Course Code: MATH202
Course Title: Linear Algebra
Semester: Spring 2023
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Pharos University
Faculty of Computer Science and Artificial Intelligence

Sheet #1

Lecture Problems:

1. If $\overline{A} = 3\underline{i} + 2\underline{j} - \underline{k}$ and $\overline{B} = 6\underline{i} + 3\underline{k}$. Find the vectors $4\overline{A}$ and $\overline{A} - 2\overline{B}$ in $\mathbb{R}^3$.
2. If $\overline{A} = 3\underline{i} + 2\underline{j} - \underline{k}$, evaluate $\left\|\overline{A}\right\|$.
3. If $\overline{A} = \langle 2, -3, a, 4\rangle$, $\overline{B} = \langle 1, b, 0, 5\rangle$ and $\overline{C} = \langle c, 3, -1, d\rangle$. Find the value of a, b, c and d such that $2\overline{A} - 3\overline{B} = \overline{C}$.
4. Find the unit vector of the resultant of two vectors $S = \langle 3, -5, 2\rangle$ and $T = \langle 1, 2, 4\rangle$.
5. Find the values of a and b which make the vectors $\begin{bmatrix} a \\ -2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ b \end{bmatrix}$ be parallelin $\mathbb{R}^4$.
6. Find a vector $\mathbf{b}$ of magnitude 8 and parallel to the vector $\mathbf{a} = 3\underline{i}+2\underline{j}-6\underline{k}$ but in the opposite direction.
7. Find the vector $\mathbf{v}$ in $\mathbb{R}^4$ comes from the point $(1, 2, -3, 4)$ going to $(7, 2, 0, 4)$.
8. If $\overline{A} = 4\underline{i} + 2\underline{j} - \underline{k}$ and $\overline{B} = 2\underline{i} - 5\underline{j} + 3\underline{k}$, evaluate $\overline{A} \cdot \overline{B}$.
9. Find the value of c making the vectors $\mathbf{a} = (c, 1, -3)$ and $\mathbf{b} = (c, -2c, 1)$ orthogonal.
10. Find the acute angle between the two vectors $\overline{A} = 5\underline{i} + 2\underline{j} - 2\underline{k}$ and $\overline{B} = 3\underline{i} - 5\underline{j} + 6\underline{k}$.
11. Calculate the work done by the force $\overline{F} = 5\underline{i} + 2\underline{j} - 2\underline{k}$ Newton to move a particle from the point to the point from $(1, 2, -3)$ to $(5, 0, 3)$ in meters.
12. If $\overline{A} = 4\underline{i} + 2\underline{j} - \underline{k}$ and $\overline{B} = 2\overline{i} - 5\overline{j} + 3\overline{k}$, find $\overline{A} \times \overline{B}$.
13. Find the vector $\overline{Q}$ of length 8 which is perpendicular on both $\overline{A} = 4\underline{i} + 10\underline{j} - \underline{k}$ and $\overline{B} = 7\underline{i} - 5\underline{j} + 2\underline{k}$.
14. Find the area of the parallelogram determined by the vectors $\mathbf{a} = \langle 1, 3\rangle$ and $\mathbf{b} = \langle 2, -4\rangle$.
15. Find the area of triangle PQR of vertices $P(1, 2, 3)$, $Q(-1, 1, 3)$ and $R(-3, 6, 5)$.
16. Find $\overline{A} \cdot (\overline{B} \times \overline{C})$ if $\overline{A} = 4\hat{i} + 10\hat{j} - \hat{k}$, $\overline{B} = 7\hat{i} - 5\hat{j} + 2\hat{k}$ and $\overline{C} = 6\hat{i} - 6\hat{j} + 2\hat{k}$.
17. Find the volume of parallelepiped determined by the vectors $\overline{A} = 4\underline{i} + 11\underline{j} - 2\underline{k}$, $\overline{B} = 3\underline{i} - 5\underline{j} + 2\underline{k}$ and $\overline{C} = 6\underline{i} - \underline{j} + 2\underline{k}$.
18. Prove that the vectors $\overline{A} = 14\underline{i} + 10\underline{j} - 2\underline{k}$, $\overline{B} = 3\underline{i} - 5\underline{j} + 2\underline{k}$ and $\overline{C} = 8\underline{i} + 20\underline{j} - 6\underline{k}$ are coplanar.

Classroom Problems:

1. State which of the following are scalars and which are vectors:

   (a) specific heat;
   (b) momentum;
   (c) distance;
   (d) speed;
   (e) magnetic field intensity;
   (f) kinetic energy.

2. An automobile travels 3 miles due north, then 5 miles northeast. Represent these displacements graphically and determine the resultant displacement: (a) graphically, (b) analytically.

3. Let $A = (1, 3, -2, 6)$, $B = (4, -3, 3, 1)$ and $C = (2, 1, 5, 0)$ be three vectors in $\mathbb{R}^4$. Evaluate the following vectors:

   (a) $A + B$;
   (b) $A + B - C$;
   (c) $7A - 2B - 3C$;
   (d) $2A + B - 3C$.

4. Find a vector $\overline{B}$ of length 8 which in the opposite direction of the vector $\overline{A} = \langle 3, 2, -6\rangle$.

5. Find a unit vector $\mathbf{u}$ parallel to the resultant $\mathbf{R}$ of vectors $r_1 = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{r}_2 = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

6. Given the radius vectors $\mathbf{r}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{r}_2 = 3\mathbf{i} - 4\mathbf{j} + 9\mathbf{k}$, and $\mathbf{r}_3 = -\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. Find the magnitudes of: (a) $\mathbf{r}_3$, (b) $\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3$, (c) $\mathbf{r}_1 - \mathbf{r}_2 + 4\mathbf{r}_3$.

7. Find a unit vector $\mathbf{u}$ parallel to the resultant $\mathbf{R}$ of vectors $a = \langle 2, -4, 5\rangle$ and $\mathbf{b} = \langle 1, 2, 3\rangle$.

8. Find the value of $(A + B) \cdot (A - B)$ if $||A|| = 3$ and $||B|| = 4$.

9. Find the angle between the vectors $\mathbf{a} = \langle 12, -5, 2, 3\rangle$ and $\mathbf{b} = \langle 3, 4, 0, 5\rangle$ in $\mathbb{R}^4$.

10. Find the angle between the vector $\overline{A} = 3\underline{i} - 5\underline{j} + 2\underline{k}$ and the y-axis.

11. Find the value of c making the vectors $\mathbf{a} = (c, 1, -3, 5, 6)$ and $\mathbf{b} = (4, -2c, 1, 0, c)$ orthogonal in $\mathbb{R}^4$.

12. Find the value of c which makes the vectors $2\underline{i} - 2\underline{j} - \underline{k}$ and $-\underline{i} + \underline{j} + c\underline{k}$ parallel.

13. Find the work done in moving an object along a straight line:

    (a) from $(3, 2, -1)$ to $(2, -1, 4)$ in a force field given by $\mathbf{F} = \langle 4, 3, -2\rangle$.
    (b) from $(3, 4, 5)$ to $(-1, 9, 9)$ in a force field given by $\mathbf{F} = \langle -3, 5, -6\rangle$.

14. Evaluate: (a) $2\underline{j} \times 3\underline{k}$, (b) $2\underline{j} \times -\underline{k}$, (c) $-3\underline{i} \times -2\underline{k}$, $2\underline{j} \times (3\underline{i} - \underline{k})$.

15. Suppose $\overline{A} = \hat{j} + 2\hat{k}$ and $\overline{B} = \hat{i} + 2\hat{j} + 3\hat{k}$. Find: (a) $A \times B$, (b) $B \times A$, (c) $(A + B) \times (A - B)$.

16. Suppose $\overline{A} = -\hat{i} + \hat{j} + \hat{k}$, $\overline{B} = \hat{i} + \hat{j} - \hat{k}$, and $\overline{C} = \hat{i} + \hat{j} - \hat{k}$. Find (a) $(A \times B) \times C$, (b) $B \times (A \times C)$.

17. Find the area of the parallelogram determined by the vectors $\mathbf{a} = <1, 3>$ and $\mathbf{b} = <2, -4>$ in $\mathbb{R}^2$.

18. Determine a unit vector perpendicular to the plane of $\overline{A} = 2\underline{i} + 6\underline{j} + 3\underline{k}$ and $\overline{B} = 4\underline{i} + 3\underline{j} + k$.

Homework Problems:

1. State which of the following are scalars and which are vectors:

   (a) electric field intensity;
   (b) entropy;
   (c) work;
   (d) centrifugal force;
   (e) temperature;
   (f) charge;
   (g) shearing stress;
   (h) frequency.

2. Suppose $\overline{A} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\overline{B} = 4\hat{i} - 2\hat{j} - 4\hat{k}$. Find: (a) $\overline{A} + \overline{B}$, (b) $-2\overline{A}$, (c) $3\overline{A} + 2\overline{B}$, (d) $3\overline{A} + 2\overline{B}\}$ (e) $(2\overline{A} + \overline{B}) \cdot (\overline{A} + 3\overline{B})$.

3. Find the angle between $\overline{A} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\overline{B} = 7\mathbf{i} + 24\mathbf{k}$.

4. Determine the value of α so that $\mathbf{a} = \begin{bmatrix} 2 \\ \alpha \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2\alpha \\ 4 \\ -5 \end{bmatrix}$ are perpendicular.

5. Prove that two vectors $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ are perpendicular.

6. Find the values of β for which vectors $\overline{A} = \langle \beta, -2, 1 \rangle$ and $\overline{B} = \langle 2\beta, \beta, -4 \rangle$ are perpendicular.

7. Determine a unit vector perpendicular to the plane of $\overline{A} = 2\underline{i} + 6\underline{j} + 3\underline{k}$ and $\overline{B} = 4\underline{i} - 3\underline{j} + \underline{k}$, $(c) A \cdot (B \times C)$.

8. Find a unit vector perpendicular to both vector $\overline{A} = 4\underline{i} - \underline{j} + 3\underline{k}$ and $\overline{B} = -2\underline{i} + \underline{j} + 2\underline{k}$.

9. Find the area of the parallelogram of the vector sides$\mathbf{a} = <2, 3, , -1>$ and $\mathbf{b} = <1, 1, 5>$.

10. Suppose $\overline{A} = \hat{\imath} - 2\hat{\jmath} - 3\hat{k}$, $\overline{B} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$, and $\overline{C} = \hat{\imath} + 3\hat{\jmath} - 2\hat{k}$. Find:

    (a) $(\overline{A} \times \overline{B}) \times \overline{C}$;
    (b) $\overline{A} \cdot (\overline{B} \times \overline{C})$;
    (c) $(\overline{A} \times \overline{B}) \times (\overline{B} \times \overline{C})$;
    (d) $\overline{A} \times (\overline{B} \times \overline{C})$;
    (e) $(\overline{A} \times \overline{B}) \cdot \overline{C}$;
    (f) $(\overline{A} \times \overline{B})(\overline{B} \cdot \overline{C})$.

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Sheet #2

Lecture Problems:

1. Given $\mathbf{R} = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 4t\mathbf{k}$. Find: (a) $\frac{d\mathbf{R}}{dt}$, (b) $\frac{d^2\mathbf{R}}{dt}$, (c) $\left\|\frac{d\mathbf{R}}{dt}\right\|$, (d) $\left\|\frac{d^2\mathbf{R}}{dt^2}\right\|$.
2. Find the velocity and acceleration vectors of a particle which moves along the curve $x = 2\sin t^2$, $y = 2\cos 3t$, $z = 8t$ at any time $t > 0$. Hence, calculate the magnitude of the velocity and acceleration.
3. A particle moves with the position vector $\mathbf{r}(t) = \cos 5t\mathbf{i} + \sin 5t\mathbf{j}$. Show that:
   (a) The velocity vector is normal to the position vector.
   (b) The acceleration is directed toward the origin and has a magnitude proportional to the distance from the origin.
4. Suppose $\mathbf{A} = 5t^2\mathbf{i} + t\mathbf{j} - t^3\mathbf{k}$ and $\mathbf{B} = \sin t\mathbf{i} - \cos t\mathbf{j}$. Find: (a) $\frac{d}{dt}(\mathbf{A}\times\mathbf{B})$, (b) $\frac{d}{dt}\|A\|^2$.
5. Evaluate the gradient for $\phi(x,y,z) = x^2 + \sin z \ln y$ at the point $\left(2, 1, \frac{\pi}{2}\right)$.
6. Let $\mathbf{B}(x,y,z) = (x^2y - \sin z)\underline{i} + (xe^{y^2}\ln z + 5)\underline{j} + 2x\tan z\underline{k}$. Find $\operatorname{div}\mathbf{B}$ at the point $(1,1,1)$.
7. Let $^{-}\overline{F} = \langle 3x^2y - z, xz^3 + y^4, -2x^3z^2\rangle$. Find $\operatorname{grad}(\operatorname{div}\overline{F})$.
8. If $\phi = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$, evaluate $\nabla^2\phi$ at the point $(1,2,1)$.
9. If $\bar{A} = xz^2\underline{i} + 2z\sin y\underline{j} - 4yz^3\underline{k}$. Find:
   (a) $\operatorname{curl}\underline{A}$; (b) $\operatorname{curl}\operatorname{curl}\bar{A}$; (c) $\operatorname{div}\operatorname{curl}\bar{A}$
10. Find the unit normal vector to the surface $x^2 + y^2 = 2\cos 2z$ at the point $(1,-1,0)$.
11. Find the directional derivative of $x^2yz + 4xz^2$ at the point $(1,-1,1)$ in the direction $2\underline{i} - \underline{j} - 2\underline{k}$.
12. In what direction from the point $(1,2,3)$ is the directional derivative of $\phi = 2xz - y^2$ maximum? and what is the magnitude of this maximum?

Classroom Problems:

1. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, and $z = 3t - 5$, where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\mathbf{i} - 3\underline{j} + 2\underline{k}$.
2. A particle moves so that its position vector is given by $\mathbf{r} = \cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}$ where ω is a constant. Show that:
   (a) the velocity $\mathbf{v}$ of the particle is perpendicular to $\mathbf{r}$,
   (b) the acceleration $\mathbf{a}$ is directed toward the origin,
   (c) $\mathbf{r}\times\mathbf{v}$ is a constant vector.
3. Consider the scalar field ϕ defined by $\phi(x,y,z) = 3x^2z^2 + xy^3 + 15$. Find ϕ at the points: (a) $(0,0,0)$, (b) $(1,-2,2)$, (c) $(-1,-2,-3)$.

4. Suppose $\phi(x, y, z) = 2xz^4 - x^2y + \ln z$. Find $\nabla\phi(\text{grad}\,\phi)$ and $\|\nabla\phi\|$ at the point $(2, -2, -1)$.

5. Suppose $\mathbf{A} = x^2z^2\mathbf{i} - 2y^2z^2\mathbf{j} + xy^2z\mathbf{k}$. Find $\nabla\cdot\mathbf{A}$ (or $\text{div}\,\mathbf{A}$) and $\nabla\times\mathbf{A}$ (or $\text{curl}\,\mathbf{A}$) at the point $P(1, -1, 1)$.

6. Suppose $\mathbf{A} = 2yz\mathbf{i} - x^2y\mathbf{j} + xz^2\mathbf{k}$, $\mathbf{B} = x^2\mathbf{i} + yz\mathbf{j} - xy\mathbf{k}$, and $\phi = 2x^2yz^3$. Find: (a) $\mathbf{A}\times\nabla\phi$, (b) $\mathbf{B}\cdot\nabla\phi$, (c) $\nabla^2\phi$.

7. Find the unit normal vector to the surface $x^2 + y^2 - 2z = 16$ at the point $(3, 3, 1)$.

8. Find the directional derivative of $\phi = 4e^{x-2y+3z}$ at the point $(1, -2, 0)$ in the direction toward the point $(3, 4, 3)$.

9. Find the directional derivative of the function $F(x, y, z) = 2z^3 - 3(x^2 + y^2)z$ at he point $(1, 1, 1)$ along the y-axis and also long the vector $\mathbf{a} = \langle 1, 2, -2, \rangle$.

10. Find the directional derivative of the scalar function $f(x, y, z) = \ln\left(\frac{x^2y^3}{z}\right)$ at the point $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$ in the direction of the vector $\mathbf{a} = \langle 6, 2, -3\rangle$.

11. Find the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$.

Homework Problems:

1. A particle moves along the curve $x = e^{-t}, y = 2\cos 3t, z = 2\sin 3t$. Find the magnitude of velocity and acceleration at $t = 0$.

2. Given the scalar field defined by $f(x, y, z) = 4yx^3 + 3xyz - z^2 - 2$. Find (a) $f(1, -1, -2)$, (b) $f(0, -3, 1)$.

3. Evaluate $\text{div}\,(2x^2z\underline{i} - xy^2z\underline{j} + 3yz^2\underline{k})$.

4. Let $f = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$. Find $\nabla^2 f$.

5. Let $U = 3x^2y$ and $V = xz^2 - 2y$. Evaluate $\text{grad}\,[(\text{grad}\,U)\cdot(\text{grad}\,V)]$.

6. Suppose $\mathbf{A} = 2x^2\underline{i} - 3yz\underline{j} + xz^2\underline{k}$ and $f = 2z - x^3y$. Find $\mathbf{A}\cdot\nabla f$ and $\mathbf{A}\times\nabla f$ at the point $(1, -1, 1)$.

7. Suppose $\phi = x2^z + e^{y/x}$ and $\psi = 2z^2y - xy^2$. Find (a) $\nabla(\phi + \psi)$ and (b) $\nabla(\phi\cdot\nabla\psi)$ at the point $(1, 0, -2)$.

8. Let $\mathbf{A} = 2xz^2\mathbf{i} - yz\mathbf{j} + 3xz^3\mathbf{k}$ and $\phi = x^2yz$. Find, at the point $(1, 1, 1)$: (a) $\nabla\times\mathbf{A}$, (b) $\text{curl}\,(\phi\mathbf{A})$, (c) $\nabla\times(\nabla\times\mathbf{A})$, (d) $\nabla[\mathbf{A}\cdot\text{curl}\,\mathbf{A}]$, (e) $\text{curl}\,\text{grad}\,(\phi\mathbf{A})$.

9. Find the rate of change of the function $g(x, y, z) = x^2yz^3 + 4xz^3$ at the point $(1, -2, -1)$ in the direction of the vector $2i - j - 2k$.

10. Find the directional derivative of $\psi = xy^2z^3$ in the direction of normal vector of the surface $x^2 + xy^2 + yz^2 = 7$ at the point $(1, 2, 1)$.

11. Find a unit vector that is perpendicular to the surface of the paraboloid of revolution $z = x^2 + y^2$ at the point $(1, 2, 5)$.

12. Find the constants a and b so that the surface $ax^2 - byz = (a + 2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$.