrier frequency. To ensure the positive values, $y(t) = (A + x(t))cos\omega_c t$, together with $x(t) \leq K$ the requirement is $A > K$. The ratio $m = K/A$ is called the modulation index.



Demodulation- tradeoffs

The envelope detector based receiver circuit requires less complex (less costly as well) hardware at the receiver end than the synchronous oscillators, but the power requirement at the transmitting end is much higher.

Typical audio frequencies range from 20 Hz to 20 KHz. When modulated using radio frequency ranging from 500 KHz to 2 MHz as done in AM radio transmission, high power transmitter and mass produced low cost receivers are quite effective.



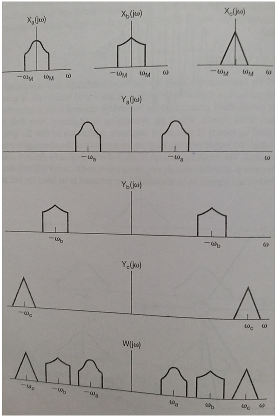
Frequency division multiplexing (FDM)

Different carriers may be used for transmitting different signals over the same wideband channel. Modulated signals do not overlap.

$w(t) = x_a(t)cos(\omega_a t) + x_b(t)cos(\omega_b t) + x_c(t)cos(\omega_c t)$

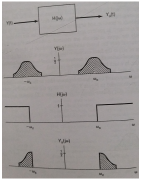
It is also easy to recover individual signals by first employing bandpass filter corresponding to specific frequency and followed by demodulation process. Different portions of the overall frequency range must be controlled for the purpose of transmission. Telephone as well as radio transmission can be done in different segments as per allotment. Receiver circuits and also attenuation varies according to the band of frequency used for the transmission and reception of specific signals.

FDM spectrum explained



Single sideband (SSB) vs Double Sideband (DSB)

Since the spectrum extends symmetrically to both sides of the origin, the bandwidth becomes doubled, which is inefficient.
Sharp cutoff highpass or bandpass filter used for removing one sideband, converting DSB to SSB, with carrier (WC) or suppressed carrier (SC).
In phase shifting technique, 90^o phase shifter can be employed.
Here, the modulator output is $y(t) = x(t)cos(\omega_c t) + x(t)h(t)sin(\omega_c t)$
To retain LSB, employ $h(t)$ so that $H(j\omega) = \pm j$ for $\omega > 0$ and $\omega < 0$
To retain USB, employ $H(j\omega) = +j$ for $\omega > 0$ and $H(j\omega) = -j$ for $\omega < 0$
Demodulation is unaffected, only modulator cost increases.



Pulse modulation - Time division multiplexing (TDM)

Modulation with pulse train carrier gives Fourier coefficients $a_k = \frac{sin\ (k\omega_c\Delta/2)}{\pi k}$ and modulated signal spectrum is

$$Y(j\omega) = \sum_{k=-\infty}^{+\infty} a_k X(j(\omega - k\omega_c))$$

.

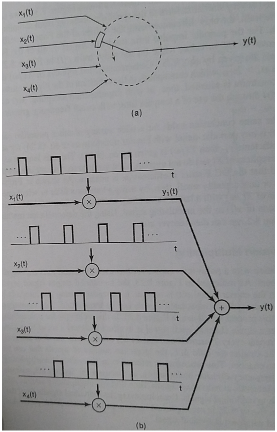
It is obvious that the spectrum replicas do not overlap as long as $\omega_c > 2\omega_M$ the Nyquist sampling rate. So using lowpass filter with cutoff frequency $> \omega_M$ and $< \omega_c - \omega_M$.

This applies to any periodic signal, yielding a sinusoidal AM signal which can then be demodulated.

This concept can be used to perform TDM. Using the pulse ON period to transmit one signal, the pulse OFF times may be used to modulate other signals using different pulses.

Each signal is thus allotted a separate time slot in the same channel for transmission. Time gating is used for receiving particular signal.

TDM schematics



Pulse amplitude modulation (PAM)

Use samples $x(nT)$ to modulate the amplitude of a sequence of pulses
Rectangular pulse of duration Δ may be used for sample and hold purpose.
TDM can be easily used for multiple PAM signals. Take
$y(t) = Ax_1(t)$ for $t = 0, \pm 3T_1, \pm 6T_1, \ldots$.
$y(t) = Ax_2(t)$ for $t = T_1, T_1 \pm 3T_1, T_1 \pm 6T_1, \ldots$.
$y(t) = Ax_3(t)$ for $t = 2T_1, 2T_1 \pm 3T_1, 2T_1 \pm 6T_1, \ldots$.
Here Intersymbol spacing is $T_1 = T/3$ leading to interference.
Distortion may be present in channel due to additive noise leading to amplitude error. Distortion due to filtering for frequency domain may lead to overlap in time domain.

Pulse coded modulation (PCM)

In frequency domain, use of bandlimited pulse is solution to distortion. In time domain design pulse such that there is zero crossing for other signals when a specific pulse is considered. $p(t) = \frac{T_1 sin(\pi t/T_1)}{\pi t}$ with samples taken at $t = kT_1$ such that $p(t - nT_1) = 0$ for $m \neq k$.
The $x[n]$ is quantized set of values having finite set of possible amplitudes, often restricted to binary. Encoding may be used to add redundancy for tackling noise that may lead to flipping of symbols.
PAM system modulated by encoded binary sequences is called PCM.

Basics of Frequency modulation (FM)

Modulating signal is used to control the frequency of the sinusoidal carrier signal.

In AM power/energy varies over large dynamic range due to amplitude fluctuation. In FM, the energy envelope is fixed so that the transmitter can always operate at peak power.

Another advantage of FM is that the distortions due to additive channel noise or fading can be eliminated easily at the receiver side. So reception of FM is better than AM.

FM requires much higher bandwidth since the frequency gets modulated to spread out the spectrum.

FM systems are also nonlinear and complex to analyze than the AM system.

Angle modulation - phase and frequency

Carrier of the form $c(t) = Acos(\omega_c t + \theta_c)$.
Here the modulation takes the form $\theta_c(t) = \theta_0 + k_p x(t)$
so the phase of carrier is function of time and called phase modulation.
Another form of modulation is Carrier of the form $c(t) = Acos\theta(t)$.
$\frac{d\theta(t)}{dt} = \omega_c + k_f x(t)$ so the frequency is offset from carrier frequency proportional to amplitude of the signal. This is called frequency modulation.
Now instantaneous frequency $\frac{d\theta(t)}{dt} = \omega_c + k_p \frac{dx(t)}{dt}$ for phase modulation or $= \omega_c + k_f x(t)$ for frequency modulation connects the two modulation types.
Frequency modulating with $x(t)$ is phase modulating with $\int x(t)$ i.e. integral of $x(t)$.
Phase modulating with $x(t)$ is frequency modulating with $\frac{dx(t)}{dt}$ i.e. derivative of $x(t)$.

Narrowband FM modulation

Consider signal $x(t) = Acos\omega_m t$
Instantaneous frequency is $\omega_i(t) = \omega_c + k_f Acos\omega_m t$ varies within $\omega_c \pm k_f A$ so that $\Delta\omega = k_f A$.
Then $y(t) = cos[\omega_c t + \int x(t)dt = cos(\omega_c t + \frac{\Delta\omega}{\omega_m} sin\omega_m t)$ with zero as integration constant.
The factor $m = \frac{\Delta\omega}{\omega_m}$ is called the FM index. Sufficiently small $m << \pi/2$ represents narrowband FM.
With some trigonometry, $y(t) \approx cos\omega_c t - m(sin\omega_m t)(sin\omega_c t)$.
Spectrum is somewhat similar to AM-DSB/WC which has $y(t) = cos\omega_c t + m(cos\omega_m t)(cos\omega_c t)$.

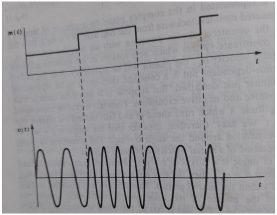
Wideband FM modulation

Now $y(t) = cos(\omega_c t)cos(msin\omega_m t) - sin(\omega_c t)sin(msin\omega_m t)$ for large m.
Fourier Coefficients for this type of periodic function is Bessel function of first kind multiplied with the carrier sinusoids.
The spectrum has impulses at $\pm\omega_c + n\omega_m$ where $n = 0, \pm 1, \pm 2, \ldots$.
For $|n| > m$ the harmonics are negligible so that there is bandlimiting to $B \approx 2m\omega_m \approx 2\Delta\omega \approx 2k_f A$.
Bandwidth of modulated signal is much larger than modulating signal.
Bandwidth of transmitted signal is directly proportional to amplitude of modulating signal and gain factor k_f.

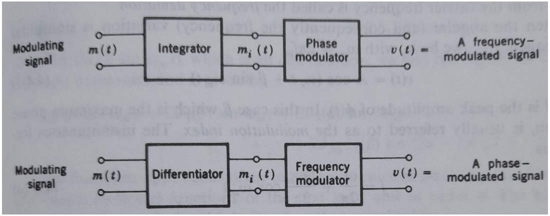
Square wave periodic modulation

Modulating signal is a periodic square wave. Let $k_f = 1$ so that $\Delta\omega = A$
Instantaneous frequency is $\omega_c + \Delta\omega$ for positive side of $x(t)$ and $\omega_c - \Delta\omega$ for negative side.
So $y(t) = r(t)cos[(\omega_c + \Delta\omega)t] + r(t - \frac{T}{2})cos[(\omega_c - \Delta\omega)t]$ where $r(t)$ is the periodic square wave.
Hence $Y(j\omega) = \frac{1}{2}[R(j\omega + j\omega_c + j\Delta\omega) + R(j\omega - j\omega_c - j\Delta\omega) + \frac{1}{2}[R_T(j\omega + j\omega_c - j\Delta\omega) + R_T(j\omega - j\omega_c + j\Delta\omega)$.
Using results for symmetric square wave,
$R(j\omega) = \Sigma \frac{2}{2k+1}(-1)^k \delta[\omega - \frac{2\pi(2k+1)}{T}] + \pi\delta(\omega)$ and
$R_T(j\omega) = R(j\omega)e^{-j\omega T/2}$.
The spectrum has two sidebands around $\omega_c \pm \Delta\omega$ alike wideband FM.

FM waveform explained



Frequency modulation and phase modulation blocks



FM demodulation

- Convert FM signal to AM signal through differentiation.
- Directly track the phase of the modulated signal.
- Directly track the frequency of the modulated signal.

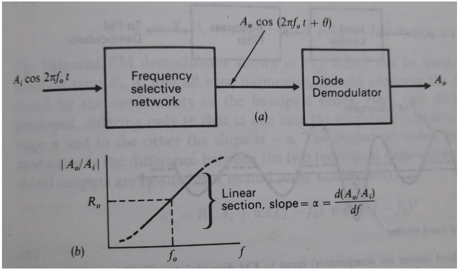
FM demodulation scheme

Waveform of frequency f_0 and input amplitude A_i applied to frequency selective network (typically LC circuit) giving output amplitude A_0. The ratio $\frac{A_0}{A_i}$ represents $|H(j\omega)|$ of the network.
This output is then applied to a diode AM demodulator which generates an output which is the peak of the Sinusoidal input that is A_0.
When the input waveform frequency is time varying or modulated, then even for fixed amplitude A_i the output amplitude is not fixed due to the frequency selectivity. Correspondingly the diode output would follow the output amplitude A_0. So the output would be amplitude modulated, to which the diode demodulator responds.

FM demodulation for linear frequency deviation

The requirement is that the instantaneous output signal A_0 should be proportional to the instantaneous frequency deviation $(f - f_0)$ of the received signal. The linear transfer characteristics with slope α and R_0 when $f = f_0$ gives $A_0 = R_0 A_i + \alpha A_i (f - f_0)$.

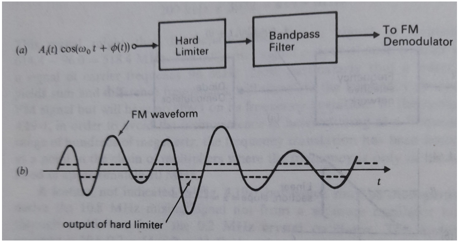
FM linear demodulation explained



FM demodulation to tackle noise

Adding a hard limiter at the input helps in providing steadiness to the input A_i in presence of additive noise. Otherwise the output starts to depend on the fluctuations of the input amplitude as well. Due to the limiter, the waveform becomes square wave. Hence a bandpass filter is inserted to extract first harmonic f_0. Then the signal is passed through the demodulator.

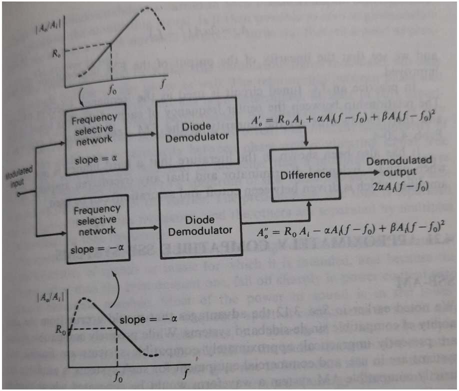
FM demodulator with limiter explained



Balanced FM demodulation

In practical scenario, the signal is $A_0 = R_0 A_i + \alpha A_i(f - f_0) + \beta A_i(f - f_0)^2 + \ldots$. This calls for balanced demodulator employing a positive and a negative slope and taking their difference. This difference output becomes $A_o = 2\alpha A_i(f - f_0)$. Thus the linearity of output is improved.

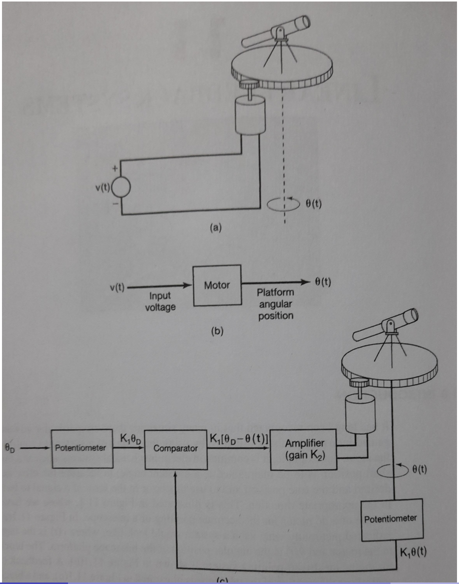
FM balanced demodulation explained



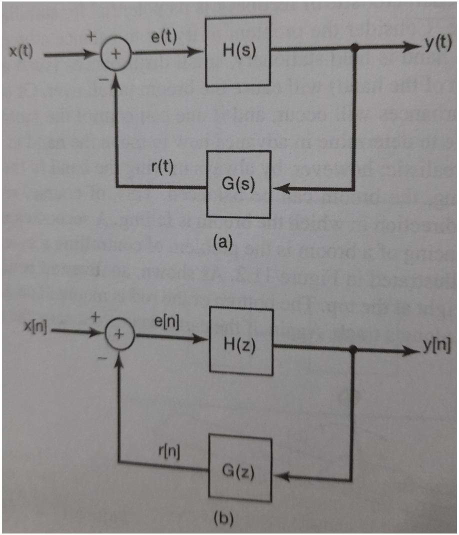
Discrete time modulation

For modulation, $y[n] = x[n]c[n]$ with $c[n] = e^{j\omega_c n}$.
Frequency spectrum of modulated signal $Y(e^{j\omega}) = X(e^{j(\omega-\omega_c)})$
Demodulation is done by translating back in frequency axis by multiplying again with $e^{-j\omega_c n}$.
With sinusoidal carrier $c[n] = cos\omega_c n$ the modulated signal replicates $X(j\omega)$ at $\omega = \pm\omega_c + k2\pi$
To avoid overlap, $\omega_c > \omega_M$ and $2\pi - \omega_c - \omega_M > \omega_c + \omega_M$ due to inherent periodicity of discrete domain signals.
Taken together, $\omega_M < \omega_c < \pi - \omega_M$ is the restriction on carrier frequency to avoid overlap.
Lowpass filtering may be used for demodulation purpose, higher order replications get removed.

Tracking control assembly



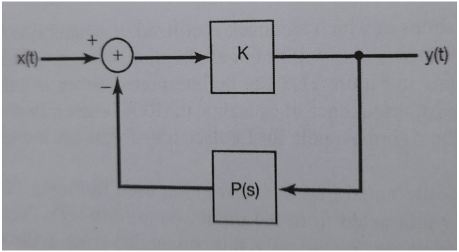
Linear feedback control



Linear feedback

$e(t) = x(t) - r(t)$ Taking LT, $E(s) = X(s) - R(s)$
$Y(s) = H(s)E(s)$ in the forward path. (Convolution)
$R(s) = G(s)Y(s)$ in the feedback path. (Convolution)
$H(s)X(s) = Y(s) + Y(s)G(s)H(s)$ after some manipulation.
Hence transfer function $Q(s) = \frac{Y(s)}{X(s)} = \frac{H(s)}{1+G(s)H(s)}$
Similarly, transfer function $Q(z) = \frac{Y(z)}{X(z)} = \frac{H(z)}{1+G(z)H(z)}$

Inverse system using feedback



Inverse system and error compensation using feedback

Inverse systems

Consider a large gain K along the forward path.

For very large gain, the transfer function of the arranged block diagram is inverse of the transfer function in the feedback path.

$Q(s) = \frac{K}{1+KP(s)} = \frac{1}{P(s)+\frac{1}{K}} \approx \frac{1}{P(s)}$

Compensation for errors:

Here $Q(j\omega) = \frac{H(j\omega)}{1+KH(j\omega)}$

When $|KH(j\omega)| \gg 1$ then $Q(j\omega) \approx \frac{1}{K}$

Stabilization of unstable systems - first order

First order system $H(s) = \frac{b}{s-a}$ and gain K

Then $Q(s) = \frac{H(s)}{1+KH(s)} = \frac{b}{(s-a+Kb)}$

Original pole on right half plane. New pole is in left half when $Kb > a$

This is called proportional feedback.

Now consider second order system $H(s) = \frac{b}{s^2+a}$ and keep $G(s) = K$.

Then $Q(s) = \frac{b}{s^2+(a+Kb)}$

This gives control only over the natural frequency part.

Stabilization of unstable systems - second order PD

Now consider $G(s) = K_p + K_d s$
that is proportional plus derivative feedback.
Then $Q(s) = \frac{b}{s^2+bK_d s+(a+K_p b)}$
This gives control over the natural frequency as well as damping.
For stability, $bK_d > 0$ and $a + K_p b > 0$.

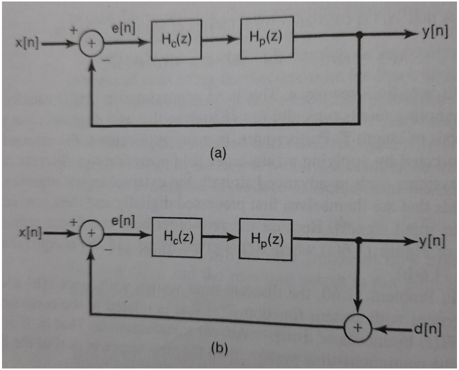
Stabilization of unstable systems - second order PI

Now consider second order system $H(s) = \frac{bs}{s^2+a}$ and keep $G(s) = K$.
Take $G(s) = K_p + K_i/s$
that is proportional plus integral feedback.
Then $Q(s) = \frac{bs}{s^2+bK_ps+(a+K_ib)}$
This gives control over the natural frequency as well as damping.
For stability, $bK_p > 0$ and $a + K_ib > 0$.

Discrete system data feedback

In Population dynamics , Take $y[n] = 2y[n-1] - 2\beta y[n-1] + x[n]$
The factor 2 resembles growth term from previous population,
The factor β is regulated depletion term proportional to previous population,
and $x[n]$ is random addition in the instant n.
Taking z transform, $Y(z) = 2(1-\beta)z^{-1}Y(z) + X(z)$
Hence the transfer function is $\frac{Y(z)}{X(z)} = \frac{1}{1-2(1-\beta)z^{-1}}$
Analysis of the above TF gives for $\beta < 1/2$ system is unstable. For $\beta > 0.5$ upto 1 it is stable as it is a fraction.

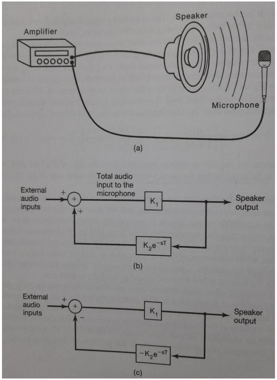
Disturbance rejection and tracking control assembly



Tracking systems

Take product of plant $H_p(z)$ and compensator $H_c(z)$ as $H(z)$.
Then $Y(z) = \frac{H(z)}{1+H(z)}X(z)$ and error TF is $Y(z) = H(z)E(z)$
Hence $E(e^{j\omega}) = \frac{1}{1+H(e^{j\omega})}X(e^{j\omega}) \approx 0$
When gain $|H(e^{j\omega})|$ is very large, error converges to zero.
Large gain needed for good tracking.
With disturbance, $Y(z) = \frac{H(z)}{1+H(z)}X(z) - \frac{H(z)}{1+H(z)}D(z)$
To reject disturbance, $H(z)$ should be small, contradicting with tracking requirements.

Positive feedback and destabilization



Destabilization due to positive feedback

Let K_f (K_2) be feedback path attenuation.
Let K_a (K_1) be amplifier gain.
Then we have $Q(s) = \frac{K_a}{1 - K_f K_a e^{-sT}}$
Here T sec time delay has been assumed.
Since $K_a K_f > 1$ this leads to positive feedback, destabilising the system.
Here K_f attenuation depends on proximity of microphone to the speaker and
Destabilising results in excessive amplification and distortions of audio signal.

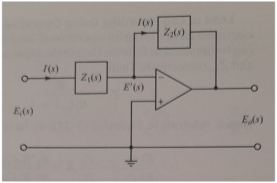
State variable representation

Let $x_1, x_2, x_3, \dots, x_n$ be the state variables of a system and the state vector is denoted as $x(t)$ in continuous time domain at time t. Also, $u(t)$ is the input variable at time t.
Then the system can be described in terms of differential equations involving the rate of change of state variables with respect to time.
$\dot{x} = Ax(t) + Bu(t)$

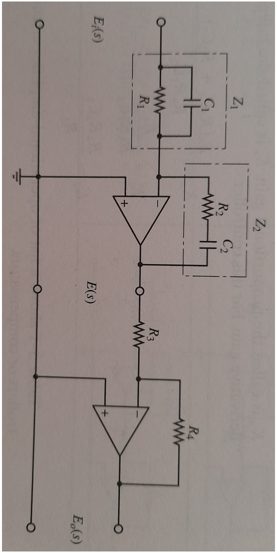
State variable - example

$L\dot{i} + Ri + v_c = v$ and $C\dot{v_c} = i$ describes the differential equations governing a LRC circuit.
Here, $(\dot{i}) = \frac{di}{dt}$ and $(\dot{v_c}) = \frac{dv_c}{dt}$ are the rate of change of current and voltage across the capacitance respectively.
Taking $x_1 = i(t)$ and $x_2 = v_c(t)$ with $u(t) = v$ applied voltage; one can write $\dot{x} = Ax(t) + Bu(t)$
$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$

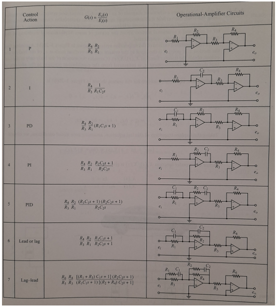
Opamp impedance based analysis



Opamp impedance based analysis



Opamp impedance based summary



	Control Action	$G(s) = \frac{E_o(s)}{E_i(s)}$	Operational-Amplifier Circuits
1	P	$\frac{R_4}{R_3}\frac{R_2}{R_1}$	
2	I	$\frac{R_4}{R_3}\frac{1}{R_1C_2s}$	
3	PD	$\frac{R_4}{R_3}\frac{R_2}{R_1}(R_1C_1s+1)$	
4	PI	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_2C_2s+1}{R_2C_2s}$	
5	PID	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{(R_1C_1s+1)(R_2C_2s+1)}{R_2C_2s}$	
6	Lead or lag	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_1C_1s+1}{R_2C_2s+1}$	
7	Lag–lead	$\frac{R_6}{R_5}\frac{R_4}{R_3}\frac{[(R_1+R_3)C_1s+1](R_2C_2s+1)}{(R_1C_1s+1)[(R_2+R_4)C_2s+1]}$	

Common example systems for simulation

Electric Motor Transfer Function: When voltage is applied, $V = L_a \frac{di}{dt} + R_a i$ in the armature.
Due to magnetic stator, electromagnetic torque proportional to the current $T = K_T i$ is produced.
Due to rotor moment of inertia, angular speed is produced as per rotational force equation $T = J\dot{\omega} + f\omega$ with moment of inertia and friction.
Now take $x_1 = i$ and $x_2 = \omega$ as state variables, $u = V$ is the applied input voltage and $y = \omega$ is the output angular speed.
$A = \begin{bmatrix} -\frac{R_a}{L_a} & 0 \\ \frac{K_T}{J} & \frac{-f}{J} \end{bmatrix}$ and $B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$ starting with zero initial condition.
Spring mass system: $\ddot{x} = -Kx - \mu\dot{x}$ where K is spring constant and μ is friction. Here take $x_1 = x$ position and $x_2 = \dot{x}$ velocity as state variables.
$A = \begin{bmatrix} 0 & 1 \\ -K & -\mu \end{bmatrix}$ The mass is pulled away from zero initial condition, resemblance of force being applied.

State Space methods in continuous domain

Consider n^{th} order differential equation describing a system:
$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-2}\ddot{y} + a_{n-1}\dot{y} + a_n y = u$
Knowledge of $y(0), \dot{y}(0), \ddot{y}(0), \ldots, y^{(n-1)}(0)$ together with $u(t)$ for $t \geq 0$ determines completely the future behaviour of the system.
Hence $y(t), \dot{y}(t), \ddot{y}(t), \ldots, y^{(n-1)}(t)$ is a set of state variables.

Choice of state variables

Mathematically convenient to express the system as a set of simultaneous first order differential equations involving $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$, . . . , $x_n = y^{(n-1)}(t)$ i.e.

$\dot{x}_1 = x_2$; $\dot{x}_2 = x_3$; $\dot{x}_{n-1} = x_n$

and $\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \ \ldots \ - a_1 x_{n-1} + u$

Now, one can express the system as $\dot{x} = Ax(t) + Bu(t)$; $y = Cx(t)$ where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

$C = [1 \ 0 \ 0 \ldots 0]$.

Choice of state variables - Example

Consider a system: $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$

Then $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$

$B = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$,

$C = [1\ 0\ 0]$

Nonuniqueness of state variables

Write the above system in s domain transfer function (TF):

$\frac{Y(s)}{U(s)} = \frac{6}{(s^3+6s^2+11^s+6)}$

Expanding partial fractions, above

$TF = \frac{3}{s+1} - \frac{6}{s+2} + \frac{3}{s+3} = \frac{X_1(s)+X_2(s)+X_3(s)}{U(s)}$

Hence, $\dot{x}_1 = -x_1 + 3u$; $\dot{x}_2 = -2x_2 - 6u$; $\dot{x}_3 = -3x_3 + 3u$ and $y = x_1 + x_2 + x_3$.

Then $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

$B = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$,

$C = [1\ 1\ 1]$

Multiple forcing function

$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}\dot{y} + a_n y =$
$b_0 u^{(n)} + b_1 u^{(n-1)} + \ldots b_{n-2}\ddot{u} + b_{n-1}\dot{u} + b_n u$
Now, for unique solution, derivatives of u must be eliminated from state equations. Define:
$x_1 = y - \beta_0 u$; $x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$;
$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$; $\ldots x_n = \dot{x}_{n-1} - \beta_{n-1} u$
Solving, we get $\beta_0 = b_0$; $\beta_1 = b_1 - a_1\beta_0$; $\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0$;
$\beta_n = b_n - a_1\beta_{n-1} - \ldots - a_{n-1}\beta_1 - a_n\beta_0$.
Hence, A = same as earlier; C = same as earlier
$B = [\beta_1; \beta_2; \ldots; \beta_{n-1}; \beta_n]^T$;$D = \beta_0 = b_0$.
Derivatives only affect the B matrix.
The Transfer function has introduced zeroes.

$$\frac{Y(s)}{U(s)} = \frac{(b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1}s + b_n)}{(s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n)}$$

Multiple forcing function - Example

$\dddot{y} + 18\ddot{y} + 192\dot{y} + 640y = 160\dot{u} + 640u$

$s^3 Y(s) + 18s^2 Y(s) + 192sY(s) + 640Y(s) = 160sU(s) + 640U(s)$

Then $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -640 & -192 & -18 \end{bmatrix}$

$B = \begin{bmatrix} 0 \\ 160 \\ -2240(640 - 18 \times 160) \end{bmatrix}$,

$C = [1\ 0\ 0]$; $D = [640]$

Solution of the homogeneous state equation

$\dot{x} = Ax$ solves to $x = exp(At)x(0) = x(0)(I + At + \ldots + \frac{1}{k!}A_k t^k + \ldots)$ converges for all t.
Now $exp(-At)$ is inverse of $exp(At)$.
Derivative of $exp(At)$ is $Aexp(At)$
Laplace transform of $\dot{x} = Ax$ is $sX(s) - x(0) = AX(s)$
Simplifying, $X(s) = (sI - A)^{-1}x(0)$
Then, taking inverse Laplace transform, $x(t) = L^{-1}[(sI - A)^{-1}]$
Now $(sI - A)^{-1} = I/s + A/s^2 + A^2/s^3 + \ldots$
Then $L^{-1}[(sI - A)^{-1}] = I + At + (1/2!)A^2t^2 + (1/3!)A^3t^3 + \ldots = exp(At)$

State transition matrix

Known as $\phi(t)$
$x(t) = \phi(t)x(0)$ i.e. $\phi(t) = exp(At)$
$\phi(t_1 + t_2) = exp(A(t_1 + t_2)) = exp(A(t_1))exp(A(t_2)) = \phi(t_1)\phi(t_2)$
$\phi(t)$ contains the Eigenvalues $exp(\lambda_i t)$ for nonrepetitive roots.
$\phi(t)$ contains $texp(\lambda_j t), \ldots, t^m exp(\lambda_j t)$ for m times repeating roots.
Also, $(\phi(t))^n = \phi(nt)$ and
$\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0) = \phi(t_1 - t_0)\phi(t_2 - t_1)$
This shows the capability of state transition matrix as it simply transforms the initial condition.

State transition matrix - Example

Let $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ Then $sI - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$

Hence $(sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & +1 \\ -2 & s \end{bmatrix}$

Hence $(sI - A)^{-1} = \begin{bmatrix} \frac{2(s+2)-(s+1)}{(s+1)(s+2)} & \frac{(s+2)-(s+1)}{(s+1)(s+2)} \\ \frac{2(s+1)-2(s+2)}{(s+1)(s+2)} & \frac{2(s+1)-(s+2)}{(s+1)(s+2)} \end{bmatrix}$

Therefore, taking partial fraction of each matrix element,

$L^{-1}[(sI - A)^{-1}] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$

Note that $\phi^{-1}(t) = \phi(-t)$ gives an easy way of inverting the state transition matrix.

Nonhomogeneous state equation

$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$
While taking inverse Laplace transform, convolution integral is required.
$x(t) = \phi(t)x(0) + \int_{t_0} exp(A(t-\tau)Bu(\tau)d(\tau)$
In the above example, take $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with $u(t) = 1$ (unit step) when $x(0) = 0$ to get Upon simplification, $x(t) = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$

Relation between time and frequency domain

$(sI - A)X(s) = BU(s)$ and $Y(s) = CX(s) + DU(s)$ so that
$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$
For example,
$\dot{x}_1 = -5x_1 - x_2 + 2u;$
$\dot{x}_2 = 3x_1 - x_2 + 5u\ ;$
and $y = x_1 + 2x_2$
From this, we get $G(s) = \frac{(12s+59)}{(s+2)(s+4)}$

Phase variable approach in discrete domain

Phase variables: $x_1(k) = x(k)$;$x_2(k) = x_1(k+1)$;. . .;$x_n(k) = x_1(k+n-1)$
$X_2(z) = zX(z)$;$X_3(z) = zX_2(z) = z^2X(z)$etc.
This gives

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -d_0 & -d_1 & -d_2 & \dots & \dots & \dots & \dots & \dots & \dots & -d_{n-1} \end{bmatrix},$$

$H^T = [0, 0...0, K_D]$; $C = [1, 0, ..., 0]$
Ex. $y(k+2) + y(k+1) + 0.16y(k) = u(k+1) + 2u(k)$
State variables: $x_1(k) = y(k)$ and $x_2(k) = x_1(k+1) - u(k)$
To consider $x(k+1) = Gx(k) + Hu(k)$ and $y(k) = Cx(k) + Du(k)$ to get

$$G = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}, \ H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \ C = [1 \ 0]; \ D = 0.$$

Connection of discrete domain with continuous domain

$T\dot{x} = x(k+1) - x(k)$ as $T \to 0$ is the connection between continuous and discrete domain.

Consider n^{th} order difference equation describing a system:

$y(k+n) + a_1 y(k+n-1) + \ldots + a_{n-1} y(k+1) + a_n y(k) =$
$b_0 u(k+n) + b_1 u(k+n-1) + \ldots + b_{n-1} u(k+1) + b_n u(k)$

Choose $x_1(k) = y(k) - h_0 u(k)$

$x_2(k) = x_1(k+1) - h_1 u(k)$

$x_3(k) = x_2(k+1) - h_2 u(k)$

$\vdots$

$x_n(k) = x_{n-1}(k+1) - h_{n-1} u(k)$

Solution for the coefficients

$h_0 = b_0$; $h_1 = b_1 - a_1 h_0$;$h_2 = b_2 - a_1 h_1 - a_2 h_0$;
$h_n = b_n - a_1 h_{n-1} \ldots - a_{n-1} h_1 - a_n h_0$

$$G = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{bmatrix}, H = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-1} \\ h_n \end{bmatrix},$$

$C = [1\ 0\ 0 \ldots 0]$ and $D = h_0 = b_0$.

Canonical variable approach in discrete domain

Start with $G(z) = Y(z)/U(z)$. Consider distinct poles p_i.
Use partial fraction method to get $G(z) = K_D + \Sigma \frac{A_i}{z-p_i}$
Now take $X_i(z) = U(z)/z - p_i$
$zX_i(z) - p_iX_i(z) = U(z)$ so that $x_i(k+1) = p_ix_i(k) + u(k)$

$$F = \begin{bmatrix} p_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & p_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & p_n \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix},$$

$C = [A_1\ A_2 \dots A_n]$ and $D = K_D$.

Canonical variable approach for repetitive poles

If the pole p_1 repeats q times, the partial fraction expansion will contain $\frac{A_{1,q}}{(z-p_1)^q}$, $\frac{A_{1,q-1}}{(z-p_1)^{q-1}}$,$\dots$, $\frac{A_{1,2}}{(z-p_1)^2}$, $\frac{A_{1,1}}{(z-p_1)}$
followed by $\frac{A_2}{(z-p_2)}$, $\frac{A_3}{(z-p_3)}$, $\dots$, $\frac{A_{n-q+1}}{(z-p_{n-q+1})}$

$X_1(z) = \frac{U(z)}{(z-p_1)}$, $X_2(z) = \frac{U(z)}{(z-p_1)^2}$, $\dots$, $X_{q-1}(z) = \frac{U(z)}{(z-p_1)^{q-1}}$,

$X_q(z) = \frac{U(z)}{(z-p_1)^q}$

$X_{q+1}(z) = \frac{U(z)}{(z-p_2)}$, $X_{q+2}(z) = \frac{U(z)}{(z-p_3)}$, $\dots$, $X_n(z) = \frac{U(z)}{(z-p_{n-q+1})}$

Canonical variable solution for repetitive poles

$$F = \begin{bmatrix} p_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & p_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & p_1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & 1 & p_1 & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & p_2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & p_3 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & p_{n-q+1} \end{bmatrix}, H = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

$C = [A_{11}\ A_{12} \dots A_{1q}\ A_2 \dots A_n]$ and $D = K_D$.
When s domain Eigenvalues are λ_i, Z domain gives $exp(\lambda_i T) = p_i$

Solution of discrete time state equations

$x(1) = Gx(0) + Hu(0);$
$x(2) = Gx(1) + Hu(1) = G^2x(0) + GHu(0) + Hu(1)$
$x(3) = G^3x(0) + G^2Hu(0) + GHu(1) + Hu(2)$
Then, $x(k) = G^kx(0) + \Sigma_j G^{k-j-1}Hu(j)$
Consider state transition matrix $\phi(k) = G^k$
$\phi(k+1) = G\phi(k)$ and $\phi(0) = I$, so that
$x(k) = \phi(k)x(0) + \Sigma\phi(k-j-1)Hu(j) = \phi(k)x(0) + \Sigma\phi(j)Hu(k-j-1)$
Output is $y(k) = C\phi(k)x(0) + C\Sigma\phi(j)Hu(k-j-1) + Du(k)$

Getting to State Variable representation from Zdomain TF

$G(z) = \frac{Y(z)}{U(z)} = \frac{Y(z)}{X(z)} \frac{(X(z)}{U(z)}$
$= \frac{(z_w + f_{w-1}z^{w-1} + ... + f_1 z + f_0) * K_D}{(z^n + d_{n-1}z^{n-1} + ... + d_1 z + d_0)}$
$K_D U(z) = z^n X(z) + d_{n-1}z^{n-1}X(z) + \ldots + d_1 zX(z) + d_0 X(z)$
$Y(z) = f_0 X(z) + f_1 zX(z) + ... + f_{w-1}z^{w-1}X(z) + z^w X(z)$
Since $Z^{-1}[z^n E(z)] = e[(k+n)T]$; taking Z^{-1} of the above expressions,
We get $x(k+n) + d_{n-1}x(k+n-1) + \ldots + d_1 x(k+1) + d_0 x(k) = K_D u(k)$
And $y(k) = f_0 x(k) + f_1 x(k+1) + \ldots + f_{w-1}x(k+w-1) + x(k+w)$

Stability analysis with Bilinear transformation

Mapping the stable region of s-plane i.e. the left half plane into the inside of the unit circle around origin of z-plane fulfils the basic stability criterion. Consider $z = \frac{1+w}{1-w}$ or $w = \frac{z-1}{z+1}$ as transformation.
Then as $T \to 0$; putting $z = e^{sT}$; $(2/T)w \to s$ as higher order terms of T vanish in the limit. So, by taking $w' = (2/T)\frac{z-1}{z+1}$, it holds $w' \approx s$.
With new $w = (2/T)\frac{e^{sT}-1}{e^{sT}+1} = (2/T)\frac{e^{sT/2}-e^{-sT/2}}{e^{sT/2}+e^{-sT/2}} = (2/T)tanh(sT/2)$ without approximations.
For s-plane imaginary axis, substitute $s = j\omega_{sp}$, we get $w = \sigma_{wp} + j\omega_{wp}$
This helps in understanding the extent of approximation validity,

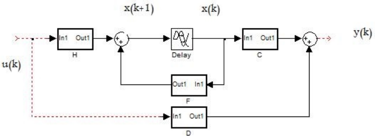
Pseudocontinuous time (PCT) control system

Pade Approximation: $F(x) \approx \frac{A_m(x)}{B_m(x)}$ is a rational fraction approximation of a differentiable power series of the form $c_0 + c_1x^1 + c_2x^2 + \ldots$ where $A_m(x)$, $B_m(x)$ are polynomials in x.
First order Pade approximation of $e^x = 1 + x + x^2/2 + \ldots$ yields $F(x) = \frac{a_0+a_1x}{b_0+b_1x} = \frac{1+x/2}{1-x/2}$ by solving for a,b by comparing coefficients.
Now, $G_{zoh}(s) = \frac{1-e^{-sT}}{s} \approx \frac{2T}{Ts+2}$ using Pade approximation of first order.
Together with the sampler gain $\frac{1}{T}$, the overall addition to the system TF is $\frac{2}{(Ts+2)}$
As $T \to 0$, $\omega_s \to \infty$; $2/(Ts+2) \to 1$.

Pseudocontinuous nature

Higher order approximation introduces more poles.
The main advantage is that all the analysis of continuous time system applies to the PCT.
Interestingly, the role of sampling time T is evident as pole introduced contains it.

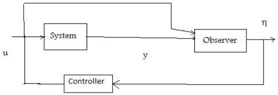
Block diagram of feedback control

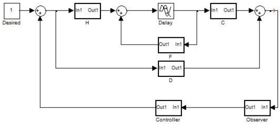


Feedback control in terms of state variables

Consider $x(k+1) = Fx(k) + Hu(k)$. Hence to take the system from arbitrary initial condition to final condition, the input sequence needs to be controlled. Instead of open loop manipulation of input sequence, the input sequence can be tuned using the state variables. This is feedback control. Negative Feedback control requires $u(k) = -Kx(k)$ where K is the gain matrix. Then the equation becomes $x(k+1) = Fx(k) - HKx(k)$. Taking Z transform. $zX(z) - x(0) = (F - HK)X(z)$.
Therefore, $X(z) = (zI - \overline{F - HK})^{-1}x(0)$. Hence the eigenvalues shift from that of $(zI - F)^{-1}$. Those poles that were outside unit circle can be brought within unit circle by taking appropriate gain matrix K.

Feedback observer and controller





Feedback control based on observations

Pole placement techniques are applied to tune the transient as well as steady state behaviour of the discrete domain systems.

For such feedback to work, the state variables must be observable.

Any measurement would be noise contaminated. Discrete domain filters must be implemented to tackle the noise. This opens up the scope of digital signal processing.

Form apriori Estimate $\hat{x}^{-}(k) = F\hat{x}^{+}(k-1)$ with F replacing $F - HK$.

Blend with measurement Z_k using proportion such that $\alpha + \beta = 1$ to get aposteriori estimate $\hat{x}^{+}(k) = \alpha\hat{x}^{-}(k) + \beta Z_k$

Choice of blending factor is part of stochastic filter design.

Block diagram of overall system in real time