5.1

The condition that characterizes the optimal amount of cake to eat in period 1 is:

$$\max_{W_2 \in [0,W_1]} u(W_1 - W_2)$$

The optimal decision is to eat the whole cake, which means $W_2 = 0$

5.2

The condition that characterizes the optimal amount of cake to leave for the next period W_3 in period 2 is: $\max_{W_3 \in [0,W_2]} u(W_2 - W_3)$

The condition that characterizes the optimal amount of cake to leave for the next period W_2 in period 1 is: $\max_{W_2 \in [0,W_1]} [u(W_1 - W_2) + \max_{W_3 \in [0,W_2]} \beta u(W_2 - W_3)]$

5.3

The condition that characterizes the optimal amount of cake to leave for the next period W_2 in period 1 is:

$\max_{W_2 \in [0,W_1]} \{u(W_1 - W_2) + \max_{W_3 \in [0,W_2]} \beta [u(W_2 - W_3) + \max_{W_4 \in [0,W_3]} \beta^2 u(W_3 - W_4)]\}$

The condition that characterizes the optimal amount of cake to leave for the next period W_3 in period 2 is: $\max_{W_3 \in [0,W_2]} [u(W_2 - W_3) + \max_{W_4 \in [0,W_3]} \beta u(W_3 - W_4)]$

The condition that characterizes the optimal amount of cake to leave for the next period W_4 in period 3 is: $\max_{W_4 \in [0,W_3]} u(W_3 - W_4)$

Since $W_1 = 1, W_4 = 0, \beta = 0.9$, we can solve the functions.
$W_2 = 0.63, W_3 = 0.30, c_1 = W_1 - W_2 = 0.37, c_2 = 0.33, c_3 = 0.3$
The figures are shown in notebook.

5.4

The condition that characterizes the optimal choice:

$$-u'\big(W_{T-1} - \psi_{T-1}(W_{T-1})\big) + \beta u'\big(\psi_{T-1}(W_{T-1})\big) = 0$$

Value function is:

$$V_{T-1}(W_{T-1}) = \max_{\psi_{T-1}(W_{T-1})} u\big(W_{T-1} - \psi_{T-1}(W_{T-1})\big) + \beta V_T(\psi_{T-1}(W_{T-1}))$$

5.5

If $V_{T-1}(\bar{W}) = V_T(\bar{W})$, since $V_T(\bar{W}) = u(\bar{W}) = ln(\bar{W})$,
we can get $V_{T-1}(\bar{W}) = \ln(\bar{W} - W_T) + \beta \ln(W_T) = ln(\bar{W})$.
We also know that $u'(\bar{W} - W_T) = \beta u'(W_T)$.
From the two functions, we can have $(W_T)^\beta = \beta + 1 > 1$
Since $\beta > 0$, $W_T \leq 1$, $(W_T)^\beta$ should be no larger than 1.
Therefore, $V_{T-1}(\bar{W}) \neq V_T(\bar{W})$.

If $\psi_{T-1}(\bar{W}) = \psi_T(\bar{W})$, since $W_{T+1} = \psi_T(\bar{W}) = 0$, we can get $\psi_{T-1}(\bar{W}) = W_T = 0$. Since we also know that $(1+\beta)W_T = \beta W_{T-1}$, $W_{T-1} = 0$.
Using the same method, we can show that all W are equal to 0, which is not possible.
Therefore, $\psi_{T-1}(\bar{W}) \neq \psi_T(\bar{W})$.

5.6
Bellman equation is:

$$V_{T-2}(W_{T-2}) = \max_{W_{T-1}} \ln(W_{T-2} - W_{T-1}) + \beta \ln\left(\frac{W_{T-1}}{1+\beta}\right) + \beta^2 \ln\left(\frac{\beta W_{T-1}}{1+\beta}\right)$$

We know that $W_{T-1} - W_T = \beta(W_{T-2} - W_{T-1})$, $W_T = \beta(W_{T-1} - W_T)$.
The value function is:

$$V_{T-2}(W_{T-2}) = \ln\left(\frac{W_{T-2}}{1+\beta+\beta^2}\right) + \beta \ln\left(\frac{\beta W_{T-2}}{1+\beta+\beta^2}\right) + \beta^2 \ln(\frac{\beta^2 W_{T-2}}{1+\beta+\beta^2})$$

5.7
The analytical solution for $\psi_{T-s}(W_{T-s})$ is:

$$\psi_{T-s}(W_{T-s}) = \frac{\sum_{i=1}^{s}\beta^i}{1+\sum_{i=1}^{s}\beta^i} W_{T-s}$$

The analytical solution for $V_{T-s}(W_{T-s})$ is:

$$V_{T-s}(W_{T-s}) = [\sum_{i=0}^{s} \beta^i \ln(\frac{\beta^i W_{T-s}}{1+\sum_{i=1}^{s}\beta^i})]$$

$$\lim_{s\to\infty} \psi_{T-s}(W_{T-s}) = \beta(W_{T-s}) = \psi(W_{T-s})$$

$$\lim_{s\to\infty} V_{T-s}(W_{T-s}) = \frac{1}{1-\beta}\ln\big((1-\beta)W_{T-s}\big) + \frac{\beta}{(1-\beta)^2}\ln\beta = V(W_{T-s})$$

5.8

$$\mathrm{V(W)} = \max_{\omega\in[0,W]} u(W-\omega) + \beta V(\omega)$$