Signals and Systems
Part-A

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November 10, 2020

Overview

1. Fundamental concepts
   - Basic concepts of a signal
   - Basic concepts of a system
2. Frequency domain representation of signals
   - Periodic continuous signals - Fourier series
   - Periodic discrete signals and Fourier series
   - Aperiodic continuous signals and Fourier Transform
   - Discrete Fourier Transform
3. Transform domain representation of systems
   - Frequency Domain interpretations
   - Laplace transform in continuous domain
   - Z transform in discrete domain

Signal and their response

- Signal is basically a function of one or more independent variables that describes a physical phenomenon.
- The input signal then passes through a system governed by some physical phenomena
- The system responds to the input signal resulting in some interesting signal characteristics at the output.
- The signal generally varies with time but system response is governed by frequency characteristics.
- Of interest in this subject is the signal, the system and the response characteristics in time and frequency domain.

Coverage of the subject

- The theory is mathematical and the applications mainly lie in fields of communication, control of dynamical systems.
- There are many diverse fields where the theory applies, making it one of the most powerful tools for grasping physical phenomena.
- Theory and Applications part are placed in two separate presentation class notes. The distinction is not of course sequential. Whenever some theory is covered in Part-A, corresponding applications would be visited in Part-B.
- Most of this study material has been prepared based on book Signals and Systems (Second Edition) by Oppenheim. Willsey and Hamid Nawab.

Continuous signal

- Signal is basically a function of one or more independent variables that describes a physical phenomenon.
- The phenomenon can be from diverse fields, like
  - Temporal variation of voltage or current across a circuit – input voltage can vary as well as there is a differential equation governing the current buildup as in capacitor or inductor.
  - Temporal variation of position or speed of a moving target – input actuation like acceleration, turn can change and output is governed by equations of motion
  - Spatial or spatiotemporal variation of brightness of picture – background and foreground changes in space and time and captured by some lens

Discrete signal

- Some phenomena are inherently discrete in nature.
- The signal value is not available as continuous function of time.
- Rather the phenomenon is defined only at (regular) discrete intervals of time.
- Population count or share market indices are only available as a discrete sequence
- In digital system, the values are generally provided in discrete form as well.

Distinguishing Continuous signal and discrete signals

- Distinguishing the two types of signals is important because the mathematical treatment and analysis differs for these two signal types.
- Otherwise it is possible to develop general mathematical concepts for treating the two signal types irrespective of the physical phenomenon they describe.
- That is precisely the goal of this subject.

Signal Energy and power

- Let the signal be represented as a function of time $x(t)$. Then the power associated with the signal is

$$\int_{t_1}^{t_2} x^2(t)dt$$

.

- The expression is kept dimensionless, so that the physical phenomenon being represented may be brought in for interpretation.
- An example could be the electrical energy when applied voltage is $v(t)$ with resistive load R the energy is

$$\frac{1}{R}\int_{t_1}^{t_2} v^2(t)dt$$

.

Independent variable concept

- While the variations of voltage or position in above examples are with respect to time, the Brightness variation is spatial and two variables are involved as independent variables.
- Apart from time and space, other physical variables can also assume the role of independent variable, like in meteorology, temperature and air pressure varies with altitude.
- In any system, it is important to find the independent variable with respect to which the signal varies.

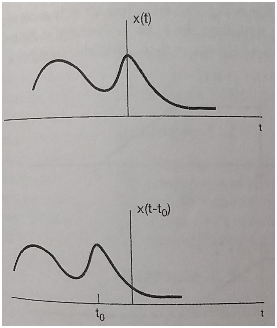
Time shift

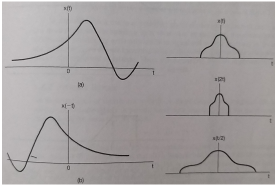
- The independent variable may undergo transformation.
- Consider $x(\alpha t + \beta)$ a signal transformed from original signal $x(t)$.
- This actually has a time shift β and a time scaling of α.
- So a signal $x(t + \beta)$ would advance (shift to the left) or delay (shift to the right when sign is negative) the signal $x(t)$ by an amount of time β.
- Now $x(-t + \beta)$ is the time reversed version of $x(t + \beta)$ obtained by reflecting the graph (plot) of the signal about the time axis.

Time scaling

- Now coming to the scaling of α the resultant signal is a compression of the original signal in time by a factor $\frac{1}{\alpha}$ of t.
- The signal is in fact linearly stretched if $|\alpha| < 1$, and compressed when $|\alpha| > 1$, and reversed in time when $\alpha < 0$.
- Such transformation is important for representing radar signals or fast forwarding audio/video recording. Several properties of signals and systems can be derived by examining the behaviour under time axis transformation.

Time scaling, shift and reflection plots





Periodicity of signal

- Period T signals in continuous time can be expressed in terms of $x(t+T) = x(t)$. In other words, the signal value does not change for a shift by an amount T.
- Natural response of systems, where energy is conserved, is periodic. Like ideal LC circuits or oscillating pendulum with no friction loss.
- Now since $x(t+mT) = x(t)$ for any m the same signal is periodic for $T, 2T, 3T \ldots$. Here smallest T for which the equation holds is called the fundamental period and rest are harmonics. If no such T can be found the signal is aperiodic.
- Analogously discrete time signals can have period N so that $x[n] = x[n+N]$.

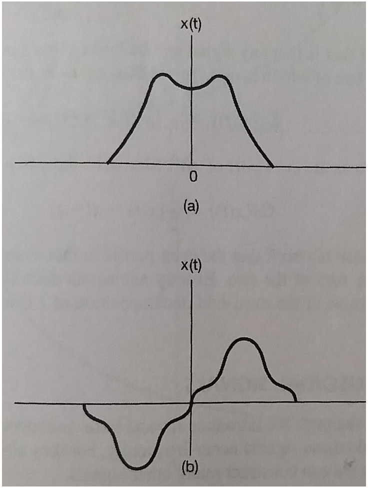
Checking periodicity

- There is a risk in determining periodicity of a signal.
- Suppose $x(t) = cos\ t$ if $t < 0$ and $= sin\ t$ for $t \geq 0$.
- Here two trigonometric functions are present, both are periodic in 2π but there is a marked discontinuity at $t = 0$ since at $t = 0^-$ $x(t) = cos0 = 1$ but at $t = 0$ $x(t) = sin0 = 0$.
- So although the periodicity recurs on either side of origin $t = 0$ the discontinuity at $t = 0$ disqualifies the signal from being periodic.

Even and Odd signals

- Signal is even if $x(-t) = x(t)$ and for discrete domain $x[-n] = x[n]$. Even signal is symmetric to its time reversed counterpart.
- Signal is odd if $x(-t) = -x(t)$ and for discrete domain $x[-n] = -x[n]$. Odd signal is symmetric on either side of the time axis.
- This implies that odd signals must cross zero at t=0 else the relation cannot hold at t=0.
- A signal can be split or decomposed into sum of an even signal and an odd signal.
- Even part $x_{even}(t) = \frac{1}{2}[x(t) + x(-t)]$ which is always even signal.
- Odd part $x_{odd}(t) = \frac{1}{2}[x(t) - x(-t)]$ which is always odd signal.

Even and Odd signal plots



Real and complex exponential signals

- Continuous time complex exponential signal $x(t) = Ce^{at}$ with C and a being complex numbers.
- When imaginary part is absent in both C, a there can be a growing exponential for +ve a and decaying exponential for -ve a. Wide variety of physical phenomena are described using these two equations.
- When a is purely imaginary, take $x(t) = e^{j\omega_0 t}$ which is periodic implies $e^{j\omega_0 t} = e^{j\omega_0 (t+T)}$.
- For periodicity to hold, $e^{j\omega_0 T} = 1$. For $\omega_0 = 1$ this is trivial. Otherwise the smallest positive value of T is $T_0 = \frac{2\pi}{|\omega_0|}$.
- Hence $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have same fundamental period. Equivalently $\omega_0 = 2\pi f_0$ being Time period and frequency respectively.
- Using Euler results, $x(t) = e^{j\omega_0 t} = cos\omega_0 t + jsin\omega_0 t$.
- Therefore the Sinusoidal signal can be expressed as $Acos(\omega_0 t + \phi) = \frac{A}{2}e^{j(\omega_0 t+\phi)} + \frac{A}{2}e^{-j(\omega_0 t+\phi)}$.

Energy of sinusoid signal

- Sinusoids describe many physical phenomena like current oscillation of LC circuit, angular position oscillation in simple pendulum.
- For constant signal, the $\omega_0 = 0$. Hence T_0 becomes undefined but frequency is zero.
- Energy of periodic signal is such that total energy is infinite but average power is finite. Over a given period,

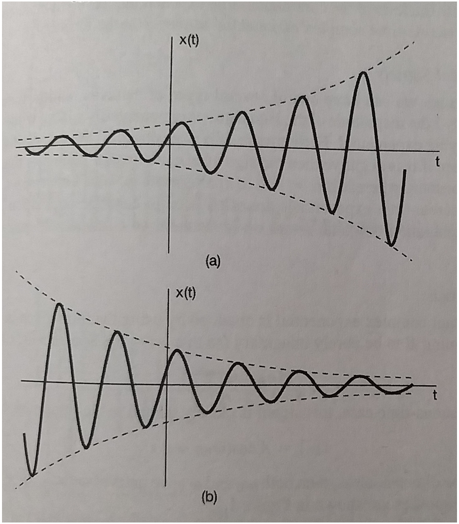
$$E = \int_0^{T_0} |e^{j\omega_0 t}|^2 dt = T_0$$

  Since amplitude modulus is unity. Therefore power over the given period is $P = \frac{1}{T_0} E = 1$.
- This average power remains unity over multiple periods that tend to infinity, although the total energy integrated over the entire time is infinite.
- Harmonically related exponential signals $\phi_k(t) = e^{jk\omega_0 t}$ for all $k = 0, \pm 1, \pm 2, \dots$
- The condition $|e^{j\omega T_0}| = 1$ holds for $\omega T_0 = 2\pi k$ so that ω is integral

General complex signal

- When both real and imaginary parts exist, the form is most general. $C = |C|e^{j\theta}$ and $a = r + j\omega_0$. So $Ce^{at} = |C|e^{rt}e^{j\omega_0 t+\theta}$.
- Thus the signal is Sinusoidal multiplied with a growing or decaying exponential.
- The exponential signal acts as an envelope for the oscillating signal.
- The decaying or damped sinusoid is found in RLC circuits or damped spring mass system where the electrical resistance or mechanical friction dissipate energy.

Periodic growing and decaying signal plots



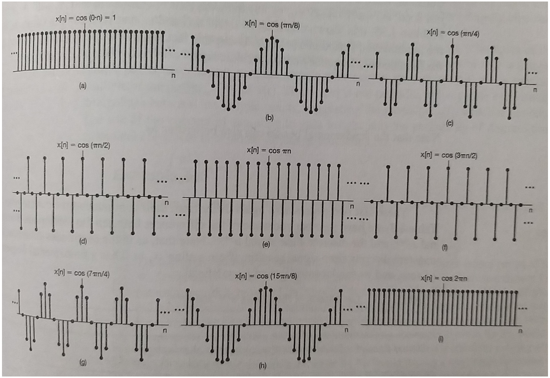
Discrete time complex signal

- For discrete time, the signal expression becomes $x[n] = C\alpha^n$ or in tune with the continuous signal counterpart $x[n] = Ce^{\beta n}$ where $\alpha = e^{\beta}$.
- With real C and α for positive values of α there is exponential growth and the signal decays exponentially for negative α.
- Again, for purely imaginary β such that $|\alpha|$ is unity, the signal is Sinusoidal. Finally for most general case expressed in polar form, $C = |C|e^{j\theta}$ and $\alpha = |\alpha|e^{j\omega_0}$ then $C\alpha^n = |C||\alpha|^n cos(\omega_0 n + \theta) + j|C||\alpha|^n sin(\omega_0 n + \theta)$.
- So the signal resembles real and imaginary part sinusoids for $|\alpha| = 1$ otherwise they represent sequence of sinusoid enveloped within growing or decaying exponential.

Periodicity properties of discrete signal

- For frequency $\omega_0 + k2\pi$ where k is an integer, the sequence $e^{j(\omega_0+2\pi k)n} = e^{j\omega_0 n}$ since $e^{j2\pi kn} = cos(2\pi kn) + jsin(2\pi kn) = 1$.
- So the sequence is not distinct, implying it is enough to consider the signal for the frequency interval $0 \leq \omega_0 < 2\pi$ or $-\pi \leq \omega_0 < +\pi$.
- Now for periodicity with period N, $e^{j\omega_0(n+N)} = e^{j\omega_0 n}$ so that $e^{j\omega_0 N} = 1$.
- For this, $\omega_0 N = 2\pi m$ with integer m. Hence $\frac{\omega_0}{2\pi} = \frac{m}{N}$.
- So the fundamental period is $N = m\frac{2\pi}{\omega_0}$.

Discrete periodic signal plots



Periodicity properties differences

The following differences exist between continuous and discrete signals:

- For distinct ω_0 signal is distinct for continuous domain but signal identical in every 2π interval for discrete domain.
- Continuous signal is periodic for any choice of ω_0 but discrete signal is periodic only for $\omega_0 = \frac{2\pi m}{N}$.
- Fundamental frequency for discrete signal is ω_0/m while that for continuous signal is ω_0.
- Fundamental period for discrete signal is likewise a multiple of the continuous domain case.

Impulse and Unit step signals

- Impulse is expressed in discrete domain as $\delta[n] = 0$ for $n \neq 0$ and $\delta[n] = 1$ for $n = 0$.
- Unit step signal in discrete domain is given by $u[n] = 0$ for $n < 0$ and $u[n] = 1$ for $n \geq 0$.
- One can write $\delta[n] = u[n] - u[n-1]$ and

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

  . Thus the two signals are related to one another. Also this brings out the sampling properties of the unit impulse.
- Continuous domain impulse and step signal can be defined similarly. The sum is replaced with integration and the difference is replaced with first order differentiation.

Systems - meaning

- Physical systems are interconnection of components, devices, subsystems.
- communication, electromechanical, industrial production process
- input signals get transformed into output signals due to system response
- signal transformation can be voltage to current in circuits or force to velocity in vehicles

Systems - continuous time

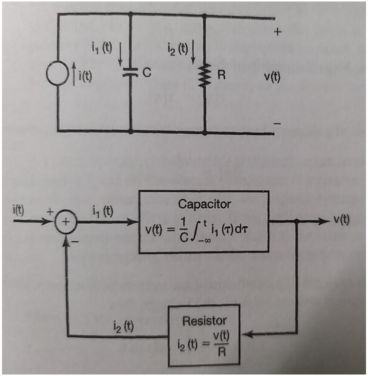
- Input signal is continuous time, output signal is also continuous time
- example can be some RC circuit with input voltage $v_s(t)$, current $i(t)$ and voltage developed across the capacitor $v_c(t)$.
  $\frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$
- example can be vehicle on which force $F(t)$ is applied and vehicle moves with output velocity $v(t)$. With friction coefficient ρ and vehicular mass m equation becomes $\frac{dv(t)}{dt} + \frac{\rho}{m} v(t) = \frac{1}{m} F(t)$
- General form $\frac{dy(t)}{dt} + ay(t) = bx(t)$ with $x(t)$ input signal and $y(t)$ output signal.

Systems - discrete time

- Input signal is discrete time, output signal is also discrete time
- Bank balance $y[n] = (1+\rho)y[n-1] + x[n]$ with rate of interest ρ and net deposit $x[n]$ for the n^{th} month.
- Digital simulation of vehicular movement in discrete time intervals $\Delta t = T$ with $\frac{dv(t)}{dt}$ evaluated at $t = nT$ is $\frac{v(nT)-v(\overline{n-1}T)}{T}$. Dropping T yields $v[n] - \frac{m}{(m+\rho T)}v[n-1] = \frac{T}{(m+\rho T)}F[n]$ for sampled force and velocity signals.
- General expression is therefore $y[n] + ay[n-1] = bx[n]$ for output signal $y[n]$ and input $x[n]$

Interconnection of systems

- Systems are built as interconnect of various subsystems.
- Complex systems can be synthesized as interconnect of basic building blocks.
- Series and parallel blocks can be cascaded to realize complex systems.
- Feedback interconnection is another very useful building block.



Realization of physical systems

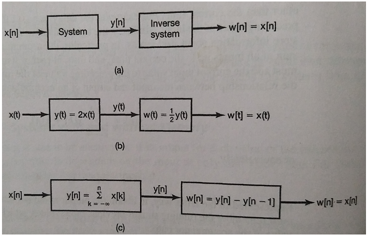
To realize physical systems like an electric motor, an electromagnetic system (producing torque depending upon applied voltage passing through inductance) and a mechanical system (producing rotational speed based on torque depending on moment of inertia) can be cascaded and feedback of speed through back emf adjusts the applied voltage. This is a second order feedback system and would be detailed later also.

Systems - memoryless and with memory

- Output depends only on present input.
- Example is Physical system that describes Ohm's law instantly describes current from voltage.
- For some systems, output depends on present as well as past inputs.
- Accumulator that keeps running total, delay system that outputs previous input after specified time are examples in discrete domain.
- A capacitor that undergoes charging is an example in analog domain.
- Any system governed by differential equations has to remember the past for present output.
- Memory often implies storage of energy. It could be electric energy in capacitor, kinetic energy in vehicles and so on.

Invertibility of systems

- Distinct inputs lead to distinct outputs.
- The inverse system when cascaded with original system would produce the input as output which implies the identity system.
- For systems that generate even powers of input, output cannot be inverted since sign is indeterminate. Also system producing sequence of zeros is noninvertible.
- Systems that use some encoding must be invertible and the decoder has to exist.



Causality of systems

- Systems that depend only upon present and past values
- Systems that do not have to anticipate any future values
- This signifies that if two inputs are identical upto a time point, the outputs are identical.
- Example systems where time is not independent variable like in image processing are non-causal.
- Systems that keep a trend (historical) rather than maintaining the entire past datasets are non-causal, like in meteorology.
- Sometimes high frequency fluctuations are deliberately suppressed by taking moving average to determine slow varying trends, as in stock market analysis resulting in non-causal system.

Stability of systems

- Systems whose response do not diverge for small inputs are stable. For bounded input, output is also bounded.
- Example of stable system is simple pendulum that swings with small deflection for small forces where the restoring force of gravity and drag forces act together. In contrast the inverted pendulum is unstable because the gravity tends to diverge any deflection produced by small forces.
- Stability results from some mechanisms that dissipate energy like a resistance in electric circuit or friction in mechanical systems.
- The bound on output can be found from system model. A constant force F applied on a vehicle would have a bounded velocity since the retarding friction would be proportional (ρV) to increase in velocity and ultimately balance the applied force when velocity reaches $V = \frac{F}{\rho}$.
- Moving average is bounded for bounded input but accumulators are unbounded as sum keeps growing.

Time invariance of systems

- System parameters do not change with time. Subtle changes like heavier mass at times or resistance changes due to higher operating temperature can render a system time varying.
- System followed by time delay is equivalent to Time delay followed by system
- $c(t) = f(r(t)) = tr(t)$ Take $r(t-\tau)$ as a delayed input and $c(t-\tau)$ as delayed output. $c_1 = delay(f(r(t))) = tr(t-\tau)$ and $c_2 = f(delay(r(t))) = (t-\tau)r(t-\tau)$ and $c_1 \neq c_2$ implies system is not time invariant.
- Time shift of input signal should result in identical time shift in output signal.
- For example, $y(t) = x(2t)$ is not time invariant. Taking $x'(t) = x(t-\tau)$ gives $y'(t) = x'(2t)$ are not identical signals.

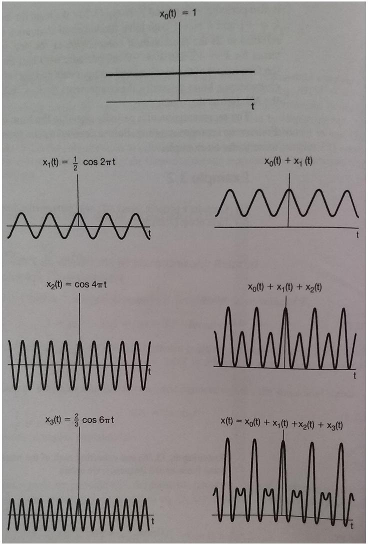
Linearity of systems

- Superposition must hold: if $c_1(t)$ is output signal when input is $r_1(t)$ and $c_2(t)$ is output signal when input is $r_2(t)$, then for input signal $a_1 r_1(t) + a_2 r_2(t)$ the output has to be $a_1 c_1(t) + a_2 c_2(t)$.
  $c(t) = f(r(t))$ is the time domain transfer function of the system.
  $c(kT) = f(a_1 r_1(kT) + a_2 r_2(kT)) = a_1 f(r1(kT)) + a_2 f(r_2(kT))$
- Ex. $c(t) = r(t) + \alpha$. Then, $c_1 = r_1 + \alpha$ and $c_2 = r_2 + \alpha$. For $r = a_1 r_1 + a_2 r_2$, $c_{actual} = a_1 r_1 + a_2 r_2 + a_1\alpha + a_2\alpha$ but for linearity $c = a_1 r_1 + a_2 r_2 + \alpha$. $c_{actual} \neq c$. Hence, the system is nonlinear.
- $c(kT) = r^2(kT)$ is nonlinear.

Linear time invariant systems- significance

- Superposition property allows arbitrary signals to be expressed as linear combination of basic signals and compute the output in terms of response to the basic signals.
- General signals can be represented as linear combination of delayed impulses which enables usage of the time invariance property.
- Studying the response to unit impulse is enough to completely characterize the system, this representation is convolution sum or integral.

Superposition of signals in time domain



Representation of periodic signals

- For some positive T, $x(t) = x(t+T)$ with $\omega_0 = 2\pi/T$ being fundamental frequency.
- In terms of harmonically related exponentials $\phi_k(t) = e^{jk\omega_0 t}$, with $k = 0, \pm 1, \pm 2, \ldots$ all have period T.
- Then,

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

coefficients a_k attached to the harmonics.

Determination of the coefficients

- $$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{j(k-n)\omega_0 t}$$

  multiplying both sides.
- Next integrating both sides from 0 to T,

  $$\int_0^T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{+\infty} a_k[\int_0^T e^{j(k-n)\omega_0 t}dt]$$

  .
- Now

  $$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T cos(k-n)\omega_0 tdt + j\int_0^T sin(k-n)\omega_0 tdt$$

Determination of the coefficients

- Value of integral is zero for $k \neq n$ due to periodic signal being integrated over its period. For $k = n$ integral is straightforward with value T.
- Consequently, the right hand side yields

$$a_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t}$$

.

- The index may be changed to k and the formula remains valid for any time period T, not just the interval $0, T$.
- The dc component is

$$a_0 = \frac{1}{T} \int_T x(t) dt$$

.

Approximation and Convergence criteria

- Error in the approximation to N terms is

$$e_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

.

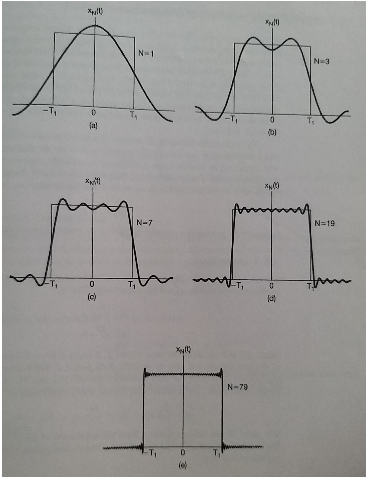
- Error criterion is based on error energy in one period

$$E_N = \int_T |e_N(t)|^2 dt$$

.

- Convergence is guaranteed for signals where $E_N \rightarrow 0$ as $N \rightarrow \infty$.
- Finite energy periodic signals guarantee such convergence. Signals need to meet the three Dirichlet conditions.

Approximating the square wave signal in time domain



Alternate Approximation and Convergence criteria

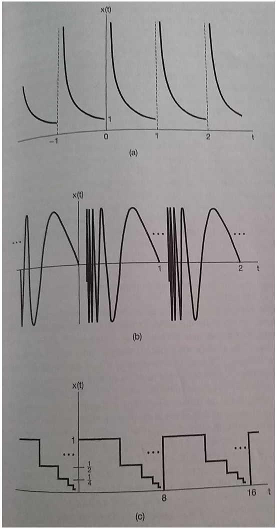
- Absolute integrability condition can also be applied.

$$\int_T |x(t)|dt < \infty$$

.

- A periodic signal that violates this is $x(t) = \frac{1}{t}$.
- Finite number of maxima and minima of the signal is a requirement. $x(t) = sin(\frac{2\pi}{t})$ in the interval $0 < t \leq 1$ fails this requirement.
- In any finite time interval, number of discontinuities has to be finite. Example is a staircase signal comprising infinite sections with the property that the width and height both become half of previous section.

Some signals that violate Dirichlet condition



Summary of Fourier series properties - continuous domain

- **Linearity:-** For Signal $\alpha x(t) + \beta y(t)$ the Fourier coefficients are $c_k = \alpha a_k + \beta b_k$. Extendable to arbitrary number of linear combinations.
- **Time shift by t_d:-** Fourier coefficient becomes (after change of variables)

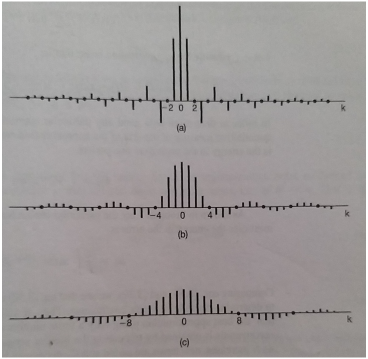
$$b_k = \frac{1}{T}\int_T x(t - t_d)e^{-jk\omega_0 t}dt = e^{-jk\omega_0 t_d}a_k$$

  Hence the magnitude remains unaltered.
- **Time reversal:-** For $y(t) = x(-t)$ Fourier coefficients are $b_k = a_{-k}$. Thus for even signal, Fourier coefficients are even, for odd signal they alter the sign.

Summary of Fourier series properties - continuous domain

- **Time scaling:-** Here
$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$
So coefficients remain unchanged, fundamental frequency has changed.
- **Multiplication:-** Product of two periodic signals is periodic and Fourier coefficients are
$$h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$
This is discrete time convolution of original coefficients.
- **Conjugation symmetry :-** For signal $x^*(t)$ coefficients are a^*_{-k} so that $a_{-k} = a^*_k$ which gives the concept of conjugate symmetry such that $|a_k| = |a_{-k}|$ for real signals.

Fourier series coefficients of square wave for various time periods



Expression for discrete domain

- Here the signal is $x[n] = x[n+N]$. Then it can be expressed as linear combination of the harmonic signals

$$x[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk(2\pi/N)n}$$

.

- Evaluated expressions for the signal at the time points are $x[0] = \Sigma_k a_k$, $x[1] = \Sigma_k a_k e^{j2\pi k/N}$, ..., $x[N-1] = \Sigma_k a_k e^{j2\pi k(N-1)/N}$.
- Since

$$\sum_n e^{jk(2\pi/N)n} = N$$

for $k = 0, \pm N, \pm 2N, \ldots$ and is zero otherwise, by premultiplying both sides with $e^{-jr(2\pi/N)n}$ and summing over all N terms ...

Expression for discrete domain

- We obtain

$$\sum_n x[n]e^{-jr(2\pi/N)n} = \sum_n \Sigma_k a_k e^{j(k-r)(2\pi/N)n}$$

so that as terms exist only when $k = r$ we get

$$a_r = \frac{1}{N}\sum_n x[n]e^{-jr(2\pi/N)n}$$

.

- Replacing $k = r$ we get the general Fourier coefficient expressions for the discrete domain periodic signals. Now $a_k = a_{k+N}$ so there is a foldback after N terms as in periodic signal.

Fourier series properties - discrete domain

- **Linearity:** For $Ax[n] + By[n]$, coefficients are $Aa_k + Bb_k$.
- **Time shift:** For signal $x[n - n_0]$ coefficients are $a_k e^{-jk(2\pi/N)n_0}$
- **Frequency shift:** For $e^{jM(2\pi/N)n}x[n]$, coefficients are a_{k-M}.
- **Conjugation:** For $x^*[n]$, coefficients are a^*_{-k}.
- **Time reversal:** For $x[-n]$, coefficients are a_{-k}.

Fourier series properties - discrete domain

- **Time scaling:** For $x[n/m]$, coefficients are $\frac{1}{m}a_k$.
- **Multiplication:** For $x[n]y[n]$, coefficients are

$$\sum_l a_l b_{k-l}$$

- **First difference:** For $x[n] - x[n-1]$, coefficients are $(1 - e^{-jk(2\pi/N)})a_k$
- **Running sum:** For the expression

$$\sum_{k=-\infty}^{n} x[k]$$

coefficients are $(\frac{1}{(1-e^{-jk(2\pi/N)})})a_k$

Parseval's relation for continuous domain

- Average power in one period of the signal can be expressed in terms of the average power in the harmonic components, which is in turn the square magnitude of Fourier coefficients.
- Expression is therefore

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

.

- Relation is

$$\frac{1}{T}\int_T |a_k e^{jk\omega_0 t}|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2$$

.

Parseval's relation for discrete domain

- Thus the total average power in the periodic signal is the sum of the average powers in all of its harmonic components.
- Can be extended to cover discrete domain as well.
- Expression for discrete domain is

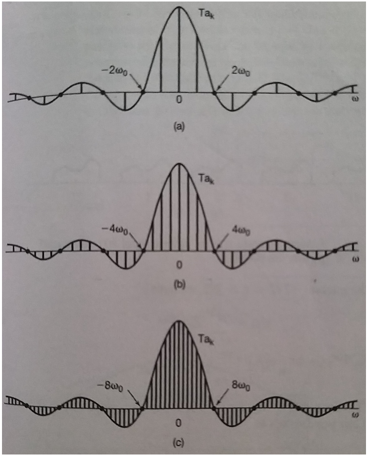
$$\frac{1}{N}\sum_{n}|x[n]|^2 = \sum_{k}|a_k|^2$$

.

Aperiodic signals - Fourier transform in continuous domain

- Aperiodic signal is viewed as periodic signal with infinite period.
- In the Fourier series, when period increases, fundamental frequency keeps decreasing so that the harmonics get closer in frequency.
- Hence as period becomes infinite, the sum in the Fourier series becomes an integral.
- Thus the argument is that aperiodic signal is also covered by the Fourier series in some form.

Envelope of Fourier coefficients of the square wave



From Square wave to aperiodic rectangular pulse

- Consider the periodic square wave to begin with. Here $x(t) = 1$ for $|t| < T_w$ and $x(t) = 0$ for $T_w < |t| < T/2$ which is symmetric about the origin with width T_w and repeats with period T.
- Fourier series coefficients are $a_k = \frac{2sin(k\omega_0 T_w)}{k\omega_0 T}$.
- Taking $\omega = k\omega_0$, the Ta_k is an envelop function with a_k interpreted as equally spaced samples of this envelope.
- Hence with arbitrarily large T, the fundamental frequency $\omega_0 = 2\pi/T$ decreases so that the samples of the envelope approach the continuum of the envelope function.
- What meanwhile happens to the square wave is an aperiodic single rectangular pulse.

Integral form of Fourier transform

In the limit, we have for an approximate periodic signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

,

$$a_k = \frac{1}{T} \int_{-T/2}^{+T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

The limits of integral can be extended to $-\infty, +\infty$ as the approximate periodic signal is replaced with actual aperiodic signal.

Integral form of Fourier transform

This leads to the Fourier transform pair of equations

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

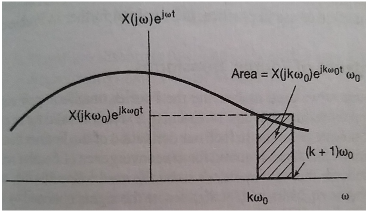
and

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

.

From this, the coefficients become $a_k = \frac{1}{T}X(j\omega)$ evaluated at $\omega = k\omega_0$ for some finite duration signal that matches with one period.

Graphical interpretation of Fourier integral



Convergence of Fourier transform

- Convergence is guaranteed in terms of the square of error in approximation of the aperiodic signal.
- The error is between the signal estimated from the Fourier transform and the actual time domain signal.
- This analysis is similar to the periodic signal case presented earlier.
- Same Dirichlet conditions also apply.

Convergence of Fourier transform - Example

$x(t) = e^{-at}u(t)$ with $a > 0$. Here

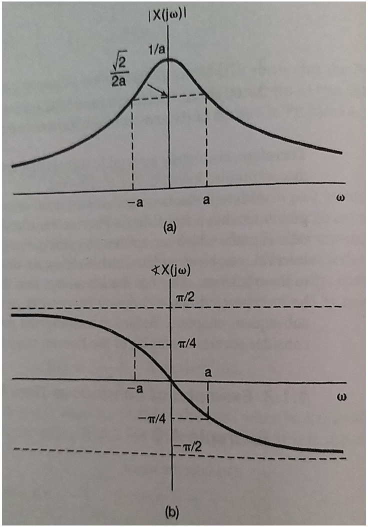
$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt$$

Which simplifies to $= X(j\omega) = \frac{1}{a+j\omega}$. This is complex.
Magnitude is $\frac{1}{\sqrt{(a^2+\omega^2)}}$ and phase is $-tan^{-1}(\frac{\omega}{a})$.
If this t extends to both sides of the origin, $X(j\omega) = \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$, Which simplifies to $\frac{2a}{a^2+\omega^2}$ that means it is real valued.
For unit impulse, we have

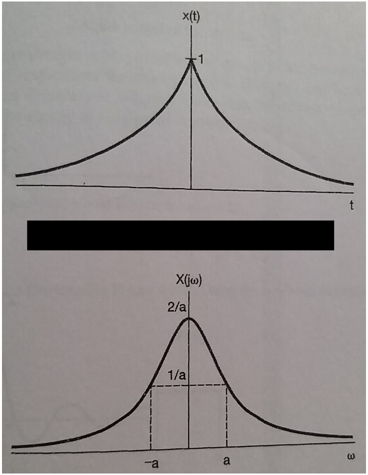
$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1$$

means unity and equal for all frequencies.

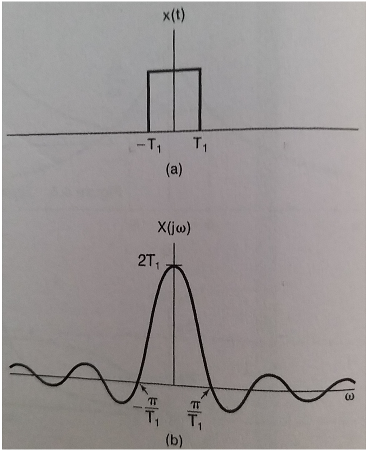
Convergence example for exponential decay $x(t) = e^{-at}u(t)$



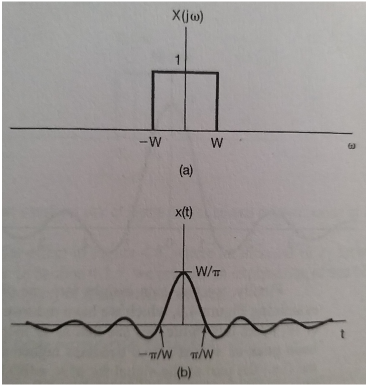
Convergence example for signal $x(t) = e^{-a|t|}$



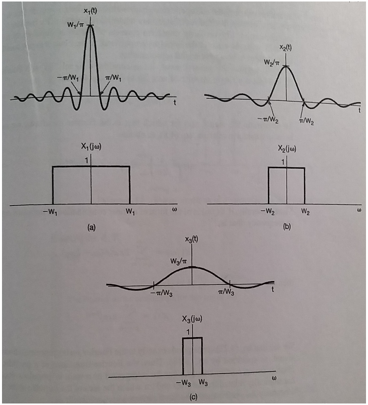
Fourier transform for rectangular pulse



Inverse Fourier transform to get rectangular pulse



Fourier transform pairs for varying bandwidths



Properties of Fourier transform

Linearity $ax(t) + by(t)$ gives $aX(j\omega) + bY(j\omega)$.
Time shift $x(t - t_d)$ gives $e^{-j\omega t_d}X(j\omega)$.
Conjugate $x^*(t)$ gives $X^*(-j\omega)$ Also $X(-j\omega) = X^*(j\omega)$
Differentiation $\frac{dx(t)}{dt}$ gives $j\omega X(j\omega)$.
Integration yields $\frac{1}{j\omega}X(j\omega)$ and additional DC or average value term $\pi X(0)\delta(\omega)$.
Time scaling $x(at)$ gives $\frac{1}{|a|}X(\frac{j\omega}{a})$ leads to frequency scaling.

Duality, convolution and multiplication

Duality exists between the Fourier transform pairs. They swap position like for example for $x(t) = \frac{sinWt}{\pi t}$ the Fourier transform is $X(j\omega) = 1$ in the interval $|\omega| < W$ and zero outside of this interval.

Parseval's relations

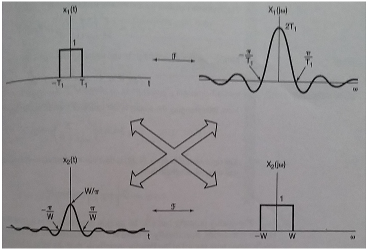
$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

.

Convolution property $Y(j\omega) = H(j\omega)X(j\omega)$ where $y(t) = h(t) * x(t)$. Convolution in time domain is multiplication in frequency domain.

Multiplication Due to duality, the reverse is also true. Hence for $r(t) = s(t)p(t)$, $R(j\omega) = \frac{1}{2\pi} \int S(j\theta)P(j(\omega - \theta))d\theta$. This is called modulation property.

Linkage between time and frequency domains



Response in continuous domain

Consider $h(t) = e^{-at}u(t)$ response to signal of the form $x(t) = e^{-bt}u(t)$.
Then $H(j\omega) = \frac{1}{a+j\omega}$ and $X(j\omega) = \frac{1}{b+j\omega}$.
Therefore $Y(j\omega) = \frac{1}{(a+j\omega)(b+j\omega)}$.
This can be written as $Y(j\omega) = \frac{1}{b-a}[\frac{1}{(a+j\omega)} - \frac{1}{(b+j\omega)}]$.
Hence $y(t) = \frac{1}{b-a}[e^{-at}u(t) - e^{-bt}u(t)]$.
When $b = a$, we have $Y(j\omega) = \frac{1}{(a+j\omega)^2} = j\frac{d}{d\omega}[\frac{1}{a+j\omega}]$.
Consequently $y(t) = te^{-at}u(t)$ applying dual of differentiation property.

Differential equation and solution

$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$

Then $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega+2}{(j\omega)^2+4j\omega+3}$

Breaking into partial fraction leads to $H(j\omega) = \frac{1/2}{j\omega+1} + \frac{1/2}{j\omega+3}$

Taking inverse transform, $h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$.

Suppose input is $x(t) = e^{-t}u(t)$

then $Y(j\omega) = [\frac{j\omega+2}{(j\omega+1)(j\omega+3)}][\frac{1}{j\omega+1}]$

Breaking into partial fraction, $Y(j\omega) = \frac{1/4}{j\omega+1} + \frac{1/2}{(j\omega+1)^2} - \frac{1/4}{j\omega+3}$ so that

$y(t) = [\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}]u(t)$.

Discrete domain expression

Following the derivations of Fourier series and continuous domain Fourier transform,

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

and

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

Example: $x[n] = a^n u[n]$ with $|a| < 1$. Then

$$X(e^{j\omega}) = \sum_{n=0}^{+\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

.

Discrete domain expression

Example: $x[n] = a^{|n|}$ with $|a| < 1$. Then

$$X(e^{j\omega}) = \sum_{n=0}^{+\infty}(a^n e^{-j\omega n}) + \sum_{n=-\infty}^{-1}(a^{-n}e^{-j\omega n})$$

. This simplified to $= \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}} = \frac{1-a^2}{1-2acos\omega+a^2}$ which is real.
Example: Rectangular pulse for width $2N$ around origin -

$$X(e^{j\omega}) = \sum_{n=-N}^{+N}(1.e^{-j\omega n})$$

This simplifies to $X(e^{j\omega}) = \frac{sin\omega(N+\frac{1}{2})}{sin(\omega/2)}$

Discrete Fourier transform property

- **Periodicity** The all important property is $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$, the discrete Fourier transform is periodic with period 2π.
- **Linearity** For $ax(t) + by(t)$ DFT is $aX(e^{j\omega}) + bY(e^{j\omega})$.
- **Time shifting** For $x[n - n_0]$, DFT is $e^{-j\omega n_0}X(e^{j\omega})$
- **Frequency shifting** For $e^{j\omega_0 n}x[n]$, DFT is $X(e^{j(\omega-\omega_0)})$.

Conjugation

- For $x^*[n]$ DFT is $X^*(e^{-j\omega})$
- When $x[n]$ is real valued, there is symmetry $X(e^{j\omega}) = X^*(e^{-j\omega})$.
- Even part of $x[n]$ corresponds to Real part $Re[X(e^{j\omega})]$
- Odd part corresponds to $jIm[X(e^{j\omega})]$.

Time reversal and expansion

- Consider $y[n] = x[-n]$. Then $Y(e^{j\omega}) = \Sigma x[-n]e^{-j(-\omega)(-n)} = X(e^{-j\omega})$.
- For $x_k[n] = x[n/k]$ for multiples $n = rk$ $X_k(e^{j\omega}) = \Sigma_r x_k[rk]e^{-j\omega rk}$
- This simplifies to $\Sigma_r x[r]e^{-j(k\omega)r} = X(e^{jk\omega})$.
- This implies that while the signal spreads out, the DFT gets compressed with period from original 2π to period $2\pi/k$.

Difference and Accumulation characteristics

- **Difference** For $x[n]-x[n-1]$, DFT is $(1-e^{-j\omega})X(e^{j\omega})$.
- **Accumulation** For
$$y[n]=\sum_{m=-\infty}^{+n}x[m]$$
, two parts are there. The term derived due to difference $y[n]-y[n-1]=x[n]$ is $\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$.
- **Average value** There is also DC component or average value that contains the impulse train
$$\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$$
.

Differentiation of frequency

- Since $X(e^{j\omega}) = \Sigma x[n]e^{-j\omega n}$,
- differentiating both sides,

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} -jnx[n]e^{-j\omega n}$$

.

- Multiplying both sides by j, we conclude that DFT of $nx[n]$ is $j\frac{dX(e^{j\omega})}{d\omega}$

Parseval relation for signal energy

- Expression is

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

.

- LHS here is total energy of the signal.
- RHS implies the energy can be obtained by integrating the energy per unit frequency over the whole interval of 2π.
- This leads to the concept of energy density spectrum of the signal.

Discrete Fourier transform convolution

- For $y[n] = x[n] * h[n]$, we have $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$.
- With impulse response $h[n] = \delta[n - n_0]$,

$$H(e^{j\omega}) = \sum_n \delta[n - n_0]X(e^{j\omega}) = e^{-j\omega n_0}$$

- This is consistent with time shift property.
- The frequency response has magnitude of unity at all frequencies and phase characteristics of $-\omega n_0$ that is linear with frequency.

DFT and lowpass filter

- For low pass filter, $H(e^{j\omega}) = 1$ within cutoff frequency ω_c.
- Then

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{sin\omega_c n}{\pi n}$$

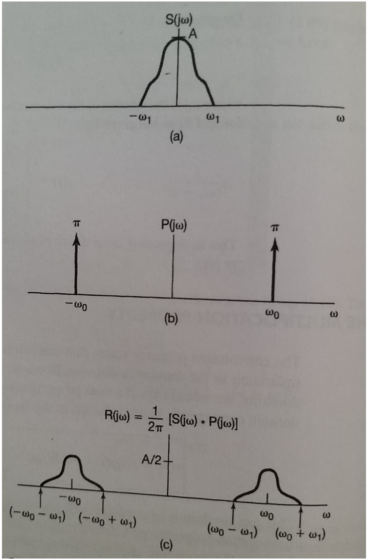
.

- Consider $h[n] = \alpha^n u[n]$ and $x[n] = \beta^n u[n]$.
- Then $Y(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}\frac{1}{1-\beta e^{-j\omega}}$.
- From this $y[n] = \frac{1}{\alpha-\beta}[\alpha^{n+1}u[n] - \beta^{n+1}u[n]]$ obtained by taking partial fraction.
- When $\alpha = \beta$, $y[n] = (n+1)\alpha^n u[n]$ applying differentiation property.

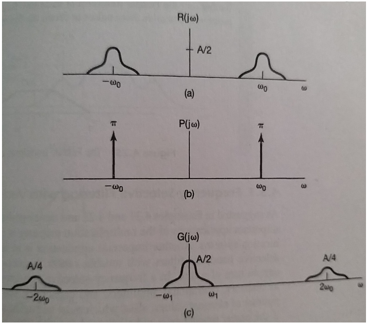
Multiplication properties

- Consider $y[n] = x_1[n]x_2[n]$.
- Since
$$x_1[n] = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta$$
we have
$$Y(e^{j\omega}) = \sum x_2[n][\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta]e^{-j\omega n}$$
- Upon interchanging order of sum and integral,
$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})[\Sigma x_2[n]e^{-j(\omega-\theta)n}]d\theta$$
$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2 e^{j(\omega-\theta)}d\theta$$
- This signifies periodic convolution of the Fourier transforms over any interval of 2π, while for aperiodic convolution, the integration limit extends to both sides of infinity.

Multiplication properties of Fourier transform explained



Multiplication properties of Fourier transform explained



Expressions of duality

Table 5.3 p. 399 has been reproduced.

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

duality

duality

duality

Difference equation based DFT expression

Class of systems

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

general linear constant coefficient difference equation based system. Here $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$.
Using linearity and time shifting properties,

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

Response of discrete systems

Ex. $y[n] - ay[n-1] = x[n]$. Here $H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$ which gives $h[n] = a^n u[n]$.

Ex. $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$.

Here $H(e^{j\omega}) = \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$.

Using partial fraction expansion, $H(e^{j\omega}) = \frac{4}{(1-\frac{1}{2}e^{-j\omega})} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})}$.

Hence, $h[n] = 4(\frac{1}{2})^n u[n] - 2(\frac{1}{4})^n u[n]$.

Now consider an input $x[n] = (\frac{1}{4})^n u[n]$. Then $X(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$

Hence $Y(e^{j\omega}) = \frac{2}{(1-\frac{1}{2}e^{-j\omega})-(1-\frac{1}{4}e^{-j\omega})^2}$.

Using partial fraction expansion,

$Y(e^{j\omega}) = \frac{4}{(1-\frac{1}{4}e^{-j\omega})} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{8}{(1-\frac{1}{2}e^{-j\omega})}$.

Then $y[n] = \{-4(\frac{1}{4})^n - 2(n+1)(\frac{1}{4})^n + 8(\frac{1}{2})^n\}u[n]$.

Magnitude phase representation

- The transform is represented with Real part and imaginary part. Often it is more meaningful to use the magnitude phase representation.
- Magnitude squared $(|X(j\omega)|^2$ signifies energy density spectrum of the signal. Relative strength of signal at different frequencies is expressed.
- Phase provides relative phases of the sinusoids without affecting the amplitudes. The amplitude of the resultant sum may cancel out if the individual sinusoids are out of phase. This leads to change in time domain characteristics.
- Auditory system is insensitive to phase unless the phase distortion is severe like the signal is wholly reversed by playing the sound backward.
- Visual image signals are phase sensitive. In particular the phase contains the entire edge information so that upon taking Fourier transform,the plot of phase can help in retrieving the image shape while the magnitude plot reveals nothing.

Frequency response

- Output transform is $Y(j\omega) = H(j\omega)X(j\omega)$,
- Magnitude is multiplicative. Hence the part $|H(j\omega)|$ is called gain of the overall system. $|Y(j\omega)| = |H(j\omega)||X(j\omega)|$,
- Phase is additive. Hence the part $\langle H(j\omega)$ is called phase shift of the system. $\langle Y(j\omega) = \langle H(j\omega)\langle X(j\omega)$,
- When the signal $x(t)$ passes through the LTI system, the response is distortion of original signal.
- Linear phase shift in frequency domain amounts to only time shift in time domain. It is equivalent to delays for discrete systems.

Magnitude and phase plots

- Magnitude becomes additive by taking log on both sides. $log|Y(j\omega)| = log|H(j\omega)| + log|X(j\omega)|$, unit of decibels (dB) is based on $20log_{10}$. Thus $6dB$ implies a gain of 2 while $0dB$ signifies unity gain.
- This representation helps in obtaining response additively for both magnitude and phase,
- The magnitude in dB and phase is generally plotted against $log_{10}\omega$ for continuous systems. This is called Bode plot. This logarithmic frequency scale is not suitable for discrete domain due to periodicity of 2π.
- For real valued $h(t)$ the magnitude is even function and phase is odd function. Hence the negative axis of ω becomes redundant so that response plots are only for positive frequency.
- Use of logarithm helps in observing the system over a large dynamic range.

Laplace transform in continuous domain

Laplace transform in continuous domain extends the Fourier transform from imaginary to the complex plane.
Taking $s = \sigma + j\omega$,

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt$$

which resembles Fourier transform of $x(t)e^{-\sigma t}$.

Example for step input with exponential profile

Considering signal $x(t) = e^{-at}u(t)$,

$$X(s) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{+\infty} e^{-(s+a)t}dt$$

Applying earlier results of Fourier transform, $X(s) = \frac{1}{s+a}$.
The region of convergence is $Re(s) > -a$
Hence for unit step with $a = 0$, $X(s) = \frac{1}{s}$.

Example for time reversed step input with exponential profile

Considering signal $x(t) = -e^{-at}u(-t)$,

$$X(s) = \int_{-\infty}^{+\infty} e^{-at}u(-t)e^{-st}dt = \int_{-\infty}^{0} e^{-(s+a)t}dt$$

Applying earlier results of Fourier transform, again $X(s) = \frac{1}{s+a}$
The region of convergence is $Re(s) < -a$

Example for linear combination

Consider $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$,
$X(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{s^2+3s+2}$
Region of convergence combines$Re(s) > -1$ and $Re(s) > -2$ and former dominates.

Example for step input with trigonometric functions

Consider $x(t) = e^{-2t}u(t) + e^{-t}(cos3t)u(t)$,
Expand $2cos3t = e^{3jt} + e^{-3jt}$
$X(s) = \frac{1}{s+2} - \frac{1}{2}\frac{1}{s+1-3j} + \frac{1}{2}\frac{1}{s+1+3j}$
$= \frac{2s^2+5s+12}{(s+2)(s^2+2s+10)}$
Region of convergence combines all three to $Re(s) > -1$.

Poles and Zeroes of transform domain

Each example shows $X(s) = \frac{N(s)}{D(s)}$ with roots of numerator as zeroes and roots of denominator as poles.
$X(s)$ is rational when $x(t)$ is combination of real or complex exponentials or for systems described by differential equations.
Poles and zeroes at infinity- depends on difference in order of numerator and denominator.

Inverse Laplace transform

$$X(\sigma + j\omega) = F[x(t)e^{-\sigma t}] = \int_{-\infty}^{+\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

Hence

$$x(t)e^{-\sigma t} = F^{-1}[X(\sigma + j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

Since σ is constant and $s = \sigma + j\omega$, we have $ds = jd\omega$.

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

Inverse Laplace transform examples

Consider $X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$
Then $x(t) = e^{-t}u(t) - e^{-2t}u(t)$.
Or $x(t) = -e^{-t}u(-t) - (-e^{-2t}u(-t))$.

Evaluation of the Fourier transform from Laplace transform

Summarized properties of Laplace transform

Linearity: $L(ax_1(t)+bx_2(t)) = aX_1(s)+bX_2(s)$
Time shifting: $L(x(t-t_0)) = e^{-st_0}X(s)$
Shifting in s domain: $L(e^{s_0 t}x(t)) = X(s-s_0)$
Shifting in frequency: $L(e^{j\omega_0 t}x(t)) = X(s-j\omega_0)$

Summarized properties of Laplace transform

Time scaling: $L(x(at)) = \frac{1}{|a|}X(\frac{s}{a})$
Time reversal: $L(x(-t)) = X(-s)$
Conjugation: $L(x^*(t)) = X^*(s^*)$ so that $X(s) = X^*(s^*)$ for real valued $x(t)$
Convolution: $L(x_1(t) * x_2(t)) = X_1(s)X_2(s)$

Summarized properties of Laplace transform

Differentiation in time domain: $L(\frac{dx(t)}{dt}) = sX(s)$
Differentiation in s domain: $L(-tx(t)) = \frac{dX(s)}{ds}$
Integration:

$$L(\int_{-\infty}^{t} x(\tau)d\tau) = \frac{1}{s}X(s)$$

Initial value theorem:

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Final value theorem:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Systems analysis through the transform

Differential equation and solutions

Butterworth filter design strategy

Butterworth filters - Nth order low pass filter frequency response
$|B(j\omega)|^2 = \frac{1}{1+(j\omega/j\omega_c)^{2N}}$
By definition, $|B(j\omega)|^2 = B(j\omega)B^*(j\omega)$
When impulse response is real, $B(-j\omega) = B^*(j\omega)$
Hence, $B(j\omega)B(-j\omega) = \frac{1}{1+(j\omega/j\omega_c)^{2N}}$
Putting $s = j\omega$, $B(s)B(-s) = \frac{1}{1+(s/j\omega_c)^{2N}}$

Butterworth filter root magnitude and phase

Roots of the denominator polynomial lie at $s = (-1)^{1/2N}(j\omega_c)$
Hence magnitude of the poles is $|s_p| = \omega_c$
and phase is $\langle s_p = \frac{\pi(2k+1)}{2N} + \frac{\pi}{2}$
So that $s_p = \omega_c exp(j[\frac{\pi(2k+1)}{2N} + \frac{\pi}{2}])$

Butterworth filter pole placement

There are $2N$ poles equally spaced in angle on a circle with radius ω_c in the s-plane.
Poles do not lie on imaginary axis and occurs on real axis for odd N, but not so for even N.
Angular spacing between the poles of $B(s)B(-s)$ is π/N radians.
Poles of $B(s)B(-s)$ occurs in $\pm s_p$ pairs, so that to construct $B(s)$ one pole is to be chosen, i.e. One semicircle of the s-plane.

Butterworth filter pole positions

For $N = 1$ we have $B(s) = \frac{\omega_c}{s+\omega_c}$
For $N = 2$ we have $B(s) = \frac{\omega_c^2}{(s+\omega_c e^{j(\pi/4)})(s+\omega_c e^{-j(\pi/4)})}$
$= \frac{\omega_c^2}{s^2+\sqrt{2}\omega_c s+\omega_c^2}$
For $N = 3$ we have $B(s) = \frac{\omega_c^3}{(s+\omega_c)(s+\omega_c e^{j(\pi/3)})(s+\omega_c e^{-j(\pi/3)})}$
$= \frac{\omega_c^3}{(s+\omega_c)(s^2+\omega_c s+\omega_c^2)}$
$= \frac{\omega_c^3}{s^3+2\omega_c s^2+2\omega_c^2 s+\omega_c^3}$

Butterworth filters in time domain

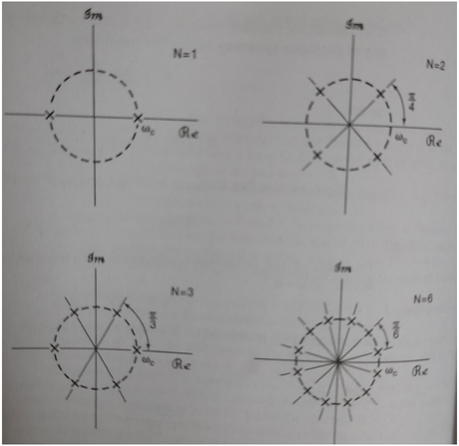
Hence the time domain differential equations can be determined.

For $N = 1$, $\frac{dy(t)}{dt} + \omega_c y(t) = \omega_c x(t)$

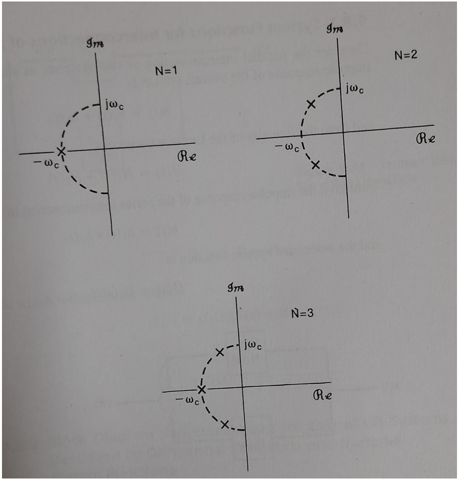
For $N = 2$, $\frac{d^2y(t)}{dt^2} + \sqrt{2}\omega_c \frac{dy(t)}{dt} + \omega_c^2 y(t) = \omega_c^2 x(t)$

For $N = 3$, $\frac{d^3y(t)}{dt^3} + 2\omega_c \frac{d^2y(t)}{dt^2} + 2\omega_c^2 \frac{dy(t)}{dt} + \omega_c^3 y(t) = \omega_c^3 x(t)$

Butterworth filter pole locations



Butterworth filter pole locations

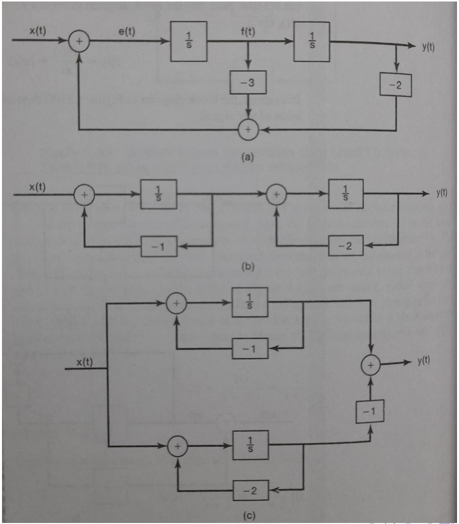


Laplace transform based block diagrams

Consider $H(s) = \frac{Y(s)}{X(s)} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$
Equivalently, $H(s) = \frac{1}{s^2+3s+2}$
Hence $s^2Y(s) + 3sY(s) + 2Y(s) = X(s)$.
In time domain, $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$.
Take $f(t) = \frac{dy(t)}{dt}$ and $e(t) = \frac{df(t)}{dt} = \frac{d^2y(t)}{dt^2}$
so that time domain expression becomes $e(t) = -3f(t) - 2y(t) + x(t)$.
Three different possible block diagrams are shown next.

Laplace transform based block diagrams

(a) Direct form; (b) Cascade form; (c) Parallel form



Pulse transfer function

$e^*(p(t)) = p(t)e(t) =$ signal $e(t)$ modulated by pulse train $p(t)$. Now

$$e^*(p(t)) = (\frac{1}{\gamma})\sum_{k=0}^{\infty} e(kT)(u(t-kT)u(t-\overline{(kT+\gamma)}))$$

where $e^*(p(t)) = e(kT/\gamma)$ for $kT \leq t < kT+\gamma$
and $e^*(p(t)) = 0$ for $kT+\gamma \leq t < (k+1)T$.
Taking transform, $E^*(p(s)) = \Sigma e(kT/\gamma)[\frac{1}{s}exp(-kTs)\frac{1}{s}exp(-\overline{kT+\gamma}s)]$.
This becomes $= \Sigma e(kT)exp(-kTs)\frac{(1-exp(-\gamma s))}{\gamma s} \approx \Sigma e(kT)exp(-kTs)$

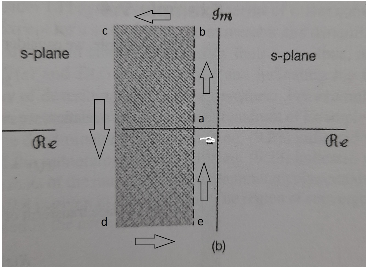
Sampling with zero order hold

Consider zero order hold $g_{zo}(t) = u(t)u(t-T)$ So the transform is $G_{zo}(s) = \frac{1}{s}(1-e^{-sT}) = G_z^*(s)G_o(s)$ with $G_o(s)$ being $1/s$.
For frequency domain analysis,
$G_{zo}(j\omega) = \frac{1-e^{-j\omega T}}{j\omega} = 2e^{-\frac{j\omega T}{2}}(e^{\frac{j\omega T}{2}} - e^{-\frac{j\omega T}{2}})/2j\omega = 2e^{-\frac{j\omega T}{2}}\frac{sin(\omega T/2)}{\omega}$
Now, using $T = 2\pi/\omega_s$ where $\omega_s = 2\pi f_s$;
$G_{zo}(j\omega) = (2\pi/\omega_s)[\frac{sin(\pi\omega/\omega_s)}{(\pi\omega/\omega_s)}]e^{-j\pi\omega/\omega_s}$
Hence $= |G_{zo}(j\omega)|\langle(\pi\omega/\omega_s) + m\pi$
Zero order polynomial passes without distortion.

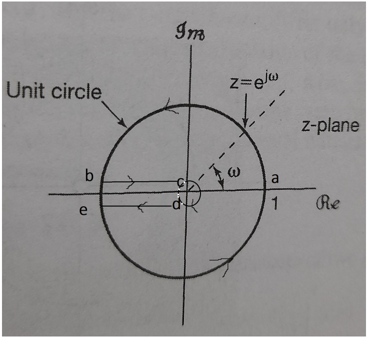
Introducing Z transform

Taking $z = exp(sT)$ or $s = (1/T)ln(z)$,
We get $E(z) = E^*(s)$ evaluated for z to be $E(z) = \Sigma e(kT)z^{-k}$
This gives a series in z, which is algebraically more convenient.
$L^{-1}(E^*p(s)) = L^{-1}(\Sigma e(kT)exp(-kTs)) \approx \Sigma e(kT)\delta(t - kT)$.
$e^*(p(t)) = e(t)\delta T(t)$ where $\delta T(t) = \Sigma\delta(t - kT)$.
Now, $s = \sigma + j\omega$ means $Re\ z = e^{\sigma T}cos(\omega T)$ and $Im\ z = e^{\sigma T}sin(\omega T)$
Z transform formula in discrete domain:
$G^*(s) = (1/T)\Sigma G(s + jn\omega_s)$;
$G^*(s) = C^*(s)/R^*(s)$; $G(z) = C(z)/R(z)$

Stable zone of s plane - left half



Stable zone of z plane - unit circle



Mapping from s domain to z domain

Interval $[a, b)$: $z = 1\langle\omega T, 0 \leq \langle\omega T \leq (180^o)$
Interval $[b, c)$: $z = z\langle(180^o), 1 \geq z > 0$
Interval $[c, d]$: $z = z\langle\pm\omega T, 0^+ \geq z > 0,$
$(180^o) \geq \langle\pm\omega T \geq (-180^o).$
Interval $(d, e]$: $z = z\langle(- 180^o), 1 \geq z > 0$
Interval (e, a) : $z = 1\langle\omega T, (-180^o) < \langle\omega T < 0^o.$

Journey through the left half stable s plane

A: origin $\sigma = 0, j\omega = 0$; $Re\ z = 1, Im\ z = 0$
Circular arc of unit radius in +ve ImZ from A to B
B:$0, +j\omega_s/2$; $Re\ z = 1, Im\ z \approx 0$ Straight line || to ReZ axis back to ImZ axis B to C
C: $\infty, +j\omega_s/2$; $Re\ z = 0, Im\ z \approx 0$Circular arc of zero radius crossing ReZ axis C to D
D: $\infty, j\omega_s/2$; $Re\ z = 0, Im\ z \approx 0$ Straight line || to ReZ axis away from ImZ axis D to E
E: $0, j\omega_s/2$; $Re\ z = 1, Im\ z \approx 0$ Circular arc of unit radius in ve ImZ back from E to A
Mapping of complementary strips similar to that of primary strip because of the trigonometric identity involving $\omega_s/2$ (Nyquist criteria for signal frequency) and the sampling frequency multiples $n\omega_s$.

Inverse Z transform

$e(kT) = (1/2\pi j) \int E(z) z^{k-1} dz$ where integration contour encompasses all singularities of $E(z)$.

$$e(kT) = \sum_i (residuesOfE(z)atSingularityz_i)$$

Z transform contains information only about e(t) at sampling instants.
Divide Num(z) by Den(z) and expand as a power series in z^{-1}
Example: $Num(z) = 11z^2 - 15z + 6$ and $Den(z) = z^3 - 4z^2 + 5z - 2$
Quotient is $11z^{-1} + 29z^{-2} + 67z^{-3} + 145z^{-4} + \ldots$

Useful Z transforms

Unit step:$Z(1(t)) = \Sigma 1(kT)z^{-k} = 1 + z^{-1} + z^{-2} + \dots$
This gives $= \frac{1}{(1-z^{-1})} = \frac{z}{(z-1)}$
Exponential decay:$x(t) = 0$ for $t < 0$ and $= exp(-at)$ for $t \geq 0$
Transform is
$Z(exp(-at)) = \Sigma e^{-akT}z^{-k} = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + \dots$
This gives $Z(exp(-at)) = \frac{1}{(1-(ze^{aT})^{-1})} = \frac{z}{(z-e^{-aT})}$

Z transform of Sinusoidal function

$Z[sin(\omega t)] = Z[\frac{(e^{j\omega t} - e^{-j\omega t})}{2j}]$
Using results for exponential function,
$Z[sin(\omega t)] = \frac{(\frac{z}{z-e^{j\omega T}} - \frac{z}{z-e^{-j\omega T}})}{2j}$
Simplifying, the transform becomes $= \frac{zsin\omega T}{(z^2 - 2zcos(\omega T) + 1)}$.

Z transform from Laplace transform

Given Laplace transform $X(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$
implies $X(z) = \frac{z}{z-1} - \frac{z}{z-e^{-T}}$.
Differentiation:
$Z[x(k+1)] = \Sigma x(k+1)z^{-k} = z\Sigma x(k+1)z^{-(k+1)} = zX(z) - zx(0)$

Z transform properties of Linearity

$Z[ag(t)] = aG(z)$ and
$Z[g_1(t) \pm g_2(t)] = G_1(z) \pm G_2(z)$
Real translation: $Z[g(t - nT)] = z^{-n}G(z)$
Note that,

$$Z[g(t - nT)] = \sum_k g(kT - nT)z^{-k}$$

Simplifying,

$$= z^{-n} \sum_k g(kT - nT)z^{-(k-n)} = z^{-n}G(z)$$

Z transform properties of time translation

$Z[g(t+nT)] = z^n G(z)$
To tackle $g(t+nT)$, take $g(kT) = 0$ for $0 \leq k \leq n-1$
Then $Z[g(t+nT)] = g(nT) + g(nT+T)z^{-1} + g(nT+2T)z^{-2} + \dots$
gives $= z^n[g(0) + g(T)z^{-1} + \dots + g(nT)z^{-n} + g(nT+T)z^{-(n+1)} + g(nT+2T)z^{-(n+2)} + \dots] = z^n G(z)$

Time Scale effect on Z domain

$Z[e^{at}g(t)] = G(e^{aT}z)$ and $Z[e^{-at}g(t)] = G(e^{-aT}z)$
$Z[e^{at}g(t)] = \Sigma[e^{anT}g(nT)z^{-n}]$
Replacing $z = e^{sT}$ and substituting $z_1 = e^{(sa)T}$
gives the transform $= ze^{aT}Z[e^{at}g(t)] = G(z_1) = G(e^{aT}z)$
Change of sign proves the other result.

Final value theorem

$$\lim_{(z \to 1)} [\frac{(z-1)}{z} G(z)] = \lim_{(t \to \infty)} [(g^*(t)]$$

provided there is no pole on or outside unit circle.
Ex: $g(t) = b(1 - e^{-at})$
$G(z) = b\frac{z}{z-1} - b\frac{z}{z-e^{-aT}}$
$G(z) = b\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})} = b\frac{z}{z-1}\frac{1-e^{-aT}}{z-e^{-aT}}$
Both values $g(t)$ and $G(z)$ tend to b in the prescribed limit.

Final value theorem - proof outline

To prove, Consider two sequences:

$$\sum_{n} f(kT)z^{-k} = f(0) + f(T)z^{-1} + \ldots + f(nT)z^{-n}$$

$$\sum_{n} f(\overline{k-1}T)z^{-k} = f(0)z^{-1} + f(T)z^{-2} + \ldots + f(\overline{n-1}T)z^{-n}$$

$$= z^{-1}(f(0) + f(T)z^{-1} + \ldots + f((n-1)T)z^{-n+1})$$

$$= z^{-1}\sum_{n-1} f(kT)z^{-k}$$

Final value theorem - proof outline

Now,

$$f(nT) = \sum_{n} f(kT) - \sum_{n-1} f(kT)$$

Then the subtraction can be written as

$$\lim_{(z\to 1)} [\sum_{n} f(kT)z^{-k} - z^{-1} \sum_{n-1} f(kT)z^{-k}]$$

In the limit,

$$\lim_{(t\to\infty)} f(nT) = \lim_{(z\to 1)} [F(z) - z^{-1}F(z)]$$

by definition of the Z transform as the sum of the series in z that runs to infinity.

Initial value theorem

$$\lim_{z\to\infty}[G(z)] = \lim_{t\to 0}[(g(t)]$$

Since $G(z) = g(0) + g(T)z^{-1} + \ldots + g(nT)z^{-n} = g(0)$ as $z \to \infty$
Now, $zG(z) = Z[g(t+T)]$ from the result $Z[x(k+1)] = zX(z)zx(0)$.
Then,

$$\lim_{(z\to\infty)}[zG(z)] = \lim_{(t\to T)}[g(t)]$$

In general,

$$\lim_{(z\to\infty)}[z^n G(z)] = \lim_{(t\to nT)}[g(t)]$$

Z transform based block diagrams

Consider $H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$
Equivalently, $H(z) = \frac{1}{-\frac{1}{8}z^{-2}+\frac{1}{4}z^{-1}+1}$
In time domain, $y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$.
Take $f[n] = y[n-1]$
and $e[n] = f[n-1] = y[n-2]$
Three different possible block diagrams are shown next.

Z transform based block diagrams

(a) Direct form; (b) Cascade form; (c) Parallel form