Counting
(Lecture – 1)

Dr. Nirnay Ghosh

Counting

- Assume we have a set of **objects with certain properties**
- **Counting** is used to determine **the number of these objects**
- **Examples:**
  - Number of available phone numbers with 7 digits in the local calling area
  - Number of possible match starters (football, basketball) given the number of team members and their positions
- Basic Counting Rules
  - **Product rule:** A count decomposes into a sequence of dependent counts ("each element in the first count is associated with all elements of the second count")
  - **Sum rule:** A count decomposes into a set of independent counts ("elements of counts are alternatives")

Product Rule

- If a count of elements can be broken down into a sequence of dependent counts where the first count yields n_1 elements, the second n_2 elements, and k^{th} count n_k elements, by the product rule the total number of elements is: $\boldsymbol{n = n_1 * n_2 * \ldots * n_k}$
- Examples:
  - A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?
  - The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?
  - How many different bit strings of length seven are there?
  - How many one-to-one functions are there from a set with m elements to one with n elements?
  - Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$

Sum Rule

- If a count of elements can be broken down into a set of independent counts where the first count yields n_1 elements, the second n_2 elements, and k^{th} count n_k elements, by the sum rule the total number of elements is: $n = \boldsymbol{n_1 + n_2 + \ldots + n_k}$
- Examples:
  - You need to travel in between city *A* and *B*. You can either fly, take a train, or a bus. There are 12 different flights in between *A* and *B*, 5 different trains and 10 buses. How many options do you have to get from *A* to *B*?
  - A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?
- **More complex counting problem**: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

The Subtraction Rule (Inclusion-Exclusion Principle)

- Used in counts where the decomposition yields two dependent count tasks with overlapping elements
  - If we use the sum rule, some element would be counted twice
- **Subtraction rule**: If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
- **Inclusion-exclusion principle**: uses a sum rule and then corrects for the overlapping elements.
- We used the principle for the cardinality of the set union.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

The Subtraction Rule (Inclusion-Exclusion Principle)

- Examples:
  - How many bit strings of length 8 start either with a bit 1 or end with 00?
  - A computer company receives 350 applications from computer graduates for a job of managing new Web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

The Division Rule

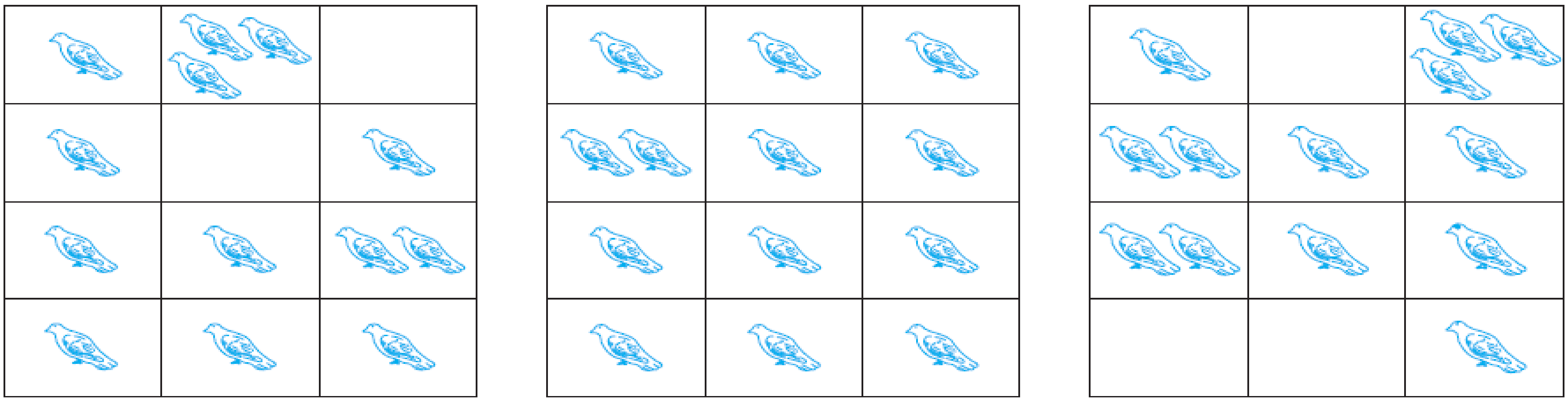
- **The division rule**: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.
- *Division rule in terms of sets*: "If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$."
- *Division rule in terms of functions*: If f is a function from A to B where A and B are finite sets, and that for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$ (in which case, we say that f is d-to-one), then $|B| = |A|/d$."
- Example:
  - How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

Tree Diagram

- Counting problems can be solved using **tree diagrams**
- **Tree:** a data structure that consists of a root, branches and leaves.
- To use trees in counting, we use a branch to represent each possible choice.
- We represent the possible outcomes by the leaves, which are the endpoints of branches not having other branches starting at them.
- Example:
  - How many bit strings of length four do not have two consecutive 1s?

Pigeonhole Principle

- **Theorem**: If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.



- **Corollary**: A function f from a set with $k + 1$ or more elements to a set with k elements is not one-to-one.

Pigeonhole Principle

- Examples:
  - Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.
  - In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.
  - How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?
- **Theorem**:

THE GENERALIZED PIGEONHOLE PRINCIPLE If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

PH Principle

- A common type of problem asks for the minimum number of objects such that at least r of these objects must be in one of k boxes when these objects are distributed among the boxes.
- When we have N objects, the generalized pigeonhole principle tells us there must be at least r objects in one of the boxes as long as $\boldsymbol{ceiling(N/k) \geq r}$.
- Problems:
  - What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades A, B, C, D, and F?
  - How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen? How many must be selected to guarantee that at least three hearts are selected?