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CHAPTER 10

Line Integrals, Double Integrals and Triple Integrals

10.1 INTRODUCTION

The concepts of line integrals, double integrals and triple integrals are basic in integral calculus and play an important role in the field of Science and Technology. The integration of a function along a curve is called the line integral of this function over this specified curve. Double integral involves integration of a function of two variables over a surface and triple integral involves integration of a function of three variables over a volume. This chapter deals with these three types of integrations of functions of more than one variable.

10.2 BASIC CONCEPTS

A *plane curve* in R^2 (two-dimensional space) is defined as a set of points:

$$C = \{(x, y) \in R^2 : x = f(t), y = g(t); a \le t \le b\}$$

where $f(t)$ and $g(t)$ are two functions of t defined in $a \le t \le b$.

This is also known as the **parametric representation** of the curve C.

If $f(t)$ and $g(t)$ are continuous in $a \le t \le b$, then the curve C is said to be *continuous*.

The curve C is said to be **closed** in $a \le t \le b$ if $f(a) = f(b)$ and $g(a) = g(b)$, *i.e.*, if the starting and end points coincide.

A point on C in called a **multiple point** if it passes through this point more than once. For example, if $f(t_1) = f(t_2)$, $g(t_1) = g(t_2)$, where $t_1 \ne t_2$, $a \le t_1, t_2 \le b$, then the points corresponding to t_1, t_2 are identical and known as a **double point** of the curve C.

The curve C is said to be **simple**, if it is continuous and does not intersect itself (*i.e.*, has no multiple points) except for a continuous closed simple curve which has only one multiple point (the end points).

The curve C is said to be **smooth**, if it is simple and $f'(t), g'(t)$ are continuous and non-zero in $a \le t \le b$, *i.e.*, it possesses a tangent that varies continuously along C and at every point it is specifying the direction of the curve C.

Note: The previous definitions can be extended to a space curve C in R^3 (three-dimensional space). In this case C is defined as a set of points:

$C = \{(x, y, z) \in R^3 : x = f(t), y = g(t), z = h(t);\ a \le t \le b\}$ where $f(t)$, $g(t)$, $h(t)$ are defined in $a \le t \le b$.

10.3 LINE INTEGRALS

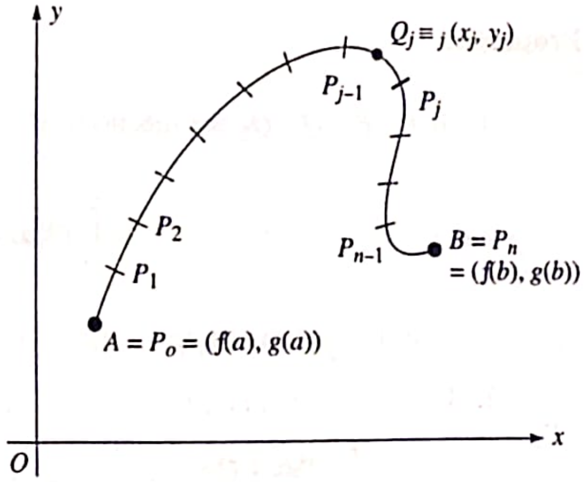
Let $\varphi(x, y)$ be a function whose domain contains a curve C defined by

$$C = \{(x,y) \in R^2 : x = f(t), y = g(t);\ a \le t \le b\}.$$

Therefore, $\varphi(x, y)$ is defined along C.

Let us divide C into n arcs having lengths Δs_j, $j = 1, 2, ..., n$. Choose an arbitrary point (x_j, y_j) on the j^{th} arc and form the sum

$$\sum_{j=1}^{n} \varphi(x_j, y_j)\Delta s_j.$$



If $\displaystyle\lim_{\substack{n\to\infty \\ \Delta s\to 0}} \sum_{j=1}^{n} \varphi(x_j, y_j)\Delta s_j$,

where $\Delta s = \max\limits_{1\le j\le n}\{|\Delta s_j|\}$, exists finitely, we call this limit the **line integral** of φ over C from $A(f(a), g(a))$, to $B(f(b), g(b))$ and write it as

$$\int_C \varphi(x, y)\, ds$$

Here $$\int_C \varphi(x, y)\, ds = \int_a^b \varphi(f(t), g(t))\sqrt{\{f'(t)\}^2 + \{g'(t)\}^2}\, dt \qquad ...(1)$$

Note:

(*i*) The previous definition can be extended to the line integral of $\varphi(x, y, z)$ over C in R^3, where $C = \{(x, y, z) : x = f(t), y = g(t), z = h(t);\ a \le t \le b\}$.

Also $$\int_C \varphi(x,y,z)\, ds = \int_a^b \varphi(f(t), g(t), h(t))\sqrt{\{f'(t)\}^2 + \{g'(t)\}^2 + \{h'(t)\}^2}\, dt. \qquad ...(2)$$

(*ii*) If the curve C is expressed as $y = F(x)$ from $x = \alpha_1$ to $x = \alpha_2$, then the line integral of $\varphi(x, y)$ over C is

$$\int_{\alpha_1}^{\alpha_2} \varphi(x, F(x))\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad ...(3)$$

Considering the equation of the curve as $x = G(y)$ from $y = \beta_1$ to $y = \beta_2$, then the line integral of $\varphi(x, y)$ over C becomes

$$\int_{\beta_1}^{\beta_2} \varphi(G(y), y)\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \qquad ...(4)$$

(*iii*) If $\varphi(x, y) = 1$, then the line integral in (1) is equal to the length of the arc C.

(*iv*) The most common form of the line integral is

$$\int_C (P\,dx + Q\,dy) \qquad ...(5)$$

where P and Q are functions of x and y and the integration is taken over the curve C.

(*v*) For a closed curve C, the symbol $\int_C$ is replaced by $\oint_C$. Thus if C is a closed curve then (5) is expressed as $\oint_C (P\,dx + Q\,dy)$.

(*vi*) The definite integral $\int_a^b f(x)\,dx$ is a special case of line integral in which the integration is taken along the x-axis from $x = a$ to $x = b$.

Properties

1. If P_1, P_2, Q_1, Q_2 are functions of x, y, then $\int_C \{k_1(P_1 dx + Q_1 dy) \pm k_2(P_2 dx + Q_2 dy)\}$

$$= k_1 \int_C (P_1 dx + Q_1 dy) \pm k_2 \int_C (P_2 dx + Q_2 dy)$$

Also $\int_C \{k_1\varphi(x,y) \pm k_2\psi(x,y)\}ds = k_1\int_C \varphi(x,y)ds + k_2\int_C \psi(x,y)ds$ where k_1, k_2 are constants.

2. If $C = C_1 \cup C_2$, *i.e.*, C is divided into two parts, C_1 and C_2, then

$$\int_C (Pdx + Qdy) = \int_{C_1} (Pdx + Qdy) + \int_{C_2} (Pdx + Qdy)$$

3. If the orientation of the path of integration is reversed, then the value of the integral changes in sign, *i.e.*,

$$\int_{AB} (Pdx + Qdy) = -\int_{BA} (Pdx + Qdy)$$

4. The area of a region bounded by a simple closed curve C is equal to $\frac{1}{2}\oint_C (xdy - ydx)$,

[see Ex. 5, art. 13.9, Chapter 13].

Note: The properties (1) – (4) are also applicable to the line integral over a curve C in R^3.

ILLUSTRATIVE EXAMPLES

Example 1: Evaluate $\int_C \varphi(x,y)ds$, where $\varphi(x,y) = \frac{x^3}{y}$ and the curve C is the arc of the parabola $y = x^2$ in the xy-plane from (0, 0) to (1,1).

Solution: Here $\int_C \varphi(x,y)ds = \int_0^1 \varphi(x,y)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$ $\qquad$ ($\because$ on C, x varies from 0 to 1)

Now, $\qquad \phi(x,y) = \frac{x^3}{y}$ and on $C : y = x^2$, so $\frac{dy}{dx} = 2x,$

$$\therefore \quad \int_C \varphi(x,y)ds = \int_0^1 \frac{x^3}{x^2}\sqrt{1+4x^2}\,dx = \int_0^1 x\sqrt{1+4x^2}\,dx \qquad ...(1)$$

$$\int x\sqrt{1+4x^2}\,dx = \frac{1}{4}\int u\,du \qquad \text{(Putting } u^2 = 1 + 4x^2 \therefore 2udu = 8xdx)$$

$$= \frac{u^2}{8} = \frac{1}{8}(1+4x^2).$$

Therefore, from (1),

$$\int_C \varphi(x,y)ds = \left[\frac{1}{8}(1+4x^2)\right]_0^1 = \frac{5}{8}-\frac{1}{8} = \frac{1}{2}.$$

Example 2: Evaluate $\int_C \{(2y+3)dx + xzdy + (yz-x)dz\}$, where C is the arc of the curve $x = 2t^2$, $y = t$, $z = t^3$ from the point (0, 0, 0) to the point (2,1,1).

Solution: The equation of the given curve C is $x = 2t^2$, $y = t$, $z = t^3$. Therefore, the points (0, 0, 0), (2,1,1) correspond to $t = 0$, $t = 1$ respectively.

$$\therefore \quad \int_C \{(2y+3)dx + xzdy + (yz-x)dz\} = \int_{t=0}^{1} \{(2t+3)d(2t^2) + 2t^5dt + (t^4-2t^2)d(t^3)\}$$

$$= \int_0^1 \{4t(2t+3) + 2t^5 + 3t^2(t^4-2t^2)\}dt$$

$$= \int_0^1 (3t^6 + 2t^5 - 6t^4 + 8t^2 + 12t)dt$$

$$= \left[3\cdot\frac{t^7}{7} + 2\cdot\frac{t^6}{6} - 6\cdot\frac{t^5}{5} + 8\cdot\frac{t^3}{3} + 12\cdot\frac{t^2}{2}\right]_0^1$$

$$= \frac{3}{7} + \frac{1}{3} - \frac{6}{5} + \frac{8}{3} + 6 = \frac{288}{35}.$$

Example 3: Evaluate $\int_C \{(5xy-6x^2)dx + (2y-4x)dy\}$, where C is the arc of the curve $y = x^3$ from the point (1,1) to (2, 8) in the xy-plane.

Solution: Here, on $C : y = x^3$. $\therefore\ dy = 3x^2dx$ and x varies from 1 to 2.

$$\therefore \int_C \{(5xy - 6x^2)dx + (2y - 4x)dy\}$$

$$= \int_{x=1}^{2} \{(5x \cdot x^3 - 6x^2)dx + (2x^3 - 4x)d(x^3)\}$$

$$= \int_{1}^{2} \{5x^4 - 6x^2 + 3x^2(2x^3 - 4x)\}dx$$

$$= \int_{1}^{2} (6x^5 + 5x^4 - 12x^3 - 6x^2)dx = \left[6 \cdot \frac{x^6}{6} + 5 \cdot \frac{x^5}{5} - 12 \cdot \frac{x^4}{4} - 6 \cdot \frac{x^3}{3}\right]$$

$$= [x^6 + x^5 - 3x^4 - 2x^3]_1^2 = 35.$$

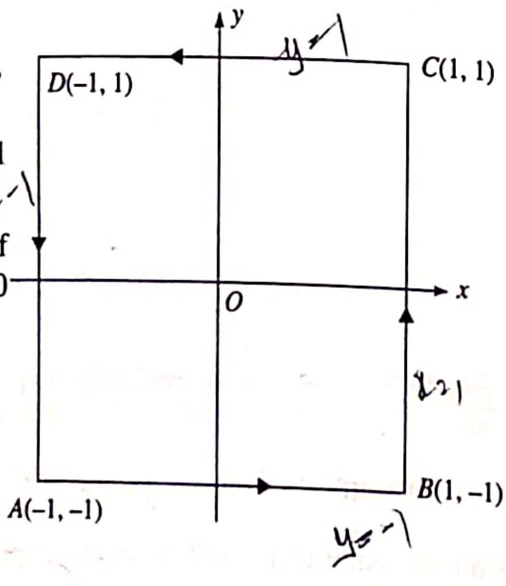
Example 4: Evaluate $\int_C \{(3x^2 + 6y)dx - 14yzdy + 20xz^2dz\}$, where C is the arc of the curve $x = t$, $y = t^2$, $z = t^3$ from the point (0,0,0) to the point (1, 1, 1). (*W.B.U.T. 2002*)

Solution: The equation of the given curve C is $x = t$, $y = t^2$, $z = t^3$. Therefore, the points (0, 0, 0), (1, 1, 1) corresponding to $t = 0$, $t = 1$ respectively.

$$\therefore \int_C \{(3x^2 + 6y)dx - 14yzdy + 20xz^2dz\} = \int_{t=0}^{1} \{(3t^2 + 6t^2)dt - 14t^5d(t^2) + 20t^7d(t^3)\}$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)dt = \left[9 \cdot \frac{t^3}{3} - 28 \cdot \frac{t^7}{7} + 60 \cdot \frac{t^{10}}{10}\right]_0^1 = 3 - 4 + 6 = 5.$$

Example 5: Evaluate $\oint_C \{(x^2 + xy)dx + (x^2 + y^2)dy\}$, where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$ with anti-clockwise orientation.



Solution: Here the path of integration is composed of the line segments AB, BC, CD and DA. On AB: $y = -1$, $dy = 0$ and x varies from −1 to 1.

On BC: $x = 1$, $dx = 0$ and y varies from −1 to 1.

On CD: $y = 1$, $dy = 0$ and x varies form 1 to −1.

On DA: $x = -1$, $dx = 0$ and y varies from 1 to −1.

$$\therefore \oint_C \{(x^2 + xy)dx + (x^2 + y^2)dy\}$$

$$= \int_{AB} (x^2 - x)\,dx + \int_{BC} (1+y^2)\,dy + \int_{CD} (x^2+x)\,dx + \int_{DA} (1+y^2)\,dy$$

$$= \int_{-1}^{1} (x^2 - x)\,dx + \int_{-1}^{1} (1+y^2)\,dy + \int_{1}^{-1} (x^2+x)\,dx + \int_{1}^{-1} (1+y^2)\,dy$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1} + \left[y + \frac{y^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{1}^{-1} + \left[y + \frac{y^3}{3}\right]_{1}^{-1}$$

$$= \frac{2}{3} + 2\left(1 + \frac{1}{3}\right) - \frac{2}{3} - 2\left(1 + \frac{1}{3}\right) = 0.$$

Example 6: Show that the line integral

$$\int_{(1,2)}^{(3,4)} \{(x^2 + 2xy)\,dx + (x^2 + y^2)\,dy\}$$

is independent of the path joining the points (1, 2) and (3, 4). Hence, evaluate the integral.

Solution: Here $(x^2 + 2xy)\,dx + (x^2 + y^2)\,dy$

$$= \frac{1}{3}(3x^2 dx + 3y^2 dy) + (2xy\,dx + x^2 dy)$$

$$= \frac{1}{3} d\,(x^3 + y^3) + d\,(x^2 y)$$

$$= d\left\{\frac{1}{3}(x^3 + y^3) + x^2 y\right\}$$

Therefore, the given integral is independent of the path and it depends only on the end points.

$$\therefore \quad \int_{(1,2)}^{(3,4)} \{(x^2 + 2xy)\,dx + (x^2 + y^2)\,dy\}$$

$$= \int_{(1,2)}^{(3,4)} d\left\{\frac{1}{3}(x^3 + y^3) + x^2 y\right\} = \left[\frac{1}{3}(x^3 + y^3) + x^2 y\right]_{(1,2)}^{(3,4)}$$

$$= \frac{1}{3}(3^3 + 4^3) + 3^2 \cdot 4 - \frac{1}{3}(1^3 + 2^3) - 1^2 \cdot 2$$

$$= \frac{1}{3}(27 + 64) + 36 - \frac{1}{3} \cdot 9 - 2 = \frac{184}{3}.$$

Note: If $Pdx + Qdy$ is a perfect differential, then $\oint_C (Pdx + Qdy) = 0$.

Example 7: Evaluate $\oint_C \{(x^2 + xy)\,dx + (x^2 - y)\,dy\}$ taken in the clockwise sense along the closed curve C formed by $y = x^2$ and $y = x$.

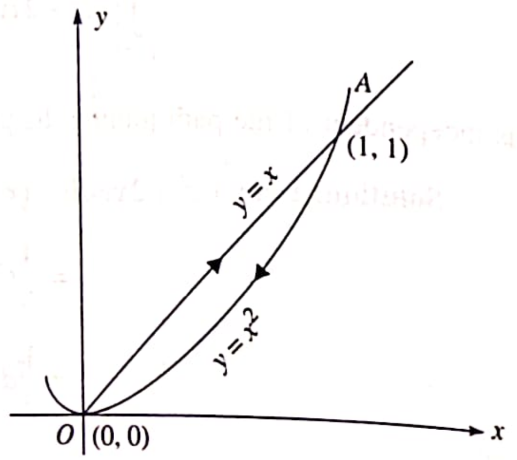
Solution: The curve C consists of the line segment OA and the arc AO of the parabola $y = x^2$.

Therefore, the given integral

$$I = \int_{OA} \{(x^2 + xy)dx + (x^2 - y)dy\} + \int_{\text{arc } AO} \{(x^2 + xy)dx + (x^2 - y)dy\}$$

$$= \int_{x=0}^{1} \{(x^2 + x^2)dx + (x^2 - x)dx\} + \int_{x=1}^{0} \{(x^2 + x^3)dx + (x^2 - x^2)2x\,dx\}$$

[$\because$ On OA : $y = x$, $dy = dx$ and x varies from 0 to 1. On arc AO : $y = x^2$, $dy = 2xdx$ and x varies from 1 to 0]



$$= \int_0^1 (3x^2 - x)\,dx + \int_1^0 (x^2 + x^3)\,dx$$

$$= \left[3\cdot\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_1^0$$

$$= 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = -\frac{1}{12}.$$

Example 8: Using line integral, find the area of the region bounded by the circle $x^2 + y^2 = a^2$ in the xy-plane.

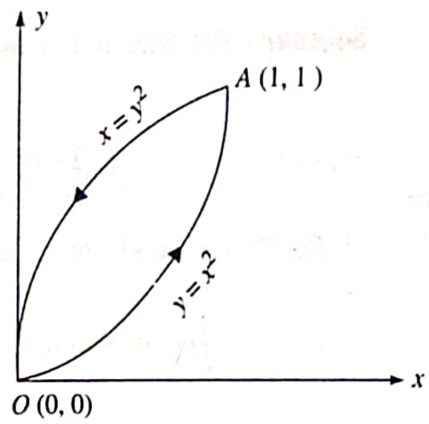
Solution: The parametric equation of the given circle is $x = a \cos t$, $y = a \sin t$, where t varies from 0 to 2π.

$\therefore$ Area of the given region.

$$= \frac{1}{2}\oint_C (x\,dy - y\,dx) = \frac{1}{2}\int_{t=0}^{2\pi} \{(a\cos t)\,d(a\sin t) - (a\sin t)\,d(a\cos t)\}$$

$$= \frac{1}{2}\int_0^{2\pi} (a^2\cos^2 t + a^2 \sin^2 t)\,dt = \frac{1}{2}a^2\int_0^{2\pi} dt = \pi a^2 \text{ square units.}$$

Example 9: Evaluate $\oint_C \{(x-y)\,dx+(x+y)\,dy\}$ taken in the counter clockwise sense along the closed curve C formed by $y=x^2$ and $x=y^2$.



Solution: The curve C consists of the arc OA and arc AO of the parabolas $y=x^2$ and $x=y^2$ respectively.

Therefore, the given integral

$$I=\int_{\text{arc}\,OA}\{(x-y)\,dx+(x+y)\,dy\}+\int_{\text{arc}\,AO}\{(x-y)\,dx+(x+y)\,dy\}$$

$$=\int_{x=0}^{1}\{(x-x^2)\,dx+(x+x^2)\,2x\,dx\}+\int_{y=1}^{0}\{(y^2-y)\,2y\,dy+(y^2+y)\,dy\}$$

[$\because$ On arc OA : $y=x^2$, $dy=2x\,dx$ and x varies from 0 to 1.
On arc AO : $x=y^2$, $dx=2y\,dy$ and y varies from 1 to 0].

$$=\int_0^1(2x^3+x^2+x)\,dx+\int_1^0(2y^3-y^2+y)\,dy$$

$$=\left[2\cdot\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1+\left[2\cdot\frac{y^4}{4}-\frac{y^3}{3}+\frac{y^2}{2}\right]_1^0=\frac{2}{3}.$$

Example 10: Evaluate $\int_C(3xy\,dx-y^2dy)$ where C is the arc of the parabola $y=2x^2$ from (0, 0) to (1, 2). *(W.B.U.T. 2007)*

Solution: Here on C : $y=2x^2$, $\therefore$ $dy=4x\,dx$ and x varies from 0 to 1.

$$\therefore\quad \int_C(3xy\,dx-y^2dy)=\int_{x=0}^{1}\{3x(2x^2)\,dx-(2x^2)^2\,4x\,dx\}$$

$$=\int_0^1(6x^3-16x^5)\,dx=6\left[\frac{x^4}{4}\right]_0^1-16\left[\frac{x^6}{6}\right]_0^1$$

$$=\frac{6}{4}-\frac{16}{6}=-\frac{7}{6}.$$

Example 11: Evaluate the line integral $\int_C(x^2\,dx+xy\,dy)$, where C is the line segment joining (1, 0) and (0,1). *(W.B.U.T. 2005)*

Solution: The equation of the line joining the points (1, 0) and (0,1) is

$$\frac{x-1}{1-0} = \frac{y-0}{0-1},$$

or $$x - 1 = -y.$$

$\therefore$ On $C: y = -x + 1$, $dy = -dx$ and x varies from 1 to 0.

$$\therefore \quad \int_C (x^2\,dx + xy\,dy) = \int_{x=1}^{0} \{x^2 dx + x(-x+1)(-dx)\}$$

$$= \int_1^0 (x^2 + x^2 - x)\,dx = \left[\frac{2x^3}{3} - \frac{x^2}{2}\right]_1^0$$

$$= -\frac{2}{3} + \frac{1}{2} = -\frac{1}{6}.$$

Example 12: Evaluate the line integral $\oint_C \{\sin y\,dx + x(1+\cos y)\,dy\}$ over a circular path C given by $x^2 + y^2 = a^2$, $z = 0$.

Solution: Along the given circular path C, $x = a\cos t$, $y = a\sin t$, $z = 0$ where t varies from 0 to 2π.

$$\therefore \quad \oint_C \{\sin y\,dx + x(1+\cos y)\,dy\} = \oint_C \{(\sin y\,dx + x\cos y\,dy) + x\,dy\}$$

$$= \oint_C \{d\,(x\sin y) + x\,dy\} = \int_0^{2\pi} [d\{a\cos t \sin(a\sin t)\} + a^2\cos^2 t\,dt]$$

[$\because$ on C, $x = a\cos t$, $y = a\sin t$, t varies from 0 to 2π, $dy = a\cos t\,dt$]

$$= [a\cos t\sin(a\sin t)]_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2t)\,dt$$

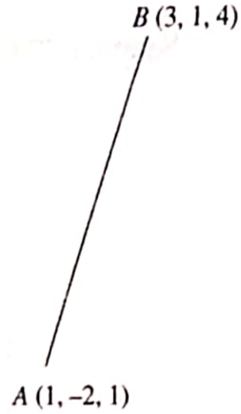
$$= 0 + \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \pi a^2.$$

Example 13: Evaluate the line integral $\int_C \{(2xy + z^3)\,dx + x^2 dy + 3xz^2 dz\}$ along the line segment joining $A(1, -2, 1)$, $B(3, 1, 4)$ and from A to B.

Solution: Equations of AB are $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-1}{3} = t$ (say), [see art. 12.5, Chapter 12]

i.e., $x = 2t + 1$, $y = 3t - 2$, $z = 3t + 1$, $0 \le t \le 1$.

At A, $t = 0$ and at B, $t = 1$.



$$\therefore \int_C \{(2xy+z^3)\,dx + x^2 dy + 3xz^2 dz\}$$

$$= \int_{t=0}^{1} [\{2(2t+1)(3t-2)+(3t+1)^3\}\,2\,dt + (2t+1)^2\,3\,dt + 3(2t+1)(3t+1)^2\,3\,dt]$$

$$= \int_0^1 \{4(6t^2-t-2)+2(3t+1)^3+3(2t+1)^2+9(18t^3+21t^2+8t+1)\}\,dt$$

$$= 4\left[6\cdot\frac{t^3}{3}-\frac{t^2}{2}-2t\right]_0^1 + \frac{2}{3}\left[\frac{(3t+1)^4}{4}\right]_0^1 + \frac{3}{2}\left[\frac{(2t+1)^3}{3}\right]_0^1 + 9\left[18\cdot\frac{t^4}{4}+21\cdot\frac{t^3}{3}+8\cdot\frac{t^2}{2}+t\right]_0^1 = 202.$$

Example 14: Show that the line integral $\int_C \{(2xy+3)\,dx + (x^2-4z)\,dy - 4y\,dz\}$ where C is any path joining (0, 0, 0) to (1, –1, 3) does not depend on the path C and evaluate the line integral.

Solution: Here $(2xy+3)\,dx+(x^2-4z)\,dy-4y\,dz$

$$= (2xy\,dx + x^2dy) + 3dx - 4\,(zdy + ydz)$$

$$= d\,(x^2y) + 3dx - 4d\,(yz)$$

$$= d\,(x^2y + 3x - 4yz)$$

Therefore, the given line integral is independent of the path C and it depends only on the end points.

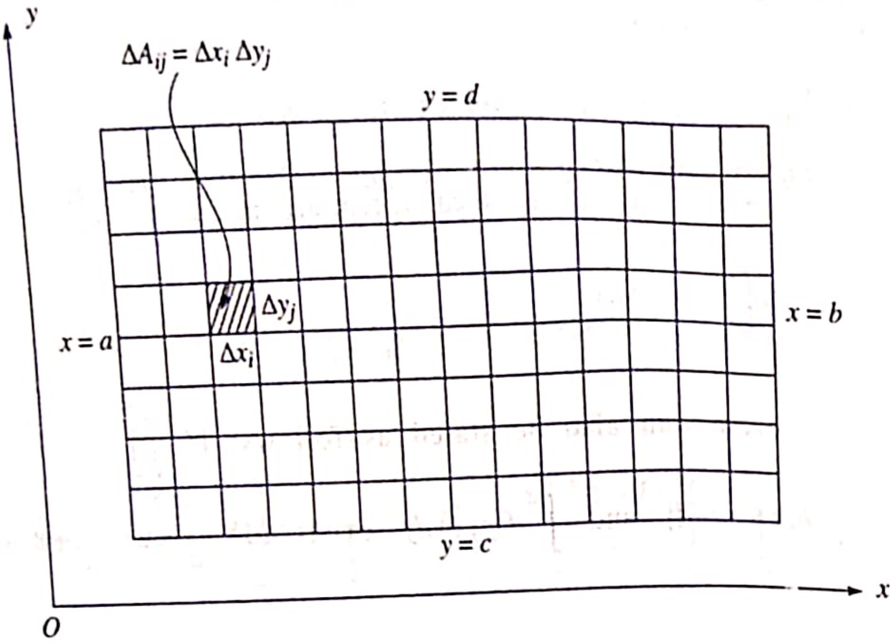
$$\therefore \int_C \{(2xy+3)\,dx+(x^2-4z)\,dy-4y\,dz\} = \int_{(0,0,0)}^{(1,-1,3)} d(x^2y+3x-4yz)$$

$$= [x^2y+3x-4yz]_{(0,0,0)}^{(1,-1,3)} = -1+3+12-0 = 14.$$

10.4 DOUBLE INTEGRALS

Double Integrals over Rectangles

Let R be a rectangular region in the xy-plane defined as a set of points $R = \{(x, y) : a \le x \le b, c \le y \le d\}$ and $f(x, y)$ be a bounded function defined over R. Divide R into mn rectangles by drawing $(m-1)$ lines parallel to x-axis and $(n-1)$ lines parallel to y-axis as shown in the figure. We denote the $(i, j)^{th}$ rectangle [formed by $(i-1)^{th}$, i^{th} lines parallel to x-axis and $(j-1)^{th}$, j^{th} lines parallel to y-axis] by A_{ij} and its area by $\Delta A_{ij} = \Delta x_i \Delta y_j$, where Δx_i, Δy_j are the dimensions of A_{ij}. Choose an arbitrary point (x_i, y_j) in the rectangle A_{ij} and form the sum



$$\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i, y_j)\,\Delta A_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i, y_j)\,\Delta x_i\,\Delta y_j$$

Let $\qquad \Delta x = \max\limits_{1 \le i \le n} \Delta x_i$

and $\qquad \Delta y = \max\limits_{1 \le i \le m} \Delta y_j$

If $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n} f(x_i, y_j)\,\Delta x_i\,\Delta y_j$ exists finitely, then we say that the double integral $\iint\limits_R f(x, y)\,dxdy$ exists and $\iint\limits_R f(x, y)\,dxdy = \iint\limits_R f(x, y)\,dR = \lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n} f(x_i, y_j)\,\Delta x_i\,\Delta y_j$.

Double Integrals over Regions other than Rectangles

Let R be a region in the xy-plane other than rectangles and $f(x, y)$ be a bounded function defined over R. Let us consider a rectangular region S in the xy-plane and S contains R and define a function $F(x, y)$ over S as follows:

$$F(x, y) = \begin{cases} f(x,y), \text{ for } (x,y) \in R \\ 0, \quad \text{elsewhere} \end{cases}$$

If $F(x, y)$ is integrable over the rectangle S, then $f(x, y)$ is said to be integrable over R and

$$\iint\limits_R f(x,y)\,dxdy = \iint\limits_S F(x,y)\,dxdy$$

Double Integral as repeated Integrals over Rectangular Region

Let us state an important theorem without proof.

Theorem: If $\iint_R f(x,y)\,dxdy$ exists over a rectangle $R = \{(x, y) : a \le x \le b,\ c \le y \le d\}$ and $\int_a^b f(x,y)\,dx$ exists for every y on $c \le y \le d$, then the repeated integral $\int_c^d dy\int_a^b f(x,y)\,dx$ exists and $\iint_R f(x,y)\,dxdy = \int_c^d dy\int_a^b f(x,y)\,dx.$

Note: This theorem can also be stated as follows: If $\iint_R f(x,y)\,dxdy$ exists on $R = \{(x,y) : a \le x \le b, c \le y \le d\}$ and $\int_c^d f(x,y)\,dy$ exists for every x on $a \le x \le b$, then $\int_a^b dx\int_c^d f(x,y)\,dy$ exists and $\iint_R f(x,y)\,dxdy = \int_a^b dx\int_c^d f(x,y)\,dy.$

Corollary 1

If $f(x,y)$ be integrable over $R = \{(x,y) : a \le x \le b, c \le y \le d\}$, $\int_a^b f(x,y)\,dx$ exists for every y on $c \le y \le d$ and $\int_c^d f(x,y)\,dy$ exists for every x on $a \le x \le b$, then $\int_a^b dx\int_c^d f(x,y)\,dy = \int_c^d dy\int_a^b f(x,y)\,dx.$

Corollary 2

If $f(x, y)$ be continuous on $R = \{(x,y) : a \le x \le b, c \le y \le d\}$, then the two repeated integrals $\int_a^b dx\int_c^d f(x,y)\,dy$ and $\int_c^d dy\int_a^b f(x,y)\,dx$ as well as the double integral $\iint_R f(x,y)\,dxdy$ exist and all are equal.

Corollary 3

If $f(x, y) = \varphi(x)\psi(y)$ defined over the rectangle $R = \{(x,y) : a \le x \le b, c \le y \le d\}$, then

$$\iint_R f(x,y)\,dxdy = \int_a^b \varphi(x)\,dx\int_c^d \psi(y)\,dy,$$

provided the double integral exists.

Double Integral as Repeated Integrals Over Domains Other than Rectangles

$$\iint_R f(x,y)\,dxdy = \int_a^b \left\{\int_{g(x)}^{h(x)} f(x,y)\,dy\right\} dx$$

provided all the integrals on the right side exist. This means that $f(x, y)$ is first integrated w.r.t. y between g and h, keeping x as constant. The resulting expression is then integrated w.r.t. x between a and b. Note that a, b are constants; g, h are either constants or functions of x.

ILLUSTRATIVE EXAMPLES

Example 1: Evaluate $\int_0^{\pi/2}\int_0^{\pi} \sin(x+y)\,dxdy$. (*W.B.U.T. 2001, 2009*)

Solution: Here

$$\int_0^{\pi/2}\int_0^{\pi}\sin(x+y)\,dxdy = \int_{y=0}^{\pi/2}\left\{\int_{x=0}^{\pi}\sin(x+y)\,dx\right\}dy$$

$$= \int_{y=0}^{\pi/2}\left[-\cos(x+y)\right]_{x=0}^{\pi}dy = \int_0^{\pi/2}\{-\cos(\pi+y)+\cos y\}\,dy$$

$$= 2\int_0^{\pi/2}\cos y\,dy = 2\,[\sin y]_0^{\pi/2} = 2\left(\sin\frac{\pi}{2}-\sin 0\right) = 2.$$

Example 2: Prove that $\int_0^1 dx\int_0^1 \frac{x-y}{(x+y)^3}dy \neq \int_0^1 dy\int_0^1 \frac{x-y}{(x+y)^3}dx$. Does $\iint\limits_R \frac{x-y}{(x+y)^3}\,dxdy$ exist, where $R=\{(x,y): 0\le x\le 1, 0\le y\le 1\}$? Justify your answer. (*W.B.U.T. 2001*)

Solution: For $x \neq 0$,

$$\varphi(x) = \int_0^1 \frac{x-y}{(x+y)^3}dy = \int_0^1 \frac{2x-(x+y)}{(x+y)^3}dy = \int_0^1\left\{\frac{2x}{(x+y)^3}-\frac{1}{(x+y)^2}\right\}dy$$

$$= \left[\frac{2x(x+y)^{-3+1}}{-3+1}-\frac{(x+y)^{-2+1}}{-2+1}\right]_{y=0}^1 = \left[-\frac{x}{(x+y)^2}+\frac{1}{x+y}\right]_{y=0}^1$$

$$= \left[-\frac{x}{(x+1)^2}+\frac{1}{x+1}+\frac{x}{x^2}-\frac{1}{x}\right] = \frac{1}{(x+1)^2}\quad (\because\ x\neq 0).$$

Here φ (0) does not exist.

$$\therefore\quad \int_0^1 dx\int_0^1\frac{x-y}{(x+y)^3}dy = \int_0^1\varphi(x)\,dx = \lim_{\varepsilon\to 0+}\int_\varepsilon^1\varphi(x)\,dx \qquad [\because\ \varphi\ (0) \text{ does not exist}]$$

$$= \lim_{\varepsilon\to 0+}\int_\varepsilon^1\frac{dx}{(x+1)^2} = \lim_{\varepsilon\to 0+}\left[\frac{(x+1)^{-2+1}}{-2+1}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0+}\left(\frac{1}{\varepsilon+1}-\frac{1}{2}\right) = \frac{1}{2}$$

Again, for $y \neq 0$,

$$\psi(y) = \int_0^1\frac{x-y}{(x+y)^3}dx = \int_0^1\frac{(x+y)-2y}{(x+y)^3}dx = \int_0^1\left\{\frac{1}{(x+y)^2}-\frac{2y}{(x+y)^3}\right\}dx$$

$$= \left[\frac{(x+y)^{-2+1}}{-2+1} - \frac{2y(x+y)^{-3+1}}{-3+1}\right]_{x=0}^{1} = \left[-\frac{1}{x+y} + \frac{y}{(x+y)^2}\right]_{x=0}^{1}$$

$$= \left[-\frac{1}{1+y} + \frac{y}{(1+y)^2} + \frac{1}{y} - \frac{y}{y^2}\right] = -\frac{1}{(1+y)^2} \quad (\because y \neq 0).$$

Here $\psi(0)$ does not exist.

$$\therefore \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\,dx = \int_0^1 \psi(y)\,dy = \lim_{\varepsilon \to 0+} \int_\varepsilon^1 \psi(y)\,dy \qquad [\because \psi\ (0) \text{ does not exist}]$$

$$= -\lim_{\varepsilon \to 0+} \int_\varepsilon^1 \frac{dy}{(1+y)^2} = -\lim_{\varepsilon \to 0+} \left[\frac{(1+y)^{-2+1}}{-2+1}\right]_\varepsilon^1 = -\lim_{\varepsilon \to 0+} \left(\frac{1}{1+\varepsilon} - \frac{1}{2}\right) = -\frac{1}{2}.$$

Thus the two iterated integrals exist but are unequal and so the double integral does not exist over R.

Here the double integral fails to exist because the integrated $f(x, y) = \dfrac{x-y}{(x+y)^3}$ is undefined at $(0, 0)$ which belongs to R.

Example 3: Evaluate $\iint_R \frac{1}{\sqrt{x^2+y^2}}\,dxdy$ where $R = \{|x| \le 1, |y| \le 1\}$. (*W.B.U.T. 2004*)

Solution: Here $R = \{|x| \le 1, |y| \le 1\} = \{-1 \le x \le 1, -1 \le y \le 1\}$.

$$\therefore \quad \iint_R \frac{1}{\sqrt{x^2+y^2}}\,dxdy = \int_{y=-1}^{1}\int_{x=-1}^{1} \frac{1}{\sqrt{x^2+y^2}}\,dxdy$$

$$= 2\int_{y=-1}^{1}\left\{\int_{x=0}^{1} \frac{dx}{\sqrt{x^2+y^2}}\right\}dy \qquad \left[\because \frac{1}{\sqrt{x^2+y^2}} \text{ is an even function of } x\right]$$

$$= 2\int_{-1}^{1}\left[\log\left|x + \sqrt{x^2+y^2}\right|\right]_{x=0}^{1} dy$$

$$= 2\int_{-1}^{1}\{\log(1+\sqrt{1+y^2}) - \log|y|\}\,dy$$

$$= 4\int_0^1 \{\log(1+\sqrt{1+y^2}) - \log y\}\,dy \qquad [\because \log(1+\sqrt{1+y^2}) - \log|y| \text{ is an even function}].$$

$$= 4\,[y\log(1+\sqrt{1+y^2})]_0^1 - 4\int_0^1 \frac{y}{(1+\sqrt{1+y^2})}\cdot\frac{y}{\sqrt{1+y^2}}\,dy - 4\,[y\log y - y]_0^1$$

$$= 4\,\{\log(1+\sqrt{2})+1\} + 4\int_0^1 \frac{y^2(1-\sqrt{1+y^2})}{y^2\sqrt{1+y^2}}\,dy$$

$$= 4\,\{\log(1+\sqrt{2})+1\} + 4\int_0^1 \left(\frac{1}{\sqrt{1+y^2}} - 1\right)dy$$

$$= 4\log(1+\sqrt{2}) + 4 + 4\,[\log\left|y+\sqrt{1+y^2}\right|]_0^1 - 4$$

$$= 8\log(1+\sqrt{2}).$$

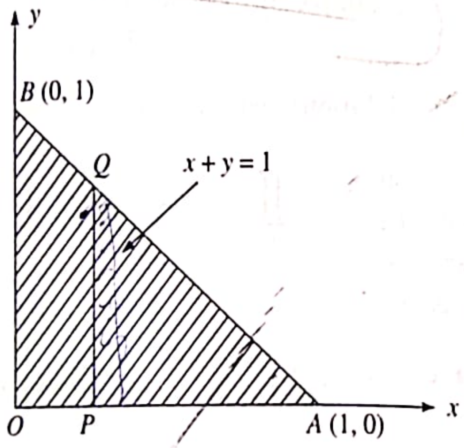
Example 4: Evaluate $\iint_R (x+y)\,dxdy$ where R is the region in the positive quadrant for which $x + y \le 1$.

Solution: The region of integration is OAB as shaded in the figure. Let us draw a line PQ parallel to y-axis, where P lies on the x-axis ($y = 0$) and Q lies on the line AB ($x + y = 1$). Therefore on PQ, y varies from 0 to $1 - x$ ($\because$ on AB, $y = 1 - x$).

Hence $$\iint_R (x+y)\,dxdy = \int_0^1\left\{\int_0^{1-x}(x+y)\,dy\right\}dx$$

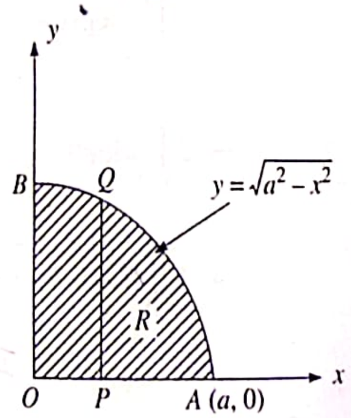
$$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_{y=0}^{1-x} dx = \int_0^1 \{x(1-x) + \frac{1}{2}(1-x)^2\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} + \frac{1}{6}(x-1)^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} + \frac{1}{6} = \frac{1}{3}.$$



Example 5: Evaluate $\iint_R xy\,dxdy$, where R is the region in the positive quadrant for which $x^2 + y^2 \le a^2$.

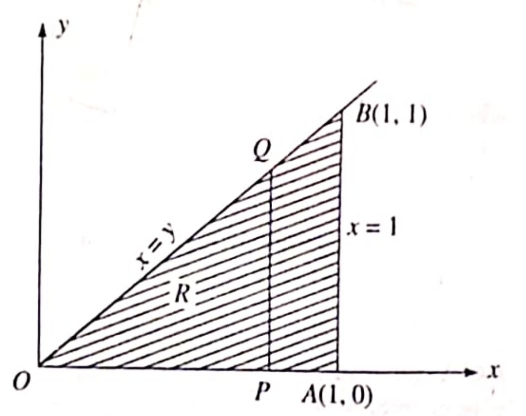
Solution: The region of integration is OAB as shaded in the figure. Let us draw a line PQ parallel to y-axis, where P lies on the x-axis ($y = 0$) and Q lies on the circle $x^2 + y^2 = a^2$ in the first quadrant. Therefore, on PQ, y varies from 0 to $\sqrt{a^2 - x^2}$.



Hence $$\iint_R xy\,dxdy = \int_0^a\left\{\int_0^{\sqrt{a^2-x^2}} xy\,dy\right\}dx = \int_0^a \left[x\frac{y^2}{2}\right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x(a^2 - x^2)\,dx = \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{a^4}{8}.$$

Example 6: Evaluate $\iint_R \sqrt{4x^2 - y^2}\,dx\,dy$ where R is the triangular region bounded by the lines $y = 0$, $x = 1$ and $y = x$. (*W.B.U.T. 2002, 2005, 2012*)



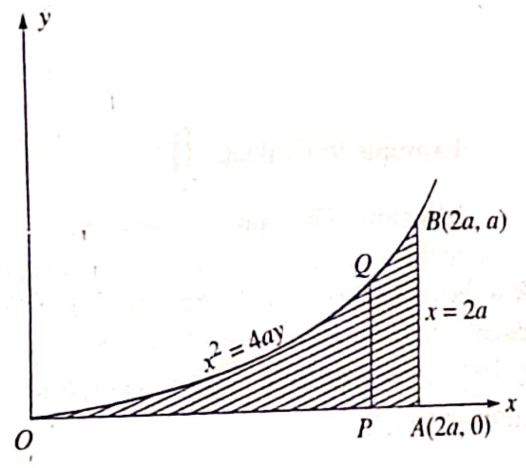
Solution: The region of integration is OAB as shaded in the figure. Let us draw a line PQ parallel to the y-axis, where P lies on the x-axis $(y = 0)$ and Q lies on the line OB $(y = x)$. Therefore, on PQ, y varies from 0 to x.

Hence $\iint_R \sqrt{4x^2 - y^2}\,dx\,dy = \int_0^1\left\{\int_0^x \sqrt{4x^2 - y^2}\,dy\right\}dx$

$$= \int_0^1\left[\frac{y\sqrt{4x^2 - y^2}}{2} + \frac{4x^2}{2}\sin^{-1}\left(\frac{y}{2x}\right)\right]_{y=0}^{x} dx = \int_0^1\left(\frac{\sqrt{3}}{2}x^2 + 2x^2\cdot\frac{\pi}{6}\right)dx$$

$$= \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)$$

Example 7: Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$, where $a > 0$.



Solution: The region of integration is OAB as shaded in the figure. Let us draw a line PQ parallel to the y-axis, where P lies on the x-axis $(y = 0)$ and Q lies on the parabola $x^2 = 4ay$ in the first quadrant. Therefore, on PQ, y varies from 0 to $\frac{x^2}{4a}$.

Hence $$\iint_R xy\,dx\,dy = \int_0^{2a} x\left\{\int_0^{\frac{x^2}{4a}} y\,dy\right\}dx$$

$$= \int_0^{2a} x\left[\frac{y^2}{2}\right]_{y=0}^{x^2/4a} dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx = \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{1}{32a^2}\cdot\frac{64a^6}{6} = \frac{a^4}{3}.$$

Example 8: Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by $y = x$ and the parabola $y = 4x - x^2$.

Solution: The given parabola is $y = 4x - x^2$, or $y - 4 = -(x-2)^2$.

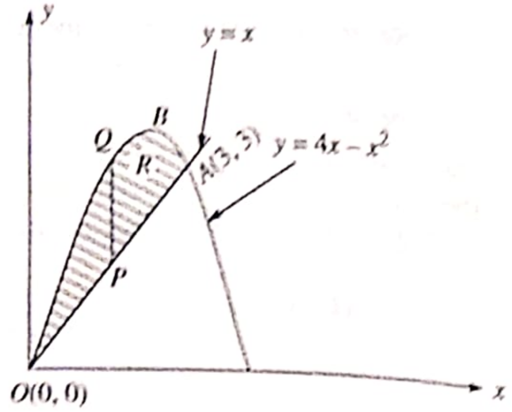
Now $\quad y = 4x - x^2,\ y = x.$

$\therefore \quad x = 4x - x^2$, or $x(x-3) = 0,\ x = 0, 3.$

when $\quad x = 0,\ y = 0$

when $\quad x = 3,\ y = 3.$

Therefore, the vertex of the given parabola is $B \equiv (2, 4)$ and the points of intersection of $y = 4x - x^2$ and $y = x$ are $O \equiv (0, 0)$ and $A \equiv (3, 3)$.



The region of integration is OAB as shaded in the figure. Let us draw a line PQ parallel to the y-axis, where P lies on the line $y = x$ and Q lies on the parabola $y = 4x - x^2$ as shown in the figure. Therefore, on PQ, y varies from x to $4x - x^2$.

Hence
$$\iint_R y\,dx\,dy = \int_{x=0}^{3}\left\{\int_{y=x}^{4x-x^2} y\,dy\right\}dx$$

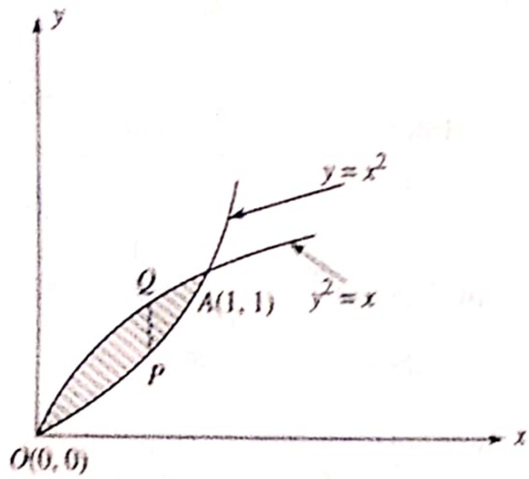
$$= \int_0^3 \left[\frac{y^2}{2}\right]_{y=x}^{4x-x^2} dx = \frac{1}{2}\int_0^3 \{(4x-x^2)^2 - x^2\}\,dx$$

$$= \frac{1}{3}\int_0^3 (x^4 - 8x^3 + 15x^2)\,dx = \frac{1}{3}\left[\frac{x^5}{5} - 8\frac{x^4}{4} + 15\frac{x^3}{3}\right]_0^3$$

$$= \frac{1}{3}\left[\frac{3^5}{5} - 2\cdot 3^4 + 5\cdot 3^3\right] = 9\left(\frac{9}{5} - 6 + 5\right) = \frac{36}{5}.$$

Example 9: Evaluate $\iint_R xy\,dx\,dy$ over the domain (R) bounded by the parabolas $y = x^2$ and $y^2 = x$.

Solution: The points of intersection of the parabolas $y = x^2$ and $y^2 = x$ are $(0, 0)$ and $(1, 1)$. The region of integration R is the shaded portion of the figure. Let us draw a line PQ parallel to the y-axis, where P lies on the parabola $y = x^2$ and Q lies on the parabola $y^2 = x$. Therefore, on PQ, y varies from x^2 to $\sqrt{x}$.



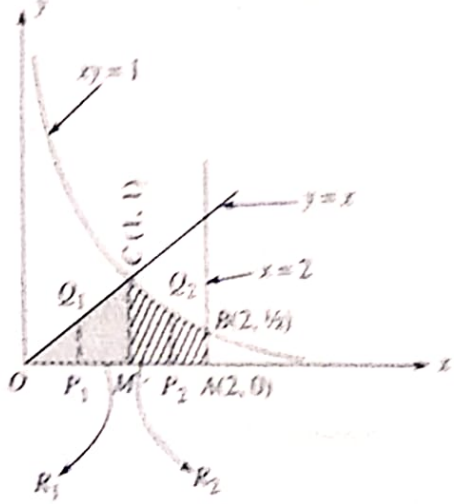
Hence
$$\iint_R xy\,dx\,dy = \int_{x=0}^{1} x\left\{\int_{y=x^2}^{\sqrt{x}} y\,dy\right\}dx$$

$$= \int_0^1 x\left[\frac{y^2}{2}\right]_{y=x^2}^{\sqrt{x}} dx$$

$$= \frac{1}{2}\int_0^1 x(x - x^4)\,dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = \frac{1}{12}.$$

Example 10: Evaluate $\iint_R (x^2 + xy)\,dx\,dy$ over the region (R) bounded by $xy = 1$, $y = 0$, $y = x$ and $x = 2$.

Solution: The region of integration R is divided into two portions R_1 and R_2 as shown in the figure. Let us draw two lines P_1Q_1 and P_2Q_2 parallel to y-axis in the regions R_1 and R_2 respectively. Here P_1, P_2 lie on the x-axis, Q_1 lies on the line $y = x$ and Q_2 lies on the hyperbola $xy = 1$. Therefore, on P_1Q_1, y varies from 0 to x and on P_2Q_2, y varies from 0 to $\frac{1}{x}$. Here CM, where $C \equiv (1,1)$, $M \equiv (1,0)$, is parallel to y-axis and CM divides the region of integration R into two regions R_1, R_2.



$$\therefore \quad \iint_R (x^2 + xy)\,dx\,dy = \iint_{R_1} (x^2 + xy)\,dx\,dy + \iint_{R_2} (x^2 + xy)\,dx\,dy$$

$$= \int_{x=0}^1 \left\{\int_{y=0}^x (x^2 + xy)\,dy\right\} dx + \int_{x=1}^2 \left\{\int_{y=0}^{1/x} (x^2 + xy)\,dy\right\} dx$$

$$= \int_0^1 \left[x^2 y + x\frac{y^2}{2}\right]_{y=0}^x dx + \int_1^2 \left[x^2 y + x\frac{y^2}{2}\right]_{y=0}^{\frac{1}{x}} dx$$

$$= \int_0^1 \left(x^3 + \frac{x^3}{2}\right) dx + \int_1^2 \left(x + \frac{1}{2x}\right) dx$$

$$= \frac{3}{2}\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^2}{2} + \frac{1}{2}\log|x|\right]_1^2$$

$$= \frac{3}{8} + 2 + \frac{1}{2}\log 2 - \frac{1}{2} = \frac{15}{8} + \frac{1}{2}\log 2.$$

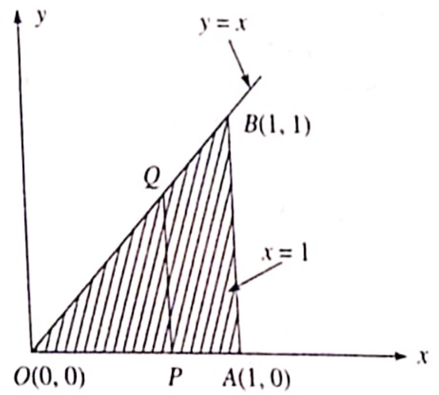
Example 11: Compute the value of $\iint_R y\,dx\,dy$ where R is the region in the first quadrant bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *(W.B.U.T. 2008)*

Solution: The region of integration is OAB as shaded in the figure. Let us draw a line PQ parallel to the y-axis, where P lies on the x-axis $(y = 0)$ and Q lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first

Hence $$I = \int_0^1 dx \int_x^{\sqrt{x}} f(x,y)\,dy = \int_0^1 dy \int_{y^2}^{y} f(x,y)\,dx.$$

Note: In some situations the evaluation of double integrals seem to be complicated, can be made easy to handle by a change in the order of integration. See the following example.

Example 17: Evaluate $\int_{y=0}^{1}\int_{x=y}^{1} e^{x^2}\,dx\,dy.$



Solution: Here we observe that $\int e^{x^2} dx$ is difficult to evaluate. Let us change the order of integration. Here x varies from $x = y$ to $x = 1$, hence the region R of integration is enclosed by the straight lines $y = 0$, $x = y$ and $x = 1$. Draw a straight line PQ parallel to y-axis, where P lies on the x-axis ($y = 0$) and Q lies on $y = x$. Therefore, on PQ, y varies from 0 to x.

Hence $$\int_{y=0}^{1}\int_{x=y}^{1} e^{x^2}\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{x} e^{x^2}\,dx\,dy$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{x} dy\right\} e^{x^2}\,dx = \int_0^1 [y]_{y=0}^{x} e^{x^2}\,dx$$

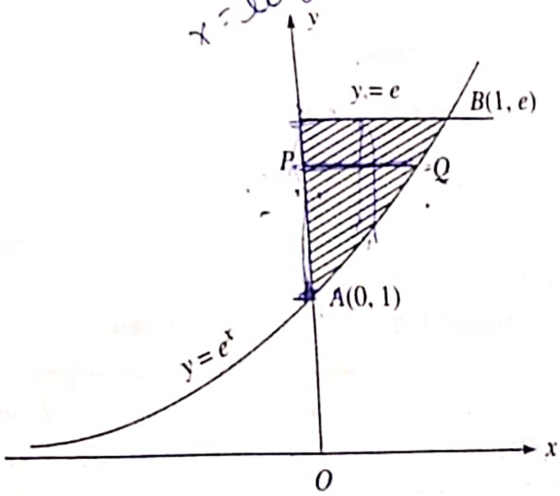
$$= \int_0^1 x e^{x^2}\,dx = \frac{1}{2}[e^{x^2}]_0^1 = \frac{1}{2}(e-1).$$

$y = e$

$y = e^x$

$x = \log y$

Example 18: Change the order of integration and hence evaluate $\int_0^1\int_{e^x}^{e} \frac{dx\,dy}{y^2 \log y}.$ *(W.B.U.T. 2007)*



Solution: Since limits of x can not be functions of x, in the given integral lower and upper limits of y are e^x and e respectively. Here the region of integration R is enclosed by the straight line $y = e$, y-axis ($x = 0$) and the curve $y = e^x$ as shown in the figure by shaded portion. Draw a straight line PQ parallel to the x-axis, where P lies on the y-axis ($x = 0$) and Q lies on $y = e^x$ (*i.e.*, $x = \log y$). Therefore, on PQ, x varies from 0 to $\log y$.

Hence $$\int_0^1\int_{e^x}^{e} \frac{dx\,dy}{y^2\log y} = \int_{y=1}^{e}\int_{x=0}^{\log y} \frac{dx\,dy}{y^2\log y} = \int_{y=1}^{e}\left\{\int_{x=0}^{\log y} dx\right\}\frac{1}{y^2\log y}\,dy$$

$$= \int_1^e [x]_{x=0}^{\log y}\frac{1}{y^2\log y}\,dy = \int_1^e \frac{dy}{y^2} = \left[-\frac{1}{y}\right]_1^e = 1-\frac{1}{e}.$$

$$\therefore \quad \iint_R xy(x+y)\,dx\,dy = \int_{x=0}^{1} x\left\{\int_{y=x^2}^{x}(xy+y^2)\,dy\right\}dx$$

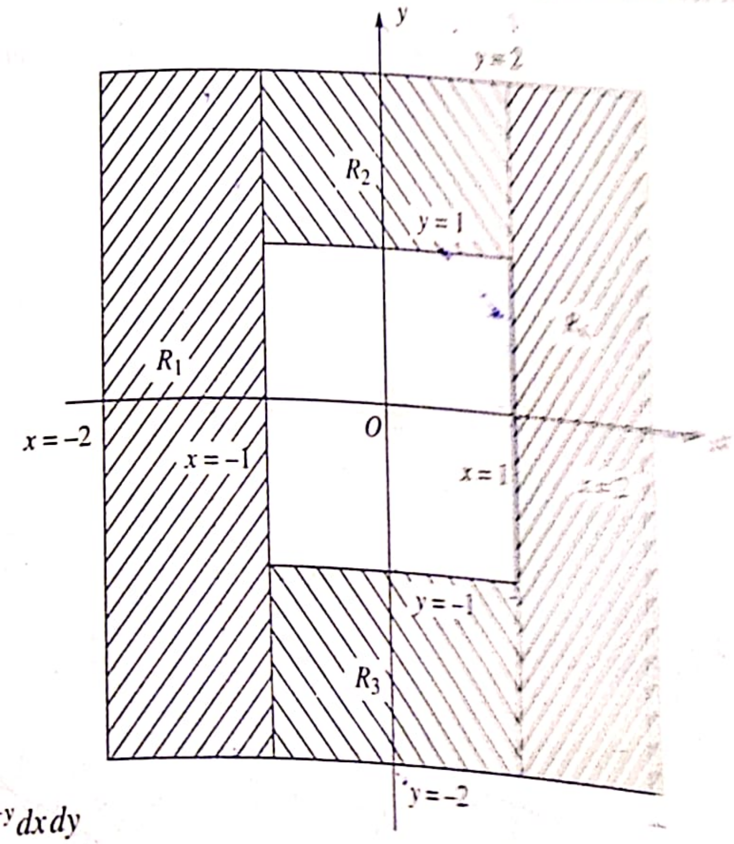
$$= \int_{x=0}^{1} x\left[x\frac{y^2}{2}+\frac{y^3}{3}\right]_{y=x^2}^{x} dx = \int_0^1 x\left(x\cdot\frac{x^2}{2}+\frac{x^3}{3}-x\cdot\frac{x^4}{2}-\frac{x^6}{3}\right)$$

$$= \int_0^1\left(5\frac{x^4}{6}-\frac{x^6}{2}-\frac{x^7}{3}\right) = \left[\frac{5}{6}\cdot\frac{x^5}{5}-\frac{1}{2}\cdot\frac{x^7}{7}-\frac{1}{3}\cdot\frac{x^8}{8}\right]_0^1$$

$$= \frac{1}{6}-\frac{1}{14}-\frac{1}{24} = \frac{3}{56}.$$

Example 14: Evaluate the double integral $\iint_R e^{x+y}\,dx\,dy$, where R is the region of integration which lies between two squares of sides 2 and 4 with centre at the origin and sides parallel to the axes.

Solution: The straight lines $x=-1$ and $x=1$ divide the region R into four subregions R_1, R_2, R_3, R_4.

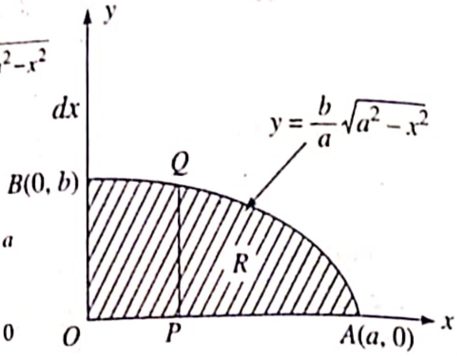


Here $\iint_R e^{x+y}\,dx\,dy$

$$= \int_{x=-2}^{-1}\int_{y=-2}^{2} e^{x+y}\,dx\,dy + \int_{x=-1}^{1}\int_{y=1}^{2} e^{x+y}\,dx\,dy + \int_{x=-1}^{1}\int_{y=-2}^{-1} e^{x+y}\,dx\,dy + \int_{x=1}^{2}\int_{y=-2}^{2} e^{x+y}\,dx\,dy$$

$$= [e^x]_{-2}^{-1}[e^y]_{-2}^{2} + [e^x]_{-1}^{1}[e^y]_{1}^{2} + [e^x]_{-1}^{1}[e^y]_{-2}^{-1} + [e^x]_{1}^{2}[e^y]_{-2}^{2}$$

quadrant. Therefore, on PQ, y varies from 0 to $\frac{b}{a}\sqrt{a^2-x^2}$.



$$\therefore \quad \iint_R y\,dx\,dy = \int_0^a \left\{\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} y\,dy\right\} dx = \int_0^a \left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx = \frac{b^2}{2a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$= \frac{b^2}{2a^2}\left(a^3-\frac{a^3}{3}\right) = \frac{ab^2}{3}.$$

Example 12: Evaluate $\int_0^3\int_1^{\sqrt{4-y}}(x+y)\,dxdy$ *(W.B.U.T. 2004)*

Solution: In the given integral lower and upper limits of x are 1 and $\sqrt{4-y}$ respectively (since limits of y can not be functions of y).

$$\therefore \quad \int_0^3\int_1^{\sqrt{4-y}}(x+y)\,dxdy = \int_0^3\left[\frac{x^2}{2}+xy\right]_{x=1}^{\sqrt{4-y}} dy$$

$$= \int_0^3\left\{\frac{1}{2}(4-y)+y\sqrt{4-y}-\frac{1}{2}-y\right\}dy = \int_0^3\left\{y\sqrt{4-y}-\frac{3}{2}y+\frac{3}{2}\right\}dy$$

$$= \int_0^3 \{4-(4-y)\}\sqrt{4-y}\,dy-\frac{3}{2}\left[\frac{y^2}{2}-y\right]_0^3$$

$$= 4\left[\frac{-(4-y)^{3/2}}{3/2}\right]_0^3-\left[-\frac{(4-y)^{5/2}}{5/2}\right]_0^3-\frac{3}{2}\left(\frac{9}{2}-3\right)$$

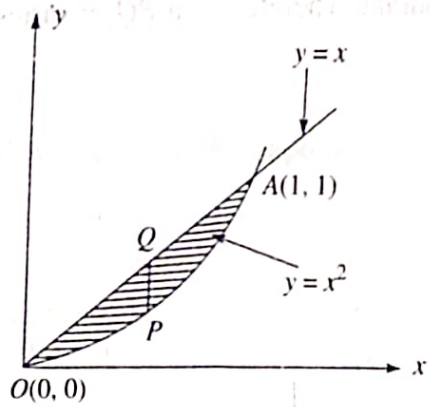
$$= \frac{8}{3}(-1+8)-\frac{2}{5}(-1+32)-\frac{9}{4} = \frac{241}{60}.$$

Example 13: Evaluate $\iint xy(x+y)\,dxdy$ over the area bounded by $y=x^2$ and $y=x$.

(W.B.U.T. 2001)

Solution: The points of intersection of the parabola $y=x^2$ and the line $y=x$ are O (0, 0) and $A(1,1)$. The region of integration R is the shaded portion of the figure. Let us draw a line PQ parallel to the y-axis, where P lies on the parabola $y=x^2$ and Q lies on the line $y=x$. Therefore, on PQ, y varies form x^2 to x.

$$\therefore \quad \iint_R xy(x+y)\,dx\,dy = \int_{x=0}^{1} x\left\{\int_{y=x^2}^{x}(xy+y^2)\,dy\right\}dx$$



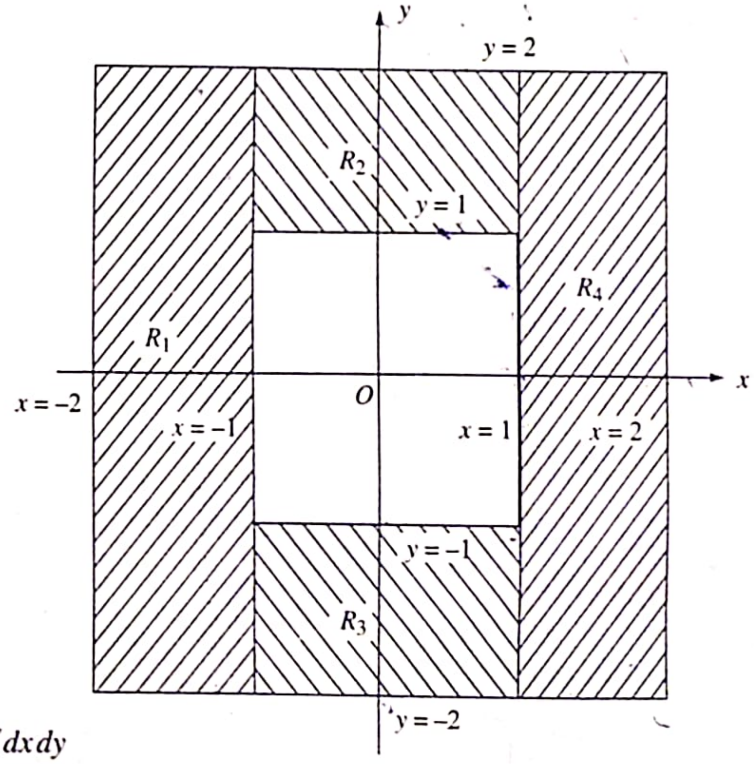
$$= \int_{x=0}^{1} x\left[x\frac{y^2}{2}+\frac{y^3}{3}\right]_{y=x^2}^{x} dx = \int_0^1 x\left(x\cdot\frac{x^2}{2}+\frac{x^3}{3}-x\cdot\frac{x^4}{2}-\frac{x^6}{3}\right)$$

$$= \int_0^1\left(5\frac{x^4}{6}-\frac{x^6}{2}-\frac{x^7}{3}\right) = \left[\frac{5}{6}\cdot\frac{x^5}{5}-\frac{1}{2}\cdot\frac{x^7}{7}-\frac{1}{3}\cdot\frac{x^8}{8}\right]_0^1$$

$$= \frac{1}{6}-\frac{1}{14}-\frac{1}{24} = \frac{3}{56}.$$

Example 14: Evaluate the double integral $\iint_R e^{x+y}\,dx\,dy$, where R is the region of integration which lies between two squares of sides 2 and 4 with centre at the origin and sides parallel to the axes.

Solution: The straight lines $x=-1$ and $x=1$ divide the region R into four regions R_1, R_2, R_3 and R_4.



Here $\iint_R e^{x+y}\,dx\,dy$

$$= \int_{x=-2}^{-1}\int_{y=-2}^{2} e^{x+y}\,dx\,dy + \int_{x=-1}^{1}\int_{y=1}^{2} e^{x+y}\,dx\,dy + \int_{x=-1}^{1}\int_{y=-2}^{-1} e^{x+y}\,dx\,dy + \int_{x=1}^{2}\int_{y=-2}^{2} e^{x+y}\,dx\,dy.$$

$$= [e^x]_{-2}^{-1}[e^y]_{-2}^{2} + [e^x]_{-1}^{1}[e^y]_{1}^{2} + [e^x]_{-1}^{1}[e^y]_{-2}^{-1} + [e^x]_{1}^{2}[e^y]_{-2}^{2}$$

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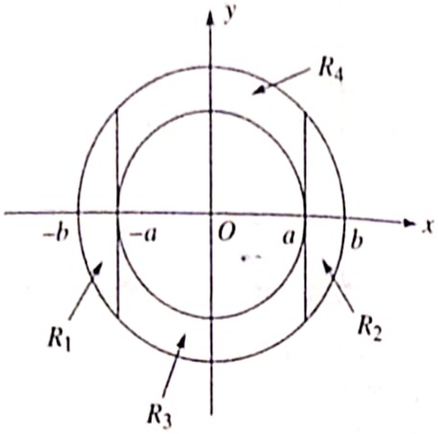
$$= (e^{-1} - e^{-2})(e^2 - e^{-2}) + (e - e^{-1})(e^2 - e) + (e - e^{-1})(e^{-1} - e^{-2}) + (e^2 - e)(e^2 - e^{-2})$$

Example 15: Evaluate the double integral $\iint_R dx\,dy$ where R is the region in the xy-plane bounded by $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ where $b > a$.



Solution: The straight lines $x = -a$ and $x = a$ divide the region R into four regions R_1, R_2, R_3 and R_4.

$$\therefore \iint_R dx\,dy$$

$$= \int_{-b}^{-a} dx \int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}} dy + \int_a^b dx \int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}} dy + \int_{-a}^{a} dx \int_{-\sqrt{b^2-x^2}}^{-\sqrt{a^2-x^2}} dy + \int_{-a}^{a} dx \int_{\sqrt{a^2-x^2}}^{\sqrt{b^2-x^2}} dy$$

$$= 2\int_{-b}^{-a} \sqrt{b^2 - x^2}\,dx + 2\int_a^b \sqrt{b^2 - x^2}\,dx + 2\int_{-a}^{a} (\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2})\,dx$$

$$= 4\int_a^b \sqrt{b^2 - x^2}\,dx + 4\int_0^a (\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2})dx \quad [\because \sqrt{b^2 - x^2}, \sqrt{a^2 - x^2} \text{ are even functions}]$$

$$= 4\left\{\int_0^a \sqrt{b^2 - x^2}\,dx + \int_a^b \sqrt{b^2 - x^2}\,dx\right\} - 4\int_0^a \sqrt{a^2 - x^2}\,dx$$

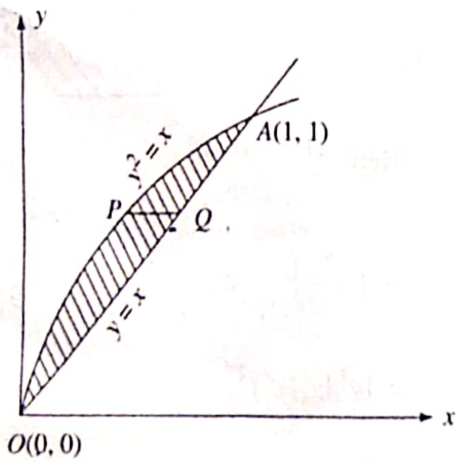
$$= 4\int_0^b \sqrt{b^2 - x^2}\,dx - 4\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= 4\left[\frac{x\sqrt{b^2 - x^2}}{2} + \frac{b^2}{2}\sin^{-1}\frac{x}{b}\right]_0^b - 4\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\cdot\frac{b^2}{2}\cdot\frac{\pi}{2} - 4\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = (b^2 - a^2)\,\pi.$$

Example 16: Change the order of integration in

$$I = \int_0^1 dx \int_x^{\sqrt{x}} f(x, y)\,dy$$

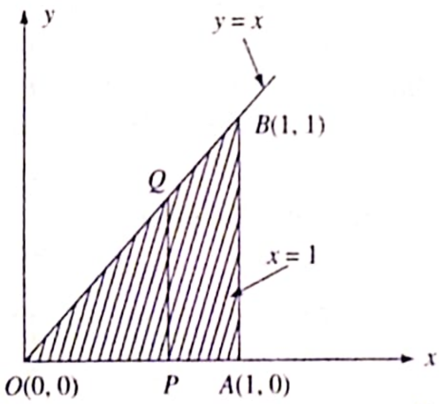


Solution: In the given integral lower and upper limits of y are x and $\sqrt{x}$ respectively (since limits of x can not be functions of x). The line $y = x$ and the parabola $y^2 = x$ cut at (0, 0) and (1,1). The domain of integration is the shaded region. Let us draw a line PQ parallel to the x-axis, where P lies on $y^2 = x$ and Q lies on $y = x$. Therefore, on PQ, x varies from y^2 to y.

Hence $$I = \int_0^1 dx \int_x^{\sqrt{x}} f(x,y)\,dy = \int_0^1 dy \int_{y^2}^{y} f(x,y)\,dx.$$

Note: In some situations the evaluation of double integrals seem to be complicated, can be made easy to handle by a change in the order of integration. See the following example.

Example 17: Evaluate $\int_{y=0}^{1}\int_{x=y}^{1} e^{x^2}\,dx\,dy.$



Solution: Here we observe that $\int e^{x^2}\,dx$ is difficult to evaluate. Let us change the order of integration. Here x varies from $x = y$ to $x = 1$, hence the region R of integration is enclosed by the straight lines $y = 0$, $x = y$ and $x = 1$. Draw a straight line PQ parallel to y-axis, where P lies on the x-axis ($y = 0$) and Q lies on $y = x$. Therefore, on PQ, y varies from 0 to x.

Hence $$\int_{y=0}^{1}\int_{x=y}^{1} e^{x^2}\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{x} e^{x^2}\,dx\,dy$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{x} dy\right\} e^{x^2}\,dx = \int_0^1 [y]_{y=0}^{x} e^{x^2}\,dx$$

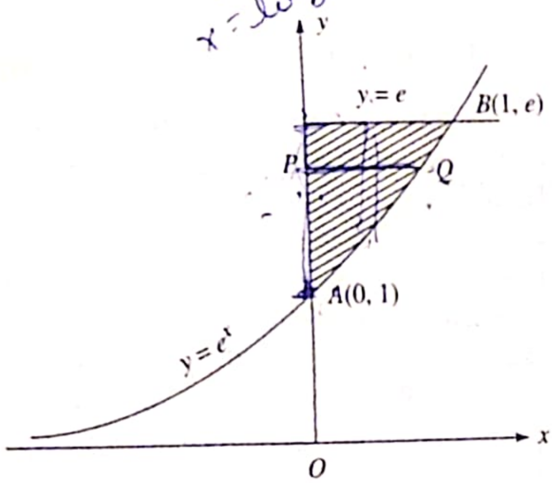
$$= \int_0^1 x e^{x^2}\,dx = \frac{1}{2}[e^{x^2}]_0^1 = \frac{1}{2}(e-1).$$

$y = e$

$y = e^x$

$x = \log y$

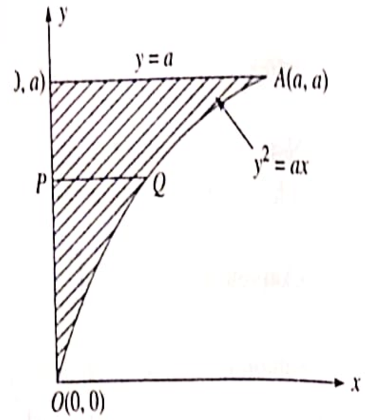
Example 18: Change the order of integration and hence evaluate $\int_0^1\int_{e^x}^{e} \frac{dx\,dy}{y^2 \log y}$. (*W.B.U.T. 2007*)



Solution: Since limits of x can not be functions of x, in the given integral lower and upper limits of y are e^x and e respectively. Here the region of integration R is enclosed by the straight line $y = e$, y-axis ($x = 0$) and the curve $y = e^x$ as shown in the figure by shaded portion. Draw a straight line PQ parallel to the x-axis, where P lies on the y-axis ($x = 0$) and Q lies on $y = e^x$ (*i.e.*, $x = \log y$). Therefore, on PQ, x varies from 0 to $\log y$.

Hence $$\int_0^1\int_{e^x}^{e} \frac{dx\,dy}{y^2\log y} = \int_{y=1}^{e}\int_{x=0}^{\log y} \frac{dx\,dy}{y^2\log y} = \int_{y=1}^{e}\left\{\int_{x=0}^{\log y} dx\right\}\frac{1}{y^2\log y}\,dy$$

$$= \int_1^e [x]_{x=0}^{\log y}\frac{1}{y^2\log y}\,dy = \int_1^e \frac{dy}{y^2} = \left[-\frac{1}{y}\right]_1^e = 1-\frac{1}{e}.$$



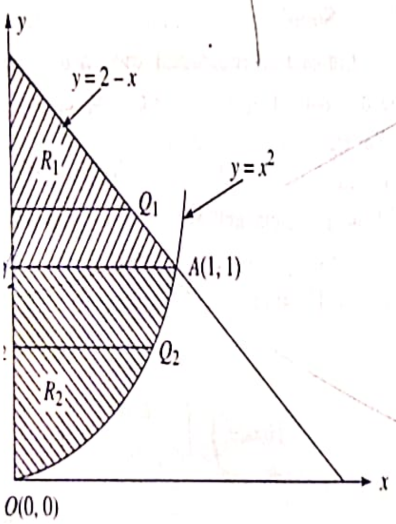
ries from 0 to $\frac{y^2}{a}$.

dy

$$\Bigg\} dy = \frac{1}{a}\int_0^a y^2\left[\sin^{-1}\left(\frac{ax}{y^2}\right)\right]_{x=0}^{=y^2/a} dy$$

$: \frac{\pi a^2}{6}$.

$^{2-x}_{x^2}$ $xy\,dx\,dy$ and hence evaluate the same.



$$y\left\{\int_0^{2-y} x\,dx\right\}dy+\int_{y=0}^{1} y\left\{\int_0^{\sqrt{y}} x\,dx\right\}dy$$

$$=\int_1^2 y\left[\frac{x^2}{2}\right]_{x=0}^{2-y}dy+\int_0^1 y\left[\frac{x^2}{2}\right]_{x=0}^{\sqrt{y}}dy$$

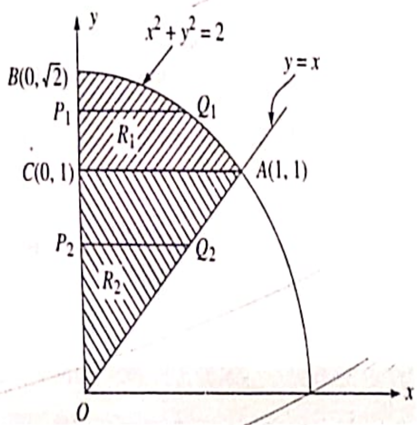
$$=\frac{1}{2}\int_1^2 y(2-y)^2\,dy+\frac{1}{2}\int_0^1 y^2\,dy=\frac{1}{2}\int_1^2 y(y^2-4y+4)\,dy+\frac{1}{2}\left[\frac{y^3}{3}\right]_0^1$$

$$=\frac{1}{2}\left[\frac{y^4}{4}-4\cdot\frac{y^3}{3}+4\cdot\frac{y^2}{2}\right]_1^2+\frac{1}{6}=\frac{3}{8}.$$

Question: Integrate by changing the order of integration $\int_0^a\int_{x^2/a}^{2a-x} xy\,dy\,dx$. (*W.B.U.T. 2010*)

Example 21: Change the order of integration in $\int_0^1\int_x^{\sqrt{2-x^2}}\frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$ and hence evaluate it.

Solution: In the given integral lower and upper limits of y are x and $\sqrt{2-x^2}$ respectively. The circle $y^2=2-x^2$, or $x^2+y^2=2$ and the line $y=x$ intersect at $A(1,1)$ in the first quadrant (we are considering the first quadrant only because x varies from 0 to 1). Here CA, where $C\equiv(0,1)$, $A\equiv(1,1)$, is parallel to x-axis and CA divides the region of integration R into two portions R_1 and R_2 as shown in the figure.



Let us draw two lines P_1Q_1 and P_2Q_2 parallel to x-axis in the regions R_1 and R_2 respectively, where P_1, P_2 lie on the y-axis ($x=0$), Q_1 lies on the circle $x^2+y^2=2$ and Q_2 lies on the line $y=x$. Therefore on P_1Q_1, x varies from 0 to $\sqrt{2-y^2}$ and on P_2Q_2, x varies from 0 to y.

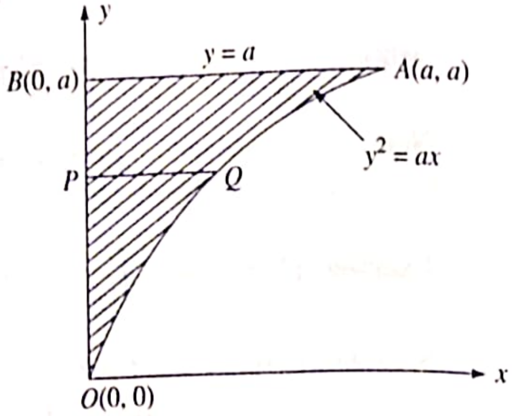
$$\therefore\quad \int_0^1\int_x^{\sqrt{2-x^2}}\frac{x\,dx\,dy}{\sqrt{x^2+y^2}}=\iint_{R_1}\frac{x\,dx\,dy}{\sqrt{x^2+y^2}}+\iint_{R_2}\frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$$=\int_{y=1}^{\sqrt{2}}\left\{\int_0^{\sqrt{2-y^2}}\frac{x\,dx}{\sqrt{x^2+y^2}}\right\}dy+\int_{y=0}^{1}\left\{\int_0^{y}\frac{x\,dx}{\sqrt{x^2+y^2}}\right\}dy$$

$$=\int_1^{\sqrt{2}}\left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}}dy+\int_0^1\left[\sqrt{x^2+y^2}\right]_{x=0}^{y}dy$$

$$=\int_1^{\sqrt{2}}(\sqrt{2}-y)\,dy+\int_0^1(\sqrt{2}-1)\,y\,dy$$

Example 19: Change the order of integration and hence evaluate $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dx\,dy}{\sqrt{y^4 - a^2x^2}}$, where $a > 0$.



Solution: In the given integral lower and upper limits of y are $\sqrt{ax}$ and a respectively (since limits of x can not be functions of x). The parabola $y^2 = ax$ and the line $y = a$ intersect at $A(a, a)$. The domain of integration R is the shaded region. Let us draw a line PQ parallel to the x-axis, where P lies on the y-axis ($x = 0$) and Q lies on the parabola $y^2 = ax$ $\left(i.e., x = \frac{y^2}{a}\right)$. Therefore, on PQ, x varies from 0 to $\frac{y^2}{a}$.

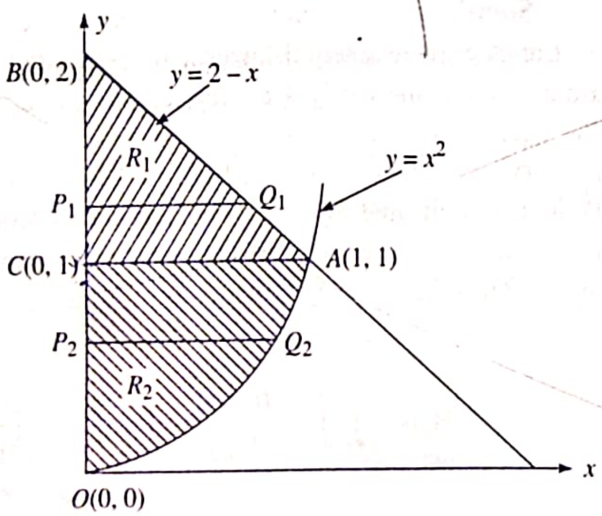
$$\therefore \int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dx\,dy}{\sqrt{y^4 - a^2x^2}} = \int_{y=0}^a y^2 \left\{\int_{x=0}^{y^2/a} \frac{dx}{\sqrt{y^4 - a^2x^2}}\right\} dy$$

$$= \frac{1}{a}\int_{y=0}^a y^2 \left\{\int_{x=0}^{y^2/a} \frac{dx}{\sqrt{(y^2/a)^2 - x^2}}\right\} dy = \frac{1}{a}\int_0^a y^2 \left[\sin^{-1}\left(\frac{ax}{y^2}\right)\right]_{x=0}^{=y^2/a} dy$$

$$= \frac{\pi}{2a}\int_0^a y^2\,dy = \frac{\pi}{2a}\left[\frac{y^3}{3}\right]_0^a = \frac{\pi a^2}{6}.$$

Example 20: Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the same.

Solution: In the given integral lower and upper limits of y are x^2 and $2 - x$ respectively. The parabola $y = x^2$ and the line $y = 2 - x$ cut at $A(1,1)$ in the first quadrant (we are considering the first quadrant only since x varies from 0 to 1). Hence CA, where $C \equiv (0,1)$, $A \equiv (1,1)$, is parallel to x-axis and CA divides the region of integration R into two portions R_1 and R_2 as shown in the figure. Let us draw two lines P_1Q_1 and P_2Q_2 parallel to x-axis in the regions R_1 and R_2 respectively, where P_1, P_2 lie on the y-axis ($x = 0$), Q_1 lies on the line $y = 2 - x$ and Q_2 lies on the parabola $y = x^2$. Therefore, on P_1Q_1, x varies from 0 to $(2 - y)$ and on P_2Q_2, x varies from 0 to $\sqrt{y}$.



$$\therefore \int_0^1\int_{x^2}^{2-x} xy\,dx\,dy = \iint_{R_1} xy\,dx\,dy + \iint_{R_2} xy\,dx\,dy = \int_{y=1}^2 y\left\{\int_0^{2-y} x\,dx\right\} dy + \int_{y=0}^1 y\left\{\int_0^{\sqrt{y}} x\,dx\right\} dy$$

$$= \int_1^2 y\left[\frac{x^2}{2}\right]_{x=0}^{2-y} dy + \int_0^1 y\left[\frac{x^2}{2}\right]_{x=0}^{\sqrt{y}} dy$$

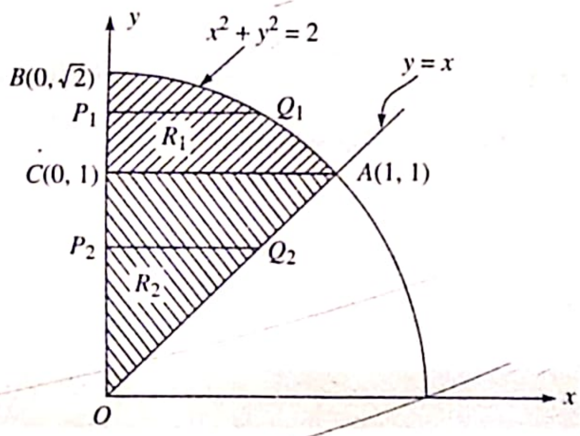
$$= \frac{1}{2}\int_1^2 y(2-y)^2 dy + \frac{1}{2}\int_0^1 y^2 dy = \frac{1}{2}\int_1^2 y(y^2-4y+4)dy + \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{y^4}{4} - 4\cdot\frac{y^3}{3} + 4\cdot\frac{y^2}{2}\right]_1^2 + \frac{1}{6} = \frac{3}{8}.$$

Question: Integrate by changing the order of integration $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$. (*W.B.U.T. 2010*)

Example 21: Change the order of integration in $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$ and hence evaluate it.

Solution: In the given integral lower and upper limits of y are x and $\sqrt{2-x^2}$ respectively. The circle $y^2 = 2 - x^2$, or $x^2 + y^2 = 2$ and the line $y = x$ intersect at $A(1,1)$ in the first quadrant (we are considering the first quadrant only because x varies from 0 to 1). Here CA, where $C \equiv (0,1)$, $A \equiv (1,1)$, is parallel to x-axis and CA divides the region of integration R into two portions R_1 and R_2 as shown in the figure.



Let us draw two lines P_1Q_1 and P_2Q_2 parallel to x-axis in the regions R_1 and R_2 respectively, where P_1, P_2 lie on the y-axis ($x = 0$), Q_1 lies on the circle $x^2 + y^2 = 2$ and Q_2 lies on the line $y = x$. Therefore on P_1Q_1, x varies from 0 to $\sqrt{2-y^2}$ and on P_2Q_2, x varies from 0 to y.

$$\therefore \quad \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} = \iint_{R_1} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} + \iint_{R_2} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$$= \int_{y=1}^{\sqrt{2}} \left\{\int_0^{\sqrt{2-y^2}} \frac{x\,dx}{\sqrt{x^2+y^2}}\right\} dy + \int_{y=0}^{1} \left\{\int_0^{y} \frac{x\,dx}{\sqrt{x^2+y^2}}\right\} dy$$

$$= \int_1^{\sqrt{2}} \left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}} dy + \int_0^1 \left[\sqrt{x^2+y^2}\right]_{x=0}^{y} dy$$

$$= \int_1^{\sqrt{2}} (\sqrt{2}-y)\,dy + \int_0^1 (\sqrt{2}-1)\,y\,dy$$

$$= \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}} + (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1$$

$$= 2 - 1 - \sqrt{2} + \frac{1}{2} + \frac{1}{2}(\sqrt{2}-1) = 1 - \frac{1}{\sqrt{2}}.$$

10.5 CHANGE OF VARIABLES IN A DOUBLE INTEGRAL

It has been observed that many multiple integrals can be more easily evaluated by changing the variables and expressing the corresponding functions and the integrals in terms of changed variables.

If $x = f(u,v)$ and $y = g(u,v)$ represents a continuous one-to-one mapping of a closed region R of the xy-plane on a region R' of the uv-plane and if the functions $f(u,v)$, $g(u,v)$ have continuous first order partial derivatives and the Jacobian

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0,$$

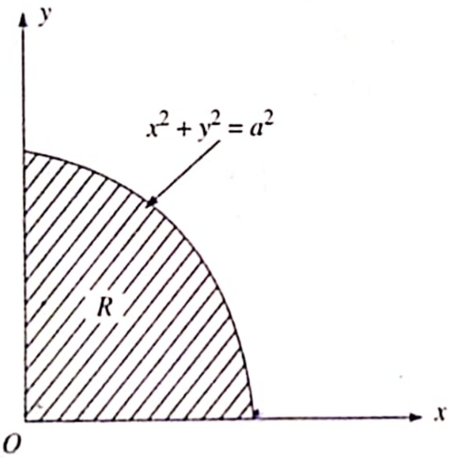
then
$$\iint_R F(x,y)\,dx\,dy = \iint_{R'} F\{f(u,v), g(u,v)\}\,|J|\,du\,dv.$$

Note: The Jacobian of transformation from Cartesian to Polar co-ordinate system is

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r, \text{ where } x = r\cos\theta,\ y = r\sin\theta.$$

ILLUSTRATIVE EXAMPLES

Example 1: Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$ by changing to polar co-ordinates. *(W.B.U.T. 2002)*



Solution: Here the limits of x are 0, $\sqrt{a^2-y^2}$ and those of y are 0, a. The domain of integration R is the shaded region. Let us change cartesian to polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \quad J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$= \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}} + (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1$$

$$= 2 - 1 - \sqrt{2} + \frac{1}{2} + \frac{1}{2}(\sqrt{2}-1) = 1 - \frac{1}{\sqrt{2}}.$$

10.5 CHANGE OF VARIABLES IN A DOUBLE INTEGRAL

It has been observed that many multiple integrals can be more easily evaluated by changing the variables and expressing the corresponding functions and the integrals in terms of changed variables.

If $x = f(u,v)$ and $y = g(u,v)$ represents a continuous one-to-one mapping of a closed region R of the xy-plane on a region R' of the uv-plane and if the functions $f(u,v)$, $g(u,v)$ have continuous first order partial derivatives and the Jacobian

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0,$$

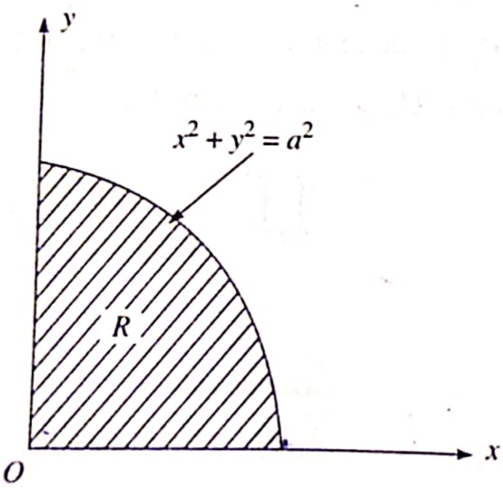
then
$$\iint_R F(x,y)\,dx\,dy = \iint_{R'} F\{f(u,v), g(u,v)\}\,|J|\,du\,dv.$$

Note: The Jacobian of transformation from Cartesian to Polar co-ordinate system is

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r, \text{ where } x = r\cos\theta,\ y = r\sin\theta.$$

ILLUSTRATIVE EXAMPLES

Example 1: Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$ by anging to polar co-ordinates. *(W.B.U.T. 2002)*



Solution: Here the limits of x are 0, $\sqrt{a^2-y^2}$ and se of y are 0, a. The domain of integration R is the shaded on. Let us change cartesian to polar co-ordinates by ng $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \quad J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

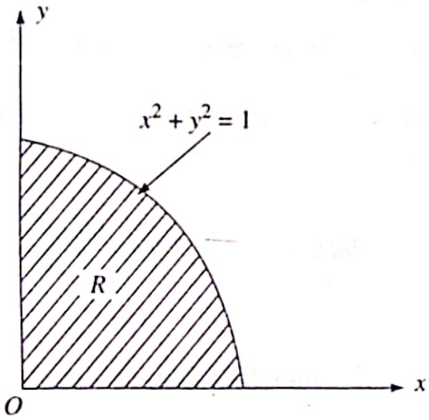
The region of integration $R = \{x^2+y^2 \le a^2, x \ge 0, y \ge 0\}$ in xy-plane is transformed to $R' = \left\{0 \le r \le a, 0 \le \theta \le \frac{\pi}{2}\right\}$ in the $r\,\theta$-plane.

$$\therefore \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} (r^2\cos^2\theta + r^2\sin^2\theta)\,|J|\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^3\,dr\,d\theta = \left[\frac{r^4}{4}\right]_0^a [\theta]_0^{\frac{\pi}{2}} = \frac{a^4}{4}\cdot\frac{\pi}{2} = \frac{\pi a^4}{8}.$$

Example 2: Evaluate $\iint \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$ over the positive quadrant, bounded by the circle $x^2+y^2=1$.

(*W.B.U.T. 2006*)



Solution: Here the domain of integration R is the shaded region. Let us change cartesian to polar co-ordinates by substituting $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \quad J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

The region of integration $R = \{x^2+y^2 \le 1, x \ge 0, y \ge 0\}$ in xy-plane is transformed to $R' = \left\{0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}\right\}$ in the $r\,\theta$-plane.

$$\therefore \quad \iint_R \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1}\sqrt{\frac{1-r^2\cos^2\theta - r^2\sin^2\theta}{1+r^2\cos^2\theta + r^2\sin^2\theta}}\,|J|\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{1}\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr\,d\theta = \left\{\int_0^{\pi/2} d\theta\right\}\int_0^1 \frac{r(1-r^2)}{\sqrt{1-r^4}}\,dr$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\int_0^1 \frac{2r\,dr}{\sqrt{1-r^4}} + \frac{1}{4}\int_0^1 (-4r^3)(1-r^4)^{-1/2}\,dr\right]$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\int_0^1 \frac{dt}{\sqrt{1-t^2}} + \frac{1}{4}\left\{\frac{(1-r^4)^{1/2}}{1/2}\right\}_0^1\right]. \qquad \text{(where } t = r^2\text{)}$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\{\sin^{-1} t\}_0^1 + \frac{1}{2}(0-1)\right] = \frac{\pi}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right).$$

Example 3: Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dx\,dy$ by changing to polar co-ordinates.

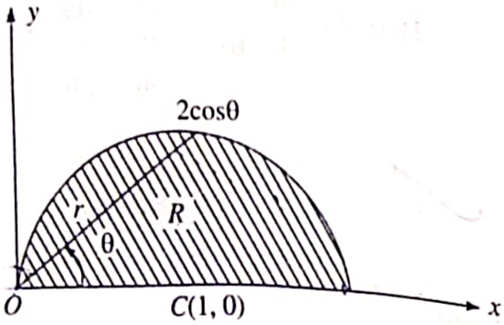
Solution: Here the limits of y are 0, $\sqrt{2x-x^2}$ and those of x are 0, 2.

$$\therefore \qquad y = \sqrt{2x-x^2}\text{, or } y^2 = 2x - x^2 \qquad \ldots(1)$$

or $x^2 - 2x + 1 + y^2 = 1$, or $(x-1)^2 + (y-0)^2 = 1^2$. It represents a circle whose centre is (1, 0) and radius is 1. The domain of integration R is the shaded region. Let us change cartesian to polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$ in (1).

$\therefore$ $r^2\sin^2\theta = 2r\cos\theta - r^2\cos^2\theta$, or $r = 2\cos\theta$.

Therefore, r varies form 0 to $2\cos\theta$ and θ varies from 0 to $\frac{\pi}{2}$.



Here
$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

The region of integration $R = \{(x-1)^2 + (y-0)^2 \le 1^2; x \ge 0, y \ge 0\}$ in xy-plane is transformed to $R' = \left\{0 \le r \le 2\cos\theta, 0 \le \theta \le \frac{\pi}{2}\right\}$ in the $r\,\theta$-plane.

$$\therefore \int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dx\,dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} (r^2\cos^2\theta + r^2\sin^2\theta)\,|J|\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left\{\int_0^{2\cos\theta} r^3\,dr\right\} d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_{r=0}^{2\cos\theta} d\theta = \frac{16}{4}\int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$= 4 \cdot \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = \frac{3\pi}{4}$$

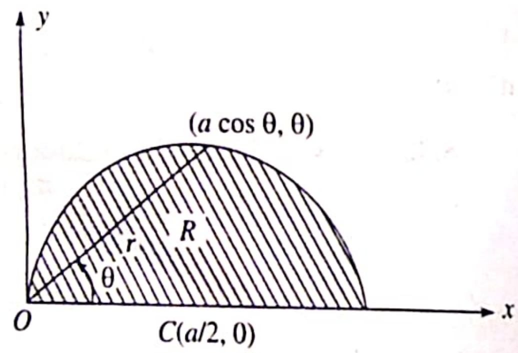
$$\left[\because \int_0^{\pi/2} \cos^n\theta\,d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \text{ where } n \text{ is even}\right]$$

Example 4: Evaluate $\iint \sqrt{a^2 - x^2 - y^2}\,dx\,dy$ over the region bounded by the semicircle $x^2 + y^2 = ax$ in the positive quadrant.

Solution: Here $x^2 + y^2 = ax$ $\qquad \ldots(1)$

or
$$x^2 - ax + \frac{a^2}{4} + y^2 = \left(\frac{a}{2}\right)^2$$

or
$$\left(x - \frac{a}{2}\right)^2 + (y-0)^2 = \left(\frac{a}{2}\right)^2.$$



It represents a circle whose centre is $(a/2, 0)$ and radius $= \frac{a}{2}\cdot$ The domain of integration R is the shaded region. Let us change cartesian to polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$ in (1).

$\therefore\ r^2\cos^2\theta + r^2\sin^2\theta = ar\cos\theta$, or $r = a\cos\theta$. Therefore, r varies from 0 to $a\cos\theta$ and θ varies from 0 to $\frac{\pi}{2}\cdot$

$$\text{Here } J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

The region of integration $R = \left\{\left(x-\frac{a}{2}\right)^2 + (y-0)^2 \le \left(\frac{a}{2}\right)^2; x \ge 0, y \ge 0\right\}$ in the xy-plane is transformed to $R' = \left\{0 \le r \le a\cos\theta, 0 \le \theta \le \frac{\pi}{2}\right\}$ in the $r\,\theta$-plane.

$$\therefore \quad \iint_R \sqrt{a^2-x^2-y^2}\,dxdy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta} \sqrt{a^2-r^2\cos^2\theta-r^2\sin^2\theta}\,|J|\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left\{\int_0^{a\cos\theta}\sqrt{a^2-r^2}\,r\,dr\right\}d\theta = -\frac{1}{2}\int_0^{\pi/2}\left\{\int_0^{a\cos\theta}(a^2-r^2)^{1/2}(-2r)\,dr\right\}d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[\frac{(a^2-r^2)^{3/2}}{3/2}\right]_{r=0}^{a\cos\theta} d\theta = -\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta - a^3)\,d\theta$$

$$= -\frac{a^3}{3}\left[\int_0^{\pi/2}\sin^3\theta\,d\theta - \int_0^{\pi/2}d\theta\right]$$

$$= -\frac{a^3}{3}\left[\frac{2}{3} - \{\theta\}_0^{\pi/2}\right] \qquad \left(\because \int_0^{\pi/2}\sin^n\theta\,d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1, \text{ when } n \text{ is odd}\right)$$

$$= \frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right).$$

Example 5: Evaluate $\iint_R \sqrt{\frac{a^2b^2 - b^2x^2 - a^2y^2}{a^2b^2 + b^2x^2 + a^2y^2}}\,dxdy$, where R is the region enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant.

Solution: Changing ellipse to a circle

Let us consider the transformation $x = au$, $y = bv$, whereby

$$J = \frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab.$$

The region of integration $R = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1; x \ge 0, y \ge 0\right\}$ in the xy-plane is transformed to $R' = \{u^2 + v^2 \le 1; u \ge 0, v \ge 0\}$ in the uv-plane and the given double integral transforms into

$$\iint_{R'} \sqrt{\frac{a^2b^2 - a^2b^2u^2 - a^2b^2v^2}{a^2b^2 + a^2b^2u^2 + a^2b^2v^2}}\, |J|\, du\, dv = ab \iint_{R'} \sqrt{\frac{1-u^2-v^2}{1+u^2+v^2}}\, du\, dv$$

Changing to polar co-ordinates

Put $u = r\cos\theta$, $v = r\sin\theta$, whereby

$$J = \frac{\partial (u,v)}{\partial (r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

and the region of integration R' is transformed to $R'' = \left\{0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}\right\}$ in the $r\,\theta$-plane. The double integral further changes to

$$ab \iint_{R''} \sqrt{\frac{1 - r^2\cos^2\theta - r^2\sin^2\theta}{1 + r^2\cos^2\theta + r^2\sin^2\theta}}\, |J|\, dr d\theta$$

$$= ab \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r\sqrt{\frac{1-r^2}{1+r^2}}\, dr d\theta$$

$$= ab \left\{\int_0^{\pi/2} d\theta\right\} \int_0^1 r\sqrt{\frac{1-r^2}{1+r^2}}\, dr$$

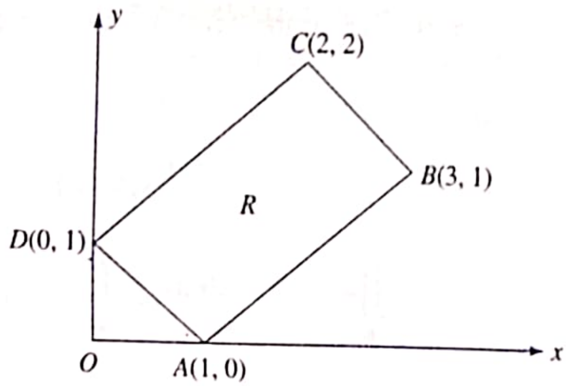
$$= \frac{\pi}{2} ab \left(\frac{\pi}{4} - \frac{1}{2}\right). \qquad \text{(see Example 2)}$$

Example 6: Evaluate $\iint_R (x+y)^2\, dx\, dy$, where R is the region enclosed by the parallelogram in the xy-plane with vertices (1, 0), (3,1), (2, 2), (0,1) using the transformation $u = x + y$ and $v = x - 2y$.

Solution: Here the points A, B, C, D in the xy-plane are transformed to A', B', C', D' respectively in the uv-plane and the parallelogram $ABCD$ is transformed to the rectangle $A'B'C'D'$ by the transformation $u = x + y$ and $v = x - 2y$.

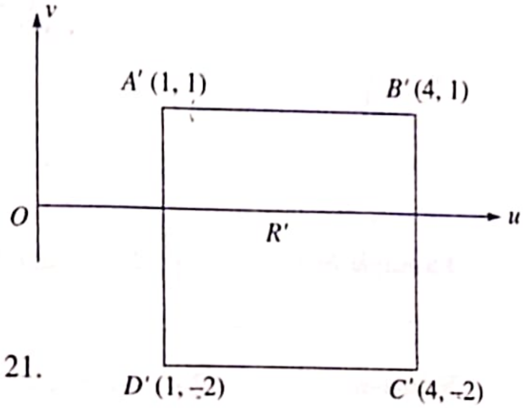
$$\therefore \qquad x = \frac{1}{3}(2u + v),$$

$$y = \frac{1}{3}(u - v).$$



Now,
$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{2}{3} & \dfrac{1}{3} \\ \dfrac{1}{3} & -\dfrac{1}{3} \end{vmatrix} = -\frac{1}{3}.$$



$$\therefore \iint_R (x + y)^2 dx\, dy = \iint_{R'} u^2 \,|J|\, du\, dv$$

$$= \frac{1}{3}\int_{v=-2}^{1}\int_{u=1}^{4} u^2 du\, dv$$

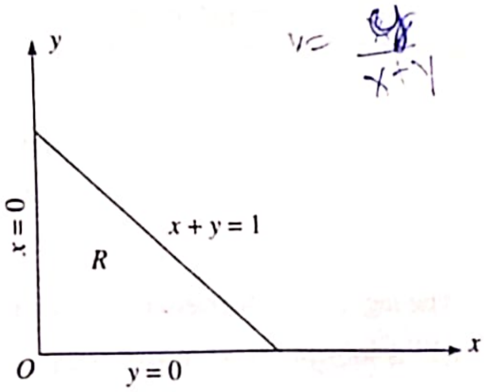
$$= \frac{1}{3}\left[\frac{u^3}{3}\right]_1^4 [v]_{-2}^{1} = \frac{1}{3}\left(\frac{63}{3}\right) \times 3 = 21.$$

Example 7: Using the transformation $x + y = u$, $y = uv$, show that

$$\iint e^{(y-x)/(y+x)} dx\, dy = \frac{1}{4}\left(e - \frac{1}{e}\right),$$

$v = \frac{y}{x+y}$

integration being taken over the area of the triangle bounded by the lines $x = 0$, $y = 0$, $x + y = 1$.

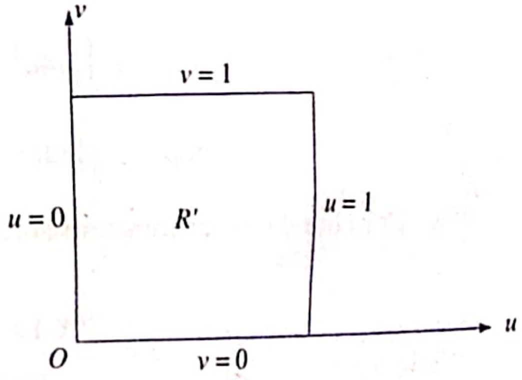


Solution: Here
$$\left.\begin{aligned} x + y &= u, & y &= uv \\ \therefore x &= u - y & &= u - uv \end{aligned}\right\} \quad \ldots(1)$$

Therefore, the Jacobian of transformation is

$$J = \frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 - v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$



Upper limit of $u = x + y$ is 1 $(\because x + y \le 1)$

$x = u - y = u - uv = u(1 - v)$, $y = uv$

$\therefore$ $x = 0$ gives $u(1 - v) = 0$, $y = 0$ gives $uv = 0$

$u = 0, v = 1$ $\qquad\qquad$ $u = 0, v = 0$

The region of integration $R = \{x+y \le 1, x \ge 0, y \ge 0\}$ in xy-plane is transformed to the region $R' = \{0 \le u \le 1, 0 \le v \le 1\}$ in uv-plane.

Also, $$y - x = uv - (u - uv) = 2uv - u,\ y + x = u \qquad \text{[from (1)]}$$

$$\therefore \quad \frac{y-x}{y+x} = 2v - 1$$

$$\therefore \quad \iint_R e^{(y-x)/(y+x)} dx\, dy = \iint_{R'} e^{2v-1} |J|\, du\, dv$$

$$= \int_0^1 \int_0^1 e^{2v-1} u\, du\, dv = \left\{\int_0^1 e^{2v-1} dv\right\} \int_0^1 u\, du$$

$$= \frac{1}{2}\left[e^{2v-1}\right]_0^1 \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{4}\left(e - \frac{1}{e}\right).$$

Example 8: Evaluate $\iint \{2a^2 - 2a(x+y) - (x^2+y^2)\}\, dx\, dy$ over the region bounded by the circle $x^2 + y^2 + 2a(x+y) = 2a^2$.

Solution: $x^2 + y^2 + 2a(x+y) = 2a^2$

or $$(x+a)^2 + (y+a)^2 = 4a^2$$

Therefore, the region of integration $R = \{(x+a)^2 + (y+a)^2 \le 4a^2\}$.

Let us use the transformation $x + a = X$, $y + a = Y$,

i.e., $$x = X - a,\ y = Y - a.$$

$$\therefore \quad J = \frac{\partial(x,y)}{\partial(X,Y)} = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

The region of integration $R = \{(x+a)^2 + (y+a)^2 \le 4a^2\}$ in the xy-plane is transformed to $R' = \{X^2 + Y^2 \le 4a^2\}$ in the XY-plane.

$$\therefore \quad \iint_R \{2a^2 - 2a(x+y) - (x^2+y^2)\}\, dx\, dy$$

$$= \iint_R \{4a^2 - (x+a)^2 - (y+a)^2\}\, dx\, dy$$

$$= \iint_{R'} (4a^2 - X^2 - Y^2)\, dX\, dY \qquad \ldots(1)$$

Now, let us use the polar transformation $X = r\cos\theta$, $Y = r\sin\theta$, whereby

$$J = \frac{\partial(X,Y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial X}{\partial r} & \frac{\partial X}{\partial \theta} \\ \frac{\partial Y}{\partial r} & \frac{\partial Y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

and the region of integration R' is transformed to $R'' = \{0 \le r \le 2a, 0 \le \theta \le 2\pi\}$ in the $r\theta$-plane. Therefore, from (1),

$$\iint_R \{2a^2 - 2a(x+y) - (x^2+y^2)\}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{2a} (4a^2 - r^2)r\,dr\,d\theta$$

$$= \left\{\int_0^{2\pi} d\theta\right\}\int_0^{2a}(4a^2-r^2)r\,dr = [\theta]_0^{2\pi}\left[4a^2\cdot\frac{r^2}{2} - \frac{r^4}{4}\right]_0^{2a}$$

$$= 2\pi(8a^4 - 4a^4) = 8\pi a^4.$$

Note: If the region of integration bounded is by the complete circle $x^2 + y^2 = a^2$ and we use the polar transformation $x = r\cos\theta$, $y = r\sin\theta$, then r varies from 0 to a and θ varies from 0 to 2π.

Example 9: Evaluate $\iint_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)dx\,dy$, where R is the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant.

Solution: Let use the transformation $x = aX$, $y = bY$.

$$J = \frac{\partial(x,y)}{\partial(X,Y)} = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

The region of integration $R = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1, x \ge 0, y \ge 0\right\}$ in the xy-plane is transformed to $R' = \{X^2 + Y^2 \le 1, X \ge 0, Y \ge 0\}$ in the XY-plane.

$$\therefore \quad \iint_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)dx\,dy = \iint_{R'} (1 - X^2 - Y^2)\,ab\,dX\,dY \qquad \text{...(1)}$$

Now let us use the polar transformation $X = r\cos\theta$, $Y = r\sin\theta$, whereby

$$J = \frac{\partial(X,Y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial X}{\partial r} & \frac{\partial X}{\partial \theta} \\ \frac{\partial Y}{\partial r} & \frac{\partial Y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

and the region of integration R' is transformed to $R'' = \left\{0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}\right\}$ in the $r\,\theta$-plane. Therefore, from (1),

$$\iint_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)dx\,dy = ab\int_{\theta=0}^{\pi/2}\int_{r=0}^{1}(1-r^2)r\,dr\,d\theta$$

$$= ab\left\{\int_0^{\pi/2} d\theta\right\}\int_r^1 (1-r^2)r\,dr = ab[\theta]_0^{\pi/2}\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1$$

$$= ab\cdot\frac{\pi}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi ab}{8}.$$

Example 10. Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy$ by changing to polar coordinates. Hence show that

$$\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

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Solution: Let us first consider the integral I_a of the function $f(x,y)=e^{-(x^2+y^2)}$ over $R=\{x^2+y^2\le a^2, x\ge 0, y\ge 0\}$ in the *xy*-plane.

Using the polar transformation $x=r\cos\theta, y=r\sin\theta$, whereby

$$J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}$$

$r=a$

$$=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$$

and the region of integration R is transformed to $R'=\left\{0\le r\le a; 0\le\theta\le\frac{\pi}{2}\right\}$ in the $r\,\theta$-plane.

$$\therefore \quad I_a=\iint_R e^{-(x^2+y^2)}dxdy$$

$$=\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} e^{-r^2}\,rdrd\theta$$

$$=[\theta]_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^a$$

$$=\frac{\pi}{4}(1-e^{-a^2})$$

Now the square $0\le x\le a, 0\le y\le a$ contains the positive guadrant of the circle $x^2+y^2=a^2$ and

$+ y^2 = a^2$ and
is contained in the positive quadrant of the circle $x^2 + y^2 = 2a^2$ as shown in the adjacent figure.

$$\therefore \qquad \frac{\pi}{4}(1-e^{-a^2}) = I_a \leq \int_0^a \int_0^a e^{-(x^2+y^2)} dxdy \leq I_{2a}$$

$$= \frac{\pi}{4}(1-e^{-4a^2})$$

Now $$\int_0^a \int_0^a e^{-(x^2+y^2)} dxdy = \int_0^a e^{-x^2} dx \int_0^a e^{-y^2} dy = \left\{\int_0^a e^{-x^2} dx\right\}^2$$

Hence $$\frac{\pi}{4}(1-e^{-a^2}) \leq \left\{\int_0^a e^{-x^2} dx\right\}^2 \leq \frac{\pi}{4}(1-e^{-4a^2})$$

Making $a \to \infty$, we get

$$\frac{\pi}{4} \leq \left\{\int_0^\infty e^{-x^2} dx\right\} \leq \frac{\pi}{4} \Rightarrow \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Triple Integ

Let V b

defined in V

F(x, y, z) in

If F(x