ME685 PROJECT

Study of Harmonic and Anharmonic Oscillator using
Euler and Modified Euler Method

By

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1 problem statement

1.1 Problem Statement 1

We will draw the phase space trajectory for 1D harmonic oscillator. We will write a code to obtain the phase space trajectory of a one-dimensional harmonic oscillator using Euler method and Modified Euler method. We have to plot:

1. Velocity and Position
2. Total Energy vs Time

Run the code from $t = 0$ to $t = 45$. We will take $k = 1$ and $m = 1$ for simplicity. We will select our initial conditions such that the total energy is equal to 1. Then, we will compare their plots.

1.2 Problem Statement 2

We will write a code to obtain the phase space trajectory of a one-dimensional Anharmonic oscillator using Euler method and Modified Euler method. The potential energy is given by:

$$U(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4$$

We have to plot:

1. Velocity and Position
2. Total Energy vs Time

We will run the code from $t = 0$ to $t = 45$. We will take $k = 1$ and $m = 1$ for simplicity. We will select our initial conditions such that the total energy is equal to 1. Then, we will compare their plots.

2 Pseudocode of Problem Statement 1

2.1 Using Euler Method

The expression for energy E is given by:

$$E = \frac{1}{2}kx^2 + \frac{1}{2m}p^2$$

where k is the stiffness constant, x is the displacement, p is the momentum, and m is the mass. Here, for simplicity, we take $k = 1$ N/m and $m = 1$ kg.

Now we initially take position and velocity such that the total energy is equal to 1, i.e. $v_{\text{initial}} = 1$ and $x_{\text{initial}} = 1$.

Firstly, we need to find acceleration that will be used in the Euler method. The expression for acceleration a is given by:

$$a = \frac{d}{dx}\left(\frac{x^2}{2}\right) = -x$$

$\text{dt} = 0.01$. Array of time is given by $t = 0 : \text{dt} : 45$. $n = \text{numel}(t)$.

$$x(1) = x_{\text{initial}}$$
$$v(1) = v_{\text{initial}}$$

Loop:

$$\begin{aligned}
&\text{for } i = 1 : n-1 \\
&\quad x_{i+1} = x_i + v_i \times \text{dt} \\
&\quad v_{i+1} = v_i + a_i \times \text{dt} \\
&\quad v_{i+1} = v_i - x_i \times \text{dt} \\
&\text{end}
\end{aligned}$$

Then we will plot position vs velocity (as velocity and momentum are the same as $m = 1$) and energy vs time.

```matlab
%%% HARMONIC OSCILLATOR WITH EULER METHOD %%%


    n = 1000 %number of time steps;
    dt = 0.01;
    x0 = 0 %initial position;
    v0 = 1; %initial velocity

    xval = zeros(1, n);
    vval = zeros(1, n);
    enval = zeros(1, n);

    for i = 1:n
        xval(i) = x0;
        vval(i) = v0;
        enval(i) = fE(x0, v0);
        x = x0;
        v = v0;
        x0 = fx(x, v, dt);
        v0 = fv(x, v, dt);
    end

    disp(['Max position:', num2str(max(xval)), ' Min position:',num2str(min(xval))]);

    t = linspace(0, (n-1)*dt, n);
    figure;
    plot(xval, vval);
    axis square;
    grid on;
    xlabel('Position');
    ylabel('Velocity');

    figure;
    plot(t, enval);
    xlabel('Time');
    ylabel('Energy');


function y = fx(x, v, dt)
    y = (x + v*dt );
end

function y = fv(x, v, dt)
    y = (v - x*dt );
end

function y = fE(x, v)
    y = (0.5*v^2 + 0.5*x^2);
end
```

2.2 Using modified Euler Method

The expression for energy E is given by:

$$E = \frac{1}{2}kx^2 + \frac{1}{2m}p^2$$

where k is the stiffness constant, x is the displacement, p is the momentum, and m is the mass. Here, for simplicity, we take $k = 1$ N/m and $m = 1$ kg.

Now we initially take position and velocity such that the total energy is equal to 1, i.e. $v_{\text{initial}} = 1$ and $x_{\text{initial}} = 1$.

Firstly, we need to find acceleration that will be used in the Euler method. The expression for acceleration a is given by:

$$a = \frac{d}{dx}\left(\frac{x^2}{2}\right) = -x$$

dt $= 0.01$. Array of time is given by $t = 0 : \text{dt} : 45$. $n = \text{numel}(t)$.

$$x(1) = x_{\text{initial}}$$
$$v(1) = v_{\text{initial}}$$

Loop:

$$\begin{aligned}
&\text{for } i = 1 : n-1 \\
&\quad x_{i+1} = x_i + v_i \times \text{dt} + a_i \times \text{dt}^2 \\
&\quad x_{i+1} = x_i + v_i \times \text{dt} - x_i \times \text{dt}^2 \\
&\quad v_{i+1} = v_i + a_i \times \text{dt} \\
&\quad v_{i+1} = v_i - x_i \times \text{dt} \\
&\text{end for}
\end{aligned}$$

Then we will plot position vs velocity (as velocity and momentum are the same as $m = 1$) and energy vs time.//

Here to calculate position we include one more term in taylor series ,thus reducing error but the velocity calculation remains same.

```matlab
%%% HARMONIC OSCILLATOR WITH MODIFIED EULER METHOD %%%


    n = 1000 %number of time steps;
    dt = 0.01;
    x0 = 0 %initial position;
    v0 = 1; %initial velocity

    xval = zeros(1, n);
    vval = zeros(1, n);
    enval = zeros(1, n);

    for i = 1:n
        xval(i) = x0;
        vval(i) = v0;
        enval(i) = fE(x0, v0);
        x = x0;
        v = v0;
        x0 = fx(x, v, dt);
        v0 = fv(x, v, dt);
    end

    disp(['Max position:', num2str(max(xval)), ' Min position:',num2str(min(xval))]);

    t = linspace(0, (n-1)*dt, n);
    figure;
    plot(xval, vval);
    axis square;
    grid on;
    xlabel('Position');
    ylabel('Velocity');

    figure;
    plot(t, enval);
    xlabel('Time');
    ylabel('Energy');


function y = fx(x, v, dt)
    y = (x + v*dt-x*dt^2 );
end

function y = fv(x, v, dt)
    y = (v - x*dt );
end

function y = fE(x, v)
    y = (0.5*v^2 + 0.5*x^2);
end
```

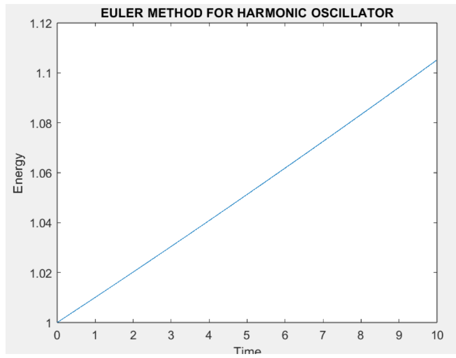

Figure 1: Energy vs Time (Euler Method)

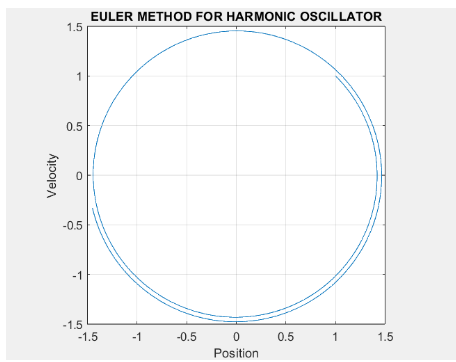

Figure 2: Position vs Velocity (Euler Method)

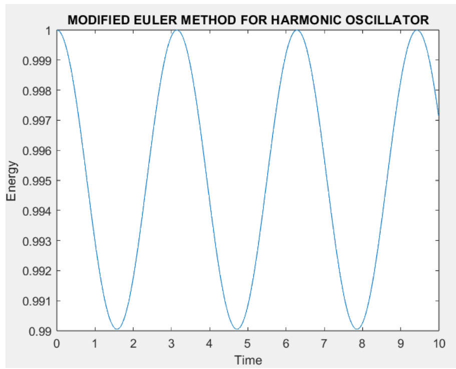

Figure 3: Energy vs Time (Modified Euler Method)

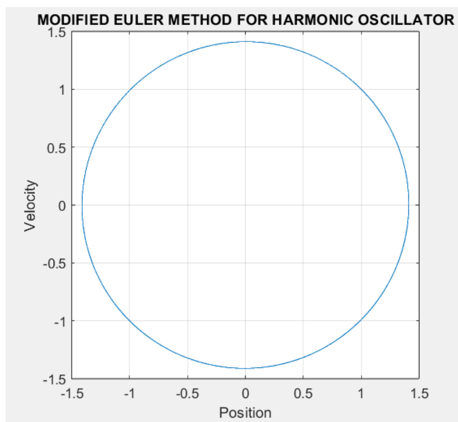

Figure 4: Position vs Velocity (Modified Euler Method)

Modified Euler method gives a solution closer to the analytical solution, as seen in the figures. The phase space trajectory of the Euler method is not accurate even with a lower value of dt.

3 Pseudocode of Problem Statement 2

3.1 Using Euler Method

The expression for energy E is given by:

$$E = 0.5 \cdot kx^2 - 0.3 \cdot kx^3 + 0.25 \cdot kx^4 + \frac{1}{2m}p^2$$

where k is the stiffness constant, x is the displacement, p is the momentum, and m is the mass. Here, for simplicity, we take $k = 1$ N/m and $m = 1$ kg.

Now we initially take position and velocity such that the total energy is equal to 1, i.e. $v_{\text{initial}} = 1$ and $x_{\text{initial}} = 1$.

Firstly, we need to find acceleration that will be used in the Euler method. The expression for acceleration a is given by:

$$a = \frac{d}{dx}\left(-0.5x^2 - 0.3x^3 + 0.25x^4\right) = -x + x^2 - x^3$$

$\mathrm{dt} = 0.01$. Array of time is given by $t = 0 : \mathrm{dt} : 45$. $n = \mathrm{numel}(t)$.

$$x(1) = x_{\text{initial}}$$
$$v(1) = v_{\text{initial}}$$

Loop:

$$\begin{aligned}
&\text{for } i = 1 : n-1 \\
&\quad x_{i+1} = x_i + v_i \times \mathrm{dt} \\
&\quad v_{i+1} = v_i + a_i \times \mathrm{dt} \\
&\quad v_{i+1} = v_i + (-x_i + x_i^2 - x_i^3 \times \mathrm{dt} \\
&\text{end}
\end{aligned}$$

Then we will plot position vs velocity (as velocity and momentum are the same as $m = 1$) and energy vs time.

```matlab
%%% ANHARMONIC OSCILLATOR WITH EULER METHOD %%%


    n = 1000 %number of time steps;
    dt = 0.01;
    x0 = 0 %initial position;
    v0 = 1.414; %initial velocity

    xval = zeros(1, n);
    vval = zeros(1, n);
    enval = zeros(1, n);

    for i = 1:n
        xval(i) = x0;
        vval(i) = v0;
        enval(i) = fE(x0, v0);
        x = x0;
        v = v0;
        x0 = fx(x, v, dt);
        v0 = fv(x, v, dt);
    end

    disp(['Max position:', num2str(max(xval)), ' Min position:',num2str(min(xval))]);

    t = linspace(0, (n-1)*dt, n);
    figure;
    plot(xval, vval);
    axis square;
    grid on;
    xlabel('Position');
    ylabel('Velocity');

    figure;
    plot(t, enval);
    xlabel('Time');
    ylabel('Energy');


function y = fx(x, v, dt)
    y = (x + v*dt );
end

function y = fv(x, v, dt)
    y = (v - x*dt + x^2*dt - x^3*dt);
end

function y = fE(x, v)
    y = (0.5*v^2 + 0.5*x^2 - 0.3333*x^3 + 0.25*x^4);
end
```

3.2 Using Modified Euler Method

The expression for energy E is given by:

$$E = 0.5 \cdot kx^2 - 0.3 \cdot kx^3 + 0.25 \cdot kx^4 + \frac{1}{2m}p^2$$

where k is the stiffness constant, x is the displacement, p is the momentum, and m is the mass. Here, for simplicity, we take $k = 1$ N/m and $m = 1$ kg.

Now we initially take position and velocity such that the total energy is equal to 1, i.e. $v_{\text{initial}} = 1$ and $x_{\text{initial}} = 1$.

Firstly, we need to find acceleration that will be used in the Euler method. The expression for acceleration a is given by:

$$a = \frac{d}{dx}\left(-0.5x^2 - 0.3x^3 + 0.25x^4\right) = -x + x^2 - x^3$$

$\mathrm{dt} = 0.01$. Array of time is given by $t = 0 : \mathrm{dt} : 45$. $n = \mathrm{numel}(t)$.

$$x(1) = x_{\text{initial}}$$
$$v(1) = v_{\text{initial}}$$

Loop:

$$\begin{aligned}
&\text{for } i = 1 : n-1\\
&\quad x_{i+1} = x_i + v_i \times \mathrm{dt} + a \times \mathrm{dt}^2\\
&\quad x_{i+1} = x_i + v_i \times \mathrm{dt} + (-x_i + x_i^2 - x_i^3) \times \mathrm{dt}^2\\
&\quad v_{i+1} = v_i + a_i \times \mathrm{dt}\\
&\quad v_{i+1} = v_i + (-x_i + x_i^2 - x_i^3) \times \mathrm{dt}\\
&\text{end}
\end{aligned}$$

Then we will plot position vs velocity (as velocity and momentum are the same as $m = 1$) and energy vs time. Here to calculate position we include one more term in taylor series ,thus reducing error but the velocity calculation remains same.

```matlab
%%% ANHARMONIC OSCILLATOR WITH MODIFIED EULER METHOD %%%


    n = 1000 %number of time steps;
    dt = 0.01;
    x0 = 0 %initial position;
    v0 = 1.414; %initial velocity

    xval = zeros(1, n);
    vval = zeros(1, n);
    enval = zeros(1, n);

    for i = 1:n
        xval(i) = x0;
        vval(i) = v0;
        enval(i) = fE(x0, v0);
        x = x0;
        v = v0;
        x0 = fx(x, v, dt);
        v0 = fv(x, v, dt);
    end

    disp(['Max position:', num2str(max(xval)), ' Min position:',num2str(min(xval))]);

    t = linspace(0, (n-1)*dt, n);
    figure;
    plot(xval, vval);
    axis square;
    grid on;
    xlabel('Position');
    ylabel('Velocity');

    figure;
    plot(t, enval);
    xlabel('Time');
    ylabel('Energy');


function y = fx(x, v, dt)
    y = (x + v*dt - x*dt^2 + x^2*dt^2 - x^3*dt^2);
end

function y = fv(x, v, dt)
    y = (v - x*dt + x^2*dt - x^3*dt);
end

function y = fE(x, v)
    y = (0.5*v^2 + 0.5*x^2 - 0.3333*x^3 + 0.25*x^4);
end
```

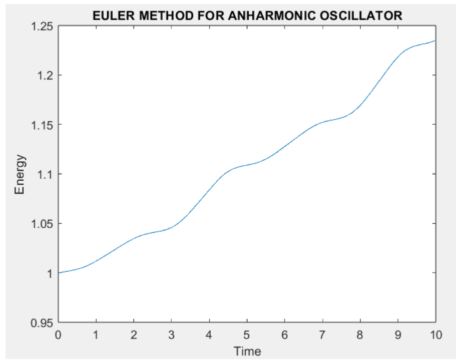

Figure 5: Energy vs Time (Euler Method)

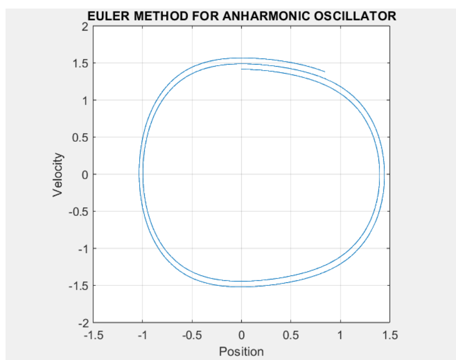

Figure 6: Position vs Velocity (Euler Method)

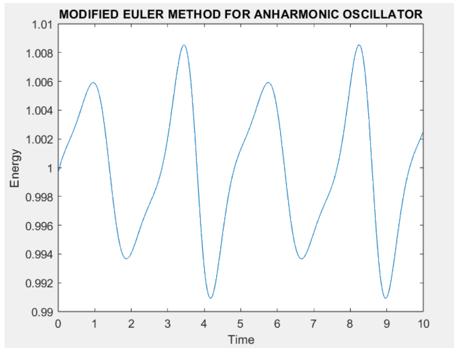

Figure 7: Energy vs Time (Modified Euler Method)

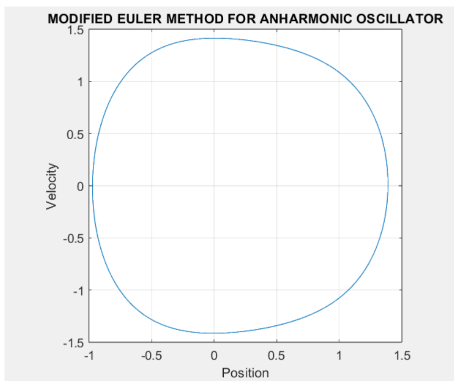

Figure 8: Position vs Velocity (Modified Euler Method)

Modified Euler method gives a solution closer to the analytical solution, as seen in the figures. The phase space trajectory of the Euler method is not accurate even with a lower value of dt.

4 CONCLUSION

We have found out the phase-space trajectory for both the Harmonic and Anharmonic oscillator using both Euler And Modified Euler Method.And it was found out that Modified Euler Method has more Accuracy Than Euler Method because In Modified Euler Method for calculating position we took taylor series upto 2nd order as compare to Euler method where we took taylor series upto 1st order.