# DIRICHLET’S CONDITIONS

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

## Dirichlet's conditions:

A function $f$ will be said to satisfy Dirichlet's condition on an interval $-\pi \leq x \leq \pi$, if the following conditions are satisfied

(i) $f(x)$ is bounded, periodic with period $2\pi$ and absolutely integrable in $-\pi \leq x \leq \pi$ i.e.

$$\int_{-\pi}^{\pi} |f(x)|\, dx < \infty$$

(ii) $f(x)$ must have a finite number of extrema in any given sub-interval

(iii) $f(x)$ must have a finite number of discontinuities in any given sub-interval.

## Convergence of Fourier series:

When $f(x)$ satisfies Dirichlet's conditions on $-\pi \leq x \leq \pi$, the Fourier series corresponding to $f(x)$

(i) converges to $\boxed{f(x)}$ at any point $x$ on $-\pi < x < \pi$, when $f(x)$ is continuous,

(ii) converges to $\boxed{\frac{1}{2}[f(x+0) + f(x-0)]}$ when there is an ordinary discontinuity at the point.

(iii) At $x = \pi$ and $x = -\pi$, it converges to $\boxed{\frac{1}{2}[f(-\pi+0) + f(\pi-0)]}$, when $f(-\pi+0)$ and $f(\pi-0)$ exists.

**Note:** $f(a+0) = \lim\limits_{x \to a+} f(x)$, $f(a-0) = \lim\limits_{x \to a-} f(x)$

① Obtain the Fourier series for the function

$$f(x) = x^2, \quad -\pi < x < \pi$$

Hence deduce that,

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

A Since $f(-x) = f(x)$ i.e. $f(x)$ is an even function, hence $b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\,dx = \boxed{\frac{2}{3}\pi^2}$$

and, $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{1}{n}x^2 \sin nx\right]_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi} 2x \sin nx\,dx$$

$$= \frac{-4}{n\pi}\left[-\frac{x}{n}\cos nx + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{4}{n^2\pi}\cdot\pi\cos n\pi = \boxed{\frac{4}{n^2}(-1)^n}$$

$\therefore$

$$x^2 \approx \frac{1}{2}\cdot\frac{2}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{1}{3}\pi^2 - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right]$$

At $x = \pi$ :

At $x=\pi$ the sum of the series converges to, $\frac{1}{2}\left[f(-\pi+0) + f(\pi-0)\right] = \frac{1}{2}\left[\pi^2+\pi^2\right] = \pi^2$

$\therefore$ we get,

$$\pi^2 = \frac{\pi^2}{3} - 4\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \cdots\right]$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\Rightarrow \boxed{\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots} \quad \text{(Proved)}$$

At $x=0$:

$x=0$ is a point of continuity, so sum of the series converges to $f(0)=0$.

$\therefore$ we get,

$$0 = \frac{1}{3}\pi^2 - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$\Rightarrow \boxed{\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots}$$ (Proved)

Adding the last two series, we get,

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2.\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{4} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$ (Proved)