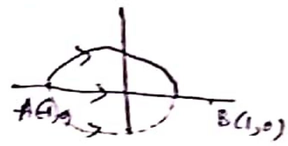
(16)

Ex Evaluate $\int_C (z^2+2z)\,dz$, where c is given by



(i) the upper half semi-circle joining A, B, ~~most~~ in the clockwise sense, where A is the point $(-1,0)$ & B is the point $(1,0)$ in the Argand plane.

(ii) Lower half semicircle joining A, B as shown in the figure

(iii) The straight line joining A, B as shown in the figure

verify that in all cases result is same.

(i) C is the curve $c: z = e^{i\theta}$, θ varies from π to 0, so that

$\frac{dz}{d\theta} = ie^{i\theta}$. [radius of the circle is 1]

Hence $\int_C (z^2+2z)\,dz = \int_\pi^0 \left[e^{2i\theta} + 2e^{i\theta}\right] ie^{i\theta}\,d\theta = i\int_\pi^0 \left[e^{3i\theta} + 2e^{2i\theta}\right] d\theta$

$$= i\left[\frac{e^{3i\theta}}{3i} + \frac{2e^{2i\theta}}{2i}\right]_\pi^0 = \frac{1}{3}\left[1 - e^{3\pi i}\right] + \left[1 - e^{2\pi i}\right]$$

$$= \frac{1}{3}\left[1 - (\cos 3\pi + i\sin 3\pi)\right] + \left[1 - (\cos 2\pi + i\sin 2\pi)\right]$$

$$= \frac{1}{3}\left[1-(-1)\right] + \left[1-1\right] = \frac{2}{3}$$

(ii) C is the curve $c: z = e^{i\theta}$, θ varies from π to 2π

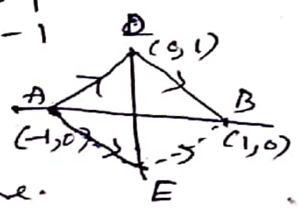
Hence $\int_C (z^2+2z)\,dz = \int_\pi^{2\pi} \left[e^{2i\theta} + 2e^{i\theta}\right] ie^{i\theta}\,d\theta$

$$= i\left[\frac{e^{3i\theta}}{3i} + \frac{2e^{2i\theta}}{2i}\right]_\pi^{2\pi} = \frac{2}{3}$$ (Calculation is as before, do yourself)

(iii) Here the curve is $c: z = x$, x varies from -1 to 1,

$\frac{dz}{dx} = 1$

$\therefore \int_C (z^2+2z)\,dz = \int_{-1}^{1} (x^2+2x)\cdot 1\,dx = \left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$.



You can verify Along ADB, also the result is same.

Along AEB also the result is same.

Actually along any curve passing through A, B from A to B, the result is $\frac{2}{3}$.

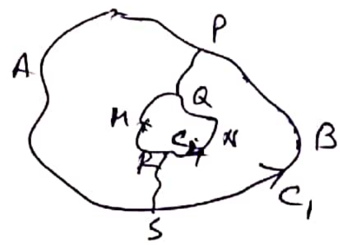
(17)

Th Cauchy–Goursat Theorem (statement only)

Let $f(z)$ be analytic function within and on a simple closed contour C. Then $\int_C f(z)\,dz = 0$.

Th Let $f(z)$ be analytic within and on an annular region bounded by two closed contours C_1 and C_2, C_2 lying completely within C_1, Then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$, the path of integration being described in the same sense.

Proof: Let us take two points P and S on C_1 and two points Q and R on C. We join PQ and RS by two polygonal arcs such that they do not intersect and do not cross C_1 and C_2. Then we get the simple closed contours $PASRMQP$ and $SBPQNRS$ and ~~calle~~ called them L_1 and L_2 respectively.



Then by Cauchy–Goursat theorem

$$\int_{L_1} f(z)\,dz = 0 = \int_{L_2} f(z)\,dz$$

$$\text{or } \int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = 0$$

$$\text{i.e. } \int_{PAS} f(z)\,dz + \cancel{\int_{SR} f(z)\,dz} + \int_{RMQ} f(z)\,dz + \cancel{\int_{QP} f(z)\,dz}$$

$$+ \int_{SBP} f(z)\,dz + \cancel{\int_{PQ} f(z)\,dz} + \int_{QNR} f(z)\,dz + \cancel{\int_{RS} f(z)\,dz} = 0$$

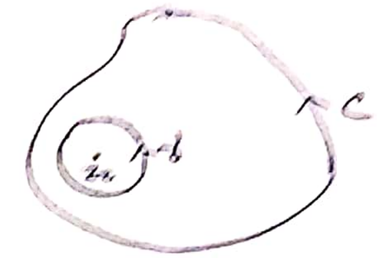
$$\text{or, } \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0 \Rightarrow \int_{C_1} f(z)\,dz = -\int_{-C_2} f(z)\,dz$$

$$= \int_{C_2} f(z)\,dz$$

18 **Cauchy's integral formula:**

Th: Let $f(z)$ be analytic within and on a positively oriented closed contour C and z_0 be a point within C, then $f(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$

Proof: Let $\epsilon>0$ be arbitrary.



Since $f(z)$ is continuous at z_0, ~~then~~ there is $\delta>0$, such that

$|f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|<\delta$ —— (1)

Let γ be a circle with centre at z_0 and radius ρ such that $\rho<\delta$ and γ lies completely within C. Then $\frac{f(z)}{z-z_0}$ is analytic in the annular region bounded by C and γ, and hence we have

$$\int_C \frac{f(z)}{z-z_0}\,dz=\int_\gamma \frac{f(z)}{z-z_0}\,dz \quad \text{—— (2)},$$ when C and γ are described in the same sense.

Now $$\int_\gamma \frac{f(z)}{z-z_0}\,dz = f(z_0)\int_\gamma \frac{dz}{z-z_0} + \int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$= f(z_0)\left[\int_0^{2\pi}\frac{i\rho e^{i\theta}}{\rho e^{i\theta}}\,d\theta\right] + \int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$= 2\pi i f(z_0) + \int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz,$$ As on γ, $z-z_0=\rho e^{i\theta}$, θ varies from 0 to 2π

$$\therefore \left|\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0)\right| = \left|\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \quad \text{—— (3)}$$

(from (2) and the result above)

Now if z lies ~~within~~ on γ, we have from (1)

$|f(z)-f(z_0)|<\epsilon$

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Also if z lies on γ, $|z-z_0| = \rho$

Therefore $\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \frac{\epsilon}{\rho}$ for z on γ

Hence $\left|\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \leq \frac{\epsilon}{\rho} \cdot 2\pi\rho = 2\pi\epsilon$

Therefore from (3), we have

$$\left|\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0)\right| \leq 2\pi\epsilon$$

or $$\left|\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz - f(z_0)\right| \leq \epsilon$$

As ϵ is arbitrary, approaching ϵ to zero, we get

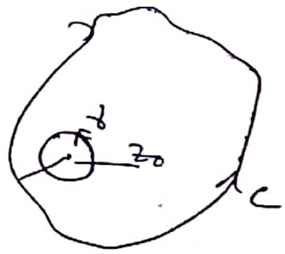
$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz.$$

Th Derivative of Analytic function

Statement: Let $f(z)$ be analytic within and on a closed contour described in the positive sense and let z_0 be any point within C;

then $$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz.$$

Proof:



Let $\delta > 0$ be the shortest distance of z_0 from C. Let h be any complex number such that $|h| < \frac{\delta}{2}$.

Then z_0+h lies within C. So by Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \quad \text{and} \quad f(z_0+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0-h}\,dz$$

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Hence $\dfrac{f(z_0+h)-f(z_0)}{h} - \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(z)}{(z-z_0)^2}\,dz$

$$= \frac{1}{2\pi i}\left[\int_C \frac{f(z)}{\cancel{h}}\left\{\frac{\cancel{h}}{(z-z_0)(z-z_0-h)}\right\}dz - \int_C \frac{f(z)}{(z-z_0)^2}\,dz\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)\;h}{(z-z_0)^2(z-z_0-h)}\,dz \quad \text{—— (1)}$$

If z lies on C, $|z-z_0| \geq \delta$ and

$|z-z_0-h| \geq |z-z_0|-|h| \geq \delta - \frac{1}{2}\delta = \frac{\delta}{2}$

and so $\left|\dfrac{1}{(z-z_0-h)(z-z_0)^2}\right| \leq \dfrac{1}{\delta^2\cdot\frac{\delta}{2}} = \dfrac{2}{\delta^3}$.

Also since $f(z)$ is continuous on C, there is $M>0$ such that $|f(z)| \leq M$ for z on C.

Hence $\left|\dfrac{f(z)}{(z-z_0-h)(z-z_0)^2}\right| \leq \dfrac{2M}{\delta^3}$

$\therefore \left|\dfrac{f(z_0+h)-f(z_0)}{h} - \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(z)}{(z-z_0)^2}\,dz\right|$

$$= \left|\frac{h}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^2(z-z_0-h)}\right| \leq \frac{|h|}{2\pi}\cdot\frac{2M}{\delta^3}\cdot L \quad \text{—— (2)}$$

where L is the length of C.

Approaching h to zero, we get

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

(21) Theorem for higher order derivative (statement only)

Statement : Let $f(z)$ be analytic within and on a closed contour C described in the positive sense and let z_0 be a point within C, then

$$f^{n}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$