2-3-4 Trees

❖ Search Cost

Critical factor determining search cost in BSTs

- worst case: length of longest path
- average case: < average path length (not all searches end at leaves)

Either way, path length (tree depth) is a critical factor

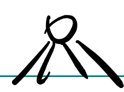
In a perfectly balanced tree, max path length = $log_2 n$

The 2 in the path length is the branching factor (*binary* search tree)

2路分支

What if branching factor > 2?

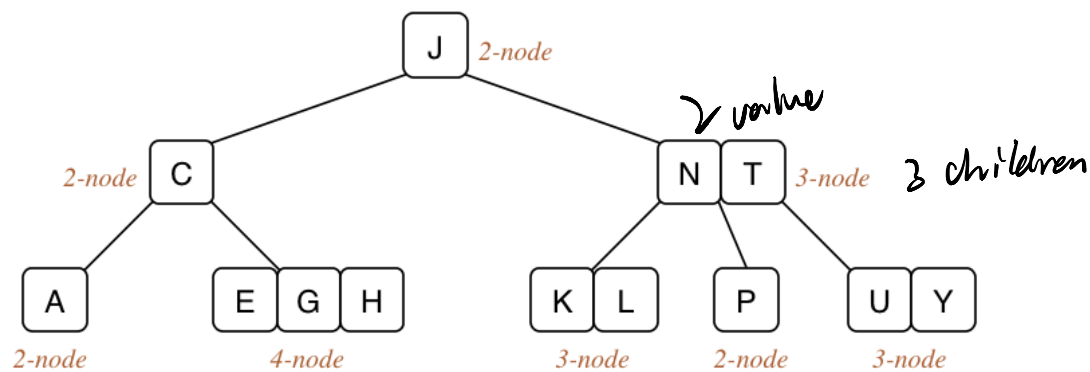
- $\log_2 4096 = 12$, $\log_4 4096 = 6$, $\log_8 4096 = 4$



❖ 2-3-4 Trees

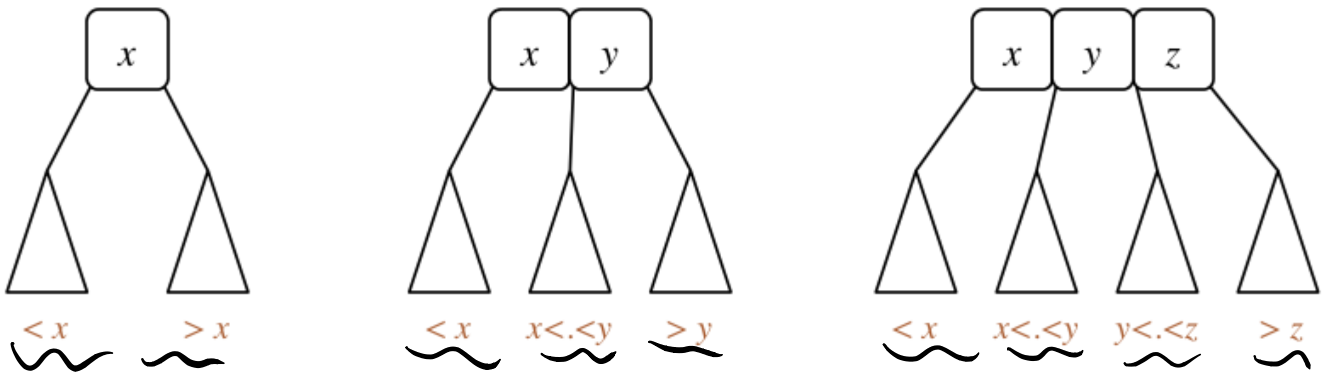
2-3-4 trees have three kinds of nodes

- 2-nodes, with two children (same as normal BSTs)
- 3-nodes, two values and three children
- 4-nodes, three values and four children



❖ ... 2-3-4 Trees

2-3-4 trees are ordered similarly to BSTs



In a balanced 2-3-4 tree:

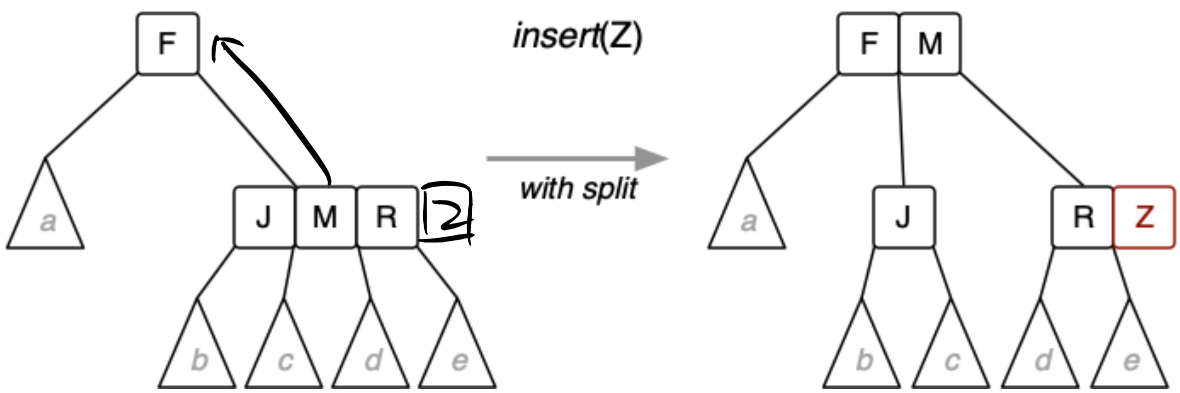
- all leaves are at same distance from the root

2-3-4 trees grow "upwards" from the leaves, via node splitting.

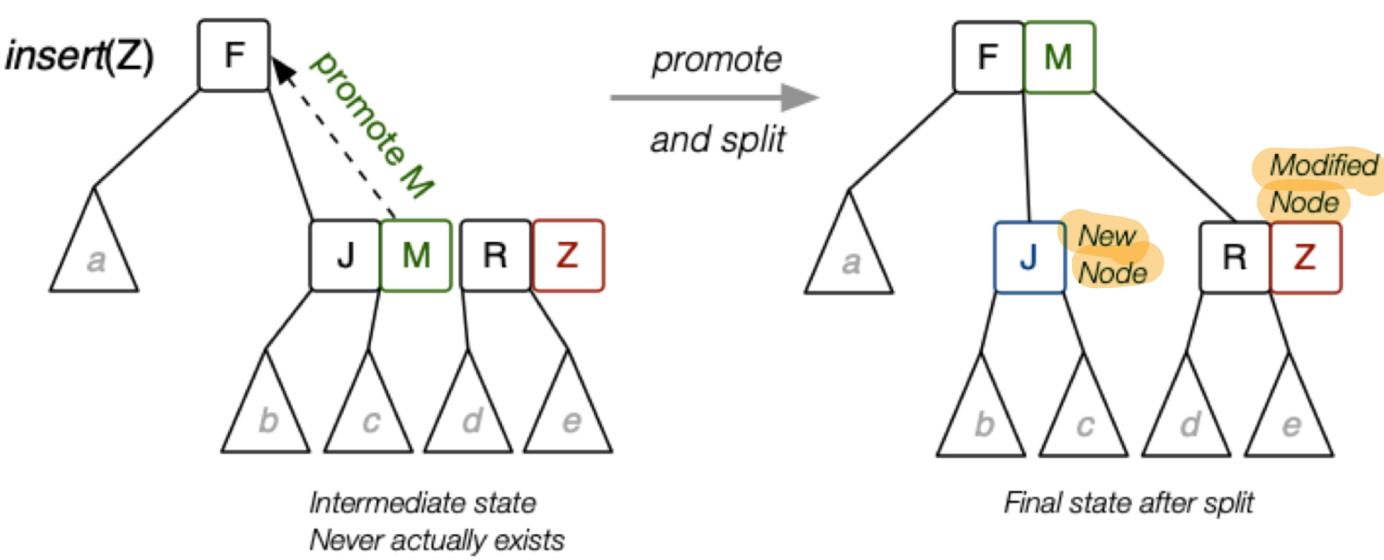
❖ Node splitting

Insertion into a full node causes a split

- middle value propagated to parent node
- values in original node split across original node and new node



Intermediate stage of insert-split:



❖ ... Node splitting

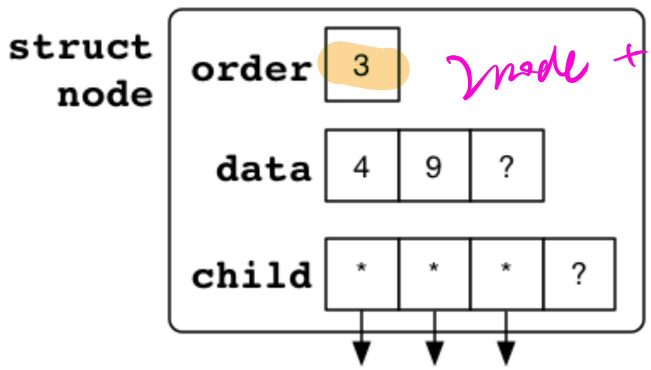
Searching in 2-3-4 trees:

```
Search(tree,item):
|   Input  tree, item
|   Output address of item if found in 2-3-4 tree
|          NULL otherwise
|
|   if tree is empty then
|      return NULL
|   else
|   |  scan tree.data to find i such that
|   |     tree.data[i-1] < item ≤ tree.data[i]
|   |  if item=tree.data[i] then    // item found
|   |     return address of tree.data[i]
|   |  else          // keep looking in relevant subtree
|   |     return Search(tree.child[i],item)
|   |  end if
|   end if
```

❖ Data Structure

Possible concrete 2-3-4 tree data structure:

```
typedef struct node {
    int            order;        // 2, 3 or 4
    int            data[3];      // items in node
    struct node *child[4];     // links to subtrees
} node;
```

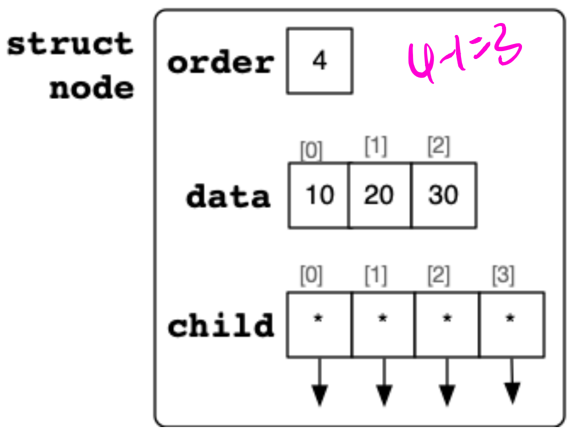


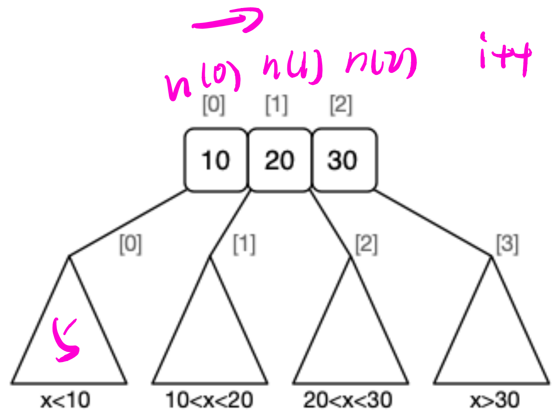
2node + 3 children

❖ ... Data Structure

Finding which branch to follow

```
// n is a pointer to a (struct node)
int i;
for (i = 0; i < n->order-1; i++) {
    if (item <= n->data[i]) break;
}
// go to the ith subtree, unless item == n->data[i]
```





❖ Search Cost Analysis

2-3-4 tree searching cost analysis:

- as for other trees, worst case determined by height h
- 2-3-4 trees are always balanced ⇒ height is $O(log\ n)$
- worst case for height: all nodes are 2-nodes
  (same case as for balanced BSTs, i.e. $h \cong log_2 n$)
- best case for height: all nodes are 4-nodes
  (balanced tree with branching factor 4, i.e. $h \cong log_4 n$)

better BST

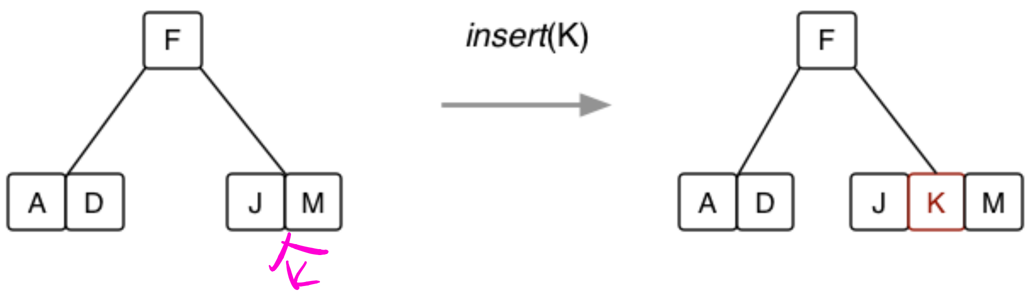
❖ Insertion into 2-3-4 Trees

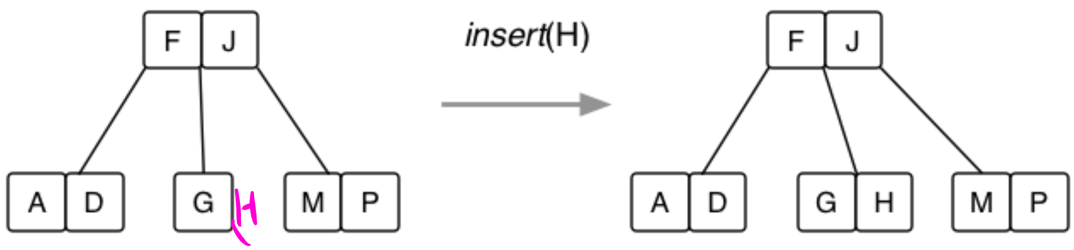
Insertion algorithm:

- find leaf node where Item belongs (via search)
- if not full (i.e. order < 4)
  - insert Item in this node, order++
- if node is full (i.e. contains 3 items)
  - split into two 2-nodes as leaves
  - promote middle element to parent
  - insert item into appropriate leaf 2-node
  - if parent is a 4-node
    - continue split/promote upwards
  - if promote to root, and root is a 4-node
    - split root node and add new root

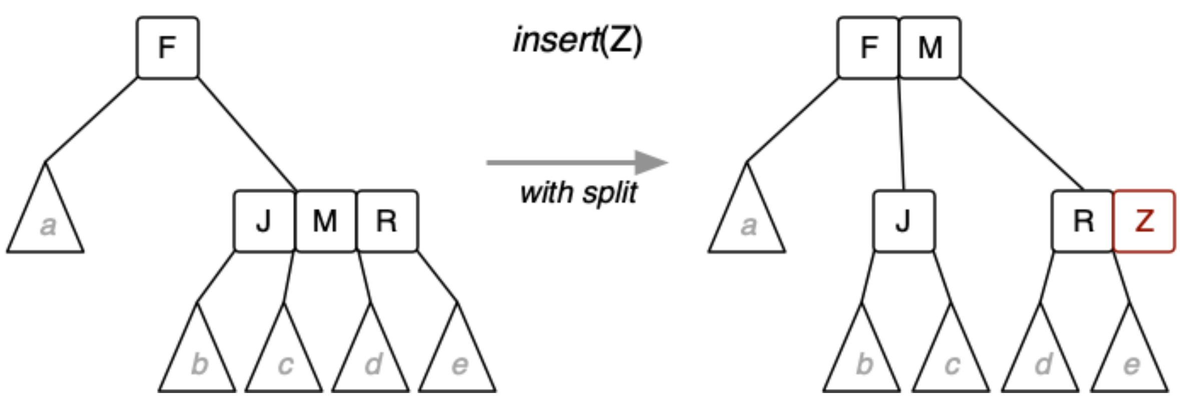
❖ ... Insertion into 2-3-4 Trees

Insertion into a 2-node or 3-node: order +1



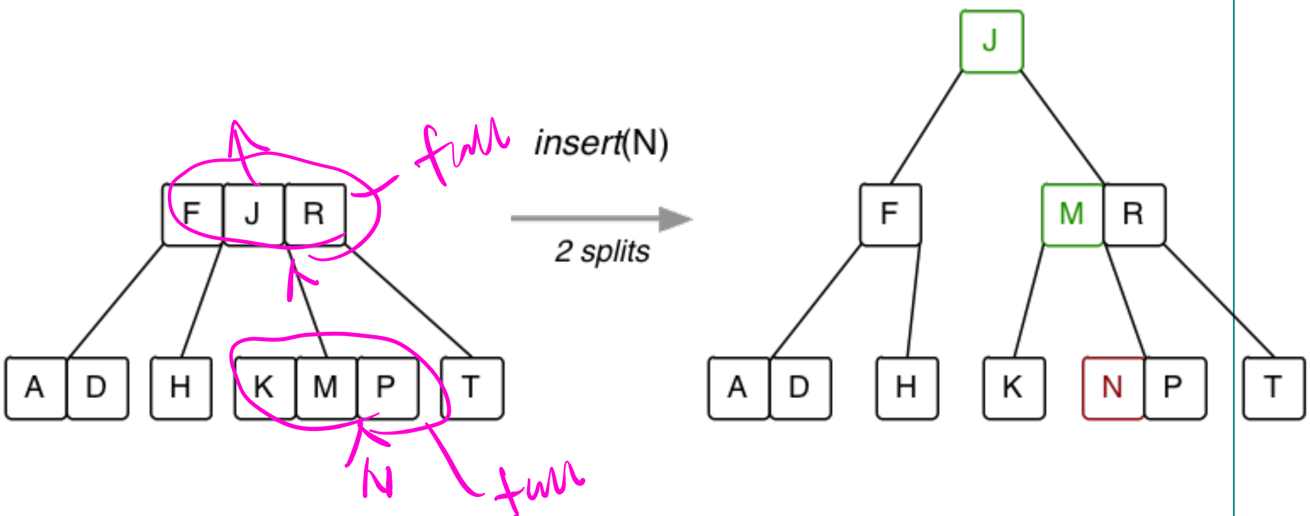


Insertion into a 4-node (requires a split): full node



❖ ... Insertion into 2-3-4 Trees

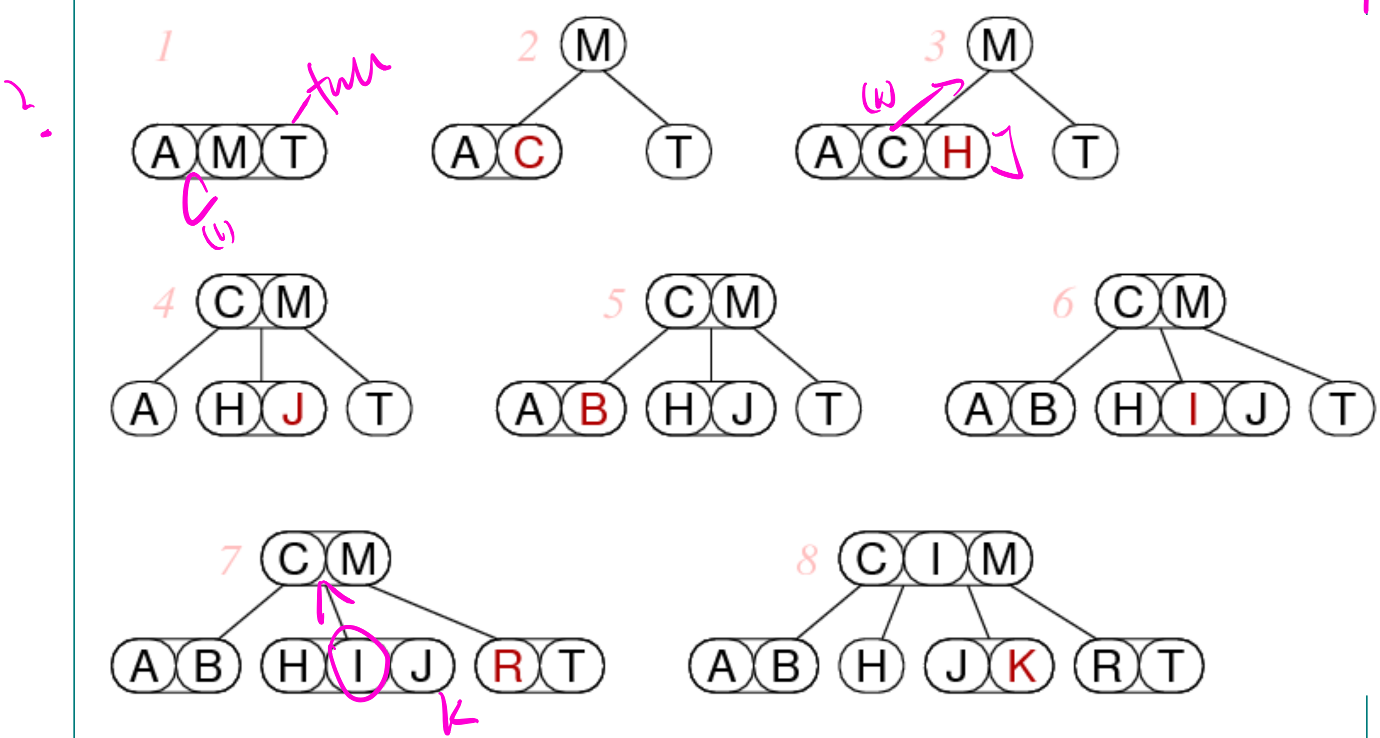
Splitting the root:



❖ ... Insertion into 2-3-4 Trees

Building a 2-3-4 tree ... 7 insertions:

will up



❖ ... Insertion into 2-3-4 Trees

Insertion algorithm:

```
insert(tree,item):
|   Input  2-3-4 tree, item
|   Output tree with item inserted
|
|   if tree is empty then
|      return new node containing item
|   end if
|   node=Search(tree,item)
|   parent=parent of node
|   if node.order < 4 then    not full
|      insert item into node
|      increment node.order   order++
|   else   full
|   |  promote = node.data[1]      // middle value
|   |  nodeL   = new node containing data[0]
|   |  nodeR   = new node containing data[2]
|   |  delete node
|   |  if item < promote then    insert 3               4= promote
|   |     insert(nodeL,item)     delet x 1 4 5
|   |  else
|   |     insert(nodeR,item)           3
|   |  end if
|   |  insert(parent,promote)
|   |  while parent.order=4 do
|   |     continue promote/split upwards
|   |  end while
|   |  if parent is root ∧ parent.order=4 then
|   |     split root, making new root
|   |  end if
|   end if
```

❖ ... Insertion into 2-3-4 Trees

Insertion cost (remembering that 2-3-4 trees are balanced ⇒ $h = \log_4 n$)

- search for leaf node in which to insert = *O(log n)*
- if node not full, insert item into node = *O(1)*
- if node full, promote middle, create two new nodes = *O(1)*
- if promotion propagates ...
  - best case: update parent = *O(1)*
  - worst case: propagate to root = *O(log n)*

Overall insertion cost = *O(log n)*

❖ 2-3-4 Variations

Variations on 2-3-4 trees ...

Variation #1: why stop at 4? why not 2-3-4-5 trees? or *M*-way trees?

- allow nodes to hold between *M/2* and *M-1* items
- 磁盘页面
  if each node is a disk-page, then we have a B-tree (databases)
  very large
- for B-trees, depending on **`Item`** size, *M > 100/200/400*

Variation #2: don't have "variable-sized" nodes

- use standard BST nodes, augmented with one extra piece of data
- implement similar strategy as 2-3-4 trees → red-black trees.

Red-black Trees

No exam

❖ Red-Black Trees

Red-black trees are a representation of 2-3-4 trees using BST nodes.

only two children

- each node needs one extra value to encode link type
- but we no longer have to deal with different kinds of nodes

Link types:

- red links ... combine nodes to represent 3- and 4-nodes
- black links ... analogous to "ordinary" BST links (child links)

Advantages:

- standard BST search procedure works unmodified
- get benefits of 2-3-4 tree self-balancing (although deeper)

❖ ... Red-Black Trees

Definition of a red-black tree

- a BST in which each node is marked **red** or **black**
- no two red nodes appear consecutively on any path
- a red node corresponds to a 2-3-4 sibling of its parent
- a black node corresponds to a 2-3-4 child of its parent

Balanced red-black tree

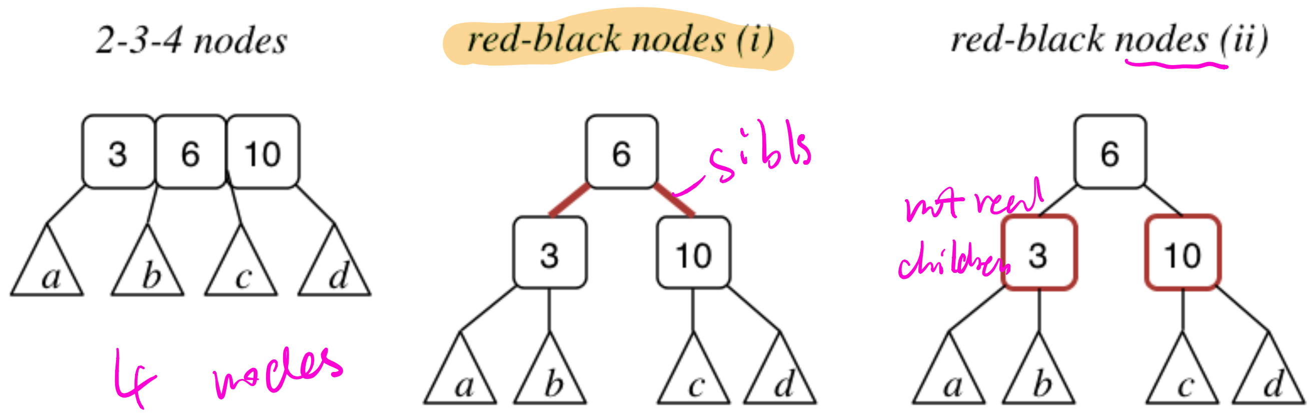
- all paths from root to leaf have same number of black nodes

Insertion algorithm: avoids worst case *O(n)* behaviour

Search algorithm: standard BST search

❖ ... Red-Black Trees

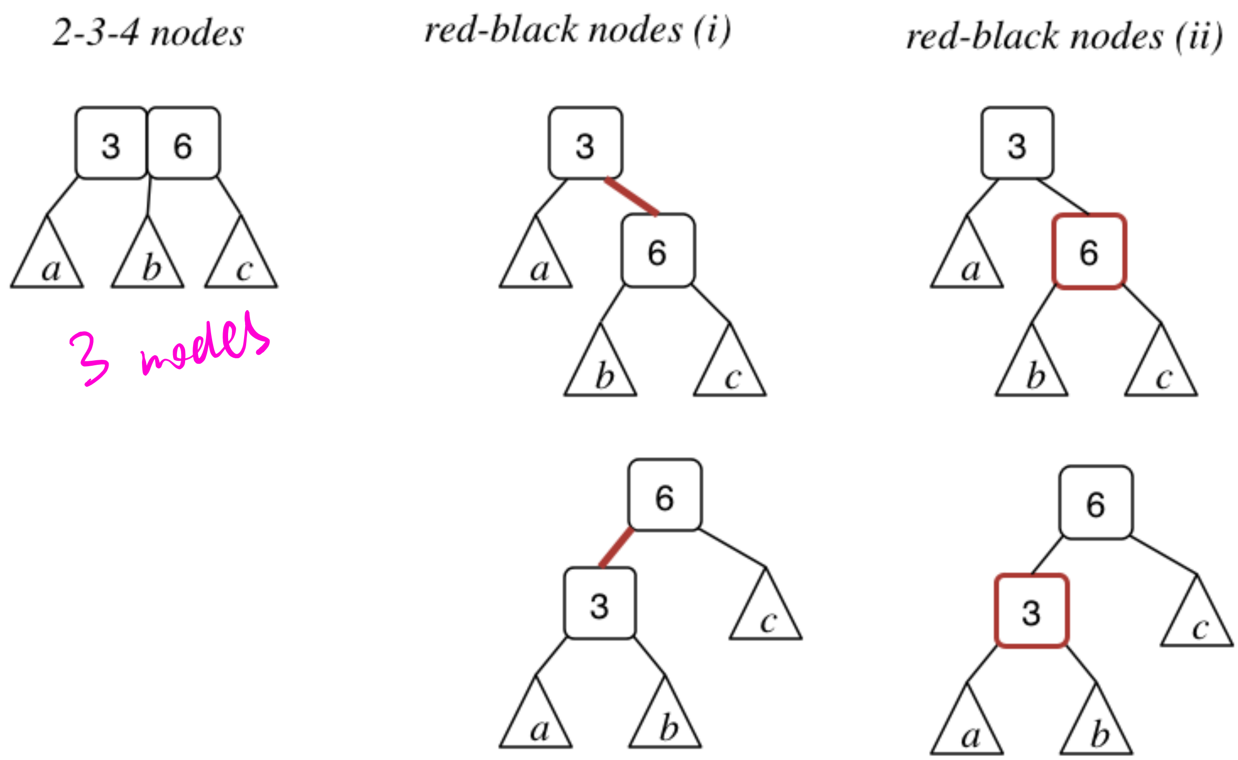
Representing 4-nodes in red-black trees:



Some texts colour the links rather than the nodes.

❖ ... Red-Black Trees

Representing 3-nodes in red-black trees (two possibilities):



❖ ... Red-Black Trees

Equivalent trees (one 2-3-4, one red-black):