Logic & Proofs (Lecture – 2)

Dr. Nirnay Ghosh

Propositional Equivalence

- An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value.
- Methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments.

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

TABLE 1 **Examples of a Tautology and a Contradiction.**

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical Equivalence

- Compound propositions that have the same truth values in all possible cases are called ***logically equivalent***
- Example: Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent

TABLE 3 Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

TABLE 2 De Morgan's Laws.

$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

- Example: Show that $p \longrightarrow q$ and $\neg p \vee q$ are logically equivalent

TABLE 4 Truth Tables for $\neg p \vee q$ and $p \to q$.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logical Equivalence

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \rightarrow q \equiv \neg p \vee q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

$p \vee q \equiv \neg p \rightarrow q$

$p \wedge q \equiv \neg(p \rightarrow \neg q)$

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing New Logical Equivalences

- Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent without using the truth table.

  $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$ (see previous example)

  $\equiv \neg(\neg p) \wedge \neg q$ (second De Morgan's law)

  $\equiv p \wedge \neg q$ (double negation law)

- Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

  $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ (second De Morgan law)

  $\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$ (first De Morgan law)

  $\equiv \neg p \wedge (p \vee \neg q)$ (double negation law)

  $\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ (second distributive law)

  $\equiv \mathbf{F} \vee (\neg p \wedge \neg q)$ (because $\neg p \wedge p \equiv \mathbf{F}$)

  $\equiv (\neg p \wedge \neg q) \vee \mathbf{F}$ (commutative law for disjunction)

  $\equiv \neg p \wedge \neg q$ (identity law for $\mathbf{F}$)

Propositional Satisfiability

- A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true.
- If a compound proposition is false for all assignments of truth values to its variables, then it is **unsatisfiable** => its negation is a *tautology*
- Determine whether each of the compound propositions $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$, and $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable.

Application of Satisfiability

- Many problems are modeled in terms of propositional satisfiability: *robotics, software testing, computer-aided design, machine vision, integrated circuit design, computer networking, genetics*, etc.
- We will discuss about modeling **Sudoku puzzles** using propositional satisfiability

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

- For each puzzle, some of the 81 cells, called **givens**, are assigned one of the numbers 1, 2, . . . , 9, and the other cells are blank.
- Assign a number to each blank cell so that *every row, every column, and every one of the nine 3 × 3 blocks contains each of the nine possible numbers*.
- Example: where to place 4?
  - One possibility: 2^{nd} row, 6^{th} column

Application of Satisfiability

- Let *p (i, j, n)* denote the proposition that is true when the number *n* is in the cell in the *i*-th row and *j*-th column
- We need to find truth assignments to 729 propositions *p (i, j, n)* with *i, j , and n* each ranging from 1 to 9 that makes the conjunction of all these compound propositions true
- Asserting every row contains every number: $\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, n)$
- Asserting every column contains every number $\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i, j, n)$
- Asserting each of the nine 3X3 blocks contains every number:

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r + i, 3s + j, n)$$

Application of Satisfiability

- Asserting each of the nine 3X3 blocks contains every number:

$$\bigwedge_{r=0}^{2}\bigwedge_{s=0}^{2}\bigwedge_{n=1}^{9}\bigvee_{i=1}^{3}\bigvee_{j=1}^{3} p(3r+i, 3s+j, n)$$

- To assert that no cell contains more than one number, we take the conjunction over all values of n, m, i, and j where each variable ranges from 1 to 9 and $n \neq m$ of $p(i, j, n) \rightarrow \neg p(i, j, m)$.

- Take conjunctions of all the listed assertions to find a solution to a given Sudoko puzzle.