* Sheet #2.

المسائل الملغية: 5, 6(C), 11

Linear Algebra
MATH 202.

- To get velocity $\vec{v}$: $\frac{d}{dx} \longrightarrow$ first derivative.
- To get acceleration: $\frac{d^2}{dx^2} \longrightarrow$ Second derivative.
  ↳ a.

→ Trigonometric differentiation.

* $\frac{d}{dx}(\sin x) = \cos x$

* $\frac{d}{dx}(\cos x) = -\sin x$

* $\frac{d}{dx}(\tan x) = \sec^2 x$

* $\frac{d}{dx}(\cot x) = -\csc^2 x$

* $\frac{d}{dx}(\sec x) = \sec x \tan x$

* $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

→ Trigonometric properties:

$\sin^2 x + \cos^2 x = 1$.

→ for partial derivative:

$\phi \rightarrow$ Scalar function.

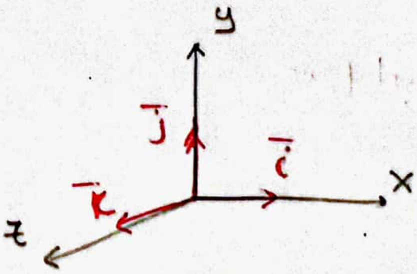
$\nabla \longrightarrow$ Nabla (used in partial derivative for vectors).

gradient ← grad $\phi = \nabla \phi$ ⇝ grad(Scalar) = vector.

$$\nabla \phi = \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k} \right)$$

diff بنعمل بالنسبة لـ x ونعتبر الباقي variables Constant

diff بنعمل بالنسبة للـ y وباقي الـ var. Constant



→ for normal vector:

$$n = \frac{\nabla \phi}{|\nabla \phi|}$$

→ **for exponential:**

$$\frac{d}{dx} e^{ax} = a e^{ax}$$

→ **for directional derivative.**

$\bar{a} = \bar{P_2} - \bar{P_1}$ → towards the vector.

$$\frac{d\phi}{d\bar{a}} = \frac{\nabla\phi \cdot \bar{a}}{|\bar{a}|} \rightarrow \text{Scalar value.}$$

$\bar{a} = \langle x, y, z \rangle$ → along the vector

→ **logarithmic derivative:**

$$\frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} \cdot f'(x).$$

→ variable vector $x\bar{i} + x\bar{j} + x\bar{k}$ في متغير في المعادلة

→ Constant vector $5\bar{i} + 5\bar{j} + 5\bar{k}$ في رقم بس في المعادلة

→ constant scalar $x^2 + x + 3x$ فيه متغير في المعادلة

→ Constant scalar $5, 6, 7$ رقم بس في المعادلة

1) A particle moves along the Curve

$x = 2t^2.$

$y = t^2 - 4t$

$z = 3t - 5$

where t is time

→ Find the Components of its velocity and acceleration at

→ time $t = 1$

→ inthe direction $\bar{i} - 3\bar{j} + 2\bar{K}$

→ Solution:

$$\boxed{\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{K}}$$

$$\bar{r}(t) = 2t^2\bar{i} + (t^2 - 4t)\bar{j} + (3t - 5)\bar{K}$$

$$\text{velocity} = \bar{v} = \frac{d\bar{r}}{dt} = 4t\bar{i} + (2t - 4)\bar{j} + 3\bar{K}$$

$\boxed{\text{at } t = 1}$

$$\bar{v}\Big|_{t=1} = 4\bar{i} + (2(1) - 4)\bar{j} + 3\bar{K}$$

$$= 4\bar{i} - 2\bar{j} + 3\bar{K} \longrightarrow \langle 4, -2, 3 \rangle$$

$$\text{acceleration} = \bar{a} = \frac{d^2\bar{r}}{dt^2} = \frac{d\bar{v}}{dt}$$

$$\bar{a} = 4\bar{i} + (2 - 0)\bar{j} + 0\bar{K} = 4\bar{i} + 2\bar{j} = \langle 4, 2, 0 \rangle$$

→ in the direction $\bar{u} = \bar{i} - 3\bar{j} + 2\bar{K} = \langle 1, -3, 2 \rangle$.

for velocity : $\bar{v} * \dfrac{\bar{u}}{|\bar{u}|}$

vector

Scalar

$$\bar{d} = \langle 4, -2, 3 \rangle * \frac{\langle 1, -3, 2 \rangle}{\sqrt{1 + 9 + 4}} = \frac{4 + 6 + 6}{\sqrt{14}} = \frac{16}{\sqrt{14}} = \underline{\underline{4.27}}$$

vector

for accelerations:

$$\bar{d} = \bar{a} \cdot \frac{\bar{u}}{|\bar{u}|^2} = \langle 4, 2, 0 \rangle * \frac{\langle 1, -3, 2 \rangle}{\sqrt{1+9+4}} = \frac{4-6}{\sqrt{14}}$$

$$= \frac{-2}{\sqrt{14}} = -0.53$$

2. A particle moves so that its position vector is given by

$$r = \cos wt\,\bar{i} + \sin wt\,\bar{j}$$

where w is a constant

show that:

a) the velocity v of the particle is perpendicular. $\hookrightarrow \bar{r}$

$\because$ Perpendicular

$$\bar{v} \perp \bar{r} = \bar{v} \cdot \bar{r} = 0$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\sin ax = a\cos ax$$

$$\frac{d}{dx}\cos ax = -a\sin ax$$

for velocity $v = \frac{d\bar{r}}{dt} = -w\sin wt\,\bar{i} + w\cos wt\,\bar{j}$

$$\bar{v} \cdot \bar{r} = (-w\sin wt\,\bar{i} + w\cos wt\,\bar{j})(\cos wt\,\bar{i} + \sin wt\,\bar{j})$$

$$= 0$$

$$= -w \cancel{\sin wt \cos wt}\,\bar{i} + w\cancel{\sin wt \cos wt}\,\bar{j} = 0$$

$\therefore v \perp \bar{r} = 0$ → لتساوي 0

b) the acceleration a is directed toward the origin.

$$\bar{a} = \frac{d^2 r}{dt^2} = \frac{dv}{dt}$$

$$= -w^2 \cos wt\,\bar{i} - w^2 \sin wt\,\bar{j}$$

$$= -w^2(\cos wt\,i + \sin wt\,\bar{j})$$

$$= -w^2 \cdot \bar{r}$$

$\therefore \vec{a}$ directed towards the origin

(لوجود $\bar{r}$ أو الماينس)

c) $r \times v$ is a constant vector.

$\bar{r} = \cos\omega t\,\bar{i} + \sin\omega t\,\bar{j}$

$\bar{v} = -\omega\sin\omega t\,\bar{i} + \omega\cos\omega t\,\bar{j}$

$$\bar{r}\times\bar{v} = \begin{vmatrix} \overset{\oplus}{\bar{i}} & \overset{\ominus}{\bar{j}} & \overset{\oplus}{\bar{k}} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix} = 0\bar{i} - 0\bar{j} + \bar{k}(\omega\cos^2\omega t + \omega\sin^2\omega t)$$

$$= \omega\bar{k}(\underbrace{\cos^2\omega t + \sin^2\omega t}_{1})$$

$$= \omega\bar{k}$$

$\therefore$ constant vector.

3) Consider the field φ defined by

$\varphi(x, y, z) = 3x^2z^2 + xy^3 + 15 \longrightarrow$ scalar

Find φ at the points a) $(0, 0, 0)$

b) $(1, -2, 2)$

c) $(-1, -2, -3)$.

we get grad $\varphi = \nabla\varphi = (6xz^2 + y^3)\bar{i} + 3y^2\bar{j} + 6x^2z\,\bar{k}$

a) $\nabla\varphi\Big|_{(0,0,0)} = (6(0)(0) + 0^3)\bar{i} + 3(0)\bar{j} + 6(0)(0)\bar{k} = \langle 0, 0, 0\rangle$

$= 0\bar{i} + 0\bar{j} + 0\bar{k}$

b) $\nabla\varphi\Big|_{(1,-2,2)} = (6(1)(2) + (-2)^3)\bar{i} + 3(-2)^2\bar{j} + 6(1)^2(2)\bar{k}$

$= 4\bar{i} + 12\bar{j} + 12\bar{k} = \langle 4, 12, 12\rangle$.

c) $\nabla\varphi\Big|_{(-1,-2,-3)} = (6(-1)(-3)^2 + (-2)^3)\bar{i} + 3(-2)^2\bar{j} + 6(-1)^2(-3)\bar{k}$

$= -62\bar{i} + 12\bar{j} - 18\bar{k} = \langle -62, -12, -18\rangle$.

4. Suppose $\phi(x, y, z) = 2xz^4 - x^2 y + \ln z$

Find $\nabla\phi$ (grad(ϕ))

$\|\nabla\phi\|$ at the point $(2, -2, -1)$.

↳ Magnitude.

$$\nabla\phi = (2z^4 - 2xy)\bar{i} + (-x^2)\bar{j} + (8xz^3 + \frac{1}{z})\bar{k}$$

$$\nabla\phi\Big|_{(2,-2,-1)} = (2(-1)^4 - 2(2)(-2))\bar{i} - (2)^2\bar{j} + (8(2)(-1)^3 + \frac{1}{-1})\bar{k}$$

$$= 10\bar{i} - 4\bar{j} - 17\bar{k} = \langle 10, -4, -17 \rangle.$$

$$\text{Magnitude} = \|\nabla\phi\| = \sqrt{10^2 + (-4)^2 + (-17)^2} = 9\sqrt{5}$$

o) suppose $A = 2yz\bar{i} - x^2y\bar{j} + xz^2\bar{K}$

$B = x^2\bar{i} + yz\bar{j} - xy\bar{K}$

$\phi = 2x^2yz^3$

Find a) $A \times \nabla\phi$ → vector x vector = vector

b) $B \cdot \nabla\phi$ → vector . vector = scalar.

$\nabla\phi = 4xyz^3\bar{i} + 2x^2z^3\bar{j} + 6x^2yz^2\bar{K}$

$$a)\ A \times \nabla\phi = \begin{vmatrix} \overset{\oplus}{\bar{i}} & \overset{\ominus}{\bar{j}} & \overset{\oplus}{\bar{K}} \\ 2yz & -x^2y & xz^2 \\ 4xyz^3 & 2x^2z^3 & 6x^2yz^2 \end{vmatrix}$$

$$= \bar{i}(-6x^4y^2z^2 - 2x^3z^5) - \bar{j}(12x^2y^2z^3 - 4x^2yz^5)$$

$$+\bar{K}(4x^2yz^4 + 4x^3y^2z^3)$$

$$b)\ B \cdot \nabla\phi = (\underline{x^2}\bar{i} + \underline{yz}\bar{j} - \underline{xy}\bar{K})(\underline{4xyz^3}\bar{i} + \underline{2x^2z^3}\bar{j} + \underline{6x^2yz^2}\bar{K})$$

$$= 4x^3yz^3 + 2x^2yz^4 - 6x^3y^2z^2\bar{K}$$

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7) Find the unit normal vector to the surface
$x^2+y^2-2z=16$ at the point $(3,3,1)$.

$\hookrightarrow$ ∴ Scalar ∴ $\phi(x,y,z) = x^2+y^2-2z-16$

$$\nabla\phi = 2x\bar{i} + 2y\bar{j} - 2\bar{k}$$

$$\nabla\phi\Big|_{(3,3,1)} = 2(3)\bar{i} + 2(3)\bar{j} - 2\bar{k}$$
$$= 6\bar{i} + 6\bar{j} - 2\bar{k} = \langle 6,6,-2\rangle .$$

normal vector $\boxed{\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}}$

$$\hat{n} = \frac{\langle 6,6,-2\rangle}{\sqrt{36+36+4}} = \left\langle \frac{6}{2\sqrt{19}}, \frac{6}{2\sqrt{19}}, \frac{-2}{2\sqrt{19}} \right\rangle$$

8) Find the directional derivative of
$\phi = 4e^{x-2y+3z}$ at the point P_1 $(1,-2,0)$ in the direction towards the point P_2 $(3,4,3)$.

to get $\bar{a} = \langle 3,4,3\rangle - \langle 1,-2,0\rangle$

مسافة

$$= \langle 2,6,3\rangle = 2\bar{i} + 6\bar{j} + 3\bar{k} .$$

$$\nabla\phi = 4e^{x-2y+3z}\,\bar{i} + (-2)(4)e^{x-2y+3z}\,\bar{j} + (4)(3)\,e^{x-2y+3z}\,\bar{k}$$
$$= 4e^{x-2y+3z}\,\bar{i} - 8e^{x-2y+3z}\,\bar{j} + 12e^{x-2y+3z}\,\bar{k}$$

$$\nabla\phi\Big|_{(1,-2,0)} = 4e^{1-2(-2)} - 8e^{1-2(-2)}\,\bar{j} + 12e^{1-2(-2)}\,\bar{k}$$
$$= 4e^5\,\bar{i} - 8e^5\,\bar{j} + 12e^5\,\bar{k}$$
$$= \langle 4e^5, -8e^5, 12e^5\rangle$$

for directional derivative :

$$\boxed{\frac{d\phi}{d\bar{a}} = \frac{\nabla\phi \cdot \bar{a}}{|\bar{a}|}} \begin{array}{l} \rightarrow \text{Scalar} \\ \rightarrow \text{Scalar} \end{array}$$

$$\frac{d\phi}{d\bar{a}} = \frac{<8e^5 - 48e^5 + 36e^5>}{\sqrt{2^2+6^2+3^2}} = \frac{-4e^5}{7} = \underline{\underline{-84.8}}$$

a) Find the directional derivative of the function $F(x,y,z) = 2z^3 - 3(x^2+y^2)z = 2z^3 - 3x^2z - 3y^2z$ at the point $(1,1,1)$ along the y-axis and also along the vector $a = <1,2,-2>$.

Next to
بجانب

a) along the y-axis : $\rightarrow <0,1,0>$.

$\bar{a} = <0,1,0>$.

$$\nabla\phi = -6xz\bar{i} - 6yz\bar{j} + (6z^2 - 3x^2 - 3y^2)\bar{k}$$

$$\nabla\phi\Big|_{(1,1,1)} = -6i - 6j + (6-3-3)\bar{k} = <-6,-6,0>.$$

$$\frac{d\phi}{d\bar{a}} = \frac{\nabla\phi \cdot \bar{a}}{|\bar{a}|} = \frac{<-6,-6,0> \cdot <0,1,0>}{\sqrt{0+1+0}} = \frac{-6}{\sqrt{1}} = \underline{\underline{-6}}$$

b) along the vector $a = \langle 1, 2, -2 \rangle$

$$\nabla \phi = \langle -6, -6, 0 \rangle$$

$$\bar{a} = \langle 1, 2, -2 \rangle.$$

$$\frac{d\phi}{d\bar{a}} = \frac{\nabla\phi \cdot \bar{a}}{|\bar{a}|} = \frac{\langle -6, -6, 0\rangle \cdot \langle 1, 2, -2\rangle}{\sqrt{1+4+4}} = \frac{-6-12}{3} = \frac{-18}{3}$$

$$= \underline{\underline{-6}}$$

10) Find the directional derivative at the scalar function $f(x, y, z) = \ln\left(\frac{x^2y^3}{z}\right)$ at the point $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$ in the direction of a vector $a = \langle 6, 2, -3 \rangle$.

باتجاه

$$\nabla f = \frac{1}{\frac{x^2y^3}{\cancel{z}}} \cdot \frac{2xy^3}{\cancel{z}} \bar{i} + \frac{1}{\frac{\cancel{x^2y^3}}{\cancel{z}}} \cdot \frac{3x^2y^2}{z} \bar{j}$$

$$+ \frac{1}{\frac{\cancel{x^2y^3}}{\cancel{z}}} \cdot \frac{-\cancel{x^2y^3}}{z^{\cancel{2}}} \bar{k}$$

$$\nabla f = \frac{2}{x}\bar{i} + \frac{3}{y}j - \frac{1}{z}\bar{k}$$

$$\nabla f \Big|_{\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)} = \frac{2}{\frac{1}{2}}\bar{i} + \frac{3}{\frac{1}{6}}\bar{j} - \frac{1}{\frac{1}{3}}\bar{k}$$

$$= 4\bar{i} + 18j - 3\bar{k} = \langle 4, 18, -3 \rangle.$$

$$\frac{d\phi}{d\bar{a}} = \frac{\nabla f \cdot \bar{a}}{|\bar{a}|} = \frac{\langle 4, 18, -3\rangle \cdot \langle 6, 2, -3\rangle}{\sqrt{6^2+2^2+(-3)^2}} = \frac{24+36+9}{\sqrt{36+4+9}}$$

$$= \boxed{\frac{69}{7}}$$