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THE FASCINATING WORLD OF GRAPH THEORY

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Preface

Mathematics rarely has the reputation we feel it deserves. To many, mathematics is an area that is, sadly, too difficult and too boring. It requires too much effort to learn and to understand. It's not as much fun as other subjects. In recent years there have been numerous articles written about how many American high-school students have been outperformed in mathematics and science by students from other nations. There have also been reports of a marked decrease in the number of Americans in colleges earning graduate degrees in mathematics. For whatever reason, not nearly enough talented American students have become sufficiently excited about mathematics. For many students, this is a missed opportunity. For the United States, this is a missed opportunity. There are many areas within mathematics and we happen to think that they are all exciting. Behind the many interesting theorems in each of these areas is a history of how these came about—a story of how some dedicated mathematicians discovered something of interest and importance. These theorems were often not only attractive to those who discovered them but in many cases unexpected to others. In many instances, these theorems turned out to be extraordinarily useful—both within and outside of mathematics. Our goal in this book is to introduce you to one of the many remarkable areas of mathematics. It is with pleasure that we invite you to enter

The Fascinating World of Graph Theory.

Like every other scholarly field, mathematics is composed of a number of areas, similar in many ways, yet each having their own distinct characteristics. The areas with which you are probably most familiar include algebra, geometry, trigonometry and calculus. Learning and understanding these subjects may very well have required some effort on your part but, hopefully, it has been interesting as well. In fact, learning any subject should be fun. But where did these and all other areas of mathematics come from? The answer to this question is that they came from people—from their curiosity, their imagination, their cleverness. Although many of these people were mathematicians, some were not. Sometimes they were students—like all of us are (or were).

It is our goal here to introduce you to a subject to which you may have had little or no exposure: the field of graph theory. While we wish to show you how interesting this area of mathematics is, we hope to convince you that mathematics itself is not only interesting but can in fact be exciting. So come with us as we take you along on what we believe will be a fascinating journey through the area of graph theory. Not only do we want to introduce you to many of the interesting topics in this area of mathematics, but it is our desire to give you an idea of how these topics may have been discovered and the kinds of problems they can be used to solve.

Among the many things we discuss here is how often a rather curious problem or question can lead not only to a mathematical solution but to an entire topic in mathematics. While it is not our intention to describe some deep or advanced mathematics here, we do want to give an idea of how we can convince ourselves that certain mathematical statements are true.

Chapter 1 begins with some curious problems, all of which can be looked at mathematically by means of the main concept of this book: graphs. Some of these problems turn out to be important historically and will be revisited when we have described enough information to solve the problems. This chapter concludes with a discussion of the fundamental concepts that occur in this area of mathematics. The last thing we do in Chapter 1 is present a theorem often called the First Theorem of Graph Theory, dealing with what happens when the degrees of all vertices of a graph are added.

Chapter 2 begins with a discussion of theorems from many areas of mathematics that have been judged among the most beautiful. We see here that not only is graph theory well represented on this list, but one mathematician in particular is especially well represented on this list. One of the theorems on this list leads us into a much-studied type of graph called regular graphs. From this, the degrees of the vertices of a graph are discussed at some length. The remainder of the chapter deals with concepts and ideas concerning the structure of graphs. And the chapter closes with a rather mysterious problem in graph theory that no one has been able to solve.

Chapter 3 discusses the most fundamental property that a graph can possess, dealing with the idea that within the graph, travel is possible between every two locations. This brings up questions of the distance between locations in a graph and those locations that are far from or close to a given location. The chapter concludes with the rather humorous concept of Erdős numbers, based on mathematicians who have done research with mathematicians who have done research with . . . who have done research with the celebrated twentieth-century mathematician Paul Erdős.

Chapter 4 concerns the simplest structure that a connected graph can possess, leading us to the class of graphs called trees—because they often look like trees. These graphs have connections to chemistry and can assist us in solving problems where certain decisions must be made at each stage in solving the problem. This chapter ends by discussing a practical problem, one that involves the least expensive highway system that would allow us to travel between any two locations in the system.

Graph theory has a rather curious history. Most acknowledge that this area began in the eighteenth century when the brilliant mathematician Leonhard Euler was introduced to and solved a problem referred to as the Königsberg Bridge Problem and then went on to describe a considerably more complex problem. This led to a class of graphs named for Euler, which we study in Chapter 5. This chapter concludes with another well-known problem, the Chinese Postman Problem, which deals with minimizing the length of a round-trip that a letter carrier might take.

Chapter 6 discusses a class of graphs named for a famous physicist and mathematician of the nineteenth century: Sir William Rowan Hamilton. Although Hamilton had very little to do with graph theory, it was he who came up with the idea of "icosian calculus", which led him to inventing a game that involved finding round-trips around a dodecahedron in which every vertex is encountered exactly once. Beginning midway through the twentieth century, an explosion of theoretical results involving this concept occurred. This chapter ends with a discussion of a problem of practical importance, that of finding a shortest or least costly round-trip that visits all locations of a certain type.

Problems that ask whether some collection of objects can be matched in some way to another collection of objects are plentiful—such as matching applicants to job openings or sometimes just people to people. These kinds of problems, discussed in Chapter 7, gave rise in the late nineteenth century to the first consideration of graph theory as a theoretical area of mathematics and no doubt led to the term "graph" being used for the structure we discuss in this book. From this topic in graph theory, we can see how different types of schedulings are possible.

Chapter 8 concerns problems of whether a graph can be divided into certain other kinds of graphs, primarily cycles. Whether some specific complete graphs can be divided into triangles in some manner was the situation encountered when, in the mid-nineteenth century, the mathematician Thomas Kirkman stated and then solved a problem often referred to as Kirkman's Schoolgirl Problem. There is a relationship between graph decomposition problems and a problem dealing with whether the vertices of a graph can be labeled with certain integers in a manner that produces a desirable labeling of its edges. This chapter ends with a tantalizing puzzle called Instant Insanity and how graphs can be utilized to solve it.

There are situations when travel involves using one-way streets and in order to model this in a graph, it is necessary to assign directions to the edges. This gives rise to the concept of an oriented graph. These structures can also be used to represent a sports tournament where assigning a direction to an edge represents the defeat of one team by another. The mathematics related to this is presented in Chapter 9. The chapter concludes with a discussion of how various voting techniques can result in often surprising outcomes.

Some interesting problems can be looked at in terms of whether certain graphs can be drawn in the plane without any of their edges crossing. This deals with the concept of planar graphs introduced in Chapter 10. There is a rich theory with these graphs, which is discussed in this chapter. One of the problems discussed here is the Brick-Factory Problem, which originated in a labor camp during World War II.

One of the most famous problems in mathematics concerns whether it's always possible to color the regions of every map with four colors so that neighboring regions are colored differently. This Four Color Problem was the idea of a young British mathematician in the mid-nineteenth century and eventually gained notoriety and interest as this problem became better known. The Four Color Problem, famous not only for the length of time it took to solve but for the controversial method used to solve it, is discussed in Chapter 11. This led to coloring the vertices of a graph and how this can be used to solve a variety of problems, from scheduling problems to traffic-light phase problems.

Not only is it of interest to consider coloring the vertices of a graph, both from practical and theoretical points of view, it is also of interest to consider coloring its edges. This is the topic of Chapter 12. Here too this can aid us in solving certain types of scheduling problems. This also leads us to consider a class of numbers in graph theory called Ramsey numbers. The chapter concludes with a curious theorem called the Road Coloring Theorem, which tells us that in certain traffic systems consisting only of one-way streets in which the same number of roads leave each location, roads can be colored so that directions can be given to arrive at some destination regardless of the location where the traveler presently resides.

While the main purpose of this book is to illustrate how interesting and intriguing (and sometimes mysterious) just one area of mathematics can be, this book can also be used as a textbook. At the end of the book is an "Exercises" section containing several exercises for all chapters in the book.

Finally, it is a pleasure to acknowledge the very professional support we received from the Princeton University Press staff, especially Vickie Kearn, Sara Lerner, Alison Anuzis, and Quinn Fusting, as well as the anonymous reviewers of our initial manuscript. Their feedback, comments, and attention to details have resulted in many improvements to this book. For this we are extremely grateful.

A.B., G.C., P.Z.

Prologue

In a traditional mathematics book, authors typically develop the subject from the bottom up, starting with basic, easier results and gradually leading to more challenging and sophisticated results. This is not what we will do here. Rather, our intention is to display what we consider as some fascinating, beautiful material in an order that we believe will keep the reader interested in the subject and wondering what might lie ahead. Sometimes we'll prove results, sometimes we won't. When we don't prove a result, we'll supply some intuition to the reader or provide a reference where more information can be found. Having just said this, however, there are lots of exercises for each chapter at the end of the book just in case a professor would like to use this as a textbook.

• • • •

So, with apologies and gratitude to composer Stephen Sondheim, we are about to introduce you to . . .

Something familiar,
Something peculiar,
Something for everybody,
Graph theory tonight!

Sometimes directed,
Often connected,
Useful and unexpected,
Graph theory tonight!

Nothing complex,
Something complete,
You can be sure that we'll be discrete!

Orientations,
New applications,
Plane-ly, four colors get it right!
Calculus tomorrow,
Graph theory tonight!

THE FASCINATING WORLD OF GRAPH THEORY

1

Introducing Graphs

The mathematical structure known as a *graph* has the valuable feature of helping us to visualize, to analyze, to generalize a situation or problem we may encounter and, in many cases, assisting us to understand it better and possibly find a solution. Let's begin by seeing how this might happen and what these structures look like.

FIRST, . . . FOUR PROBLEMS

We begin with four problems that have a distinct mathematical flavor. Yet any attempt to solve these problems doesn't appear to use any mathematics you may have previously encountered. However, all of the problems can be analyzed and eventually solved with the aid of a relatively new sort of mathematical object and that object is a graph. The graph we're referring to is not the kind of graph you've seen before. For example, Figure 1.1 shows the graph of the function $y = \sin x$. That is not the kind of graph we're referring to.

The Problem of the Five Princes

Once upon a time, there was a kingdom ruled by a king who had five sons. It was his wish that upon his death, this kingdom should be divided into five regions, one region for each son, such that each region would have a common boundary with each of the other four regions. Can this be done?

Figure 1.2 illustrates an unsuccessful attempt to satisfy the king's wishes. Every two of the five regions, numbered 1, 2, 3, 4, 5, share some common boundary, except regions 4 and 5.

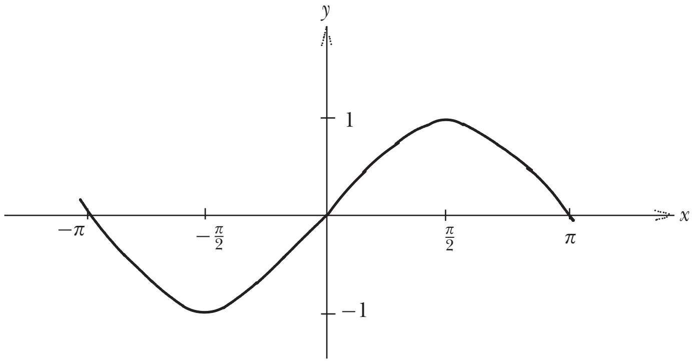


Figure 1.1. Not the sort of graph we're talking about.

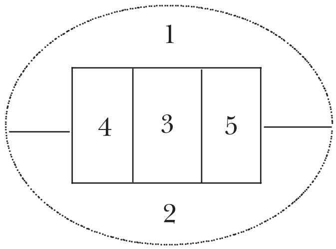


Figure 1.2. Attempting to satisfy the king's wishes.

If the kingdom can be divided into five regions in the manner desired by the king, then something else would have to be true. Place a point in each region and join two points by a line or curve if the regions containing these points have a common boundary. If A and B are two adjacent regions in the kingdom and C and D are two other adjacent regions, then it's always possible to connect each pair of points by a line in such a way that these two lines don't cross.

What we have just encountered is a graph (*our* type of graph) for the first time. A *graph* G is a collection of points (called *vertices*) and lines (called *edges*) where two vertices are joined by an edge if they are related in some way. In particular, the division of the kingdom into the five regions shown in Figure 1.2 gives rise to the graph G shown in Figure 1.3.

In order to have a solution to the king's wishes, the resulting graph must have five vertices, every two joined by an edge. Such a graph is called a *complete graph* of order 5 and expressed as K_5. Furthermore,

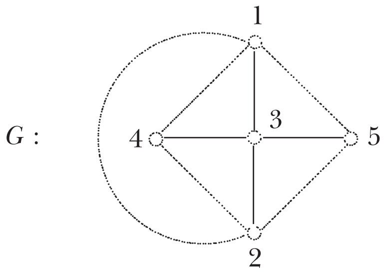


Figure 1.3. The graph representing the regions in Figure 1.2.

it must be possible to draw K_5 without any of its edges crossing. Since there is no edge joining vertices 4 and 5 in Figure 1.3, the division of the kingdom into regions shown in Figure 1.2 does not represent a solution. In Chapter 10 we will visit the Problem of the Five Princes again when we will be able to give a complete solution to this problem.

The Three Houses and Three Utilities Problem

Three houses are under construction and each house must be provided with connections to each of three utilities, namely water, electricity and natural gas. Each utility provider needs a direct line from the utility terminal to each house without passing through another provider's terminal or another house along the way. Furthermore, all three utility providers need to bury their lines at the same depth underground without any lines crossing. Can this be done?

Figure 1.4 shows a failed attempt to solve this problem, where the three houses are labeled A, B and C. Not only can this problem be looked at in terms of graphs, but in terms of graphs this problem is extremely similar to the Problem of the Five Princes. We can represent this situation by a graph with six vertices, three representing the three houses A, B and C and three representing the three utilities water (W), electricity (E) and natural gas (NG). Two vertices are joined by an edge when one vertex represents a house and the other represents a utility. This graph then has nine edges. This graph is denoted by $K_{3,3}$, indicating that there are two sets of three vertices each where each vertex in one set is joined to all vertices in the other set. To solve the Three Houses and Three Utilities

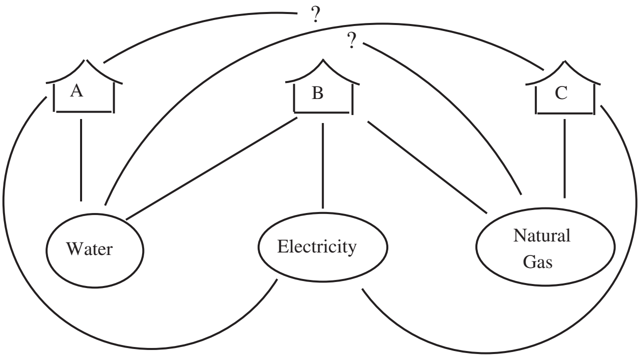


Figure 1.4. The Three Houses and Three Utilities Problem.

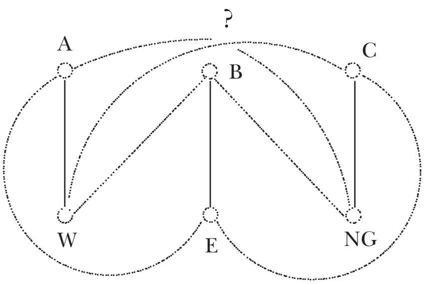


Figure 1.5. The graph representing the situation in Figure 1.4.

Problem, we need to know whether $K_{3,3}$ can be drawn without any edges crossing. The attempted solution of the Three Houses and Three Utilities Problem in Figure 1.4 gives rise to the graph shown in Figure 1.5.

We will visit the Three Houses and Three Utilities Problem as well in Chapter 10 and explain how to solve the problem.

In our next problem a graph will be introduced whose vertices represent people. Here we assume every two people are friends or strangers.

The Three Friends or Three Strangers Problem

What is the smallest number of people that must be present at a gathering to be certain that among them three are mutual friends or three are mutual strangers?

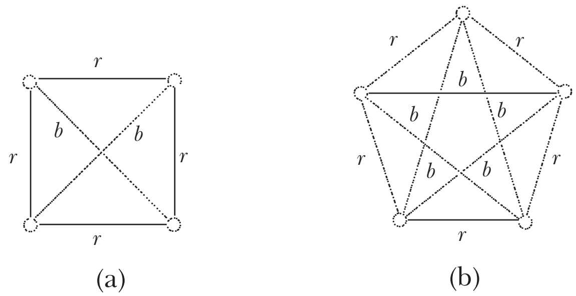


Figure 1.6. The answer to the Three Friends or Three Strangers Problem is neither four nor five.

Here too the situation can be represented by a graph, in fact by a complete graph. Suppose that four people are present at a gathering. Then we have a graph with four vertices, corresponding to the four people. We join every two vertices by an edge to indicate that these two people are friends or are strangers, resulting in the complete graph K_4 with four vertices and six edges. To indicate whether two people are friends or are strangers, we color the edge red (r) if the two people are friends and color the edge blue (b) if the two people are strangers. Thus three mutual friends would be represented by a red triangle in our graph and three mutual strangers would be represented by a blue triangle. The situation shown in Figure 1.6a shows that with four people it is possible to avoid having three mutual friends or three mutual strangers. Likewise, when we color the complete graph K_5 as in Figure 1.6b, we see that this situation can even be avoided with five people.

It turns out that the answer to the Three Friends or Three Strangers Problem is six, however. In fact, we believe that we can convince you of this, even so early in our discussion. We state this as a theorem.

Theorem 1.1: *The answer to the Three Friends or Three Strangers Problem is six. That is, among any six people, there must be three mutual friends or three mutual strangers.*

Proof: We've already seen that the answer is not five. So what we must do is consider the complete graph K_6 with six vertices where each edge is colored red or blue and show that there are three vertices where all three edges joining them have the same color.

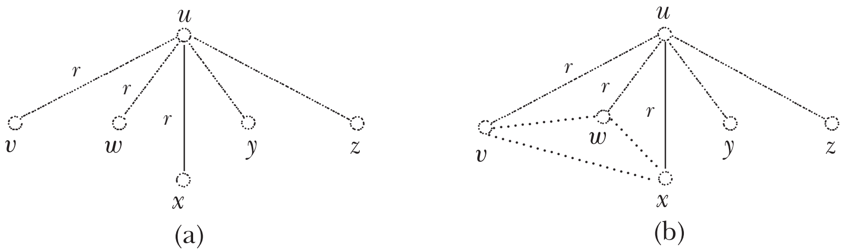


Figure 1.7. Proving Theorem 1.1.

Let's denote the vertices of K_6 by u, v, w, x, y, z and look at u, say. Then there are five edges leading from u to the other five vertices. At least three of these five edges must be colored the same, say red. Suppose that three red edges lead to v, w and x as shown in Figure 1.7a. It's not important what the colors are of the edges leading u to y and z.

There are three edges joining the pairs of vertices among v, w and x. If even one of these edges is red—say the edge between v and w is red—then u, v and w represent three friends at the gathering, represented by the red triangle uvw. On the other hand, if no edge joining any two of the vertices v, w and x is red, then all three of these edges are blue, implying that v, w and x are mutual strangers at the gathering, represented by the blue triangle vwx, which is shown in Figure 1.7b where the edges of the blue triangle vwx are drawn with dashed lines. ■

Although the next problem is not well known historically, it is a practical problem and shows how graphs can be used to analyze a problem that we all might encounter.

A Job-Hunters Problem

A counselor in a high school has contacted a number of business executives she knows for the purpose of finding summer jobs for six hard-working students: Harry, Jack, Ken, Linda, Maureen, Nancy. She found six companies, each of which is willing to offer a summer position to a qualified student who is interested in the business. The six business areas are architecture, banking, construction, design, electronics, financial. The six students apply for these positions as follows:

Harry:	architecture, banking, construction;
Jack:	design, electronics, financial;
Ken:	architecture, banking, construction, design;
Linda:	architecture, banking, construction;
Maureen:	design, electronics, financial;
Nancy:	architecture, banking, construction.

(a) How can this situation be represented by a graph?

(b) Can each student obtain a job for which he or she has applied?

SOLUTION:

(a) We construct a graph G with 12 vertices, 6 of which represent the 6 students, which we denote by H, J, K, L, M, N (the first letters of their first names), and the other 6 vertices represent the 6 positions a, b, c, d, e, f, representing architecture, banking, construction, design, electronics, financial. An edge joins two vertices if one vertex represents a business and the other represents a student who applied for a position in that business area. (See Figure 1.8.)

(b) Yes. The edges Ha, Je, Kd, Lb, Mf, Nc within the graph G show that this is possible. (See Figure 1.9.) In this situation, Ken will have a summer job in the area of design. If this business decides that they would rather hire someone other than Ken, will all six students still be able to have a summer job for which they applied? ♦

We'll see more about these kinds of "matching" problems in Chapter 7.

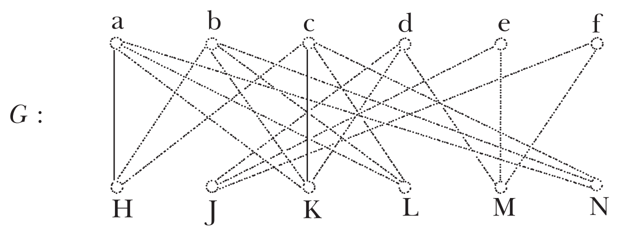


Figure 1.8. Modeling job applications by means of a graph.

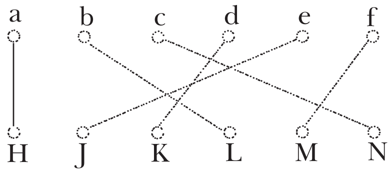


Figure 1.9. Illustrating the job situation.

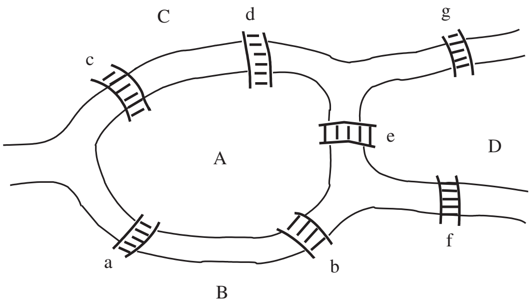


Figure 1.10. A famous problem concerning Königsberg and its seven bridges.

NEXT, . . . FOUR FAMOUS PROBLEMS

We now look at four problems that are not only important in the history of graph theory (which we will describe later in the book) but which led to new areas within graph theory.

In 1736 the city of Königsberg was located in Prussia (in Europe). The River Pregel flowed through the city dividing it into four land areas. Seven bridges crossed the river at various locations. Figure 1.10 shows a map of Königsberg where the four land regions are A, B, C, D and the bridges are a, b, . . . , g.

The Königsberg Bridge Problem

Is it possible to walk about Königsberg crossing each of its seven bridges exactly once?

Königsberg and this problem can be represented by a graph G—well, not exactly a graph. There are four vertices in G, one for each land region and two vertices are joined by a number of edges equal to the number of bridges joining these two land regions. What we get here is called a *multigraph* because more than one edge can join the same

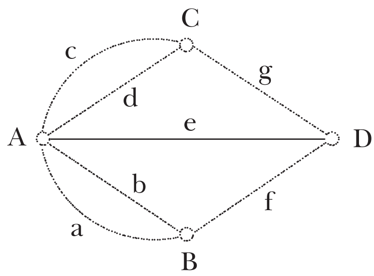


Figure 1.11. The multigraph representing the Königsberg Bridge Problem.

pair of vertices. This multigraph G is shown in Figure 1.11. In terms of this multigraph, solving the Königsberg Bridge Problem is the same as determining whether it is possible to walk about G and use each edge exactly once. Actually, there are two problems here, depending on whether we are asking whether there is a walk in Königsberg that ends where it began or whether there is a walk that ends in a land region different from the one where it began. A solution to both problems will be provided in Chapter 5.

In 1852 it was observed that in a map of England, the counties could be colored with four colors in such a way that every two counties sharing a common boundary are colored differently. This led to a much more general problem.

The Four Color Problem

In a map consisting of regions, can the regions be colored with four or fewer colors in such a way that every two regions sharing a common boundary are colored differently?

The map in Figure 1.12 is divided into 10 regions. These regions are colored with four colors, where the colors are 1, 2, 3, 4. It turns out that the regions of this map cannot be colored with three colors so that every two regions sharing a common boundary are colored differently, however.

This example, and the Four Color Problem in general, can be looked at in terms of graphs. A point is placed in each region and, like the Problem of the Five Princes, two points are joined by a line if the regions have a common boundary. Every graph constructed in this way can be drawn without any edges crossing. Instead of coloring regions, we can color the vertices of the resulting graph so that every two vertices joined by an

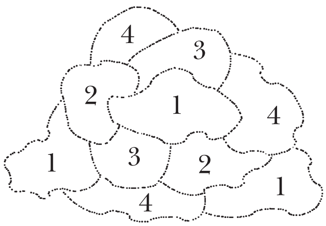


Figure 1.12. A map whose regions can be colored with four colors.

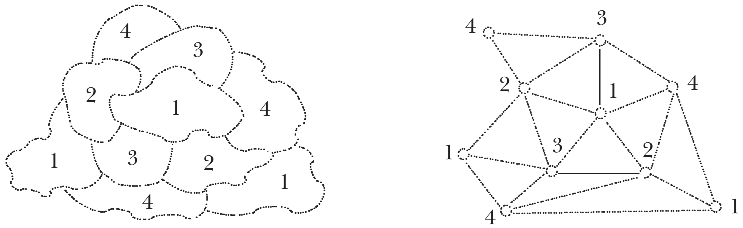


Figure 1.13. A coloring of the vertices of the graph representing the map in Figure 1.12.

edge are colored differently. This is illustrated in Figure 1.13 for the map in Figure 1.12. The Four Color Problem will be discussed in more detail in Chapter 11.

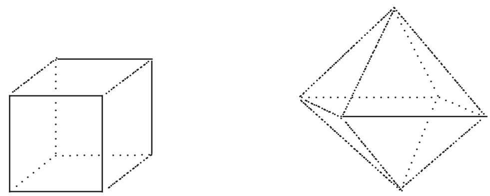
In geometry, a *polyhedron* (the plural is *polyhedra*) is a three-dimensional solid where the boundary of each face is a polygon. Figure 1.14 shows two polyhedra: the cube and the octahedron. It is common to represent the number of vertices of a polyhedron by V, the number of edges by E and the number of faces by F. These numbers for the cube and the octahedron are also given in Figure 1.14. In both cases, $V - E + F = 2$.

In 1750 the problem occurred as to whether $V - E + F = 2$ was a formula for every polyhedron.

The Polyhedron Problem

For a polyhedron with V vertices, E edges and F faces, is $V - E + F = 2$?

Every polyhedron can be represented by a graph whose edges do not cross. The graphs corresponding to the cube and the octahedron are shown in Figure 1.15. Here the number n of vertices of the graph is the number V of vertices of the polyhedron, the number m of edges of the



cube

$V = 8, E = 12, F = 6$

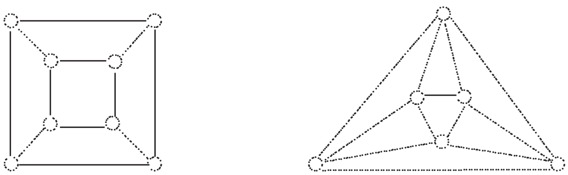
$V - E + F = 8 - 12 + 6 = 2$

octahedron

$V = 6, E = 12, F = 8$

$V - E + F = 6 - 12 + 8 = 2$

Figure 1.14. The cube and octahedron.



graph of the cube

$n = 8, m = 12, r = 6$

$n - m + r = 8 - 12 + 6 = 2$

graph of the octahedron

$n = 6, m = 12, r = 8$

$n - m + r = 6 - 12 + 8 = 2$

Figure 1.15. The graphs of the cube and octahedron.

graph is the number E of edges of the polyhedron and the number r of regions of the graph (including the outside region) is the number F of faces of the polyhedron. If it could be shown that $n - m + r = 2$ for all such graphs, then the Polyhedron Problem would be solved. This too will be discussed in Chapter 10.

Another polyhedron is the dodecahedron, shown in Figure 1.16. For this polyhedron, $V = 20$, $E = 30$ and $F = 12$ and, once again, $V - E + F = 2$. It was observed in 1856 that a round-trip can be made along edges of the dodecahedron passing through each vertex exactly once. Determining such a round-trip is known as an Around the World Problem. Consequently, a round-trip can be made along edges of the graph of the dodecahedron passing through each vertex exactly once. The graph corresponding to the dodecahedron is shown in Figure 1.17 where a trip around the world on this graph can be found by following

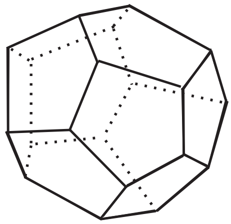


Figure 1.16. A dodecahedron.

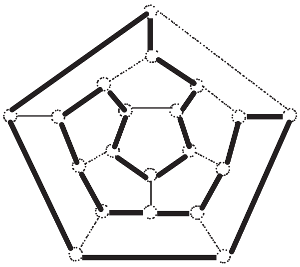


Figure 1.17. Around the world on the graph of the dodecahedron.

the edges drawn in bold. The question occurs then as to which graphs have this property.

The Around the World Problem

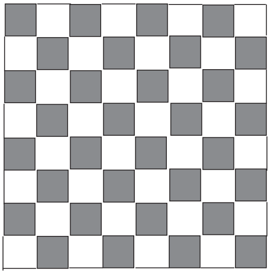
Which graphs have the property that there is a round-trip along edges of the graph that passes through each vertex of the graph exactly once?

This problem will be discussed in Chapter 6.

GRAPHS, GAMES, GALLERIES AND GRIDLOCK

The game of chess has always been considered to be a mathematical game. Perhaps then it comes as no surprise that there are puzzles and problems involving chess that have connections to graph theory. The first of these has been traced back to the year 840.

A knight is a chess piece that can move from one square to another square that is two squares forward, backward, left or right and one square



60	11	56	07	54	03	42	01
57	08	59	62	31	64	53	04
12	61	10	55	06	41	02	43
09	58	13	32	63	30	05	52
34	17	36	23	40	27	44	29
37	14	33	20	47	22	51	26
18	35	16	39	24	49	28	45
15	38	19	48	21	46	25	50

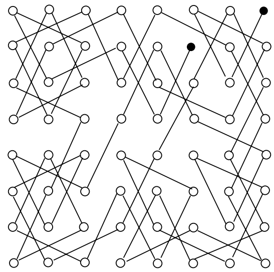


Figure 1.18. A solution of the Knight's Tour Puzzle.

perpendicular to it. A knight therefore always moves to a square whose color is different from the square where it started.

The Knight's Tour Puzzle

Following the rules of chess, is it possible for a knight to tour an 8×8 chessboard, visiting each square exactly once, and return to the starting square?

Figure 1.18 shows (1) a chessboard, (2) the solution of the Knight's Tour Puzzle given in 840, where the numbers on the squares indicate the order in which the squares are visited and (3) this solution given in terms of a graph, where each square of the chessboard is a vertex and where two vertices are joined by an edge if this indicates a move of the knight. This problem therefore has a great deal of similarity to the Around the World Problem mentioned in the preceding section.

The next chess problem concerns a different chess piece: the queen. The queen can move in any direction (horizontally, vertically or diagonally), any number of vacant squares. A queen is said to capture or attack a square (or a chess piece on the square) if the queen can reach that square through a single legal move. It is known that there is no way to place four queens on the squares of a chessboard so that every vacant square can be captured by a queen. The following problem asks whether this can be done with five.

The Five Queens Puzzle

Can five queens be placed on an 8×8 chessboard so that every vacant square can be captured by at least one of these queens?

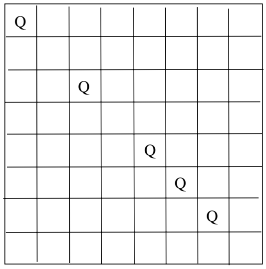


Figure 1.19. Five queens that can capture every vacant square on a chessboard.

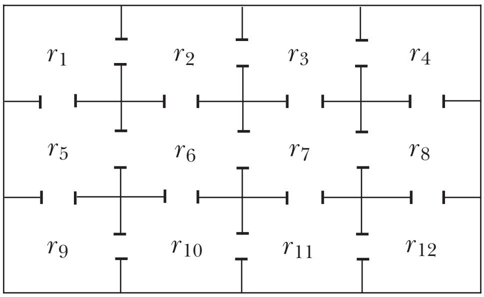


Figure 1.20. The 12 rooms in an art gallery.

The answer to this problem is yes. One possible placement of five such queens on a chessboard is shown in Figure 1.19. Once again, let G be the graph whose 64 vertices are the squares of the chessboard, where an edge joins 2 vertices if a queen can move between these two squares in a single move. The Five Queens Puzzle tells us that this graph contains 5 vertices such that each of the remaining 59 vertices is joined to at least one of these 5 vertices. We will say more about this when we discuss graph domination in Chapter 3.

Example 1.2: *A famous art gallery contains* 12 *rooms* $r_1, r_2, \ldots, r_{12}$ *(see Figure 1.20) in which expensive paintings are on display. Every room has exits leading to neighboring rooms.*

(a) Represent this situation by a graph.

(b) Do there exist four rooms where security guards may be placed so that every room either contains a guard or is a neighboring room of some room containing a guard?

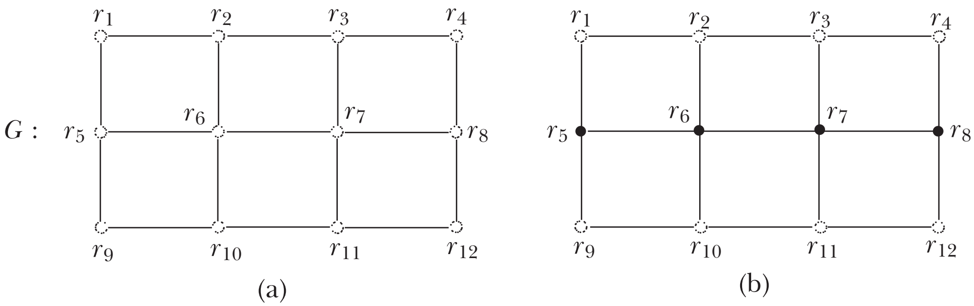


Figure 1.21. Modeling the art gallery by means of a graph.

SOLUTION:

(a) Let G be a graph of order 12 where $V = \{r_1, r_2, \ldots, r_{12}\}$ and two vertices are adjacent if they represent neighboring rooms (see Figure 1.21a).

(b) If four guards are stationed in rooms r_5, r_6, r_7, r_8, then every room either contains a guard or is a neighboring room of some room containing a guard. In the graph G, this means that every vertex is either one of the vertices r_5, r_6, r_7, r_8 or is adjacent to one of these (see Figure 1.21b). This situation may suggest two other questions:

(1) By placing the guards in rooms r_5, r_6, r_7, r_8, the eight rooms without guards are neighboring rooms of exactly one room with a guard. It would be helpful if some of these rooms were near to more than one guard. Is it possible to place four guards in rooms in such a way that the number of rooms without guards and which are neighboring rooms of exactly one room with a guard is less than eight?

(2) Is it possible to place fewer than four security guards in the rooms so that every room either contains a guard or is a neighboring room of a room containing a guard? ♦

Example 1.3: *Figure 1.22 shows an intersection of two streets where there is often heavy traffic. There are seven traffic lanes* L1, L2, ..., L7 *where vehicles can enter the intersection of these two streets. A traffic light is located*

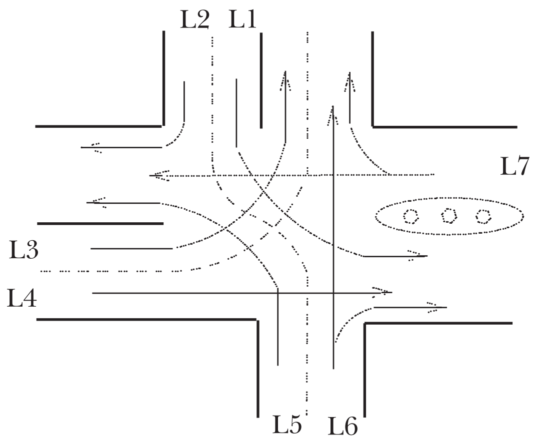


Figure 1.22. The traffic lanes at an intersection.

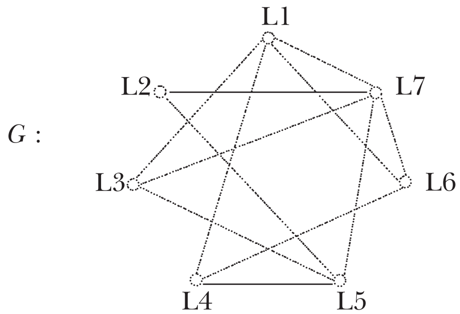


Figure 1.23. Modeling traffic lanes at an intersection by means of a graph.

at this intersection. During a certain phase of this traffic light, those cars in lanes for which the light is green may proceed safely through the intersection.

(a) Represent this situation by a graph.

(b) Determine whether it is possible, with four phases, for cars in all lanes to proceed safely through the intersection?

SOLUTION:

(a) Let G be a graph with vertex set $V = \{L1, L2, \ldots, L7\}$, where two vertices (lanes) are joined by an edge if vehicles in these two lanes cannot safely enter the intersection at the same time, as there is the possibility of an accident. (See Figure 1.23.)

(b) Since cars in lanes L1 and L2 may proceed safely through the intersection at the same time, the traffic light can be green for both lanes at the same time. The same is true for L3 and L4, for L5 and L6 and for L7. We may represent this as {L1, L2}, {L3, L4}, {L5, L6}, {L7}. This can also be accomplished as {L1, L5}, {L2, L6}, {L3}, {L4, L7}. ♦

The question asked in (b) above suggests another question. Is it possible for cars in all lanes to proceed safely through the intersection where there are fewer than four phases for the traffic light? Questions of this type will be studied and answered in Chapter 11.

There are occasions when we might choose to give the edges an "orientation" to indicate a direction or perhaps a preference relation. Orienting all the edges of the complete graph K_n results in a "tournament" with n players where the orientation of an edge indicates the winner of a match played between two players. In Chapter 9, we'll discover the following amazing result:

> For any tournament with n players, there is always a way to number the players in such a way that Player 1 beat Player 2, Player 2 beat Player 3, Player 3 beat Player 4 and so up to Player $n-1$ beat Player n.

Figure 1.24a shows the complete graph K_5 where the labels u, v, w, x, y indicate five players where every two will participate in some sports match. Figure 1.24b shows the outcome of these 10 matches. By referring to the players u, v, w, x, y as 2, 3, 5, 1, 4, respectively, we see that Player 1 beat Player 2, Player 2 beat Player 3 and so on, as shown in Figure 1.24c.

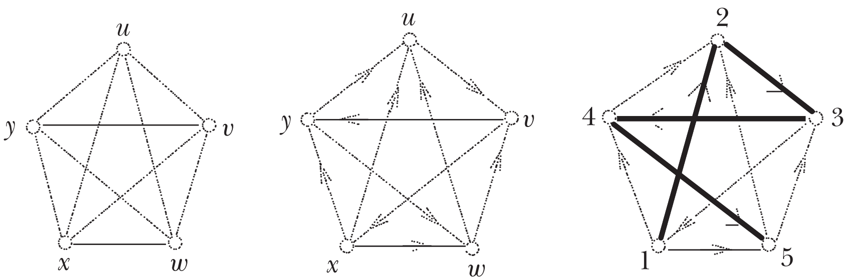


Figure 1.24. The outcome of a five-player tournament.

THE ARRIVAL OF GRAPH THEORY

While the games, puzzles, problems and results we've mentioned were not initially part of graph theory as there was not yet an area of mathematics called graph theory, all this changed in 1891 when the first purely theoretical article dealing with graphs as mathematical objects was written by the Danish mathematician Julius Petersen (1839–1910). That he called these objects by the name "graph" was perhaps the deciding factor in that name being used from that time on. While we have used this term a number of times, we have yet to give a formal definition of a graph and describe some of the basic terminology associated with graphs. It is now time to do this. Although there are some variations on how mathematicians define these terms, the definitions we are about to present are among the most common.

As we have noted and the reader will have already experienced, a graph can be represented by a diagram, like the one in Figure 1.25. Associated with a graph, there are two sets, a vertex set V and an edge set E. For instance, the graph H in Figure 1.25 has $V = \{u, v, w, x, y, z\}$ and $E = \{uv, ux, uz, vw, vx, vz, xz\}$. In every graph, these sets are finite so they cannot have an infinite number of points. Also, each element of E consists of two different elements of V, where the order does not matter. (For instance, the edge uv is the same as the edge vu.) Mathematicians refer to the elements of E as 2-element subsets of V.

The formal definition of a graph adopted by mathematicians goes as follows. A *graph* G is a finite nonempty set V of objects called *vertices* (the singular is *vertex*) together with a set E consisting of 2-element subsets of V. Each element of E is called an *edge* of G. The sets V and E are called the *vertex set* and *edge set* of G. In fact, G is often written as $G = (V, E)$. Sometimes the vertex set and edge set of a graph G are expressed as $V(G)$

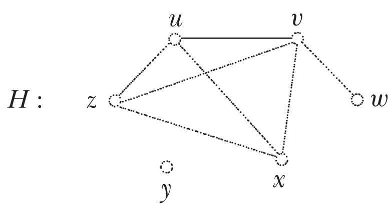


Figure 1.25. A graph.

and $E(G)$, respectively, to emphasize that the graph G is involved.

The number of vertices in a graph G is called the *order* of G and the number of edges in G is its *size*.

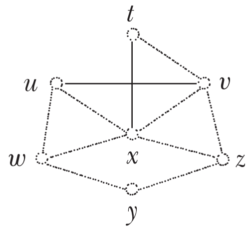
The order and size of a graph are typically denoted by n and m, respectively. The order of the graph H of Figure 1.25 is $n = 6$ and its size is $m = 7$. We often represent a graph G by means of a diagram (and refer to the diagram as the graph), where each vertex is indicated by a small circle (and called a vertex) and an edge ab is indicated by placing a straight-line segment or a curve between the vertices a and b. Although the edges ux and vz intersect in the diagram of the graph H of Figure 1.25, the point of intersection is *not* a vertex of H.

Since $e = uv$ is an edge of the graph H of Figure 1.25, the vertices u and v are said to be *adjacent* and e is said to *join* the vertices u and v. The vertices u and w are *nonadjacent vertices*. Since u and v are adjacent vertices, they are *neighbors* of each other. Because $e = uv$ is an edge of H, the vertex u and the edge e are *incident*, as are v and e. Since uv and vw are incident with the same vertex v, they are *adjacent edges*. The edges ux and vz are not adjacent.

When a graph G is given in terms of a diagram and we want to refer to the vertex set of G or discuss particular vertices in G, it is useful to assign each vertex of G a label. In this case, G is called a *labeled graph*. On the other hand, if there is no particular advantage to labeling the vertices of G, then G is an *unlabeled graph*. The graph F of Figure 1.26 is labeled, while the graph J of Figure 1.26 is unlabeled.

A graph with exactly one vertex is a *trivial graph*. Thus a *nontrivial graph* has order at least 2. A graph with no edges is an *empty graph*. A *nonempty graph* therefore has at least one edge.

F :



J :

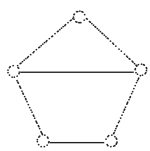


Figure 1.26. A labeled graph and an unlabeled graph.

THE FIRST THEOREM OF GRAPH THEORY

The number of edges incident with a vertex v in a graph G is called the *degree* of v and denoted by $\deg_G v$. When the graph G under consideration is understood, we write the degree of v more simply as $\deg v$. If v is a vertex in a graph G with n vertices, then v can have at most $n-1$ neighbors; that is, if v is a vertex in a graph G of order n, then $0 \leq \deg v \leq n-1$. In the graph H of Figure 1.25,

$$\deg y = 0, \deg w = 1, \deg u = 3, \deg x = 3, \deg z = 3, \deg v = 4.$$

A vertex of degree 0 is an *isolated vertex*, while a vertex of degree 1 is called an *end-vertex*. The vertex y is therefore an isolated vertex and w is an end-vertex. The *minimum degree* of a vertex of G is denoted by $\delta(G)$ or simply δ while the *maximum degree* is denoted by $\Delta(G)$ or Δ, where δ and Δ are the lowercase and uppercase Greek letter delta, respectively. For the graph H of Figure 1.25, $\delta = 0$ and $\Delta = 4$. Notice that when the degrees of the vertices of any graph G are added, each edge of G is counted twice. This gives us the following result, namely a theorem that is sometimes called the *First Theorem of Graph Theory* since it is felt that if anyone were to study graph theory on his or her own, this would likely be the first result he or she would discover.

> In any graph, the sum of the degrees of the vertices is twice of the number of edges.

More formally, and reinforcing your vocabulary, we state this fact as follows.

> Theorem 1.4: *Let G be a graph of order n and size m with vertices $v_1, v_2, \ldots, v_n$. Then*
>
> $$\deg v_1 + \deg v_2 + \cdots + \deg v_n = 2m.$$

So how many edges can a graph of order n have? For a graph to have maximum size, it would have to be complete, where every vertex is adjacent to all other vertices and so every vertex would have degree $n-1$. Applying Theorem 1.4 then tells us that

$$(n-1) + (n-1) + \cdots + (n-1) = 2m$$

and therefore $m = n(n-1)/2$. Another way that we may answer this question is to observe that for an edge e, there are n choices for one vertex in e and $n-1$ choices for the other vertex, or $n(n-1)$ choices in all. But since vu is the same as uv for every edge uv, the number $n(n-1)$ counts each edge twice and so $m = n(n-1)/2$.

Some people have referred to Theorem 1.4 as the *Handshaking Lemma*. (A lemma is a mathematical result, usually not of primary interest but useful in proving some theorem of greater interest.) Suppose that there was a gathering of people, some pairs of whom shook hands with each other, followed by an inquiry as to how many hands each person shook. If all of these numbers were added, we would arrive at an even number, namely twice the total number of handshakes that took place.

A vertex is called *even* or *odd* according to whether its degree is even or odd. The graph H in Figure 1.25 then has two even vertices and four odd vertices. That the graph H has an even number of odd vertices is a consequence of Theorem 1.4.

Corollary 1.5: *Every graph has an even number of odd vertices.*

Proof: According to Theorem 1.4, when we add the degrees of the even vertices and the degrees of the odd vertices of any graph, the result is always an even number. Thus the sum of the degrees of all odd vertices is even, implying that the graph must have an even number of odd vertices. ■

Degrees of vertices will be discussed in considerable detail in Chapter 2.

2

Classifying Graphs

Many of the topics and problems that we will encounter and that can be represented by graphs deal with the degrees of the vertices of these graphs. Some of these occur in unexpected ways. To see an example of this, we look at a curious and amusing mathematics article that was published more than a quarter of a century ago.

David Wells, a British mathematician who is the author of many books on mathematics, puzzles, games and mathematics education, has displayed concern over the way mathematics is often taught to high-school students as well as to beginning college students. What Wells saw was the apparent goal of packing as much material as possible into each course while possibly neglecting aspects of mathematics that he (and many others) felt were important. At a number of universities, advanced mathematics students are able to participate in seminars, where they can see what the research faculty is studying. At these seminars, students are exposed to new mathematical ideas that can lead to interesting discussions. Wells believed that there are certain things that even beginning university students should have the opportunity to wonder about. Why is the topic being discussed important? Where did the idea for this problem come from? What is the background of this subject? In beginning university mathematics courses, questions such as these were rarely answered—or even asked. In other words, there should be more emphasis on thinking and asking questions. Of course, this could and should be applied to all subjects (not just mathematics), including, perhaps, one's job.

Wells felt that students were missing something important. They were missing out on the beauty of mathematics. Surely, you've heard the statement "Beauty is in the eye of the beholder". This means that what is beautiful is subjective. It depends more on the viewer than on what is being viewed. It's a matter of taste or opinion. And whether something is more beautiful than something else is, well, a matter of degree.

While the statement above is perhaps most commonly applied to art and music, it can be applied to anything—including mathematics. Mathematicians think that concepts, proofs and theorems can be beautiful. Observing something in mathematics that is unexpected, makes use of something unanticipated or has a surprising application is part of what makes mathematics beautiful. Wells asked,

> In mathematics is the process (of discovery and proof) more important than the outcome?

Such a question has no (simple) answer and so we are left—happily—with the endless richness of mathematics and mathematicians' responses to it.

BEAUTY IN MATHEMATICS

The Fall 1988 issue of the mathematics journal the *Mathematical Intelligencer* contained an article written by Wells and titled "Which Is the Most Beautiful?" in which Wells asked its readers to evaluate 24 theorems he listed, on a scale of 0 to 10, to determine just how beautiful these theorems are. Two years later he reported his findings in another article (titled "Are These the Most Beautiful?"). Let's look at the 12 theorems that the readers voted as the top half:

(1) $e^{\pi i} = -1$.

(2) The Euler Polyhedron Formula: $V - E + F = 2$.

(3) The number of primes is infinite.

(4) There are five regular polyhedra.

(5) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$.

(6) A continuous mapping of a closed unit disk into itself has a fixed point.

(7) There is no rational number whose square is 2. (The number $\sqrt{2}$ is irrational.)

(8) The number π is transcendental.

(9) Every plane map can be colored with four colors (the Four Color Theorem).

(10) Every prime number of the form $4n+1$ is the sum of the squares of two integers in exactly one way.

(11) The order of a subgroup of a finite group divides the order of the group.

(12) Any square matrix satisfies its characteristic equation.

You may not understand what all these theorems say and if you don't, this is neither surprising nor important. But you probably recognize some of these. In fact, theorems 2 and 9 were discussed in Chapter 1.

It is noteworthy that both the top two theorems were due to the same mathematician: Leonhard Euler (1707–1783). We'll be seeing a lot of Euler in this book, but for now let's just say that if a list were created of the most important mathematicians, he would be on everyone's top 10 list (and probably most would put him in the top 5, among such luminaries as Euclid, Archimedes, Newton and Gauss). The theorem at the top of the list can also be written as

$$e^{\pi i}+1=0,$$

which shows that there is a simple relationship among five of the most famous numbers in mathematics:

0: the additive identity ($0+r=r$ for every real number r);

1: the multiplicative identity ($1\cdot r=r$ for every real number r);

π: the constant occurring with circles;

e: the base of natural logarithms;

i: the basic imaginary number.

Indeed, it was Euler who introduced the base of natural logarithms and denoted it by e.

Euler was also responsible for theorem 5 on the list. Euler verified this surprising result in 1735, the year before the appearance of his paper that marked the beginning of graph theory: his solution of the Königsberg Bridge Problem. The series $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$ had come up during the seventeenth century. While mathematicians knew at that time its value was approximately 8/5, it was Euler who determined the exact value. This was totally unexpected. In fact, the American mathematical historian William Dunham wrote,

> When it happened in 1735, the splash was Euler's. The answer was not only a mathematical tour de force but a genuine surprise.... This highly non-intuitive result made the solution all the more spectacular and its solver all the more famous.

Irregular Graphs

Of the 12 theorems that made up the first half of Wells's list of beautiful theorems, three have close connections with graph theory (theorems 2, 4, 9). Of these 24 theorems, one other can be interpreted as a theorem in graph theory.

(20) At any party, there is a pair of people who have the same number of friends present.

While this theorem doesn't rank very high among the 24, it did make the list. Even though this theorem may not sound like a theorem in graph theory or even in mathematics, it is closely related to our earlier discussion of the degrees of the vertices of a graph. Let us assume that at a party, every two people are either friends or they're not. We construct a graph G whose vertices are the people present at the party and where an edge joins two vertices if these two people are friends. Then the degree of a vertex in G is the number of friends this person has at the party.

What theorem 20 says in terms of graph theory is that every graph with at least two vertices has at least two vertices of the same degree. This is directly related to a rather curious concept in graph theory. A graph G of order 2 or more is *irregular* if every two vertices of G have different degrees. What makes these graphs so unusual is that there aren't any!

Theorem 2.1: *No graph is irregular.*

Proof: Suppose, to the contrary, that there is some irregular graph G of order $n \geq 2$. Since G is irregular, the degrees of its vertices are all different. Thus G must have a vertex of every degree from 0 to $n-1$. But this is impossible since a graph with more than one vertex can't have both a vertex of degree $n-1$ (adjacent to all other vertices) and a vertex of degree 0 (adjacent to no other vertices). ■

Often in mathematics when one question is answered, another question is suggested. For each integer $n \geq 2$, does there exist an almost irregular graph of order n? A graph G is *almost irregular* if it has exactly one pair of vertices of the same degree. The answer to this question turns out to be yes, and for small values of n, it is relatively easy to give examples of such graphs. Figure 2.1 shows the six smallest examples.

By the *complement* $\overline{G}$ of a graph G, we mean the graph having the same vertex set as G, that is, $V(\overline{G}) = V(G)$, and where two vertices u and v of $\overline{G}$ are adjacent if and only if u and v are *not* adjacent in G. A graph G of order 6 and its complement $\overline{G}$ are shown in Figure 2.2.

Notice that if v is a vertex in a graph G of order n, then

$$\deg_{\overline{G}} v = n - 1 - \deg_G v.$$

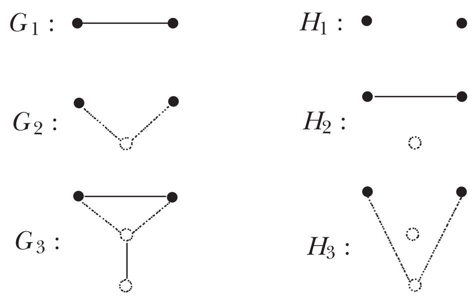


Figure 2.1. Six graphs having exactly one pair of vertices of the same degree.

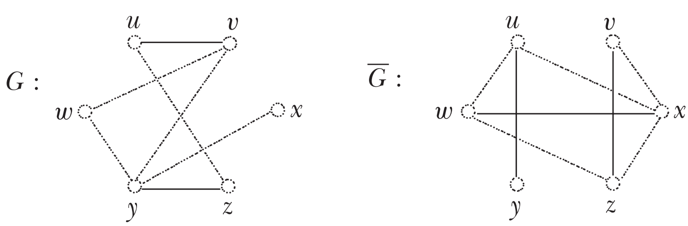


Figure 2.2. A graph and its complement.

This says that two vertices u and v have the same degree in a graph $\overline{G}$ if and only if u and v have the same degree in G. Notice that the graphs in Figure 2.1 come in complementary pairs: $H_1 = \overline{G}_1$, $H_2 = \overline{G}_2$ and $H_3 = \overline{G}_3$. This illustrates the next theorem.

Theorem 2.2: *For each integer $n \geq 2$, there are exactly two almost irregular graphs of order n and they are complements of each other.*

Although this theorem is more difficult to verify than Theorem 2.1, let's look at how Theorem 2.2 might be proved.

Idea of Proof of Theorem 2.2: Figure 2.1 shows all almost irregular graphs with fewer than five vertices. How do we create an almost irregular graph with five vertices? Easy. Look at the graph G_3 in Figure 2.1. This graph has four vertices whose degrees are 1, 2, 2, 3. Add an isolated vertex to G_3. This gives us an almost irregular graph H_4 whose five vertices have degrees 0, 1, 2, 2, 3. Thus the complement of H_4 (call this graph G_4) will have five vertices with degrees 4, 3, 2, 2, 1 and therefore is also almost irregular (see Figure 2.3).

Continuing in this manner, we can create other pairs of almost irregular graphs of the same order, namely G_5 and H_5, G_6 and H_6 and so on. (See Figure 2.4.)

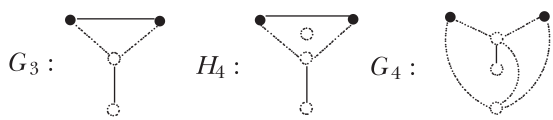


Figure 2.3. Creating almost irregular graphs of order 5.

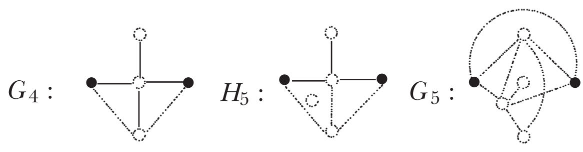


Figure 2.4. Creating almost irregular graphs of order 6.

How do we know that G_4 and H_4 are the *only* almost irregular graphs of order 5? Suppose that there was another almost irregular graph G having five vertices. Since G can't contain both a vertex of degree 0 and a vertex of degree 4, it must have either

(a) vertices of degree $0, 1, 2, 3$ (with one of these degrees repeated) or

(b) vertices of degree $1, 2, 3, 4$ (with one of these degrees repeated).

Suppose that G is of type (a). Then it contains only one isolated vertex. (It can't contain two isolated vertices since if one were removed, then an irregular graph of order 4 would result, which is impossible.)

Removing the single isolated vertex from G results in a graph with degrees 1, 2, 3 (with one of these degrees repeated). But we know that G_3 is the only graph with this property and so G must be H_4. Similarly, if G is of type (b), then its complement $\overline{G}$ has an isolated vertex and so the previous argument shows that $\overline{G}$ is H_4 and therefore G is $\overline{H}_4$, which is G_4. Continuing this argument shows that G_5 and H_5 are the only almost irregular graphs of order 6, G_6 and H_6 are the only almost irregular graphs of order 7 and so on. ■

We saw in Theorem 2.2 that for each integer $n \geq 2$, there is exactly one almost irregular graph of order $n+1$ whose vertices have the degrees $1, 2, \ldots, n$ (where two of the vertices have the same degree). So, for example, for $n = 4$, there is a graph of order 5 containing a vertex of each of the degrees $1, 2, 3, 4$. The one graph with this property is shown in Figure 2.5.

Actually, as it turns out, there is a graph of order 5 whose vertices have as their degrees any prescribed positive integers with largest integer 4. (See Figure 2.6.)

In fact, the following is true.

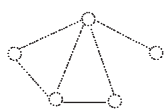


Figure 2.5. The almost irregular graph whose vertices have degrees $1, 2, 3, 4$.

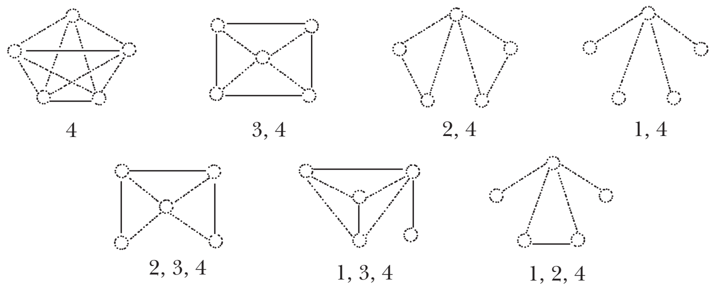


Figure 2.6. Graphs of order 5 with positive degrees having maximum degree 4.

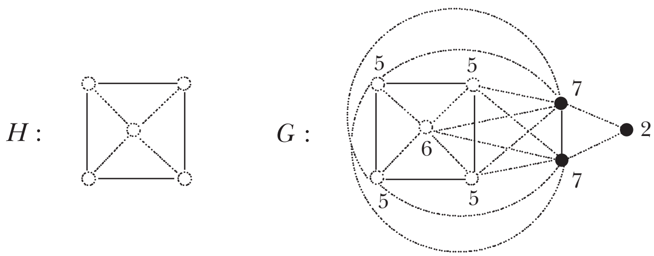


Figure 2.7. Constructing a graph of order 8 whose vertices have degrees 2, 5, 6, 7.

Theorem 2.3: *For every given set of positive integers whose largest integer is n, there is a graph of order $n+1$, the degrees of whose vertices are precisely these integers.*

Idea of Proof: We have seen that this is true for $n=4$. It is even easier to show that this is true for $n=3$ and not all that difficult to show that this is true for $n=5$ and $n=6$. Let's see how we can use this information to show that if we have any set of positive integers the largest of which is 7, then there is a graph G of order 8 whose vertices have exactly these degrees. For example, consider a, b, c, d where $(a, b, c, d) = (2, 5, 6, 7)$. We know that there is a graph H of order $(c-a)+1=5$ whose vertices have degrees $b-a=5-2=3$ and $c-a=6-2=4$. To this graph H, we add $d-c=7-6=1$ isolated vertices (isolated vertex in this case). We then join the vertices of the graph K_a (that is, K_2 here) to H and this isolated vertex. The resulting graph G has order 8 and its degrees are 2, 5, 6, 7. (See Figure 2.7.) ■

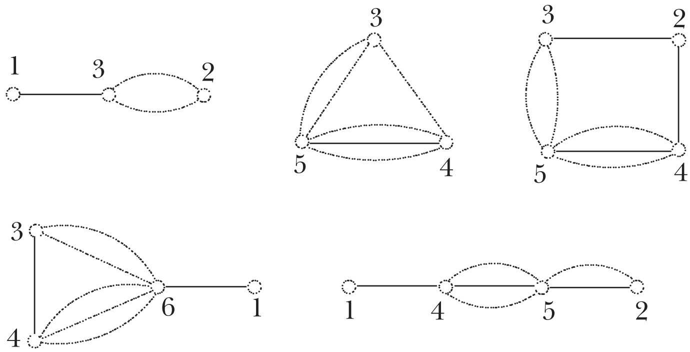


Figure 2.8. Irregular multigraphs.

IRREGULAR MULTIGRAPHS AND WEIGHTED GRAPHS

While there are no irregular graphs of order $n \geq 2$, it turns out that there do exist irregular multigraphs of order n for every $n \geq 3$. We define a multigraph to be *irregular* if its vertices have distinct degrees (where the *degree* of a vertex in a multigraph is the number of edges incident with the vertex). For example, all of the multigraphs in Figure 2.8 are irregular. Each vertex is labeled with its degree.

Multigraphs with many parallel edges joining the same pair of vertices can be difficult to draw. These structures can be looked at in a different way, however. We can replace all those parallel edges joining the same pair u, v of vertices by the single edge $e = uv$ and assign to e the number of edges joining u and v in the multigraph. This number is often called the *weight* of e and produces what is called a *weighted graph*. The weighted graphs obtained from the multigraphs in Figure 2.8 are shown in Figure 2.9. The *degree* of a vertex v in a weighted graph is then the sum of the weights of the edges incident with v. A weighted graph G is *irregular* if its vertices have distinct degrees.

If, for a given multigraph H, we replace all those parallel edges joining the same pair of vertices by a single edge, then the graph G obtained in this way is referred to as the *underlying graph* of H. While, according to Theorem 2.1, no graph of order 2 or more is irregular, it is possible to begin with *nearly* any graph G of order 3 or more and by appropriately assigning weights to the edges of G, we can convert G into an irregular weighted graph. We say "nearly" every graph since a graph G with two isolated vertices must have two vertices of degree 0 regardless of how the

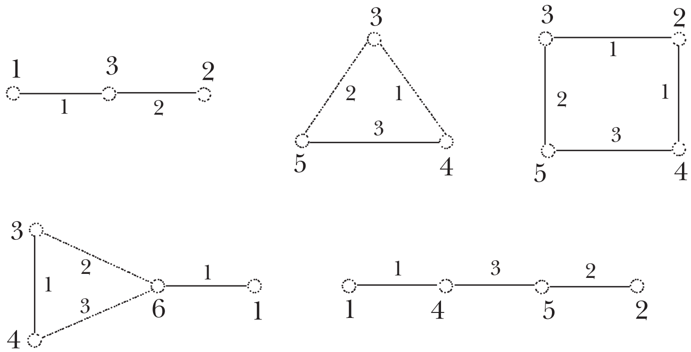


Figure 2.9. Irregular weighted graphs.

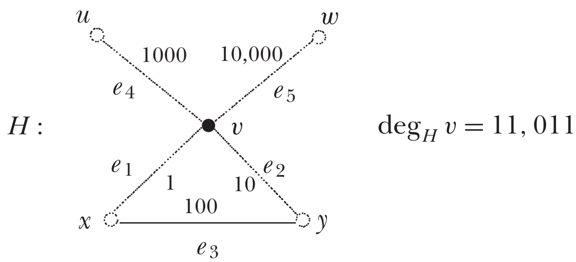


Figure 2.10. An irregular weighted graph.

edges of G are assigned weights and if G contains two adjacent vertices u and v of degree 1, then the degrees of u and v will be the weight of uv for any weight assigned to the edge.

Theorem 2.4: *For every graph G of order 3 or more having at most one isolated vertex and not containing two adjacent end-vertices, there is an irregular weighted graph H whose underlying graph is G.*

Proof Strategy: If G has m edges, assign them the weights $1, 10, 100, 1000, \ldots, 10^{m-1}$. For example, if v is incident with the edges e_1, e_2, e_4 and e_5, as in the graph H of Figure 2.10, then $\deg_H v = 10^0 + 10^1 + 10^3 + 10^4 = 11,011$. Since every two vertices of H have different degrees, H is irregular. ■

Assigning different powers of 10 to the edges of G in order to produce an irregular weighted graph H gives rather large degrees to the vertices of H. This makes one wonder if it is possible to assign smaller weights to the edges of G and still produce an irregular weighted graph.

A proof of Theorem 2.4 can be simplified somewhat by assigning different powers of 2 to the edges of G, that is, assign the smaller weights $1, 2, 4, 8, \ldots, 2^{m-1}$ to the edges of G rather than $1, 10, 100, 1000, \ldots, 10^{m-1}$. In the graph H of Figure 2.10, in which v is incident with the edges e_1, e_2, e_4 and e_5, the degree of v in H would then be $2^0 + 2^1 + 2^3 + 2^4 = 27$, a much smaller number. Of course, this immediately suggests the problem of attempting to minimize the largest weight that needs to be assigned to any edge of G to produce an irregular weighted graph (or irregular multigraph). For a graph G, the smallest positive integer k such that the weights can be chosen from the set $\{1, 2, \ldots, k\}$ to produce an irregular weighted graph is called the *irregularity strength* of G and is denoted by $s(G)$.

While it is not difficult to give examples of graphs whose irregularity strength is large, there is a related problem having a decidedly different outcome. Suppose, instead of demanding that all vertices have different degrees, we require, by assigning weights to the edges of a graph, only that every two adjacent vertices in the resulting weighted graph have distinct degrees. Here too no graph being considered can contain two adjacent end-vertices. One would expect that the largest weight we would need to assign any edge of a graph G would be less than $s(G)$. Indeed, for the graph G_1 in Figure 2.11, every edge can be assigned the weight 1 and for the graph G_2 each edge can be assigned the weight 1 or 2. While it is impossible to assign each edge of the graph G_3 either the weight 1 or 2 to produce a graph in which every two adjacent vertices have distinct degrees, this can be accomplished with the weights 1, 2, 3. These observations serve to illustrate a problem having a catchy name.

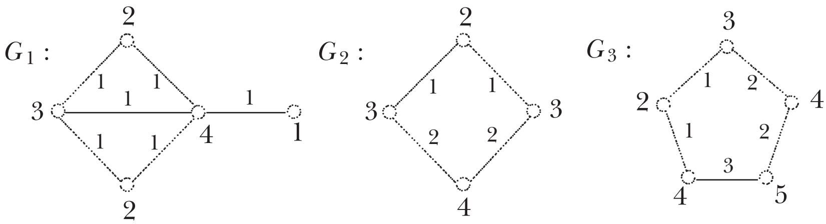


Figure 2.11. Weighted graphs in which every two adjacent vertices have different degrees.

The 1-2-3 Problem

For every graph G not containing adjacent end-vertices, can the weights 1, 2, 3 be assigned to the edges of G in such a manner that every two adjacent vertices of the resulting weighted graph have different degrees?

While no one has ever found such a graph for which the answer to this question is no, Louigi Addario-Berry, Robert Aldred, Ketan Dalal and Bruce Reed have shown that the answer is always yes if we can use the weights 1, 2, 3, 4.

REGULAR GRAPHS

While there is no graph of order $n \geq 2$ all of whose vertices have different degrees, there are many graphs whose vertices have the same degree. Furthermore, many of these graphs are among the most often encountered graphs in graph theory. A graph G is called *regular* if every vertex of G has the same degree. If this degree is r, then G is r*-regular*.

There are two regular graphs of order 2 and these are complements of each other. One is 0-regular and the other is 1-regular. There are two regular graphs of order 3 and these too are complements of each other. One of these is 0-regular and the other is 2-regular. There is no 1-regular graph of order 3 as no graph contains an odd number of odd vertices (see Corollary 1.5). There is a 0-regular, 1-regular, 2-regular and 3-regular graph of order 4. All of these are shown in Figure 2.12. This figure also shows a 0-regular, 2-regular and 4-regular graph of order 5.

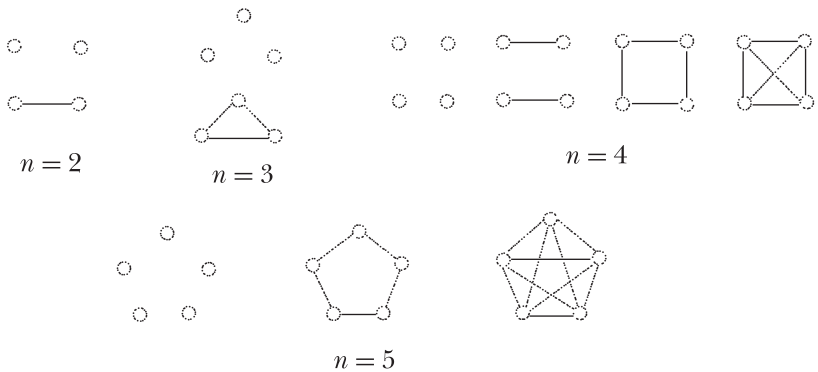
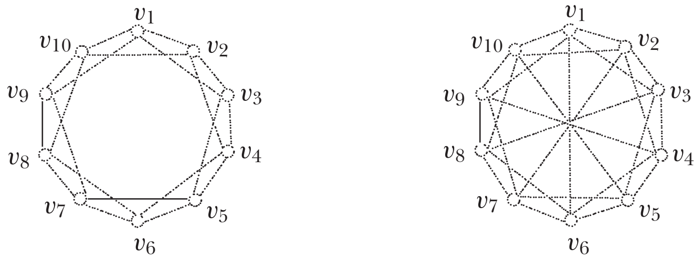


Figure 2.12. Small regular graphs.



(a) (b)

Figure 2.13. Regular graphs of order 10.

Again by Corollary 1.5, there is neither a 1-regular nor a 3-regular graph of order 5.

If G is an r-regular graph of order n, then $0 \le r \le n - 1$. Since no graph can contain an odd number of odd vertices, no r-regular graph of order n can exist if r and n are both odd. With the exception of this restriction, it is always possible to construct an r-regular graph of order n.

Theorem 2.5: *For every two integers r and n, not both odd, with $0 \le r \le n - 1$, there exists an r-regular graph of order n.*

Proof: Begin by placing the n vertices $v_1, v_2, \ldots, v_n$ evenly spaced around a circle. We consider two cases, according to whether r is even or odd.

Case 1. *The integer r is even, say $r = 2k$.* Then for each vertex v_i, draw an edge to the r nearest vertices on the circle (k forward and k backward). The resulting graph is an r-regular graph of order n. (When $r = 4$ and $n = 10$, see Figure 2.13a.)

Case 2. *The integer r is odd, say $r = 2k + 1$.* Therefore, n must be even. We proceed as in Case 1 but also join every two diametrically opposite vertices on the circle, that is, join v_i and $v_{i+(n/2)}$. This also produces an r-regular graph of order n. (When $r = 5$ and $n = 10$, see Figure 2.13b.) ■

The study of regular graphs goes back to 1891. In fact, these graphs constituted the main area of study in the first theoretical article ever written on graph theory. This article was written by the Danish mathematician

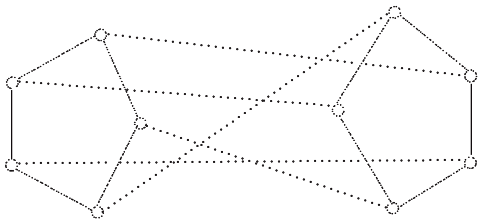


Figure 2.14. Petersen's drawing of the Petersen graph.

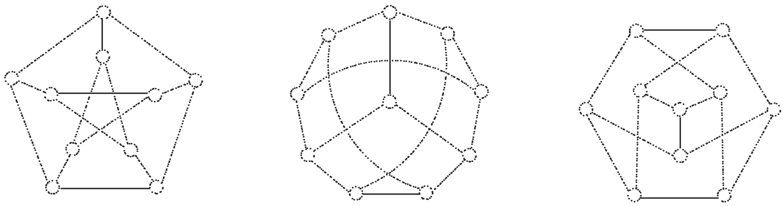


Figure 2.15. Three drawings of the Petersen graph.

Julius Petersen. There is a graph encountered by Petersen during his research that has become famous and is named for him. The *Petersen graph* is a 3-regular graph of order 10. While Petersen drew this graph as shown in Figure 2.14, it is normally drawn in one of the three ways shown in Figure 2.15, especially the way it is drawn in the leftmost figure.

CLASSES OF REGULAR GRAPHS

While the Petersen graph is a regular graph that is one of the most famous graphs encountered in graph theory, there are many well-known classes of regular graphs.

We saw in Chapter 1 that if every two vertices of a graph of order n are adjacent, then the graph is called *complete* and is denoted by K_n. The complete graph K_n is $(n-1)$-regular and its size is then $n(n-1)/2$. The complement $\overline{K}_n$ of K_n is therefore the 0-regular empty graph of order n. The complete graphs K_n, $1 \leq n \leq 6$ are shown in Figure 2.16.

One of the simplest classes of regular graphs is the class of cycles. For $n \geq 3$, the *cycle* C_n is the graph of order n whose vertices can be labeled $v_1, v_2, \ldots, v_n$ and whose edges are $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$ and v_1v_n. The cycle C_n is also referred to as an *n-cycle*. The cycles C_n $(3 \leq n \leq 6)$ are shown in Figure 2.17. The cycle C_3 is also called a *triangle*.

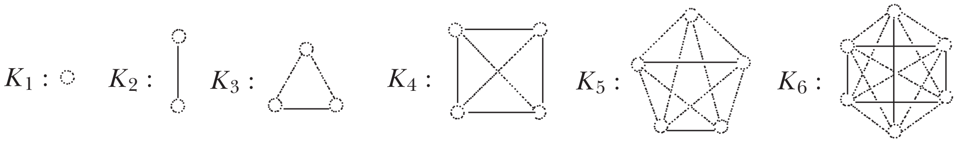


Figure 2.16. Some complete graphs.

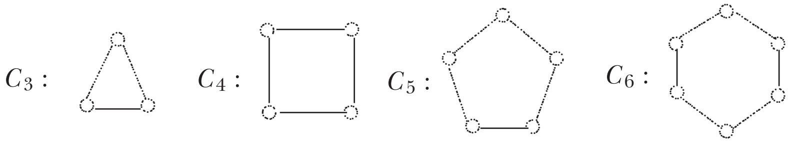


Figure 2.17. Some cycles.

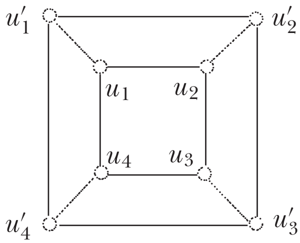


Figure 2.18. The 3-cube Q_3.

The cycle C_4 is a member of two other well-known classes of regular graphs. The graph K_2 is sometimes denoted by Q_1 and called the 1-cube, while C_4 is also denoted by Q_2 and called the 2-cube. The 3-cube Q_3 consists of two copies of Q_2 (see Figure 2.18), one whose vertices are labeled consecutively as u_1, u_2, u_3, u_4 and the other u_1', u_2', u_3', u_4', together with the edges u_1u_1', u_2u_2', u_3u_3', u_4u_4'. If the vertices of Q_3 are then labeled $v_1, v_2, \ldots, v_8$ and Q_3' is another copy of Q_3 with corresponding vertices $v_1', v_2', \ldots, v_8'$, then the 4-cube Q_4 is formed by adding the edges v_iv_i'. Continuing in this way, we obtain the *n-cube*, which is also called a *hypercube*. The n-cube Q_n is an n-regular graph of order 2^n.

Another class of regular graphs of which C_4 is a member consists of those graphs denoted by $K_{r,r}$ for integers $r \geq 1$. The graph $K_{r,r}$ has order $2r$ and vertex set consisting of two disjoint sets U and W of r vertices each such that every vertex of U is adjacent to every vertex of W. The graph $K_{r,r}$ is then r-regular. (An even more general class of graphs that contains

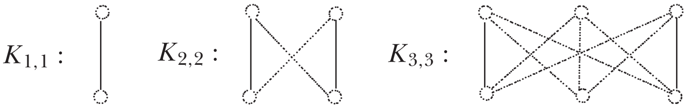


Figure 2.19. Some graphs $K_{r,r}$.

$K_{r,r}$ as a member will be discussed in Chapter 3.) The graphs $K_{r,r}$ are shown in Figure 2.19 for $r = 1, 2, 3$. Notice that $K_{1,1}$ is K_2, the graph $K_{2,2}$ is C_4 and $K_{3,3}$ is the graph we encountered when discussing the Three Houses and Three Utilities Problem in Chapter 1.

SUBGRAPHS

While many problems in graph theory deal with degrees of vertices, there are many others that concern the structure of graphs that lie within a given graph. A graph H is called a *subgraph* of a graph G if every vertex and edge of H is a vertex and edge, respectively, of G. Using set notation, this says that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph G is a subgraph of itself; all other subgraphs are *proper subgraphs* of G. If H is a subgraph of G having the same vertices as G, then H is a *spanning subgraph* of G.

For a nonempty set S of vertices of G, the subgraph $G[S]$ of G *induced* by S has vertex set S where two vertices u and v are adjacent if and only if u and v are adjacent in G. A subgraph H of G is an *induced subgraph* of G if H is $G[S]$ for some nonempty subset S of $V(G)$. For the graph G of Figure 2.20, H_1 and H_2 are subgraphs of G. While H_1 is an induced subgraph, H_2 is not since it fails to contain the edge vw. The subgraph H_3 is a spanning subgraph of G.

A special kind of induced subgraph is obtained by the removal of a single vertex. For a vertex v in a graph G, the subgraph $G - v$ of G is obtained by removing v from G and, necessarily, removing all edges incident with v. In Figure 2.20, the graph H_4 is the subgraph $G - x$. Any subgraph of G obtained by removing a vertex from G is called a *vertex-deleted subgraph* of G.

While a graph G may not be regular, there is always some regular graph F containing G as an induced subgraph. This observation was made by Dénes König, who included this result in his 1936 book. The challenge is to create F by adding vertices to G but without adding any edges between vertices of G.

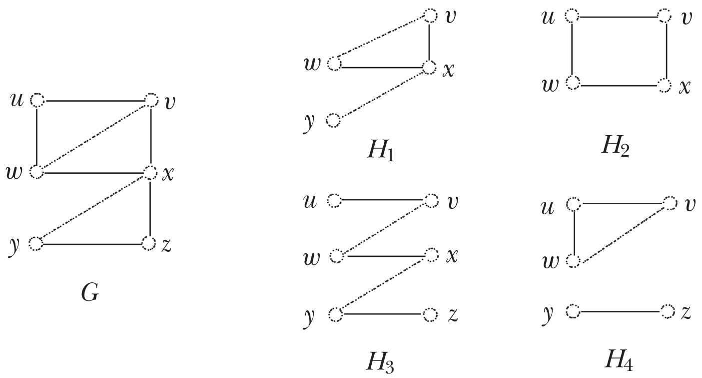


Figure 2.20. Subgraphs of a graph.

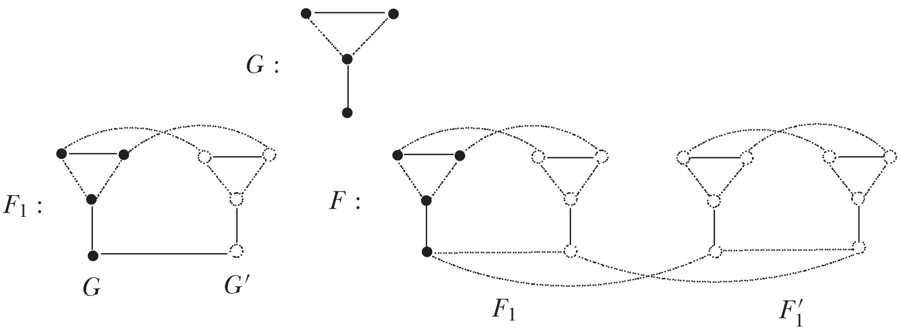


Figure 2.21. The construction of a 3-regular graph F containing G as an induced subgraph.

Theorem 2.6: *For any graph G, there exists a regular graph F containing G as an induced subgraph.*

Proof Strategy: Consider the graph G of Figure 2.21. It's certainly not regular since its maximum degree is $\Delta = 3$ and its minimum degree is $\delta = 1$. We create a new graph F_1 consisting of two copies of G, where the second copy of G is denoted by G'.

We then join every vertex v in G to its corresponding vertex in G' unless v already has degree Δ. Note that in our example every vertex in G that had degree 1 or 2 will now have degree 2 or 3, respectively, as will their corresponding vertices in G', so F_1 is closer to being regular and contains G as an induced subgraph. Applying the

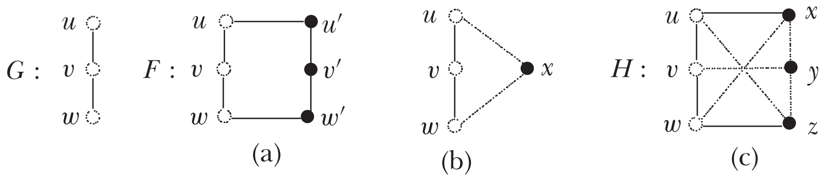


Figure 2.22. The graphs in Example 2.7.

same process to F_1 gives us a new graph F where every vertex will have degree 3 and still contains G as an induced subgraph. ■

In fact, if $r \geq \Delta(G)$, then this method can be used to construct an r-regular graph F containing G as an induced subgraph. The graph F produced need not be an r-regular graph of minimum order containing G as an induced subgraph. For a general graph G, the construction described above will take $\Delta(G) - \delta(G)$ steps and the resulting graph F will be r-regular where $r = \Delta(G)$.

Example 2.7: *(a) Consider the graph G of Figure 2.22. Use the construction described in Theorem 2.6 to find a 2-regular graph containing G as an induced subgraph. Find a smaller graph that has this property.*

(b) What is the smallest order of a 3-regular graph containing the graph G of Figure 2.22 as an induced subgraph?

SOLUTION:

(a) The construction described in Theorem 2.6 produces the 2-regular graph F of order 6 in Figure 2.22a containing G as an induced subgraph. By adding a new vertex x to G and joining x to u and w, we produce the 2-regular graph of order 4 in Figure 2.22b containing G as an induced subgraph. Thus the minimum order of a graph with this property is 4.

(b) Since G contains the end-vertices u and w, at least two vertices must be added to G to raise the degrees of u and w from 1 to 3. Because there is no 3-regular graph of order 5, it is necessary to add at least three vertices to G. This is sufficient, however, as the graph H of Figure 2.22c shows. Thus the minimum order of a graph with this property is 6. ♦

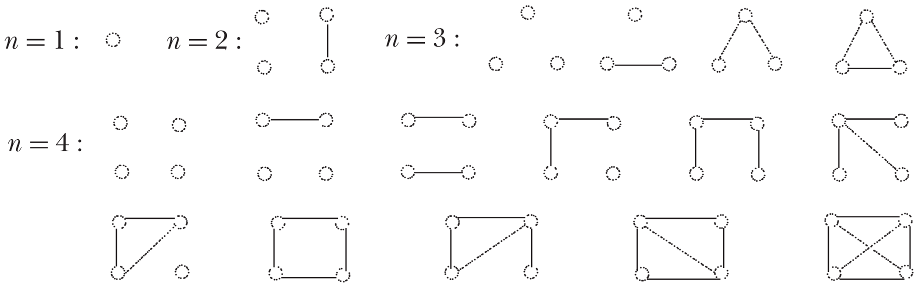


Figure 2.23. The graphs of order n for $n = 1, 2, 3, 4$.

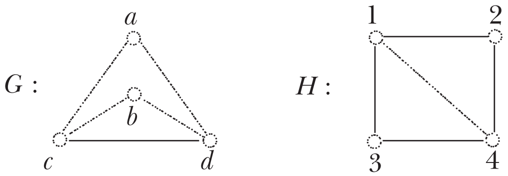


Figure 2.24. Isomorphic graphs.

ISOMORPHIC GRAPHS

In Figure 2.12 all of the regular graphs of order 5 or less are displayed. Figure 2.23 shows *all* graphs of order 4 or less.

The fact that we've referred to the graphs in Figure 2.23 as *all* of the graphs of order 4 or less suggests that we mean these graphs are all different, which, in fact, is the case. This brings up the question as to when two graphs are considered to be the same. Two graphs are considered to be the same if they have the same structure. The technical term for this is that two such graphs are *isomorphic*. This word comes from the Greek, meaning "same shape". Formally, two graphs G and H are isomorphic graphs if the vertices of G can be relabeled to produce H. For example, the graphs G and H of Figure 2.24 are isomorphic since if we relabel the vertices a, b, c, d in G with (respectively) 2, 3, 1, 4, we obtain H. If G and H are isomorphic, then this is indicated by writing $G \cong H$.

If two graphs are isomorphic, then this fact does not depend on how the graphs are drawn (if, in fact, they are drawn at all) or how the vertices of the graphs are labeled. For example, the graphs G_1 and G_2 in Figure 2.25 are isomorphic. This can be shown by relabeling the vertices

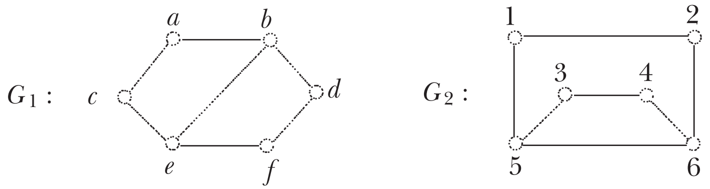


Figure 2.25. Isomorphic graphs.

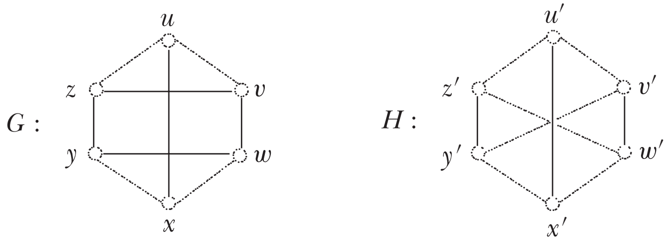


Figure 2.26. Two 3-regular nonisomorphic graphs.

a, b, c, d, e, f with 2, 6, 1, 4, 5, 3, respectively. (By the way, the function F that assigns $F(a) = 2$, $F(b) = 6$ and so on is called an *isomorphism*.)

Both graphs G_1 and G_2 in Figure 2.25 have the same order 6, the same size 7 and the same degrees of their vertices, namely 3, 3, 2, 2, 2, 2. That these numbers are the same is not surprising as these graphs, being isomorphic, are considered the same.

Observation 2.8: *If G and H are isomorphic graphs, then their orders are the same and their sizes are the same, as are the degrees of their vertices.*

While Observation 2.8 provides necessary conditions for two graphs to be isomorphic, these conditions are *not* sufficient. That is, two graphs G and H can satisfy all of these conditions and yet not be isomorphic. For example, the graphs G and H in Figure 2.26 have order 6, size 9 and are 3-regular; yet $G \ncong H$. To see this, notice that G contains a copy of K_3 (with vertices u, v and z) while H does not.

AN UNSOLVED PROBLEM: GRAPH RECONSTRUCTION

In Chapter 1 we discussed the Four Color Problem, which was the best known unsolved problem in graph theory during much of the nineteenth and twentieth centuries even though graph theory was still in its infancy during the nineteenth century. In fact, during portions of the twentieth century, the Four Color Problem was one of the best known unsolved problems in all of mathematics. That graph theory was growing in popularity during the twentieth century was likely due, at least in part, to the notoriety of the Four Color Problem and to the many mathematicians who attempted to solve it. Even though this problem was eventually solved and graph theory had developed into a substantial area of mathematics, it had lost its famous unsolved problem. While no other unsolved problem in graph theory achieved the status of the Four Color Problem, a number of other interesting unsolved problems in graph theory appeared.

Stanislaw Ulam (1909–1984) was born in Poland and as a youngster became interested in astronomy, physics and mathematics. In 1927 Ulam entered the Polytechnic Institute in Lvov. One of his professors there was Kazimierz Kuratowski (whom we will encounter in Chapter 10). Ulam received his PhD in 1933. In 1940 Ulam became an assistant professor of mathematics at the University of Wisconsin. Paul J. Kelly (1915–1985) was a graduate student there, completing his PhD. A problem in graph theory is believed to have originated in 1941 and is often attributed jointly to Kelly and Ulam.

For the graph G of order 6 in Figure 2.27, all six vertex-deleted subgraphs are shown. If only the six vertex-deleted subgraphs of G are given and not G itself, can the graph G be determined from these subgraphs? The answer to this question is most definitely yes as one can see by starting with $G - v_1$, say, and then using the other graphs to reinsert the edges v_1v_2, v_1v_3, v_1v_4 and v_1v_5.

However, what if all vertex labels are removed from the vertex-deleted subgraphs and these six unlabeled graphs are drawn a bit differently (say as in Figure 2.28)? This makes the problem more challenging.

The problem of Kelly and Ulam is then the following: If the unlabeled vertex-deleted subgraphs of a graph G are given, can G be uniquely determined from this information? That is, if G and H are graphs having the same vertex-deleted subgraphs, then is it true that $G \cong H$? Any graph G that can be uniquely determined in this manner is said to be *reconstructible*. We can now state the problem of Kelly and Ulam, which remains unsolved to this day.

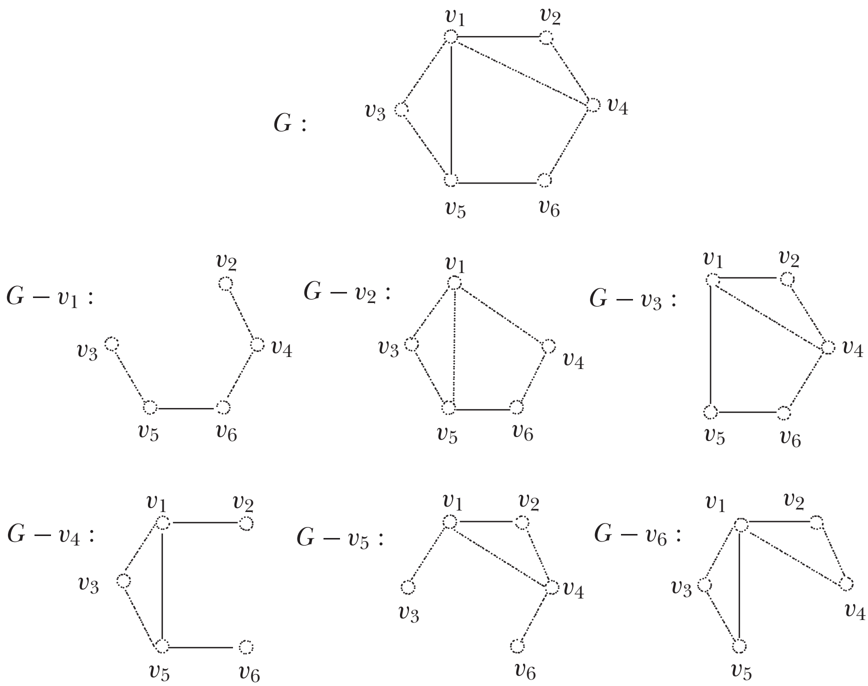


Figure 2.27. The vertex-deleted subgraphs of a graph.

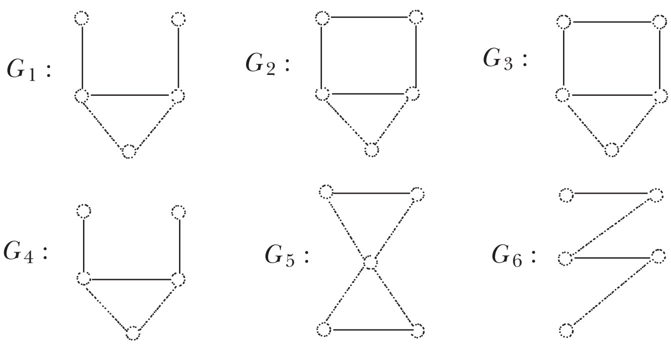


Figure 2.28. The vertex-deleted subgraphs of a graph.

The Reconstruction Problem

Is it true that every graph with more than two vertices is reconstructible?

Note that the two graphs K_2 and $\overline{K}_2$ can't be determined from their vertex-deleted subgraphs since each vertex-deleted subgraph of both

Figure 2.29. The graphs K_2 and $\overline{K}_2$.

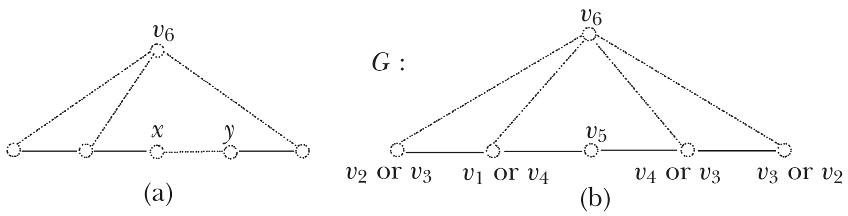


Figure 2.30. The graph G having the vertex-deleted subgraphs in Figure 2.28.

graphs consists of a single isolated vertex. (See Figure 2.29.) Nobody knows whether there are any other such graphs.

Let's try to find the graph G that has the vertex-deleted subgraphs $G_1, G_2, \dots, G_6$ in Figure 2.28. Naturally, since each vertex-deleted subgraph has five vertices, G must contain $n = 6$ vertices, which we'll call $v_1, v_2, \dots, v_6$. In fact, let's say $G - v_i$ generates the graph G_i.

We can determine the number of edges in G by noting that every edge uv will be deleted twice (in $G - u$ and in $G - v$) and so each edge will appear $n - 2 = 4$ times among $G_1, G_2, \dots, G_6$. If m_i denotes the size of G_i, then $m_1 + m_2 + \cdots + m_6 = 5 + 6 + 6 + 5 + 6 + 4 = 32$ and the number of edges in G is $32/4 = 8$. Furthermore, the degree of each vertex v_i must be $m - m_i$ and so the respective degrees of $v_1, v_2, \dots, v_6$ are 3, 2, 2, 3, 2, 4.

Now let's reconstruct G. Since v_6 has degree 4, it must be adjacent to both vertices of degree 1 (for otherwise G would contain a vertex of degree 1) and two of the three vertices of degree 2. So G must contain the subgraph shown in Figure 2.30a. Thus v_6 must be adjacent to x or y. Since G_5 is a subgraph of G, v_6 must be adjacent to y and G must have the structure shown in Figure 2.30b. That is, G is reconstructible. There are four ways to label the vertices in Figure 2.30, but all of these graphs are isomorphic to each other. They all have the shape of the graph G in Figure 2.30b.

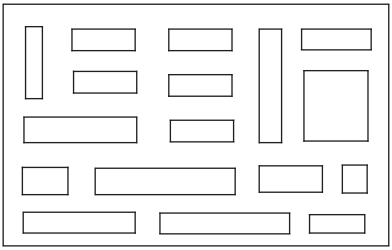
We are still left with the unanswered question as to whether there are two nonisomorphic graphs (with more than two vertices) having the same (unlabeled) vertex-deleted subgraphs.

3

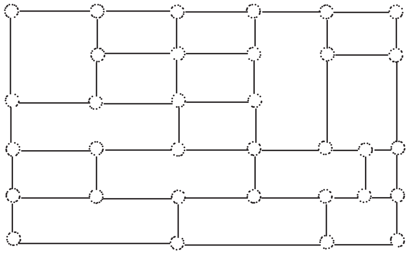
Analyzing Distance

Distance has been fundamental to civilizations for centuries. Over time many questions involving distance have arisen. What is the distance for a ship to travel between two ports (and how much time would it take a ship to travel that distance)? What is the distance between two cities (by highway or by air)? What is the distance between Earth and Mars? What is the distance for a taxicab to travel between two locations in a major city? For this last question, the distance between two street intersections A and B can be defined as the smallest number of blocks a taxicab would have to drive to travel between A and B. This distance is often called the taxicab distance or Manhattan distance between A and B. That is, when one talks about the distance between A and B, we need to know not only what A and B represent but what kinds of routes are possible to travel between A and B.

We have seen many examples in which graphs (or multigraphs) have been used to model a variety of situations but, even more naturally, graphs can be used to model locations within certain street systems. For example, the street system of a town shown in Figure 3.1 can be modeled by a graph, also shown in Figure 3.1. Suppose that a decision has been made to construct a building at some street intersection in the town that would house an emergency facility operating an ambulance service. The question is, where is an ideal location for this emergency facility to be built? It seems reasonable that this facility should be located at an intersection that allows any ambulance to reach its destination as quickly as possible. Therefore, the facility should not be located too far from a possible emergency. This appears to be a matter of minimizing distance. Since the graph models the town, we can attempt to answer this question by considering distance in graphs. In preparation for doing this, we first introduce some fundamental concepts in graph theory.



A town T



Its graph G_T

Figure 3.1. A town and its graph.

CONNECTED GRAPHS

If a major interest of ours concerns the distance between every two locations in a certain region that is being represented by a graph G whose vertices are the locations, then we certainly must be able to travel from one vertex to another in G. This brings us to one of the most important properties that a graph may possess—namely that of being connected.

Consider the graph H in Figure 3.2. How can you get from vertex x to vertex y by always moving along edges and never repeating any vertices? There are four ways to do this. One way is to go from x to z to w to y, denoted by $P = (x, z, w, y)$. Can you find the other three ways? Here P is called an $x - y$ path of length 3. Formally, a *path* P in a graph G is a subgraph of G whose vertices can be listed in some order, say as

$$P = (u = v_0, v_1, \ldots, v_k = v), \tag{3.1}$$

such that $v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$ are all edges of P. Then P is called a $u - v$ *path* in G. So if G contains a $u - v$ path, it is possible to start at u and successively proceed along adjacent edges to arrive at v, never repeating vertices along the way. A $u - v$ path in a graph G therefore describes a means to travel between two vertices u and v. The number of edges in a path or a cycle is its *length*. In the graph H of Figure 3.2, $P' = (s, v, u, x, z, y)$ is an $s - y$ path of length 5, while $P'' = (v, s, t, u, w, z, x)$ is a $v - x$ path of length 6.

A graph G is *connected* if it's possible to walk from any vertex to any other vertex of G along edges of G; or, equivalently, G contains an $x - y$

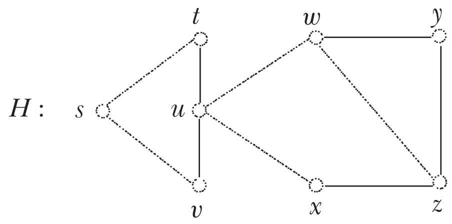


Figure 3.2. Paths in a graph.

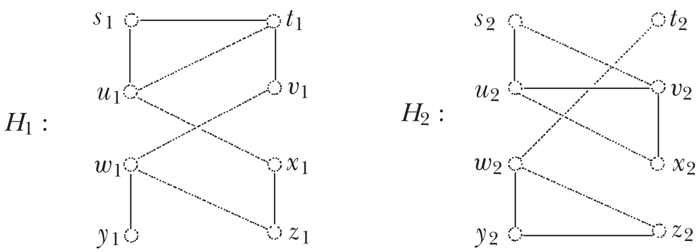


Figure 3.3. A connected graph and a disconnected graph.

path for every two vertices x and y of G. Otherwise, G is *disconnected*. Consequently, the graph H of Figure 3.2 is connected. The graph H_1 of Figure 3.3 is also connected, while the graph H_2 of Figure 3.3 is disconnected. For example, there is no $u_2 - w_2$ path in H_2.

A disconnected graph can be broken into two or more connected pieces, called *components*. For example, the disconnected graph H_2 of Figure 3.3 contains the two components H_2' and H_2'' shown in Figure 3.4. Formally, a *component* of G is a connected subgraph of a graph G that is not a proper subgraph of any other connected subgraph of G. A connected graph G has only one component, namely G itself.

We have seen that there is a class of graphs called cycles, where a cycle with n vertices is denoted by C_n. In addition, a cycle can be a subgraph of a graph. Recall too that a cycle of length 3 is called a triangle. For example, in the graph H of Figure 3.2, (x, u, w, z, x) is a cycle of length 4 and (w, y, z, w) is a triangle. A cycle is *even* or *odd*, according to whether its length is even or odd. In the graph H of Figure 3.2, the 4-cycle (s, t, u, v, s) is an even cycle and the 5-cycle (u, x, z, y, w, u) is an odd cycle.

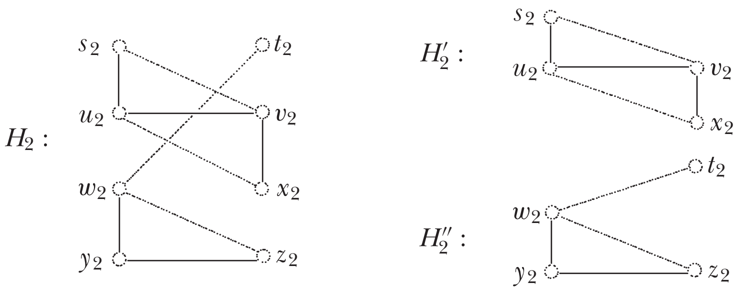


Figure 3.4. The components of a disconnected graph.

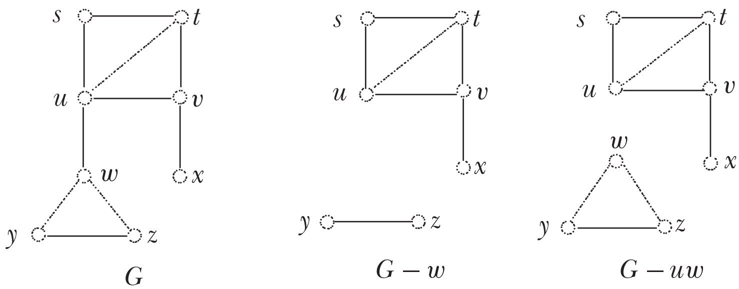


Figure 3.5. Subgraphs obtained by the deletion of a vertex or edge.

CUT-VERTICES AND BRIDGES

Sometimes a connected graph G contains a vertex w with the property that the only way to travel between two vertices u and v (neither of which is w) is to pass through w. Or perhaps G contains an edge e and two vertices u and v with the property that to travel between u and v, we must use e. It is these kinds of vertices w and edges e that we will now discuss.

For a vertex v in a graph G, recall that the vertex-deleted subgraph $G-v$ is obtained by removing v and all edges incident with v. For an edge e of G, the subgraph $G-e$ has the same vertex set as G and consists of all edges of G except e. These concepts are illustrated for the graph G in Figure 3.5.

A vertex v in a graph G is a *cut-vertex* of G if $G-v$ has more components than G has. In particular, v is a cut-vertex in a connected graph G if $G-v$ is disconnected. An edge e in a graph G is a *bridge* of G if $G-e$ has more components than G has. In fact, if an edge $e=uv$

is a bridge in a connected graph, then $G-e$ consists of two components, one component containing u and the other containing v. In the graph G of Figure 3.5, u, v and w are cut-vertices of G while uw and vx are bridges.

It is relatively easy to determine which edges of a graph are bridges.

Theorem 3.1: *Let G be a connected graph. An edge e in G is a bridge if and only if e does not belong to a cycle of G.*

Since this theorem is stated with the phrase "if and only if", this means that there are two statements we must verify. We must show that if e is a bridge of G, then e belongs to no cycle. Also, we must show that if e is not on any cycle of G, then e is a bridge. We verify each statement using a proof by contradiction.

Proof of Theorem 3.1: Let $e = uv$. Suppose first that e is a bridge of G but e belongs to a cycle C of G. Since e is a bridge, $G-e$ is a disconnected graph consisting of two components, one containing u and the other containing v. However, since $e = uv$ lies on C, there is a $u-v$ path along C that does not contain e, which is a contradiction.

For the converse, suppose that e does not belong to a cycle but e is not a bridge. Then $G-e$ is connected and so $G-e$ contains a $u-v$ path P, necessarily of length 2 or more. Adding e to P produces a cycle containing e, which is a contradiction. ■

There are two similar theorems that tell us which edges are bridges and which vertices are cut-vertices.

Theorem 3.2: *Let G be a connected graph. An edge e is a bridge of G if and only if there are vertices u and v in G such that e is on every $u-v$ path in G.*

Theorem 3.3: *Let G be a connected graph. A vertex w is a cut-vertex of G if and only if there are vertices u and v in G, both different from w, such that w is on every $u-v$ path in G.*

DISTANCE IN GRAPHS

We have seen that in a connected graph G, there is at least one path between every two vertices in G. In fact, there may be several paths between two vertices in a graph. For example, the graph F in Figure 3.6 contains many $u-v$ paths, among them

$$\begin{aligned} P_4 &= (u, w, x, v), \\ P_5 &= (u, r, s, t, v), \\ P_6 &= (u, r, y, z, t, v), \\ P_7 &= (u, w, x, y, z, t, v), \\ P_8 &= (u, r, s, t, z, y, x, v). \end{aligned}$$

So P_4 has length 3, P_5 has length 4 and so on. That is, for each integer i with $4 \le i \le 8$, the path P_i has length $i-1$. There is no $u-v$ path of length 1 or 2 in F, however. Therefore, the smallest length of a $u-v$ path in F is 3.

The *distance* $d_G(x, y)$ (or $d(x, y)$) between vertices x and y in a connected graph G is the minimum length of an $x-y$ path in G. Consequently, $d_F(u, v) = 3$ for the vertices u and v in the graph F in Figure 3.6.

Let's return to the town T and the graph G_T in Figure 3.1 that models T. We asked where an emergency facility in this town should be built. Regardless of where we decide to build it, it is important to know how far from that location an emergency may occur. This thought suggests the following concept in graphs.

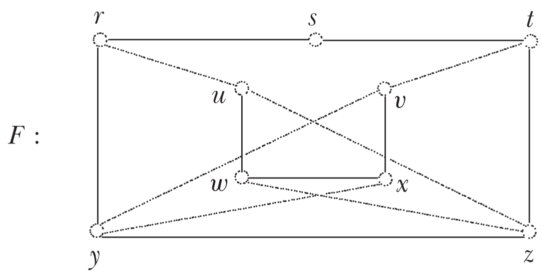


Figure 3.6. Distance in a graph.

G_T :

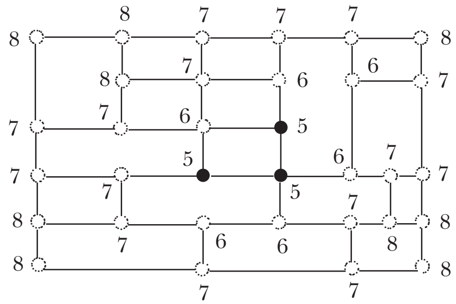


Figure 3.7. The eccentricities of the vertices of G_T.

For a vertex v in a connected graph G, the distance from v to a vertex farthest from v is referred to as the *eccentricity* of v. The vertices of G_T of Figure 3.1 are labeled with their eccentricities in Figure 3.7.

A vertex of minimum eccentricity in a connected graph G is called a *central vertex* of G. A central vertex v then has the property that its distance to a vertex farthest from v is as small as possible.

The graph G_T of Figure 3.7 has three central vertices, which are indicated by solid circles in that figure. In a certain sense then, this corresponds to the three optimal locations in the town T to place an emergency facility. There is some question, however, as to whether these are in fact the logical locations for an emergency facility.

Representing the distance between two intersections u and v in town T by the distance between the corresponding vertices u and v in the graph G_T has some disadvantages. First, this distance in G_T interprets every two blocks in T the same—the same length and the same amount of time to travel along the blocks. As far as time is concerned, this can vary, depending on traffic and other factors. However, computing the distance in a graph is considerably easier and does provide a possible representation of distance in the town T. There is, however, a possibly more fundamental difficulty in constructing an emergency facility at one of the central vertices in the graph G_T.

Consider the graph H of order 10 in Figure 3.8, where the eccentricity of each vertex is indicated in bold. Since w is the only vertex of H having the minimum eccentricity, w is the unique central vertex of H. So thinking of the vertices of H as representing locations in some town, w corresponds to the unique optimal location to build an emergency facility. The choice of w for the emergency facility is greatly influenced by the

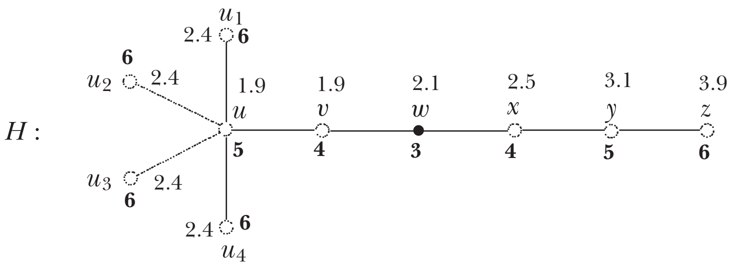


Figure 3.8. Eccentricities and average distances of vertices.

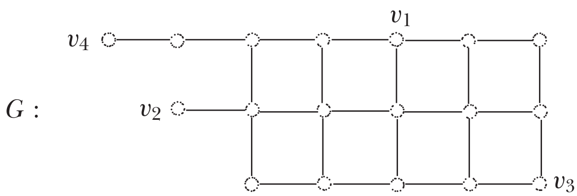


Figure 3.9. Eccentric vertices.

location of a single vertex, namely z. There are, however, five vertices at distance 3 from w. So if an emergency should occur at distance 3 from w, it is much more likely to occur at one of u_1, u_2, u_3 or u_4 than at z. Thus the choice of w for the emergency facility may be influenced by the location of z (and perhaps y) more than it should be. There may be a better way to choose the location of an emergency facility.

Suppose that we were to calculate the *average distance* of the vertices of H from each vertex. To compute the average distance of the vertices from a given vertex p of H, we sum the distances from p to all 10 vertices and divide this sum by 10. These averages are also shown in Figure 3.8. Since the minimum average distance is 1.9, it may seem logical to construct the emergency facility at u or v. Of course, should an emergency occur at z, an ambulance located at u or v has a greater distance to travel to arrive at z than if the ambulance were located at w.

For a vertex u in a connected graph G, a vertex v is an *eccentric vertex* of u if v is a vertex that is farthest from u. For example, in the connected graph G in Figure 3.9, v_2 is an eccentric vertex of v_1 but (perhaps surprisingly) v_1 is not an eccentric vertex of v_2. The vertex v_3 is an eccentric vertex of v_2 but v_2 is not an eccentric vertex of v_3. Finally, v_4 is an eccentric vertex of v_3 (*and* v_3 is an eccentric vertex of v_4).

BIPARTITE GRAPHS

For positive integers s and t, the *complete bipartite graph* $K_{s,t}$ has order $s+t$ and its vertex set can be divided into subsets U and W with s and t vertices, respectively, such that every vertex of U is adjacent to every vertex of W. In $K_{s,t}$, every vertex of U has degree t and every vertex of W has degree s. When $s=t=r$, this is the r-regular graph $K_{r,r}$ introduced in Chapter 2. More generally, a graph G of order $n \geq 2$ is *bipartite* if the vertex set of G can be partitioned into two sets U and W in such a way that every edge of G joins a vertex of U and a vertex of W. The graphs H, C_6, Q_3, $K_{3,3}$ and $K_{2,4}$ shown in Figure 3.10 are all bipartite, where C_6, Q_3 and $K_{3,3}$ are regular bipartite graphs. We have colored the vertices so that each vertex in U is colored black and each vertex in W is white. Since each of these graphs is bipartite, each edge joins vertices of opposite color.

The graph F in Figure 3.11 is not bipartite, however. To see this, suppose that F is bipartite. Then its vertex set can be partitioned into two sets U and W such that every edge of G joins a vertex of U and a vertex of W. We can assume that $u \in U$. Then necessarily, $w \in W$, $x \in U$, $y \in W$ and $v \in U$. However, uv is an edge of F and both u and v belong to U. This is impossible. You might observe that the contradiction was caused because F contains the odd cycle $C = (u, w, x, y, v, u)$. Notice that none of the bipartite graphs in Figure 3.10 contains an odd cycle. In fact,

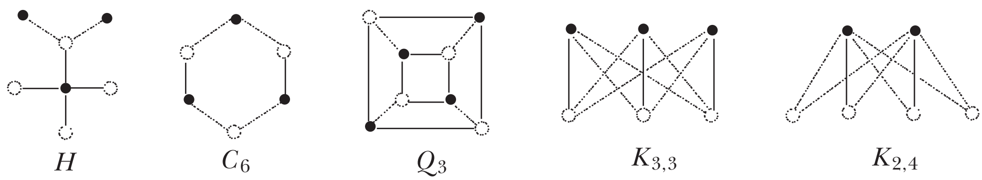


Figure 3.10. Bipartite graphs.

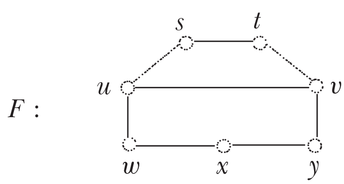


Figure 3.11. A graph that is not bipartite.

this is the key characteristic that determines whether a graph is bipartite. We have seen that if G has an odd cycle, then it can't be bipartite. But are odd cycles the only enemy of bipartiteness? Surprisingly, yes, as the next theorem shows.

Theorem 3.4: *A graph G is bipartite if and only if G contains no odd cycles.*

Proof: We have already seen that if a graph contains an odd cycle, then it is not bipartite. For the converse, assume that G is a nontrivial connected graph that contains no odd cycles. (If G is not connected, then the argument below will show that each component is bipartite, so G is too.) We show that G is bipartite. For a vertex u in G, let

$$U = \{x \in V(G) : d(u, x) \text{ is even}\} \text{ and } W = \{x \in V(G) : d(u, x) \text{ is odd}\}.$$

Therefore, $u \in U$ (since $d(u, u) = 0$) and every vertex adjacent to u belongs to W. If no edge joins two vertices of U and no edge joins two vertices of W, then G is bipartite. Suppose, however, that this is not the case. Assume that there are two vertices in either U or W that are adjacent, say in W. Then W contains two adjacent vertices x and y. Since $d(u, x) = a$ and $d(u, y) = b$ are both odd, there is a $u - x$ path P' of length a and a $u - y$ path P'' of length b. Let z be the last vertex on P' that is also on P'', where possibly $z = u$. Suppose that $d(u, z) = c$. The cycle C consisting of the $z - x$ subpath of P', the $z - y$ subpath of P'' and the edge xy has length $(a - c) + (b - c) + 1 = (a + b + 1) - 2c$, which is odd, contradicting the fact that G has no odd cycles. Note that if x and y are both in U, we arrive at the same contradiction since a and b are both even. ■

LOCATING SETS

In the past, distance in graphs has been used to model and study a number of real-life problems. As an illustration of one of these types of problems, suppose that a certain facility consists of five rooms R_1, R_2, R_3, R_4, R_5 (shown in Figure 3.12). Two of these rooms are adjacent

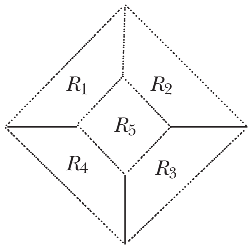


Figure 3.12. A facility consisting of five rooms.

(and the distance between them is 1) if they share a common wall (not just a corner). The distance between two rooms is 2 if they are not adjacent but there is a room that is adjacent to both (and so on). For example, the distance between the rooms R_1 and R_3 in Figure 3.12 is 2 and the distance between R_2 and R_4 is also 2. The distance between all other pairs of distinct rooms is 1. The distance between a room and itself is 0.

Suppose that a certain sensor is placed in one of the rooms. Should a fire take place in one of the rooms, this sensor is able to detect the distance from the room with the sensor to the room containing the fire. Suppose, for example, that the sensor is placed in R_1. If a fire occurs in R_3, then the sensor alerts us that a fire has occurred in a room at distance 2 from R_1; that is, the fire is in R_3 since R_3 is the only room at distance 2 from R_1. If the fire is in R_1, then the sensor indicates that the fire has occurred in a room at distance 0 from R_1; that is, the fire is in R_1. However, if the fire is in any of the other three rooms, then the sensor tells us that there is a fire in a room at distance 1 from R_1. But with this information, we cannot determine the precise room in which the fire has occurred. In fact, there is *no* room in which the sensor can be placed to identify the exact location of a fire in every instance.

On the other hand, if we were to place two sensors (one red and the other blue) in two rooms, where, say, the red sensor is placed in R_1 and the blue sensor in R_2, and a fire occurs in R_4, say, then the red sensor in R_1 tells us that there is a fire in a room at distance 1 from R_1, while the blue sensor tells us that the fire is in a room at distance 2 from R_2; that is, we can assign the ordered pair (1, 2) to the room R_4. In this example, the ordered pairs are distinct for all rooms, so the minimum number of sensors required to detect the exact location of any fire is 2. Even though the minimum number of sensors needed to detect the location of any

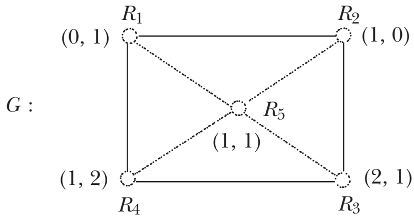


Figure 3.13. A graph representing a facility consisting of five rooms.

fire is 2, care must be taken as to where the two sensors are placed. For example, we cannot place the two sensors in R_1 and R_3 since, in this case, the ordered pairs of R_2, R_4 and R_5 are all (1, 1), and we cannot distinguish the precise location of the fire. The facility that we have just described can be modeled by the graph G in Figure 3.13, whose vertices are the rooms. Each vertex v of G is assigned a distance "vector" (a, b), where a is the distance from v to R_1 and b is the distance from v to R_2. Since every two vertices of G have distinct distance vectors, the vertices of G can be uniquely identified by these vectors.

The idea of determining an ordered set $S = \{w_1, w_2, \ldots, w_k\}$ of vertices in a connected graph G so that for every two vertices u and v of G, there is some vertex w_i in S whose distance to u is different from its distance to v is due to Peter Slater. He referred to such a set S as a *locating set* and the minimum number of vertices in a locating set as the *location number* of G. So the graph G of Figure 3.13 has location number 2 and $S = \{R_1, R_2\}$ is a *minimum* locating set. Thus a locating set S in a graph G has the characteristic of being able to uniquely identify each vertex of G by its distances to the vertices of S. In the past these ideas were useful when working with the United States sonar and Coast Guard LORAN (LOng RAnge aid to Navigation) stations.

DOMINATING SETS

We mentioned in Chapter 1, "Graphs, Games, Galleries and Gridlock", that on a standard 8×8 chessboard the queen is the only chess piece that can move any number of squares horizontally, vertically or diagonally and that the queen can attack or capture each vacant square that can be reached by moving the queen in one of these directions. The queen is

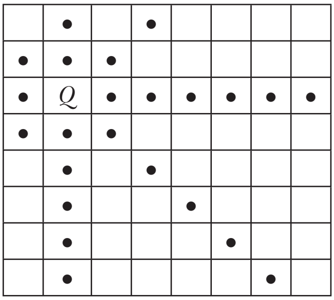


Figure 3.14. Squares dominated by a queen Q.

also said to dominate these squares as well as the square containing the queen. This is illustrated in Figure 3.14.

In 1862 Carl Friedrich de Jaenisch considered the problem of determining the minimum number of queens that can be placed on a chessboard so that every square is dominated by at least one of these queens. It had been thought for some time that this minimum number is 5. This, in fact, led to a problem encountered in Chapter 1, “Graphs, Games, Galleries and Gridlock”.

The Five Queens Problem

Can five queens can be placed on an 8×8 chessboard so that every vacant square can be captured by at least one of these queens?

Figure 3.15 shows a solution of the Five Queens Problem. This chessboard problem can be considered the origin of the topic of dominating sets in graphs.

A vertex v in a graph G is said to *dominate* a vertex u if u is adjacent to v or $u = v$, that is, v dominates u if the distance from v to u is at most 1. Thus v dominates each of its neighbors as well as itself. A set S of vertices in a graph G is a *dominating set* of G if every vertex of G is dominated by at least one vertex of S. That is, every vertex of G either belongs to S or is adjacent to some vertex in S. Figure 3.16 shows three different dominating sets,

$$S_1 = \{r, t, x, z\},\ S_2 = \{s, u, w, y\} \text{ and } S_3 = \{s, t, y\},$$

of a graph H.

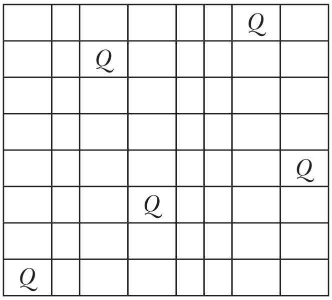
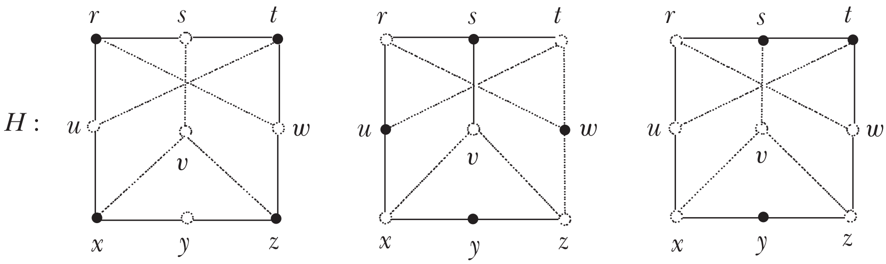


Figure 3.15. A solution to the Five Queens Problem.



$S_1 = \{r, t, x, z\}$ $S_2 = \{s, u, w, y\}$ $S_3 = \{s, t, y\}$

Figure 3.16. Dominating sets for a graph.

Two of the three dominating sets shown for the graph H in Figure 3.16 consist of four vertices, while the third dominating set consists of three vertices. Of course, the entire vertex set of a graph is always a dominating set, but the interesting question here concerns the *smallest* number of vertices in a dominating set.

The minimum number of vertices in a dominating set of a graph G is called the *domination number* of G and is denoted by $\gamma(G)$. (Here γ is the lowercase Greek letter gamma.) Since the graph H in Figure 3.16 has a dominating set with three vertices, it follows that $\gamma(H) \leq 3$. On the other hand, every vertex of H has degree at most 3, implying that no vertex can dominate more than four vertices and so two vertices of H can dominate at most eight vertices of H. Since the order of H is 9, every dominating set of H must contain at least three vertices. Therefore, $\gamma(H) = 3$.

In 1958 the French mathematician Claude Berge (1926–2002) wrote *Théorie des graphes et ses applications*, the second book ever written on

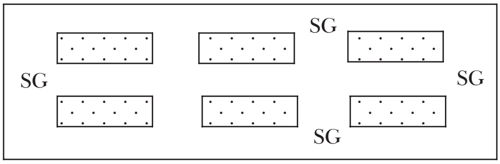


Figure 3.17. A city map.

G :

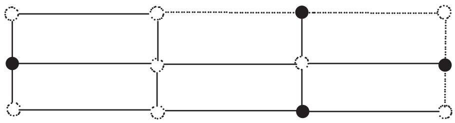


Figure 3.18. A graph modeling a city map.

graph theory. In this book he defined for the first time the concept of the domination number of a graph (although he didn't use this terminology). Four years later in 1962, Oystein Ore wrote *Theory of Graphs*, the first book written in English on graph theory. In this book he introduced the terminology *dominating set* and *domination number*. The publication of the 1977 survey paper "Towards a Theory of Domination in Graphs" by Ernest Cockayne and Stephen Hedetniemi served as the impetus for domination becoming one of the areas of study within graph theory.

Let's look at a practical example involving domination. Figure 3.17 shows a portion of a city, consisting of six city blocks, determined by three horizontal streets and four vertical streets. A security agency has been retained to watch over the street intersections. A security guard stationed at an intersection can observe the intersection where he is located as well as all intersections *up to one block away* in a straight-line view from this intersection. The question is, what is the minimum number of security guards needed to guard all 12 intersections? Figure 3.17 shows 4 intersections where security guards can be placed (labeled by SG) so that all 12 intersections are under observation.

This situation can be modeled by the graph G of Figure 3.18. The street intersections are the vertices of G and two vertices are adjacent if the vertices represent intersections on the same street at opposite ends of a city block. Looking for the smallest number of security guards in the city of Figure 3.17 is the same problem as seeking the domination number of the graph G in Figure 3.18. The solid vertices in Figure 3.18 correspond to the placement of the security guards in Figure 3.17.

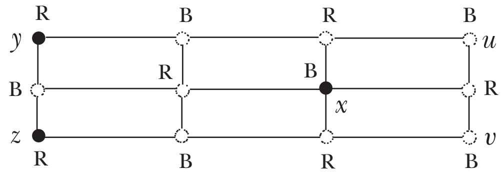


Figure 3.19. Determining the domination number of a graph.

Example 3.5: *For the graph G in Figure 3.18, determine $\gamma(G)$.*

SOLUTION:

We claim that $\gamma(G) = 4$. Since the four solid vertices in Figure 3.18 form a dominating set of G, it follows that $\gamma(G) \leq 4$. To verify that $\gamma(G) \geq 4$, it is necessary to show that there is no dominating set with three vertices in G.

The graph G has twelve vertices, two of which have degree 4 and six have degree 3. The remaining four vertices have degree 2. Therefore, there are two vertices that dominate five vertices each and six vertices that dominate four vertices each. Conceivably then, there could be some set of three vertices that together dominate all twelve vertices of G. Notice that the vertices of G can be colored with two colors, say red (R) and blue (B), so that no two vertices of the same color are adjacent. Without loss of generality, we can assume that the vertices of G are colored as in Figure 3.19. Necessarily, the neighbors of each vertex have a color that is different from the color assigned to this vertex.

Assume, to the contrary, that G has a dominating set S consisting of only three vertices. At least two vertices of S are colored the same. If all three vertices of S are colored the same, say red, then only three of the six red vertices will be dominated. Therefore, exactly two vertices of S are colored the same, say red, with the third vertex colored blue. If the blue vertex of S has degree at most 3, then it can dominate at most three red vertices and S dominates at most five red vertices of G, which is impossible. Hence S must contain x (see Figure 3.19) as its only blue vertex. Since y and z are the only two red vertices not dominated by x, it follows that $S = \{x, y, z\}$. However, u and v are not dominated by any vertex of S, which cannot occur. Therefore, $\gamma(G) = 4$. ♦

A	B	C
0	0	0

(a) Lights out

A	B	C
1	1	1

(b) Lights on

Figure 3.20. Lights out and lights on.

0	0	0		0	0	1		0	1	0
1	0	0		1	0	1		1	1	0

0	1	1
1	1	1

Figure 3.21. The possible light arrangements of the three offices.

THE LIGHTS OUT PUZZLE

On a certain floor of a business building, a firm occupies three offices A, B and C located in a row. Each office has a large ceiling light and a light switch which, when pressed, reverses the light in that office (on to off or off to on) as well as the light in each adjacent office. So if we begin the day, as in Figure 3.20a, with all lights off and push the light switch in the central office B, then we arrive at the situation in Figure 3.20b, where all lights are on. Of course, pressing the light switch in office B returns us to the situation in Figure 3.20a.

Each light arrangement of the three offices can be represented by an ordered triple (a, b, c) or abc, where a, b and c can be 0 or 1, with 0 meaning that the light is off in the particular office and 1 meaning that the light is on. The eight possibilities are shown in Figure 3.21.

This situation can be represented by a graph G of order 8, whose vertices are the ordered triples abc, where $a, b, c \in \{0, 1\}$. If we can change from one light arrangement to another by pressing a single light switch, then we draw an edge between the two vertices representing these arrangements. The graph G is shown in Figure 3.22. The graph G of Figure 3.22 shows that, beginning with lights out in all three offices, we can obtain any light pattern we desire, although it may require pressing as many as three switches. Notice that this graph is the 3-cube Q_3.

The situation that we have just described can be interpreted in terms of graphs from the beginning. Consider the graph G in Figure 3.23, where the vertices are drawn as solid vertices, indicating that all lights are on.

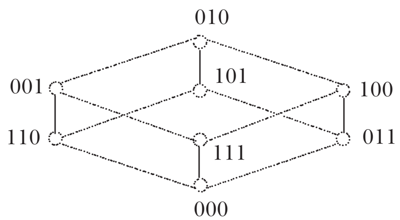


Figure 3.22. The graph of light arrangements.

Figure 3.23. The graphs G and H.

If we "press" the solid vertex in the middle, this causes all lights to go out and we obtain the graph H.

So the general situation might go something like this: Let G be a connected graph where there is a light as well as a light switch at each vertex. For each vertex, the light at that vertex is either on or off. When the light switch at that vertex is pressed, it reverses the light (changing it from on to off or from off to on) not only at that vertex but at all vertices adjacent to that vertex. That is, when the light switch at a vertex v in a graph G is pressed, it reverses the light at each vertex that is dominated by v. Therefore, if a vertex u of G is dominated by k vertices and the light switches at these k vertices are pressed, the light at u is reversed k times. There is a variety of questions that can be asked here, but our chief question concerns the following.

The Lights Out Puzzle

Let G be a graph. If all vertex lights of G are on, does there exist a collection of light switches which when pressed will turn all vertex lights out? (If so, what is the smallest number of light switches in such a collection?)

There is an electronic puzzle called Lights Out marketed by Tiger Electronics of Hasbro, Inc., that gave rise to the more general graph theory puzzle mentioned above. Earlier manufactured as a cube, the current Lights Out game is played on a 5×5 checkerboard with multicolored

LEDs and digitized sound. Indeed, there are interactive websites where various versions of the puzzle can be played.

In terms of the Lights Out Puzzle on a graph G mentioned above, what we are asking is whether there is a dominating set S such that every vertex of G is dominated by an odd number of vertices in S. (If so, then pressing the vertices in S will successfully turn all lights off.) The following (possibly unexpected) result of Klaus Sutner says that this puzzle always has a solution.

Theorem 3.6: *If G is a connected graph all of whose vertex lights are on, then there exists a set S of vertices of G such that if the light switch is pressed at each vertex of S, then all vertex lights of G will be out.*

ERDŐS NUMBERS

Many mathematicians do mathematical research, ordinarily attempting to discover some new results in mathematics. If a mathematician discovers a new result (theorem), he or she often writes up the work, including proofs, in the form of an article called a mathematical paper. This paper is then submitted to a mathematics magazine or a journal that publishes such articles. If all goes well, this paper will be published and the mathematician will be the author (or one of the authors) of the paper.

A curious graph in which distance plays an interesting role in the world of mathematical papers is the *collaboration graph*. This graph has mathematicians as its vertices. Two vertices (mathematicians) of the collaboration graph are joined by an edge if they are coauthors of some mathematical paper. Years ago a mathematician would ordinarily work on mathematics by himself or herself. In fact, prior to World War II over 90% of the mathematical papers were written by a single author and virtually no papers had three or more coauthors. Perhaps due to such things as the increase in mathematics conferences and the ease of communicating electronically, a much higher percentage of multiple-authored mathematical papers have occurred. Multiple collaboration in the laboratory sciences is even more common. Indeed, there are many scientific papers with more than 500 coauthors. Consequently, mathematical research has often become more of a collaborative endeavor.

Many major undertakings have involved people working successfully together, not only on scientific projects such as space exploration but on artistic projects such as Broadway shows and motion pictures. In fact, the people at the Pixar Animation Studios have said that much of their success is due to the creative collaboration that exists there.

The mathematician best known for the number of people with whom he collaborated is Paul Erdős (1913–1996), a prolific Hungarian mathematician. There is, in fact, a humorous concept associated with Erdős. For each mathematician A, the *Erdős number* of A is the distance from A to Erdős in the collaboration graph. Consequently, Erdős is the only mathematician having Erdős number 0. Any mathematician who has coauthored a paper with Erdős has Erdős number 1. In general, a mathematician has Erdős number $k \geq 2$ if he or she does not have Erdős number less than k but has coauthored a paper with someone who has Erdős number $k - 1$. The more than 1500 papers authored or coauthored by Erdős have resulted in over 500 mathematicians having Erdős number 1 and over 6000 having Erdős number 2. At one time, a mathematician was known to have Erdős number 13. Those mathematicians from whom there is no path to the vertex Erdős in the collaboration graph have no Erdős number. One thing about an Erdős number is that it is a function of time. In particular, a mathematician's Erdős number can decrease with time but never increase.

Similar to the problem of finding the Erdős number of a mathematician is that of determining the Bacon number of a film actor. The Bacon number here refers to the movie actor Kevin Bacon. In this case, a graph is constructed whose vertices are those actors who have appeared in feature films. Two vertices (actors) are joined by an edge if they have appeared in the same feature film. The distance from an actor A to Kevin Bacon in this graph is then the *Bacon number* of A. Both the Erdős number and Bacon number are related to the so-called six degrees of separation (also the title of a play written by John Guare), dealing with the notion that every two people can be connected by a path of length 6 or less, where consecutive vertices in the path refer to two people who are friends or are acquainted.

When one thinks of authors or coauthors of mathematical papers, the tendency is to think of mathematicians rather than writers. Typically, the term *author* is more likely thought of as referring to someone who writes a story. Of the many well-known authors, two of the best known are Mark Twain and Lewis Carroll. Mark Twain created the characters Huckleberry Finn and Tom Sawyer, while Lewis Carroll created Alice from *Alice's Adventures in Wonderland*.

Although the stories that Mark Twain and Lewis Carroll wrote are considerably different, these two authors have something in common—namely, Mark Twain and Lewis Carroll are *not* the real names of these two people. The real name of Mark Twain is Samuel Langhorne Clemens and the real name of Lewis Carroll is Charles Lutwidge Dodgson. Although it is not all that unusual for writers of stories to use a pen name, this is nearly unheard of for mathematicians—*but* it has happened and was meant to be humorous.

In 1979 a paper titled "Maximum Antichains of Rectangular Arrays" was published with the listed author G. W. Peck. As it turned out, there is no such person. Indeed, the six letters of this name were taken from the first letter of the last names of its six coauthors: Ronald Graham, Douglas West, George B. Purdy, Paul Erdős, Fan Chung Graham, Daniel Kleitman. In particular, this meant that Douglas West and Paul Erdős were coauthors of a paper having an imaginary author. Since West had never coauthored a paper with Erdős (at that time), it was determined by some that West should have the imaginary Erdős number i.

G. W. Peck is not the only imaginary mathematician to write papers—nor is "he" the most famous imaginary mathematician. Another twentieth-century mathematician who made a major impact on the development of graph theory was William Tutte. Born in England, Tutte (1917–2002) entered Trinity College, Cambridge in 1935 to study chemistry. While chemistry was Tutte's major area of study, it was not his favorite subject—mathematics was. He was an active member of the Trinity Mathematical Society, where he met three undergraduates, who became lifelong friends of his: Cedric Smith, Rowland Leonard Brooks, Arthur Stone. The four students wrote a paper using electrical networks to solve a geometric problem, which became a standard reference in the field.

These four students were sometimes referred to as the "Trinity four". Smith learned of a group of mathematicians who called themselves "Hector Pétard" and had written a mathematics paper under that name. Smith thought that was an interesting idea and the Trinity four set about finding a name for themselves. The first letters of their first names (Bill, Leonard, Arthur and Cedric) gave BLAC. The idea of using

"The Reverend Cornelius Black" occurred to them but Brooks objected. The color BLACk became the more cheerful color BLAnChe ("white" in French). The phrase "carte blanche" then suggested the name "Blanche Descartes" (René Descartes was a famous mathematician). So over the years Blanche Descartes became the name of various subsets of the Trinity four. Under that name, some thirty papers were published, some lighthearted, some serious mathematics. Even though many knew who Ms. Descartes was, Tutte would never acknowledge this. Writing under the name Blanche Descartes permitted Tutte to conceal his identity and say whatever he imagined.

4

Constructing Trees

In an underdeveloped region of a country, several settlements have grown into villages and the village leaders have decided that it is time to construct paved roads between certain pairs of villages so that it is possible to travel by vehicle between all villages along paved roads. The question is, what is a good way to accomplish this so that the cost involved is kept as low as possible? Figure 4.1a shows a map indicating these villages, which are denoted by $v_1, v_2, \ldots, v_8$, along with all practical locations of paved roads and estimated costs (in thousands of dollars).

Figure 4.1b shows one possible way of selecting roads to pave. The decision regarding which roads should be paved was made by successively selecting the least expensive available road between two villages if no paved route between these villages has already been selected. Deciding, in this manner, which roads to pave is often referred to as a greedy method, which involves making what appears to be the optimal choice at each step. Do you think this is a good method to use in this case? The surprising answer to this question will be given later in the chapter. Quite clearly, the roads to pave can be modeled by a graph, namely the graph G of Figure 4.1c.

The system of roads chosen to pave, shown in Figure 4.1b, brings up a number of additional questions. For example, for a person in village v_2 to drive to village v_6 along paved roads, it is necessary to drive through v_3, v_5 and v_4. Equivalently, the only path from v_2 to v_6 in the graph G of Figure 4.1c is $(v_2, v_3, v_5, v_4, v_6)$. It would be more convenient if additional roads were built. This, however, would defeat the goal of attempting to keep the construction costs down. As we mentioned in the preceding paragraph, there is also the question as to whether the roads that were chosen to be paved in Figure 4.1b were the best choices. Questions of this type will be discussed in the section on "The Minimum Spanning Tree Problem", later in this chapter. Regardless of how one

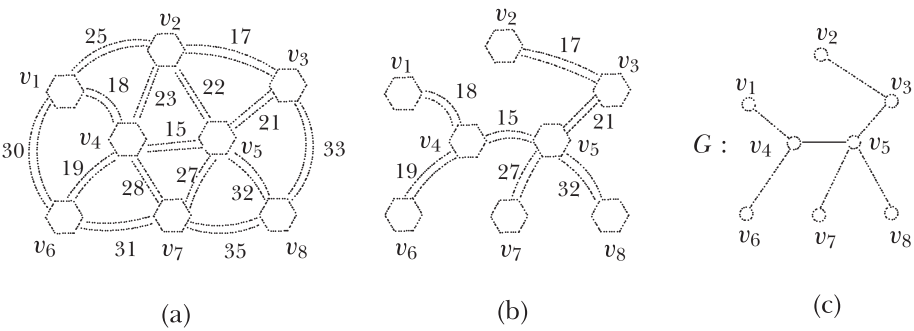


Figure 4.1. A proposed minimum cost solution and a graph representing the paved roads.

chooses which roads to be paved, what we need to do is to construct as few roads as possible so that the graph G representing the road system is connected. Also, this graph G should contain no cycles because, according to Theorem 3.1, deleting any edge e belonging to a cycle in G results in a graph that is still connected.

This brings us to the major topic of this chapter: connected graphs containing no cycles.

AN INTRODUCTION TO TREES

A connected graph that contains no cycles is called a *tree*. Figure 4.2 shows three trees: T_1, T_2 and T_3. (It is common to denote a tree by T rather than G.) The trees shown in Figure 4.2 probably explain why these graphs are called trees. Indeed, a tree has humorously been defined as a graph that looks like a tree.

Since a tree T is a connected graph, every two vertices of T are connected by a path. Even more can be said.

Theorem 4.1: *A graph G is a tree if and only if every two vertices of G are connected by only one path.*

Proof: Suppose first that G is a graph in which every two vertices are connected by a single path. Then certainly G is connected.

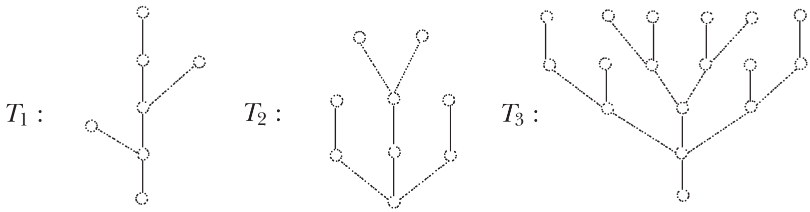


Figure 4.2. Trees.

To show that G is a tree, it remains only to show that G contains no cycles. If G did contain a cycle C, then any two vertices on C would be connected by two paths on C, which would contradict our assumption. Therefore, G contains no cycles and so G is a tree.

For the converse, assume that G is a tree. We show that every two vertices of G are connected by a unique path in G. Suppose, though, that there are two vertices u and v in G connected by two paths, say P and P'. Since P and P' are not the same path, there must be a first vertex x on both P and P', where the vertex that follows x on P is not the same vertex that follows x on P'. Since P and P' both terminate at v, there is a first vertex w after x on P that is also on P'. Then the $x - w$ subpaths of P and P' have no vertex in common except for x and w and so P and P' form a cycle in the tree G, which is impossible. ■

It's a consequence of Theorem 4.1 that if u and v are any two nonadjacent vertices of a tree T, then T contains exactly one $u - v$ path P. If we were to add the edge uv to T, resulting in the graph $T + uv$, then this graph is not a tree as it contains a cycle—indeed a single cycle C, formed from the path P and the edge uv.

LEAVES IN TREES

When dealing with trees, a vertex of degree 1 is typically called a *leaf* rather than an end-vertex. For example, the tree T_1 in Figure 4.2 has four leaves. The next theorem is proved by a method called an *extremal argument*.

Theorem 4.2: *Every tree with at least two vertices has at least two leaves.*

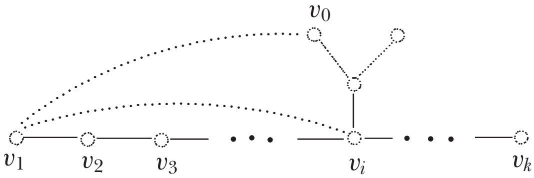


Figure 4.3. If $P = (v_1, v_2, \ldots, v_k)$ is a longest path in a tree, then v_1 can only be adjacent to v_2

Proof: This is obvious if a tree has exactly two vertices. So suppose that T is a tree of order $n \geq 3$ and let P be a path of maximum length in T, say $P = (v_1, v_2, \ldots, v_k)$. We claim that v_1 and v_k are leaves in T. To prove that v_1 is a leaf (the proof for v_k is similar), we must show that v_1 is adjacent only to v_2. Certainly, v_1 can't be adjacent to any other vertex on P for this would create a cycle in T. Nor could v_1 be adjacent to any other vertex, say v_0, not on P for then $(v_0, v_1, v_2, \ldots, v_k)$ would be a path longer than P, contradicting our assumption that P has maximum length. (See Figure 4.3.) ■

Notice that if u is a leaf of the tree T, then the graph $T - u$ obtained by removing u from T is connected and contains no cycles, that is, $T - u$ is also a tree. This fact allows us to "inductively" prove many important theorems about trees. In addition, if a new vertex v is added to T and joined to a single vertex of T, then the resulting graph T' is also connected and contains no cycles; so T' is a tree.

Observe that the tree T_1 of Figure 4.2 has order 7 and size 6 while the tree T_2 has order 9 and size 8. The tree T_3 has order 17 and size 16. In each of these examples, the size of the tree is one less than its order. This is true of every tree.

Theorem 4.3: *Every tree with n vertices has $n - 1$ edges.*

Idea of Proof: Figure 4.4 shows all trees of order 6 or less. In each tree the size m is one less than its order n. The vertices of two trees of order 6 are labeled with their degrees. Suppose that T is a tree of order 7 and v is a leaf of T. The vertex v is adjacent to one vertex in T, say u. Since $T - v$ is a tree of order 6 (necessarily one of those

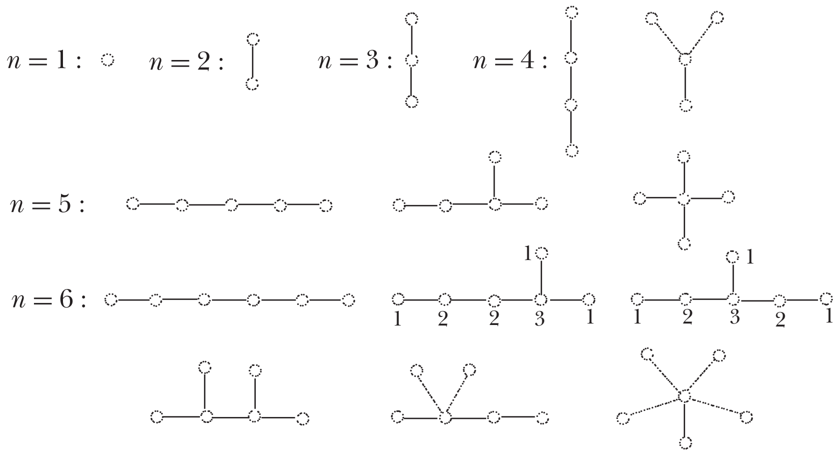


Figure 4.4. The trees with six or fewer vertices

shown in Figure 4.4) and so has five edges, the edges of T consist of these five edges together with the edge uv, or six edges in all.

Repeating this argument shows that every tree with eight vertices has seven edges, every tree with nine vertices has eight edges and so on. ■

From this theorem and the First Theorem of Graph Theory, we have the following.

Corollary 4.4: *If T is a tree of order n and size m whose vertices are $v_1, v_2, \ldots, v_n$, then*

$$\deg v_1 + \deg v_2 + \cdots + \deg v_n = 2m = 2(n-1) = 2n-2.$$

If we know the number of vertices of degree 3 or more in a tree, then the number of leaves can be determined.

Corollary 4.5: *Suppose that T is a tree of order $n \geq 2$ such that Δ is the maximum degree of T and n_i is the number of vertices of degree i for $i = 1, 2, \ldots, \Delta$. Then the number n_1 of leaves in T is*

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta. \tag{4.1}$$

Proof: By the First Theorem of Graph Theory,

$$1n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + \cdots + \Delta n_\Delta = 2n - 2.$$

Since $n = n_1 + n_2 + n_3 + \cdots + n_\Delta$, it follows that

$$1n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + \cdots + \Delta n_\Delta = 2(n_1 + n_2 + \cdots + n_\Delta) - 2. \tag{4.2}$$

Solving equation (4.2) for n_1 gives us the formula in (4.1). ■

Let's now see an example of how Corollary 4.5 can be used to tell us how many leaves a tree contains if we know the number of vertices of every other degree.

Example 4.6: *Suppose that T is a tree with two vertices of degree 3, three vertices of degree 4 and one vertex of degree 6. No other vertices of T have degree 3 or more. How many leaves does T have?*

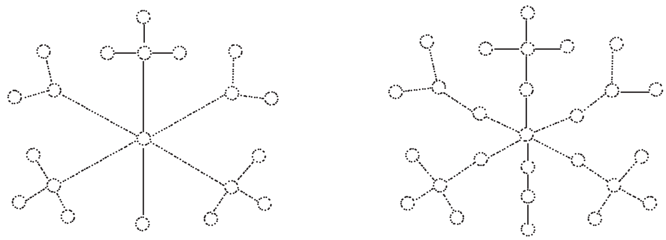
SOLUTION:

Let n_i be the number of vertices of degree i in T, where $1 \leq i \leq 6$. According to Corollary 4.5, the number n_1 of leaves in T is

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + 4n_6 = 2 + 2 + 2 \cdot 3 + 3 \cdot 0 + 4 \cdot 1 = 14.$$

Although there is no way to determine precisely which tree T we may be referring to, one possibility for T is shown in Figure 4.5a. You might notice that the formula (4.1) for the number n_1 of leaves in T does not involve the number n_2 of vertices of degree 2. In fact, the number of leaves in a tree is not affected by the number of vertices of degree 2 in a tree. While the tree T in Figure 4.5a has no vertex of degree 2, the tree in Figure 4.5b has seven vertices of degree 2 and satisfies all degree conditions stated in Example 4.6. ♦

We saw that Figure 4.4 shows all of the different (nonisomorphic) trees of order 6 or less. The two trees in this figure having six vertices that include one vertex of degree 3, two vertices of degree 2 and three leaves, are not isomorphic because, for example, the vertices of degree 2 are adjacent in one tree but not in the other. This illustrates the fact that



(a) (b)

Figure 4.5. Trees satisfying the degree conditions of Example 4.6.

even for trees, knowing the exact number of vertices of each degree does not necessarily tell us precisely which graph we might be referring to.

According to Theorem 4.2 and Corollary 4.4, if T is a tree of order $n \geq 2$ and $d_1, d_2, \ldots, d_n$ are the degrees of its vertices, then $\sum_{i=1}^{n} d_i = 2n - 2$ and at least two of the degrees $d_1, d_2, \ldots, d_n$ are 1. How about the converse? For example, if you are given six positive integers that sum to 10, must there be a tree T that has these six integers as the degrees of its vertices? The number 10 can be written as the sum of six positive integers (say from largest to smallest) in five ways:

$$10 = 2+2+2+2+1+1, \qquad 10 = 3+2+2+1+1+1,$$

$$10 = 3+3+1+1+1+1, \qquad 10 = 4+2+1+1+1+1,$$

$$10 = 5+1+1+1+1+1.$$

For each sum, there is a tree of order 6 (in Figure 4.4) having these degrees for its vertices. In fact, the first sum listed is the first tree of order 6 listed in Figure 4.4.

Theorem 4.7: *If $d_1, d_2, \ldots, d_n$ are $n \geq 2$ positive integers whose sum is $2n-2$, then these integers are the degrees of the vertices of some tree T of order n.*

Idea of Proof: We have already seen that the theorem is true for $n = 6$. How can we use this fact to show that the theorem is true for $n = 7$? Let $d_1, d_2, \ldots, d_7$ be seven positive integers whose sum is $2n-2=2(7)-2=12$, that is, $d_1+d_2+\cdots+d_7=12$. At least one of these numbers, say d_1, is 2 or more. On the other hand, if all seven of these numbers were 2 or more, then the sum would be at least 14; so at least one number is 1, say $d_7=1$. Now, ignoring d_7 and decreasing d_1 by 1, we arrive at six positive integers that sum to 10, namely $(d_1-1)+d_2+\cdots+d_6=10$. But since we know the theorem is true for $n=6$, there is a tree T' of order 6 with six vertices $v_1, v_2, \ldots, v_6$ with respective degrees $d_1-1, d_2, \ldots, d_6$. Now appending the vertex v_7 to T' and joining it to v_1 produces a tree T whose seven vertices have the desired property.

For example, consider the seven numbers 3, 3, 2, 1, 1, 1, 1 which sum to 12. We create a tree T with seven vertices whose degrees are these seven numbers. Reducing the largest number by 1 and deleting the least number gives us the six integers 2, 3, 2, 1, 1, 1, which sum to 10. Locating a tree in Figure 4.4 having $n=6$ whose vertices have these degrees (there are two of them) and attaching a leaf to any vertex of degree 2 produces a desired tree T.

Now repeating this process for $n=8$, $n=9$ and so on establishes the theorem. ■

Example 4.8: *The sum of the 17 integers* 5, 4, 3, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 *is* 32. *Since* $32=2\cdot 17-2$, *it follows by Theorem 4.7 that there is a tree of order* 17, *the degrees of whose vertices are these* 17 *integers. An example of such a tree T is given in Figure 4.6a.*

One must be careful not to read too much into Theorem 4.7. Not every graph whose vertices have the degrees mentioned in Example 4.8 must be a tree. For example, the vertices of the graph G in Figure 4.6b have the same degrees as the vertices of T but G is not a tree.

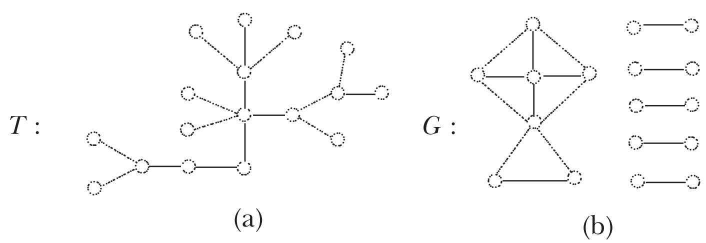


Figure 4.6. Two graphs of the same order whose vertices have the same degrees.

TREES AND SATURATED HYDROCARBONS

The British mathematician Arthur Cayley (1821–1895) was one of the great mathematicians of the nineteenth century. In fact, Cayley wrote 967 research papers and is one of the most prolific mathematicians of all time. Paul Erdős wrote more than 1525 papers, Euler 886 and Augustin-Louis Cauchy 789. As we saw in Chapter 3, many papers of Erdős were coauthored. The papers of Cayley and Euler are noted for their volume. While Cayley was best known for his work in modern algebra, he also spent several years as a lawyer. Although Cayley's connections with graph theory were few in number, they were nevertheless important.

In 1857, while trying to solve a problem in differential calculus, Cayley was led to the problem of counting all *rooted trees*—trees in which a particular vertex has been designated as the *root*. In 1875 Cayley would encounter another tree-counting problem when he studied the counting of special kinds of molecules called saturated hydrocarbons (also referred to as alkanes or paraffins). These molecules consist of carbon atoms and hydrogen atoms. In a saturated hydrocarbon, every carbon atom has valency 4 and every hydrogen atom has valency 1. The chemical formula for each such molecule is C_nH_{2n+2}, indicating that each molecule has n carbon atoms and $2n+2$ hydrogen atoms for some positive integer n. For $n = 1, 2, 3$, these molecules are called methane, ethane and propane, respectively. The diagrams for these three molecules, as shown in Figure 4.7, are actually trees, where each "carbon vertex" has degree 4 and each "hydrogen vertex" is a leaf. Not only is the valency of an atom in

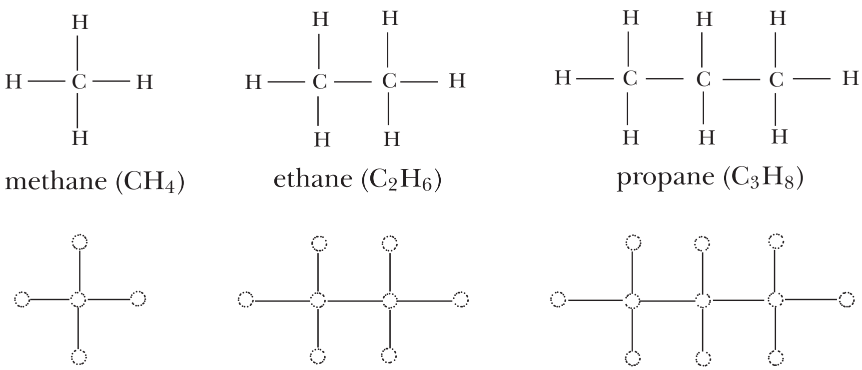


Figure 4.7. The molecules C_nH_{2n+2} for $n = 1, 2, 3$.

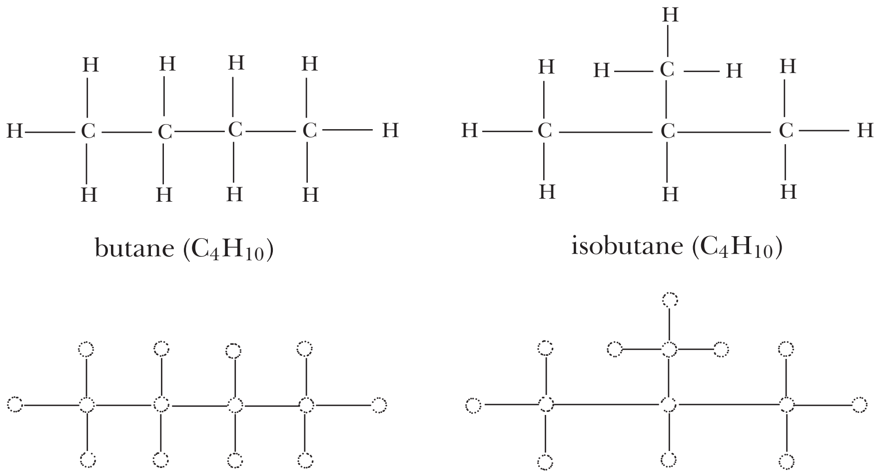


Figure 4.8. The molecules C_4H_{10}.

a molecule the same as the degree of the vertex in the tree representing the molecule, there were times when some mathematicians referred to the degree of a vertex in a graph as the valency of that vertex.

The saturated hydrocarbons butane and isobutane (see Figure 4.8) have the same formula, C_4H_{10}, indicating that each molecule has four carbon atoms and ten hydrogen atoms. Nevertheless, they have different properties. These are examples of what are called isomers, which are molecules with different properties having the same formula.

While there are only two isomers whose formula is C_4H_{10}, determining the number i_n of isomers having the formula C_nH_{2n+2} for $n \geq 5$ is considerably more complicated. Cayley was able to compute the numbers i_n correctly for $1 \leq n \leq 11$:

n	1	2	3	4	5	6	7	8	9	10	11
i_n	1	1	1	2	3	5	9	18	35	75	159

Counting the number of saturated hydrocarbons is the same as counting the number of certain kinds of nonisomorphic trees. If the hydrogen atoms are removed from the tree diagram of a saturated hydrocarbon, another tree is obtained—a so-called carbon tree, in which all vertices have degree 4 or less. So the number of saturated hydrocarbons with n carbon atoms equals the number of trees with n vertices where the degree of each vertex is 4 or less. From the trees drawn in Figure 4.4, we see that the number t_n of such trees with n vertices is that given below.

n	1	2	3	4	5	6
t_n	1	1	1	2	3	5

This, of course, agrees with the first portion of the table for i_n discovered by Cayley.

CAYLEY'S TREE FORMULA

In 1889 Cayley turned his attention to yet another counting problem, one that involved counting labeled trees. Suppose that T' and T'' are two trees of order n, each of whose vertices are labeled with the positive integers $1, 2, \ldots, n$. Then each tree has $n-1$ edges. The labeled trees T' and T'' are considered different if they don't contain the same edges. For example, the vertices of the three trees T_1, T_2 and T_3 of order 3 in Figure 4.9 are labeled with 1, 2, 3. While these three trees are isomorphic, as labeled trees they are all different. For example, T_1 is the only tree not containing the edge 12.

Figure 4.10 shows the 16 labeled trees with four vertices. Let's see why there are 16 such trees. First, there are only two nonisomorphic trees with four vertices, namely the "path graph" and the "star graph". There are exactly four ways to label the vertices of the star graph with the numbers $1, 2, 3, 4$ as the labeling is completely determined by the label given to the central vertex. There are $4 \cdot 3 \cdot 2 \cdot 1 = 4!$ ways to label the vertices of the path graph from one end of the path to the other but each labeling is equivalent to the reverse labeling starting from the other end of the path, so there are only $4!/2 = 12$ distinct ways to label the vertices of the path graph. Consequently, there are $4 + 12 = 16$ ways to label all trees of order 4.

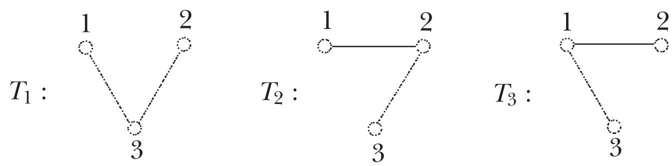


Figure 4.9. The three different trees of order 3.

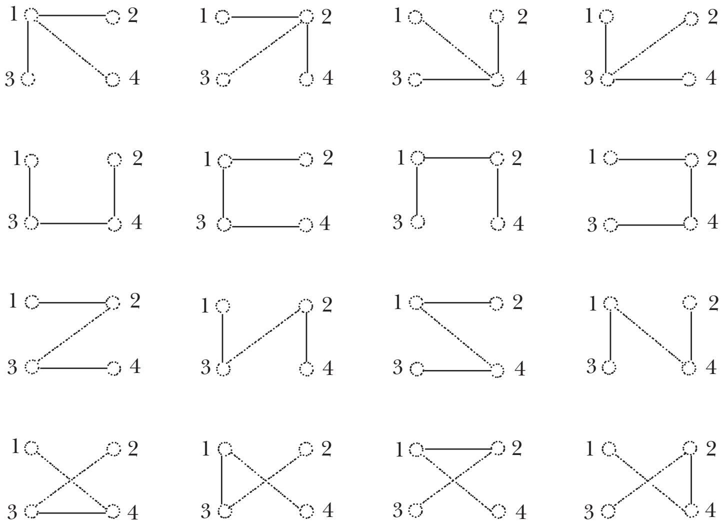


Figure 4.10. Labeled trees with four vertices.

There are three nonisomorphic trees with five vertices, namely the star graph, the path graph and the "broomstick graph". As above, there are five ways to label the star graph and $5!/2 = 60$ ways to label the path graph. For the broomstick graph, there are five choices for the vertex of degree 3 and four choices for the vertex of degree 2. Since there are three choices for the leaf adjacent to the vertex of degree 2, there are $5 \cdot 4 \cdot 3 = 60$ ways to label this graph in all, or $5 + 60 + 60 = 125$ labeled trees of order 5.

It is an exercise to show that there are 1296 ways to label the trees of order 6. Observing that $3 = 3^1$, $16 = 4^2$, $125 = 5^3$ and $1296 = 6^4$, it is impossible not to think that the number of ways to label the trees of order n is n^{n-2}, as Cayley thought. While Cayley was the one who stated the general formula for the number of labeled trees of a given order and, using the same set of labels, illustrated it for trees of order 6, he did not give a complete proof.

Cayley's Tree Formula

There are n^{n-2} different labeled trees whose vertices are labeled with the same set of n labels.

Figure 4.11. Two trees whose vertices have degrees 3, 2, 2, 1, 1, 1.

There is another result concerning labeled trees—this one dealing with the degrees of the vertices. According to Theorem 4.7, there is a tree of order 6 whose vertices have degrees 3, 2, 2, 1, 1, 1. In fact, there are two nonisomorphic trees whose vertices have degrees 3, 2, 2, 1, 1, 1, both shown in Figure 4.11.

A related result is the following.

Theorem 4.9: *If $d_1, d_2, \ldots, d_n$ are n positive integers whose sum is $2n - 2$, then the number of labeled trees of order n where vertex 1 has degree d_1, vertex 2 has degree d_2 and so on is*

$$\frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!}.$$

For example, the degree sequence

$$5, 4, 3, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1$$

in Example 4.8 can be represented by

$$\frac{(17-2)!}{(5-1)!(4-1)!(3-1)!(3-1)!(3-1)!} = 1{,}135{,}134{,}000$$

different labeled trees, in which the vertex of degree 5 is labeled 1, the vertex of degree 4 is labeled 2, the three vertices of degree 3 are labeled 3, 4, 5, the two vertices of degree 2 are labeled 6 and 7 and the ten leaves are labeled with $8, 9, \ldots, 17$. The degree sequence 3, 2, 2, 1, 1, 1 for the two trees in Figure 4.11 can therefore be represented by

$$\frac{(6-2)!}{(3-1)!(2-1)!(2-1)!(1-1)!(1-1)!(1-1)!} = \frac{24}{2} = 12$$

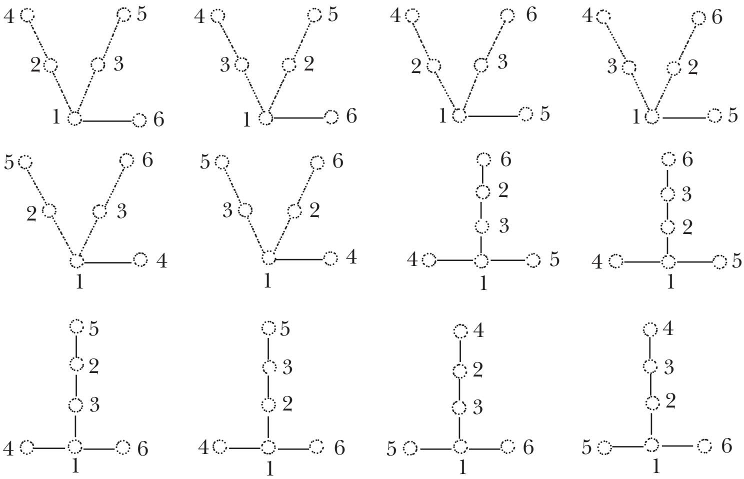


Figure 4.12. The 12 labeled trees with degree sequence $3, 2, 2, 1, 1, 1$, whose vertex of degree 3 is labeled 1, whose vertices of degree 2 are labeled $2, 3$ and whose leaves are labeled $4, 5, 6$.

labeled trees, where the vertex of degree 3 is labeled 1, the two vertices of degree 2 are labeled 2 and 3 and the leaves are labeled 4, 5, 6. These 12 trees are shown in Figure 4.12.

PRÜFER CODES

The first complete proof of Cayley's Tree Formula was given in 1918 by the German mathematician Heinz Prüfer (1896–1934). Although the verification of Prüfer's technique is a bit complicated to explain, the technique itself is not difficult to describe.

Consider the tree T of order 9 shown in Figure 4.13, whose vertices are labeled with $1, 2, \ldots, 9$. Associated with the tree T is a sequence of length 7 (2 less than the order of T), each of whose terms is one of the labels $1, 2, \ldots, 9$. This sequence is called the *Prüfer sequence* or the *Prüfer code* of the tree.

We now describe how to compute the Prüfer code of the tree T of Figure 4.13. First, let $T_0 = T$. In T_0, the five leaves have the labels 3, 4, 5, 6 and 8. The smallest label of a leaf of T_0 is therefore 3. The neighbor of this vertex is labeled 9. This is the first term of the Prüfer code of T.

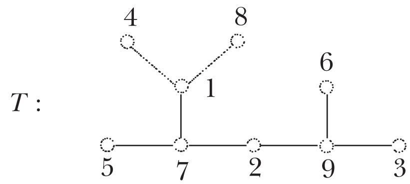


Figure 4.13. A labeled tree of order 9.

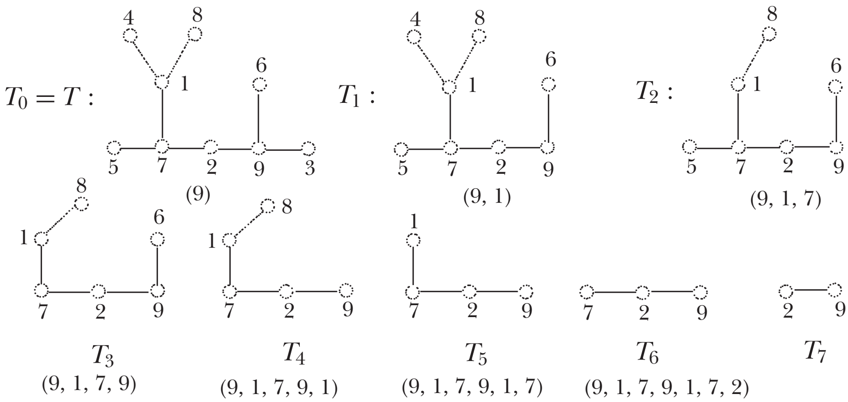


Figure 4.14. Constructing the Prüfer code for a tree.

This leaf is then deleted from T_0, producing a tree T_1 of order 8. The neighbor of the leaf of T_1 having the smallest label is the second term of the Prüfer code for T. Here T_1 has leaves with labels 4, 5, 6, 8 and the neighbor of leaf 4 is vertex 1. Thus the second term is 1. This is continued until we arrive at $T_7 = K_2$, resulting in the Prüfer code (9, 1, 7, 9, 1, 7, 2) (see Figure 4.14). In general, the Prüfer code of a tree of order n (whose vertices are labeled with $1, 2, \ldots, n$) is a sequence of length $n-2$, each of whose terms is an element of $\{1, 2, \ldots, n\}$.

We now consider the converse question. That is, for some integer $n \geq 3$, suppose that we have a sequence of length $n-2$, each of whose terms is an element of the set $\{1, 2, \ldots, n\}$. Which labeled tree T of order n has this sequence as its Prüfer code? For example, suppose that $n = 9$ and that (9, 1, 7, 9, 1, 7, 2) is the given sequence of length 7. The smallest element of $\{1, 2, \ldots, 9\}$ not in this sequence is 3. We join vertex 3 to vertex 9 (the first element of the sequence). The first term of the sequence is then deleted, producing the reduced sequence (1, 7, 9, 1, 7, 2). The element 3 is removed from the set $\{1, 2, \ldots, 9\}$,

Sequence	Set
(9, 1, 7, 9, 1, 7, 2)	{1, 2, 3, 4, 5, 6, 7, 8, 9}
(1, 7, 9, 1, 7, 2)	{1, 2, 4, 5, 6, 7, 8, 9}
(7, 9, 1, 7, 2)	{1, 2, 5, 6, 7, 8, 9}
(9, 1, 7, 2)	{1, 2, 6, 7, 8, 9}
(1, 7, 2)	{1, 2, 7, 8, 9}
(7, 2)	{1, 2, 7, 9}
(2)	{2, 7, 9}
	{2, 9}

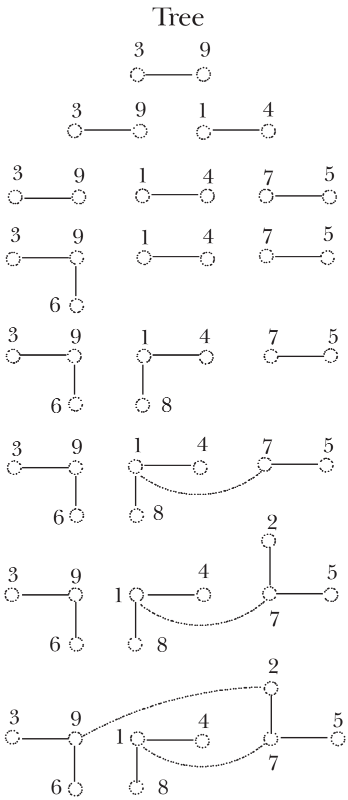
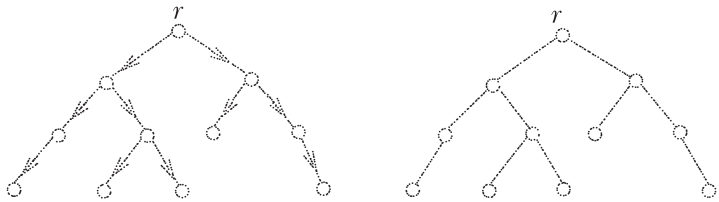


Figure 4.15. Constructing a labeled tree from a sequence.

giving us the set $\{1, 2, 4, 5, 6, 7, 8, 9\}$. The smallest element of this set not in the sequence is 4, which is joined to vertex 1 (the first element of the reduced sequence). We continue this procedure until no sequence remains and the final two elements of the set are 2 and 9. These two vertices are joined, producing the tree T of Figure 4.13. This procedure is illustrated in Figure 4.15. As we saw, the Prüfer code of this tree is (9, 1, 7, 9, 1, 7, 2).

What Heinz Prüfer was able to prove is that each labeled tree has a Prüfer code that no other labeled tree has and that every Prüfer code corresponds to a unique labeled tree. Since a Prüfer code has the appearance $(a_1, a_2, \ldots, a_{n-2})$ and each term is one of the elements $1, 2, \ldots, n$, the total number of Prüfer codes is obtained by computing the product $n \cdot n \cdot \ldots \cdot n$ (a total of $n - 2$ terms), arriving at n^{n-2}. That is, there



(a) (b)

Figure 4.16. A rooted binary tree.

are n^{n-2} different labeled trees of order n, where the trees are labeled with the same set of n elements.

DECISION TREES

We saw that a rooted tree is a tree T in which some vertex r is designated as the root. Typically, each edge of T is directed away from r and so T becomes a directed tree. It is customary to draw T with r at the top and all arcs of T directed downward. In a directed rooted tree, every vertex v different from the root has exactly one arc incident with v that is directed into v. If every vertex u of a directed rooted tree T has at most two arcs directed away from u, then T is called a *binary tree*, such as the rooted tree in Figure 4.16a. Since it is understood that all arcs of a rooted tree are directed downward, rooted trees are often drawn without arrowheads. Thus the binary tree in Figure 4.16a can also be drawn as in Figure 4.16b.

Rooted trees can be used to represent a variety of situations. They can be especially useful when encountering problems in which solutions can be found by means of successive comparisons where each vertex v of the tree corresponds to a decision which when made directs us via an arc (v, u) to another decision to be made at the vertex u. We continue doing this until we reach a leaf, indicating that a solution to the problem has been found. For this reason, such a rooted tree is called a *decision tree*. There is one type of problem for which a procedure for finding a solution can be conveniently described by means of a decision tree.

Suppose that we have a collection of coins, all of which look the same and all but one of which are authentic coins. The remaining coin is a fake coin, however, and weighs slightly less than each of the authentic coins. What we have available is a balance scale (see Figure 4.17) consisting of

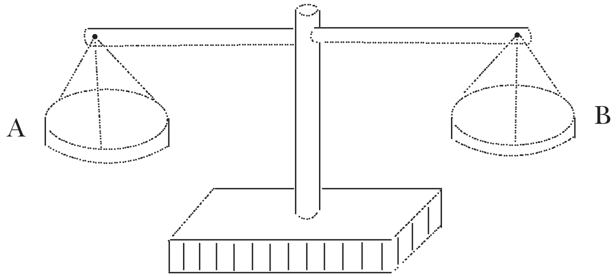


Figure 4.17. A balance scale.

balance pans A and B. By placing coins on the two balance pans, it is then possible to determine whether the coins on pan A weigh less than, more than or the same as the coins on pan B. The goal is to use this scale but nothing else to determine which coin is fake and to accomplish this with the minimum number of weighings.

> Example 4.10: *There are four coins numbered* 1, 2, 3, 4, *three of which are authentic and one is fake. The fake coin weighs slightly less than each of the authentic coins. What is the minimum number of weighings needed to determine which coin is the fake coin and how can the fake coin be found?*

SOLUTION:

First, it is impossible to determine which coin is the fake coin by means of a single weighing. Of course, it is always possible to stumble upon a solution with one weighing. For example, if we happen to place coin 1 on A and coin 2 on B (indicated by writing 1, 2) and coin 1 happens to be the fake coin, then we can see that the contents of pan A weigh less than the contents of pan B and the problem is solved. If, however, either coin 3 or 4 is the fake coin, then we don't know this from the weighing.

Suppose, however, that coins 1 and 2 are placed on pan A and 3 and 4 are placed on pan B (see Figure 4.18). There is no possibility that these pans will balance. One of A and B must be lighter than the other and therefore contains the fake coin. If the contents of pan A are lighter than those of pan B, then either 1 or 2 is the fake coin; otherwise, either 3 or 4 is the fake coin. Determining which coin is the fake one can be learned by means of one more weighing (again, see Figure 4.18). ♦

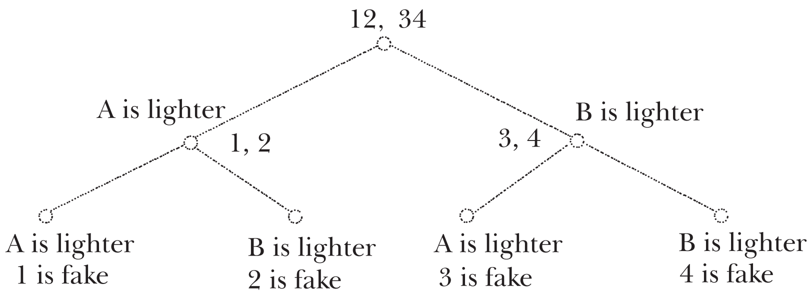


Figure 4.18. Identifying the fake coin in Example 4.10 by means of a decision tree.

The approach described in Example 4.10 to determine the fake coin is not the only one in which a decision tree can be used to answer this question. (Indeed, we could begin by comparing coins 1 and 2. If the pans balance, then we compare coins 3 and 4. While this procedure also always solves the problem within two weighings, it may solve it with a single weighing.)

Another question we might ask is, what happens if we have six coins rather than four?

Example 4.11: *Suppose that six coins are numbered* $1, 2, \ldots, 6$*, five of which are authentic and one is fake. The fake coin weighs slightly less than each authentic coin. What is the minimum number of weighings needed to determine which coin is the fake?*

SOLUTION:

Even with two more coins, the fake coin can be discovered in only two weighings. A decision tree showing how to do this is given in Figure 4.19. ♦

THE MINIMUM SPANNING TREE PROBLEM

At the beginning of this chapter we were introduced to a situation concerning eight villages $v_1, v_2, \ldots, v_8$ (see Figure 4.20a) and the problem of constructing paved roads between certain pairs of villages so that it becomes possible to travel by paved roads between every two villages, and

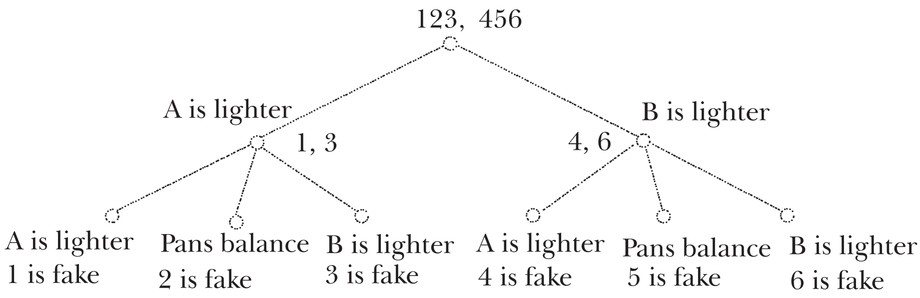


Figure 4.19. Identifying the fake coin in Example 4.11 by means of a decision tree.

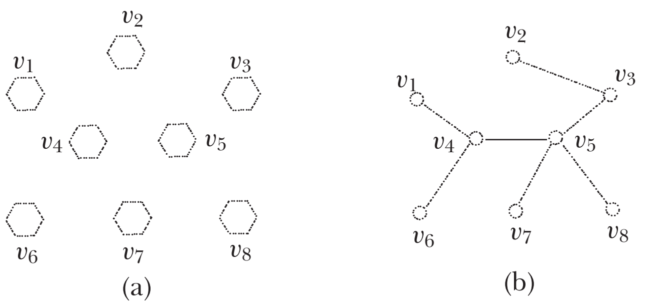


Figure 4.20. A tree representing a possible road system in a region.

accomplishing this as economically as possible. We saw that a solution to this problem could be represented by a tree of order 8. This tree is repeated in Figure 4.20b. In fact, every spanning subgraph T of a connected graph H where T is a tree is called a *spanning tree* of H.

One disadvantage with the road system constructed in Figure 4.20b is that if a person were to travel by paved roads between villages v_1 and v_2, for example, then it would be necessary to pass through (and likely stop at) villages v_4, v_5 and v_3. This would undoubtedly make for a longer, more time-consuming trip between v_1 and v_2. However, constructing a paved road directly between v_1 and v_2 would be more costly. Building a road system of minimum cost connecting these eight villages would necessarily mean constructing a tree with vertices $v_1, v_2, \ldots, v_8$. But how do we know that the road system constructed in Figure 4.20b is the least expensive? This brings us to a well-known problem in graph theory.

Recall that a weighted graph is a graph in which each edge e is assigned a number (usually a positive number) called the weight of the edge and denoted by $w(e)$. If G is a weighted graph and H is a subgraph of G, then the *weight* $w(H)$ of H is the sum of the weights of the edges of H. What we're interested in then is determining, for a connected weighted graph G, a connected spanning subgraph H of G having minimum weight. Necessarily, such a subgraph would have to be a tree. Such a tree is called a *minimum spanning tree* of G and the problem of finding such a tree is called the *Minimum Spanning Tree Problem*.

The Minimum Spanning Tree Problem goes back to 1926 when Otakar Borůvka was seeking the most economical layout of a power-line network. While he provided the first method of solving this problem, the best known solution occurred in 1956 and is due to Joseph Bernard Kruskal (1928–2010).

Kruskal's Algorithm

Let G be a connected weighted graph of order n. A minimum spanning tree T of G can be constructed as follows: Select any edge e_1 of G of minimum weight for T followed by any edge e_2 of $G - e_1$ of minimum weight. Once k edges $e_1, e_2, \ldots, e_k$ $(2 \le k \le n-2)$ have been selected for T, select any other edge e_{k+1} of minimum weight such that no cycle is formed from the edges $e_1, e_2, \ldots, e_{k+1}$.

This method, known as a *greedy algorithm*, should sound familiar for this was the method we used at the beginning of the chapter. This method always produces a minimum spanning tree. The proof that follows exploits the fact that inserting an edge into a tree creates a unique cycle.

Theorem 4.12: *Kruskal's algorithm produces a minimum spanning tree in every connected weighted graph.*

Idea of Proof: We prove the special case when all weights are distinct although the result is true even when some weights may be repeated.

Suppose, to the contrary, that there is some connected weighted graph G of order $n \ge 2$ such that the spanning tree T created by Kruskal's algorithm is not a minimum spanning tree of G. Let $e_1, e_2, \ldots, e_{n-1}$ be the $n-1$ edges of T, where the edges have been selected in that order. Therefore, $w(e_1) < w(e_2) < \cdots < w(e_{n-1})$ and

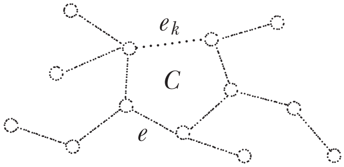


Figure 4.21. What $G^* = T^* + e_k$ might look like.

the weight of T is

$$w(T) = w(e_1) + w(e_2) + \cdots + w(e_{n-1}).$$

Since T is not a minimum spanning tree of G, there must be a minimum spanning tree T^* whose weight is smaller than that of T. Necessarily, T contains one or more edges that do not belong to T^*. Let e_k be the first edge on the list $e_1, e_2, \ldots, e_{n-1}$ not belonging to T^*. (This means that both T and T^* contain the edges $e_1, e_2, \ldots, e_{k-1}$.) Let $G^* = T^* + e_k$ be the graph obtained by adding the edge e_k to T^*. (See Figure 4.21.) As we observed earlier, the graph G^* has a single cycle C, necessarily containing e_k.

Since T is a tree and contains no cycles, there must be some edge e on C that does not belong to T. What can we say about the weight of this edge e? Well, after choosing the edges e_1 through e_{k-1}, Kruskal's algorithm could have chosen e (since the edges $e_1, e_2, \ldots, e_{k-1}$ and e are all in the tree T^*) but it chose e_k instead. Since the edge weights are distinct, this means that $w(e_k) < w(e)$. Suppose that we now remove the edge e from G^*. This produces a new tree T^{**} whose weight is

$$w(T^{**}) = w(T^*) + w(e_k) - w(e) < w(T^*).$$

This, however, contradicts the assumption that T^* is a minimum spanning tree of G. ■

If the edges in the weighted graph G in Theorem 4.12 had some repeated weights, this situation could be handled in a couple of ways. First, we could make some minor adjustments in repeated weights (like replacing a weight 25 by 25.001) so that Kruskal's algorithm wouldn't be off by any appreciable amount. Alternatively, to avoid a contradiction in the argument above, we must have $w(e) = w(e_k)$. In this case, replacing

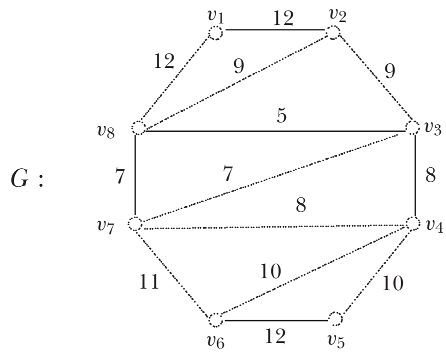


Figure 4.22. The weighted graph G in Example 4.13.

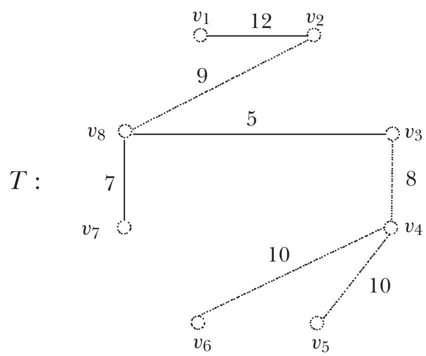


Figure 4.23. Constructing a minimum spanning tree by Kruskal's algorithm.

e with e_k results in T^{**} having the same weight as T^* (and so is also a minimum spanning tree of G) but has more edges in common with T. Eventually, T^* is transformed into T by this method.

We now illustrate how a minimum spanning tree of a connected weighted graph can be found with the aid of Kruskal's algorithm.

Example 4.13: *Determine a minimum spanning tree of the connected weighted graph G shown in Figure 4.22.*

SOLUTION:

We construct a minimum spanning tree T of G by beginning with the only edge v_3v_8 of minimum weight 5. There are two edges of weight 7. Since these two edges form a cycle with v_3v_8, we can select only one of these for T. Since it doesn't matter which of these edges we select, we choose v_7v_8. There are two edges of weight 8. Only one of these can be selected for T, say v_3v_4 is selected. There are two edges of weight 9. Again, only one can be selected for T. We select v_2v_8. There are two edges of weight 10. In this case, both can be selected for T. There is one edge of weight 11. However, it would produce a cycle with those edges already selected. Thus the edge v_6v_7 is not selected. There are three edges of weight 12. The edge v_5v_6 would produce a cycle and thus cannot be selected. Selecting either v_1v_2 or v_1v_8 produces a spanning tree T. The edge v_1v_2 is selected. By Kruskal's algorithm, T is a minimum spanning tree (see Figure 4.23). Also, $w(T) = 61$. ♦

5

Traversing Graphs

To pass the time while attending a business meeting or a lecture, some people like to scribble designs with a pencil on a pad of paper (or on an electronic tablet with a drawing pad app). For example, we might place our pencil on the paper (at a point designated *A* in Figure 5.1a) and, without lifting our pencil from the paper, arrive at the drawings in Figures 5.1a–d until completing the drawing in Figure 5.1e.

This rather innocent activity brings up a question that a curious person might ask.

> Given some pencil drawing, can one determine whether it's possible to construct this drawing by starting at some point, and completing the entire drawing in one continuous motion without lifting one's pencil from the paper? If so, can it be determined where this drawing might have started—and ended?

For the drawing of Figure 5.1e, we know, of course, that the answer is yes. But what about the drawings in Figure 5.2?

For the pencil drawings in Figures 5.2a, b and c, the answers are yes, yes and no, respectively. Explanations for these answers were known many years ago—indeed a few centuries ago. This is the period that marked the beginning of graph theory, a subject of mathematics that is considered to have begun with a person, a place, a problem and a paper. The person here is Leonhard Euler, the famous Swiss mathematician whom we've already encountered.

THE KÖNIGSBERG BRIDGE PROBLEM

The place in this case is the city of Königsberg, founded in 1255 by the Teutonic knights under the leadership of Bohemian King Ottoker II.

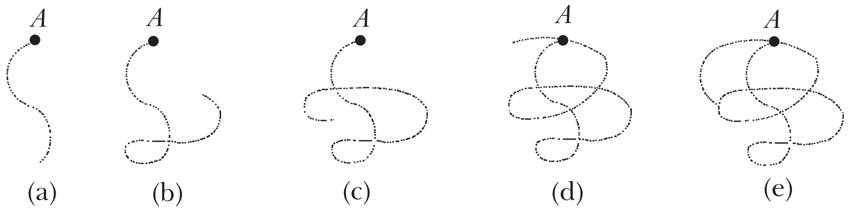


Figure 5.1. A pencil drawing.

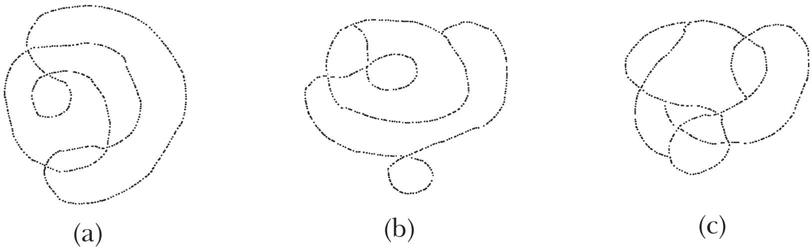


Figure 5.2. Three pencil drawings.

Königsberg was the capital of German East Prussia. During the Middle Ages, Königsberg was an important trading center because of its location on the banks of the River Pregel. Seven bridges were built across the river, five of which were connected to the island of Kneiphof (see Figure 5.3). As the map in Figure 5.3 shows, the river divides Königsberg into four land regions.

The Prussian royal castle was located in Königsberg but it was destroyed during World War II, as was much of the city. After the war, it was decided at the Potsdam Conference in 1945 that a region located between Poland and Lithuania, containing the city of Königsberg, should be made part of Russia. In 1946 Königsberg was renamed Kaliningrad after Mikhail Kalinin, the formal leader of the Soviet Union during 1919–1946. After the fall of the USSR, Lithuania and other former Soviet republics became independent and Kaliningrad was no longer part of Russia. Nevertheless, none of the attempts to change its name back to Königsberg have been successful.

Now—the problem. The story goes that early in the eighteenth century many of the citizens of Königsberg spent their Sunday afternoons strolling about the city. The problem arose as to whether it was possible

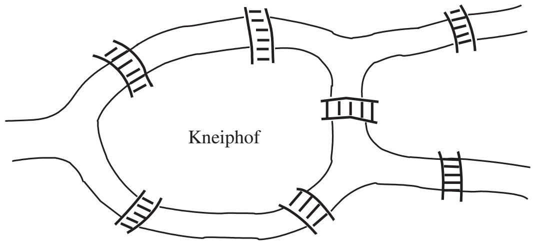


Figure 5.3. A map of the city of Königsberg with its seven bridges crossing the River Pregel.

to walk about the city and cross each of the seven bridges exactly once. This problem became known as the *Königsberg Bridge Problem*.

So what is Euler's connection with the Königsberg Bridge Problem? Actually, it's not entirely clear how Euler became aware of this problem. Carl Leonhard Ehler was the mayor of Danzig, a city in Prussia located some 80 miles west of Königsberg. Danzig is now the city of Gdansk in Poland. Ehler and Euler corresponded during the period 1735–1742. A letter that Ehler wrote to Euler on 9 March 1736 made it clear that they had discussed the Königsberg Bridge Problem. This letter, written on behalf of himself and a local mathematician, Heinrich Kühn, said, in part,

> You would render to me and our friend Kühn a most valuable service, putting us greatly in your debt, most learned Sir, if you would send us the solution, which you know well, to the problem of the seven Königsberg bridges, together with a proof. It would prove to be an outstanding example of the calculus of position worthy of your great genius. I have added a sketch of the said bridges.

As indicated, the letter from Ehler to Euler contained a drawing of where the River Pregel flows through Königsberg and the locations on the river where the seven bridges were situated. The reason for this inquiry was the desire by Kühn and Ehler to encourage mathematical growth within Prussia.

Four days later, on 13 March 1736, Euler wrote a letter to the Italian mathematician Giovanni Marinoni. In this letter, Euler described the problem and, even though he considered the problem rather simple, explained that he had solved the problem and what drew him to the problem. This letter said, in part,

> A problem was posed to me about an island in the city of Königsberg, surrounded by a river spanned by seven bridges, and I was asked whether someone could traverse the separate bridges in a connected walk in such a way that each bridge is crossed only once. I was informed that hitherto no-one had demonstrated the possibility of doing this, or shown that it is impossible. This question is so banal, but seemed to me worthy of attention in that geometry, nor algebra, nor even the art of counting was sufficient to solve it. In view of this, it occurred to me to wonder whether it belonged to the geometry of position, which Leibniz had once so much longed for. And so, after some deliberation, I obtained a simple, yet completely established, rule with whose help one can immediately decide for all examples of this kind, with any number of bridges in any arrangement, whether such a round-trip is possible.

On 3 April 1736, Euler responded to Ehler's letter. This letter from Euler to Ehler said, in part,

> Thus you see, most noble Sir, how this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others. In the meantime, most noble Sir, you have assigned this question to the geometry of position, but I am ignorant as to what this new discipline involves, and so to which types of problem Leibniz and Wolff expected to see expressed in this way.

This letter refers to Leibniz, Wolff and a proof technique called the geometry of position. Christian Wolff was an eminent German philosopher while Gottfried Leibniz was a famous German mathematician and philosopher. In 1670 Leibniz had written a letter to Christiaan Huygens, a Dutch mathematician and astronomer, that read, in part,

> I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful constructions of geometry.

> Consequently, in view of this, I consider that we need yet another kind of analysis, geometric or linear, which deals directly with position, as algebra deals with magnitude.

Much later, in 1833, the brilliant German mathematician Carl Friedrich Gauss reflected on the "geometry of position" when he wrote,

> Of the geometry of position, which Leibniz initiated and to which only two geometers, Euler and Vandermonde, have given a feeble glance, we know and possess, after a century and half, very little more than nothing.

The "feeble glance" which Euler gave to the so-called geometry of position consisted of a single paper.

EULER'S KÖNIGSBERG PAPER

The first solution of the Königsberg Bridge Problem, showing that it was impossible to stroll through the city of Königsberg and cross each bridge exactly once, was presented by Euler to the members of the Petersburg Academy on 26 August 1735. The following year (in 1736) Euler wrote a paper giving his solution to the Königsberg Bridge Problem. This paper, titled "Solutio problematis ad geometriam situs pertinentis" ("The Solution to a Problem Relating to the Geometry of Position") was published in the proceedings of the Petersburg Academy (the *Commentarii*).

In his paper (written in Latin), consisting of 21 paragraphs, Euler does not begin by stating the problem he is about to solve but rather explains in the first paragraph that he has been introduced to a problem whose solution uses the geometry of position to which Leibniz had referred:

> 1. In addition to that branch of geometry which is concerned with magnitudes, and which has always received the greatest attention, there is another branch, previously almost unknown, which Leibniz first mentioned, calling it the geometry of position. This branch is concerned only with the determination of position and its properties; it does not involve measurements, nor calculations made with them. It has not yet been satisfactorily determined what kinds of problems are relevant to

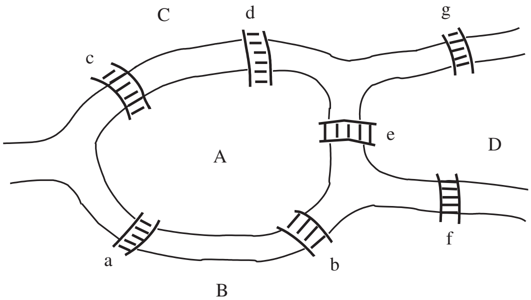


Figure 5.4. A map of the city of Königsberg with its seven bridges crossing the River Pregel.

> this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned, which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned with the geometry of position—especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this kind of problem, as an example of the geometry of position.

In the second paragraph of his paper, Euler states the problem and goes on to say that he has described a more general problem:

> 2. The problem, which I am told is widely known, is as follows: in Königsberg in Prussia, there is an island A, called the Kneiphof; the river which surrounds it is divided into two branches as can be seen in [Figure 5.4] and these branches are crossed by seven bridges, a, b, c, d, e, f and g. Concerning these bridges, it was asked whether anyone could arrange a route in such a way that he would cross each bridge once and only once. I was told that some people asserted that this was impossible, while others were in doubt: but no one would actually assert that it could be done. From this, I have formulated the general problem: whatever be the arrangement and division of the river into branches, and however many bridges there be, can one find out whether or not it is possible to cross each bridge exactly once?

Euler then goes on to say what it would mean if it were possible to cross each of the seven Königsberg bridges exactly once. In this case, a walk through Königsberg could be represented as a sequence, each term of which is one of the letters A, B, C, D. Two consecutive letters in the sequence would indicate that at some point in the walk, the traveler had reached the land area of the first letter and had then crossed a bridge that led him or her to the land area represented by the second letter. Since there are seven bridges, the sequence must consist of eight terms.

Because there are five bridges leading into (or out of) land area A (the island Kneiphof), each occurrence of the letter A must indicate that either the walk started at A, ended at A or had progressed to and then exited A. Necessarily then, A must appear three times in the sequence. Because three bridges enter or exit each of B, C and D, these three letters must appear twice each in the sequence. This, however, implies that the sequence must contain nine terms, not eight, which produces a contradiction. Therefore, there can be no walk about Königsberg that crosses each bridge exactly once, which solves the Königsberg Bridge Problem.

EULERIAN GRAPHS

In the second paragraph of his paper, Euler mentioned that he had "formulated the general problem". In order to describe Euler's general problem and his solution to it, it is now convenient to bring graph theory into the picture. Before doing this, however, we note that the term "graph" never appears in Euler's paper. Indeed, evidently the term "graph" (as used in this manner) was first used in print in 1878 by the British mathematician James Joseph Sylvester. The reasoning that Euler used to solve not only the Königsberg Bridge Problem but the more general problem he described, was essentially that of a problem involving graphs whose vertices are the land regions and whose edges are the bridges. In fact, the map of Königsberg shown in Figure 5.3 can be represented by the graph shown in Figure 5.5 (actually a multigraph since some pairs of vertices are joined by more than one edge). In terms of graphs, what Euler showed is that there is no way to proceed about the multigraph of Figure 5.5 in a manner that uses each edge exactly once.

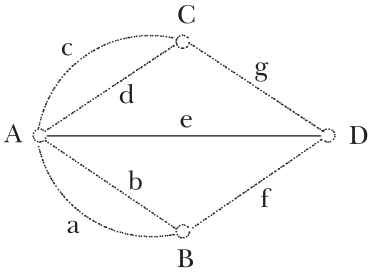


Figure 5.5. The multigraph of Königsberg.

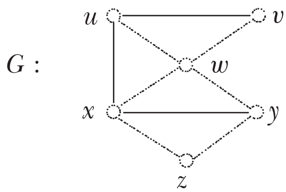


Figure 5.6. Circuits and trails in a graph.

In order to state Euler's general result in a more precise manner, it is necessary to introduce some additional terminology. Let G be a nontrivial connected graph. By a *circuit* C in a graph G, we mean a sequence $C = (u_1, u_2, \ldots, u_k, u_{k+1} = u_1)$ of vertices of G such that $u_i u_{i+1}$ is an edge of G for each i $(1 \leq i \leq k)$ but no edge in C is repeated. Vertices can be repeated in a circuit, however. For example, in the graph G in Figure 5.6, $C = (u, v, w, y, x, w, u)$ is a circuit. While every cycle in a graph is a circuit, a circuit need not be a cycle. Indeed, the circuit C just described is not a cycle as the vertex w is encountered more than once in C.

Another concept in graph theory that describes a manner in which one can proceed about a graph has a western flair to it. By a *trail* in a graph G, we mean a sequence $T = (w_1, w_2, \ldots, w_k)$ of vertices of G such that $w_i w_{i+1}$ is an edge of G for each integer i with $1 \leq i \leq k-1$, but no edge in T is repeated. The trail T is *open* if $w_1 \neq w_k$ and *closed* if $w_1 = w_k$. A closed trail is therefore a circuit. The trail (u, x, z, y, x) in the graph G in Figure 5.6 is therefore an open $u-x$ trail.

A circuit in a connected graph G that contains every edge of G is called an *Eulerian circuit* (named for Euler, of course), while an open trail that contains every edge of G is an *Eulerian trail*. (Some refer to an Eulerian circuit as an *Euler tour*.) These terms are defined in exactly the same way if G is a connected multigraph. As we mentioned, the map of Königsberg in Figure 5.3 can be represented by the multigraph shown in Figure 5.5. Then the Königsberg Bridge Problem can be reformulated:

> Does the multigraph shown in Figure 5.5 contain either an Eulerian circuit or an Eulerian trail?

As Euler showed (although not using this terminology, of course), the answer to this question is no.

A connected graph G is itself called *Eulerian* if G contains an Eulerian circuit. The following theorem determining exactly which graphs are Eulerian graphs is attributed to Euler.

Theorem 5.1: *A connected graph (or multigraph) G is Eulerian if and only if every vertex of G has even degree.*

Proof: It is actually rather easy to verify one direction of Theorem 5.1, namely to show that if G is an Eulerian graph, then every vertex of G has even degree. If G is an Eulerian graph, then G has an Eulerian circuit C. Suppose that C begins at vertex v and thus ends at v as well. If u is a vertex of G that is different from v, then each time that C enters u it exits u. Thus each occurrence of u on C results in a contribution of 2 to the degree of u. This implies that u has even degree. As far as v is concerned, the circuit C begins by leaving v, contributing 1 to its degree. Since C ends at v, this also contributes 1 to its degree. If C encounters v elsewhere on C, then each such encounter contributes 2 to its degree and so v as well has even degree.

It remains to show that if G is a connected graph all of whose vertices are even, then G contains an Eulerian circuit. Let v_1 be a vertex of G and let v_1v_2 be an edge of G. We begin a trail T starting at v_1 and followed by v_2. At v_2, we select an edge incident with v_2 that does not lie on T. Let v_2v_3 be such an edge. We continue to extend this trail until we reach a vertex x where every edge incident with x belongs to T. Since every vertex of G is even, the vertex x must be v_1 and T is necessarily a circuit, which we now denote by C.

If every edge of G belongs to C, then C is an Eulerian circuit. If this is not the case, then C contains only some of the edges of G.

Let H be the subgraph of G obtained by removing the edges of C from G. Since every vertex of G is even and every vertex of C is even, every vertex of H is even. Let H' be a nontrivial component of H. Because G is connected, there must a vertex u on C that belongs to H'. As above, we construct a trail T' in H' starting with u until we reach a vertex x' where all edges incident with x' belong to T'. As before, T' is a circuit, say C'. As we proceed about C, we insert C' when reaching u, which produces a circuit C'' containing the edges of both C and C'. If C'' is an Eulerian circuit, then the proof is complete; otherwise, we proceed as above until an Eulerian circuit is obtained. ■

With the aid of Theorem 5.1, all connected graphs possessing an Eulerian trail can also be determined.

Corollary 5.2: *A connected graph (or multigraph) G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at the other.*

In paragraph 20 (the next-to-last paragraph) of Euler's paper, Euler actually wrote (again an English translation),

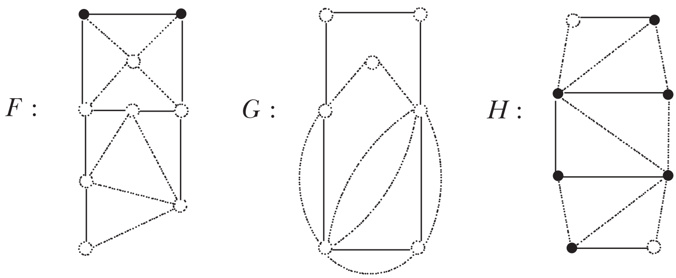
> 20. So whatever arrangement may be proposed, one can easily determine whether or not a journey can be made, crossing each bridge once, by the following rules:
>
> > If there are more than two areas to which an odd number of bridges lead, then such a journey is impossible.
> >
> > If, however, the number of bridges is odd for exactly two areas, then the journey is possible if it starts in either of these areas.
> >
> > If, finally, there are no areas to which an odd number of bridges lead, then the required journey can be accomplished from any starting point.
>
> With these rules, the given problem can also be solved.

Euler ended his paper by writing,

21. When it has been determined that such a journey can be made, one still has to find how it should be arranged. For this I use the following rule: let those pairs of bridges which lead from one area to another be mentally removed, thereby considerably reducing the number of bridges; it is then an easy task to construct the required route across the remaining bridges, and the bridges which have been removed will not significantly alter the route found, as will become clear after a little thought. I do not therefore think it worthwhile to give any further details concerning the finding of the routes.

In Euler's paper therefore, he actually showed only that every vertex of an Eulerian graph is even and that there are exactly two odd vertices in a graph containing an Eulerian trail. He did not verify the converse. Euler apparently thought that the converse was rather self-evident. The first proof of the converse was not published until 137 years later, in an 1873 paper authored by Carl Hierholzer, who was born in 1840, received his PhD in 1870 and died in 1871. Thus his paper was published two years after his death. He had told colleagues what he had done but died before he could write a paper containing this work. In an act of respect and unselfishness, his colleagues wrote the paper on his behalf and had it published for him.

As a consequence of Euler's work, it follows that the graph F of Figure 5.7a has an Eulerian trail, starting at one odd vertex and ending at the other (where the odd vertices are drawn as solid circles), the multigraph G of Figure 5.7b is Eulerian and the graph H of Figure 5.7c has neither an Eulerian circuit nor an Eulerian trail.



(a) (b) (c)

Figure 5.7. Three graphs.

THE CHINESE POSTMAN PROBLEM

We know that it's not possible to take a round-trip about Königsberg and cross each bridge exactly once. But, is it possible to take a round-trip about Königsberg and cross each bridge *at least* once? This question is rather easy to answer. It's yes. This suggests the following question:

> What is the minimum number of bridges (counting multiplicities) in Königsberg that must be crossed in a round-trip that traverses every bridge at least once?

This, in turn, suggests an even more general problem, which can also be placed in a graph theory setting. To do this, some additional terminology is useful.

A *walk* in a graph G is a sequence $W = (v_1, v_2, \ldots, v_k)$ of vertices such that $v_i v_{i+1}$ is an edge of G for $i = 1, 2, \ldots, k-1$. In a walk, no restriction is placed on repetition of edges or vertices. Both are possible. The *length* of a walk is the number of edges traversed on it, including multiplicities. Consequently, the length of the walk W above is $k-1$. If $v_1 = v_k$, then W is a *closed walk*, while if $v_1 \neq v_k$, then W is an *open walk*. If W is a walk in which no edges are repeated, then W is a trail (a circuit if the walk is closed). If, in addition, W is an open walk in which no vertex is repeated, then W is a path.

In the graph G of Figure 5.8, $W_1 = (u, v, w, y, w, u)$ is a closed walk of length 5, $W_2 = (y, w, x, w, z)$ is an open walk of length 4 and $W_3 = (w, u, v, w, y, z, w)$ is a closed walk of length 6. Indeed, W_3 is a circuit that is not a cycle.

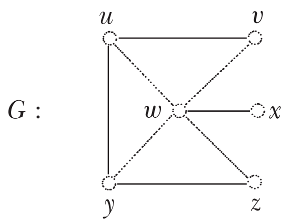


Figure 5.8. Walks in a graph.

In 1962 the Chinese mathematician Meigu Guan (often known as Mei-Ko Kwan) introduced a problem that a postman may encounter. Suppose that a postman starts from the post office and has mail to deliver to the houses along each street on his mail route. When he has completed delivering the mail, he returns to the post office. The problem is to find the minimum length of a round-trip that accomplishes this. Since Kwan's article referred to optimizing a postman's route and the article was authored by a Chinese mathematician, Alan Goldman coined the term *Chinese Postman Problem* for such a problem. At the time, Goldman worked for the United States National Bureau of Standards (now the National Institute of Standards and Technology) but spent the latter part of his career as a faculty member at Johns Hopkins University.

The Chinese Postman Problem

Determine the minimum length of a round-trip that traverses every road at least once in a mail route.

As it turns out, there is more than one interpretation of "minimum length" in the statement of the problem. We consider two of these.

The Chinese Postman Problem described above can be represented by a connected graph G whose vertices are street intersections and whose edges are the streets on the mail delivery route. One possible interpretation of the Chinese Postman Problem is to determine the minimum number of roads traversed (including multiplicities) that uses each road at least once. This is equivalent to finding the minimum length of a closed walk in the graph G that uses every edge of G at least once. Certainly, such a closed walk exists in G because if, for every edge e in G, an edge e' is added joining the same pair of vertices as e, then a multigraph H is obtained in which every edge of G is duplicated and every vertex in H has even degree; in fact, $\deg_H v = 2\deg_G v$ for every vertex v of G. Since H is Eulerian, there is a closed walk in G that traverses each edge of G twice. That is, if G is a connected graph of size m, then G contains a closed walk of length $2m$ that traverses every edge of G at least once (in fact, exactly twice). However, there may be a closed walk in G of shorter length that traverses every edge of G at least once. By an *Eulerian walk* in G is meant a closed walk of minimum length that traverses each edge of G at least once.

The following is then a graph-theoretical statement of this problem.

The Chinese Postman Problem

Let G be a connected graph. What is the length of an Eulerian walk in G?

Although the Chinese Postman Problem concerns mail delivery, there are many other applications and interpretations of this problem, including

- snow removal along the streets of a certain route;
- garbage collection along the streets of a certain route;
- police patrol of streets in a certain route.

In the first two of these, snow removal or garbage collection (and perhaps mail delivery as well) may take place only along one side of a street at a time and so it would be necessary to traverse each street in the route at least twice. Since we know such a round-trip can be made, traversing each street exactly twice, this problem has an immediate solution. Consequently, we assume that traversing each street at least once is sufficient for our purposes.

Naturally, if G is an Eulerian graph of size m, then the Chinese Postman Problem is easy to solve since any Eulerian walk in G is an Eulerian circuit and its length is m. If G is not Eulerian, then the trick to solving the problem is to duplicate as few edges of G as possible so that the resulting multigraph becomes Eulerian. Specifically, if the minimum number of edges of G that need to be duplicated so that every vertex has even degree in the resulting multigraph is k, then the length of an Eulerian walk in G is $m + k$.

For example, the graph G of size 19 in Figure 5.9 has two odd vertices, namely p and y. By duplicating the edges pq, qu and uy, we obtain the

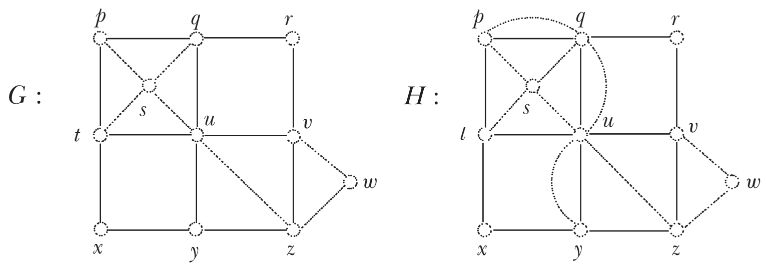


Figure 5.9. An Eulerian multigraph of a graph.

Eulerian multigraph H in that figure since all vertices of H have even degree. Therefore, an Eulerian circuit of H has length 22 and this gives us an Eulerian walk in G of length 22, one example of which is

$$W = (p, q, r, v, w, z, y, x, t, p, s, t, u, y, u, z, v, u, q, u, s, q, p).$$

In general, if a connected graph G contains exactly two odd vertices u and v with $d(u, v) = k$, and P is a $u - v$ path of length k, then a closed walk in G that traverses every edge of P twice and all other edges of G once is an Eulerian walk in G.

If uv is a bridge in a connected graph G (and so, by Theorem 3.1, uv is not on a cycle of G), any closed walk that contains uv, say from u to v, must eventually return to u by traversing v to u. This gives us the following.

Theorem 5.3: *Every Eulerian walk in a connected graph G must traverse every bridge of G twice.*

The connected graph G in Figure 5.10 has two bridges, namely xy and yz. Therefore, to construct an Eulerian multigraph H from G, the edges xy and yz must be duplicated. Since the multigraph obtained from doing only this is already Eulerian, no other edges of G need be duplicated. Hence the size of this Eulerian multigraph H is 11 and the length of an Eulerian walk in G is 11.

In the Chinese Postman Problem that we have introduced, the goal is to determine the minimum number of roads (counting multiplicities) that need be traversed in a round-trip so that each road is used at least

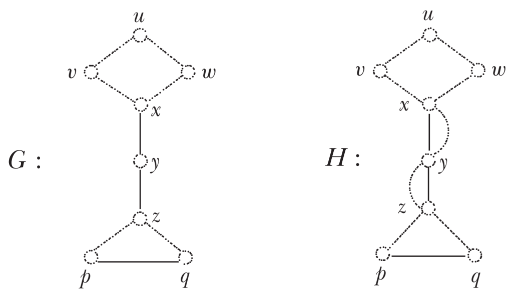


Figure 5.10. Constructing an Eulerian walk in a connected graph with bridges.

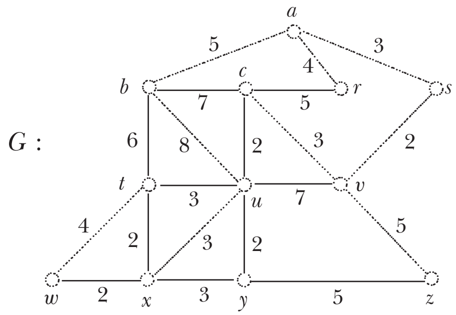


Figure 5.11. Finding an Eulerian walk in a weighted graph.

once. More than likely, this approach oversimplifies the situation because this interprets every two roads in the mail route the same. In practice, no two roads will have the same length or require the same amount of time to drive along them. In this case, the situation can more appropriately be represented by a weighted graph G where the weight assigned to each edge of G is the length of the road represented by the edge. In this setting, the goal of the Chinese Postman Problem is to minimize the total distance driven on a round-trip that traverses each edge at least once in the mail route. By an *Eulerian walk* in a weighted graph G is meant a closed walk of minimum length that contains every edge of G at least once, where the length of the walk is the sum of the weights of each edge encountered (again including multiplicities). The corresponding problem in graph theory is the following.

The Chinese Postman Problem

Let G be a connected weighted graph. What is the length of an Eulerian walk in G?

If G is a Eulerian graph, then here too the problem can be solved immediately. The length of an Eulerian walk in this case is the length of an Eulerian circuit in G, which is $w(G)$, the sum of the weights of all the edges of G. We can therefore assume that G contains odd vertices. The procedure here is essentially the same as when G is not weighted. In the simplest case, suppose that G contains exactly two odd vertices, say u and v. Then what we need to determine is the length of a shortest weighted path from u to v, which we denote by $d_G(u, v)$. By duplicating the edges on that path, an Eulerian circuit in the resulting multigraph gives rise to an Eulerian walk in G of length $w(G) + d_G(u, v)$.

The weighted graph in Figure 5.11 has weight $w(G) = 81$. This graph G has exactly two odd vertices, namely a and y. Since (a, s, v, c, u, y) is an $a - y$ path of minimum length and so $d_G(a, y) = 12$, it follows that the length of an Eulerian walk in G is $81 + 12 = 93$. From this, an Eulerian walk of length 93 can be constructed in G. When the weighted graph G has more than two odd vertices, the procedure used to determine the length of an Eulerian walk in G is essentially the same as when G is not a weighted graph.

Suppose that a connected graph G of size m has four odd vertices, say u, v, w, x. Then there are three ways to divide these four vertices into two pairs, namely $\{u, v\}, \{w, x\}$; $\{u, w\}, \{v, x\}$; $\{u, x\}, \{v, w\}$. We then compute $d(u, v) + d(w, x)$, $d(u, w) + d(v, x)$ and $d(u, x) + d(v, w)$. If s is the smallest of these three numbers, then the length of an Eulerian walk in G is $m + s$.

6

Encircling Graphs

Inspired by the Königsberg Bridge Problem, the problem of determining conditions under which a graph G has a circuit containing every edge of G (necessarily exactly once) was introduced, discussed and solved in the preceding chapter. Under the assumption that G is connected, such a circuit not only traverses every edge of G exactly once, it traverses every vertex of G but, quite likely, more than once—possibly many times. This brings up the question of when a round-trip can be made in a graph that traverses every vertex of the graph exactly once except, of course, that the terminal vertex of the trip is the same as the initial vertex.

SIR WILLIAM ROWAN HAMILTON

One of the most brilliant mathematicians and physicists of the nineteenth century was William Rowan Hamilton (1805–1865). Born in Dublin, Ireland, Hamilton was a gifted child both in mathematics and languages. His accomplishments in physics would lead to his being knighted in 1835 and thus becoming *Sir* William Rowan Hamilton. Also in 1835 Hamilton realized that complex numbers could be represented as ordered pairs of real numbers. That is, a complex number $a + b\mathbf{i}$ (where a and b are real numbers) could be treated as the ordered pair (a, b). Here the number $\mathbf{i}$ has the property that $\mathbf{i}^2 = -1$. Consequently, while the equation $x^2 = -1$ has no real number solutions, this equation has two solutions that are complex numbers, namely $\mathbf{i}$ and $-\mathbf{i}$.

While the real numbers correspond to points on the real line, the two-dimensional complex plane is a geometric representation of the complex numbers. Hamilton spent years trying to invent three-dimensional numbers that satisfied certain properties that would represent three-dimensional Euclidean space but he failed.

However, on 16 October 1843, while walking with his wife along the Royal Canal in Dublin, Hamilton suddenly discovered a collection of four-dimensional numbers,

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},$$

called *quaternions*, that possessed the properties he was striving for. In these numbers, a, b, c and d are real numbers and the numbers **i**, **j** and **k** satisfy the equations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Hamilton was so pleased with his discovery that he carved these equations into the stone of the Brougham Bridge in Dublin. Even today, tourists visit this bridge for the purpose of seeing what Hamilton carved on the bridge. Since the numbers **i**, **j** and **k** also satisfy

$$\mathbf{ij} = \mathbf{k}, \mathbf{ji} = -\mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{kj} = -\mathbf{i}, \mathbf{ki} = \mathbf{j}, \mathbf{ik} = -\mathbf{j},$$

the quaternions are not commutative. The quaternions do satisfy the distributive laws, however. With the aid of these laws and the properties of **i**, **j** and **k** listed above, it follows that

$$(b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = -b^2 - c^2 - d^2.$$

From this, it follows that if b, c and d are any three real numbers such that $b^2 + c^2 + d^2 = 1$, then $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is a solution of $x^2 = -1$. That is, among the quaternions, the equation $x^2 = -1$ has infinitely many solutions.

One of Hamilton's best friends was the mathematician John Graves, who may have inspired Hamilton to invent the quaternions. Two months after Hamilton discovered the quaternions, Graves wrote to Hamilton that he had discovered eight-dimensional numbers with desirable properties. Graves called these numbers *octonions*. Hamilton observed that the octonions were not associative, meaning that if a, b, c were octonions, it was not necessarily true that $(ab)c = a(bc)$. In fact, it was about this time that Hamilton introduced the term "associative".

One of Hamilton's last important ideas was that of the "icosian calculus", which was another consequence of his friendship with Graves. In August 1856, Hamilton attended the British Association meeting in Cheltenham and stayed at the home of Graves. Hamilton always enjoyed

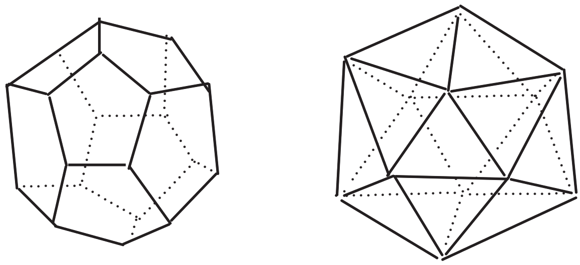


Figure 6.1. The dodecahedron and icosahedron.

talking to Graves and reading books from his extensive collection. While there, Graves posed some puzzles to Hamilton. Whether it was Graves or the books Hamilton was reading at Graves's home, Hamilton began thinking about regular polyhedra. Among the regular polyhedra are the *dodecahedron*, which has 20 vertices, 12 faces and 30 edges, and the *icosahedron*, which has 12 vertices, 20 faces and 30 edges. These two polyhedra are *duals* of one another (see Figure 6.1) in the sense that if we start with either polyhedron, replace each face by a vertex and join every two vertices by an edge if the corresponding faces have an edge in common, then the other polyhedron can be obtained. When Hamilton returned to Dublin, he thought about the *symmetry group* of the icosahedron, which is an algebraic structure associated with this polyhedron. Hamilton saw that there was a connection between his new icosian calculus and traveling along the edges of a dodecahedron, visiting every vertex exactly once and returning to the starting point.

THE ICOSIAN GAME

As it turned out, Hamilton's icosian calculus could be illustrated by means of a game with a game board containing the "graph of the dodecahedron" with holes at the vertices where numbered pegs could be placed to help keep track of a cyclic route that traveled about every vertex of the dodecahedron.

It is not entirely clear how Hamilton thought of connecting his icosian calculus to the problem of traveling along the edges of a dodecahedron, visiting each vertex exactly once and returning to the starting point,

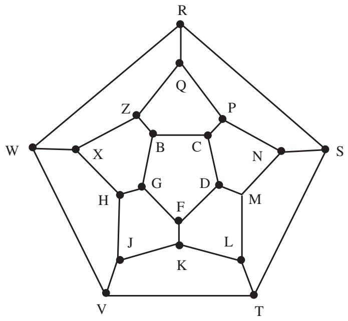


Figure 6.2. Hamilton's Icosian Game.

but this is what led to the concept of a "Hamiltonian cycle" in a graph. A cycle in a graph G that contains every vertex of G is called a *Hamiltonian cycle* of G and a graph containing a Hamiltonian cycle is called a *Hamiltonian graph*. Calling such cycles "Hamiltonian cycles" and calling such graphs "Hamiltonian graphs" was not Hamilton's idea of course. In fact, Hamilton was not the first person to think of searching for a cycle that passed though all vertices of a polyhedron. A few months earlier, the Reverend Thomas Penyngton Kirkman had already studied questions such as these.

Hamilton also invented a two-player game where the first player takes the first five steps (a path of order 5) on the graph of a dodecahedron in any way whatsoever and the other player is required to extend this path to a Hamiltonian cycle. In 1859 a friend of Graves manufactured a version of this game in the form of a small table consisting of a game board with legs, which he sent to Hamilton. Graves put Hamilton in contact with John Jacques, a London toy maker, to whom Hamilton sold the rights. Now called Jacques of London, this company was established in 1795 and is one of the oldest and most influential sports and games manufacturers in the world. In 1893 it popularized the game of chess by creating an easily produced and instantly recognizable set of chess pieces. In 1851 Jacques introduced the game of croquet to England and in 1901 Jacques turned the parlor game of Gossima into "Ping-Pong" which later became known as table tennis. Two versions of Hamilton's game were manufactured by Jacques, one for the parlor, played on a flat board, and another for a "traveler", played on an actual dodecahedron. Hamilton called his game the *Icosian Game* (see Figure 6.2).

The second (traveler) version of Hamilton's Icosian Game was labeled

NEW PUZZLE

TRAVELLER'S DODECAHEDRON

or

A VOYAGE ROUND THE WORLD.

In this game, the 20 vertices of the dodecahedron were labeled with the 20 consonants of the English alphabet, which stood for 20 cities of the world:

B. Brussels	C. Canton	D. Delhi	F. Frankfort
G. Geneva	H. Hanover	J. Jeddo	K. Kashmere
L. London	M. Moscow	N. Naples	P. Paris
Q. Quebec	R. Rome	S. Stockholm	T. Toholsk
V. Vienna	W. Washington	X. Xenia	Z. Zanzibar

The idea was thus to construct a round-trip around the world where each of the 20 cities would be visited on the trip exactly once. It is easy enough to find such a trip if there are no restrictions (see Figure 6.3) but it's more of a challenge when a trip has to satisfy extra conditions.

Hamilton was also involved in marketing the game. He even wrote the preface to the instruction pamphlet, which began as follows:

In this new Game (invented by

Sir WILLIAM ROWAN HAMILTON, LL.D., & c., of Dublin,

and by him named Icosian from a Greek word signifying "twenty") a player is to place the whole or part of a set of twenty numbered pieces or men upon the points or in the holes of a board, represented by the diagram above drawn, in such a manner as always to proceed along the lines of the figure, and also to fulfil certain other conditions, which may in various ways be assigned by another player. Ingenuity and skill may thus be exercised in proposing as well as in resolving problems of the game. For example, the first of

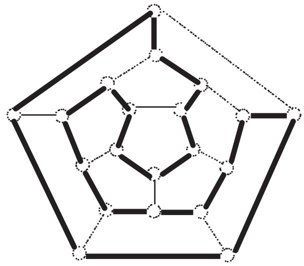


Figure 6.3. A Hamiltonian cycle in the graph of the dodecahedron.

> the two players may place the first five pieces in any five consecutive holes, and then require the second player to place the remaining fifteen men consecutively in such a manner that the succession may be cyclical, that is, so that No. 20 may be adjacent to No. 1; and it is always possible to answer any question of this kind.

The problems proposed by Hamilton in his Icosian Game not only gave rise to new concepts in graph theory, they gave rise to a popular area of study by mathematicians. As we noted, the graph of the dodecahedron is Hamiltonian.

A KNIGHT'S TOUR

While Hamilton's Icosian Game and the resulting problem of finding a cycle on the dodecahedron that visits every vertex gave rise to the concepts of Hamiltonian cycle and Hamiltonian graph, these ideas occurred (indirectly) long before the 1850s. We noted that a popular version of Hamilton's Icosian Game was manufactured by Jacques of London, a well-known manufacturer of games and famous for its production of chess sets. One chess piece is the knight, typically represented by a horse's head. A single move of a knight situated on a square of a chessboard consists of moving the knight two squares horizontally or vertically followed by one square perpendicular to this. A knight always moves from one square of a chessboard to a square of the other color on the board. (See Figure 6.4.)

The Knight's Tour Puzzle consists of finding a round-trip of a knight on a chessboard that visits every square exactly once.

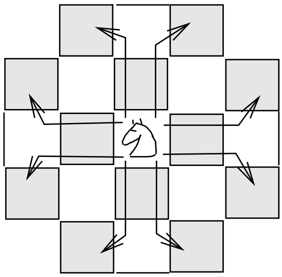


Figure 6.4. A knight's moves.

The Knight's Tour Puzzle

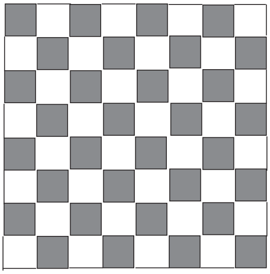
Following the rules of chess, is it possible for a knight to tour an 8×8 chessboard, visiting each square exactly once and return to the starting square?

Such knight's tours exist. One of the first mathematicians to show an interest in this puzzle was Leonhard Euler. In the 1750s, he wrote the following:

> I found myself one day in a company where, on the occasion of a game of chess, someone proposed the question:
>
> To move with a knight through all the squares of the chessboard, without ever moving two times to the same square, and beginning with a given square.

Euler mentioned a *knight's tour* to the German mathematician Christian Goldbach in a letter written in 1757 and two years later Euler wrote an article about this.

Associated with the Knight's Tour Puzzle is a graph theory problem. Let G be a graph of order 64 whose vertices are the 64 squares of an 8×8 chessboard. Two vertices u and v of G are adjacent if it is possible for a knight on square u to move to square v by a single move. This graph G is called the *Knight's graph*. Solving the Knight's Tour Puzzle is then equivalent to showing that G is Hamiltonian. In a certain sense, the concept of Hamiltonian graphs goes back long before the times of Euler and Hamilton. The question proposed to Euler was actually answered in the affirmative some 900 years earlier (in 840) by al-Adli ar-Rumi of Baghdad. Figure 6.5 shows his solution of the Knight's Tour Puzzle.



60	11	56	07	54	03	42	01
57	08	59	62	31	64	53	04
12	61	10	55	06	41	02	43
09	58	13	32	63	30	05	52
34	17	36	23	40	27	44	29
37	14	33	20	47	22	51	26
18	35	16	39	24	49	28	45
15	38	19	48	21	46	25	50

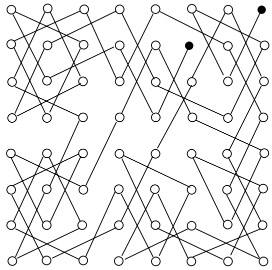


Figure 6.5. The knight's tour of al-Adli ar-Rumi.

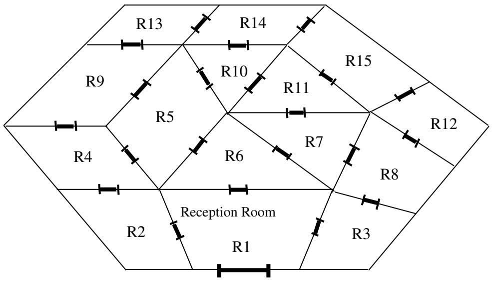


Figure 6.6. A diagram of the exhibition rooms in a museum.

The fact that the Knight's graph is Hamiltonian provides a solution to the Knight's Tour Puzzle.

There are numerous other problems that can be modeled by a graph and whose solution depends on whether the resulting graph is Hamiltonian.

Example 6.1: *Figure 6.6 shows a diagram of a modern art museum that is divided into* 15 *exhibition rooms. At the end of each day, a security officer enters the reception room by the front door and checks each exhibition room to make certain that everything is in order. It would be most efficient if the officer could visit each room only once and return to the reception room. Can this be done?*

SOLUTION:

This question can be rephrased in terms of graphs. A graph G can be associated with this museum where the vertices of G are the exhibition rooms and two vertices (rooms) are joined by an edge if there is a

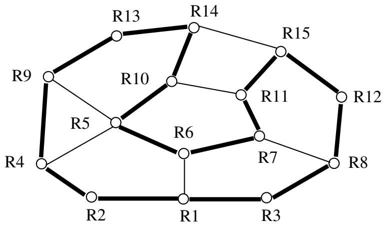


Figure 6.7. The graph G that models the exhibition rooms and doorways of the museum of Figure 6.6.

doorway between the two rooms. This graph G is shown in Figure 6.7. The question above can now be asked as follows: Does the graph G of Figure 6.7 have a cycle that contains every vertex of G, that is, does the graph G contain a Hamiltonian cycle? It does. Indeed,

$$C = (R1, R2, R4, R9, R13, R14, R10, R5, R6, R7, R11, R15,$$
$$R12, R8, R3, R1)$$

is such a Hamiltonian cycle. Therefore, the graph G is Hamiltonian and so the officer can start in the reception room, visit each exhibition room exactly once and return to the reception room. ♦

WHEN IS A GRAPH HAMILTONIAN?

We have seen that there are problems of a recreational nature whose solution depends on whether a certain graph is Hamiltonian. During the past several decades, however, the study of Hamiltonian graphs developed into a theoretical area of graph theory whose central question is the following: Under what conditions is a given graph Hamiltonian? Of course, a graph is Hamiltonian if it contains a Hamiltonian cycle. Therefore, every Hamiltonian graph G can be drawn as a cycle containing all vertices of G (such as indicated in Figure 6.8) with possibly some other edges as well.

But how does one determine whether a given graph G is Hamiltonian? Unlike the situation for Eulerian graphs, there is no theorem that tells us precisely which graphs are Hamiltonian. The first result of a mathematical nature dealing with Hamiltonian graphs occurred in 1952

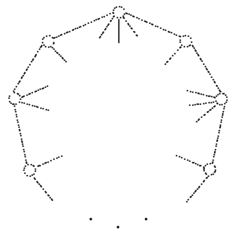


Figure 6.8. A Hamiltonian graph.

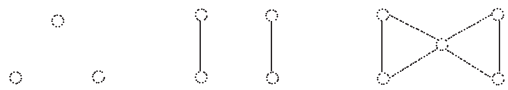


Figure 6.9. Graphs of order $n = 3, 4, 5$ whose vertices have degrees $n-3$ or more but are not Hamiltonian.

when the Danish mathematician Gabriel Andrew Dirac (1925–1984) described a sufficient condition for a graph to be Hamiltonian. That is, if any graph were to satisfy this condition, then it must be Hamiltonian.

Before stating Dirac's theorem, notice that every Hamiltonian graph must contain at least three vertices. Also, such a graph must be connected. These are examples of necessary conditions for a graph to be Hamiltonian. That is, every Hamiltonian graph must have these properties. If G is a graph of order $n \geq 3$ and every vertex has degree $n-1$, then G is the complete graph K_n, which is necessarily a Hamiltonian graph. In fact, if every vertex of a graph G of order $n \geq 3$ has degree $n-2$ or more, then it turns out that G must also be Hamiltonian. If every vertex of G has degree $n-3$ or more, then G need not be Hamiltonian, as the graphs in Figure 6.9 show.

Dirac found a condition on the degrees of the vertices of a graph G of order n that forces G to be Hamiltonian regardless of the value of n.

Theorem 6.2: *If G is a graph of order $n \geq 3$ such that $\deg v \geq n/2$ for each vertex v of G, then G is Hamiltonian.*

Proof: We prove this by using an extremal argument. Suppose that $P = (v_1, v_2, \ldots, v_k)$ is a path of *greatest* length in G. Then $k \leq n$. Since P is a longest path in G, every vertex adjacent to v_1 must be

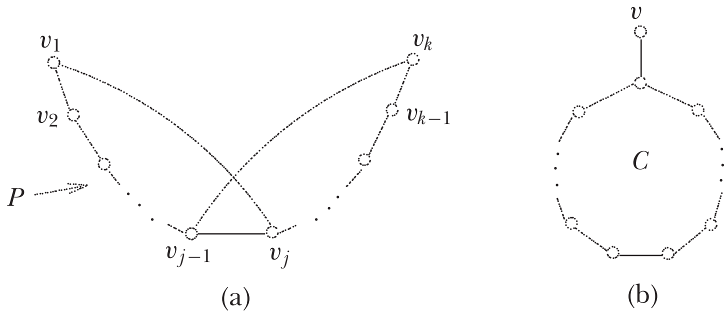


Figure 6.10. The cycle C in the proof of Theorem 6.2.

on P, as is every vertex adjacent to v_k. Since $n \geq 3$ and $\deg v_1 \geq n/2$, it follows that $k \geq n/2 + 1$ and so $k \geq 3$ (since if $n = 3$, then $3/2 + 1 = 2.5$).

We next show that G contains a cycle C whose vertices are precisely those on P. Of course, this is obvious if v_1 is adjacent to v_k. Suppose then that v_1 is not adjacent to v_k. There must be a vertex v_j to which v_1 is adjacent, $3 \leq j \leq k-1$, such that v_{j-1} is adjacent to v_k (see Figure 6.10a); for if this were not the case, then for each of the $n/2$ or more vertices to which v_1 is adjacent, there are $n/2$ or more vertices on P to which v_k is not adjacent. This, however, is impossible due to the fact that $\deg v_k \geq n/2$. So, as we claimed, there is some vertex v_j to which v_1 is adjacent such that v_{j-1} is adjacent to v_k. This then implies that $C = (v_1, v_j, v_{j+1}, \ldots, v_k, v_{j-1}, v_{j-2}, \ldots, v_1)$ is a cycle in G. If $k = n$, then C is a Hamiltonian cycle. If $k < n$, then there is a vertex v of G that is not on C. Since C contains at least $n/2 + 1$ vertices, at most $n/2 - 1$ vertices are not on C, so v is adjacent to some vertex on C (see Figure 6.10b). However, this says that G contains a path longer than P, which is impossible because P is a path of *greatest* length in G. So G contains a Hamiltonian cycle. ■

Gabriel Dirac had a famous stepfather, namely Paul Adrien Maurice Dirac. Paul Dirac was a recipient of a Nobel Prize in Physics in 1933. Curiously, much of Paul Dirac's work dealt with quantum mechanics and was based on the work of William Rowan Hamilton who, recall, was knighted because of his accomplishments in physics.

That Theorem 6.2 provides only a sufficient condition for a graph G of order n to be Hamiltonian means that it is not necessary for a

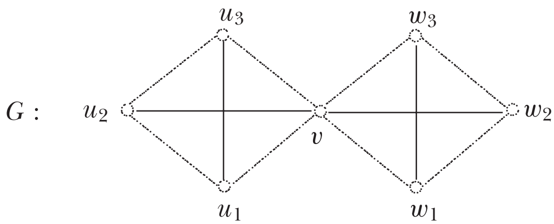


Figure 6.11. A non-Hamiltonian graph of order $n = 7$ in which every vertex has degree at least $(n-1)/2$.

Hamiltonian graph to satisfy the condition that $\deg v \geq n/2$ for every vertex v of G. Indeed, for $n \geq 5$, the cycle C_n is certainly Hamiltonian yet every vertex of C_n has degree 2 and so every vertex of C_n has degree less than $n/2$.

There is another interesting aspect of Theorem 6.2. If the degree requirement $n/2$ in the statement of this theorem were to be lowered even slightly, say to $(n-1)/2$, then there is no longer any guarantee that the graph must be Hamiltonian. For example, the graph G of Figure 6.11 has order $n = 7$, six vertices of which have degree 3 and one has degree 6. So every vertex of G has degree $(n-1)/2 = 3$ or more. However, this graph is not Hamiltonian since it contains a cut-vertex. Let's discuss this a bit more. Suppose that G were to contain a Hamiltonian cycle C. Then the vertex immediately following v on C is one of the vertices u_1, u_2, u_3 or one of the vertices w_1, w_2, w_3. Because of the similarity of these vertices, we can assume that it's u_1 that immediately follows v on C. Then the first vertex that immediately precedes any of w_1, w_2, w_3 on C is v, but this means that C encounters v twice, which is impossible. Thus no graph with a cut-vertex can be Hamiltonian.

In 1960 the mathematician Oystein Ore (1899–1968), well known for his work in number theory (the study of integers) and his interest in mathematical history, discovered a theorem which improved Dirac's result.

Theorem 6.3: *If G is a graph of order $n \geq 3$ such that $\deg u + \deg v \geq n$ for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian.*

Consider the graph G of order $n = 8$ in Figure 6.12. Each of the vertices x_1, x_2, x_3 and x_4 has degree 5, each of y_1 and y_2 has degree 7 and each of z_1 and z_2 has degree 3. Despite the fact that not every vertex of G has degree at least $n/2$ and we cannot conclude that G is Hamiltonian by

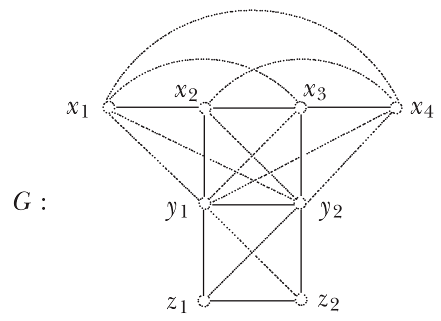


Figure 6.12. A Hamiltonian graph of order 8.

Dirac's theorem, the graph G is, in fact, Hamiltonian. The vertices z_1 and z_2 are adjacent. Therefore, for each pair u, v of nonadjacent vertices of G,

$$\deg u + \deg v \geq 8 = n.$$

Consequently, according to Ore's theorem, the graph G is Hamiltonian. In this case, it is not all that difficult to find a Hamiltonian cycle in G. In particular,

$$C = (x_1, y_1, z_1, z_2, y_2, x_4, x_3, x_2, x_1)$$

is a Hamiltonian cycle in G.

While Ore's theorem (Theorem 6.3) improves on Dirac's theorem (Theorem 6.2) and a similar proof can be used to prove Ore's theorem, we do not include it. In fact, several theorems have been obtained that improve on Ore's theorem but, for the most part, are more difficult to apply.

We end this section with a necessary condition for a graph to be Hamiltonian. We saw earlier that if G is a Hamiltonian graph, then it must be connected and can't have any cut-vertices. In other words, if you delete any vertex (and its incident edges), then the resulting graph is still connected. A more general theorem is worth knowing, although we won't prove it here.

Theorem 6.4: *If G is a Hamiltonian graph and we delete any k vertices of G, then the resulting graph has at most k components.*

Equivalently, we have the contrapositive statement.

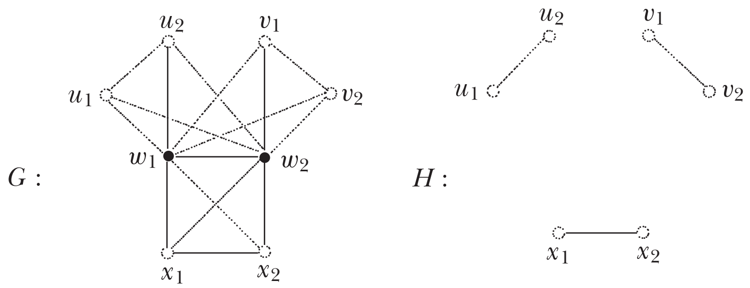


Figure 6.13. A non-Hamiltonian graph.

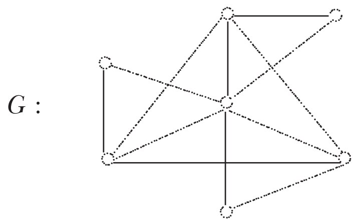


Figure 6.14. A non-Hamiltonian graph that satisfies the conditions in Theorem 6.4.

Theorem 6.5: *For any graph G, if there is a positive integer k such that deleting k vertices results in a graph with more than k components, then G is not Hamiltonian.*

For example, if the two vertices w_1 and w_2 are removed from the graph G of Figure 6.13, then the resulting graph H has three components. By Theorem 6.5, G is not Hamiltonian.

While Theorem 6.4 provides a necessary condition for a graph to be Hamiltonian, it is not a sufficient condition for a graph to be Hamiltonian. For example, it can be shown that the graph G of Figure 6.14 satisfies the conditions in Theorem 6.4 but is not Hamiltonian.

THE TRAVELING SALESMAN PROBLEM

While a network of highways connecting cities can quite easily be modeled by a graph (or perhaps a multigraph), graphs can also be used to represent a collection of airline routes connecting pairs of cities. One may ask whether it's possible, by taking appropriate flights, to begin at one of these cities, visit each city exactly once and return to the starting city. However, there is a more interesting and considerably more challenging question. Suppose that there is a collection of cities with flights existing between certain pairs of these cities. Furthermore, suppose that we know the distance between each pair of cities connected by an airline route (or perhaps the cost of such a flight). What is the minimum total distance (or the minimum total cost) of such a trip? Problems of this type go by a certain name.

The Traveling Salesman Problem

A salesman wishes to make a round-trip that visits a certain number of cities. He knows the distance between all pairs of cities. If he is to visit each city exactly once, then what is the minimum total distance of such a round-trip?

The study of such problems can be made by means of weighted graphs. Recall that a weighted graph is a graph G in which every edge e of G is assigned a real number (ordinarily a positive number) called the weight of e, denoted by $w(e)$. The Traveling Salesman Problem can be modeled by a weighted graph G whose vertices are the cities and where two vertices u and v are joined by an edge having weight r if the distance between u and v is known and this distance is r. The *weight of a cycle* C in G is the sum of the weights of the edges of C. To solve this Traveling Salesman Problem, we need to determine the minimum weight of a Hamiltonian cycle in G. Certainly, G must contain a Hamiltonian cycle for this problem to have a solution. However, if G is complete (that is, if we know the distance between every pair of cities), then there are many Hamiltonian cycles in G if its order n is large. Since every city must lie on every Hamiltonian cycle of G, we can think of a Hamiltonian cycle starting (and ending) at a city c.

It turns out that the remaining $n-1$ cities can follow c on the cycle in any of $(n-1)!$ orders. Indeed, if we have one of the $(n-1)!$ orderings of these $n-1$ cities, then we need to add the distances between consecutive cities in the sequence, as well as the distance between c and the last city in the sequence. We then need to compute the minimum of these $(n-1)!$ sums. Actually, we need *only* find the minimum of $(n-1)!/2$ sums since we would get the same sum if a sequence was traversed in reverse order. Unfortunately, $(n-1)!/2$ grows very, very fast. For example, when $n = 10$, then $(n-1)!/2 = 181{,}400$.

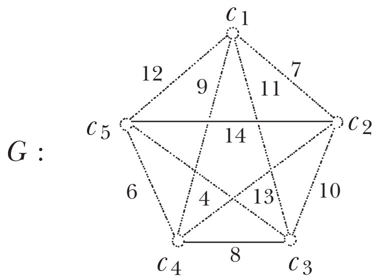


Figure 6.15. The graph G in Example 6.6.

Example 6.6: *A salesman plans a round-trip to visit a certain number of cities and this situation is modeled by the complete weighted graph G in Figure 6.15. What is the minimum total distance of such a round-trip?*

SOLUTION:

Since the order of G is 5, there are $(5-1)!/2 = 12$ Hamiltonian cycles in G. These are listed together with their weights:

Hamiltonian cycle	Weight of cycle
$s_1 = (c_1, c_2, c_3, c_4, c_5, c_1)$	$7+10+8+6+12=43$
$s_2 = (c_1, c_2, c_3, c_5, c_4, c_1)$	$7+10+4+6+9=36$
$s_3 = (c_1, c_2, c_4, c_3, c_5, c_1)$	$7+13+8+4+12=44$
$s_4 = (c_1, c_2, c_4, c_5, c_3, c_1)$	$7+13+6+4+11=41$
$s_5 = (c_1, c_2, c_5, c_3, c_4, c_1)$	$7+14+4+8+9=42$
$s_6 = (c_1, c_2, c_5, c_4, c_3, c_1)$	$7+14+6+8+11=46$
$s_7 = (c_1, c_3, c_2, c_4, c_5, c_1)$	$11+10+13+6+12=52$
$s_8 = (c_1, c_3, c_2, c_5, c_4, c_1)$	$11+10+14+6+9=50$
$s_9 = (c_1, c_3, c_4, c_2, c_5, c_1)$	$11+8+13+14+12=58$
$s_{10} = (c_1, c_3, c_5, c_2, c_4, c_1)$	$11+4+14+13+9=51$
$s_{11} = (c_1, c_4, c_2, c_3, c_5, c_1)$	$9+13+10+4+12=48$
$s_{12} = (c_1, c_4, c_3, c_2, c_5, c_1)$	$9+8+10+14+12=53$

Thus the minimum weight of a Hamiltonian cycle is 36. To obtain this weight, the vertices of G should be visited in the order listed in the sequence s_2, that is, $c_1, c_2, c_3, c_5, c_4, c_1$ (or $c_1, c_4, c_5, c_3, c_2, c_1$). ♦

The importance of the Traveling Salesman Problem is due to a number of related but useful applications, including, but definitely not limited to, the following:

(1) Each morning a school bus leaves school to pick up students at a number of bus stops. It would be useful to know a route that would use the least time (to get the students to school quickly and to minimize the cost of gasoline for the bus).

(2) Late afternoon each day, a van leaves a restaurant to deliver "meals on wheels" to customers who prefer to have meals delivered to them.

(3) Each day a mail truck leaves the post office to pick up mail that has been left in mailboxes.

Indeed, there are numerous applications that have nothing to do with "traveling" but involve performing activities in some cyclic sequence that is least costly or most time efficient.

The Traveling Salesman Problem is, in general, an extraordinarily complicated problem. Despite this, there have been instances when a Traveling Salesman Problem has been solved for a large number of cities. For example, in 1998 David Applegate, Robert Bixby, Vašek Chvátal and William Cook solved a Traveling Salesman Problem for the 13,509 largest cities in the United States (those whose population exceeded 500 at that time). They also solved a Traveling Salesman Problem for 15,113 German cities in 2001 and for 24,978 Swedish cities in 2004. Their ultimate goal was to solve the Traveling Salesman Problem for every registered city or town in the world plus a few research bases in Antarctica (1,904,711 locations in all). In 2007 Applegate, Bixby, Chvátal and Cook wrote a book titled *The Traveling Salesman Problem*, in which they described the history of the Traveling Salesman Problem as well as the method they used to solve a range of large-scale problems. In 2012 Cook wrote a book titled *In Pursuit of the Traveling Salesman* for a more general audience.

7

Factoring Graphs

Suppose that in a certain graduate class in mathematics, there are seven students, namely Alice, Bob, Carla, David, Emma, Frank and Gina, whom we denote by a, b, c, d, e, f and g, respectively. The professor of this class assigns seven challenging problems to them. He tells the class that they can work on each problem in study groups of three students each such that every pair of students belongs only to one of these study groups. Is this even possible? The answer is yes. In fact, the students in the class have divided themselves into the following seven groups where Student Group 1, denoted by S_1, is working on Problem 1 and so on:

$$S_1 = \{b, e, g\}, \quad S_2 = \{c, f, g\}, \quad S_3 = \{a, e, f\}, \quad S_4 = \{a, d, g\},$$
$$S_5 = \{c, d, e\}, \quad S_6 = \{b, d, f\}, \quad S_7 = \{a, b, c\}.$$

Later the professor tells the class that each group must select one representative from the group to describe their solution of the problem to the class. Furthermore, every student in the class must give one presentation. Once again, we are faced with the question, is this even possible? And once again, the answer is yes. One way to accomplish this is the following:

$$S_1 : b, \quad S_2 : g, \quad S_3 : a, \quad S_4 : d, \quad S_5 : e, \quad S_6 : f, \quad S_7 : c.$$

This situation can be placed in a graph theory setting. Let G be a bipartite graph of order 14 with partite sets $U = \{S_1, S_2, \ldots, S_7\}$ and $W = \{a, b, \ldots, g\}$. That is, one partite set of G is the set of the seven study groups and the other partite set consists of the seven students. A vertex j in W is adjacent to a vertex (set S_i) in U if j is contained in S_i, that is, if student j is in the ith study group. This graph G is shown in Figure 7.1a. The solution to the problem described above corresponds to the seven

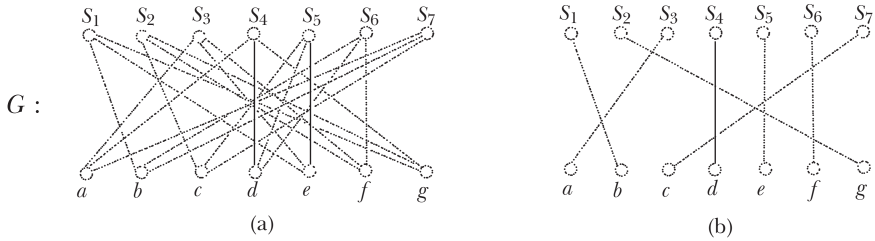


Figure 7.1. Matching seven students with seven study groups.

edges of G shown in Figure 7.1b. That is, a solution consists of pairing off or "matching" the seven students in the class with the seven study groups.

This problem is related to a theorem that we are about to describe and that was introduced and proved by a mathematician known for his work in abstract algebra.

SYSTEMS OF DISTINCT REPRESENTATIVES

Suppose that $\{S_1, S_2, \ldots, S_n\}$ is a collection of $n \geq 2$ finite nonempty sets. This collection is said to have a *system of distinct representatives* if there exist n distinct elements $x_1, x_2, \ldots, x_n$ such that $x_i \in S_i$ for each integer i $(1 \leq i \leq n)$. In other words, x_i is a representative of the set S_i. The subgraph of the graph G in Figure 7.1a shown in Figure 7.1b indicates that the seven sets $S_1, S_2, \ldots, S_7$ of students described above has a system of distinct representatives.

In order for a collection $\{S_1, S_2, \ldots, S_n\}$ of n sets to have a system of distinct representatives, the sets must certainly have at least n elements, that is, $|S_1 \cup S_2 \cup \cdots \cup S_n| \geq n$. However, even if $|S_1 \cup S_2 \cup \cdots \cup S_n| \geq n$, there is no guarantee that $\{S_1, S_2, \ldots, S_n\}$ will have a system of distinct representatives. For example, suppose that $S_1 = \{a\}$, $S_2 = \{a, c, d\}$, $S_3 = \{a\}$ and $S_4 = \{b, d\}$. Then $S_1 \cup S_2 \cup S_3 \cup S_4 = \{a, b, c, d\}$ and $|S_1 \cup S_2 \cup S_3 \cup S_4| = 4$ but $\{S_1, S_2, S_3, S_4\}$ does not have a system of distinct representatives since a is the only possible representative for each of S_1 and S_3.

A condition under which collections of sets contain a system of distinct representatives was discovered in 1935 and is referred to as *Hall's theorem*.

Theorem 7.1 (Hall's Theorem): *A collection $\{S_1, S_2, \ldots, S_n\}$ of n nonempty finite sets has a system of distinct representatives if and only if, for each integer k with $1 \leq k \leq n$, the union of any k of these sets contains at least k elements.*

For the sets $S_1 = \{a\}$, $S_2 = \{a, c, d\}$, $S_3 = \{a\}$ and $S_4 = \{b, d\}$ described above, the union of the two sets S_1 and S_3 does not contain at least two elements and so, by Hall's theorem, the sets S_1, S_2, S_3, S_4 do not have a system of distinct representatives.

The person responsible for obtaining this theorem was Philip Hall (1904–1982), a British mathematician well known for his work in abstract algebra, especially for his research in group theory. He acquired a faculty position at the University of Cambridge shortly before proving this theorem. He spent some time during World War II at Bletchley Park, the site of the United Kingdom's main establishment of code breakers. Hall was known for his writing and the concern he had for his students.

MATCHINGS

We have already hinted that Hall's theorem has a connection with graph theory. Dénes König was a Hungarian mathematician whose 1936 book *Theorie der endlichen und unendlichen Graphen* (*Theory of Finite and Infinite Graphs*) was the first book written entirely on graph theory. In his book, graph theory was presented for the first time as an organized area of mathematics. Prior to the publication of his book, there was little interest in graph theory by mathematicians in English-speaking countries. Although König's book was written in German and little graph theory was done anywhere during World War II, things would soon change after the war ended.

It was, in fact, König who noticed that Hall's theorem could be stated in the setting of graph theory. Suppose that $U = \{S_1, S_2, \ldots, S_n\}$ is a collection of nonempty finite sets and $W = S_1 \cup S_2 \cup \cdots \cup S_n$. We wish to determine whether the collection U has a system of distinct representatives. Certainly we can assume that $|W| \geq n$, for otherwise, as we saw before, U does not contain such a system. We now construct a bipartite graph G with partite sets U and W. That is, the vertices in U are the sets $S_1, S_2, \ldots, S_n$ and the vertices in W are the elements belonging to $S_1 \cup S_2 \cup \cdots \cup S_n$. Therefore, $|U| = n$ and $|W| \geq n$. Consequently, the collection $\{S_1, S_2, \ldots, S_n\}$ of sets has a system of distinct representatives if and only if the graph G has n edges, no two of which are adjacent.

A collection of edges, no two adjacent, in a bipartite graph G is called a *matching*. A matching of size n in G then takes $2n$ vertices of G and divides them into n pairs, where each pair of vertices are adjacent in G. Thus the collection $\{S_1, S_2, \ldots, S_n\}$ has a system of distinct representatives if and only if G has a matching of size n. Hall's theorem leads us to a theorem dealing with matchings in bipartite graphs. Let G be a bipartite graph with partite sets U and W such that $|U| \leq |W|$. For a nonempty set X of U, the *neighborhood* $N(X)$ of X is the set of all vertices in W that are adjacent to at least one vertex of X.

Theorem 7.2 (Hall's Theorem): *Let G be a bipartite graph with partite sets U and W such that $r = |U| \leq |W|$. Then G contains a matching of size r if and only if $|N(X)| \geq |X|$ for every nonempty subset X of U.*

Example 7.3: *As a result of doing well on an exam, six students Ashley (A), Bruce (B), Charles (C), Duane (D), Elke (E) and Faith (F) have earned the right to receive a complimentary text in either algebra (a), calculus (c), differential equations (d), geometry (g), history of mathematics (h), programming (p) or topology (t). There is only one book on each of these subjects. The preferences of the students are*

$$A : d, h, t; \quad B : g, p, t; \quad C : a, g, h; \quad D : h, p, t; \quad E : a, c, t; \quad F : c, d, p.$$

Can each of the students receive a book he or she likes?

SOLUTION:

This situation can be modeled by the bipartite graph G of Figure 7.2a having partite sets $U = \{A, B, C, D, E, F\}$ and $W = \{a, c, d, g, h, p, t\}$. We are asking whether G contains a matching with six edges. Here we don't need to use Hall's theorem. In fact, we can observe that such a matching does exist, as shown in Figure 7.2b. From this matching, we see how six of the seven books can be paired off with the six students. ♦

While we didn't need to use Hall's theorem in the preceding example, it is very useful in the next example.

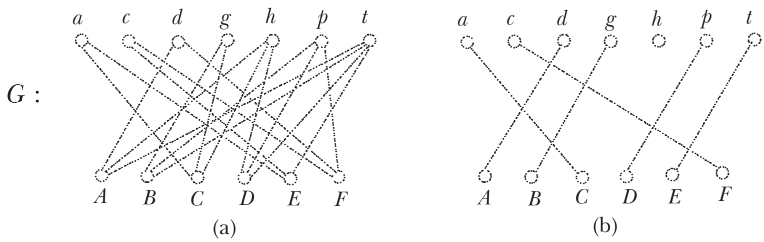


Figure 7.2. A matching in a bipartite graph.

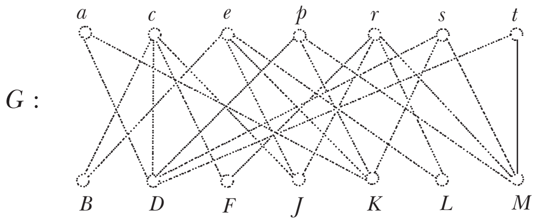


Figure 7.3. A graph modeling the situation in Example 7.4.

Example 7.4: *Seven seniors Ben* (B)*, Don* (D)*, Felix* (F)*, June* (J)*, Kim* (K)*, Lyle* (L) *and Maria* (M) *are looking for positions after they graduate. The university Placement Office has posted open positions for an accountant* (a)*, consultant* (c)*, editor* (e)*, programmer* (p)*, reporter* (r)*, secretary* (s) *and teacher* (t)*. Each of the seven students has applied for some of these positions:*

$$B : c, e; \quad D : a, c, p, s, t; \quad F : c, r; \quad J : c, e, r;$$

$$K : a, e, p, s; \quad L : e, r; \quad M : p, r, s, t.$$

Is it possible for each student to be hired for a job for which he or she has applied?

SOLUTION:

This situation can be modeled by the bipartite graph G of Figure 7.3, where one partite set $U = \{B, D, F, J, K, L, M\}$ is the set of students and the other partite set $W = \{a, c, e, p, r, s, t\}$ is the set of positions. A vertex $u \in U$ is joined to a vertex $w \in W$ if u has applied for position w.

For the subset $X = \{B, F, J, L\}$, we have $N(X) = \{c, e, r\}$. Since $|N(X)| < |X|$, it follows by Theorem 7.2 that there is no matching of size 7. Therefore, the answer to this question is no and the set X proves that such a matching is impossible. ♦

Example 7.5: *A standard deck of playing cards consists of* 52 *cards. These* 52 *cards are divided into four types of* 13 *cards each. The types are referred to as suits, called hearts (♡), diamonds (◇), clubs (♣) and spades (♠). In each suit, there is one card of each of the following values: ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king. This deck of cards is shuffled (mixed up) and the* 52 *cards are placed in a rectangular array of* 4 *rows and* 13 *columns. Show that, regardless of how these cards are spread out in such a manner, it is always possible to select one card from each of the* 13 *columns so that the* 13 *selected cards have different values.*

SOLUTION:

A bipartite graph G is constructed with partite sets

$$U = \{1, 2, \ldots, 13\} \quad \textit{and} \quad W = \{\text{ace, two, three}, \ldots, \text{queen, king}\},$$

where the vertices in U correspond to the 13 columns. A vertex $u \in U$ is joined to a vertex $w \in W$ if column u contains a card having value w. Since each subset X of U of k columns has $4k$ cards in these columns and at most 4 of these cards have the same value, at least k values occur in these $4k$ cards, that is, $|N(X)| \geq k = |X|$. By Hall's theorem, G has a matching of size 13, giving the desired result. ♦

Perhaps the best known version of Hall's work is a statement referred to as the Marriage Theorem. This theorem is also implicit in the work of the German mathematician Ferdinand Georg Frobenius.

The Marriage Theorem

In a collection of r women and r men, a total of r marriages between acquainted couples is possible if and only if, for each integer k with $1 \leq k \leq r$, every subset of k women is collectively acquainted with at least k men.

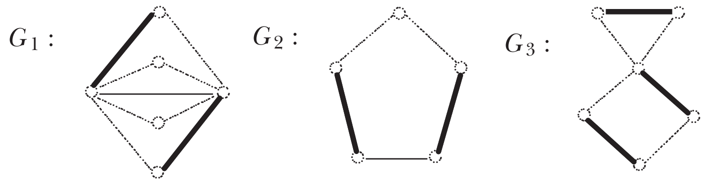


Figure 7.4. Maximum matchings and perfect matchings.

WILLIAM TUTTE

Bipartite graphs aren't the only graphs in which matchings are of interest. Matchings can occur in any graph, bipartite or not. Our primary interest in matchings is those of maximum size. In fact, a matching M in a graph G is a *maximum matching* if M has the maximum size among all matchings in G. A *perfect matching* in a graph G is a matching M in which every vertex of G is incident with some edge of M. Necessarily, the size of a perfect matching in a graph of order n is $n/2$ and so n must be even. Furthermore, every perfect matching is a maximum matching. The graphs G_1 and G_2 of Figure 7.4 contain maximum matchings (indicated by bold edges) that are not perfect matchings. The graph G_3 of Figure 7.4, however, has a perfect matching (also indicated by bold edges).

A theorem that describes precise conditions under which a graph contains a perfect matching was obtained by another Englishman, who was only a graduate student when he discovered the theorem.

One of the best known mathematicians in the area of graph theory was William Thomas Tutte (1917–2002). Born in England, Tutte was brought up in the village of Cheveley, where Tutte's father worked as a gardener at a hotel. Tutte attended the school there from ages 6 to 11. At age 10, he did so well in a competition that he was awarded a scholarship to attend a school in Cambridge. Because the distance from home to Cambridge was so great, Tutte's parents kept him at home. However, the next year, he took the exam again, did well again and this time his parents permitted him to attend school in Cambridge.

At age 18, Tutte entered Trinity College, Cambridge where he majored in chemistry even though this was not his favorite subject—mathematics was. Tutte's college career was interrupted by World War II. In January 1941 Tutte was invited to Bletchley Park, the organization of code breakers in the United Kingdom. In October of that year, Tutte encountered machine-coded messages from Berlin named Fish, used only by the German army. The Bletchley code breakers had been successful in deciphering naval and air-force versions of Enigma codes but had no success with the army version. The Fish code was used for high-level communications. With only samples of messages to go by, Tutte discovered the structure of machines that generated these codes. Tutte's work has been described as the "greatest intellectual feat of the whole war".

After World War II, Tutte returned to Cambridge, but this time as a graduate student in mathematics. Tutte completed his PhD at the University of Cambridge but before earning his PhD, he determined conditions under which a graph (bipartite or not) has a perfect matching.

Let $k(G)$ denote the number of components in a graph G and $k_o(G)$ denote the number of odd components (the components of odd order) in G.

Theorem 7.6 (Tutte's Theorem): *A graph G contains a perfect matching if and only if*

$$k_o(G - S) \leq |S|$$

for every proper subset S of V(G).

Stating one direction of Tutte's theorem in its contrapositive form tells us that if there is a proper subset S of vertices in a graph G such that $k_o(G - S) > |S|$, then G does not contain a perfect matching. In particular, if G is a graph of odd order and S is the empty set, then $k_o(G - S) \geq 1 > 0 = |S|$, implying that it's impossible for G to have a perfect matching. In addition, neither the graph G_1 nor the graph G_2 of Figure 7.5 has a perfect matching. If we let $S_1 = \{u\}$ in G_1 and $S_2 = \{x, y\}$ in G_2, then both $k_o(G_1 - S_1) = 3 > |S_1|$ and $k_o(G_2 - S_2) = 4 > |S_2|$. The conclusion then follows from Tutte's theorem.

JULIUS PETERSEN

The first paper on graph theory of a purely theoretical nature occurred in 1891 and was written by Peter Christian Julius Petersen (1839–1910). Petersen was born in Sorø, Denmark. He attended a private school and

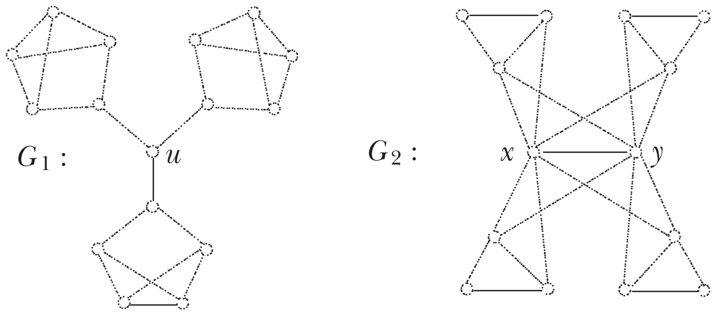


Figure 7.5. Two graphs without perfect matchings.

then the Sorø Academy, which was founded by King Frederick II of Denmark in 1586.

Young Julius left school in 1854 because his parents couldn't afford the expense of his education. He then went to work for his uncle. When his uncle died, enough money was left for Julius for him to attend the Polytechnical School at Copenhagen. As a teenager, Julius had published a book on logarithms. Even though Julius had passed the first part of a civil engineering examination, he decided to study mathematics at the university. Because the money that his uncle had left him was running out, he decided to take a job as a teacher. During the next few years, Julius had a heavy teaching load, was married and had a family—but he continued to study mathematics. While teaching, he recognized his talent for writing and wrote five textbooks on geometry. By the time he reached 30 years old, he had started working on a doctoral dissertation. Two years later, Petersen earned a PhD in mathematics from the University of Copenhagen.

Soon afterward, Petersen became a faculty member at the University of Copenhagen, where he was recognized as an outstanding teacher. There is a humorous story about an occasion when he was teaching and was baffled by the textbook he was using where it was written "it is easy to see" and this book was written by Petersen. While Petersen was an excellent writer, there were times when the exposition of his research took precedence over its rigor. Petersen liked to discover things on his own. The unfortunate part of this is that he would often not read the work of others and obtained results that were already known. He was also a bit casual when it came to referencing the work of others.

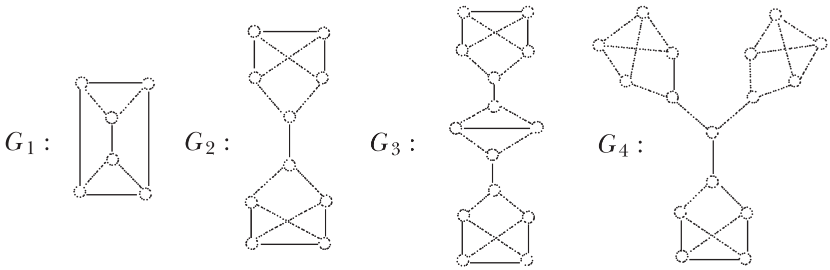


Figure 7.6. Cubic graphs and perfect matchings.

Although Petersen worked in several areas of mathematics, he is known only for his work in graph theory. Indeed, he is essentially known only for a single paper he wrote in 1891, namely "Die Theorie der regulären Graphen" ("The Theory of Regular Graphs"). It's obvious when 1-regular and 2-regular graphs contain perfect matchings. Every 1-regular graph *is* essentially a perfect matching. A 2-regular graph G has a perfect matching if and only if every component of G is an even cycle. Determining which 3-regular (cubic) graphs contain a perfect matching is considerably more complicated, however. Some cubic graphs contain perfect matchings and some don't. The cubic graphs G_1, G_2, G_3 of Figure 7.6 have perfect matchings while G_4 does not.

In Petersen's 1891 paper, he was able to prove a theorem that is commonly named for him.

Theorem 7.7 (Petersen's Theorem): *Every bridgeless cubic graph contains a perfect matching.*

Since the graph G_1 of Figure 7.6 is cubic and bridgeless, it contains a perfect matching. Actually, Petersen obtained an even stronger result. He showed that every cubic graph with at most two bridges contains a perfect matching. Therefore, it is not surprising that the graphs G_2 and G_3 of Figure 7.6 also contain perfect matchings even though they both have bridges. Since G_4 does not contain a perfect matching, obviously it does not follow that every cubic graph with at most three bridges contains a perfect matching. Therefore, what Petersen proved cannot be improved upon.

THE PETERSEN GRAPH

There are certain subgraphs that correspond to perfect matchings. A subgraph F of a graph G without isolated vertices and whose edge set $E(F)$ is a perfect matching of G is necessarily a 1-regular spanning subgraph of G. Such a subgraph F is called a 1-*factor* of G. According to Petersen's theorem (Theorem 7.7), every cubic bridgeless graph G contains a 1-factor F_1. If the edges of the 1-factor F_1 are removed from G, then the resulting graph H is necessarily 2-regular. A possible question here is whether H also has a 1-factor. If it does, then H has two edge-disjoint 1-factors F_2 and F_3 and so G has three 1-factors in all, namely F_1, F_2 and F_3, no two of which have an edge in common. This brings up another concept.

A graph G is 1-*factorable* if G contains 1-factors $F_1, F_2, \ldots, F_r$ such that $\{E(F_1), E(F_2), \ldots, E(F_r)\}$ is a partition of $E(G)$. In this case, $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$ is called a 1-*factorization* of G. Necessarily then, every 1-factorable graph is an r-regular graph of even order for some positive integer r. For example, the 4-regular graph G in Figure 7.7 is 1-factorable (into four 1-factors). A 1-factorization of this graph is shown in Figure 7.7 as well.

This suggests a question: Is every cubic bridgeless graph 1-factorable? Petersen answered this question in a follow-up paper he wrote in 1898. In this paper, he gave an example of a cubic bridgeless graph which he drew as in Figure 7.8a. Petersen was not the first person to

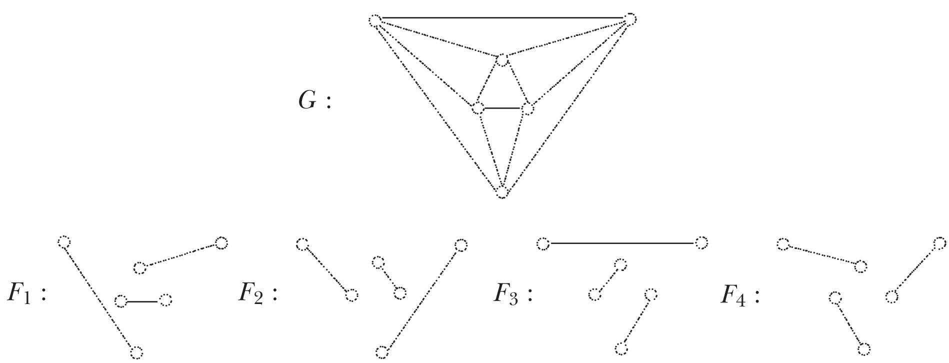


Figure 7.7. A 1-factorization of a 4-regular graph.

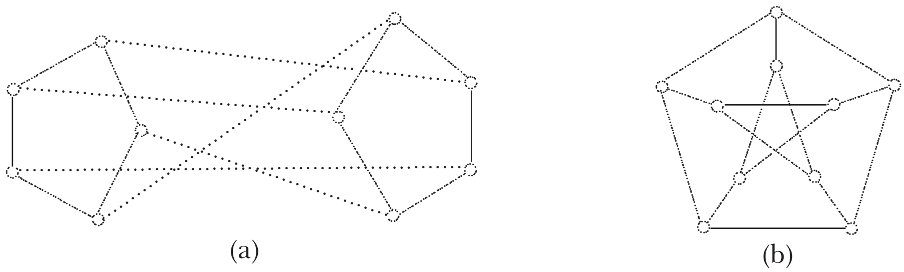


Figure 7.8. The Petersen graph.

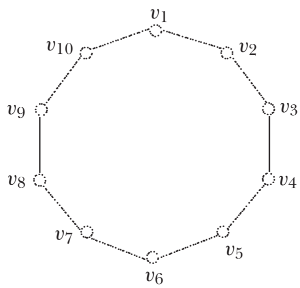


Figure 7.9. A step in the proof of Theorem 7.8.

consider this graph. In 1886 this graph had appeared in a paper written by Alfred Bray Kempe. (We'll meet this person again in Chapter 11.) This graph is usually drawn as shown in Figure 7.8b. We have seen this graph before and it's called the Petersen graph, named for Julius Petersen.

This graph has a number of interesting properties. In particular, the length of a smallest cycle in the Petersen graph is 5. In fact, the Petersen graph is the unique cubic graph of smallest order whose smallest cycle has length 5. For $k \geq 3$, a graph G is a *k-cage* if G is a cubic graph of smallest order whose smallest cycle has length k. So the Petersen graph is the unique 5-cage. Another interesting property of the Petersen graph is given next.

Theorem 7.8: *The Petersen graph is not Hamiltonian.*

Proof: Suppose that the Petersen graph is Hamiltonian. Then P has a Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{10}, v_1)$. (See Figure 7.9.) Since P is cubic, every vertex of P is incident with exactly one edge of P that is not on C.

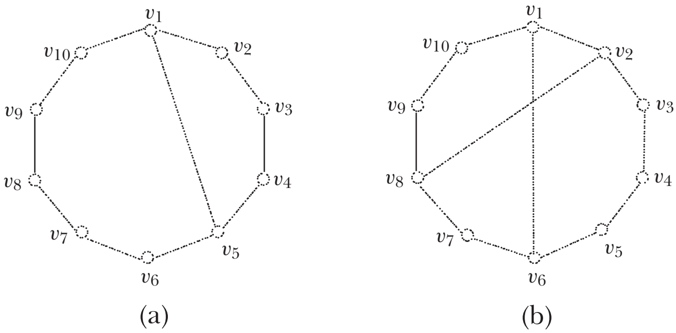


Figure 7.10. A step in the proof of Theorem 7.8.

Because v_1 is incident with one edge that is not on C, and P contains no triangles or 4-cycles, v_1 must be adjacent to either v_5, v_6 or v_7. Since v_1 being adjacent to v_5 or to v_7 is, by symmetry, the same case, there are only two different cases.

Case 1. *The vertex v_1 is adjacent to v_5.* (See Figure 7.10a.) The vertex v_6 can only be adjacent to v_2 or to v_{10}. For example, if v_6 is adjacent to v_2, then P would contain the 4-cycle $(v_1, v_2, v_6, v_5, v_1)$. In either case, a cycle of length less than 5 is produced, which is impossible.

Case 2. *The vertex v_1 is adjacent to v_6.* Necessarily, v_2 must be adjacent to v_8, for otherwise, P contains a cycle of length less than 5. (See Figure 7.10b.) However, then we are essentially back to Case 1, which we saw was impossible. ■

While we have now seen that the Petersen graph is not Hamiltonian and that the smallest cycle in this graph is a 5-cycle, the reason Petersen discussed this graph in his 1898 paper is because of another property it does not have.

Theorem 7.9: *The Petersen graph is not* 1*-factorable.*

Proof: Since the Petersen graph P is a cubic bridgeless graph, it has a 1-factor F by Petersen's theorem. If the edges of F are removed from P, then what remains is a 2-regular graph H. Therefore, every component of H is a cycle. Since P is not Hamiltonian by

Theorem 7.8, it follows that H must have at least two components. Since P has neither a triangle nor a 4-cycle, the only possibility is that H consists of two 5-cycles. Since a 5-cycle does not have a 1-factor, the Petersen graph is not 1-factorable. ∎

The fact that Petersen introduced the Petersen graph to show that a cubic bridgeless graph need not be 1-factorable and the Petersen graph also illustrates the fact that a cubic bridgeless graph need not be Hamiltonian are, as it turns out, just two of the common reasons why the Petersen graph appears so often throughout graph theory. In fact, this graph has often been a counterexample to conjectures or an example to illustrate why some statement is not true for all graphs under consideration.

1-FACTORABLE GRAPHS

Even though the Petersen graph is not 1-factorable, there are some important regular graphs that are.

Theorem 7.10: *For every even integer $n \geq 2$, the complete graph K_n is 1-factorable.*

Proof: It is easy to see that K_n is 1-factorable if $n = 2$ or $n = 4$. So assume that $n \geq 6$ is an even integer and let the vertices of K_n be $v_0, v_1, v_2, \ldots, v_{n-1}$. Let $v_1, v_2, \ldots, v_{n-1}$ be equally spaced about a circle and place v_0 in the center of the circle. Draw each edge of K_n as a straight-line segment. From this, we can construct $n-1$ 1-factors $F_1, F_2, \ldots, F_{n-1}$ resulting in a 1-factorization $\{F_1, F_2, \ldots, F_{n-1}\}$ of K_n. For $1 \leq i \leq n-1$, let the 1-factor F_i consist of the edge v_0v_i together with all edges of K_n that are perpendicular to v_0v_i. ∎

Following the proof of Theorem 7.10, the 1-factors F_1 and F_2 of K_6 are shown in Figure 7.11, where the edges of F_1 are drawn as solid edges and the edges of F_2 are drawn as dashed edges.

The fact that K_n is 1-factorable when n is even can be used to solve certain kinds of scheduling problems.

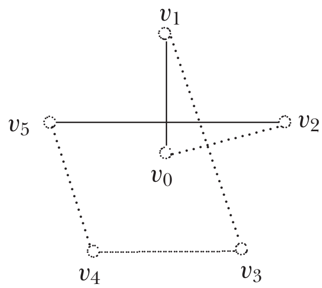


Figure 7.11. Two 1-factors F_1 and F_2 of K_6.

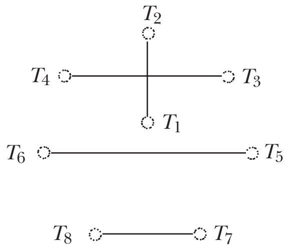


Figure 7.12. Constructing a 1-factorization of K_8.

Example 7.11: *From a group of eight tennis players $T_1, T_2, \ldots, T_8$, we'd like to schedule four tennis matches per day for seven days such that no one has two tennis matches on the same day and everyone plays a tennis match against all of the other seven players. Is this possible?*

SOLUTION:

According to Theorem 7.10, such a schedule must exist. Following the proof of Theorem 7.10, we describe one way to construct such a schedule. Place the seven vertices (tennis players) $T_2, T_3, \ldots, T_8$ equally spaced on a circle (see Figure 7.12) and then place T_1 at the center of the circle. Join every two vertices by a straight-line segment to construct the complete graph K_8. A 1-factor F_1 of K_8 can be constructed by taking the edge T_1T_2 and all edges perpendicular to this edge (again, see Figure 7.12). More generally, for $1 \leq i \leq 7$, a 1-factor F_i can be constructed by the edge T_1T_{i+1} and all edges perpendicular to this edge. This produces a 1-factorization of K_8.

G :

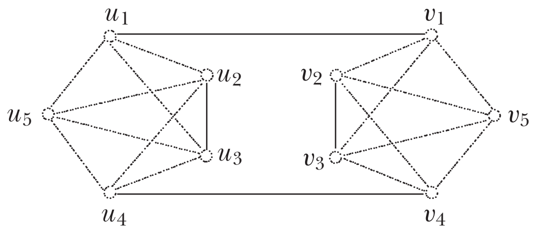


Figure 7.13. A 4-regular graph of order 10 that is not 1-factorable.

From this factorization, a schedule of tennis matches can be constructed, where the matches occurring on day i $(1 \leq i \leq 7)$ are the edges belonging to the 1-factor F_i:

Sunday:	T_1–T_2,	T_3–T_4,	T_5–T_6,	T_7–T_8;
Monday:	T_1–T_3,	T_2–T_5,	T_4–T_7,	T_6–T_8;
Tuesday:	T_1–T_4,	T_2–T_6,	T_3–T_8,	T_5–T_7;
Wednesday:	T_1–T_5,	T_3–T_7,	T_2–T_8,	T_4–T_6;
Thursday:	T_1–T_6,	T_4–T_8,	T_2–T_7,	T_3–T_5;
Friday:	T_1–T_7,	T_5–T_8,	T_3–T_6,	T_2–T_4;
Saturday:	T_1–T_8,	T_6–T_7,	T_4–T_5,	T_2–T_3.

♦

We have already seen that the Petersen graph is a 3-regular graph of order 10 that is not 1-factorable. In fact, the graph G shown in Figure 7.13 is a 4-regular graph of order 10 that is also not 1-factorable.

There is, however, no example of a 5-regular graph of order 10 that is not 1-factorable. In fact, a conjecture in this area was made by Amanda Chetwynd and Anthony Hilton and possibly by Gabriel Dirac as well. This conjecture provides conditions under which a regular graph must be 1-factorable.

The 1-Factorization Conjecture

Let G be an r-regular graph of order $2k$.

(a) If k is odd and $r \geq k$, then G is 1-factorable.

(b) If k is even and $r \geq k - 1$, then G is 1-factorable.

Therefore, if the 1-Factorization Conjecture is true and G is an r-regular graph of order n where $r \geq n/2$, then G is 1-factorable.

2-FACTORABLE GRAPHS

We have seen that a 1-factor of a graph G is a 1-regular spanning subgraph of G, while G is 1-factorable if G contains a collection of 1-factors such that each edge of G belongs to exactly one of these 1-factors. Furthermore, a 1-factorable graph G must be r-regular for some positive integer r, in which case, there is a 1-factorization of G into r 1-factors.

A 2-*factor* of a graph G is, not surprisingly, a 2-regular spanning subgraph of G. A graph G is 2-*factorable* if G contains a collection of 2-factors such that each edge of G belongs to exactly one of these 2-factors. Necessarily, every 2-factorable graph must be r-regular for some positive even integer r. Julius Petersen showed that this necessary condition for a graph to be 2-factorable is sufficient as well. This theorem appeared in the same paper as Theorem 7.7.

Theorem 7.12: *A graph G is 2-factorable if and only if G is r-regular for some positive even integer r.*

Since the graph G of Figure 7.14 is 4-regular, it follows by Theorem 7.12 that G is 2-factorable. A 2-factorization $\{F_1, F_2\}$ of G is shown in Figure 7.14, where the 2-factor F_1 consists of two triangles and F_2 consists of a single cycle, namely a Hamiltonian cycle.

Every 2-factor in a graph is a union of cycles. If each 2-factor in a 2-factorization of a graph G is a single cycle, then there is a factorization

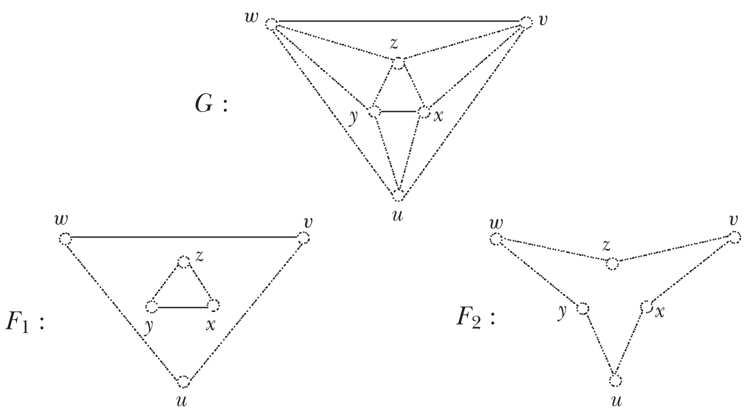


Figure 7.14. A 2-factorization of a 4-regular graph.

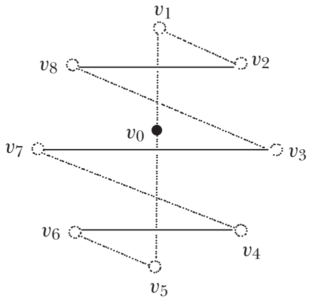


Figure 7.15. Constructing a Hamiltonian factorization of K_9.

of G into Hamiltonian cycles. A graph with this property is called *Hamiltonian-factorable*. Since every Hamiltonian-factorable graph is also 2-factorable, every Hamiltonian-factorable graph is r-regular for some positive even integer r.

While every Hamiltonian-factorable graph is 2-factorable, the converse is not true. For example, the 2-factorable graph G in Figure 7.13 is not Hamiltonian-factorable. To see this, suppose that G is Hamiltonian-factorable. Then G contains two Hamiltonian cycles C and C' that have no edges in common. One of these cycles, say C, contains the edge u_1v_1. In fact, we may assume that cycle C begins with u_1, v_1. Since C terminates at u_1, the cycle C must contain the edge v_4u_4. Since the 2-regular graph H obtained from G by removing the edges of C is disconnected, H cannot contain a Hamiltonian cycle and so G is not Hamiltonian-factorable.

One well-known class of Hamiltonian-factorable graphs is the class of complete graphs of odd order. A proof of this fact was given in 1890 by Walecki.

Theorem 7.13: *For every odd integer $n \geq 3$, the complete graph K_n is Hamiltonian-factorable.*

For example, to see that K_9 is Hamiltonian-factorable, we can place the eight vertices $v_1, v_2, \ldots, v_8$ cyclically as the vertices of a regular 8-gon (see Figure 7.15) and place v_0 at some convenient location within this 8-gon. Then a Hamiltonian cycle can be constructed as shown in Figure 7.15. By performing a clockwise rotation of this cycle three times through an angle of $45°$, three additional Hamiltonian cycles are formed that produce a Hamiltonian factorization of K_9.

8

Decomposing Graphs

There are many examples of mathematicians who were very young when they made their most famous discoveries. In fact, it is thought by many that the best work of mathematicians occurs during their early years. While this may very well be true of many mathematicians, it is certainly not true of all. One of the major figures in nineteenth-century combinatorics was Thomas Penyngton Kirkman. To many, Kirkman was thought to be an amateur mathematician whose contribution to mathematics consisted of a single problem he invented that dealt with 15 schoolgirls. Kirkman was no amateur mathematician, however. Indeed, he authored some 60 papers that made significant contributions to mathematics, all of which were written when he was 40 years of age or older.

Thomas Kirkman (1806–1895) was born in a small town near Manchester, England. He was by far the best student in the grammar school he attended. Kirkman's father was a cotton dealer of modest wealth and insisted that his son should work in the family business. Kirkman was forced to leave school at age 14 and for the next nine years worked in his father's office. At the age of 23 he broke away from his father and entered Trinity College in Dublin, Ireland. After four years, in 1833, he had earned his undergraduate degree, studying mathematics, philosophy, science and the classics. He returned to England that year and entered the Church of England, spending five years as a parish priest.

In 1839 Kirkman became pastor of a parish in Lancashire, a position he held for the next 52 years. By the time he was approaching 40 years old, his wife had inherited some property and, although not a wealthy man, Reverend Kirkman now held a respected position and was financially secure. While there was no evidence that he had any special interest in mathematics, Kirkman clearly had mathematical ability.

All that was lacking was a stimulus that would draw him into mathematics. Kirkman found it with a problem he encountered in a magazine.

A popular journal during the 1840s was the *Lady's and Gentleman's Diary*. In 1844 the editor Wesley Woolhouse stated what was called Prize Question No. 1733 in the magazine:

> Determine the number of combinations that can be made of n symbols, p symbols in each; with this limitation, that no combination of q symbols which may appear in any one of them shall be repeated in any other.

The 1845 volume of the journal contained many attempted but unsuccessful solutions to this rather awkwardly worded problem. After a year, the problem was replaced by the special case where $p = 3$ and $q = 2$. The editor drew attention to the difficulties of the problem, pointing out that when $n = 10$, it is impossible to find a system of triples ($p = 3$) in which each pair ($q = 2$) occurs exactly once.

On 15 December 1846 Kirkman presented a paper dealing with this substitute Prize Question to the Literary and Philosophical Society of Manchester. Shortly afterward (in 1847) an article by him was published in the *Cambridge and Dublin Mathematical Journal*. In it, he addressed the following problem:

> How many triples can be formed with x symbols in such a way that no pair of symbols occurs more than once in the triple?

STEINER TRIPLE SYSTEMS

For an integer $n \geq 3$, a system S_n of triples of n symbols is called a *Steiner triple system* if every pair of symbols occurs in exactly one triple.

Trivially, the system S_3 of any three symbols consists of a single triple, where, of course, each pair of symbols belongs to this triple. In the preceding chapter, we actually saw an example of a Steiner triple system S_7 of seven symbols. In a graduate class of seven students, denoted by a, b, c, d, e, f, g, the students were to divide themselves into seven study groups of three students each (seven triples) such that every pair of students belongs to exactly one of these study groups. It was asked whether this was even possible. It was. In fact,

$$S_7 = \{\{b, e, g\}, \{c, f, g\}, \{a, e, f\}, \{a, d, g\}, \{c, d, e\}, \{b, d, f\}, \{a, b, c\}\}$$

is a Steiner triple system of seven symbols.

Steiner triple systems can be looked at in terms of graphs. Suppose that we have a collection of $n \geq 3$ symbols, say $\{1, 2, \ldots, n\}$. We then construct the complete graph K_n of order n with vertex set $\{1, 2, \ldots, n\}$. The number of edges in K_n equals the number of pairs of vertices in K_n. This number is commonly denoted by $\binom{n}{2}$ and called *n choose* 2 since it counts the number of ways to choose 2 vertices among n, where order is not important. Thus, the size of K_n is $\binom{n}{2} = n(n-1)/2$. If it is possible to find a collection of triangles in K_n such that every edge of K_n belongs to exactly one of these triangles, then we have a Steiner triple system. For example, the complete graph K_7 with vertex set $\{a, b, c, d, e, f, g\}$, shown in Figure 8.1, contains seven triangles (seven study groups) such that every edge of K_7 belongs to exactly one of these triangles. This is a Steiner triple system.

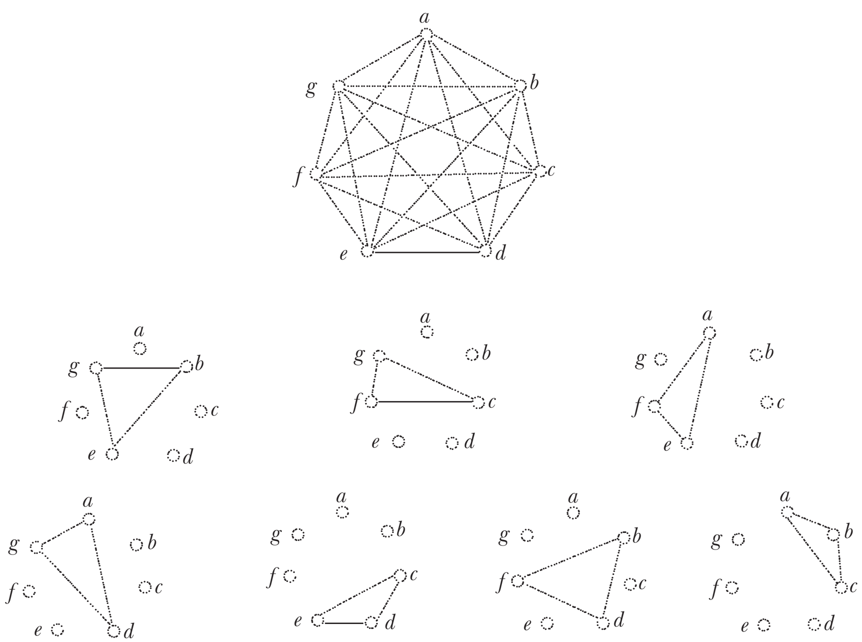


Figure 8.1. A Steiner triple system S_7.

We have now seen that there is both a Steiner triple system S_3 and a Steiner triple system S_7. This brings up a question: If S_n is a Steiner triple system, then what do we know about n?

Theorem 8.1: *If S_n is a Steiner triple system, then either $n \geq 3$ and $n = 6q + 1$, or $n = 6q + 3$ for some integer q.*

Proof: Suppose that S_n is a Steiner triple system. Then the complete graph K_n has a collection of triangles such that every edge of K_n belongs to exactly one of these triangles. Since the size of K_n is $n(n-1)/2$ and a triangle contains three edges, it follows that 3 divides $n(n-1)/2$ and so $n(n-1)/6$ is an integer. Furthermore, the degree of every vertex of K_n is $n-1$. Since every vertex of K_n is incident with two edges in each triangle containing the vertex, $n-1$ must be even and so n is odd. When an integer n is divided by 6, a quotient q and a remainder r are obtained. This implies that every integer n can be expressed as $n = 6q + r$, where $0 \leq r \leq 5$. Since n is odd, either $n = 6q + 1$, $n = 6q + 3$ or $n = 6q + 5$. Should $n = 6q + 5$, then the fact that 6 divides $n(n-1)/2$ says that 3 divides $(6q + 5)(3q + 2)$. However, this is not possible since 3 divides neither $6q + 5$ nor $3q + 2$. So n cannot equal $6q + 5$. Thus $n = 6q + 1$ or $n = 6q + 3$. ■

What Kirkman was able to prove in his 1847 paper is that the converse of Theorem 8.1 is also true, that is, if $n \geq 3$ and $n = 6q + 1$ or $n = 6q + 3$ for some integer q, then there is a Steiner triple system on n symbols.

The history of this problem did not end with Kirkman's interesting paper and the solution it contained. Some six years later, in 1853, the famous geometer Jakob Steiner (1796–1863) wrote a short note in which he brought up the question of the existence of such triples, obviously being unaware of Kirkman's paper. Six years after Steiner's note was published, Steiner's question was answered by M. Reiss. Both Steiner's note and Reiss's paper were published in *Journal für die reine und angewandte Mathematik* (*Journal for Pure and Applied Mathematics*). This journal was founded by August Leopold Crelle and he was its editor until he died in 1855. In fact, this journal was commonly called *Crelle's Journal*. It was one of the first major mathematics journals that wasn't the proceedings of an academy.

What Reiss did then was to answer Steiner's question 12 years after Kirkman had posed and solved the very same problem. This clearly annoyed Kirkman and caused him to write,

> How did the *Cambridge and Dublin Mathematical Journal* Vol. II, p. 191, contrive to steal so much from a later paper in *Crelle's Journal* Vol. LVI, p. 326, on exactly the same problem on combinations?

Even though the triple systems described by Kirkman occurred in print much earlier, they were eventually named for Jacob Steiner, giving rise to the concept of Steiner triple systems. This is certainly not the only example where credit was given to the wrong person.

As we mentioned, Steiner triple systems can be described in terms of graphs. A graph G is called *K_3-decomposable* if G contains pairwise edge-disjoint triangles $T_1, T_2, \ldots, T_k$ such that every edge of G belongs to (exactly) one of these triangles. Obviously, if G is K_3-decomposable, then G has size $3k$ for some positive integer k. Consequently, there is a Steiner triple system S_n if and only if K_n is K_3-decomposable. The concept of a graph being K_3-decomposable is a special case of a more general concept.

A graph G is said to be *decomposable* into the subgraphs $H_1, H_2, \ldots, H_k$ if every edge of G belongs to exactly one of these subgraphs. These subgraphs result in a *decomposition* of G. If all of the subgraphs $H_1, H_2, \ldots, H_k$ are isomorphic to the same graph H, then the graph G is called *H-decomposable* and the decomposition is an *H-decomposition*. If a graph G of size m is H-decomposable where H has size m', then m' divides m. So if $H = K_3$ and $G = K_n$ for some integer $n \geq 3$, then we're back to a Steiner triple system S_n.

After the publication of Kirkman's 1847 paper, Kirkman noticed that the Steiner triple system S_{15} had an interesting additional property. Suppose that the 15 symbols of S_{15} are denoted by the seven integers 1, 2, 3, 4, 5, 6, 7 and the eight letters a, b, c, d, e, f, g, h. Since the size of K_{15} is $\binom{15}{2} = 15 \cdot 14/2 = 105$, the $105/3 = 35$ triples of this Steiner triple system can be divided into seven sets of five triples in such a way that each of the 15 symbols occurs exactly once in each of these five triples. This is shown in Figure 8.2.

In the *Lady's and Gentleman's Diary* for 1850, Kirkman challenged the readers to discover such an arrangement for themselves by stating the problem in a unique way.

123	145	167	357	346	256	247
$4ae$	$2bd$	$2ac$	$1eb$	$1cd$	$1ef$	$1gh$
$5cg$	$3fh$	$3eg$	$2fg$	$2eh$	$3bc$	$3ad$
$6bh$	$6ag$	$4bf$	$4ch$	$5af$	$4dg$	$5be$
$7df$	$7ce$	$5dh$	$6de$	$7bg$	$7ah$	$6cf$

Figure 8.2. An arrangement of the 35 triples in a Steiner triple system S_{15}.

Kirkman's Schoolgirl Problem

Fifteen young ladies in a school walk out three abreast for seven days in succession; it is required to arrange them daily, so that no two shall walk twice abreast.

The British mathematician Norman Biggs stated that this clever wording of Kirkman's Schoolgirl Problem may have had a negative effect on how history has viewed Kirkman:

> It is unfortunate that such a trifle should overshadow the many more significant contributions which its author was to make to mathematics. Nevertheless, it is his most lasting memorial.

CYCLE DECOMPOSITIONS

Of course, decomposing a complete graph K_n into triangles is the same as decomposing K_n into 3-cycles. A necessary condition for K_n to be C_3-decomposable is that n is odd and 3 divides $\binom{n}{2}$. As we have seen, Kirkman proved that these conditions are sufficient as well. At the other extreme of decomposing K_n into cycles of smallest length is decomposing K_n into cycles of largest length. In this case, we are referring to decomposing the complete graph of order n into Hamiltonian cycles C_n. Any complete graph K_n with this property is Hamiltonian-factorable. A necessary condition for K_n to be Hamiltonian-factorable is that n is odd and n divides $\binom{n}{2}$. However, when n is odd, n always divides $\binom{n}{2}$. That K_n is Hamiltonian-factorable when n is odd has been known for a long time. We mentioned that this was verified by Walecki in 1880. Evidently, little is known about this individual, however.

We have now seen that if $n \geq 3$ is any odd integer such that 3 divides $\binom{n}{2}$, then K_n is C_3-decomposable and if $n \geq 3$ is any odd integer, then K_n is C_n-decomposable (or Hamiltonian-factorable). In 1981 Brian Alspach conjectured that these results may be special cases of a more general result.

Alspach's Conjecture

Suppose that $n \geq 3$ is an odd integer and that $m_1, m_2, \ldots, m_t$ are integers such that $3 \leq m_i \leq n$ for each i $(1 \leq i \leq t)$ and $m_1 + m_2 + \cdots + m_t = \binom{n}{2}$. Then K_n can be decomposed into the cycles $C_{m_1}, C_{m_2}, \ldots, C_{m_t}$. In particular, if m is an integer with $3 \leq m \leq n$, where n is odd and m divides $\binom{n}{2}$, then K_n is C_m-decomposable.

In 2012 Darryn Bryant, Daniel Horsley and William Pettersson not only verified Alspach's Conjecture but a related conjecture of Alspach as well.

Theorem 8.2 (Bryant, Horsley and Pettersson's Theorem):

(i) If $n \geq 3$ is an odd integer, $m_1, m_2, \ldots, m_t$ are integers such that $3 \leq m_i \leq n$ for each i $(1 \leq i \leq t)$ and $m_1 + m_2 + \cdots + m_t = \binom{n}{2} = n(n-1)/2$, then K_n can be decomposed into the cycles $C_{m_1}, C_{m_2}, \ldots, C_{m_t}$.

(ii) If $n \geq 4$ is an even integer, M is a perfect matching in K_n, $m_1, m_2, \ldots, m_t$ are integers such that $3 \leq m_i \leq n$ for each i $(1 \leq i \leq t)$ and $m_1 + m_2 + \cdots + m_t = \binom{n}{2} - \frac{n}{2} = n(n-2)/2$, then $K_n - M$ can be decomposed into the cycles $C_{m_1}, C_{m_2}, \ldots, C_{m_t}$.

Example 8.3: *Suppose that $n = 5$. Then $\binom{n}{2} = \binom{5}{2} = 10$ and $5 + 5 = 3 + 3 + 4 = 10$. Decompositions of K_5 into C_5, C_5 and C_3, C_3, C_4 are shown in Figures 8.3a, b. If $n = 6$, then $\binom{n}{2} - \frac{n}{2} = \binom{6}{2} - \frac{6}{2} = 12$ and $3 + 3 + 3 + 3 = 4 + 4 + 4 = 3 + 3 + 6 = 12$, for example. Decompositions of $K_6 - M$ for a perfect matching M of K_6 into C_3, C_3, C_3, C_3 and C_4, C_4, C_4 as well as C_3, C_3, C_6 are shown in Figures 8.3c–e. (The matching M removed from K_6 is denoted by dashed lines.)*

Now, suppose that a connected graph G can be decomposed into cycles in some manner. Since each cycle in the decomposition containing

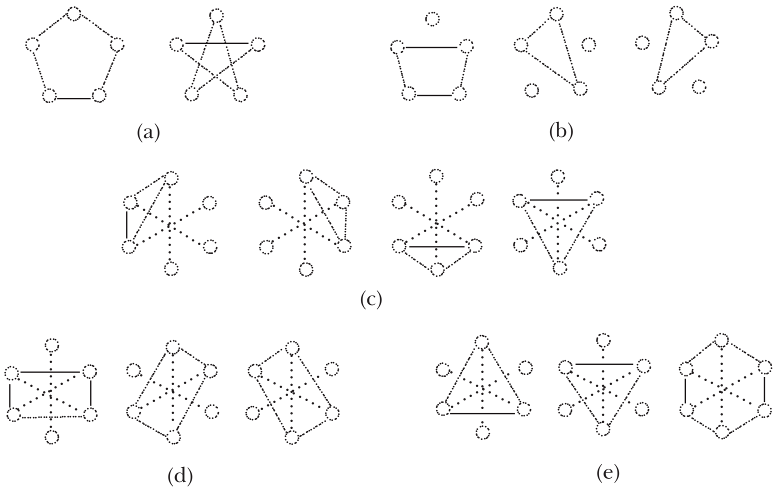


Figure 8.3. Cycle decompositions of K_5 and $K_6 - M$.

a vertex v of G contributes 2 to the degree of v, it follows that v has even degree. Therefore, every vertex of G is even and so G is Eulerian. Consequently, every connected graph that has a cycle decomposition is Eulerian. Oswald Veblen (1880–1960) showed that the converse is true. In 1922 Veblen wrote a book titled *Analysis situs*, a major early book devoted to a mathematical area that would become topology. Chapter 1 of this book was titled "Linear Graphs" and dealt with graph theory. That is, Veblen's book, containing a chapter on graph theory, was published years before the first published book entirely devoted to graph theory.

Theorem 8.4 (Veblen's Theorem): *Every Eulerian graph has a cycle decomposition.*

Proof: In an Eulerian graph G, every vertex has even degree and since there are no isolated vertices, the degree of every vertex is at least 2. Such a graph must contain at least one cycle C. The new graph H obtained by removing the edges of C from G still has the property that every vertex has even degree and so the components of H are Eulerian. This process can be repeated on each component of H until eventually all edges of G are contained in a cycle. ■

GRACEFUL GRAPHS

While a great deal of attention has been focused on decomposing complete graphs of odd order into cycles, particularly cycles of the same length, there has also been considerable interest in decomposing complete graphs into copies of a graph that is not necessarily a cycle. Much of the interest in this topic can be traced to a 1967 paper of Alexander Rosa dealing with labeling the vertices of a graph.

For a graph G of order n and size m, a labeling of the vertices of G with distinct elements of $\{0, 1, \ldots, m\}$ was called a *β-valuation* (*beta valuation*) of G by Rosa if, in the resulting labeling of the edges of G that assigns the label $|a - b|$ to edge uv if u and v have been labeled a and b, distinct edges receive distinct labels. In 1972 Solomon Golomb referred to a β-valuation as a *graceful labeling* and it is this terminology that has been adopted since then. A graph that has a graceful labeling is called a *graceful graph*. For example, the three graphs shown in Figure 8.4 are all graceful. A graceful labeling is exhibited in each case, where the vertex label is placed within the vertex and the resulting edge label in each case is placed next to the edge.

If a graph G of size m has a graceful labeling f (that is, vertex v gets label $f(v)$), then the labeling g that assigns the label $g(v) = m - f(v)$ to each vertex v of G is also a graceful labeling of G, called the *complementary labeling* of f. That the labeling g is graceful follows because

$$|g(u) - g(v)| = |(m - f(u)) - (m - f(v))| = |f(u) - f(v)|$$

for each edge uv of G.

Not only is the complete graph K_3 of order 3 graceful (as shown in Figure 8.4), so too is K_4. However, such is not the case for K_5.

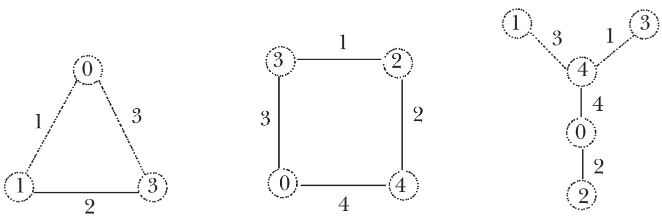


Figure 8.4. Three graceful graphs.

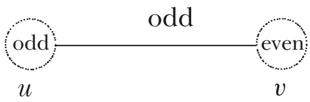


Figure 8.5. Showing that K_5 is not graceful.

Example 8.5: *The complete graph K_5 is not graceful.*

SOLUTION:

If K_5 were graceful, then because the size of K_5 is 10, it would be possible to label the vertices with five of the integers $0, 1, \ldots, 10$ in such a way that the edges are labeled $1, 2, \ldots, 10$. In particular, five edges of K_5 must be labeled with the odd integers 1, 3, 5, 7, 9. The only way that an edge uv of K_5 can be labeled with an odd integer is for one of u and v to be labeled with an odd integer and the other to be labeled with an even integer (see Figure 8.5). If all five vertices of K_5 are labeled with odd integers (or even integers), then no edge has an odd label. If exactly one vertex of K_5 has an odd label (or an even label), then exactly four edges receive an odd label. If exactly two vertices of K_5 have odd labels (or even labels), then exactly six edges receive an odd label. Therefore, it is not possible to label the vertices so that exactly five edges are labeled with odd integers. ♦

While many questions exist concerning which graphs are graceful, there is one well-known class of graphs, every member of which is believed to be graceful. The following conjecture is due to Gerhard Ringel and Anton Kotzig, who was the doctoral advisor of Alexander Rosa.

The Graceful Tree Conjecture

Every tree is graceful.

One of the major interests in graceful graphs lies in a theorem discovered by Rosa which states that if H is a graceful graph of size m, then the complete graph K_{2m+1} is H-decomposable. In fact, he showed that K_{2m+1} has a special kind of H-decomposition called a *cyclic decomposition*. That is, a copy of the graph H can be found in the graph K_{2m+1} and by a sequence of rotations, an H-decomposition of K_{2m+1} can be produced.

Theorem 8.6: *If H is a graceful graph of size m, then the complete graph K_{2m+1} has a cyclic H-decomposition.*

Proof: Since H is a graceful graph of size m, the vertices of H can be labeled with distinct elements of $\{0, 1, \ldots, m\}$ in such a way that the resulting edge labels are $1, 2, \ldots, m$. Let the vertex set of the graph K_{2m+1} be $\{v_0, v_1, \ldots, v_{2m}\}$ and place these vertices equally spaced on a circle. Every two vertices are joined by a straight-line segment, resulting in the complete graph K_{2m+1}. Thus $C = (v_0, v_1, \ldots, v_{2m}, v_0)$ is a Hamiltonian cycle in K_{2m+1}.

We now consider a particular copy of H in K_{2m+1}. If some vertex of H is labeled i $(0 \leq i \leq m)$ in the graceful labeling of H, then place this vertex at v_i in K_{2m+1}. This is done for each vertex of H. This copy of H in K_{2m+1} is denoted by H_1.

Each edge $v_s v_t$ in K_{2m+1} $(0 \leq s, t \leq 2m$, where $s \neq t)$ can be thought of as joining these two vertices in C. We label the edge $v_s v_t$ by the distance $d_C(v_s, v_t)$ between v_s and v_t on C. Since $1 \leq d_C(v_s, v_t) \leq m$, every edge of K_{2m+1} is assigned one of the labels $1, 2, \ldots, m$. In fact, there are $2m + 1$ edges of K_{2m+1} receiving each of these labels and H_1 has exactly one edge with each of these labels.

If we now rotate H_1 (clockwise, say) through an angle of $360^\circ/(2m + 1)$, we obtain another copy of H, edge disjoint with H_1, which we denote by H_2. Performing this rotation $2m$ times produces a cyclic H-decomposition of K_{2m+1} into $2m + 1$ copies $H_1, H_2, \ldots, H_{2m+1}$ of H. ■

To illustrate the proof of Theorem 8.6, consider the graph H of size $m = 4$ shown in Figure 8.6a. This graph is graceful and a graceful labeling is shown in the figure. This graceful labeling of H results in the copy H_1 of H in K_9 shown in Figure 8.6b. Rotating H_1 clockwise through an angle of $360^\circ/9 = 40^\circ$ produces the copy H_2 of H in K_9 shown in Figure 8.6c.

INSTANT INSANITY

The Guinness Book of World Records is well known for the multitude of records it lists on a wide variety of topics, including the top-selling

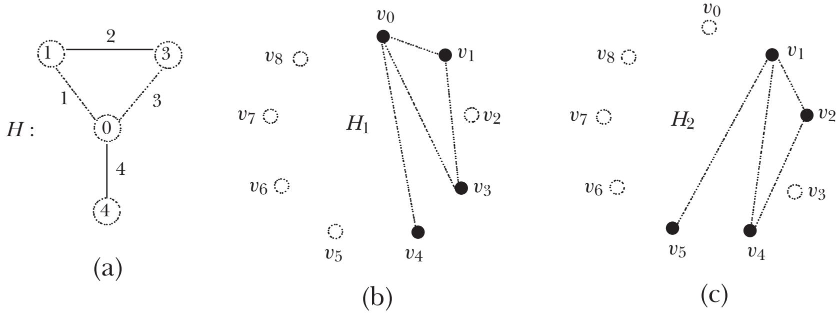


Figure 8.6. A cyclic H-decomposition of K_9.

toys, games and puzzles. For many years the board game Monopoly monopolized these lists. It is the top-selling board game of all time. Over 200 million of these games have been sold and over 500 million people have played it. It remains popular in various forms even today. While Rubik's cube, invented in 1974 by the Hungarian sculptor and professor of architecture Emő Rubik, became the top-selling puzzle game of all time, during 1966–1967 Monopoly was outsold by another puzzle. In 1966 the Guinness Book of World Records listed a puzzle known as Instant Insanity as the year's best seller. Over 12 million of these puzzles were sold during 1966–1967. What is this puzzle, what made it so popular and what is its connection with graph theory?

The puzzle Instant Insanity consists of four multicolored plastic cubes. Each of the six faces of each cube is colored with one of four colors, say red, blue, green and yellow. The object of the puzzle is to stack the four cubes on top of one another so that all four colors appear on all four sides of the stacked cubes.

What are the chances of solving this puzzle if we were to use a trial-and-error approach? Suppose that we refer to these four cubes as Cube 1, Cube 2, Cube 3 and Cube 4. Since we are only interested in the colors of the sides of the cube, we can assume that in a stacking of the cubes, Cube 1 is on the bottom, followed by Cube 2 and then Cubes 3 and 4. There are three ways of placing Cube 1 on a table depending on which pair of opposite faces are "buried", that is, serve as the top and bottom of this cube. Select one of the four faces appearing on the side of Cube 1 as the front of the cube. Next, Cube 2 is placed on top of Cube 1. One of the six faces of Cube 2 is chosen to be placed directly above the front face of Cube 1 and once this face is chosen, Cube 2 can be rotated in one of four positions. Therefore, there are $6 \cdot 4 = 24$ ways to place Cube 2 on top of Cube 1. Doing this for Cubes 3 and 4 as well, there are then $3(24)^3 = 41{,}472$ ways to stack the four cubes. If, in fact, the faces of these cubes have been colored in such a way that there is only one way to stack the cubes so that all four colors appear on all four sides, then the probability that we found the right way to stack the cubes is less than 0.000025. Therefore, finding the correct way to solve this puzzle by trial-and-error is likely to result in. . . *instant insanity*.

So where did this puzzle come from? It was conceived in 1900 by Fredrick Schossow using the four suits of playing cards (hearts, diamonds, clubs, spades) on the faces. He introduced another version during World War I where the flags of the allied nations were used to decorate the blocks. In 1900 this puzzle was called The Great Tantalizer. There are many other versions of this puzzle, known by such names as The Katzenjammer Puzzle, Symington's Puzzle, Frantic, Diabolical, The Cat Puzzle and Crazy Cubes. However, by far the best known and most popular version of the puzzle is one based on a design by Franz (Frank) Armbruster in mid-1965 and consists of four plastic cubes with each face having one of four different colors. (In those days, the colors red, blue, green and white were used.)

Frank Armbruster began as an educational consultant in 1960. While working on teaching machine designs, he saw similarities between what a teaching machine does and what a game does. In particular, if the rules of the game are structured from the rules of the subject matter, then the game will teach. Each game has a set of structured rules, a goal and an opportunity for strategies.

Armbruster had been interested in puzzles for much of his life and started designing games as teaching tools in 1965. He saw Instant Insanity as a great aid for teaching permutations and combinations at the high-school level. Originally the cubes were made of wood. Thinking that the grain of wood used was giving an unintended clue to the solver, he turned to constructing the cubes from plastic. Armbruster was able to schedule a lunch with a representative of Macy's in San Francisco to discuss his puzzle. This was the beginning of the puzzle as a commercial enterprise. The first version of his famous puzzle Instant Insanity was licensed to the Parker Brothers Game Company. Parker Brothers has manufactured such well-known games as Monopoly, Clue,

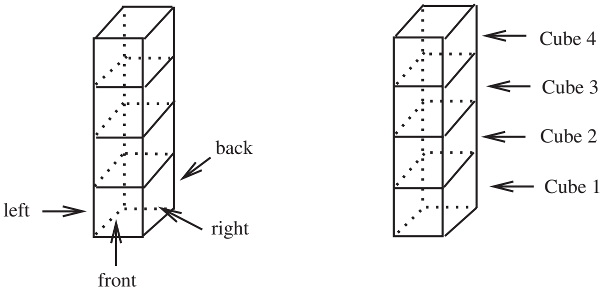


Figure 8.7. The stacking of four cubes.

Sorry! and Trivial Pursuit. In 1991 Parker Brothers was bought out by Hasbro.

Let us return to Instant Insanity.

> Open the package. Notice that there are four different colors showing on each side of this stack of blocks. You may NEVER, EVER see them this way again. Now mix them up and then restack them so that there are again four colors, all different, showing on each side.

What is written above appeared on an insert within packaging that contains the four multicolored cubes that make up Instant Insanity, which, as we mentioned, was later manufactured by Hasbro. Each face of each cube is colored with one of the four colors red (R), blue (B), green (G) and yellow (Y). As we stated, the object of the puzzle is to stack the cubes as in Figure 8.7, one on top of another, so that all four colors appear on each of the four sides.

On the reverse side of the insert is written, "Give up?". An address is supplied where a solution of the puzzle can be obtained. Reading all of this can be quite intimidating. Indeed, even before we attempt to solve the puzzle, we are being informed, if not warned, that it is very unlikely that we will be successful. After all, we stated that the number of ways to stack all four cubes on the top of one another is 41,472.

Graph theory can help us to solve this tantalizing puzzle. Let's see how this can be done. For this purpose, it is convenient to have a way of representing a cube and the locations of the colors on its faces. (See Figure 8.8.)

We are now prepared to present an example.

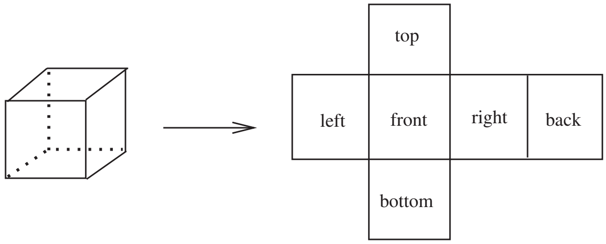


Figure 8.8. The six faces of a cube.

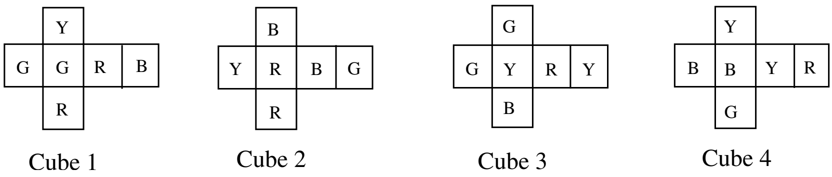


Figure 8.9. The four cubes in an Instant Insanity puzzle.

Example 8.7: *Consider the four multicolored cubes given in Figure 8.9.*

With each of the four cubes of Figure 8.9, we associate a multigraph. Actually in this case we permit an edge to join a vertex to itself. An edge of this type is called a *loop*. Each of the four multigraphs that we are about to describe has order 4 and size 3. The vertices of each multigraph are the four colors R, B, G, Y and there is an edge joining one color to another whenever there is a pair of opposite faces having these colors. Since there are three opposite pairs of faces, the multigraph has three edges. The four multigraphs corresponding to the four cubes of Figure 8.9 are shown in Figure 8.10.

Next, a "composite multigraph" M of order 4 (with vertices R, B, G, Y) and size 12 is constructed. The edges of the multigraph M are all the edges in the four multigraphs shown in Figure 8.10.

In order to distinguish which edges of M come from which cube, those three edges of M that come from Cube i $(i = 1, 2, 3, 4)$ are labeled i in M. So the composite multigraph M has three edges labeled 1, three edges labeled 2 and so on. The multigraph M constructed from the multigraphs of Figure 8.10 is shown in Figure 8.11.

We now pause in our solution of the example to make some general observations. First, let's review what our goal is. We wish to stack the four cubes on top of one another so that all four colors appear on all four

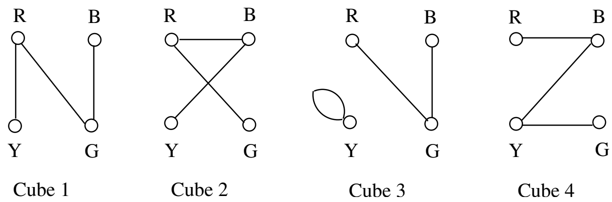


Figure 8.10. The four multigraphs in Example 8.7.

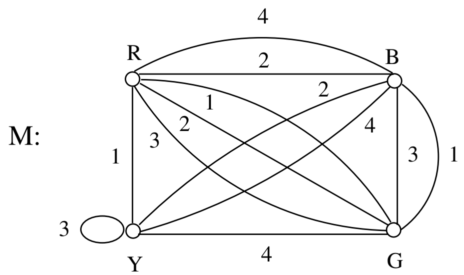


Figure 8.11. The composite multigraph of Example 8.7.

sides. Of course, all four colors must appear on both the front and the back of the stack. If the front face of Cube i $(i = 1, 2, 3, 4)$ is colored a and the opposite face of this cube (on the back of the stack) is colored b, then there must be an edge labeled i joining a and b in the multigraph M.

For example, suppose that a, b, c, d are the four colors red, blue, green, yellow in some order and in a stacking of the cubes that provides a solution to the puzzle, the front and back appear as in Figure 8.12. Then the composite multigraph M contains an edge bc labeled 1 because the front and back faces of Cube 1 are colored b and c. Similarly, M contains an edge db labeled 2, a loop at a labeled 3 and an edge cd labeled 4.

Since each color appears exactly once on the front and exactly once on the back of the stacked cubes, every vertex of the spanning submultigraph M' of M produced from the front and back has degree 2 (where a loop is considered to have degree 2) and exactly one edge labeled with each of the four numbers 1, 2, 3 and 4.

Similarly, corresponding to the right and left sides of the stack, there is also a spanning submultigraph M'' of M where every vertex of M'' has degree 2 and where no edge of M'' belongs to M'. If a solution to Instant Insanity exists, then we must find a way to stack the cubes so that all four colors appear on both the front and back and all four colors appear on

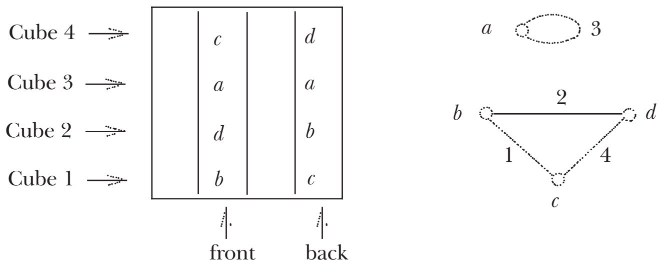


Figure 8.12. Some edges in the composite multigraph.

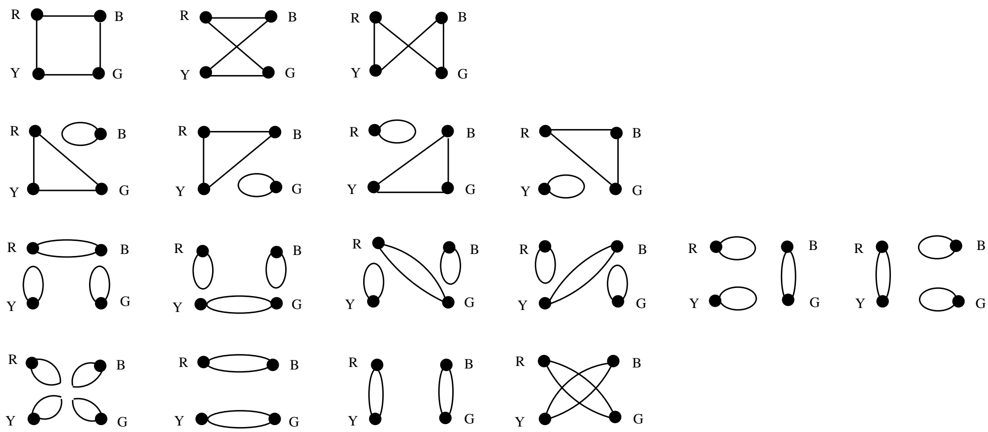


Figure 8.13. The 17 possible 2-regular multigraphs.

both the left side and right side. This says that there must be a way of decomposing M in some manner such that two of the multigraphs in the decomposition are two 2-regular spanning submultigraphs M' and M'' of M where the four edges in each of M' and M'' are labeled $1, 2, 3, 4$. The remaining multigraph (corresponding to the top and bottom sides of the stack) in the decomposition of M is of no interest to us and can be ignored.

If such a pair M', M'' of multigraphs does not exist, then the puzzle cannot have a solution, which would be very unfair to the person trying to solve the puzzle. If such a pair M', M'' of multigraphs exists, then we will see how these can be used to solve the puzzle, that is, to stack the cubes appropriately. Finding such a pair M', M'' of submultigraphs of M becomes the challenge then of finding a solution of Instant Insanity by this method. There are 17 2-regular multigraphs of order 4 with vertex set {R, B, G, Y}, which are shown in Figure 8.13.

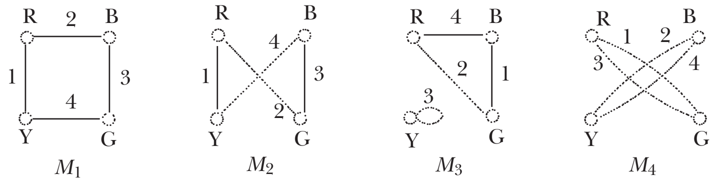


Figure 8.14. The four spanning 2-regular multigraphs of the multigraph M of Figure 8.11 whose edges are labeled $1, 2, 3, 4$.

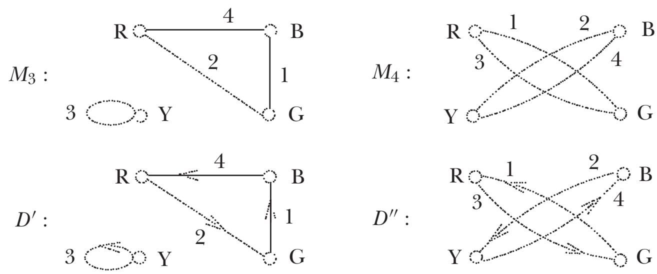


Figure 8.15. Two 2-regular spanning submultigraphs for Example 8.7.

Therefore, any 2-regular spanning submultigraph of M must be one of these 17 multigraphs. The multigraph M, however, contains only 4 of these multigraphs as spanning submultigraphs whose edges are labeled $1, 2, 3, 4$, namely the first and third multigraphs in the first row and the final multigraph in each of the second and fourth rows. These four multigraphs are shown in Figure 8.14 where they are denoted by M_1, M_2, M_3, M_4, respectively.

We now return to the puzzle in Example 8.7. Observe that the multigraph M of Figure 8.11 contains the two edge-disjoint submultigraphs M_3 and M_4 shown in Figure 8.15, where the edges of these two multigraphs are labeled 1, 2, 3 and 4. (Again, recall that finding such edge-disjoint submultigraphs is sometimes challenging.) Suppose that the multigraph M_3 corresponds to the front and back of the stack to be produced and M_4 corresponds to the right and left sides. (We could reverse what M_3 and M_4 stand for if we wish.)

For the purpose and convenience of stacking the cubes, we direct the edges of each component of M_3 and M_4 so that "directed cycles" result. Thus two "directed multigraphs" D' and D'' are produced, as shown in Figure 8.15.

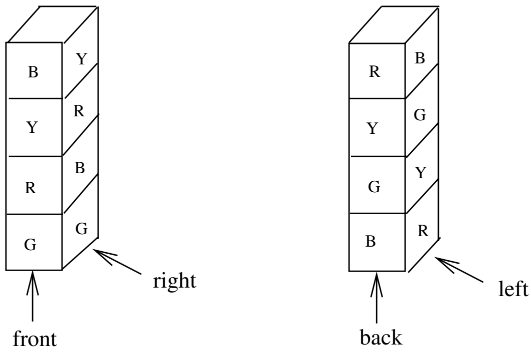


Figure 8.16. A solution for the puzzle in Example 8.7.

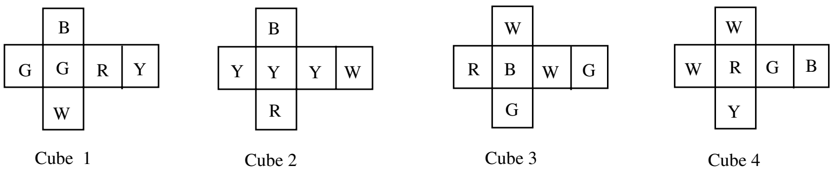


Figure 8.17. The four cubes in an Instant Insanity puzzle.

With the aid of the directed multigraphs D' and D'' of Figure 8.15, we now stack the cubes. Since the directed edge from G to B is labeled 1 in D', we place Cube 1 so that a green face appears on the front and a blue face on the back. Since the directed edge from G to R is labeled 1 in D'', we rotate this cube if necessary (keeping a green face on the front and a blue face on the back) until we have a green face on the right and a red face on the left. We now proceed in the same way with the other three cubes and... Voila! The puzzle has been solved (see Figure 8.16). ♦

The Instant Insanity puzzle in Example 8.7 is an actual puzzle that has been sold by the game company Hasbro. However, several other Instant Insanity puzzles could be created ourselves. Of course, if each face of the four cubes is colored with one of four colors at random, then there is no guarantee that there will be a solution.

Example 8.8: *In the Instant Insanity puzzle in Figure 8.17, each face of each cube is colored with one of the five colors red* (R)*, blue* (B)*, green* (G)*, yellow* (Y) *and white* (W)*. Show that it is possible to stack these four cubes in such a way that no color is repeated on any side.*

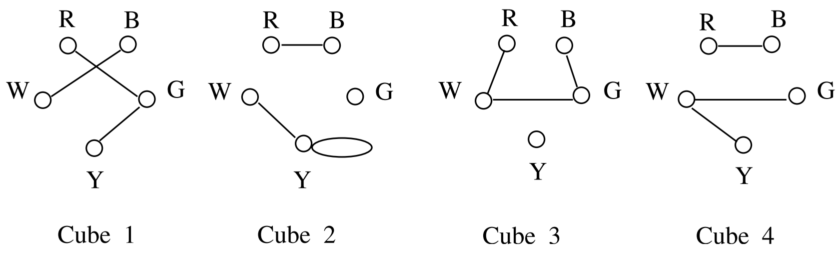


Figure 8.18. The four multigraphs in Example 8.8.

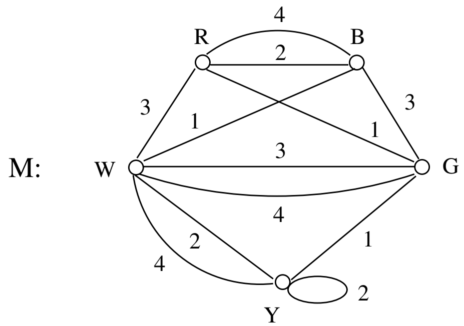


Figure 8.19. The composite multigraph in Example 8.8.

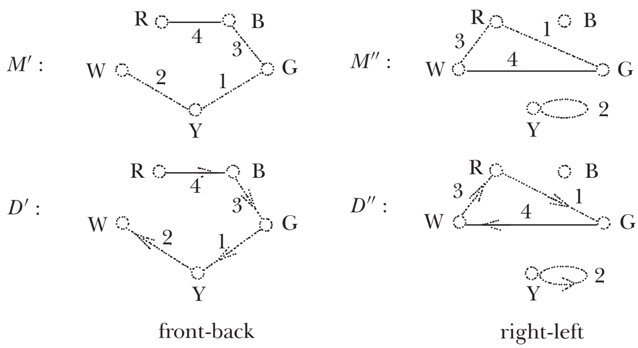


Figure 8.20. Two submultigraphs and directed multigraphs of M for Example 8.8.

SOLUTION:

With each of the four cubes in Figure 8.17, we associate a multigraph of order 5 and size 3 in Figure 8.18.

Next we construct a composite multigraph M of order 5 and size 12 whose edge set is the union of the edge sets of the four multigraphs in Figure 8.19. Each edge of M is labeled with the number of the cube containing this edge.

In the multigraph M of Figure 8.19 we seek two edge-disjoint submultigraphs M' and M'' of size 4 such that each vertex of M' and M'' has degree at most 2. Necessarily, each component of M' and M'' will have degree at most 2. Two such submultigraphs M' and M'' are shown in Figure 8.20. These submultigraphs are now converted into directed multigraphs D' and D'', respectively, by assigning directions to the edges so that each component that is a path is a directed path and each component that is a cycle is a directed cycle.

The directed edges of the directed multigraph D' correspond to the front and back of the stack and the directed edges of the directed multigraph D'' correspond to the right and left sides of the stack. ♦

9

Orienting Graphs

While Harvard University is well known for its academic reputation, there have been occasions when it was also known for its athletic achievements. In 1931 Harvard University's football team was led by all-American quarterback Barry Wood. He was one of the most prominent players of his time and appeared on the cover of the 23 November 1931 issue of *Time* magazine. On 17 October of that year, Harvard played Army and by the end of the first half, Harvard unexpectedly trailed 13–0. During halftime, Harvard President A. Lawrence Lowell, visibly upset, told Lieutenant Colonel (at the time) Robert C. Richardson, Jr., commandant of cadets at the U.S. Military Academy, that while Army was showing that they could defeat Harvard in football, Harvard could win any contest of a more scholarly nature. Richardson accepted Lowell's challenge and the decision was later made to have a mathematics competition between Army and Harvard.

The choice of mathematics for the subject of the competition was probably due to the fact that this was a subject taught at both schools and because George Putnam (a relative of President Lowell) was interested in mathematics and helped make arrangements for the competition. The contest was named for a relative of George Putnam: William Lowell Putnam. As it turned out, Barry Wood led Harvard in the second half to score two touchdowns and defeat Army 14–13. In fact, Harvard would remain undefeated until 21 November when it lost to Yale, 3–0.

HERBERT ROBBINS

An incoming freshman at Harvard during fall semester 1931 was Herbert Ellis Robbins (1915–2001). He had no mathematics knowledge beyond quadratic equations. Because of his mathematical deficiency, he decided to enroll in a course in analytic geometry and calculus. There were two textbooks for this course, *Analytic Geometry* and *Calculus*, both written by the distinguished Harvard Professor William Fogg Osgood. The mathematics class that Robbins took was taught by a junior instructor. Being occupied with other interests, Robbins cut many classes but did well on the exams. At the end of Robbins's freshman year (in May 1932), he was invited to be a member of the mathematics team that would represent Harvard in the forthcoming competition to take place against Army. Since Army cadets took only two years of mathematics, the Harvard team was restricted to students who would be completing their second year of mathematics at the time the exam was given. Because Robbins was selected as a member of the team, he decided to take a second year of mathematics at Harvard.

The Department of Mathematics at Harvard decided that Professor Marston Morse would coach the mathematics team. The team met with Morse during Robbins's sophomore year but because it was assumed that Harvard would easily defeat Army, much of their meeting time was spent with conversations rather than solving problems.

Robbins, however, was interested in learning mathematics and was impressed by Morse. The two had several conversations outside of the Putnam sessions. Morse had been in low spirits because his wife had left him to marry Professor Osgood; this eventually led Osgood to leave Harvard. Robbins had still not decided on a major at Harvard.

During the spring of 1933, the Harvard mathematics team traveled to West Point for the competition. The exam was divided into two parts, a morning portion and an afternoon portion. Robbins felt that the exam required little originality. The highlight for Robbins that weekend was a side trip to New York City where he had a date with a girl he had met the previous summer. After the team returned to Harvard, they were embarrassed to learn that they had lost to Army. Robbins did well on the exam, however, and decided to major in mathematics, this decision based not only on his Putnam experience but on how much he enjoyed working with Professor Morse. In 1938 the "Putnam exam" would become an annual competition given throughout the United States and Canada.

Shortly afterward, Morse left Harvard to go to the Institute for Advanced Study at Princeton but Morse told Robbins to continue his studies, to get a PhD in mathematics at Harvard and then to get in touch with him. Five years later, in 1938, Robbins defended his dissertation at Harvard. Robbins had had no contact with Morse for five years but he sent Morse a telegram: "Have Ph.D. in mathematics." Morse responded with, "You are my assistant starting September 1." At that time Robbins felt that the reason he probably decided to become a mathematics major was that most Harvard mathematics professors were rather pompous know-it-alls and he wanted to show them that any reasonably bright person could do mathematics.

By the time Robbins arrived at the Institute for Advanced Study, he was in great need of money as he was supporting his mother, his sister and himself. While Robbins had a one-year position working for Marston Morse at the Institute for Advanced Study, he needed a permanent position. One of the great mathematicians during that time was Richard Courant, a German mathematician who just a few years earlier had taken on a position at New York University. Courant visited Morse at the Institute, looking for someone to hire for New York University. Morse recommended Robbins to Courant. Robbins accepted Courant's offer and stayed at New York University during the period 1939–1942, making an annual salary of $2500. Courant offered Robbins an additional $700–$800 if he would help him turn some course materials of his into a book. Robbins needed the money and the project appealed to him, so he accepted the offer. Robbins was putting in so much time on the book that it was interfering with his research. This continued for about two years with the two exchanging drafts of chapters.

After it became clear how helpful Robbins had become in writing the book, Courant proposed joint authorship, which Robbins agreed to. Since Robbins had became a coauthor, Courant no longer paid Robbins for working on the book. This book had the title *What Is Mathematics?*, which became a mathematical best seller and was a literary work that discussed mathematics as it existed at that time. However, a major problem occurred shortly after the book was written. While Robbins was reading the final page proofs, the title page read "What Is Mathematics? by Richard Courant". Moreover, the preface thanked Robbins for his collaboration and the book was dedicated to Courant's children. Courant copyrighted the book in his name only and told Robbins that he would give him a portion of the royalties from time to time. Robbins neither knew how many copies of the book were sold nor how much money Courant had received.

While Courant was already famous and Robbins was to become famous, both may be best known for this book, which was published in 1941. When the book was reviewed in *Mathematical Reviews*, it was stated,

> The book is an elementary approach to modern mathematics... and is a model of lucid explanation... it is a work of extraordinary perfection.

Even Albert Einstein complimented the book:

> A lucid representation of the fundamental concepts and methods of the whole field of mathematics.... Easily understandable.

As we mentioned, Robbins earned his PhD from Harvard in 1938 with his research in the mathematical area of topology. He was 23 years old at that time.

By the time Robbins completed his stay at New York University in 1942, the United States had entered World War II and Robbins joined the Navy. After serving in the Navy, he found himself 32 years old and unemployed. He was offered a position teaching in the newly formed Department of Statistics at the University of North Carolina in Chapel Hill. Despite having little knowledge of statistics, he accepted the position. Robbins's interests then turned to mathematical statistics. He would become a well-known statistician of great stature. Robbins had a wide range of opinions and interests. Among some of his quotes are the following:

- The public has a terrible fear of mathematics. Most people haven't the faintest idea of what mathematicians do, how they think or what they contribute to society.
- Good researchers are often poor teachers; bad researchers are almost always poor teachers. The reason that you have poor teachers is that you have... individuals who have nothing to offer students except the subject matter itself. They have no joie de vivre, enthusiasm or curiosity for learning. I like to think that I'm a teacher by profession; research is what I do for fun.
- There seems to be a regression (of students) toward mediocrity: lots of fairly good students, but not many really bright or as many really dumb students.

- I have three children who spend a lot of time watching television and damn little time reading books. When I was their age, I used to go down to the public library after school and come back with an armful of books. I must have read every book in the library.

STRONG ORIENTATIONS OF GRAPHS

Robbins had doctoral students of his own, one of whom (another Herbert) was Herbert Wilf, who was well known for his work in combinatorics. Indeed, one of Wilf's students was Fan Chung Graham, well known for her work in graph theory and the mathematics of the Internet. Robbins's PhD advisor at Harvard was Hassler Whitney, who obtained several important theorems in graph theory. Robbins authored some 150 publications, primarily in mathematical statistics. His second paper and his first published paper after receiving his PhD was his only paper in graph theory—a note in the *American Mathematical Monthly*. In this note, Robbins investigated a problem in traffic control:

> Let us suppose that week-day traffic in our city is not particularly heavy, so that all streets are two-way, but that we wish to be able to repair any one street at a time and still detour traffic around it so that any point in the city may be reached from any other point. On week-ends no repairing is done, so that all streets are available, but due to the heavy traffic (perhaps it is a college town with a noted football team) we wish to make all streets one way and still be able to get from any point to any other point without violating the law.

What Robbins was able to show is that if the streets are suitable for weekday traffic, they are also suitable for weekend traffic, and conversely. Figure 9.1 shows a street system.

The street system in Figure 9.1 can be modeled by the graph G of Figure 9.2. The vertices of G correspond to the street intersections and two vertices are joined by an edge if, for the corresponding two intersections, it is possible to travel from one to the other without passing through a third intersection.

In order to state Robbins's theorem in terms of graph theory, some new terminology is needed. If each edge of a graph G is assigned a direction, then the resulting structure is called an *oriented graph* or an *orientation*

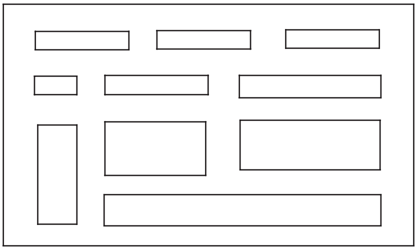


Figure 9.1. A street system.

G :

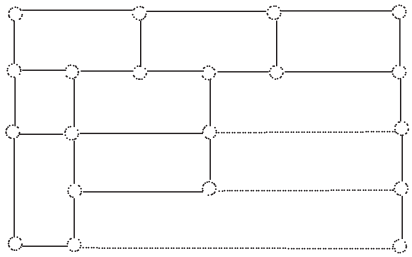


Figure 9.2. A graph modeling the two-way street system.

of G. If u and v are two vertices in an oriented graph, there can be either one or no directed edges between u and v, depending on whether the given graph G contains the edge uv. If this edge is directed from u to v, then the resulting directed edge is denoted by (u, v). A directed edge is also commonly called an *arc*. The arc (u, v) is also represented by $u \to v$ or $v \leftarrow u$. In this case, u is said to be *adjacent to* v, while v is *adjacent from* u.

An orientation D of G is *strong* if, for every two vertices u and v, D contains both a directed $u - v$ path and a directed $v - u$ path. Recall also that a *bridge* in a connected graph G is an edge whose removal from G results in a disconnected graph.

Theorem 9.1 (Robbins's Theorem): *A graph G has a strong orientation if and only if G is connected and contains no bridges.*

Since the graph G in Figure 9.2 is connected and contains no bridges, it follows by Robbins's theorem that G has a strong orientation. One possible strong orientation is shown in Figure 9.3.

This strong oriented graph in Figure 9.3 provides a way of converting the streets in the system of Figure 9.1 to one-way streets so that in the

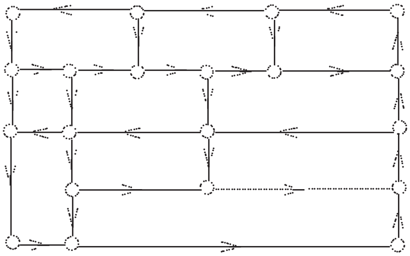


Figure 9.3. A strong orientation.

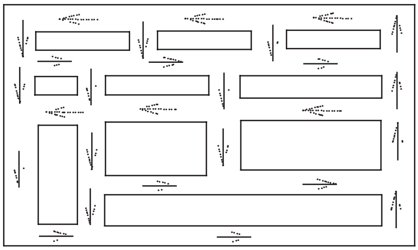


Figure 9.4. A one-way street system.

resulting system, it is possible to drive (legally) from any location in the town to any other location. This is shown in Figure 9.4.

According to Robbins's theorem, it is possible to repair any one street and still be able to travel between any two points in the city discussed by Robbins if and only if it is possible to convert all streets of the city to one-way streets and travel (legally) between any two points.

TOURNAMENTS

By far, the best known class of oriented graphs is the orientations of complete graphs. A *round-robin tournament* is a competition, among n teams say, such that every two teams play exactly one game against each other. If no ties are permitted, then this competition can be represented by an oriented graph that itself is called a *tournament*. This tournament T has as its vertices the teams involved in the competition and T contains the directed edge (u, v) if team u defeats team v.

Probably the first theorem on tournaments was one obtained in 1934 by the Hungarian mathematician and former schoolteacher László Rédei.

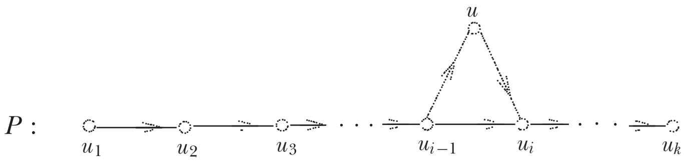


Figure 9.5. A step in the proof of Theorem 9.2.

Theorem 9.2: *Every tournament contains a directed Hamiltonian path.*

Proof: Let T be a tournament of order n and let $P = (u_1 \to u_2 \to \cdots \to u_k)$ be a directed path in T of greatest length (see Figure 9.5). If $k = n$, then P is a directed Hamiltonian path and the proof is complete. Therefore, we can assume that $k < n$. This means that there is at least one vertex of T that does not belong to P. Let u be one of these vertices.

Since P is a longest directed path in T, it follows that (u_1, u) and (u, u_k) are arcs of T; otherwise, T contains a directed path longer than P. Now let u_i be the first vertex on P such that (u, u_i) is an arc of T (see Figure 9.5). Thus (u_{i-1}, u) and (u, u_i) are both arcs of T. However, then

$$P' = (u_1 \to u_2 \to \cdots \to u_{i-1} \to u \to u_i \to \cdots \to u_k)$$

is a directed path longer than P, which produces a contradiction. ■

What this theorem says is that in every round-robin tournament (in which no ties are permitted), the teams can be listed as $t_1, t_2, \ldots, t_n$ so that t_1 has defeated t_2, t_2 has defeated t_3 and so on. In terms of graph theory, Rédei's theorem says that the vertices of every tournament T can always be listed, as $v_1, v_2, \ldots, v_n$ say, so that $v_i \to v_{i+1}$ is an arc of T for $1 \le i \le n-1$. In the tournament of order 8 in Figure 9.6, the path

$$P : v_1 \to v_2 \to v_6 \to v_5 \to v_8 \to v_4 \to v_7 \to v_3$$

is a directed Hamiltonian path.

Not only can the teams participating in a round-robin tournament (in which no ties are permitted) be listed in an order $t_1, t_2, \ldots, t_n$ so that each

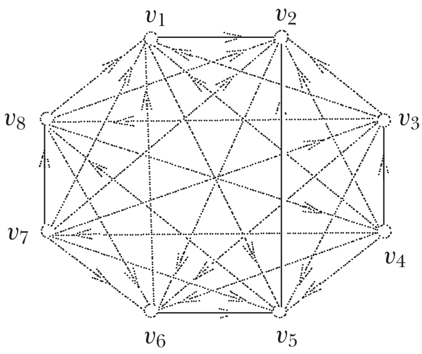


Figure 9.6. A tournament of order 8.

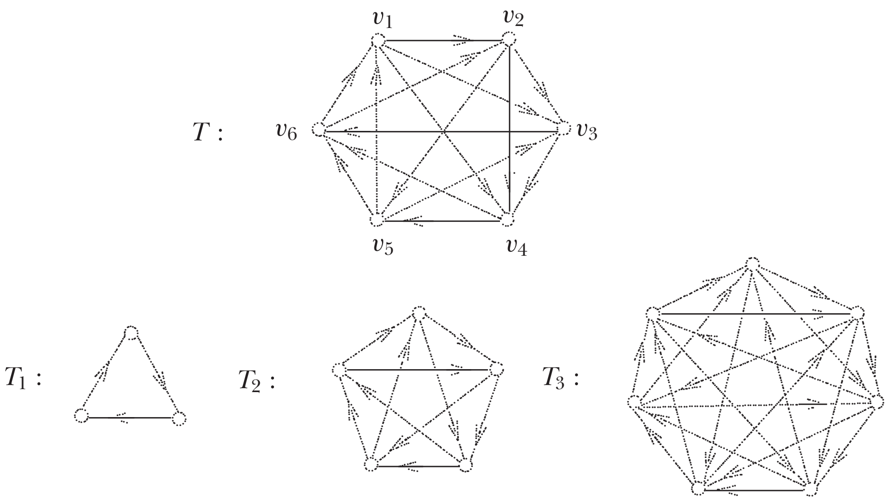


Figure 9.7. Tournaments of order 6, 3, 5, 7.

team on the list defeats the team immediately following it, this fact turns out to be a special case of something much more general.

A path P in a tournament is *antidirected* if every two consecutive arcs of P are directed oppositely. For example, the path $P : v_1 \to v_3 \leftarrow v_2 \to v_5 \leftarrow v_4 \to v_6$ is an antidirected path of order 6 in the tournament T of Figure 9.7 as is $P' : v_1 \leftarrow v_5 \to v_6 \leftarrow v_3 \to v_4 \leftarrow v_2$. None of the tournaments T_1, T_2 and T_3 in Figure 9.7 have antidirected paths of order n for $n = 3, 5, 7$, respectively, however. The mathematician Branko Grünbaum, who spent much of his career at the University of Washington, showed that these three tournaments are the only exceptions.

Theorem 9.3: *With the exception of the three tournaments T_1, T_2 and T_3 in Figure 9.7, every tournament of order n has an antidirected path of order n. In particular, every tournament of order $n \geq 8$ contains an antidirected path of order n.*

What this says, for example, is that if we have a round-robin tournament with eight teams, then there is some ordering of the teams, as $t_1, t_2, \ldots, t_8$ say, such that for each of the teams t_i, $2 \leq i \leq 7$, either t_i defeats both t_{i-1} and t_{i+1} or loses to both.

In 2000 the French mathematicians Frédéric Havet and Stéphan Thomassé proved an even more remarkable result.

Theorem 9.4: *Every tournament of order n contains each oriented path of order n except that the tournaments T_1, T_2 and T_3 in Figure 9.7 do not contain antidirected paths of orders 3, 5 and 7, respectively.*

For example, there is a path of order 8 in every tournament of order 8 whose arcs have the directions

$$\leftarrow \quad \leftarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \quad \leftarrow \quad \rightarrow .$$

In particular, in the round-robin tournament shown in Figure 9.8, we have

$$u_1 \leftarrow u_3 \leftarrow u_6 \rightarrow u_5 \rightarrow u_2 \rightarrow u_8 \leftarrow u_4 \rightarrow u_7.$$

While a tournament T of order 8 or more has an oriented path containing all vertices such that the arcs are directed in any order we

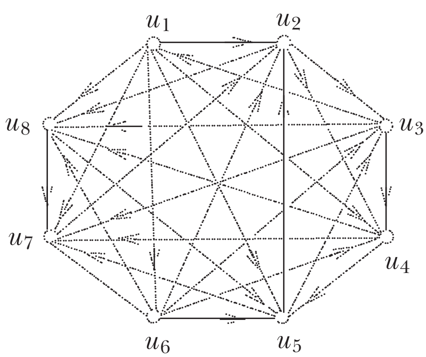


Figure 9.8. A path of order 8 in a tournament of order 8.

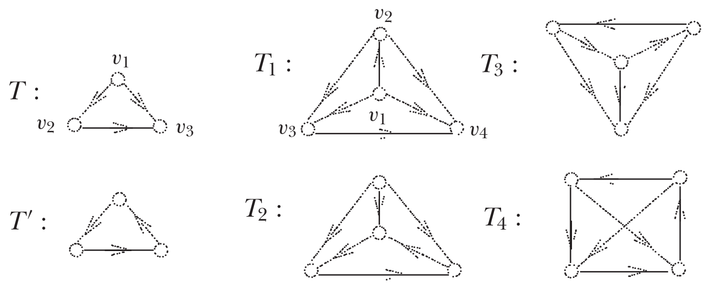


Figure 9.9. The nonisomorphic tournaments of orders 3 and 4.

desire, this is not the case for cycles in T. A tournament T of order n is called *transitive* if the vertices of T can be listed in the order $v_1, v_2, \ldots, v_n$ such that for each vertex v_i on the list, the arc is directed from v_i to v_j if $j > i$. For example, the tournaments T and T_1 in Figure 9.9 are both transitive. The reason that these two tournaments are called transitive is that if (u, v) and (v, w) are arcs, then so is (u, w). This is the transitive property of mathematics. Transitive tournaments not only do not contain directed Hamiltonian cycles, they don't contain directed cycles of any length. The tournaments T and T' are the only two (nonisomorphic) tournaments of order 3 while T_1, T_2, T_3 and T_4 are the only tournaments of order 4. The tournaments T_2 and T_3 are neither transitive nor strong.

The tournaments T' and T_4 in Figure 9.9 contain directed Hamiltonian cycles and so are Hamiltonian tournaments. While it should be clear that every Hamiltonian tournament is strong, it is no doubt less clear that the converse is true. This unexpected fact was verified in 1959 by Paul Camion.

Theorem 9.5: *A tournament of order at least* 3 *is Hamiltonian if and only if it is strong.*

As Figure 9.9 indicates, there are 2 tournaments of order 3 and 4 tournaments of order 4. There are 12 tournaments of order 5 and 56 tournaments of order 6. As expected, the number of nonisomorphic tournaments of order n grows rapidly as n grows. In fact, there are 154,108,311,168 nonisomorphic tournaments of order 12.

KINGS IN TOURNAMENTS

It is not all that unusual for people belonging to some organization (such as a government agency or business) to experience some sort of *pecking order*. That is, there is often a hierarchy or power structure in organizations where for every pair of people, one may dominate the other in some manner.

The concept of "pecking order" comes from what occurs within a flock of chickens. For each pair of chickens in the flock, one will dominate (or peck) the other. This then describes a pecking order among the chickens, which can be modeled by a tournament T whose vertices are the chickens and where (u, v) is an arc of T if chicken u pecks chicken v.

In a flock of chickens, a chicken K is called a *king* if, for every other chicken C in the flock, either $K \to C$ or there is some chicken C' such that $K \to C' \to C$. This leads to a concept in oriented graphs.

A vertex u in an oriented graph D is called a *king* if, for any other vertex w of D, either $u \to w$ or there is some vertex v in D such that $u \to v \to w$. That is, D contains a directed $u - w$ path of length at most 2. For the graph G of order 5 and size 8 in Figure 9.10, the vertex u in the orientation R of G is a king. So too are v and x. Neither w nor y is a king in R, however. If the kings u, v and x could talk, they might say (or even sing),

We three kings of orientation R
Are the only kings of orientation R.
To those not royal, we are loyal
And from us, you're not afar.

Not all oriented graphs contain kings. For example, neither orientation of the 4-cycle C_4 shown in Figure 9.11 contains a king.

By the *outdegree*, denoted by od v, of a vertex v in an oriented graph D is meant the number of vertices adjacent from v. The number of vertices

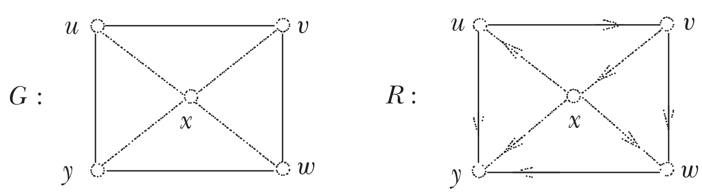


Figure 9.10. An oriented graph with three kings.

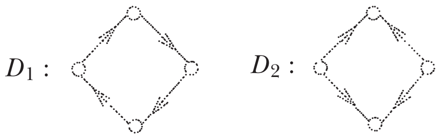


Figure 9.11. Oriented graphs without kings.

in D adjacent to v is the *indegree* of v, denoted by $\mathrm{id}\, v$. For the oriented graph R in Figure 9.10, the king u has outdegree 2 while $\mathrm{od}\, v = 2$ and $\mathrm{od}\, x = 3$. Also, $\mathrm{od}\, w = 1$ and $\mathrm{od}\, y = 0$. Furthermore, $\mathrm{id}\, u = \mathrm{id}\, v = \mathrm{id}\, x = 1$, $\mathrm{id}\, w = 2$ and $\mathrm{id}\, y = 3$. The following result is immediate and is the directed version of the First Theorem of Graph Theory (Theorem 1.4).

Theorem 9.6: *For an oriented graph D of order n and size m with $V(D) = \{v_1, v_2, \ldots, v_n\}$,*

$$\sum_{i=1}^{n} \mathrm{od}\, v_i = \sum_{i=1}^{n} \mathrm{id}\, v_i = m.$$

Obviously, a vertex having outdegree $n - 1$ in an oriented graph of order n is a king. As we saw in Figure 9.11, not all orientations of a connected graph contain a king. In the case of complete graphs, however, every orientation contains a king. Since the pecking order of a flock of chickens is represented by a tournament, the following theorem often has a rather amusing name.

Theorem 9.7 (The King Chicken Theorem): *Every tournament contains a king.*

Proof: Let w be a vertex having the maximum outdegree in a tournament T of order n. We show that w is a king. Suppose then that $\mathrm{od}\, w = k$ and let $w_1, w_2, \ldots, w_k$ be the vertices of T that are adjacent from w. If w is not a king, then there is a vertex v for which there is no $w - v$ path of length 1 or 2. Then v is adjacent to w as well as to $w_1, w_2, \ldots, w_k$ (see Figure 9.12). Therefore, $\mathrm{od}\, v \geq k + 1 = \mathrm{od}\, w + 1$, contradicting the maximality of w. ∎

Since a tournament also represents a round-robin tournament among sports teams, Theorem 9.7 says that every round-robin tournament

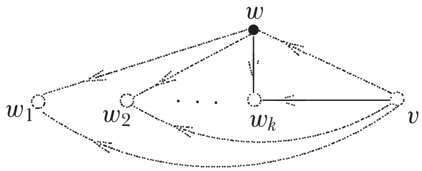


Figure 9.12. A step in the proof of Theorem 9.7.

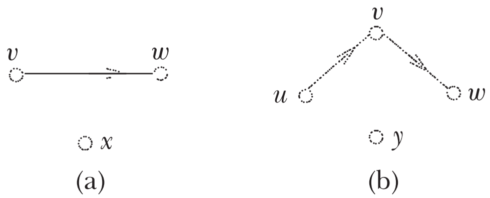


Figure 9.13. Steps in the proof of Theorem 9.8.

contains a team A with the property that for every other team B, either A has defeated B or A has defeated a team that defeated B.

A vertex of outdegree $n-1$ in a tournament of order n is called an *emperor*. If a tournament has an emperor, then it is the only king. There is, however, no tournament containing exactly two kings.

Theorem 9.8: *If T is a tournament of order $n \geq 3$ that does not contain an emperor, then T contains at least three kings.*

Proof: By Theorem 9.7, T contains a king w. Since $\operatorname{od} w < n-1$, there are vertices adjacent to w. Let v be one of these vertices having maximum outdegree. We claim that v is also a king of T. Suppose, to the contrary, that v is not a king. Then there is some vertex x of T such that there is no directed $v-x$ path of length 2 or less (see Figure 9.13a). Since v is not adjacent to x, it follows that x is adjacent to v. Furthermore, since there is no directed $v-x$ path of length 2, every vertex adjacent from v is also adjacent from x. However, then x is adjacent to w and $\operatorname{od} x > \operatorname{od} v$, which is impossible. Thus, as claimed, v is a king of T.

Next, let u be a vertex of maximum outdegree that is adjacent to v (see Figure 9.13b). We claim that u too is a king of T. If u is not a king, then there is some vertex y of T for which there is no directed $u-y$ path of length 2 or less. Now y is adjacent to u and every vertex adjacent from u is also adjacent from y. Since y is adjacent to u and $\text{od}\, y > \text{od}\, u$, a contradiction is produced. So u is also a king of T. ■

VOTING PROCEDURES

While tournaments can be used to represent a round-robin tournament in which no ties are permitted, they can also be used to represent a variety of other situations. For example, tournaments can also be used to represent "paired comparisons". Suppose that we have a collection of objects. These objects are the vertices of a tournament in which (a, b) is an arc if a is preferred to b.

For example, suppose that a person prefers a red car to a blue car, and a blue car to a silver car, and a silver car to a white car. This situation can be represented by the tournament T in Figure 9.14.

The tournament T is transitive. This is not surprising, for if u is preferred to v and v is preferred to w, then it's expected that u is preferred to w. This is also a common property when one is deciding which candidate to vote for in an election. For example, if the three individuals John Adams, James Buchanan and Calvin Coolidge are candidates for an office and a voter prefers Adams over Buchanan and Buchanan over Coolidge, then quite obviously Adams is preferred over Coolidge for this voter. This situation can be represented by the transitive tournament in Figure 9.15.

While it is clear who the preferred candidate is among Adams, Buchanan and Coolidge and how all three candidates rank according to this voter, it can be anything but clear when there are more candidates and more voters. For example, suppose that there are five students,

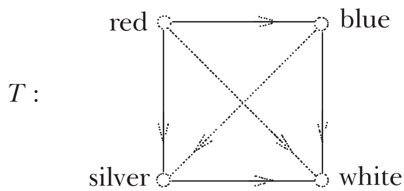


Figure 9.14. A tournament of paired comparisons.

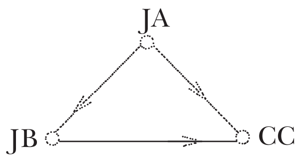


Figure 9.15. A transitive tournament showing the preferences of a voter.

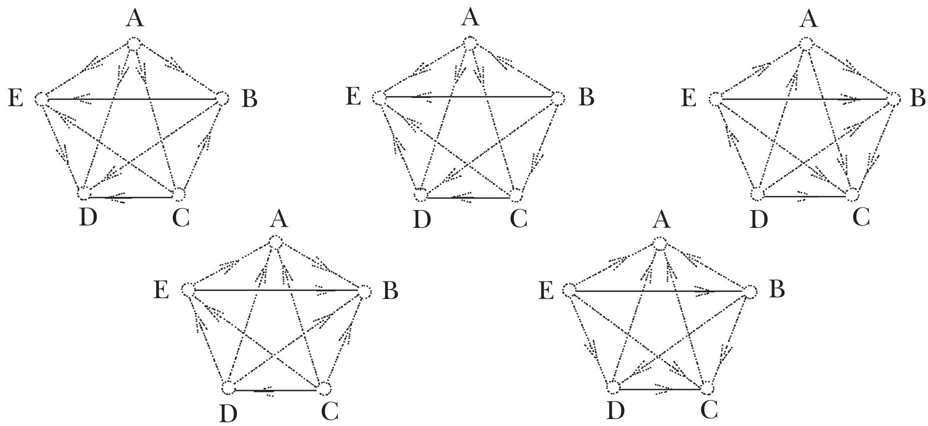


Figure 9.16. The preferences of the five committee members.

namely A, B, C, D and E, being considered for most outstanding senior and there is a faculty committee of five whose responsibility it is to make this decision. It is possible that we could have the five tournaments in Figure 9.16 that correspond to the preferences of the five committee members. As we can see, there is not total agreement of who should be selected as most outstanding senior.

The five tournaments in Figure 9.16 can be merged into a single tournament. For example, since three of the five committee members prefer A over B, we draw the arc $A \to B$ in the combined tournament. This is done for all 10 pairs of candidates, producing the combined tournament in Figure 9.17. Thus A is preferred over B, B is preferred over C, C is preferred over D, D is preferred over E—and E is preferred over A! Who should be selected as the most outstanding senior?

The situation could be even more complicated. Suppose that there are four candidates for a particular office and 408 people are voting in the election. In this election, each voter is required to vote for one of the

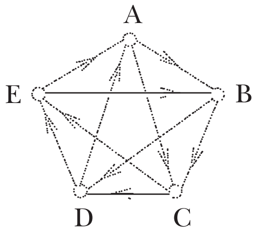


Figure 9.17. The combined tournament.

12	11	28	10	27	26	11	10	25	9	24	29
A	A	A	A	A	A	B	B	B	B	B	B
B	B	C	C	D	D	A	A	C	C	D	D
C	D	B	D	B	C	C	D	A	D	A	C
D	C	D	B	C	B	D	C	D	A	C	A

10	9	22	12	21	20	11	8	21	7	20	25
C	C	C	C	C	C	D	D	D	D	D	D
A	A	B	B	D	D	A	A	B	B	C	C
B	D	A	D	A	B	B	C	A	C	A	B
D	B	D	A	B	A	C	B	C	A	B	A

Figure 9.18. The outcome of an election.

$4! = 24$ ranked lists of the four candidates. The outcome of this election is shown in Figure 9.18.

Suppose that we were to determine the outcome of the election by one of the following methods:

(1) Count only the first choice of each voter.

(2) Eliminate the two candidates who received the smallest number of votes in (1) and then recount the votes of the two remaining candidates.

(3) Eliminate the candidate who received the smallest number of votes in (1) and then recount the votes of the three remaining candidates.

(4) Construct the tournament of paired comparisons of the four candidates.

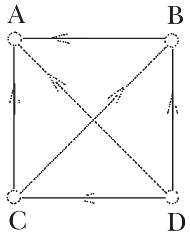


Figure 9.19. The combined tournament.

Since the first-place votes are A: 114, B: 108, C: 94, D: 92, it follows that A wins in (1) with B finishing second, C third and D fourth. In (2), B wins and in (3), C wins. In the case of (4), we have the tournament in Figure 9.19. So not only is D preferred over all other candidates, but C is preferred over A and B, and B is preferred over A. That is, D wins, C is second, B is third and A fourth—just the opposite of what occurs in (1).

It might seem a bit far-fetched to have the outcome of a vote based on a voting system in which each voter is required to vote for his or her preferred list of all candidates (the so-called Preferential Voting System), but this has happened in practice. For the Academy of Motion Picture Arts and Sciences, the first Academy Award ceremony was held in 1929 to honor movies made in 1927 and 1928 but there was no Best Picture award. The silent movie *Wings* was chosen as the most outstanding production; there were 3 nominated pictures. There were 5 nominated pictures the next three years, 8 the following year and then 10. There were 12 nominated pictures in 1934 and 1935. It went back to 10 through the 1943 awards when the movie *Casablanca* won. Starting with the 1944 award (when *Going My Way* won), there were 5 nominated pictures. The Preferential Voting System was employed during 1934–1945. From 1946 to 2008 each Academy member could vote for only one picture and the one with the most votes won. In 2010 there were 10 nominations and the Preferential Voting System returned. Each Academy member was therefore required to turn in a ranked list of the 10 nominated pictures for Best Picture. If some picture was the first choice of more than 50% of the voters, then this was selected Best Picture. Otherwise, the picture receiving the smallest number of votes was eliminated and for those votes that listed that one as the best, the votes went to the second picture on each such list. This was continued until some picture received more than 50% of the votes and this one was declared the winner. By this method, *The Hurt Locker* won the 2009 Academy Award for Best Picture; the movie *The King's Speech* won in 2010.

After receiving complaints that some of the 10 nominated pictures may not have been worthy of being nominated, the Academy of Motion Picture Arts and Sciences changed the rules again for the Best Picture award, beginning with the 2011 award. In fact, what they did this time was change the nomination process, which resulted in a variable number of pictures being nominated instead of a specific number. Each voter (member of the Academy) was given a nomination ballot consisting of five blank slots. The member was then asked to fill in this ballot with a ranked list of 5 pictures. Any picture that received at least 5% of the first choices of the Academy members received a nomination. Those ballots listing such a picture as a first choice were then looked at for their second choice which was then assigned a partial vote. This procedure continued. It has been determined that this system results in 5 to 10 pictures being nominated. Once the (unranked) list of nominated pictures has been determined, the Preferential Voting System is applied to determine the recipient of the Best Picture award. This system resulted in 9 pictures being nominated for the 2011 Best Picture award, for example, which was won by *The Artist*. Perhaps using the tournament of paired comparisons to make such a decision isn't such a bad idea after all!

10

Drawing Graphs

Decades ago, a puzzle appeared in many books and magazines that has been known by many names. One of the most common names for this puzzle is the *Three Houses and Three Utilities Problem*.

The Three Houses and Three Utilities Problem

Three houses A, B and C are under construction and each house must be provided with connections to each of three utilities, namely water, electricity and natural gas. (See Figure 10.1.) Each utility provider needs a direct line from the utility terminal to each house without passing through another provider's terminal or another house along the way. Furthermore, all three utility providers need to bury their lines at exactly the same depth underground without any lines crossing. Can this be done?

Henry Ernest Dudeney was a well-known British puzzle maker and proponent of recreational mathematics. This mathematician and writer belonged to a literary circle in England that included the famous Sir Arthur Conan Doyle, creator of the fictional detective Sherlock Holmes. In 1917 Dudeney referred to the Three Houses and Three Utilities Problem in his book *Amusements in Mathematics*. Dudeney also stated there that this problem is as "old as the hills".

Among several problems that are equivalent to the Three Houses and Three Utilities Problem is one stated in the book *Graphs and Their Uses* by Oystein Ore:

> Three houses have been built on a piece of land and three wells have been dug for the use of the occupants. The nature of the land and the climate is such that one or another of the wells frequently runs dry; it is therefore important that the people of each house have access to each of the three wells. After a while the residents develop

rather strong dislikes of one another and decide to construct paths to the three wells in such a manner that they avoid meeting each other on their way to and from the wells. Is such an arrangement possible?

PLANAR GRAPHS

While the statement of the Three Houses and Three Utilities Problem is deceptively simple, it turns out that its solution is not. Quite clearly, this problem can be represented as a problem in graph theory. Indeed, Figure 10.1 is essentially a graph, which is repeated in a more typical manner in Figure 10.2, where the vertices W, E and NG stand for water, electricity and natural gas, respectively.

Consequently, the Three Houses and Three Utilities Problem can be modeled by the graph $K_{3,3}$, where three of the six vertices of $K_{3,3}$ are the

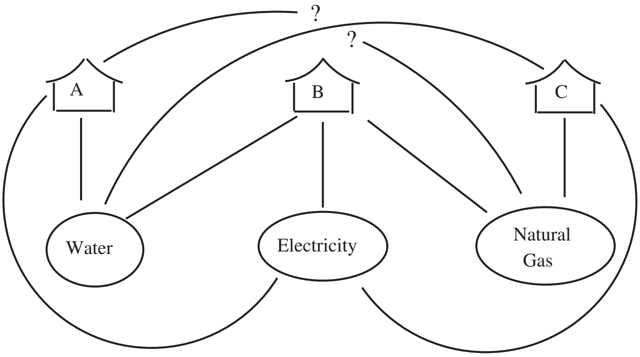


Figure 10.1. Three Houses and Three Utilities Problem.

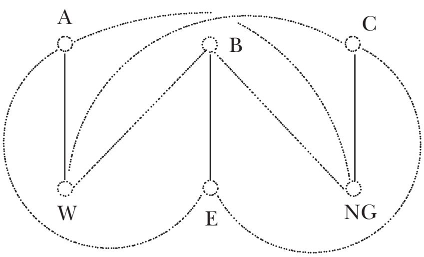
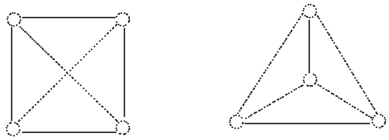


Figure 10.2. A graphical representation of the Three Houses and Three Utilities Problem.



(a) (b)

Figure 10.3. Showing that K_4 is a planar graph.

three houses and the other three vertices are the three utilities. Solving the Three Houses and Three Utilities Problem is therefore equivalent to the problem of determining whether the graph $K_{3,3}$ can be drawn in the plane without any of its edges crossing. Attempting to draw $K_{3,3}$ in this manner most likely results in difficulties, as Figure 10.2 shows. In fact, it is probably not clear how to draw $K_{3,3}$ in the plane in such a manner or to convince yourself that any drawing of $K_{3,3}$ requires some edges to cross. This leads to the consideration of an important class of graphs.

A graph G is a *planar graph* if G can be drawn in the plane so that no two of its edges cross. Figure 10.3a shows a drawing of the complete graph K_4. In this drawing, two edges cross. Nevertheless, K_4 *can* be drawn without any edges crossing, as in Figure 10.3b. Therefore, K_4 is planar.

Solving the Three Houses and Three Utilities Problem is therefore equivalent to determining whether $K_{3,3}$ is a planar graph. A graph that is not planar is a *nonplanar graph*. We'll return to the Three Houses and Three Utilities Problem in the section "Kuratowski's Theorem", later in this chapter.

Suppose that G is a connected planar graph. Then it is possible to draw G in the plane without any of its edges crossing. If we interpret a vertex as a point in the plane and an edge as a line segment or curve and remove the vertices and edges of G from the plane, then certain connected pieces of the plane remain, referred to as the *regions* of G. There is always one unbounded region, called the *exterior region* of G. For the planar graph G of Figure 10.4 (drawn in the plane without edges crossing), there are six regions, namely $R_1, R_2, \ldots, R_6$. The exterior region is R_6.

THE EULER IDENTITY

A *polyhedron* is a three-dimensional object whose boundary consists of polygonal plane surfaces. These surfaces are typically called the *faces*

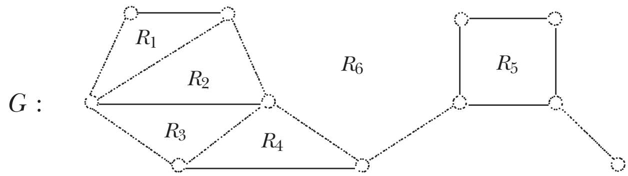
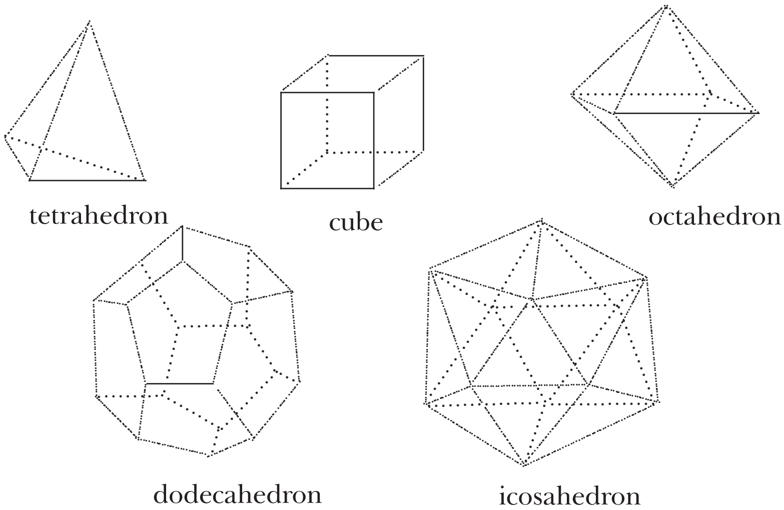


Figure 10.4. The regions of a planar graph.



Platonic solid	V	E	F
tetrahedron	4	6	4
cube	8	12	6
octahedron	6	12	8
dodecahedron	20	30	12
icosahedron	12	30	20

Figure 10.5. The five Platonic solids.

of the polyhedron. The boundary of a face consists of the edges and vertices of the polygon. In this setting, the total number of faces in the polyhedron is commonly denoted by F, the total number of edges in the polyhedron by E and the total number of vertices by V. The best known polyhedra are the so-called Platonic solids: the *tetrahedron*, *cube* (*hexahedron*), *octahedron*, *dodecahedron* and *icosahedron*. These five polyhedra are shown in Figure 10.5, together with the values of V, E and F for each.

During the eighteenth century, many letters (over 160) were exchanged between Leonhard Euler (who, as we saw, essentially introduced graph theory to the world when he solved and then generalized the Königsberg Bridge Problem) and Christian Goldbach (well known for stating the conjecture that every even integer greater than or equal to 4 can be expressed as the sum of two primes). In a letter that Euler wrote to Goldbach on 14 November 1750, he stated a relationship that existed among the numbers V, E and F for a polyhedron and which would later bear his name.

The Euler Polyhedron Formula

If a polyhedron has V vertices, E edges and F faces, then $V - E + F = 2$.

That Euler was evidently the first mathematician to observe this formula (which is actually an identity rather than a formula) may be somewhat surprising in light of the fact that Archimedes (287 BC–212 BC) and René Descartes (1596–1650) both studied polyhedra long before Euler. A possible explanation as to why others had apparently overlooked this identity might be due to the fact that geometry had primarily been a study of distances.

The Euler Polyhedron Formula appeared in print two years later (in 1752) in two papers by Euler. In the first of these two papers, Euler stated that he had been unable to prove the formula. However, in the second paper, he presented a proof by dissecting polyhedra into tetrahedra. Although his proof was clever, he nonetheless made some missteps. The first generally accepted complete proof of this identity was obtained by the French mathematician Adrien-Marie Legendre.

By appropriately positioning a point, a polyhedron and a plane in 3-space, each polyhedron can be projected onto the plane from the point to produce a map from which a planar graph can be obtained which represents the polyhedron. This is illustrated in Figure 10.6 for the case where the polyhedron is a cube.

The planar graphs obtained from the five Platonic solids are shown in Figure 10.7. (Notice that the graph of the tetrahedron is K_4, which we visited in Figure 10.3.) The faces of the polyhedra thus become the regions of the planar graphs.

A planar graph G that has been drawn in the plane without any of its edges crossing is said to be *embedded in the plane*, resulting in a

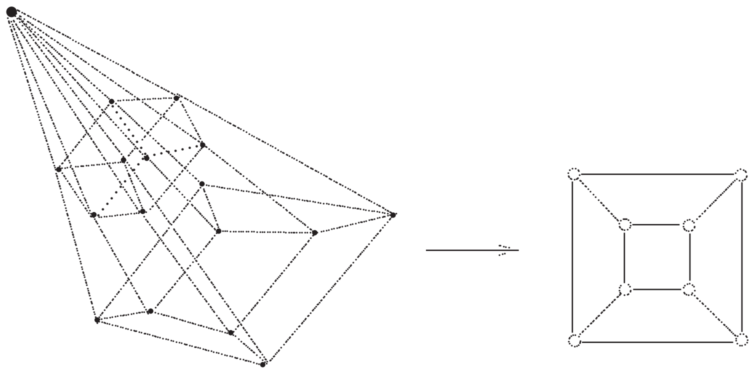


Figure 10.6. A planar graph representation of a polyhedron.

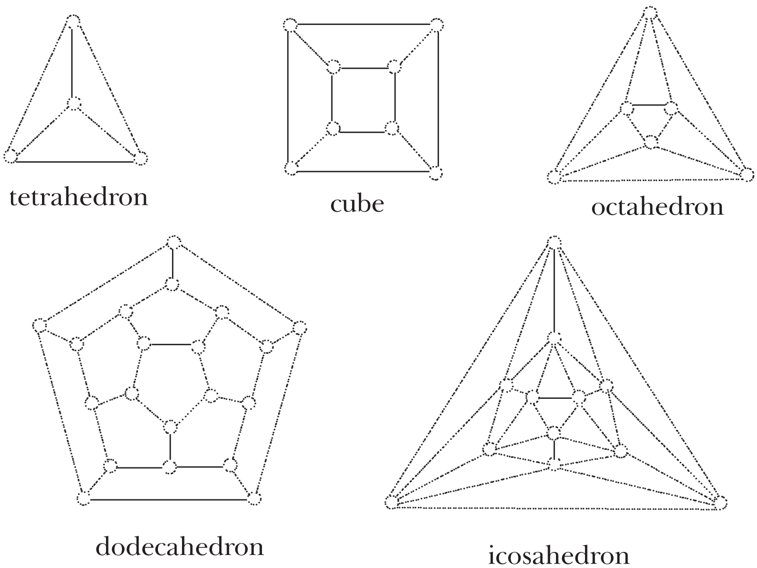


Figure 10.7. The graphs of the five Platonic solids.

planar embedding of G. The vertices and edges incident with a region R in a planar embedding of a graph G form a subgraph of G called the *boundary* of R. The boundaries of the six regions of the planar graph G in Figure 10.4 are shown in Figure 10.8. Observe that every edge of G lies on the boundary of exactly two regions unless the edge is a bridge, in which case it is on the boundary of exactly one region.

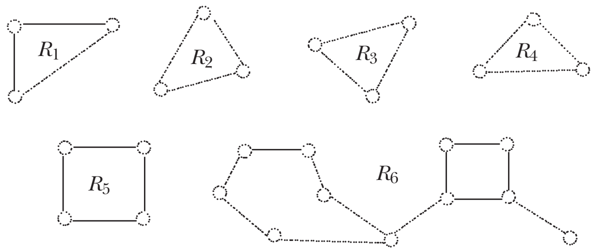


Figure 10.8. The boundaries of the regions of a graph embedded in the plane.

We have seen that if a polyhedron has V vertices, E edges and F faces, then $V - E + F = 2$ by the Euler Polyhedron Formula. This produces a planar graph with, say, n vertices, m edges and r regions. It therefore follows that $n - m + r = 2$. It turns out that this formula (or identity) holds for every planar graph, not just those planar graphs obtained from polyhedra.

Theorem 10.1 (The Euler Identity): *If G is a connected graph of order n and size m embedded in the plane and resulting in r regions, then $n - m + r = 2$.*

Idea of Proof: First, if G is a tree of order n and size m, then $m = n - 1$ (by Theorem 4.3) and there is only one region, so $r = 1$. Hence in this case, $n - m + r = n - (n-1) + 1 = 2$. Therefore, we can restrict our attention to connected graphs embedded in the plane that are not trees. Figure 10.9 shows all connected graphs of size 5 or less that are not trees and that are embedded in the plane. In each case, the Euler Identity holds.

We now show that if G is any connected graph of size 6 that is not a tree and embedded in the plane, the Euler Identity holds. Suppose that G has order n and there are r regions in the embedding. Since G is not a tree, G contains a cycle C. Let e be an edge on C and let H be the graph $G - e$ obtained by removing e from G. Thus H has size 5. Since e is not a bridge of G (see Theorem 3.1), H is connected and is either a tree or one of the graphs of size 5 in Figure 10.9. Since e was on the boundary of two regions in G, these two regions merged into a single region in H. So H has order n, size 5 and $r - 1$ regions. Since the Euler Identity holds for H,

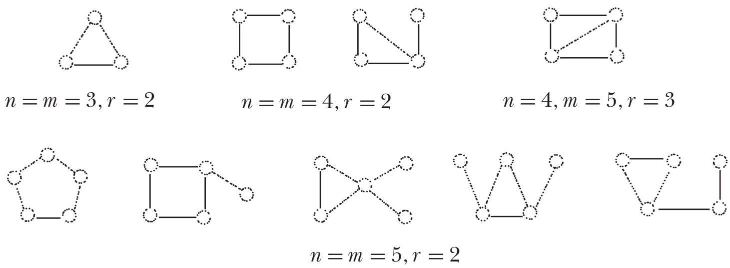


Figure 10.9. Verifying the Euler Identity for connected graphs of size 5 or less that are not trees.

it follows that $n - 5 + (r - 1) = 2$ and so $n - 6 + r = 2$, that is, the Euler Identity holds for G as well. In the same way, it can be shown that the Euler Identity holds when G has seven edges, then eight edges and so on. ■

By Theorem 10.1, if G is a connected planar graph of order n and size m resulting in r regions, then $n - m + r = 2$. If $m \geq 3$, then the boundary of every region of G must contain at least three edges. In fact, if $n \geq 3$, then there is a relationship involving n and m only.

Theorem 10.2: *If G is a planar graph of order $n \geq 3$ and size m, then*

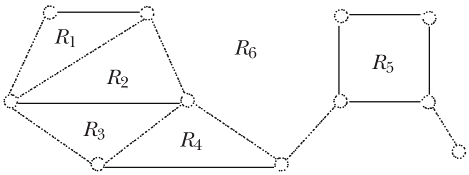
$$m \leq 3n - 6.$$

Proof: We may assume that G is connected for if G is disconnected, then edges can be added to result in a connected planar graph. Since the result is obvious if $n = 3$, we can assume that $n \geq 4$ and so $m \geq 3$. Suppose that a planar embedding of G results in r regions. Then by Theorem 10.1, $n - m + r = 2$. Let N be the result obtained when the number of edges on the boundary of a region is summed over all regions of G. For example, let's compute N for the graph G in Figure 10.4, which is redrawn in Figure 10.10.

Since each boundary contains at least three edges and each edge is counted at most twice, it follows that $3r \leq N \leq 2m$. Therefore,

$$6 = 3n - 3m + 3r \leq 3n - 3m + 2m$$

and so $m \leq 3n - 6$. ■



region	edges on boundary
R_1	3
R_2	3
R_3	3
R_4	3
R_5	4
R_6	12
	$28 = N$

Figure 10.10. Illustrating a step in the proof of Theorem 10.2.

With the aid of Theorem 10.2, it can be shown that there are always vertices in a planar graph whose degrees are not large.

Corollary 10.3: *Every planar graph contains a vertex of degree* 5 *or less.*

Proof: This is obvious for planar graphs of order 6 or less. Suppose, however, that there is a planar graph G of order $n \geq 7$ and size m in which no vertex of G has degree 5 or less. If $v_1, v_2, \ldots, v_n$ are the vertices of G, then $\deg v_i \geq 6$ for every vertex v_i of G. By the First Theorem of Graph Theory,

$$2m = \deg v_1 + \deg v_2 + \cdots + \deg v_n \geq 6n$$

and so $m \geq 3n$, contradicting Theorem 10.2. ■

Stating Theorem 10.2 in its contrapositive form, we have the following.

Theorem 10.4: *If G is a graph of order $n \geq 3$ and size m such that $m > 3n - 6$, then G is nonplanar.*

Theorem 10.4 therefore provides us with a sufficient condition for a graph to be nonplanar. Our major question remains though, how can we determine whether a given graph is planar?

KURATOWSKI'S THEOREM

As an illustration of Theorem 10.4, consider the complete graph K_5 shown in Figure 10.11. This graph has order $n = 5$ and size $m = 10$. Since $10 = m > 3n - 6 = 9$, it follows that K_5 is nonplanar.

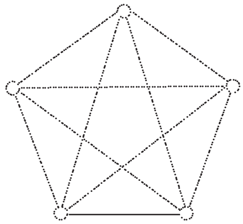


Figure 10.11. The complete graph K_5.

Of course, another graph where the question of planarity interests us is $K_{3,3}$, for determining whether this graph is planar will provide us with a solution to the Three Houses and Three Utilities Problem. First, the order of $K_{3,3}$ is $n = 6$ and its size is $m = 9$. However, $9 = m < 3n - 6 = 12$, which by Theorem 10.2 provides us with no information. The proof of Theorem 10.2 uses the fact that the boundary of every region in a connected planar graph of size $m \geq 3$ contains at least three edges. If G is bipartite and $m \geq 4$, then the boundary of every region must contain at least four edges. Hence $4r \leq N \leq 2m$ and so $2r \leq m$. Thus, by doubling the Euler Identity, we obtain

$$4 = 2n - 2m + 2r \leq 2n - 2m + m$$

and so $m \leq 2n - 4$. For the graph $K_{3,3}$, however, $9 = m > 2n - 4 = 8$. Therefore, $K_{3,3}$ is nonplanar, which solves the Three Houses and Three Utilities Problem. Consequently, it is *not* possible to provide connections with each house to the three utilities without lines crossing.

We now know of two graphs, namely K_5 and $K_{3,3}$ (shown in Figure 10.12), that are nonplanar. It should be clear that if G is a graph containing a nonplanar subgraph, then G itself is nonplanar. Consequently, any graph that contains K_5 or $K_{3,3}$ as a subgraph is nonplanar.

Consider the graph G in Figure 10.13. This graph has order $n = 7$ and size $m = 16$. Since $m = 16 > 15 = 3n - 6$, it follows by Theorem 10.4 that G is nonplanar. It turns out, however, that this graph contains neither K_5 nor $K_{3,3}$ as a subgraph. Let's see why.

First, suppose that $H = K_5$ is a subgraph of G. Then H must have five vertices of degree 4. Necessarily, H contains the vertex y, for if this were not the case, then H contains at least two of u, v and w, which have degree

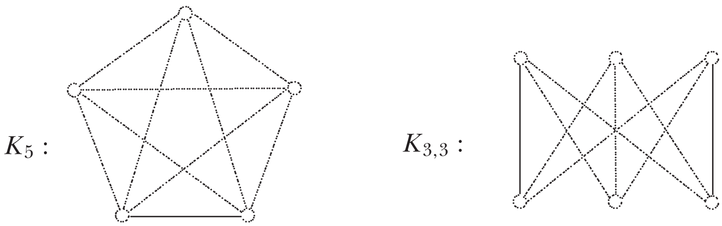


Figure 10.12. Two nonplanar graphs.

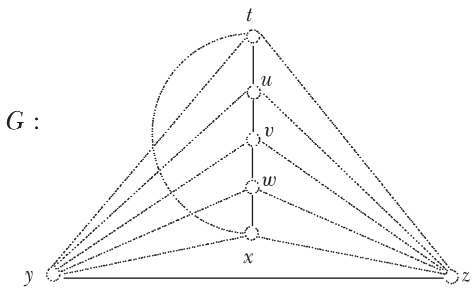


Figure 10.13. A nonplanar graph.

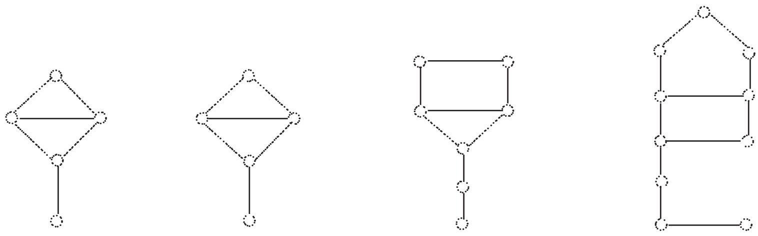
at most 3 in H. For the same reason, H contains the vertex z. However, y and z must be adjacent to all three vertices of a triangle in G. Since no such triangle exists among t, u, v, w and x, it follows that G does not contain K_5 as a subgraph.

Next, suppose that $F = K_{3,3}$ is a subgraph of G. So F contains every vertex of G except one. We consider four cases.

Case 1. *The subgraph F does not contain y.* Since t has degree 3 in F and is adjacent to u, x and z, the two partite sets of F must be $\{t, v, w\}$ and $\{u, x, z\}$. However, v is not adjacent to x, which is impossible. (If F does not contain z, this is essentially the same case.)

Case 2. *The subgraph F does not contain t.* Since u has degree 3 in F and is adjacent to v, y and z, the two partite sets of F must be $\{u, w, x\}$ and $\{v, y, z\}$. However, v is not adjacent to x, a contradiction. (If F does not contain x, this is essentially the same case.)

Case 3. *The subgraph F does not contain u.* Since t has degree 3 in F and is adjacent to x, y and z, the two partite sets of F must be $\{t, v, w\}$ and



G H_1 H_2 H_3

Figure 10.14. Subdivisions of a graph.

$\{x, y, z\}$. Here too, v is not adjacent to x, a contradiction. (If F does not contain w, the situation is similar.)

Case 4. *The subgraph F does not contain v.* Since u has degree 3 in F and is adjacent to t, y and z, the two partite sets of F must be $\{u, w, x\}$ and $\{t, y, z\}$. However, t and w are not adjacent, which is a contradiction.

Therefore, as we claimed, the graph G of Figure 10.13 contains neither K_5 nor $K_{3,3}$ as a subgraph. Consequently, a graph can be nonplanar without containing either K_5 or $K_{3,3}$ as a subgraph. Despite this, these two graphs play a central role in why this graph G (indeed, in why *any* graph) is nonplanar.

A graph H is a *subdivision* of a graph G if either $H = G$ or H can be obtained from G by inserting vertices of degree 2 into the edges of G. For the graph G of Figure 10.14, all of the graphs H_1, H_2 and H_3 are subdivisions of G. Indeed, H_3 is a subdivision of H_2.

Certainly, a subdivision H of a graph G is planar if and only if G is planar. Therefore, not only are K_5 and $K_{3,3}$ nonplanar, every subdivision of K_5 or $K_{3,3}$ is nonplanar. This observation provides a necessary condition for a graph to be planar.

Theorem 10.5: *If a graph G is planar, then G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

The remarkable feature about this necessary condition for a graph to be planar is that this condition is also sufficient. The first published proof of this fact occurred in 1930. This theorem is due to the Polish topologist Kazimierz Kuratowski (1896–1980), who first announced this theorem in 1929. The title of Kuratowski's paper is "Sur le problème des courbes gauches en topologie" ("On the Problem of Skew Curves in Topology"), which suggests, and rightly so, that the setting of his theorem was in topology—not graph theory. Nonplanar graphs were sometimes called *skew graphs* during that period.

The publication date of Kuratowski's paper was critical to having the theorem credited to him, for, as it turned out, later in 1930 two American mathematicians, Orrin Frink and Paul Althaus Smith, submitted a paper containing a proof of this theorem as well but withdrew it after they became aware that Kuratowski's proof had preceded theirs, although just barely. They did publish a one-sentence announcement of what they had accomplished in the *Bulletin of the American Mathematical Society* and, as the title of their note indicates ("Irreducible Non-planar Graphs"), the setting for their proof was graph theoretical in nature.

It is believed by some that a proof of this theorem may have been discovered somewhat earlier by the Russian topologist Lev Semenovich Pontryagin, who was blind his entire adult life. Because the first proof of this theorem may have occurred in Pontryagin's unpublished notes, in Russia and elsewhere, this result is sometimes referred to as the Pontryagin–Kuratowski theorem. However, since the possible proof of this theorem by Pontryagin did not satisfy the established practice of appearing in print in an accepted refereed journal, the theorem is generally recognized as Kuratowski's theorem.

Theorem 10.6 (Kuratowski's Theorem): *A graph G is planar if and only if G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Since the graph G in Figure 10.13 is known to be nonplanar and is known to contain neither K_5 nor $K_{3,3}$ as a subgraph, we can conclude from Theorem 10.6 that this graph G must contain a subdivision of one of these graphs that is not actually K_5 or $K_{3,3}$. In fact, the graph G contains both a subdivision of K_5 (Figure 10.15a) *and* a subdivision of $K_{3,3}$ (Figure 10.15b)!

CROSSING NUMBERS

Since the graphs K_5 and $K_{3,3}$ are nonplanar, they cannot be drawn in the plane without edges crossing. Actually, both graphs can be drawn with just a single crossing, however (see Figure 10.16).

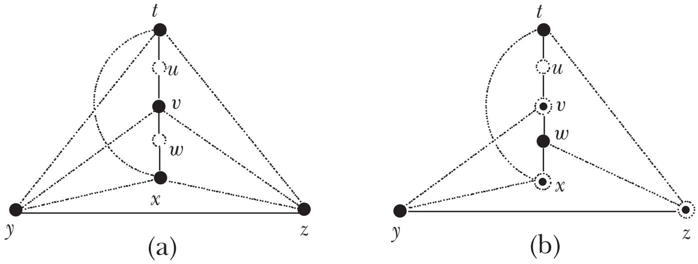


Figure 10.15. Subdivisions of K_5 and $K_{3,3}$ in the graph G of Figure 10.13.

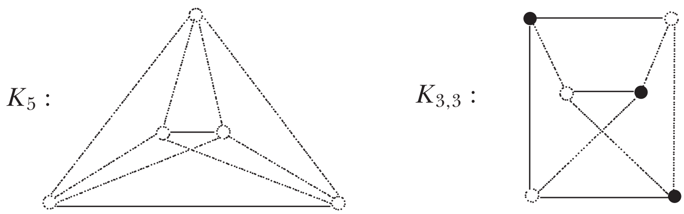


Figure 10.16. Drawings of K_5 and $K_{3,3}$ with one crossing.

The problem of determining the minimum number of crossings in any drawing of a graph G in the plane is a difficult problem that can be traced back to a difficult period of world history. In 1977 the Hungarian mathematician Paul Turán wrote the following that describes an event in his life that occurred when he was being held in a labor camp during World War II.

> In July, 1944 the danger of deportation was real in Budapest and a reality outside Budapest. We worked near Budapest, in a brick factory. There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. The bricks were carried on small wheeled trucks to the storage yards. All we had to do was to put the bricks on the trucks at the kilns, push the trucks to the storage yards, and unload them there. We had a reasonable piece rate for the trucks, and the work itself was not difficult; the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short this caused a lot of trouble and loss of time which was precious to all of us. We were all

> sweating and cursing at such occasions, I too; but nolens volens the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized. But what is the minimum number of crossings? I realized after several days that the actual situation could have been improved, but the exact solution of the general problem with m kilns and n storage yards seemed to be very difficult.

What Turán was asking for then in his *Brick-Factory Problem* is the minimum number of crossings among all drawings in the plane of the complete bipartite graph $K_{m,n}$. This eventually led to a more general concept in graph theory.

The *crossing number* $\text{cr}(G)$ of a graph G is the minimum number of crossings of its edges among all drawings of G in the plane. Here it is always assumed that at most two edges can cross at a single point in the plane. Of course, $\text{cr}(G) = 0$ if and only if G is a planar graph. Furthermore, since K_5 and $K_{3,3}$ are nonplanar and a drawing of each exists with a single crossing, it follows that $\text{cr}(K_5) = 1$ and $\text{cr}(K_{3,3}) = 1$.

What Turán was interested in, therefore, was the value of $\text{cr}(K_{m,n})$ for all positive integers m and n. At one time the Polish mathematician Kazimierz Zarankiewicz believed that he had discovered a formula for $\text{cr}(K_{m,n})$ but, as it turned out, he had only verified the upper bound

$$\text{cr}(K_{m,n}) \le \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \qquad (10.1)$$

where $\lfloor m/2 \rfloor$, for example, denotes the floor of $m/2$ (the greatest integer less than or equal to $m/2$). So $\lfloor m/2 \rfloor = m/2$ if m is even and $\lfloor m/2 \rfloor = (m-1)/2$ if m is odd. The upper bound for $\text{cr}(K_{m,n})$ in (10.1) has been shown to be the exact value of $\text{cr}(K_{m,n})$ when $m \le n$ and either (1) $m \le 6$ or (2) $m = 7$ and $n \le 10$. Turán's Brick-Factory Problem remains an unsolved problem and is obviously very challenging.

While determining the crossing number of a nonplanar graph is, in general, very difficult, there is a bound for the crossing number that is useful at times. As we saw in Theorem 10.4, if G is a connected graph of order $n \ge 3$ and size m such that $m > 3n - 6$, then G is nonplanar; that is, if $m - 3n + 6 > 0$, then G is nonplanar.

Theorem 10.7: *If G is a graph of order $n \geq 3$ and size m, then*

$$\mathrm{cr}(G) \geq m - 3n + 6.$$

Proof: By Theorem 10.2, if G is planar, then $m \leq 3n - 6$ and so $m - 3n + 6 \leq 0$. Certainly, $\mathrm{cr}(G) \geq m - 3n + 6$ in this case. Hence we may assume that G is nonplanar. Let $\mathrm{cr}(G) = c \geq 1$ and let G be drawn in the plane with c crossings. At each of the c crossings, introduce a new vertex. This results in a planar graph G' of order $n' = n + c$ and size $m' = m + 2c$. By Theorem 10.2, $m' \leq 3n' - 6$ and so

$$m + 2c \leq 3(n + c) - 6.$$

From this, it follows that $\mathrm{cr}(G) = c \geq m - 3n + 6$. ■

Let's see how Theorem 10.7 can be used to compute the crossing number of a particular nonplanar graph.

Example 10.8: *Determine the crossing number of the complete graph K_6.*

SOLUTION:

Let c be the crossing number of K_6. Since K_6 contains the nonplanar graph K_5 as a subgraph, K_6 is also nonplanar and so $c \geq 1$. In fact, K_6 has order $n = 6$ and size $m = \binom{n}{2} = \binom{6}{2} = 15$. So by Theorem 10.7, $c \geq 15 - 3 \cdot 6 + 6 = 3$. Thus $\mathrm{cr}(K_6) \geq 3$. Since there is a drawing of K_6 with three crossings (see Figure 10.17), it follows that $\mathrm{cr}(K_6) = 3$. ♦

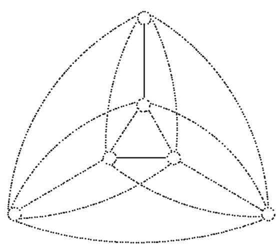


Figure 10.17. A drawing of K_6 with three crossings.

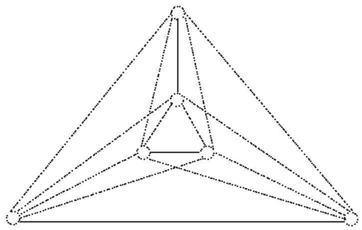


Figure 10.18. A straight-line drawing of K_6 with three crossings.

For complete graphs in general, it has been shown by the combined efforts of a number of mathematicians that whenever $1 \le n \le 12$,

$$\operatorname{cr}(K_n) = \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor, \qquad (10.2)$$

and that the expression in (10.2) is an upper bound for $\operatorname{cr}(K_n)$ for every positive integer n.

In a drawing of a graph G in the plane, an edge of G can be any curve, including a straight-line segment. By a *straight-line drawing* of a graph G (planar or not) is meant a drawing of G in the plane in which every edge is a straight-line segment. The *rectilinear crossing number* $\overline{\operatorname{cr}}(G)$ of a graph G is the minimum number of crossings among all straight-line drawings of G in the plane. Consequently, $\operatorname{cr}(G) \le \overline{\operatorname{cr}}(G)$ for every graph G. That $\overline{\operatorname{cr}}(K_6) = 3$ follows by observing that the drawing of K_6 in Figure 10.17 can be altered slightly to arrive at the straight-line drawing in Figure 10.18.

While the numbers $\operatorname{cr}(G)$ and $\overline{\operatorname{cr}}(G)$ are the same for many graphs G, they aren't the same for every graph G. For example, it has been shown that $\operatorname{cr}(K_8) = 18$ and $\overline{\operatorname{cr}}(K_8) = 19$. However, no pair m, n of positive integers is known for which $\operatorname{cr}(K_{m,n}) \ne \overline{\operatorname{cr}}(K_{m,n})$.

There is an interesting fact that states there is always a straight-line drawing of every planar graph in the plane such that no two edges cross. While this result was discovered independently by István Fáry, Sherman K. Stein and Klaus Wagner, it is commonly known as Fáry's theorem.

Theorem 10.9 (Fáry's Theorem): *If G is a planar graph, then $\overline{\operatorname{cr}}(G) = 0$.*

Although we are not presenting a proof of Fáry's theorem, a common proof of it uses the geometric fact that within the interior of each triangle,

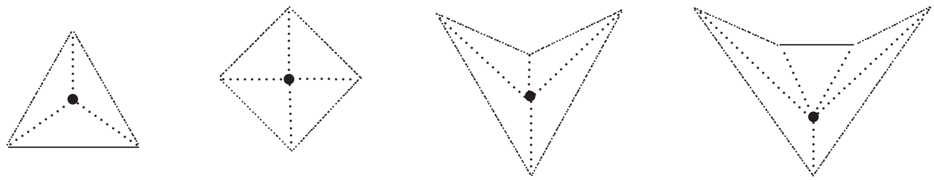


Figure 10.19. Straight-line segments joining the vertices of certain polygons from a point in the interior of the polygons.

quadrilateral and pentagon in the plane, a point can be placed that can be joined by straight-line segments to the vertices of the polygon so that each line segment lies in the interior of the polygon. (See Figure 10.19 for some illustrations of this.) These facts can be used to solve a popular problem from geometry.

THE ART GALLERY PROBLEM

Suppose that an art gallery consists of a single large room with n walls on which paintings are hung. What is the minimum number of security guards that must be stationed in the gallery to guarantee that for every painting hung on a wall there is a guard who has a straight-line view of the artwork?

This problem was posed in 1973 by the geometer Victor Klee after a discussion he had with Vašek Chvátal. It was shown by Chvátal that no more than $\lfloor n/3 \rfloor$ guards are needed and that examples exist where $\lfloor n/3 \rfloor$ guards are actually required.

EMBEDDING GRAPHS ON SURFACES

The solution of the Three Houses and Three Utilities Problem tells us that it is impossible to produce a planar embedding of $K_{3,3}$ (that is, in every drawing of $K_{3,3}$ in the plane, some edges must cross). It turns out that there is a planar embedding of a graph G if and only if there is a spherical embedding of G, that is, only planar graphs can be drawn on the surface of a sphere without edges crossing.

To see this, suppose that a graph G is drawn on (the surface of) a sphere with no edges crossing—a *spherical embedding* of G. Select a point p on the sphere which is neither a vertex of G nor lies on an edge of G. Let q be the point on the sphere that is diametrically opposite to p. Then place the sphere on a plane so that it is tangent to the plane at q. Hence p can now be considered the north pole of the sphere and q the south pole. For each point x on the sphere which is either a vertex of G or lies on an edge of G, a straight line is drawn from p through x until it intersects the plane at the point x'. An embedding of G on the plane results. This projection of a graph G on the sphere to a graph G on the plane is called a *stereographic projection*. This procedure can be reversed, beginning with a planar embedding of a (planar) graph G to produce a spherical embedding of G. Therefore, a graph G is planar if and only if G can be embedded on a sphere. This is illustrated in Figure 10.20.

There are other, more complex surfaces, however, on which some graphs may be drawn without any edges crossing. One such surface is the *torus*, which is a doughnut-shaped surface (see Figure 10.21a). While the graph $K_{3,3}$ cannot be drawn in the plane without edges crossing and therefore cannot be drawn on a sphere without edges crossing, $K_{3,3}$ can be drawn on the torus, resulting in a *toroidal embedding* of $K_{3,3}$ (see Figure 10.21b).

There is another way to see that $K_{3,3}$ can be embedded on a torus, which may be easier to visualize. Suppose that we have a rectangular piece of flexible material as in Figure 10.22a and that we identify the two vertical sides s and s' of this rectangle, resulting in the cylinder-shaped surface shown in Figure 10.22c. The top and bottom circles c and c' of this

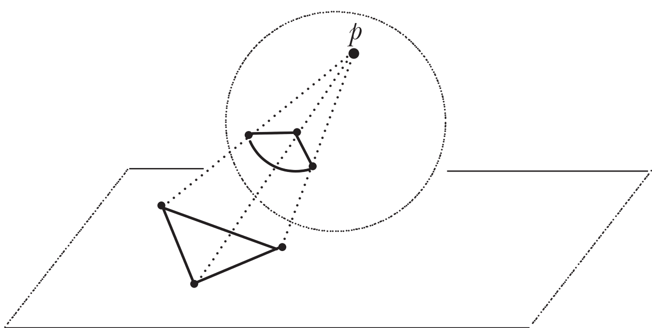


Figure 10.20. Stereographic projection.

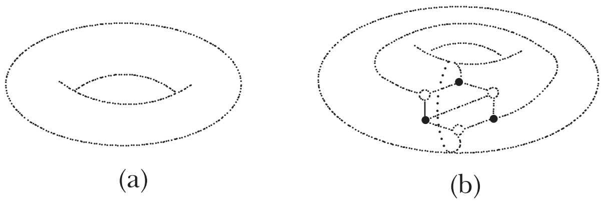


Figure 10.21. A torus and a toroidal embedding of $K_{3,3}$.

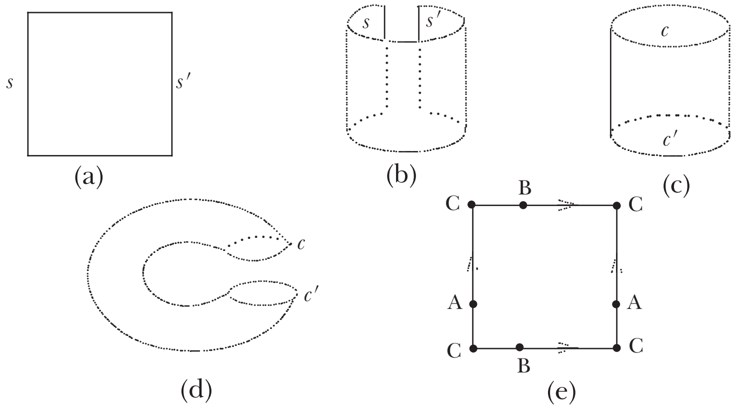


Figure 10.22. Constructing a torus.

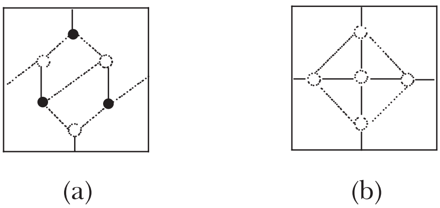


Figure 10.23. Embedding $K_{3,3}$ and K_5 on a torus.

cylinder are then also identified, giving us a torus. Consequently, in the rectangular representation of the torus of Figure 10.22e, the two points labeled A are the same point on the torus, as are the two points labeled B and the four points labeled C.

Using this representation of the torus, we see in Figure 10.23a a toroidal embedding of $K_{3,3}$. Figure 10.23b shows that K_5 can also be embedded on the torus. In fact, both K_6 and K_7 can be embedded on the torus but K_8 cannot.

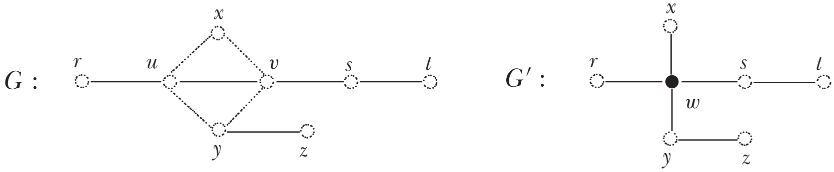


Figure 10.24. The contraction of an edge.

GRAPH MINORS

Kuratowski's theorem (Theorem 10.6) provides a method of determining whether a graph is planar; namely, a graph G is planar if and only if G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$. There is yet another way of deciding which graphs are planar, which goes back to a theorem obtained by the German mathematician Klaus Wagner (1910–2000) in 1937. While Wagner's theorem is related to and resembles Kuratowski's theorem in its statement, it is quite different.

By a *contraction* of an edge uv in a graph G is meant a merging of uv into a single vertex w where w is adjacent to every vertex in G that is adjacent to u or v in G. This results in a graph G' whose order is one less than that of G. For example, when the edge uv is contracted in the graph G of Figure 10.24, the graph G' is produced.

A graph H is called a *minor* of a graph G if either H is isomorphic to G or H is isomorphic to a graph that can be obtained from G by a succession of contractions, edge deletions or vertex deletions in any order. The graph H of Figure 10.25 is a minor of the graph G in the same figure, as H can be obtained from G by deleting z, then deleting st, then deleting xy and finally by contracting uv.

An important observation concerning minors is the following.

Theorem 10.10: *If a graph G is a subdivision of a graph H, then H is a minor of G.*

While the Petersen graph has a subgraph that is a subdivision of $K_{3,3}$, it has no subgraph that is a subdivision of K_5 (see Exercise 5). The Petersen graph, however, contains K_5 as a minor, for contracting the edges $u_1v_1, u_2v_2, u_3v_3, u_4v_4$ and u_5v_5 in Figure 10.26 produces K_5.

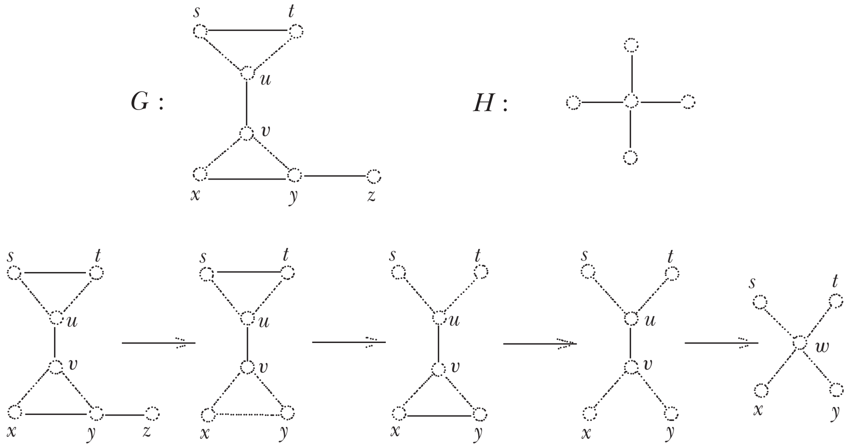


Figure 10.25. A minor of a graph.

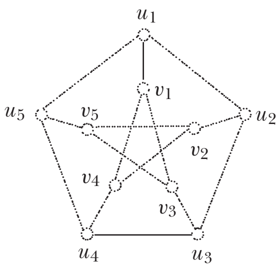


Figure 10.26. Showing that K_5 is a minor of the Petersen graph.

The fact that the Petersen graph contains K_5 as a minor but has no subdivision of K_5 shows that the converse of Theorem 10.10 is not true.

By Kuratowski's theorem and Theorem 10.10, it follows that if G is a nonplanar graph, then either K_5 or $K_{3,3}$ is a minor of G. What Wagner was able to prove was that the converse is true as well. That is, a graph G is nonplanar if and only if K_5 or $K_{3,3}$ is a minor of G. The contrapositive of this statement provides us with another way of determining whether a graph is planar.

Theorem 10.11 (Wagner's Theorem): *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.*

Around the time that Wagner proved his theorem, he evidently made a conjecture concerning minors (although this conjecture did not appear in print until 1970).

Wagner's Conjecture

For any infinite collection of graphs, there is a graph in the collection that is a minor of another graph in the collection.

Wagner's Conjecture was verified in 2004 by Neil Robertson and Paul Seymour after more than two decades of work on this problem. Robertson spent his academic career at Ohio State University while Seymour spent the latter part of his career at Princeton University. Once Wagner's Conjecture was shown to be true, this became known as the Robertson–Seymour theorem, as well as the Graph Minor Theorem.

Theorem 10.12 (The Graph Minor Theorem): *For any infinite collection of graphs, there is a graph in the collection that is a minor of another graph in the collection.*

Some sets S of graphs have the property that every minor of a graph in S also belongs to S. A set of graphs with this property is called *minor-closed*. The class of planar graphs is minor-closed. The set of graphs that can be embedded on the torus is also minor-closed.

One characteristic of a minor-closed set S of graphs is that there is always some finite set M of graphs that are *forbidden minors* in the sense that a graph G belongs to a minor-closed set S if and only if no minor of G belongs to M. For example, for the set S of planar graphs, it follows by Wagner's theorem that $M = \{K_5, K_{3,3}\}$ is a set of forbidden minors. The next result is then a consequence of the Graph Minor Theorem.

Theorem 10.13: *Let S be a minor-closed set of graphs. Then there exists some finite set M of graphs such that a graph G belongs to S if and only if no graph in M is a minor of G.*

In particular, for the set of graphs that can be embedded on the torus, there is a finite set M of forbidden minors. That is, a graph G can be embedded on the torus if and only if no graph in M is a minor of G. No one knows what such a set M is, except M must contain more than 80 graphs.

11

Coloring Graphs

Over the past few centuries, many fascinating mathematics problems have emerged, some quite easy to understand but notoriously difficult to solve.

One of the famous mathematicians of the seventeenth century was the Frenchman Pierre Fermat. He wrote that for each integer $n \geq 3$, there are no positive integers a, b and c such that $a^n + b^n = c^n$. Of course, there are many positive integer solutions when $n = 2$. For example, $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$ and $8^2 + 15^2 = 17^2$. A triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$ is called a Pythagorean triple. This term is used because there is then a right triangle with sides of lengths a and b and hypotenuse of length c, thereby satisfying the Pythagorean theorem. Fermat's assertion was discovered after his death, unproved, in the margin of a page of one of his books. In fact, he wrote that there was insufficient space in the margin to contain his remarkable proof. This statement became known as Fermat's Last Theorem but its truth would remain in question until 1995 when an article by the British mathematician Andrew Wiles appeared containing his own remarkable proof of this theorem.

An integer $p \geq 2$ is a prime if its only positive integer divisors are 1 and p. The first ten primes are 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29. Many well-known mathematicians have introduced problems involving primes. A Fermat number is an integer of the form $F_t = 2^{2^t} + 1$, where t is a nonnegative integer. In particular,

$$F_0 = 3,\ F_1 = 5,\ F_2 = 17,\ F_3 = 257,\ F_4 = 65,537.$$

All five of these integers are primes. In 1640 Fermat wrote to many that he believed every Fermat number was a prime. (He didn't call these numbers Fermat numbers, of course.) Fermat couldn't prove this though.

Nearly one century later (in 1739), the brilliant mathematician Leonhard Euler proved that the Fermat number $F_5 =$ 4,294,967,297 is divisible by 641 and is therefore not a prime. In fact, in recent decades, other Fermat numbers have been shown not to be prime. Many mathematicians now have the opposite belief that the only Fermat numbers that are primes are F_0, F_1, F_2, F_3 and F_4.

One of the most famous problems involving primes is due to the German mathematician Christian Goldbach who, around 1742, stated that he believed that every even integer greater than 2 is the sum of two primes. For example, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 5 + 3$ and $10 = 7 + 3 = 5 + 5$. It is still not known whether this is true.

Each of the three problems stated above involves what is called a conjecture, a guess that some mathematical statement is true. As we just saw, some conjectures turn out to be true, some conjectures turn out to be false and others still remain a mystery. From the examples above, one might have the impression that famous mathematical problems are due to famous mathematicians. And, for the most part, this may be true—but not always.

THE ORIGIN OF THE FOUR COLOR PROBLEM

Augustus De Morgan (1806–1871) was a British mathematician of the nineteenth century, probably best known for laws in logic and sets named for him. De Morgan taught mathematics at University College London for a number of years. During fall 1852, one of De Morgan's students was the teenager Frederick Guthrie. Frederick would go on to become a distinguished physics professor and founder of the Physical Society in London. One of the areas that Frederick studied was the science of thermionic emission, first reported by him in 1873. He discovered that a red-hot iron sphere with a positive charge would lose its charge. This effect was rediscovered in 1880 by the famous American inventor Thomas Edison.

Frederick had an older brother, Francis Guthrie (1831–1899), and while Frederick was in De Morgan's class, Francis noticed, when coloring the counties of a map of England, that he could color them with four colors in such a way that every two counties sharing a common boundary were colored with different colors. This made Francis wonder whether this might be true of all maps. This led to what would become a famous conjecture.

The Four Color Conjecture

The regions of every map can be colored with four or fewer colors in such a way that every two regions sharing a common boundary are colored differently.

The *Four Color Problem* became the problem of determining whether the Four Color Conjecture is true.

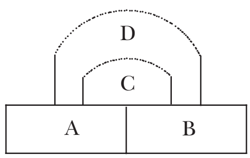
In 1850, two years before he had thought of the Four Color Problem, Francis earned a bachelor of arts degree from University College London and then a bachelor of laws degree in 1852. He would later become a mathematics professor himself at the University of Cape Town in South Africa.

Francis Guthrie attempted to prove the Four Color Conjecture and although he thought he may have been successful, he was not completely satisfied with his proof. Francis discussed his discovery with Frederick. With Francis's approval, Frederick mentioned this problem to Professor De Morgan, who expressed pleasure with it and believed it to be new. Evidently, Frederick asked Professor De Morgan if he was aware of an argument that would establish the truth of the conjecture.

This led De Morgan to write a letter on 23 October 1852 to his friend, the famous Irish mathematician, Sir William Rowan Hamilton in Dublin. These two mathematical giants had corresponded for years, although apparently had met only once. De Morgan wrote, in part,

> My dear Hamilton:
>
> A student of mine asked me to day to give him a reason for a fact which I did not know was a fact – and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary lines are differently coloured – four colours may be wanted but not more – the following is his case in which four are wanted.

A B C D are names of colours



> Query cannot a necessity for five or more be invented....
>
> My pupil says he guessed it colouring a map of England.... The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did.

In De Morgan's letter to Hamilton, he refers to the "Sphynx": While the "sphinx" is an ancient-Egyptian male statue of a lion with the head of a human, which guards the entrance to a temple, the Greek "Sphinx" is a female creature of bad luck who sat atop a rock posing the following riddle to all those who pass by:

> What animal is that which in the morning goes on four feet, at noon on two, and in the evening upon three?

Those who did not solve the riddle were killed. Only Oedipus (the title character in *Oedipus Rex* by Sophocles, a play about how people do not control their own destiny) answered the riddle correctly as "man", who in childhood (the morning of life) creeps on hands and knees, in manhood (the noon of life) walks upright and in old age (the evening of life) walks with the aid of a cane. Upon learning that her riddle had been solved, the Sphinx cast herself from the rock and perished, a fate De Morgan had envisioned for himself if his riddle (the Four Color Problem) had an easy and immediate solution.

In De Morgan's letter to Hamilton, he attempted to explain why the problem appeared to be difficult. De Morgan followed this explanation by writing,

> But it is tricky work and I am not sure of all convolutions – What do you say? And has it, if true been noticed?

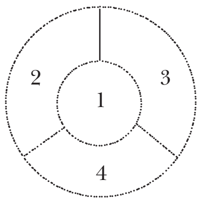
Since De Morgan had shown an interest in Hamilton's research, it is likely that De Morgan expected an enthusiastic reply to his letter. Such was not the case, however. Indeed, three days later, on 26 October 1852, Hamilton gave a quick but probably unexpected response:

> I am not likely to attempt your "quaternion" of colours very soon.

There was a bit of a pun in Hamilton's response as he used the term "quaternion", the name that was given to a class of four-dimensional numbers that Hamilton had discovered. However, Hamilton's response did nothing to diminish De Morgan's interest in the Four Color Problem as De Morgan would remain interested in this problem the rest of his life.

Since De Morgan's letter to Hamilton did not mention Frederick Guthrie by name, there may be reason to question whether Frederick was in fact the student to whom De Morgan was referring in his letter to Hamilton and that it was Frederick's older brother Francis who was the originator of the Four Color Problem. However, in 1880 Frederick wrote the following which removed any doubt as to who initiated the Four Color Problem:

> Some thirty years ago, when I was attending Professor De Morgan's class, my brother, Francis Guthrie, who had recently ceased to attend then (and who is now professor of mathematics at the South African University, Cape Town), showed me the fact that the greatest necessary number of colours to be used in colouring a map so as to avoid identity colour in lineally contiguous districts is four. I should not be justified, after this lapse of time, in trying to give his proof, but the critical diagram was as in the margin.



> With my brother's permission I submitted the theorem to Professor De Morgan, who expressed himself very pleased with it; accepted it as new; and, as I am informed by those who subsequently attended his classes, was in the habit of acknowledging where he had got his information.
>
> If I remember rightly, the proof which my brother gave did not seem altogether satisfactory to himself; but I must refer to him those interested in the subject.

The first statement in print of the Four Color Problem evidently occurred in an anonymous review written in the 14 April 1860 issue of the literary journal *Athenaeum*. Although the author of the review was not identified, De Morgan was quite clearly the writer. This review led to the Four Color Problem becoming known in the United States. Between 1852 (when the Four Color Problem was first stated) and 1878, there was apparently little interest in and publicity for this problem.

Arthur Cayley was a famous British mathematician of the nineteenth century, well known for his work in algebra. On 13 June 1878, while attending a meeting of the London Mathematical Society, Cayley asked a question that brought renewed attention to this problem:

> Has a solution been given of the statement that in colouring a map of a country, divided into counties, only four distinct colours are required, so that no two adjacent counties should be painted in the same colour?

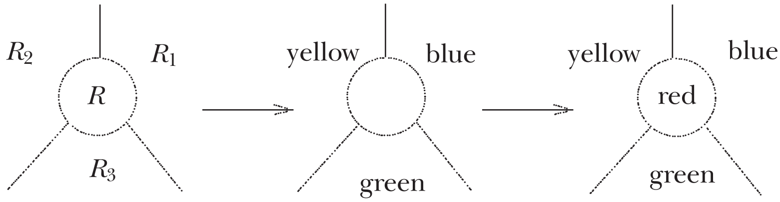
This question appeared in the proceedings of the society's meeting. In the April 1879 issue of the *Proceedings of the Royal Geographical Society*, Cayley reported,

> I have not succeeded in obtaining a general proof; and it is worth while to explain wherein the difficulty consists.

Cayley observed that if a map with a certain number of regions has been colored with four colors and a new map is obtained by adding a new region, then there is no guarantee that the new map can be colored with four colors—without first recoloring the original map. This showed that any attempted proof of the Four Color Conjecture would not be straightforward.

ALFRED BRAY KEMPE

Among those who studied under Arthur Cayley was Alfred Bray Kempe (1849–1922), who, despite having great enthusiasm for mathematics, chose a career in the legal profession. Kempe attended the meeting in which Cayley had inquired about the status of the Four Color Problem. Kempe turned his attention to this problem and on 17 July 1879, an announcement appeared in the British journal *Nature* stating that Kempe had solved the Four Color Problem and that the countries of every map could in fact be colored with four or fewer colors so that every two countries having a boundary in common are colored differently. Kempe's



(a) in M (b) in M' (c) in M

Figure 11.1. A region surrounded by three neighboring regions in a map.

proof of this result was published in an 1879 issue of the *American Journal of Mathematics*.

The method that Kempe used in his proof of the Four Color Conjecture was clever and others would use this technique in the years that followed to provide their own proofs of this famous conjecture. His method can be described as follows. Suppose that the Four Color Conjecture is false. Then there are maps that cannot be colored only with four colors. Among these maps is a map M with a minimum number k of regions. This means that every map with $k-1$ or fewer regions can be colored with four colors. Since maps with relatively few regions can certainly be colored with four colors, k cannot be all that small.

Kempe observed that this map M cannot contain a region that has three or fewer neighboring regions. For example, suppose that M has a region R that has three neighboring regions R_1, R_2 and R_3 (see Figure 11.1a). If we consider the map M' that does not contain the region R, then M' has $k-1$ regions and can therefore be colored with four colors. Let there be given a coloring of M' with four colors, say red, blue, green and yellow. Since the regions R_1, R_2 and R_3 are three mutually adjacent regions in M', these three regions are colored differently in M', say blue, green and yellow, as indicated in Figure 11.1b. However, we can use this coloring of M' to color the region R red (see Figure 11.1c) in M to produce a coloring of M with four colors. Since we know that M cannot be colored with four colors, M cannot contain a region surrounded by three regions.

Suppose, however, that M contains no region surrounded by three or fewer regions but does contain a region R surrounded by four regions, say R_1, R_2, R_3 and R_4 (see Figure 11.2a). Let M' be the map obtained

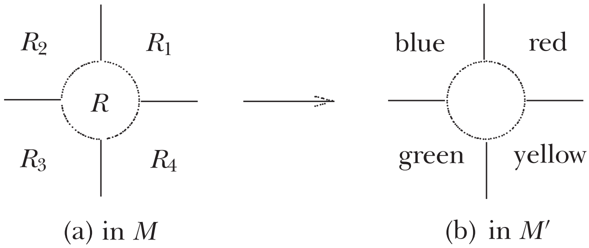


Figure 11.2. A region surrounded by four neighboring regions in a map.

from M by excluding R. Then M' has $k-1$ regions and can be colored with four colors. If R_1, R_2, R_3 and R_4 in M' are colored with two or three colors, then we can return to M, where there is a color available for R. This implies that M can be colored with four colors, which is impossible. This says that if M contains a region R surrounded by four neighboring regions, then these regions in M' must be colored with all four colors as shown in Figure 11.2b.

In Kempe's paper, he considered the question of whether there was a chain of regions in M' beginning with R_1 and ending at R_3 that are alternately colored red and green. Such a chain would later be called a red–green *Kempe chain*. Suppose first that no such chain from R_1 to R_3 exists, that is, there is no red–green Kempe chain in M' beginning with R_1 that contains R_3. The colors red and green can then be interchanged in all red–green Kempe chains beginning at R_1. This produces another coloring of the regions in M' in which no two neighboring regions are colored the same. But in this coloring, the region R_1 is colored green. By returning to M and then coloring the region R red, a coloring of M is produced using four colors. This, however, is impossible, which means that there must be a red–green Kempe chain in M' from R_1 to R_3. In this case, however, there is no blue–yellow Kempe chain of regions in M' from R_2 to R_4. So in M', we have a situation as shown in Figure 11.3 (where r, b, g, y stand for red, blue, green, yellow). We can then interchange the colors blue and yellow in all blue–yellow Kempe chains beginning at R_2. After doing this, both R_2 and R_4 are colored yellow and the color blue is available for R in M, which implies that M can be colored with four colors. Since this too is impossible, there can be no region in M surrounded by four or fewer neighboring regions.

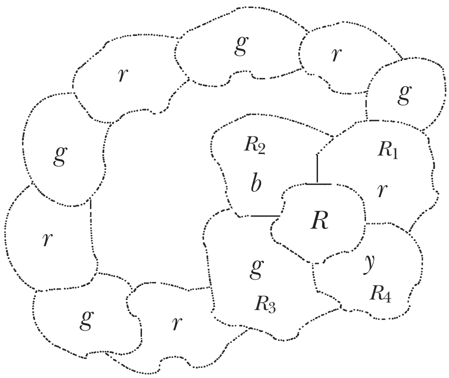


Figure 11.3. A red–green chain of regions from R_1 to R_3.

It can be shown that every map contains a region that is surrounded by five or fewer neighboring regions. Since M cannot contain a region surrounded by four or fewer neighboring regions, M must contain a region R surrounded by exactly five neighboring regions. The map M' obtained by removing R from M has $k-1$ regions and can be colored with four colors. Kempe applied his "Kempe chain" approach to M' and convinced himself that no matter how the regions of M' are colored with four colors, they can be recolored in such a manner that only three colors are used for the five regions surrounding R. This leaves a color available for R and so M can be colored with four colors. Since, once again, this is impossible, Kempe's proof was complete.

Kempe was honored for his accomplishment, including being elected as a fellow of the Royal Society in 1881. During that time, others had become interested in the Four Color Problem, some of whom had given their own proofs. One of the people with an interest in this problem was Charles Lutwidge Dodgson, better known under his pen name Lewis Carroll, author of *Alice's Adventures in Wonderland*. Another well-known individual with mathematical interests was Frederick Temple, bishop of London, who would later become the archbishop of Canterbury. Temple proved that it was impossible for any map to contain five countries, every two of which shared a common boundary, from which he concluded that no map would require five colors. Although Temple was correct that no map could contain five such countries (recall the Problem of the Five Princes in Chapter 1), he was incorrect in thinking that this would result in a solution of the Four Color Problem.

PERCY JOHN HEAWOOD

No matter how careful a mathematics student tries to be, it is always possible to make a mistake. This happens to mathematicians too. And it happened to Alfred Bray Kempe. Percy John Heawood (1861–1955) was a faculty member and administrator at Durham College in England for more than fifty years. Heawood read Kempe's solution of the Four Color Problem and in 1889 discovered an error in Kempe's solution, an error so serious that it couldn't be corrected. Heawood himself then wrote an article on this subject, published in 1890. In this article, Heawood gave an example of a map that showed that Kempe's method of proof was incorrect.

Heawood's example did not show that the Four Color Conjecture was false, however. In fact, it was relatively easy to color the countries of Heawood's map with four colors. Indeed, what Heawood's article did was return the Four Color Problem to its original status as an unsolved problem. Even though Kempe was not successful in showing that every map could be colored with four colors, Heawood was able to use Kempe's proof technique to show that every map could be colored with five or fewer colors although no one was aware of a map that actually required five colors.

Theorem 11.1 (The Five Color Theorem): *The regions of every map can be colored with five or fewer colors in such that a way that every two regions sharing a common boundary are colored differently.*

Proof: As with Kempe's approach for attempting to prove the Four Color Theorem, suppose that there are maps that cannot be colored with five colors. Among these maps, there is a map M with a minimum number k of regions. So every map with $k-1$ or fewer regions can be colored with five colors. If M contains a region R^* that is surrounded by four or fewer regions, then the map M^* not containing R^* has $k-1$ regions and can be colored with five or fewer colors. So a color is available for R^* in any coloring of M^* with five colors. This is impossible. Therefore, M has no region R^* surrounded by four or fewer neighboring regions.

As we noted earlier, M must contain a region R surrounded by five regions $R_1, R_2, \ldots, R_5$, as in Figure 11.4. Since the map M' not containing R has $k-1$ regions, it can be colored with five or fewer

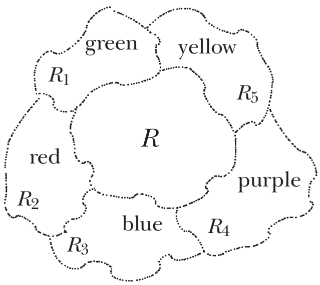


Figure 11.4. A region R surrounded by exactly five neighboring regions.

colors. Let such a coloring of M' be given. If only four or fewer colors are used for $R_1, R_2, \ldots, R_5$, there is a color available for R. So M can be colored with five colors, which is impossible. Therefore, all five colors must be used for these five regions, as shown in Figure 11.4.

If there is no green–blue Kempe chain from R_1 to R_3 in M', then we can interchange the colors green and blue in all Kempe chains beginning with R_1 and color the region R green. This produces a coloring of M with five colors, an impossibility. So there must be a green–blue Kempe chain from R_1 to R_3 in M'. Then there is no red–yellow Kempe chain from R_2 to R_5 in M'. In this case, the colors red and yellow can be interchanged in all red–yellow Kempe chains beginning with R_2. The region R can then be colored red, resulting in a coloring of M with five colors. Once again, this is impossible. ■

In addition to Kempe's attempted proof of the Four Color Theorem, Kempe's paper contained a number of interesting observations. One of these was to notice that if a piece of tracing paper was placed over a map and a point was marked on the tracing paper over each country of the map and two points were joined by a line segment whenever the corresponding countries had a common boundary, then a graph was obtained—in fact a planar graph (even though the term "graph" hadn't been introduced yet); see Figure 11.5. What this says is that the Four Color Problem could be looked at in an entirely different way. Instead of trying to show that the regions of every map could be colored with four or fewer colors, one could instead attempt to show that the vertices of every planar graph can be colored with four or fewer colors so that adjacent vertices are colored differently. In terms of graphs, the Four Color Conjecture can then be restated as follows.

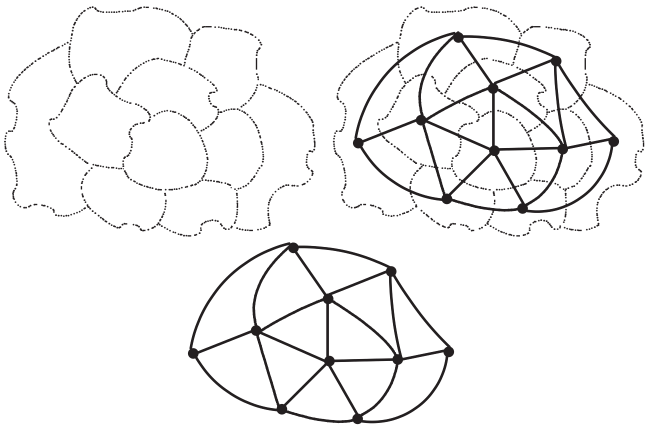


Figure 11.5. A map and corresponding planar graph.

The Four Color Conjecture

The vertices of every planar graph can be colored with four or fewer colors in such a way that every two adjacent vertices are colored differently.

While mathematical history might very well remember Alfred Bray Kempe as the individual who announced that he had solved the Four Color Problem, then wrote an article "proving" this and was later proved to be wrong, this characterization of what he did is itself incorrect—and unfair. The technique that Kempe developed was clever and later proved to be useful. He went on to have many accomplishments in mathematics. Indeed, what everyone can learn from Kempe is that we should be willing to try out new ideas and not be afraid of making mistakes. There is much to be learned from the mistakes of others and those that we make ourselves.

THE FOUR COLOR PROBLEM IN THE TWENTIETH CENTURY

In 1951 the Danish mathematician Gabriel Andrew Dirac stated the following:

> The colouring of abstract graphs is a generalization of the colouring of maps, and the study of the colouring of abstract graphs . . . opens a new chapter in the combinatorial part of mathematics.

The fact that every planar graph contains a vertex of degree 5 or less (see Corollary 10.3) implies that every map contains a region surrounded by a ring of five or fewer neighboring regions as we noted earlier. We saw that if *M* were a map possessing a minimum number of regions that cannot be colored with four colors, then Kempe could obtain a contradiction in his attempted proof if *M* contained a region surrounded by a ring of four or fewer neighboring regions. However, if *M* contained no such region, then *M* must contain a region surrounded by a ring of five regions. In that case, Kempe was not successful in producing a contradiction. In fact, no one was able to produce a contradiction in that case.

Many mathematicians who tried to solve the Four Color Problem during the twentieth century used the Kempe approach with one major difference. Instead of considering the case where the map *M* in question contained a region surrounded by a ring of five neighboring regions, those attempting to solve the problem considered a large number of possible configurations of regions, at least one of which *M* must contain. Since *M* has the minimum number of regions among those maps that cannot be colored with four colors, each map consisting of the regions on and outside the ring of regions surrounding any of these configurations can be colored with four colors. If it could be shown that for every coloring of each of these smaller maps with four colors, there was also a coloring with four colors of the regions of the configuration within the ring, resulting in a coloring of the entire map with four colors, then a contradiction would be obtained.

However, such an approach had many difficulties associated with it. First, it had to be shown that whatever the map *M* might look like, it must contain one of the configurations being considered, that is, these configurations were unavoidable. Second, even though the map consisting of the regions lying on and outside the ring surrounding each configuration could be colored with four colors, it was necessary to show that regardless of how this smaller map could be colored with four colors, there was some way that the regions in the configuration could be colored that resulted in a coloring of *M* with four colors, thereby obtaining a contradiction. After many years during which this approach was attempted, two mathematicians were finally successful. In 1976 Kenneth Appel (1932–2013) and Wolfgang Haken (1928–) of the University of Illinois announced that they had succeeded in locating 1936 unavoidable configurations with the required property that each coloring of the map outside of each configuration with four colors led to a coloring of the map M itself with four colors. To show that each possible coloring led to a coloring of M was a monumental undertaking, something that could not be done by any person. In fact, this was accomplished by means of a computer program written exclusively for this purpose. What resulted then was a highly controversial solution of the Four Color Problem, one that relied heavily on computers. This "proof" initiated a great number of philosophical questions, including the following:

(1) Is this really a proof of the Four Color Theorem?

(2) Is it necessary that a person should be able to read a proof?

(3) What is a proof of a mathematical theorem?

Because so many were not certain that a proof so complicated and relying so heavily on computers was accurate, in 1996 Neil Robertson, Daniel Sanders, Paul Seymour and Robin Thomas took it upon themselves to construct their own proof of the Four Color Theorem. While their proof required only 633 unavoidable configurations, it too relied heavily on computers and used the same overall approach. But the truth of the Four Color Conjecture had been independently verified!

Consequently, the Four Color Problem, due to the young mathematician Francis Guthrie, took 124 years to solve. While one can always wonder whether it was worth such tremendous effort to solve this problem, one response to this is that this is what mathematicians do. If there's an interesting and challenging problem, they want to be able to solve it and to learn what makes it so difficult. In the case of the Four Color Problem, however, there was an even more important reason why it was so worthwhile to work on this problem. The popularity of the Four Color Problem and the mathematics that was developed in the process of trying to solve it led to graph theory becoming a major area of mathematics and coloring becoming a major topic within graph theory. In fact, many of the theorems in graph theory can be traced back to problems involving coloring and thus back to the Four Color Problem.

Graph theory is an area of mathematics whose past is always present.

This therefore brings us to the area of coloring vertices in graphs—not only planar graphs but graphs in general.

VERTEX COLORINGS

By a *vertex coloring*, or simply a *coloring*, of a graph G is meant an assignment of colors to the vertices of G, one color to each vertex, and whose colors are chosen from a set S of colors, such that every two adjacent vertices of G are colored differently. Although there are vertex colorings of a graph that do not require adjacent vertices to be assigned distinct colors, the definition above is the common one and a coloring with this property is called a *proper coloring* of G. The set S of colors can be any set. If the number of colors involved is small, it is not unusual for S to consist of actual colors, such as red, blue, green and yellow. Typically, however, positive integers are used for colors; that is, we may take $S = \{1, 2, \ldots, k\}$ for some positive integer k. Using this as our set S of colors makes it easier to keep track of the number of colors being used. In fact, we are often interested in the minimum number of colors needed to color a graph G. This minimum number is called the *chromatic number* of G and is denoted by $\chi(G)$. (The symbol χ is the Greek letter chi.) A coloring of a graph G that uses k colors is called a *k-coloring* of G. The minimum integer k for which G has a k-coloring is then $\chi(G)$. From the Four Color Theorem, it follows that $\chi(G) \leq 4$ for every planar graph G.

There are certain classes of graphs whose chromatic number is quite easy to determine. First, since every two vertices of the complete graph K_n of order n are adjacent,

$$\chi(K_n) = n.$$

Because no two vertices of the empty graph $\overline{K}_n$ of order n are adjacent,

$$\chi(\overline{K}_n) = 1.$$

For cycles, it is easy to see the following.

Proposition 11.2: *For an integer $n \geq 3$,*

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

The following observation might seem rather obvious but is quite useful at times.

Proposition 11.3: *If H is a subgraph of a graph G, then*

$$\chi(H) \leq \chi(G).$$

As a consequence of this proposition, if a graph G contains a complete subgraph of order k, then $\chi(G) \geq k$. Also, if G contains an odd cycle, then $\chi(G) \geq 3$.

Those graphs having chromatic number 2 comprise a well-known class.

Proposition 11.4: *A graph G has chromatic number 2 if and only if G is a nonempty bipartite graph.*

Proof: First, suppose that G is a nonempty bipartite graph. Since G is nonempty, $\chi(G) \geq 2$. Since G is bipartite, G contains partite sets U and W where every edge of G joins a vertex of U and a vertex of W. Assigning the color 1 to the vertices of U and the color 2 to the vertices of W results in a 2-coloring of G and so $\chi(G) = 2$.

For the converse, suppose that G is a graph having chromatic number 2. Then G is nonempty and contains no odd cycles. By Theorem 3.4, G is bipartite. ■

In general, it is often extraordinarily difficult to determine the chromatic number of a graph. There is no formula for it. However, if G is a graph of order n, then

$$1 \leq \chi(G) \leq n.$$

Of course, we know exactly when G has chromatic number 1, 2 or n. If we know the degrees of the vertices of a graph G, then we can say more about how large the chromatic number of G might be. Recall that $\Delta(G)$ represents the largest degree of a vertex in G.

Theorem 11.5: *For every graph G, $\chi(G) \leq 1 + \Delta(G)$.*

Proof: Suppose that G has order n and the n vertices of G are listed in the order $v_1, v_2, \ldots, v_n$. First, assign the color 1 to v_1. If v_2 is adjacent to v_1, then assign the color 2 to v_2; otherwise, v_2 is

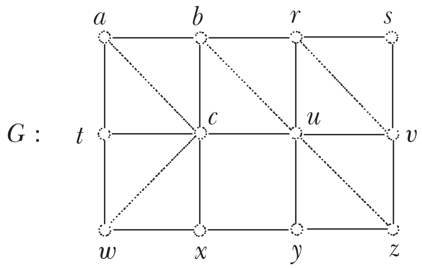


Figure 11.6. A graph in Example 11.7.

colored 1 as well. More generally, suppose now that $v_1, v_2, \ldots, v_k$ have been colored from colors in the set $\{1, 2, \ldots, \Delta(G)+1\}$, where $1 \leq k < n$. We then assign to v_{k+1} the smallest color (positive integer) that has not been used to color a neighbor of v_{k+1} that belongs to the set $\{v_1, v_2, \ldots, v_k\}$. Since at most $\deg v_{k+1}$ neighbors of v_{k+1} are in this set and since $\deg v_{k+1} \leq \Delta(G)$, some color in the set $\{1, 2, \ldots, \Delta(G)+1\}$ is available for v_{k+1}. Hence the vertices of G can be colored using at most $\Delta(G)+1$ colors. ■

We have seen graphs G for which $\chi(G) = \Delta(G)+1$. For example, $\chi(K_n) = n = 1 + \Delta(K_n)$ and if $n \geq 3$ is an odd integer, then $\chi(C_n) = 3 = 1 + \Delta(C_n)$. The British mathematician Rowland Leonard Brooks (1916–1993) proved that no other connected graphs have this property.

Theorem 11.6 (Brooks's Theorem): *If G is a connected graph of order n, then $\chi(G) \leq \Delta(G)$ unless $G = K_n$ or $n \geq 3$ is odd and $G = C_n$.*

When attempting to determine the chromatic number of a graph G, there are some important facts to be aware of. Suppose that we want to show that $\chi(G) = k$. Then what we need to do is (1) show that there is a k-coloring of G and (2) show that there is no $(k-1)$-coloring of G.

Example 11.7: *Determine the chromatic number of the graph G in Figure 11.6.*

SOLUTION:

Since the three vertices a, b and c are mutually adjacent (thereby producing an odd cycle of length 3), $\chi(G) \geq 3$. We may assume that a is colored 1, b is colored 2 and c is colored 3. Suppose that $\chi(G) = 3$. Then all vertices of G can be colored with the colors 1, 2 and 3. This forces the vertices of G to be colored as follows:

$$a - 1,\ b - 2,\ c - 3,\ t - 2,\ w - 1,\ x - 2,$$
$$u - 1,\ r - 3,\ v - 2,\ s - 1,\ z - 3,\ y - 2.$$

However, x and y are then adjacent vertices that are both colored 2. This is impossible. Therefore, $\chi(G) > 3$. Since G is planar, $\chi(G) \leq 4$ and so $\chi(G) = 4$. (Furthermore, if we color the vertex y, say, with the color 4 and all other vertices of G as above, then we have a proper coloring of G and so $\chi(G) = 4$.) ♦

APPLICATIONS OF VERTEX COLORINGS

There are many problems that can be represented by a graph and whose solution involves finding the chromatic number of this graph. We present two examples in this section that should be reminiscent of examples that appeared in Chapter 1.

Example 11.8: *The mathematics department of a certain college plans to schedule the classes Graph Theory* (GT)*, Statistics* (S)*, Linear Algebra* (LA)*, Advanced Calculus* (AC)*, Geometry* (G) *and Modern Algebra* (MA) *this summer. Ten students have indicated the courses they plan to take.*

Anden: LA, S;	*Brynn:* MA, LA, G;
Chase: MA, G, LA;	*Denise:* G, LA, AC;
Everett: AC, LA, S;	*François:* G, AC;
Greg: GT, MA, LA;	*Harper:* LA, GT, S;
Irene: AC, S, LA;	*Jennie:* GT, S.

With this information, use graph theory to determine the minimum number of time periods needed to offer these courses so that every two classes having a student in common are taught at different time periods during the day. Of course, two classes having no students in common can be taught during the same period.

SOLUTION:

First, we construct a graph H whose vertices are the six subjects. Two vertices (subjects) are joined by an edge if some student is taking classes in these two subjects (see Figure 11.7). The minimum number of time periods is $\chi(H)$. Since H contains the odd cycle (GT, S, AC, G, MA, GT), it follows that three colors are needed to color the vertices on this cycle. Since LA is adjacent to all vertices of this cycle, a fourth color is needed for LA. Thus $\chi(H) \geq 4$. However, there is a 4-coloring of H shown in Figure 11.7 and so $\chi(H) = 4$. This also tells us one way to schedule these six classes during four time periods, namely period 1: Graph Theory, Advanced Calculus; period 2: Geometry; period 3: Statistics, Modern Algebra; period 4: Linear Algebra. ♦

Example 11.9: *Figure 11.8 shows the nine traffic lanes* L1, L2, . . . , L9 *at the intersection of two busy streets. A traffic light is located at this intersection.*

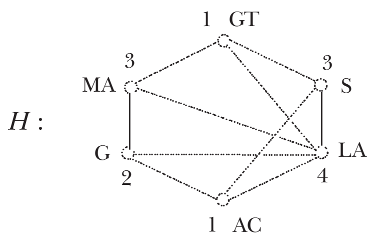


Figure 11.7. The graph of Example 11.8.

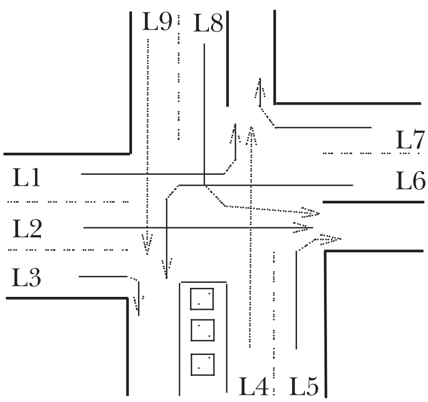


Figure 11.8. Traffic lanes at street intersections in Example 11.9.

G :

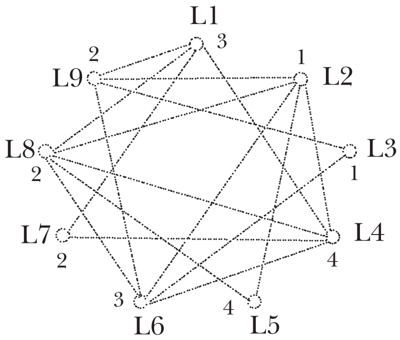


Figure 11.9. The graph in Example 11.9.

During a certain phase of the traffic light, those cars in lanes for which the light is green may proceed safely through the intersection. What is the minimum number of phases needed for the traffic light so that (eventually) all cars may proceed through the intersection?

SOLUTION:

First, a graph G is constructed that models this situation, where $V(G) = \{L1, L2, \dots, L9\}$ and two vertices (lanes) are joined by an edge if vehicles in these two lanes cannot safely enter the intersection at the same time, as there is the possibility of an accident (see Figure 11.9).

Answering this question requires determining the chromatic number of the graph G in Figure 11.9. First, notice that the four vertices L2, L4, L6 and L8 are mutually adjacent and so four colors are needed to color these vertices. Thus $\chi(G) \geq 4$. Since there exists a proper coloring of G using the four colors 1, 2, 3, 4, as indicated in Figure 11.9, $\chi(G) = 4$.

Consequently, the minimum number of phases for the traffic light is four and vehicles in lanes with the same color may proceed through the intersection at the same time once the traffic light turns green for that phase. ♦

12

Synchronizing Graphs

Among the interests of the Scottish physicist Peter Guthrie Tait (1831–1901) were mathematics and golf. His interest in golf carried over to his son Frederick (better known as Freddie Tait). Indeed, Frederick became the finest amateur golfer of his time.

Like many others, Peter Tait played a role in the history of the Four Color Problem. In fact, Tait came up with several solutions of the problem himself—unfortunately, all incorrect. One of Tait's approaches to solve the Four Color Problem was a new idea, one he believed would lead to a different solution. As it turned out, his idea did not lead to a solution but it did lead to a new type of graph coloring: *edge coloring*.

It had been known already that the regions of every planar graph could be colored with four or fewer colors if the regions of every 3-regular bridgeless planar graph could be colored with four or fewer colors. What Tait was able to prove is that the regions of every 3-regular bridgeless planar graph could be colored with four or fewer colors if and only if the *edges* of such a graph could be colored with three colors so that every two adjacent edges are colored differently. Since Tait didn't think coloring the edges of such graphs with three colors would be all that difficult, he felt that he had found a new way to solve the Four Color Problem. Such was not the case, however. Nevertheless, this did bring up the idea of coloring the edges of a graph.

THE CHROMATIC INDEX OF A GRAPH

Just as the major interest in vertex colorings of a graph G concerns minimizing the number of colors so that every two adjacent vertices are assigned different colors, the major interest in edge colorings of a graph

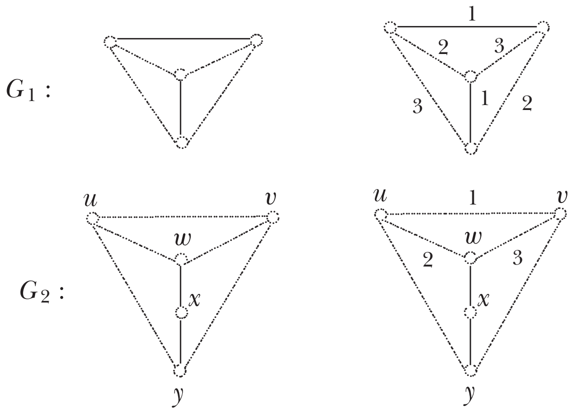


Figure 12.1. Two graphs with maximum degree 3.

G deals with minimizing the number of colors so that every two adjacent edges are assigned different colors. These are called *proper edge colorings*. Since no two adjacent edges of a graph G are colored the same in a proper edge coloring of G, the edges assigned any particular color form a matching of G. The minimum number of colors needed in a proper edge coloring of a graph G is called the *chromatic index* of G and is denoted by $\chi'(G)$.

In a proper edge coloring of a graph G, the number of colors needed to color the edges incident with each vertex v of G is $\deg v$. Therefore, coloring all of the edges of G requires at least $\Delta(G)$ colors, that is, $\chi'(G) \geq \Delta(G)$. For example, for the graphs G_1 and G_2 shown in Figure 12.1, $\Delta(G_1) = \Delta(G_2) = 3$ and so both $\chi'(G_1) \geq 3$ and $\chi'(G_2) \geq 3$. Since there is a proper edge coloring of G_1 with three colors, it follows that $\chi'(G_1) = 3$.

For the graph G_2 of Figure 12.1, the situation is different. Suppose that $\chi'(G_2) = 3$. Then there is a proper edge coloring of G_2 with the colors 1, 2 and 3. Since the vertices u, v and w form a triangle, the edges uv, uw and vw must be assigned distinct colors, say 1, 2 and 3, respectively (see Figure 12.1). This implies that uy is colored 3 and vy is colored 2. Therefore, wx and xy are both colored 1, which is impossible since these edges are adjacent. Therefore, $\chi'(G_2) \neq 3$. By assigning wx the color 1 and xy the color 4, an edge coloring of G with four colors is produced and so $\chi'(G_2) = 4$.

The chromatic index of some of the best known graphs is determined next.

Proposition 12.1: *For $n \geq 2$,*

$$\chi'(K_n) = \begin{cases} n-1 & \textit{if } n \textit{ is even,} \\ n & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof: Since $\Delta(K_n) = n-1$, it follows that $\chi'(K_n) \geq n-1$. If n is even, then K_n is 1-factorable by Theorem 7.10. Let $F_1, F_2, \ldots, F_{n-1}$ be $n-1$ 1-factors in a 1-factorization of K_n. By assigning the color i to each edge in F_i for $i = 1, 2, \ldots, n-1$, we have a proper coloring of the edges of K_n using $n-1$ colors and so $\chi'(K_n) = n-1$.

Suppose next that n is odd. Since no matching in K_n can contain more than $(n-1)/2$ edges, at most $(n-1)/2$ edges of K_n can be assigned the same color. Because K_n has $n(n-1)/2$ edges, at least n colors are needed to color the edges of K_n. If we add a new vertex v to K_n and join v to each vertex of K_n, then K_{n+1} is produced. Since $n+1$ is even, $\chi'(K_{n+1}) = n$, that is, the edges of K_{n+1} (and so the edges of K_n as well) can be colored with n colors. Therefore, $\chi'(K_n) = n$ if n is odd. ■

According to Proposition 12.1 then, the chromatic index of every complete graph is always an odd integer.

VIZING'S THEOREM

The Russian mathematician Vadim Vizing was born in 1937. His family was forced to move to Siberia after World War II because his mother was half German. After completing his undergraduate degree, Vizing was sent to the famous Steklov Institute in Moscow to study for a PhD in the area of function approximation. Because Vizing did not like this area and was not permitted to change areas, he left the Institute and went to Novosibirsk where he had lived as a youngster. He then studied at the Mathematical Institute of the Academy of Sciences in Academgorodoc, where he obtained a PhD without the aid of a formal supervisor. Nevertheless, he met a professor there, Alexander Zykov, who became a mentor to Vizing. Zykov's interest was graph theory, which led to Vizing becoming interested in graph theory.

While in Novosibirsk, Vizing became interested in a problem involving coloring the wires of a network, which led to his investigation of more theoretical questions. We have already observed that $\chi'(G) \geq \Delta(G)$ for every nonempty graph G. In fact, for every graph G we have seen in this chapter, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. Vizing proved that there is no other possibility; namely, if G is a graph with maximum degree Δ, it is always possible to color the edges of G with $\Delta + 1$ colors in such a way that no adjacent edges have the same color. In symbols, this can be stated as follows.

Theorem 12.2 (Vizing's Theorem): *For every graph G,*

$$\chi'(G) \leq \Delta(G) + 1.$$

Vizing wrote a research paper containing this remarkable result and submitted it for publication to a prestigious Russian mathematics journal, only to have it rejected when the referee found the result to be uninteresting. The paper was eventually published in a local journal. This only goes to show that even talented mathematicians and scientists experience disappointments but many, including Vizing, go on to make major contributions in their areas of interest.

According to Vizing's theorem, the chromatic index of every graph G is either $\Delta(G)$ or $\Delta(G) + 1$. This fact divides the graphs into two classes. Graphs G for which $\chi'(G) = \Delta(G)$ are called *class one* graphs and graphs G for which $\chi'(G) = \Delta(G) + 1$ are called *class two* graphs. So every graph is either a class one or a class two graph. In particular, the graph G_1 of Figure 12.1 is a class one graph and the graph G_2 of Figure 12.1 is a class two graph. The main question with respect to proper edge colorings is therefore, which graphs are class one graphs and which graphs are class two graphs? Of course, Vizing's theorem guarantees that there is a proper edge coloring of every graph G using the colors $1, 2, \ldots, \Delta(G), \Delta(G) + 1$. Although the general problem of determining the class of a graph is hard, there is a condition under which a graph of odd order must be a class two graph.

Theorem 12.3: *Suppose that a graph G has odd order n and size m. If*

$$m > \frac{(n-1)\Delta(G)}{2},$$

then G is a class two graph.

Proof: Suppose, to the contrary, that G is a class one graph. Then $\chi'(G) = \Delta(G)$ and there is a proper edge coloring of G using the colors $1, 2, \ldots, \Delta(G)$. As we observed, the edges assigned any one of these colors is a matching in G. Since n is odd, no matching can contain more than $(n-1)/2$ edges. Therefore, this coloring can color at most $(n-1)\Delta(G)/2$ edges. Since $m > (n-1)\Delta(G)/2$, not all edges of G can be colored, which is a contradiction. ■

In the case where G is an r-regular graph of odd order n and size m, the degree r of every vertex is even and $m = rn/2$. Since

$$m = \frac{rn}{2} > \frac{r(n-1)}{2} = \frac{(n-1)\Delta(G)}{2},$$

it follows by Theorem 12.3 that G is a class two graph. What if G is an r-regular graph of *even* order, however? If G is a class one graph, that is, if $\chi'(G) = \Delta(G) = r$, then every vertex of G must be incident with an edge of each color. This says that a regular graph G is a class one graph if and only if G is 1-factorable. These observations therefore provide us with an alternative explanation for Proposition 12.1. A regular graph of even order need not be a class one graph though. For example, we saw in Theorem 7.9 that the Petersen graph (a 3-regular graph of order 10) is not 1-factorable and so it is a class two graph.

APPLICATIONS OF EDGE COLORINGS

Edge colorings are often useful in solving certain kinds of scheduling problems.

Example 12.4: *Alvin* (A) *has invited three married couples to his summer home for a week: Bob* (B) *and Carrie* (C) *Hanson, David* (D) *and Edith* (E) *Irwin and Frank* (F) *and Gena* (G) *Jackson. Since all six guests enjoy playing tennis, he decides to set up some tennis matches. Each of his six guests will play a tennis match against every other guest except his/her spouse. In addition, Alvin will play a match against each of David, Edith, Frank and Gena. If no one is to play two matches on the same day, what is a schedule of matches over the smallest number of days?*

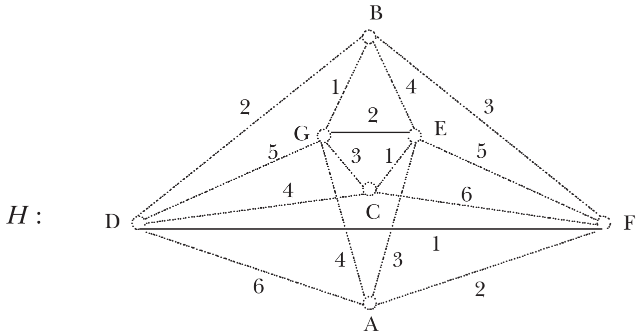


Figure 12.2. The graph H in Example 12.4.

SOLUTION:

First, we construct a graph H whose vertices are the people at Alvin's summer home; so $V(H) = \{A, B, C, D, E, F, G\}$, where two vertices of H are adjacent if the two vertices (people) are to play a tennis match. (The graph H is shown in Figure 12.2.) To answer the question, we determine the chromatic index of H.

First, observe that $\Delta(H) = 5$. By Vizing's theorem (Theorem 12.2), $\chi'(H) = 5$ or $\chi'(H) = 6$. Also, the order of H is $n = 7$ and its size is $m = 16$. Since

$$m = 16 > 15 = \frac{(7-1)\cdot 5}{2} = \frac{(n-1)\Delta(H)}{2},$$

it follows from Theorem 12.3 that $\chi'(H) = 6$. Figure 12.2 gives an edge coloring of H with six colors. The edges colored the same give a schedule of matches for a given day. So a possible schedule of matches is the following:

Day 1: Bob–Gena, Carrie–Edith, David–Frank;
Day 2: Alvin–Frank, Bob–David, Edith–Gena;
Day 3: Alvin–Edith, Bob–Frank, Carrie–Gena;
Day 4: Alvin–Gena, Bob–Edith, Carrie–David;
Day 5: David–Gena, Edith–Frank;
Day 6: Alvin–David, Carrie–Frank.

These matches take place over the smallest number of days (namely six). ♦

Example 12.5: *Five individuals have been invited to a bridge tournament (bridge is a game of cards): Allen (*A*), Brian (*B*), Charles (*C*), Doug (*D*) and Ed (*E*). A game of bridge is played between two two-person teams. Every two-person team* {X, Y} *is to play against all other two-person teams* {W, Z}*, where, of course, neither* W *nor* Z *is* X *or* Y*. If the same team cannot play bridge more than once on the same day, what is the fewest number of days needed for all possible games of bridge to be played. Set up a schedule for doing this in the smallest number of days. What graph models this situation?*

SOLUTION:

We construct a graph G whose vertices consist of all two-person teams, where we denote a vertex {X, Y} by XY for simplicity. Two vertices (two-person teams) XY and WZ are adjacent in G if they will be playing a game of bridge. The graph G is shown in Figure 12.3. Observe that G is the famous Petersen graph.

We have already seen that the Petersen graph G is not 1-factorable and so G is a class two graph. Thus $\chi'(G) = 4$. An edge coloring of G with four colors is shown in Figure 12.3. This creates the following schedule of games which takes place over the smallest number of days:

Day 1: AB–DE, AE–BC, AC–BE, AD–CE;
Day 2: AB–CE, AC–DE, AE–BD, AD–BC, BE–CD;
Day 3: AB–CD, BC–DE, BD–CE;
Day 4: AC–BD, AD–BE, AE–CD.

♦

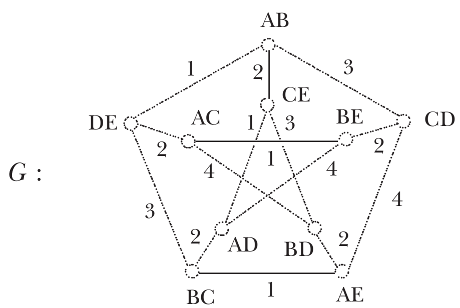


Figure 12.3. The graph in Example 12.5.

While proper edge colorings are the best known and most studied edge colorings of graphs, there are edge colorings which are not proper that have resulted in curious problems. We now look at some of these.

RAMSEY NUMBERS

Except for a three-year period during World War II, the William Lowell Putnam mathematical competition for undergraduates has taken place every year since 1938. This exam, administered by the Mathematical Association of America, consists of (since 1962) 12 challenging mathematical problems. We saw in Chapter 9 that this competition was designed to stimulate a healthy rivalry in colleges and universities throughout the United States and Canada. The 1953 exam contained the following problem.

> **Problem A2** The complete graph with 6 points (vertices) and 15 edges has each edge colored red or blue. Show that we can find 3 points such that the 3 edges joining them are the same color.

This problem deals with the topic of Ramsey numbers named for Frank Plumpton Ramsey (1903–1930), a British philosopher, economist and mathematician who died at the age of 26. Ramsey proved a remarkable theorem, a restricted version of which is stated next.

> Theorem 12.6 (Ramsey's Theorem): *For any $k+1 \geq 3$ positive integers $t, n_1, n_2, \ldots, n_k$, there exists a positive integer n such that if each of the t-element subsets of the set $\{1, 2, \ldots, n\}$ is assigned one of the k colors $1, 2, \ldots, k$, then for some integer i with $1 \leq i \leq k$, there is a subset S of $\{1, 2, \ldots, n\}$ containing n_i elements such that every t-element subset of S is colored i.*

First, let's see what this rather complicated-sounding theorem of Ramsey says when $t = 1$. Let $n_1, n_2, \ldots, n_k$ be $k \geq 2$ positive integers. Then there exists a positive integer n such that if each element of the set $\{1, 2, \ldots, n\}$ is assigned one of the k colors $1, 2, \ldots, k$, then for some integer i with $1 \leq i \leq k$, there is a subset S of $\{1, 2, \ldots, n\}$ containing n_i elements each of which is colored i. In this case, it is not difficult to find an integer n that satisfies this condition. In fact,

$$n = 1 + \sum_{i=1}^{k}(n_i - 1)$$

is such an integer. What we have just seen is actually a version of the *Pigeonhole Principle*.

The integers in Ramsey's theorem can be the objects of any set. For example, suppose that $k = 3$ and $n_1 = 5$, $n_2 = 4$ and $n_3 = 3$. If we had a collection of

$$n = 1 + (n_1 - 1) + (n_2 - 1) + (n_3 - 1) = 1 + (5 - 1) + (4 - 1) + (3 - 1) = 10$$

Olympic medals, each of which is gold, silver or bronze, then Ramsey's theorem says that either there are 5 gold medals, 4 silver medals or 3 bronze medals. (If we had only 9 medals, there might be 4 gold, 3 silver and 2 bronze, not what is desired.)

Also, if $k = 2$ and $n_1 = n_2 = 3$ and we had a collection of $1 + (n_1 - 1) + (n_2 - 1) = 1 + (3 - 1) + (3 - 1) = 5$ edges, each of which is colored red or blue, then either three edges are colored red or three edges are colored blue.

What Ramsey's theorem says when $t = 2$ is even more intriguing, however. Here we are talking about the 2-element subsets of some set. In this case, for any $k \geq 2$ positive integers $n_1, n_2, \ldots, n_k$, there exists a positive integer n such that if each 2-element subset of the set $S = \{1, 2, \ldots, n\}$ is assigned one of the k colors $1, 2, \ldots, k$, then for some integer i with $1 \leq i \leq k$, there is a subset T of S with $|T| = n_i$ such that every 2-element subset of T is colored i. In terms of graph theory, the integers in Ramsey's theorem can be interpreted as vertices in a complete graph and so the 2-element subsets are edges. Then there exists a positive integer n such that if every edge of K_n is assigned one of the k colors $1, 2, \ldots, k$, then for some integer $i \in \{1, 2, \ldots, k\}$, there is a complete subgraph of order n_i of K_n such that every edge of this subgraph is colored i. For example, in the case where $k = 2$ and $n_1 = n_2 = 3$, there exists a positive integer n such that if each edge of the graph K_n is colored 1 or 2, then there is either a complete subgraph K_3 all of whose edges are colored 1 or a complete subgraph K_3 all of whose edges are colored 2. If we interpret color 1 as red and color 2 as blue, then

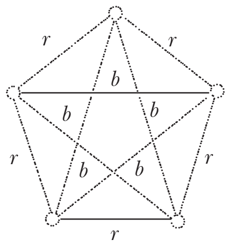


Figure 12.4. A coloring of K_5 in which no triangle has its edges colored the same.

Problem A2 of the 1953 Putnam exam asks the student to show that $n = 6$ satisfies the condition. Let's see why.

Suppose that every edge of K_6 is colored red or blue. We show that either K_6 contains a triangle whose three edges are colored red or a triangle whose three edges are colored blue. Let v be any vertex of K_6. Then v has degree 5 and so is incident with five edges. By the Pigeonhole Principle, three of these five edges are colored the same, say vv_1, vv_2 and vv_3 are colored red. If either v_1v_2, v_1v_3 or v_2v_3 is colored red, then K_6 has a triangle all three edges of which are colored red. Otherwise, these three edges are colored blue, producing a triangle all three edges of which are colored blue.

On the other hand, $n = 5$ does not satisfy the condition. If the edges of K_5 are colored as shown in Figure 12.4 (where we write r for red and b for blue), then there is no triangle all of whose edges are colored the same.

More generally, by a *red–blue coloring* of a graph G is meant an edge coloring of G in which every edge is colored red or blue. Adjacent edges may very well be colored the same. A subgraph F of G in which every edge is colored red is called a *red* F and a subgraph H of G in which every edge is colored blue is called a *blue* H.

It is a consequence of Ramsey's theorem that for every two graphs F and H, there is always a positive integer n such that for every red–blue coloring of K_n, either a red F or a blue H results. The smallest such positive integer n with this property is called the *Ramsey number* of F and H and is denoted by $R(F, H)$. You might be reminded of the following question.

> What is the smallest number of individuals in a gathering of people that guarantees that there are three mutual friends or three mutual strangers?

In Chapter 1, this question was asked in the Three Friends or Three Strangers Problem and we saw that the answer is six. We saw that a gathering of n people could be represented by the complete graph K_n, whose vertices are the people and an edge uv is colored red if u and v are friends and colored blue if u and v are strangers. We now see that the answer to the question above is the Ramsey number $R(K_3, K_3) = 6$.

The most studied Ramsey numbers $R(F, H)$ are those in which both F and H are complete graphs. In this case, they are called *classical Ramsey numbers*. Perhaps surprisingly, there is only a handful of pairs $s, t \geq 3$ of integers for which $R(K_s, K_t)$ is known. For example,

$$\begin{array}{lll} R(K_3, K_3) = 6, & R(K_3, K_6) = 18, & R(K_3, K_9) = 36, \\ R(K_3, K_4) = 9, & R(K_3, K_7) = 23, & R(K_4, K_4) = 18, \\ R(K_3, K_5) = 14, & R(K_3, K_8) = 28, & R(K_4, K_5) = 25. \end{array}$$

In particular, the classical Ramsey number $R(K_5, K_5)$ is unknown. In the following example, the classical Ramsey number $R(K_3, K_4)$ is verified.

Example 12.7: $R(K_3, K_4) = 9$.

SOLUTION:

To establish the inequality $R(K_3, K_4) \geq 9$, we show that there is a red–blue coloring of K_8 for which there is no red K_3 and no blue K_4. Consider the red–blue coloring of K_8 shown in Figure 12.5, where the red subgraph of K_8 is shown in Figure 12.5a and the blue subgraph of K_8 is shown

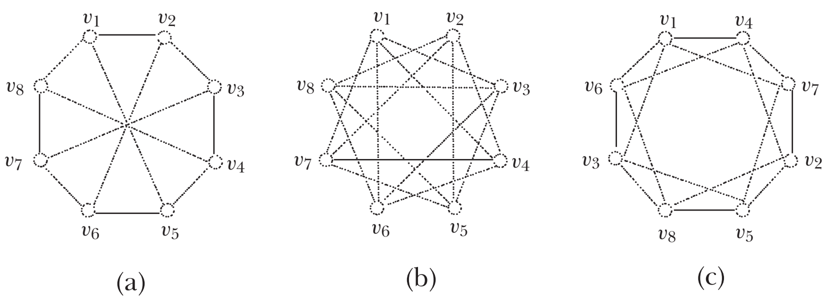


Figure 12.5. A red–blue coloring of K_8 that avoids a red K_3 and a blue K_4.

in Figure 12.5b (redrawn in Figure 12.5c). Since the red subgraph of K_8 in Figure 12.5a contains no red K_3 and the blue subgraph of K_8 in Figure 12.5b contains no blue K_4 (seen more clearly in Figure 12.5c), $R(K_3, K_4) \geq 9$.

It remains to show that $R(K_3, K_4) \leq 9$. To establish this inequality, we show that every red–blue coloring of K_9 contains either a red K_3 or a blue K_4. Let a red–blue coloring of K_9 be given. First, observe that the resulting red subgraph of K_9 cannot be 3-regular as no graph can contain an odd number of odd vertices. Therefore, some vertex v of K_9 is not incident with exactly three red edges. We consider two cases.

Case 1. *The vertex v is incident with four or more red edges*. Thus there exist vertices v_1, v_2, v_3 and v_4 of K_9 such that vv_1, vv_2, vv_3 and vv_4 are all red. If any two vertices in the set $A = \{v_1, v_2, v_3, v_4\}$ are joined by a red edge, a red K_3 results; otherwise, every two vertices in A are joined by a blue edge, producing a blue K_4.

Case 2. *The vertex v is incident with two or fewer red edges*. Thus v is incident with six or more blue edges. Hence there exist vertices $u_1, u_2, \ldots, u_6$ of K_9 such that all of the edges $vu_1, vu_2, \ldots, vu_6$ are blue. Since $R(K_3, K_3) = 6$, the subgraph $H = K_6$ with vertex set $\{u_1, u_2, \ldots, u_6\}$ contains either a red K_3 or a blue K_3. If H contains a red K_3, so does K_9. If H contains a blue K_3, then since v is joined to every vertex of H by a blue edge, K_9 contains a blue K_4.

Therefore, in both cases, the red–blue coloring of K_9 results in either a red K_3 or a blue K_4. ♦

We now consider two Ramsey numbers that are not classical Ramsey numbers.

Example 12.8: $R(P_3, K_3) = 5$.

SOLUTION:

First, we show that $R(P_3, K_3) \geq 5$. The red–blue coloring of K_4 shown in Figure 12.6 (where each red edge of K_4 is drawn as a bold edge) avoids both a red P_3 and a blue K_3 and so $R(P_3, K_3) \geq 5$.

It remains to show that $R(P_3, K_3) \leq 5$. Let a red–blue coloring of K_5 be given. Consider a vertex v_1 in K_5. If v_1 is incident with two red edges,

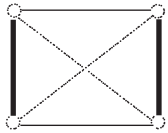


Figure 12.6. A red–blue coloring of K_4 that avoids a red P_3 and a blue K_3.

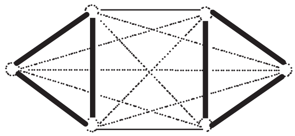


Figure 12.7. A red–blue coloring of K_6 that avoids a red $K_{1,3}$ and a blue K_3.

then a red P_3 is produced. Otherwise, v_1 is incident with at most one red edge. So there are three blue edges incident with v_1, say v_1v_2, v_1v_3 and v_1v_4. If there is a blue edge joining any two of the vertices v_2, v_3 and v_4, a blue K_3 is produced. Otherwise, v_2v_3 and v_3v_4 are red edges, producing a red P_3. Therefore, $R(P_3, K_3) \leq 5$. ♦

Example 12.9: $R(K_{1,3}, K_3) = 7$.

SOLUTION:

First, we show that $R(K_{1,3}, K_3) \geq 7$. Consider the red–blue coloring of K_6 shown in Figure 12.7, where again each red edge of K_6 is drawn as a bold edge. Since the red subgraph consists of two disjoint copies of K_3 and the blue subgraph is the complete bipartite graph $K_{3,3}$, there is neither a red $K_{1,3}$ nor a blue K_3 in this coloring and so $R(K_{1,3}, K_3) \geq 7$.

Next we show that $R(K_{1,3}, K_3) \leq 7$. Let a red–blue coloring of K_7 be given. Consider a vertex v_1 in K_7. If v_1 is incident with three red edges, then a red $K_{1,3}$ is produced. Otherwise, v_1 is incident with four blue edges, say v_1v_2, v_1v_3, v_1v_4 and v_1v_5. If any edge joining two of the vertices in $\{v_2, v_3, v_4, v_5\}$ is blue, then a blue K_3 is produced. Otherwise, all edges joining any two of the vertices in $\{v_2, v_3, v_4, v_5\}$ are colored red. In particular, the edges v_2v_3, v_2v_4 and v_2v_5 are colored red and a red $K_{1,3}$ is produced. ♦

The two Ramsey numbers we just considered are of the type $R(F, H)$, where F is a tree and H is a complete graph. In the first issue of the *Journal of Graph Theory* (published in 1977), Vašek Chvátal established a simple formula for every such Ramsey number.

Theorem 12.10: *For every tree T of order $m \geq 2$ and the complete graph K_n of order $n \geq 2$,*

$$R(T, K_n) = (m-1)(n-1) + 1.$$

As Ramsey's theorem states, Ramsey numbers are not limited to two colors. There is one nontrivial multicolor case of Ramsey's theorem for which the exact value of the resulting classical Ramsey number is known, namely the case when $k = 3$ and $n_1 = n_2 = n_3 = 3$. In particular, the Ramsey number $R(K_3, K_3, K_3)$ is the minimum positive integer n such that if every edge of K_n is colored with one of the colors red, blue and green, then there is a K_3 all of whose edges are colored the same.

Example 12.11: $R(K_3, K_3, K_3) = 17$.

SOLUTION:

First, we show that $R(K_3, K_3, K_3) \leq 17$. Let there be given a red–blue–green coloring of K_{17}, that is, each edge of K_{17} is colored red, blue or green. Let v be a vertex of K_{17}. Then $\deg v = 16$. By the Pigeonhole Principle, v is incident with six edges of the same color, say the edges vv_i $(i = 1, 2, \ldots, 6)$ are all colored green. If any two vertices of $S = \{v_1, v_2, \ldots, v_6\}$ are joined by a green edge, then K_{17} contains a green K_3. On the other hand, if no two vertices of S are joined by a green edge, then every two vertices of S are joined by a red edge or a blue edge. Since $R(K_3, K_3) = 6$, the complete graph K_6 with vertex set S contains either a red K_3 or a blue K_3. Thus, as claimed, $R(K_3, K_3, K_3) \leq 17$.

It turns out that the graph K_{16} is H-decomposable, where H is the 5-regular graph of order 16 shown in Figure 12.8. This graph H is referred to as the *Clebsch graph* and has the property that it contains no triangles. There are three copies H_1, H_2, H_3 of H in an H-decomposition of K_{16}. By coloring each edge of H_1 red, each edge of H_2 blue and each edge of H_3 green, there is a red–blue–green coloring of K_{16} containing

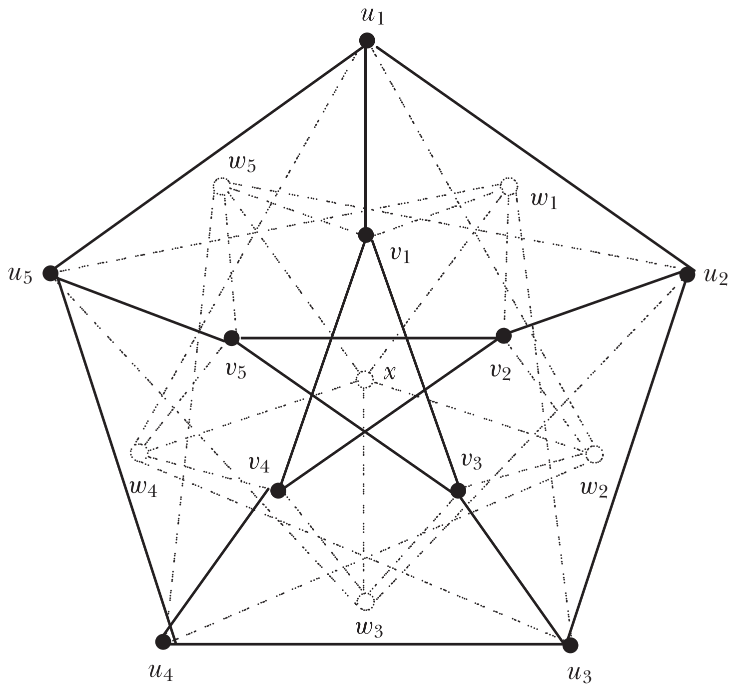
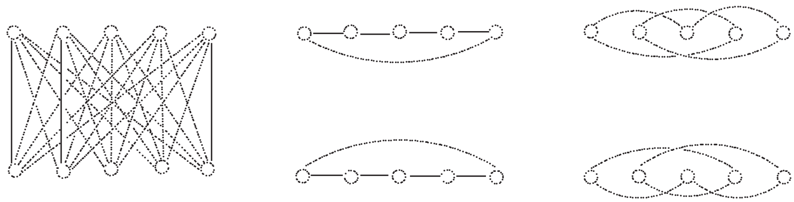


Figure 12.8. The Clebsch graph.

no K_3 whose edges are colored the same. Thus $R(K_3, K_3, K_3) \geq 17$ and so $R(K_3, K_3, K_3) = 17$. ♦

A graph H whose edges are colored the same is called a *monochromatic* H. So $R(K_3, K_3, K_3) = 17$ says that any edge coloring of K_{17} with three colors results in a monochromatic K_3, while there is an edge coloring of K_{16} with three colors resulting in no monochromatic K_3. A graph F, no two edges of which are colored the same, is called a *rainbow* F. There is a multicolor Ramsey number that deals with both monochromatic and rainbow graphs. The *rainbow Ramsey number* $RR(K_3, K_3)$ is the minimum positive integer n such that every red–blue–green coloring of K_n produces either a rainbow K_3 or a monochromatic K_3.

The graph K_{10} can be decomposed into the graph $F_1 = K_{5,5}$, the graph F_2 consisting of two disjoint 5-cycles and F_3 also consisting of two disjoint 5-cycles, as shown in Figure 12.9. By coloring each edge of F_1 red, each edge of F_2 blue and each edge of F_3 green, there is neither a rainbow K_3



(a) F_1 (b) F_2 (c) F_3

Figure 12.9. A decomposition of K_{10}.

nor a monochromatic K_3. So $RR(K_3, K_3) \geq 11$. It is more challenging to show that $RR(K_3, K_3) \leq 11$. Nevertheless, $RR(K_3, K_3) = 11$.

THE ROAD COLORING PROBLEM

There is a curious edge coloring problem that concerns the concept of finite-state automata, a topic that is often encountered in computer science. Prior to discussing this problem, let's consider the following example.

Example 12.12: A motel has a laundry room where guests can wash and dry their clothes. In addition to a number of washing machines and dryers in this room, there is a vending machine and a money changer where dollar bills and five dollar bills can be converted into all quarters or all half-dollars. The vending machine dispenses small bottles of laundry detergent and small bottles of fabric softener, each bottle costing 75¢. This machine accepts quarters and half-dollars only and immediately returns the change if more than 75¢ is deposited. The machine can be in one of four states s_0, s_1, s_2 and s_3, as indicated below.

s_0: Nothing has been deposited.

s_1: 25¢ has been deposited.

s_2: 50¢ has been deposited.

s_3: 75¢ has been deposited.

The vending machine contains two buttons, labeled LD (laundry detergent) and FS (fabric softener). If the vending machine is in any of the states s_0, s_1 or s_2 and either of these two buttons is pressed, nothing happens. On the other hand, if the vending machine is in the state s_3, then either the bottle of laundry detergent or the bottle of fabric softener is dispensed, depending on whether the button LD or FS is pressed. Once this occurs, the vending machine then returns to state s_0.

One day Matthew, one of the guests at the motel, decides to wash a load of clothes. Since he has no fabric softener or laundry detergent, he decides to use the vending machine to purchase a bottle of each. Matthew has two half-dollars and two quarters. He first inserts the two half-dollars (receiving a quarter in change when the second half-dollar is deposited) and presses the button labeled FS, obtaining a bottle of fabric softener. He next inserts the three quarters he now has and presses the button labeled LD. A bottle of laundry detergent is then dispensed for him. The table in Figure 12.10 describes what happens in each case after (1) the input of each half-dollar followed by pressing the FS button and then after (2) the input of each quarter followed by pressing the LD button.

The possible actions of this vending machine can be modeled by a *directed graph* D, also called a *digraph*, in which parallel *directed edges* and *directed loops* are permitted. The vertex set of D is $V(D) = \{s_0, s_1, s_2, s_3\}$. This is shown in Figure 12.11. Here LD represents the laundry detergent button, FS represents the fabric softener button and N indicates no output. For example, there is an arc from state s_0 to state s_1 labeled (25, N). This means that if we deposit 25¢ at state s_0, then the state changes to s_1 and there is no output. There is an arc from state s_3 to state s_0 labeled (FS, softener). This means that if we were to press the FS button at state s_3, a bottle of fabric softener is dispensed and the vending machine is returned to state s_0. The directed loop at s_3 labeled (50, 50) means that if

State	s_0	s_2	s_3
Input	50¢	50¢	FS
Output	nothing	25¢	softener

State	s_0	s_1	s_2	s_3
Input	25¢	25¢	25¢	LD
Output	nothing	nothing	nothing	detergent

Figure 12.10. A table illustrating what Matthew did at the vending machine.

$D:$

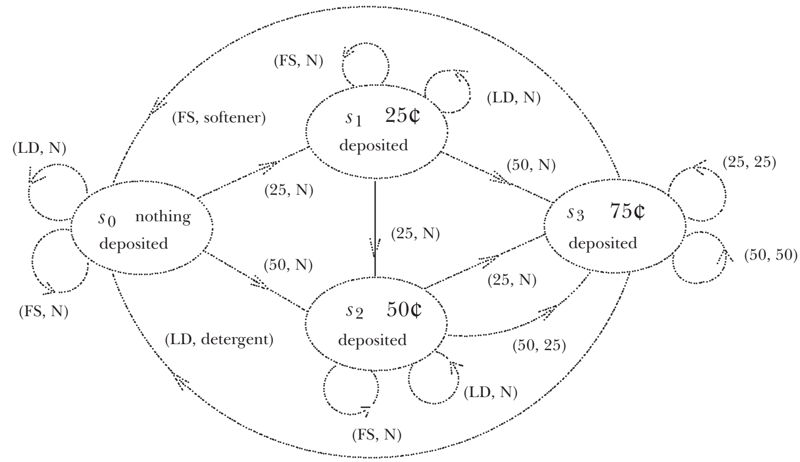


Figure 12.11. The digraph D representing the actions of the vending machine in Example 12.12.

we deposit 50¢ at state s_3, then 50¢ is returned and the machine remains at state s_3. ♦

This type of digraph is also referred to as a *finite-state machine*. Observe that the outdegree of each vertex of the digraph D in Figure 12.11 is 4. This is because at each state, one of four actions can take place: 25¢ is deposited, 50¢ is deposited, the LD button is pressed, the FS button is pressed. It is not unusual for the vertices of digraphs representing finite-state machines to have the same outdegree. A directed loop at a vertex contributes 1 to both the outdegree and the indegree of that vertex. Not every vertex of D has indegree 4, however; in fact, $\text{id}\, s_0 = \text{id}\, s_2 = 4$ but $\text{id}\, s_1 = 3$ and $\text{id}\, s_3 = 5$.

There are finite-state machines that produce no output. These are called *finite-state automata*. The singular of automata is *automaton*. For example, a finite-state automaton might model the road network of a town (see Figure 12.12). Suppose that the states are the entrances to various street intersections and the input values are *turn left and drive one block* (ℓ), *turn right and drive one block* (r) and *go straight ahead for one block* (s). Whichever state one is in, applying one of the input values will lead us to a new state. An input string, such as $rss\ell ss\ell$ (turn right and drive one block, go straight ahead for two blocks, turn left and drive one block, go straight ahead for two blocks, turn left and drive one block) provides directions for how to drive from a given state (location in this case)

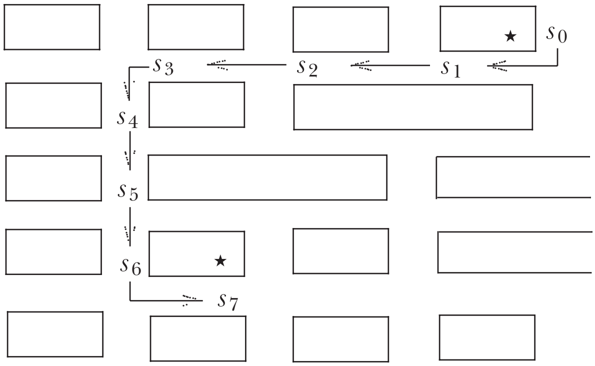


Figure 12.12. A diagram modeling the road network of a town.

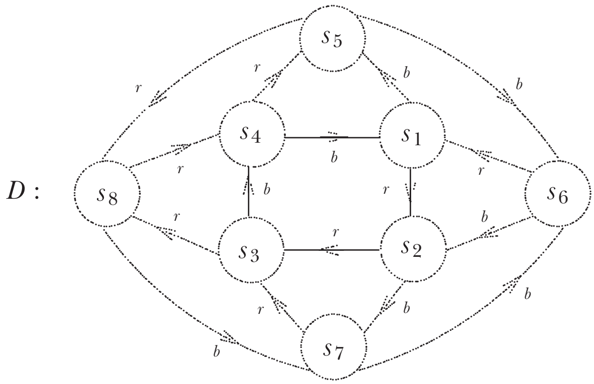


Figure 12.13. A digraph representing a finite-state automaton.

to another state (our destination). This is illustrated in the diagram shown in Figure 12.12.

Suppose, for example, that we have a finite-state automaton A with eight states, where, this time, we denote the states by $s_1, s_2, \ldots, s_8$. In this instance, we have two input values, which we denote by r and b. The digraph D representing A is shown in Figure 12.13. Since A has two input values, there are two actions that can be performed at each state and so every vertex of D has outdegree 2.

Suppose that we were to consider the list of input values (called an *input string*)

$$brrbrrbrr \tag{12.1}$$

and we apply this with s_5, say, chosen as the initial state. The input value b changes the state from s_5 to s_6; so we are now at the state s_6. Applying the next input value r changes state s_6 to s_1. Continuing this, we arrive at the following directed walk:

$$(s_5, s_6, s_1, s_2, s_7, s_3, s_8, s_7, s_3, s_8).$$

That is, the input string (12.1) results in a directed $s_5 - s_8$ walk. Next, suppose that we apply the same input string with the initial state s_4, say. This results in the directed walk

$$(s_4, s_1, s_2, s_3, s_4, s_5, s_8, s_7, s_3, s_8).$$

In this case, the terminal state is s_8 once again. In fact, if we were to apply this same input string with any state chosen as the initial state, the terminal state would always be s_8.

Let us now explain what the input values r and b are meant to designate. These represent colors, namely red and blue, while the states $s_1, s_2, \ldots, s_8$ represent locations. The arcs represent one-way roads. An arc labeled r is a *red road* and an arc labeled b is a *blue road*. Hence if we were to start at the location s_5 and take the blue road, then this would lead us to s_6. If we then take the red road (from s_6), we would be led to the location s_1. Taking the red road out of s_1 moves us to s_2, and so on. Indeed, the string (12.1) can be interpreted as driving directions. Following the driving directions (12.1), we can drive to s_8, regardless of where we begin our trip. Therefore, if someone wanted directions for how to drive to s_8, the input string (12.1) serves as driving directions—and it is not necessary to know where the person is currently located.

Next, suppose that we consider a new input string, say

$$bbrbbrbbr, \tag{12.2}$$

and apply this, once again using s_5 as the initial state. In this case, we obtain the directed walk

$$(s_5, s_6, s_2, s_3, s_4, s_1, s_2, s_7, s_6, s_1).$$

Hence a directed $s_5 - s_1$ walk is obtained in this case. If we apply the input string (12.2) with the initial state s_6, say, we obtain the directed walk

$$(s_6, s_2, s_7, s_3, s_4, s_1, s_2, s_7, s_6, s_1).$$

So in this case as well, s_1 is the terminal state of this directed walk. In fact, if we were to apply the input string (12.2) with any initial state, then the terminal state of this resulting directed walk is once again s_1.

Perhaps surprisingly, if we were to select any state s in $V(D)$, then there is always an input string (driving directions) such that applying this string to any initial state results in a directed walk having the final state s. Hence the digraph D of Figure 12.13 has a likely unanticipated property. This leads us to a more general question.

There are several not particularly useful responses to questions involving providing directions for how to get from "here" to "there", including the pessimistic response

"You can't get there from here"

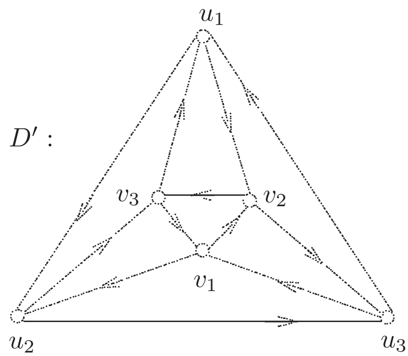
and the puzzling response (due to the former baseball player Yogi Berra)

"When you come to a fork in the road, take it".

If the vertices of a digraph D have the same outdegree, then D is said to be *out-regular* or have *uniform outdegree*. As expected, a digraph D is *strong* if, for every two vertices u and v of D, there is a directed $u-v$ path as well as a directed $v-u$ path in D. A digraph D is *periodic* if $V(D)$ can be divided into $k \geq 2$ subsets $V_1, V_2, \ldots, V_k$ such that for every arc (u, v) of D, it follows that $u \in V_i$ and $v \in V_{i+1}$ for some i with $1 \leq i \leq k$, where $V_{k+1} = V_1$. In this case, D is called *k-periodic*. A digraph D is *aperiodic* if it is not periodic.

The digraph D of Figure 12.13 is strong, aperiodic and has uniform outdegree 2. While the digraph D' of Figure 12.14a is also strong and has uniform outdegree 2, it is periodic, however. In fact, D' is 3-periodic with subsets V_1, V_2, V_3, where $V_1 = \{u_1, v_1\}$, $V_2 = \{u_2, v_2\}$ and $V_3 = \{u_3, v_3\}$. This periodic nature of D' is more evident in the redrawing of D' shown in Figure 12.14b.

If D is a strong digraph with uniform outdegree Δ, then for every vertex of D, there are Δ arcs directed away from that vertex. Suppose, for every vertex w of D, that the Δ arcs directed away from w are colored with the Δ distinct colors of the set $S = \{1, 2, \ldots, \Delta\}$. This coloring is called a *proper Δ-arc coloring* of D. Let $u = u_0$ be a vertex of D and let $s = a_1 a_2 \cdots a_k$ be a finite sequence of colors in S. There is exactly one arc



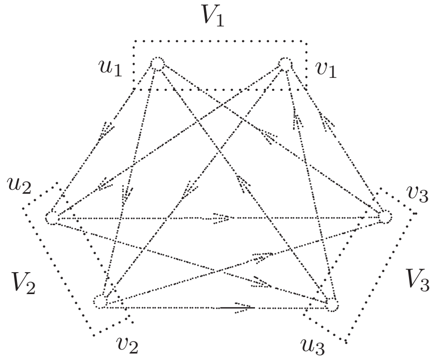


Figure 12.14. A periodic digraph.

with initial vertex u_0 whose color is a_1, say (u_0, u_1), and exactly one arc with initial vertex u_1 whose color is a_2, say (u_1, u_2), and so on. That is, s determines a unique directed walk

$$W = (u = u_0, u_1, \ldots, u_k = v)$$

of length k, where the arc (u_{i-1}, u_i) is colored a_i for $1 \le i \le k$. Consequently, s determines a unique terminal vertex v of a directed walk with initial vertex u.

For a strong digraph D with uniform outdegree Δ, a proper Δ-arc coloring c of D is said to be *synchronizing* if, for every vertex v of D, there exists a sequence s_v of colors such that for every vertex u of D, the directed walk with initial vertex u determined by s_v has terminal vertex v. In this case, the sequence s_v is called a *synchronizing sequence* for the vertex v. No periodic strong digraph with uniform outdegree Δ can possess a synchronizing Δ-arc coloring. For example, in Figure 12.14b, the length of any directed walk from u_1 to v_1 must be divisible by 3. Consequently, the length of any synchronizing sequence for v_1 must also be divisible by 3. However, then any directed walk with initial vertex u_2 determined by such a sequence must terminate at u_2 or v_2. This implies that there is no synchronizing sequence for v_1 and so there is no synchronizing 2-arc coloring of D'.

In 1970 Roy L. Adler and Benjamin Weiss posed a problem on this topic in the context of symbolic dynamics and coding theory. In 1977 this problem was stated by Adler, L. Wayne Goodwyn and Weiss in terms of digraphs. Thinking of the vertices as representing locations, each arc representing a one-way road from one location to another and an arc colored i $(1 \leq i \leq k)$ as an i-colored road, this problem can be stated in the following manner.

Suppose that there is a network of one-way roads between a collection of cities such that

(1) every city in the network is reachable from every other city in the network;

(2) the same number $\Delta \geq 2$ of roads leave each city and

(3) the network is aperiodic; that is, the cities cannot be divided into $k \geq 2$ sets $S_1, S_2, \ldots, S_k, S_{k+1} = S_1$ such that every road leaving a city in S_i proceeds to a city in S_{i+1} for each i $(1 \leq i \leq k)$.

Is it possible to color exactly one road leaving each city with one of the colors $1, 2, \ldots, \Delta$ in such a way that each city A in the network can be assigned universal driving directions (a sequence s_{A} of integers from the set $\{1, 2, \ldots, \Delta\}$) such that if we start at any city B in the network and follow the driving directions in s_{A}, then the trip terminates at city A? Thus, if "there" refers to city A and "here" is any city B, then following the driving directions gets us from here to there.

In terms of digraphs, this 1970 problem by Weiss and Adler gained an appealing name, which is stated as follows.

The Road Coloring Problem

Does every strong aperiodic digraph with uniform outdegree $\Delta \geq 2$ have a synchronizing Δ-arc coloring?

Over the years a number of mathematicians attempted unsuccessfully to solve this problem. However, as announced in the *New York Times* in 2008, the Russian-born Israeli mathematician Avraham Trahtman (1944–) was successful in obtaining a proof.

Theorem 12.13 (The Road Coloring Theorem): *Every strong aperiodic digraph with uniform outdegree $\Delta \geq 2$ has a synchronizing Δ-arc coloring.*

For example, the digraph D of Figure 12.13 representing a finite-state automaton is strong, aperiodic and has uniform outdegree 2. By the Road Coloring Theorem, there exists a synchronizing 2-arc coloring of D.

In fact, the coloring of the arcs of D shown in Figure 12.13 is a proper 2-arc coloring. Actually, we saw that

$$brrbrrbrr$$

in (12.1) is a synchronizing sequence for the vertex s_8 and that

$$bbrbbrbbr$$

in (12.2) is a synchronizing sequence for the vertex s_1. Since the digraph D is strong, there is a directed walk (indeed, a directed path) from s_1 to each vertex of D. In particular, (s_1, s_2, s_3) is a directed $s_1 - s_3$ path in D. Therefore, if we were at location s_1, took the red road to location s_2 and then took the red road out of s_2, we would arrive at s_3. Thus if we were to append rr to the sequence $bbrbbrbbr$, giving us $bbrbbrbbrrr$, we would also have a synchronizing sequence for s_3. This gives rise to a more general observation.

Once a Δ-arc coloring is found for a strong aperiodic digraph D with uniform outdegree Δ and a synchronizing sequence s_v is found for some vertex v of D, then this is a synchronizing Δ-arc coloring of D. To see this, let u be some other vertex of D. Because D is strong, there is a directed $v - u$ path $P = (v = v_0, v_1, \ldots, v_k = u)$ in D. Let $a_i \in \{1, 2, \ldots, \Delta\}$ be the color of the arc (v_{i-1}, v_i) for $i = 1, 2, \ldots, k$. Then $s_u = s_v a_1 a_2 \cdots a_k$ is a synchronizing sequence for u. Of course, it may be of greater interest to find a synchronizing sequence for a vertex of D that is not obtained merely by appending a sequence to a synchronizing sequence of some other vertex. For example,

$rbrrbrrbr$ is a synchronizing sequence for s_3

in the 2-arc colored digraph D in Figure 12.13. Moreover,

$brbbrbbrb$ is a synchronizing sequence for s_5,
$rbbrbbrbb$ is a synchronizing sequence for s_6 and
$rrbrrbrrb$ is a synchronizing sequence for s_7.

Nevertheless, for a given strong aperiodic digraph D with uniform outdegree Δ, the challenge is to find a Δ-arc coloring of D that is synchronizing.

Epilogue

Graph Theory: A Look Back—The Road Ahead

Now in its third century, the mathematical area of graph theory had a most humble beginning. The city of Königsberg, located in what was East Prussia in the eighteenth century, became the subject of a question of whether it was possible to stroll about this city and cross each of its seven bridges exactly once. Leonhard Euler, one of the great mathematicians of all time, saw that this problem and a generalization of it could possibly be solved with the aid of a technique called the geometry of position, originated by Gottfried Leibniz, one of the developers of calculus. A 1736 research paper by Euler containing his solution is recognized as the beginning of graph theory. This led to the subject of Eulerian graphs in graph theory. Indeed, as one looks back at many games and puzzles of the past, it can be seen that there are elements of graph theory in many of these. In a number of these games and puzzles, other famous mathematicians were involved. Sir William Rowan Hamilton, renowned mathematician, physicist and inventor of the class of numbers called quaternions, observed that there was a connection between his icosian calculus and cycles on a dodecahedron that pass through each of its vertices. Despite the fact that the Reverend Thomas Kirkman had earlier considered cycles on polyhedra containing each of their vertices, this led to the subject of Hamiltonian graphs in graph theory.

The problem introduced in the nineteenth century that had the greatest impact on the development of graph theory was the famous Four Color Problem. This problem, created by a young Francis Guthrie in October 1852, asked whether it was possible to color the regions of every map with four or fewer colors in such a way that every two regions with a common boundary are colored differently. While this problem may very well have been thought by many to be rather frivolous initially, Augustus De Morgan, the first well-known mathematician to encounter this problem, obviously saw this problem as both interesting and challenging and is credited for others becoming aware of it. This problem may still have faded into history, however, had it not been for the famous mathematician Arthur Cayley. In 1878, while attending a meeting of the London Mathematical Society, Cayley asked for the status of the Four Color Problem, which made this problem well known, even outside of Europe. Alfred Bray Kempe, a former student of Cayley, believed that he had solved the problem and a paper written by him that contained his proposed proof was published in 1879, only for it to emerge a decade later that Kempe's proof contained an error that could not be corrected. This mistake was discovered by Percy John Heawood, who was able to use Kempe's technique to show that the regions of every map could be colored with five or fewer colors, although neither he nor anyone else was able to give an example of a map that actually required five colors. This essentially marked the beginning of numerous assaults by numerous mathematicians to solve this problem. While a controversial computer-aided solution of the Four Color Problem finally surfaced in 1976, what was more important than the resulting Four Color Theorem was all the graph theory and its applications that were developed as this problem was being tackled.

While games, puzzles and other recreational problems may have led to the beginning of graph theory, it was an 1891 research paper by Julius Petersen that showed that graph theory was, in fact, a theoretical branch of mathematics. This marked the beginning of substantial growth in graph theory, a subject that was greatly aided by many research mathematicians who showed great interest in and made major contributions to graph theory. One of these people was Denés König, an advocate of graph theory from Budapest, Hungary who, in the years leading up to World War II, had students who were to make their own mark in graph theory. In 1936 it was König who wrote the first book entirely devoted to graph theory. After World War II, this book, written in German, made graph theory considerably more well known. In 1958 a second book on graph theory, written by Claude Berge in French, would make graph theory even better known in Europe. This was followed by two books on graph theory, written in English and published in the United States, one in 1962 by Oystein Ore of Yale University and one in 1969 by Frank Harary of the University of Michigan.

One of König's students was Paul Erdős, who became perhaps the best known mathematician during the second half of the twentieth century.

He was a tireless mathematician who did research with hundreds of others. The fact that one of his major interests was graph theory and that Erdős traveled so much and lectured so often throughout the world only contributed to graph theory becoming better known.

Another important mathematician who contributed significantly to graph theory was William Tutte. As a young British student, Tutte played a major role in breaking German codes during World War II. He would later discover important theorems in graph theory as a graduate student. After earning his PhD, he would move to Canada and become one of the major figures in graph theory during the latter half of the twentieth century.

Since the 1960s, there has been an explosive growth in the number of (1) mathematicians throughout the world interested in and/or working on graph theory, (2) meetings and conferences around the world where graph theory is one of the major topics, (3) research journals where graph theory is one of the main topics and (4) books and monographs where graph theory is either the main subject or one of the main subjects. As the world moved into the digital age and now is firmly implanted in the age of technology, more and more applications of graph theory dealing with communication and social networks and the Internet in general have blossomed. In particular, the web graph has the web pages as its vertices and its edges (or directed edges) correspond to the links between pages. This graph has already been studied extensively. Certainly graph theory is destined to play a central role as more information is sought about the mathematics of the Internet.

Thus the curious little problem about Königsberg, as well as the question concerning the number of colors needed to color a map, has developed into an area of mathematics that shows no letup in its growth. Perhaps William Tutte (under his pen name Blanche Descartes) described the situation best when in 1969 he reminded us of graph theory's past and hinted at its future when he wrote the following poem titled "The Expanding Unicurse":

Some citizens of Königsberg
Were walking on the strand
Beside the river Pregel
With its seven bridges spanned.

“O Euler, come and walk with us”,
Those burghers did beseech.
“We’ll roam the seven bridges o’er,
And pass but once by each.”

“It can’t be done”, thus Euler cried.
“Here comes the Q. E. D.
Your islands are but vertices,
And four have odd degree.”

From Königsberg to König’s book,
So runs the graphic tale,
And still it grows more colorful,
In Michigan and Yale.

(This poem was published in *Proof Techniques in Graph Theory*, ed. Frank Harary, p. 25, Copyright Academic Press (1969).)

Exercises

EXERCISES FOR CHAPTER 1

(1) If a king had four sons and wanted to divide his kingdom into four regions so that each region had a common boundary with the other three, could this be done? Show how this situation can be represented by a graph.

(2) Once there was a king who had six sons—in fact, three sets of twins. Upon his death, he wanted his kingdom to be divided into six regions, one for each son, in such a way that every region shares some common boundary with each of the other regions except that of the son's twin. Can this be done? Use graphs to answer this question.

(3) Suppose that four houses are under construction and each house must be provided with a connection to each of two utilities. Under the same conditions as the Three Houses and Three Utilities Problem, can these conditions be satisfied for this Four Houses and Two Utilities Problem? Use graphs to answer this question.

(4) Figure 1 shows a map consisting of nine regions.

   (a) Construct a graph G whose vertices are the regions and where two vertices are adjacent if they correspond to two regions sharing a common boundary.
   (b) Show that the regions of the map can be colored with four colors such that two regions sharing a common boundary are colored differently.

(5) For the two polyhedra shown in Figure 2, determine the number V of vertices, the number E of edges and the number F of faces. Show that the Euler Polyhedron Formula holds in each case.

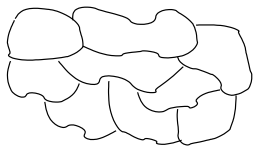


Figure 1. The map in Exercise 4.

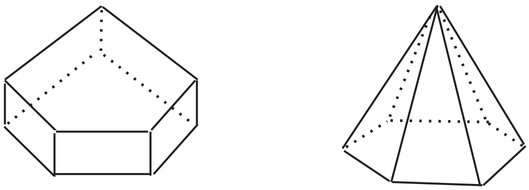


Figure 2. The two polyhedra in Exercise 5.

(6) Figure 3 shows a city (an imaginary city) through which a river flows and across which are eight bridges. Represent this situation by a graph (or multigraph) G where each land region in the city is a vertex in G and whenever two land regions are joined by a bridge there is an edge.

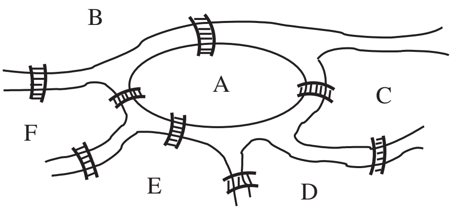


Figure 3. The city map in Exercise 6.

(7) Is it possible to find a closed route on a cube that passes through each vertex exactly once?

(8) We have seen that a knight's tour is possible on the standard 8×8 chessboard. Show that no such knight's tour is possible on

(a) a 4×4 chessboard;
(b) a 3×5 chessboard;
(c) a 3×4 chessboard.

(9) Alexander, Carver, Dennis, Jordan, Perkins and Thomas were all present at a business meeting. Several of these people had never met. In particular, Alexander had never met Carver, Perkins and Thomas, while Carver had never met Dennis and Jordan. In addition, Dennis and Jordan had never met, as was the case with Perkins and Thomas. Those pairs of people who had never met shook hands with each other. Represent this situation by a graph and use the First Theorem of Graph Theory (the Handshaking Lemma) to compute the total number of handshakes that took place.

(10) Eight coins are placed on the nine squares of a 3×3 chessboard, at most one coin per square. Therefore, exactly one square does not contain a coin. The nine possible configurations are shown in Figure 4, where the configuration labeled (i, j) indicates that the square without a coin is located in row i and column j. If a coin in configuration (i, j) can be moved horizontally or vertically one square to the vacant square in that configuration to produce the configuration (k, ℓ), then we say that configuration (i, j) can be transformed into configuration (k, ℓ). Observe that if configuration (i, j) can be transformed into configuration (k, ℓ), then configuration (k, ℓ) can also be transformed into configuration (i, j). Show that this situation can be represented by a graph.

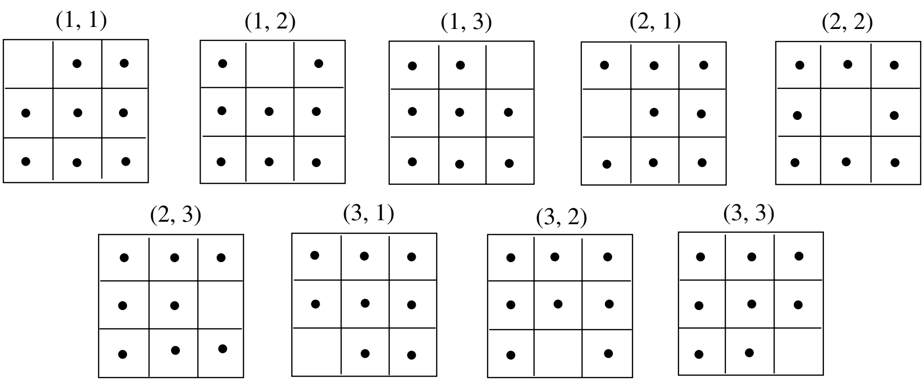


Figure 4. Eight coins on a chessboard, from Exercise 10.

(11) In Exercise 10, eight coins are placed on eight of the nine squares of a 3×3 chessboard. Suppose instead that only one coin is placed on

one of these squares. The configuration (i, j), where $i, j \in \{1, 2, 3\}$, indicates that the coin is on the square in row i and column j. The coin in configuration (i, j) can be moved one square horizontally or one square vertically to be transformed into configuration (k, ℓ). Draw the graph that represents this situation.

(12) Five friends Al (a), Bob (b), Charlie (c), Dave (d) and Ed (e) occasionally get together for meals and activities. They decide to divide themselves into all possible pairs (teams) to play golf against every other team (not containing a common person of course). There are ten teams in all, namely

$$\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}, \{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}, \{d, e\}.$$

For example, the team $\{a, b\}$ will therefore play golf against the teams $\{c, d\}$, $\{c, e\}$ and $\{d, e\}$. In fact, every team will play golf against three other teams. Show that this situation can be represented by a graph.

(13) Four college students Fred (F), Lou (L), Matt (M) and Pete (P) are watching a football game on television at a local sports restaurant. During halftime, they get into a discussion of which football teams they have seen play in person among the New England Patriots (NE), New York Giants (NG), Dallas Cowboys (DC) and Chicago Bears (CB). Here's what they learn:

F: NE, NG, CB; L: NE, DC, CB; M: NE, NG, DC;
P: NG, DC, CB.

Show that this situation can be represented by a graph.

(14) During the holidays, a community has three decorated trees located in a row on a platform where the lights on each tree are either all blue or all silver. Every minute the lights on one of these trees change color (from all blue to all silver or from all silver to all blue). Draw a graph that represents the situation.

(15) Suppose that in Exercise 14 the lights on each tree are either all blue, all silver or all red. As in Exercise 14, each minute the lights on one

of these three trees change color. What is the order and size of the graph that represents this situation?

(16) An outpatient area of a hospital requires each patient to set aside at least 30 minutes during 1:30–4:00 pm when checking in for a medical procedure. During a particular afternoon eight patients, denoted by $P_1, P_2, \ldots, P_8$, have set aside the following time periods:

P_1: 1:30–2:15 pm; P_2: 1:40–2:35 pm;

P_3: 1:45–2:20 pm; P_4: 2:00–2:45 pm;

P_5: 2:25–3:00 pm; P_6: 2:30–3:15 pm;

P_7: 2:50–3:30 pm; P_8: 3:10–4:00 pm.

The hospital scheduler is interested in knowing those pairs of patients whose time periods overlap. Represent this situation by a graph whose vertices are the patients and where there is an edge joining two vertices if the corresponding time periods of the patients overlap.

(17) A certain airline has flights into and out of a large number of cities. Eight of these cities are denoted by $C_1, C_2, \ldots, C_8$. This airline has direct flights between certain pairs of cities. This information is given in the 8×8 matrix

$$\mathbf{A} = [a_{ij}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $a_{ij} = 1$ means that there is a direct flight between the cities C_i and C_j. Describe this situation by means of a graph.

(18) A representative of a company finds it convenient to travel to eight cities, denoted by $c_1, c_2, \ldots, c_8$. Her favorite airline has direct routes between various pairs of these cities. In particular, there are direct routes from c_1 to c_3, c_4, c_7 and back again, which we denote by c_1: c_3, c_4, c_7. The complete set of direct routes is

$$c_1\colon\ c_3, c_4, c_7;\ \ c_2\colon\ c_4, c_8;\ \ c_3\colon\ c_6, c_8;\ \ c_4\colon\ c_5, c_7;\ \ c_5\colon\ c_6, c_7.$$

Model this situation by a graph G whose vertex set consists of these eight cities. Two vertices (cities) of G are adjacent if there is a direct flight between these two cities. What are the order and size of this graph?

(19) Figure 5 shows the traffic lanes at the intersection of two busy streets. When a vehicle approaches this intersection, it could be in one of the nine lanes L1, L2, ..., L9. This intersection has a traffic light that informs drivers in vehicles in the various lanes when they are permitted to proceed through the intersection. To be sure, there are pairs of lanes containing vehicles that should not enter the intersection at the same time, such as L1 and L7. However, there would be no difficulty for vehicles in L1 and L5, for example, to drive through this intersection at the same time. Represent this situation by a graph.

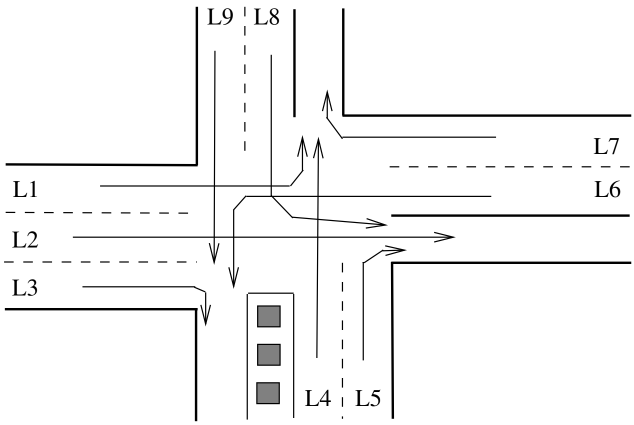


Figure 5. Traffic lanes at street intersections in Exercise 19.

(20) At a university dance class, the instructor wants to pair off the six boys in class (Paul, Quin, Ron, Sam, Tim, Walt) with the six girls in class (Alice, Betty, Carla, Donna, Edith, Fran). The instructor believes each boy dances best with only certain girls, namely

Paul:	Betty, Donna, Edith;
Quin:	Alice, Carla, Fran;
Ron:	Betty, Donna, Edith;
Sam:	Alice, Betty, Carla, Donna, Edith, Fran;
Tim:	Betty, Donna, Edith, Fran;
Walt:	Betty, Donna, Edith.

(a) Show that this situation can be represented by a graph.
(b) Show that the instructor can divide the 12 students into six compatible pairs of dancers.
(c) How many different dancing partners are there for Fran among all possible six pairs of compatible dancers?

EXERCISES FOR CHAPTER 2

(1) For every integer $n \geq 2$, there is exactly one graph of order n containing a vertex of degree $n-1$ and containing exactly one pair of vertices having the same degree. What are these equal degrees?

(2) Ten college students are attending a party and each student has at least one friend present. When Alphonso asks each of the other nine how many friends of theirs are at the party, each gives a different answer. How many of Alphonso's friends are at the party?

(3) Figure 2.1 shows the two graphs of orders 2, 3 and 4 containing exactly one pair of vertices with the same degree. Determine the graphs of orders 5 and 6 with this property.

(4) There is no graph of order 4 containing exactly one vertex of the degrees 0, 1, 2 and 3. What is the smallest order of a graph containing exactly one vertex of each of these degrees?

(5) Suppose that the graph G of size $m \geq 2$ in Figure 6 is converted into an irregular weighted graph by assigning a weight to each edge. What is the smallest possible degree of the vertex v in this weighted graph?

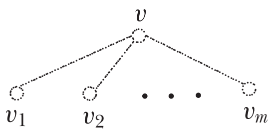


Figure 6. The graph G in Exercise 5.

(6) Determine the irregularity strength of the three graphs in Figure 7.

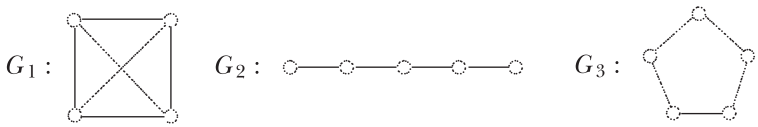


Figure 7. Three graphs in Exercise 6.

(7) Determine the irregularity strength of the two graphs in Figure 8.

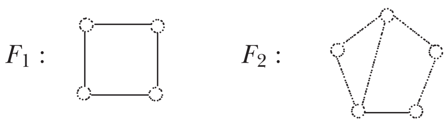


Figure 8. Two graphs in Exercise 7.

(8) Show that each edge in each graph in Figure 9 can be assigned one of the weights 1, 2, 3 so that every two adjacent vertices in the resulting weighted graph have different degrees.

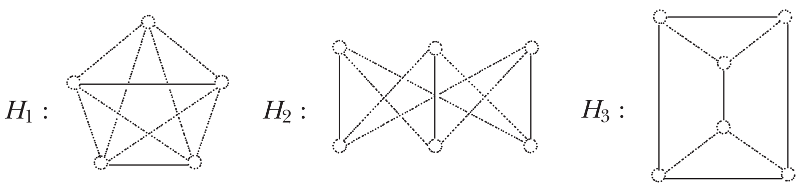


Figure 9. The graphs H_1, H_2 and H_3 in Exercise 8.

(9) Draw an r-regular graph of order 9 for all possible values of r.

(10) Draw the 4-regular graphs $K_{4,4}$ and Q_4.

(11) Suppose that G is a graph of order 8 where $V(G) = \{(a, b, c) : a, b, c \in \{0, 1\}\}$. Two vertices $u = (r, s, t)$ and $v = (x, y, z)$ are adjacent if $|r - x| + |s - y| + |t - z| = 1$, that is, if u and v differ in exactly one coordinate. Draw the graph. What is this graph?

(12) Determine all of the induced subgraphs of the graph G in Figure 10.

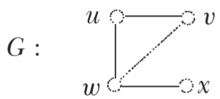


Figure 10. The graph G in Exercise 12.

(13) Find a 3-regular graph H of minimum order containing the graph G in Figure 2.21 as an induced subgraph.

(14) In a committee of four men, two are not friends and refuse to shake hands with each other. All other pairs of men agree to shake hands. There is a group S of men who are happy to shake hands with anyone. What is the minimum number of members of the group S that can be added to the committee so that in the revised committee everyone can shake hands with exactly three others?

(15) What is the minimum order of a 3-regular graph that contains the graph G of Figure 11 as an induced subgraph?

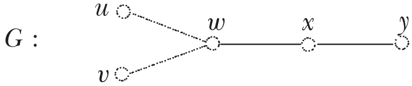


Figure 11. The graph G in Exercise 15.

(16) Give an example of two nonisomorphic 2-regular graphs of the same order and show that they are not isomorphic.

(17) Determine whether the graphs G and H of Figure 12 are isomorphic.

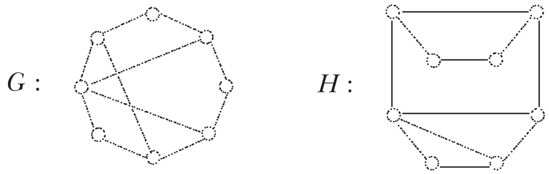


Figure 12. Graphs G and H in Exercise 17.

(18) Prove that every regular graph is reconstructible.

(19) At a gathering of seven people, some pairs of people know each other and some pairs do not. Each pair of people (whether they know each other or not) asks the other five people how many pairs among them know each other. All of these numbers are recorded. Is it possible to determine how many pairs of people among the seven know each other?

EXERCISES FOR CHAPTER 3

(1) For the graph F of Figure 3.6, is there a $u - v$ path of length greater than 7?

(2) Give an example of a graph of order 10 with three components.

(3) Give an example of a connected graph of order 8 containing two cut-vertices and three bridges.

(4) Give an example of a connected graph G of order 10 containing a vertex v such that $G - v$ has four components.

(5) Prove Theorem 3.2: *Let G be a connected graph. An edge e is a bridge of G if and only if there are vertices u and v in G such that e is on every $u - v$ path in G.*

(6) Prove Theorem 3.3: *Let G be a connected graph. A vertex w is a cut-vertex of G if and only if there are vertices u and v in G, both different from w, such that w is on every $u - v$ path in G.*

(7) Find the central vertices in the graph G of Figure 13.

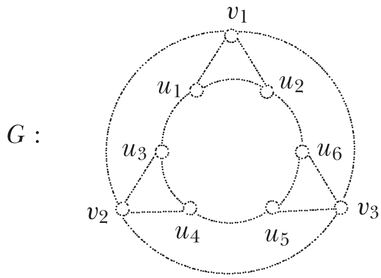


Figure 13. The graph G in Exercise 7.

(8) The length of a longest path in a certain connected graph G is 6. Show that if G contains two paths P and P' of length 6, then P and P' must have at least one vertex in common.

(9) Show that for every positive integer k, there exists a connected graph G containing a central vertex u and a vertex v having minimum average distance such that the distance between u and v is k.

(10) Give an example of a connected graph G and three vertices u, v and w of G such that

(a) v and w are eccentric vertices of u,
(b) w is an eccentric vertex of v and
(c) v is not an eccentric vertex of w.

(11) A pirate encounters a diagram in the desert in the shape of a graph G (see Figure 14). The pirate is at vertex p and locates a message in a bottle that says, "Go to the vertex farthest from here and read the message". At that vertex, there is a message in a bottle that says the same thing. This is repeated at the next vertex. After following these directions, the pirate finds a treasure map. At which vertex was the map located?

(12) For the bipartite graph G of Figure 15, determine

(a) the distance from z to all vertices of G;
(b) all numbers that are the lengths of some cycles in G.

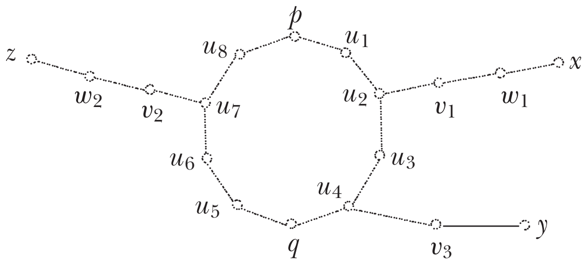


Figure 14. The graph G in Exercise 11.

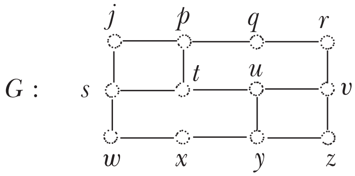


Figure 15. The graph G in Exercise 12.

(13) Let G be an r-regular bipartite graph where $r \geq 1$. Then the vertex set of G can be divided into disjoint subsets U and W such that every edge of G joins a vertex of U and a vertex of W. Show in this case that the sets U and W contain the same number of vertices.

(14) A connected graph G of order $n \geq 3$ has the property that for every two vertices u and v of G, if P is a $u - v$ path of length k and P' is a $u - v$ path of length k', then k and k' are either both even or both odd. Show that G is a bipartite graph.

(15) A certain facility consists of nine rooms $R_1, R_2, \ldots, R_9$ (shown in Figure 16). A sensor placed in one of these rooms has the capability of detecting the distance between this room and the room where a fire has occurred.

(a) What is the minimum number of sensors needed to detect the exact room in which a fire has occurred?
(b) Draw a graph that models this facility. For a set S consisting of a minimum number of rooms where sensors can be placed to detect the exact location of a fire, assign distance vectors (that is, an ordered pair, ordered triple, etc.) to the vertices indicating

the distance from each vertex (room) to the vertices (rooms) in the set S.

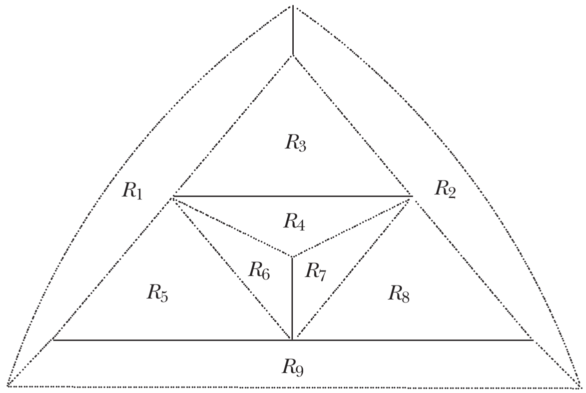


Figure 16. A graph representing the facility consisting of nine rooms in Exercise 15.

(16) For each integer $n \geq 3$, is there a graph of order n that has location number 1?

(17) Determine the domination number of the graph G shown in Figure 17.

G :

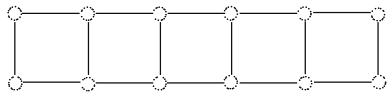


Figure 17. The graph in Exercise 17.

(18) A portion of a city consisting of seven city blocks is shown in Figure 18.

(a) What is the minimum number of security guards needed to guard all intersections, assuming that each guard has a straight-line view of all intersections up to one block away?
(b) What is the minimum number of security guards needed to view all intersections if each guard is within one block of some other guard?

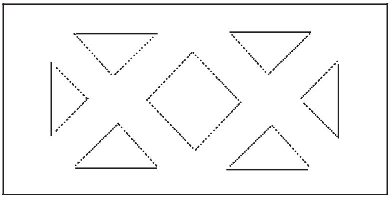


Figure 18. The city map in Exercise 18.

(19) Use Theorem 3.6 to show that every graph G contains a dominating set S so that every vertex of G is dominated by an odd number of vertices in S.

(20) Find the minimum number of light switches which when pressed will change the lights in the graph G in Figure 19 from all on to all off.

G :

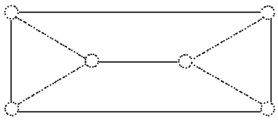


Figure 19. The graph G in Exercise 20.

(21) Find the minimum number of light switches which when pressed will change the lights in the graph G in Figure 20 from all on to all off.

G :

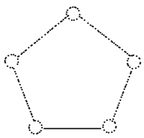


Figure 20. The graph G in Exercise 21.

(22) For the graph G in Figure 21, determine the minimum number of light switches which when pressed will change the lights from all on to all off.

G :

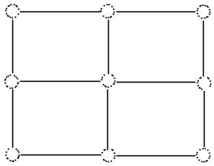


Figure 21. The graph G in Exercise 22.

(23) For which connected graphs G of order $n \geq 2$ having all lights on is a graph with all lights off produced if *every* light switch is pressed?

(24) Give an example of a graph G where some lights are on and some lights are off but for which it is impossible to turn all the lights off.

(25) Show that for every graph G where some lights are on and some are off, it is always possible to turn at least half the lights off.

(26) Suppose that we define the book collaboration graph as the graph G whose vertices are the mathematicians who have authored or coauthored a mathematics book. Two vertices A and B are adjacent if mathematicians A and B have coauthored a mathematics book. For mathematicians A and B, the mathematician B has A-number k if and only if mathematician A has B-number k if and only if the distance between A and B in the book collaboration graph is k.

(a) What does an isolated vertex C in G refer to?
(b) What does it mean for a vertex D in G to have degree d?
(c) Find an example of two mathematicians A and B such that B has A-number 2.

EXERCISES FOR CHAPTER 4

(1) Give an example of a graph G such that every two distinct vertices of G are connected by exactly two paths.

(2) Prove that the size of every connected graph of order n is at least $n-1$.

(3) Prove Corollary 4.4: *If T is a tree of order n and size m whose vertices are $v_1, v_2, \ldots, v_n$, then* $\deg v_1 + \deg v_2 + \cdots + \deg v_n = 2m = 2(n-1) = 2n-2$.

(4) If a tree T has exactly one vertex of degree i for $i = 2, 3, \dots, \Delta$, then how many leaves does T have?

(5) Draw the different (nonisomorphic) trees of order 7.

(6) According to Theorem 4.7, does there exist a tree whose vertices have degrees

$$5, 5, 4, 4, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1?$$

If so, draw such a tree.

(7) We have seen that every carbon atom has valency 4 and each hydrogen atom in a saturated hydrocarbon has valency 1. Show that the chemical formula for every saturated hydrocarbon is C_nH_{2n+2} for some positive integer n. That is, show that if T is a tree with only vertices of degree 1 and degree 4 such that T has n vertices of degree 4, then T has $2n+2$ leaves.

(8) Draw the trees corresponding to the saturated hydrocarbons with n carbon atoms for $n = 5, 6$.

(9) Show that there are 1296 different ways to label the trees of order 6 with the labels $1, 2, \dots, 6$.

(10) Find the Prüfer code of the tree in Figure 22.

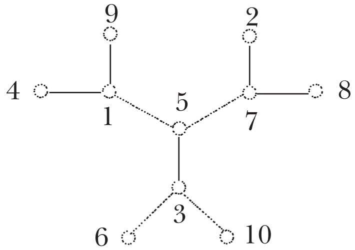


Figure 22. The labeled tree of order 10 in Exercise 10.

(11) What labeled trees have the following Prüfer codes?

(a) (1, 2, 3, 4, 5, 6, 7) (b) (3, 3, 3, 3, 3, 3, 3)
(c) (1, 2, 3, 4, 3, 2, 1) (d) (2, 3, 2, 3, 2, 3, 2, 3)

(12) Solve the problem in Example 4.10 in two weighings by first placing coin 1 on pan A and coin 2 on pan B. Draw an accompanying decision tree.

(13) Suppose that there are three coins: two authentic coins and a fake coin. The fake coin weighs slightly less than an authentic coin. What is the minimum number of weighings needed to determine which coin is the fake? Draw an accompanying decision tree.

(14) There are four coins, three of which are authentic and one is fake. The fake coin does not weigh the same as an authentic coin; it may weigh slightly less or slightly more than an authentic coin. What is the minimum number of weighings needed to determine which coin is the fake coin and how can the fake coin be found? Draw an accompanying decision tree.

(15) Two of four coins numbered 1, 2, 3, 4 are authentic and weigh the same as each other. The other two coins are fake, one of which weighs slightly less and the other weighs slightly more than an authentic coin but the sum of the weights of the two fake coins equals the sum of the weights of the two authentic coins. Show, in three weighings, that the fake coins can be discovered as well as which is the lighter coin.

(16) (a) Solve the problem described in Examples 4.10 and 4.11 where this time there are eight coins, seven of which are authentic and one is fake, and the fake coin weighs slightly less than an authentic coin. Draw an accompanying decision tree.
(b) What happens if there are nine coins rather than eight, where one of the nine coins is a fake coin weighing slightly less than each of the eight authentic coins?

(17) Suppose that there is a region having the property that the cost of constructing railroad tracks between any two cities in the region is

proportional to the distance between the two cities. Figure 23 shows four cities A, B, C and D in this region, together with the distance between certain pairs of these cities. (Thus A, B, C and D form the vertices of a rectangle R.)

(a) What is the cost of a minimum spanning tree in the graph G with vertices A, B, C and D?
(b) Suppose that E is the location situated at the intersection of the diagonals AC and BD in the rectangle R. What is the cost of a minimum spanning tree in the graph with vertices A, B, C, D and E?
(c) What observations can you make from the results in (a) and (b)?
(d) Answer the questions in (a) and (b) if the distances between B and C and between A and D had been 40 miles instead of 30 miles.

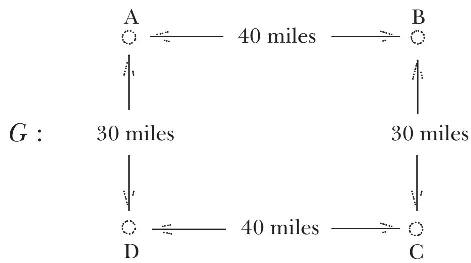


Figure 23. A graph G modeling the region with cities A, B, C and D in Exercise 17.

(18) In a region where the cost of building railroad tracks between any two cities in the region is proportional to the distance between the two cities, there are four cities A, B, C and D that are the vertices of a rectangle R. Suppose that the distance between A and B and between C and D is a, while the distance between A and D and between B and C is b, where $a \leq b$. Let E be the intersection of the two diagonals of the rectangle R. Under what conditions is the weight of a minimum spanning tree with vertices A, B, C, D and E less than the weight of a minimum spanning tree with vertices A, B, C and D?

(19) In a region where the cost of building railroad tracks between any two cities in the region is proportional to the distance between the

two cities, there are four cities A, B, C and D that are the vertices of a rectangle R, where the distance between A and B and between C and D is 96 miles and the distance between A and D and between B and C is 40 miles. Let E be the intersection of the two diagonals of R.

(a) Determine the weight of a minimum spanning tree with vertices A, B, C and D.
(b) Determine the weight of a minimum spanning tree with vertices A, B, C, D and E.
(c) Determine the weight of a minimum spanning tree with vertices A, B, C, D, X and Y (see Figure 24), where the cost of building railroad tracks between any two of these six locations is proportional to the distance between them.

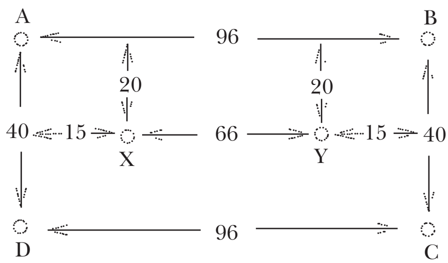


Figure 24. The region with cities A, B, C, D, X and Y in Exercise 19.

(20) Find all trees whose complement is also a tree.

(21) After a long perilous journey, a professor of archaeology and his team have reached the tomb of a king of an ancient civilization. According to the map he has, the tomb consists of several interior compartments (see Figure 25). The professor wants to inspect each compartment by creating as few holes as possible in the walls.

(a) What is the minimum number of holes that must be inserted in the walls of the tomb so that each compartment can be entered?
(b) Give an example of a minimum number of locations where holes can be made so that each compartment can be visited.
(c) Construct a graph H whose vertices are the compartments of the tomb together with the exterior of the tomb such that two

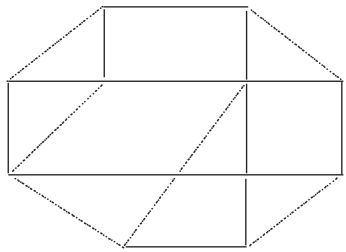


Figure 25. A diagram of the tomb in Exercise 21.

vertices are joined by an edge if the vertices correspond to either (i) two compartments sharing a common wall in which a hole has been created or (ii) a compartment and the exterior of the tomb where a hole has been placed in the wall leading to the compartment from outside the tomb. What is the graph H?

EXERCISES FOR CHAPTER 5

(1) Suppose that it was possible to go for a walk about Königsberg and traverse each bridge exactly once. Then the walk must start at one of the land regions A, B, C or D and end at one of these land regions (possibly the same one where the walk started). Let S be the land region where this walk started and T the land region where the walk terminated. Let R be a land region that is neither S nor T. Thus R is encountered only during the interior of the walk. Show that this is impossible.

(2) Prove Corollary 5.2: *A connected graph (or multigraph) G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at the other.*

(3) Figure 26 shows the downtown district of a town with nine street intersections, denoted by $I_1, I_2, \ldots, I_9$. Is it possible for a mail carrier to deliver mail along the streets of this district and drive along each street exactly once?

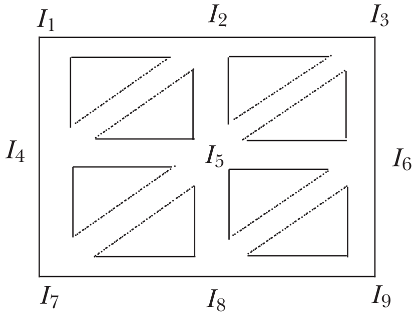
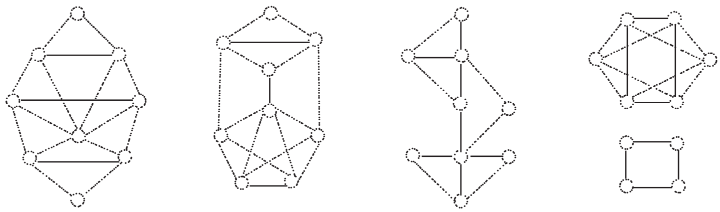


Figure 26. The downtown district of the town in Exercise 3.

(4) Determine which of the graphs in Figure 27 contain either an Eulerian circuit or an Eulerian trail.



(a) (b) (c) (d)

Figure 27. Four graphs in Exercise 4.

(5) Show how Theorem 5.1 and Corollary 5.2 can be used to help answer the question on the first page of Chapter 5 concerning the drawings in Figure 5.2.

(6) The diagram of Figure 28 shows the nine rooms on the second floor of a large house with doorways between various rooms. Is it possible to start in some room and go for a walk so that each doorway is passed through exactly once? How is this question related to graph theory? Explain.

(7) Inspector House is investigating the mysterious death of the evil Count Gilbertson III. This apparent crime occurred at the estate of the count (see Figure 29). The butler states that he saw a suspicious

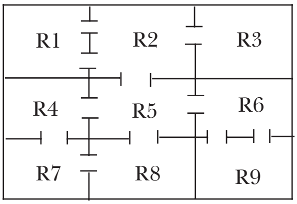


Figure 28. A diagram of the second floor of the large house in Exercise 6.

person enter the Computer Room (where the body was found) and then leave the room by the very same door. When the inspector questions the count's business partner Mr. Garfield Floyd, he admits entering the estate by the front door and exiting by the rear door. However, Floyd says that he went through each doorway exactly once and so he could not have been the person the butler saw. Is someone lying?

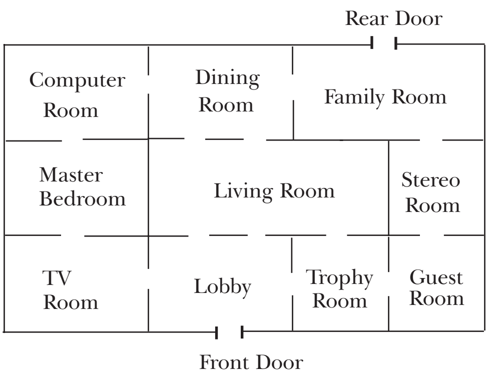


Figure 29. The estate of Count Gilbertson III in Exercise 7.

(8) Figure 30 is a diagram of the Hall of Mirrors at an amusement park. After each visitor passes through the entrance door and through each door thereafter, the door automatically shuts and locks behind the visitor. Assuming that you can eventually find your way out of any room if not all the doors in the room are locked, determine whether it is always possible to escape from the Hall of Mirrors, or whether you might become trapped in one of its rooms. . . forever.

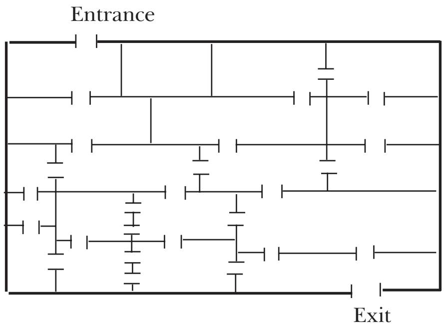


Figure 30. The Hall of Mirrors in Exercise 8.

(9) A connected graph G of even size contains exactly four odd vertices. Therefore, G contains neither an Eulerian circuit nor an Eulerian trail.

(a) Show that G contains two trails T_1 and T_2 such that every edge of G belongs to exactly one of these trails.
(b) Show that G contains two trails T_1' and T_2', each of even size, such that every edge of G belongs to exactly one of these trails.

(10) In the lobby of a large hotel, a waterway has been constructed that surrounds eight land areas A, B, C, D, W, X, Y, Z. At certain locations ten bridges have been built over the water, denoted by a, b, c, d, e, f, g, h, j, k. (See Figure 31.)

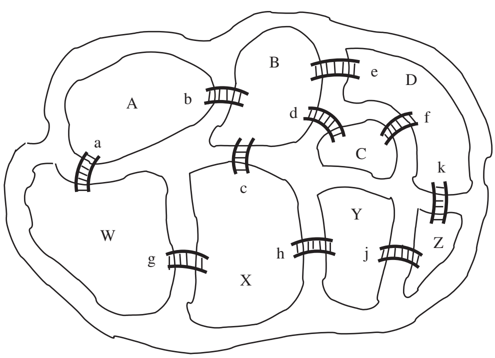


Figure 31. The lobby of the large hotel in Exercise 10.

(a) Is it possible to go for a walk over the land regions in the lobby such that each bridge is crossed exactly once?
(b) Is it possible to take a boat ride through a portion of the waterway so that the boat goes under each bridge exactly once?

(11) What is the minimum number of bridges in Königsberg that must be traversed (counting multiplicities) to conduct a round-trip in Königsberg that crosses each of the seven bridges at least once?

(12) Determine the length of an Eulerian walk in the graph of Figure 32 and describe an Eulerian walk for this graph.

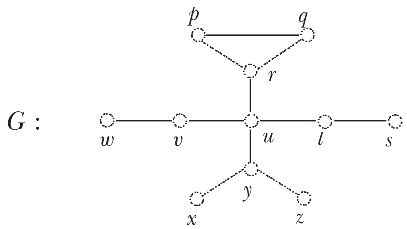


Figure 32. The graph in Exercise 12.

(13) By Theorem 5.3, every Eulerian walk in a connected graph G must traverse each bridge of G twice. Suppose that G is a connected graph containing exactly one bridge e. Is it possible that G has an Eulerian walk that traverses e twice and all other edges of G once?

(14) What is the length of an Eulerian walk in a tree of order $n \geq 2$?

(15) Determine the length of an Eulerian walk in the weighted graph of Figure 33 and describe an Eulerian walk for this weighted graph.

(16) In Figure 34 a weighted graph G is shown that represents the streets in a gated community. The vertices of G represent street intersections, the edges represent the streets and the weight of an edge represents the typical time (in minutes) it takes to drive along the street, inspecting the houses and grounds on both sides.

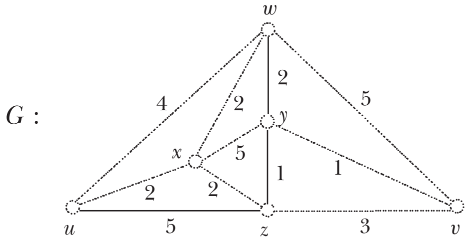


Figure 33. The graph G with two odd vertices in Exercise 15.

A security guard has been hired to watch over the community. Each night, he starts at intersection a and drives along each street at least once and returns to a. What is the minimum amount of time required to do this?

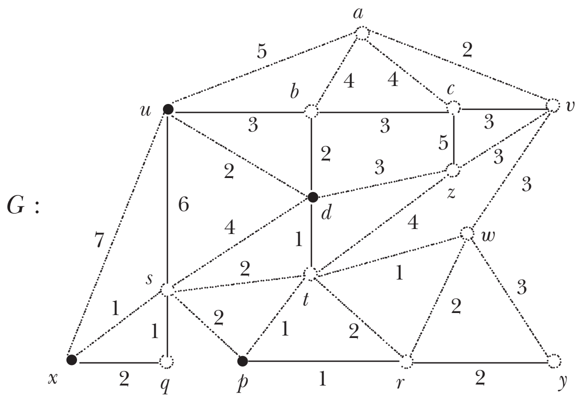


Figure 34. A graph representing the streets in the gated community in Exercise 16.

(17) A college student likes to begin each morning by running around the neighborhood where he lives. He likes to run along each road at least once. This area can be represented by the weighted graph in Figure 35 where the weight of each edge represents the time (in seconds) it takes him to run along that road. What is the least amount of time it would take him to run a round-trip that traverses each road at least once?

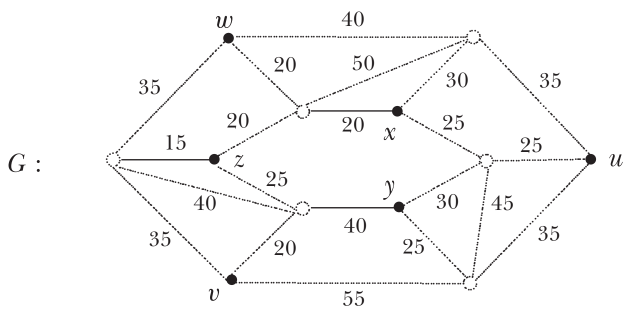


Figure 35. A graph representing the neighborhood in Exercise 17.

EXERCISES FOR CHAPTER 6

(1) Suppose that C is a cycle that visits each vertex on a dodecahedron. How many edges of the dodecahedron do not lie on C?

(2) There is a Hamiltonian cycle C on an icosahedron. How many edges of the icosahedron do not lie on C?

(3) Figure 36 shows a diagram of an art exhibit consisting of seven rooms labeled A, B, C, D, R, S, T. A visitor enters the art exhibit by going from the outside into room A.

(a) Is it possible to walk through the exhibit, entering each room exactly once, returning to room A, and then exiting?

(b) Once a visitor has entered room A, is it possible to stroll through the art exhibit, passing through each doorway exactly once, returning to room A, and then exiting?

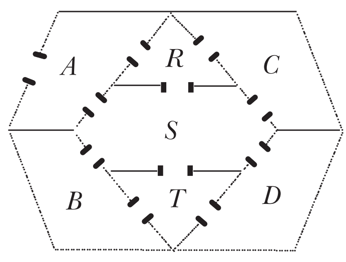


Figure 36. Strolling through the art exhibit in Exercise 3.

(4) Figure 37 shows a 6×6 maze (consisting of 36 squares). Is it possible to start at one of the squares, say the one in the upper-left corner, proceed to each square exactly once, and return to the starting square?

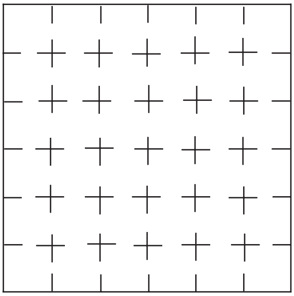


Figure 37. The 6×6 maze in Exercise 4.

(5) Figure 38 shows a 5×5 chessboard. Suppose that a certain chess piece on one of the squares can move either horizontally or vertically to an adjacent square. If the chess piece is placed on the middle square, is it possible to move this chess piece about all the squares so that each square is visited exactly once and the chess piece returns to the middle square?

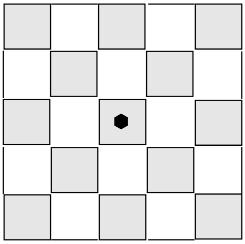


Figure 38. The 5×5 chessboard in Exercise 5.

(6) (a) Show that there is no knight's tour on the 5×5 chessboard.
(b) Is there a modified knight's tour that visits every square of the 5×5 chessboard except the middle square?

(7) Figure 39 shows a map of an area of an amusement park, referred to as Motor World. Each portion of Motor World is named for a major American city. When a youngster enters Motor World, he or she is given a small car to drive around this area. Is it possible for a youngster to begin his or her trip in some "city" and visit every city exactly once and return to the starting city in such a way that every time the youngster drives out of a city containing an even number of letters, he or she must go to a neighboring city with an odd number of letters (and vice versa)?

Figure 39. Motor World in Exercise 7.

(8) Figure 40 shows a graph G of order 18. Is G Hamiltonian?

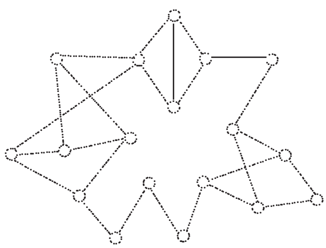


Figure 40. The graph G of order 18 in Exercise 8.

(9) A graph G of order 20 has the integers $1, 2, \ldots, 20$ as its vertices. Two vertices i and j are adjacent if $i + j$ is odd. Is G Hamiltonian?

(10) A graph G of order $n \geq 4$ has $\deg v \geq (n+1)/2$ for each vertex v of G.

(a) Show that G is Hamiltonian.
(b) If v is a vertex of G, is $G - v$ Hamiltonian?

(11) Determine whether the graph G of Figure 41 is Hamiltonian.

G :

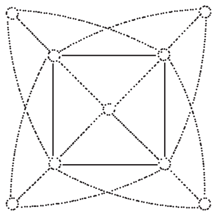


Figure 41. The graph G in Exercise 11.

(12) Figure 42 shows a view of a polyhedron. A vertex is placed in every face (except the hidden face at the bottom) and each added vertex is joined to the three vertices on the triangle surrounding the face, resulting in a graph G. Is G Hamiltonian?

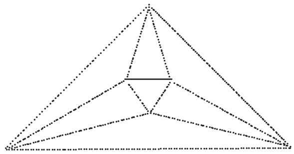


Figure 42. The polyhedron in Exercise 12.

(13) A businessman in New York City learns that he will need to visit four branch offices (in Miami, Houston, Minneapolis and Los Angeles) of his company for one week each next year. When he checks into the cost of constructing trips between these cities, he learns that it is less expensive to purchase a round-trip ticket between any two cities than to purchase a one-way ticket in either direction. The costs of

these trips are

NYC–MIA: 232; MIA–MIN: 279; NYC–HOU: 333;
MIA–LOS: 322; NYC–MIN: 325; HOU–MIN: 292;
NYC–LOS: 315; HOU–LOS: 552; MIA–HOU: 285;
MIN–LOS: 260.

(These were the actual prices for flying on a certain airline several years ago.)

(a) In what order should he visit the cities and then return to New York City to minimize the cost?
(b) Later, it's determined that it will not be necessary for him to visit Minneapolis after all. How will this affect this trip and the cost?
(c) What interesting observation is there after comparing the 12 possible round-trips in (a) with the 3 possible round-trips in (b)?

EXERCISES FOR CHAPTER 7

(1) Determine whether the following sets have a system of distinct representatives:

$$S_1 = \{c, f\}, \quad S_2 = \{a, b, f\}, \quad S_3 = \{b, e, g\}, \quad S_4 = \{d, g\},$$
$$S_5 = \{a, b, f\}, \quad S_6 = \{c, d, e\}, \quad S_7 = \{d, f\}.$$

(2) Determine whether the following sets have a system of distinct representatives:

$$S_1 = \{3, 5\}, \quad S_2 = \{1, 2, 4, 6\}, \quad S_3 = \{3, 4, 5\},$$
$$S_4 = \{4, 5\}, \quad S_5 = \{3, 4\}, \quad S_6 = \{1, 3, 5, 6\}.$$

(3) In a collection $\{S_1, S_2, \ldots, S_n\}$ of $n \geq 2$ nonempty sets, no two sets have the same number of elements. Show that this collection has a system of distinct representatives

(a) by using Hall's theorem;

(b) without using Hall's theorem.

(4) Let $\{S_1, S_2, S_3, S_4, S_5\}$ be a collection of five nonempty finite sets. For each integer k $(1 \leq k \leq 5)$, there exist k of these subsets whose union contains at least k elements. Does this collection of sets have a system of distinct representatives?

(5) A high school has openings for six teachers, with one teacher needed for each of these areas: mathematics, chemistry, physics, biology, psychology and ecology. In order for a teacher to be hired in any particular area, he or she must have either majored or minored in that subject. There are six applicants for these positions, namely Mr. Arrowsmith (major: physics; minor: chemistry), Mr. Beckman (major: biology; minors: physics, psychology, ecology), Miss Chase (major: chemistry; minors: mathematics, physics), Mrs. Deerfield (majors: chemistry, biology; minors: psychology, ecology), Mr. Evans (major: chemistry; minor: mathematics), Ms. Form (major: mathematics; minor: physics). What is the largest number of applicants the school can hire?

(6) Two young children have been given 100 cards. On the top half of each card is a circle and on the bottom half is a square. Each child has a box of ten crayons, each crayon a different color. One child colors the inside of all 100 circles, ten circles with each of the ten colors. All 100 cards are mixed up and given to the other child, who then colors the inside of all 100 squares, ten squares with each color. Show that no matter how this is done, the 100 cards can be divided into ten groups of ten cards each, where in each group the circles are colored differently and the squares are colored differently.

(7) Determine whether the graphs G_1 and G_2 of Figure 43 contain a 1-factor.

(8) Does there exist a cubic graph with three bridges that contains a 1-factor?

(9) Determine all 3-cages and 4-cages.

(10) Show that the Petersen graph is the unique 5-cage.

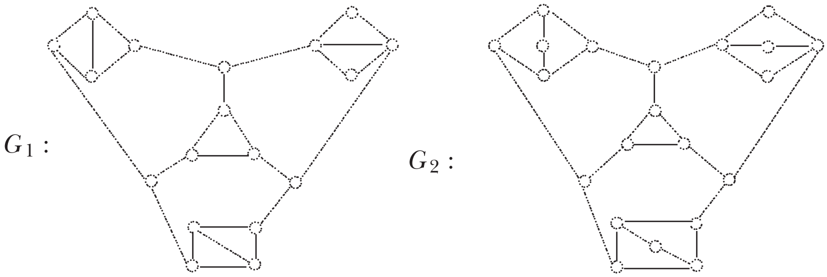


Figure 43. The graphs G_1 and G_2 in Exercise 7.

(11) From a group of six tennis players $t_1, t_2, \ldots, t_6$, construct a five-day schedule of matches in which no one has two tennis matches on the same day and everyone plays a match against each of the other five.

(12) Suppose that n teams $1, 2, \ldots, n$ are involved in a softball tournament in which every two teams play each other exactly once. For $n = 10$ and $n = 9$, set up a schedule of games that takes place during the smallest number of days so that no team plays more than one game per day.

(13) Show that the graph G of Figure 7.13 is not 1-factorable.

(14) Show that if a cubic bridgeless graph is Hamiltonian, then it is also 1-factorable.

(15) Determine whether the 6-regular graph G of Figure 44 is Hamiltonian-factorable.

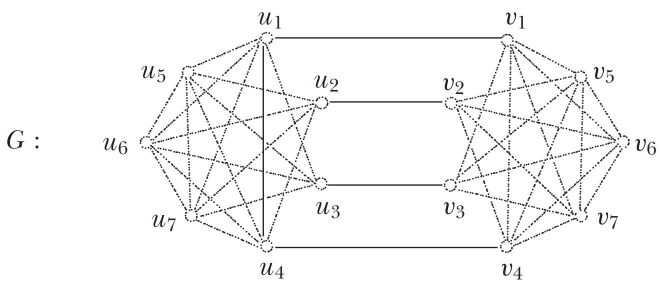


Figure 44. The 6-regular graph in Exercise 15.

(16) A committee of seven students has a luncheon meeting three times during the semester. When they meet, they sit at a round table with seven chairs. Show that it is possible for the committee to meet so that every two students sit next to each other during only one of the three meetings.

(17) Figure 7.14 shows a 2-factorization of a 4-regular graph where one of the 2-factors is not a Hamiltonian cycle. Is this graph Hamiltonian-factorable?

(18) A group of 11 professors have been invited to a five-day research conference (Monday through Friday) to discuss ways that undergraduates can be introduced to research in mathematics. A round table seating 11 people has been set up so that these professors can also meet at lunchtime each of these five days. Show that it's possible to have a seating arrangement for these five lunches so that no professor sits next to anyone twice during the week.

EXERCISES FOR CHAPTER 8

(1) Show that if a graph G of size m contains a subgraph H of size m' where m' divides m, then G need not be H-decomposable.

(2) Show that if S_n is a Steiner triple system, then S_{2n+1} is also a Steiner triple system.

(3) Show that K_7 can be decomposed into cycles of different lengths.

(4) Nine executives of a company are attending a six-day conference. Show that it is possible to set up six round tables, each seating at least three but at most nine people and no two tables seating the same number of people, such that each table is used for a luncheon meeting on only one of the six days and such that each executive attends lunch on four of these six days and never sits next to the same person for two lunches.

(5) Show that the complete graph K_4 is graceful.

(6) Show that neither of the graphs G_1 and G_2 in Figure 45 is graceful.

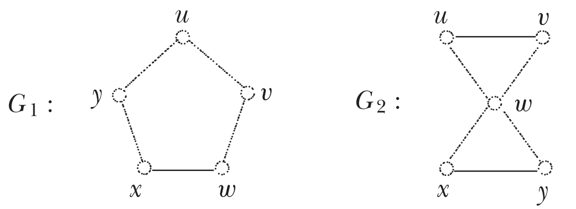


Figure 45. The two graphs in Exercise 6.

(7) (a) Show that the path P_5 of order 5 is graceful.
(b) Find a cyclic P_5-decomposition of K_9.

(8) (a) Show that the graph $K_{1,m}$ is graceful for every positive integer m.
(b) According to Theorem 8.6, the complete graph K_{2m+1} is H-decomposable for $H = K_{1,m}$. Show that K_{2m} is also H-decomposable.

(9) For the graph $H = K_{1,3}$, determine all positive integers $n \leq 7$ such that K_n is H-decomposable.

(10) For the graph $H = K_{1,6}$, determine all positive integers $n \leq 13$ such that K_n is H-decomposable.

(11) Solve the following 27-schoolgirl problem: A school mistress has 27 schoolgirls whom she wishes to take on a daily walk. The girls are to walk in nine rows of three girls each. Show that such walks can be made for 13 days without two girls walking in the same row twice. [Hint: Number the 27 schoolgirls 0, 1, 2, . . . , 26. Place 26 equally spaced vertices on a circle, with the vertices labeled clockwise $1, 2, \ldots, 26$ to produce the cycle $C = (1, 2, \ldots, 26, 1)$. Each vertex of C has distance $1, 2, \ldots, 12$ to two vertices on C and distance 13 to one vertex on C. By joining each pair of vertices on C by a straight-line segment, the complete graph K_{26} is produced. Now place vertex 0 at some convenient location and join it to all vertices of C, producing K_{27}. Now consider the triangle in K_{27} with vertices {1, 0, 14} and the four pairs of triangles with the vertices {2, 26, 18},

{5, 13, 15}; {3, 4, 9}, {16, 17, 22}; {6, 10, 25}, {12, 19, 23}; {7, 21, 24}, {8, 11, 20}. Next, replace {1, 0, 14} by {2, 0, 15} and perform an appropriate clockwise rotation of the four pairs of triangles.]

(12) Figure 8.16 shows the solution of the Instant Insanity puzzle that was given in Example 8.7. Once the four cubes have been stacked as in Figure 8.16, take the top cube off the stack and place it on a table so that the top of this cube is still on the top. Continue this for the next three cubes. Look at the colors of the tops of these four cubes. Now do the same thing for the bottoms of these cubes. What does this tell you? Is this typical?

(13) Solve the Instant Insanity puzzle in Figure 46 by providing

(a) the multigraphs for each cube;
(b) the composite multigraph for these four cubes;
(c) the related submultigraphs (front–back and right–left);
(d) a solution.

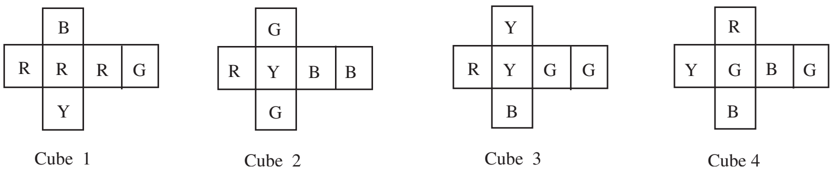


Figure 46. Cubes for Exercise 13.

(14) Figure 47 shows five cubes where each face of each cube is colored with one of the five colors red (R), blue (B), green (G), yellow (Y) and white (W). Is it possible to stack these five cubes on top of one another so that all five colors appear on each side of the stacked cubes?

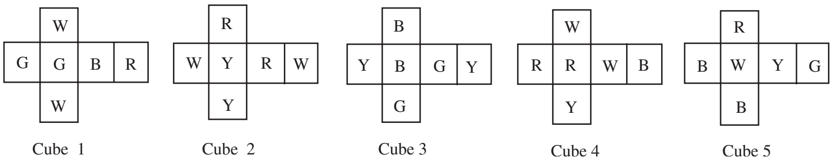


Figure 47. Cubes for Exercise 14.

EXERCISES FOR CHAPTER 9

(1) Model the street system shown in Figure 48 by a graph G, find a strong orientation D of G and use D to find a way to convert the streets into one-way streets so that it is possible to drive (legally) from any location in the town to any other location.

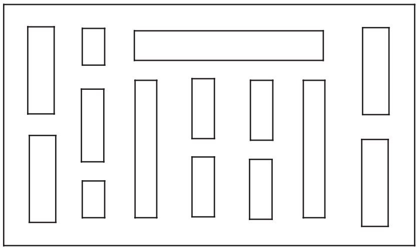


Figure 48. The street system in Exercise 1.

(2) Find a strong orientation of each graph in Figure 49 if such an orientation exists.

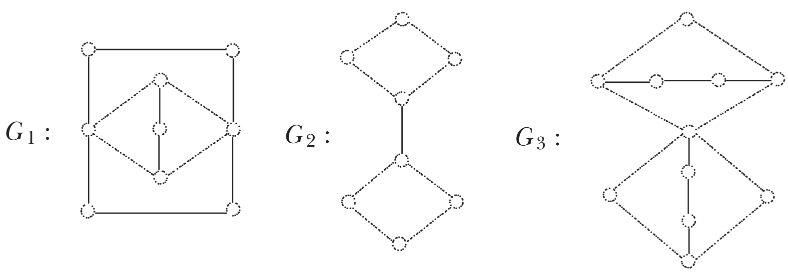


Figure 49. The graphs in Exercise 2.

(3) The graph G in Figure 50 is connected and contains no bridges. Find a strong orientation of G.

(4) Suppose that D is an orientation of a connected graph G such that for each vertex v of G, some edge is directed toward v and some edge is directed away from v. Is D a strong orientation of G?

G :

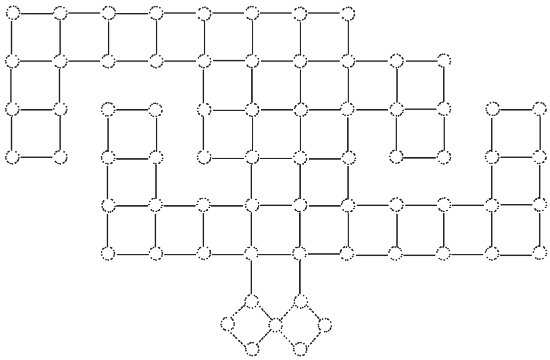
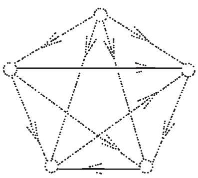


Figure 50. The graph G in Exercise 3.

(5) Find directed Hamiltonian paths in the tournaments in Figure 51.

T_1 :



T_2 :

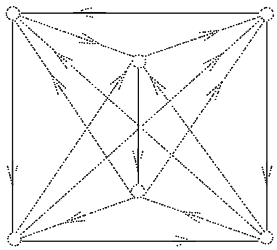
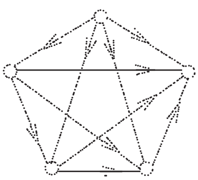


Figure 51. The tournaments in Exercise 5.

(6) Give examples of antidirected paths of orders 5 and 6, respectively, in the tournaments T' and T'' in Figure 52.

T' :



T'' :

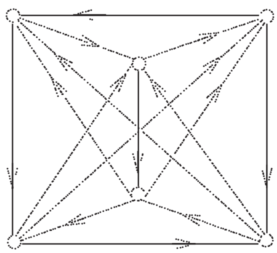


Figure 52. The tournaments T' and T'' in Exercise 6.

(7) First, have a coin available. We now construct a tournament T of order 8 with vertex set $\{v_1, v_2, \ldots, v_8\}$. For the vertex v_1, consider the remaining vertices in the order $v_2, v_3, \ldots, v_8$. Flip the coin. If it comes up heads, direct the edge v_1v_2 from v_1 to v_2; if it comes up tails, direct it from v_2 to v_1. Flip the coin again. If it comes up heads, direct the edge v_1v_3 from v_1 to v_3; if it comes up tails, direct it from v_3 to v_1. Continue this until the edge v_1v_8 is directed. Now consider v_2 along with the vertices $v_3, v_4, \ldots, v_8$. Once the coin has been flipped 28 times, a tournament T of order 8 is obtained. Now consider the list

$$p \,\_\_\_\, q \,\_\_\_\, r \,\_\_\_\, s \,\_\_\_\, t \,\_\_\_\, x \,\_\_\_\, y \,\_\_\_\, z$$

of 8 letters and 7 slots. Now flip the coin 7 times. If the first flip results in heads, we write $p \to q$; otherwise, we write $p \leftarrow q$. This is continued until the list is completed. Show that the tournament T just constructed contains vertices denoted by p, q, r, s, t, x, y and z and the oriented path just described.

(8) Give an example of

(a) two nonisomorphic strong tournaments of order 5;
(b) two nonisomorphic tournaments of order 5 neither of which is transitive or strong.

(9) Is there an orientation of C_4 that contains a king?

(10) After the completion of a round-robin tournament involving $n \geq 5$ teams, the teams are listed in order according to the number of victories they have—the team with the most victories in first place and the team with the least victories in last place. Show that it's possible that the team in last place is a king in the tournament.

(11) Give an example of a tournament of order more than 3 having exactly three kings.

(12) According to Theorem 9.8, every tournament of order n containing no vertex of outdegree $n-1$ has at least three kings. Show that in

such a tournament, every vertex is adjacent from at least one of the kings.

(13) Suppose that a college is having an election for student president and this year there are three candidates: Alice, Bruce and Charles. In order to have the full input of the students, each student is asked to cast his/her vote by making one of the following choices:

□	□	□	□	□	□
A	A	B	B	C	C
B	C	C	A	B	A
C	B	A	C	A	B

For example, checking the box in the third column in the list would mean that the first choice of the person voting is Bruce (B), the second choice is Charles (C) and the third choice is Alice (A). The voting takes place with the following outcome:

100	500	75	425	50	350
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

Suppose that we were to determine the outcome of the election by one of the following:

(i) Count only the first choice of each voter.
(ii) Eliminate the candidate who received the smallest number of votes in (i) and then recount the votes of the two remaining candidates.
(iii) Construct the tournament of paired comparisons of the three candidates.

Determine the winner in each case.

(14) The preferences of 98 voters for three candidates (A, B and C) are shown:

18	17	16	13	18	16
A	A	B	B	C	C
B	C	C	A	B	A
C	B	A	C	A	B

(a) If the candidate who is the first choice of most voters wins, then who would win?
(b) Construct the tournament of paired comparisons of the three candidates. According to this tournament, which candidate should win?

(15) Suppose that A, B and C are three candidates for a certain position and each of 45 voters is required to vote for one of the six ranked lists of the three candidates. Give an example of a possible number of votes for each of these lists that results in (i) A is first, B is second and C is third when only the first choices of the voters are counted, (ii) C is first, B is second and A is third when the tournament of preferences is constructed and (iii) B wins when the candidate who received the smallest number of votes in (i) is eliminated and the votes of the two remaining candidates are recounted.

EXERCISES FOR CHAPTER 10

(1) Solve the Five Houses and Two Utilities Problem.

(2) Show that the Euler Polyhedron Formula holds for the polyhedron (prism) shown in Figure 53.

(3) Show that the graph G of Figure 54 can be embedded in the plane and then show that the Euler Identity holds for this planar embedding.

(4) Show that Corollary 10.3 cannot be improved by giving an example of a planar graph that contains no vertex of degree 4 or less.

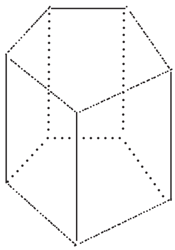


Figure 53. The polyhedron (prism) in Exercise 2.

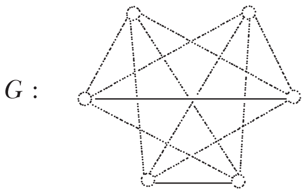


Figure 54. The planar graph G in Exercise 3, not embedded in the plane.

(5) (a) Show that the Petersen graph does not contain a subdivision of K_5.
(b) Show that the Petersen graph is nonplanar.

(6) Does there exist a 4-regular planar graph of order 7?

(7) Find all graphs G of order $n \geq 5$ and size $m = 3n - 5$ such that $G - e$ is planar for every edge e of G.

(8) Find all cycles C_n of order $n \geq 3$ for which $\overline{C_n}$ is a nonplanar graph.

(9) Find a positive integer k and a tree T of order k such that $\overline{T}$ is planar while the complement of every tree whose order is more than k is nonplanar.

(10) (a) In Chapter 1, the following problem was stated.
The Problem of the Five Princes. There was once a king with five sons. In his will he stated that after his death, his kingdom should be divided into five regions for his sons in such a way that each region should have a common boundary with the other four. Can the terms of the king's will be satisfied?
Show that the conditions of the king's will cannot be satisfied.

(b) **The Problem of the Five Palaces**. The king additionally required each of his five sons to build a palace in his region, and the sons should link each pair of palaces by roads so that no two roads cross. Can this be done?

(11) We have seen that $\mathrm{cr}(K_{3,4}) = 2$. Show that $\mathrm{cr}(K_{3,4}) = \overline{\mathrm{cr}}(K_{3,4})$.

(12) Show, for the graph G of Figure 54, that $\overline{\mathrm{cr}}(G) = 0$.

(13) Figure 55 shows a graph G drawn in the plane with two crossings. Is $\mathrm{cr}(G) = 2$?

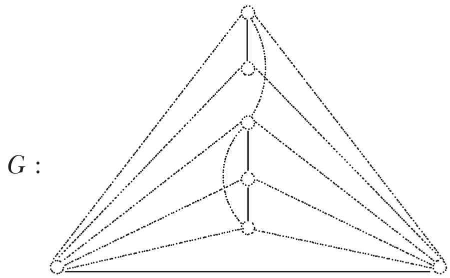


Figure 55. A drawing of the graph with two crossings in Exercise 13.

(14) For the pentagons shown in Figure 56, show that there is a point P in the interior for which the straight-line segment joining P and each vertex of the pentagon lies within the interior of the pentagon.

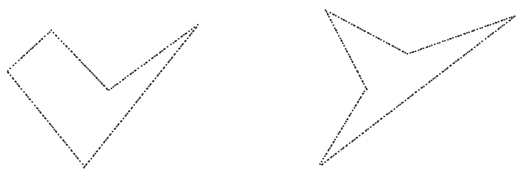


Figure 56. Pentagons in Exercise 14.

(15) Give an example of a hexagon (6-gon) for which no point in the interior can be joined to the vertices of the hexagon by a straight-line segment lying in the interior of the hexagon.

(16) Show that no more than $\lfloor n/3 \rfloor$ security guards are needed to solve the Art Gallery Problem.

(17) Show that the complete graph K_6 can be embedded on the torus.

(18) Show that both K_5 and $K_{3,3}$ are minors of the graph G in Figure 10.13.

(19) Suppose that a connected graph H is a minor of a tree T. Show that H is also a tree.

(20) It has been mentioned that for the set of graphs that can be embedded on the torus, there is a finite set M of forbidden minors. Therefore, there is a theorem that begins as follows: *A graph G can be embedded on the torus if and only if....* How does this theorem end?

EXERCISES FOR CHAPTER 11

(1) We have seen that a Pythagorean triple is a triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$.

- (a) Show that if (a, b, c) is a Pythagorean triple, then (ka, kb, kc) is a Pythagorean triple for every positive integer k.
- (b) Show that there are infinitely many 4-tuples (a, b, c, d) of positive integers such that $a^2 + b^2 + c^2 = d^2$ where the only positive integer that divides all of a, b, c and d is 1.

(2) For the map M in Figure 57, what is the minimum number of colors needed to color the regions of M so that the regions sharing a common boundary are colored differently?

(3) Figure 58 shows a map of some of the states in the western United States. Construct a planar graph G from this map by associating a vertex with each state and joining two vertices by an edge if the corresponding states have a boundary (not just a single point) in common. Show that the vertices of this graph G can be colored with four colors, but not with three colors, in such a way that adjacent vertices are colored differently.

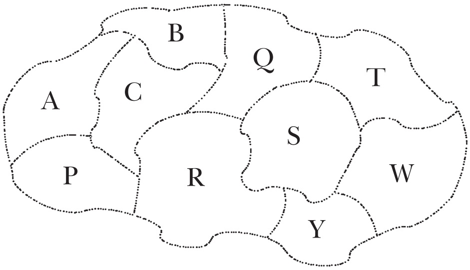


Figure 57. The map M in Exercise 2.

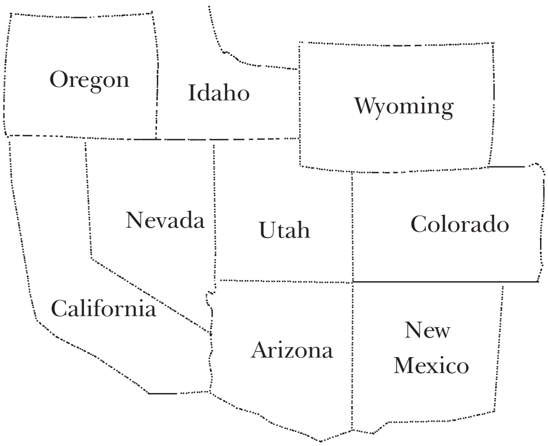


Figure 58. Western United States in Exercise 3.

(4) Show that the countries in a map of South America (see Figure 59) can be colored with four colors in such a way that every two countries sharing a common boundary are colored differently. Show that these countries cannot be colored with three colors, however.

(5) Give an example of a map that does not contain four mutually neighboring countries but yet requires four colors to color it.

(6) Figure 60 shows two maps M_1 and M_2. The map M_1 has 8 regions while M_2 has 10 regions, including the exterior region. By the Four Color Theorem, we know that these maps can be colored with four colors so that regions sharing a common boundary (not just a single point) are colored differently.

Figure 59. Map of South America in Exercise 4

(a) Show that the vertices in the map M_1 can be colored with three colors but the vertices in the map M_2 cannot.
(b) Show that the regions of M_1 and M_2 can be colored with two colors.

(7) Give an example of two Eulerian planar graphs where the minimum numbers of colors needed to color the vertices of the resulting maps are different.

(8) Figure 61 shows a map M containing 16 regions, denoted by $R, R_1, R_2, \ldots, R_{15}$. Each of the 15 regions $R_1, R_2, \ldots, R_{15}$ is colored

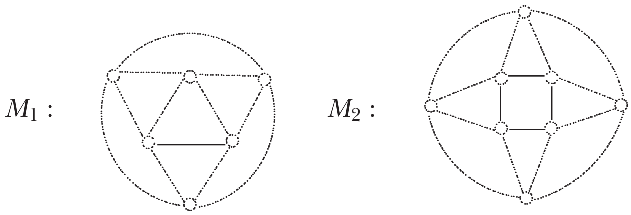


Figure 60. Two maps M_1 and M_2 in Exercise 6.

with one of the four colors red (r), blue (b), green (g) and yellow (y) in such a way that every two regions having a common boundary (not just a common point) are colored differently.

(a) Does M contain a red–green Kempe chain beginning in R_1 and ending in R_3? If so, describe such a Kempe chain.
(b) Does M contain a yellow–blue Kempe chain beginning in R_5 and ending in R_2? If so, describe such a Kempe chain.

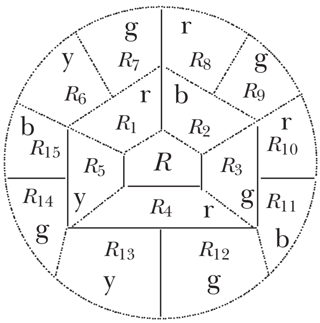


Figure 61. The map M in Exercise 8.

(9) Let a be the smallest chromatic number and b the largest chromatic number of any 5-regular graph. Show that if k is any integer such that $a \le k \le b$, then there is a 5-regular graph G_k such that $\chi(G_k) = k$.

(10) Eight mathematics majors at a small college are permitted to attend a meeting dealing with undergraduate research during final exam week provided they make up all the exams missed on the Monday

after they return. The possible time periods for these exams on Monday are

(i) 8:00–10:00; (ii) 10:15–12:15; (iii) 12:30–2:30;
(iv) 2:45–4:45; (v) 5:00–7:00; (vi) 7:15–9:15.

Use graph theory to determine the earliest time on Monday that all eight students can finish their exams if two exams cannot be given during the same time period if some student must take both exams. The eight students and the courses [Advanced Calculus (AC), Differential Equations (DE), Geometry (G), Graph Theory (GT), Linear Programming (LP), Modern Algebra (MA), Statistics (S), Topology (T)] each student is taking are

Alicia: AC, DE, LP; Brian: AC, G, LP; Carla: G, LP, MA;
Diane: GT, LP, MA; Edward: DE, GT, LP; Faith: DE, GT, T;
Grace: DE, S, T; Henry: AC, DE, S.

(11) Eight chemicals are to be shipped across country by air express. The cost of doing this depends on the number of containers shipped. The cost of shipping one container is \$125. For each additional container the cost increases by \$85. Some chemicals interact with one another and it is too risky to ship them in the same container. The chemicals are labeled $c_1, c_2, \ldots, c_8$ and chemicals that interact with a given chemical are

$c_1 : c_2, c_5, c_6$ $c_2 : c_1, c_3, c_5, c_7$; $c_3 : c_2, c_4, c_7$;
$c_4 : c_3, c_6, c_7, c_8$; $c_5 : c_1, c_2, c_6, c_7, c_8$; $c_6 : c_1, c_4, c_5, c_8$;
$c_7 : c_2, c_3, c_4, c_5, c_8$; $c_8 : c_4, c_5, c_6, c_7$.

What is the minimum cost of shipping the chemicals and how should the chemicals be packed into containers?

(12) The road intersection shown in Figure 62 needs a traffic signal to handle the traffic flow. If cars from two different lanes could collide, then cars from these two lanes will not be permitted to enter the

intersection at the same time. What is the minimum number of signal phases that are needed to ensure safe traffic?

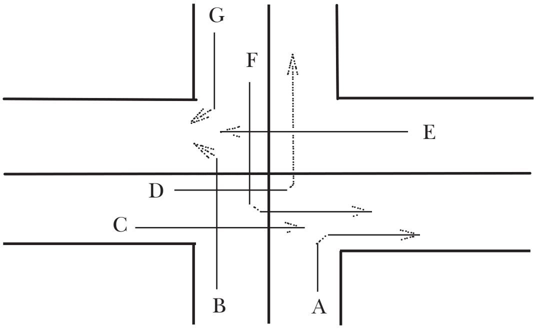


Figure 62. The road intersection in Exercise 12.

EXERCISES FOR CHAPTER 12

(1) Determine the chromatic index of each graph in Figure 63.

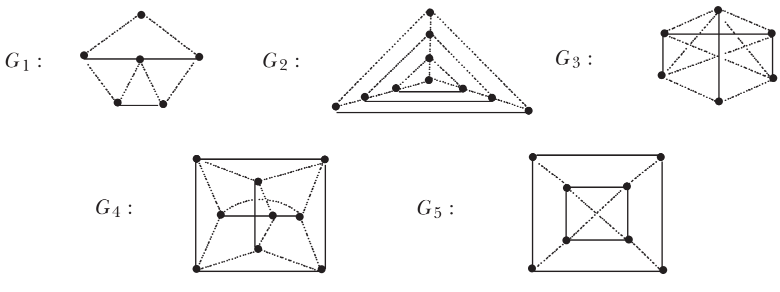


Figure 63. The graphs in Exercise 1.

(2) For a positive integer k, let H be a $2k$-regular graph of order $4k+1$. Let G be obtained from H by removing a matching of size $k-1$ from H. Prove that $\chi'(G) = \Delta(G) + 1$.

(3) There are two nonisomorphic 3-regular graphs of order 6. Are these graphs both class one graphs, both class two graphs, or one of each?

(4) Is the 3-regular graph in Figure 64 a class one graph or a class two graph?

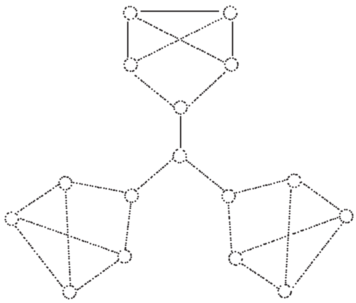


Figure 64. The 3-regular graph in Exercise 4.

(5) Give an example of three graphs F, G and H such that F is a subgraph of G, G is a subgraph of H, F and H are class one graphs and G is a class two graph.

(6) Seven softball teams from Atlanta, Boston, Chicago, Denver, Louisville, Miami and Nashville have been invited to participate in a tournament, where each team is scheduled to play a certain number of the other teams:

Atlanta:	Boston, Chicago, Miami, Nashville;
Boston:	Atlanta, Chicago, Nashville;
Chicago:	Atlanta, Boston, Denver, Louisville;
Denver:	Chicago, Louisville, Miami, Nashville;
Louisville:	Chicago, Denver, Miami;
Miami:	Atlanta, Denver, Louisville, Nashville;
Nashville:	Atlanta, Boston, Denver, Miami.

No team is to play more than one game each day. Set up a schedule of games over the smallest number of days.

(7) Determine the Ramsey number $R(P_3, P_3)$.

(8) Determine the Ramsey number $R(K_{1,3}, P_3)$.

(9) Suppose that in a group of people attending a party, every two people are either acquaintances or strangers. How many people must be present at the party for some person to be an acquaintance of at least three people or for some person to be a stranger to at least three people?

(10) For integers $n \geq 2$ and $m \geq 2$, suppose that $R(K_m, K_n) = p$. Show that if every edge of K_{p-1} is colored red or blue, then there is either a red K_{m-1} or a blue K_{n-1}.

(11) Show that if every edge of the complete graph K_{66} is colored red, blue, green or yellow, then there is a monochromatic triangle.

(12) For the synchronizing 2-arc coloring of D in Figure 12.13, find synchronizing sequences for s_2 and s_4 that are not obtained by appending sequences to known synchronizing sequences.

(13) The strong digraph D of Figure 65 has uniform outdegree 2.

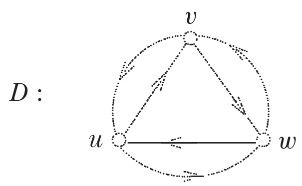


Figure 65. The strong aperiodic digraph in Exercise 13.

(a) Show that D is aperiodic.
(b) Find a synchronizing coloring of D and a synchronizing sequence for each vertex of D.
(c) Show that there exists a proper 2-arc coloring of D that is not synchronizing.

(14) Let D be the strong aperiodic digraph of Figure 66 with uniform outdegree 2.

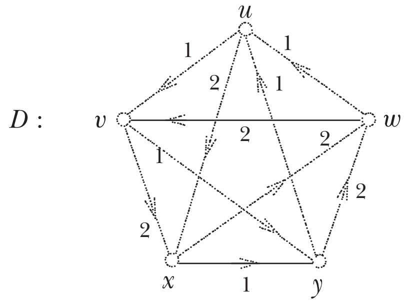


Figure 66. The strong aperiodic digraph in Exercise 14.

(a) For the proper 2-arc coloring c of D shown in Figure 66 (using the colors 1 and 2), show that 11221122 is a synchronizing sequence for the vertex w.
(b) Find a synchronizing sequence for the vertex v.
(c) Is the coloring c synchronizing?

(15) Let D be the strong aperiodic digraph of Figure 67 with uniform outdegree 3. A proper 3-arc coloring c of D is shown in Figure 67.

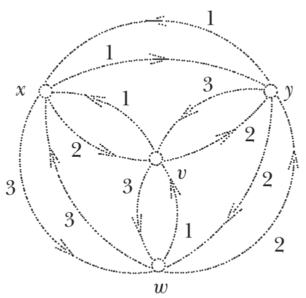


Figure 67. The strong aperiodic digraph D in Exercise 15.

(a) Is 112233 a synchronizing sequence for any vertex of D?
(b) Find a synchronizing sequence for the vertex x.
(c) Is the coloring c synchronizing?

(16) There is a large walking area near a housing development that has four buildings and eight one-way walking paths (shown in Figure 68). Two of these walking paths leave each building, one of which is called Ostrich Lane, clearly marked and indicated by a narrow line in the figure, and the other is called Sparrow Trail, also clearly marked and indicated by a bold line in the figure.

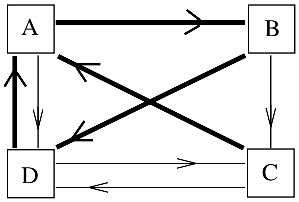


Figure 68. The housing development in Exercise 16.

Past experience has shown that walkers using this area have often lost items such as cell phones, keys and sunglasses. For this reason, a lost and found room has been placed in one of these buildings. The person in charge of building maintenance has comically placed a sign in each building that reads,

If you lose or find anything, just remember: SOS.

In which building is the lost and found room located?

(17) One of the areas at the World of Fun Amusement Park is called NOW EIGHT. The guests at the park can drive to the NOW EIGHT parking area, which is located at one of the buildings N, O, W, E, I, G, H, T (shown in Figure 69).

Once in a building, a person can take one of the fast moving cars: the rocket car, the arrow car or the danger car. Each of these cars takes one to a new building, where a person leaves his or her car and gets in one of three cars having the same three names. The goal is to try to keep track of where you are. It's not all that unusual after a while for a person to forget where his or her car was parked. For that reason, each building has a phone to assist a person to return to the

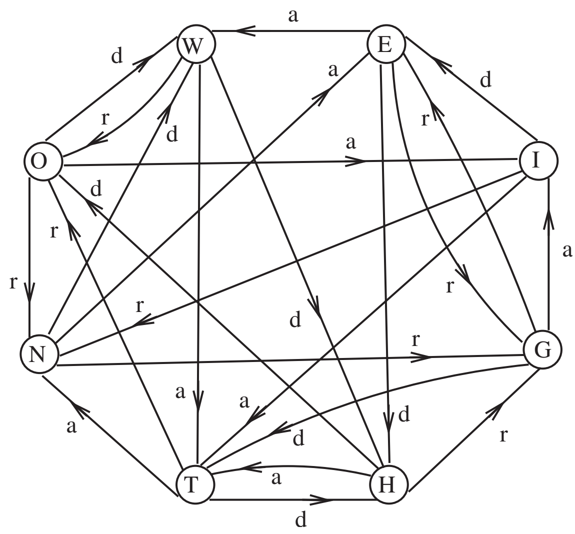


Figure 69. The World of Fun Amusement Park in Exercise 17.

parking lot. The message on each phone says the same thing:

No need to fear
If you've lost your car.
Just start from here
And use radar.

Where is the parking lot located?

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