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Engineering Mathematics-II

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AS PER THE 2010 REVISED SYLLABUS

With 3 Years' Solved Question Papers

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Engineering Mathematics-II

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In memory of

my parents

Smt. Manohari Devi and Sri Makhan Lal

Contents

Preface

All branches of Engineering, Technology and Science require mathematics as a tool for the description of their contents. Therefore, a thorough knowledge of various topics in mathematics is essential to pursue courses in Engineering, Technology and Science. The aim of this book is to provide students with sound mathematics skills and their applications. Although the book is designed primarily for use by engineering students, it is also suitable for students pursuing bachelor degrees with mathematics as one of the subjects and also for those who prepare for various competitive examinations. The material has been arranged to ensure the suitability of the book for class use and for individual self study. Accordingly, the contents of the book have been divided into seven chapters covering the complete syllabus prescribed for B.Tech. Semester-II of West Bengal University of Technology. A number of examples, figures, tables and exercises have been provided to enable students to develop problem-solving skills. The language used is simple and lucid. Suggestions and feedback on this book are welcome.

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Babu Ram

Symbols and Basic Formulae

1 Greek Letters

α	alpha	ϕ	phi
β	beta	Φ	capital phi
γ	gamma	ψ	psi
Γ	capital gamma	Ψ	capital psi
δ	delta	ξ	xi
Δ	capital delta	η	eta
ε	epsilon	ζ	zeta
ι	iota	χ	chi
θ	theta	π	pi
λ	lambda	σ	sigma
μ	mu	Σ	capital sigma
ν	nu	τ	tau
ω	omega	ρ	rho
Ω	capital omega	κ	kapha

2 Algebraic Formulae

(i) Arithmetic progression a, $a + d$, $a + 2d$,
nth term $T_n = a + (n - 1)\,d$

Sum of n terms $= \dfrac{n}{2}[2a + (n-1)d]$

(ii) Geometrical progression: a, ar, ar^2,
nth term $T_n = ar^{n-1}$

Sum of n terms $= \dfrac{a(1-r^n)}{1-r}$

(iii) Arithmetic mean of two numbers a and b is $\dfrac{1}{2}(a+b)$

(iv) Geometric mean of two numbers a and b is $\sqrt{ab}$

(v) Harmonic mean of two numbers a and b is $\dfrac{2ab}{a+b}$

(vi) If $ax^2 + bx + c = 0$ is quadratic, then

(a) its roots are given by $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(b) the sum of the roots is equal to $-\dfrac{b}{a}$

(c) product of the roots is equal to $\dfrac{c}{a}$

(d) $b^2 - 4ac = 0 \Rightarrow$ the roots are equal

(e) $b^2 - 4ac > 0 \Rightarrow$ the roots are real and distinct

(f) $b^2 - 4ac < 0 \Rightarrow$ the roots are complex

(g) if $b^2 - 4ac$ is a perfect square, the roots are rational

3 Properties of Logarithm

(i) $\log_a 1 = 0$, $\log_a 0 = -\infty$ for $a > 1$, $\log_a a = 1$
$\log_e 2 = 0.6931$, $\log_e 10 = 2.3026$,
$\log_{10} e = 0.4343$

(ii) $\log_a p + \log_a q = \log_a pq$

(iii) $\log_a p + \log_a q = \log_a \dfrac{p}{q}$

(iv) $\log_a p^q = q \log_a p$

(v) $\log_a n = \log_a b \cdot \log_b n = \dfrac{\log_b n}{\log_b a}$

4 Angles Relations

(i) $1 \text{ radian} = \dfrac{180°}{\pi}$

(ii) $1° = 0.0174$ radian

5 Algebraic Signs of Trigonometrical Ratios

(a) First quadrant: All trigonometric ratios are positive

(b) Second quadrant: $\sin\theta$ and $\operatorname{cosec}\theta$ are positive, all others negative

(c) Third quadrant: $\tan\theta$ and $\cot\theta$ are positive, all others negative

(d) Fouth quadrant: $\cos\theta$ and $\sec\theta$ are positive, all others negative

6 Commonly Used Values of Trigonometrical Ratios

$$\sin\frac{\pi}{2} = 1,\ \cos\frac{\pi}{2} = 0,\ \tan\frac{\pi}{2} = \infty$$

$$\operatorname{cosec}\frac{\pi}{2} = 1,\ \sec\frac{\pi}{2} = \infty,\ \cos\frac{\pi}{2} = 0$$

$$\sin\frac{\pi}{6} = \frac{1}{2},\ \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2},\ \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\operatorname{cosec}\frac{\pi}{6} = 2,\ \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}},\ \cot\frac{\pi}{6} = \sqrt{3}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},\ \cos\frac{\pi}{3} = \frac{1}{2},\ \tan\frac{\pi}{3} = \sqrt{3}$$

$$\text{cosec}\frac{\pi}{3} = \frac{2}{\sqrt{3}}, \sec\frac{\pi}{3} = 2, \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \tan\frac{\pi}{4} = 1$$

$$\text{cosec}\frac{\pi}{4} = \sqrt{2}, \sec\frac{\pi}{4} = \sqrt{2}, \cot\frac{\pi}{4} = 1$$

7 Trigonometric Ratios of Allied Angles

(a) $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$
$\tan(-\theta) = -\tan\theta$
$\text{cosec}(-\theta) = -\text{cosec}\,\theta$, $\sec(-\theta) = \sec\theta$
$\cot(-\theta) = -\cot\theta$

(b) Any trigonometric ratio of $(n.90 \pm \theta) =$

$$\begin{cases} \pm \text{ same trigonometric ratio of } \theta \\ \text{when } n \text{ is even} \\ \pm \text{ co-ratio of } \theta \text{ when } n \text{ is odd} \end{cases}$$

For example: $\sin(4620) = \sin[90°(52) - 60°]$
$= \sin(-60°) = -\sin 60° = -\frac{\sqrt{3}}{2}$.
Similarly, $\text{cosec}(270° - \theta) = \text{cosec}(90°(3) - \theta)$
$= -\sec\theta$.

8 Transformations of Products and Sums

(a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(b) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(c) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(d) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(e) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

(f) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

(g) $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$

(h) $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$
$= 2\cos^2 A - 1 = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(i) $\tan 2A = \dfrac{\sin 2A}{\cos 2A} = \dfrac{2\tan A}{1-\tan^2 A}$

(j) $\sin 3A = 3\sin A - 4\sin^3 A$

(k) $\cos 3A = 4\cos^3 A - 3\cos A$

(l) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

(m) $\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

(n) $\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

(o) $\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

(p) $\cos A - \cos B = 2\sin\dfrac{A+B}{2}\sin\dfrac{B-A}{2}$

(q) $\sin A\cos B = \dfrac{1}{2}[\sin(A+B) + \sin(A-B)]$

(r) $\cos A\sin B = \dfrac{1}{2}[\sin(A+B) - \sin(A-B)]$

(s) $\cos A\cos B = \dfrac{1}{2}[\cos(A+B) + \cos(A-B)]$

(t) $\sin A\sin B = \dfrac{1}{2}[\cos(A-B) - \cos(A+B)]$

9 Expressions for $\sin\frac{A}{2}$; $\cos\frac{A}{2}$ and $\tan\frac{A}{2}$

(a) $\sin\frac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$

(b) $\cos\frac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$

(c) $\tan\frac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$

(d) $\sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1+\sin A}$

(e) $\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1-\sin A}$

10 Relations Between Sides and Angles of a Triangle

(a) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ (sine formulae)

(b) $$\left.\begin{aligned} \cos A &= \frac{b^2+c^2-a^2}{2bc} \\ \cos B &= \frac{c^2+a^2-b^2}{2ca} \\ \cos C &= \frac{a^2+b^2-c^2}{2ab} \end{aligned}\right\} \text{cosine formulae}$$

(c) $$\left.\begin{aligned} a &= b\cos C + c\cos B \\ b &= c\cos A + a\cos C \\ c &= a\cos B + b\cos A \end{aligned}\right\} \text{Projection formulae.}$$

11 Permutations and Combinations Formulae

$${}^nP_r = \frac{n!}{(n-r)!},$$

$${}^nC_r = \frac{n!}{r!(n-r)!} = {}^nC_{n-r},$$

$${}^nC_0 = {}^nC_n = 1$$

12 Differentiation Formulae

(a) $\frac{d}{dx}(\sin x) = \cos x$

(b) $\frac{d}{dx}(\cos x) = -\sin x$

(c) $\frac{d}{dx}(\tan x) = \sec^2 x$

(d) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(e) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(f) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(g) $\frac{d}{dx}(e^x) = e^x$

(h) $\frac{d}{dx}(a^x) = a^2 \log_e a$

(i) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$

(j) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(k) $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$

(l) $\frac{d^n}{dx^n}(ax+b)^m = m(m-1)(m-2)$
$\ldots(m-n+1)(ax+b)^{m-n}$

(m) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(n) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

(o) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(p) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(q) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(r) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

(s) $\frac{d}{dx}(\sinh x) = \cosh x$

(t) $\frac{d}{dx}(\cosh x) = \sinh x$

(u) $D^n(uv) = D^n u + nc_1 D^{n-1} uDv + nc_2 D^{n-2} uD^2 v$
$+ \cdots + {}^nC_r D^{n-r} uD^r v + \cdots + {}^nC_n uD^n v$

(Leibnitz's Formula)

13 Integration Formulae

(a) $\int \sin x \, dx = -\cos x$

(b) $\int \cos x \, dx = \sin x$

(c) $\int \tan x \, dx = -\log \cos x$

(d) $\int \cot x \, dx = \log \sin x$

(e) $\int \sec x \, dx = \log(\sec x + \tan x)$

(f) $\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x)$

(g) $\int \sec^2 x \, dx = \tan x$

(h) $\int \operatorname{cosec}^2 x \, dx = -\cot x$

(i) $\int e^x dx = e^x$

(j) $\int a^x dx = \frac{a^x}{\log_e a}$

(k) $\int \frac{1}{x} dx = \log_e x$

(l) $\int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1$

(m) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

(n) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log_e \frac{a+x}{a-x}$

(o) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log_e \frac{x-a}{x+a}$

(p) $\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

(q) $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}$

(r) $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}$

(s) $\int \sqrt{a^2+x^2}\, dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$

(t) $\int\sqrt{x^2-a^2}dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$

(u) $\int\sqrt{a^2-x^2}dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

(v) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

(w) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

(x) $\int_0^{\frac{\pi}{2}}\sin^n x\,dx = \int_0^{\frac{\pi}{2}}\cos^n x\,dx$

$$= \begin{cases} \frac{(n-1)(n-3)(n-5)\dots}{n(n-2)(n-4)\dots}\cdots \text{if } n \text{ is odd} \\ \frac{(n-1)(n-3)(n-5)\dots}{n(n-2)(n-4)\dots}\frac{\pi}{2} \text{ if } n \text{ is even} \end{cases}$$

(y) $\int_0^{\frac{\pi}{2}}\sin^m x\cos^n x\,dx$

$$= \begin{cases} \frac{(m-1)(m-3)\dots(n-1)(n-3)\dots}{(m+n)(m+n-2)(m+n-4)\dots} \\ \quad \text{if } m \text{ and } n \text{ are not simultaneously even} \\ \frac{(m-1)(m-3)\dots(n-1)(n-3)\dots}{(m+n)(m+n-2)(m+n-4)\dots}\frac{\pi}{2} \\ \quad \text{if both } m \text{ and } n \text{ are even} \end{cases}$$

14 Beta and Gamma Functions

(a) $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ converges for $m, n > 0$

(b) $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$ converges for $n > 0$

(c) $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(n+1) = n!$ if n is positive integer

(d) $\Gamma(1) = 1 = \Gamma(2)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(e) $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

(f) $\int_0^{\frac{\pi}{2}}\sin^p x\cos^q x\,dx = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

(g) $\int_0^{\frac{\pi}{2}}\sqrt{\tan\theta\,d\theta} = \int_0^{\frac{\pi}{2}}\sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta\,d\theta = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\,\Gamma(1)}$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

Roadmap to the Syllabus

W.B. UNIVERSITY OF TECHNOLOGY, KOLKATA

MODULE I

Ordinary differential equations (ODE) — First order and first degree: Exact equations, Necessary and sufficient condition of exactness of a first order and first degree ODE (statement only), Rules for finding Integrating factors, Linear equation, Bernoulli's equation. General solution of ODE of first order and higher degree (different forms with special reference to Clairaut's equation).

MODULE II

ODE — Higher order and first degree: General linear ODE of order two with constant coefficients, C.F. and P.I., D-operator methods for finding P.I., Method of variation of parameters, Cauchy-Euler equations, Solution of simultaneous linear differential equations.

☞ **REFER** **Chapter 1**

MODULE III

Basics of Graph Theory: Graphs, Digraphs, Weighted graph, Connected and disconnected graphs, Complement of a graph, Regular graph, Complete graph, Subgraph, Walks, Paths, Circuits, Euler Graph, Cut sets and cut vertices, Matrix representation of a graph, Adjacency and incidence matrices of a graph, Graph isomorphism, Bipartite graph.

MODULE IV

Tree: Definition and properties, Binary tree, Spanning tree of a graph, Minimal spanning tree, properties of trees, Algorithms: Dijkstra's Algorithm for shortest path problem, Determination of minimal spanning tree using DFS, BFS, Kruskal's and Prim's algorithms.

☞ **REFER** **Chapter 2**

MODULE V

Improper Integral: Basic ideas of improper integrals, working knowledge of Beta and Gamma functions (convergence to be assumed) and their interrelations. Laplace Transform (LT): Definition and existence of LT, LT of elementary functions, First and second shifting properties, Change of scale property; LT of $f(t)$ t, LT of ( ) n t f t, LT of derivatives of $f(t)$, LT of $\int f(u)du$. Evaluation of improper integrals using LT, LT of periodic and step functions, Inverse – LT: Definition and its properties; Convolution Theorem (statement only) and its application to the evaluation of inverse LT, Solution of linear ODE with constant coefficients (initial value problem) using LT.

☞ **REFER** **Chapters 3–7**

1 Ordinary Differential Equations

Differential equations play an important role in engineering and science. Many physical laws and relations appear in the form of differential equations. For example, the current I in an LCR circuit is described by the differential equation $LI'' + RI + \frac{1}{C}I = E$, which is derived from Kirchhoff's laws. Similarly, the displacement y of a vibrating mass m on a spring is described by the equation $my'' + ky = 0$. The study of differential equations involves *formation of differential equations*, the *solutions of differential equations,* and the *physical interpretation* of the solution in terms of the given problem.

1.1 DEFINITIONS AND EXAMPLES

Definition 1.1. A *differential equation* is an equation which involves derivatives.
For example,

(a) $\frac{d^2x}{dt^2} + n^2\, x = 0$, (b) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

(c) $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ and (d) $\frac{dy}{dx} = x + 1$

are differential equations.

We note that a differential equation may have the variables present only in the derivatives. For example in (b), the variables are present only in derivatives. Moreover, a differential equation may have more than one dependent variable. For example,

$$\frac{d\phi}{dt} + \frac{d\psi}{dt} = \phi + \psi$$

has two dependent variables ϕ and ψ and one independent variable t.

Definition 1.2. A differential equation involving derivatives with respect to a single independent variable is called an *ordinary differential equation*.
For example,

$$\frac{d^2y}{dx^2} + 3y = 0$$

is an ordinary differential equation.

Definition 1.3. A differential equation involving partial derivatives with respect to two or more independent variables is called a *partial differential equation*.
For example,

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

are partial differential equations.

Definition 1.4. The order of the highest derivative appearing in a differential equation free from radicals is called the *order* of that differential equation. For example the, order of the differential equation $y'' + 4y = 0$ is two, the order of the differential equation $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$ is two, and order of the differential equation $y = x\frac{dy}{dx} + \frac{x}{dy/dx}$ is one.

Definition 1.5. The degree or exponent of the highest derivative appearing in a differential equation free from radicals and fractions is called the *degree* of the differential equation.
For example, the degree of the differential equation

$$y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + 1$$

is two. Similarly, the degree of the differential equation

$$\left(\frac{d^3y}{dx^3}\right)^{2/3} = 1 + 2\frac{dy}{dx}$$

is two because the given equation can be written as

$$\left(\frac{d^3y}{dx^3}\right)^2 = \left(1 + 2\frac{dy}{dx}\right)^3.$$

1.2 FORMULATION OF DIFFERENTIAL EQUATION

The derivation of differential equations from physical or other problems is called *modelling*. The modelling involves the successive differentiations and elimination of parameters present in the given sytem.

EXAMPLE 1.1

Form the differential equation for *"free fall"* of a stone dropped from the height y under the action of force due to gravity g.

Solution. We know that equation of motion of the free fall is

$$y = \frac{1}{2}gt^2.$$

Differentiating with respect to t, we get

$$\frac{dy}{dt} = gt.$$

Differentiating once more, we get

$$\frac{d^2y}{dt^2} = g,$$

which is the desired differential equation representing the free fall of a stone.

EXAMPLE 1.2

Form the differential equation of simple harmonic motion given by $x = \text{A} \cos(\omega t + \phi)$, where A and ϕ are constants.

Solution. To get the required differential equation, we have to differentiate the given relation and eliminate the constants A and ϕ. Differentiating twice, we get

$$\frac{dx}{dt} = -A\omega \sin(\omega t + \phi),$$

$$\frac{d^2x}{dt^2} = -A\omega^2 \cos(\omega t + \phi) = -\omega^2 x. \qquad (1)$$

Hence

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

is the differential equation governing simple harmonic motion. Equation (1) shows that acceleration varies as the distance from the origin.

EXAMPLE 1.3

Find the differential equation governing the motion of a particle of mass m sliding down a frictionless curve.

Solution. Velocity of the particle at the starting point $\text{P}(x, y)$ is zero since it starts from rest. Let $\phi(x, u)$ be some intermediate point during the motion (see Figure 1.1). Let the origin O be the lowest point of the curve and let the length of the arc OQ be s.

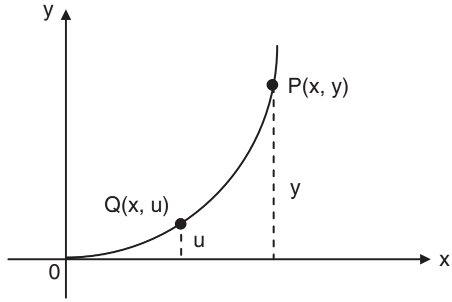


Figure 1.1

By Law of Conservation of energy, we have

potential energy at P + kinetic energy at P
= potential energy at Q + kinetic energy at Q,

and so

$$mgy + 0 = mgu + \frac{1}{2}m\left(\frac{ds}{dt}\right)^2.$$

Hence

$$\left(\frac{ds}{dt}\right)^2 = 2g(y - u)$$

is the required differential equation.

If the duration T_0 of descent is independent of the starting point, the solution of this differential equation comes out to be the equation of a *cycloid*. Thus the shape of the curve is a cycloid. This curve is called *Tautochrone Curve*.

EXAMPLE 1.4

Derive the differential equation governing a mass-spring system.

Solution. Let m be the mass suspended by a spring that is rigidly supported from one end (see Figure 1.2). Let

(i) rest position be denoted by $x = 0$, downward displacement by $x > 0$ and upward displacement be denoted by $x < 0$.

(ii) $k > 0$ be spring constant and $a > 0$ be damping constant.

(iii) $a\frac{dx}{dt}$ be the damping force due to medium (damping force is proportional to the velocity).

(iv) $f(t)$ be the external impressed force on m.

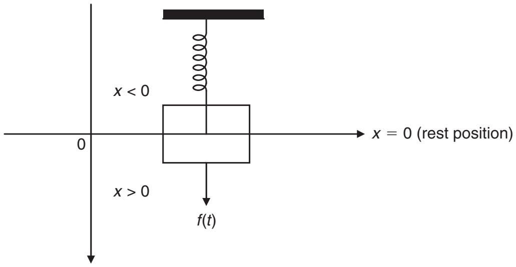


Figure 1.2

Then, by Newton's second law of motion, the sum of force acting on m is $m\frac{d^2x}{dt^2}$ and so

$$m\frac{d^2x}{dt^2} = -kx - a\frac{dx}{dt} + f(t),$$

that is,

$$\frac{d^2x}{dt^2} + \frac{a}{m}\frac{dx}{dt} + \frac{k}{m}x = f(t),$$

which is the required differential equation governing the system.

EXAMPLE 1.5

Find the differential equation of all circles of radius a and centre (h, k).

Solution. We know that the equation of the circle with radius a and centre (h, k) is

$$(x-h)^2 + (y-k)^2 = a^2 \tag{2}$$

Differentiating twice, we get

$$x - h + (y-k)\frac{dy}{dx} = 0, \tag{3}$$

$$1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \tag{4}$$

From (4), we have

$$y - k = -\frac{1 + \left(\frac{dy}{dx}\right)^2}{d^2y/dx^2}$$

and then (3) yields

$$x - h = -(y-k)\frac{dy}{dx} = \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{d^2y/dx^2}.$$

Substituting the values of $x - h$ and $y - k$ in (2), we get

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2,$$

which is the required differential equation. It follows that

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} = a,$$

and so the radius of curvature of a circle at any point is constant.

EXAMPLE 1.6

Form the differential equation from the equation $xy = Ae^x + B\,e^{-x}$.

Solution. Differentiating twice, we get

$$x\frac{dy}{dx} + y = Ae^x - B\,e^{-x}$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = Ae^x + B\,e^{-x} = xy.$$

Hence

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$$

is the required differential equation.

1.3 SOLUTION OF DIFFERENTIAL EQUATION

Definition 1.6. A solution of a differential equation is a functional relation between the variables involved such that this relation and the derivatives obtained from it satisfy the given differential equation.
For example, $x^2 + 4y = 0$ is a solution of the differential equation

$$\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0. \tag{5}$$

In fact, differentiating $x^2 + 4y = 0$, we get

$$2x + 4\frac{dy}{dx} = 0$$

and so $\frac{dy}{dx} = -\frac{x}{2}$. Hence

$$\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = \frac{x^2}{4} + x\left(-\frac{x}{2}\right) - y$$
$$= \frac{x^2}{4} - \frac{x^2}{2} - \left(-\frac{x^2}{4}\right) = 0.$$

Hence $x^2 + 4y = 0$ is a solution of (5).

Definition 1.7. A solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation is called the *general* (or *complete*) *solution* of the differential equation. For example $xy = Ae^x + Be^{-x}$ is the general solution of

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0.$$

Definition 1.8. A solution obtained from the general solution of a differential equation by giving particular values to the arbitrary constants is called a *particular solution* of that differential equation.

Definition 1.9. A problem involving a differential equation and one or more supplementary conditions, relating to one value of the independent variable, which the solution of the given differential equation must satisfy is called an *initial-value problem*. For example,

$$\frac{d^2y}{dt^2} + y = \cos 2t,$$

$$y(0) = 1,\ y'(0) = -2$$

is an initial-value problem. Similarly, the problem

$$\frac{d^2y}{dx^2} + y = 0,$$
$$y(1) = 3, y'(1) = 4$$

is also an initial-value problem.

Definition 1.10. A problem involving a differential equation and one or more supplementary conditions, relating to more than one values of the independent variable, which the solution of the differential equation must satisfy, is called a *boundary-value problem*.

For example,

$$\frac{d^2y}{dx^2} + y = 0$$
$$y(0) = 2, \quad y\left(\frac{\pi}{2}\right) = 4$$

is a boundary value problem.

1.4 DIFFERENTIAL EQUATIONS OF FIRST ORDER

We consider first the differential equations of first order. Let D be a open connected set in $\mathbb{R}^2$ and let $f: \mathrm{D} \to \mathbb{R}$ be continuous. We discuss the problem of determining solution in D of the first order differential equation

$$\frac{dy}{dx} = f(x,\ y).$$

Definition 1.11. A real-valued function f: D $\to \mathbb{R}$ defined on the connected open set D in $\mathbb{R}^2$ is said to satisfy a *Lipschitz condition* in y on D with Lipschitz constant M if and only if

$$|f(x, y_2) - f(x, y_1)| \le M|y_2 - y_1|$$

for all (x, y_1) and $(x, y_2) \in$ D.

Regarding existence of solutions of first order differential equation, we have the following theorems.

Theorem 1.1. (Picard's Existence and Uniqueness Theorem). Let f: $D \to \mathbb{R}$ be continuous on open connected set D in $\mathbb{R}^2$ and satisfy Lipschitz condition in y on D. Then for every $(x_0, y_0) \in D$, the initial-value problem $\frac{dy}{dx} = f(x,\ y)$ has a solution passing through (x_0, y_0).

The solution obtained in Theorem 1.1 is unique. The Lipschitz condition in the hypothesis of Picard's theorem cannot be dropped because continuity of f without this condition will not yield unique solution. For example, consider the equation

$$\frac{dy}{dx} = y^{2/3}, \quad y(0) = 0.$$

Clearly $\phi_1(x) \equiv 0$ is a solution to this equation. Further substituting $y = \sin^3\theta$, we have $dy = 3\sin^2\theta \cos\theta\, d\theta$ and so

$$\frac{3\sin^2\theta\cos\theta\, d\theta}{dx} = \sin^2\theta,$$

which yields $3\cos\theta\, d\theta = dx$ and $x = 3\sin\theta$. Thus $x^3 = 27\sin^3\theta = 27y$ and hence $y = \frac{x^3}{27}$ is also a solution. Therefore, the given initial-value problem does not have unique solution. The reason is that it does not satisfy Lipschitz condition.

Theorem 1.2. (Peano's Existence Theorem). Let f be a continuous real-valued function on a non-empty open subset D of the Euclidean space $\mathbb{R}^2$ and let $(x_0, y_0) \in D$. Then there is a positive real number α such that the first order differential equations $\frac{dy}{dx} = f(x, y)$ has a solution ϕ in the interval $[x_0, x_0 + \alpha]$ which satisfies the boundary condition $\phi(x_0) = y_0$.

Clearly, Peano's theorem is merely an existence theorem and not a uniqueness theorem.

We now consider certain basic types of first order differential equations for which an exact solution may be obtained by definite procedures. The most important of these types are separable equations, homogeneous equations, exact equations, and linear equations. The corresponding methods of solution involve various devices. We take these types of differential equations one by one.

1.5 SEPARABLE EQUATIONS

The first order differential equation

$$\frac{dy}{dx} = f(x, y) \tag{6}$$

is separable if f may be expressed as

$$f(x, y) = \frac{M(x)}{N(y)}, \quad (x, y) \in D \tag{7}$$

The function M(x) and N(y) are real-valued functions of x and y, respectively. Thus (6) becomes

$$N(y)\frac{dy}{dx} = M(x) \tag{8}$$

The equation (8) is solved by integrating with respect to x. Thus the solution is

$$\int N(y)dy = \int M(x)dx + \text{C}.$$

EXAMPLE 1.7

Solve

$$\frac{dy}{dx} = e^{x+y}, \ y(1) = 1. \text{ Find } y(-1).$$

Solution. We have

$$\frac{dy}{dx} = \frac{e^x}{e^{-y}}$$

and so

$$e^{-y}dy = e^x \, dx.$$

Integrating both sides

$$\int e^{-y} \, dy = \int e^x dx + C \quad \text{or} \quad -e^{-y} = e^x + C.$$

Using initial condition $y\,(1) = 1$, we get

$$-e^{-1} = e + C$$

and so $C = -\left(\frac{1+e^2}{e}\right)$.

Thus the solution is

$$-e^{-y} = e^x - \left(\frac{1+e^2}{e}\right).$$

Hence $y(-1)$ is given by

$$y(-1) = e^{-1} - \left(\frac{1+e^2}{e}\right)$$

$$= \frac{1}{e} - \frac{1+e^2}{e} = -e,$$

that is

$$-e^{-y} = -e$$

or

$$y = -1.$$

EXAMPLE 1.8

Solve $\frac{dy}{dx} = (4x + y + 1)^2$, $y(0) = 1$.

Solution. Substituting $4x + y + 1 = t$, we get

$$\frac{dy}{dx} = \frac{dt}{dx} - 4$$

Hence, the given equation reduces to

$$\frac{dt}{dx} - 4 = t^2$$

or

$$\frac{dt}{dx} = t^2 + 4$$

or

$$\frac{dt}{t^2 + 4} = dx.$$

Integrating both sides, we have

$$\int \frac{dt}{t^2 + 4} \, dt = \int dx + C$$

or

$$\frac{1}{2}\tan^{-1}\frac{t}{2} = x + C$$

or

$$\frac{1}{2}\tan^{-1}\frac{4x+y+1}{2} = x + C$$

or

$$4x + y + 1 = 2\tan 2(x + C)$$

Using given initial conditions $x = 0$, $y = 1$, we get

$$\frac{1}{2}\tan^{-1} 1 = C$$

which gives C $= \frac{\pi}{8}$. Hence the solution is

$$4x + y + 1 = 2\tan(2x + \pi/4).$$

EXAMPLE 1.9

Solve

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right).$$

Solution. The given equation can be written as

$$\frac{dy}{dx}(a + x) = y(1 - ay)$$

or

$$\frac{dy}{y(1-ay)} = \frac{dx}{a+x}.$$

Integrating both sides, we have

$$\log y - \log(1 - ay) = \log(a + x) + C$$

or

$$\log\frac{y}{(a+x)\,(1-ay)} = C.$$

Hence

$$y = K(a + x)(1 - ay), \text{ K constant}$$

is the general solution.

EXAMPLE 1.10

Solve

$$x(1 + y^2)dx + y(1 + x^2)dy = 0.$$

Solution. Dividing the given equation throughout by $(1 + x^2)\,(1 + y^2)$, we get

$$\frac{x}{1+x^2}dx + \frac{y}{1+y^2}\,dy = 0.$$

Integrating both sides, we have

$$\int\frac{x}{1+x^2}\,dx + \int\frac{y}{1+y^2}\,dy = C$$

or

$$\frac{1}{2}\int\frac{2x}{1+x^2}\,dx + \frac{1}{2}\int\frac{2y}{1+y^2}\,dy = C$$

or

$$\frac{1}{2}\log(1 + x^2) + \frac{1}{2}\log(1 + y^2) = C$$

or

$$\frac{1}{2}\log(1 + x^2)(1 + y^2) = C$$

or

$$\log(1 + x^2)(1 + y^2) = 2C = \log C$$

Hence

$$(1 + x^2)(1 + y^2) = K \text{ (constant)}$$

is the required solution.

EXAMPLE 1.11

Solve

$$\frac{dy}{dx} = e^{x-y} + x^2e^{-y}.$$

Solution. We have

$$\frac{dy}{dx} = \frac{e^x}{e^y} + \frac{x^2}{e^y}$$

or

$$e^y\frac{dy}{dx} = e^x + x^2$$

or

$$e^y dy = (e^x + x^2)dx = e^x dx + x^2 dx.$$

Integrating both sides, we get

$$e^y = e^x + \frac{x^3}{3} + C \text{ (constant)}.$$

EXAMPLE 1.12

Solve

$$16y\frac{dy}{dx} + 9x = 0.$$

Solution. We are given that

$$16y\frac{dy}{dx}+9x=0$$

or

$$16y\ dy=-9x\ dx.$$

Integrating both sides, we have

$$16\frac{y^2}{2}=-9\frac{x^2}{2}+C$$

or

$$\frac{x^2}{16}+\frac{y^2}{9}=\text{K (constant)},$$

which is the required solution and represents a family of ellipses.

EXAMPLE 1.13

Solve

$$\frac{dy}{dx}=(x+y)^2.$$

Solution. Substituting $z=x+y$, we get $\frac{dz}{dx}=1+\frac{dy}{dx}$. Therefore, the given equation reduces to

$$\frac{dz}{dx}-1=z^2$$

or

$$\frac{dz}{dx}=1+z^2$$

or

$$\frac{dz}{1+z^2}=dx.$$

Integrating, we get

$$\tan^{-1}z=x+C.$$

Putting back the value of z, we get

$$\tan^{-1}(x+y)=x+C.$$

Hence

$$x+y=\tan(x+C).$$

EXAMPLE 1.14

Solve

$$(2x-4y+5)\frac{dy}{dx}=x-2y+3.$$

Solution. We have

$$2(x-2y+\frac{5}{2})\frac{dy}{dx}=x-2y+3.$$

Substituting $z=x-2y$, we get $\frac{dz}{dx}=1-2\frac{dy}{dx}$. The equation becomes

$$(2z+5)\frac{dz}{dx}=4z+11$$

or

$$(4z+10)\frac{dz}{dx}=2(4z+11)$$

or

$$\frac{4z+10}{4z+11}dz=2\ dx$$

or

$$\frac{4z+11-1}{4z+11}\ dz=2dx$$

or

$$\left(1-\frac{1}{4z+11}\right)dz=2dx$$

Integrating both sides, we have

$$z-\frac{1}{4}\log|4z+11|=2x+\mathrm{C}.$$

Putting back the value of z, we have

$$4x+8y+\log|4x-8y+11|=C.$$

EXAMPLE 1.15

Solve $(x+2y)(dx-dy)=dx+dy$.

Solution. We have

$$(x+2y)(dx-dy)=dx+dy$$

or

$$(x+2y-1)dx=(x+2y+1)dy$$

or

$$\frac{dy}{dx}=\frac{x+2y-1}{x+2y+1}.$$

Substituting $x+2y=z$, we have $1+2\frac{dy}{dx}=\frac{dz}{dx}$ and so

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{dz}{dx}-1\right).$$

Hence the above equation becomes

$$\frac{1}{2}\left(\frac{dz}{dx}-1\right)=\frac{z-1}{z+1}$$

or

$$\frac{dz}{dx}=\frac{3z-1}{z+1}.$$

Separating the variables, we have

$$\frac{(z+1)\ dz}{3z-1}=dx$$

or

$$\frac{1}{3}\left(\frac{3z-1+4}{3z-1}\right)dz=dx$$

or

$$\frac{1}{3}\left(1+\frac{4}{3z-1}\right)dz=dx.$$

Integrating both sides, we get

$$\frac{1}{3}\int\left(1+\frac{4}{3z-1}\right)\ dz=\int dx+C$$

or

$$\frac{1}{3}\left(z+\frac{4}{3}\log(3z-1)\right)=x+C$$

or

$$3z+4\log(3z-1)=9x+k\ \text{(constant)}$$

or

$$3(y-x)+2\log(3x+6y-1)=\frac{k}{2}=\mathrm{K}\ \text{(constant)}.$$

1.6 HOMOGENEOUS EQUATIONS

Definition 1.12. An expression of the form

$$a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\ldots+a_ny^n,$$

in which every term is of the n^{th} degree, is called a *homogeneous function* of degree n.

Definition 1.13. A differential equation of the form

$$\frac{dy}{dx}=\frac{f(x,\ y)}{\phi(x,\ y)},$$

where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of the same degree in x and y is called an *homogeneous equation*.

A homogeneous differential equation can be solved by substituting $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Putting the value of y and $\frac{dy}{dx}$ in the given equation, we get a differential equation in which variables can be separated. Integration then yields the solution in terms of v, which we replace with $\frac{y}{x}$.

EXAMPLE 1.16

Solve

$$(x^2+y^2)dx-2xy\,\mathrm{dy}=0.$$

Solution. We have

$$(x^2+y^2)dx-2xy\ dy=0$$

and so

$$\frac{dy}{dx}=\frac{x^2+y^2}{2xy}\tag{9}$$

This equation is homogeneous in x and y. So, put $y = vx$. We have $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Hence (9) becomes

$$v+x\frac{dv}{dx}=\frac{x^2+v^2x^2}{2vx^2}=\frac{1+v^2}{2v}$$

or

$$x\frac{dv}{dx}=\frac{1+v^2}{2v}-v=\frac{1-v^2}{2v}$$

or

$$\frac{2v\ dv}{1-v^2}=\frac{dx}{x}$$

or

$$\left(\frac{1}{1-v}-\frac{1}{1+v}\right)\ dv=\frac{dx}{x}.$$

Integrating both sides, we get

$$-\log(1-v)-\log(1+v)=\log x+C$$

or

$$-\log(1-v^2)=\log x+C$$

or

$$\log x+\log(1-v^2)=C$$

or

$$\log x(1 - v^2) = C$$

or

$$\log x\left(1 - \left(\frac{y}{x}\right)^2\right) = C$$

or

$$x\left(1 - \frac{y^2}{x^2}\right) = C.$$

Hence

$$x^2 - y^2 = Cx$$

is the general solution of the given differential equation.

EXAMPLE 1.17

Solve

$$x\frac{dy}{dx} = y + \sqrt{x^2 + y^2}.$$

Solution. We have

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x},$$

which is homogeneous in x and y. Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Hence the equation takes the form

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x}$$
$$= v + \sqrt{1 + v^2}.$$

Thus

$$x\frac{dv}{dx} = \sqrt{1 + v^2}$$

and so

$$\frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}.$$

Integrating both sides, we get

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x} + C$$

or

$$\log(v + \sqrt{1 + v^2}) = \log x + \log C$$
$$= \log x\ C.$$

Hence

$$v + \sqrt{1 + x^2} = x\ C.$$

Substituting the value of v, we get

$$\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = x\ C,$$

which is the required solution.

EXAMPLE 1.18

Solve

$$(y^2 - 2xy)dx = (x^2 - 2xy)dy.$$

Solution. The given differential equation can be expressed by

$$\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - 2xy}.$$

Clearly it is a homogeneous equation. Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Hence

$$v + x\frac{dy}{dx} = \frac{v^2x^2 - 2x^2v}{x^2 - 2x^2v} = \frac{v^2 - 2v}{1 - 2v}.$$

or

$$x\frac{dv}{dx} = \frac{v^2 - 2v}{1 - 2v} - v.$$
$$= \frac{v^2 - 2v - v + 2v^2}{(1 - 2v)}$$
$$= \frac{3v^2 - 3v}{1 - 2v}.$$

Thus

$$\frac{1 - 2v}{3v^2 - 3v}dv = \frac{dx}{x}$$

or

$$\frac{1 - 2v}{3(v^2 - v)} = \frac{dx}{x}.$$

Integrating, we get

$$-\frac{1}{3}\log(v^2 - v) = \log x + C$$
$$-\frac{1}{3}\log\left(\frac{y^2}{x^2} - \frac{y}{x}\right) = \log x + C$$
$$-\frac{1}{3}\log\left(\frac{y^2 - xy}{x^2}\right) = \log x + C$$
$$\log x^3\left(\frac{y^2 - xy}{x^2}\right) = \log C.$$

Hence

$$x(y^2 - xy) = C.$$

EXAMPLE 1.19

Solve

$$x\frac{dy}{dx}+\frac{y^2}{x}=y.$$

Solution. We have

$$x\frac{dy}{dx}=y-\frac{y^2}{x}=\frac{xy-y^2}{x}$$

or

$$\frac{dy}{dx}=\frac{xy-y^2}{x^2},$$

which is homogeneous in x and y. Putting $y=vx$, we have $\frac{dy}{dx}=v+x\frac{dv}{dx}$ Hence the equation becomes

$$v+x\frac{dy}{dx}=\frac{xy-y^2}{x^2}=\frac{vx^2-v^2x^2}{x^2}=v-v^2$$

or

$$x\frac{dv}{dx}=-v^2$$

or

$$\frac{dv}{v^2}=-\frac{dx}{x}.$$

Integrating, we get

$$\frac{1}{v}=-\log x+\log C.$$

Putting $v=\frac{y}{x}$, we get

$$\frac{x}{y}=-\log x\ C.$$

EXAMPLE 1.20

Solve

$$(x^2-y^2)dx=2xy\,dy.$$

Solution. We have

$$\frac{dy}{dx}=\frac{x^2-y^2}{2xy}$$

so that the given equation is homogeneous in x and y. Putting $y=vx$, the equation takes the form

$$v+x\frac{dv}{dx}=\frac{x^2-v^2x^2}{2vx^2}=\frac{1-v^2}{2v}.$$

Therefore,

$$x\frac{dv}{dx}=\frac{1-v^2}{2v}-v=\frac{1-v^2-2v^2}{2v}=\frac{1-3v^2}{2v}.$$

Now, variables separation yields

$$\frac{2v}{1-3v^2}\ dv=\frac{dx}{x}.$$

Integrating both sides, we get

$$-\frac{1}{3}\int\frac{-6v}{1-3v^2}\ dv=\int\frac{dx}{x}+C$$

or

$$-\frac{1}{3}\log(1-3v^2)=\log x+C$$

or

$$-\log(1-3v^2)=\log x^3+3C$$

or

$$\log x^3(1-3v^2)=-3C$$

or

$$x^3\left(1-\frac{3y^2}{x^2}\right)=K$$

or

$$x(x^2-3y^2)=\mathrm{K}.$$

EXAMPLE 1.21

Solve

$$(1+e^{x/y}dx+e^{x/y}(1-\frac{x}{y})dy=0.$$

Solution. We have

$$\frac{dx}{dy}=-\frac{e^{x/y}(1-\frac{x}{y})}{1+e^{x/y}}.$$

Putting $x=vy$, we have $\frac{dx}{dy}=v+y\frac{dv}{dy}$ and so the above equation reduces to

$$v+y\frac{dv}{dy}=-\frac{e^v(1-v)}{1+e^v}=\frac{e^v(v-1)}{1+e^v}.$$

Hence

$$y\frac{dv}{dy}=\frac{e^v(v-1)}{1+e^v}-v=-\frac{v+e^v}{1+e^v}$$

and so separation of variable gives

$$\frac{1+e^v}{v+e^v}\ dv=-\frac{dy}{y}.$$

Integrating both sides, we get

$$\int\frac{1+e^v}{v+e^v}=-\int\frac{dy}{y}+C$$

or

$$\log(v + e^v) = -\log y + \log k$$

or

$$\log(v + e^v) = \log \frac{k}{y}.$$

Thus

$$v + e^v = \frac{k}{y}.$$

But $v = \frac{x}{y}$. Hence

$$x + ye^{x/y} = k \text{ (constant)}.$$

1.7 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

Equations of the form

$$\frac{dy}{dx} = \frac{ax + by + C}{a'x + b'y + C'}$$

can be reduced, by substitution, to the homogeneous form and then solved. Two cases arise:

(i) If $\frac{a}{a'} \neq \frac{b}{b'}$, then the substitution $x = X + h$ and $y = \mathrm{Y} + k$, where h and k are suitable constants, makes the given equation homogeneous in X and Y.

(ii) If $\frac{a}{a'} = \frac{b}{b'}$, then the substitution $ax + by = z$ serves our purpose.

EXAMPLE 1.22

Solve

$$\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}.$$

Solution. We observe that the condition $\frac{a}{a'} \neq \frac{b}{b'}$ is satisfied in the present case. So, we put

$$x = X + h \quad \text{and} \quad y = Y + k.$$

Therefore, $dx = d\mathrm{X}$ and $dy = d\mathrm{Y}$ and the given equation reduces to

$$\frac{dY}{dX} = \frac{2(X + h) - (Y + k) + 1}{X + h + 2(Y + k) - 3}$$
$$= \frac{2X - Y + 2h - k + 1}{X + 2Y + h + 2k - 3}.$$

We choose h and k such that

$$2h - k + 1 = 0, \text{ and}$$
$$h + 2k - 3 = 0.$$

Solving these two equations, we get $h = \frac{1}{5}$, $k = \frac{7}{5}$. Hence

$$\frac{dY}{dX} = \frac{2X - Y}{X + 2Y},$$

which is homogeneous in X and Y. So put $\mathrm{Y} = v\mathrm{X}$. Then

$$v + X\frac{dv}{dX} = \frac{2X - vX}{X + 2vX} = \frac{2 - v}{1 + 2v}$$

and so

$$X\frac{dv}{dX} = \frac{2 - v}{1 + 2v} - v = \frac{-2v^2 - 2v + 2}{1 + 2v}.$$

Now separation of variables yields

$$\frac{1 + 2v}{-2(v^2 + v - 1)}\,dv = \frac{dX}{X}.$$

Integrating both sides, we get

$$-\frac{1}{2}\int \frac{1 + 2v}{v^2 + v - 1}\,dv = \int \frac{dX}{X} + C$$

or

$$-\frac{1}{2}\log(v^2 + v - 1) = \log X + \mathrm{C}$$

or

$$-\log(v^2 + v - 1) = 2\log X + \mathrm{C} = \log \mathrm{X}^2 + \mathrm{C}$$

or

$$\log X^2(v^2 + v - 1) = \log k$$

or

$$X^2(v^2 + v - 1) = k$$

or

$$X^2\left(\frac{Y^2}{X^2} + \frac{Y}{X} - 1\right) = k$$

or

$$Y^2 + YX - X^2 = k$$

or

$$\left(y + \frac{7}{5}\right)^2 + \left(y + \frac{7}{5}\right)\left(x + \frac{1}{5}\right) - \left(x + \frac{1}{5}\right)^2 = k$$

or

$$y^2+xy-x^2+\left(\frac{14}{5}+\frac{1}{5}\right)y+\left(-\frac{2}{5}+\frac{7}{5}\right)x=k$$

or

$$y^2+xy-x^2+3y+x=k.$$

EXAMPLE 1.23

Solve

$$\frac{dy}{dx}=\frac{2x+3y+4}{4x+6y+5}.$$

Solution. We note that the condition $\frac{a}{a'}=\frac{b}{b'}$ is satisfied in the present case. Hence put $2x+3y=z$ so that $2+3\frac{dy}{dx}=\frac{dz}{dx}$. Hence the given equation reduces to

$$\frac{dz}{dx}=\frac{7z+22}{2z+5}$$

or

$$\frac{2z+5}{7z+22}\,dz=dx$$

Integrating both sides, we get

$$\int\frac{2z+5}{7z+22}dz=\int dx+C$$

or

$$\frac{2}{7}z-\frac{9}{49}\log(7z+22)=x+C.$$

Substituting $z=2x+3y$, we get

$$14(2x+3y)-9\log(14x+21y+22)=49x+C$$

or

$$21x-42y+9\log(14x+21y+22)=C$$

EXAMPLE 1.24

Solve

$$(x+2y+1)\,dx=(2x+4y+3)dy.$$

Solution. We have

$$\frac{dy}{dx}=\frac{x+2y+1}{2x+4y+3}=\frac{ax+by+C}{a'x+b'y+C}.$$

We observe that $\frac{a}{a'}=\frac{b}{b'}=\frac{1}{2}$. So we put $x+2y=z$ and have $1+2\frac{dy}{dx}=\frac{dz}{dx}$. Then $\frac{dy}{dx}=\frac{dz/dx-1}{2}$. The given equation now reduces to

$$\frac{(dz/dx)-1}{2}=\frac{z+1}{2z+3}$$

or

$$\frac{dz}{dx}=\frac{2z+2}{2z+3}+1=\frac{4z+5}{2z+3}$$

or

$$\left(\frac{2z+3}{4z+5}\right)dz=dx.$$

Integrating both sides, we have

$$\int\frac{2z+3}{4z+5}\,dz=\int dx+C$$

or

$$\int\left[\frac{1}{2}+\frac{1}{2(4z+5)}\right]dz=\int dx+C$$

or

$$\frac{1}{2}z+\frac{1}{8}\log(4z+5)=x+C$$

or

$$4z+\log(4z+5)=4x+k\ \text{(constant)}$$

or

$$4(x+2y)+\log(4x+8y+5)=8x+k$$

or

$$4(2y-x)+\log(4x+8y+5)=k.$$

1.8 LINEAR DIFFERENTIAL EQUATIONS

Definition 1.14. A differential equation is said to be *linear* if the dependent variable and its derivatives occur in the first degree and are not multiplied together.

Thus a linear differential equation is of the form

$\frac{dy}{dx}+\text{Py}=\text{Q}$, where P and Q are functions of x only

or

$\frac{dx}{dy}+\text{Px}=\text{Q}$, where P and Q are funtions of y only.

A linear differential equation of the first order is called *Leibnitz's linear equation*.

To solve the linear differential equation

$$\frac{dy}{dx}+Py=Q, \tag{10}$$

we multiply both sides by $e^{\int Pdx}$ and get

$$\frac{dy}{dx}\,e^{\int Pdx}+Pye^{\int Pdx}=Q\,e^{\int Pdx} \tag{11}$$

But

$$\frac{d}{dx}\left(y\, e^{\int P dx}\right) = \frac{dy}{dx} e^{\int P dx} + P\, y e^{\int P dx}.$$

Hence (11) reduces to

$$\frac{d}{dx}\left(y\, e^{\int P dx}\right) = Q\, e^{\int P dx},$$

which an integration yields

$$y\, e^{\int P dx} = \int Q\, e^{\int P dx} dx + C,$$

as the required solution.

The factor $e^{\int P dx}$ is called an *integrating factor* (I.F.) of the differential equation.

EXAMPLE 1.25

Solve

$$(x + 2y^3)\frac{dy}{dx} = y.$$

Solution. The given differential equation can be written as

$$y\frac{dx}{dy} = x + 2y^3$$

or

$$y\frac{dx}{dy} - x = 2y^3$$

or

$$\frac{dx}{dy} - \frac{1}{y}x = 2y^2.$$

Comparing with $\frac{dx}{dy}$ + Px = Q, we have P $= -\frac{1}{y}$ and Q $= 2y^4$. The integrating factor is

$$\text{I.F.} = e^{\int P\, dy} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1}.$$

Therefore, the solution of the differential equation is

$$xy^{-1} = \int (2y^2)y^{-1}\, dy + C$$
$$= \int 2y\, dy + C = y^2 + C.$$

EXAMPLE 1.26

Solve

$$(1 + y^2)\frac{dx}{dy} = \tan^{-1} y - x.$$

Solution. The given equation can be expressed as

$$\frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2}$$

and so is Leibnitz's linear equation in x. Comparing with $\frac{dx}{dy}$ + Px = Q, we get P $= \frac{1}{1+y^2}$ and Q $= \frac{\tan^{-1} y}{1+y^2}$. Therefore

$$\text{I.F.} = e^{\int P\, dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}.$$

Hence the solution of the differential equation is

$$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1 + y^2} e^{\tan^{-1} y} dy + C$$
$$= \int t\, e^t\, dt + C, \ \tan^{-1} y = t$$
$$= t\, e^t - e^t + C \quad \text{(integration by parts)}$$
$$= (\tan^{-1} y - 1)\, e^{\tan^{-1} y} + C.$$

Hence

$$x = \tan^{-1} y - 1 + C\, e^{-\tan^{-1} y}.$$

EXAMPLE 1.27

Solve

$$\sin 2x \frac{dy}{dx} = y + \tan x.$$

Solution. We have

$$\frac{dy}{dx} - \frac{1}{\sin 2x} y = \frac{\tan x}{\sin 2x} = \frac{\sin x}{2\cos^2 x \sin x} = \frac{1}{2}\sec^2 x.$$

Thus, the given equation is linear in y. Now

$$\text{I.F.} = e^{\int -\text{cosec}\, 2x\, dx} = e^{-\frac{1}{2}\log \tan x}$$
$$= e^{\log(\tan x)^{-1/2}} = (\tan x)^{-1/2}.$$

Hence, the solution of the given differential equation is

$$y(\tan x)^{-\frac{1}{2}} = \int \frac{1}{2}\sec^2 x(\tan x)^{-1/2} dx + C$$
$$= \frac{1}{2}\int (\sec^2 x)\ (\tan x)^{-1/2} dx + C$$
$$= \frac{1}{2}\frac{(\tan x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$
$$= (\tan x)^{1/2} + C,$$

which can be expressed as

$$y = \tan x + C\sqrt{\tan x}.$$

1.9 EQUATIONS REDUCIBLE TO LINEAR DIFFERENTIAL EQUATIONS

Definition 1.15. An equation of the form

$$\frac{dy}{dx} + Py = Qy^n, \tag{12}$$

where P and Q are functions of x is called *Bernoulli's equation*.

The Bernoulli's equation can be reduced to Leibnitz's differential equation in the following way:

Divide both sides of (12) by y^n to get

$$y^{-n}\frac{dy}{dx} + P\,y^{1-n} = Q \tag{13}$$

Put $y^{1-n} = z$ to give

$$(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$$

or

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}.$$

Hence (13) reduces to

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

or

$$\frac{dz}{dx} + P(1-n)z = Q(1-n),$$

which is Leibnitz's linear equation in z and can be solved by finding the appropriate integrating factor.

EXAMPLE 1.28

Solve

$$\frac{dy}{dx} + x\sin 2y = x^2\cos^2 y.$$

Solution. Dividing throughout by $\cos^2 y$, we have

$$\sec^2 y\frac{dy}{dx} + \frac{2x\sin y\cos y}{\cos^2 y} = x^2$$

or

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3.$$

Putting $\tan y = z$, we have $\sec^2 y\,\frac{dy}{dx} = \frac{dz}{dx}$. Hence, the given equation reduces to

$$\frac{dz}{dx} + 2xz = x^3, \tag{14}$$

which is Leibnitz-equation in z and x. The integrating factor is given by

$$\text{I.F.} = e^{\int P dx} = e^{\int 2x\,dx} = e^{x^2}.$$

Hence solution of the equation (14) is

$$ze^{x^2} = \int x^3\cdot e^{x^2}dx + C = \int x(x^2e^{x^2})dx + C$$

$$= \frac{1}{2}\int 2x(x^2e^{x^2})dx + C = \frac{1}{2}\int te^t dt + C,\ x^2 = t$$

$$= \frac{1}{2}(x^2-1)e^{x^2} + C$$

Putting back the value of z, we get

$$\tan y\ e^{x^2} = \frac{1}{2}(x^2-1)\ e^{x^2} + C$$

or

$$\tan y = \frac{1}{2}(x^2-1) + C\ e^{-x^2}.$$

EXAMPLE 1.29

Solve

$$\frac{dy}{dx} + y = xy^3.$$

Solution. Dividing throughout by y^3, we get

$$y^{-3}\frac{dy}{dx} + y^{-2} = x.$$

Put $y^{-2} = z$. Then $\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$ and, therefore, the above differential equation reduces to

$$-\frac{1}{2}\frac{dz}{dx} + z = x$$

or

$$\frac{dz}{dx} - 2z = -2x,$$

which is Leibnitz's equation in z. We have

$$\text{I.F.} = e^{-\int 2\,dx} = e^{-2x}.$$

Therefore, the solution of the equation in z is

$$z\ e^{-2x} = \int -2x.e^{-2x}dx + C = \frac{1}{2}e^{-2x}(2x+1) + C$$

or

$$z = x + \frac{1}{2} + C\, e^{2x}$$

or

$$y^{-2} = x + \frac{1}{2} + C\, e^{2x}.$$

EXAMPLE 1.30

Solve

$$x\frac{dy}{dx} + y = x^3 y^6.$$

Solution. Dividing throughout by y^6, we have

$$y^{-6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2.$$

Putting $y^{-5} = z$, we have $-5y^{-6}\frac{dy}{dx} = \frac{dz}{dx}$ and so the preceeding equation transfers to

$$-\frac{1}{5}\frac{dz}{dx} + \frac{z}{x} = x^2$$

or

$$\frac{dz}{dx} - \frac{5z}{x} = -5x^2.$$

Now

$$\text{I.F.} = e^{-5\int \frac{1}{x}dx} = e^{-5\log x} = x^{-5}.$$

Therefore, the solution the above differential equation in z is

$$z.x^{-5} = \int -5x^2 x^{-5}\, dx + C$$

or

$$y^{-6}.x^{-5} = -5\int x^{-3}\, dx + C = \frac{-5}{-2}x^{-2} + C$$

or

$$1 = (10 + Cx^2)x^3\, y^5.$$

1.10 EXACT DIFFERENTIAL EQUATION

Definition 1.16. A differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is called an *exact differential equation* if there exists a function $U \equiv U(x, y)$ of x and y such that $M(x, y)dx + N(x, y)\, dy = d\, U$.

Thus, a differential eqution of the form $M(x, y)dx + N(x, y)dy = 0$ is an exact differential equation if $Mdx + Ndy$ is an exact differential.

For example, the differential equation $y^2\, dx + 2xy\, dy = 0$ is an exact equation because it is the total differential of $U(x, y) = xy^2$. In fact the coefficient of dy is $\frac{\partial F}{\partial y}(xy^2) = 2xy$.

The following theorem tells us whether a given differential equation is exact or not.

Theorem 1.3. A necessary and sufficient condition for the differential equation $Mdx + N\, dy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

where M and N are functions of x and y having continuous first order derivative at all points in the rectangular domain.

Proof: (1) *Condition is necessary*. Suppose that the differential equation $Mdx + Ndy = 0$ is exact. Then there exists a function $U(x, y)$ such that

$$M\, dx + N\, dy = dU.$$

But, in term of partial derivatives, we have

$$dU = \frac{\partial U}{\partial x}\, dx + \frac{\partial U}{\partial y}\, dy.$$

Therefore,

$$M\, dx + N\, dy = \frac{\partial U}{\partial x}\, dx + \frac{\partial U}{\partial y}\, dy.$$

Equating coefficients of dx and dy, we get

$$M = \frac{\partial U}{\partial x} \quad \text{and} \quad N = \frac{\partial U}{\partial y}.$$

Now

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y\, \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 U}{\partial x\, \partial y}.$$

Since partial derivatives of M and N are continuous, we have

$$\frac{\partial^2 U}{\partial y\, \partial x} = \frac{\partial^2 U}{\partial x\, \partial y}.$$

Hence

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

(2) *Condition is sufficient*. Suppose that M and N satisfy

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Let

$$\int M\,dx = U,$$

where y is treated as a constant while integrating M. Then

$$\frac{\partial}{\partial x}\left(\int M\,dx\right) = \frac{\partial U}{\partial x}$$

or

$$M = \frac{\partial U}{\partial x}.$$

Therefore,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y\,\partial x}$$

and so

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y\,\partial x} = \frac{\partial N}{\partial x}.$$

Also, by continuity of partial derivatives,

$$\frac{\partial^2 U}{\partial y\,\partial x} = \frac{\partial^2 U}{\partial x\,\partial y}.$$

Thus

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y\,\partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 U}{\partial x\,\partial y}.$$

Integrating both sides of $\frac{\partial N}{\partial x} = \frac{\partial^2 U}{\partial x\,\partial y}$ with respect to x, we get

$$N = \frac{\partial U}{\partial y} + f(y).$$

Thus,

$$\begin{aligned} Mdx + Ndy &= \frac{\partial U}{\partial x}\,dx + \left[\frac{\partial U}{\partial y} + f(y)\right]dy \\ &= \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}dy + f(y)dy \\ &= d\,U + f(y)dy \\ &= d[U + \int f(y)dy] \end{aligned}$$

Thus, M dx + N dy is the exact differential of U + $\int f(y)\,dy$ and, hence, the differential equation M dx + N $dy = 0$ is exact.

1.11 THE SOLUTION OF EXACT DIFFERENTIAL EQUATION

In the proof of Theorem 1.3, we note that if M dx + N $dy = 0$ is exact, then

$$M\,dx + N\,dy = d(U + \int f(y)dy).$$

Therefore,

$$d(U + \int f(y)\,dy) = 0$$

or

$$dU + f(y)dy = 0$$

Integrating, we get the required solution as

$$U + \int f(y)\,dy = C(\text{constant})$$

or

$$\int_{y\text{ constant}} M\,dx + \int f(y)\,dy = C$$

or

$$\int_{y\text{ constant}} M\,dx + \int (\text{terms of N not containing } x)dy = C.$$

EXAMPLE 1.31

Solve

$$(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy = 0$$

Solution. Comparing with M dx + N $dy = 0$, we get

$$\mathrm{M} = 2x\cos y + 3x^2y,$$
$$\mathrm{N} = x^3 - x^2\sin y - y.$$

Then

$$\frac{\partial \mathrm{M}}{\partial y} = -2x\sin y + 3x^2 = \frac{\partial \mathrm{N}}{\partial x}.$$

Hence the given equation is exact. Therefore, the solution of the equation is given by

$$\int_{y\text{ constant}} \mathrm{M}\,dx + \int (\text{terms of N not containing } x)dy = \mathrm{C}$$

or

$$\int_{y\text{ constant}} (2x\ \cos y + 3x^2y)dx + \int -ydy = \mathrm{C}$$

or

$$x^2 \cos y + x^3 y - \frac{y^2}{2} = C.$$

EXAMPLE 1.32

Solve
$(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0.$

Solution. Comparing with Mdx + Ndy = 0, we observe that

$$\mathrm{M} = 2xy + y - \tan y, \text{ and}$$
$$\mathrm{N} = x^2 - x\tan^2 y + \sec^2 y.$$

Therefore,

$$\frac{\partial \mathrm{M}}{\partial y} = 2x - \sec^2 y + 1,$$
$$\frac{\partial \mathrm{N}}{\partial x} = 2x - \tan^2 y.$$

Hence, $\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$ and so the equation is exact. Its solution is given by

$$\int_{y \text{ constant}} (2xy + y - \tan y)\ dx + \int \sec^2 y\ dy = \mathrm{C}$$

or

$$x^2 y + xy - x\tan y + \tan y = \mathrm{C}.$$

EXAMPLE 1.33

Solve

$$(2xy\cos x^2 - 2xy + 1)\, dx + (\sin x^2 - x^2)\, dy = 0.$$

Solution. Comparing with M dx + N dy = 0, we note that

$$\mathrm{M} = 2xy\cos x^2 - 2xy + 1, \mathrm{N} = \sin x^2 - x^2.$$

Then

$$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x, \frac{\partial N}{\partial y} = 2x\cos x^2 - 2x.$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and, therefore, the given equation is exact. The solution of the given equation is

$$\int_{y \text{ constant}} (2xy\cos x^2 - 2xy + 1)\ dx + \int 0\ dy = \mathrm{C}$$

or

$$y\int 2x\cos x^2 dx - 2y\int x\ dx + \int dx = 0$$

or

$$y\int \cos t\ dt - x^2 y + x = C, \quad x^2 = t$$

or

$$y\sin x^2 - x^2 y + x = \mathrm{C}.$$

EXAMPLE 1.34

Solve

$$\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$$

Solution. The given differential equation is

$(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$

Comparing with M dx + N dy = 0, we get

$$\mathrm{M} = y\cos x + \sin y + y,$$
$$\mathrm{N} = \sin x + x\cos y + x,$$

and so

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1,$$
$$\frac{\partial N}{\partial x} = \cos x + \cos y + 1.$$

Therefore, the given equation is exact and its solution is

$$\int_{y \text{ constant}} (y\cos x + \sin y + y)\ dx + \int 0\ dy = \mathrm{C}$$

or

$$y\sin x + x\sin y + xy = \mathrm{C}.$$

EXAMPLE 1.35

Solve

$$(\sec x\ \tan x\ \tan y - e^x)dx + \sec x\ \sec^2 y\, dy = 0.$$

Solution. Comparing with M dx + N dy = 0, we have

$$\mathrm{M} = \sec x \tan x \tan y - e^x,$$
$$\mathrm{N} = \sec x \sec^2 y.$$

Therefore,

$$\frac{\partial M}{\partial y} = \sec x \tan x \sec^2 y,$$
$$\frac{\partial N}{\partial x} = \sec x \tan x \sec^2 y.$$

Hence the equation is exact and its solution is given by

$$\int_{y \text{ constant}} (\sec x \tan x \tan y \; - e^x)\, dx + \int 0 \; dx = \mathrm{C}$$

or

$$\tan y \int \sec x \tan x \; dx - e^x = \mathrm{C}$$

or

$$\tan y \sec x - e^x = 0$$

EXAMPLE 1.36

Solve

$$(1 + e^{x/y})\, dx + e^{x/y}(1 - \frac{x}{y})\, dy = 0.$$

Solution. Comparing with M dx + N $dy = 0$, we get

$$\mathrm{M} = 1 + e^{x/y}, \qquad \mathrm{N} = e^{x/y}\left(1 - \frac{x}{y}\right),$$
$$\frac{\partial M}{\partial y} = e^{x/y}\left(\frac{-x}{y^2}\right) = -\frac{x}{y^2}\; e^{x/y},$$
$$\frac{\partial N}{\partial x} = e^{x/y}\left(\frac{1}{y}\right)\left(1 - \frac{x}{y}\right) + e^{x/y}\left(-\frac{1}{y}\right) = -\frac{x}{y^2} e^{x/y}.$$

Hence the given differential equation is exact and its solution is

$$\int_{y\,\text{constant}} (1 + e^{x/y}) dx + \int 0 \; dx = \mathrm{C}$$

or

$$x + y \; e^{x/y} = \mathrm{C}.$$

EXAMPLE 1.37

Show that the differential equation

$$(ax + hy + g)dx + (hx + by + f)dt = 0$$

is the differential equation of a family of conics.

Solution. Comparing the given differential equation with Mdx + Ndy = 0, we get

$$\mathrm{M} = ax + hy + g, \;\; \mathrm{N} = hx + by + f,$$
$$\frac{\partial M}{\partial y} = h, \;\; \frac{\partial N}{\partial x} = h.$$

Hence the given differential equation is exact and its solution is

$$\int_{y \text{ constant}} (ax + hy + g)dx + \int (by + f)dy = \mathrm{C}$$

or

$$a\frac{x^2}{2} + hyx + gx + b\frac{y^2}{2} + fy = \mathrm{C}$$

or

$$ax^2 + bx^2 + 2hxy + 2gx + 2fy + k = 0,$$

which represents a family of conics.

1.12 EQUATIONS REDUCIBLE TO EXACT EQUATION

Differential equations which are not exact can sometimes be made exact on multiplying by a suitable factor called an *integrating factor*.

The integrating factor for Mdx + Ndy = 0 can be found by the following rules:

1. If Mdx + Ndy = 0 is a homogeneous equation in x and y, then $\frac{1}{Mx+Ny}$ is an integrating factor, provided Mx + N$y \neq 0$.
2. If the equation Mdx + Ndy = 0 is of the form $f(xy)y\, dx + \phi(xy)x\, dy = 0$, then $\frac{1}{Mx-Ny}$ is an integrating factor, provided Mx – N$y \neq$ 0.
3. Let M dx + N dy = 0 be a differential equation. If $\frac{(\partial M/\partial y)-(\partial N/\partial x)}{N}$ is a function of x only, say $f(x)$, then $e^{\int f(x)\, dx}$ is an integrating factor.
4. Let M dx + N dy = 0. If $\frac{(\partial N/\partial x)-(\partial M/\partial y)}{M}$ is a function of y only, say $f(y)$, then $e^{\int f(y)\, dy}$ is an integrating factor.

5. For the equation
$x^a y^b (mydx + nxdy) + x^{a'} y^{b'} (m'ydx + n'xdy) = 0$
the integrating factor is $x^h\, y^k$, where h and k are such that
$$\frac{a+h+1}{m} = \frac{b+k+1}{n},$$
$$\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}.$$

EXAMPLE 1.38

Solve
$$y\,dx - x\,dy + \log x\,dx = 0.$$

Solution. The given equation is not exact. Dividing by x^2, we get
$$\frac{y}{x^2}dx - \frac{x\,dy}{x^2} + \frac{1}{x^2}\log x\ dx = 0$$
or
$$\frac{ydx - xdy}{x^2} + \frac{1}{x^2}\log x\,dx = 0$$
or
$$-d\left(\frac{y}{x}\right) + d\left(\int \frac{1}{x^2}\log x\ dx\right) = 0$$
or
$$d\left(\int \frac{1}{x^2}\log xdx - \frac{y}{x}\right) = 0$$
Thus $\frac{1}{x^2}$ is an integrating factor and on integration, we get the solution as
$$\int \frac{1}{x^2}\ \log\ x\,dx - \frac{y}{x} = C$$
On integration by parts, we have
$$-\ \frac{\log x}{x} + \int \frac{1}{x^2}\ dx = C + \frac{y}{x}.$$
or
$$-\ \frac{1}{x}\log x - \frac{1}{x} = C + \frac{y}{x}.$$

EXAMPLE 1.39

Solve
$$xdx + ydy = \frac{a^2(x\,dy - y\,dx)}{x^2+y^2}.$$

Solution. We know that
$$d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2+y^2}.$$
Therefore, the given differential equation is
$$xdx + ydy\ -\ a^2 d\left(\tan^{-1}\frac{y}{x}\right)$$
Integrating, we get
$$\frac{x^2}{2} + \frac{y^2}{2} - a^2\tan^{-1}\frac{y}{x} = C$$
or
$$x^2 + y^2 - 2a^2\tan^{-1}\frac{y}{x} = k\ (\text{constant}).$$

EXAMPLE 1.40

Solve
$$x\,dy - y\,dx + a(x^2+y^2)\,dx = 0.$$

Solution. Dividing throughout by $x^2 + y^2$, we get
$$\frac{x\ dy}{x^2+y^2} - \frac{y\ dx}{x^2+y^2} + a\ dx = 0$$
or
$$\frac{x\ dy - y\ dx}{x^2+y^2} + adx = 0$$
or
$$d\left(\tan^{-1}\frac{y}{x}\right) + adx = 0.$$
Integrating, we get
$$\tan^{-1}\frac{y}{x} + ax = C.$$

EXAMPLE 1.41

Solve
$$(x^2\,y - 2xy^2)dx = (x^3 - 3x^2y)dy = 0.$$

Solution. The given equation is homogeneous. Comparing it with M dx + N $dy = 0$, we have
$$M = x^2y - 2xy^2, N = -x^3 + 3x^2y,$$
$$\frac{\partial M}{\partial y} = x^2 - 4xy,\ \frac{\partial N}{\partial x} = -3x^2 + 6xy.$$
Therefore, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$and so the given equation is not exact. Further
$$\mathrm{M}x + \mathrm{N}y = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0.$$
Hence the integrating factor is $\frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$. Multiplying the given equation by $\frac{1}{x^2y^2}$, we have
$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \qquad (15)$$

Since

$$\frac{\partial}{\partial y}\left(\frac{1}{y}-\frac{2}{x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{y^2}-\frac{3}{y}\right) = -\frac{1}{y^2},$$

the equation (15) is exact and so its solution is

$$\int_{y \text{ constant}}\left(\frac{1}{y}-\frac{2}{x}\right) dx - \int\left(-\frac{3}{y}\right) dy = C$$

or

$$\frac{x}{y} - 2\log x + 3\log y = C.$$

EXAMPLE 1.42

Solve

$$y\,dx + 2x\,dy = 0.$$

Solution. The given equation is of type Mdx +N$dy = 0$ and is homogeneous. Further

$$M = y,\ N = 2x,$$

$$\frac{\partial M}{\partial y} = 1,\ \frac{\partial N}{\partial x} = 2,\ \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Thus the equation is not exact. But

$$Mx + N\,y = xy + 2xy = 3xy \neq 0.$$

Therefore, $\frac{1}{3xy}$ is the integrating factor. Multiplying the given equation throughout by $\frac{1}{3xy}$, we get

$$\frac{1}{3xy}y\,dx + \frac{2x}{3xy}dy = 0$$

or

$$\frac{1}{3x}dx + \frac{2}{3y}dy = 0.$$

The solution is

$$\frac{1}{3}\int\frac{1}{x}\,dx + \frac{2}{3}\int\frac{1}{y}\,dy = C$$

or

$$\frac{1}{3}\log x + \frac{2}{3}\log y = C$$

or

$$\log xy^2 = k = \log p$$

or

$$xy^2 = p \text{ (constant)}.$$

EXAMPLE 1.43

Solve

$$x^2y\,dx - (x^3 + y^3)\,dy = 0.$$

Solution. The given equation is homogeneous and comparing with Mdx + N $dy = 0$, we get

$$M = x^2y, N = -x^3 - y^3.$$

Then

$$Mx + Ny = x^3y - x^3y - y^4 = -y^4 \neq 0.$$

Thus the integrating factor is $\frac{1}{-y^4}$. Multiplying the given differential equation throughout by $-\frac{1}{y^4}$, we get

$$-\frac{1}{y^4}\,x^2ydx + \frac{1}{y^4}\,(x^3 + y^3)dy = 0$$

or

$$-\frac{x^2}{y^3}dx + \left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy = 0,$$

which is exact. Hence the required solution is

$$-\frac{1}{y^3}\int x^2\,dx + \int\frac{1}{y}\,dy = C$$

$$-\frac{x^3}{3y^3} + \log y = C.$$

EXAMPLE 1.44

Solve

$$y(xy + 2x^2\,y^2)\,dx + x(xy - x^2\,y^2)\,\mathrm{d}y = 0.$$

Solution. The given differential equation is of the form

$$f(xy)ydx + \phi(xy)xdy = 0.$$

Also comparing with M dx + Ndy = 0, we get

$$M = y(xy + 2x^2y^2), N = x(xy - x^2y^2).$$

Therefore,

$$Mx - Ny = 3x^3y^3 \neq 0.$$

Thus, $\frac{1}{3x^3y^3}$ is the integrating factor. Multiplying throughout by $\frac{1}{3x^3y^3}$, we have

$$\frac{1}{3x^3y^3}(y(xy+2x^2y^2))dx + \frac{1}{3x^3y^3}[x(xy-x^2y^2)]dy = 0.$$

or

$$\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2}-\frac{1}{3y}\right)x\,dy = 0.$$

Since

$$\frac{\partial}{\partial y}\left(\frac{1}{3x^2y}+\frac{2}{3x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{3xy^2}-\frac{1}{3y}\right),$$

the above equation is exact and its solution is

$$\int_{y\text{ constant}}\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\int\left(-\frac{1}{3y}\right)dy=\text{C}$$

or

$$-\frac{1}{3xy}+\frac{2}{3}\log x-\frac{1}{3}\log y=\text{C}$$

or

$$-\frac{1}{xy}+2\log x-\log y=k\text{ (constant).}$$

EXAMPLE 1.45

Solve

$$(1+xy)\,y\,dx+(1-xy)\,x\,dy=0.$$

Solution. The given differential equation is of the form

$$f(xy)\;ydx+\phi(xy)\;x\;dy=0.$$

Comparing with M dx + N$dy = 0$, we have

$$\text{M}=y+xy^2,\;\text{N}=x-x^2y.$$

Therefore,

$$\text{M}x-\text{N}y=2x^2y^2\neq 0.$$

Therefore, the integrating factor is $\frac{1}{2x^2y^2}$. Multiplying the given differential equation throughout by $\frac{1}{2x^2y^2}$, we get

$$\frac{1}{2x^2y^2}(y+xy^2)dx+\frac{1}{2x^2y^2}\;(x-x^2y)=0$$

or

$$\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\left(\frac{1}{2xy^2}-\frac{1}{2y}\right)dy=0.$$

We note that this equation is exact. Hence its solution is

$$\int_{y\text{ constant}}\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\int\left(-\frac{1}{2y}\right)dy=\text{C}$$

or

$$\frac{1}{2y}\left(-\frac{1}{x}\right)+\frac{1}{2}\log x-\frac{1}{2}\log y=C$$

or

$$\log\frac{x}{y}-\frac{1}{xy}=k\text{ (constant).}$$

EXAMPLE 1.46

Solve

$$(x^2y^2+xy+1)\,y\,dx+(x^2\,y^2-xy+1)\,x\,dy=0.$$

Solution. The given differential equation is of the form

$$f(xy)ydx+\phi(xy)xdx=0.$$

Moreover, comparing the given equation with M dx + N $dy = 0$, we get

$$\text{M}=x^2y^3+xy^2+y,\;N=x^3y^2-x^2y+x.$$

Therefore

$$\begin{aligned}\text{M}x-\text{N}y&=x^3y^3+x^2y^2+xy-x^3y^3+x^2y^2-xy\\&=2x^2y^2\neq 0.\end{aligned}$$

Therefore, the integrating factor is $\frac{1}{2x^2y^2}$. Multiplying the given differential equation throughout by $\frac{1}{2x^2y^2}$, we get

$$\frac{1}{2x^2y^2}\;(x^2y^3+xy^2+y)\;dx+\frac{1}{2x^2y^2}\;(x^3y^2-x^2y+x)\;dy=0$$

or

$$\left(\frac{y}{2}+\frac{1}{2x}+\frac{1}{2x^2y}\right)dx+\frac{1}{2}\left(x-\frac{1}{y}+\frac{1}{xy^2}\right)dy=0,$$

which is exact. Hence the solution of the equation is

$$\int_{y\text{ constant}}\left(\frac{y}{2}+\frac{1}{2x}+\frac{1}{2x^2y}\right)dx+\frac{1}{2}\int\left(-\frac{1}{y}\right)dy=\text{C}$$

or

$$\left(\frac{xy}{2}+\frac{1}{2}\log x-\frac{1}{2yx}\right)-\frac{1}{2}\log y=k$$

or

$$xy-\frac{1}{xy}+\log\frac{x}{y}=k\text{ (constant).}$$

EXAMPLE 1.47

Solve

$$y(2xy+1)\,dx+x(1+2xy-x^3\,y^3)dy=0.$$

Solution. The differential equation in question is of the form

$$f(xy)ydx+\phi(xy)xdy=0.$$

Further comparing the given equation with M dx + N $dy = 0$, we get

$$M = 2xy^2 + y, N = x + 2x^2y - x^4y^3.$$

Therefore,

$$Mx - Ny = 2x^2y^2 + xy - xy - 2x^2y^2 - x^4y^4$$
$$= -x^4y^4.$$

Thus the integrating factor is $\frac{1}{-x^4y^4}$. Multiplying the given differential equation throughout by $-\frac{1}{x^4y^4}$, we get

$$-\frac{1}{x^4y^4}(2xy^2 + y)dx - \frac{1}{x^4y^4}(x + 2x^2y - x^4y^3)\,dy = 0.$$

or

$$\left(-\frac{2}{x^3y^2} - \frac{1}{x^4y^3}\right)dx + \left(-\frac{1}{x^3y^4} - \frac{2}{x^2y^3} + \frac{1}{y}\right)dy = 0,$$

which is exact. Hence the solution of the equation is

$$\int_{y \text{ constant}} \left(\frac{-2}{x^3y^2} - \frac{1}{x^4y^3}\right) dx + \int \frac{1}{y}\, dy = C$$

or

$$\frac{1}{x^2y^2} + \frac{1}{3x^3y^3} + \log y = C.$$

EXAMPLE 1.48

Solve

$$(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0.$$

Solution. Comparing the given equation with Mdx + N$dy = 0$, we get

$$M = x^2 + y^2 + 2x, \; N = 2y$$

which gives

$$\frac{\partial M}{\partial y} = 2y, \; \frac{\partial N}{\partial x} = 0.$$

Thus the equation is not exact. We have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y}{2y} = 1 = x^0 \text{ (function of } x).$$

Therefore, $e^{\int 1\,dx} = e^x$ is the integrating factor. Multiplying the given differential equation throughout by e^x, we get

$$(x^2 + y^2 + 2x)e^x dx + 2ye^x dy = 0,$$

which is exact. Hence the required solution is

$$\int_{y \text{ constant}} (x^2 + y^2 + 2x)\, e^x\, dx + \int 0\, dy = C,$$

which yield

$$(x^2 + y^2)\, e^x = C.$$

EXAMPLE 1.49

Solve

$$(xy^2 - e^{x^{1/3}})dx - x^2\,y\,dy = 0.$$

Solution. Comparing the given equation with M dx + N $dy = 0$, we get

$$M = xy^2 - e^{x^{1/3}}, \quad N = -x^2y,$$
$$\frac{\partial M}{\partial y} = 2xy, \quad \frac{\partial N}{\partial x} = -2xy.$$

Therefore,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x},$$

which is a function of x only. Hence $e^{\int -\frac{4}{x}\,dx} = e^{-4\log x} = \frac{1}{x^4}$ is the integrating factor. Multiplying the equation throughout by $\frac{1}{x^4}$, we get

$$\left(\frac{y^2}{x^3} - \frac{1}{x^4}\, e^{x^{1/3}}\right) dx - \frac{y}{x^2}\, dy = 0,$$

which is exact. The required solution is, therefore,

$$\int_{y \text{ constant}} \left(\frac{y^2}{x^3} - \frac{1}{x^4}\, e^{x^{1/3}}\right) dx = C$$

or

$$-\frac{y^2}{2x^2} + \frac{1}{3}\int -\frac{3}{x^4}\, e^{x^{1/3}} dx = C$$

or

$$-\frac{y^2}{2x^2} + \frac{1}{3}\int e^t\, dt = C,\; t = x^{\frac{1}{3}}.$$

or

$$-\frac{y^2}{2x^2} + \frac{1}{3}e^t = C$$

or

$$-\frac{3y^2}{2x^2} + 2e^{x^{1/3}} = k \text{ (constant)}.$$

EXAMPLE 1.50

Solve

$$(xy^3 + y)dx + 2(x^2y^2 + x + y^4)\,dy = 0.$$

Solution. Comparing with M dx + N $dy = 0$, we get

$$M = xy^3 + y, \qquad N = 2x^2y^2 + 2x + 2y^4,$$

$$\frac{\partial M}{\partial y} = 3xy^2 + 1, \qquad \frac{\partial N}{\partial x} = 4xy^2 + 2.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact. It is also not homogeneous. It is also not of the form $f(x\,y)\,y\,dx + \phi\,(x\,y)x\,dy = 0$. We note that

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{xy^2+1}{y(xy^2+1)} = \frac{1}{y} \text{ (functionof } y \text{ alone).}$$

Hence, the integrating factor is $e^{\int \frac{1}{y}dy} = e^{\log y} = y$. Multiplying the given equation throughout by y, we get

$$y(xy^3 + y)dx + 2y(x^2y^2 + x + y^4)dy = 0$$
$$(xy^4 + y^2)dx + (2x^2y^3 + 2xy + y^5)dy = 0,$$

which is exact. Therefore, its solution is

$$\int_{y\text{ constant}} (xy^4 + y^2)dx + \int y^5 dy = C$$

or

$$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{6} = C.$$

EXAMPLE 1.51

Solve

$$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$$

Solution. The given differential equation is neither homogeneous nor is of the type $f(x\,y)\,y\,dx + \phi(x\,y)\,x\,dy = 0$. Comparing with M$dx$ + N$dy = 0$, we get

$$M = y^4 + 2y, \; N = xy^3 + 2y^4 - 4x.$$

Therefore,

$$\frac{\partial M}{\partial y} = 4y^3 + 2, \quad \frac{\partial N}{\partial x} = y^3 - 4,$$

which shows that the given equation is not exact. Further,

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-3y^3 - 6}{y^4 + 2y} = \frac{-3(y^2+2)}{y(y^3+2)} = -\frac{3}{y},$$

which is a function of y alone. Therefore, the integrating factor is $e^{\int -\frac{3}{y}dy} = e^{-3\log y} = y^{-3} = \frac{1}{y^3}$. Multiplying the given differential equation throughout by $\frac{1}{y^3}$, we get

$$\frac{1}{y^3}(y^4 + 2y)dx + \frac{1}{y^3}(xy^3 + 2y^4 - 4x)dy = 0$$

or

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0,$$

which is exact. The solution of the given differential equation is, therefore,

$$\int_{y\text{ constant}} \left(y + \frac{2}{y^2}\right)dx + \int 2y\,dy = 0$$

or

$$\left(y + \frac{2}{y^2}\right)x + y^2 = C.$$

EXAMPLE 1.52

Solve

$$y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0.$$

Solution. The given equation can be written as

$$xy(\,ydx + xdy) + x^2y^2(2ydx - xdy) = 0.$$

Comparing it with

$$x^a y^b(mydx + nxdy) + x^{a'}y^{b'}(m'y\,dx + n'x\,dy) = 0,$$

we note that $a = b = 1$, $a' = b' = 2$, $m = n = 1$ and $m' = 2$, $n' = -1$. Then the integrating factor is $x^h y^k$, where

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \quad \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'},$$

that is,

$$\frac{h+2}{1} = \frac{k+2}{1}, \quad \frac{3+h}{2} = \frac{3+k}{-1}$$

or

$$h+2=k+2, \quad -3-k=6+2k$$

or

$$h-k=0, \quad h+2k=-9.$$

Solving for h and k, we get $h=k=-3$. Hence, the integrating factor is $\frac{1}{x^3y^3}$. Multiplying throughout by $\frac{1}{x^3y^3}$, we get

$$\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\left(\frac{1}{xy^2}-\frac{1}{y}\right)\,dy\ =0,$$

which is exact. Therefore, the solution is

$$\int\limits_{y\text{ constant}}\left(\frac{1}{x^2y}+\frac{2}{x}\right)\,dx+\int\left(-\frac{1}{y}\right)\,dy=C$$

or

$$\frac{1}{y}\left(-\frac{1}{x}\right)+2\log x-\log y=C$$

or

$$-\frac{1}{xy}+2\log x-\log y=C.$$

EXAMPLE 1.53

Solve

$$(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0.$$

Solution. The given equation can be written as

$$x^0y(y\ dx-x\ dy)+x^2y^0(2y\ dx+2x\ dy)=0.$$

Comparing this with

$$x^ay^b(mydx+nxdy)+x^{a'}y^{b'}(m'ydx+n'xdy)=0,$$

we get

$$a=0,\ b=1,\ a'=2,\ b'=0,$$
$$m=1,\ n=-1,\ m'=2,\ n'=2.$$

Then the integrating factor is x^hy^k, where

$$\frac{a+h+1}{m}=\frac{b+k+1}{n},\ \frac{a'+h+1}{m'}=\frac{b'+k+1}{n'},$$

or

$$\frac{h+1}{1}=\frac{2+k}{-1},\ \frac{3+h}{2}=\frac{k+2}{2}$$

or

$$h+k=-3 \quad \text{and} \quad 2h-2k=-4.$$

Solving for h and k, we get $h=-\frac{5}{2}\ k=-\frac{1}{2}$. Hence, $\frac{1}{x^{5/2}\ y^{1/2}}$ is the integrating factor. Multiplying the given differential equation by $\frac{1}{x^{5/2}\ y^{1/2}}$, we get

$$(x^{-5/2}\ y^{3/2}+2x^{-1/2}y^{1/2})dx + (2x^{1/2}y^{1/2}+2x^{-3/2}y^{1/2})dy\ =0,$$

which is exact. Therefore, the required solution is

$$\int\limits_{y\text{ constant}}(x^{-5/2}y^{3/2}+2x^{-1/2}y^{1/2})dx+\int 0\ dy\ =0$$

or

$$-\frac{2}{3}y^{3/2}x^{-3/2}+4y^{1/2}x^{1/2}=\ C$$

or

$$-y^{3/2}x^{-3/2}+6x^{1/2}y^{1/2}\ =C$$

or

$$6\sqrt{xy}-\left(\frac{y}{x}\right)^{3/2}=C.$$

EXAMPLE 1.54

Solve

$$(3xy-2ay^2)+(x^2-2axy)\,dy=0.$$

Solution. The given equation can be written as

$$xy^0(3ydx+xdy)+yx^0(-2aydx-2axdy)=0.$$

Comparing this with

$$x^ay^b(mydx+nxdy)+x^{a'}y^{b'}(m'ydx+n'xdy)=0,$$

we get

$$a=1,\ b=0,\quad a'=0,\ b'=1,$$
$$m=3,\ n=1,\ m'=-2a,\ n'=-2a.$$

Then the integrating factor is $x^h\ y^k$, where

$$\frac{a+h+1}{m}=\frac{b+k+1}{m},\ \frac{a'+h+1}{m'}=\frac{b'+k+1}{n'}$$

or

$$\frac{2+h}{3}=\frac{k+1}{1},\ \frac{h+1}{-2a}=\frac{k+2}{-2a}$$

or

$$h-3k\ =1,\ h-k=1.$$

Thus $h=1$, $k=0$. Therefore, the integrating factor is x. Multiplying the given differential equation throughout by x, we get

$$(3x^2y - 2axy^2)dx + (x^3 - 2ax^2y)dy = 0.$$

Let

Then $M = 3x^2y - 2axy^2, \quad N = x^3 - 2ax^2y.$

$$\frac{\partial M}{\partial y} = 3x^2 - 4axy, \quad \frac{\partial N}{\partial x} = 3x^2 - 4axy.$$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the transformed equation is exact.

The required solution is

$$\int_{y \text{ constant}} (3x^2y - 2axy^2)dx + \int 0\, dy = 0$$

or

$$yx^3 - ax^2y^2 = C.$$

EXAMPLE 1.55

Solve

$$(2x^2y^2 + y)dx + (-x^3y + 3x)\,dy = 0.$$

Solution. Comparing the given differential equation with $Mdx + Ndy = 0$, we get

$$M = 2x^2y^2 + y, \quad N = -x^3y + 3x,$$

$$\frac{\partial M}{\partial y} = 4x^2y + 1, \quad \frac{\partial N}{\partial x} = 3 - 3x^2y.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the given equation is not exact. However, the given equation can be written in the form

$$x^2y(2ydx - xdy) + x^0y^0(ydx + 3xdy) = 0.$$

Comparing it with

$$x^ay^b(my\,dx + nx\,dy) + x^{a'}\,y^{b'}\,(m'ydx + n'xdy) = 0,$$

we get

$$a = 2,\ b = 1,\ a' = 0,\ b' = 0,$$
$$m = 2,\ n = -1,\ m' = 1,\ n' = 3.$$

The integrating factor is $x^h\,y^k$, where

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \quad \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

or

$$h + 2k = -7, \quad 3h - k = -2.$$

Solving these equations for h and k, we get $h = -\frac{11}{7}$ and $k = -\frac{19}{7}$. Thus the integrating factor is $x^{-11/7}$ $y^{-19/7}$. Multiplying the given equation throughout by $x^{-11/7}\,y^{-19/7}$, we get

$$(2x^{3/7}y^{-5/7} + x^{-11/7}y^{-12/7})dx$$
$$- (x^{10/7}y^{-12/7} - 3x^{-4/7}y^{-10/7})dy = 0.$$

This transformed equation is exact and its solution is

$$\int_{y \text{ constant}} (2x^{3/7}y^{-5/7} + x^{-11/7}y^{-12/7})\,dx = C$$

or

$$\frac{7}{5}\,x^{10/7}y^{-5/7} - \frac{7}{4}x^{-4/7}y^{-12/7} = C.$$

1.13 APPLICATIONS OF FIRST ORDER AND FIRST DEGREE EQUATIONS

The aim of this section is to form differential equations for physical problems like flow of current in electric circuits, Newton law of cooling, heat flow and orthogonal trajectories, and to find their solutions.

(A) Problems Related to Electric Circuits

Consider the RCL circuit shown in the Figure 1.3 and consisting of resistance, capacitor, and inductor connected to a battery.

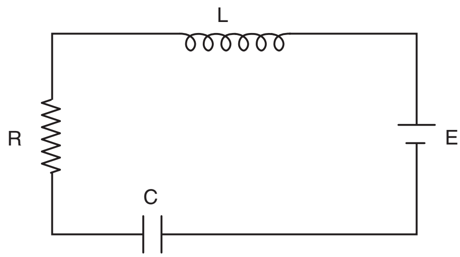


Figure 1.3

We know that the resistance is measured in ohms, capacitance in farads, and inductance in henrys. Let I denote the current flowing through the circuit and Q denote the charge. Since the current is rate of flow of charge, we have $I = \frac{dQ}{dt}$. Also, by Ohm's law, $\frac{V}{I} =$ R (resistance). Therefore, the voltage drop across a resistor R is RI. The voltage drop across the inductor L is L $\frac{dI}{dt}$ and the voltage drop across a capacitor is $\frac{Q}{C}$. If E is the voltage (e.m.f.) of the battery, then by Kirchhoff's law, we have

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t), \qquad (16)$$

where L, C, and R are constants. Since $I = \frac{dQ}{dt}$, we have $Q = \int_0^t I(u)\, du$ and so (16) reduces to

$$L\,\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(u)\; du \;= E(t).$$

The forcing function (input function), E(*t*), is supplied by the battery (voltage source). The system described by the above differential equation is known as *harmonic oscillator*.
The equation (16) can be written as

$$\frac{dI}{dt} + \frac{R}{L}I + \frac{Q}{LC} = \frac{E}{L}(t) \qquad (17)$$

EXAMPLE 1.56

Given that $I = 0$ at $t = 0$, find an expression for the current in the LR circuit shown in the Figure 1.4.

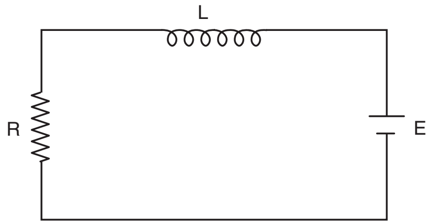


Figure 1.4

Solution. By Kirchhoff's law, we have

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}, \qquad (18)$$

which is Leibnitz's linear equation. Its integrating factor is $e^{\int \frac{R}{L}\,dt} = e^{Rt/L}$. Hence the solution of (18) is

$$I\, e^{Rt/L} = \int \frac{E}{L}\, e^{Rt/L}\, dt + C \;=\; \frac{E}{L}\int e^{Rt/L} dt + C$$
$$= \frac{E}{L}\frac{e^{Rt/L}}{R/L} + C \;=\; \frac{E}{R}\, e^{Rt/L} + C.$$

Thus

$$I \;= \frac{E}{R} + \; Ce^{-Rt/L}. \qquad (19)$$

But I(0) = 0, therefore, (19) yields $0 = \frac{E}{R} + C$ and so $C = -\frac{E}{R}$. Hence

$$I \;=\; \frac{E}{R} - \frac{E}{R}\; e^{-Rt/L} = \frac{E}{R}(1 - e^{-Rt/L}).$$

Clearly *I* increases with time *t* and attains its maximum value, E/R.

EXAMPLE 1.57

Find the time *t* when the current reaches half of its theoretical maximum in the circuit of Example 1.56.

Solution. From Example 1.56, we have

$$I = \frac{E}{R}(1 - e^{-Rt/L}).$$

The maximum current is $\frac{E}{R}$. By the requirement of the problem, we must have

$$\frac{1}{2}\frac{E}{R} \;=\; \frac{E}{R}(1 - e^{-Rt/L})$$

or

$$\frac{1}{2} = e^{-Rt/L}$$

or

$$-\frac{Rt}{L} \;= \log\frac{1}{2} \;= -\log 2$$

or

$$t \;= \frac{L}{R}\;\log 2.$$

EXAMPLE 1.58

In an LR circuit, an e.m.f. of 10 sin *t* volts is applied. If I(0) = 0, find the current in the circuit.

Solution. In an LR circuit the current is governed by the differential equation

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}.$$

We are given that E = 10 sin *t*. Therefore,

$$\frac{dI}{dt} + \frac{R}{L}I \;=\; \frac{10}{L}\;\sin t.$$

This is Leibnitz's linear equation with integrating factor as $e^{\int \frac{R}{L}\,dt} \;=\; e^{Rt/L}$. Therefore, its solution is

$$\begin{aligned} I\, e^{Rt/L} &= \int \frac{10}{L}\sin t\; e^{Rt/L} dt + C \\ &= \frac{10}{L}\int e^{Rt/L}\sin t\; dt \;+\; C \end{aligned} \qquad (20)$$

But we know (using integration by parts) that

$$\int e^{at}\;\sin\;bt\;dt \;=\; \frac{e^{at}}{a^2+b^2}\;(a\sin bt \;-\; b\cos bt)$$
$$= \frac{e^{at}}{a^2+b^2}\;\sin\left\{bt - \tan^{-1}\frac{b}{a}\right\}.$$

Therefore (20) reduces to

$$I\, e^{Rt/L} = \frac{10}{L}\left[\frac{e^{Rt/L}}{(R^2/L^2)+1}\left(\frac{R}{L}\sin t - \cos t\right)\right] + C$$

$$= \frac{10L^2}{L^2}\left[\frac{e^{Rt/L}}{R^2+L^2}(R\sin t - L\cos t)\right] + C$$

$$= \frac{10\, e^{Rt/L}}{R^2+L^2}(R\sin t - L\cos t) + C.$$

Hence

$$I = \frac{10}{R^2+L^2}\,(R\sin t - L\cos t) + Ce^{-Rt/L}.$$

Using the initial condition $I(0) = 0$, we get $C = \frac{10L}{R^2+L^2}$. Hence

$$I = \frac{10}{R^2+L^2}(R\sin t - L\cos t + Le^{-Rt/L}).$$

EXAMPLE 1.59

If voltage of a battery in an LR circuit is E_0 sin t, find the current I in the circuit under the initial condition $I(0) = 0$.

Solution. Proceeding as in Example 1.58, we get

$$I = \frac{E_0}{R^2+L^2}\, e^{-Rt/L} + \frac{E_0 L}{R^2+L^2}\cos t + \frac{E_0 R}{R^2+L^2}\sin t.$$

EXAMPLE 1.60

Find the current in the following electric circuit containing condenser C, resistance R, and a battery of e.m.f. $E_0 \sin \omega t$ with the initial condition $I(0) = 0$.

Solution. The RC circuit of the problem is shown in Figure 1.5.

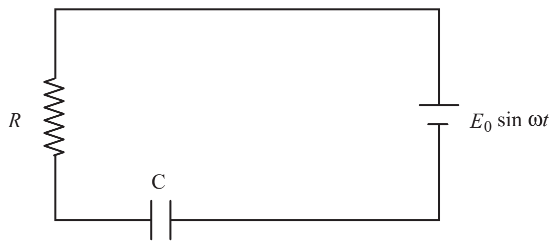


Figure 1.5

Using Kirchhoff's law, we have

$$RI + \frac{1}{C}\int_0^t I(u)\, du = E_0 \sin \omega t.$$

Differentiating both sides with respect to t, we have

$$R\,\frac{dI}{dt} + \frac{I}{C} = \omega E_0 \sin \omega t$$

or

$$\frac{dI}{dt} + \frac{I}{RC} = \frac{\omega E_0}{R}\cos \omega t,$$

which is Leibnitz's linear equation. The integrating factor is $e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$. Therefore, the solution of the above first order equation is

$$I\, e^{\frac{t}{RC}} = \int \frac{\omega E_0}{R}\cos \omega t . e^{\frac{t}{RC}} dt + C$$

$$= \frac{\omega\, E_0}{R}\int \cos\ \omega t\ e^{\frac{t}{RC}} dt + C$$

Using $\int e^{ax}\cos bx\; dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$, we have

$$Ie^{\frac{t}{RC}} = \frac{\omega E_0}{R}\left[\frac{e^{\frac{t}{RC}}}{\left(\frac{1}{RC}\right)^2+\omega^2}\left(\frac{1}{RC}\cos\omega t + \omega\sin\omega t\right)\right] + C$$

$$= \frac{\omega R^2 C^2 E_0}{R^2 C}\left[\frac{e^{\frac{t}{RC}}}{1+R^2\omega^2C^2}(\cos\omega t + RC\omega\sin\omega t)\right] + C$$

$$= \frac{\omega C E_0\, e^{\frac{t}{RC}}}{1+R^2\omega^2C^2}(\cos\omega t + RC\omega\sin\omega t) + C.$$

But $I(0) = 0$ implies $C = -\frac{\omega C E_0}{1+R^2\omega^2C^2}$. Hence

$$I = \frac{\omega C E_0}{1+R^2\omega^2C^2}\ (\cos\omega t + RC\omega\sin\omega t - e^{-t/RC}).$$

EXAMPLE 1.61

A voltage E e^{-at} is applied at $t = 0$ to a LR circuit. Find the current at any time t.

Solution. The differential equation governing the LR circuit is

$$\frac{d\,I}{dt} + \frac{R}{L}I = \frac{e.m.f.}{L} = \frac{E\ e^{-at}}{L}.$$

As in Example 1.58, the integrating factor is $e^{\frac{Rt}{L}}$. Therefore, the solution of the above equation is

$$I\ e^{\frac{Rt}{L}} = \int \frac{E\ e^{-at}}{L}\ e^{\frac{Rt}{L}} dt + C\ =\ \frac{E}{L}\int e^{\frac{Rt}{L}-at} dt + C$$
$$= \frac{E\, e^{\frac{Rt}{L}-at}}{L\,\frac{R}{L}-a} + C\ =\ \frac{E}{R-aL}\ [e^{\frac{Rt}{L}-at}]\ + C$$

and so

$$\mathrm{I} = \frac{E}{R-aL}\ e^{-at} + C\ e^{-\frac{Rt}{L}}.$$

Using the initial condition $I(0) = 0$, we get $C = -\frac{R}{R-aL}$. Hence

$$I = \frac{E}{R-aL}[e^{-at} - e^{-\frac{Rt}{L}}].$$

EXAMPLE 1.62

An RC circuit has an e.m.f. given in volt by 400 cos 2t, a resistance of 100 ohms, and a capacitance of 10^{-2} Farad. Initially there is no charge on the capacitor. Find the current in the circuit at any time t.

Solution. The equation governing the circuit is

$$\mathrm{RI} + \frac{Q}{C} = \mathrm{E}.$$

We are given that $\mathrm{R} = 100$ ohms, $\mathrm{C} = 10^{-2}$ farad and $\mathrm{E} = 400\cos 2t$. Thus, we have

$$\mathrm{I} + \mathrm{Q} = 4\cos 2t$$

or

$$\frac{dQ}{dt} + Q\ =\ 4\ \cos 2t,\ \text{since } \mathrm{I} = \frac{dQ}{dt}.$$

The integrating factor is $e^{\int 1\,dt} = e^t$. Therefore, the solution is

$$Q.e^t = 4\int \cos 2t\ e^t\ dt$$
$$= 4\left[\frac{e^t}{5}[\cos 2t + 2\sin 2t]\right] + \mathrm{C}$$
$$= \frac{4}{5}\ e^t\cos 2t + \frac{8}{5}e^t\sin 2t + C.$$

Thus

$$\mathrm{Q} = \frac{4}{5}\cos 2t + \frac{8}{5}\sin 2t + C\ e^{-t}.$$

Using the initial condition $Q(0) = 0$, we get $C = -\frac{4}{5}$. Therefore,

$$\mathrm{Q} = \frac{4}{5}\cos 2t + \frac{8}{5}\sin 2t - \frac{4}{5}e^{-t}.$$

Hence

$$\mathrm{I} = \frac{dQ}{dt} = -\frac{8}{5}\sin 2t + \frac{16}{5}\cos 2t + \frac{4}{5}\ e^{-t}.$$

(B) Problems Related to Newton's Law of Cooling

Newton's law of cooling states that *the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium*.

Let T be the temperature of the body at any time t and T_0 be the temperature of the surrounding at that particular time. Then, according to Newton's Law of Cooling,

$$\frac{dT}{dt}\ \propto\ (T - T_0)$$

and so

$$\frac{dT}{dt} = -\mathrm{k}(T - T0), \tag{21}$$

the negative sign with the constant of proportionality is required to make $\frac{dT}{dt}$negative in cooling process when T is greater than T_0, and positive in a heating process when T is less than T_0.

Equation (21) is first order differential equation and can be solved for T.

EXAMPLE 1.63

A body at a temperature of 50°F is placed outdoors, where the temperature is 100°F. If after 5 minutes, the temperature of the body is 60°F, find the time required by the body to reach a temperature of 75°F.

Solution. By Newton's law of cooling, we have

$$\frac{dT}{dt}\ =\ -k(T - T_0)$$

or

$$\frac{dT}{dt} + k\ \mathrm{T} = k\ \mathrm{T}_0.$$

But $T_0 = 100$ F. Therefore,

$$\frac{dT}{dt} + kT = 100k,$$

which is linear. The integrating factor is $e^{\int k\,dt} = e^{kt}$. Hence the solution is

$$T.e^{kt} = 100\int k\, e^{kt}dt + C = 100e^{kt} + \int C$$

or

$$T = C\, e^{-kt} + 100.$$

When $t = 0$, $T = 50$, therefore, C = –50. Hence

$$T = -50e^{-kt} + 100.$$

Now it is given that $T = 60$ at $t = 5$. Hence

$$60 = -50e^{-5k} + 100 \text{ or } e^{-5k} = \frac{4}{5}.$$

Taking log, we get

$$k = -\frac{1}{5}\log\frac{4}{5} = -\frac{1}{5}\ (-0.223) = 0.045.$$

Hence $T = -50e^{0.045t} + 100$. When $T = 75$, we get $e^{-0.045\,t} = \frac{1}{2}$, which yields $-0.045t = \log\frac{1}{2}$ and so $t = 15.4$ minutes.

EXAMPLE 1.64

A body originally at 80° C cools down to 60° C in 20 minutes, the temperature of the air being 40° C. Find the temperature of the body after 40 minutes from the original.

Solution. By Newton's law of cooling, we have

$$\frac{dT}{dt} = -k(T - T_0)$$

and so variable separation gives

$$\frac{dT}{T - T_0} = -kdt.$$

Integrating, we have

$$\log(T - T_0) = -kt + \log C$$

or

$$T - T_0 = C\, e^{-kt} \quad \text{or} \quad T - 40 = C\, e^{-kt}.$$

But when $t = 0$, T = 80°C. Therefore, C = 80 – 40 = 40 and we have

$$T - 40 = 40e^{-kt}.$$

Now when $t = 20$, T = 60°. Therefore

$$20 = 40\, e^{-20\,k} \text{ or } e^{-20\,k} = \frac{1}{2}$$

or

$$-20\,k = \log\frac{1}{2},\ \text{which yields } k = \frac{1}{20}\log 2.$$

Hence

$$T - 40 = 40e^{-\left(\frac{1}{20}\log 2\right)t}.$$

When $t = 40$, we have

$$T - 40 = 40e^{-2\log 2}$$

or

$$T = 40 + 40\, e^{\log\frac{1}{4}} = 40 + \frac{40}{4} = 50°\text{C}.$$

(C) Problems Relating to Heat Flow

The fundamental principles of heat conduction are:

(i) Heat always flow from a higher temperature to a lower temperature.

(ii) The quantity of heat in a body is proportional to its mass and temperature ($Q = mst$), where m is the mass, s is the specific heat, and t is the time.

(iii) The rate of heat flow across an area is proportional to the area and to the rate of change of temperature with respect to its distance normal to the area.

Let Q be the quantity of heat flow per second across a slab of area A and thickness δx and whose faces are kept at temperature T and T + δT. Then, by the above principles

$$Q \propto A\frac{dT}{dx}.$$

or

$$Q = -k\, A\frac{dT}{dx} \tag{22}$$

where k is a constant, called the *coefficient of thermal conductivity* and depends upon the material of the body. Negative sign has been taken since T decreases as x increases. The relation (22) is called the *Fourier's law of conductivity*.

EXAMPLE 1.65

The inner and outer surfaces of a spherical shell are maintained at temperature T_0 and T_1, respectively. If the inner and outer radii of the shell are r_0 and r_1, respectively and k is the thermal conductivity, find the amount of heat lost from the shell per unit time. Also find the temperature distribution through the shell.

Solution. We have

$$Q = -k\,(4\pi x^2)\frac{dT}{dx},$$

where x is the radius. Thus

$$dT = -\frac{Q}{4\pi k}\cdot\frac{dx}{x^2}.$$

Integrating, we get

$$T = -\frac{Q}{4\pi k}\left(-\frac{1}{x}\right) + C = \frac{Q}{4\pi xk} + C \qquad (23)$$

Now $T = T_0$ when $x = r_0$ and $T = T_1$ when $x = r_1$. Therefore, we have

$$T_0 = \frac{Q}{4\pi r_0 k} + C \text{ and } T_1 = \frac{Q}{4\pi r_1 k} + C.$$

Subtracting, we get

$$T_0 - T_1 = \frac{Q}{4\pi k}\left(\frac{1}{r_0} - \frac{1}{r_1}\right) = \frac{Q}{4\pi k}\left(\frac{r_1 - r_0}{r_0\, r_1}\right)$$

or

$$Q = \frac{4\pi k\, r_0\, r_1 (T_0 - T_1)}{r_1 - r_0}.$$

When $x = r_0$, $T = T_0$, then (23) gives

$$\begin{aligned} T_0 &= \frac{Q}{4\pi r_0 k} + C \\ &= \frac{4\pi k r_0 r_1 (T_0 - T_1)}{4\pi\, r_0 k (r_1 - r_0)} + C \\ &= \frac{r_1 (T_0 - T_1)}{r_1 - r_0} + C. \end{aligned}$$

Thus

$$\begin{aligned} C &= T_0 - \frac{r_1\,(T_0 - T_1)}{r_1 - r_0} \\ &= \frac{T_0(r_1 - r_0) - r_1(T_0 - T_1)}{r_1 - r_0} = \frac{r_1 T_1 - r_0 T_0}{r_1 - r_0} \end{aligned}$$

Hence (23) transforms to

$$\begin{aligned} T &= \frac{4\pi k r_0 r_1 (T_0 - T_1)}{4\pi xk(r_1 - r_0)} + \frac{r_1 T_1 - r_0 T_0}{r_1 - r_0} \\ &= \frac{1}{r_1 - r_0}\left[\frac{(T_0 - T_1) r_0 r_1}{x} + r_1 T_1 - r_0 T_0\right]. \end{aligned}$$

EXAMPLE 1.66

A spherical shell of inner and outer radii 10 cm and 15 cm, respectively, contains steam at 150°C. If the temperature of the outer surface of the shell is 40°C and thermal conductivity $k = 0.0025$, find the temperature half-way through the thickness of the shell under steady state conditions.

Solution. With the notation of Example 1.65, we have

$$r_0 = 10\text{cm}, r_1 = 15\text{cm}, \; x = 12.5\text{cm},$$
$$T_0 = 150°\text{C}, \; T_1 = 40°\text{C}.$$

Hence

$$\begin{aligned} T &= \frac{1}{r_1 - r_0}\left[\frac{(T_0 - T_1) r_0\, r_1}{x} + r_1 T_1 - r_0\, T_0\right] \\ &= \frac{1}{5}\left[\frac{(150 - 40)150}{12.5} + 600 - 1500\right] \\ &= \frac{1}{5}\left[\frac{16500}{12.5} - 900\right] = \frac{1}{5}\,[1320 - 900] = 84°\text{C}. \end{aligned}$$

EXAMPLE 1.67

A long hollow pipe has a inner radius of r_0 cm and outer radius of r_1 cm. The inner surface is kept at a temperature T_0 and the outer surface at the temperature T_1. If thermal conductivity is k, find the heat lost per second of 1 cm length of the pipe. Also find the temperature distribution through the thickness of the pipe.

Solution. Let Q cal/sec be the constant quantity of heat flowing out radially through the surface of the pipe having radius x cm and length 1 cm. Then the area of the lateral surface is $2\pi x$. Therefore, by

Fourier law,

$$Q = -kA\frac{dT}{dx} = -k(2\pi x)\frac{dT}{dx}$$

and so

$$dT = -\frac{Q}{2\pi k}.\frac{dx}{x}.$$

Integrating, we get

$$T = -\frac{Q}{2\pi k}\log x + C. \tag{24}$$

When $x = r_0$, $T = T_0$, and so

$$T_0 = -\frac{Q}{2\pi k}\log r_0 + C \tag{25}$$

When $x = r_1$, $T = T_1$ and we have

$$T_1 = -\frac{Q}{2\pi k}\log r_1 + C. \tag{26}$$

Subtracting (26) from (25), we have

$$T_0 - T_1 = \frac{Q}{2\pi k}[\log r_0 - \log r_1] = \frac{Q}{2\pi k}\log\frac{r_1}{r_0}.$$

Thus,

$$\frac{Q = 2\pi k(T_0 - T_1)}{\log\frac{r_1}{r_0},} \tag{27}$$

which gives the heat lost per second in 1 cm of the pipe. Further subtracting (25) from (24) and using (27), we get

$$\begin{aligned} T - T_0 &= -\frac{Q}{2\pi k}[\log x - \log r_0] = -\frac{Q}{2\pi k}\log\frac{x}{r_0} \\ &= -\frac{2\pi k(T_0 - T_1)}{2\pi k\log\frac{r_1}{r_0}}\log\frac{x}{r_0} \\ &= \frac{(T_1 - T_0)}{\log\frac{r_1}{r_0}}\log\frac{x}{r_0}. \end{aligned}$$

Hence,

$$T = T_0 + \frac{(T_1 - T_0)}{\log\frac{r_1}{r_0}}\log\frac{x}{r_0} \tag{28}$$

which gives the required temperature distribution through the thickness of the pipe.

EXAMPLE 1.68

A pipe 20 cm in diameter contains steam at 150°C and is protected with a covering 5 cm thick. If thermal conductivity is 0.0025 and the temperature of the outer surface of the covering is 40°C, find the temperature half-way through the covering under steady-state conditions.

Solution. Using the notation of Example 1.67, we have

$$r_0 = 10cm, \quad r_1 = 15\ cm, \quad x = 12.5,$$
$$T_0 = 150°C, \quad T_1 = 40°C, \quad k = 0.0025.$$

Hence using (28), we have

$$\begin{aligned} T &= 150 + \frac{40 - 150}{\log\frac{15}{10}}\log\frac{12.5}{10} \\ &= 150 - 110\log\frac{1.25}{1.5} = 89.5°C. \end{aligned}$$

EXAMPLE 1.69

A long hollow pipe has an inner diameter of 10 cm and outer diameter of 20 cm. The inner surface is kept at 200°C and the outer surface at 50°C. The thermal conductivity is 0.12. How much heat will be lost per second from a portion of 1 cm of the pipe and what is the temeprature at a distance of 7.5 cm from the centre of the pipe?

Solution. From Example 1.67, we have

$$\begin{aligned} Q &= \frac{2\pi k(T_0 - T_1)}{\log\frac{r_1}{r_0}} \\ &= \frac{2\pi k(200 - 50)}{\log\frac{10}{5}} = \frac{300\pi k}{\log 2} \\ &= \frac{300\pi(0.12)}{\log 2} = 163\text{cal/sec}. \end{aligned}$$

Also

$$\begin{aligned} T &= T_0 + \frac{(T_1 - T_0)}{\log\frac{r_1}{r_0}}\log\frac{x}{r_0} \\ &= 200 + \frac{(50 - 200)}{\log 2}\log\frac{7.5}{5} \\ &= 200 - 150\log\frac{1.5}{2} = 200 - 150(.58) = 113°C. \end{aligned}$$

(D) Rate Problems

In some problems, the rate at which a quantity changes is a known function of the amount present and/or the time and it is desired to find the quantity itself. Radioactive nuclei decay, population growth, and chemical reactions are some of the phenomenon of this kind.

Let x be the amount of radioactive nuclei present after t years. Then $\frac{dx}{dt}$ represents the rate of decay. Since the nuclei decay at a rate proportional to the amount present, we have

$$\frac{dx}{dt} = kx, \qquad (29)$$

where k is constant of proportionality.

The law of chemical reaction states that the *rate of change of chemical reaction is proportional to the amount of substance present at that instant*. Thus, the differential equation (29) governs the chemical reaction of first order.

Moreover, if the rate at which amount of a substance increases or decreases is found to be jointly proportional to two factors, each factor being a linear function of x, then the chemical reaction is said to be of *second order*. For example, if a solution contains two substances whose amounts at the beginning are a and b respectively and if equal amount x of each substance changes in time t, then the amounts of the substance left in the solution at time t are $(a-x)$ and $(b-x)$ and, therefore, we have

$$\frac{dx}{dt} = k(a-x)(b-x).$$

Taking the case of population growth, we assume that the population is a continuous and differentiable function of time. Let x be the number of individuals in a population at time t. Then *rate of change of population is proporitional to the number of individuals in it at any time*. Thus equation (29) is valid for population growth also.

EXAMPLE 1.70

If 10% of 50 mg of a radioactive material decays in 2 hours, find the mass of the material left at any time t and the time at which the material has decayed to one-half of its initial mass.

Solution. Let x denote the amount of material present at time t. Then the equation governing the decay is

$$\frac{dx}{dt} = kx.$$

Variable separation gives

$$\frac{dx}{x} = kdt.$$

Integrating, we get

$$\log x = kt + \log C \quad \text{or} \quad x = Ce^{kt}.$$

At $t = 0$, $x = 50$. Therefore, $C = 50$ and so $x = 50\ e^{kt}$. At $t = 2$, 10% of the mass present is decayed. Thus 5 mg of the substance has been decayed and 45 gm still remains. Therefore, $45 = 50\ e^{2k}$. Thus

$$k = \frac{1}{2}\log\frac{45}{50} = -0.053.$$

Hence mass of the material left at any time t is

$$x = 50^\circ e^{-0.053\,t}.$$

Further, when half of the material is decayed, we have $x = 25$ mg and so

$$25 = 50^\circ e^{-0.053\,t}$$

or

$$-0.053t = \log\frac{1}{2}$$

or

$$t = 13 \text{ hours.}$$

EXAMPLE 1.71

A tank contains 1000 litres of fresh water. Salt water which contains 150 gm of salt per litre runs into it at the rate of 5 litres/min and well stirred mixture runs out of it at the same rate. When will the tank contain 5000 gm of salt?

Solution. Let x denote the amount of salt in the tank at time t. Then

$$\frac{dx}{dt} = \text{IN} - \text{OUT}.$$

The brine flows at the rate of 5 litres/min and each litre contains 150 gm of salt. Thus

$$\text{IN} = 5 \times 150 = 750 \text{ gm/min}$$

Since the rate of outflow equals the rate of inflow, the tank contains 1000 litres of mixture at any time t. This 1000 litres contains x gm of salt at time t and so the concentration of the salt at time t is $\frac{x}{1000}$ gm/litres. Since mixture flows out at the rate of 5 litres/min, we have

$$\text{OUT} = \frac{x}{1000} \times 5 = \frac{x}{200} \text{ gm/litres.}$$

Thus the differential equation for x becomes

$$\frac{dx}{dt} = 750 - \frac{x}{200}. \qquad (30)$$

Since initially there was no salt in the tank, we have the initial condition $x(0) = 0$. The equation (30) is linear and separable. We have in fact

$$\frac{dx}{150000 - x} = \frac{dt}{200}.$$

Integrating, we get

$$-\log(150000 - x) = \frac{t}{200} + C. \qquad (31)$$

Using the initial condition $x(0) = 0$, we have

$$C = -\log 150000.$$

Hence (31) yields

$$\frac{t}{200} = \log 150000 - \log(150000 - x)$$

or

$$t = 200 \log \frac{150000}{150000 - x}.$$

If $x = 5000$ gm, then

$$t = 200 \ \log \frac{150000}{145000} = 200 \ \log \frac{30}{29} = 6.77\text{min}.$$

EXAMPLE 1.72

If the population of a city gets doubled in 2 years and after 3 years the population is 15,000, find the initial population of the city.

Solution. Let x denote the population at any time t and let x_0 be the initial population of the city. Then

$$\frac{dx}{dt} = kx,$$

which has the solution as

$$x = C\, e^{kt}.$$

At $t = 0$, $x = x_0$. Hence $C = x_0$. Thus

$$x = x_0 e^{kt} \qquad (32)$$

But at $t = 2$, $x = 2x_0$. Therefore

$$2x_0 = x_0 e^{2k} \text{ or } e^{2k} = 2$$

or

$$k = \frac{1}{2} \log 2 = 0.347$$

Hence (32) reduces to

$$x = x_0\, e^{0.347\,t}.$$

At $t = 3$, $x = 15{,}000$ and so

$$15,000 = x_0\, e^{(0.347)\,(3)} = x_0(2.832)$$

Hence

$$x_0 = \frac{15000}{2.832} = 5297.$$

(E) Falling Body Problems

Consider a body of mass m falling under the influence of gravity g and an air resistance, which is proportional to the velocity of the falling body. Newton's second law of motion states that the *net force acting on a body is equal to the time rate of change of the momentum of the body*.
Thus

$$F = \frac{d}{dt}(mv).$$

If m is assumed to be constant, then

$$F = m\frac{dv}{dt}, \qquad (33)$$

where F is the net force on the body and v is the velocity of the body at time t.

The falling body is under the action of two forces: (i) Force due to gravity which is given by the weight mg of the body (ii) the force due to resistance of air and that is $-kv$, where $k \geq 0$ is a constant of proportionality. Thus (33) yields

$$mg - kv = m\frac{dv}{dt}$$

or

$$\frac{dv}{dt} + \frac{k}{m}v = g, \qquad (34)$$

which is *equation of motion for the falling body*.

If air resistance is negligible, then $k = 0$ and we have

$$\frac{dv}{dt} = g. \qquad (35)$$

The differential equation (35) is separable and we have

$$dv = g\,dt.$$

Integrating, we get

$$v = gt + C.$$

But when $t = 0$, $v = 0$ and so C = 0. Hence

$$v = gt. \tag{36}$$

Also velocity is time rate of change of displacement x and so

$$\frac{dx}{dt} = gt \text{ or } dx = gt\,dt.$$

Integrating, we get

$$x = \frac{1}{2}gt^2 + k \text{ (constant)}.$$

But at $t = 0$, the displacement is 0. Therefore $k = 0$. Hence

$$x = \frac{1}{2}gt^2. \tag{37}$$

EXAMPLE 1.73

A body of mass 16 kg is dropped from a height of 625 ft. Assuming that there is no air resistance, find the time required by the body to reach the ground.

Solution. By (37), we have (with $g = 32$ ft/sec^2)

$$x = \frac{1}{2}gt^2 = 16t^2,$$

therefore,

$$t^2 = \sqrt{\frac{x}{16}} = \sqrt{\frac{625}{16}} = \frac{25}{4} = 6.25 \text{ sec.}$$

(F) Orthogonal Trajectories

Recall that a curve which cuts every member of a given family of curves according to some definite law is called a *trajectory* of the family.

A curve which cuts every member of a given family of curves at right angles is called *orthogonal trajectories*. Further, two families of curves are said to be orthogonal if every member of either family cuts each member of the other family at right angles.

Consider a one-parameter family of curves in the xy-plane defined by

$$f(x, y, c) = 0, \tag{38}$$

where c denotes the parameter. Differentiating (38), with respect to x and eliminating c between (38) and the resulting equation, we get the differential equation of the family in question. Let the differential equation be

$$F\left(x,\ y,\ \frac{dy}{dx}\right) = 0. \tag{39}$$

To obtain the equation of the orthogonal trajectory, we replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and get the differential equation of orthogonal trajectory as $F\left(x, y, -\frac{dx}{dy}\right)$. Solution of this differential equation will yield the equation of the orthogonal trajectory.

In case of polar curves

$$f(r, \theta, c) = 0. \tag{40}$$

Differentiating (40) and eliminating c between (40) and the resulting equations, we get the differential equation of the family represented by (40). Let the differential equation be

$$F\,(r\,, \theta\,, \frac{dr}{d\theta}) = 0. \tag{41}$$

Replacing $\frac{dr}{d\theta}$ *by* $-\ r^2\,\frac{d\theta}{dr}$ in (41), we get the differential equation of the orthogonal trajectory as

$$F\,(r\,, \theta - r^2\,\frac{d\theta}{dr}) = 0. \tag{42}$$

Solution of (42) will then yield the equation of the required orthogonal trajectory.

EXAMPLE 1.74

Find the orthogonal trajectories of the family of curves $x^2 + y^2 = cx$.

Solution. We have

$$x^2 + y^2 - cx = 0. \tag{43}$$

Differentiating, we get

$$2x + 2y\frac{dy}{dx} = c. \tag{44}$$

Eliminating c between (43) and (44) yields

$$2x + 2y\frac{dy}{dx} = \frac{x^2 + y^2}{x}$$

or

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}. \tag{45}$$

The equation (45) is the differential equation of the family represented by (43). Therefore, the differential equation of the orthogonal trajectory is

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}. \tag{46}$$

This is an homogeneous equation. Substituting $y = vx$, and separating variables, we get

$$\frac{dx}{x} + \left(-\frac{1}{v} + \frac{2v}{v^2 + 1}\right)dv = 0.$$

Integrating, we get

$$\log x - \log v + \log(v^2 + 1) = C$$

or

$$x(v^2 + 1) = kv, \; (C = \log k).$$

Substituting $v = \frac{y}{x}$, we get

$$x^2 + y^2 = ky.$$

EXAMPLE 1.75

Find the orthogonal trajectory of the family of the curves $xy = C$.

Solution. The equation of the given family of curves is

$$xy = C. \tag{47}$$

Differentiating, we get

$$x\frac{dy}{dx} + y = 0$$

or

$$\frac{dy}{dx} = -\frac{y}{x}. \tag{48}$$

Therefore, the differential equation of the family of orthogonal trajectory is

$$\frac{dy}{dx} = \frac{x}{y} \tag{49}$$

or

$$x\,dx - y\,dy = 0.$$

Integrating, we get

$$x^2 - y^2 = k,$$

which is the equation of orthogonal trajectories called *equipotential lines* (shown in Figure 1.6).

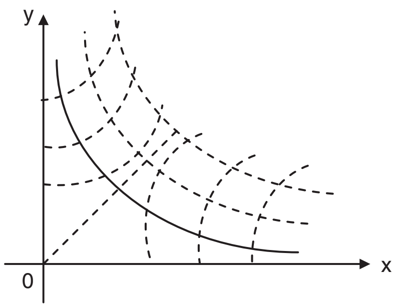


Figure 1.6

EXAMPLE 1.76

Find the orthogonal trajectories of the family of curves $y = ax^2$.

Solution. The given family represented by the equation

$$y = ax^2 \tag{50}$$

is a family of parabolas symmetric about y-axis with vertices at (0, 0). Differentiating with respect to x, we get

$$\frac{dy}{dx} = 2ax. \tag{51}$$

Eliminating a between (50) and (51), we get

$$\frac{dy}{dx} = \frac{2xy}{x^2} = \frac{2y}{x}.$$

Therefore, differential equation of the orthogonal trajectory is

$$\frac{dy}{dx} = -\frac{x}{2y} \tag{52}$$

or

$$2y\,dy + x\,dx = 0.$$

Integrating, we get

$$2\frac{y^2}{2} + \frac{x^2}{2} = C$$

or

$$\frac{x^2}{2} + \frac{y^2}{1} = C. \tag{53}$$

The orthogonal trajectories represented by (53) are ellipses (shown in the Figure 1.7)

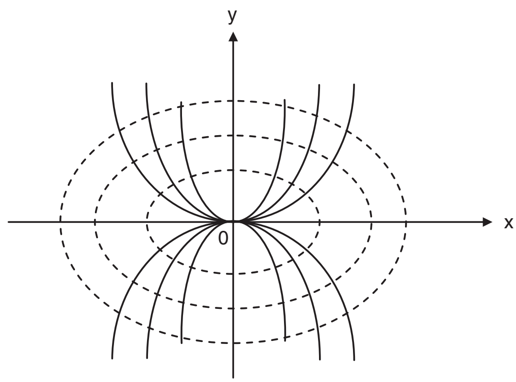


Figure 1.7

EXAMPLE 1.77

Show that the system of confocal and coaxial parabolas is self-orthogonal.

Solution. The equation of the family of confocal parabolas having x-axis as their axis is of the form

$$y^2 = 4a(x+a). \tag{54}$$

Differentiating, we get

$$y\frac{dy}{dx} = 2a$$

or

$$\frac{dy}{dx} = \frac{2a}{y}. \tag{55}$$

From (55), we have $a = \frac{y\,dy}{2\,dx}$. Substituting this value in (54), we get

$$y^2 = 2y\frac{dy}{dx} + \left(x + \frac{1}{2}y\frac{dy}{dx}\right)$$

or

$$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0, \tag{56}$$

which is the differential equation of the given family. Replacing $\frac{dy}{dx}$ *by* $-\frac{dx}{dy}$ in (56) we obtain (56) again. Hence, each member of family (54) cuts every other member of the same family orthogonally.

EXAMPLE 1.78

Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$.

Solution. The equation of the family of the given curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1. \tag{57}$$

Differentiating with respect to x, we get

$$\frac{x}{a^2} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0. \tag{58}$$

From (57) and (58) we have respectively

$$b^2+\lambda = \frac{a^2y^2}{a^2-x^2} \quad \text{and} \quad b^2+\lambda = -\frac{a^2y}{x}\frac{dy}{dx}.$$

Hence

$$\frac{a^2y^2}{a^2-x^2} = -\frac{a^2y}{x}\frac{dy}{dx}$$

or

$$\frac{dy}{dx} = \frac{-xy}{a^2-x^2}.$$

Therefore, differential equation of the orthogonal trajectory is

$$\frac{dy}{dx} = \frac{a^2-x^2}{xy}$$

or

$$y\,dy = \frac{a^2-x^2}{x}dx = \frac{a^2}{x}dx - x\,dx.$$

Integrating, we get

$$\frac{y^2}{2} = a^2\log x - \frac{x^2}{2} + C$$

or

$$x^2 + y^2 = 2a^2\log x + k(\text{constant}),$$

which is the equation of the required orthogonal trajectories.

EXAMPLE 1.79

Find the orthogonal trajectory of the cardioid $r = a(1 - \cos\theta)$.

Solution. The equation of the family of given cardioide is

$$r = a(1-\cos\theta). \tag{59}$$

Differentiating with respect to θ, we have

$$\frac{dr}{d\theta} = a\sin\theta. \tag{60}$$

Dividing (60) by (59), we get the differential equation of the given family as

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = \cot\frac{\theta}{2} \tag{61}$$

Replacing $\frac{dr}{d\theta}$ *by* $-r^2\frac{d\theta}{dr}$ in (61), we get

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot\frac{\theta}{2}$$

or

$$\frac{dr}{r} + \tan\frac{\theta}{2}\,d\theta = 0, \tag{62}$$

which is the equation of the family of orthogonal trajectories. Integrating (62), we get

$$\log r - 2\log\cos\frac{\theta}{2} = \log C$$

or

$$\log r = \log C + \log\cos^2\frac{\theta}{2} = \log C\cos^2\frac{\theta}{2}$$

or

$$r = C\cos^2\frac{\theta}{2} = C(1 + \cos\theta),$$

which is the equation of orthogonal trajectory of the given family.

EXAMPLE 1.80

Find the orthogonal trajectories of the family of curves $x^2 + y^2 = c^2$

Solution. The equation of the given family of curves is

$$x^2 + y^2 - c^2 = 0 \tag{63}$$

Differentiating (63), we get

$$2x + 2y\frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = -\frac{x}{y}, \tag{64}$$

which is the differential equation representing the given curves. Therefore the differential equation of the required family of orthogonal trajectories is

$$\frac{dy}{dx} = \frac{y}{x}$$

or

$$\frac{dy}{y} = \frac{dx}{x} \tag{65}$$

Integrating, we get

$$\log y = \log x + \log k$$

or

$$y = kx,$$

which is the equation of the orthogonal trajectories, which are straight lines through the origin as shown in the Figure 1.8.

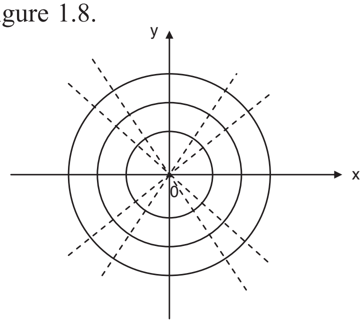


Figure 1.8

EXAMPLE 1.81

Find the orthogonal trajectories of the curves $r^2 = a^2\cos 2\theta$.

Solution. We are given that

$$r^2 = a^2\cos 2\theta = a^2(1 - 2\sin^2\theta). \tag{66}$$

Differentiating (66) w.r.t. θ, we have

$$2r\frac{dr}{d\theta} = -2a^2\sin 2\theta. \tag{67}$$

Dividing (67) by (66), we get

$$\frac{2}{r}\frac{dr}{d\theta} = \frac{-2\sin 2\theta}{\cos 2\theta}.$$

Replacing $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$, we get

$$-2r\frac{d\theta}{dr} = -\frac{\sin 2\theta}{\cos 2\theta} = -2\tan 2\theta$$

or

$$\frac{dr}{r} = \cot 2\theta.$$

Integrating, we get

$$\log r = \frac{1}{2}\log\sin 2\theta + \log C$$

or

$$2\log r = 2\log C + \log\sin 2\theta$$

or

$$\log r^2 = \log C^2 + \log\sin 2\theta$$

or

$$r^2 = C^2\sin 2\theta.$$

1.14 LINEAR DIFFERENTIAL EQUATIONS

Definition 1.17. A differential equation in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together is called a *linear differential equation*.

Thus, a linear differential equation of n^{th} order is of the form

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = F(x) \tag{68}$$

where $a_0, a_1, \ldots, a_n$ and F(x) are functions of x alone.

If $a_0, a_1, \ldots, a_n$ are constants, then the above equation is called a *linear differential equation with constant* coefficients.

If F is identically zero, then the equation (68) reduces to

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = 0 \tag{69}$$

and is called a *homogeneous linear differential equation* of order n.

Definition 1.18. If $f_1, f_2, \ldots, f_n$ are n given functions and $c_1, c_2, \ldots, c_n$ are n constants, then the expression

$$c_1 f_1 + c_2 f_2 + \ldots + c_n f_n$$

is called a linear combination of $f_1, f_2, \ldots, f_n$.

Definition 1.19. The set of functions $\{f_1, f_2, \ldots, f_n\}$ is said to be linearly independent on $[a, b]$ if the relation

$$c_1 f_1 + c_2 f_2 + \ldots + c_n f_n = 0$$

for all $x \in [a, b]$ implies that

$$c_1 = c_2 = \ldots = c_n = 0.$$

Definition 1.20. The symbol D $= \frac{d}{dx}$ is called a differential operator. Similarly, $D^2 = \frac{d^2}{dx^2}, D^3 = \frac{d^3}{dx^3}, \ldots, D^n = \frac{d^n}{dx^n}$ are also regarded as operators. In terms of these symbols, the equation (68) takes the form

$$(a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n) y = F(x)$$

or

$$f(D) y = F(x),$$

where

$$f(D) = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n.$$

We note that

$$\begin{aligned} D(u+v) &= Du + Dv \\ D(\lambda u) &= \lambda D(u) \\ (D^m D^n)(u) &= D^{m+n}(u) \\ (D^m D^n)(u) &= (D^n D^m)(u). \end{aligned}$$

Theorem 1.4. Any linear combination of linearly independent solutions of the homogeneous linear differential equation is also a solution (in fact, complete solution) of that equation.

Proof: Let $y_1, y_2, \ldots, y_n$ be the solution of the homogeneous linear differential equation

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n) y = 0 \quad (70)$$

Therefore,

$$\left.\begin{aligned} &D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \ldots + a_n y_1 = 0 \\ &D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_2 + \ldots + a_n y_2 = 0 \\ &\ldots\ldots\ldots\ldots \\ &D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + \ldots + a_n y_n = 0 \end{aligned}\right\} \quad (71)$$

Let

$$u = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

Then

$$\begin{aligned} &D^n u + a_1 D^{n-1} u + a_2 D^{n-2} u + \ldots + a_n u \\ &= D^n (c_1 y_1 + c_2 y_2 + \ldots + c_n y_n) \\ &\quad + a_1 D^{n-1} (c_1 y_1 + c_2 y_2 + \ldots + c_n y_n) \\ &\quad + a_2 D^{n-2} (c_1 y_1 + c_2 y_2 + \ldots + c_n y_n) \\ &\quad + \ldots + a_n (c_1 y_1 + c_2 y_2 + \ldots + c_n y_n) \\ &= c_1 (D^n y_1 + a_1 D^{n-1} y_1 + \ldots + a_n y_1) \\ &\quad + c_2 (D^n y_2 + a_1 D^{n-1} y_2 + \ldots + a_n y_2) \\ &\quad + \ldots + c_n (D^n y_n + a_1 D^{n-1} y_n + \ldots + a_n y_n) \\ &= 0 + 0 + .. + 0 = 0 \text{ using } (71). \end{aligned}$$

Hence, $u = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ is also a solution of the homogeneous linear differential equation (68). Since this solution contains n arbitrary constants, it is a general or a complete solution of (70).

EXAMPLE 1.82

Show that $c_1 \sin x + c_2 \cos x$ is a solution of $\frac{d^2y}{dx^2} + y = 0$.

Solution. Let

$$y_1 = \sin x, \quad y_2 = \cos x.$$

Then

$$\frac{dy_1}{dx} = \cos x, \quad \frac{dy_2}{dx} = -\sin x$$

$$\frac{d^2 y_1}{dx^2} = -\sin x, \quad \frac{d^2 y^2}{dx^2} = -\cos x.$$

We note that

$$\frac{d^2 y_1}{dx^2} + y_1 = -\sin x + \sin x = 0$$

and

$$\frac{d^2 y_2}{dx^2} + y_2 = -\cos x + \cos x = 0.$$

Hence, $\sin x$ and $\cos x$ are solutions of the given equation. These two solutions are linearly independent. Therefore, their linear combination $c_1 \sin x + c_2 \cos x$ is also a solution of the given equation.

Theorem 1.5. If y_1 is a complete solution of the homogeneous equation $f(D)y = 0$ and y_2 is a particular solution containing no arbitrary constants of the differential equation $f(D)y = F(x)$, then $y_1 + y_2$ is the complete solution of the equation $f(D)y = F(x)$.

Proof: Since y_1 is a complete solution of the homogeneous differential equation $f(D)y = 0$, we have

$$f(D)y_1 = 0 \qquad (72)$$

Further, since y_2 is a particular solution of linear differential equation $f(D)y = F(x)$, we have

$$f(D)y_2 = F(x). \qquad (73)$$

Adding (70) and (71), we get

$$f(D)y_1 + f(D)y_2 = F(x)$$

or

$$f(D)(y_1 + y_2) = F(x).$$

Hence, $y_1 + y_2$ satisfies the equation $f(D)y = F(x)$ and so is the complete solution since it contains n arbitrary constants.

Definition 1.21. Let $f(D)y = F(x)$ be a linear differential equation with constant coefficients. If y_1 is a complete solution of $f(D)y = 0$ and y_2 is a particular solution of $f(D)y = F(x)$, then $y_1 + y_2$ is a complete solution of $f(D)y = F(x)$ and then y_1 is called the *complementary function* and y_2 is called the *particular integral* of the differential equation $f(D)y = F(x)$.

Consider the homogeneous differential equation $f(D)y = 0$. Then

$$(D^n + a_1D^{n-1} + a_2^{n-2} + ... + a_n)y = 0. \qquad (74)$$

Let $y = e^{mx}$ be a solution of (74). Then

$$Dy = me^{mx}, D^2y = m^2 e^{mx}, \ldots, \quad D^ny = m^n e^{mx}$$

and so (74) transforms to

$$(m^n + a_1m^{n-1} + ... + a_n)e^{mx} = 0.$$

Since $e^{mx} \neq 0$, we have

$$m^n + a_1m^{n-1} + a_2m^{n-2} + ... + a_n = 0. \qquad (75)$$

It follows, therefore, that if e^{mx} is a solution of $f(D)y = 0$, then equation (75) is satisfied.

The equation (75) is called *auxiliary equation* for the differential equation $f(D)y = 0$.

1.15 SOLUTION OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Consider the homogeneous linear differential equation

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}\frac{dy}{dx} + a_ny = 0.$$

The symbolic form of this equation is

$$(D^n + a_1D^{n-1} + ... + a_{n-1}D + a_n)y = 0, \qquad (76)$$

where $a_1, a_2, \ldots, a_n$ are constants. If $y = e^{mx}$ is a solution of (76), then

$$m^n + a_1m^{n-1} + ... + a_{n-1}m + a_n = 0. \qquad (77)$$

Three cases arise, according as the roots of (77) are *real and distinct, real and repeated* or *complex*.

Case I. Distinct Real Roots

Suppose that the auxiliary equation (77) has n distinct roots $m_1, m_2, \ldots, m_n$. Therefore, (77) reduces to

$$(m - m_1)(m - m_2)\ldots(m - m_n) = 0 \qquad (78)$$

Equation (78) will be satisfied by the solutions of the equations

$$(D - m_1)y = 0, (D - m_2)y = 0, \ldots,$$
$$(D - m_n) = 0.$$

We consider $(D - m_1)y = 0$. This can be written as

$$\frac{dy}{dx} - m_1y = 0,$$

which is linear differential equation with integrating factor as e^{-m_1x}. Therefore, its solution is

$$y.e^{-m_1x} = \int 0 \,.\, e^{-m_1x}dx + c_1$$

or

$$y = c_1\, e^{m_1x}.$$

Similarly,

the solution of $(D - m_2)y = 0$ *is* $c_2\, e^{m_2x}$,
the solution of $(D - m_3)y = 0$ *is* $c_3\, e^{m_3x}$,
...
...
the solution of $(D - m_y)y = 0$ *is* $c_ne^{m_nx}$.

Hence, the complete solution of homogeneous differential equation (76) is

$$y = c_1\, e^{m_1x} + c_2e^{m_2x} + \ldots + c_ne^{m_nx}. \qquad (79)$$

Case II. Repeated Real Roots

Suppose that the roots m_1 and m_2 of the auxiliary equation are equal. Then the solution (79) becomes

$$\begin{aligned} y &= c_1\, e^{m_1x} + c_2\, e^{m_1x} + c_3\, e^{m_3x} + \ldots + c_n\, e^{m_nx} \\ &= (c_1 + c_2)\, e^{m_1x} + c_3\, e^{m_3x} + \ldots + c_n\, e^{m_nx}. \end{aligned}$$

Since it contains $n - 1$ arbitrary constants, it is not a complete solution of the given differential equation. We shall show that the part of the solution corresponding to equal roots m_1 and m_2 is $(c_1x + c_2)\, e^{m_1x}$. To prove it, consider the equation

$$(D - m_1)^2 y = 0,$$

that is,

$$(D - m_1)(D - m_1)y = 0.$$

Substituting $(D - m_1)y = U$, the above equation becomes

$$(D - m_1)U = 0$$

or

$$\frac{dU}{dx} - m_1 U = 0$$

or

$$\frac{dU}{U} = m_1 dx.$$

Integrating, we get

$$\log U = m_1 x + \log C_1$$

or

$$\log \frac{U}{c_1} = m_1 x \, or \, \frac{U}{c_1} = e^{m_1x}$$

and so

$$U = c_1\, e^{m_1x}.$$

Hence

$$(D - m_1)y = c_1\, e^{m_1x}$$

or

$$\frac{dy}{dx} - m_1 y = c_1\, e^{m_1x},$$

which is again a linear equation with integrating factor e^{-m_1x}. Hence, the solution is

$$y.e^{-m_1x} = \int c_1 e^{m_1x}.e^{-m_1x} + c_2$$
$$= c_1 x + c_2,$$

which yields

$$y = (c_1x + c_2)e^{m_1x}.$$

Hence, the complete solution of the given differential equation is

$$y = (c_1x + c_2)e^{m_1x} + c_3e^{m_3x} + \ldots + c_n\, e^{m_nx}.$$

Remark 1.1. If three roots of the auxiliary equation are equal, that is, $m_1 = m_2 = m_3$, then the complete solution will come out to be

$$y = (c_1x^2 + c_2x + c_3)e^{m_1x} + c_4e^{m_4x} + \ldots + c_n\, e^{m_nx}.$$

In general, if $m_1 = m_2 = \ldots = m_k$, then the complete solution of the differential equation shall be

$$y = (c_1x^{k-1} + c_2x^{k-2} + \ldots + c_k) + c_{k+1}\, e^{m_{k+1}x} + \ldots + c_n\, e^{m_nx}.$$

Case III. Conjugate Complex Roots

(a) Suppose that the auxiliary equation has a non-repeated complex root $\alpha + i\beta$. Then, since the coefficients are real, the conjugate complex number $\alpha - i\beta$ is also a non-repeated root. Thus, the solution given in (79) becomes

$$\begin{aligned} y &= c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx} \\ &= e^{\alpha x}(c_1e^{i\beta x} + c_2e^{-i\beta x}) + c_3e^{m_3x} + \ldots + c_ne^{m_nx} \\ &= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)] \\ &\quad + c_3e^{m_3x} + \ldots + c_ne^{m_nx} \\ &= e^{\alpha x}[(c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x] \\ &\quad + c_3e^{m_3x} + \ldots + c_ne^{m_nx} \\ &= e^{\alpha x}[k_1\cos\beta x + k_2\sin\beta x] + c_3e^{m_3x} + \ldots + c_ne^{m_nx}, \end{aligned}$$

where $k_1 = c_1 + c_2$, $k_2 = i(c_1 - c_2)$.

(b) If two pairs of imaginary roots are equal, then

$$m_1 = m_2 = \alpha + i\beta \quad and \quad m_3 = m_4 = \alpha - i\beta.$$

Using Case II, the complete solution is

$$= e^{\alpha x}[(c_1x + c_2)\cos\beta x + (c_3x + c_4)\sin\beta x] + c_5\, e^{m_5x} + \ldots + c_ne^{m_nx}.$$

EXAMPLE 1.83

Solve

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$$

Solution. The symbolic form of the given equation is

$$(D^3 + D^2 + 4D + 4)y = 0.$$

Therefore its auxiliary equation is

$$m^3 + m^2 + 4m + 4 = 0.$$

By inspection -1 is a root. Therefore, $(m + 1)$ is a factor of $m^3 + m^2 + 4m + 4$. The synthetic division by $m+1$ gives

$$\begin{array}{r|rrrr} -1 & 1 & 1 & 4 & 4 \\ & & -1 & 0 & -4 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

Therefore, the auxiliary equation is

$$(m+1)(m^2+4)=0$$

and so

$$m=-1 \quad \textit{and} \quad m=\pm 2i.$$

Hence, the complementary solution is

$$\begin{aligned} y &= c_1 e^{-x} + e^{0x}(c_2\cos 2x + c_3 \sin 2x) \\ &= c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x. \end{aligned}$$

EXAMPLE 1.84

Solve

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0.$$

Solution. The symbolic form of the given equation is

$$(D^3 - 3D^2 + 3D - 1)y = 0.$$

Therefore, the auxiliary equation is

$$m^3 - 3m^2 + 3m - 1 = 0.$$

By inspection 1 is a root. Then synthetic division yields

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 3 & -1 \\ & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

Therefore, the auxiliary equation is

$$(m-1)(m^2-2m+1)=0 \text{ or } (m-1)^3 = 0.$$

Hence the roots are 1, 1, 1 and so the solution of the given equation is

$$y = (c_1 + c_2 x + c_3 x^2)e^x.$$

EXAMPLE 1.85

Find the general solution of

$$\frac{d^4y}{dx^4} - 5\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0.$$

Solution. The auxiliary equation is

$$m^4 - 5m^3 + 6m^2 + 4m - 8 = 0,$$

whose roots are 2, 2, 2, and –1. Hence the general solution is

$$y = (c_1 + c_2 x + c_3 x^2)e^{2x} + c_4 e^{-x}.$$

EXAMPLE 1.86

Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$ subject to the conditions $y(0)=4$, $y'(0)=1$.

Solution. The symbolic form of the given differential equation is

$$(D^2 - 2D + 10)y = 0.$$

Therefore, the auxiliary equation is

$$m^2 - 2m + 10 = 0,$$

which yields

$$m = 1 \pm 3i.$$

Therefore, the solution is

$$y = e^x(c_1 \cos 3x + c_2 \sin 3x).$$

Now,

$$\begin{aligned} y' &= c_1[e^x \cos 3x - 3e^x \sin 3x] \\ &\quad + c_2[e^x \sin 3x + 3e^x \cos 3x] \\ &= e^x \cos 3x(c_1 + 3c_2) + e^x \sin x(c_2 - 3c_1). \end{aligned}$$

The initial conditions $y(0)=4$ and $y'(0)=1$ yield

$$4 = c_1 \text{ and } 1 = c_1 + 3c_2$$

and so $c_1 = 4$ and $c_2 = -1$. Hence the solution is

$$y = e^x(4\cos 3x - \sin 3x).$$

EXAMPLE 1.87

Solve

$$\frac{d^3y}{dx^3} + y = 0.$$

Solution. The auxiliary equation is

$$m^3 + 1 = 0$$

or

$$(m+1)(m^2 - m + 1) = 0.$$

Thus the roots are –1, $\frac{1\pm\sqrt{3}i}{2}$. Hence, the general solution of the equation is

$$y = c_1 e^{-x} + e^{\frac{1}{2}x}(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x).$$

EXAMPLE 1.88

Solve
$$\frac{d^4y}{dx^4} = m^4 y.$$

Solution. The auxiliary equation for the given differential equation is

$$s^4 - m^4 = 0$$

or

$$(s+m)(s-m)(s^2+m^2) = 0$$

and so $s = m, -m, \pm mi$. Hence the solution is

$$\begin{aligned} y &= c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx \\ &= c_1[\cosh mx + \sinh mx] + c_2[\cosh mx - \sinh mx] \\ &\quad + c_3 \cos mx + c_4 \sin mx \\ &= (c_1 + c_2)\cosh mx + (c_1 - c_2)\sinh mx \\ &\quad + c_3 \cos mx + c_4 \sin mx \\ &= C_1 \cosh mx + C_2 \sinh mx + c_3 \cos mx \\ &\quad + c_4 \sin mx, \end{aligned}$$

where

$$C_1 = c_1 + c_2 \text{ and } C_2 = c_1 - c_2.$$

EXAMPLE 1.89

Solve $4\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$

Solution. The auxiliary equation for the given differential equation is

$$4m^3 + 4m^2 + m = 0$$

or

$$m(4m^2 + 4m + 1) = 0.$$

Thus the roots are $m = 0, -\frac{1}{2}$, and $-\frac{1}{2}$. Hence the solution is

$$\begin{aligned} y &= c_1 e^{0x} + (c_2 + c_3 x)\, e^{-x/2} \\ &= c_1 + (c_2 + c_3 x) e^{-x/2}. \end{aligned}$$

1.16 COMPLETE SOLUTION OF LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

We now discuss the methods of finding particular integral of a linear differential equation with constant coefficients so that complete solution of the equation may be found.

Definition 1.22. $\frac{1}{D}F(x)$ is that function of x which when operated upon by D yields $F(x)$.

Similarly, $\frac{1}{f(D)}F(x)$ is that function of x, free from arbitrary constant, which when operated upon by f (D) yields $F(x)$.

Thus, $\frac{1}{D}$ is the inverse operator of D and $\frac{1}{f(D)}$ is the inverse operator of f (D).

Theorem 1.6. $\frac{1}{D}F(x) = \int F(x)\,dx.$

Proof: Let

$$\frac{1}{D}F(x) = v. \tag{80}$$

Operating both sides of (80) by D, we get

$$D.\frac{1}{D}F(x) = Dv$$

or

$$F(x) = Dv = \frac{dv}{dx}$$

or

$$dv = F(x)dx.$$

Integrating, we get

$$v = \int F(x)\,dx,$$

where no constant of integration is added since $\frac{1}{D}$F(x) contains no constant. Thus,

$$\frac{1}{D}F(x) = \int F(x)\,dx.$$

Hence, $\frac{1}{D}$ stands for integration.

Theorem 1.7. $\frac{1}{D-\alpha}F(x) = e^{\alpha x}\int F(x)e^{-\alpha x}\,dx.$

Proof: Let

$$\frac{1}{D-\alpha}F(x) = y. \tag{81}$$

Operating both sides of (81) by D – α, we have

$$\begin{aligned} F(x) &= (D-\alpha)y = Dy - \alpha y \\ &= \frac{dy}{dx} - \alpha y. \end{aligned}$$

Therefore,

$$\frac{dy}{dx} - \alpha y = F(x),$$

which is a linear differential equation with integrating factor $e^{-\alpha x}$. Therefore, its solution is

$$y.e^{-\alpha x} \int F(x)\, e^{-\alpha x}\, dx$$

or

$$y = e^{\alpha x} \int F(x)\, e^{-\alpha x}\, dx$$

or

$$\frac{1}{D-\alpha} F(x) = e^{\alpha x} \int F(x)\, e^{-\alpha x}\, dx.$$

Theorem 1.8. $\frac{1}{f(D)}F(x)$ is the particular integral of $f(D)y = F(x)$.

Proof: The given linear differential equation is

$$f(D)y = F(x) \tag{82}$$

Substituting $y = \frac{1}{f(D)}\mathrm{F}(x)$ in (82), we have $\mathrm{F}(x) = \mathrm{F}(x)$, which is true. Hence $y = \frac{1}{f(D)}\,\mathrm{F}(x)$ is a solution of (82).

1.16.1 Standard Cases of Particular Integrals

Consider the linear differential equation

$$f(D)y = F(x).$$

By Theorem 1.8, its particular integral is

$$\text{P.I} = \frac{1}{f(D)} F(x).$$

Case I. When *F(x)* $= e^{ax}$

We have

$$f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n.$$

Therefore,

$$\begin{aligned} f(D)e^{ax} &= (D^n + a_1 D^{n-1} + \ldots + a_n)e^{ax} \\ &= D^n e^{ax} + a_1 D^{n-1} e^{ax} + \ldots + a_{n-1} D e^{ax} + a_n e^{ax} \\ &= a^n e^{ax} + a_1 a^{n-1} e^{ax} + \ldots + a_{n-1} a e^{ax} + a_n e^{ax} \\ &= (a^n + a_1 a^{n-1} + \ldots + a_{n-1} a + a_n) e^{ax} \\ &= f(a) e^{ax}. \end{aligned}$$

Operating both sides by $\frac{1}{f(D)}$ yields

$$\begin{aligned} e^{ax} &= \frac{1}{f(D)}[f(a)e^{ax}] \\ &= f(a) \frac{1}{f(D)} e^{ax}. \end{aligned}$$

Hence

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{ provided } f(a) \neq 0. \tag{83}$$

If $f(a) = 0$, then $D - a$ is a factor of $f(D)$. So, let

$$f(D) = (D-a)\phi(D). \tag{84}$$

Then

$$\begin{aligned} \frac{1}{f(D)} e^{ax} &= \frac{1}{(D-a)\phi(D)} e^{ax} = \frac{1}{(D-a)} \left[\frac{1}{\phi(D)} e^{ax} \right] \\ &= \frac{1}{D-a} \left[\frac{1}{\phi(a)} e^{ax} \right] \text{ using (83) since } \phi(a) \neq 0 \\ &= \frac{1}{\phi(a)} \left[\frac{1}{D-a} e^{ax} \right] \\ &= \frac{1}{\phi(a)} e^{ax} \int e^{ax}.e^{-ax}\, dx, \text{ by Theorem 1.7} \\ &= x \frac{e^{ax}}{\phi(a)} \end{aligned} \tag{85}$$

Differentiating (84) with respect to D gives

$$f'(D) = \phi(D) + (D-a)\phi'(D).$$

Putting $D = a$, we get $f'(a) = \phi(a)$. Therefore, (85) reduces to

$$\begin{aligned} \frac{1}{f(D)} e^{ax} &= \mathrm{x}\, \frac{e^{ax}}{\phi(a)} = \mathrm{x}\, \frac{e^{ax}}{\left[\frac{d}{dD} f(D)\right]_{D=a}} \\ &= \mathrm{x}\, \frac{e^{ax}}{f'(a)}, \text{ provided } f'(a) \neq 0. \end{aligned} \tag{86}$$

If $f'(a) = 0$, then the rule can be repeated to give

$$\frac{1}{f(D)} \mathrm{e}^{\mathrm{ax}} = x^2 \frac{e^{ax}}{f''(a)}\, e^{ax},$$

provided $f''(a) \neq 0$ and so on.

Case II. When $F(x) = \sin(ax+b)$ or $\cos(ax+b)$

We have

$$\begin{aligned} D\sin(ax+b) &= a\cos(ax+b) \\ D^2\sin(ax+b) &= -a^2\sin(ax+b) \\ D^3\sin(ax+b) &= -a^3\cos(ax+b) \\ D^4\sin(ax+b) &= a^4\sin(ax+b) \\ &\cdots \\ &\cdots \end{aligned}$$

We note in general that

$$(D^2)^n\sin(ax+b) = (-a^2)^n\sin(ax+b).$$

Hence

$$f(D^2)\sin(ax+b) = f(-a^2)\sin(ax+b).$$

Operating on both sides by $\frac{1}{f(D^2)}$, we get

$$\sin(ax+b) = f(-a^2)\frac{1}{f(D^2)}\sin(ax+b).$$

Dividing both sides by $f(-a^2)$, we have

$$\frac{1}{f(-a^2)}\sin(ax+b) = \frac{1}{f(D^2)}\sin(ax+b).$$

Hence

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b), \quad (87)$$

provided $f(-a^2) \neq 0$.
Similarly,

$$\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b), \quad (88)$$

provided $f(-a^2) \neq 0$.

If $f(-a^2) = 0$, then (87) and (88) are not valid. In such a situation, we proceed as follows:

By Euler's formula

$$e^{i(ax+b)} = \cos(ax+b) + i\sin(ax+b).$$

Thus

$$\frac{1}{f(D^2)}e^{i(ax+b)} = \frac{1}{f(D^2)}[\cos(ax+b) + i\sin(ax+b)]$$

or, by 86,

$$x.\frac{1}{f'(D^2)}e^{i(ax+b)} = \frac{1}{f(D^2)}[\cos(ax+b) + i\sin(ax+b)]$$

or

$$\begin{aligned} x.\frac{1}{f'(D^2)}&[\cos(ax+b) + i\sin(ax+b)] \\ &= \frac{1}{f(D^2)}[\cos(ax+b) + i\sin(ax+b)] \end{aligned}$$

Equating real and imaginary parts, we have

$$\frac{1}{f(D^2)}\cos(ax+b) = x.\frac{1}{[f'(D^2)]_{D^2=-a^2}}\cos(ax+b) \quad (89)$$

provided that $f'(-a^2) \neq 0$, and

$$\frac{1}{f(D^2)}\sin(ax+b) = x\frac{1}{[f'(D^2)]_{D^2=-a^2}}\sin(ax+b), \quad (90)$$

provided $f'(-a^2) \neq 0$.

If $f'(-a^2) = 0$, then repeating the above process, we have

$$\frac{1}{f(D^2)}\cos(ax+b) = x^2\frac{1}{[f''(D^2)]_{D^2=-a^2}}\cos(ax+b)$$

provided $f''(-a^2) \neq 0$
and

$$\frac{1}{f(D^2)}\sin(ax+b) = x^2\frac{1}{[f''(D^2)]_{D^2=-a^2}}\sin(ax+b)$$

provided $f''(-a^2) \neq 0$.

Case III. When $F(x) = x^n$, n being positive integer

Since in this case

$$\text{P.I.}\frac{1}{f(D)}F(x) = \frac{1}{f(D)}x^n,$$

we make the coefficient of the leading term of $f(D)$ unity, take the denominator in numerator and then expand by Binomial theorem. Operate the resulting expansion on x^n.

Case IV. When $F(x) = e^{ax}Q(x)$, where $Q(x)$ is some function of x

Let G is a function of x, we have

$$\begin{aligned} D[e^{ax}G] &= e^{ax}DG + a\,e^{ax}G = e^{ax}(D+a)G \\ D^2[e^{ax}G] &= e^{ax}(D+a)^2G \\ &\cdots \\ &\cdots \\ D^n[e^{ax}G] &= e^{ax}(D+a)^nG \end{aligned}$$

Hence

$$f(D)[e^{ax}G] = e^{ax}f(D+a)G$$

Operating both sides by $\frac{1}{f(D)}$, we get

$$e^{ax}G = \frac{1}{f(D)}[e^{ax}f(D+a)]G$$

Putting $f(\mathrm{D}+a)\mathrm{G} = \mathrm{Q}$, we have $\mathrm{G} = \frac{Q}{f(D+a)}$ and so we have

$$e^{ax}.\frac{1}{f(D+a)}Q = \frac{1}{f(D)}(e^{ax}Q)$$

or

$$\frac{1}{f(D)}(e^{ax}Q(x)) = e^{ax}\frac{1}{f(D+a)}Q \tag{91}$$

Case V. When *F(x)* = *x Q(x)*

Resolving f(D) into linear factors, we have

$$f(D) = (D-m_1)(D-m_2)\dots(D-m_n).$$

Then, using partial fractions and Theorem 1.7, we have

$$\begin{aligned}
\text{P.I.} &= \frac{1}{f(D)}F(x)\\
&= \frac{1}{(D-m_1)(D-m_2)\dots(D-m_n)}F(x)\\
&= \left[\frac{A_1}{D-m_1}+\frac{A_2}{D-m_2}+\dots+\frac{A_n}{D-m_n}\right]F(x)\\
&= A_1e^{m_1x}\int F(x)\,e^{-m_1x} + A_2e^{m_2x}\int F(x)\,e^{-m_2x}\\
&\quad +\dots+A_ne^{m_nx}\int F(x)e^{-m_nx}.
\end{aligned}$$

EXAMPLE 1.90

Solve $4\frac{d^2y}{dx^2}+4\frac{dy}{dx}-3y = e^{2x}$.

Solution. The symbolic form of the given differential equation is

$$(4D^2+4D-3)y = e^{2x}$$

and so the auxiliary equation is

$$4m^2+4m-3 = 0.$$

The roots of A.E. are $m = \frac{1}{2}, -\frac{3}{2}$. Therefore, the complementary function is

$$\text{C.F.} = c_1e^{x/2}+c_2e^{-3x/2}.$$

Since 2 is not a root of the auxiliary equation, by (82), we have

$$\text{P.I} = \frac{1}{f(2)}e^{2x} = \frac{1}{4(4)+4(2)-2}\,e^{2x} = \frac{e^{2x}}{21}.$$

Hence the complete solution of the given equation is

$$y = \text{C.F.}+\text{P.I} = c_1e^{x/2}+c_2e^{-x/2}+\frac{e^{2x}}{21}.$$

EXAMPLE 1.91

Solve $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y = e^{3x}$.

Solution. The symbolic form of the equation is

$$(D^2-5D+6)y = e^{3x}.$$

The auxiliary equation is

$$m^2-5m+6 = 0$$

or

$$(m-3)(m-2)-0.$$

Therefore $m = 2, 3$. Then

$$\text{C.F.} = c_1\,e^{3x}+c_2\,e^{2x}.$$

Since 3 is a root of auxiliary equation, we use (85) and get

$$\text{P.I} = x\frac{e^{3x}}{\phi(3)} = x\frac{e^{3x}}{(3-2)} = xe^{3x}.$$

Hence, the complete solution of the given equation is

$$y = \text{C.F.}+\text{P.I.} = c_1e^{3x}+c_2e^{2x}+x\,e^{3x}.$$

Remark 1.2. (a) In the above example, if we use (86), then

$$\begin{aligned}
\text{P.I.} &= x\frac{e^{ax}}{\left(\frac{d}{dD}f(D)\right)_{D=a}}\\
&= x\frac{e^{3x}}{[2D-5]_{D=3}} = x\frac{e^{3x}}{6-5} = x\,e^{3x}.
\end{aligned}$$

(b) We can also find the particular integral in the above case by using Theorem 1.7. In fact, we have

$$\text{P.I.} = \frac{1}{f(D)}F(x) = \frac{1}{(D-3)(D-2)}\,e^{3x}$$
$$= \frac{1}{D-3}e^{2x}\int F(x)\,e^{-2x}\,dx$$
$$= \frac{1}{D-3}e^{2x}\int e^{3x}\,e^{-2x}\,dx = \frac{1}{D-3}\,e^{3x}$$
$$= e^{3x}\int e^{3x}\,.\,e^{-3x}\,dx = e^{3x}\int e^{0}dx = x\,e^{3x}.$$

EXAMPLE 1.92

Solve
$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \cos hx.$$

Solution. The auxiliary equation is
$$m^2 - 3m + 2 = 0,$$
which yields $m = 1, 2$. Therefore,
$$\text{C.F.} = c_1e^x + c_2e^{2x}.$$
Now
$$\cosh x = \frac{e^x + e^{-x}}{2}.$$
Therefore,
$$\text{P.I.} = \frac{1}{f(D)}F(x) = \frac{1}{(D-1)(D-2)}\left[\frac{e^x+e^{-x}}{2}\right]$$
$$= \frac{1}{2}.\frac{1}{D^2-3D+2}e^x + \frac{1}{2}.\frac{1}{D^2-3D+2}e^{-x}$$
$$= \frac{1}{2}\frac{1}{D^2-3D+2}e^x + \frac{1}{2}\frac{1}{1+3+2}e^{-x}$$
$$= \frac{1}{2}x\frac{1}{\frac{d}{dD}[D^2-3D+2]_{D=1}}\,e^x + \frac{1}{12}e^{-x}\text{since} f(1)=0$$
$$= \frac{x}{2}\frac{1}{[2D-3]_{D=1}}e^x + \frac{1}{12}e^{-x}$$
$$= -\frac{x}{2}e^x + \frac{1}{12}e^{-x}.$$
Hence, the complete solution of the equation is
$$y = \text{C.F.} + \text{P.I.} = c_1e^x + c_2e^{2x} - \frac{x}{2}e^x + \frac{1}{12}e^{-x}.$$

EXAMPLE 1.93

Solve $\frac{d^2y}{dx^2} - 4y = e^x + \sin 2x$.

Solution. The auxiliary equation for the given differential equation is
$$m^2 - 4 = 0,$$
which yields $m = \pm 2$. Hence
$$\text{C.F.} = c_1\,e^{2x} + c_2e^{-2x}.$$
Now, using (82) and (87), we have
$$\text{P.I.} = \frac{1}{D^2-4}(e^x+\sin 2x) = \frac{1}{D^2-4}e^x + \frac{1}{D^2-4}\sin 2x$$
$$= \frac{e^x}{1^2-4} + \frac{1}{-4-4}\sin 2x = -\frac{1}{3}e^x - \frac{1}{8}\sin 2x.$$
Hence, the complete solution of the given differential equation is
$$y = \text{C.F.} + \text{P.I.} = c_1e^{2x} + c_2e^{-2x} - \frac{1}{3}e^x - \frac{1}{8}\sin 2x.$$

EXAMPLE 1.94

Solve $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x$.

Solution. The auxiliary equation is
$$m^3 + m^2 - m - 1 = 0,$$
whose roots are 1, –1, –1. Therefore,
$$\text{C.F.} = c_1e^x + (c_2 + c_3x)e^{-x}$$
Further,
$$\text{P.I.} = \frac{1}{f(D)}F(x) = \frac{1}{D^3+D^2-D-1}\cos 2x$$
$$= \frac{1}{DD^2-D^2-D-1}\cos 2x$$
$$= \frac{1}{D(-4)+(-4)-D-1}\cos 2x$$
$$= \frac{1}{-5D-5}\cos 2x = -\frac{1}{5(D+1)}\cos 2x$$
$$= -\frac{(D-1)}{5(D^2-1)}\cos 2x = -\frac{1}{5}(D-1)\frac{1}{-4-1}\cos 2x$$
$$= \frac{1}{25}(D-1)\cos 2x = \frac{1}{25}(D\cos 2x - \cos 2x)$$
$$= \frac{1}{25}(-2\sin 2x - \cos 2x) = -\frac{1}{25}(2\sin 2x + \cos 2x).$$
Hence the complete solution is
$$y = c_1e^x + (c_2 + c_2x)e^{-x} - \frac{1}{25}(2\sin 2x + \cos 2x).$$

EXAMPLE 1.95

Solve $\frac{d^2y}{dx^2} - 4y = x \sin hx$.

Solution. The auxiliary equation is $m^2 - 4 = 0$ and $m = \pm 2$. Therefore,

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}.$$

Further,

$$\text{P.I.} = \frac{1}{D^2-4} x \sinh x = \frac{1}{D^2-4} x \left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4} x e^x - \frac{1}{D^2-4} x e^{-x}\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^2-4} x - e^{-x} \frac{1}{(D-1)^2-4} x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^2+2D-3} x - e^{-x} \frac{1}{D^2-2D-3} x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{-3\left(1-\frac{2D}{3}-\frac{D^2}{3}\right)} x - e^{-x} \frac{1}{-3\left(1+\frac{2D}{3}-\frac{D^2}{3}\right)} x\right]$$

$$= -\frac{1}{6}\left[e^x \left\{1 - \left(\frac{2D}{3} + \frac{D^2}{3}\right)\right\}^{-1} x \right.$$

$$\left. - e^{-x} \left\{1 + \left(\frac{2D}{3} - \frac{D^2}{3}\right)\right\}^{-1} x\right]$$

$$= -\frac{1}{6}\left[e^x \left(1 + \frac{2D}{3} + \ldots\right) x - e^{-x} \left(1 - \frac{2D}{3} + \ldots\right) x\right]$$

$$= -\frac{1}{6}\left[e^x \left(x + \frac{2}{3}\right) - e^{-x} \left(x - \frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= -\frac{x}{3} \sin hx - \frac{2}{9} \cos hx.$$

Hence the solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3} \sin hx - \frac{2}{9} \cos hx.$$

EXAMPLE 1.96

Solve $\frac{d^2y}{dx^2} - 4y = x^2$.

Solution. The auxiliary equation is $m^4 - 4 = 0$ and so $m = \pm 2$. Therefore,

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}.$$

Further,

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{D^2-4} x^2$$

$$= -\frac{1}{4-D^2} x^2 = -\frac{1}{4\left(1-\frac{D^2}{4}\right)} x^2$$

$$= -\frac{1}{4}\left(1 - \frac{D^2}{4}\right)^{-1} x^2$$

$$= -\frac{1}{4}\left[1 + \frac{D^2}{4} + \ldots\right] x^2$$

$$= -\frac{1}{4} x^2 - \frac{1}{16} D^2(x^2)$$

$$= -\frac{1}{4} x^2 - \frac{1}{16}(2) = -\frac{1}{4} x^2 - \frac{1}{8}.$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4} x^2 - \frac{1}{8}.$$

EXAMPLE 1.97

Solve $\frac{d^2y}{dx^2} + 4y = e^x + \sin 3x + x^2$.

Solution. The auxiliary equation is $m^2 + 4 = 0$ and so $m = \pm 2i$. Therefore,

$$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x.$$

Further,

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{D^2+4}(e^x + \sin 3x + x^2)$$

$$= \frac{1}{D^2+4} e^x + \frac{1}{D^2+4} \sin 3x + \frac{1}{D^2+4} x^2$$

$$= \frac{1}{5} e^x + \frac{1}{-9+4} \sin 3x + \frac{1}{4}\left(1 + \frac{D^2}{4}\right)^{-1} x^2$$

$$= \frac{1}{5} e^x - \frac{1}{5} \sin 3x + \frac{1}{4}\left(1 - \frac{D^2}{4} + \ldots\right) x^2$$

$$= \frac{1}{5} e^x - \frac{1}{5} \sin 3x + \frac{x^2}{4} - \frac{1}{16}.2$$

$$= \frac{1}{5} e^x - \frac{1}{5} \sin 3x + \frac{1}{4} x^2 - \frac{1}{8}.$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^x - \frac{1}{5}\sin 3x + \frac{1}{4}x^2 - \frac{1}{8}.$$

EXAMPLE 1.98

Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$.

Solution. The auxiliary equation of the given differential equation is

$$m^2 - 2m + 1 = 0,$$

which yields $m = 1, 1$. Hence

$$\text{C.F.} = (c_1 + c_2 x)e^x.$$

The particular integral is

$$\text{P.I.} = \frac{1}{f(D)}F(x) = \frac{1}{(D-1)^2}xe^x \sin x$$

$$= e^x \frac{1}{(D+1-1)^2}x \sin x = e^x \frac{1}{D^2}x \sin x$$

$$= e^x \frac{1}{D}\int x \sin x \, dx = e^x \frac{1}{D}(-x\cos x + \sin x)$$

$$= e^x \int (-x \cos x + \sin x)\, dx$$

$$= e^x[-x \sin x - \cos x - \cos x]$$

$$= -e^x(x \sin x + 2\cos x).$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= (c_1 + c_2 x)e^x - e^x(x \sin x + 2\cos x).$$

EXAMPLE 1.99

Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x$.

Solution. The auxiliary equation is

$$m^2 - 4m + 3 = 0,$$

which yields $m = 3, 1$. Therefore,

$$\text{C.F.} = c_1 e^{3x} + c_2 e^x.$$

Further

$$\text{P.I.} = \frac{1}{D^2-4D+3}[\sin 3x \cos 2x]$$

$$= \frac{1}{D^2-4D+3}[\frac{1}{2}2\sin 3x \cos 2x]$$

$$= \frac{1}{D^2-4D+3}[\frac{1}{2}(\sin 5x + \sin x)]$$

$$= \frac{1}{2}\cdot\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{2}\cdot\frac{1}{D^2-4D+3}\sin x$$

$$= \frac{1}{2}\left[\frac{1}{-25-4D+3}\sin 5x + \frac{1}{-1-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-22-4D}\sin 5x + \frac{1}{2-4D}\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2(11+2D)}\sin 5x + \frac{1}{2(1-2D)}\sin x\right]$$

$$= \frac{1}{4}\left[-\frac{11-2D}{121-4D^2}\sin 5x + \frac{1+2D}{1-4D^2}\sin x\right]$$

$$= \frac{1}{4}\left[-\frac{11-2D}{121-4(-25)}\sin 5x + \frac{1+2D}{1-4(-1)}\sin x\right]$$

$$= \frac{1}{4}\left[-\frac{11-2D}{221}\sin 5x + \frac{1+2D}{5}\sin x\right]$$

$$= \frac{1}{4}\left[-\frac{1}{221}[11\sin 5x - 2D\sin 5x] + \frac{1}{5}(\sin x + 2D\sin x)\right]$$

$$= \frac{1}{4}\left[-\frac{11}{221}\sin 5x + \frac{10}{221}\cos 5x + \frac{1}{5}\sin x + \frac{2}{5}\cos x\right]$$

$$= -\frac{11}{884}\sin 5x + \frac{10}{884}\cos 5x + \frac{1}{20}\sin x + \frac{1}{10}\cos x.$$

Hence the complete solution is

$$y = c_1 e^{3x} + c_2 e^x - \frac{11}{884}\sin 5x + \frac{10}{884}\cos 5x + \frac{1}{20}\sin x + \frac{1}{10}\cos x.$$

EXAMPLE 1.100

Solve $(D^2 + 1)y = \operatorname{cosec} x$.

Solution. The auxiliary equation is $m^2 + 1 = 0$, which yields $m = \pm i$. Thus,

$$\text{C.F.} = c_1 \cos x + c_2 \sin x.$$

Now

$$\text{P.I.} = \frac{1}{f(D)}F(x) = \frac{1}{D^2+1}\text{cosec } x$$

$$= \frac{1}{(D+i)(D-i)} \text{ cosec } x$$

$$= \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] \text{cosec } x$$

$$= \frac{1}{2i}\left[\frac{1}{D-i}\text{cosec } x - \frac{1}{D+i}\text{cosec } x\right].$$

But, by Theorem 1.7,

$$\frac{1}{D-i}\text{cosec } x = e^{ix}\int \text{cosec}\, x\; e^{-ix}dx$$

$$= e^{ix}\int \text{cosec}\, x\;(\cos x - i\sin x)\,dx$$

$$= e^{ix}\int (\cot x - i)\,dx$$

$$= e^{ix}(\log\sin x - ix).$$

Similarly,

$$\frac{1}{D+i}\text{cosec}\, x = e^{-ix}(\log\sin x + ix).$$

Therefore,

$$\text{P.I.} = \frac{1}{2i}[e^{ix}(\log\sin x - ix) - e^{-ix}(\log\sin x + ix)]$$

$$= \log\sin x\left(\frac{e^{ix}-e^{-ix}}{2i}\right) - x\left(\frac{e^{ix}+e^{-ix}}{2}\right)$$

$$= (\log\sin x)\sin x - x\cos x.$$

Hence the complete solution is

$$y = c_1\cos x + c_2\sin x + \sin x\log\sin x - x\cos x.$$

EXAMPLE 1.101

Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x$.

Solution. The auxiliary equation is $m^2+5m+6=0$, which yields

$$m = \frac{-5\pm\sqrt{25-24}}{2} = -2, -3.$$

Therefore,

$$\text{C.F.} = c_1e^{-2x} + c_2e^{-3x}.$$

Further

$$\text{P.I.} = \frac{1}{f(D)}\;F(x)$$

$$= \frac{1}{D^2+5D+6}e^{-2x}\sin 2x$$

$$= e^{-2x}\frac{1}{(D-2)^2+5(D-2)+6}\sin 2x$$

$$= e^{-2x}\frac{1}{D^2+D}\;\sin 2x$$

$$= e^{-2x}\frac{1}{D-4}\sin 2x$$

$$= e^{-2x}.\frac{D+4}{D^2-4}\;\sin 2x$$

$$= e^{-2x}\frac{D+4}{-8}\;\sin 2x$$

$$= \frac{e^{-2x}}{-8}[D\sin 2x + 4\sin 2x]$$

$$= \frac{e^{-2x}}{-8}[2\cos 2x + 4\sin 2x].$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1e^{-2x} + c_2e^{-3x} - \frac{1}{8}e^{-2x}[2\cos 2x + 4\sin 2x].$$

EXAMPLE 1.102

Solve $\frac{d^3y}{dx^2} + y = \sin 3x - \cos^2\frac{x}{2}$.

Solution. The auxiliary equation of the given differential equation is

$$m^3+1=0$$

or

$$(m+1)(m^2-m+1)=0$$

and so $m = -1, \frac{1}{2}\pm\frac{\sqrt{3}}{2}\,i$. Therefore,

$$\text{C.F.} = c_1e^{-x}$$

$$+ e^{x/2}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right].$$

On the other hand,

$$\text{P.I.}=\frac{1}{D^3+1}\left[\sin 3x-\frac{1}{2}(1+\cos x)\right]$$
$$=\frac{1}{D^3+1}\sin 3x-\frac{1}{2(D^3+1)}(1+\cos x)$$
$$=\frac{1}{DD^2+1}\sin 3x-\frac{1}{2(1+D^3)}[1+\cos x]$$
$$=\frac{1}{-9D+1}\sin 3x-\frac{1}{2(1+D^3)}[1+\cos x]$$
$$=\frac{1}{1-9D}\sin 3x-\frac{1}{2}(1+D^3)^{-1}(1)-\frac{1}{2(1+D^3)}\cos x$$
$$=\frac{1+9D}{1-81D^2}\sin 3x-\frac{1}{2}(1-D^3+..)(1)-\frac{1}{2+2DD^2}\cos x$$
$$=\frac{1+9D}{1-81(-9)}\sin 3x-\frac{1}{2}-\frac{1}{2-2D}\cos x$$
$$=\frac{1+9D}{730}\sin 3x-\frac{1}{2}-\frac{2+2D}{4-4D^2}\cos x$$
$$=\frac{1}{730}(\sin 3x+27\cos 3x)-\frac{1}{2}-\frac{2+2D}{8}\cos x$$
$$=\frac{1}{730}(\sin 3x+27\cos 3x)-\frac{1}{2}-\frac{1}{4}\cos x+\frac{1}{4}\sin x$$
$$=\frac{1}{730}(\sin 3x+27\cos 3x)-\frac{1}{2}-\frac{1}{4}(\cos x-\sin x).$$

Hence the complete solution is

$$y=c_1e^{-x}+e^{x/2}(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x)$$
$$+\frac{1}{730}(\sin 3x+27\cos 3x)-\frac{1}{2}$$
$$-\frac{1}{4}(\cos x-\sin x).$$

EXAMPLE 1.103

Solve $\frac{d^3y}{dx^3}-3\frac{dy}{dx}+2y=x^2e^x$.

Solution. The symbolic form of the given equation is

$$(D^3-3D+2)y=x^2e^x.$$

Its auxiliary equation is

$$m^3-3m+2=0,$$

which yields $m = 1, 1, -4$. Therefore,

$$\text{C.F.}=(c_1+c_2x)e^x+c_3\,e^{-2x}.$$

Further

$$\text{P.I.}=\frac{1}{D^3-3D+2}x^2e^x$$
$$=e^x\frac{1}{(D+1)^3-3(D+1)+2}x^2$$
$$=e^x\frac{1}{D^3+3D^2}x^2=e^x\frac{1}{3D^2\left(1+\frac{D}{3}\right)}x^2$$
$$=e^x\frac{1}{3D^2}\left[1-\frac{D}{3}+\left(\frac{D}{3}\right)^2-..\right]x^2$$
$$=\frac{e^x}{3D^2}\left[1-\frac{D}{3}+\frac{D^2}{9}-\ldots\right]x^2$$
$$=\frac{e^x}{3D^2}\left(x^2-\frac{Dx^2}{3}+\frac{D^2x^2}{9}\right)$$
$$=\frac{e^x}{3D^2}\left(x^2-\frac{2x}{3}+\frac{2}{9}\right)$$
$$=\frac{e^x}{3D}\left[\int x^2dx-\frac{2}{3}\int x\,dx+\frac{2}{9}\int dx\right]$$
$$=\frac{e^x}{3D}\left[\frac{x^3}{3}-\frac{x^2}{3}+\frac{2}{9}x\right]$$
$$=\frac{e^x}{3}\left[\int\frac{x^3}{3}dx-\int\frac{x^2}{3}dx+\frac{2}{9}\int x\,dx\right]$$
$$=\frac{e^x}{3}\left[\frac{x^4}{12}-\frac{x^3}{9}+\frac{x^2}{9}\right]=\frac{e^x}{108}[3x^4-4x^3+4x^2]$$
$$=\frac{x^2e^x}{108}[3x^2-4x+4].$$

Hence complete solution is

$$y=\text{C.F.}+\text{P.I.}$$
$$=(c_1+c_2x)e^x+c_3e^{-2x}+\frac{x^2e^x}{108}[3x^2-4x+4].$$

EXAMPLE 1.104

Solve $(D^2-1)y=x\sin 3x+\cos x$.

Solution. The auxiliary equation for the given differential equation is $m^2-1=0$, and so $m=\pm 1$. Therefore,

$$\text{C.F.}=c_1e^x+c_2e^{-x}.$$

Further,

$$\text{P.I.} = \frac{1}{D^2-1}\,(x\sin 3x + \cos x)$$

$$= \frac{1}{D^2-1}x(I.P.\ of\ e^{3ix}) + \frac{1}{D^2-1}\cos x$$

$$= \text{I.P. of}\frac{1}{D^2-1}x\,e^{3ix} + \frac{1}{(-1^2)-1}\cos x$$

$$= \text{I.P.of}\left[e^{3ix}\frac{1}{(D+3i)^2-1}\,x\right] - \frac{1}{2}\cos x$$

$$= \text{I.P.of}\left[e^{3ix}\frac{1}{D^2+6iD-10}\,x\right] - \frac{\cos x}{2}$$

$$= \text{I.P.of}\left[e^{3ix}\frac{1}{-10\left(1-\frac{6}{10}\,iD-\frac{D^2}{10}\right)}x\right] - \frac{1}{2}\cos x$$

$$= -\frac{1}{10}\text{I.P.of}\left[e^{3ix}\left(1-\left(\frac{3}{5}iD+\frac{D^2}{10}\right)\right)^{-1}x\right] - \frac{1}{2}\cos x$$

$$= -\frac{1}{10}\text{I.P.of}\left[e^{3ix}\left(1+\frac{3}{5}\,iD+\frac{D^2}{10}+..\right)x\right] - \frac{1}{2}\cos x$$

$$= -\frac{1}{10}\text{I.P.of}\left[e^{3ix}\left(x+\frac{3}{5}i\right)\right] - \frac{1}{2}\cos x$$

$$= -\frac{1}{10}\text{I.P.of}\left[(\cos 3x + i\sin 3x)\left(x+\frac{3}{5}\,i\right)\right] - \frac{1}{2}\cos x$$

$$= -\frac{1}{10}\text{I.P.of}[x\cos 3x + ix\sin 3x + \frac{3i}{5}\cos 3x - \frac{3}{5}\sin 3x] - \frac{1}{2}\cos x$$

$$= -\frac{1}{10}\text{I.P.of}\left[x\cos 3x - \frac{3}{5}\sin 3x + i\left(\frac{3}{5}\cos 3x + x\sin 3x\right)\right] - \frac{1}{2}\cos x$$

$$= -\frac{1}{10}\left(\frac{3}{5}\cos 3x + x\sin 3x\right) - \frac{1}{2}\cos x.$$

Hence the complete solution of the given differential equation is

$$y = \text{C.F.} + \text{P.I.} = c_1e^{-x} + c_2e^{-x} - \frac{1}{10}\left(\frac{3}{5}\cos 3x + x\sin 3x\right) - \frac{1}{2}\cos x.$$

1.17 METHOD OF VARIATION OF PARAMETERS TO FIND PARTICULAR INTEGRAL

Definition 1.23. Let $y_1(x)$, $y_2(x)$, …, $y_n(x)$ be the functions defined on $[a, b]$ such that each function possesses $n-1$ derivatives on $[a, b]$. Then the determinant

$$W(y_1, y_2 \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is called the *Wronskian* of the set $\{y_1, y_2, \ldots, y_n\}$.

If the Wronskian of a set of n functions on $[a, b]$ is non-zero for atleast one point in $[a, b]$, then the set of n functions is linearly independent.

If the Wronskian is identically zero on $[a, b]$ and each of the function is a solution of the same linear differential equation, then the set of functions is linearly dependent.

The method of variation of parameters is applicable to the differential equation of the form

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = \text{F}(x), \tag{92}$$

where p, q, and F are functions of x.

Let the complementary function of (92) be $y=c_1y_1+c_2y_2$. Then y_1 and y_2 satisfy the equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0 \tag{93}$$

Replacing c_1 and c_2 (regarded as parameters) by unknown functions $u(x)$ and $v(x)$, we assume that particular integral of (92) is

$$y = uy_1 + vy_2. \tag{94}$$

Differentiating (94) with respect to x, we get

$$y' = uy_1' + xy_2' + u'y_1 + x'y_2$$
$$= uy_1' + xy_2', \tag{95}$$

under the assumption that

$$u'y_1 + v'y_2 = 0 \tag{96}$$

Differentiating (95) with respect to x, we get

$$y'' = uy_1'' + vy_2'' + u'y_1 + v'y_2'. \tag{97}$$

Substituting the values of y, y', and y'' from (94), (95), and (97) in (92), we have

$$uy_1'' + vy_2'' + u'y_1' + v'y_2' + p(uy_1' + vy_2') + q(uy_1 + vy_2) = F(x)$$

or

$$u(y_1''+py_1' + qy_1)+v(y_2''+py_2'+qy_2) + u'y_1'+v'y_2' = F(x).$$

Since y_1, y_2 satisfy (93), the above expression reduces to

$$u'y_1'+v'y_2' = F(x). \qquad (98)$$

Solving (96) and (98), we get

$$u' = -\frac{y_2F(x)}{W} \text{ and } v' = \frac{y_1F(x)}{W},$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2.$$

Integrating, we have

$$u = -\int \frac{y_2F(x)}{W}\,dx,\ v = \int \frac{y_1\,F(x)}{W}\,dx.$$

Substituting the value of u and v in (94), we get

$$\text{P.I.} = y = -y_1\int \frac{y_2F(x)}{W}\,dx + y_2\int \frac{y_1F(x)}{W}\,dx.$$

EXAMPLE 1.105

Using method of variation of parameters, solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^3x}{x^2}$.

Solution. The auxiliary equation for the given differential equation is

$$m^2 - 6m + 9 = 0,$$

which yields $m = 3, 3$. Therefore,

$$\text{C.F.} = (c_1+c_2x)\,e^{3x}.$$

Thus, we get

$$y_1 = e^{3x} \text{ and } y_2 = x\,e^{3x}.$$

The Wronskian of y_1, y_2 is

$$\text{W} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & x\,e^{3x} \\ 3e^{3x} & (3x+1)e^{3x} \end{vmatrix} = e^{6x}.$$

Therefore,

$$\begin{aligned} \text{P.I.} &= -y_1\int \frac{y_2F(x)}{W}\,dx + y_2\int \frac{y_1F(x)}{W}\,dx \\ &= -e^{3x}\int \frac{x\,e^{3x}.e^{3x}}{x^2\,e^{6x}}\,dx + xe^{3x}\int \frac{e^{3x}.e^{3x}}{x^2\,e^{6x}} \\ &= -e^{3x}\int \frac{dx}{x} + xe^{3x}\int \frac{1}{x^2}\,dx \\ &= -e^{3x}\log x + x\,e^{3x}\left(-\frac{1}{x}\right) = -e^{3x}(\log x + 1). \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} = (c_1 + c_2x)e^{3x} - e^{3x}(\log x + 1) \\ &= [k+c_2x-\log x]e^{3x} \text{ where } k = c_1-1. \end{aligned}$$

EXAMPLE 1.106

Using method of variation of parameters, solve $\frac{d^2y}{dx^2} + y = \sec x$.

Solution. The auxiliary equation for the given differential equation is $m^2 + 1 = 0$ and so $m = \pm i$. Thus

$$\text{C.F.} = c_1\cos x + c_2\sin x.$$

To find P.I., let

$$y_1 = \cos x \text{ and } y_2 = \sin x.$$

Then

$$\text{W} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

Therefore,

$$\begin{aligned} \text{P.I.} &= -y_1\int \frac{y_2F(x)}{W}dx + y_2\int \frac{y_1F(x)}{W}dx \\ &= -\cos x\int \frac{\sin x\sec x}{1}dx + \sin x\int \frac{\cos x\sec x\,dx}{1} \\ &= \cos x\log\cos x + x\sin x. \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} = c_1\cos x + c_2\sin x \\ &\quad + \cos x\log\cos x + x\sin x. \end{aligned}$$

EXAMPLE 1.107

Solve the given equation using method of variation of parameters

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x.$$

Solution. The symbolic form of the differential equation is

$$(D^2+1)y = \operatorname{cosec} x.$$

Its auxiliary equation is $m^2 + 1 = 0$ and so $m = \pm\, i$. Therefore,

$$\text{C.F.} = c_1 \cos x + c_2 \sin x.$$

To find P.I., let

$$y_1 = \cos x \quad \text{and} \quad y_2 = \sin x.$$

Then Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos \end{vmatrix} = 1.$$

Therefore,

$$\text{P.I.} = -y_1 \int \frac{y_2 F(x)}{W}\, dx + y_2 \int \frac{y_1 F(x)}{W} dx$$

$$= -\cos \mathrm{x} \int \sin x \operatorname{cosec} x\, dx$$

$$+ \sin x \int \cos x \operatorname{cosec} x\ dx$$

$$= -\cos \mathrm{x} \int dx + \sin x \int \frac{\cos x}{\sin x} dx$$

$$= -x \cos x + \sin x \log \sin x.$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x$$
$$- x \cos x + \sin x \log \sin x.$$

EXAMPLE 1.108

Solve the given equation using method of variation of parameters

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x.$$

Solution. The auxiliary equation is $m^2 - 2m = 0$ or $m(m - 2) = 0$ and so $m = 0,2$. Hence

$$\text{C.F.} = c_1 + c_2 e^{2x}.$$

Now let

$$y_1 = 1 \text{ and } y_2 = e^{2x}.$$

Then the Wronskian of y_1, y_2 is

$$W = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x}.$$

Therefore,

$$\text{P.I.} = -y_1 \int \frac{y_2 F(x)}{W}\, dx + y_2 \int \frac{y_1\, F(x)}{W}\ dx$$

$$= -\int \frac{e^{2x}.e^x \sin x}{2e^{2x}}\, dx + e^{2x} \int \frac{e^x \sin x}{2e^{2x}} dx$$

$$= -\frac{1}{2} \int e^x \sin x\, dx + \frac{e^{2x}}{2} \int e^{-x} \sin x\, dx$$

$$= -\frac{1}{2} e^x \sin x.$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x.$$

EXAMPLE 1.109

Solve $y'' - 2y' + 2y = e^x \tan x$.

Solution. The auxiliary equation is $m^2 - 2m + 2 = 0$ and so $m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$. Hence,

$$\text{C.F.} = e^x (c_1 \cos x + c_2 \sin x).$$

Let

$$y_1 = e^x \cos x \text{ and } y_2 = e^x \sin x.$$

Then the Wronskian of y_1, y_2 is

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\cos x + \sin x) \end{vmatrix}$$
$$= e^{2x}.$$

Therefore

$$\text{P.I.} = -y_1 \int \frac{y_2 F(x)}{W}\, dx + y_2 \int \frac{y_1\, F(x)}{W}\ dx$$

$$= -e^x \cos x \int \frac{e^x \sin x e^x \tan x}{e^{2x}}\ dx$$

$$+ e^x \sin x \int \frac{e^x \cos x e^x \tan x}{e^{2x}}\ dx$$

$$= -e^x \cos x \int (\sec x - \cos x)\, dx$$

$$+ e^x \sin x \int \sin x\, dx$$

$$= -e^{-x} \cos x[\log(\sec x + \tan x) - \sin x]$$
$$- e^x \sin x \cos x$$
$$= -e^{-x} \cos x \log(\sec x + \tan x).$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^x (c_1 \cos x + c_2 \sin x)$$
$$- e^{-x} \cos x\ \log(\sec x + \tan x).$$

EXAMPLE 1.110

Using method of variation of parameters, solve the differential equation

$$\frac{d^2y}{dx^2}+4y=\tan 2x.$$

Solution. The symbolic form of the given differential equation is

$$(D^2+4)y=\tan 2x.$$

Its auxiliary equation is $m^2+4=0$, which yields $m=\pm 2i$. Thus,

$$\text{C.F.} = c_1\cos 2x+c_2\sin 2x.$$

To find P.I., let

$$y_1=\cos 2x \text{ and } y_2=\sin 2x.$$

Then Wronskian W is

$$W=\begin{vmatrix}\cos 2x & \sin 2x\\ -2\sin 2x & 2\cos 2x\end{vmatrix}=2.$$

Hence,

$$\begin{aligned}\text{P.I.} &= -y_1\int\frac{y_2F(x)}{W}\,dx+y_2\int\frac{y_1F(x)}{W}\,dx\\ &=-\frac{\cos 2x}{2}\int\sin 2x\tan 2x\,dx\\ &\quad+\frac{\sin 2x}{x}\int\cos 2x\tan 2x\,dx\\ &=-\frac{\cos 2x}{2}\int\frac{\sin^2 2x}{\cos 2x}\,dx\\ &\quad+\frac{\sin 2x}{2}\int\cos 2x\tan 2x\,dx\\ &=-\frac{\cos 2x}{2}\int\frac{1-\cos^2 2x}{\cos 2x}\,dx\\ &\quad+\frac{\sin 2x}{2}\int\cos 2x\tan 2x\,dx\\ &=-\frac{1}{2}\cos 2x\int(\sec 2x-\cos 2x)dx\\ &\quad+\frac{1}{2}\sin 2x\int\sin 2x\,dx\\ &=-\frac{1}{4}\cos 2x[\log(\sec 2x+\tan 2x)-\sin 2x]\\ &\quad-\frac{1}{4}\sin 2x\cos 2x\\ &=-\frac{1}{4}\cos 2x\log(\sec 2x+\tan 2x).\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}y=\text{C.F.}+\text{P.I.} &=c_1\cos 2x+c_2\sin 2x\\ &\quad-\frac{1}{4}\cos 2x\log(\sec 2x+\tan 2x).\end{aligned}$$

EXAMPLE 1.111

Solve the equation using the method of variation of parameters

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=e^x\log x.$$

Solution. The auxiliary equation is $m^2-2m+1=0$ and so $m=1, 1$. Thus,

$$\text{C.F.}=(c_1+c_2x)e^x.$$

To find P.I., let $y_1=e^x$, $y_2=xe^x$. Then

$$W=\begin{vmatrix}e^x & x\,e^x\\ e^x & (x+1)e^x\end{vmatrix}=e^{2x}.$$

Therefore,

$$\begin{aligned}\text{P.I.} &= -y_1\int\frac{y_2F(x)}{W}\,dx+y_2\int\frac{y_1F(x)}{W}\,dx\\ &=-e^x\int\frac{x\,e^x.e^x}{e^{2x}}\log x\,dx+xe^x\int\frac{e^x.e^x\log x}{e^{2x}}\,dx\\ &=-e^x\int x\log x\,dx+xe^x\int\log x\,dx\\ &=-e^x\left[\frac{x^2}{2}\log x-\int\frac{x^2}{2x}\,dx\right]\\ &\quad+x\,e^x\left[x\log x-\int\frac{x}{x}\,dx\right]\\ &=-e^x\left(\frac{x^2}{2}\log x-\frac{x^2}{4}\right)+x^2e^x\log x-x^2e^x\\ &=-e^x\frac{x^2}{2}\log x+\frac{x^2}{4}e^x+x^2e^x\log x-x^2e^x\\ &=\frac{1}{2}e^xx^2\log x-\frac{3}{4}x^2e^x=\frac{1}{4}x^2e^x(2\log x-3).\end{aligned}$$

Hence the complete solution is

$$y=\text{C.F.}+\text{P.I.}=(c_1+c_2x)e^x+\frac{1}{4}x^2e^x(2\log x-3).$$

1.18 DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

In this section, we shall consider differential equations with variable coefficients which can be reduced to linear differential equations with constant coefficients.

The general form of a linear equation of second order is

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R,$$

where P, Q and R are functions of x only. For example,

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}+x^2y=\cos x$$

is a linear equation of second order.

If the coefficients P and Q are constants, then such type of equation can be solved by finding complementary function and particular integral as discussed earlier. But, if P and Q are not constants but variable, then there is no general method to solve such problem. In this section, we discuss three special methods to handle such problems.

(A) Method of Solution by Changing Independent Variable

Let

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R, \tag{99}$$

be the given linear equation of second order. We change the independent variable x to z by taking $z=f(x)$. Then

$$\frac{dy}{dx}=\frac{dy}{dz}.\frac{dz}{dx},$$

and

$$\frac{d^2y}{dx^2}=\frac{dy}{dz}\left(\frac{d^2z}{dx^2}\right)+\frac{d^2y}{dz^2}\left[\frac{dz}{dx}\right]^2.$$

Substituting these values in (99), we get

$$\frac{d^2y}{dz^2}\left[\frac{dz}{dx}\right]^2+\frac{dy}{dz}\left(\frac{d^2z}{dx^2}\right)+P\frac{dy}{dz}.\frac{dz}{dx}+Q\,y=R$$

or

$$\frac{d^2y}{dz^2}\left[\frac{dz}{dx}\right]^2+\frac{dy}{dz}\left[\frac{d^2z}{dx^2}+p\frac{dz}{dx}\right]+Q\,y=R$$

or

$$\frac{d^2y}{dz^2}+\frac{dy}{dz}\left[\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\right]+\frac{Q}{\left[\frac{dz}{dx}\right]^2}y=\frac{R}{\left[\frac{dz}{dx}\right]^2}$$

or

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1,$$

where

$$P_1=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left[\frac{dz}{dx}\right]^2},$$

$$Q_1=\frac{Q}{\left[\frac{dz}{dx}\right]^2},$$

and

$$R_1=\frac{R}{\left[\frac{dz}{dx}\right]^2}.$$

Using the functional relation between z and x, it follows that P_1, Q_1 and R_1 are functions of x. Choose z so that $P_1=0$, that is,

$$\frac{d^2z}{dx^2}+P\frac{dz}{dx}=0,$$

which yields

$$\frac{dz}{dx}=e^{-\int P dx}$$

or

$$z=\int e^{-\int P dx}dx.$$

If for this value of z, Q_1 becomes constant or a constant divided by z^2, then the equation (99) can be integrated to find its solution.

EXAMPLE 1.112

Solve $\frac{d^2y}{dx^2}+\cot x\frac{dy}{dx}+4y\operatorname{cosec}^2x=0$.

Solution. Comparing with the standard form, we get

$$P=\cot x,\quad Q=4\operatorname{cosec}^2x\quad\text{and}\quad R=0.$$

If we choose z such that $P_1=0$, then z is given by

$$\begin{aligned}z&=\int e^{-\int P dx}dx\\&=\int e^{-\int \cot x\,dx}dx\\&=\int e^{\log\,\operatorname{cosec} x}dx\\&=\int \operatorname{cosec} x\,dx=\log\,\tan\frac{x}{2}.\end{aligned}$$

Then

$$Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=\frac{4\operatorname{cosec}^2x}{\operatorname{cosec}^2x}=4,$$

and

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{0}{\operatorname{cosec}^2 x} = 0.$$

Therefore the given equation reduces to

$$\frac{d^2y}{dz^2} + 4y = 0,$$

or, in symbolic form,

$$(D^2 + 4)y = 0.$$

The auxiliary equation for this differential equation is $m^2 + 4 = 0$, which yields $m = \pm 2i$. Therefore the solution of the given equation is

$$\begin{aligned} y &= c_1 \cos 2z + c_2 \sin 2z \\ &= c_1 \cos\left(2\ \log\tan\frac{x}{2}\right) + c_2 \sin\left(2\ \log\tan\frac{x}{2}\right). \end{aligned}$$

EXAMPLE 1.113

Solve $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = x^5$.

Solution. Dividing throughout by x, the given differential equation reduces to

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = x^4.$$

Comparing with the standard form, we have

$$P = -\frac{1}{x}, \quad Q = -4x^2, \quad \text{and} \quad R = x^4.$$

Choose z so that $P_1 = 0$. Then

$$\begin{aligned} z &= \int e^{-\int P dx} dx = \int e^{\int \frac{1}{x} dx}\, dx = \int e^{\log x} dx \\ &= \int x\, dx = \frac{x^2}{2}. \end{aligned}$$

Further,

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-4x^2}{x^2} = -4,$$

and

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{x^2} = x^2.$$

Hence the given equation reduces to

$$\frac{d^2y}{dx^2} - 4y = x^2 = 2z$$

or

$$(D^2 - 4)y = 2z.$$

The auxiliary equation for this differential equation is $m^2 - 4 = 0$ and so $m = \pm 2$. Therefore

$$\text{C.F} = c_1 e^{2z} + c_2 e^{-2z} = c_1 e^{x^2} + c_2 e^{-x^2}.$$

Now

$$\begin{aligned} \text{P.I} &= \frac{1}{D^2 - 4}(2z) = \frac{-2}{4}\left(1 - \frac{D^2}{4}\right)^{-1}(z) \\ &= -\frac{1}{2}\left[1 + \frac{D^2}{4} + \ldots\right](z) \\ &= -\frac{z}{2} = -\frac{x^2}{4}. \end{aligned}$$

Hence the complete solution of the given differential equation is

$$y = \text{C.F} + \text{P.I} = c_1 e^{x^2} + c_2 e^{-x^2} - \frac{1}{4}x^2$$

EXAMPLE 1.114

Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3$.

(This equation is Cauchy–Euler equation and has also been solved in Example 1.121 by taking $t = \log x$).

Solution. Dividing the given equation throughout by x^2, we get

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2}{x^2}y = x.$$

Comparing with the standard form, we have

$$P = -\frac{2}{x}, \quad Q = \frac{2}{x^2} \quad \text{and} \quad R = x.$$

Choosing z such that $P_1 = 0$, we have

$$\begin{aligned} z &= \int e^{-\int P dx} dx = \int e^{2\int \frac{1}{x} dx} dx \\ &= \int e^{2\log x} dx = \int x^2\, dx = \frac{x^3}{3}. \end{aligned}$$

Therefore the given equation reduces to

$$\frac{d^2y}{dz^2} + Q_1 y = R_1,$$

where

$$Q_1 = \frac{Q}{\left[\frac{dz}{dx}\right]^2} = \frac{\frac{2}{x^2}}{(x^2)^2} = \frac{2}{x^6} = \frac{2}{9z^2}$$

and

$$R_1 = \frac{R}{\left[\frac{dz}{dx}\right]^2} = \frac{x}{x^4} = \frac{1}{x^3} = \frac{1}{3z}.$$

Hence the equation reduces to

$$\frac{d^2y}{dx}+\frac{2y}{9z^2}=\frac{1}{3z}$$

or

$$9z^2\frac{d^2y}{dz^2}+2y=3z.$$

Let $X=\log z$ so that $z=e^X$ and (see Example 1.121), $z^2\frac{d^2y}{dz^2}=D(D-1)y$. Thus the equation now reduces to

$$[9D(D-1)+2]y=3e^X$$

or

$$(9D^2-9D+2)y=3e^X.$$

The auxiliary equation for this symbolic equation is

$$m^2-9m+2=0,$$

which yields $m=\frac{2}{3},\frac{1}{3}$. Hence the complementary function is given by

$$\begin{aligned}\text{C.F}&=c_1e^{\frac{2}{3}X}+c_2e^{\frac{1}{3}X}\\&=c_1e^{\frac{2}{3}\log z}+c_2e^{\frac{1}{3}\log z}\\&=c_1z^{\frac{2}{3}}+c_2z^{\frac{1}{3}}=c_3x^2+c_4x.\end{aligned}$$

Further,

$$\begin{aligned}\text{P.I}&=3\frac{1}{9D^2-9D+2}e^X\\&=3\frac{1}{9-9+2}e^X=\frac{3}{2}e^X=\frac{3}{2}e^{\log z}\\&=\frac{3}{2}z=\frac{3}{2}\left(\frac{x^3}{3}\right)\\&=\frac{1}{2}x^3.\end{aligned}$$

Hence the complete solution of the given differential equation is

$$y=\text{C.F}+\text{P.I}=c_3x^2+c_4x+\frac{1}{2}x^3.$$

(B) Method of Solution by Changing the Dependent Variable

This method is also called the ‘‘Method of Removing First Derivative’’. Let

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \tag{100}$$

be the given linear equation of second order. We change the dependent variable y by taking $y=vz$. Then

$$\frac{dy}{dx}=v\frac{dz}{dx}+z\frac{dv}{dx},$$

$$\frac{d^2y}{dx^2}=v\frac{d^2z}{dx^2}+2\frac{dv}{dx}.\frac{dz}{dx}+z\frac{d^2v}{dx^2}.$$

Therefore equation (100) reduces to

$$z\frac{d^2v}{dx^2}+\left(2\frac{dz}{dx}+Pz\right)\frac{dv}{dx}+\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}+Qz\right)v=R$$

or

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{z}\frac{dz}{dx}\right)\frac{dv}{dx}+\frac{1}{z}\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}+Qz\right)v=\frac{R}{z}$$

or

$$\frac{d^2v}{dx^2}+P_1\frac{dv}{dx}+Q_1v=R_1, \tag{101}$$

where

$$P_1=P+\frac{2}{z}\frac{dz}{dx},$$

$$Q_1=\frac{1}{z}\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}+Qz\right),\quad R_1=\frac{R}{z}.$$

By making proper choice of z, any desired value can be assigned to P_1 or Q_1. In particular, if $P_1=0$, then

$$P+\frac{2}{z}.\frac{dz}{dx}=0$$

or

$$\frac{dz}{dx}=-\frac{Pz}{2}$$

or

$$\frac{dz}{z}=-\frac{1}{2}Pdx.$$

Integrating, we get

$$\log z=-\frac{1}{2}\int Pdx$$

or

$$z=e^{-\frac{1}{2}\int Pdx}.$$

Therefore,

$$\frac{dz}{dx}=-\frac{P}{2}e^{-\frac{1}{2}\int Pdx}$$

and

$$\frac{d^2z}{dx^2} = e^{-\frac{1}{2}\int P dx}\left[\frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx}\right].$$

Putting these values in Q_1, we get

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}.$$

Hence the equation (101) reduces to

$$\frac{d^2v}{dx^2} + \left(Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}\right)v = R_1. \qquad (102)$$

The equation (102) is called the *normal form* of the given differential equation (100) and we observe that this equation *does not have first derivative*. That is why, the present method is called the ''Method of Removing First Derivative''. The normal form (102) can be solved by using already discussed methods.

EXAMPLE 1.115

Solve $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} - 5y = 0$ by the method of removing first derivative.

Solution. Comparing the given equation with standard form, we have

$$P = -2\tan x \ , \ Q = -5 \ , \ R = 0.$$

Therefore,

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = -5 + \sec^2 x - \tan^2 x$$
$$= -5 + (1 + \tan^2 x) - \tan^2 x = -4,$$

and

$$z = e^{-\frac{1}{2}\int P dx} = e^{\int \tan x\, dx} = \sec x.$$

The normal form of the given equation is

$$\frac{d^2v}{dx^2} + Q_1 v = R_1$$

or

$$\frac{d^2v}{dx^2} - 4v = 0$$

or

$$(D^2 - 4)v = 0.$$

The auxiliary equation for this symbolic form is $m^2 - 4 = 0$. Therefore $m = \pm 2$ and so

$$v = c_1 e^{2x} + c_2 e^{-2x}$$

Hence the required solution is

$$y = vz = (c_1 e^{2x} + c_2 e^{-2x})\sec x.$$

EXAMPLE 1.116

Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)y = e^{x^2}$.

Solution. We have

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$
$$= 4x^2 - 3 - \frac{1}{2}(-4) - 4x^2$$
$$= -1$$

and

$$z = e^{-\frac{1}{2}\int P dx} = e^{2\int x\, dx} = e^{x^2}.$$

The normal form of the given equation is

$$\frac{d^2v}{dx^2} - v = \frac{e^{x^2}}{z} = \frac{e^{x^2}}{e^{x^2}} = 1$$

or

$$(D^2 - 1)v = 1.$$

Therefore A.E is $m^2 - 1 = 0$, which yields $m = \pm 1$. Thus,

$$\text{C.F} = c_1 e^x + c_2 e^{-x}.$$

Moreover,

$$\text{P.I} = -1.$$

Hence,

$$v = \text{C.F} + \text{P.I} = c_1 e^x + c_2 e^{-x} - 1.$$

But $y = vz$. Therefore

$$y = (c_1 e^x + c_2 e^{-x} - 1)e^{x^2}.$$

(c) Method of Undetermined Coefficients

This method is used to find Particular integral of the differential equation $F(D) = X$, where the input (forcing) function X consists of the sum of the terms, each of which possesses a finite number of essentially different derivatives. A trial solution consisting of terms in X and their finite derivatives is considered. Putting the values of the derivatives of the trial solution in $f(D)$ and comparing the coefficient on both sides of $f(D) = X$, the P I can be found. Obviously the method fails if X consists of terms like $\sec x$ and $\tan x$ having infinite number of different derivatives. Further, if any term in the trial solution is a part of the complimentary function, then that term should be multiplied by x and then tried.

EXAMPLE 1.117

Solve $\frac{d^2y}{dx^2} + y = e^x + \sin x$.

Solution. The symbolic form of the given differential equation is

$$(D^2 + 1)y = e^x + \sin x.$$

The auxiliary equation is $m^2 + 1 = 0$. Therefore $m = \pm i$ and so

$$\text{C.F} = c_1 \cos x + c_2 \sin x.$$

The forcing function consists of terms e^x and $\sin x$. Their derivatives are e^x and $\cos x$. So consider the trial solution.

$$y = ae^x + bx \sin x + c\,x \cos x.$$

Then,

$$\frac{dy}{dx} = a\,e^x + b(x \cos x + \sin x) + c(\cos x - x \sin x)$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= a\,e^x + b[\cos x - x \sin x + \cos x] \\ &\quad + c[-\sin x - \sin x - x \cos x] \\ &= a\,e^x + 2b \cos x - bx \sin x - 2c \sin x \\ &\quad - c\,x \cos x \\ &= a\,e^x + (2b - c\,x) \cos x - (bx + 2c) \sin x.\end{aligned}$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given equation, we get

$$2a\,e^x + 2b \cos x - 2c \sin x = e^x + \sin x.$$

Comparing corresponding coefficients, we have

$$2a = 1 \;,\; 2b = 0 \quad \text{and} \quad 2c = -1.$$

Thus $a = \frac{1}{2}$, $b = 0$, $c = -\frac{1}{2}$ and so

$$\text{P.I} = \frac{1}{2}e^x - \frac{1}{2}x \cos x.$$

Hence the solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2}e^x - \frac{1}{2}x \cos x.$$

EXAMPLE 1.118

Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x \sin x$.

Solution. The symbolic form of the given differential equation is

$$m^2 - 4m + 4 = 0$$

which yields $m = 2, 2$. Hence

$$\text{C.F} = (c_1 + c_2x)e^{2x}.$$

The forcing function is $e^x \sin x$. Its derivative is $e^x \cos x + e^x \sin x$. Therefore, we consider

$$y = ae^x \sin x + be^x \cos x$$

as the trial solution. Then

$$\begin{aligned}\frac{dy}{dx} &= a(e^x \cos x + e^x \sin x) + b(e^x \cos x - e^x \sin x) \\ &= (a + b)e^x \cos x + (a - b)e^x \sin x,\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= (a + b)[e^x \cos x - e^x \sin x] \\ &\quad + (a - b)[e^x \cos x + e^x \sin x] \\ &= 2a\,e^x \cos x - 2be^x \sin x.\end{aligned}$$

Substituting the values of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y in the given differential equation, we get

$$\begin{aligned}&2a\,e^x \cos x - 2b\,e^x \sin x - 4(a + b)e^x \cos x \\ &\quad - 4(a - b)e^x \sin x + 4ae^x \sin x + 4be^x \cos x \\ &= e^z \sin x\end{aligned}$$

or

$$-2ae^x \cos x + 2be^x \sin x = e^x \sin x.$$

Comparing coefficients, we get $2b = 1$ or $b = \frac{1}{2}$. Hence

$$\text{P.I} = \frac{1}{2}e^x \cos x$$

and so the complete solution of the given differential equation is

$$y = \text{C.F} + \text{P.I} = (c_1 + c_2x)e^{2x} + \frac{1}{2}e^x \cos x.$$

(D) Method of Reduction of Order

This method is used to find the complete solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q\,y = R$, where P, Q and R are function of x only, and when part of complementary function is known. So, let u, a function of x, be a part of the complementary function of the above differential equation. Then,

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Q\,U = 0. \qquad (103)$$

Let $y = u\,v$ be the complete solution of the given differential equation, where v is also a function of x.

Then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx},$$

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2v}{dx^2}.$$

Substituting these values of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y in the given equation, we get

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2v}{dx^2} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Quv = R$$

or

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\cdot\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

or, using (103),

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} = R$$

or, division by u yields,

$$\frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} = \frac{R}{u}$$

or, taking $\frac{dv}{dx} = z$, we get

$$\frac{dz}{dx} + \left(\frac{2}{u}\frac{du}{dx} + P\right)z = \frac{R}{u}, \qquad (104)$$

which is a first order differential equation in z and x. The integration factor for (104) is

$$I.F = e^{\int\left(\frac{2}{u}\frac{du}{dx}+P\right)dx}$$
$$= e^{\left(\int\frac{2}{u}dx+\int pdx\right)}$$
$$= u^2e^{\int pdx}.$$

Therefore, the solution of (104) is

$$z\,u^2\,e^{\int Pdx} = \int\frac{R}{u}\left(u^2e^{\int pdx}\right)dx + c_1$$

or

$$z = \frac{1}{u^2e^{\int Pdx}}\left[\int\frac{R}{u}\left(u^2e^{\int Pdx}\right)dx + e\right]$$

or

$$\frac{dv}{dx} = \frac{1}{u^2}e^{-\int Pdx}\left[\int Ru\,e^{\int Pdx}dx + c_1\right]. \qquad (105)$$

Integrating (105) with respect to x, we get

$$v = \int\frac{1}{u^2}e^{-\int Pdx}\left[\int Ru\,e^{\int Pdx}dx + c_1\right] + c_2,$$

where c_1 and c_2 are constants of integration. Hence the complete solution of the given differential equation is

$$y = uv$$
$$= u\left\{\int\frac{1}{u^2}e^{-\int Pdx}\left[\int Rue^{\int Pdx}dx + c_1\right]\right\} + c_2u.$$

EXAMPLE 1.119

Find the complete solution of

$$\frac{d^{2y}}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 2)y = 0,$$

if $y = e^{x^2}$ is an integral included in the complementary solution.

Solution. Let $y = uv$, where $u = e^{x^2}$ be the complete solution of the given equation. Then

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = 0,$$

where

$$P = -4x\,,\ Q = 4x^2 - 2$$

and $R = 0$.
Thus

$$\frac{d^2v}{dx^2} + \left[-4x + \frac{2}{e^{x^2}}(2xe^{x^2})\right]\frac{dv}{dx} = 0$$

or

$$\frac{d^2v}{dx^2} + [4x - 4x]\frac{dv}{dx} = 0$$

or

$$\frac{d^2v}{dx^2} = 0.$$

Integrating, we get

$$\frac{dv}{dx} = c_1.$$

Integrating once more, we get

$$v = c_1x + c_2.$$

Hence the complete solution is

$$y = uv = e^{x^2}[c_1x + c_2].$$

EXAMPLE 1.120

Solve $\sin^2 x \frac{d^2y}{dx^2} = 2y$, given that $y = \cot x$ is an integral included in the complementary function.

Solution. The given equation is

$$\frac{d^2y}{dx^2} - (2\text{cosec}^2 x)y = 0.$$

Comparing with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q\,y = R,$$

we get

$$P = 0,\ Q = -2\ \text{cosec}^2 x,\ R = 0.$$

Therefore, putting $y = uv = (\cot x)v$, the reduced equation is

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = 0,$$

that is,

$$\frac{d^2v}{dx^2} + \left[\frac{2}{\cot x}(-\text{cosec}^2 x)\right]\frac{dv}{dx} = 0$$

or

$$\cot x\frac{d^2v}{dx^2} - 2\text{cosec}^2 x\frac{dv}{dx} = 0. \qquad (106)$$

Let $\frac{dv}{dx} = z$. Then (106) reduces to

$$\cot x\frac{dz}{dx} - (2\text{cosec}^2 x)z = 0$$

or

$$\frac{dz}{z} = 2\frac{\text{cosec}^2 x}{\cot x}dx. \qquad (107)$$

Integrating (107), we get

$$\log z = -2\log\cot x + \log c_1$$

or

$$\log z + \log\cot^2 x = \log c_1$$

or

$$z = c_1\tan^2 x$$

or

$$\frac{dv}{dx} = c_1\tan^2 x = c_1(\sec^2 x - 1) \qquad (108)$$

Integrating (108), we get

$$v = c_1(\tan x - x) + c_2,$$

where c_1 and c_2 are constants of integration.

Hence the complete solution is

$$y = u\,v = \cot x[c_1(\tan x - x) + c_2]$$
$$= c_1(1 - x\cot x) + c_2\cot x.$$

(E) Cauchy–Euler Homogeneous Linear Equation

Consider the following differential equation with variable coefficients:

$$x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}x\frac{dy}{dx} + a_n = F(x), \qquad (109)$$

where a_i are constant and F is a function of x. This equation is known as *Cauchy–Euler homogeneous linear equation* (or *equidimensional equation*). The Cauchy–Euler homogeneous linear equation can be reduced to linear differential equation with constant coefficients by putting $x = e^t$ or $t = \log x$. Then,

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{1}{x}$$

or

$$x\frac{dy}{dx} = \frac{dy}{dt} = Dy.$$

Now,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\cdot\frac{dt}{dx}$$
$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

and so

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} = \mathrm{D}_2y\text{–}\mathrm{D}y = \mathrm{D}(\mathrm{D} - 1)y.$$

Similarly,

$$x^3\frac{d^3y}{dx^2} = \mathrm{D}(\mathrm{D} - 1)(\mathrm{D} - 2)y$$

and so on. Putting these values in (109), we obtain a linear differential equation with constant coefficients which can be solved by using the methods discussed already.

EXAMPLE 1.121

Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3$.

Solution. This is a Cauchy–Euler equation. Putting $x = e^t$ or $t = \log x$, we have $x\frac{dy}{dx} = \mathrm{D}y, x^2\frac{d^2y}{dx^2} = \mathrm{D}(\mathrm{D}-1)y$. Hence the given equation transforms to

$$(\mathrm{D}(\mathrm{D}-1) - 2\mathrm{D} + 2)\mathrm{y} = e^{3t}$$

or

$$(\mathrm{D}^2 - 3\mathrm{D} + 2)y = e^{3t},$$

which is a linear differential equation with constant coefficient. The auxiliary equation is $m^2 - 3m + 2 = 0$ and so $m = 1, 2$. Therefore

$$\text{C.F.} = c_1\, e^t + c_2\, e^{2t}.$$

The particular integral is

$$\text{P.I.} = \frac{1}{f(D)}F(x) = \frac{1}{D^2 - 3D + 2}\, e^{3t}$$
$$= \frac{1}{9 - 9 + 2}e^{3t} = \frac{1}{2}\, e^{3t}.$$

Hence the complete solution is

$$y = c_1 e^t + c_2 e^{2t} + \frac{1}{2}\, e^{3t}$$

Returning back to the variable x, we have

$$y = c_1 x + c_2 x^2 + \frac{1}{2}x^3.$$

EXAMPLE 1.122

Solve the Cauchy–Euler equation

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x.$$

Solution. Putting $x = e^t$, we have

$$x\frac{dy}{dx} = \mathrm{D}y \text{ and } x^2\frac{d^2y}{dx^2} = \mathrm{D}(\mathrm{D}-1)y$$

and so the equation transforms to

$$(\mathrm{D}(\mathrm{D}-1) - \mathrm{D} + 1)y = t$$

or

$$(\mathrm{D}-1)^2 y = t.$$

The complementary function is

$$\text{C.F.} = (c_1 + c_2 t)e^t.$$

Now

$$\text{P.I.} = \frac{1}{(D-1)^2}\, t = (1-D)^{-2}\, t$$
$$= (1 + 2\mathrm{D} + \ldots)t = t + 2.$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 t)e^t + t + 2.$$

Returning back to x, we get

$$y = (c_1 + c_2 \log x)x + \log x + 2.$$

EXAMPLE 1.123

Solve Cauchy–Euler equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x).$$

Solution. Putting $x = e^t$, this equation transforms to

$$(\mathrm{D}^2 + 1)y = t\sin t.$$

The complementary function is

$$\text{C.F.} = c_1\cos t + c_2\sin t.$$

Further

$$\text{P.I.} = \frac{1}{D^2+1}\, t\sin t = \text{I.P. of } \frac{1}{D^2+1}\, te^{it}$$
$$= \text{I.P. of } e^{it}\frac{1}{(D+i)^2 + 1}$$
$$= \text{I.P. of } e^{it}\frac{1}{2iD\left(1 + \frac{D}{2i}\right)}\, t$$
$$= \text{I.P. of } e^{it}.\frac{1}{2i}.\frac{1}{D}\left(1 - \frac{iD}{2}\right)^{-1} t$$
$$= \text{I.P. of } e^{it}\left(-\frac{it^2}{4} + \frac{t}{4}\right)$$
$$= \text{I.P. of } (\cos t + i\sin t)\left(\frac{-it^2}{4} + \frac{t}{4}\right)$$
$$= -\frac{t^2}{4}\cos t + \frac{t}{4}\sin t.$$

Therefore, the complete solution is

$$y = c_1\cos t + c_2\sin t - \frac{t^2}{4}\cos t + \frac{t}{4}\sin t.$$

Returning back to x, we get

$$y=c_1\cos(\log x)+c_2\sin(\log x)-\frac{1}{4}(\log x)^2\cos(\log x)+\frac{1}{4}\log x\sin(\log x).$$

EXAMPLE 1.124

Solve the Cauchy–Euler equation

$$x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y=x\log x.$$

Solution. Putting $x = e^t$, the equation reduces to

$$(\mathrm{D}(\mathrm{D}-1)-\mathrm{D}+2)y = t\,e^t$$

or

$$(\mathrm{D}^2-2\mathrm{D}+2)y = t\,e^t$$

The C.F. for this equation is

$$\text{C.F.} = e^t(c_1\cos t+c_2\sin t).$$

The particular integral is

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2-2D+2}t\,e^t\\ &= e^t\frac{1}{(D+1)^2-2(D+1)+2}t\\ &= e^t\frac{1}{D^2+1}t = e^t(1-D^2)^{-1}t = e^t(t-0) = t\,e^t.\end{aligned}$$

Therefore, the complete solution is

$$\begin{aligned}y = \text{C.F.}+\text{P.I.} &= e^t(c_1\cos t+c_2\sin t)+t\,e^t\\ &= x(c_1\cos(\log x)\\ &\quad + c_2\sin(\log x)+x\log x.\end{aligned}$$

EXAMPLE 1.125

Solve $x^2\frac{d^2y}{dx^2}-2y=x^2+\frac{1}{x}$.

Solution. Putting $x = e^t$, the given equation reduces to $(\mathrm{D}(\mathrm{D}-1)-2)y = e^{2t}+\frac{1}{e^t}$.

The auxiliary equation is $m^2 - m - 2 = 0$ and so $m = 2, -1$. Therefore,

$$\text{C.F.} = c_1\,e^{2t}+c_2\,e^{-t}.$$

Moreover,

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2-D-2}\left(e^{2t}+\frac{1}{e^t}\right)\\ &= \frac{1}{D^2-D-2}e^{2t}+\frac{1}{D^2-D-2}(e^{-t})\\ &= t\frac{1}{[2D-1]_{D=2}}e^{2t}+t\frac{1}{(2D-1)_{D=-1}}e^{-t}\\ &= \frac{t}{3}e^{2t}-\frac{1}{3}te^{-t}.\end{aligned}$$

Thus the complete solution is

$$\begin{aligned}y = \text{C.F.}+\text{P.I.} &= c_1e^{2t}+c_2\,e^{-t}+\frac{t}{3}e^{2t}-\frac{t}{3}e^{-t}\\ &= c_1x^2+\frac{c_2}{x}+\frac{1}{3}\left(x^2-\frac{1}{x}\right)\log x.\end{aligned}$$

EXAMPLE 1.126

Solve the Cauchy–Euler equation

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-20y=(x+1)^2.$$

Solution. Putting $x = e^t$, the given equation transforms into

$$(\mathrm{D}(\mathrm{D}-1)+2\mathrm{D}-20)y = e^{2t}+2e^t+1$$

or

$$(\mathrm{D}^2+\mathrm{D}-20)y = e^{2t}+2e^t+1.$$

The auxiliary equation is $m^2 + m - 20 = 0$ and so $m = -5, 4$. Therefore,

$$\text{C.F.} = c_1\,e^{-5t}+c_2e^{4t}.$$

Now,

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2+D-20}e^{2t}+\frac{2}{D^2-D-20}e^t\\ &\quad+\frac{1}{D^2-D-20}x^{0\,t}\\ &= -\frac{1}{14}e^{2t}+\frac{2}{-20}e^t-\frac{1}{20}.\end{aligned}$$

Thus the complete solution is

$$\begin{aligned}y = \text{C.F.}+\text{P.I.} &= c_1e^{-5t}+c_2\,e^{4t}-\frac{1}{14}e^{2t}-\frac{1}{10}e^t-\frac{1}{20}\\ &= c_1x^{-5}+c_2x^4-\frac{1}{14}x^2-\frac{x}{10}-\frac{1}{20}\end{aligned}$$

(F) Legendre's Linear Equation

An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + a_1 (ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_n y = \mathrm{F}(x) \qquad (110)$$

where a_n are constants and F is a function of x, is called *Legendre's linear equation*.

To reduce the Legendre's equation to a linear differential equation with constant coefficient, we put $ax + b = e^t$ or $t = \log(ax + b)$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{a}{ax+b}\frac{dy}{dt}$$

or

$$(ax+b)\frac{dy}{dx} = a\mathrm{D}y.$$

Further

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dt}\right) = \frac{a^2}{(ax+b)^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

and so

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2(\mathrm{D}^2 - \mathrm{D})y = a^2\mathrm{D}(\mathrm{D}-1)y.$$

Similarly,

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3\mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2)y$$

and so on.

Putting these values in (110), we get a linear differential equation with constant coefficients which can be solved by usual methods.

EXAMPLE 1.127

Solve $(2x+3)^2\frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6\mathrm{x}$.

Solution. Putting $2x + 3 = e^t$ or $t = \log(2x + 3)$, the given equation reduces to

$$(4(\mathrm{D}^2 - \mathrm{D}) - 4\mathrm{D} - 12)y = 6\frac{e^t - 3}{2} = 3e^t - 9$$

or

$$(4\mathrm{D}^2 - 8\mathrm{D} - 12)y = 3e^t - 9.$$

The auxiliary equation is

$$4m^2 - 8m - 12 = 0,$$

which yields $m = 3, -1$. Therefore,

$$\mathrm{C.F.} = c_1 e^{-t} + c_2 e^{3t}.$$

Now

$$\mathrm{P.I.} = \frac{1}{4D^2 - 8D - 12}(3e^t - 9) = \frac{3}{-16}e^t + \frac{3}{4}.$$

Hence the complete solution is

$$y = c_1 e^{-t} + c_2 e^{3t} - \frac{3}{16}e^t + \frac{3}{4}$$

$$= \frac{c_1}{(2x+3)} + c_2(2x+3)^3 - \frac{3}{16}(2x+3) + \frac{3}{4}.$$

EXAMPLE 1.128

Solve the Legendre's equation

$$(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x).$$

Solution. Putting $x + 1 = e^t$ or $t = \log(x + 1)$, the given equations transforms to $(1^2(\mathrm{D}^2 - \mathrm{D}) + 1\,\mathrm{D} + 1)y = 4\cos t$

or

$$(\mathrm{D}^2 + 1)y = 4\cos t.$$

The auxiliary equation is $m^2 + 1 = 0$ and so $m = \pm i$. Therefore,

$$\mathrm{C.F.} = c_1\cos t + c_2\sin t.$$

The particular integral is

$$\mathrm{P.I.} = \frac{4}{D^2+1}\cos t = t\frac{4(2D)}{[4D^2]_{D^2=-1}}\cos t$$

$$= \frac{8t}{-4}D\cos t = -2t(-\sin t) = 2t\sin t.$$

Thus the complete solution is

$$y = c_1\cos t + c_2\sin t + 2t\sin t$$

$$= c_1\cos(\log(x+1)) + c_2\sin(\log(x+1)) + 2\log(x+1)\sin(\log(x+1)).$$

EXAMPLE 1.129

Solve the Legendre's equation

$$(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin(2\log(1+x)).$$

Solution. As in Example 1.128, putting $x+1=e^t$, the given equation reduces to

$$(D^2+1)y=\sin(2t).$$

The complementary function is

$$\text{C.F.} = c_1\cos t+c_2\sin t$$

Further,

$$\text{P.I.} = \frac{1}{D^2+1}\sin[2t] = \frac{1}{-4+1}\sin 2t = -\frac{1}{3}\sin 2t.$$

Therefore, the complete solution is

$$\begin{aligned} y &= c_1\cos t+c_2\sin t-\frac{1}{3}\sin 2t \\ &= c_1\cos(\log(x+1))+c_2\sin(\log(x+1) \\ &\quad -\frac{1}{3}\sin(2\log(x+1)). \end{aligned}$$

EXAMPLE 1.130

Solve $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=2\sin(\log(x+1))$.

Solution. As in Example 1.128, putting $x+1=e^t$, the given equation transforms to

$$(D^2+1)y=2\sin t.$$

Its complementary function is given by

$$\text{C.F.} = c_1\cos t+c_2\sin t.$$

Its particular integral is given by

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2+1}(2\sin t) = \frac{2}{D^2+1}\sin t \\ &= \frac{2t}{[2D]_{D^2=-1}}\sin t = \frac{D\sin t}{-1} = -t\cos t. \end{aligned}$$

Therefore, the complete solution is

$$\begin{aligned} y &= c_1\cos t+c_2\sin t-t\cos t \\ &= c_1\cos(\log(x+1))+c_2\sin(\log(x+1)) \\ &\quad -\log(x+1)\cos(\log(x+1)). \end{aligned}$$

1.19 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Differential equations, in which there are two or more dependent variables and a single independent variable are called *simultaneous linear equations*. The aim of this section is to solve a system of linear differential equations with constant coefficients. The solution is obtained by eliminating all but one of the dependent variables and then solving the resultant equations by usual methods.

EXAMPLE 1.131

Solve the simultaneous equations

$$\frac{dx}{dt}=7x-y,\ \frac{dy}{dt}=2x+5y.$$

Solution. In symbolic form, we have

$$(D-7)x+y=0 \tag{111}$$

$$(D-5)y-2x=0 \tag{112}$$

Multiplying (111) by (D − 5) and subtracting (112) from it, we get

$$(D-5)(D-7)x+2x=0$$

or

$$(D^2-12D+35+2)x=0$$

or

$$(D^2-12D+37)x=0.$$

Its auxiliary equation is $m^2-12m+37=0$, which yields $m=6\pm i$. Therefore, its complete solution is $x=e^{6t}(c_1\cos t+c_2\sin t)$.

Now

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}(c_1e^{6t}\cos t)+\frac{d}{dt}(c_2e^{6t}\sin t) \\ &= c_1[6e^{6t}\cos t-e^{6t}\sin t]+c_2[6e^{6t}\sin t+e^{6t}\cos t] \\ &= e^{6t}[(6c_1+c_2)\cos t+(6c_2-c_1)\sin t] \end{aligned}$$

Putting the values of x and $\frac{dx}{dt}$ in (101), we get

$$e^{6t}[(6c_1+c_2)\cos t+(6c_2-c_1)\sin t] - 7e^{6t}[c_1\cos t+c_2\sin t]+y=0.$$

Therefore,

$$\begin{aligned} y &= 7e^{6t}(c_1\cos t+c_2\sin t) \\ &\quad - e^{6t}[(6c_1+c_2)\cos t+(6c_2-c_1)\sin t] \\ &= e^{6t}[(c_1-c_2)\cos t+(c_1+c_2)\sin t]. \end{aligned}$$

Hence the solution is

$$x=e^{6t}(c_1\cos t+c_2\sin t)$$

$$y=e^{6t}[(c_1-c_2)\cos t+(c_1+c_2)\sin t].$$

EXAMPLE 1.132

Solve $\frac{d^2y}{dt^2}+\frac{dy}{dt}-2y=\sin t,\ \frac{dx}{dy}+x-3y=0.$

Solution. The given system of equations are

$$(D^2+D-2)y = \sin t \qquad (113)$$

$$(D+1)x - 3y = 0 \qquad (114)$$

The auxiliary equation for (113) is

$$m^2+m-2=0$$

and so $m = -2, 1$. Therefore,

$$\text{C.F.} = c_1 e^t + c_2 e^{-2t}.$$

$$\text{P.I.} = \frac{1}{D^2+D^2-2}\sin t = \frac{1}{D-3}\sin t$$
$$= \frac{D+3}{D^2-9}\sin t = -\frac{1}{10}(D+3\sin t)$$
$$= -\frac{3}{10}\sin t - \frac{1}{10}\cos t.$$

Therefore, the complete solution of (113) is

$$y = c_1 e^t + c_2 e^{-2t} - \frac{1}{10}(\cos t + 3\sin t).$$

Putting this value of y in (114), we get

$$\frac{dx}{dt}+x = 3[c_1e^t + c_2e^{-2t} - \frac{1}{10}(\cos t + 3\sin t)]$$

The integrating factor is $e^{\int 1\,dt} = e^t$. Therefore,

$$x.e^t = 3\int e^t[c_1e^t + c_2e^{-2t} - \frac{1}{10}(\cos t + 3\sin t)]dt,$$

which yields

$$x = \tfrac{3}{2}c_1e^{2t} - 3e_2e^{-t} - \tfrac{3}{10}e^t(\cos t - 2\sin t).$$

Therefore, the solution is

$$x = \frac{3}{2}c_1e^{2t} - 3c_2e^{-t} - \frac{3}{10}e^t(\cos t - 2\sin t)$$

$$y = c_1e^t + c_2e^{-2t} - \frac{1}{10}(\cos t + 3\sin t).$$

EXAMPLE 1.133

A mechanical system with two degrees of freedom satisfies the equation

$$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4, 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0$$

under the condition that $x, y, \frac{dx}{dt}, \frac{dy}{dt}$ all vanish at $t = 0$. Find x and y.

Solution. The given equations are

$$2D^2x + 3Dy = 4$$

$$2D^2y - 3Dx = 0 \qquad (115)$$

or

$$4D^3x + 6D^2y = 2D(4) = 0$$
$$-9Dx + 6D^2y = 0.$$

Subtracting, we get

$$4D^3x+9Dx = 0$$

or

$$(4D^3+9D)x = 0.$$

Auxiliary equation is

$$4m^3 + 9m = 0, \text{and so } m = 0, \pm\frac{3}{2}i.$$

Hence

$$x = c_1 + c_2\cos\frac{3}{2}t + c_3\sin\frac{3}{2}t.$$

At $t = 0$, $x = 0$. Therefore, $0 = c_1 + c_2$ or $c_1 = -c_2$. Also

$$\frac{dx}{dt} = -\frac{3}{2}c_2\sin\frac{3}{2}t + \frac{3}{2}c_3\cos\frac{3}{2}t$$

At $t = 0$, $\frac{dx}{dt} = 0$ and so $0 = \frac{3}{2}c_3$ or $c_3 = 0$. Thus

$$x = c_1 - c_1\cos\frac{3}{2}t.$$

Therefore,

$$\frac{dx}{dt} = \frac{3}{2}c_1\sin\frac{3}{2}t \text{ and } \frac{d^2x}{dt^2} = \frac{9}{4}c_1\cos\frac{3}{2}t.$$

Putting this value of $\frac{dx}{dt}$ in (115), we get

$$2D^2y - \frac{9}{2}c_1\sin\frac{3}{2}t = 0 \text{ or } D^2y = \frac{9}{4}c_1\sin\frac{3}{2}t.$$

Integrating, we get

$$Dy = -\frac{3}{2}c_1\cos\frac{3}{2}t + k.$$

Using initial condition $\frac{dy}{dt} = 0$ at $t = 0$, we get $k = \frac{3}{2}c_1$. Therefore,

$$Dy = -\frac{3}{2}c_1\cos\frac{3}{2}t + \frac{3}{2}c_1.$$

Integrating again, we have

$$y = -c_1\sin\frac{3}{2}t + \frac{3}{2}c_1t + k.$$

When $t = 0$, $y = 0$. So we have $k = 0$. Hence

$$y = -c_1\sin\frac{3t}{2} + \frac{3}{2}c_1t.$$

Further, putting the value of $\frac{d^2x}{dt^2}$ and $\frac{dy}{dt}$ in the first of the given equations, we get $c_1 = \frac{8}{9}$. Hence

$$x = \frac{8}{9}\left(1 - \cos\frac{3t}{2}\right), \; y = \frac{4}{3}t - \frac{8}{9}\sin\frac{3t}{2}.$$

1.20 APPLICATIONS OF LINEAR DIFFERENTIAL EQUATIONS

Linear differential equations play an important role in the analysis of electrical, mechanical and other linear systems. Some of the applications of these equations are discussed below.

(A) Electrical Circuits

Consider an LCR circuit consisting of inductance L, capacitor C, and resistance R. Then, using Kirchhoff's law [see (16)], the equation governing the flow of current in the circuit is

$$\mathrm{L}\frac{dI}{dt} + RI + \frac{Q}{C} = \mathrm{E}(t),$$

where I is the current flowing in the circuit, Q is the charge, and E is the e.m.f. of the battery. Since $I = \frac{dQ}{dt}$, the above equation reduces to

$$\mathrm{L}\frac{d^2Q}{dt} + R\frac{dQ}{dt} + \frac{Q}{C} = \mathrm{E}(t) \qquad (116)$$

If we consider LC circuit without having e.m.f. source, then the differential equation describing the circuit is

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

or

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$$

or

$$\frac{d^2Q}{dt^2} + \mu^2 Q = 0, \mu^2 = \frac{1}{LC}.$$

This equation represents *free electrical oscillations* of the current having period

$$\mathrm{T} = \frac{2\pi}{\mu} = 2\pi\sqrt{LC}.$$

EXAMPLE 1.134

In an LCR circuit, an inductance L of one henry, resistance of 6 ohm, and a condenser of 1/9 farad have been connected through a battery of e.m.f. $\mathrm{E} = \sin t$. If $I = Q = 0$ at $t = 0$, find charge Q and current I.

Solution. The differential equation for the given circuit is

$$\mathrm{L}\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = \mathrm{E}(t).$$

Here $L = 1$, $\mathrm{R} = 6$, $C = \frac{1}{9}$, $\mathrm{E}(t) = \sin t$. Thus, we have

$$\frac{d^2Q}{dt^2} + 6\frac{dQ}{dt} + 9Q = \sin t$$

subject to $Q(0) = 0$, $Q'(0) = 0 = \mathrm{I}(0)$. The auxiliary equation for this differential equation is $m^2 + 6m + 9 = 0$ and so $m = -3, -3$. Thus

$$\text{C.F.} = (c_1 + c_2 t)e^{-3t}.$$

Now

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 + 6D + 9}\sin t = \frac{1}{6D + 8}\sin t \\ &= \frac{6D - 8}{36D^2 - 64}\sin t = \frac{6D - 8}{-100}\sin t \\ &= -\frac{1}{100}[6D\sin t - 8\sin t] \\ &= -\frac{6}{100}\cos t + \frac{8}{100}\sin t.\end{aligned}$$

Hence the complete solution is

$$\mathrm{Q} = (c_1 + c_2 t)e^{-3t} - \frac{6}{100}\cos t + \frac{8}{100}\sin t.$$

Now $\mathrm{Q}(0) = 0$ gives $0 = c_1 - \frac{6}{100}$ and so $c_1 = \frac{6}{100}$. Also

$$\begin{aligned}\frac{dQ}{dt} &= -3c_1 e^{-3t} + c_2(e^{-3t} - 3te^{-3t}) \\ &\quad + \frac{6}{100}\sin t + \frac{8}{100}\cos t.\end{aligned}$$

Therefore $\frac{dQ}{dt} = 0$ at $t = 0$ yields

$$0 = -3c_1 + c_2 + \frac{8}{100} = -\frac{18}{100} + c_2 + \frac{8}{100}$$

and so $c_2 = \frac{1}{10}$. Hence

$$\begin{aligned}\mathrm{Q} &= \left(\frac{6}{100} + \frac{t}{10}\right)e^{-3t} - \frac{6}{100}\cos t + \frac{8}{100}\sin t \\ &= \frac{e^{-3t}}{50}(5t + 3) - \frac{3}{50}\cos t + \frac{2}{25}\sin t.\end{aligned}$$

Since $I = \frac{dQ}{dt}$, we have

$$I = \frac{5e^{-3t}}{50} - \frac{3}{50}(5t+3)e^{-3t} + \frac{3}{50}\sin t + \frac{2}{25}\cos t$$
$$= -\frac{e^{-3t}}{50}(15t+4) + \frac{3}{50}\sin t + \frac{2}{25}\cos t.$$

EXAMPLE 1.135

Find the frequency of *free vibrations* in a closed electrical circuit with inductance L and capacity C in series.

Solution. Since there is no applied e.m.f., the differential equation governing this LC circuit is

$$\mathrm{L}\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

or

$$\frac{d^2Q}{dt^2} = -\frac{Q}{LC} = -\omega^2\mathrm{Q},$$

where $\omega^2 = \frac{1}{LC}$. Thus the equation represents oscillatory current with period.

$$\mathrm{T} = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}.$$

Then

Frequency $= \frac{1}{T} = \frac{1}{2\pi\sqrt{LC}}$ per second

$$= \frac{60}{2\pi\sqrt{LC}} = \frac{30}{\pi\sqrt{LC}} \text{ per minute.}$$

EXAMPLE 1.136

The differential equation for a circuit in which self-inductance and capacitance neutralize each other is

$$\mathrm{L}\frac{d^2i}{dt^2} + \frac{i}{C} = 0.$$

Find the current i as a function of t, given that I is maximum current and $i = 0$ when $t = 0$.

Solution. We have

$$\frac{d^2i}{dt^2} + \frac{i}{LC} = 0.$$

The auxiliary equation is $m^2 + \frac{1}{LC} = 0$ and so $m = \pm\frac{i}{\sqrt{LC}}$. Hence the solution is

$$i = c_1\cos\frac{1}{\sqrt{LC}}t + c_2\sin\frac{1}{\sqrt{LC}}t.$$

Since $i = 0$ at $t = 0$, we have $c_1 = 0$ and so

$$i = c_2\sin\frac{t}{\sqrt{LC}}.$$

For maximum current I, we have $I = c_2$ max sin $\frac{t}{\sqrt{LC}} = c_2$. Hence $i = I\sin\frac{t}{\sqrt{LC}}$.

EXAMPLE 1.137

An LCR circuit with battery e.m.f. E sin pt is tuned to resonance so that $p^2 = \frac{1}{LC}$. Show that for small value of $\frac{R}{L}$, the current in the circuit at time t is given by $\frac{Et}{2L}\sin pt$.

Solution. The differential equation governing the LCR circuit is $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = \mathrm{E}(t) = \mathrm{E}\sin pt$.
The auxiliary equation is

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0,$$

which yields

$$m = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$
$$= \frac{-\frac{R}{L} \pm \sqrt{-\frac{4}{LC}}}{2} \quad \text{since } \frac{R}{L} \text{ is small}$$
$$= -\frac{R}{2L} \pm \frac{1}{\sqrt{LC}}i$$
$$= -\frac{R}{2L} \pm pi \text{ since } p^2 = \frac{1}{LC}.$$

Therefore,

$$\text{C.F.} = e^{-\frac{Rt}{2L}}(c_1\cos pt + c_2\sin pt)$$
$$= \left(1 - \frac{Rt}{2L}\right)(c_1\cos pt + c_2\sin pt)$$

rejecting higher power of $\frac{R}{L}$.

Further,

$$\text{P.I.} = \frac{1}{LD^2 + RD + \frac{1}{C}}(E\sin pt)$$
$$= \frac{E}{-Lp^2 + RD + \frac{1}{C}}\sin pt$$
$$= \frac{E}{-\frac{L}{LC} + \frac{1}{C} + RD}\sin pt$$
$$= \frac{E}{RD}\sin pt = \frac{E}{R}\frac{D}{D^2}\sin pt$$
$$= -\frac{E}{Rp^2}D\sin pt$$
$$= -\frac{E}{Rp^2}p\cos pt = -\frac{E}{Rp}\cos pt.$$

Thus, the complete solution is

$$q = \left(1 - \frac{Rt}{2L}\right)(c_1 \cos pt + c_2 \sin pt) - \frac{E}{Rp}\cos pt.$$

Using the initial condition $q = 0$ for $t = 0$, we get

$$0 = c_1 - \frac{E}{Rp} \text{ or } c_1 = \frac{E}{Rp}.$$

Also

$$i = \frac{dq}{dt} = \left(1 - \frac{Rt}{2L}\right)(-c_1 \sin pt + c_2 \cos pt)p$$
$$-\frac{R}{2L}(c_1 \cos pt + c_2 \sin pt) + \frac{E}{R}\sin pt.$$

Using the initial condition $i = 0$ for $t = 0$, we get

$$0 = pc_2 - \frac{Rc_1}{2L}$$
$$= pc_2 - \frac{R}{2L}\left(\frac{E}{Rp}\right)$$
$$= pc_2 - \frac{RE}{2Lp}$$

and so $c_2 = \frac{E}{2Lp^2}$. Hence, the solution is

$$i = \left(1 - \frac{Rt}{2L}\right)\left(-\frac{E}{Rp}\sin pt + \frac{2}{2Lp^2}\cos pt\right)p$$
$$-\frac{R}{2L}\left(\frac{E}{Rp}\cos pt + \frac{E}{2Lp^2}\sin pt\right) + \frac{E}{R}\sin pt$$
$$= \frac{Et}{2L}\sin pt$$

1.21 MASS-SPRING SYSTEM

In Example 1.4, we have seen that the differential equation governing a Mass-spring system is

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = f(t)$$

where m is the mass, $a\,\frac{dx}{dt}$ the damping force due to the medium, k is spring constant, and x represents the displacement of the mass.

This is exactly the same differential equation which occurs in LCR electric circuits. When $a = 0$, the motion is called *undamped* whereas if $a \neq 0$, the motion is called *damped*. If $f(t) = 0$, then the motion is called *forced*.

EXAMPLE 1.138

A mass of 10 kg is attached to a spring having spring constant 140 N/m. The mass is started in motion from the equilibrium position with a velocity of 1 m/sec in the upward direction and with an applied external force $f(t) = 5 \sin t$. Find the subsequent motion of the mass if the force due to air resistance is $-90\,\dot{x}$ N.

Solution. The describing differential equation is

$$\frac{d^2x}{dt^2} + \frac{a}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{1}{m}f(t).$$

Here $a = 90$, $k = 140$, and $m = 10$. Therefore, we have

$$\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 14x = \frac{1}{2}\sin t.$$

The auxiliary equation is $m^2 + 9m + 14 = 0$, and so $m = -2, -7$. Therefore,

$$\text{C.F.} = c_1e^{-2t} + c_2e^{-7t}.$$

Further

$$\text{P.I.} = \frac{1}{D^2+9D+14}\left(\frac{1}{2}\sin t\right) = \frac{1}{2(-1+9D+14)}\sin t$$
$$= \frac{1}{2}\frac{1}{(9D+13)}\sin t = \frac{1}{2}\frac{9D-13}{(-81-169)}\sin t$$
$$= -\frac{1}{500}(-13\sin t + 9\cos t)$$
$$= \frac{13}{500}\sin t - \frac{9}{500}\cos t.$$

Hence, the complete solution is

$$x = c_1e^{-2t} + c_2e^{-7t} + \frac{13}{500}\sin t - \frac{9}{500}\cos t.$$

Using the initial condition $x(0) = 0$, we get $0 = c_1 + c_2 - \frac{9}{500}$ and so $c_1 + c_2 = \frac{9}{500}$. Since $\frac{dx}{dt}(0) = -1$ (initial velocity in upper direction), we get

$$-1 = -2c_1 - 7c_2 + \frac{13}{500} \text{ or } 2c_1 + 7c_2 = \frac{513}{500}.$$

Solving for c_1 and c_2, we get $c_1 = -\frac{90}{500}$, $c_2 = \frac{99}{500}$. Hence

$$x = \frac{1}{500}(-90e^{-2t} + 99e^{-7t} + 13\sin t - 9\cos t).$$

EXAMPLE 1.139

If in a mass spring system, mass = 4kg, spring constant = 64, $f(t) = 8 \sin 4t$, and if there is no air resistance and initial velocity, then find the subsequent motion of the weight. Show that the resonance occurs in this case.

Solution. The governing equation is

$$\frac{d^2x}{dt^2} + \frac{a}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{1}{m}f(t).$$

Therefore, we have

$$\frac{d^2x}{dt^2}+16x= 2\sin 4t.$$

The auxiliary equation is $m^2 + 16 = 0$ and so $m = \pm 4i$. Therefore,

$$\text{C.F.} = c_1\cos 4t+c_2\sin 4t.$$

Now

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2+16}(2\sin 4t) = \frac{2t}{2D}(\sin 4t)\\ &= \frac{t\,D(\sin 4t)}{-16} = -\frac{t}{16}\,(4\cos 4t)\\ &= -\frac{t}{4}\cos 4t.\end{aligned}$$

Hence the complete solution is

$$x = c_1\cos 4t+c_2\sin 4t - \frac{t}{4}\cos 4t.$$

Using initial condition $x(0) = 0$, we have $0 = c_1$. Differentiating w.r.t. t, we get

$$\begin{aligned}\frac{dx}{dt} &= -4c_1\sin 4t+4c_2\cos 4t\\ &\quad -\frac{1}{4}\,[\cos 4t - 4t\sin 4t].\end{aligned}$$

Now $\frac{dx}{dt} = 0$ at $t = 0$. Therefore, $0 = 4c_2 - \frac{1}{4}$ which gives $c_2 = \frac{1}{16}$. Hence

$$x = \frac{1}{16}\sin 4t - \frac{t}{4}\cos 4t.$$

We observe that $x(t) \to \infty$ as $t \to \infty$ due to the presence of the term $t\cos 4t$. This term is called a *secular term*. The presence of secular term causes resonance because the solution becomes unbounded.

1.22 SIMPLE PENDULUM

The system in which a heavy particle (bob) is attached to one end of a light inextensible string, the other end of which is fixed, and oscillates under the action of gravity force in a vertical plane is called a *simple pendulum*. To describe its motion, let m be the mass of the particle, l be the length of the string, and O be the fixed point of the string (Figure 1.9).

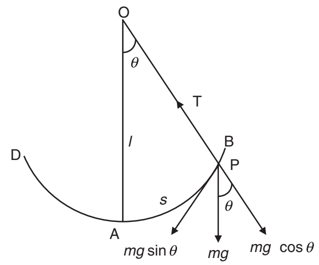


Figure 1.9

Let P be the position of the heavy particle at any time t and let $\angle AOP = \theta$, where OA is vertical line through O. Then the force acting on the bob are
(a) weight mg acting vertically downward
(b) tension T in the string.
Resolving mg, we note that tension is balanced by $mg\cos\theta$. The equation of motion along the tangent is

$$m\frac{d^2s}{dt^2} = -mg\sin\theta$$

or

$$\frac{d^2}{dt^2}(l\theta) = -g\sin\theta$$

or

$$\begin{aligned}\frac{d^2\theta}{dt^2} &= -\frac{g}{l}\sin\theta\\ &= -\frac{g}{l}\left[\theta - \frac{\theta^3}{3!}+ ..\right] \text{ (using expansion for } \sin\theta)\\ &= -\frac{g\theta}{l} \text{ to the first approximation.}\end{aligned}$$

Thus the differential equation describing the motion of bob is

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0,$$

where $\omega^2 = \frac{g}{l}$. The auxiliary equation of this equation is

$$m^2 + \omega^2 = 0$$

and so $m = \pm\omega i$. Therefore, the solution is

$$\begin{aligned}\theta &= c_1\cos\omega t + c_2\sin\omega t\\ &= c_1 cos\sqrt{\frac{g}{l}}t + c_2\sin\sqrt{\frac{g}{l}}t.\end{aligned}$$

The motion in case of simple pendulum is *simple harmonic motion* where time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}.$$

The motion of the bob from one extreme position to the other extreme position on the other side of A is called *a beat* or a *swing*. Therefore,

$$\text{Time for one swing} = \tfrac{1}{2}T = \pi\sqrt{\tfrac{l}{g}}.$$

The pendulum which beats once every second is called a *second's pendulum*. Therefore, the number of beats in a second pendulum in one day is equal to the number of seconds in a day. Thus, a second pendulum beats 86,400 times a day.

Now, since, the time for 1 beat in a second's pendulum is 1 sec, we have

$$1 = \pi\sqrt{\frac{l}{g}} = \pi\sqrt{\frac{l}{981}}$$

and so $l = 99.4$ cm is the length of second's pendulum.

EXAMPLE 1.140

The differential equation of a simple pendulum is $\frac{d^2x}{dt^2} + \omega_0^2 x = F_0 \sin nt$, where ω_0 and F_0 are constants. If initially $\frac{dx}{dt} = 0$, determine the motion when $\omega_0 = n$

Solution. We have

$$\frac{d^2x}{dt^2} + n^2x = \mathrm{F}_0 \sin nt \text{ since}, \omega_0 = n.$$

The auxiliary equation is $m^2 + n^2 = 0$ and so $m = \pm ni$. Therefore,

$$C.F. = c_1 \cos nt + c_2 \sin nt.$$

Now

$$\text{P.I.} = \frac{F_0}{D^2+n^2}\sin nt = \frac{F_0 t}{2D}\sin nt$$
$$= \frac{F_0 t}{2}\int \sin nt\, dt = -\frac{F_0 t}{2n}\cos nt.$$

Thus, the complete solution is

$$x = c_1 \cos nt + c_2 \sin nt - \frac{F_0 t}{2n}\cos nt.$$

Initially $x = 0$ for $t = 0$, so $c_1 = 0$.

Also

$$\frac{dx}{dt} = -nc_1 \sin nt + nc_2 \cos nt - \frac{F_0}{2n}[\cos nt - \text{tn} \sin nt].$$

But for $t = 0$, $\frac{dx}{dt} = 0$. Therefore, $0 = nc_2 - \frac{F_0}{2n}$ and so $c_2 = \frac{F_0}{2n^2}$.

Hence

$$x = \frac{F_0}{2n^2}\sin nt - \frac{F_0 t}{2n}\cos nt$$
$$= \frac{F_0}{2n^2}(\sin nt - nt\cos nt).$$

1.23 SOLUTION IN SERIES

The method of series solution of differential equations is applied to obtain solutions of *linear differential equations with variable coefficients*.

Consider the differential equation

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0, \qquad (117)$$

where $P_0(x)$, $P_1(x)$, and $P_2(x)$ are polynomials in x. This equation can be written as

$$\frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)}\frac{dy}{dx} + \frac{P_2(x)}{P_0(x)}y = 0. \qquad (118)$$

The point $x = a$ is called an *ordinary point* of the equation (117) or (118) if the functions $\frac{P_1(x)}{P_0(x)}$ and $\frac{P_2(x)}{P_0(x)}$ are analytic at $x = a$. In other words, $x = a$ is an ordinary point of (117) if $P_0(a) \neq 0$.

If either (or both) of $\frac{P_1(x)}{P_0(x)}$ or/and $\frac{P_2(x)}{P_0(x)}$ is (are) not analytic at $x = a$, then $x = a$ is called a *singular point* of the equation (117) or (118). Thus $x = a$ is a singular point of (117) if $P_0(a) = 0$.

Further, let

$$Q_1(x) = (x-a)\frac{P_1(x)}{P_0(x)},$$
$$Q_2(x) = (x-a)^2\frac{P_2(x)}{P_0(x)}.$$

If Q_1 and Q_2 are both analytic at $x = a$, then $x = a$ is called a *regular singular point* of (117) otherwise it is called *irregular point* of (117).

For example, consider the equation

$$9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0.$$

We have

$$\frac{P_1(x)}{P_0(x)} = \frac{-12}{9x(1-x)} \text{ and } \frac{P_2(x)}{P_0(x)} = \frac{4}{9x(1-x)},$$

which are not analytic at $x = 0$ and $x = 1$. Hence $x = 0$ and $x = 1$ are singular points of the given equation. Further, at $x = 0$,

$$Q_1(x) = \frac{-12}{9(1-x)},\ Q_2(x) = \frac{4}{-(1-x)},$$

which are analytic at $x = 0$. Thus $x = 0$ is a regular singular point.

For $x = 1$, we have

$$Q_1(x) = \frac{12}{9x},\ Q_2(x) = \frac{-4}{9x},$$

which are analytic if $x = 1$. Hence $x = 1$ is also regular.

1.23.1 Solution About Ordinary Point

If $x = a$ is an ordinary point of the differential equation (117), then its every solution can be expressed in the form

$$\begin{aligned} y = a_0 + a_1(x-a) + a_2(x-a)^2 \\ + a_3(x-a)^3 + \dots , \end{aligned} \qquad (119)$$

where the power series converges in some interval $|x - a| < R$ about a. Thus the series may be differentiated term by term on this interval and we have

$$\frac{dy}{dx} = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \dots ,$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3(x-a) + \dots .$$

Substituting the values of y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ is (117), we get an equation of the type

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \dots = 0, \qquad (120)$$

where the coefficients c_0, c_1, and c_2 are functions of a. Then (120) will be valid for all x in $|x - a| < R$ if all $c_1, c_{2,\dots}$ are zero. Thus

$$c_0 = c_1 = c_2 = \dots .0. \qquad (121)$$

The coefficients a_i of (119) are obtained from (121). In case (120) is expressed in powers of x, then equating to zero the coefficients of the various powers of x will determine $a_{2,}\ a_{3,}\ a_4, \dots$ in terms of a_0 and a_1. The relation obtained by equating to zero the coefficient of x^n is called the *recurrence relation*.

1.23.2 Solution About Singular Point (Forbenious Method)

If $x = a$ is a regular singular point of (117), then the equation has at least one non-trivial solution of the form

$$y = (x-a)^m[a_0 + a_1(x-a) + a_2(x-a)^2 + \dots], \quad (122)$$

where m is a definite constant (real or complex) and the series on the right converges at every point of the interval of convergence with centre a. Differentiating (122) twice, we get

$$\frac{dy}{dx} = ma_0(x-a)^{m-1} + (m+1)a_1(x-a)^m + \dots$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= m(m-1)a_0(x-a)^{m-2} \\ &\quad + m(m+1)a_1(x-a)^{m-1} + \dots . \end{aligned}$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (117), we get an equation of the form

$$\begin{aligned} c_0(x-a)^{m+k} + c_1(x-a)^{m+k+1} \\ + c_2(x-a)^{m+k+2} + \dots = 0, \end{aligned} \qquad (123)$$

where k is an integer and the coefficients c_i are functions of m and a_i. In order that (123) be valid in $|x - a| < R$, we must have

$$c_0 = c_1 = c_2 = \dots = 0. \qquad (124)$$

On equating to zero the coefficient c_0 in (123), we get a quadratic equation in m, called *indicial equation*, which gives the value of m. The two roots m_1 and m_2 of the indicial equation are called *exponents of the differential equation* (117). The coefficients $a_1, a_2, a_3, \dots$ are obtained in terms of a_0 from $c_1 = c_2 = .. = 0$. Putting the values of $a_1, a_2, \dots$in (122), the solution of (117) is obtained.

If $m_1 - m_2 \neq 0$ or a positive integer, then the complete solution of equation (117) is

$$y = c_1(y)_{m_1} + c_2(y)_{m_2}.$$

If $m_1 - m_2 = 0$, that is, the roots of indicial equation are equal, then the two independent solutions are obtained by substituting the value of m in y and $\frac{\partial y}{\partial m}$. Thus, in this case,

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

If $m_1 - m_2$ is a positive integers making a coefficient of y infinite when $m = m_2$, then the form of y is modified by replacing a_0 by $k(m - m_2)$. Two independent solutions of the differential equation (117) are then

$$y = c_1(y)_{m_2} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_2}$$

If $m_1 - m_2$ is a positive integer making a coefficient of y indeterminate when $m = m_2$, then the complete solution of (117) is

$$y = c_1(y)_{m_2}.$$

EXAMPLE 1.141

Find the power series solution of the equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

in powers of x, that is, about $x = 0$.

Solution. Let

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots.$$

Differentiating twice we get

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots + na_nx^{n-1} + \dots$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + \dots + n(n-1)a_nx^{n-2} + \dots.$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation, we get

$$\begin{aligned}&(1 - x^2)[2a_2 + 6a_3x + \dots + n(n-1)a_nx^{n-2} + \dots]\\ &\quad - 2x[a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots]\\ &\quad + 2(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots) = 0\end{aligned}$$

or

$$\begin{aligned}&2(a_2 + a_0) + (6a_3 - 2a_1)x\\ &\quad + (12a_4 - 2a_2 - 4a_2 + 2a_2)x^2\\ &\quad + \dots + [(n+2)(n+1)a_{n+2}\\ &\quad - n(n-1)a_n - 2na_n + 2a_n]x^n + \dots = 0\end{aligned}$$

or

$$\begin{aligned}&2(a_2 + a_0) + 6a_3x + (12a_4 - 4a_2)x^2\\ &\quad + \dots + [(n^2 + 3n + 2)a_{n+2} - (n^2 - n)a_n\\ &\quad + 2a_n(1 - n)]x^n + \dots = 0.\end{aligned}$$

Equating to zero the coefficients of the various powers of x, we get

$$a_2 = -a_0,\ a_3 = 0,\ a_4 = -\frac{1}{3}a_{0,\ \dots}$$

$$a_{n+2}[n^2 + 3n + 2] + a_n(-n^2 - n + 2) = 0.$$

Taking $n = 3, 4, \dots$

$$\begin{aligned}&20a_5 - 10a_3 = 0 \Rightarrow a_5 = 0\\ &30a_6 - 18a_4 = 0 \Rightarrow a_6 = \tfrac{18}{30}a_4 = -\tfrac{1}{5}a_0\\ &\dots\\ &\dots\end{aligned}$$

Therefore,

$$\begin{aligned}y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots\\ &= a_0 + a_1x - a_0x^2 - \frac{1}{3}a_0x^4 - \frac{1}{5}a_0x^6 + \dots\\ &= a_0\left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \dots\right) + a_1x.\end{aligned}$$

EXAMPLE 1.142

Find the solution in series of the equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2y = 0$$

about $x = 0$.

Solution. The point $x = 0$ is a regular point of the given differential equation. So, let the required solution be

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots + a_nx^n + \dots.$$

Then differentiating twice, we get

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots + na_nx^{n-1} + \dots$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3\\ &\quad + \dots + n(n-1)a_nx^{n-2} + \dots.\end{aligned}$$

Substituting the values of y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$\begin{aligned}&[2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3\\ &\quad + \dots + n(n-1)a_nx^{n-2} + \dots]\\ &\quad + x[a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3\\ &\quad + \dots + na_nx^{n-1} + \dots]\\ &\quad + x^2[a_0 + a_1x + a_2x^2 + a_3x^3\\ &\quad + a_4x^4 + \dots + a_nx^n + \dots] = 0\end{aligned}$$

or

$$2a_2+(6a_3+a_1)x+(12a_4+2a_2+a_0)x^2$$
$$+(20a_5+3a_3+a_1)x^3+(30a_6+4a_4+a_2)x^4$$
$$+(42a_7+5a_5+a_3)x^5+\dots$$
$$+\dots[(n+2)(n+1)a_{n+2}+na_n+a_{n-2}]x^n+\dots=0.$$

Equating to zero the coefficients of the various powers of x, we get

$$a_2=0, a_3=-\frac{1}{6}a_1,$$

$12a_4+2a_2+a_0=0$, which yields $a_4=-\frac{1}{12}a_0$

$20a_5+3a_3+a_1=0$ which yields $a_5=-\frac{1}{40}a_1$

$30a_6+4a_4+a_2=0$ which yields $a_6=\frac{1}{90}a_0$, and so on.

Hence

$$y=a_0+a_1x-\frac{1}{6}a_1x^3-\frac{1}{12}a_0x^4$$
$$-\frac{1}{40}a_1x^5+\frac{1}{90}a_0x^6\dots$$
$$=a_0\left(1-\frac{1}{12}x^4+\frac{1}{90}x^6-\dots\right)$$
$$+a_1\left(x-\frac{1}{6}x^3-\frac{1}{40}x^5-\dots\right).$$

EXAMPLE 1.143

Find power series solution of the equation

$$\frac{d^2y}{dx}+xy=0$$

in powers of x, that is, about $x=0$.

Solution. We note that $x=0$ is a regular point of the given equation. Therefore, its solution is of the form

$$y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6$$
$$+\dots+a_nx^n+\cdots.$$

Differentiating twice successively, we get

$$\frac{dy}{dx}=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4$$
$$+6a_6x^5+\dots+na_nx^{n-1}+\dots$$
$$\frac{d^2y}{dx^2}=2a_2+6a_3x+12a_4x^2+20a_5x^3$$
$$+30a_6x^4+\dots+n(n-1)x^{n-2}+\dots$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$in the given equation, we get

$$2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4$$
$$+\dots+n(n-1)a_nx^{n-2}+\dots$$
$$+x[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\dots]=0$$

or

$$2a_2+(6a_3+a_0)x+(12a_4+a_1)x^2+(20a_5+a_2)x^3$$
$$+\dots+[(n+2)(n+1)a_{n+2}+a_{n+1}]x^n+\dots.$$

Equating to zero the coefficients of various powers of x, we get

$a_2=0,\ 6a_3+a_0=0$ which yields $a_3=-\frac{1}{6}a_0$

$12a_4+a_1=0$ which yields $a_4=-\frac{1}{12}a_1$

$20a_5+a_2=0$ which yields $a_5=0$

$(n+2)(n+1)a_{n+2}+a_{n-1}=0.$

Putting $n=4, 5, 6, 7, \dots$, we get

$30a_6+a_3=0$ which yields $a_6=-\frac{a_3}{30}=\frac{1}{180}a_0$

$42a_7+a_4=0$ which yields $a_7=-\frac{a_4}{42}=\frac{1}{504}a_1$,

and so on.

Hence

$$y=a_0\left(1-\frac{1}{6}x^3+\frac{1}{180}x^6-\dots\right)$$
$$+a_1\left(x-\frac{1}{12}x^4+\frac{1}{504}x^7-\dots\right).$$

EXAMPLE 1.144

Find series solution of the differential equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0$$

about $x=0$.

Solution. The point $x=0$ is a regular singular point of the given equation. So, let

$$y=a_0x^m+a_1x^{m+1}+a_2x^{m+2}+\dots,\quad a_0\neq 0.$$

Differentiating twice in succession, we get

$$\frac{dy}{dx}=\sum_{n=0}^{\infty}(n+m)a_nx^{n+m-1}$$

and

$$\frac{d^2y}{dx^2}=\sum_{n=0}^{\infty}(n+m)(n+m-1)a_nx^{n+m-2}.$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$x\sum_{n=0}^{\infty}(n+m)(n+m-1)a_nx^{n+m-2}+\sum_{n=0}^{\infty}(n+m)a_nx^{n+m-1}-\sum_{n=0}^{\infty}a_nx^{n+m}=0$$

or

$$\sum_{n=0}^{\infty}(n+m)(n+m-1)a_nx^{n+m-1}+\sum_{n=0}^{\infty}(n+m)a_nx^{n+m-1}-\sum_{n=0}^{\infty}a_nx^{n+m}=0$$

or

$$\sum_{n=0}^{\infty}(n+m)^2a_nx^{n+m-1}-\sum_{n=0}^{\infty}a_nx^{n+m}=0$$

or

$$\sum_{n=-1}^{\infty}(n+m+1)^2a_{n+1}x^{n+m}-\sum_{n=0}^{\infty}a_nx^{n+m}=0$$

or

$$m^2a_0x^{m-1}+\sum_{n=0}^{\infty}(n+m+1)^2a_{n+1}x^{n+m}-\sum_{n=0}^{\infty}a_nx^{n+m}=0$$

or

$$m^2a_0x^{m-1}+\sum_{n=0}^{\infty}x^{n+m}\times[(n+m+1)^2a_{n+1}-a_n]=0 \quad (125)$$

Therefore, the indicial equation is $m^2=0$, which yields $m=0,0$. Equating to zero other coefficients in (125), we get

$$(n+m+1)^2a_{n+1}=a_n,\quad n\geq 0.$$

Therefore,

$$a_1=\frac{1}{(m+1)^2}a_0,$$
$$a_2=\frac{1}{(m+2)^2}a_1=\frac{1}{(m+1)^2(m+2)^2}a_0,$$
$$a_3=\frac{1}{(m+3)^2}a_2=\frac{1}{(m+1)^2(m+2)^2(m+3)^2}a_0,$$

and so on. Thus

$$y=a_0x^m+\frac{1}{(m+1)^2}a_0x^{m+1}+\frac{1}{(m+1)^2(m+2)^2}a_0x^{m+2}+\frac{1}{(m+1)^2(m+2)^2(m+3)^2}a_0x^{m+3}+\dots$$
$$=a_0x^m\left[1+\frac{1}{(m+1)^2}x+\frac{1}{(m+1)^2(m+2)^2}x^2+\frac{1}{(m+1)^2(m+2)^2(m+3)^2}x^3\right].$$

Putting $m=0$, we get one solution of the given differential equation as

$$y_1=c_1\left[1+x+\frac{1}{4}x^2+\frac{1}{36}x^3+\dots\right].$$

Further,

$$\frac{\partial y}{\partial m}=a_0x^m\log x\left[1+\frac{1}{(m+1)^2}x+\frac{1}{(m+1)^2(m+2)^2}x^2+\frac{1}{(m+1)^2(m+2)^2(m+3)^2}x^3+\dots\right]$$
$$+a_0x^m\left[-\frac{2x}{(m+1)^3}-\frac{1}{(m+1)^2(m+2)^2}\times\left(\frac{2}{(m+1)}+\frac{2}{(m+2)^2}\right)x^2+\dots\right].$$

Therefore,

$$\left(\frac{\partial y}{\partial m}\right)_{m=0}=a_0\log\left[1+x+\frac{1}{4}x^2+\frac{1}{36}x^3+\dots\right]+a_0\left[-2x-\frac{1}{4}(2+1)x^2+\dots\right].$$

Hence the solution of the given equations is

$$y = c_1(y)_{m=0} + c_2\left(\frac{\partial y}{\partial m}\right)_{m=0}$$

$$= (c_1 + c_2 \log x)\left[1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \dots\right]$$

$$- 2c_2\left[x + \frac{1}{4}\left(1 + \frac{1}{2}\right)x^2 + \dots\right].$$

EXAMPLE 1.145

Find the series solution near $x = 0$ of the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2y = 0.$$

(This equation can also the written as $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$ and is known as *Bessel's Equation of order zero*.)

Solution. The point $x = 0$ is a regular singular point of the given equation. So, let

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}.$$

Differentiating twice in succession, we get

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+m)a_n x^{n+m-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+m)(n+m-1)a_n x^{n-m-2}.$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$x\sum_{n=0}^{\infty}(n+m)(n+m-1)a_n x^{n+m-2}$$

$$+ \sum_{n=0}^{\infty}(n+m)a_n x^{n+m-1} + x\sum_{n=0}^{\infty}a_n x^{n+m} = 0$$

or

$$\sum_{n=0}^{\infty}(n+m)(n+m-1)a_n x^{n+m-1}$$

$$+ \sum_{n=0}^{\infty}(n+m)a_n x^{n+m-1} + \sum_{n=0}^{\infty}a_n x^{n+m+1} = 0$$

or

$$\sum_{n=0}^{\infty}(n+m)^2 a_n x^{n+m-1} + \sum_{n=0}^{\infty}a_n x^{n+m+1} = 0$$

or

$$\sum_{n=-1}^{\infty}(n+m+1)^2 a_{n+1}x^{n+m} + \sum_{n=0}^{\infty}a_n x^{n+m+1} = 0$$

or

$$m^2a_0x^{m-1} + \sum_{n=0}^{\infty}(n+m+1)^2a_{n+1}x^{n+m}$$

$$+ \sum_{n=0}^{\infty}a_n x^{n+m+1} = 0.$$

Therefore, the indicial equation is $m^2 = 0$, which yields $m = 0, 0$. Equating to zero the coefficients of powers of x^m , x^{m+1} $x^{m+2}, \dots,$ we get

$$(n+m+1)^2a_1 = 0 \text{ which yields } a_1 = 0,$$

$$(m+2)^2a_2 + a_0 = 0 \text{ and so } a_2 = -\frac{a_0}{(m+2)^2},$$

$$(m+3)^2a_3 + a_1 = 0 \text{ and so } a_3 = 0,$$

$$(m+4)^2a_4 + a_2 = 0 \text{ and so } a_4 = \frac{-a_2}{(m+4)^2}$$

$$= \frac{a_0}{(m+2)^2(m+4)^2},$$

and so on. Hence,

$$y = a_0x^m - \frac{a_0}{(m+2)^2}x^{m+2} + \frac{a_0}{(m+2)^2(m+4)^2}x^{m+4} - \dots$$

$$= a_0x^m\left[1 - \frac{1}{(m+2)^2}x^2 + \frac{1}{(m+2)^2}x^4 - \dots\right].$$

Further,

$$\frac{\partial y}{\partial m} = a_0x^m\log x\left[1 - \frac{1}{(m+2)^2}x^2\right.$$

$$\left. + \frac{1}{(m+2)^2(m+4)^2}x^4 - \dots\right]$$

$$+ a_0x^m\left[\frac{2x^2}{(m+2)^3} - \frac{x^4}{(m+2)^2(m+4)^2}\right.$$

$$\left.\times\left(\frac{2}{m+2} + \frac{2}{m+4}\right) + \dots\right].$$

Therefore,

$$\left(\frac{\partial y}{\partial m}\right)_{m=0} = a_0\log x\left[1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \dots\right]$$

$$+ a_0\left[\frac{x^2}{4} - \frac{3x^4}{2.4.16} + \dots\right].$$

Hence the solution of the given equation is

$$\begin{aligned}y &= c_1(y)_{m=0} + c_2\left(\frac{\partial y}{\partial m}\right)_{m=0}\\ &= c_1\left[1-\frac{1}{2^2}x^2+\frac{1}{2^2.4^2}x^4-\frac{1}{2^2.4^2.6^2}x^6+\ldots\right]\\ &\quad + c_2\log x\left[1-\frac{1}{2^2}x^2+\frac{1}{2^2.4^2}x^4-\frac{1}{2^2.4^2.6^2}x^6+\ldots\right]\\ &\quad + a_0\left[\frac{1}{2^2}x^2-\frac{1}{2^2.4^2}\left(1+\frac{1}{2}\right)x^4\right.\\ &\quad \left.+\frac{1}{2^2.4^2.6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^6-\ldots\right]\\ &= (c_1+c_2\log x)\left[1-\frac{1}{2^2}x^2+\frac{1}{2^2.4^2}x^4\right.\\ &\quad \left.-\frac{1}{2^2.4^2.6^2}x^6+\ldots\right]+a_0\left[\frac{1}{2^2}x^2-\frac{1}{2^2.4^2}\left(1+\frac{1}{2}\right)x^4\right.\\ &\quad \left.+\frac{1}{2^2.4^2.6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^6+\ldots\right].\end{aligned}$$

The solution

$$c_1(y)_{m=0} = c_1\left[1--\frac{1}{2^2}x^2+\frac{1}{2^2.4^2}x^4-\frac{1}{2^2.4^2.6^2}x^6+\ldots\right]$$

is called *Bessel function of the first kind of order zero* and is represented by $J_0(x)$, where as the solution.

$$\begin{aligned}c_2\left(\frac{\partial y}{\partial m}\right)_{m=0} &= c_2\log x\left[1-\frac{1}{2^2}x^2+\frac{1}{2^2.4^2}x^4\right.\\ &\quad \left.-\frac{1}{2^2.4^2.6^2}x^6-\ldots\right]\\ &\quad + a_0\left[\frac{1}{2^2}x^2-\frac{1}{2^2.4^2}\left(1+\frac{1}{2}\right)x^4\right.\\ &\quad \left.+\frac{1}{2^2.4^2.6^2}\left(1+\frac{1}{2}+\frac{1}{5}\right)x^6-\ldots\right]\end{aligned}$$

is called *Neumann function* or *Bessel function of second kind of order zero* and is denoted by $Y_0(x)$.

EXAMPLE 1.146

Find series solution about $x = 0$ of the differential equations

$$2x(1-x)\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+3y=0.$$

Solution. Evidently $x = 0$ is a regular singular point of the given equation. So, let

$$y=\sum_{n=0}^{\infty}a_n x^{n+m}, \quad a_n \neq 0.$$

Differentiating twice in succession, we get

$$\frac{dy}{dx}=\sum_{n=0}^{\infty}(n+m)a_n x^{n+m-1},$$

$$\frac{d^2y}{dx^2}=\sum_{n=0}^{\infty}(n+m)(n+m-1)a_n x^{n+m-2}.$$

Substituting the values of y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$2x(1-x)\sum_{n=0}^{\infty}(n+m)(n+m-1)a_n x^{n+m-2}$$

$$+(1-x)\sum_{n=0}^{\infty}(n+m)a_n x^{n+m-1}+3\sum_{n=0}^{\infty}a_n x^{n+m}=0.$$

or

$$2\sum_{n=0}^{\infty}(n+m)(n+m-1)a_n x^{n+m-1}$$

$$-2\sum_{n=0}^{\infty}(n+m)(n+m-1)a_n x^{n+m}$$

$$+\sum_{n=0}^{\infty}(n+m)a_n x^{n+m-1}-\sum_{n=0}^{\infty}(n+m)a_n x^{n+m}$$

$$+3\sum_{n=0}^{\infty}a_n x^{n+m}=0$$

or

$$\sum_{n=0}^{\infty}(n+m)(2n+2m-1)a_n x^{n+m-1}$$

$$-\sum_{n=0}^{\infty}[(n+m)(2n+2m-1)-3]a_n x^{n+m}=0$$

or

$$\sum_{n=-1}^{\infty}(n+m+1)(2n+2m+1)a_{n+1}x^{n+m}$$

$$-\sum_{n=0}^{\infty}[(n+m)(2n+2m-1)-3]a_n x^{n+m}=0$$

or

$$m(2m-1)a_0x^{m-1} + \sum_{n=0}^{\infty}(n+m+1)(2n+2m+1)a_{n+1}x^{n+m} - \sum_{n=0}^{\infty}[(n+m)(2n+2m-1)-3]a_nx^{n+m} = 0.$$

Therefore, the indicial equation is

$$m(2m-1) = 0, \text{ which yields } m = 0\ ,\ \frac{1}{2}.$$

Equating to zero the other coefficients, we get

$$\begin{aligned}(n+m+1)(2n+2m+1)a_{n+1} &= [(n+m)(2n+2m-1)-3]a_n \\ &= [2n^2+2m^2+4nm-m-n-3]a_n\end{aligned}$$

or

$$a_{n+1} = \frac{2n^2+2m^2+4nm-m-n-3}{(n+m+1)(2n+2m+1)}a_n.$$

For $m = 0$, we get

$$\begin{aligned}a_{n+1} &= \frac{2n^2-n-3}{(n+1)(2n+1)}a_n = \frac{(n+1)(2n-3)}{(n+1)(2n+1)}a_n \\ &= \frac{2n-3}{2n+1}a_n, n \geq 0.\end{aligned}$$

Putting $n = 0, 1, 2, \ldots$, we have

$$a_1 = -3a_0, a_2 = -\frac{1}{3}a_1 = a_0,$$
$$a_3 = \frac{1}{5}a_2 = \frac{1}{5}a_0, a_4 = \frac{3}{7}a_3 = \frac{3}{35}a_0, \text{ and so on.}$$

Therefore, the solution for $m = 0$ is

$$\begin{aligned}y_1 &= a_0x^0 + a_1x + a_2x^2 + a_3x^3 + \ldots \\ &= a_0 - 3a_0x + a_0x^2 + \frac{1}{5}a_0x^3 + \frac{3}{35}a_0x^4 + \ldots \\ &= a_0\left[1 - 3x + x^2 + \frac{1}{5}x^3 + \frac{3}{35}x^4 + \ldots\right].\end{aligned}$$

For $m = \frac{1}{2}$, we have

$$\begin{aligned}a_{n+1} &= \frac{2n^2+2m^2+4mn-m-n-3}{(n+m+1)(2n+2m+1)}a_n \\ &= \frac{2n^2+\frac{1}{2}+2n-\frac{1}{2}-n-3}{\left(n+\frac{3}{2}\right)(2n+2)}a_n \\ &= \frac{2n^2+n-3}{(2n+3)(n+1)}a_n \\ &= \frac{(2n+3)(n-1)}{(2n+3)(n+1)}a_n = \frac{n-1}{n+1}a_n.\end{aligned}$$

Putting $n = 0, 1, 2, \ldots$, we get

$$a_1 = -a_0\ ,\quad a_2 = 0,$$
$$a_3 = \frac{1}{4}a_2 = 0\ ,\quad a_4 = \frac{1}{2}a_3 = 0,$$

and so on. Therefore, the solution corresponding to $m = \frac{1}{2}$ is

$$\begin{aligned}y_2 &= a_0x^{\frac{1}{2}} + a_1x^{\frac{3}{2}} + a_2x^{\frac{5}{2}} + a_3x^{\frac{7}{2}} + \ldots \\ &= a_0x^{\frac{1}{2}} - a_0x^{\frac{3}{2}} = a_0x^{\frac{1}{2}}(1-x).\end{aligned}$$

Hence, the general solution of the given equation is

$$\begin{aligned}y &= c_1y_1 + c_2y_2 \\ &= c_1\left[1 - 3x + x^2 + \frac{1}{5}x^3 + \frac{3}{35}x^4 + \ldots\right] \\ &\quad + c_2x^{\frac{1}{2}}(1-x).\end{aligned}$$

EXAMPLE 1.147

Find series solution about $x = 0$ of the differential equation

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - \left(x^2 + \frac{5}{4}\right)y = 0.$$

Solution. The point $x = 0$ is a regular singular point of the given equation. So, let

$$y = \sum_{n=0}^{\infty} a_nx^{n+m}, \quad a_0 \neq 0.$$

Differentiating twice in succession, we have

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+m)a_nx^{n+m-1},$$

$$\frac{d^2y}{dx} = \sum_{n=0}^{\infty}(n+m)(n+m-1)a_nx^{n+m-2}.$$

Putting the values of y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given differential equation, we have

$$\sum_{n=0}^{\infty}(n+m)(n+m-1)a_n x^{n+m} - \sum_{n=0}^{\infty}(n+m)a_n x^{n+m} - \sum_{n=0}^{\infty}x^{n+m+2} - \frac{5}{4}\sum_{n=0}^{\infty}a_n x^{n+m} = 0$$

or

$$\sum_{n=0}^{\infty}\left[(n+m)(n+m-1)-(n+m)-\frac{5}{4}\right]a_n x^{n+m} - \sum_{n=0}^{\infty}a_n x^{n+m+2} = 0$$

or

$$\sum_{n=0}^{\infty}\left[(n+m)(n+m-2)-\frac{5}{4}\right]a_n x^{n+m} - \sum_{n=2}^{\infty}a_{n-2}x^{n+m} = 0$$

or

$$\left[m(m-2)-\frac{5}{4}\right]a_0x^m + \left[(m+1)(m-1)-\frac{5}{4}\right]a_1x^{m+1} - \sum_{n=2}^{\infty}\left[\left((n+m)(n+m-2)-\frac{5}{4}\right)a_n - a_{n-2}\right]x^{n+m} = 0.$$

Therefore, the indicial equation is

$$m(m-2)-\frac{5}{4}=0, \text{ which yields } m=\frac{5}{2},\ -\frac{1}{2}.$$

Equating to zero the other coefficients, we get

$$\left[(m+1)(m-1)-\frac{5}{4}\right]a_1 = 0 \text{ and so } a_1 = 0$$

and

$$\left[(n+m)(n+m-2)-\frac{5}{4}\right]a_n = a_{n-2} \text{ for } n \geq 2.$$

For $m=-\frac{1}{2}$, we get

$$\left[\left(n-\frac{1}{2}\right)\left(n-\frac{5}{2}\right)-\frac{5}{4}\right]a_n = a_{n-2},\ n\geq 2 \quad (126)$$

or

$$a_n = \frac{1}{n(n-3)}a_{n-2},\ n\geq 2,\ n\neq 3 \qquad (127)$$

Hence, for $m=-\frac{1}{2}$, we have, from (127), $a_2=-\frac{1}{2}a_0$. Putting $n=3$ in (126), we have

$$\left[\left(3-\frac{1}{2}\right)\left(3-\frac{5}{2}\right)-\frac{5}{4}\right]a_3 = a_1 \quad \text{or} \quad 0.\ a_3 = a_1$$

and so a_3 may be any constant. Further, from (127), we have

$$a_4 = -\frac{1}{8}a_0, \quad a_5 = \frac{a_3}{10}, \quad a_6 = \frac{-a_0}{144},$$
$$a_7 = \frac{a_3}{280}, \text{ and so on.}$$

Hence for $m=-\frac{1}{2}$, the required solution is

$$y = a_0x^{-\frac{1}{2}}\left(1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{144}-\dots\right) + a_3x^{-\frac{1}{2}}\left(x^3+\frac{x^5}{10}+\frac{x^7}{280}+\dots\right). \qquad (128)$$

Since this solution contains two constants a_0 and a_3 and $a_0\neq 0$, this is general solution of the given differential equation. By taking $m=\frac{5}{2}$, the solution is

$$y = x^{\frac{5}{2}}\left(1+\frac{x^2}{10}+\frac{x^4}{280}+\dots\right).$$

Hence, (128) is general solution.

EXAMPLE 1.148

Find series solution about $x=0$ of the differential equation

$$9x(1-x)\frac{d^2y}{dx} - 12\frac{dy}{dx} + 4y = 0.$$

Solution. The point $x=0$ is a regular singular point of the given equation. Let

$$y = \sum_{n=0}^{\infty}a_n x^{n+m},\ a_0\neq 0.$$

Differentiation of y with respect to x yields

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+m)a_n x^{n+m-1},$$
$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+m)(n+m-1)a_n x^{n+m-2}.$$

Substituting the values of y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$9\sum_{n=0}^{\infty}(n+m)(n+m-1)a_nx^{n+m-1}$$
$$-9\sum_{n=0}^{\infty}(n+m)(n+m-1)a_nx^{n+m}$$
$$-12\sum_{n=0}^{\infty}(n+m)a_nx^{n+m-1}+4\sum_{n=0}^{\infty}a_nx^{n+m}=0$$

or

$$\sum_{n=0}^{\infty}(n+m)(9n+9m-21)a_nx^{n+m-1}$$
$$-\sum_{n=0}^{\infty}[9(n+m)(n+m-1)-4]a_nx^{n+m}=0$$

or

$$\sum_{n=-1}^{\infty}(n+m+1)(9n+9m-12)a_{n+1}x^{n+m}$$
$$-\sum_{n=0}^{\infty}[9(n+m)(n+m-1)-4]a_nx^{n+m}=0$$

or

$$m(9m-21)a_0x^{m-1}+\sum_{n=0}^{\infty}[(n+m+1)(9n+9m-12)a_{n+1}$$
$$-(9(n+m)(n+m-1)-4)a_n]x^{n+m}=0.$$

Therefore, the indicial equation is

$$m(9m-21)=0,\ \text{which yields } m=0,\ \frac{7}{3}.$$

We note that the roots are distinct and their difference is not an integer. Equating other coefficients of the powers of x to zero, we get

$$(n+m+1)(9n+9m-12)a_{n+1}$$
$$=[9(n+m)(n+m-1)-4]a_n$$

or

$$a_{n+1}=\frac{9(n+m)(n+m-1)-4}{(n+m+1)(9n+9m-12)}a_n,\ n\geq 0.$$

For $m=0$, we get

$$a_{n+1}=\frac{3m+1}{3m+3}a_n\ ,\ n\geq 0.$$

Putting $n=0, 1, 2, 3,\ldots$, we obtain

$$a_1=\frac{1}{3}a_0,\ a_2=\frac{2}{3}a_1=\frac{2}{9}a_0,$$
$$a_3=\frac{7}{9}a_2=\frac{14}{81}a_0,\ a_4=\frac{5}{6}a_3=\frac{35}{243}a_0,$$

and so on. Thus, the solution corresponding to $m=0$ is

$$y_1=a_0\left[1+\frac{x}{3}+\frac{2}{9}x^2+\frac{14}{81}x^3+\frac{35}{243}x^4+\ldots\right].$$

For $m=-\frac{7}{3}$, we have

$$a_{n+1}=\frac{9\left(n-\frac{7}{3}\right)\left(n-\frac{7}{3}-1\right)}{3\left(n-\frac{7}{3}+1\right)(3n-7-4)}=\frac{3n-6}{3n-4}a_n,\ n\geq 0.$$

Putting $n=0, 1, 2, 3,\ldots$, we get

$$a_1=\frac{3}{2}a_0,\ a_2=3a_1=\frac{9}{2}a_0,$$
$$a_3=0,\ a_4=0,\ a_5=0,\ldots.$$

Thus, the solution corresponding to $m=-\frac{7}{3}$ is

$$y_2=a_0x^{\frac{7}{3}}\left(1+\frac{3}{2}x+\frac{9}{2}x^2\right).$$

Hence, the general solution of the given differential equation is

$$y=c_1y_1+c_2y_2$$
$$=c_1\left[1+\frac{x}{3}+\frac{2}{9}x^2+\frac{14}{81}x^3+\frac{35}{243}x^4+\ldots\right]$$
$$+c_2x^{\frac{7}{3}}\left(1+\frac{3}{2}x+\frac{9}{2}x^2\right).$$

EXAMPLE 1.149

Find series solution about $x=0$ of the differential equation

$$(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0.$$

Solution. Since $x=0$ is a regular point of the given equation, so let

$$y=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots.$$

Then,

$$\frac{dy}{dx}=a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots+na_nx^{n-1}+\ldots$$

$$\frac{d^2y}{dx^2}=2a_2+6a_3x+12a_4x^2+20a_5x^3+\ldots+n(n-1)a_nx^{n-2}+\ldots.$$

Substituting the values of y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$(1+x^2)[2a_2+6a_3x+12a_4x^2+20a_5x^3+\ldots+n(n-1)a_nx^{n-2}]$$
$$+x[a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots+na_nx^{n-1}+\ldots]$$
$$-[a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots]=0.$$

Equating to zero the coefficients of various powers of x, we get

$$2a_2-a_0=0 \text{ and so } a_2=\frac{a_0}{2},$$
$$6a_3+a_1-a_1=0 \text{ and so } a_3=0,$$
$$2a_2+12a_4+2a_2-a_2=0 \text{ and so}$$
$$a_4=-\frac{1}{4}a_2=-\frac{1}{8}a_0,$$
$$6a_3+20a_5+3a_3+a_3=0 \text{ and so } a_5=0,$$

and in general,

$$n(n-1)a_n+(n+2)(n+1)a_{n+2}+na_n-na_n=0$$

or

$$a_{n+2}=\frac{n(n+1)}{(n+1)(n+2)}a_n.$$

Putting $n=4, 5, \ldots$, we get

$$a_6=\frac{12}{30}a_4=\frac{2}{5}a_4=-\frac{1}{20}a_0$$

and so on. Thus, the required solution is

$$y=a_0\left(1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{20}x^6-\ldots\right)+a_1x.$$

1.24 BESSEL'S EQUATION AND BESSEL'S FUNCTION

The equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0,$$

where n is a non-negative real number, is called Bessel's equation of order n. It occurs in problems related to vibrations, electric fields, heat conduction, etc. For $n=0$, we have already found its solution in Example 1.136. Now we find series solution of Bessel's equation of order n. This equation can be written as

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)=0.$$

Since $\frac{1}{x}$ and $\left(1-\frac{n^2}{x^2}\right)$ are not analytic at 0, it follows that 0 is a singular point of the given equation. But $x\left(\frac{1}{x}\right)$ and $x^2\left(1-\frac{n^2}{x^2}\right)$ are analytic at 0. Therefore, $x=0$ is a regular singular point of the equation. So, let

$$y=\sum_{m=0}^{\infty}a_mx^{m+r},\ a_m\neq 0$$

be the series solution of the equation about $x=0$. Differentiating twice in succession, we get

$$\frac{dy}{dx}=\sum_{m=0}^{\infty}(m+r)a_mx^{m+r-1}$$

$$\frac{d^2y}{dx^2}=\sum_{m=0}^{\infty}(m+r)(m+r-1)a_mx^{m+r-2}.$$

Putting the values of y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given differential equation, we get

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_mx^{m+r}+\sum_{m=0}^{\infty}(m+r)a_mx^{m+r}+\sum_{m=0}^{\infty}a_mx^{m+r+2}-n^2\sum_{m=0}^{\infty}a_mx^{m+r}=0$$

or

$$\sum_{m=0}^{\infty}[(m+r)^2-n^2]a_mx^{m+r}+\sum_{m=0}^{\infty}a_mx^{m+r+2}=0$$

or

$$\sum_{m=0}^{\infty}[(m+r)^2-n^2]a_mx^{m+r}+\sum_{m=2}^{\infty}a_{m-2}x^{m+r}=0$$

or

$$(r^2-n^2)a_0x^r+[(r+1)^2-n^2]a_1x^{r+1}+\sum_{m=2}^{\infty}[\left((m+r)^2-n^2\right)a_m+a_{m-2}]x^{m+r}=0.$$

Equating to zero the coefficient of lower power of x, we get the indicial equation as

$$r^2-n^2=0, \text{ which yields } r=n,\ -n.$$

Equating other coefficients to zero, we get

$$[(r+1)^2-n^2]a_1=0 \text{ or } a_1=0$$

and

$$[(m+r)^2 - n^2]a_m + a_{m-2} = 0,$$

which yields

$$a_m = \frac{a_{m-2}}{(m+r)^2 - n^2}, \quad m \geq 2.$$

Putting $m = 2, 3, 4, 5, 6, \ldots$, we get

$$a_3 = a_5 = a_7 = \ldots = 0$$

and

$$a_2 = -\frac{a_0}{(r+2)^2 - n^2}$$

$$a_4 = \frac{(-1)^2 a_0}{\left\{(r+2)^2 - n^2\right\}\left\{(r+4)^2 - n^2\right\}}$$

$$a_6 = \frac{(-1)^3 a_0}{\left\{(r+2)^2 - n^2\right\}\left\{(r+4)^2 - n^2\right\}\left\{(r+6)^2 - n^2\right\}}$$

and so on. For $r = n$, we have

$$a_1 = 0, \; a_2 = -\frac{a_0}{(n+2)^2 - n^2} = -\frac{a_0}{4(n+1)},$$

$$a_4 = \frac{a_0}{4^2 2\,!(n+1)(n+2)},$$

$$a_6 = \frac{-a_0}{4^3 3\,!(n+1)(n+2)(n+3)}, \quad \text{and so on,}$$

where as $a_2 = a_5 = a_7 = \ldots = 0$. Thus the solution corresponding to $r = n$ is

$$y_1 = a_0 x^n \left[1 - \frac{1}{4(n+1)}x^2 + \frac{1}{4^2.2\,!(n+1)(n+2)}x^4 - \frac{1}{4^3.3\,!(n+1)(n+2)(n+3)}x^6 + \ldots\right]. \quad (129)$$

Similarly, for $r = -n$, the solution is

$$y_2 = a_0 x^{-n} \left[1 - \frac{1}{4(1-n)}x^2 + \frac{1}{4^2.2\,!(1-n)(2-n)}x^4 - \frac{1}{4^3.3\,!(1-n)(2-n)(3-n)}x^6 + \ldots\right]. \quad (130)$$

We observe that $y_1 = y_2$ for $n = 0$. Further, y_1 is meaningless if n is a negative integer and y_2 is meaningless if n is a positive integer. Hence if n is non-zero and non-integer, then the general solution of the Bessel's equation of order n is

$$y = c_1 y_1 + c_2 y_2.$$

But y_1 can be expressed is

$$y_1 = a_0 x^n \Gamma(n+1)\left[\frac{1}{\Gamma(n+1)} - \frac{1}{2^2(n+1)\Gamma(n+1)}x^2 + \frac{1}{2^4.2\,!(n+1)(n+2)\Gamma(n+1)}x^4 + \frac{1}{2^6.3\,!(n+1)(n+2)(n+3)\Gamma(n+1)}x^6 + \ldots\right]$$

$$= a_0 x^n \Gamma(n+1)\left[\frac{1}{\Gamma(n+1)} - \frac{1}{2^2\,\Gamma(n+2)}x^2 + \frac{1}{2^4.2\,!\Gamma(n+3)}x^4 + \frac{1}{2^6.3\,!\,\Gamma(n+4)}x^6 + \ldots\right]$$

$$= a_0 x^n \Gamma(n+1)\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}.m!\,\Gamma(n+m+1)}$$

$$= a_0 2^n \Gamma(n+1)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m+n}$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m+n}, \quad (131)$$

where $a_0 = \frac{1}{2^n\,\Gamma\,(n+1)}$. The solution (131) is called the *Bessel's function of the first kind of order n* and is denoted by $J_n(x)$. Thus,

$$J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\;\Gamma\;(n+m+1)}\left(\frac{x}{2}\right)^{2m+n}. \quad (132)$$

Replacing n by $-n$ in $J_n(x)$, we get

$$J_{-n}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m\,!\,\Gamma\,(-n+m+1)}\left(\frac{x}{2}\right)^{2m-n}, \quad (133)$$

which is called *Bessel's function of the first kind* of order $-n$. Thus, the complete solution of the Bessel's equation of order n may be expressed as

$$y = c_1 J_n(x) + c_2 J_{-n}(x), \quad (134)$$

whose n is not an integer.

When n is an integer, let $y = u(x)J_n(x)$ be a solution of the Bessel's equation of order n. Then

$$\frac{dy}{dx} = u'(x)J_n(x) + u(x)J_n'(x)$$

$$\frac{d^2y}{dx^2} = u''(x)J_n(x) + J_n'(x)u'(x) + u'(x)J_n'(x) + u(x)J_n''(x)$$

$$= u''(x)J_n(x) + 2u'(x)J_n'(x) + u)x)J_n''(x).$$

Putting the values of y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$u(x)[x^2J_n''(x) + xJ_n'(x) + (x^2 - n^2)J_n(x)]$$
$$+ x^2u''(x)J_n(x) + 2x^2u'(x)J_n'(x) + xu'(x)J_n(x) = 0.$$

Since $J_n(x)$ is a solution of the given equation, we have

$$x^2J_n''(x) + xJ_n'(x) + (x^2 - n^2)J_n(x) = 0.$$

Therefore, the above expression reduces to

$$x^2u''(x)J_n(x) + 2x^2u'(x)J_n'(x) + xu'(x)J_n(x) = 0$$

or

$$\frac{u''(x)}{u'(x)} + 2\frac{J_n'(x)}{J_n(x)} + \frac{1}{x} = 0$$

or

$$\frac{d}{dx}(\log u'(x)) + 2\frac{d}{dx}(\log J_n(x)) + \frac{d}{dx}(\log x) = 0$$

or

$$\frac{d}{dx}[\log(xu'(x)J_n^2(x))] = 0.$$

Integrating this expression, we get

$$\log xu'(x)J_n^2(x) = \log B \text{ so that } xu'(x)J_n^2(x) = B.$$

Thus

$$u'(x) = \frac{B}{xJ_n^2(x)} \text{ and so } u(x) = B\int \frac{dx}{xJ_n^2(x)} + A.$$

Hence, the complete solution is

$$y = u(x)J_n(x) = \left[A + B\int \frac{dx}{xJ_n^2(x)}\right]J_n(x)$$
$$= AJ_n(x) + BJ_n(x)\int \frac{dx}{xJ_n^2(x)} = AJ_n(x) + BY_n(x),$$

where

$$Y_n(x) = J_n(x)\int \frac{dx}{xJ_n^2(x)}$$

is called the Bessel function of the second kind of order n or the Neumann function.

EXAMPLE 1.150

Show that

$$J_{-n}(x) = (-1)^n J_n(x).$$

Solution. We have

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m+n}$$

and

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(-n+m+1)}\left(\frac{x}{2}\right)^{2m-n}.$$

But for positive integer n,

$$\Gamma(n+m+1) = (n+m)! \text{ and}$$
$$\Gamma(-n+m+1) = (m-n)!$$

Therefore,

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!}\left(\frac{x}{2}\right)^{2m+n}$$

and

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m-n)!}\left(\frac{x}{2}\right)^{2m-n}.$$

Since, (-n)! is infinite for $n > 0$, we have

$$J_{-n}(x)$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m-n)!}\left(\frac{x}{2}\right)^{2m-n}$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{m!(n+m)!}\left(\frac{x}{2}\right)^{2m+n}, \text{ by changing } m \text{ to } m+n$$
$$= (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!}\left(\frac{x}{2}\right)^{2m+n} = (-1)^n J_n(x).$$

EXAMPLE 1.151

Show that

(i) $\frac{d}{dx}[x^nJ_n(x)] = x^nJ_{n-1}(x)$

(ii) $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

(iii) $J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$

(iv) $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$.

These results are known as *recurrence formulae for the Bessel's function* $J_n(x)$.

Solution. (i) We know that

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\,(n+m+1)}\left(\frac{x}{2}\right)^{2m+n}.$$

Therefore,

$$x^n J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\,(n+m+1)}\frac{x^{2(m+n)}}{2^{2m+n}}$$

and so

$$\frac{d}{dx}(x^n J_n(x)) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)}\cdot\frac{x^{2(m+n)-1}}{2^{2m+n}}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n-1+m+1)}\left(\frac{x}{2}\right)^{n-1+2m}$$

$$= x^n J_{n-1}(x).$$

(ii) Multiplying the expression for $J_{-n}(x)$ by x^{-n} throughout and differentiating, we get (ii)
(iii) Part (i) implies

$$x^n J_n'(x) + nx^{n-1}J_n(x) = x^n J_{n-1}(x)$$

and so

$$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x). \tag{135}$$

Similarly, part (ii) yields

$$-J_n'(x) + \frac{n}{x}J_n(x) = J_{n+1}(x). \tag{136}$$

Adding (135) and (136), we obtain

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

or

$$J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$$

or

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x).$$

(iv) Subtracting (136) from (135) yields

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

or

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)].$$

EXAMPLE 1.152

Show that $e^{\frac{1}{2}x\left(z-\frac{1}{z}\right)}$ is the *generating function of the Bessel's functions.*

Solution. We have

$$e^{\frac{1}{2}x\left(z-\frac{1}{z}\right)} = e^{\frac{1}{2}xz}.\quad e^{-\frac{1}{2}xz^{-1}}$$

$$= \left[1 + \left(\frac{1}{2}x\right)z + \left(\frac{1}{2}x\right)^2\frac{z^2}{2!} + \ldots + \frac{\left(\frac{1}{2}x\right)^r z^r}{r!} + \ldots\right]$$

$$\times\left[1 + \left(-\frac{1}{2}x\right)(z^{-1}) + \left(-\frac{1}{2}x\right)^2\frac{z^{-2}}{2!} + \ldots + \frac{\left(-\frac{1}{2}x\right)^r z^{-r}}{r!}\right].$$

The coefficient of z^n in this expansion is

$$\frac{\left(\frac{1}{2}x\right)^n}{n!} + \frac{\left(\frac{1}{2}x\right)^{n+1}\left(-\frac{1}{2}x\right)^2}{(n+1)!} + \frac{\left(\frac{1}{2}x\right)^{n+2}\left(-\frac{1}{2}x\right)^2}{(n+2)!\;2!} + \ldots$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\;(m+n)!}\left(\frac{1}{2}x\right)^{2m+n}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\,(n+m+1)}\left(\frac{x}{2}\right)^{2m+n} = J_n(x).$$

Thus

$$e^{\frac{1}{2}x\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n J_n(x).$$

Hence, $e^{\frac{1}{2}x\left(z-\frac{1}{z}\right)}$ is the *generating function of the Bessel's function*.

EXAMPLE 1.153

Show that

(i) $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$

(ii) $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

(iii) $\int xJ_0^2(x)dx = \frac{1}{2}x^2[J_0^2(x) + J_1^2(x)]$.

Solution. (i) Putting $n = \frac{1}{2}$ in the expression for $J_n(x)$, we have

$$J_{\frac{1}{2}}(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma\left(\frac{1}{2}+m+1\right)}\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}$$
$$=\left(\frac{x}{2}\right)^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma\left(m+\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{2m}$$
$$=\left(\frac{x}{2}\right)^{\frac{1}{2}}\left[\frac{1}{\Gamma\left(\frac{3}{2}\right)}-\frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^2+\frac{1}{2!\,\Gamma\left(\frac{7}{2}\right)}\left(\frac{x}{2}\right)^4-\ldots\right]$$
$$=\left(\frac{x}{2}\right)^{\frac{1}{2}}\left[\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}-\frac{1}{\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2\right.$$
$$\left.+\frac{1}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4-\ldots\right]$$
$$=\left(\frac{x}{2}\right)^{\frac{1}{2}}\frac{2}{\Gamma\left(\frac{1}{2}\right)}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\ldots\right]$$
$$=\frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right]=\sqrt{\frac{2}{\pi x}}\sin x.$$

(ii) Putting $n = \frac{1}{2}$ in the expression for $J_{-n}(x)$ and proceeding as in part (i), we obtain part (ii).

(iii) We have

$$\int xJ_0^2(x)dx$$
$$=\frac{x^2}{2}J_0^2(x)-\int\frac{x^2}{2}2J_0(x)J_0'(x)dx\,(\text{integration by parts})$$
$$=\frac{x^2}{2}J_0^2(x)+\int x^2J_0(x)J_1(x)dx\;\text{since}\,J_0'(x)=-J_1(x)$$
$$=\frac{x^2}{2}J_0^2(x)+\int xJ_1(x).\frac{d}{dx}(xJ_1(x))dx\;\;\text{since}$$
$$\frac{d}{dx}(xJ_1(x))=xJ_0(x)$$
$$=\frac{x^2}{2}J_0^2(x)+\frac{1}{2}(xJ_1(x))^2=\frac{x^2}{2}[J_0^2(x)+J_1^2(x)].$$

EXAMPLE 1.154

Show that $J_n(\lambda_i x)$ is a solution of

$$x^2\frac{d^2y}{dx}+x\frac{dy}{dx}+(\lambda_i^2x^2-n^2)y=0.$$

Solution. We know that $J_n(z)$ is a solution of the Bessel's equation

$$z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}+(z^2-n^2)y=0. \qquad (137)$$

Put $z = \lambda_i x$. Then

$$\frac{dy}{dz}=\frac{dy}{dx}\cdot\frac{dx}{dz}=\frac{1}{\lambda_i}\frac{dy}{dx}$$

and

$$\frac{d^2y}{dz^2}=\frac{1}{\lambda_i^2}\frac{d^2y}{dx^2}.$$

Putting the values of z, $\frac{dy}{dz}$, and $\frac{d^2y}{dz^2}$ in (137), we get

$$\lambda_i^2x^2\left(\frac{1}{\lambda_i^2}\frac{d^2y}{dx^2}\right)+\lambda_i x\left(\frac{1}{\lambda_i}\frac{dy}{dx}\right)+(\lambda_i^2x^2-n^2)y=0.$$

or

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\lambda_i^2x^2-n^2)y=0. \qquad (138)$$

Hence $J_n(\lambda_i x)$ is a solution of (138).

EXAMPLE 1.155

If $\lambda_1, \lambda_2, \ldots$ are the roots of $J_n(ax)=0$, $a \,\varepsilon\, \Re$, show that the system

$$J_n(\lambda_1 x),\; J_n(\lambda_2 x),\; J_n(\lambda_3 x), \ldots$$

of Bessel's functions is an orthogonal system, with weight function x, on $[0, a]$, $a \varepsilon\, \Re$.

Solution. Let

$$u_i = J_n(\lambda_i x)\;\text{and}\; u_j = J_n(\lambda_j x), \quad i \neq j.$$

Then, by Example 1.154, we have

$$x^2u_i''+xu_i'+(\lambda_i^2x^2-n^2)u_i=0 \qquad (139)$$

and

$$x^2u_j''+xu_j'+(\lambda_j^2x^2-n^2)u_j=0. \qquad (140)$$

Multiplying (139) by u_J and (140) by u_i and subtracting, we get

$$x^2\left(u_ju_i''-u_iu_j''\right)+x\left(u_ju_i'-u_iu_j'\right)+x^2u_iu_j\left(\lambda_i^2-\lambda_j^2\right)=0$$

or

$$x\left(u_ju_i''-u_iu_j''\right)+\left(u_ju_i'-u_iu_j'\right)=x\left(\lambda_j^2-\lambda_i^2\right)u_iu_j$$

or

$$x\frac{d}{dx}\left(u_ju_i' - u_iu_j'\right) + \left(u_ju_i' - u_iu_j'\right) = x\left(\lambda_j^2 - \lambda_i^2\right)u_iu_j$$

or

$$\frac{d}{dx}\left[x\left(u_ju_i' - u_iu_j'\right)\right] = x\left(\lambda_j^2 - \lambda_i^2\right)u_iu_j. \quad (141)$$

Integrating (141) with respect to x in $[0, a]$, we get

$$\left[x\left(u_ju_i' - u_iu_j'\right)\right]_0^a = \left(\lambda_j^2 - \lambda_i^2\right)\int_0^a xu_iu_j dx$$

or

$$a\left[\lambda_i J_n\left(\lambda_j a\right)J_n'(\lambda_i a) - \lambda_j J_n(\lambda_i a)J_n'\left(\lambda_j a\right)\right]$$

$$= \left(\lambda_j^2 - \lambda_i^2\right)\int_0^a xu_iu_j dx. \quad (142)$$

Since $J_n\left(\lambda_j a\right) = J_n(\lambda_i a) = 0$, it follows that

$$\left(\lambda_j^2 - \lambda_i^2\right)\int_0^a xu_iu_j = 0, \quad \lambda_i \neq \lambda_j$$

or

$$\int_0^a xJ_n(\lambda_i x)J_n\left(\lambda_j x\right)dx = 0, \quad i \neq j.$$

Hence the system $\{J_n(\lambda_i x)\}$ forms an orthogonal system with weight x.

Remark 1.3. From (142), we have

$$\int_0^a xJ_n(\lambda_i x)J_n(\lambda_j x)dx$$

$$= \frac{a[\lambda_i J_n(\lambda_j a)J'_n(\lambda_i a) - \lambda_j J_n(\lambda_i a)J'_n(\lambda_j a)]}{\lambda_j^2 - \lambda_i^2}.$$

If $\lambda_i = \lambda_j$, then right hand side of the above expression is of $\frac{0}{0}$ form. We assume that λ_i is a root of $J_n(ax) = 0$ and $\lambda_j \to \lambda_i$. Then we have

$$\lim_{\lambda_j\to\lambda_i}\int_0^a xJ_n(\lambda_i x)J_n(\lambda_j x)dx$$

$$= \lim_{\lambda_j\to\lambda_i}\frac{a\lambda_i J_n(\lambda_j a)J'_n(\lambda_i a)}{\lambda_j^2 - \lambda_i^2}$$

Or, using L'Hospital's Rule,

$$\int_0^a xJ_n^2(\lambda_i x)dx = \lim_{\lambda_j\to\lambda_i}\frac{a^2\lambda_i J'_n(\lambda_j a)J'_n(\lambda_i a)}{2\lambda_j}$$

$$= \frac{a^2}{2}[J'_n(\lambda_i a)]^2$$

$$= \frac{a^2}{2}J_{n+1}^2(\lambda_i a).$$

Hence

$$\frac{2}{a^2 J_{n+1}^2(\lambda_i a)}\int_0^a xJ_n^2(\lambda_i x)dx = 1.$$

EXAMPLE 1.156

Show that $x^n J_n(x)$ is a solution of the differential equation

$$x\frac{d^2y}{dx^2} + (1-2n)\frac{dy}{dx} + xy = 0.$$

Solution. Let $y = x^n J_n(x)$. Then

$$\frac{dy}{dx} = \frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}, \text{ and}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(x^n J_{n-1}) = x^n J_{n-1}' + nx^{n-1}J_{n-1}.$$

Putting the values of y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$x\frac{d^2y}{dx^2} + (1-2n)\frac{dy}{dx} + xy$$

$$= x^{n+1}\left[J'_{n-1} - \frac{n-1}{x}J_{n-1}\right] + x^{n+1}J_n$$

$$= x^{n+1}(-J_n) + x^{n+1}J_n = 0.$$

EXAMPLE 1.157

Express $J_4(x)$ is term of $J_0(x)$ and $J_1(x)$.

Solution. We know that

$$J_n(x) = \frac{x}{2n}[J_{n-1},(x) + J_{n+1}(x)] \text{ (recurrence formula)}$$

or

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x).$$

Putting $n = 1, 2$, and 3, we get

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$$

$$\begin{aligned}J_3(x) &= \frac{4}{x}J_2(x) - J_1(x)\\ &= \frac{4}{x}\left[\frac{2}{x}J_1(x) - J_0(x)\right] - J_1(x)\\ &= \frac{8}{x^2}J_1(x) - \frac{4}{x}J_0(x) - J_1(x)\\ &= \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x),\end{aligned}$$

$$\begin{aligned}J_4(x) &= \frac{6}{x}J_3(x) - J_2(x)\\ &= \frac{6}{x}\left[\left(\frac{8}{x} - 1\right)J_1(x) - \frac{4}{x}J_0(x)\right]\\ &\quad - \left[\frac{2}{x}J_1(x) - J_0(x)\right]\\ &= \frac{48}{x^3}J_1(x) - \frac{6}{x}J_1(x) - \frac{24}{x^2}J_0(x) - \frac{2}{x}J_1(x) + J_0(x)\\ &= \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x).\end{aligned}$$

EXAMPLE 1.158

Solve

$$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + (x^2 - 64)z = 0.$$

Solution. The given equation is evidently Bessel's equation of order 8, which is an integer. Therefore, its general solution is

$$z = c_1J_8(x) + c_2Y_8(x),$$

where c_1 and c_2 are arbitrary constants.

EXAMPLE 1.159

Solve the differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{1}{4}y = 0.$$

Solution. Let $z = \sqrt{x}$. Then

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{1}{2\sqrt{x}}, \text{ and}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{dy}{dz}\cdot\frac{1}{2}\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) + \frac{1}{2\sqrt{x}}\left(\frac{d^2y}{dz^2}\cdot\frac{1}{2\sqrt{x}}\right)\\ &= \frac{1}{4x}\frac{d^2y}{dz^2} - \frac{1}{4x^{\frac{3}{2}}}\frac{dy}{dz} = \frac{1}{4z^2}\frac{d^2y}{dz^2} - \frac{1}{4z^3}\frac{dy}{dz}.\end{aligned}$$

Putting these values of the partial derivatives in the given equation, we get

$$z^2\left[\frac{1}{4z^2}\frac{d^2y}{dz^2} - \frac{1}{4z^3}\frac{dy}{dz}\right] + \frac{1}{2z}\frac{dy}{dz} + \frac{1}{4}y = 0$$

or

$$\frac{d^2y}{dz^2} - \frac{1}{z}\frac{dy}{dz} + \frac{2}{z}\frac{dy}{dz} + y = 0$$

or

$$z\frac{d^2y}{dz^2} + \frac{dy}{dz} + zy = 0$$

or

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - 0^2)y = 0,$$

which is Bessel's equation of order 0. Therefore, its general solution is

$$y = c_1J_0(z) + c_2y_0(z) = c_1J_0(\sqrt{x}) + c_2Y_0(\sqrt{x}).$$

EXAMPLE 1.160

Show that

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_0(x) + \left(z - \frac{1}{z}\right)J_1(x) + \left(z^2 + \frac{1}{z^2}\right)J_2(x) + \left(z^3 - \frac{1}{z^3}\right)J_3(x) + \dots$$

Hence or otherwise show that

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\theta)d\theta.$$

Solution. We know that

$$\begin{aligned}e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} &= \sum_{n-\infty}^{\infty} z^nJ_n(x) \text{ (generating function)}\\ &= J_0 + \sum_{n=1}^{\infty}(J_nz^n + J_{-n}z^{-n})\\ &= J_0 + \sum_{n=1}^{\infty}(J_nz^n + (-1)^nJ_nz^{-n})\\ &= J_0 + \sum_{n=1}^{\infty}[z^n + (-1)^nz^{-n}]J_n\\ &= J_0 + \left(z - \frac{1}{z}\right)J_1 + (z^2 + z^{-2})J_2\\ &\quad + (z^3 - z^{-3})J_3 + \dots \qquad (143)\end{aligned}$$

Putting $z = e^{i\phi}$ we get $z - \frac{1}{z} = 2i\sin\phi$, $z^2 + \frac{1}{z^2} = 2\cos 2\phi$, and so on. Therefore, (143) reduces to

$$e^{i(x\sin\phi)} = J_0 + (2i\sin\phi)J_1 + (2\cos 2\phi)J_2 + \ldots$$

or

$$\cos(x\sin\phi) + i\sin(x\sin\phi)$$
$$= J_0 + (2i\sin\phi)J_1 + (2\cos 2\phi)J_2 + \ldots .$$

Separating real and imaginary parts, we get

$$\cos(x\sin\phi) = J_0 + (2\cos 2\phi)J_2 + (2\cos 4\phi)J_4 + \ldots \quad (144)$$

and

$$\sin(x\sin\phi) = (2\sin\phi)J_1 + (2\sin 3\phi)J_3 + (2\sin 5\phi)J_5 + \ldots, \quad (145)$$

which are called *Jacobi's series.*

Putting $\phi = \frac{\pi}{2} - \theta$ in (144), we get

$$\cos(x\cos\theta) = J_0 - (2\cos 2\theta)J_2 + \ldots .$$

Therefore,

$$\int_0^{\pi} \cos(x\cos\theta)d\theta$$
$$= \int_0^{\pi} J_0 d\theta - \int_0^{\pi} (2\cos 2\theta)J_2 d\theta + \ldots = \pi J_0.$$

Hence

$$J_0 = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)d\theta.$$

EXAMPLE 1.161

Solve the differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - ixy = 0.$$

Solution. The given equation can be written as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - ix^2y = 0.$$

Putting $z = \sqrt{-i}\,x$, we have

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \sqrt{-i}\,\frac{dy}{dz},$$

$$\frac{d^2y}{dx^2} = -i\frac{d^2y}{dz^2}.$$

Thus, the given equation reduces to

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dx} + (z^2 - 0)y = 0,$$

which is Bessel's equation of order zero. Its solution is

$$y = J_0(z) = J_0(\sqrt{-i}\,x) = J_0\left(i^{\frac{3}{2}}x\right)$$
$$= 1 - \frac{i^3x^2}{2^2} + \frac{i^6x^4}{2^4(2!)^2} - \frac{i^9x^6}{2^6(3!)^2} + \frac{i^{12}x^8}{2^8(4!)^2} - \ldots$$
$$= \left[1 - \frac{x^4}{2^2.4^2} + \frac{x^8}{2^2.4^2.6^2.8^2} - \ldots\right]$$
$$+ i\left[\frac{x^2}{2^2} - \frac{x^6}{2^2.4^2.6^2} + \frac{x^{10}}{2^2.4^2.6^2.8^2.10^2} - \ldots\right]$$
$$= ber\ x + i\ bei\ x,$$

where

$$ber\ x = 1 + \sum_{m=1}^{\infty}(-1)^m \frac{x^{4m}}{2^2.4^2.6^2\ldots(4m)^2}$$

is called *Bessel's real (or ber) function* and

$$bei\ x = -\sum_{m=1}^{\infty}(-1)^m \frac{x^{4m-2}}{2^2.4^2.6^2\ldots(4m-2)^2}$$

called *Bessel's imaginary* (or *bei) function.*

1.25 FOURIER–BESSEL EXPANSION OF A CONTINUOUS FUNCTION

We have proved in Example 1.155 that Bessel's functions $J_n(\lambda x)$ form an orthogonal set with weight x. This property allows us to expand a continuous function f in Fourier–Bessel series in a range 0 to a. So, let

$$f(x) = c_1J_n(\lambda_1 x) + c_2J_n(\lambda_2 x) + \cdots + c_nJ_n(\lambda_n x) \quad (146)$$

where λ_i are the roots of the equation $J_n(\lambda a) = 0$. To determine the coefficients c_i, we multiply both sides of the equation (146) by $xJ_n(\lambda_i x)$ and integrate with respect to x between the limits 0 to a. Using orthogonal property and the remark cited above, we have

$$\int_0^a xf(x)J_n(\lambda_i x)dx = c_i \int_0^a xJ_n(\lambda_i x)dx$$
$$= c_i\left[\frac{a^2}{2}J_{n+1}^2(\lambda_i a)\right].$$

Hence

$$c_i = \frac{2}{a^2 J_{n+1}^2(\lambda_i a)}\int_0^a xf(x)J_n(\lambda_i x)dx.$$

Substituting the values of c_i in the expression (146), we get the Fourier–Bessel series of f.

EXAMPLE 1.162

Expand $f(x) = x^2\ (0 < x < a)$ in Fourier–Bessel series in $J_2(\lambda_n x)$ if $\lambda_n a$ are the positive roots of $J_2(x) = 0$.

Solution. Let the Fourier–Bessel series of the function $f(x) = x^2\ (0 < x < a)$ be

$$x^2 = \sum_{n=1}^{\infty} c_n J_2(\lambda_n x).$$

We know that

$$c_i = \frac{2}{a^2 J_{n+1}^2(\lambda_i a)}\int_0^a xf(x)J_n(\lambda_i x)dx$$
$$= \frac{2}{a^2 J_3^2(\lambda_i a)}\int_0^a x^3 J_2(\lambda_i x)dx$$
$$= \frac{2}{a^2 J_3^2(\lambda_i a)}\left[\frac{x^3 J_3(\lambda_i x)}{\lambda_i}\right]_0^a$$
$$= \frac{2a}{\lambda_i J_3(\lambda_i a)}.$$

Hence the Fourier–Bessel series for $f(x) = x^2$ is

$$x^2 = 2a\sum_{n=1}^{\infty}\frac{J_2(\lambda_n x)}{\lambda_n J_3(\lambda_n a)}.$$

EXAMPLE 1.163

Expand $f(x) = 1$ in Fourier–Bessel series in $J_0(\lambda_n x)$ if $\lambda_1, \lambda_2, ..., \lambda_n$ are the roots of $J_0(x) = 0$.

Solution. Let the Fourier–Bessel series of $f(x) = 1$ be

$$1 = \sum_{1}^{\infty} c_n J_0(\lambda_n x).$$

Since $f(x) = 1$ and $a = 1$, the coefficients c_i are given by

$$c_i = \frac{2}{J_1^2(\lambda_i)}\int_0^1 xJ_0(\lambda_i x)dx$$
$$= \frac{2}{J_1^2(\lambda_i)}\left[\frac{xJ_1(\lambda_i x)}{\lambda_i}\right]_0^1$$
$$= \frac{2}{\lambda_i J_1(\lambda_i)}.$$

Hence the required Fourier–Bessel series is

$$1 = 2\sum_{n=1}^{\infty}\frac{J_0(\lambda_n x)}{\lambda_n J_1(\lambda_n)}.$$

1.26 LEGENDRE'S EQUATION AND LEGENDRE'S POLYNOMIAL

The differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad (147)$$

is called Legendre's equation of order n, where n is a real number. The equation can be written is the form

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{n(n+1)}{1-x^2}y = 0.$$

Since $-\frac{2x}{1-x^2}$ and $\frac{n(n+1)}{1-x^2}$ are analytic at 0, the point $x = 0$ is a regular point of the Legendre's equation. So let

$$y = \sum_{m=0}^{\infty} a_m x^m$$

be the power series solution of the given Legendre's equation about $x = 0$. Then

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} m a_m x^{m-1}, \text{ and}$$

$$\frac{d^2y}{dx^2} = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}.$$

Putting the values of y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

$$-2x\sum_{m=1}^{\infty} m a_m x^{m-1} + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

or

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m$$

$$-2\sum_{m-1}^{\infty} m a_m x^m + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

or

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m-2}^{\infty} m(m-1)a_m x^n$$

$$-2\sum_{m=1}^{\infty} m a_m x^m + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

or

$$[2a_2 + n(n+1)a_0] + [6a_3 - 2a_1 + n(n+1)a_1]x$$

$$+\sum_{m=2}^{\infty} [(m+2)(m+1)a_{m+2}$$

$$+(n-m)(n+m+1)a_m]x^m = 0.$$

Equating to zero the coefficient of powers of x, we get

$$2a_2 + n(n+1)a_0 = 0, \text{ which yields}$$

$$a_2 = -\frac{n(n+1)}{2!}a_0,$$

$$6a_3 - 2a_1 + n(n+1)a_1 = 0, \text{ which gives}$$

$$a_3 = -\frac{(n-1)(n+2)}{3!}a_1,$$

and in general

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+1)(m+2)}a_m, \quad m \geq 2.$$

Putting $m = 0$, 1, we observe that a_2 and a_3 are the same as found above. Therefore,

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+1)(m+2)}a_m, \quad m \geq 0,$$

which is called the *recurrence solution of the Legendre's equation*. Putting $m = 2, 3, \ldots$ we have

$$a_4 = \frac{(n-2)n(n+1)(n+3)}{4!}a_0,$$

$$a_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1$$

and so on. Hence

$$y = a_0 + a_1x - \frac{n(n+1)}{2!}a_0x^2 - \frac{(n-1)(n+2)}{3!}x^3$$
$$+\frac{(n-2)n(n+1)(n+3)}{4!}a_0x^4$$
$$+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1x^5 + \ldots$$
$$= a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 + \ldots\right]$$
$$+a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{4!}x^5 + \ldots\right]$$
$$= a_0y_1(x) + a_1y_2(x), \text{say}.$$

Thus, y_1 contains only even powers of x while y_2 contains odd powers of x. Also y_1 and y_2 are linearly independent.

If n is even, then $y_1(x)$ is a polynomial of degree n and if n is odd, then $y_2(x)$ is a polynomial of degree n. Moreover if $a_n = \frac{(2n)!}{2^n(n!)^2}$, then

$$P_n(x) = \begin{cases} a_0 + a_2x^2 + \ldots + a_nx^n \text{ if } n \text{ is even} \\ a_1x + a_3x^3 + \ldots + a_nx^n \text{ if } n \text{ is odd} \end{cases}$$

is called the *Legendre polynomial of degree n*.

We note that

$$P_0(x) = 1,\ P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1),\ P_3(x) = \frac{1}{2}(5x^2 - 3x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \text{ and so on.}$$

Evidently, the value of the polynomial $P_n(x)$, $n = 0$, 1, 2,... is 1 for $x = 1$.

EXAMPLE 1.164

Show that

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2 - 1)^n \text{ (Rodrigue's formula).}$$

Solution. Putting $u = (x^2 - 1)^n$, we have

$$\frac{du}{dx} = 2nx(x^2-1)^{n-1} = \frac{2nxu}{x^2-1}$$

and so

$$(1-x^2)\frac{du}{dx} + 2nxu = 0. \qquad (148)$$

Differentiating (148), $(n + 1)$ times by Leibnitz's theorem, we get

$$(1-x^2)u_{n+2} - 2(n+1)xu_{n+1} - n(n+1)u_n + 2n[xu_{n+1} + (n+1)u_n] = 0$$

or

$$(1-x^2)u_{n+2} - 2xu_{n+1} + n(n+1)u_n = 0$$

or

$$(1-x^2)\frac{d^2u_n}{dx^2} - 2x\frac{du_n}{dx} + n(n+1)u_n = 0.$$

Thus, $u_n = \frac{d^n}{dx^n}(x^2-1)^n$ satisfies the Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0.$$

Since the solution of Legendre's equation is $P_n(x)$, we have

$$P_n(x) = cu_n = c\frac{d^n}{dx^n}(x^2-1)^n. \qquad (149)$$

Putting $x = 1$ in (149), we get

$$1 = c\left[\frac{d^n}{dx^n}(x-1)^n(x+1)^n\right]_{x=1}$$

$$= c[n!\ (x+1)^n + \text{term containing } x-1 \text{ along with its powers}]_{x=1}$$

$$= c\ n!\ 2^n,$$

and so $c = \frac{1}{n!\,2^n}$. Thus,

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n.$$

EXAMPLE 1.165

Show that

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x),\ |x| < 1,\ |t| < \frac{1}{3}.$$

[Thus $(1-2xt+t^2)^{-\frac{1}{2}}$ is a *generating function of the Legendre's polynomials*.]

Solution. We have, by binomial expansion,

$$[1-(2xt-t^2)]^{\frac{1}{2}}$$

$$= 1 + \frac{1}{2}(2xt-t^2) + \frac{1.3}{2^2.2!}(2xt-t^2)^2 + \ldots + \frac{1.3\ldots(2n-5)}{2^{n-2}(n-2)!}(2xt-t^2)^{n-2} + \frac{1.3\ldots(2n-3)}{2^{n-1}(n-1)!}(2xt-t^2)^{n-1} + \frac{1.3\ldots(2n-1)}{2^n.n!}(2xt-t^2)^n + \ldots$$

$$= 1 + xt + \left(-\frac{1}{2} + \frac{3}{8}.4x^2\right)t^2 + \ldots + \left[\frac{1.3\ldots(2n-1)}{2^n.n!}(2x)^n - \frac{1.3\ldots(2n-3)}{2^{n-1}(n-1)!}(n-1)(2x)^{n-2} + \frac{1.3\ldots(2n-5)}{2^{n-2}(n-2)!}\frac{(n-2)(n-3)}{1.2}(2x)^{n-4} + \ldots\right]t^n + \ldots$$

$$= 1 + xt + \frac{1}{2}(3x^2-1)t^2 + \ldots + \frac{1.3\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} + \ldots\right]t^n + \ldots$$

$$= P_0(x) + tP_1(x) + t^2P_2(x) + \ldots + t^nP_n(x) + \ldots$$

$$= \sum_{n=0}^{\infty} t^n P_n(x).$$

Hence, $(1-2xt+t^2)^{-\frac{1}{2}}$ generates Legendre polynomials $P_n(x)$.

EXAMPLE 1.166

Express the polynomial $x^3 + 2x^2 - x - 3$ in terms of Legendre's polynomials.

Solution. We know that

$$P_0(x) = 1 \quad , \quad P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}(3x^2-1) \text{ and so } x^2 = \frac{2}{3}P_2(x) + \frac{1}{3},$$

$$P_3(x) = \frac{1}{2}(5x^3-3x) \text{ and so } x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x).$$

Hence,

$$\begin{aligned}
x^3 &+ 2x^2 - x - 3 \\
&= \left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] + 2\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] \\
&\quad - P_1(x) - 3P_0(x) \\
&= \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{2}{5}P_1(x) - \frac{7}{3}P_0(x).
\end{aligned}$$

EXAMPLE 1.167

Show that Legendre polynomials are *orthogonal* on the interval [– 1, 1].

Solution. Let P_m and P_n be Legendre polynomials for $m \neq n$. We want to show that

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0, \; m \neq n.$$

Since P_m and P_n satisfy Legendre's equation, we have

$$(1-x^2)P_n''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0 \quad (150)$$

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0. \quad (151)$$

Multiplying (150) by $P_n(x)$ and (151) by $P_m(x)$ and subtracting, we get

$$\begin{aligned}
&(1-x^2)[P_n(x)P_m''(x) - P_m(x)P_n''(x)] \\
&\quad - 2x[P_n(x)P_m'(x) + P_m(x)P_n'(x)] \\
&= (n^2 - m^2 + n - m)P_m(x)P_n(x)
\end{aligned}$$

or

$$\begin{aligned}
&\frac{d}{dx}[(1-x^2)(P_n(x)P_m'(x) - P_m(x)P_n'(x))] \\
&= (n-m)(n+m+1)P_m(x)P_n(x).
\end{aligned}$$

Integration on the interval [–1, 1] yields

$$\begin{aligned}
&[(1-x^2)(P_n(x)P_m'(x) - P_m(x)P_n'(x)]_{-1}^{1} \\
&= (n-m)(n+m+1)\int_{-1}^{1} P_m(x)P_n(x)dx
\end{aligned}$$

or

$$0 = (n-m)(n+m+1)\int_{-1}^{1} P_m(x)P_n(x)dx$$

or

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0, \quad m \neq n.$$

Hence, $\{P_n(x)\}$ forms an orthogonal system.

EXAMPLE 1.168

Show that $\left\{\sqrt{\frac{2n+1}{2}}P_n(x)\right\}$ forms an *orthonormal system*.

Solution. The system of polynomials $\left\{\sqrt{\frac{2n+1}{2}}P_n(x)\right\}$ will form an orthonormal system if

$$\frac{2n+1}{2}\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n. \end{cases}$$

By Example 1.167, it follows that

$$\frac{2n+1}{2}\int_{-1}^{1} P_m(x)P_n(x)dx = 0, \; m \neq n.$$

If remains to show that $\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}$.

In this direction, we have by generating functions,

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{m=0}^{\infty} t^m P_m(x)$$

and

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x).$$

Multiplying these two expressions, we get

$$(1 - 2xt + t^2)^{-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} t^{m+n}P_m(x)P_n(x).$$

Integrating w.r.t. x on [– 1, 1], we have

$$\begin{aligned}
&\int_{-1}^{1} (1 - 2xt + t^2)^{-1}dx \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} t^{m+n}P_m(x)P_n(x)dx.
\end{aligned}$$

Since $\int_{-1}^{1} P_m(x)P_n(x)dx = 0$ for $m \neq n$, the above expression reduces to

$$\int_{-1}^{1} (1-2xt+t^2)^{-1}dx = \sum_{n=0}^{\infty} \int_{-1}^{1} t^{2n}P_n^2(x)dx.$$

Thus

$$\sum_{n=0}^{\infty} \int_{-1}^{1} t^{2n}P_n^2(x)dx = \left[-\frac{1}{2t}\log(1+t^2-2xt\right]_{x=-1}^{1}$$

$$= -\frac{1}{2t}\log\frac{(1-t)^2}{(1+t)^2} = \frac{1}{t}\log\frac{1+t}{1-t}$$

$$= \frac{1}{t}\left[2\left(t+\frac{t^3}{3}+\frac{t^5}{5}+\ldots\right)\right]$$

$$= 2\left(1+\frac{t^2}{3}+\frac{t^4}{5}+\ldots+\frac{t^{2n}}{2n+1}+\ldots\right).$$

Comparing the coefficients of t^{2n}, we have

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}$$

or

$$\frac{2n+1}{2}\int_{-1}^{1} P_n^2(x)dx = 1.$$

EXAMPLE 1.169

Using Rodrigue's formula, find the polynomial expression for $P_4(x)$.

Solution. The Rodrigue's formula is

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

Therefore,

$$P_4(x) = \frac{1}{2^n 4!}\frac{d^4}{dx^4}(x^2-1)^4 = \frac{1}{384}\frac{d^4}{dx^4}(x^2-1)^4.$$

So let

$$y = (x^2-1)^4 = x^8 - 4x^6 + 6x^4 - 4x^2 + 1.$$

Then,

$$\frac{dy}{dx} = 8x^7 - 24x^5 + 24x^3 - 8x$$

$$\frac{d^2y}{dx^2} = 56x^6 - 120x^4 + 72x^2 - 8$$

$$\frac{d^3y}{dx^3} = 336x^5 - 480x^3 + 144x$$

$$\frac{d^4y}{dx^4} = 1680x^4 - 1440x^2 + 144.$$

Therefore,

$$P_4(x) = \frac{1}{384}[1680x^4 - 1440x^2 + 144]$$

$$= \frac{1}{8}[35x^4 - 30x^2 + 3].$$

EXAMPLE 1.170

Establish the following recurrence relations for Legendre polynomial.

(i) $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$
(ii) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$
(iii) $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$
(iv) $(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x)$
(v) $(1-x^2)P_n'(x) = n(P_{n-1}(x) - xP_n(x))$
(vi) $(1-x^2)P_n'(x) = (n+1)(xP_n(x) - P_{n+1})$.

Solution. (i) The generating formula for $P_n(x)$ is

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x).$$

Differentiating w.r.t. t, we get

$$-\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

or

$$(x-t)(1-2xt+t^2)^{-\frac{1}{2}} = (1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

or

$$(x-t)\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)\sum_{n=0}^{\infty} nt^{n-1}P_n(x).$$

Comparing coefficients of t^n, we get

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2nxP_n(x) + (n-1)P_{n-1}(x)$$

or

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

(ii) Differentiating the generating formula w.r.t. x, we get

$$-\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2t) = \sum_{n=0}^{\infty} t^n P_n'(x)$$

or

$$t(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} t^n P_n'(x). \quad (152)$$

On the other hand, differentiation of the generating formula w.r.t. t yields

$$(x-t)(1-2tx+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} nt^{n-1} P_n(x). \quad (153)$$

Dividing (153) by (152), we get

$$\frac{x-t}{t} = \frac{\sum_{n=0}^{\infty} nt^{n-1} P_n(x)}{\sum_{n=0}^{\infty} t^n P_n'(x)}$$

or

$$\sum_{n=0}^{\infty} nt^n P_n(x) = (x-t) \sum_{n=0}^{\infty} t^n P_n'(x).$$

Comparing the coefficients of t^n on both sides, we have

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x).$$

(iii) From (i), we have

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

Differentiating w.r.t. x, we get

$$(n+1)P_{n+1}'(x) = (2n+1)P_n(x) + (2n+1)xP_n'(x) - nP_{n-1}'(x). \quad (154)$$

But from part (ii)

$$xP_n'(x) = nP_n(x) + P_{n-1}'(x).$$

Therefore, (154) reduces to

$$(n+1)P_{n+1}'(x) = (2n+1)P_n(x) + (2n+1)x[nP_n(x) + P_{n-1}'(x)] - nP_{n-1}'(x)$$

or

$$(2n+1)P_n(x) = P_{n+1}' - P_{n-1}'.$$

(iv) From part (ii), we have

$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$$

and from part (ii), we have

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x).$$

Subtracting, we obtain

$$(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x).$$

(v) From part (iv), we have

$$(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x).$$

Replacing n by $n - 1$, we get

$$nP_{n-1}(x) = P_n'(x) - xP_{n-1}'(x). \quad (155)$$

Also, from part (ii), we have

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x)$$

or

$$nxP_n(x) = x^2P_n'(x) - xP_{n-1}'(x). \quad (156)$$

Subtracting (156) from (155), we get

$$nP_{n-1}(x) - nxP_n(x) = (1-x^2)P_n'(x)$$

or

$$(1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)].$$

(vi) From (155), we have

$$nxP_{n-1}(x) = xP_n'(x) - x^2P_{n-1}'(x) \quad (157)$$

and from part (ii), we have

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x). \quad (158)$$

Subtracting (158) from (157), we obtain

$$n[xP_{n-1}(x) - P_n(x)] = (1-x^2)P_{n-1}'(x).$$

Replacing n by $n + 1$, we get

$$(1-x^2)P_n'(x) = (n+1)[xP_n(x) - P_{n+1}(x)].$$

EXAMPLE 1.171

Show that if n is odd, then

$$P_n'(x) = (2n-1)P_{n-1}(x) + (2n-5)P_{n-3}(x) + (2n-9)P_{n-5}(x) + \ldots + 3P_1(x).$$

Solution. We have

$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$$

(recurrence formula).

Changing n to $(n - 1)$, $(n - 3)$, $(n - 5)$,. . ., we get

$$(2n-1)P_{n-1}(x) = P_n'(x) - P_{n-2}'(x)$$
$$(2n-5)P_{n-3}(x) = P_{n-2}'(x) - P_{n-4}'(x)$$
$$(2n-9)P_{n-5}(x) = P_{n-4}'(x) - P_{n-6}'(x)$$
$$\ldots$$
$$\ldots$$
$$3P_1(x) = P_2'(x) - P_0'(x) \text{ if } n \text{ is odd.}$$

Adding these equations, we get

$$(2n-1)P_{n-1}(x)+(2n-5)P_{n-3}(x)$$
$$+(2n-9)P_{n-5}(x)+\ldots+3P_1(x)$$
$$=P_n'(x)-P_0'(x)=P_n'(x)-0$$
$$=P_n'(x),\ \text{since } P_0(x)=1.$$

EXAMPLE 1.172

Show that

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx=\frac{2n}{4n^2-1}.$$

Solution. We know that

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x).$$

Changing n to $n-1$, we get

$$nP_n(x)=(2n-1)xP_{n-1}(x)-(n-1)P_{n-2}(x)$$

or

$$xP_{n-1}(x)=\frac{1}{2n-1}[nP_n(x)+(n-1)P_{n-2}(x)].$$

Therefore,

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx$$
$$=\frac{1}{2n-1}\int_{-1}^{1}[nP_n(x)+(n-1)P_{n-2}(x)]P_n(x)dx$$
$$=\frac{n}{2n-1}\int_{-1}^{1}P_n^2(x)dx+\frac{n-1}{2n-1}\int_{-1}^{1}P_n(x)P_{n-2}(x)dx$$
$$=\frac{1}{2n-1}\int_{-1}^{1}P_n^2(x)dx+0 \text{ using orthogonal property}$$
$$=\frac{n}{2n-1}\left[\frac{2}{2n+1}\right]=\frac{2n}{4n^2-1}.$$

EXAMPLE 1.173

Show that

(i) $P_n(-x)=(-1)^nP_n(x)$

(ii) $P_n(1)=1$

(iii) $P_n(-1)=(-1)^n$.

Solution. (i) Generating function formula is

$$(1-2xt+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}t^nP_n(x).$$

Therefore,

$$\sum_{n=0}^{\infty}t^nP_n(-x)=(1+2xt+t^2)^{-\frac{1}{2}}$$
$$=[1-2x(-t)+t^2]^{-\frac{1}{2}}$$
$$=\sum_{n=0}^{\infty}(-t)^nP_n(x)=\sum_{n=0}^{\infty}(-1)^nt^nP_n(x)$$

and so $P_n(-x)=(-1)^n P_n(x)$.

(ii) We have

$$\sum_{n=0}^{\infty}P_n(1)t^n=(1-2t+t^2)^{-\frac{1}{2}}=(1-t)^{-1}$$
$$=1+t+t^2+\ldots+t^n+\ldots$$
$$=\sum_{n=0}^{\infty}t^n.$$

Hence $P_n(1)=1$.

(iii) We note that

$$\sum_{n=0}^{\infty}t^nP_n(-1)=(1+2t+t^2)^{-\frac{1}{2}}$$
$$=(1+t)^{-1}=\sum_{n=0}^{\infty}(-1)^nt^n.$$

Hence $P_n(-1)=(-1)^n$.

Second Method. From part (i)

$$P_n(-x)=(-1)^nP_n(x).$$

Therefore,

$$P_n(-1)=(-1)^nP_n(1)=(-1)^n \text{ using(ii)}.$$

1.27 FOURIER–LEGENDRE EXPANSION OF A FUNCTION

We have established in Examples 1.167 and 1.168 that $\sqrt{\frac{2n+1}{2}}\{P_n(x)\}$ constitutes an orthonormal system. Therefore, as in the case of Fourier expansion, the expansion in terms of $P_n(x)$ of a given function $f(x)$ is possible. Let

$$f(x)=\sum_{n=0}^{\infty}c_nP_n(x). \tag{159}$$

Multiplying both sides by $P_n(x)$ and integrating both sides by [–1, 1], we get

$$\int_{-1}^{1} f(x)P_n(x)dx = c_n \int_{-1}^{1} P_n^2(x)dx = c_n\left(\frac{2}{2n+1}\right)$$

and so

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx\,.$$

The coefficients c_n are called Fourier–Legendre coefficients and the expansion (159) is called Fourier–Legendre expansion of $f(x)$.

EXAMPLE 1.174

Expand

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1. \end{cases}$$

in Fourier–Legendre series.

Solution. We have

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx = \frac{2n+1}{2}\int_{-1}^{1} P_n(x)dx.$$

Therefore,

$$c_0 = \frac{1}{2}\int_0^1 P_0(x)dx = \frac{1}{2}\int_0^1 dx = \frac{1}{2},$$

$$c_1 = \frac{3}{2}\int_0^1 P_1(x)dx = \frac{3}{2}\int_0^1 xdx = \frac{3}{2}\left(\frac{1}{2}\right) = \frac{3}{4},$$

$$c_2 = \frac{5}{2}\int_0^1 P_2(x)dx = \frac{5}{2}\int_0^1 \frac{3x^2-1}{2}dx = \frac{5}{4}[x^3-x]_0^1 = 0,$$

$$c_3 = \frac{7}{2}\int_0^1 P_3(x)dx = \frac{7}{2}\int_0^1 \frac{5x^3-3x}{2}dx$$

$$= \frac{7}{4}\left[5\frac{x^4}{4} - 3\frac{x^2}{2}\right]_0^1 = -\frac{7}{16},$$

$$c_4 = \frac{9}{2}\int_0^1 P_4(x)dx = \frac{9}{2}\int_0^1 \frac{35x^4-30x^2+3}{8}dx = \frac{27}{16},$$

and so on. Hence, Fourier–Legendre expansion of the given function is

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$
$$= \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{27}{16}P_4(x) + \ldots$$

1.28 MISCELLANEOUS EXAMPLES

EXAMPLE 1.175

Solve the differential equation

$$x\,dy - y\,dx = (x^2+y^2)\,dx$$

Solution. We have

$$x\,dy - y\,dx = (x^2+y^2)dx$$

or

$$\frac{x\,dy - y\,dx}{x^2+y^2} = dx$$

or

$$d\left(\tan^{-1}\frac{y}{x}\right) = dx.$$

Integrating, we get

$$\tan^{-1}\frac{y}{x} = x + C.$$

EXAMPLE 1.176

Solve $(1+y^2)dx = (e^{\tan^{-1}y} - x)dy$.

Solution. The given equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2},$$

which is of the form

$$\frac{dx}{dy} + Px = Q,$$

where P and Q are function of y only. The integrating factor is

$$I.F = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}.$$

Therefore the solution of the differential equation is

$$x\, e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2}.e^{\tan^{-1}y}dy + c$$

$$= \int \frac{e^{2\tan^{-1}y}}{1+y^2}dy + c = \frac{e^{2\tan^{-1}y}}{2} + c$$

or

$$2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + c_1.$$

EXAMPLE 1.177

Solve the initial value problem $e^x(\cos y dx - \sin y dy) = 0$, $y(0) = 0$.

Solution. The given equation is of the form $Mdx + Ndy = 0$, where

$$M = e^x \cos y \quad \text{and} \quad N = -e^x \sin y.$$

Therefore

$$\frac{\partial M}{\partial y} = -e^x \sin y \quad \text{and} \quad \frac{\partial N}{\partial x} = -e^x \sin y.$$

Hence the given equation is exact. The solution of the given differential equation is

$$\int_{y\text{constant}} Mdx + \int (\text{terms in } N \text{ not containing } x)dy = k$$

or

$$e^x \cos y = k.$$

Using the initial condition $y(0) = 0$, we get $k = e^0 \cos 0 = 1$. Hence the required solution is

$$e^x \cos y = 1.$$

EXAMPLE 1.178

Solve the following equation
$(xy^2 - e^{\frac{1}{x^3}})dx - x^2y\, dy = 0.$

Solution. The equation here is

$$\left(xy^2 - e^{\frac{1}{x^3}}\right)dx \quad - x^2ydy = 0$$

Comparing it with $Mdx + Ndy = 0$, we have

$$M = xy^2 - e^{\frac{1}{x^3}} \quad \text{and} \quad N = -x^2y$$

Therefore

$$\frac{\partial M}{\partial y} = 2xy \quad \text{and} \quad \frac{\partial N}{\partial x} = -2xy.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the given equation is not exact. But

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x},$$

which is a function of x only. Hence $e^{\int \frac{-4}{x}dx} = e^{-4\log x} = \frac{1}{x^4}$ is the integrating factor. So multiplying the given equation throughout by $\frac{1}{x^4}$, we get

$$\left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{\frac{1}{x^3}}\right)dx - \frac{y}{x^2}dy = 0,$$

which is an exact equation. Therefore the required solution is

$$\int_{y\text{ constant}} \left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{\frac{1}{x^3}}\right) \; dx = c$$

y constant

or

$$-\frac{y^2}{2x^2} + \int \frac{1}{3}e^{\frac{1}{x^3}}\left(\frac{-3}{x^4}\right) \; dx = \text{c}$$

or

$$-\frac{y^2}{2x^2} + \frac{1}{3}e^{\frac{1}{x^3}} = \text{c}.$$

EXAMPLE 1.179

Solve the differential equation

$$(y \;\cos\; x + 1)\; dx + \sin\; x\; dy \;= 0.$$

Solution. The given differential equation is of the form $Mdx + Ndy = 0$, where

$$M = y \;\cos +1, \quad N = \sin x.$$

Now

$$\frac{\partial M}{\partial y} = \cos x, \qquad \frac{\partial N}{\partial x} = \cos x.$$

Therefore the given equation is exact and its solution is

$$\int_{\text{y constant}} (y \;\cos +1)dx + \int 0dy = C$$

or

$$y \;\sin x + x = C.$$

EXAMPLE 1.180

If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 24 minutes.

Solution. Let T be the temperature of the body at any instant t. By Newton's law of cooling, we have

$$\frac{dT}{dt} = k(T - T_0)$$

and so variable separation yields

$$\frac{dT}{T - T_0} = -\,kdt.$$

Integrating, we get

$$\log(T - T_0) = -kt + \log C$$

or

$$T - T_0 = C\,e^{-kt}$$

or

$$T - 30 = c\,e^{-kt}.$$

But when $t = 0$, $T = 80°$. Therefore $C = 80 - 30 = 50$ and we have

$$T - 30 = 50\,e^{-kt}.$$

Now when $t = 12$, $T = 60°$. Therefore

$$60 - 30 = 50\,e^{-12k}$$

or

$$30 = 50\,e^{-12\,k}$$

or

$$-12k = \log\frac{3}{5} \quad \text{which yields} \quad k = \frac{1}{12}\log\frac{5}{3}.$$

Hence

$$T = 30 + 50\,e^{-\left(\frac{1}{12}\log\frac{5}{3}\right)\,t}$$

When $t = 24$, we have

$$\begin{aligned} T &= 30 + 50\,e^{-2\log\frac{5}{3}} \\ &= 30 + 50\,e^{\log\frac{9}{25}} \\ &= 30 + 50\left(\frac{9}{25}\right) = 30 + 18 = 48°\text{C}. \end{aligned}$$

EXAMPLE 1.181

Suppose that an object is heated to 300°F and allowed to cool in a room whose air temperature is 80°F. If after 10 minutes the temperature of the object is 250°F, what will be its temperature after 20 minutes?

Solution. By Newton's law of cooling, we have

$$\frac{dT}{dt} = -k(T - T_0),$$

or

$$\frac{dT}{dt} = -k\ dt.$$

Integrating, we get

$$\log(T - T_0) = -kt + \log C$$

or

$$T - T_0 = Ce^{-kt}. \tag{160}$$

Equation (160) gives

$$T - 80 = C\,e^{-\,kt}$$

When $t = 0$, $T = 300°$. Therefore $C = 300 - 80 = 220°$. Thus

$$T = 80 + 220\,e^{-kt}.$$

But when $t = 10$, $T = 250°$. Therefore

$$250 = 80 + 220\,e^{-10k},$$

which yields

$$e^{-10k} = \frac{250 - 80}{220} = \frac{17}{22}$$

or

$$k = \frac{1}{10}\log\frac{29}{7} = 0.02578.$$

Hence

$$T = 80 + 220\,e^{-0.2578t}.$$

Therefore temperature of the body after 20 minutes is

$$T = 80 + 220\,e^{-5.156}.$$

EXAMPLE 1.182

A bacterial culture growing exponentially increases 200 to 500 grams in the period from 6 AM to 9 AM. How many grams will it be at noon?

Solution. The governing differential equation of this problem is

$$\frac{dx}{dt} = kx,$$

which has the solution as

$$x = C\,e^{kt}. \tag{161}$$

When $t = 0$, $x = 200$. Therefore (161) gives $C = 200$. Thus

$$x = 200\,e^{kt}. \tag{162}$$

When $t = 3$ hours, $x = 500$. Therefore (162) yields

$$500 = 200\,e^{3k}$$

or

$$e^{3k} = \frac{5}{2} \tag{163}$$

At noon $t = 6$ hour, using (163), we have

$$x = 200\, e^{6k} = 200(e^{3k})^2 = 200\left(\frac{5}{2}\right)^2 = 1250 \text{ grams.}$$

EXAMPLE 1.183

Find the orthogonal trajectories of the family

$$\frac{2a}{r} = 1 - \cos\theta.$$

Solution. We have

$$\frac{2a}{r} = 1 - \cos\theta,$$

or

$$r = \frac{2a}{1-\cos\theta}. \tag{164}$$

Differentiating (164) with respect to θ, we get

$$\frac{dr}{d\theta} = \frac{(1-\cos\theta)\,(0) - 2a(\sin\theta)}{(1-\cos\theta)^2}$$

$$= -\frac{2a\sin\theta}{(1-\cos\theta)^2}. \tag{165}$$

Dividing (165) by (164), we get

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{1-\cos\theta} = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$= -\cot\frac{\theta}{2}. \tag{166}$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (166), we get

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\cot\frac{\theta}{2}$$

or

$$r\frac{d\theta}{dr} = \cot\frac{\theta}{2}$$

or

$$\frac{dr}{r} = \tan\frac{\theta}{2}d\theta = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}d\theta.$$

Integrating, we get

$$\log r = -2\log\cos\frac{\theta}{2} + \log c$$

or

$$\log\ r + 2\log\ \cos\ \frac{\theta}{2} = \log\ c$$

or

$$\log r + \log\cos^2\frac{\theta}{2} = \log c$$

or

$$r\cos^2\frac{\theta}{2} = c$$

or

$$\frac{r}{2}(1+\cos\theta) = c$$

or

$$r = \frac{2c}{1+\cos\theta},$$

which is the required orthogonal trajectory.

EXAMPLE.1.184

Solve $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$

Solution. The auxiliary equation for the given non-homogeneous equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$

or

$$(m-1)(m-2)(m-3) = 0,$$

which yields $m = 1, 2, 3$. Therefore

$$\text{C.F.} = c_1e^x + c_2\,e^{2x} + c_3\,e^{3x}.$$

The particular integral is given by

$$\text{P.I} = \frac{1}{(D-1)(D-2)(D-3)}e^{-2x} + \frac{1}{(D-1)(D-2)(D-3)}e^{-3x}$$

$$= \frac{1}{(-2-1)(-2-2)(-2-3)}\,e^{-2x} + \frac{1}{(-3-1)(-3-2)(-3-3)}\,e^{-3x}.$$

$$= -\frac{1}{60}\,e^{-2x} - \frac{1}{120}\,e^{-3x}.$$

Hence the complete solution of the given differential equation is

$$y = \text{C.F.} + \text{P.I}$$

$$= C_1e^x + C_2e^{2x} + c_3e^{3x} - \frac{1}{60}e^{-2x} - \frac{1}{120}e^{-3x}.$$

EXAMPLE.1.185

Solve $(D^2+4D+13)y = e^{2x}$

Solution. The auxiliary equation for the given differential equation is

$$m^2 - 4m + 13 = 0,$$

which yields $m = \frac{4\pm\sqrt{16-52}}{2} = 2 \pm 3i$. Therefore

$$\text{C.F} = (c_1 \cos 3x + c_2 \sin 3x)e^{2x}.$$

The particular integral is given by

$$\text{P.I} = \frac{1}{f(D)}F(x) = \frac{1}{D^2-4D+13}e^{2x}$$
$$= \frac{1}{2^2-4(2)+13}e^{2x} = \frac{1}{9}e^{2x}.$$

Hence the complete solution is

$$y = \text{C.F} + \text{P.I} = (c_1 \cos 3x + c_2 \sin 3x)e^{2x} + \frac{1}{9}e^{2x}.$$

EXAMPLE.1.186

Solve $(D^2+1)y = \sin x\ \sin 2x + e^x x^2$.

Solution. The auxiliary equation for the given differential equation is

$$m^2 + 1 = 0, \quad \text{which yields} \quad m = \pm 2.$$

Therefore

$$\text{C.F} = c_1 \cos x + c_2 \sin x.$$

The particular integral is given by

$$\text{P.I} = \frac{1}{D^2+1}\sin x \sin 2x + \frac{1}{D^2+1}e^x x^2$$
$$= \frac{1}{D^2+1}\left[\frac{1}{2}(\cos x - \cos 3x)\right] + \frac{1}{D^2+1}e^x x^2$$
$$= \frac{1}{2(D^2+1)}\cos x - \frac{1}{2(D^2+1)}\cos 3x$$
$$+ e^x \frac{1}{(D+!)^2+1}x^2$$
$$= \frac{x}{4}\sin x - \frac{1}{2(-9+1)}\cos 3x$$
$$+ \frac{e^x}{2}\left(1 + D + \frac{D^2}{2}\right)^{-1} x^2$$
$$= \frac{x}{4}\sin x + \frac{1}{16}\cos 3x + \frac{1}{2}e^x(x^2 - 2x + 1).$$

Hence the complete solution is

$$y = \text{C.F} + \text{P.I} = c_1 \cos x + c_2 \sin x + \frac{1}{4}x \sin x$$
$$+ \frac{1}{16}\cos 3x + \frac{1}{2}e^x(x^2 - 2x + 1).$$

EXAMPLE 1.187

Solve $(D^2+1)y = \sin^2 x$.

Solution. We have

$$(D^2+1)y = \sin^2 x.$$

The auxiliary equation is $m^2 + 1 = 0$, which yields $m = \pm i$. Therefore

$$\text{C.F.} = c_1 \cos\ x + c_2 \sin\ x.$$

Now

$$\text{P.I} = \frac{1}{f(D)}F(x) = \frac{1}{D^2+1}\left[\frac{1-\cos 2x}{2}\right]$$
$$= \frac{1}{2(D^2+1)}[1 - \cos 2x]$$
$$= \frac{1}{2} - \frac{1}{2(D^2+1)}\cos\ 2x = \frac{1}{2} - \frac{1}{2(-4+1)}\cos\ 2x$$
$$= \frac{1}{2} + \frac{1}{6}\cos 2x.$$

Hence the solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} + \frac{1}{6}\cos 2x.$$

EXAMPLE 1.188

Find the particular integral of $(D^2 - 2D + l)y = \cosh\ x$.

Solution. The required particular integral is

$$\text{P.I} = \frac{1}{D^2-2D+1}\ \cosh\ x$$
$$= \frac{1}{D^2-2D+1}\left[\frac{e^x+e^{-x}}{2}\right]$$
$$= \frac{1}{2}\cdot\frac{1}{D^2-2D+1}e^x + \frac{1}{2}\cdot\ \frac{1}{D^2-2D+1}\ e^{-x}$$
$$= \frac{1}{2}x.\ \frac{1}{2D-2}\ e^x + \frac{1}{8}e^{-x}$$
$$= \frac{1}{2}x^2.\ \frac{1}{2}e^x + \frac{1}{8}e^{-x}$$
$$= \frac{1}{4}x^2\ e^x + \frac{1}{8}e^{-x}.$$

EXAMPLE 1.189

Solve the differential equation

$$\frac{d^2y}{dx^2}+4y=x\ \sin\ x.$$

Solution. The auxiliary equation for the given differential equation is

$$m^2+4=0,$$

which yields $m=\pm 2i$. Therefore,

$$\text{C.F}=c_1\cos 2x+c_2\sin 2x.$$

Further,

$$\begin{aligned}
\text{P.I}&=\frac{1}{D^2+4}(x\sin x)=\frac{1}{D^2+4}x[\text{I.P. of } e^{ix}]\\
&=\text{I.P of }\frac{1}{D^2+4}x\,e^{ix}=\text{I.P. of }\left[e^{ix}\frac{1}{(D+i)^2+4}x\right]\\
&=\text{I.P. of }\left[e^{ix}\frac{1}{D^2+2iD-1+4}x\right]\\
&=\text{I.P. of }\left[e^{ix}\frac{1}{D^2+2iD-1+3}x\right]\\
&=\text{I.P. of }\left[e^{ix}\frac{1}{3\left(1+\frac{2}{3}iD+\frac{D^2}{3}\right)}x\right]\\
&=\frac{1}{3}\text{I.P. of }\left[e^{ix}\left(1+\frac{2}{3}iD+\frac{D^2}{3}\right)^{-1}x\right]\\
&=\frac{1}{3}\text{I.P of }\left\{e^{ix}\left[1-\frac{2}{3}iD-\frac{D^2}{3}\right.\right.\\
&\qquad\left.\left.+\left(\frac{2}{3}iD+\frac{D^2}{3}\right)^2-\cdots\right]x\right\}\\
&=\frac{1}{3}\text{I.P of }\left\{e^{ix}\left(x-\frac{2}{3}i\right)\right\}\\
&=\frac{1}{3}\text{I.P. of }\left\{(\cos x+i\sin x)\left(x-\frac{2}{3}i\right)\right\}\\
&=\frac{1}{3}\text{I.P. of }\left\{x\cos x+ix\sin x-\frac{2}{3}i\cos+\frac{2}{3}\sin x\right\}\\
&=\frac{1}{3}\text{I.P. of }\left\{x\cos x+\frac{2}{3}\sin x+i\left(x\sin x-\frac{2}{3}\cos x\right)\right\}\\
&=\frac{1}{3}\left[x\sin x-\frac{2}{3}\cos x\right].
\end{aligned}$$

Hence the complete solution of the given differential equation is

$$y=c_1\cos 2x+c_2\sin 2x+\frac{1}{3}\left[x\sin x-\frac{2}{3}\cos x\right].$$

EXAMPLE 1.190

Solve: $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=xe^{3x}+\sin\ 2x.$

Solution. The auxiliary equation for the given differential equation is $m^2-3m+2=0$, which yields $m=1,\ 2$. Hence

$$\text{C.F.}=c_1e^x+c_2e^{2x}.$$

Further,

$$\begin{aligned}
\text{P.I}&=\frac{1}{f(D)}F(x)=\frac{1}{D^2-3D+2}[xe^{3x}+\sin 2x]\\
&=\frac{1}{D^2-3D+2}(xe^{3x})+\frac{1}{D^2-3D+2}(\sin 2x)\\
&=e^{3x1}\frac{1}{(D+3)^2-3(D+3)+2}x\\
&\quad+\frac{1}{-4-3D+2}(\sin 2x)\\
&=e^{3x}\frac{1}{D^2+3D+2}x-\frac{2-3D}{(2+3D)(2-3D)}(\sin 2x)\\
&=\frac{1}{2}e^{3x}\left(1+\frac{D^2+3D}{2}\right)^{-1}x-\frac{2-3D}{4-9D^2}(\sin 2x)\\
&=\frac{1}{2}e^{3x}\left(1-\frac{3D}{2}-\frac{D^2}{2}+\cdots\right)x-\frac{2-3D}{4+36}(\sin 2x)\\
&=\frac{1}{2}e^{3x}\left(x-\frac{3}{2}\right)-\frac{1}{40}[2\sin 2x-3D(\sin 2x)]\\
&=\frac{1}{2}xe^{3x}-\frac{3}{4}e^{3x}-\frac{1}{20}\sin 2x+\frac{6}{40}\cos 2x\\
&=\frac{1}{4}e^{3x}(2x-3)+\frac{1}{20}(3\cos 2x-\sin 2x).
\end{aligned}$$

Hence the complete solution of the given differential equation is

$$y=c_1e^x+c_2e^{2x}+\frac{1}{4}e^{3x}(2x-3)+\frac{1}{20}(3\cos 2x-\sin 2x).$$

EXAMPLE 1.191

Solve:
$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 13x = 0$ with $x(0) = 0$, $\frac{dx(0)}{dt} = 2$.

Solution. The auxiliary equation for the given equation is

$$m^2 - 4m + 13 = 0,$$

which yield $m = 2 \pm 3i$. Therefore the general solution is

$$x = e^{2t}[c_1 \cos 3t + c_2 \sin 3t].$$

When $t = 0$ we have $x = 0$. Therefore $c_1 = 0$. Further,

$$\frac{dx}{dt} = e^{2t}[-3c_1 \sin 3t + 3c_2 \cos 3t] + 2e^{2t}[c_1 \cos 3t + c_2 \sin 3t]$$

When $t = 0$, $\frac{dx}{dt} = 2$ and so we have

$$2 = 3c_2 + 2c_1 \Rightarrow c_2 = \frac{2}{3}.$$

Hence

$$x = \frac{2}{3}e^{2t} \sin 3t.$$

EXAMPLE 1.192

Solve $(x^2D^2 + 4xD + 2)y = x \log x$.

Solution. The given Cauchy–Euler equation is

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x \log x.$$

Proceeding as in Example 1.115, we put $x = e^t$ and reduce the equation to the form

$$[D(D-1) + 4D + 2]y = t\,e^t.$$

The C.F for this equation is

$$\text{C.F} = c_1e^{-t} + c_2e^{-2t}.$$

The particular integral is

$$\begin{aligned}
\text{P.I} &= \frac{1}{D^2 + 3D + 2}t\,e^t \\
&= e^t\frac{1}{(D+1)^2 + 3(D+1) + 2}t \\
&= e^t\frac{1}{D^2 + 5D + 6}t \\
&= e^t\frac{1}{6\left(1 + \frac{5}{6}D + \frac{1}{6}D^2\right)}t \\
&= e^t\frac{1}{6}\left(1 + \frac{5}{6}D + \frac{1}{6}D^2\right)^{-1}t \\
&= \frac{1}{6}e^t\left[1 - \frac{5}{6}D - \frac{1}{6}D^2 + \cdots\right]t \\
&= \frac{1}{6}e^t\left[t - \frac{5}{6}\right] = \frac{1}{6}t\,e^t - \frac{5}{36}e^t.
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
y &= c_1e^{-t} + c_2e^{-2t} + \frac{1}{6}t\,e^t - \frac{5}{36}e^t \\
&= \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{6}x\log x - \frac{5}{6}x.
\end{aligned}$$

EXAMPLE 1.193

Convert $(x^3D^2 + 3x^2D + 5x)y = 2$ into a differential equation with constant co-efficients.

Solution. Given equation can be written as

$$(x^2D^2 + 3xD + 5)y = \frac{2}{x},$$

which is Cauchy's homogeneous linear difference equation, Let $x = e^t$. Then, we have $x\frac{dy}{dx} = Dy$ and $x^2\frac{d^2y}{dx^2} = D(D-1)y$, where $D = \frac{d}{dt}$. Hence given equation reduces to

$$[D(D-1) + 3D + 5]y = 2.e^{-t} \quad \text{or}$$
$$(D^2 + 2D + 5)y = 2e^{-t}.$$

EXAMPLE 1.194

Solve

$$(xD^2 + D)y \equiv 0.$$

Solution. The given differential equation with variable coefficients is

$$x\,D^2y + Dy = 0.$$

Multiplying throughout by x, we get

$$x^2\ D^2y + x\ Dy = 0,$$

which is Cauchy–Euler equation. Putting $x = e^t$ so that $t = \log x$, we have

$$x\frac{dy}{dx} = Dy \quad \text{and} \quad x^2\frac{d^2y}{dx^2} = D(D-1)y.$$

Consequently, the given equation transforms to $[D(D-1)+D]\ \ y = 0$ or $D^2y = 0$. Integrating twice, we get

$$y = at + b = a\log x + b.$$

EXAMPLE 1.195

Solve $(x^2D^2 - 3xD + 4)y = x^2$ given that $y(1) = 1$ and $y'(1) = 0$.

Solution. We are given the differential equation

$$(x^2D^2 - 3x\ D + 4)\ y = x^2,$$

which is a Cauchy–Euler homogeneous equation. Putting $x = e^t$ so that $t = \log x$, we have

$$x\frac{dy}{dx} = Dy \text{ and } x^2\frac{d^2y}{dx^2} = [D(D-1)]y.$$

Therefore the given equation transforms to

$$[D(D-1) - 3D + 4]y = e^{2t}$$

or

$$\left[D^2 - 4D + 4\right]y\ = e^{2t} \qquad (167)$$

The auxiliary equation for the equation (167) is

$$m^2 - 4m + 4 = 0, \quad \text{which yields} \quad m = 2, 2.$$

Therefore

$$\text{C.F.} = (c_1 + c_2t)e^{2t}.$$

Further

$$\begin{aligned}\text{P. I} &= \frac{1}{D^2-4D+4}e^{2t}\\ &= t\frac{1}{2D-4}e^{2t} = t^2.\ \frac{1}{2}e^{2t}\\ &= \frac{1}{2}t^2\ e^{2t}.\end{aligned}$$

Hence the required solution is

$$\begin{aligned}y &= \text{C. F} + \text{P. I}\\ &= (c_1 + c_2\log\ x)\ x^2 + \frac{x^2}{2}\ (\log\ x)^2.\end{aligned}$$

EXAMPLE 1.196

Solve the differential equations:

$$(x^2D^2 - xD - 3)\ y = x^2\ (\log\ x)^2$$

Solution. The given equation

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2(\log x)^2$$

is a Cauchy–Euler homogeneous linear equation. So put $x = e^t$ so that $t = \log x$. Then

$$x\frac{dy}{dx} = Dy \quad \text{and} \quad x^2\frac{d^2y}{dx^2} = D(D-1)y,$$

and the given equation transforms to

$$[D(D-1) - D - 3]y = t^2e^{2t}$$

or

$$(D^2 - 2D - 3)y = t^2e^{2t}.$$

The roots of the auxiliary equation $m^2 - 2m - 3 = 0$ are $m = 3$ and $m = -1$. Therefore

$$\text{C.F} = c_1e^{3t} + c_2e^{-t}.$$

Now

$$\begin{aligned}\text{P.I} &= \frac{1}{D^2-2D-3}(t^2e^{2t})\\ &= e^{2t}\frac{1}{[(D+2)^2 - 2(D+2) - 3]}t^2\\ &= e^{2t}\frac{1}{D^2+2D-3}t^2\\ &= -\frac{e^{2t}}{3}\frac{1}{\left(1-\frac{2}{3}D-\frac{D^2}{3}\right)}t^2\\ &= -\frac{1}{3}e^{2t}\left[1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right]^{-1}t^2\\ &= -\frac{1}{3}e^{2t}\left[1+\frac{2D}{3}+\frac{D^2}{3}+\frac{4D^2}{9}+\cdots\right]t^2\\ &= -\frac{1}{3}e^{2t}\left[1+\frac{2D}{3}+\frac{7D^2}{9}+\cdots\right]t^2\\ &= -\frac{1}{3}e^{2t}\left[t^2+\frac{2}{3}D(t^2)+\frac{7}{9}D^2(t^2)\right]\\ &= -\frac{1}{3}e^{2t}\left[t^2+\frac{4}{3}t+\frac{14}{9}\right].\end{aligned}$$

Therefore the complete solution is

$$y = \text{C.F} + \text{P.I}$$
$$= c_1e^{3t} + c_2e^{-t} - \frac{1}{3}e^{2t}\left[t^2 + \frac{4}{3}t + \frac{14}{9}\right]$$
$$= c_1x^3 + \frac{c_2}{x} - \frac{x^2}{3}\left[(\log x)^2 + \frac{4}{3}\log x + \frac{14}{9}\right].$$

EXAMPLE 1.197

Solve the differential equation

$$\frac{x^2d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x.$$

Solution. The given Cauchy–Euler homogeneous linear equation is

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x.$$

Substitute $x = e^t$ so that $t = \log x$. Then

$$x\frac{dy}{dx} = Dy \quad \text{and} \quad \frac{d^2y}{dx^2} = D(D-1)y.$$

Therefore the given equation reduces to

$$D(D-1)y - 2Dy - 4y = e^{2t} + 2t$$

or

$$(D^2 - 3D - 4)y = e^{2t} + 2t.$$

The characteristic equation for the equation is

$$m^2 - 3m - 4 = 0, \quad \text{which yields} \quad m = 4, \ -1.$$

Thus

$$\text{C.F} = c_1e^{4t} + c_2e^{-t}.$$

The particular integral is given by

$$\text{P.I} = \frac{1}{D^2 - 3D + 4}e^{2t} + \frac{2\,t}{D^2 - 3D + 4}e^{-t}.$$
$$= \frac{e^{2t}}{4-6-4} - \frac{2t}{4\left(1 - \frac{D-3D}{4}\right)}$$
$$= -\frac{e^{2t}}{6} - \frac{1}{2}\left(1 - \frac{D^2 - 3D}{4}\right)^{-1}t$$
$$= -\frac{1}{6}e^{2t} - \frac{1}{2}\left(1 + \frac{D^2 - 3D}{4} + \cdots\right)t$$
$$= -\frac{1}{6}e^{2t} - \frac{1}{2}t - \frac{1}{2}\left[\frac{D^2}{4} - \frac{3D}{4}\right]t$$
$$= -\frac{1}{6}e^{2t} - \frac{1}{2}t + \frac{3}{8}.$$

Hence the complete solution of the given differential equation is

$$y = \text{C.F} + \text{P.I} = c_1e^{4t} + c_2e^{-t} - \frac{1}{6}e^{2t} - \frac{1}{2}t + \frac{3}{8}$$
$$= c_1x^4 + \frac{c_2}{x} - \frac{x^2}{6} - \frac{1}{2}\log x + \frac{3}{8}.$$

EXAMPLE 1.198

Solve: $(2x+3)^2\frac{d^2y}{dx^2} + 5(2x+3)\frac{dy}{dx} + y = 4x.$

Solution. We have

$$(2x+3)^2\frac{d^2y}{dx^2} + 5(2x+3)\frac{dy}{dx} + y = 4x,$$

which is Legendre's linear equation. So putting $2x + 3 = e^t$ or $t = \log(2x+3)$, the given equation reduces to

$$\left[4(D^2 - D) + 10D + 1\right]y = 4\frac{e^t - 3}{2}$$

or

$$(4D^2 + 6D + 1)y = 2e^t - 6.$$

The auxiliary equation is

$$4m^2 + 6m + 1 = 0,$$

which yields $m = \frac{-6\pm\sqrt{36-16}}{8} = \frac{-6\pm2\sqrt{5}}{8} = \frac{-3\pm\sqrt{5}}{4}$. Therefore

$$\text{C.F} = c_1e^{\left(\frac{-3+\sqrt{5}}{4}\right)t} + c_2e^{\left(\frac{-3\sqrt{5}}{4}\right)t},$$

Further,

$$\text{P.I} = \frac{1}{4D^2 + 6D + 1}(2e^t - 6) = \frac{2}{11}e^t - 6$$
$$= \frac{2}{11}(2x+3) - 6.$$

Hence the solution is

$$y = \text{C.F} + \text{P.I}$$
$$= c_1(2x+3)^{\frac{-3+\sqrt{5}}{4}} + c_2(2x+3)^{\frac{-3-\sqrt{5}}{4}}$$
$$+ \frac{2}{11}(2x+3) - 6$$

EXAMPLE 1.199

Solve by using the method of variation of parameters $(D^2 + 9)y = \cot 3x$.

Solution. The auxiliary equation is $m^2+9=0$, which yields $m=\pm 3i$. Therefore,

$$\text{C.F} = c_1\cos 3x + c_2\sin 3x.$$

To find P. I, let $F(x)=\cot 3x$, $y_1=\cos 3x$. and $y_2=\sin 3x$.
Then Wronskian

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3.$$

Therefore

$$\begin{aligned}
\text{P.I} &= -y_1\int \frac{y_2F(x)}{W}dx + y_2\int \frac{y_1F(x)}{W}dx \\
&= \frac{-1}{3}\cos 3x\int \sin 3x\cot 3x dx \\
&\quad + \frac{1}{3}\sin 3x\int \cos 3x\cot 3x dx \\
&= \frac{-1}{3}\cos 3x\int \cos 3x dx + \frac{1}{3}\sin 3x\int \frac{\cos^2 3x}{\sin 3x}dx \\
&= \frac{-1}{9}\cos 3x\sin 3x + \frac{1}{3}\sin 3x\left[\int \frac{1-\sin^2 3x}{\sin 3x}dx\right] \\
&= -\frac{1}{9}\cos 3x\sin 3x + \frac{1}{3}\sin 3x \\
&\quad \times\int(\text{cosec}3x - \sin 3x)dx \\
&= -\frac{1}{9}\cos 3x\sin 3x + \frac{1}{3}\sin 3x \\
&\quad \times\left[\frac{1}{3}\log(\text{cosec}3x - \cot 3x) + \frac{\cos 3x}{3}\right] \\
&= \frac{1}{9}\sin 3x[\log(\text{cosec}3x - \cot 3x)]
\end{aligned}$$

EXAMPLE 1.200

Solve $(D^2+a^2)y=\tan ax$ by the method of variation of parameters.

Solution. We have $(D^2+a^2)y=\tan ax$. It's auxiliary equation is $m^2+a^2=0$, which yields $m=\pm\, ai$.
Therefore

$$\text{C. F} = c_1\cos ax + c_2\sin ax.$$

To find particular integral, let

$$y_1=\cos ax \text{ and } y_2=\sin ax.$$

Then

$$y_1'=-a\ \sin\ ax \textit{ and } y_2'=a\ \cos\ ax.$$

Thus the wronskian W is given by

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos\ ax & \sin\ ax \\ -a\ \sin\ ax & a\ \cos\ ax \end{vmatrix} = a.$$

Hence

$$\begin{aligned}
\text{P. I} &= -y_1\int \frac{y_2\,F(x)}{W}dx + y_2\int \frac{y_1\,F(x)}{W}dx \\
&= \frac{-\cos\ ax}{a}\int \sin\ ax\ \tan\ ax\, dx \\
&\quad + \frac{\sin\ ax}{a}\int \cos\ ax\ \tan\ ax\, dx \\
&= \frac{-\cos\ ax}{a}\int \frac{\sin^2\ ax}{\cos\ ax}dx \\
&\quad + \frac{\sin ax}{a}\int \cos\ ax\ \tan\ ax\, dx \\
&= \frac{-\cos\ ax}{a}\int \frac{1-\cos^2 ax}{\cos\ ax}dx \\
&\quad + \frac{\sin\ ax}{a}\int \cos\ ax\ \tan\ ax\, dx \\
&= \frac{-\cos\ ax}{a}\int (\sec\ ax\ - \cos\ ax)dx \\
&\quad + \frac{\sin\ ax}{a}\int \sin\ ax\, dx \\
&= \frac{-\cos\ ax}{a^2}\ [\log\ (\sec\ ax\ + \tan\ ax) - \sin\ ax] \\
&\quad - \frac{1}{a^2}\ \sin\ ax\ \cos\ ax \\
&= \frac{-\cos\ ax}{a^2}\ [\log\ (\sec\ ax\ +\ \tan\ ax)].
\end{aligned}$$

Hence the complete solution of the given differential equation is

$$\begin{aligned}
y &= \text{C.F} + \text{P. I} = c_1\cos ax + c_2\sin ax - \frac{\cos\ ax}{a^2} \\
&\quad \times[\log(\sec ax + \tan ax)]
\end{aligned}$$

EXAMPLE 1.201

Solve the differential equation by the method of variations of parameters:

$$y''+y=\sec^2 x$$

Solution. The given differential equation is

$$y''+y=\sec^2 x.$$

The symbolic form of the equation is

$$(D^2+)y = \sec^2 x.$$

The auxiliary equation is $m^2 + 1 = 0$, which yields $m = \pm i$. Therefore

$$\text{C.F} = c_1 \cos x + c_2 \sin x.$$

To find particular integral, let

$$y_1 = \cos x \quad \text{and} \quad y_2 = \sin x.$$

Then

$$y_1' = -\sin x \quad \text{and} \quad y_2' = \cos x.$$

The wronskian W is given by

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Therefore

$$\begin{aligned}
\text{P.I} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\
&= -\cos x \int \sin x \sec^2 x \, dx \\
&\quad + \sin x \int \cos x \sec^2 x \, dx \\
&= -\cos x \int \sec x \tan x \, dx \\
&\quad + \sin x \int \sec x \cos x \, dx \\
&= -\cos x \sec x + \sin x \log(\sec x + \tan x) \\
&= -1 + \sin x \log(\sec x + \tan x).
\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}
y &= \text{C.F} + \text{P.I} \\
&= c_1 \cos x + c_2 \sin x - 1 + \sin x \log(\sec x \\
&\quad + \tan x).
\end{aligned}$$

EXAMPLE 1.202

Find the general solution of the equation $y'' + 16y = 32 \sec 2x$, using method of variation of parameters.

Solution. The symbolic form of the given differential equation is

$$(D^2 + 16)y = 32 \sec 2x.$$

The roots of the auxiliary equation $m^2 + 16 = 0$ are $m = \pm 4i$. Therefore the complementary function is

$$\text{C.F} = c_1 \cos 4x + c_2 \sin 4x.$$

To find the particular integral, let $F(x) = 32 \sec 2x$, $y_1 = \cos 4x$, $y_2 = \sin 4x$. Then the Wronskian W is given by

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} - \begin{vmatrix} \cos 4x & \sin 4x \\ -4 \sin 4x & 4 \cos 4x \end{vmatrix} = 4.$$

By the method of variation of parameters, we have

$$\begin{aligned}
\text{P.I} &= -y_1 \int \frac{y_2 F(x)}{w} dx + y_2 \int \frac{y_1 F(x)}{w} dx \\
&= -\cos 4x \int \frac{\sin 4x (32 \sec 2x)}{4} dx \\
&\quad + \sin 4x \int \frac{\cos 4x (32 \sec 2x)}{4} dx \\
&= -8 \cos 4x \int \sec 2x (2 \sin 2x \cos 2x) \, dx \\
&\quad + 8 \sin 4x \int (2\cos^2 2x - 1) \sec 2x \, dx \\
&= -16 \cos 4x \int \sin 2x \, dx + 8 \sin 4x \\
&\quad \times \int (\cos 2x - \sec 2x) dx \\
&= -16 \cos 4x \left(\frac{-\cos 2x}{2} \right) + 8 \sin 4x \\
&\quad \times \left[\sin 2x - \frac{1}{2} \log (\sec 2x + \tan 2x) \right] \\
&= 8 \cos 4x \cos 2x + 8 \sin 4x \sin 2x \\
&\quad - 4 \sin 4x \log (\sec 2x + \tan 2x) \\
&= 8 \cos (4x - 2x) \\
&\quad - 4 \sin 4x \log (\sec 2x + \tan 2x) \\
&= 8 \cos 2x - 4 \sin 4x \log (\sec 2x + \tan 2x).
\end{aligned}$$

Hence the required solution is

$$\begin{aligned}
y &= \text{C.F} + \text{P.I} \\
&= c_1 \cos 4x + c_2 \sin 4x + 8 \cos 2x \\
&\quad - 4 \sin 4x \log(\sec 2x + \tan 2x)
\end{aligned}$$

EXAMPLE 1.203

(a) Solve: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = \frac{1}{1-e^x}$ by using the method of variation of parameters.

(b) Solve $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=e^{-x}\log x$ by using the method of variation of parameters.

Solution.

(a) The symbolic form of the given differential equation is

$$(D^2+D-2)y=\frac{1}{1-e^x}.$$

The corresponding auxiliary equation is

$$m^2+m-2=0,$$

which yields $m=1\ ,\ -2$. Hence

$$\text{C.F}=c_1e^x+c_2e^{-2x}.$$

Let

$$y_1=e^x\ ,\ y_2=e^{-2x}\quad\text{so that}\quad y_1'=e^x\quad\text{and}\quad y_2'=-2e^{-2x}.$$

Let $F(x)=\frac{1}{1-e^x}$. Then Wronskian W is given by

$$W=\begin{vmatrix} y_1 & y_2\\ y'_1 & y'_2\end{vmatrix}=\begin{vmatrix} e^x & e^{-2x}\\ e^x & -2e^{-2x}\end{vmatrix}$$
$$=-3e^xe^{-2x}.$$

Therefore,

$$\begin{aligned}\text{P.I}&=-y_1\int\frac{y_2F(x)}{W}dx+y_2\int\frac{y_1F(x)}{W}dx\\ &=-e^x\int\frac{e^{-2x}}{-3e^xe^{-2x}(1-e^x)}dx\\ &\quad+e^{-2x}\int\frac{e^x}{-3e^xe^{-2x}(1-e^x)}dx\\ &=\frac{1}{3}e^x\int\frac{e^{-x}}{1-e^x}dx-\frac{1}{3}e^{-2x}\int\frac{e^{2x}}{1-e^x}dx.\end{aligned}$$

But,

$$\begin{aligned}\int\frac{e^{-x}}{1-e^x}dx&=\int\frac{1}{z^2(1-z)}dz\ ,\ \ e^x=z\\ &=\int\left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{1-z}\right]dz\\ &=\log z-\frac{1}{z}-\log(1-z)\\ &=\log\frac{z}{1-z}-\frac{1}{z}=\log\frac{e^x}{1-e^x}-e^{-x}\end{aligned}$$

and

$$\begin{aligned}\int\frac{e^{2x}}{1-e^x}dx&=\frac{1}{2}\int\frac{dz}{1-z}=-\frac{1}{2}\log(1-z)\\ &=-\frac{1}{2}\log(1-e^x).\end{aligned}$$

Therefore,

$$\text{P.I.}=\frac{1}{3}e^x\left[\log\frac{e^x}{1-e^x}-e^{-x}\right]+\frac{1}{6}e^{-2x}\log(1-e^x).$$

Hence the complete solution is

$$\begin{aligned}y&=c_1e^x+c_2e^{-2x}+\frac{1}{3}e^x\left[\log\frac{e^x}{1-e^x}-e^{-x}\right]\\ &\quad+\frac{1}{6}e^{-2x}\log(1-e^x).\end{aligned}$$

(b) The auxiliary equation for the given differential equation is

$$m^2+2m+1=0,$$

Which yields $m=-1\ ,\ -1$. Thus

$$\text{C.F}=(c_1+c_2x)e^{-x}.$$

To find P. I, let

$$y_1=e^{-x}\quad\text{and}\quad y_2=x\,e^{-x}.$$

Then,

$$W=\begin{vmatrix} e^{-x} & x\,e^{-x}\\ -e^{-x} & -x\,e^{-x}+e^{-x}\end{vmatrix}=e^{-2x}.$$

Therefore,

$$\begin{aligned}\text{P.I}&=-y_1\int\frac{y_2F(x)}{W}dx+y_2\int\frac{y_1F(x)}{W}dx\\ &=-e^{-x}\int\frac{xe^{-x}.e^{-x}\log x}{e^{-2x}}dx\\ &\quad+xe^{-x}\int\frac{e^{-x}.e^{-x}\log x}{e^{-2x}}dx\\ &=-e^{-x}\int x\log x\,dx+x\,e^{-x}\int\log x\,dx\\ &=-e^{-x}\left[\frac{x^2}{2}\log x-\frac{x^2}{4}\right]+x\,e^{-x}[x\log x-x].\end{aligned}$$

EXAMPLE 1.204

Solve: $(D^2-3DD'+2D'^2)z=\sin\ x\ \cos\ y.$

Solution. Replacing D by m and D' by 1, the auxiliary equation is

$$m^2-3m+2=0,$$

Which yields $m = 1, 2$. Therefore

$$C.F = \phi_1(y+x) + \phi_2\ (y+2x).$$

Further,

$$\begin{aligned}
P.I &= \frac{1}{D^2 - 3DD' + 2D'^2} \sin x \cos y \\
&= \frac{1}{D^2 - 3DD' + 2D'^2} \\
&\quad \times \left\{ \frac{1}{2} [\sin (x+y) + \sin(x-y)] \right\} \\
&= \frac{1}{2} \left[\frac{1}{D^2 - 3DD' + 2D'^2} \sin(x+y) \right. \\
&\quad \left. + \frac{1}{D^2 - 3DD' + 2D'^2} \sin (x-y) \right] \\
&= \frac{1}{2} \left[\frac{1}{(D-2D')\ (D-D')} \sin (x+y) \right] \\
&\quad + \frac{1}{2} \left[\frac{1}{(D-2D')\ (D-D')} \sin (x-y) \right] \\
&= \frac{1}{2} \left[\frac{1}{(1-2)(D-D')} \right. \\
&\quad \left. \times \int \cos u\, du, \text{ where } u = x+y \right] \\
&\quad + \frac{1}{2} \left[\frac{1}{(D-2D')(1+1)} \right. \\
&\quad \left. \times \int \sin u\, du, \text{ where } u = x-y \right] \\
&= -\frac{1}{2(D-D')} \sin (x+y) \\
&\quad - \frac{1}{4(D-2D')} \cos (x-y) \\
&= -\frac{1}{2} \frac{x^1}{1^2.1!} \sin (x+y) - \frac{1}{4} \frac{x^1}{1.1!} \cos (x-y) \\
&= -\frac{1}{2} x \sin (x+y) - \frac{1}{4} x \cos(x-y).
\end{aligned}$$

EXAMPLE 1.205

A simple pendulum of length l is oscillating through a small angle θ in a medium in which the resistance is proportional to velocity. Find the differential equation of its motion. Discuss the motion and find the period of oscillation.

Solution. Please refer to Figure 1.9 of Article 1.22. We have $\underline{|AOP} = \theta$, $AP = s = l\theta$, $OA = l$. The resistance is proportional to the velocity and so it is $\lambda \frac{ds}{dt}$, where λ is a constant. Therefore the equation of motion along the tangent is

$$m \frac{d^2 s}{dt^2} = -mg \quad \sin\theta - \lambda \frac{ds}{dt}.$$

or

$$\frac{d^2(l\theta)}{dt^2} + \frac{\lambda}{m} \frac{d(l\theta)}{dt} + g \sin \theta = 0$$

or

$$\frac{d^2\theta}{dt^2} + \frac{\lambda}{m} \frac{d\theta}{dt} + \frac{g}{l} \sin\theta = 0$$

Thus, to the first approximation of $\sin\theta$, we have

$$\frac{d^2\theta}{dt^2} + 2k \frac{d\theta}{dt} + \frac{g\theta}{l} = 0, \tag{168}$$

where $\frac{\lambda}{m} = 2k$. The differential equation (168) describes the motion of the bob. Its auxiliary equation is

$$m^2 - 2km + \frac{g}{l} = 0,$$

whose roots are $m = -k \pm \sqrt{k^2 - \omega^2}$, where $\omega^2 = \frac{g}{l}$. Since the oscillatory motion of the bob is possible only if $k < \omega$, the roots are $-k \pm i\sqrt{\omega^2 - k^2}$. Therefore the solution of differential equation (168) is

$$\begin{aligned}
\theta = e^{-kt} \Big[& C_1 \cos \left(\sqrt{\omega^2 - k^2} \right) t \\
& + C_2 \sin \left(\sqrt{\omega^2 - k^2} \right) t \Big],
\end{aligned}$$

which gives a vibratory motion of period $\frac{2\pi}{\sqrt{\omega^2 - k^2}}$.

EXAMPLE 1.206

(a) Solve the simultaneous equations $\frac{dx}{dt} - y = t$ and $\frac{dy}{dt} + x = t^2$.

(b) Solve the following simultaneous equations $\frac{dx}{dt} = 3x + 2y$ and $\frac{dy}{dt} = 5x + 3y$.

(c) Solve the following simultaneous equations $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$.

Solution. (a) We have

$$Dx - y - t = 0, \tag{169}$$

$$x + Dy - t^2 = 0. \tag{170}$$

Multiplying (169) by D and adding to (170), we get

$$D^2 x - Dt - t^2 = 0$$

or

$$D^2x - 1 - t^2 = 0$$

or

$$\frac{D^2x}{dt^2} = 1 + t^2.$$

Therefore

$$\frac{dx}{dt} = t + \frac{t^3}{3} + c_1$$

and

$$x = \frac{t^2}{2} + \frac{t^4}{12} + c_1x + c_2.$$

Putting the value of $\frac{dx}{dt}$ in (169), we get

$$t = \frac{t^3}{3} + c_1 - y - t = 0$$

or

$$y = \frac{t^3}{3} + c_1.$$

Thus the required solution is

$$x = \frac{t^2}{2} + \frac{t^4}{12} + c_1x + c_2,$$

$$y = \frac{t^3}{3} + c_1.$$

(b) The given equations are

$$\frac{dx}{dt} = 3x + 2y \quad \text{and} \quad \frac{dy}{dt} = 5x + 3y.$$

In symbolic form, we have

$$(D-3)x - 2y = 0, \qquad (171)$$

$$-5x + (D-3)y = 0$$

or

$$(D-3)^2x - 2(D-3)y = 0 \qquad (172)$$

$$-10x + 2(D-3)y = 0 \qquad (173)$$

Adding (172) and (173), we get

$$[(D-3)^2x - 10]x = 0$$

or

$$(D^2 - 6D - 1)x = 0 \qquad (174)$$

The auxiliary equation of (174) is

$$m^2 - 6m - 1 = 0,$$

which gives $m = 3 \pm \sqrt{10}$. Therefore

$$x = e^{3t}[c_1 \cosh\sqrt{10}t + c_2 \sin\sqrt{10}t] \qquad (175)$$

From equation (171), we have (using (175))

$$\begin{aligned} 2y &= \frac{dx}{dt} - 3x \\ &= e^{3t}\sqrt{10}[c_1 \sinh\sqrt{10}t + c_2\cosh\sqrt{10}t] \\ &\quad + 3\,e^{3t}\sqrt{10}[c_1\cosh\sqrt{10}t + c_2\sinh\sqrt{10}t] \\ &\quad - 3\,e^{3t}\sqrt{10}[c_1\cosh\sqrt{10}t + c_2\sinh\sqrt{10}t] \\ &= e^{3t}[c_1\sinh\sqrt{10}t + c_2\cosh\sqrt{10}t]. \end{aligned}$$

Hence

$$y = \frac{1}{2}[e^{3t}\sqrt{10}[c_1\sinh\sqrt{10}t + c_2\cosh\sqrt{10}t].$$

(c) The symbolic forms of the equations are

$$(D+5)x - 2y = t \qquad (176)$$

$$2x + (D+1)y = 0 \qquad (177)$$

Operating (176) by $(D+1)$ and multiplying (177) by 2 and then adding, we get

$$(D^2 + 6D + 9)x = 1 + t.$$

The auxiliary equation is

$$m^2 + 6D + 9 = 0$$

and so $m = -3, -3$ and hence

$$\text{C.F} = (c_1 + c_2t)e^{-3t}$$

Further,

$$\begin{aligned} \text{P.I} &= \frac{1}{(D+3)^2}(1+t) = \frac{1}{9}\left(1 + \frac{D}{3}\right)^{-2}(1+t) \\ &= \frac{1}{9}\left[1 - \frac{2D}{3} + \cdots\right](1+t) = \frac{1}{9}\left(t + \frac{1}{3}\right). \end{aligned}$$

Hence,

$$x = \text{C.F} + \text{P.I} = (c_1 + c_2t)e^{-3t} + \frac{1}{9}\left(t + \frac{1}{3}\right) \qquad (178)$$

Therefore,

$$\frac{dx}{dt} = -3(c_1 + c_2t)e^{-3t} + c_2e^{-3t} + \frac{1}{9}.$$

Putting in the first (given) equation, we get

$$y = (c_1 + c_2t)e^{-3t} + \frac{1}{2}c_2e^{-3t} - \frac{2}{9}t + \frac{4}{27} \qquad (179)$$

Hence the solution is given by (178) and (179).

EXAMPLE 1.207

Solve the following simultaneous equations

$$\frac{dx}{dt} + 2x + 3y = 0, \; 3x + \frac{dy}{dt} + 2y = 2e^t$$

Solution. In symbolic representation, we have

$$(D+2)x+3y=0 \tag{180}$$

and

$$3x+(D+2)y=2e^t. \tag{181}$$

Multiplying (180) by $(D+2)$ and (181) by 3 and subtracting, we get

$$(D+2)^2x-9x=-6e^t$$

or

$$(D^2+4D-5)x=-6e^t \tag{182}$$

The auxiliary equation for (182) is $m^2+4m-5=0$, whose roots are 1 and -5. Therefore

$$\text{C.F} = c_1e^t+c_2e^{-5t}.$$

Further,

$$\begin{aligned}\text{P.I} &= \frac{1}{D^2+4D-5}(-6e^t)\\ &= t\frac{1}{2D+4}(-6e^t)=\frac{t}{6}(-6e^t)\\ &= -te^t.\end{aligned}$$

Hence the complete solution of (182) is

$$x=c_1e^t+c_2e^{-5t}-te^t \tag{183}$$

Differentiating (183) with respect to t, we have

$$\frac{dx}{dt}=c_1e^t-5c_2\,e^{-5t}-[e^t+te^t].$$

Putting the value of $\frac{dx}{dt}$ in the equation $\frac{dx}{dt}+2x+3y=0$, we get

$$\begin{aligned}3y &= -\frac{dx}{dt}-2x=-\left[c_1e^t-5c_2e^{-5t}-e^t-te^t\right]\\ &\quad -2[c_1e^t+c_2e^{-5t}-te^t]\\ &= -3c_1e^t+4c_2e^{-5t}+e^t+3te^t\end{aligned}$$

or

$$y=\frac{4}{3}c_2e^{-5t}-c_1e^t+\frac{1}{3}e^t+te^t \tag{184}$$

Expressions (183) and (184) represent the required solution.

EXAMPLE.1.208

(a) Show that $\int_{-1}^{1}(1-x^2)[P'_n(x)]^2dx=\frac{2n(n+1)}{2n+1}$.

(b) Show that $\int_{-1}^{1}f(x)P_n(x)dx=\frac{(-1)^n}{2^nn!}\int_{-1}^{1}(x^2-1)^n f^{(n)}(x)dx$.

(c) Show that $J_3(x)+3J_0'(x)+4J_0'''(x)=0$.

(d) $P_{2n}(0)=\frac{(-1)^2(2n)!}{2^{2n}(n!)^2}$.

Solution.

(a) We have (see solved Exercise 45, chapter 1)

$$\frac{d}{dx}[(1-x^2)P_n'(x)]P_n(x)+n(n+1)P_n(x)=0$$

or

$$\begin{aligned}&n(n+1)P_n(x)\\ &\quad=-\frac{d}{dx}[(1-x^2)P_n'(x)]P_n(x)\end{aligned} \tag{185}$$

Multiplying both sides of (185) by $P_n(x)$ and integrating between the limits –1 and 1, we get

$$\begin{aligned}n(n+1)\int_{-1}^{1}P_n^2(x)dx &= -\int_{-1}^{1}[(1-x^2)P_n'(x)]'P_n(x)dx\\ &= \{-P_n(x)[(1-x^2)P_n'(x)]\}_{-1}^{1}\\ &\quad+\int_{-1}^{1}P_n'(x)(1-x^2)P_n'(x)dx\\ &= \int_{-1}^{1}(1-x^2)[P_n'(x)]^2dx\end{aligned} \tag{186}$$

But

$$\int_{-1}^{1}P_n^2(x)dx=\frac{2}{2n+1}.$$

Therefore (186) implies

$$\int_{-1}^{1}(1-x^2)(P_n'(x)]^2dx=\frac{2n(n+1)}{2x+1}.$$

(b) By Rodrigue's formula, we have

$$P_n(x)=\frac{1}{2^nn!}\frac{d^n}{dx^n}(x^2-1)^n.$$

Therefore integrating by parts , we have

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{1}{2^n n!}\int_{-1}^{1} f(x)\frac{d^n}{dx^n}(x^2-1)^n dx$$

$$= \frac{1}{2^n n!}\left[\left\{f(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right\}_{-1}^{1} - \int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx\right]$$

$$= \frac{(-1)}{2^n n!}\int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)dx$$

$$= \frac{(-1)^2}{2^n n!}\int_{-1}^{1} f(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$$

(using integration by parts second time)

....

....

....

$$= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(x)(x^2-1)^n dx$$

(using integration by parts nth time)

(c) We know that

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

(Please see Example 1.151 (ii))

Putting $n=0$, we have

$$J_0'(x) = -J_1(x).$$

Therefore

$$J_0''(x) = -J_1'(x) = -\frac{1}{2}[J_0 - J_2]$$

(Please see Example 1.151 (iv))

and

$$\begin{aligned} J_0'''(x) &= -\frac{1}{2}[J_0'(x) - J_2'(x)] \\ &= -\frac{1}{2}J_0'(x) + \frac{1}{2}\left[\frac{1}{2}\{J_1(x) - J_0(x)\}\right] \\ &= -\frac{1}{2}J_0'(x) + \frac{1}{4}J_1(x) - \frac{1}{4}J_3(x) \\ &= -\frac{1}{2}J_0'(x) + \frac{1}{4}[-J_0'(x)] - \frac{1}{4}J_3(x) \\ &= -\frac{3}{4}J_0'(x) - \frac{1}{4}J_3(x). \end{aligned}$$

Hence

$$4J_0'''(x) + 3J_0'(x) + J_3(x) = 0.$$

(d) Since $(1-2xt+t^2)^{-\frac{1}{2}}$ is generating function of the Legendre's polynomial, we have

$$(1-2x+t^2)^{-\frac{1}{2}} = \Sigma t^n P_n(x)$$

or

$$\sum t^n P_n(0) = 1 - \frac{1}{2}t^2 + \frac{1.3}{2.4}t^4 - \cdots (-1)^r \frac{1.3.5\cdots(2r-1)}{2.4.6\cdots(2r)}t^{2r} + \cdots$$

Equating coefficients of t^{2n} on both sides, we get

$$\begin{aligned} P_{2n}(0) &= (-1)^n \frac{1.3.5\cdots(2n-1)}{2.4.6\cdots(2n)} \\ &= (-1)^n \frac{1.2.3.4\cdots(2n-1)2^n}{[2.4.6\cdots(2n)]^2} \\ &= (-1)^n \frac{(2n)!}{[2^n(1.2\cdots n)]^2} = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}. \end{aligned}$$

EXAMPLE 1.209

Solve the simultaneous equations

$$\frac{dx}{dt} + y = \sin t,\ \frac{dy}{dt} + x = \cos t,$$

given that $x=2$, $y=0$ when $t=0$

Solution. We have

$$\frac{dx}{dy} + y = \sin t \quad \text{and} \quad \frac{dy}{dt} + x = \cos t,$$

under the condition that $x = 2, y = 0$ when $t = 0$. In symbolic form, we have

$$Dx + y = \sin t \tag{187}$$

$$Dy + x = \cos t. \tag{188}$$

Multiplying (187) by D and subtracting (188) from the resultant equation, we have

$$D^2 x - x = \cos t - \cos t = 0.$$

Its Auxiliary equation is

$$m^2 - 1 = 0, \quad \text{which yields} \quad m = \pm 1.$$

Hence

$$x = c_1 e^t + c_2 e^{-t}. \tag{189}$$

When $t = 0$, we have $x = 2$. Therefore

$$c_1 + c_2 = 2 \tag{190}$$

Now putting the value of x from (189) in (187), we get

$$D(c_1 e^t + c_2 e^{-t}) + y = \sin t$$

or

$$c_1 e^t - c_2 e^{-t} + y = \sin t.$$

or

$$y = -c_1 e^t + c_2 e^{-t} + \sin t \tag{191}$$

When $t = 0$, $y = 0$. Therefore

$$0 = -c_1 + c_2 \tag{192}$$

Solving (190) and (192), we have $c_1 = 1$, $c_2 = 1$. Hence the required solution is

$$x = e^t + e^{-t}, \quad y = -e^t + e^{-t} + \sin t.$$

EXAMPLE 1.210

Solve the following differential equation in series:

$$2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x - 5)y = 0.$$

Solution. Comparing the given equation with

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0,$$

we get

$$P(x) = -\frac{1}{2x} \quad , \quad Q(x) = \frac{x - 5}{2x^2}.$$

At $x = 0$, both $P(x)$ and $Q(x)$ are not analytic. Thus $x = 0$ is a singular point. Further

$$xP(x) = -\frac{1}{2} \quad \text{and} \quad x^2 Q(x) = \frac{x - 5}{2},$$

which are analytic at $x = 0$. Hence $x = 0$ is a regular singular point. So, let

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}, \quad a_n \neq 0.$$

Differentiating twice, we get

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (n + m) a_n x^{n+m-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (n + m)(n + m - 1) a_n x^{n+m-2}.$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we have

$$\sum_{n=0}^{\infty} 2(n+m)(n+m-1)a_n x^{n+m} - \sum_{n=0}^{\infty} (n+m)a_n x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m+1} - \sum_{n=0}^{\infty} 5a_n x^{n+m} = 0$$

or

$$\sum_{n=0}^{\infty} [2(n + m)(n + m - 1) - (n + m) - 5] a_n x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m+1} = 0$$

or

$$\sum_{n=0}^{\infty} [2n^2 + 2m^2 + 4mn - 3n - 3m - 5] a_n x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m+1} = 0$$

or

$$\sum_{n=0}^{\infty} [2n^2 + 2m^2 + 4mn - 3n - 3m - 5] a_n x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m+1} = 0$$

or

$$\sum_{n=0}^{\infty} [2n^2 + 2m^2 + 4mn - 3n - 3m - 5] a_n x^{n+m} + \sum_{n=1}^{\infty} a_{n-1} x^{n+m} = 0$$

or

$$(2m^2 - 3m - 5)a_0 x^m + \sum_{n=1}^{\infty} [(2n^2 + 2m^2 + 4mn - 3n - 3m - 5)a_n + a_{n-1}]x^{m+n} = 0.$$

Therefore the indicial equation is

$$2m^2 - 3m - 5 = 0, \quad \text{which yields} \quad m = \frac{5}{2}\ , \ -1.$$

Equating to zero the other coefficients, we have

$$(2n^2 + 2m^2 + 4mn - 3n - 3m - 5)a_n = -a_{n-1} \quad \text{for} \quad n \geq 1. \tag{193}$$

For $m = \frac{5}{2}$, we have from (193),

$$(2n^2 + 7n)a_n = -a_{n-1}\ \ ,\ \ a \geq 1$$

or

$$a_n = -\frac{a_{n-1}}{2n^2 + 7n}\ \ ,\ \ n \geq 1.$$

Thus, putting $n = 1\ ,\ 2\ ,\ 3, \cdots$ we get

$$a_1 = -\frac{a_0}{9}$$
$$a_2 = \frac{a_0}{198}$$
$$a_3 = -\frac{a_0}{7722}$$

and so on. Therefore

$$y_1 = a_0 x^{\frac{5}{2}} \left[1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \cdots\right].$$

For $m = -1$, the relation (1) yields

$$(2n^2 - 7n)a_n = -a_{n-1}\ \ ,\ \ n \geq 1$$

or

$$a_n = \frac{-a_{n-1}}{2n^2 - 7n}\ \ ,\ \ n \geq 1.$$

Therefore,

$$a_1 = -\frac{a_0}{-5} = \frac{a_0}{5}$$
$$a_2 = -\frac{a_1}{-6} = \frac{a_0}{30}$$
$$a_3 = -\frac{a_2}{-3} = \frac{a_0}{90}\ ,$$

and so on. Hence

$$y_2 = a_0 x^{-1}\left[1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \cdots\right]$$

Therefore the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= A x^{\frac{5}{2}}\left[1 + \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \cdots\right] \\ &\quad + B x^{-1}\left[1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \cdots\right]. \end{aligned}$$

EXAMPLE 1.211

For Bessel's polynomial $J_n(x)$ of degree n, show that

(a) $J_2'(x) = \left(1 - \frac{4}{x^2}\right)J_1(x) + \frac{2}{x}J_0(x)$

(b) $J_0^2 + 2\left(J_1^2 + J_2^2 + J_3^2 + \cdots\right) = 1.$

Solution.

(a) We know that

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)].$$

Therefore

$$J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]. \tag{194}$$

Also

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x).$$

Therefore

$$\begin{aligned} J_3(x) &= \frac{4}{x}J_2(x) - J_1(x) \\ &= \frac{4}{x}\left[\frac{2}{x}J_1(x) - J_0(x)\right] - J_1(x) \\ &= \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x). \end{aligned}$$

Putting this value in (194), we get

$$J_2'(x) = \left(1 - \frac{4}{x^2}\right)J_1(x) + \frac{2}{x}J_0(x).$$

(b) From example 1.151, the Jacobi's series are

$$J_0 + 2J_2\cos 2\phi + 2J_4\cos 4\phi + \cdots = \cos(x\sin\phi)$$

and

$$2J_1\sin\phi + 2J_3\sin 3\phi + 2J_5\sin 5\phi + \cdots = \sin(x\sin\phi).$$

Squaring these equations and integrating with respect to ϕ in the interval $(0, \pi)$, we get

$$\pi J_0^2 + 2\pi J_2^2 + 2\pi J_4^2 + \cdots = \int_0^{\pi} \cos^2(x \sin\phi)d\phi \qquad (195)$$

and

$$2\pi J_1^2 + 2\pi J_3^2 + 2\pi J_5^2 + \cdots = \int_0^{\pi} \sin^2(x \sin\phi)d\phi \qquad (196)$$

Adding (195) and (196) we get

$$\pi J_0^2 + 2\pi J_1^2 + 2\pi J_2^2 + \cdots = \int_0^{\pi} d\phi = \pi.$$

Hence

$$J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots = 1.$$

EXERCISES

1. Form differential equation from the following equations
   (a) $y = ae^{2x} + be^{-3x} + ce^x$
   Ans. $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$
   (*b*) $x = A\cos(nt + \alpha)$ **Ans.** $\frac{d^2x}{dt^2} + n^2x = 0$
2. Solve the separable equations
   (*a*) $(x - 4)y^4\,dx - x^3(y^3 - 3)\,dy = 0.$
   Ans. $-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c$
   (b) $\frac{dy}{dx} = e^{2x-3y} + 4x^2e^{-3y}$
   Ans. $3e^x - 2e^{3y} + 8x^3 = c$
   (c) $\frac{dy}{dx} - x\tan(y - x) = 1.$
   Ans. $\log\sin(y - x) = \frac{1}{2}x^2 + c$
   (d) $(x - y)^2\frac{dy}{dx} = a^2.$
   Ans. $a\log\left(\frac{x-y-a}{x-y+a}\right) = 2y + k$
   (e) $\frac{dx}{dt} = x^2 - 2x + 2$
   Ans. $x = 1 + \tan(t + c)$
   (f) $x\frac{dy}{dx} + \cot y = 0$ if $y(\sqrt{2}) = \pi/4.$
   Ans. $x = 2\cos y$
3. Solve the homogeneous equations :
   (a) $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$ **Ans.** $2x\tan^{-1}(cx)$
   (b) $ydx - xdy = \sqrt{x^2 + y^2}\,dx$
   Ans. $y + \sqrt{x^2 + y^2} = c$
   (c) $2xy\frac{dy}{dx} = 3y^2 + x^2$ **Ans.** $x^2 + y^2 = cx^3$
   (d) $y' = \frac{y+x}{x}$ **Ans.** $y = x\log|cx|$
4. Reduce the following equations to homogeneous form and solve them
   (a) $(2x + y - 3)dy = (x + 2y - 3)dx$
   Ans. $(y - x)^3 = k(y + x - 2)$
   (b) $(x + 2y)(dx - dy) = dx + dy$
   Ans. $\log(x + y + \frac{1}{3}) + \frac{3}{2}(y - x) + k$
   (c) $\frac{dy}{dx} + \frac{ax + hy + g}{hx + by + f} = 0.$
   Ans. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
5. Solve the following linear equations :
   (a) $x^2\frac{dy}{dx} = 3x^2 - 2yx + 1$
   Ans. $y = x + \frac{1}{x} + \frac{c}{x^2}.$
   (b) $(x+1)\frac{dy}{dx} - y = e^{3x}(x + 1)^2.$
   Ans. $y = (x+1)(\frac{1}{3}e^{3x} + c)$
   (c) $\frac{dy}{dx} = -\frac{x + y\cos x}{1 + \sin x}$
   Ans. $y(1 + \sin x) = c - \frac{x^2}{2}$
   (d) $y\log y\,dx + (x - \log y)\,dy = 0$
   Ans. $x = \frac{1}{2}\log y + \frac{c}{\log y}$
   (e) $x\log x\frac{dy}{dx} + y = \log x^2$. **Ans.** $y = \log x + \frac{c}{\log x}$
6. Solve the following equations
   (a) $(x^3y^2 + xy)\,dx = dy.$
   Ans. $\frac{1}{y} = x^2 - 2 + ce^{-x^2/2}$
   (b) $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)e^x\sec y$
   Ans. $\sin y = (1 + x)(e^x + c)$
   (c) $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x}(\log z)^2$
   Ans. $\frac{1}{\log z} = 1 + cx$
   (d) $\frac{dy}{dx} + y\tan x = y^3\sec x$
   Ans. $\cos^2 x = y^2(c + 2\sin x)$
7. Solve the exact equations
   (a) $[y(1 + \frac{1}{x}) + \cos y]dx + (x + \log x - x\sin y)dy = 0$
   Ans. $y(x + \log x) + x\cos y = 0$
   (b) $[5x^4 + 3x^2y^2 - 2xy^3]dx = (2x^3y - 3x^2y^2 - 5y^4)dy = 0$
   Ans. $x^5 + x^3y^2 - x^2y^3 - y^5 = 0$
   (c) $\frac{2x}{y^3}\,dx + \frac{y^2 - 3x^2}{y^4}\,dy = 0$
   Ans. $x^2 - y^2 = cy^3$
   (*d*) $(x^2 + y^2 - a^2)\,x\,dx + (x^2 - y^2 - b^2)y\,dy = 0.$
   Ans. $x^4 + 2x^2y^2 - y^4 - 2a^2x^2 - 2b^2y^2 = c$

8. Solve the following equations which are reducible to exact equations:

   (a) $(y^2 + 2x^2y)\,dx + (2x^3 - xy)dy = 0.$

   Ans. $6\sqrt{xy} - \left(\frac{y}{x}\right)^{3/2} = c$

   (b) $(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0.$

   Ans. $x^2 - \frac{1}{x} - \frac{y^2}{x} = c$

   (c) $(xy\,e^{x/y} + y^2)dx - x^2\,e^{x/y}\,dy = 0$

   Ans. $3\log x - 2\log y + \frac{y}{x} = c$

   (*d*) $x\,dy - y\,dx + a(x^2 + y^2)dx = 0$

   Ans. $ax + \tan^{-1}\frac{y}{x} = c$

   (e) $x\,dy - y\,dx = xy^2\,dx$ **Ans.** $\frac{x^2}{2} + \frac{x}{y} = c$

   (f) $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$ **Ans.** $3\log x - \left(\frac{y}{x}\right)^3 = c$

9. In RC circuit containing steady voltage V, find the change at any time *t*. Also derive expression for the current flowing through the circuit.

   Ans. $Q = CV(1 - e^{-\frac{t}{RC}}$ and $I = \dfrac{V}{R}e^{-\frac{t}{RC}}$

10. An RL circuit has an e.m.f. of 3 sin 2*t* volts, a resistance of 10 ohms, an inductance of 0.5 henry, and an initial current of 6 amp. Find the current in the circuit at any time *t*.

    Ans. $I = \frac{609}{101}e^{-20t} + \frac{30}{101}\sin 2t - \frac{3}{101}\cos 2t$

11. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 min, find the temperature of the body after 24 min. **Ans.** 48°C

12. A pipe 20 cm in diameter contains steam at 200°C. It is covered by a layer of insulation 6 cm thick and thermal conductivity 0.0003. If the temperature of the outer surface is 30°C, find the heat loss per hour from 2 metre length of the pipe. **Ans.** 490,000 cal.

13. A steam pipe 20 cm in diameter is covered with an insulating sheath 5 cm thick, the conductivity of which is 0.00018. If the pipe has the constant temperature 100°C, and outer surface of the sheath is kept at 30°C, find the temperature of the sheath as a function of the distance *x* from the axis of the pipe. How much heat is lost per hour through a section 1 metre long.

    Ans. T = 497.5 – 172.6 log *x* and $Q = 70300$ cal.

14. A tank contains 5000 litres of fresh water. Salt water which contains 100 gm of salt per litre flows into it at the rate of 10 l/min and the mixture kept uniform by stirring, runs out at the same rate. When will the tank contain 2,00,000 gm of salt? How long will it take for the quantity of salt in the tank to increase from 1,50,000 gm to 2,50,000 gm?

    Ans. 4 h 15.52 min, 2 h 48.23 min

15. The amount *x* of a substance present in a certain chemical reaction at time *t* is given by $\frac{dx}{dt} + \frac{x}{10} = 2 - 1.5\,e^{-\frac{t}{10}}$. If at $t = 0, x = 0.5$, find *x* at $t = 10$. **Ans.** $20 - \frac{69}{2e}$

16. Radium decomposes at a rate proportional to the amount present. If a fraction *p* of the original amount disappears in 1 year, how much will it remain at the end of 21 years?

    Hint: use $x = x_0\,e^{kt}$. Using given data $k = \log\frac{1}{p}$]

    Ans. $(1 - \frac{1}{p})^{21}$ times the original amount

17. Find the orthogonal trajectories of the family of curves $r = a(1+\cos\theta)$.

    Ans. $r = c(1 - \cos\theta)$

18. Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$.

    Ans. $2x^2 + y^2 = c$

19. Find the orthogonal trajectories of the family of curves $ay^2 = x^3$.

    Ans. $3y^2 + 2x^2 = c^2$

20. Solve the following differential equation

    (a) $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 14\frac{d^2y}{dx^2} - 20\frac{dy}{dx} + 25y = 0$

    Hint: Roots of A.E. are $1+2i, 1-2i, 1+2i, 1-2i$.

    Ans. $y = e^x[(c_1 + c_2x)\sin 2x + (c_3 + c_4x)\cos 2x$

    (b) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0,\ y(0) = -3,\ y'(0) = -1.$

    Ans. $y = e^{3x}(2\sin 4x - 3\cos 4x)$

    (c) $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0.$

    Ans. $y = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$

    (d) $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$

    Ans. $x = (c_1 + c_2t)\,e^{-3t}$

    (e) $\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = 0$

    Ans. $y = c_1e^{-ax} + c_2e^{-bx}$

    (f) $(D^3 + D^2 + 4D + 4)\,y = 0$

Ans. $y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x$

(g) $\frac{d^3y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$. Are the solutions linearly independent?

Ans. $y = c_1e^x + c_2e^x$, Wronskian

$$W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0,$$

Therefore, e^x, e^{2x} are linearly independent.

21. Solve the following differential equations:

(a) $\frac{d^2y}{dx^2} + a^2y = \tan ax$

Ans. $y = c_1 \cos ax + c_2 \sin ax$
$-\frac{1}{a^2}\cos ax \log(\sec ax + \tan ax)$

(b) $(D^2 - 3D + 2)\, y = 6\, e^{-3x} + \sin 2x$.

Ans. $y = c_1 e^x + c_2e^{2x}$
$+\frac{3}{10}e^{-3x} + \frac{1}{20}(3\cos 2x - \sin 2x)$

(c) $\frac{d^2y}{dx^2} - 4y = x^2 + 2x$.

Ans. $y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4}(x^2 + \frac{1}{2})$

(d) $(D^2 - 4D + 4)y = 8x^2\, e^{2x} \sin 2x$.

Ans. $y = (c_1 + c_2x)e^{2x}$
$- e^{2x}[4x\cos 2x + (2x^2 - 3)\sin 2x$

(e) $(D^2 - 1)y = x \sin x + (1 + x^2)e^x$.

Ans. $y = c_1e^x + c_2e^{-x} - \dfrac{1}{2}(x \sin x + \cos x)$
$+\dfrac{1}{12}xe^x(2x^2 - 3x + 9)$

(f) $(D - 1)^2 (D + 1)^2\, y = \sin^2 \frac{x}{2} + e^x + x$.

Ans. $y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x}$
$+\frac{1}{2} - \frac{1}{8}\cos x + e^x + x$

(g) $(D^2 + 4)y = e^x + \sin 2x$.

Ans. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^x - \frac{x}{4}\cos 2x$

(h) $\frac{d^3y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$.

Ans. $y = c_1e^{-x} + c_2e^{-2x} + 1$
$+\frac{1}{10}(3\sin 2x - \cos 2x)$

(i) $\frac{d^3y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$.

Ans. $y = c_1e^x + c_2e^{2x} + \frac{1}{4}e^{3x}(2x - 3)$
$+\frac{1}{20}(3\cos 2x - \sin 2x)$

(j) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2e^{2x} + \sin^2 x$.

Ans. $y = c_1 + (c_2 + c_3x)e^{-x} + \frac{e^{2x}}{18}(x^2 - \frac{7}{8}x + \frac{11}{6})$
$+\dfrac{1}{100}(3\sin 2x + 4\cos 2x)$

(k) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$.

Ans. $y = c_1e^{-x} + c_2e^{-2x} + e^{-2x}e^{e^x}$

(l) $(D^2 - D)y = 2x + 1 + 4\cos x + 2e^x$.

Ans. $y = c_1 + c_2\, e^x + c_3\, e^{-x}$
$+ x\, e^x - (x^2 + x) - 2\sin x$

22. Solve the following differential equations of second order by changing the independent variable.

(a) $x^4\frac{d^2y}{dx^2} + 2x^3\frac{dy}{dx} + 4y = 0$

Ans. $y = c_1 \cos\frac{2}{x} + c_2 \sin\left(\frac{2}{x}\right)$

(b) $x^6\frac{d^2y}{dx^2} + 3x^5\frac{dy}{dx} + a^2y = \frac{1}{x^2}$

Ans. $y = c_1 \cos\frac{a}{2x^2} - c_2 \sin\frac{a}{2x^2} + \frac{1}{a^2x^2}$

(c) $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Ans. $y = (c_1 + c_2\log x)x + \log x + 2$

(d) $\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - (2\cos^3 x)y = 2\cos^5 x$

Ans. $y = c_1e^{\sqrt{2}\sin x} + c_2e^{-\sqrt{2}\sin x} + \sin^2 x$.

23. Solve the following differential equations of second order by changing the dependent variable:

(a) $\frac{d^2y}{dx^2} + 2n\cot nx\frac{dy}{dx} + (m^2 - n^2)y = 0$

Ans. $y = \frac{1}{\sin nx}[c_1\cos mx + c_2\sin mx]$.

(b) $x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$

Ans. $y = c_1x^2 + \frac{c_2}{x} + \frac{1}{3}\left(x^2 - \frac{1}{x}\right)\log x$.

(c) $\cos^2 x\frac{d^2y}{dx^2} - (2\sin x\cos x)\frac{dy}{dx} + (\cos^2 x)y = 0$

Ans. $y = [c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x)]\sec x$.

(d) $\left[\frac{d^2y}{dx^2} + y\right]\cot x + 2\left[y\tan x + \frac{dy}{dx}\right] = 0$

Ans. $y = (c_1 + c_2x)\cos x$.

24. Solve by method of undetermined coefficients:

(a) $\frac{d^2y}{dx^2} + y = 2e^x + \cos x$

Ans. $y = c_1\cos x + c_2\sin x + e^x + \frac{1}{2}x\sin x$

(b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^x$

Ans. $y = (c_1 + c_2x)e^x + \frac{x^4}{12}e^x$

(c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x - e^x$

Ans. $y = (c_1 + c_2x)e^{-x} + x - 2 - \frac{e^x}{4}$.

25. Solve by method of reduction of order.

(a) $x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$ if $y = x$ is a solution of the C. F.

Ans. $y = x\left[-\frac{x}{2} + \frac{c_1}{2}e^{2x} + c_2\right]$.

(b) $(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x(1-x^2)^{\frac{3}{2}}$ if $y = x$ is a part of C. F.

Ans. $y = \frac{-x(1-x^2)^{\frac{3}{2}}}{9} - c_1\left[\sqrt{1-x^2} + x\sin^{-1}x\right] + c_2x.$

26. Solve the following equations using the method of variation of parameters:

(a) $\frac{d^2y}{dx^2} + a^2y = \sec ax.$

Ans. $y = c_1\cos ax + c_2\sin ax + \frac{1}{a^2}\cos ax\log(\cos ax) + \frac{1}{a}x\sin ax$

(b) $\frac{d^2y}{dx^2} + y = x\sin x.$

Ans. $y = c_1\cos x + c_2\sin x + \frac{x}{4}\sin x - \frac{x^2}{4}\cos x$

(c) $\frac{d^2y}{dx^2} + 4y = \tan 2x.$

Ans. $y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}\cos 2x\log(\sec 2x + \tan 2x)]$

(d) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x\tan x.$

Ans. $y = e^x(c_1\cos x + c_2\sin x) - e^x\cos x\log(\sec x + \tan x)$

(e) $(D^2+1)y = \frac{1}{1+\sin x}.$

Ans. $y = c_1\cos x + c_2\sin x + \sin x\log(1+\sin x) - x\cos x - 1$

(f) $(D^2+1)y = x\sin x.$

Ans. $y = c_1\cos x + c_2\sin x + \frac{x}{4}\sin x - \frac{x^2}{4}\cos x$

27. Solve the following Cauchy–Euler equations :

(a) $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right).$

Ans. $y = \frac{c_1}{x} + x[c_2\cos(\log x) + c_3\sin(\log x)$

(b) $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2\sin(\log x)$

Ans. $y = x^2[c_1\cos(\log x) + c_2\sin(\log x)] - \frac{1}{2}x^2\log x\cos(\log x)$

(c) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x$

Ans. $y = \frac{c_1}{x} + c_2x^4 - \frac{x^6}{6} - \frac{1}{2}\log x + \frac{3}{8}$

(d) $x^2\frac{d^3y}{dx^3} - 4x\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 4$

Hint: Write the given equation in the form

$$x^3\frac{d^3y}{dx^3} - 4x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} = 4x$$

Ans. $y = c_1 + c_2x^3 + c_3x^4 + \frac{2}{3}x$

(e) $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^2$

Ans. $y = \frac{1}{x}(c_1 + c_2\log x) + \frac{1}{x}\log\frac{x}{1-x}$

(f) $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \log\frac{\sin(\log x)+1}{x}$

Ans. $y = c_1x^{\sqrt{3}+2} + c_2x^{2-\sqrt{3}} + \frac{1}{61x}\log x[5\sin(\log x + 6\cos(\log x)] + \frac{2}{61}[21\sin(\log x) + 191\cos(\log x)] + \frac{1}{6x}(1+\log x)$

(g) $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos(\log x)$

Ans. $y = c_1x^{-2} + x[c_2\cos\sqrt{3}(\log x) + c_3\sin(\sqrt{3}\log x)] + 8cos(\log x) - \sin(\log x)$

28. Solve the following Legendre's linear equations:

(a) $(3x+2)\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1.$

Ans. $y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2\log(3x+2) + 1$

(b) $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4).$

Ans. $y = c_1 + c_2\log(x+1) + [\log(x+1)]^2 + x^2 + 8x$

(c) $(1+2x)^2\frac{d^2y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$

Ans. $(1+2x)^2[c_1 + c_2\log(1+2x) + \log(1+2x)]$

29. Solve the following simultaneous equations:

(a) $\frac{d^2x}{dt^2} + 4x + 5y = t^2,\ \frac{d^2y}{dt^2} + 5x + 4y = t + 1$

Ans. $x = c_1e^t + c_2e^{-t} + c_3\cos 3t + c_4\sin 3t - \frac{1}{9}\left(4t^2 - 5t + \frac{37}{9}\right),$

$y = -c_1e^t - c_2e^{-t} + c_3\cos 3t + c_4\sin 3t + \frac{1}{9}\left(5t^2 - 4t + \frac{44}{9}\right)$

(b) $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = \sin t,\ \frac{dx}{dt} + x - 3y = 0.$

Ans. $x = \frac{3}{2} c_1 e^{2t} - 3c_2 e^{-t}$
$- \frac{3}{10} e^{t}(\cos t - 2\sin t),$
$y = c_1 e^t + c_2 e^{-2t} - \frac{1}{10}(\cos t + 3\sin t)$

(c) $\frac{dx}{dt} + 2x + 3y = 0,\ \frac{dy}{dt} + 3x + 2y = 2e^{2t}.$

Ans. $x = c_1 e^t + c_2 e^{-5t} + \frac{6}{7} e^{2t},$
$y = c_2 e^{-5t} - c_1 e^t + \frac{8}{7} e^{2t}$

(d) $\frac{dx}{dt} + y = \sin t,\ \frac{dy}{dt} + x = \cos t$ subject to $x = 2$ and $y = 0$ when $t = 0$.

Ans. $x = e^{-t} + e^{t},\ y = e^{-t} - e^{t} + \sin t$

(e) $2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4,\quad 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0$ subject to $x(0) = y(0) = 0,\ x'(0) = y'(0) = 0.$

Ans. $x = \frac{8}{9}\left(1 - \cos\frac{3t}{2}\right), y = \frac{4}{3}t - \frac{8}{9}\sin\frac{3t}{2}$

30. An uncharged condenser of capacity C is charged by applying an e.m.f. E sin $\frac{t}{\sqrt{LC}}$ through leads of self inductance L and negligible resistance. Then find the charge at the condenser plate at time t.

Ans. $\frac{EC}{2}\left[\sin\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right]$

31. A circuit consists of an inductance L and a capacitor of capacity C in series. An alternating e.m.f. E sin nt is applied to the circuit at time $t = 0$, the initial current and charge on the condenser being zero. Prove that the current at time t is $\frac{nE}{L(n^2-\omega^2)}$ $(\cos\omega t - \cos\omega t)$,where CL $\omega^2 = 1$. If $n = \omega$, show that current at time t is $\frac{Et\sin\omega t}{2L}$.

Hint: Solve $L\frac{d^2Q}{dt^2} + \frac{Q}{C} = E\sin nt$ under the conditions I(0) = 0, Q(0) = 0. Then use $I = \frac{dQ}{dt}$.

32. A pendulum of length l hangs against a wall at an angle θ to the horizontal. Show that the time of complete oscillation is $2\pi\sqrt{\frac{l}{g\sin\theta}}$.

33. A simple pendulum has a period T. When the length of the string is increased by a small fraction $\frac{1}{n}$ of its original length, the period is T'. Show that (approximately) $\frac{1}{n} = \frac{2(T'-T)}{T}$.

34. How many seconds a clock would lose per day if the length of its pendulum were increased in the ratio 900:901.

Ans. 48 s/day

Solution about regular points

35. Solve the following differential equations about $x = 0$.

(i) $\frac{d^2y}{dx^2} + x^2 y = 0$

Ans. $y = a_0\left[x - \frac{x^4}{4.3} + \frac{x^8}{8.7.4.3} - \frac{x^{12}}{12.11.8.7.4.3} + \ldots\right]$
$+ a_1\left[x - \frac{x^5}{5.4} + \frac{x^9}{9.8.5.4} - \frac{x^{13}}{13.12.9.8.5.4} + \ldots\right]$

(ii) $(x^2 + 1)\frac{d^2y}{dx^2} + xy\frac{dy}{dx} = xy = 0$

Ans. $y = a_0\left(1 - \frac{1}{6}x^3 - \frac{3}{40}x^5 + \ldots\right)$
$+ a_1\left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 + \ldots\right)$

(iii) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (2x^2 + 1)y = 0$

Ans. $y = a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \ldots\right)$
$+ a_1\left(x - \frac{x^3}{3} - \frac{x^5}{30} - \ldots\right)$

(iv) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (3x + 2)y = 0$

Ans. $y = a_0\left(1 - x^2 - \frac{x^3}{2} + \frac{x^4}{3} + \frac{11x^5}{40} + \ldots\right)$
$+ a_1\left(x - \frac{x^3}{2} - \frac{x^4}{4} + \frac{x^5}{8} + \ldots\right)$

36. Find the power series solution in the powers of $(x - 1)$ of the initial value problem.

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 0,\ \ y(1) = 1,\ y'(1) = 2.$$

Hint: $x = 1$ is the ordinary point. So consider $y = \sum_{n=0}^{\infty} a_n(x-1)^n$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, put in the given equation. Use the initial conditions in finding a_0 and a_1 by putting $x = 1$ in y and $\frac{dy}{dx}$. Other coefficients to be found by equating to zero the powers of $(x - 1)$.

Ans. $y = 1 + 2(x-1) - 2(x-1)^2$
$+ \frac{2}{3}(x-1)^3 - \frac{1}{6}(x-1)^4 + \ldots$

Solution about regular singular point

37. Solve the following differential equations about $x = 0$

(i) $x(1-x)\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} - y = 0$

Ans. $y = a_0(1 + \log x)(1 + x + \frac{2x^2}{4}$
$+ \frac{2.5}{4.9}x^3 + \ldots) + a_0(-2x - x^2 - \ldots)$

(ii) $x(1-x)\frac{d^2y}{dx^2}-(1+3x)\frac{dy}{dx}-y=0$

Ans. $y=(A+B\log x)(1.2x^2+2.3x^3$
$+3.4x^4+\ldots)$
$+B(-1+x+2x^2+11x^3+\ldots)$

(iii) $2x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-1)y=0$

Ans. $y=c_1x\left(1-\frac{x^2}{14}+\frac{x^4}{616}+\ldots\right)$
$+c_2\sqrt{x}\left(1-\frac{x^2}{2}+\frac{x^4}{40}+\ldots\right)$

(iv) $(2x+x^3)\frac{d^2y}{dx^2}-\frac{dy}{dx}-6xy=0$

Ans. $y=c_1\left(1+3x^2+\frac{3x^4}{5}+\ldots\right)$
$+c_2x^{\frac{3}{2}}\left(1+\frac{3x^2}{8}-\frac{3x^4}{128}+\ldots\right)$

(v) $4x\frac{d^2y}{dx^2}+2(1-x)\frac{dy}{dx}-y=0$

Ans. $y=c_1\left(1+\frac{x}{2!}+\frac{x^2}{2^2.2\,!}+\frac{x^3}{2^3.3\,!}+\ldots\right)$
$+c_2x^{\frac{1}{2}}\left(1+\frac{x}{1.3}+\frac{x^2}{1.3.5}+\frac{x^3}{1.3.5.7}+\ldots\right)$

(vi) $x\frac{d^2y}{dx^2}+\frac{dy}{dx}+2y=0$

Ans. $y=(c_1+c_2\log x)\left(1-2x+\frac{2^2x^2}{(2\,!)^2}-\ldots\right)$
$+c_2(4x-3x^2+\ldots)$

Bessel's Equation and Bessel Function

38. Write the general solution of the differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-\tfrac{9}{16}\right)y=0 \text{ in terms of } J_n(x).$$

Ans. $y=c_1J_{\frac{3}{4}}(x)+c_2J_{-\frac{3}{4}}(x)$.

39. Express $J_5(x)$ in terms of $J_0(x)$and $J_1(x)$.

Ans. $J_5(x)=\left(\frac{384}{x^4}-\frac{72}{x^2}-1\right)J_1(x)+\left(\frac{12}{x}-\frac{192}{x^3}\right)J_0(x)$

40. Solve $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+\frac{1}{2}xy=0$ in terms of Bessel's function.

Ans. $y=\sqrt{x}\left[c_1J_{\frac{1}{2}}\left(\frac{x}{\sqrt{2}}\right)++c_2J_{-\frac{1}{2}}\left(\frac{x}{\sqrt{2}}\right)\right]$

41. Show that $\int J_3(x)dx=-J_2(x)-\frac{2}{x}J_0(x)+c$

42. Show that $J_0^2(x)+2J_1^2(x)+2J_2^2(x)+2J_3^2(x)+\ldots=1$.

Hint: Squaring and integrating the Jocobi series

$\cos(x\sin\phi)=J_0+2J_2\cos 2\phi+2J_4\cos 4\phi+\ldots,$
$\sin(x\sin\phi)=2J_1\sin\phi+2J_3\sin 3\phi+2J_5\sin 5\phi+\ldots,$

we get

$$\int_0^\pi \cos^2(x\sin\phi)d\phi=\pi(J_0^2+2J_2^2+2J_4^2+\ldots)$$
$$\int_0^\pi \sin^2(x\sin\phi)d\phi=\pi(2J_1^2+2J_3^2+2J_5^2+\ldots).$$

Adding these expressions, we get the required result

43. Show that

(i) $\cos x=J_0-2J_1+2J_4+\ldots$
$=J_0+2\sum_{n=1}^{\infty}(-1)^nJ_{2n}$,

(ii) $\sin x=2J_1-2J_3+2J_5+\ldots$
$=2\sum_{n=1}^{\infty}(-1)^nJ_{2n+1}$

Hint: Put $\phi=\frac{\pi}{2}$ in the Jocobi series

$\cos(x\sin\phi)=J_0+2J_2\cos 2\phi+2J_4\cos 4\phi+\ldots$

and

$\sin(x\sin\phi)=2J_1\sin\phi+2J_3\sin 3\phi$
$+2J_5\sin 5\phi+\ldots$

44. Show that

$$\frac{d}{dx}[J_n^2(x)]=\frac{x}{2n}[J_{n-1}^2(x)-J_{n+1}^2(x)].$$

Legendre Equation and Legendre Polynomial

45. Express in the terms of Legendre polynomial

(i) $4x^3-2x^2-3x+8$

Ans. $\frac{8}{5}P_3(x)-\frac{4}{3}P_2(x)-\frac{3}{5}P_1(x)+\frac{22}{3}P_0(x)$

(ii) $x^4+3x^3-x^2+5x-2$

Ans. $\frac{8}{35}P_4(x)+\frac{6}{5}P_3(x)-\frac{2}{21}P_2(x)+\frac{34}{5}P_1(x)$
$-\frac{224}{105}P_0(x)$

(iii) $4x^2-3x+2$

Ans. $\frac{8}{3}P_2(x)-3P_1(x)+\frac{10}{3}P_0(x)$

46. Show that

$$P_n(x)=P'_{n+1}(x)-2xP'_n(x)+P'_{n-1}(x)$$

Hint: Use recurrence formulae

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \qquad \text{(i)}$$

and

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x) \qquad \text{(ii)}$$

$$\begin{aligned}\text{From } (i), P_n(x) &= P'_{n+1}(x) - P'_{n-1}(x) - 2nP_n(x) \\ &= P'_{n+1}(x) - P'_{n-1}(x) - 2xP'_n(x) \\ &\quad + 2P'_{n-1}(x) \\ &= P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x)\end{aligned}$$

47. Show that $\int\limits_{-1}^{1} P_n(x)dx = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1. \end{cases}$

48. Show that

$$\int\limits_{-1}^{1} xP_n(x)P'_n(x)dx = \tfrac{2n}{2n+1}\,.$$

Hint: Use integration by parts and $\int\limits_{-1}^{1} P_n^2(x)dx = \tfrac{2}{2n+1}$.

49. Using Rodrigue's formula, show that $P_n(x)$ satisfies the differential equation

$$\tfrac{d}{dx}\left[(1-x^2)\,\tfrac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = 0\,.$$

Hint: Let $y = (x^2-1)^n$. Find $\frac{dy}{dx}$ and differentiate once more to get $(x^2-1)\frac{d^2y}{dx^2} + 2xy_1 = 2n(xy_1 + n)$.

Then use Leibnitz's theorem to get

$$(1-x^2)y_{n+2} - 2xy_{n+1} + n(n+1)y_n = 0$$

or

$$[(1-x^2)y_{n+1}] + n(n+1)y_n = 0$$

or

$$\tfrac{d}{dx}\left[(1-x^2)\tfrac{d}{dx}\left(\tfrac{d^n}{dx^n}(x^2-1)^n\right)\right] + n(n+1)\tfrac{d^n}{dx^n}(x^2-1)^n = 0$$

50. Expand

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$$

in Fourier–Legendre series.

Ans. $f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_3(x) + \ldots$

2 Graphs

The interest in the study of graph theory has increased due to its applicability in so many fields like artificial intelligence, electrical engineering, transportation system, scheduling problems, economics, chemistry and operations research.

2.1 DEFINITIONS AND BASIC CONCEPTS

Definition 2.1. A **graph** $G=(V, E)$ is a mathematical structure consisting of two finite sets V and E. The elements of V are called **vertices (or nodes)** and the elements of E are called **edges**. Each edge is associated with a set consisting of **either one** or **two vertices** called its **endpoints**.

The correspondence from edges to endpoints is called **edge-endpoint function**. This function is generally denoted by γ. Due to this function, some authors denote graph by $G=(V, E, \gamma)$.

Definition 2.2. A graph consisting of one vertex and no edges is called a **trivial graph**.

Definition 2.3. A graph whose vertex and edge sets are empty is called a **null graph**.

Definition 2.4. An edge with just one endpoint is called a **loop** or a **self-loop**.

Thus, a loop is an edge that joins a single endpoint to itself.

Definition 2.5. An edge that is not a self-loop is called a **proper edge**.

Definition 2.6. If two or more edges of a graph G have the same vertices, then these edges are said to be **parallel** or **multi-edges**.

Definition 2.7. Two vertices that are connected by an edge are called **adjacent**.

Definition 2.8. An endpoint of a loop is said to be **adjacent to itself**.

Definition 2.9. An edge is said to be **incident** on each of its endpoints.

Definition 2.10. Two edges incident on the same endpoint are called **adjacent edges**.

Definition 2.11. The number of edges in a graph G which are incident on a vertex is called the **degree of that vertex**.

Definition 2.12. A vertex of degree zero is called an **isolated vertex**.

Thus, a vertex on which no edges are incident is called **isolated**.

Definition 2.13. A graph without multiple edges **(parallel edges)** and loops is called **simple graph**.

Notations: In pictorial representations of a graph, the vertices will be denoted by dots and edges by line segments.

EXAMPLE 2.1

(a) Let

$$V=\{1, 2, 3, 4\} \quad \text{and} \quad E=\{e_1, e_2, e_3, e_4, e_5\}.$$

Let γ be defined by

$$\gamma(e_1)=\gamma(e_5)=\{1, 2\}, \quad \gamma(e_2)=\{4, 3\},$$
$$\gamma(e_3)=\{1, 3\}, \quad \gamma(e_4)=\{2, 4\}.$$

We note that both edges e_1 and e_5 have same endpoints $\{1, 2\}$. The endpoints of e_2 are $\{4, 3\}$, the endpoints of e_3 are $\{1, 3\}$ and endpoints of e_4 are $\{2, 4\}$. Thus the graph is as shown in the Figure 2.1.

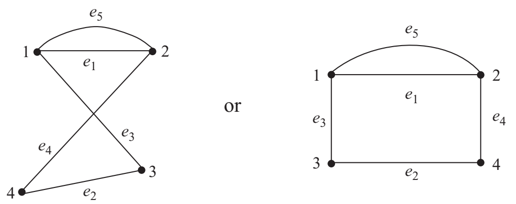


Figure 2.1

(b) For the graph pictured in the Figure 2.2,

(i) Write down the degree of the vertices A, B, D.

(ii) Which of the edges are parallel?

(iii) Which vertex is isolated?

(iv) Point out whether it is a simple graph or a multi-graph.

(v) Point out one pair of adjacent vertices and one pair of adjacent edges.

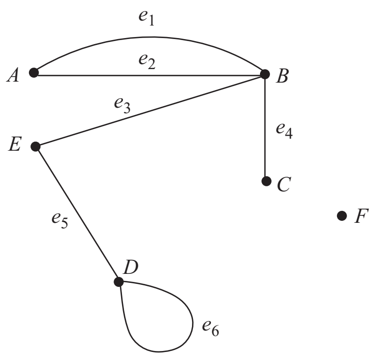


Figure 2.2

In this graph,

(i) We notice that

Degree of the vertex $A = 2$

Degree of the vertex $B = 4$

Degree of the vertex $D = 3$

(because loop e_6 has degree 2).

The vertex D has degree 3 (because e_6 is a loop and so has degree 2)

(ii) The edges e_1 and e_2 are parallel

(iii) The vertex F is isolated because no edge is incident on F

(iv) It is a multi-graph because it has multi-edges and a loop.

(v) A and B are adjacent vertices because they are connected by the edge e_2 or e_1 whereas A and E are not adjacent.

The edges e_2 and e_3 are adjacent edges because they are incident on the same vertex B.

(c) Consider the graph with the vertices A, B, C, D and E pictured in Figure 2.3.

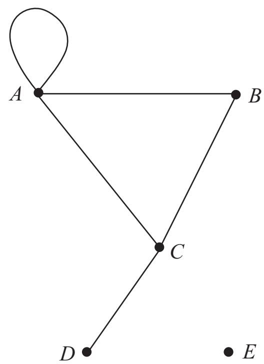


Figure 2.3

In this graph, we note that

Number of edges $= 5$

Degree of vertex $A = 4$

Degree of vertex $B = 2$

Degree of vertex $C = 3$

Degree of vertex $D = 1$

Degree of vertex $E = 0$

Sum of the degree of vertices

$= 4 + 2 + 3 + 1 + 0 = 10.$

Thus, we observe that

$$\sum_{i=1}^{5} \deg(v_i) = 2e,$$

where $\deg(v_i)$ denotes the degree of vertex v_i and e denotes the number of edges.

Euler's Theorem 2.1. (The First Theorem of Graph Theory)

The sum of the degrees of the vertices of a graph G is equal to twice the number of edges in G.
Thus, total degree of a graph is even.

Proof: Each edge in a graph contributes a count of 1 to the degree of two vertices (endpoints of the edge), that is, each edge contributes 2 to the degree sum. Therefore, the sum of degrees of the vertices is equal to twice the number of edges.

Corollary 2.1. There can be only an even number of vertices of odd degree in a given graph G.

Proof: We know, by the fundamental theorem, that

$$\sum_{i=1}^{n} \deg(v_i) = 2 \times \text{number of edges}.$$

Thus the right-hand side is an even number. Hence to make the left-hand side an even number there can be only even number of vertices of odd degree.

Remarks 2.1.

(i) A vertex of degree d is also called a ***d*-valent vertex**.
(ii) The degree (or valence) of a vertex v in a graph G is the number of proper edges incident on v plus twice the number of self-loops.

Theorem 2.2. A nontrivial simple graph G must have at least one pair of vertices whose degrees are equal.

Proof: Let the graph G has n vertices. Then there appear to be n possible degree values, namely $0, 1, \ldots, n-1$. But there cannot be both a vertex of degree 0 and a vertex of degree $n-1$ because if there is a vertex of degree 0 then each of the remaining $n-1$ vertices is adjacent to at most $n-2$ other vertices. Hence the n vertices of G can realize at most $n-1$ possible values for their degrees. Hence, the pigeonhole principle implies that at least two of the vertices have equal degree.

EXAMPLE 2.2

Is there a graph with eight vertices of degree 2, 2, 3, 6, 5, 7, 8, 4?

Solution. The answer is No. In such a graph, there will be three vertices of odd degree which is impossible.

We can also argue as follows: Total degree of the graph (if possible) is equal to $2+2+3+6+5+7+8+4=37$, which is odd. This contradicts the first theorem of graph theory, according to which, total degree of a graph is always even.

2.2 SPECIAL GRAPHS

Definition 2.14. A graph G is said to **simple** if it has no parallel edges or loops. In a simple graph, an edge with endpoints v and w is denoted by $\{v, w\}$.

Definition 2.15. For each integer $n \geq 1$, let D_n denote the graph with n **vertices** and **no edges**. Then D_n is called the **discrete graph on** n **vertices**.

For example, we have

• • • and • • • • •
D_3 D_5

Definition 2.16. Let $n \geq 1$ be an integer. Then a simple graph with n vertices in which there is an edge between each pair of distinct vertices is called the **complete graph** on n vertices. It is denoted by K_n.

For example, the complete graphs K_2, K_3 and K_4 are shown in Figure 2.4.

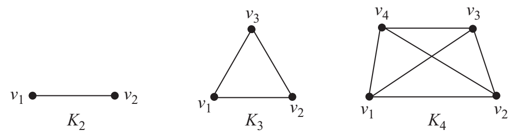


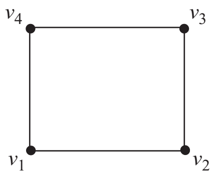
Figure 2.4

Definition 2.17. If each vertex of a graph G has the same degree as every other vertex, then G is called a **regular graph**.

A ***k*-regular graph** is a regular graph whose common degree is k.

For example, consider K_3. The degree of each vertex in K_3 is 2. Hence K_3 is regular. Similarly, K_4 is regular. Also the graph shown in Figure 2.5 is regular because degree of each vertex here is 2.

But this graph is not complete because v_2 and v_4 have not been connected through an edge. Similarly, v_1 and v_3 are not connected by any edge.



2-regular graph

Figure 2.5

Thus, **a complete graph is always regular** but **a regular graph need not be complete**.

EXAMPLE 2.3

The oxygen molecule O_2, made up of two oxygen atoms linked by a double bond can be represented by the regular graph shown in Figure 2.6.

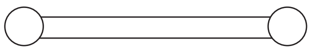


Figure 2.6

Definition 2.18. Let $n \geq 1$ be an integer. Then a graph L_n with n vertices $\{v_1, v_2, \ldots, v_n\}$ and with edges $\{v_i, v_{i+1}\}$ for $1 \leq i < n$ is called a **linear graph on** n **vertices**.

For example, the linear graphs L_2 and L_4 are shown in Figure 2.7.
It is also called **path graph** denoted by P_n.

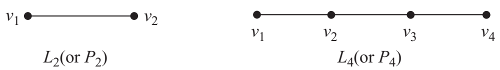


Figure 2.7

Definition 2.19. A **bipartite graph** G is a graph whose vertex set V can be partitioned into two subsets U and W, such that each edge of G has one endpoint in U and one endpoint in W.

The pair (U, W) is called a **vertex bipartition of** G and U and W are called the bipartition subsets. Obviously, a bipartite graph cannot have any self-loop.

EXAMPLE 2.4

If vertices in U are solid vertices and vertices in W are hollow vertices, then the graphs shown in Figure 2.8 are bipartite graphs.

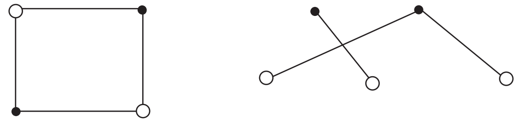
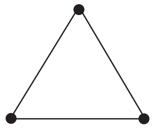


Figure 2.8

EXAMPLE 2.5

The smallest possible simple graph that is not bipartite is the complete graph K_3 shown in Figure 2.9.



K_3

Figure 2.9

Definition 2.20. **A complete bipartite graph** G is a simple graph whose vertex set V can be partitioned into two subsets $U=\{v_1, v_2, \ldots, v_m)$ and $W=\{w_1, w_2, \ldots, w_n\}$ such that for all i, k in $\{1, 2, \ldots, m\}$ and j, l in $\{1, 2, \ldots, n\}$

(i) There is an edge from each vertex v_i to each vertex w_j.
(ii) There is not an edge from any vertex v_i to any other vertex v_k.
(iii) There is not an edge from any vertex w_j to any other vertex w_l.

A complete bipartite graph on (m, n) vertices is denoted by $K_{m,n}$.

EXAMPLE 2.6

The complete bipartite graphs $K_{3,2}$ and $K_{3,4}$ are shown in Figure 2.10.

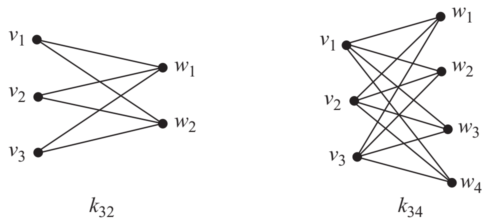


Figure 2.10

Definition 2.21. A **cycle graph** is a single vertex with a self-loop or a simple connected graph C with $|V_c|=|E_c|$, that can be drawn so that all of its vertices and edges lie on a circle.

An n-vertex cycle graph is denoted by C_n. The cycle graphs C_1, C_2, and C_4 are shown in Figure 2.11.

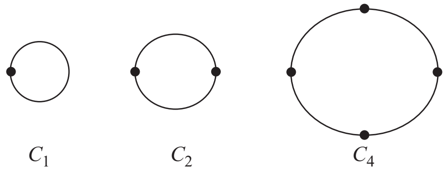
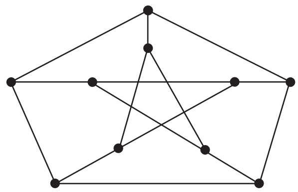


Figure 2.11

Definition 2.22. The **Petersen graph** is the 3-regular graph shown in Figure 2.12.



(Petersen graph)

Figure 2.12

This graph is frequently used to establish theorems and to test conjectures.

Definition 2.23. The **hypercube graph** Q_n is the n-regular graph whose vertex set is the set of bit string of length n and such that there is an edge between two vertices if and only if they differ in exactly one bit.

For example, 8-vertex cube graph is a hypercube graph Q_3 shown in Figure 2.13.

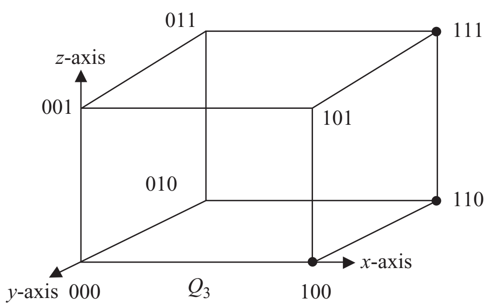


Figure 2.13

Definition 2.24. A graph consisting of two concentric n-cycles in which each of the n pairs of the corresponding vertices is joined by an edge is called the **circular ladder graph** CL_n.

For example, the graphs CL_2 and CL_4 are shown in the Figure 2.14.

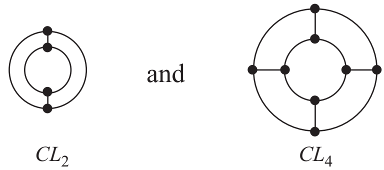


Figure 2.14

Definition 2.25. A graph consisting of a single vertex having n loops is called a **bouquet** and is denoted by B_n.

For example, the graphs shown in Figure 2.15 represent bouquet B_2 and bouquet B_4.

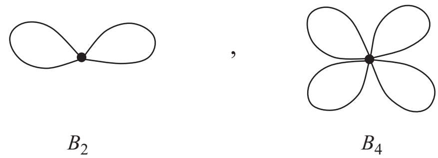


Figure 2.15

Definition 2.26. A graph consisting of two vertices and n edges joining them is called a **dipole**. It is denoted by D_n.

For example, Figure 2.16 represents D_2, D_3 and D_4.

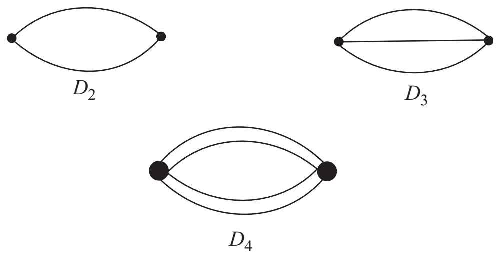


Figure 2.16

EXAMPLE 2.7

Draw all simple graphs with vertices $\{u, v, x, y\}$ and two edges, one of which is $\{u, v\}$.

Solution. In a simple graph, an edge corresponds to a subset of two vertices. Since there are four vertices, we can have $4c_2=6$ such subsets in all and they are $\{u, v\}$, $\{u, x\}$, $\{u, y\}$, $\{v, x\}$, $\{v, y\}$ and $\{x, y\}$. But one edge of the graph is specified to be $\{u, v\}$. Hence the possible simple graphs are as shown in Figure 2.17.

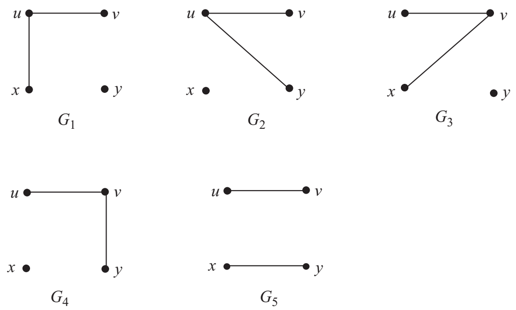


Figure 2.17

2.3 SUBGRAPHS

Definition 2.27. A graph H is said to be a subgraph of a graph G if and only if every vertex in H is also a vertex in G, every edge in H is also an edge in G and every edge in H has the same endpoints as in G.

We may also say that G is a supergraph of H. For example, the graphs (Figure 2.18).

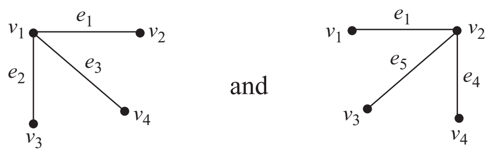


Figure 2.18

are subgraphs of the graph given in Figure 2.19.

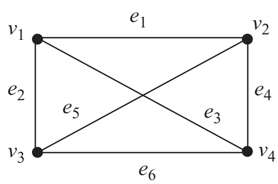


Figure 2.19

Similarly, the graph (Figure 2.20)

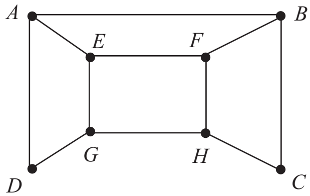


Figure 2.20

is a subgraph of the graph given in the Figure 2.21.

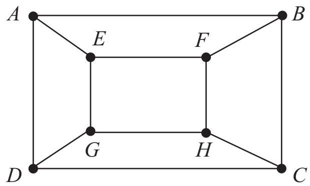


Figure 2.21

Definition 2.28. A subgraph H is said to be a **proper subgraph** of a graph G if vertex set V_H of H is a proper subset of the vertex set V_G of G or edge set E_H is a proper subset of the edge set E_G.

For example, the subgraphs in the above examples are proper subgraphs of the given graphs.

Definition 2.29. A subgraph H is said to **span** a graph G if $V_H = V_G$.

Thus H is a spanning subgraph of graph G if it contains all the vertices of G.

For example, the subgraph (Figure 2.22)

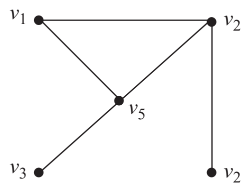


Figure 2.22

spans the graph give in Figure 2.23.

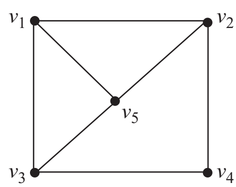


Figure 2.23

Definition 2.30. Let $G=(V, E)$ be a graph. Then the **complement of a subgraph** $G'=(V', E')$ with respect to the graph G is another subgraph $G''=(V'', E'')$ such that $E''=E-E'$ and V'' contains only the vertices with which the edges in E'' are incident.

For example, the subgraph (Figure 2.24)

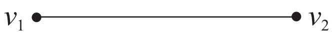


Figure 2.24

is the complement of the subgraph (Figure 2.25).

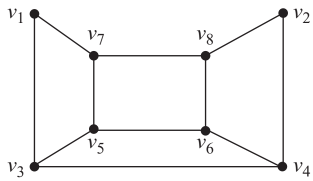


Figure 2.25

with respect to the graph G shown in Figure 2.26.

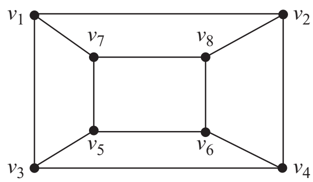


Figure 2.26

Definition 2.31. If G is a simple graph, the **complement of** G (**edge complement**), denoted by G' or G^c is a graph such that

(i) The vertex set of G' is identical to the vertex set of G, that is, $V_{G'}=V_G$.
(ii) Two distinct vertices v and w of G' **are connected by** an edge if and only if v and w **are not connected** by an edge in G.

For example, consider the graph G shown in Figure 2.27.

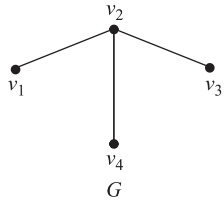


Figure 2.27

Then complement G' of G is the graph shown in Figure 2.28.

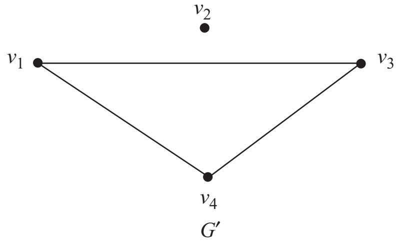


Figure 2.28

EXAMPLE 2.8

Find the complement of the graphs:

(a)

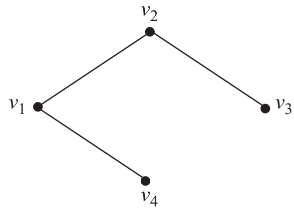
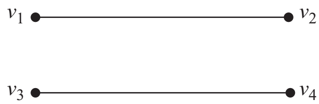


Figure 2.29

(b)



(c) Complete graph K_4:

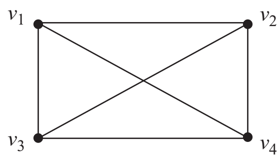


Figure 2.31

Solution.

(a)

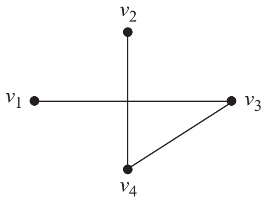


Figure 2.32

(b)

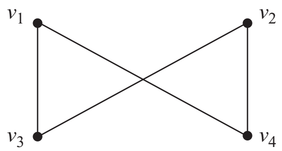


Figure 2.33

(c) Null graph.

EXAMPLE 2.9

Find the edge complement of the graph G shown in Figure 2.34.

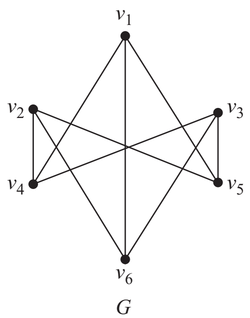


Figure 2.34

Solution. The edge complement of G is the following graph G^c (Figure 2.35).

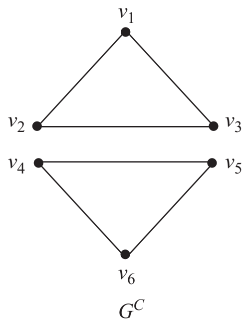


Figure 2.35

Definition 2.32. If a new vertex v is joined to each of the pre-existing vertices of a graph G, then the resulting graph is called the **join of** G **and** v or the **suspension of** G **from** v. It is denoted by $G+v$.

Thus, a graph obtained by joining a new vertex v to each of the vertices of a given graph G is called the **join of** G **and** v or the **suspension of** G **from** v. For example, consider the graph given below (Figure 2.36).

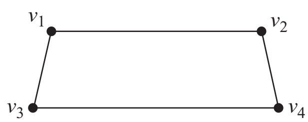


Figure 2.36

Let v be a vertex. Then the graph shown in the Figure 2.37 is the join of G to v.

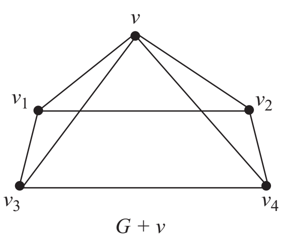


Figure 2.37

2.4 ISOMORPHISMS OF GRAPHS

We know that shape or length of an edge and its position in space are not part of specification of a graph. For example, the Figures 2.38 represent the same graph.

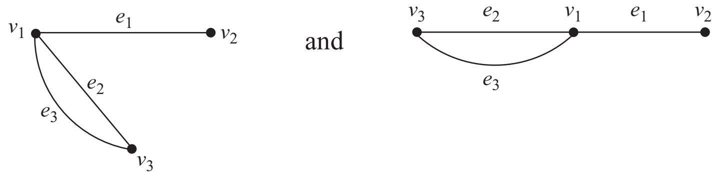


Figure 2.38

Definition 2.33. Let G and H be graphs with vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$, respectively. Then G **is said to isomorphic to** H if there exist one-to-one correspondences g: $V(G) \to V(H)$ and h: $E(G) \to E(H)$ such that for all $v \in V(G)$ and $e \in E(G)$,

v is an endpoint of $e \Leftrightarrow g(v)$ is an endpoint of $h(e)$.

Definition 2.34. The property of mapping endpoints to endpoints is called **preserving incidence** or **the continuity rule** for graph mappings.

As a consequence of this property, a self-loop must map to a self-loop.

Thus, two isomorphic graphs are same except for the labelling of their vertices and edges. For example, the graphs shown in the Figure 2.39 are isomorphic.

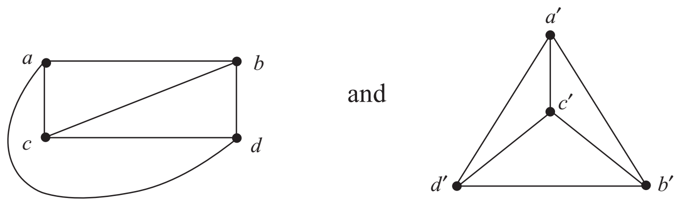


Figure 2.39

Similarly, the hypercube graph Q_3 and circular ladder CL_4 shown in Figure 2.40 are isomorphic.

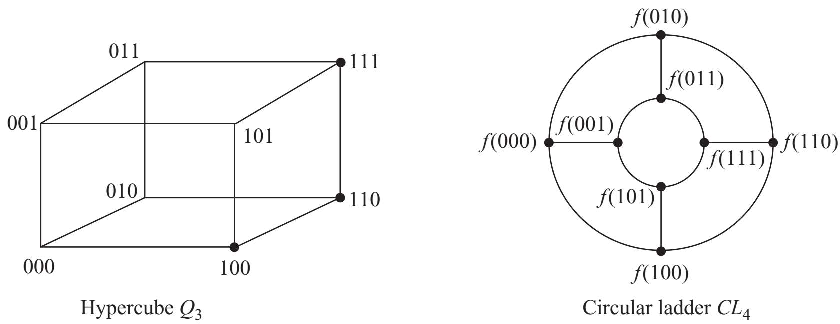
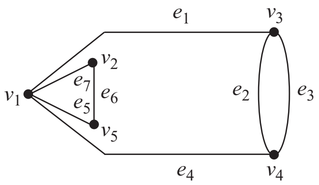


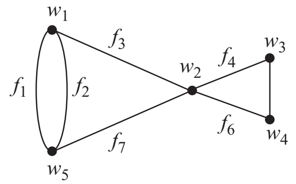
Figure 2.40

EXAMPLE 2.10

Show that the graphs shown in Figure 2.41 are isomorphic.



and



H

Figure 2.41

Solution. To solve this problem, we have to find g: $V(G) \to V(H)$ and h: $E(G) \to E(H)$ such that for all $v \in V(G)$ and $e \in E(G)$,

v is an endpoint of $e \Leftrightarrow g(v)$ is an endpoint of $h(e)$.

Since e_2 and e_3 are parallel (have the same endpoints), $h(e_2)$ and $h(e_3)$ must also be parallel. Thus we have

$$h(e_2)=f_1 \text{ and } h(e_3)=f_2 \quad \text{or} \quad h(e_2)=f_2 \text{ and } h(e_3)=f_1.$$

Also the endpoints of e_2 and e_3 must correspond to the endpoints of f_1 and f_2 and so

$$g(v_3)=w_1 \text{ and } g(v_4)=w_5 \quad \text{or} \quad g(v_3)=w_5 \text{ and } g(v_4)=w_1.$$

Further, we note that v_1 is the endpoint of four distinct edges e_1, e_7, e_5 and e_4 and so $g(v_1)$ should be the endpoint of four distinct edges. We observe that w_2 is the vertex having four edges and so $g(v_1)=w_2$. If $g(v_3)=w_1$, then since v_1 and v_3 are endpoints of e_1 in G, $g(v_1)=w_2$ and $g(v_3)=w_1$ must be endpoints of $h(e_1)$ in H. This implies that $h(e_1)=f_3$.

Continuing this way, we can find g and h to define the isomorphism between G and H.

One such pair of functions (of course there exist several) is shown in Figure 2.42.

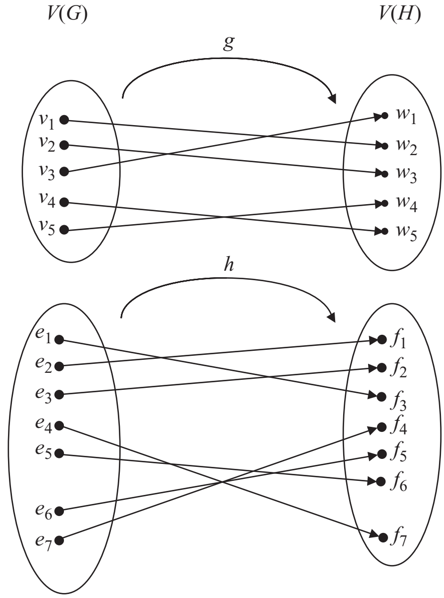


Figure 2.42

Remark 2.2. Each of the following properties of a graph is invariant under graph isomorphism, where n, m and k are all non-negative integers:

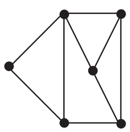
1. Has n vertices
2. Has m edges
3. Has a vertex of degree k
4. Has m vertices of degree k
5. Has a circuit of length k
6. Has a simple circuit of length k
7. Has m simple circuits of length k,
8. Is connected
9. Has an Euler circuit
10. Has a Hamiltonian circuit

(5–10) } to be studied later on in this chapter

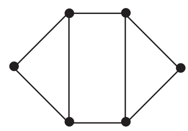
EXAMPLE 2.11

Examine for isomorphism

(a)

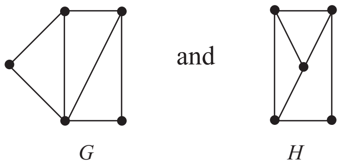


and



G *H*

(b)



(c)

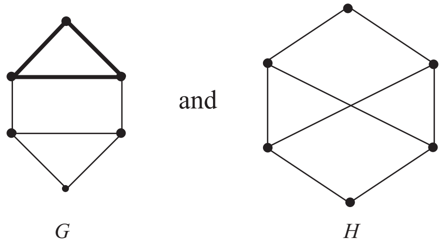


Figure 2.43

Solution. We note in Figure 2.43 that

(a) G has nine edges whereas H has only eight. Hence, G is not isomorphic to H.
(b) G has a vertex v of degree 4, whereas H has no vertex of degree 4. Hence, G is not isomorphic to H.
(c) G has a simple circuit of length 3 (**marked dark**) whereas H does not have. Hence, G is not isomorphic to H.

2.5 WALKS, PATHS AND CIRCUITS

Definition 2.35. In a graph G, **a walk from vertex v_0 to vertex v_n** is a finite alternating sequence $\{v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n\}$ of vertices and edges such that v_{i-1} and v_i are the endpoints of e_i.

The **trivial walk** from a vertex v to v consists of the single vertex v.

Definition 2.36. In a graph G, a **path** from the vertex v_0 to the vertex v_n is a walk from v_0 to v_n that does not contain a repeated edge.

Thus a **path** from v_0 to v_n is a walk of the form

$$\{v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n\},$$

where all the edges e_i are distinct.

Definition 2.37. In a graph, a **simple path** from v_0 to v_n is a path that does not contain a repeated vertex.

Thus a simple path is a walk of the form

$$\{v_0, e_1, v_1, e_2, v_2, \ldots, v_{i-1}, e_n, v_n\},$$

where all the e_i and all the v_i are distinct.

Definition 2.38. A walk in a graph G that starts and ends at the same vertex is called a **closed walk**.

Definition 2.39. A closed walk that does not contain a repeated edge is called a **circuit**.

Thus, a closed path is called a circuit (or a **cycle**) if it is a walk of the form

$$\{v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n\},$$

where $v_0 = v_n$ and all the e_i are distinct.

Definition 2.40. A **simple circuit** is a circuit that does not have any other repeated vertex except the first and the last.

Thus, a simple circuit is a walk of the form

$$\{v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n\},$$

where all the e_i are distinct and all the v_j are distinct except that $v_0 = v_n$.

EXAMPLE 2.12

In the graph given in the Figure 2.44, determine whether the following walks are paths, simple paths, closed walks, circuits, simple circuits or are just walks.

1. $v_1\, e_2\, v_2\, e_3\, v_3\, e_4\, v_4\, e_5\, v_2\, e_2\, v_1\, e_1\, v_0$,
2. $v_5\, v_4\, v_2\, v_1$,
3. $v_2\, v_3\, v_4\, v_5\, v_2$,
4. $v_4\, v_2\, v_3\, v_4\, v_5\, v_2\, v_4$,
5. $v_2\, v_1\, v_5\, v_2\, v_3\, v_4\, v_2$,
6. $v_0\, v_5\, v_2\, v_3\, v_4\, v_2\, v_1$.

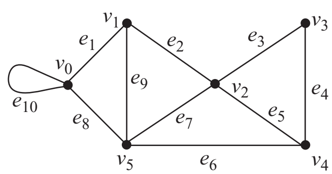


Figure 2.44

Solution

1. It is **just a walk** because it has repeated edges and repeated vertices.

2. It is a **simple path**, because here no vertex is repeated and no edge is repeated.
3. It is **closed walk** in which no edge is repeated and hence it is a **circuit**. It starts and ends at the same vertex, and therefore is a **simple circuit**.
4. It is a closed walk in which no edge is repeated but vertices v_2 and v_4 are repeated. Hence it is a **circuit**.
5. It is a closed walk in which no edge is repeated. Hence it is a circuit. Only one vertex is repeated twice, hence it is **not a simple circuit**.
6. It is a walk in which no edge is repeated but the vertex v_2 is repeated. Hence it is a **path**.

EXAMPLE 2.13

Consider the graph shown in Figure 2.45.

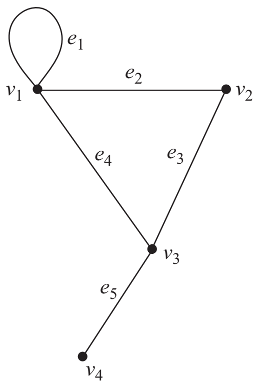


Figure 2.45

We note that e_3, e_5 is a path. The walk e_1, e_2, e_3, e_5 is a path but it is not a simple path because the vertex v_1 is repeated (e_1 being a self-loop). The walk e_2, e_3, e_4 is a circuit. The walk e_2, e_3, e_4, e_1 is a circuit but it is not simple circuit because vertex v_1 repeats twice (or we may write that v_1 is met twice).

Definition 2.41. In a graph the number of edges in the path $\{v_0, e_1, v_1, e_2, \ldots, e_n, v_n\}$ from v_0 to v_n is called the **length of the path**.

Definition 2.42. A cycle with k-edges is called a **k-cycle** or **cycle of length k**.

For example, loop is a cycle of length 1. On the other hand, a pair of parallel edges e_1 and e_2, shown in the Figure 2.46, is a cycle of length 2.

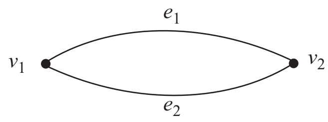


Figure 2.46

EXAMPLE 2.14

Consider the graph shown in Figure 2.47.

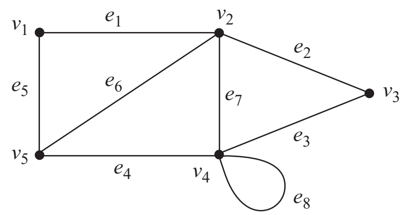


Figure 2.47

In this graph, we observe that

(i) The path $v_4\, e_8\, v_4$ is a cycle of length 1.
(ii) The path $v_5\, e_4\, v_4\, e_8\, v_4\, e_3\, v_3$ from v_5 to v_3 (Figure 2.48) is a path of length 3.

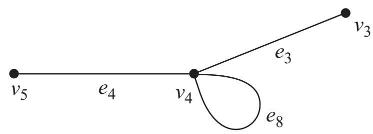


Figure 2.48

(iii) The path $v_1\, e_1\, v_2\, e_6\, v_5\, e_5\, v_1$ (Figure 2.49) is a

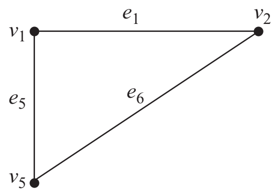


Figure 2.49

cycle of length 3, since it consists of three edges.

Definition 2.43. A graph is said to be **acyclic** if it contains no cycle.

For example, the graphs shown in Figure 2.50 are acyclic.

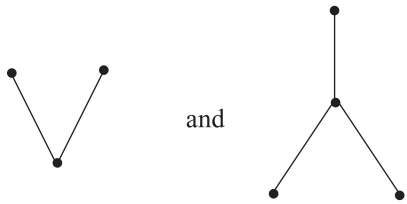


Figure 2.50

Theorem 2.3. If there is a path from vertex v_1 to v_2 in a graph with n vertices, then there does not exist a simple path of more than $n-1$ edges from vertex v_1 to v_2.

Proof: Suppose there is a path from v_1 to v_2. Let $v_1, \ldots, v_i, \ldots, v_2$ be the sequence of vertices which the path meets between the vertices v_1 and v_2. Let there be m edges in the simple path. Then there will be $m+1$ vertices in the sequence. Therefore if $m > n-1$, then there will be more than n vertices in the sequence. But the graph is with n vertices. Therefore some vertex, say v_k, appears more than once in the sequence. So the sequence of vertices shall be $v_1, \ldots, v_i, \ldots, v_k, \ldots, v_k, \ldots, v_2$. Deleting the edges in the path that lead v_k back to v_k we have a path from v_1 to v_2 that has fewer edges than the original one. This argument is repeated until we get a path that has $n-1$ or fewer edges.

Definition 2.44. Two vertices v_1 and v_2 of a graph G are said to be **connected** if and only if there is a walk from v_1 to v_2.

Definition 2.45. A graph G is said to be **connected** if and only if given any two vertices v_1 and v_2 in G, there is a walk from v_1 to v_2.

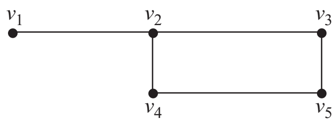
Thus, a graph G is connected if there exists a walk between every two vertices in the graph.

Definition 2.46. A graph which is not connected is called **disconnected graph**.

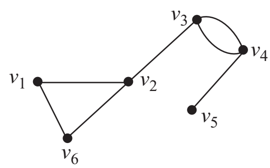
EXAMPLE 2.15

Which of the graphs shown below are connected?

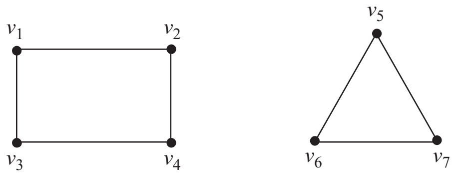
(a)



(b)



(c)



(d)

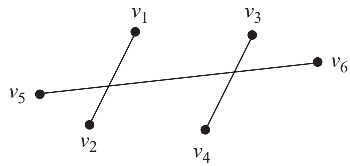


Figure 2.51

Solution. In Figure 2.51, we observe that

(a) Since every two vertices in the given graph are connected by a path, therefore the given graph is connected.
(b) This graph is also connected.
(c) The graph in (c) is disconnected. It has two connected components.
(d) In this graph, the edge (v_5, v_6) cross the edges (v_1, v_2) and (v_3, v_4) at points which are not vertices. Therefore the graph can be redrawn as shown in Figure 2.52.

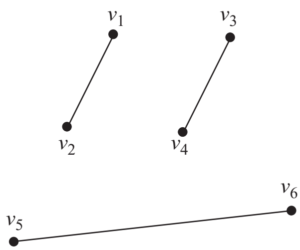


Figure 2.52

There is no path from v_2 to v_4, etc. Hence the given graph is disconnected and has three connected components.

EXAMPLE 2.16

Which of the graphs shown in Figure 2.53 are connected?

(a) (b)

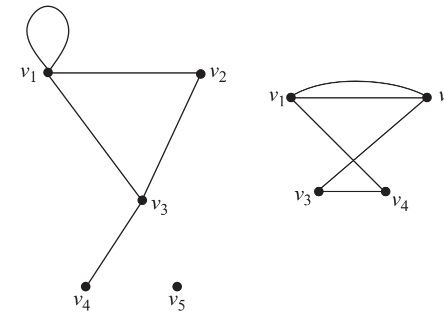


Figure 2.53

Solution. Graph (a) is not connected as there is no walk from any of v_1, v_2, v_3, v_4 to the vertex v_5.

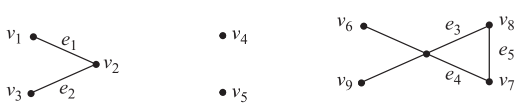
The graph (b) is clearly connected.

Definition 2.47. If a graph G is disconnected, then the various connected pieces of G are called the **connected components of the graph**.

EXAMPLE 2.17

Find the connected components of the graphs given below (Figure 2.54).

(a)



(b)

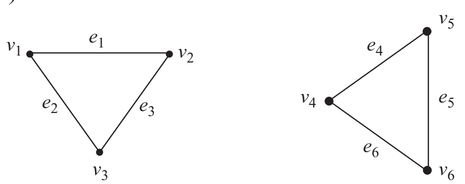


Figure 2.54

Solution

(a) The graph in (a) has four connected components (Figures 2.55 and 2.56)

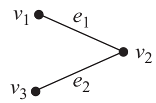


Figure 2.55

H_1: with vertex set $\{v_1, v_2, v_3\}$ and edge set $\{e_1, e_2\}$
H_2: with vertex set $\{v_4\}$ and edge set ϕ
H_3: with vertex set $\{v_5\}$ and edge set ϕ
H_4: with vertex set $\{v_6, v_7, v_8, v_9\}$ and edge set $\{e_3, e_4, e_5\}$

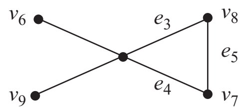


Figure 2.56

(b) The graph in (b) is disconnected and have two connected components (Figures 2.57(a), 2.57(b))

H_1:

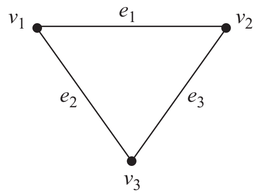


Figure 2.57(a)

with vertex set $\{v_1, v_2, v_3\}$ and edge set $\{e_1, e_2, e_3\}$

H_2:

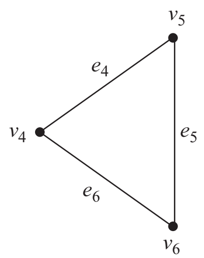


Figure 2.57(b)

with vertex set $\{v_4, v_5, v_6\}$ and edge set $\{e_4, e_5, e_6\}$.

EXAMPLE 2.18

Find the number of connected components in the graph shown below (Figure 2.58).

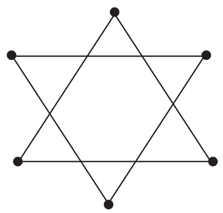


Figure 2.58

Solution. The connected components are shown in Figure 2.59.

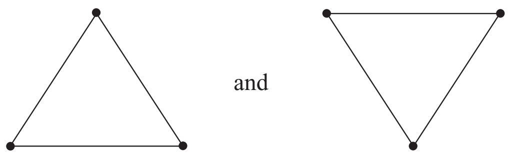


Figure 2.59

Remark 2.3. If a connected component has n vertices, then degree of any vertex cannot exceed $n-1$.

2.6 EULERIAN PATHS AND CIRCUITS

Definition 2.48. A path in a graph G is called an **Euler path** if it includes **every edge exactly once**.

Definition 2.49. A circuit in a graph G is called an **Euler circuit** if it includes every edge exactly once. Thus, an Euler circuit (Eulerian trail) for a graph G is a sequence of adjacent vertices and edges in G that starts and ends at the same vertex, uses **every vertex of G at least once**, and uses **every edge of G exactly once**.

Definition 2.50. A graph is called **Eulerian graph** if there exists an Euler circuit for that graph.

EXAMPLE 2.19

Find which of the following are Euler paths and Euler circuits in the graph given below (Figure 2.60)

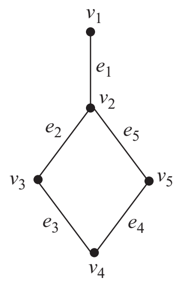


Figure 2.60

(a) $v_1\, e_1\, v_2\, e_2\, v_3\, e_3\, v_4\, e_4\, v_5\, e_5\, v_2$
(b) $v_1\, e_1\, v_2\, e_2\, v_3\, e_3\, v_4\, e_4\, v_5\, e_5\, v_2\, e_1\, v_1$
(c) $v_2\, e_2\, v_3\, e_3\, v_4\, e_4\, v_5\, e_5\, v_2$

Solution

(a) The walk in (a) is an Euler path but it is not Euler circuit because it is not closed.
(b) It is not an Euler circuit because the edge e_1 is covered twice.
(c) It is neither an Euler path nor an Euler circuit because the vertex v_1 of G has not been used.

Theorem 2.4. If a graph has an Euler circuit, then every vertex of the graph has even degree.

Proof: Let G be a graph which has an Euler circuit. Let v be a vertex of G. We shall show that degree of v is even. By definition, Euler circuit contains every edge of graph G. Therefore the Euler circuit contains all edges incident on v. We start a journey beginning in the middle of one of the edges adjacent to the start of Euler circuit and continue around the Euler circuit to end in the middle of the starting edge. Since Euler circuit uses every edge exactly once, the edges incident on v occur in entry/exist pair and hence the degree of v is a multiple of 2. Therefore, the degree of v is even. This completes the proof of the theorem.

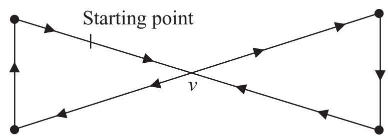


Figure 2.61

We know that ***contrapositive of a conditional statement is logically equivalent to the statement***. Thus, Theorem 2.4 is equivalent to the following:

Theorem 2.5. If a vertex of a graph is not of even degree, then it does not have an Euler circuit.

Thus,
If some vertex of a graph has odd degree, then that graph does not have an Euler circuit.

EXAMPLE 2.20

Show that the graphs shown in Figure 2.62 do not have Euler circuits.

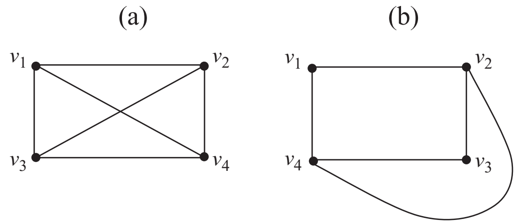


Figure 2.62

Solution. In graph (a), degree of each vertex is 3. Hence this **does not** have an Euler circuit.

In graph (b), we have

$$\deg(v_2)=3 \quad \text{and} \quad \deg(v_4)=3.$$

Since there are vertices of odd degree in the given graph, therefore it **does not** have an Euler circuit.

Remark 2.4. The converse of Theorem 2.4 is not true. There exist graphs in which every vertex has even degree but the Euler circuits do not exist.

For example,

and

Figure 2.63

are graphs (Figure 2.63) in which each vertex has degree 2 but these graphs do not have Euler circuits since there is no path which uses each vertex at least once.

Theorem 2.6. If G is a connected graph and every vertex of G has even degree, then G has an Euler circuit.

Proof: Let every vertex of a connected graph G has even degree. If G consists of a single vertex v, the trivial walk from v to v is an Euler circuit. So suppose that G consists of more than one vertices. We start from any vertex v of G. Since the degree of each vertex of G is even, if we reach each vertex other than v by travelling on one edge, the same vertex can be reached by travelling on another previously unused edge. Thus a sequence of distinct adjacent edges can be produced indefinitely as long as v is not reached. Since the number of edges of the graph is finite (by definition of graph), the sequence of distinct edges will terminate. Thus the sequence must return to the starting vertex. We thus obtain a sequence of adjacent vertices and edges starting and ending at v without repeating any edge. Thus we get a circuit C.

If C contains every edge and vertex of G, then C is an Euler circuit.

If C does not contain every edge and vertex of G, remove all edges of C from G and also any vertices that become isolated when the edges of C are removed. Let the resulting subgraph be G'. We note that when we removed edges of C, an even number of edges from each vertex have been removed. Thus degree of each remaining vertex remains even.

Further, since G is connected, there must be at least one vertex common to both C and G'. Let it be w (in fact there are two such vertices). Pick any sequence of adjacent vertices and edges of G' starting and ending at w without repeating an edge. Let the resulting circuit be C'.

Join C and C' together to create a new circuit C''. Now, we observe that if we start from v and follow C all the way to reach w and then follow C' all the way to reach back to w. Then continuing travelling along the untravelled edges of C, we reach v.

If C'' contains every edge and vertex of C, then C'' is an Euler circuit. If not, then we again repeat our process. Since the graph is finite, the process must terminate.

The process followed has been described in the graph G shown below (Figure 2.64).

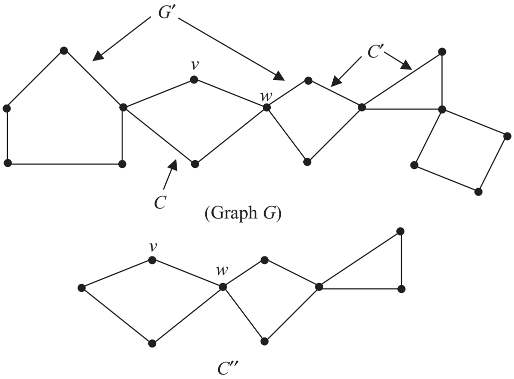


Figure 2.64

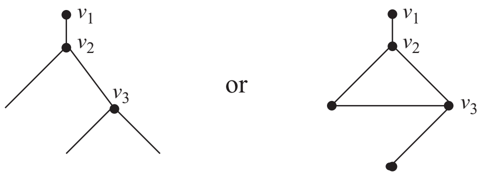
Theorems 2.4 and 2.6 taken together imply:

Theorem 2.7 (Euler's Theorem). A finite connected graph G has an Euler circuit if and only if every vertex of of G has even degree.

Thus, finite connected graph is Eulerian if and only if each vertex has even degree.

Theorem 2.8. If a graph G has more than two vertices of odd degree, then there can be no Euler path in G.

Proof: Let v_1, v_2 and v_3 be vertices of odd degree. Since each of these vertices had odd degree, any possible Euler path must leave (arrive at) each of v_1, v_2, v_3 with no way to return (or leave). One vertex of these three vertices may be the beginning of Euler path and another the end, but this leaves the third vertex at one end of an untravelled edge. Thus, there is no Euler path.



(Graphs having more than two vertices of odd degree).

Figure 2.65

Theorem 2.9. If G is a connected graph and has exactly two vertices of odd degree, then there is an Euler path in G. Further, any Euler path in G must begin at one vertex of odd degree and end at the other.

Proof: Let u and v be two vertices of odd degree in the given connected graph G (see Figure 2.66)

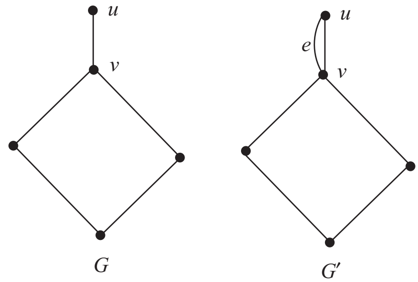


Figure 2.66

If we add the edge e to G, we get a connected graph G' all of whose vertices have even degree. Hence there will be an Euler circuit in G'. If we omit e from Euler circuit, we get an Euler path beginning at u (or v) and ending at v (or u).

EXAMPLE 2.21

Has the graph given in Figure 2.67 an Eulerian path?

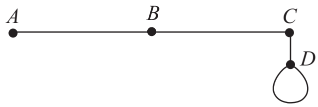


Figure 2.67

Solution. In the given graph,

$$\deg(A)=1,\quad \deg(B)=2,$$
$$\deg(C)=2,\quad \deg(D)=3.$$

Thus the given connected graph has exactly two vertices of odd degree. Hence, it has an Eulerian path.

If it starts from A (vertex of odd degree), then it ends at D (vertex of odd degree). If it starts from D (vertex of odd degree), then it ends at A (vertex of odd degree).

But, on the other hand, let the graph be as shown below (Figure 2.68)

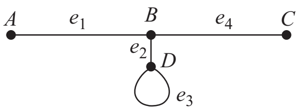


Figure 2.68

Then deg(A)=1, deg(B)=3, deg(C)=1, degree of D=3 and so we have four vertices of odd degree. Hence it does not have an Euler path.

EXAMPLE 2.22

Does the graph given in Figure 2.69 possess an Euler circuit?

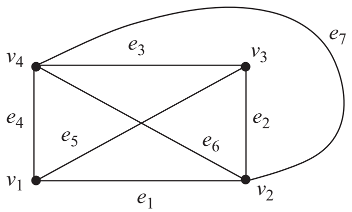


Figure 2.69

Solution. The given graph is connected. Further

$$\deg(v_1)=3, \quad \deg(v_2)=4,$$
$$\deg(v_3)=3, \quad \deg(v_4)=4.$$

Since this connected graph has vertices with odd degree, it cannot have Euler circuit. But this graph has Euler path, since it has exactly two vertices of odd degree. For example, $v_3\, e_2\, v_2\, e_7\, v_4\, e_6\, v_2\, e_1\, v_1\, e_4\, v_4\, e_3\, v_3\, e_5\, v_1$ is an Euler path.

EXAMPLE 2.23

Consider the graph given below (Figure 2.70):

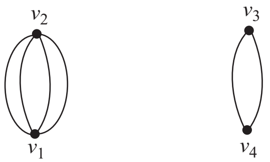


Figure 2.70

Here, deg(v_1)=4, deg(v_2)=4, deg(v_3)=2, deg(v_4)=2. Thus degree of each vertex is even. But the graph is not Eulerian since it is **not connected**.

EXAMPLE 2.24

Is it possible to trace the graph in Figure 2.71 without lifting the pencil?

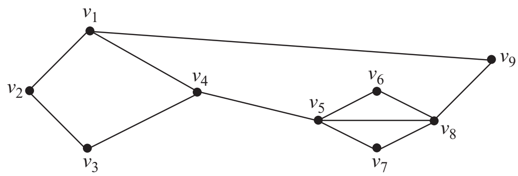


Figure 2.71

Solution. The problem is equivalent to say that “Is it possible for this graph to have an Eulerian circuit”? We observe that

$$\deg(v_1)=3,\ \deg(v_2)=2,\ \deg(v_3)=2,\ \deg(v_4)=3,$$
$$\deg(v_5)=4,$$
$$\deg(v_6)=2,\ \deg(v_7)=2,\ \deg(v_8)=4,\ \deg(v_9)=2.$$

Since the graph contains vertex of odd degree, therefore it cannot have Euler’s circuit and therefore can not be traced without lifting the pencil.

EXAMPLE 2.25

Find the Euler’s circuit for the graph given below (Figure 2.72):

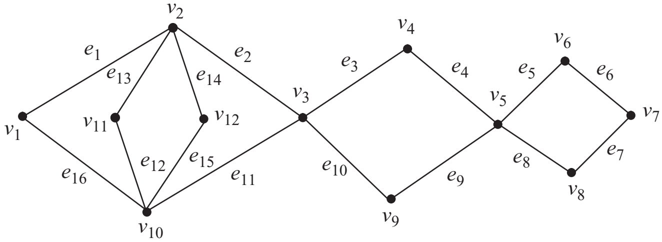


Figure 2.72

Solution. The given connected graph has 12 vertices and degree of each vertex is even. Hence, by Euler's theorem, this has an Euler's circuit. For example, we observe that $v_1\ e_1\ v_2\ e_{13}\ v_{11}\ e_{12}\ v_{10}\ e_{15}\ v_{12}\ e_{14}\ v_2\ e_2\ v_3\ e_3\ v_4\ e_4\ v_5\ e_5\ v_6\ e_6\ v_7\ e_7\ v_8\ e_8\ e_9\ v_9\ e_{10}\ v_3\ e_{11}\ v_{10}\ e_{16}\ v_1$ is an Euler's circuit. In short we can represent it by e_1, e_{13}, e_{12}, e_{15}, e_{14}, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{16}.

EXAMPLE 2.26 (The bridges of Konigsberg)

The graph theory began in 1736 when Leonhard Euler solved the problem of seven bridges on Pregel River in the town of Konigsberg in Prussia (now Kaliningrad in Russia). The two islands and seven bridges are shown in Figure 2.73.

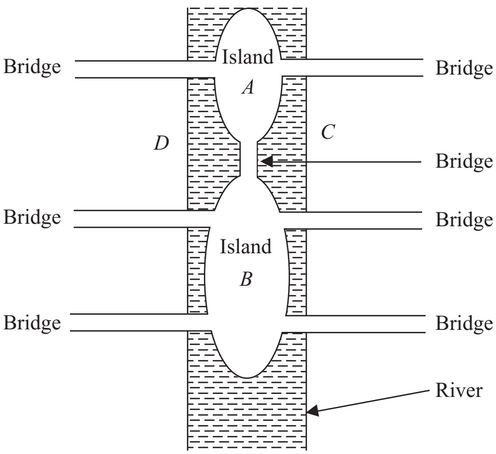


Figure 2.73

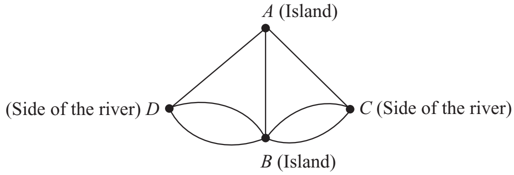
The people of Konigsberg posed the following question to famous Swiss Mathematician Leonhard Euler:

"Beginning anywhere and ending any where, can a person walk through the town of Konigsberg crossing all the seven bridges exactly once?"

Euler showed that such a walk is impossible. He replaced the islands A, B and the two sides (banks) C and D of the river by vertices and the bridges as edges of a graph. We note then that

$$\deg(A)=3,\quad \deg(B)=5,$$
$$\deg(C)=3,\quad \deg(D)=3.$$

Thus the graph of the problem is as shown in Figure 2.74.



(Euler's graphical representation of seven bridges problem)

Figure 2.74

The problem then reduces to

"Is there any Euler's path in the above diagram?".

To find the answer, we note that there are more than two vertices having odd degree. Hence there exists no Euler path for this graph.

EXAMPLE 2.27

The floor plan shown in Figure 2.75 is for a house that is open for public viewing. Each room is connected to every room with which it has a common wall and to the outside along each wall. Is it possible to begin in a room or outside and take a walk which goes through each door exactly once?

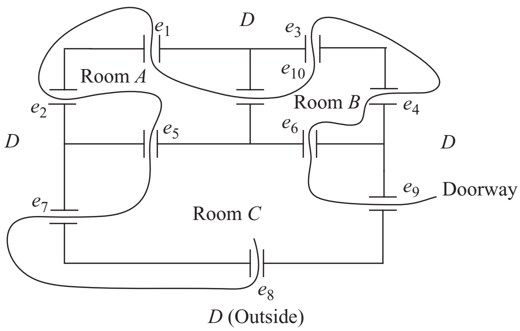


Figure 2.75

If each room and out side constitute a vertex and each door corresponds to an edge, then the floor plan converts into the graph. There are two edges e_1 and e_2 from A to D, two edges e_3 and e_4 from B to D, three edges e_7, e_8, e_9 from C to D, one edge e_{10} from A to B, one edge e_5 from A to C, one edge e_6 from B to C. Hence the graph is as shown below (Figure 2.76):

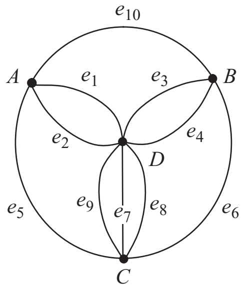


Figure 2.76

We note that

$$\deg(A) = \deg(B) = 4, \quad \deg(C) = 5, \quad \deg(D) = 7.$$

Since the graph is connected and degree of exactly two vertices C and D is odd, there exists an Euler's path and that path should start from one vertex of odd degree and end to the other vertex of odd degree. Therefore, either the path will begin from C and end at D or it will begin from D and end at C. An Euler's path is shown (which begins from C).

Definition 2.51. An edge in a connected graph is called a **bridge** or a **cut edge** if deleting that edge creates a disconnected graph.

For example, consider the graph shown below (Figure 2.77):

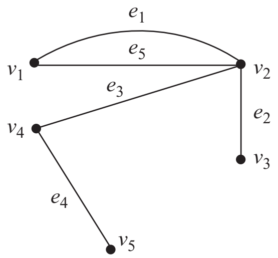


Figure 2.77

In this graph, if we remove the edge e_3, then the graph breaks into two connected components given below (Figure 2.78):

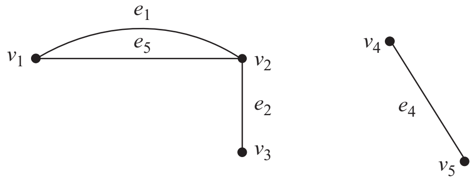


Figure 2.78

Hence the edge e_3 is a bridge in the given graph.

2.6.1 Methods for Finding Euler Circuit

Method 1. We know that if every vertex of a non-empty connected graph has even degree, then the graph has an Euler circuit. We shall make use of this result to find an Euler path in a given graph.

Consider the graph shown in Figure 2.79.

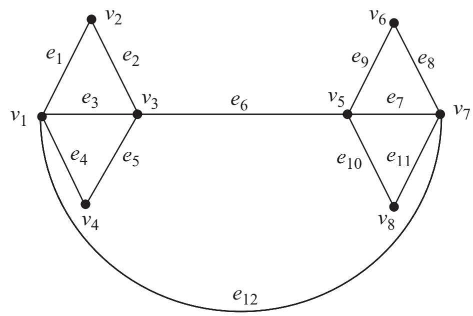


Figure 2.79

We note that

$$\deg(v_2) = \deg(v_4) = \deg(v_6) = \deg(v_8) = 2,$$
$$\deg(v_1) = \deg(v_3) = \deg(v_5) = \deg(v_7) = 4.$$

Hence all vertices have even degree. Also the given graph is connected. Hence the given graph has an Euler circuit. We start from the vertex v_1 and let C be

$$C: v_1\ v_2\ v_3\ v_1.$$

Then C is not an Euler circuit for the given graph but C intersects the rest of the graph at v_1 and v_3.

Let C' be

$$C': v_1 v_4\ v_3\ v_5\ v_7\ v_6\ v_5\ v_8\ v_7\ v_1.$$

(In case we start from v_3, then C' will be $v_3\ v_4\ v_1\ v_7\ v_6\ v_5\ v_7\ v_8\ v_5$).

Path C' into C and obtain

$$C'': v_1 v_2\, v_3\, v_1\, v_4\, v_3\, v_5\, v_7\, v_6\, v_5\, v_8\, v_7\, v_1,$$

that is,

$$C'': e_1 e_2\, e_3\, e_4\, e_5\, e_6\, e_7\, e_8\, e_9\, e_{10}\, e_{11}\, e_{12}.$$

(If we had started from v_2, then C'': $v_1 v_2\, v_3\, v_4\, v_1\, v_7\, v_6\, v_5\, v_7\, v_8\, v_5\, v_3\, v_1$ **or** $e_1 e_2\, e_5\, e_4\, e_{12}\, e_8\, e_9\, e_7\, e_{11}\, e_{10}\, e_6\, e_3$). In C'' all edges are covered exactly once. Also every vertex has been covered at least once. Hence C'' is an Euler circuit.

Method 2: Fleury's Algorithm: This algorithm is used to find Euler's circuit for a connected graph with no vertices of odd degree.

We shall illustrate the method with the help of the following example:

Let G be the connected graph as shown in the Figure 2.80.

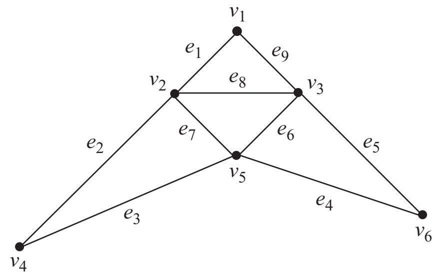


Figure 2.80

Step 1. We begin from any vertex, say v_1

Step 2. We then construct the following table:

Current Path	*Next Edge*	*Reason*
v_1	$\{v_1, v_2\}$	No edge from v_1 is a bridge, so choose any edge, say $\{v_1, v_2\}$
$v_1\, v_2$	$\{v_2, v_4\}$	(v_2, v_5) a bridge, so choose $\{v_2, v_4\}$
$v_1\, v_2\, v_4$	$\{v_4, v_5\}$	only one edge $\{v_4, v_5\}$from v_4 remains
$v_1\, v_2\, v_4\, v_5$	$\{v_5, v_6\}$	Since $\{v_2, v_5\}$ and $\{v_5, v_3\}$ are bridges, choose $\{v_5, v_6\}$
$v_1\, v_2\, v_4\, v_5\, v_6$	$\{v_6, v_3\}$	only one edge $\{v_6, v_3\}$ remains
$v_1\, v_2\, v_4\, v_5\, v_6\, v_3$	$\{v_3, v_5\}$	Since $\{v_1, v_3\}$ is a bridge, choose either $\{v_3, v_2\}$or $\{v_3, v_5\}$
$v_1\, v_2\, v_4\, v_5\, v_6\, v_3\, v_5$	$\{v_5, v_2\}$	only one edge $\{v_5, v_2\}$ remains
$v_1\, v_2\, v_4\, v_5\, v_6\, v_3\, v_5 v_2$	$\{v_2, v_3\}$	only one edge $\{v_2, v_3\}$ remains
$v_1\, v_2\, v_4\, v_5\, v_6\, v_3\, v_5\, v_2\, v_3$	$\{v_3, v_1\}$	only one edge $\{v_3, v_1\}$ remains

Hence one possible Euler circuit is

$$C: v_1\, v_2\, v_4\, v_5\, v_6\, v_3\, v_5\, v_2\, v_3\, v_1.$$

In term of edges, this Euler circuit can be expressed as

$$e_1\, e_2\, e_3\, e_4\, e_5\, e_6\, e_7\, e_8\, e_9.$$

2.7 HAMILTONIAN CIRCUITS

Definition 2.52. A **Hamiltonian path** for a graph G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once.

Definition 2.53. A **Hamiltonian circuit** for a graph G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last which are the same.

Definition 2.54. A graph is called **Hamiltonian** if it admits a Hamiltonian circuit.

EXAMPLE 2.28

The wooden graph shown in Figure 2.81 and constructed by William Hamilton in the shape of a regular dodecahedron is a Hamiltonian circuit.

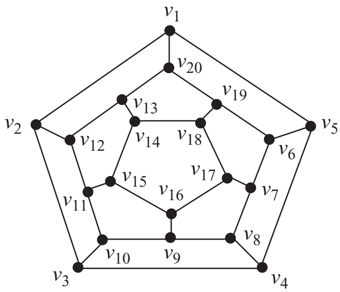


Figure 2.81

The Hamilton circuit is $v_1\, v_2\, v_3 \ldots v_{18}\, v_{19}\, v_{20}\, v_1$.

EXAMPLE 2.29

A complete graph K_n has a Hamiltonian circuit. In particular, the graphs shown in Figure 2.82 are Hamiltonian.

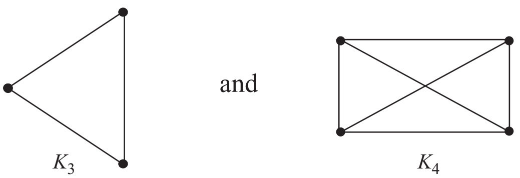


Figure 2.82

EXAMPLE 2.30

The graph shown below does not have a Hamiltonian circuit.

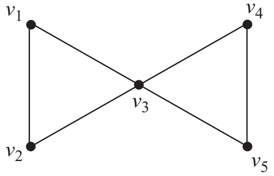


Figure 2.83

EXAMPLE 2.31

The graph shown in Figure 2.84 does not have a Hamiltonian circuit

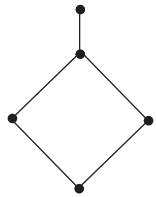


Figure 2.84

Remark 2.5. It is clear that **only connected graphs can have Hamiltonian circuit**. However, there is no simple criterion to tell us whether or not a given graph has Hamiltonian circuit. The following results give us some sufficient conditions for the existence of Hamiltonian circuit/path.

Theorem 2.10. Let G be a linear graph of n vertices. If the sum of the degrees for each pair of vertices in G is greater than or equal to $n-1$, then there exists a Hamiltonian path in G.

Theorem 2.11. Let G be a connected graph with n vertices. If $n \geq 3$ and $\deg(v) \geq n$ for each vertex v in G, then G has a Hamiltonian circuit.

Theorem 2.12. Let G be a connected graph with n vertices and let u and v be two vertices of G that are not adjacent. If $\deg(u)+\deg(v) \geq n$, then G has a Hamiltonian circuit.

Corollary 2.2. Let G be a connected graph with n vertices. If each vertex has degree greater than or equal to $n/2$, then G has a Hamiltonian circuit.

Proof: It is given that degree of each vertex is greater than or equal to $n/2$. Hence the sum of the degree of any two vertices is greater than or equal to $n/2+n/2=n$. So, by the above theorem, the graph G has a Hamiltonian circuit.

Theorem 2.13. Let n be the number of vertices and m be the number of edges in a connected graph G. If $m \geq \frac{1}{2}(n^2-3n+6)$, then G has a Hamiltonian circuit.

The following example shows that the above conditions are not necessary for the existence of Hamiltonian path.

EXAMPLE 2.32

Let G be the connected graph shown in the Figure 2.85.

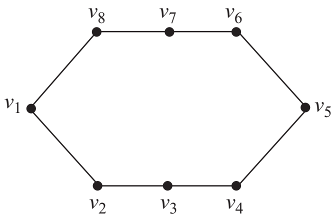


Figure 2.85

We note that

$$n = \text{Number of vertices in } G = 8,$$
$$m = \text{Number of edges in } G = 8,$$
$$\text{Degree of each vertex} = 2.$$

Thus, if u and v are non-adjacent vertices, then

$$\deg(u) + \deg(v) = 2 + 2 = 4 \ngeq 8.$$

Also,

$$\frac{1}{2}(n^2 - 3n + 6) = \frac{1}{2}(64 - 24 + 6) = 23.$$

Clearly $m \ngeq \frac{1}{2}(n^2 - 3n + 6)$. Therefore the above two theorems fail. But the given graph has Hamiltonian circuit. For example, $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_1$ is an Hamiltonian circuit for the graph.

Proposition 2.1. Let G be a graph with at least two vertices. If G has a Hamiltonian circuit, then G has a subgraph H with the following properties:

1. H contains every vertex of G
2. H is connected
3. H has the same number of edges as vertices
4. Every vertex of H has degree 2

The contrapositive of this proposition is

"If a graph G with at least two vertices does not have a subgraph H satisfying (1) – (4), then G does not have a Hamiltonian circuit".

Also we know that contrapositive of a statement is logically equivalent to the statement. Therefore the above result can be used to show non-existence of a Hamiltonian circuit.

EXAMPLE 2.33

Consider the graph G shown below (Figure 2.86):

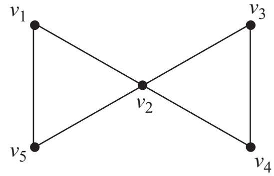


Figure 2.86

Let us verify whether this graph has a Hamiltonian circuit.

Suppose that G has a Hamiltonian circuit. Then it has a subgraph H such that

1. H contains every vertex of G, i.e., H has 5 vertices v_1, v_2, v_3, v_4, v_5
2. H is connected
3. H has 5 edges
4. Every vertex of H has degree 2

Since the degree of v_2 in G is 4 and every vertex of H has degree 2, two edges incident on v_2 must be removed from G to create H. But we note that the edge $\{v_1, v_2\}$ cannot be removed, because the removal of $\{v_1, v_2\}$ will decrease the degree of v_1 to 1, which is less than 2. Similarly the edge $\{v_2, v_3\}$, $\{v_2, v_4\}$ and $\{v_2, v_5\}$ cannot be removed. Hence the degree of v_2 in H must stand at 4 which contradicts the condition that every vertex in H has degree 2 in H. Therefore, no such subgraph H exists. Hence G does not have a Hamiltonian circuit.

EXAMPLE 2.34

Does the graph G given below (Figure 2.87) have Hamiltonian circuit?

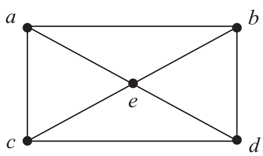


Figure 2.87

Solution. The given graph has

$$n = \text{Number of vertices} = 5$$
$$m = \text{Number of edges} = 8$$

$\deg(a) = \deg(b) = \deg(c) = \deg(d) = 3$
$\deg(e) = 4$

We observe that

(i) Degree of each vertex is greater than $n/2$
(ii) The sum of degrees of any non-adjacent pair of vertices is greater than n
(iii) $\frac{1}{2}(n^2 - 3n + 6) = \frac{1}{2}(25 - 15 + 6) = 8.$

Thus the condition $m \geq \frac{1}{2}(n^2 - 3n + 6)$ is satisfied.

(iv) The sum of degrees of each pair of vertices in the given graph is greater than $n - 1 = 5 - 1 = 4$.

Thus four sufficiency conditions are satisfied (whereas one condition out of these four conditions is sufficient for the existence of Hamiltonian path/circuit). Hence the graph has a Hamiltonian circuit.

For example, the following circuits in G are Hamiltonian (Figure 2.88).

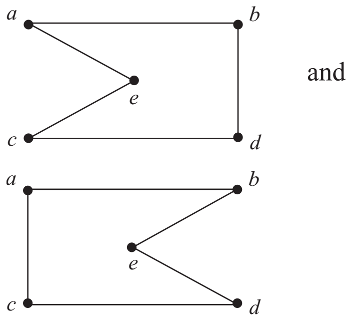


Figure 2.88

EXAMPLE 2.35

Does the graph shown below (Figure 2.89) has Hamiltonian circuit?

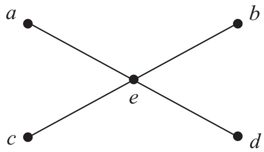


Figure 2.89

Solution.

Here

Number of vertices $(n) = 5$
Number of edges $(m) = 4$
$\deg(a) = \deg(b) = \deg(c) = \deg(d) = 1$
$\deg(e) = 4$

We note that

(i) $\deg(a) = \deg(b) = \deg(c) = \deg(d) \ngeq \frac{5}{2}$
(ii) $\deg(a) + \deg(b) = 2 \ngeq 5$, that is sum of any non-adjacent pair of vertices is not greater than 5
(iii) $\frac{1}{2}(n^2 - 3n + 6) = \frac{1}{2}(25 - 15 + 6) = 8$

Therefore the condition $m \geq \frac{1}{2}(n^2 - 3n + 6)$ is **not satisfied**,

(iv) $\deg(a) + \deg(b) = 2 \ngeq 4$, i.e., the condition that sum of degrees of each pair of vertices in the graph is not greater than or equal to $n - 1$.

Hence no sufficiency condition is satisfied. So we try the Proposition 2.1. Suppose that G has a Hamiltonian circuit. Then G should have a subgraph which contains every vertex of G, and number of vertices and number of edges in H should be same. Thus, H should have five vertices a, b, c, d, e and five edges. Since G has only four edges, H cannot have more than four edges. Hence no such subgraph is possible. Hence, the given graph does not have Hamiltonian circuit.

EXAMPLE 2.36

Does the graph shown below (Figure 2.90) possess a Hamiltonian circuit?

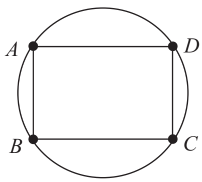


Figure 2.90

Solution. In the given graph

Number of vertices $(n) = 4$
Number of edges $(m) = 8$
Degree of each vertex $= 4$

Thus we see that

(i) Degree of each vertex is greater than $n/2$
(ii) Sum of degree of each pair of vertices is greater than $n-1$,

(iii) $\frac{1}{2}(n^2-3n+6)=\frac{1}{2}(16-12+6)=5$ and so $m \geq \frac{1}{2}(n^2-3n+6)$.

Hence the given graph has a Hamiltonian path. For example, *ABCDA* is a Hamiltonian path in the given graph (see Figure 2.91).

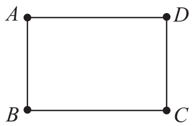


Figure 2.91

EXAMPLE 2.37

Is the graph shown in Figure 2.92 a Hamiltonian?

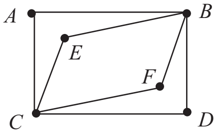


Figure 2.92

Solution. We note that

$$\deg(A)=\deg(D)=\deg(E)=\deg(F)=2,$$
$$\deg(B)=\deg(C)=4.$$

Further,

Number of vertices $(n)=6$
Number of edges $(m)=8$.

So,

$$\frac{1}{2}(n^2-3n+6)=\frac{1}{2}(36-18+6)=12.$$

Thus the conditions

(i) Degree of each vertex is greater than or equal to $n/2$ is not satisfied
(ii) $m \geq \frac{1}{2}(n^2-3n+6)$ is not satisfied
(iii) $\deg(A)+\deg(E)=2+2=4$ and so the condition that "the sum of degrees of non-adjacent vertices is greater than or equal to n" is not satisfied
(iv) "The sum of degrees of any pair of vertices is greater than or equal to $n-1$" is not satisfied.

So suppose that G has a Hamiltonian circuit. Then it should have a connected subgraph H containing six vertices, six edges and degree of each vertex should be 2. To have degree of each vertex equal to 2, we should remove two edges from C and two edges from B.

For example, now, if we remove CE from C, then $\deg(E)\neq 2$, If we remove CF, then $\deg(F)\neq 2$. So, we cannot remove CE and CF. If we remove CD, then $\deg(D)\neq 2$. If we remove CA, then $\deg(A)\neq 2$. Hence no such subgraph H exists. So, G cannot have a Hamiltonian circuit.

Remark 2.6. Since the degree of each vertex in the above graph is even, it has Eulerian circuit. Thus the graph in EXAMPLE 2.37 is Eulerian but not Hamiltonian.

EXAMPLE 2.38

Is the graph given in Figure 2.93 Hamiltonian?

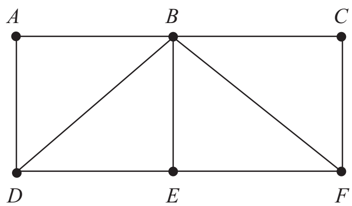


Figure 2.93

Solution. In this graph,

Number of vertices $(n)=6$
Number of edges $(m)=9$.

So,

$$\frac{1}{2}(n^2-3n+6) = \frac{1}{2}(36-18+6)=12,$$
$$\deg(A)=\deg(C)=2,$$
$$\deg(D)=\deg(F)=3=\deg(E),$$
$$\deg(B)=5.$$

Thus, no sufficient condition is satisfied in this case. **But, the graph has Hamiltonian circuit** *ADEFCBA* shown in the Figure 2.94.

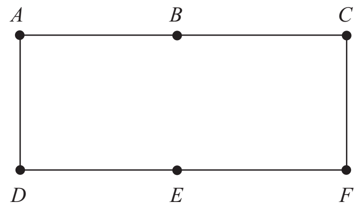


Figure 2.94

EXAMPLE 2.39

Show that the graph G shown in Figure 2.95 is not Hamiltonian.

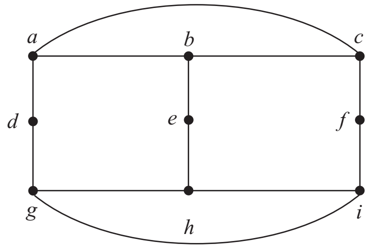


Figure 2.95

Solution. If G is Hamiltonian, then it has a subgraph H that

1. Contains every vertex of G
2. Is connected
3. Has the same number of edges as vertices
4. Is such that every vertex has degree 2

Thus, if such a graph exists, then it will have vertices $(a, b, c, d, e, f, g, h, i)$, will be connected, will have six edges and the degree of each vertex shall be 2. We note that $\deg(d)=\deg(e)=\deg(f)=2$. The degree of the vertex h in G is 3, one edge incident on h must be deleted from G to create H. The edge $\{h, e\}$ is required since otherwise $\deg(e)\neq 2$. So we have to delete either $\{g, h\}$ or $\{h, i\}$. If we delete $\{h, i\}$ and retain $\{g, h\}$, then $\{g, i\}$ has to be deleted to keep g of degree 2. In such a case, there is only one edge $\{i, f\}$ on i and so there is a contradiction to (4). So we cannot remove $\{h, i\}$. Similarly, we see that $\{g, h\}$ cannot be deleted.

Hence no such subgraph H exists and so G does not have a Hamiltonian circuit.

Definition 2.55. A **weighted graph** is a graph for which each edge or each vertex or both is (are) labelled with a numerical value, called its **weight**.

For example, if vertices in a graph denote recreational sites of a town and weights of edges denote the distances in kilometers between the sites, then the graph shown in Figure 2.96 is a weighted graph.

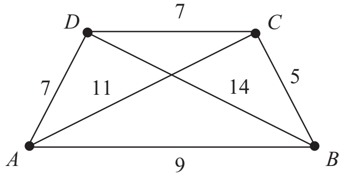


Figure 2.96

Definition 2.56. The **weight of an edge** (v_i, v_j) is called **distance between the vertices** v_i **and** v_j.

Definition 2.57. A vertex u is a **nearest neighbour** of vertex v in a graph if u and v are adjacent and no other vertex is joined to v by an edge of lesser weight than (u, v).

For example, in the above example, B is the nearest neighbour of C, whereas A and C are both nearest neighbours of the vertex D. **Thus, nearest neighbour of a set of vertices is not unique.**

Definition 2.58. A vertex u is a nearest neighbour of a set of vertices $\{v_1, v_2, \ldots, v_n\}$ in a graph if u is adjacent to some member v_i of the set and no other vertex adjacent to member of the set is joined by an edge of lesser weight than (u, v_i).

In the above example, if we have set of vertices as $\{B, D\}$, C is the neatest neighbour of $\{B, D\}$ because the edge (C, B) has weight 5 and no other vertex adjacent to $\{B, D\}$ is linked by an edge of lesser weight than (C, B).

Definition 2.59. The **length of a path** in a graph is the sum of lengths of edges in the path.

Definition 2.60. Let $G=(V, E)$ be a graph and let l_{ij} denote the length of edge (v_i, v_j) in G. Then a **shortest path** from v_i to v_k is a path such that the sum $l_{12}+l_{23}+\cdots+l_{k-1,k}$ of lengths of its edges is **minimum**, that is, total edge weight is minimum.

2.7.1 Travelling Salesperson Problem

This problem requires the determination of a **shortest Hamiltonian circuit** in a given graph of cities and lines of transportation to minimize the total fare for a travelling person who wants to make a tour of n cities visiting each city exactly once before returning home.

The weighted graph model for this problem consists of vertices representing cities and edges with weight as distances (fares) between the cities. The salesman starts and ends his journey at the same city and visits each of $n-1$ cities once and only once. We want to find minimum total distance.

We discuss the case of five cities and so consider the weighted graph shown in the Figure 2.97.

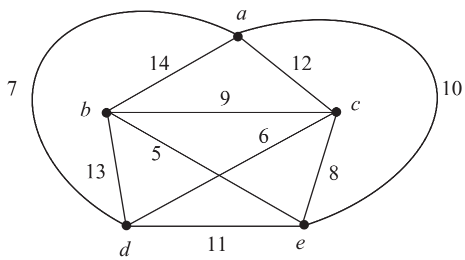


Figure 2.97

We shall use **nearest neighbour** algorithm to solve the problem:

Algorithm: Nearest neighbour (closest insertion)
Input: A weighted complete graph G.
Output: A sequence of labelled vertices that forms a Hamiltonian cycle.

Start at any vertex v.
Initialize $l(v)=0$.
Initialize $i=0$.

While there are unlabelled vertices

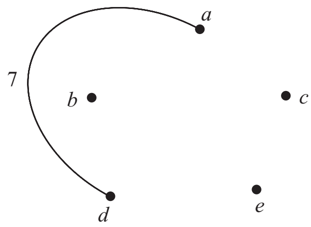
$$i:=i+1.$$

Traverse the cheapest edge that join v to an unlabelled vertex, say w

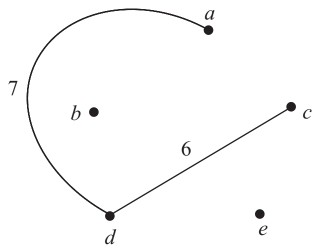
$$\text{Set } l(w)=i.$$
$$v:=w.$$

For the present example,

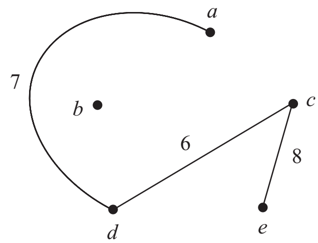
(i) Let us choose a as the starting vertex. Then d is the nearest vertex and then (a, d) is the corresponding edge. Thus we have the figure



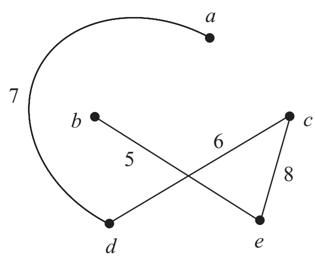
(ii) From d, the nearest vertex is c, so we have a path shown below:



(iii) From c, the nearest vertex is e. So we have the path as shown below:



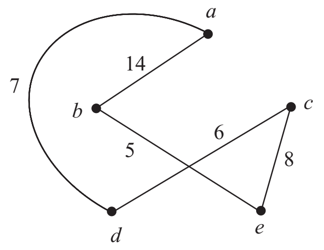
(iv) From e, the nearest vertex is b and so we have the path



(v) Now, from b, the only vertex to be covered is a to **form Hamiltonian circuit.** Thus we have a

Hamiltonian circuit as given below. The length of this Hamiltonian circuit is

$$7+6+8+5+14=40.$$



However, **this is not Hamiltonian circuit of minimal length.**

The **total distance** of a **minimum Hamiltonian** circuit (Figure 2.98) **is 37**.

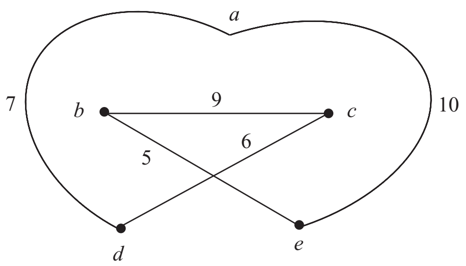


Figure 2.98

Total length $=7+6+9+5+10=37$.

Remark 2.7. Unless otherwise stated, try to start from a vertex of largest weight.

EXAMPLE 2.40

Find a Hamiltonian circuit of minimal weight for the graph given below (Figure 2.99)

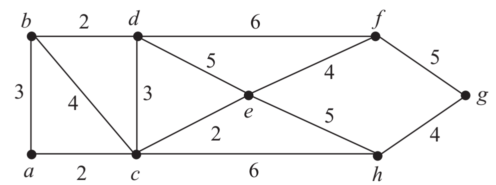


Figure 2.99

Solution. Starting from c and applying nearest neighbour method, we have the required Hamiltonian circuit as $c\ a\ b\ d\ e\ f\ g\ h\ c$ with total length as

$$2+3+2+5+4+5+4+6=31.$$

EXAMPLE 2.41

Find a Hamiltonian circuit of minimal weight for the graph shown below (Figure 2.100)

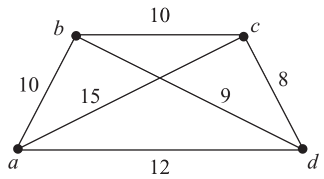


Figure 2.100

Solution. Starting from the point a and using nearest neighbour method, we have the required Hamiltonian circuit as $a\ b\ c\ d\ a$ with total length as

$$10+10+8+12=40.$$

Definition 2.61. A k**-factor of a graph** is a spanning subgraph of the graph **with the degree of its vertices being** k.

Consider the graph shown in Figure 2.101.

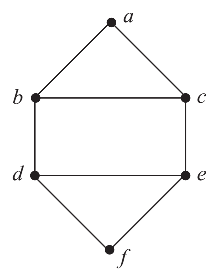


Figure 2.101

Then the graph shown in Figure 2.102 shows a 1-factor of the given graph.

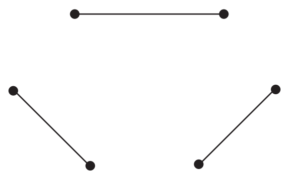


Figure 2.102

Also then, (see Figure 2.103)

Figure 2.103

is a 2-factor of the given graph.

2.8 MATRIX REPRESENTATION OF GRAPHS

A graph can be represented inside a computer by using the adjacency matrix or the incidence matrix of the graph.

Definition 2.62. Let G be a graph with n ordered vertices $v_1, v_2, \ldots, v_n$. Then the **adjacency matrix of** G is the $n \times n$ matrix $A(G)=(a_{ij})$ over the set of non-negative integers such that

$$a_{ij} = \text{the number of edges connecting } v_i \text{ and } v_j \text{ for all } i, j = 1, 2, \ldots, n.$$

We note that if G has no loop, then there is no edge joining v_i to v_i, $i=1, 2, \ldots, n$.Therefore, in this case, all the entries on the main diagonal will be 0.

Further, if G has no parallel edge, then the entries of $A(G)$ are either 0 or 1.

It may be noted that adjacent matrix of a graph is symmetric.

Conversely, given a $n \times n$ symmetric matrix $A(G)=(a_{ij})$ over the set of non-negative integers, we can associate with it a graph G, whose adjacency matrix is $A(G)$, by letting G have n vertices and joining v_i to vertex v_j by a_{ij} edges.

EXAMPLE 2.42

Find the adjacency matrix of the graph shown below (Figure 2.104):

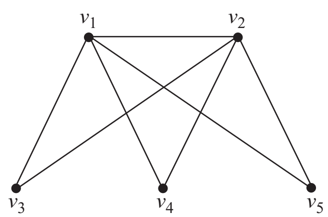


Figure 2.104

Solution. The adjacency matrix $A(G)=(a_{ij})$ is the matrix such that

$$a_{ij} = \text{Number of edges connecting } v_i \text{ and } v_j.$$

So, for the given graph, the adjacency matrix is

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

EXAMPLE 2.43

Find the adjacency matrix of the graph given below:

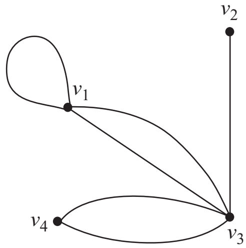


Figure 2.105

Solution. We note that there is a loop at v_1 and parallel edges between v_1, v_3, and v_3, v_4. So the adjacency matrix $A(G)$ is given by the following 4×4 matrix:

$$A(G) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

EXAMPLE 2.44

Find the graph that has the following adjacency matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 0 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

Solution. We note that there is a loop at v_1 and a loop at v_3. There are parallel edges between v_1, v_2; v_1, v_4; v_2, v_1; v_2, v_3; v_3, v_2; v_4, v_1. Thus the graph is as shown in Figure 2.106.

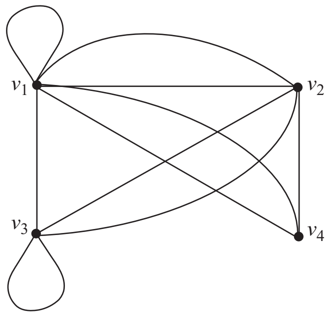


Figure 2.106

The following theorem is stated without proof.

Theorem 2.14. Let G be a graph with n vertices $v_1, v_2, \ldots, v_n$ and let $A(G)$ denote the matrix of G. If $B=(b_{ij})$ is the matrix

$$B=A+A^2+\cdots+A^{n-1},$$

then the graph G is a connected graph if and only if $b_{ij}\neq 0$ for $i\neq j$, that is, B has no zero entry off the main diagonal.

EXAMPLE 2.45

A graph G has the following adjacency matrix. Verify whether it is connected.

$$A=\begin{bmatrix}0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&1\\0&1&0&0&1\\0&0&1&1&0\end{bmatrix}.$$

Solution. We have

$$A^2=\begin{bmatrix}1&0&0&0&1\\0&1&0&0&1\\0&0&2&1&0\\0&0&1&2&0\\1&1&0&0&2\end{bmatrix},$$

$$A^3=\begin{bmatrix}0&0&2&1&0\\0&0&1&2&0\\2&1&0&0&3\\1&2&0&0&3\\0&0&3&3&0\end{bmatrix}$$

$$A^4=\begin{bmatrix}2&1&0&0&3\\1&2&0&0&3\\0&0&5&4&0\\0&0&4&5&0\\3&3&0&0&6\end{bmatrix}.$$

Therefore,

$$B=A+A^2+A^3+A^4=\begin{bmatrix}3&1&3&1&4\\1&3&1&3&4\\3&1&7&5&4\\1&3&5&7&4\\4&4&4&4&8\end{bmatrix}.$$

Since B has no zero entry off the main diagonal, the graph is connected.

Definition 2.63. Suppose a graph G has n vertices $v_1, v_2, \ldots, v_n$ and t edges $e_1, e_2, \ldots, e_t$. The **incidence matrix** $B(G)$ of G is the $n\times t$ matrix $B(G)=(b_{ij})$, where

$b_{ij}=$the number of times that the vertex v_i is incident with the edge e_j,

that is,

$$b_{ij}=\begin{cases}0 & \text{if } v_i \text{ is not end of } e_j\\ 1 & \text{if } v_i \text{ is an end of the non-loop } e_j\\ 2 & \text{if } v_i \text{ is an end of the loop } e_j\end{cases}$$

2.9 PLANAR GRAPHS

Definition 2.64. A graph which can be drawn in the plane so that its edges do not cross is said to be **planar.**

For example, the graph shown in Figure 2.107 is planar:

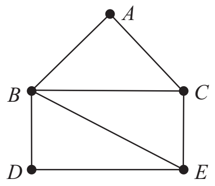


Figure 2.107

Also the complete graph K_4 shown below (Figure 2.108) is planar.

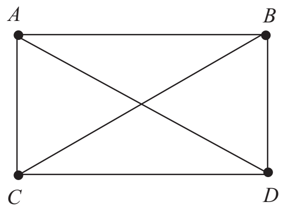


Figure 2.108

In fact, it can be redrawn as shown below in Figure 2.109 so that no edges cross.

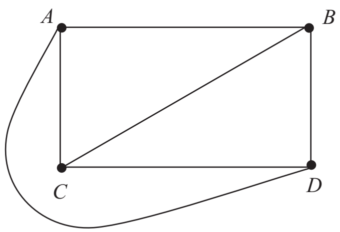


Figure 2.109

But the map K_5 is not planar because in this case, the edges cross each others (see Figure 2.110).

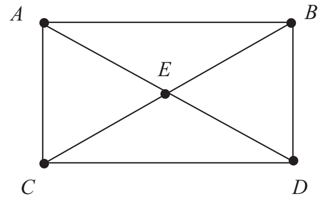


Figure 2.110

Definition 2.65. An area of the plane that is bounded by edges of the planar graph and is not further subdivided into subareas is called a **region** or **face** of the planar graph.

A face is characterized by the cycle that forms its boundary.

Definition 2.66. A region is said to be **finite** if its area is finite and **infinite** if its area is infinite. Clearly a planar graph has exactly one infinite region.

For example, consider the graphs shown in Figure 2.111.

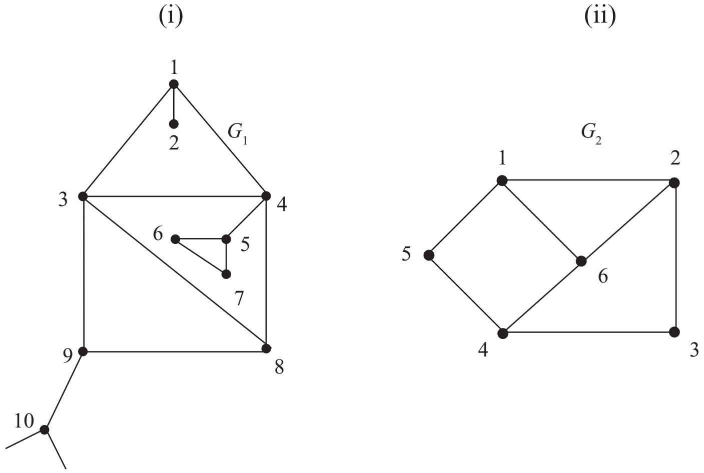
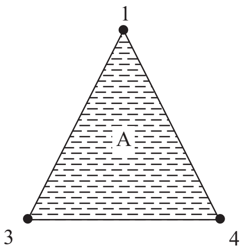
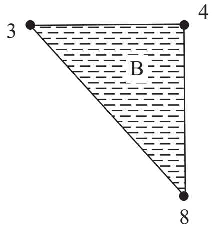


Figure 2.111

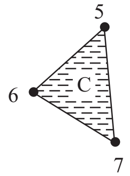
We note that in the graph G_1, there are five regions **A, B, C, D, E** as shown below:



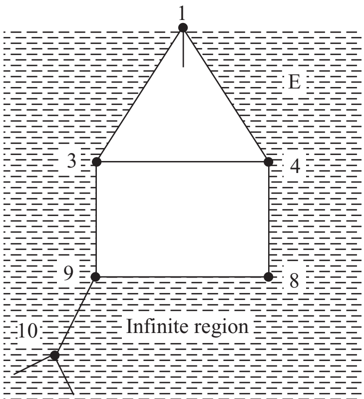
Finite region



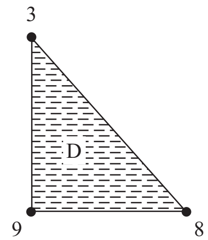
Finite region



Finite region

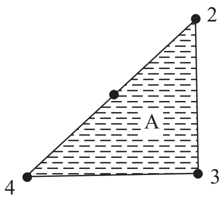


Infinite region

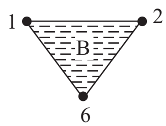


Infinite region

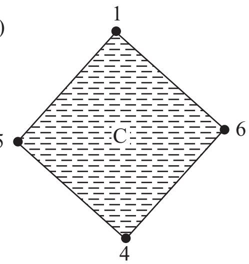
In graph $G_{2,}$ there are four region **A, B, C, D** (Figure 2.112)



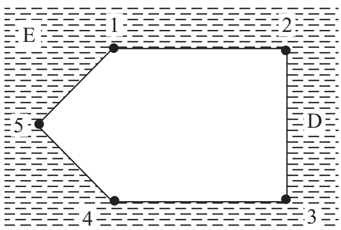
Finite region



Finite region



Finite region



Infinite region

Figure 2.112

Definition 2.67. Let f be a face (region) in a planar graph. The length of the cycle (or closed walk) which borders f is called the **degree of the region** f. It is denoted by $\deg(f)$.

In a planar graph we note that **each edge either borders two regions or is contained in a region and will occur twice in any walk along the border of the region.** Thus we have the following:

Theorem 2.15. The sum of the degrees of the regions of a map is equal to twice the number of edges.

For example, in the graph G_2, discussed above, we have

$$\deg(A)=4, \quad \deg(B)=3, \quad \deg(C)=4, \quad \deg(d)=5.$$

The sum of degrees of all regions $=4+3+4+5=16$.

$$\text{Number of edges in } G_2=8.$$

Hence the sum of degrees of regions is twice the number of edges.

Theorem 2.16. (Euler's Formula for Connected Planar Graphs) If G is a connected planar graph with e edges, v vertices and r regions, then

$$v-e+r=2.$$

Proof: We shall use induction on the number of edges. Suppose that $e=0$. Then the graph G consists of a single vertex, say P. Thus, G is as shown below:

$$\bullet\, P$$

and we have $e=0$, $v=1$, $r=1$. Thus $1-0+1=2$ and the formula holds in this case.

Suppose that $e=1$. Then the graph G is one of the two graphs shown below:

$e = 1, v = 2, r = 1$

,

$e = 1, v = 1, r = 2$

We see that, in either case, the formula holds.

Suppose that the formula holds for connected planar graph with n edges. We shall prove that this holds for graph with $n+1$ edges. So, let G be the graph with $n+1$ edges. Suppose first that G contains no cycles. Choose a vertex v_1 and trace a path starting at v_1. Ultimately, we will reach a vertex a with degree 1, that we cannot leave (see Figure 2.113).

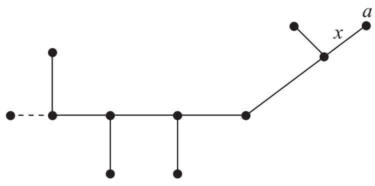


Figure 2.113

We delete "a" and the edge x incident on "a" from the graph G. The resulting graph G' (Figure 2.114) has n edges and so by induction hypothesis, the formula holds for G'. Since G has one more edge than G', one more vertex than G' and the same number of faces as G', it follows that the formula $v-e+r=2$ holds also for G.

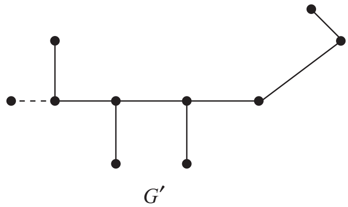


Figure 2.114

Now suppose that G contains a cycle. Let x be an edge in a cycle as shown in Figure 2.115.

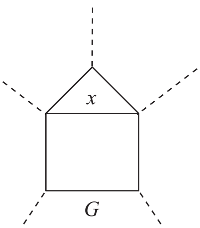


Figure 2.115

Now the edge x is part of a boundary for two faces. We delete the edge x but no vertices to obtain the graph G' as shown in Figure 2.116.

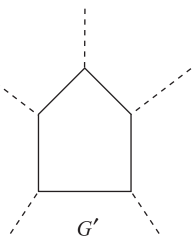


Figure 2.116

Thus G' has n edges and so by induction hypothesis the formula holds. Since G has one more face (region) than G', one more edge than G' and the same number of vertices as G', it follows that the formula $v-e+r=2$ also holds for G. Hence, by mathematical induction, the theorem is true.

Remark 2.8. Planarity of a graph is not affected if

(i) An edge is divided into two edges by the insertion of new vertex of degree 2 (Figure 2.117).

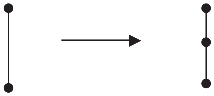


Figure 2.117

(ii) Two edges that are incident with a vertex of degree 2 are combined as a single edge by the removal of that vertex (Figure 2.118).

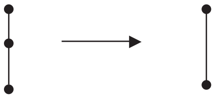
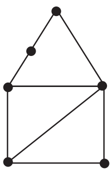


Figure 2.118

Definition 2.68. Two graphs G_1 and G_2 are said to be **isomorphic to within vertices of degree 2 (or homeomorphic)** if they are isomorphic or if they can be transformed into isomorphic graphs by repeated insertion and/or removal of vertices of degree 2.

Definition 2.69. The repeated insertion/removal of vertices of degree 2 is called **sequence of series reduction**.

For example, the graphs shown in Figure 2.119 are isomorphic to within vertices of degree 2.



and

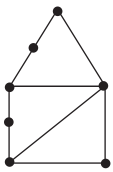
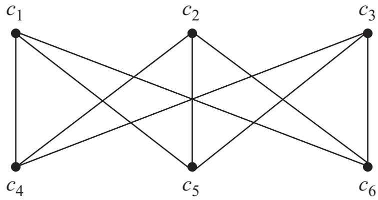


Figure 2.119

If we define a relation R on the set of graphs by $G_1 \; R \; G_2$ if G_1 and G_2 are homeomorphic, then R is an equivalence relation. Each equivalence class consists of a set of mutually homeomorphic graphs.

EXAMPLE 2.46

Show that the graph $K_{3,3}$, given in Figure 2.120 below, is not planar.



$K_{3,3}$

Figure 2.120

A problem based on this example can be stated as "Three cities c_1, c_2 and c_3 are to be directly connected by express ways to each of three cities c_4, c_5 and c_6. Can this road system be designed so that the express ways do not cross?" This example shows that it cannot be done.

Solution. Suppose that $K_{3,3}$ is planar. Since every cycle in $K_{3,3}$ has at least four edges, each face (region) is bounded by at least four edges. Thus the number of edges that bound regions is at least $4r$. Also, in a planar graph each edge belongs to at most two bounding cycles. Therefore,

$2e \geq 4r$ (sums of degrees of region is equal to twice the number of edges)

But, by Euler's formula for planar graph,

$$r = e - v + 2.$$

Hence,

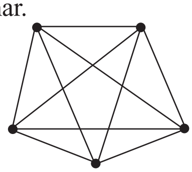
$$2e \geq 4(e-v+2). \tag{1}$$

In case of $K_{3,3}$ we have $e=9$, $v=6$ and so (1) yields

$$18 \geq 4\,(9-6+2) = 20,$$

which is a contradiction. Therefore $K_{3,3}$ is not planar.

Remark 2.9. By a argument similar to the above example, we can show that the graph K_5 (Figure 2.121) is not planar.



(Non-planar graph K_5)

Figure 2.121

We observe that if a graph contains $K_{3,3}$ or K_5 as a subgraph, then it cannot be planar.

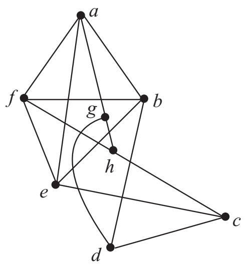
The following theorem, which we state without proof, gives necessary and sufficient condition for a graph to be planar.

Kuratowski's Theorem 2.17. A graph G is planar if and only if G does not contain a **subgraph** homeomorphic to $K_{3,3}$ or K_5.

The complete graph K_5 and the complete bipartite graph $K_{3,3}$ are called the **Kuratowski graphs.**

EXAMPLE 2.47

Using Kuratowski's theorem show that the graph G, shown in Figure 2.122, is not planar.

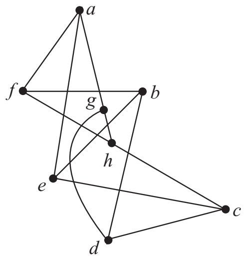


G

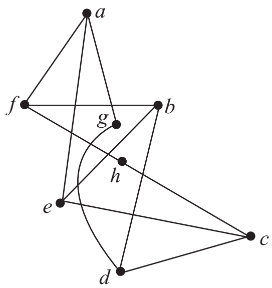
Figure 2.122

Solution. Let us try to find $K_{3,3}$ in the graph G. We know that in $K_{3,3}$, each vertex has degree 3. But we note that in G, the degree of a, b, f and e each is 4. So we eliminate the edges (a, b) and (f, e) so that all vertices have degree 3. If we eliminate one more edge, we will obtain two vertices of degree 2 and we can then carry out series reduction. The resulting graph will have nine edges.

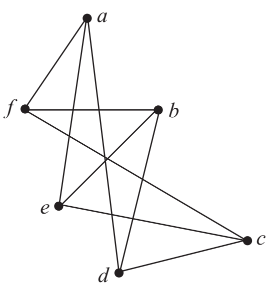
Also we know that $K_{3,3}$ **has nine edges**. So this approach seems promising. Using trial and error, we find that the edge (g, h) should be removed. Then g and h have degree 2.



(Graph obtained by deleting edges (a, b) and (f, e)).



(Graph obtained by eliminating the edge (g, h). Performing series reduction now, we obtain an isomorphic copy of $K_{3,3}$ (Figure 2.123).



(Isomorphic copy of $K_{3,3}$, obtained by series reduction)

Figure 2.123

Hence, by Kurtowski's theorem, the given graph G is not planar.

EXAMPLE 2.48

If the graph given in Figure 2.124 is planar, redraw it so that no edges cross, otherwise find a subgraph homeomorphic to either K_5 or $K_{3,3}$.

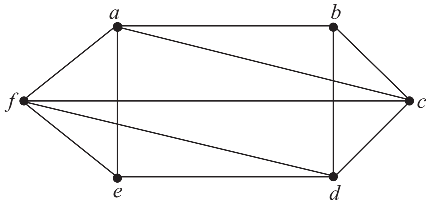


Figure 2.124

Solution. The given graph can be redrawn as shown in the Figure 2.125.

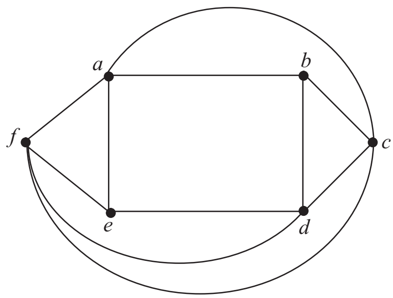


Figure 2.125

We then observe that no edges cross. Hence the given graph is planar.

Theorem 2.18. Let G be a connected planar graph with v vertices, e edges, where $v \geq 3$. Then $e \leq 3v-6$ or $6 \leq 3v - e$.

(Note that the theorem is not true for K_1, where $v=1$ and $e=0$ and is not true for K_2, where $v=2$ and $e=1$).

Proof: Let r be the number of regions in a planar representation of the graph G. The sum of the degrees of the regions is equal to $2e$. But each region has degree greater than or equal to 3. Hence $2e \geq 3r$, that is, $r \leq \frac{2e}{3}$. But, by Euler's formula,

$$v-e+r=2.$$

So,

$$2=v-e+r \leq v-e+\frac{2e}{3}=v-\frac{e}{3}.$$

Hence,

$$6 \leq 3v-e \quad \text{or} \quad e \leq 3v-6$$

Remark 2.10. Using above theorem, let us consider planarity of K_5. Here $v=5$, $e=10$. Suppose K_5 is planar, then by the above theorem,

$$6 \leq 3(5)-10=5,$$

which is impossible. Hence K_5 is non-planar.

Definition 2.70. Any graph homeomorphic either to K_5 or to $K_{3,3}$ is called **Kuratowski subgraph**.

EXAMPLE 2.49

Find Kuratowski subgraph of the following graph (Figure 2.126):

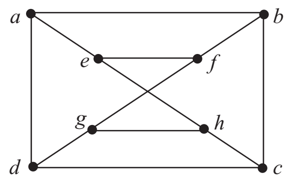


Figure 2.126

Solution. In the given graph,

Number of vertices$=8$

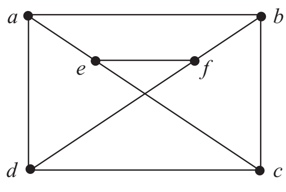
Number of edges$=12$.

Firstly, we remove the edge (g, h). Then $\deg(g)=2$, $\deg(h)=2$ and so the vertices g and h can be removed. Then, we have

Number of vertices$=6$

Number of edges$=9$

and the graph reduces to



,

which is homeomorphic to $K_{3,3}$ (Kuratowski graph) shown in Figure 2.127.

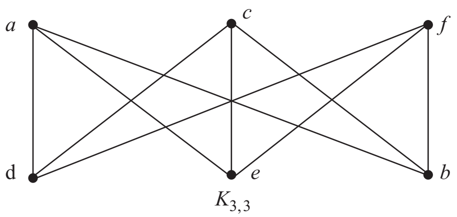


Figure 2.127

Thus the given graph is non-planar by Kuratowski theorem.

EXAMPLE 2.50

Find a subgraph homeomorphic to either K_5 or $K_{3,3}$ in the graph given below (Figure 2.128):

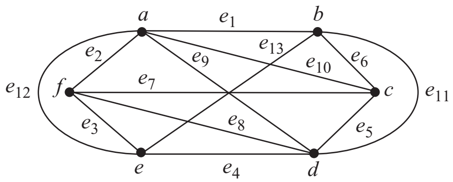


Figure 2.128

Solution.

Here,

Number of vertices $=6$

Number of edges $=13$

$\deg(a)=5, \quad \deg(f)=4, \quad \deg(c)=4, \quad \deg(d)=5$

But, in $K_{3,3}$, the degree of each vertex is 3. We delete the edges (a, c), (f, d), (a, e), (b, d), then degree of each vertex a, f, c and d becomes 3. As such, the subgraph becomes as given below (Figure 2.129), which has six vertices each with degree 3 and has 9 edges.

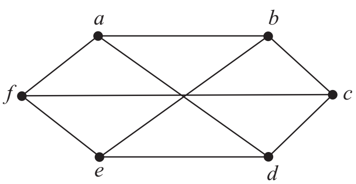


Figure 2.129

This subgraph is homeomorphic to $K_{3,3}$ as shown in Figure 2.130.

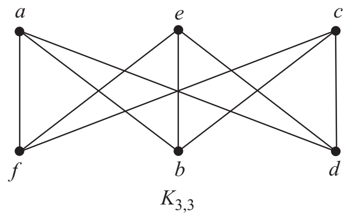


Figure 2.130

This also shows, by Kuratowski theorem, that the given graph is not planar.

2.10 COLOURING OF GRAPH

Definition 2.71. Let G be a graph. The assignment of colours to the vertices of G, one colour to each vertex, so that the adjacent vertices are assigned different colours is called **vertex colouring** or **colouring of the graph** G.

Definition 2.72. A graph G is n-**colourable** if there exists a colouring of G which uses n colours.

Definition 2.73. The minimum number of colours required to paint (colour) a graph G is called the **chromatic number of** G and is denoted by $\chi(G)$.

EXAMPLE 2.51

Find the chromatic number for the graph shown in the Figure 2.131.

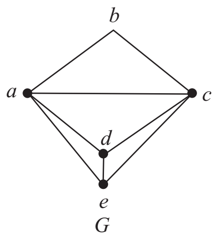


Figure 2.131

Solution. The triangle $a\ b\ c$ needs three colours. Suppose that we assign colours c_1, c_2, c_3 to a, b and c respectively. Since d is adjacent to a and c, d will have different colour than c_1 and c_3. So we paint d by c_2. Then e must be painted with a colour different from those of a, d and c, that is, we cannot colour e with c_1, c_2 or c_3. Hence, we have to give e a fourth colour c_4. Hence

$$\chi(G)=4.$$

Welsh–Powell algorithm to determine upper bound to the chromatic number of a given graph.
The input is a given graph G.

1. Order the vertices of G according to decreasing degree.
2. Assign the first colour, say c_1, to the first vertex and then, in sequential order, assign c_1 to each vertex, which is not adjacent to a previous vertex assigned c_1.
3. Repeat Step 2 with a second colour c_2 and the subsequence of the remaining non-painted vertices.
4. Repeat Step 3 with a third colour c_3, then a fourth colour c_4 and so on until all vertices are coloured.
5. Exit.

EXAMPLE 2.52

Use Welsh-Powell algorithm to determine an upper bound to the chromatic number of the "Wheel" graph shown in Figure 2.132.

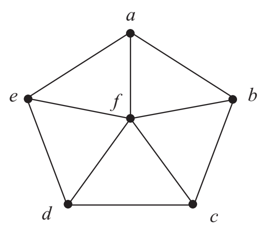


Figure 2.132

Solution. We note that

$$\deg(f)=5,$$
$$\deg(a)=\deg(b)=\deg(c)=\deg(d)=\deg(e)=3.$$

Step 1. Ordering the vertices according to decreasing degree yields

$$f, a, b, c, d, e$$

Step 2. Paint f with colour c_1.
Step 3. Paint a, d with colour c_2.
Step 4. Paint b, e with colour c_3.
Step 5. Paint c with colour c_4.

Hence $\chi(G)\leq 4$. Also since there is a triangle in the given graph so we have $\chi(G)\geq 3$. Thus $3\leq\chi(G)\leq 4$. But we do not yet know exactly what the chromatic number is. We try to build a 3-colouring of G.

Let us start colouring the triangle abf with the colours c_1, c_2, c_3, respectively. Since c is adjacent to the vertices b and f of colour c_2 and c_3, respectively, c is forced to be coloured c_1 and then d is forced to be c_2. However, now the adjacent vertices a and e cannot both have colour c_1. Thus the graph cannot be 3-coloured. But using a fourth colour c_4 for e gives us 4-colouring of G. Hence, $\chi(G)=4$.

EXAMPLE 2.53

Use Welsh-Powell algorithm to colour the graph shown in Figure 2.133.

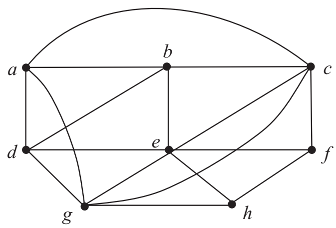


Figure 2.133

Solution. Ordering the vertices according to decreasing degrees, we get the sequence e, c, g, a, b, d, f, h.

Use the colour c_1 to colour (paint) e and a.
Use the colour c_2 to paint c, d and h.
Use third colour c_3 to paint vertices g, b and f.

Thus all the vertices are painted such that no adjacent vertices get the same colour. Hence $\chi(G)=3$.

Some rules for colouring a graph

1. $\chi(G) \leq |V|$, where $|V|$ is the number of vertices of G.
2. A triangle always requires three colours, that is, $\chi(K_3)=3$. Similarly $\chi(K_n)=n$, where K_n is the complete graph of n vertices.
3. If some subgraph of G requires k-colours, then $\chi(G) \geq k$.
4. If $\deg(v)=n$, then at most n colours are required to colour the vertices adjacent to v.
5. $\chi(G)=\max\{\chi(C)$: C is a connected component of $G\}$.
6. Every n-colourable graph has at least n vertices v such that $\deg(v) \geq n-1$.
7. $\chi(G) \leq 1+\Delta(G)$, where $\Delta(G)$ is the largest degree of any vertex of G.
8. The following statements are equivalent:
   (i) A graph G is 2-colourable
   (ii) G is bipartite
   (iii) Every cycle of G has even length
9. If $\delta(G)$ is the smallest degree of any vertex of G, then

$$\chi(G) \geq \frac{|v|}{|v|-\delta(G)}.$$

EXAMPLE 2.54

Find chromatic number of the graphs shown in Figure 2.134.

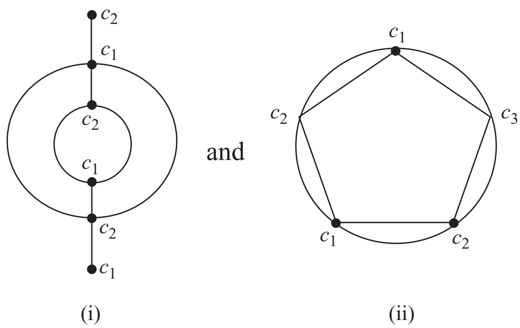


Figure 2.134

Solution. The graph (i) is 2-colourable whereas the graph (ii) is 3-colourable.

Theorem 2.19. Bipartite graph is 2-colourable unless G is edgeless.

Proof: A two colouring is obtained by assigning one colour to every vertex in one of the bipartition parts and another colour to every vertex in the other partition part as shown in Figure 2.135.

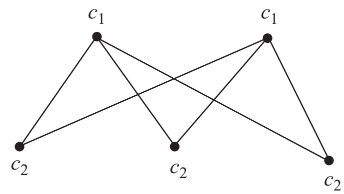


Figure 2.135

Corollary 2.3. Even cycle graphs C_{2n} have $\chi(C_{2n})=2$.

Proof: An even cycle graph is bipartite and therefore, by Theorem 2.19, its chromatic number is 2.

For example consider the graph shown in Figure 2.136.

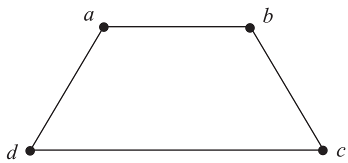


Figure 2.136

It is equivalent to the following bipartite graph shown in Figure 2.137.

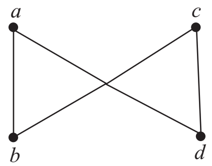


Figure 2.137

Therefore, it is 2-colourable.

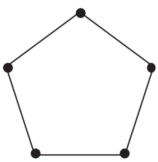
Proposition 2.2. Odd cycle graph C_{2n+1} has $\chi(C_{2n+1})=3$.

Proof: Let $v_1, v_2, \ldots, v_{2n}, v_{2n+1}$ be vertices of a cycle graph c_{2n+1}. If two colours were sufficient, then they would have to alternate around the cycle. Thus, the odd subscripted would have to be one colour and the even subscripted vertices have to be second colour but vertex v_{2n+1} is adjacent to v_1, and so according to this scheme two adjacent vertices v_1 and v_{2n+1} have the same colour. This is a contradiction and so C_{2n+1} is not 2-colourable but 3-colourable.

It follows, therefore, that "The chromatic number of a cycle is either two or three, depending on whether its length is even or odd."

Definition 2.74. The **join** $G+H$ of the graphs G and H is obtained from the graph $G \cup H$ by adding an edge between each vertex of H.

For example, if we have a graph as shown below (Figure 2.138):



(G)

Figure 2.138

and a graph

$$\bullet\ K_1,$$

then the join $G+K_1$ of the graphs G and K_1 is the graph as shown below (Figure 2.139):

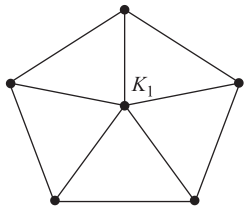


Figure 2.139

Regarding the chromatic number of the join $G+H$ of the graphs G and H we have the following:

Proposition 2.3. The join of a graph G and H has chromatic number

$$\chi(G+H)=\chi(G)+\chi(H).$$

Definition 2.75. The n vertex wheel graph W_n is called an **odd-order wheel** if n is odd and an **even-order wheel** if n is even.

Proposition 2.4. Odd-order wheel graph has

$$\chi(W_{2m+1})=3, \quad \text{for all } m \geq 1.$$

Proof: Using the fact that the wheel graph W_{2m+1} is the join $C_{2m}+K_1$, it follows that

$$\chi(W_{2m+1})=\chi(C_{2m})+\chi(K_1)=2+1=3.$$

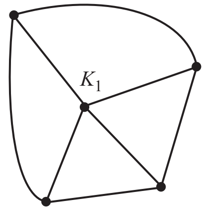


Figure 2.140 (W_5)

Proposition 2.5. Even-order wheel graph has $\chi(W_{2m})=4$.

Proof: Using the fact that $W_{2m}=c_{2m-1}+K_1$, it follows that

$$X(W_{2m})=\chi(c_{2m-1})+\chi(K_1)=3+1=4.$$

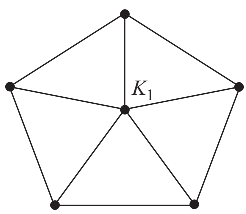


Figure 2.141 (W_6)

EXAMPLE 2.55

Find chromatic number of the graph shown in Figure 2.142:

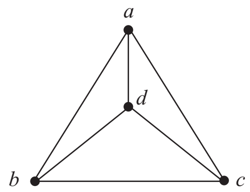


Figure 2.142

Solution. Here degree of each vertex is 3. Therefore, $\chi(G)\leq 1+3=4$. Since it has a triangle, $\chi(G)\geq 3$. Thus, $3\leq\chi(G)\leq 4$.

We first colour the triangle *abd* with colours c_1, c_2, c_3, respectively. Since c is adjacent to each of a, b and d, therefore a different colour c_4 have to be given to it. Hence the graph is 4-colourable and so $\chi(G)=4$.

Note: If we apply rule 9, then $\chi \geq \dfrac{4}{(4-3)} = 4.$ Hence $\chi(G) = 4$.

Theorem 2.20 (Four-Colour Theorem of Appel and Haken)

Any planar graph is 4-colourable.

(Proof of the theorem is out of the scope of this book).

2.11 DIRECTED GRAPHS

Definition 2.76. A **directed graph** or **digraph** consists of two finite sets:

(i) A set V of vertices (or nodes or points).

(ii) A set E of directed edges (or arcs), where each edge is associated with an ordered pair (v, w) of vertices called its endpoints. If edge e is associated with the ordered pair (v, w), then e is said to be **directed edge from** v **to** w.

The directed edges are indicated by arrows.

We say that edge $e=(v, w)$ is incident from v and is incident into w.

The vertex v is called **initial vertex** and the vertex w is called the **terminal vertex** of the directed edge (v, w).

Definition 2.77. Let G be a directed graph. The **outdegree of a vertex** v **of** G is the number of edges beginning at v. It is denoted by outdeg(v).

Definition 2.78. Let G be a directed graph. The **indegree of a vertex** v **of** G is the number of edges ending at v. It is denoted by indeg(v).

EXAMPLE 2.56

Consider the directed graph shown below (Figure 2.143):

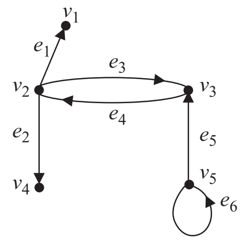


Figure 2.143

Here edge e_1 is (v_2, v_1) whereas e_6 is denoted by (v_5, v_5) and is called a loop. The indegree of v_2 is 1, outdegree of v_2 is 3.

Definition 2.79. A vertex with 0 indegree is called a **source**, whereas a vertex with 0 outdegree is called a **sink**.

For instance, in the above example, v_1 is a sink.

Definition 2.80. If the edges and/or vertices of a directed graph G are labelled with some type of data, then G is called a **labelled directed graph.**

Definition 2.81. Let G be a **directed graph** with ordered vertices $v_1, v_2, \ldots, v_n$. **The adjacency matrix of** G **is the matrix** $A=(a_{ij})$ over the set of non-negative integers such that

a_{ij}=the number of arrows from v_i to v_j, $i, j=1, 2, \ldots, n$.

EXAMPLE 2.57

Find the adjacency matrices for the graphs given below (Figure 2.144):

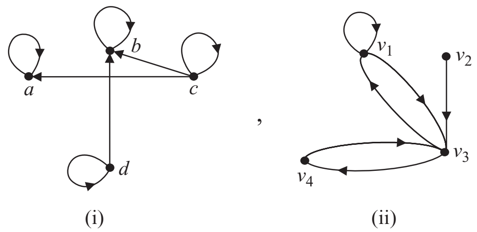


Figure 2.144

Solution. The edges in the directed graph are (a, a), (b, b), (c, c), (d, d), (c, a), (c, b) and (d, b). Therefore the adjacency matrix $A=(a_{ij})$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

(iii) The edges in the graph in (ii) are (v_2, v_3), (v_1, v_1), (v_1, v_3), (v_3, v_1), (v_3, v_4), (v_4, v_3). Hence the adjacency matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

EXAMPLE 2.58

Find the directed graph represented by the adjacency matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution. We observe that $a_{12}=1$, $a_{23}=1$, $a_{34}=1$, $a_{35}=1$, $a_{41}=1$, $a_{42}=1$. Hence the digraph is as shown in Figure 2.145.

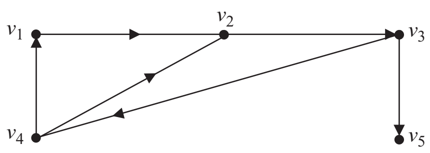


Figure 2.145

Definition 2.82. In a directed graph, if there is no more than one directed edge in a particular direction between a pair of vertices, then it is called **simple directed graph**.

For example, the graph shown in Figure 2.146 is a simple directed graph.

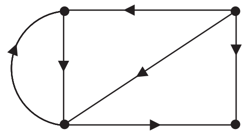


Figure 2.146

A directed graph which is not simple is called **directed multi-graph**.

Definition 2.83. A simple digraph is said to be **strongly connected** if for each pair of vertices v, w, there are path, from v to w and w to v.

For example, the graphs show in Figure 2.147 are strongly connected.

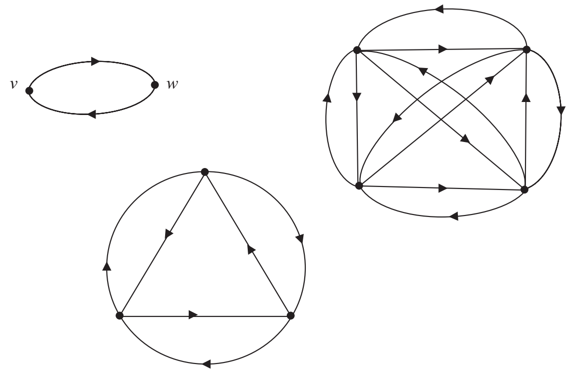


Figure 2.147

Definition 2.84. Let G be a simple digraph with n vertices. Then a $n \times n$ matrix $P=(p_{ij})$ such that

$$p_{ij} = \begin{cases} 1 & \text{if there is a path from } v_i \text{ to } v_j \\ 0 & \text{otherwise} \end{cases}$$

is called **path matrix** or **reachability matrix** of G.

The path matrix can be given in term of adjacency matrix:

$$P=A \vee A^{(2)} \vee A^{(3)} \vee \cdots \vee A^{(n)}$$

and

$$A^{(2)}=A \odot A, \quad A^{(r)}=A^{(r-1)} \odot A,$$

where $\vee$ represents Boolean addition and $\odot$ denotes the Boolean matrix multiplication.

Warshall's algorithm for finding path matrix from adjacency matrix

We discuss this algorithm with the help of the following example.

EXAMPLE 2.59

Find the adjacency matrix and the path matrix for the digraph shown in Figure 2.148.

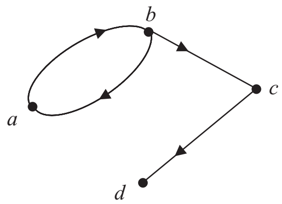


Figure 2.148

Solution. The adjacency matrix of this digraph is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the path matrix $P=(p_{ij})$ is found as follows: We take

$$P_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A \text{ itself.}$$

Now we find P_1 by consulting column 1 and row 1. We note that P_0 has 1 in location **2** of column 1 and location **2** of row 1. Thus the element p_{22} is replaced by 1 in P_0. Thus we have

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To find P_2 we consult column 2 and row 2 of P_1. We note that P_1 has 1 in locations 1 and 2 of column 2 and locations 1, 2 and 3 of row 2. Thus, to obtain P_2 we should put 1 in positions $p_{11}, p_{12}, p_{13}, p_{21}, p_{22}$ and p_{23} of matrix P_1. We thus have

$$P_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We now consult column 3 and row 3 of P_2. The matrix P_2 has 1 in locations 1 and 2 of column 3 and 1 in location 4 of row 3. Thus we shall obtain P_3 by putting 1 in positions p_{14}, p_{24}. Thus

$$P_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, we consult column 4 and row 4 of P_3. We have 1 in location 1, 2, 3 of column 4 and there is no 1 in row 4. Hence $P_4=P_3$. Thus the path matrix is

$$P = (p_{ij}) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Remark 2.11. Given an adjacency matrix of a graph G, we first draw digraph of G and then apply definition of path matrix. We will obtain path matrix. Now, we state a result without proof.

Theorem 2.21. Let A be the adjacency matrix of a graph G with n vertices and let $B_n=A+A^2+A^3+\cdots+A^n$. Then the path matrix P and B_n have the same non-zero entries.

Let G be a strongly connected directed graph. Then for any pair of vertices v and w in G, there is a path from v to w and from w to v. Accordingly, G is **strongly connected if and only if the path matrix** P **of** G **has no zero entries**.

Thus, in view of the above theorem, we have:

Theorem 2.22. Let A be adjacency matrix of a graph G with n vertices and let $B_n=A+A^2+\ldots+A^n$. Then G is strongly connected if and only if B_n has no zero entries.

EXAMPLE 2.60

A directed graph has the following adjacency matrix. Check whether it is strongly connected.

$$A(G) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution. To find the path matrix, we first consider column 1 and row 1 of

$$P_0 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = A(G).$$

In column 1, the entry 1 is located at 2 and 3 positions and 1 is at locations 2 and 3 in row 1. Thus put 1 at $p_{22}, p_{23}, p_{32}, p_{33}$ and get

$$P_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

We note that 1 is located in column 2 of P_1 at position 1, 2, 3, and 1 is located at position 1, 2, 3 of row 2. Thus put 1 at $p_{11}, p_{12}, p_{13}, p_{21}, p_{22}, p_{23}, p_{31}, p_{32}, p_{33}$, and get

$$P_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The next step will not change the position. Hence,

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{(all non-zero entries).}$$

Therefore, G is strongly connected.
The graph of G is shown in Figure 2.149.

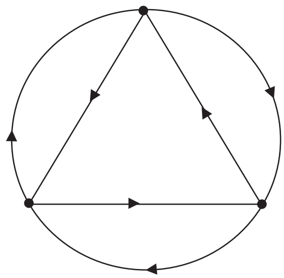


Figure 2.149

Remark 2.12. In the above example, we also note that

$$\begin{aligned} B_3 &= A + A^2 + A^3 \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 5 & 5 \\ 5 & 4 & 5 \\ 5 & 5 & 4 \end{bmatrix} \quad \text{(all non-zero entries)} \end{aligned}$$

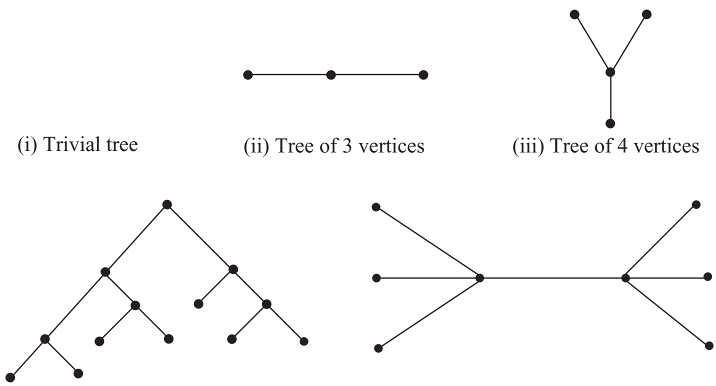
showing that G is strongly connected.

2.12 TREES

Definition 2.85. A graph is said to be a **tree** if it is a connected acyclic graph.

A **trivial tree** is a graph that consists of a single vertex. An **empty tree** is a tree that does not have any vertices or edges.

For example, the graphs shown in Figure 2.150 are all trees.



(i) Trivial tree (ii) Tree of 3 vertices (iii) Tree of 4 vertices

(iv) Tree of 13 vertices (v) Tree of 8 vertices

Figure 2.150

But the graphs shown in Figure 2.151 are not trees:

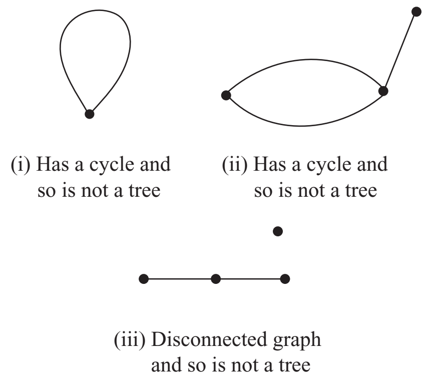


Figure 2.151

Definition 2.86. A collection of disjoint trees is called a **forest**.

Thus a graph is a forest if and only if it is circuit free.

Definition 2.87. A vertex of degree 1 in a tree is called a **leaf** or a **terminal node** or a **terminal vertex**.

Definition 2.88. A vertex of degree greater than 1 in a tree is called a **branch node** or **internal node** or **internal vertex**.

Consider the tree shown in Figure 2.152.

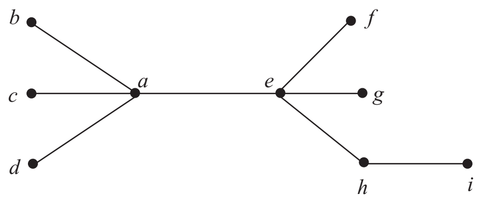


Figure 2.152

In this tree, the vertices b, c, d, f, g, i are leaves whereas the vertices a, e, h are branch nodes.

2.12.1 Characterization of Trees

We have the following interesting characterization of trees:

Lemma 2.1. A tree that has more than one vertex has at least one vertex of degree 1.

Proof: Let T be a particular but arbitrary chosen tree having more than one vertex (Figure 2.153).

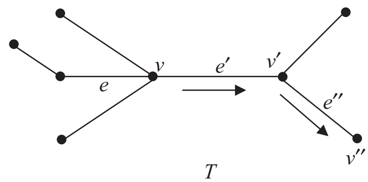


Figure 2.153

1. Choose a vertex v of T. Since T is connected and has at least two vertices, v is not isolated and there is an edge e incident on v.
2. If $\deg(v) > 1$, there is an edge $e' \neq e$ because there are, in such a case, at least two edges incident on v. Let v' be the vertex at the other end of e'. This is possible because e' is not a loop by the definition of a tree.
3. If $\deg(v') > 1$, then there are at least two edges incident on v'. Let e'' be the other edge different from e' and v'' be the vertex at other end of e''. This is again possible because T is acyclic.
4. If $\deg(v'') > 1$, repeat the above process. Since the number of vertices of a tree is finite and T is circuit free, the process must terminate and we shall arrive at a vertex of degree 1.

Remark 2.13. In the proof of the Lemma 2.1, after finding a vertex of degree 1, if we return to v and move along a path outward from v starting with e, we shall reach to a vertex of degree 1 again. Thus it follows that **"Any tree that has more than one vertex has at least two vertices of degree 1".**

Lemma 2.2. There is a unique path between every two vertices in a tree.

Proof: Suppose on the contrary that there are more than one path between any two vertices in a given tree T. Then T has a cycle which contradicts the definition of a tree because T is acyclic. Hence the lemma is proved.

Lemma 2.3. The number of vertices is one more than the number of edges in a tree.

Equivalently, we may state this lemma as

"For any positive integer n, a tree with n vertices has $n-1$ edges".

Proof: We shall prove the lemma by mathematical induction.

Let T be a tree with **one** vertex. Then T has no edges, that is, T has 0 edge. But $0=1-1$. Hence the lemma is true for $n=1$.

Suppose that the lemma is true for $k>1$. We shall show that it is then true also for $k+1$. Since the lemma is true for k, the tree has k vertices and $k-1$ edges. Let T be a tree with $k+1$ vertices. Since k is +ve, $k+1\geq 2$ and so T has more than one vertex. Hence, by Lemma 2.1, T has a vertex v of degree 1. Also there is another vertex w and so there is an edge e connecting v and w. Define a subgraph T' of T so that

$$V(T')=V(T)-\{v\},$$
$$E(T')=E(T)-\{e\}.$$

Then number of vertices in $T'=(k+1)-1=k$ and since T is circuit free and T' has been obtained on removing one edge and one vertex, it follows that T' is acyclic. Also T' is connected. Hence T' is a tree having k vertices and therefore by induction hypothesis, the number of edges in T' is $k-1$.

But then

$$\begin{aligned}\text{Number of edges in } T &= \text{number of edges in } T'+1\\ &=k-1+1=k.\end{aligned}$$

Thus the Lemma is true for tree having $k+1$ vertices. Hence the lemma is true by mathematical induction.

Corollary 2.4. Let $C(G)$ denote the number of components of a graph. Then a forest G on n vertices has $n-C(G)$ edges.

Proof: Apply Lemma 2.3 to each component of the forest G.

Corollary 2.5. Any graph G on n vertices has at least $n-C(G)$ edges.

Proof: If G has cycle-edges, remove them one at a time until the resulting graph G^* is acyclic. Then G^* has $n-C(G^*)$ edges by Corollary 2.4. Since we have removed only circuit, $C(G^*)=C(G)$. Thus G^* has $n-C(G)$ edges. Hence G has at least $n-C(G)$ edges.

Lemma 2.4. A graph in which there is a unique path between every pair of vertices is a tree

(This lemma is converse of Lemma 2.2).

Proof: Since there is a path between every pair of points, therefore the graph is connected. Since a path between every pair of points is unique, there does not exist any circuit because existence of circuit implies existence of distinct paths between pair of vertices. Thus the graph is connected and acyclic and so is a tree.

Lemma 2.5. (Converse of Lemma 2.3)

A connected graph G with $e=v-1$ is a tree

First Proof: The given graph is connected and $e=v-1$. To prove that G is a tree, it is sufficient to show that G is acyclic. Suppose on the contrary that G has a cycle. Let m be the number of vertices in this cycle. Also, we know that **number of edges in a cycle is equal to number of vertices in that cycle**. Therefore number of edges in the present case is m. Since the graph is connected, every vertex of the graph which is not in cycle must be connected to the vertices in the cycle (see Figure 2.154).

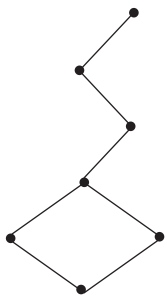


Figure 2.154

Now each edge of the graph that is not in the cycle can connect only one vertex to the vertices in the cycle. There are $v-m$ vertices that are not in the cycle. So the graph must contain at least $v-m$ edges that are not in the cycle. Thus we have $e\geq v-m+m=v$, which is a contradiction to our hypothesis. Hence there is no cycle and so the graph is a tree.

Second Proof: We shall show that a connected graph with v vertices and $v-1$ edges is a tree. It is sufficient to show that G is acyclic. Suppose on the contrary that G is not circuit free and has a nontrivial circuit C. If we remove one edge of C from the graph G, we obtain a graph G' which is connected (see Figure 2.155).

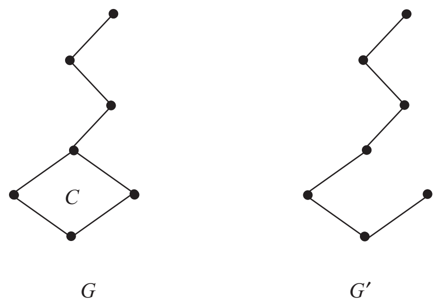


Figure 2.155

If G' still has a nontrivial circuit, we repeat the above process and remove one edge of that circuit obtaining a new connected graph. Continuing this process, we obtain a connected graph G^* which is circuit free. Hence G^* is a tree. Since no vertex has been removed, the tree G^* has v vertices. Therefore, by Lemma 2.3, G^* has $v-1$ edges. But at least one edge of G has been removed to form G^*. This means that G^* has not more than $v-1-1=v-2$ edges. Thus, we arrive at a contradiction. Hence our supposition is wrong and G has no cycle. Therefore G is connected and cycle free and so is a tree.

Lemma 2.6. A graph G with $e=v-1$, that has no circuit is a tree.

Proof: It is sufficient to show that G is connected. Suppose G is not connected and let $G', G'', \ldots$ be the connected components of G. Since each of $G', G'', \ldots$ is connected and has no cycle, they all are tree. Therefore, by Lemma 2.3,

$$\begin{gathered} e'=v'-1, \\ e''=v''-1, \\ \rule{2cm}{0.4pt} \\ \rule{2cm}{0.4pt}, \end{gathered}$$

where $e', e'', \ldots$ are the number of edges and $v', v'', \ldots$ are the number of vertices in $G', G'', \ldots$, respectively. We have, on adding

$$e'+e''+\cdots=(v'-1)+(v''-1)+\cdots$$

Since

$$\begin{gathered} e=e'+e''+\cdots, \\ v=v'+v''+\cdots, \end{gathered}$$

we have

$$e<v-1,$$

which contradicts our hypotheses. Hence G is connected. So G is connected and acyclic and is therefore a tree.

EXAMPLE 2.61

Construct a graph that has six vertices and five edges but is not a tree.

Solution. We have

$$\begin{gathered} \text{Number of vertices}=6 \\ \text{Number of edges}=5 \end{gathered}$$

So the condition $e=v-1$ is satisfied. Therefore, to construct graph with six vertices and five edges that is not a tree, we should keep in mind that the graph should not be connected. The graph shown in Figure 2.156 has six vertices and five edges but is not connected. Hence it is not a tree.

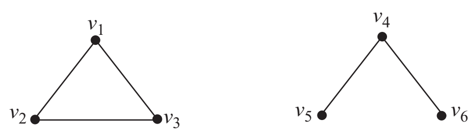


Figure 2.156

Definition 2.89. A directed graph is said to be a **directed tree** if it becomes a tree when the direction of edges are ignored.

For example, the graph shown in Figure 2.157 is a directed tree.

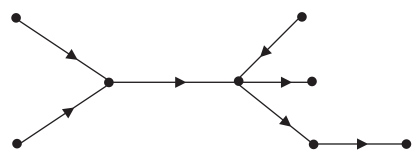


Figure 2.157

Definition 2.90. A directed tree is called a **rooted tree** if there is exactly one vertex whose incoming degree is 0 and the incoming degrees of all other vertices are 1.

The vertex with incoming degree 0 is called the **root** of the rooted tree.
A tree T with root v_0 will be denoted by (T, v_0).

Definition 2.91. In a rooted tree, a vertex, whose outgoing degree is 0 is called a **leaf** or **terminal node**, whereas a vertex whose outgoing degree is non-zero is called a **branch node** or an **internal node**.

Definition 2.92. Let u be a branch node in a rooted tree. Then a vertex v is said to be **child** (**son** or **offspring**) of u if there is an edge from u to v. In this case, u is called **parent** (**father**) of v.

Definition 2.93. Two vertices in a rooted tree are said to be **siblings** (**brothers**) if they are both children of same parent.

Definition 2.94. A vertex v is said to be a **descendent** of a vertex u if there is a unique directed path from u to v.

In this case, u is called the **ancestor** of v.

Definition 2.95. The **level** (or **path length**) of a vertex u in a rooted tree is the number of edges along the unique path between u and the root.

Definition 2.96. The **height** of a rooted tree is the maximum level to any vertex of the tree.

As an example of these terms, consider the rooted tree shown in Figure 2.158:

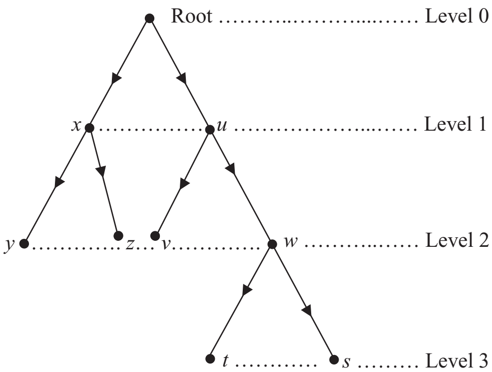


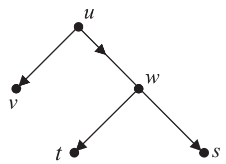
Figure 2.158

Here y is a child of x; x is the parent of y and z. Thus y and z are siblings. The descendents of u are v, w, t and s. Levels of vertices are shown in the figure. The height of this rooted tree is 3.

Definition 2.97. Let u be a branch node in the tree $T=(V, E)$. Then the subgraph $T'=(V', E')$ of T such that the vertices set V' contains u and all of its descendents and E' contains all the edges in all directed paths emerging from u is called a **subtree** with u as the root.

Definition 2.98. Let u be a branch node. By a subtree of u, we mean a subtree that has child of u as root.

In the above example, we note that the Figure 2.159 is a subtree of T,



,

Figure 2.159

whereas the Figure 2.160 is a subtree of the branch node u.

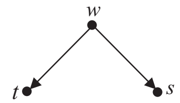


Figure 2.160

EXAMPLE 2.62

Let

$$V=\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$$

and let

$$E=\{(v_2, v_1), (v_2, v_3), (v_4, v_2), (v_4, v_5), (v_4, v_6), (v_6, v_7), (v_5, v_8)\}.$$

Show that (V, E) is rooted tree. Identify the root of this tree.

Solution. We note that

Incoming degree of $v_1=1$,
Incoming degree of $v_2=1$,
Incoming degree of $v_3=1$,
Incoming degree of $v_4=0$,
Incoming degree of $v_5=1$,
Incoming degree of $v_6=1$,

Incoming degree of $v_7 = 1$,
Incoming degree of $v_8 = 1$.

Since incoming degree of the vertex v_4 is 0, it follows that v_4 is root.

Further,

Outgoing degree of $v_1 = 0$,

Outgoing degree of $v_3 = 0$,

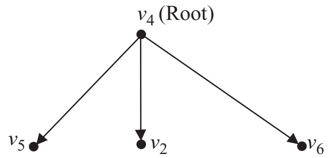
Outgoing degree of $v_7 = 0$,

Outgoing degree of $v_8 = 0$.

Therefore, v_1, v_3, v_7, v_8 are leaves. Also,

Outgoing degree of $v_2 = 2$,
Outgoing degree of $v_4 = 3$,
Outgoing degree of $v_5 = 1$,
Outgoing degree of $v_6 = 1$.

Now the root v_4 is connected to v_2, v_5 and v_6. So, we have the following diagram:



Now v_2 is connected to v_1 and v_3, v_5 is connected to v_8, v_6 is connected to v_7. Thus, we have the graph shown in Figure 2.161.

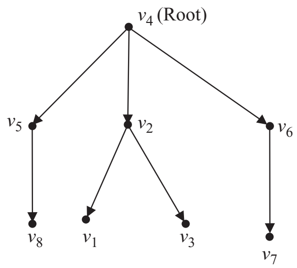


Figure 2.161

We thus have a connected acyclic graph and so (V, E) is a rooted tree with root v_4.

EXAMPLE 2.63

Let F be the set of all female descendent of a lady v_0. We define a relation T on F as follows: if $v_1, v_2, \in F$, then $v_1\ T\ v_2$ if v_1 is mother of v_2. Show that the relation T on F is a rooted tree with root v_0. Is this relation an equivalence relation?

Solution. The graph of T is cycle free, because descendents cannot be parents of their parents. The graph is also connected. Hence T is rooted tree. We further note that

(i) Since v_1 is not mother of itself, $(v_1, v_1) \notin T$.
(ii) If $(v_1, v_2) \in T$, then v_1 is mother of v_2. But then v_2 cannot be mother of v_1. Hence $(u_1, v_2) \in T$ does not imply $(v_2, v_1) \in T$.
(iii) If $(v_1, v_2) \in T$, $(v_2, v_3) \in T$, then

$$v_1 \text{ is mother of } v_2 \quad \text{and} \quad v_2 \text{ is mother of } v_3.$$

This does not imply that v_1 is mother of v_3, i.e., $(v_1, v_3) \notin T$. Thus T is irreflexive, asymmetric and is not transitive. Hence T is not an equivalence relation.

Consider the rooted trees T and T' shown in the Figure 2.162, where T is family tree of a man who has two sons, with the elder son having no child and younger having three children.

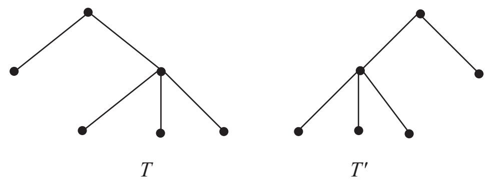


Figure 2.162

Although T'(as a graph) is isomorphic to T, it could be the family tree of another man whose elder son has three children and whose younger son has no child.

The above discussion motivates the following definitions:

Definition 2.99. A rooted tree in which the edges incident from each branch node are labelled with integers 1, 2, 3, … is called an **ordered tree**.

2.13 ISOMORPHISM OF TREES

Definition 2.100. Two ordered trees are said to be **isomorphic** if (i) there exists a one-to-one correspondence between their vertices and edges and that preserves the incident relation (ii) labels of the corresponding edges match.

In view of this definition, the ordered trees shown in Figure 2.163 are not isomorphic.

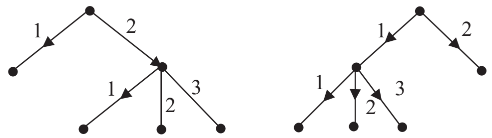


Figure 2.163

EXAMPLE 2.64

Show that the tree T_1 and T_2 shown in the diagram below are isomorphic.

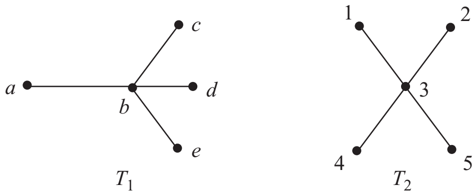


Figure 2.164

Solution. We observe that in the tree T_1,

$$\deg(b)=4.$$

In the tree T_2,

$$\deg(3)=4.$$

Further, $\deg(a)=\deg(1)=1$, $\deg(c)=\deg(2)$, $\deg(d)=\deg(4)=\deg(e)=1=\deg(5)$ and $\deg(b)=\deg(3)=4$. Thus we may define a function f from the vertices of T_1 to the vertices of T_2 by

$$f(a)=1,\quad f(b)=3,\quad f(c)=2,\quad f(d)=4,\quad f(e)=5.$$

This is a one-to-one and onto function. Also, adjacency relation is preserved because if v_i and v_j are adjacent vertices in T_1, then $f(v_i)$ and $f(v_j)$ are adjacent vertices in T_2. Hence T_1 is isomorphic to T_2.

EXAMPLE 2.65

Show that the tree T_1 and T_2, shown in the Figure 2.165 are isomorphic

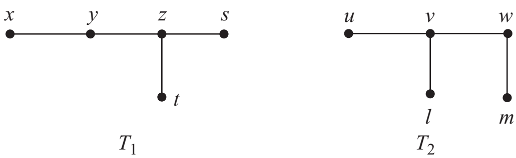


Figure 2.165

Solution. Let f be a function defined by

$$f(z)=v,\quad f(y)=w,\quad f(x)=m,$$
$$f(s)=u,\quad f(t)=l.$$

Then f is an one-one, onto mapping which preserves adjacency. Hence, T_1 and T_2 are isomorphic.

Remark 2.14. There are three non-isomorphic trees (Figure 2.166) with five vertices:

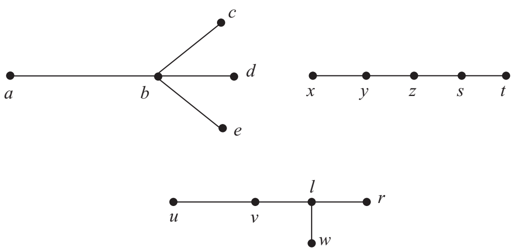


Figure 2.166

Definition 2.101. Let T_1 and T_2 be rooted trees with roots r_1 and r_2, respectively. Then T_1 and T_2 are **isomorphic** if there exists a one-one, onto function f from the vertex set of T_1 to the vertex set of T_2 such that

(i) Vertices v_i and v_j are adjacent in T_1 if and only if the vertices $f(v_i)$ and $f(v_j)$ are adjacent in T_2,
(ii) $f(r_1)=r_2$.

The function f is then called an **isomorphism**.

EXAMPLE 2.66

Show that the trees T_1 and T_2 shown in Figure 2.167 are isomorphic.

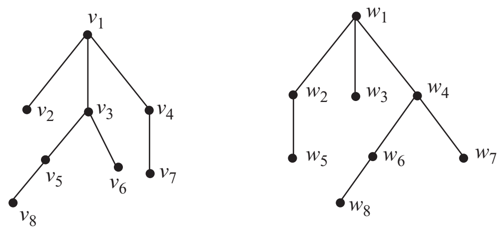


Figure 2.167

Solution. We observe that T_1 and T_2 are rooted trees. Define f: (vertex set of T_1)$\rightarrow$(vertex set of T_2) by

$$f(v_1)=w_1,\quad f(v_2)=w_3,\quad f(v_3)=w_4$$
$$f(v_4)=w_2,\quad f(v_5)=w_6,\quad f(v_6)=w_7$$
$$f(v_7)=w_5,\quad f(v_8)=w_8.$$

Then f is one-to-one and adjacency relation is preserved. Hence f is an isomorphism and so the rooted trees T_1 and T_2 are isomorphic

EXAMPLE 2.67

Show that the rooted trees shown in Figure 2.168 are not isomorphic:

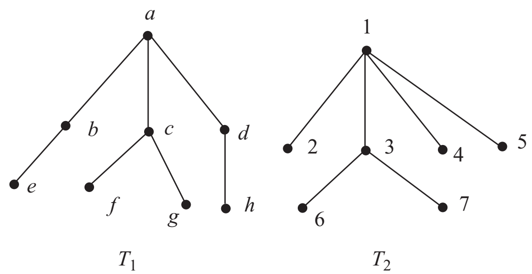


Figure 2.168

Solution. We observe that the degree of root in T_1 is 3, whereas the degree of root in T_2 is 4. Hence T_1 is not isomorphic to T_2.

Definition 2.102. An ordered tree in which every branch node has at most n offspring is called a n-**ary tree** (or n-**tree**).

Definition 2.103. An n-ary tree is said to be **fully** n-**ary tree** (**complete** n-**ary tree** or **regular** n-**ary tree**) if every branch node has exactly n offspring.

Definition 2.104. An ordered tree in which every branch node has at most two offspring is called a **binary tree** (or **2-tree**).

Definition 2.105. A binary tree in which every branch node (internal vertex) has exactly two offspring is called a **fully binary tree**.

For example, the tree given in Figure 2.169 is a binary tree,

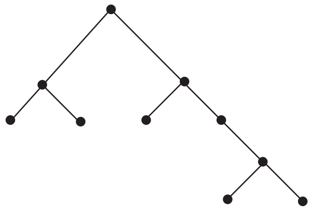


Figure 2.169

whereas the tree shown in Figure 2.170 is a fully binary tree.

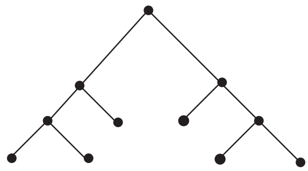


Figure 2.170

Definition 2.106. Let T_1 and T_2 be binary trees with roots r_1 and r_2, respectively. Then T_1 and T_2 are **isomorphic** if there is a one-one, onto function f from the vertex set of T_1 to the vertex set of T_2 satisfying

(i) Vertices v_i and v_j are adjacent in T_1 if and only if the vertices $f(v_i)$ and $f(v_j)$ are adjacent in T_2
(ii) $f(r_1)=r_2$
(iii) v is a left child of w in T_1 if and only if $f(v)$ is a left child of $f(w)$ in T_2
(iv) v is a right child of w in T_1 if and only if $f(v)$ is a right child of $f(w)$ in T_2

The function f is then called an **isomorphism** between binary trees T_1 and T_2.

EXAMPLE 2.68

Show that the trees given in Figure 2.171 are isomorphic.

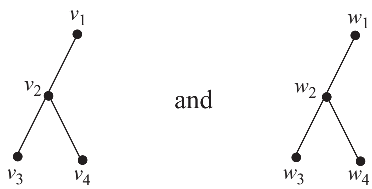


Figure 2.171

Solution. Define f by $f(v_i)=w_i$, $i=1, 2, 3, 4$. Then f satisfies all the properties for isomorphism. Hence T_1 and T_2 are isomorphic.

EXAMPLE 2.69

Show that the trees given in Figure 2.172 are not isomorphic.

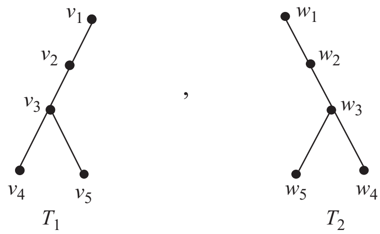


Figure 2.172

Solution. Since the root v_1 in T_1 has a left child but the root w_1 in T_2 has no left child, the binary trees are not isomorphic.

Definition 2.107. Let v be a branch node of a binary tree T. The **left subtree** of v is the binary tree whose root is the left child of v, whose vertices consists of the left child of v and all its descendents and whose edges consists of all those edges of T that connects the vertices of the left subtree together.

The **right subtree** can be defined analogously.

For example, the left subtree and the right subtree of v in the tree (Figure 2.173)

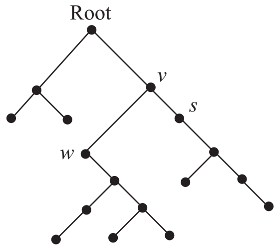


Figure 2.173

are, respectively the trees shown in Figure 2.174.

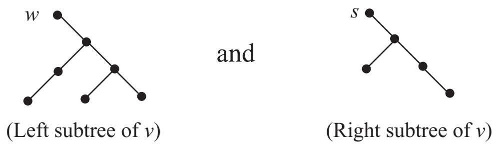


Figure 2.174

2.14 REPRESENTATION OF ALGEBRAIC EXPRESSIONS BY BINARY TREES

Binary trees are used in computer science to represent algebraic expressions involving parentheses. For example, the binary trees (Figure 2.175) represent the expressions, $a+b$, a/b and $b * (c * d)$, respectively.

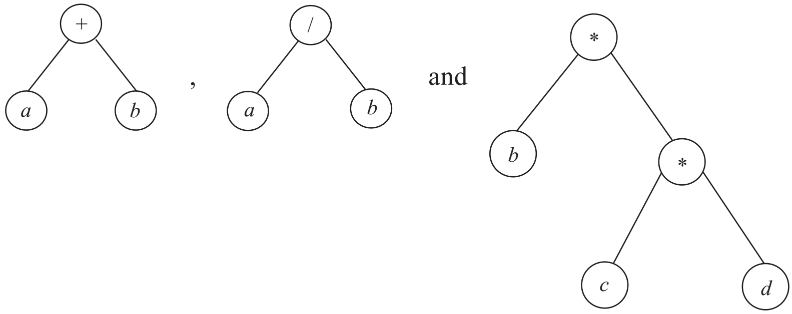


Figure 2.175

Thus, the **central operator acts as root of the tree.**

EXAMPLE 2.70

Draw a binary tree to represent

(i) $(2-(3\times x))+((x-3)-(2+x))$

(ii) $ab-(c/(d+e))$.

Solution

(i) In this expression + is the central operator. Therefore the root of tree is + . The binary tree is shown in Figure 2.176.

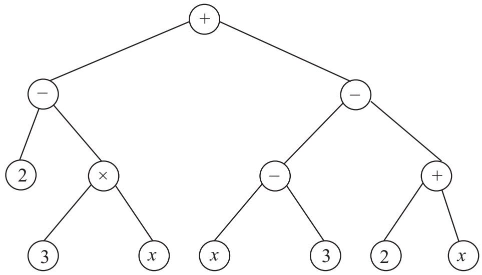


Figure 2.176

(ii) Here the central operator is −. Therefore it is the root of the tree. We have the following binary tree (Figure 2.177) to represent this expression:

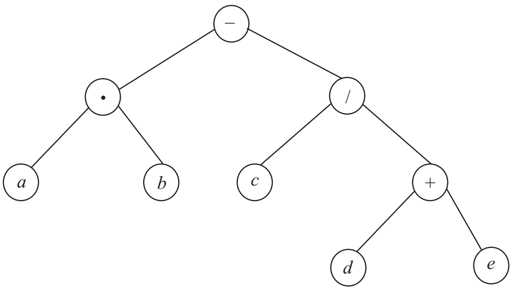


Figure 2.177

EXAMPLE 2.71

Consider a single elimination tournament with eight players in a wrestling game. How many rounds will be there to declare the champion and how many games are to be played?

Solution. In this example, the leaves of the tree shown in Figure 2.178 will represent player, the branch nodes will represent winners in the first round and the second round. Then the tree showing the tournament shall be as given below: The graph of single elimination tournament is a full binary tree. The champion shall be at the root.

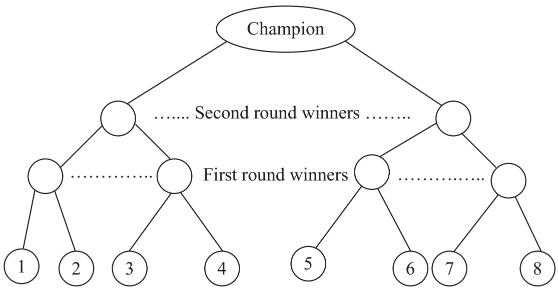


Figure 2.178

We note that there shall be three rounds to declare the champion. Let

$$
\begin{aligned}
i &= \text{Number of games played} \\
&= \text{number of internal vertices} \\
p &= \text{Number of leaves} = \text{number of players.}
\end{aligned}
$$

Then, we see that

$$i = p - 1 = 7.$$

In general, for complete n-ary tree, we will have

$$(n-1)\, i = p - 1.$$

To derive this formula we first prove the following result:

Theorem 2.23. If T is a full binary tree with i internal vertices, then T has $i+1$ terminal vertices (leaves) and $2i+1$ total vertices.

Proof: The vertices of T consist of the vertices that are children (of some parent) and the vertices that are not children (of any parent). There is a non-child (the root). Since there are i internal vertices, each having two children, there are $2i$ children. Thus the total number of vertices of T is $2i+1$ and the number of terminal vertices is

$$(2i+1) - i = i + 1.$$

This completes the Proof.

In the context of above example, we have

$$
\begin{aligned}
\text{Number of leaves} &= p = i + 1 \\
&\Rightarrow i = p - 1.
\end{aligned}
$$

Remark 2.15. In case of full n-ary tree, if i denotes the number of branch nodes, then total number of vertices of T is $ni+1$ and the number of terminal vertices is

$$ni + 1 - i = i(n-1) + 1.$$

If p is the number of terminal vertices, then

$$p = i(n-1) + 1,$$

which yields

$$(n-1)\, i = p - 1.$$

EXAMPLE 2.72

Find the minimum number of extension cords, each having four outlets, required to connect 22 bulbs to a single electric outlet.

Solution. Clearly, the graph of the problem is a regular quaternary tree with 22 leaves.

Let i denote the internal vertices and p denote the number of leaves, then using the expression $(n-1)i=p-1$, we have

$$(4-1)i = 22-1 \quad \text{or} \quad i = \frac{21}{3} = 7.$$

Thus seven extension cords, as shown in Figure 2.179, are required.

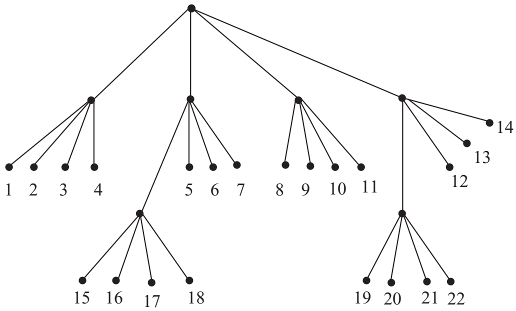


Figure 2.179

EXAMPLE 2.73

A computing machine has been given instruction which computes the sum of three numbers. How many times the addition instruction will be executed to perform the sum of 11 numbers?

Solution. In this problem, we will have a regular ternary tree with 11 leaves. Thus $n=3$, $p=11$ and so the relation $(n-1)i=p-1$ yields

$$i = \frac{p-1}{n-1} = \frac{10}{2} = 5.$$

Theorem 2.24. If a binary tree of height h has p leaves, then

$$\log_2 p \le h \quad \text{or} \quad p \le 2^h.$$

Proof: It is sufficient to prove that

$$p \le 2^h, \tag{1}$$

because then taking logarithm to the base 2 of both sides, we will get

$$\log_2 p \le h.$$

We shall prove the inequality (1) by induction on h.

If $h=0$, then the binary tree consists of a single vertex and $p=1$ and so (1) is satisfied.

Now suppose that (1) holds for a binary tree whose height is less than h. We shall prove that it holds for a tree of height h. So let T be a tree with height $h>0$ with p leaves. We consider the following two cases:

(a) Suppose that the root of T has only one child (Figure 2.180).

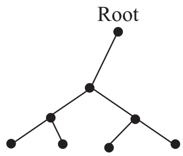


Figure 2.180

Eliminating the root and the edge incident on the root, we get a tree T' of height $h-1$ in which number of leaves is same as in T. Using induction hypothesis on T', we get $p \le 2^{h-1}$. Since $2^{h-1} < 2^h$, we have $p \le 2^h$ and the inequality holds in this case.

(b) Suppose the root of T has offspring's v and w (Figure 2.181).

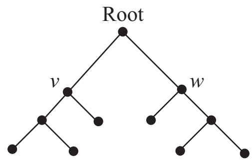


Figure 2.181

Let T_1 be the subtree rooted at v whose height is h_1 and has p_1 leaves. Similarly, let T_2 be the subtree rooted at w whose height is h_2 and has p_2 leaves. Using mathematical induction, to these trees, we have $p_1 \le 2^{h_1}$ and $p_2 \le 2^{h_2}$. But the leaves of T consist of the leaves of T_1 and T_2. Hence,

$$p = p_1 + p_2 \le 2^{h_1} + 2^{h_2}$$

$$\le 2^{h-1} + 2^{h-1} = 2^{h-1}(1+1) = 2^h.$$

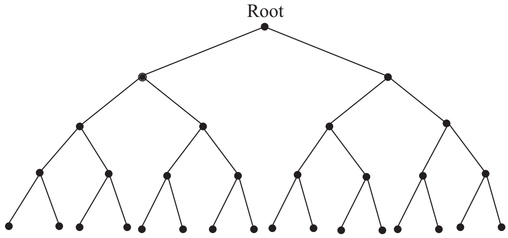
Hence, by mathematical induction, the result holds.

It can be proved on the same lines that an n-**ary tree of height** h **has atmost** n^h **leaves**.

EXAMPLE 2.74

Find the number of terminal vertices in a regular binary tree of height 4.

Solution. Let p be the number of terminal vertices. We are given that height (h) of the tree is 4. So, we must have $p \le 2^4 = 16$. Since the tree is regular, we have $p = 16$.



(A regular binary tree of height 4 with 16 leaves)

Figure 2.182

EXAMPLE 2.75

Does there exist a full binary tree with 12 internal vertices and 15 leaves?

Solution. We know that if i is the number of branch nodes in a full binary tree, then the number of leaves is $i+1$. Therefore for a tree with 12 branch nodes, the number of leaves should be 13 and not 15. Hence such tree does not exist.

EXAMPLE 2.76

Is there a binary tree with height 6 and 65 leaves?

Solution. No, since

$$p \le 2^h = 2^6 = 64.$$

We now state some results without proof on n-ary tree.

Theorem 2.25. The number b_n of different binary trees on n vertices is given by

$$b_n = \frac{1}{n+1}\binom{2n}{n}.$$

Theorem 2.26. A regular n-ary tree of height h has **at least** $n+(n-1)\,(h-1)$ leaves.

Theorem 2.27. If in a rooted regular n-ary tree,

I = Sum of the path length of all branch nodes

E = Sum of the path length of all leaves

i = Number of branch nodes

then

$$E = (n-1)I + n\,i$$

Thus, in particular, for a full binary tree,

$$E = I + 2i.$$

EXAMPLE 2.77

How many binary trees are possible on three vertices?

Solution. We know that the number of different trees on n vertices is given by

$$b_n = \frac{1}{n+1}\binom{2n}{n}.$$

So we have

$$b_n = \frac{1}{3+1}\binom{6}{3} = \frac{1}{4}\left(\frac{6\times5\times4\times3\times2\times1}{3\times2\times1\times3\times2\times1}\right) = 5 \text{ trees}$$

and those are shown in Figure 2.183.

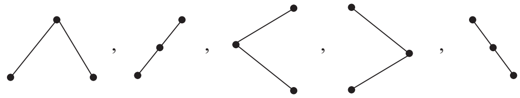


Figure 2.183

2.15 SPANNING TREE OF A GRAPH

Definition 2.108. Let G be a graph, then a subgraph of G which is a tree is called **tree of the graph**.

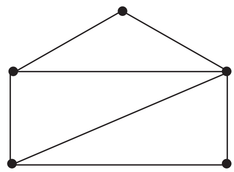
Definition 2.109. A **spanning tree** for a graph G is a subgraph of G that contains every vertex of G and is a tree.

Thus,

"A **spanning tree** for a graph G is a spanning subgraph of G which is a tree".

EXAMPLE 2.78

Determine a tree and a spanning tree for the connected graph given below (Figure 2.184):



G

Figure 2.184

Solution. The given graph G contains circuits and we know that removal of the circuits gives a tree. So, we note that the Figure 2.185 is a tree.

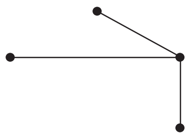


Figure 2.185

And the Figure 2.186 is a spanning tree of the graph G.

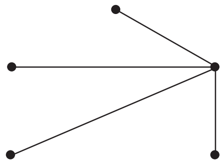


Figure 2.186

EXAMPLE 2.79

Find all spanning trees for the graph G shown below (Figure 2.187).

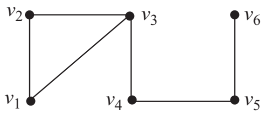


Figure 2.187

Solution. The given graph G has a circuit $v_1\ v_2\ v_3\ v_1$. We know that removal of any edge of the circuit gives a tree. So the spanning trees of G are shown in Figure 2.188.

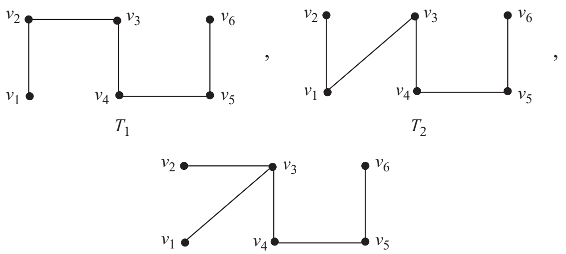


Figure 2.188

Remark 2.16. We know that a tree with n vertices has exactly $n-1$ edges. Therefore if G is a connected graph with n vertices and m edges, a spanning tree of G must have $n-1$ edges. Hence the number of edges that must be removed before a spanning tree is obtained must be

$$m-(n-1)=m-n+1.$$

For illustration, in EXAMPLE 2.79, $n=6$, $m=6$, so, we had to remove one edge to obtain a spanning tree.

Definition 2.110. A **branch** of a tree is an edge of the graph that is in the tree.

Definition 2.111. A **chord** (or a **link**) of a tree is an edge of the graph that is not in the tree.

It follows from the above remark that the number of chords in a tree is equal to $m-n+1$, where n is the number of vertices and m is the number of edges in the graph related to the tree.

Definition 2.112. The set of the chords of a tree is called the **complement of the tree**.

EXAMPLE 2.80

Consider the graph discussed in the above example. We note that the edge (v_2, v_3) is a branch of the tree T_1, whereas (v_1, v_3) is a chord of the tree T_1.

Theorem 2.28. A graph G has a spanning tree if and only if G is connected.

Proof: Suppose first that a graph G has a spanning tree T. If v and w are vertices of G, then they are also vertices in T and since T is a tree there is a path from v to w in T. This path is also a path in G. Thus every two vertices are connected in G. Hence G is connected.

Conversely, suppose that G is connected. If G is acyclic, then G is its own spanning tree and we are done. So suppose that G contains a cycle C_1. If we remove an edge from the cycle, the subgraph of G so obtained is also connected. If it is acyclic, then it is a spanning tree and we are done. If not, it has at least one circuit, say C_2. Removing one edge from C_2, we get a subgraph of G which is connected. Continuing in this way, we obtain a connected circuit free subgraph T of G. Since T contains all vertices of G, it is a spanning tree of G.

Theorem 2.29. (Cayley's Formula). The number of spanning trees of the complete graph K_n, $n \geq 2$ is n^{n-2}.

(Proof of this formula is out of scope of this book)

EXAMPLE 2.81

Find all the spanning trees of K_4 (Figure 2.189).

Solution. According to Cayley's formula, K_4 has $4^{4-2}=4^2=16$ different spanning trees.

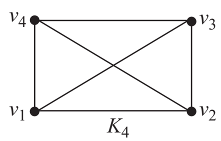


Figure 2.189

Here $n=4$, so the number of edges in any tree should be $n-1=4-1=3$. But here number of edges is equal to 6. So to get a tree, we have to remove three edges of K_4. The 16 spanning trees so obtained are shown in Figure 2.190.

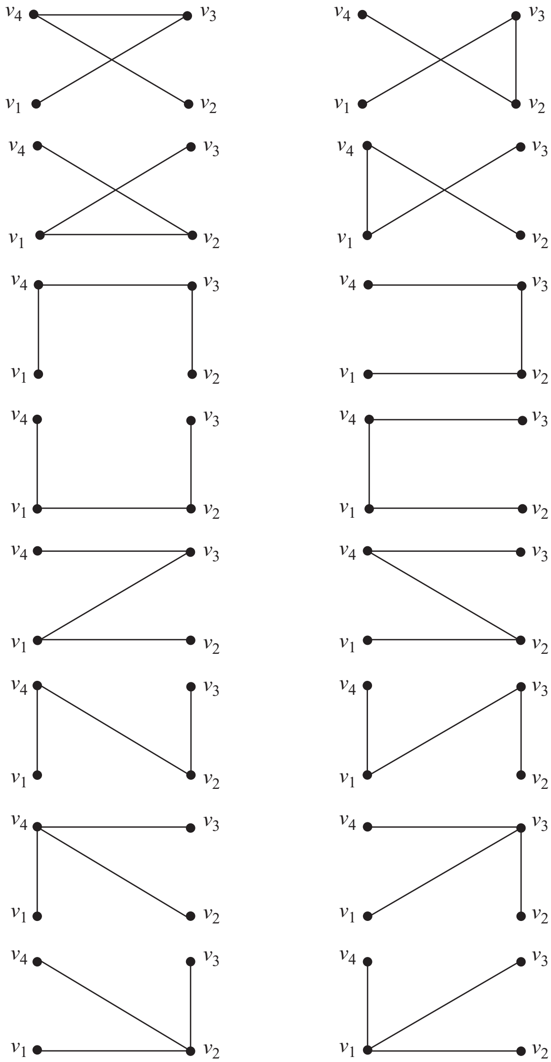


Figure 2.190

2.16 SHORTEST PATH PROBLEM

Let s and t be two vertices of a connected weighted graph G. Shortest path problem is to find a **path from s to t whose total edge weight is minimum.**

We now discuss algorithm due to $E \cdot W$. Dijkstra which efficiently solves the shortest path problem. The idea is to grow a Disjkstra tree, starting at the vertex s, by adding, at each iteration, a frontier edge, whose non-tree endpoint is as close as possible to s. The algorithm involves assigning labels to vertices.

For each tree vertex x, let dist $[x]$ denote the distance from vertex s to x and for each edge e in the given weighted graph G, let $w(e)$ be its edge-weight.

After each iteration, the vertices in the Dijkstra tree (the labelled vertices) are those to which the shortest paths from s have been found.

Priority of the Frontier Edges: Let e be a frontier edge and let its P value be given by

$$P(e)=\text{dist }[x]+w(e),$$

where x is the labelled endpoint of e and $w(e)$ is the edge-weight of e. Then,

(i) The edge with the smallest P value is given the **highest priority**.
(ii) The P value of this highest priority edge e gives the distant from the vertex s to the unlabelled endpoint of e.

We are now in a position to describe Dijkstra's shortest path algorithm.

2.16.1 Dijkstra's Shortest Path Algorithm

Input: A connected weighted graph G with non-negative edge-weights and a vertex s of G.
Output: A spanning tree T of G, rooted at the vertex s, whose path from s to each vertex v is a shortest path from s to v in G and a vertex labelling giving the distance from s to each vertex.

Initialize the Dijkstra tree T as vertex s.
Initialize the set of frontier edges for the tree T as empty.

$$\text{dist: }[s]=0.$$

Write label 0 on vertex s.
While Dijkstra tree T does not yet span G.
For each frontier edge e for T,

Let x be the labelled endpoint of edge e.
Let y be the unlabelled endpoint of edge e.
Set $P(e) = \text{dist }[x] + w(e)$
Let e be a frontier edge for T that has smallest P value
Let x be the labelled endpoint of edge e
Let y be the unlabelled endpoint of edge e
Add edge e (and vertex y) to tree T

$$\text{dist }[y]: P(e)$$

Write label dist $[y]$ on vertex y.

Return Dijkstra tree T and its vertex labels.

EXAMPLE 2.82

Apply Dijkstra algorithm to find shortest path from s to each other vertex in the graph given in Figure 2.191.

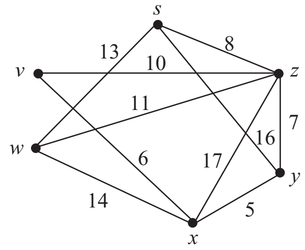


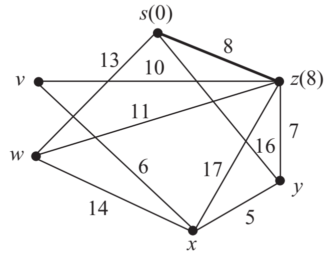
Figure 2.191

If t is the labelled endpoint of edge e, then P values are given by

$$P(e) = \text{dist }[t] + w(e),$$

where dist $[t]$ = distance from s to t and $w(e)$ is the edge weight of edge e. For each vertex v, dist $[v]$ appears in the parenthesis. Iteration tree at the end of each iteration is **drawn in dark line**.

Iteration 1



dist $[s]=0$ $\quad P(sw)=13$ (minimum)

dist $[z]=8$ $\quad P(zy)=8+7=15$

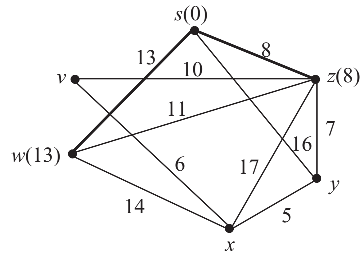
$P(sy)=16$

$P(zv)=8+10=18$

$P(zw)=8+11=19$

$P(zx)=8+17=25$

Iteration 2



dist $[s]=0$ $\quad P(zy)=8+7=15$ (minimum)

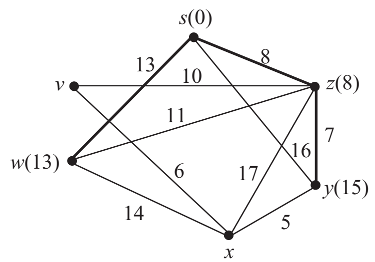
dist $[z]=8$ $\quad P(zx)=8+17=25$

dist $[w]=13$ $\quad P(zv)=8+10=18$

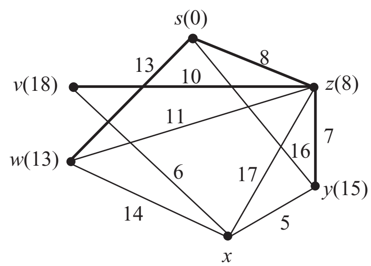
$P(sy)=16$

$P(wx)=13+14=27$

Iteration 3



Iteration 4



dist $[s]=0$ $\quad P(zv)=18$ (minimum)

dist $[z]=8$ $\quad P(zx)=8+17=25$

dist $[w]=13$ $\quad P(wx)=13+14=27$

dist $[y]=15$ $\quad P(yx)=15+5=20$

Dist $[s]=0$ $\quad P(yx)=20$ (minimum)

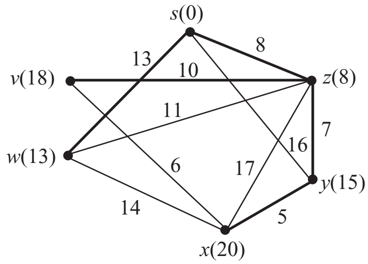
Dist $[z]=8$ $\quad P(zx)=8+17=25$

Dist $[w]=13$ $\quad P(vx)=18+6=24$

Dist $[y]=15$ $\quad P(wx)=13+14=27$

Dist $[v]=18$

Iteration 5



dist $[s]=0$

dist $[z]=8$

dist $[w]=13$

dist $[y]=15$

dist $[v]=18$

dist $[x]=20$,

which are the required shortest paths from s to any other point. The Dijkstra tree is shown in dark lines in Figure 2.192.

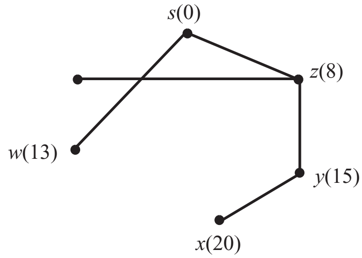


Figure 2.192

EXAMPLE 2.83

Apply Dijkstra algorithm to find the shortest path from s to t in the graph given below (Figure 2.193).

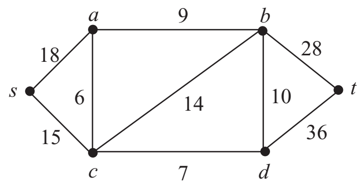
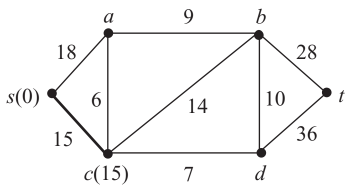


Figure 2.193

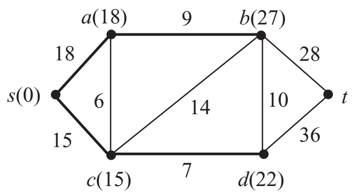
Solution. Let x be the labelled endpoint of the edge e, then P values are given by $P(e)=\text{dist }[x]+w(e)$, where dist $[x]$ denotes the distance of x from s and $w(e)$ is the edge weight of e.

For each vertex v, dist $[v]$ appears in the bracket. Iteration tree at the end of each iteration is shown by dark lines:

Iteration 1



Iteration 4



dist $[s]=0$ $\quad P(s\ a)=18$ (minimum)

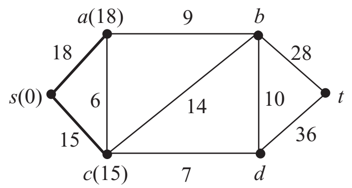
dist $[c]=15$ $\quad P(c\ a)=15+6=21$

$P(c\ b)=15+14=29$

$P(c\ d)=15+7=22$

$P(c\ t)=\infty$

Iteration 2



dist $[s]=0$ $\quad P(a\ b)=18+9=27$

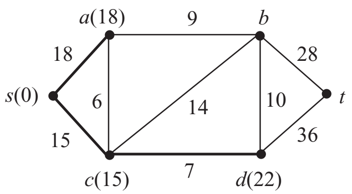
dist $[c]=15$ $\quad P(c\ b)=15+14=29$

dist $[a]=18$ $\quad P(c\ d)=15+7=22$ (minimum)

$P(c\ t)=\infty$

$P(a\ t)=\infty$

Iteration 3



dist $[s]=0$ $\quad P(c\ b)=15+14=29$

dist $[c]=15$ $\quad P(a\ b)=18+9=27$ (minimum)

dist $[a]=18$ $\quad P(d\ b)=22+10=32$

dist $[d]=22$ $\quad P(d\ t)=22+36=58$

dist $[s]=0$ $\quad P(d\ t)=22+36=58$

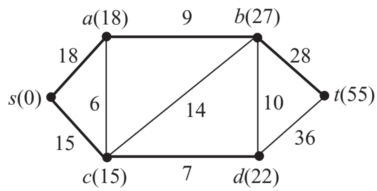
dist $[c]=15$ $\quad P(b\ t)=27+28=55$ (minimum)

dist $[a]=18$

dist $[d]=22$

dist $[b]=27$

Iteration 5



dist $[s]=0$

dist $[c]=15$

dist $[a]=18$

dist $[s]=0$

dist $[c]=15$

dist $[a]=18$

dist $[d]=22$

dist $[b]=27$

dist $[t]=55$

Hence, **the length of the shortest path (*s*, *a*, *b*, *t*) from *s* to *t* is 55** and the Dijkstra's tree is shown in Figure 2.194.

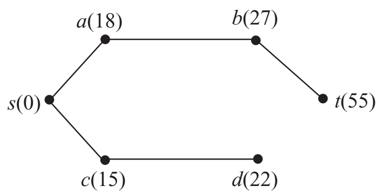


Figure 2.194

EXAMPLE 2.84

Find a shortest path from s to t and its length for the graph given below:

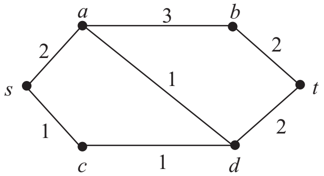


Figure 2.195

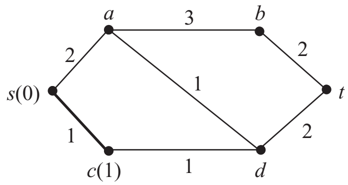
Solution. Let x be the labelled endpoint of edge e, then P values are given by

$$P(c) = \text{dist}\,[x] + w(e),$$

where dist $[x]$ denotes the distance from s to x and $w(e)$ is the weight of the edge e.

For each vertex v, dist $[v]$ appears in the bracket. Iteration tree at the end of each iteration is shown in dark lines.

Iteration 1



dist $[s]=0$ $P(c\ d)=2$

dist $[c]=1$ $P(s\ a)=2$ (minimum)

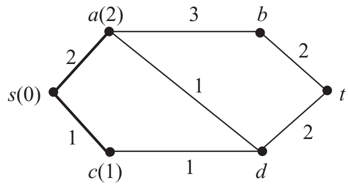
$P(s\ b)=\infty$

$P(s\ t)=\infty$

$P(d\ t)=\infty$

$P(b\ t)=\infty$

Iteration 2



dist $[s]=0$ $P(a\ b)=2+3=5$

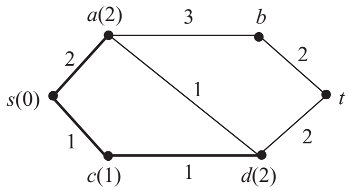
dist $[c]=1$ $P(c\ d)=1+1=2$ (minimum)

dist $[a]=2$ $P(d\ t)=\infty$

$P(b\ t)=\infty$

$P(a\ d)=3$

Iteration 3



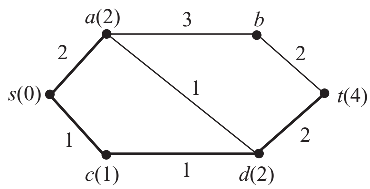
dist $[s]=0$ $P(d\ t)=2$ (minimum)

dist $[c]=1$ $P(a\ b)=5$

dist $[a]=2$ $P(b\ t)=\infty$

dist $[d]=2$

Iteration 4



dist $[s]=0$ $P(a\ b)=5$ (minimum)

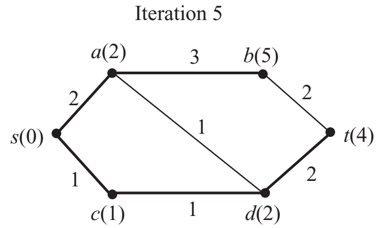
dist $[c]=1$ $P(b\ t)=\infty$

dist $[a]=2$

dist $[d]=2$

dist $[t]=4$

dist $[s]=0$
dist $[c]=1$
dist $[a]=2$
dist $[d]=2$
dist $[t]=4$
dist $[b]=5$



Thus, the Dijkstra tree is

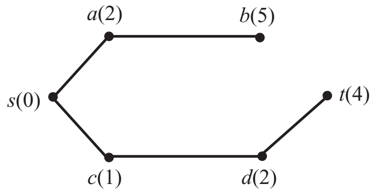


Figure 2.196

Thus the shortest path is *scdt* and its length is 4.

2.16.2 Shortest Path if All Edges Have Length 1

If all edges in a connected graph G have length 1, then a **shortest path** $v_1 \rightarrow v_k$ is the path that has the smallest number of edges among all paths $v_1 \rightarrow v_k$ in the given graph G.

Moore's Breadth First Search Algorithm

This method of finding shortest path in a connected graph G from a vertex s to a vertex t is used when all edges have length 1.

Input: Connected graph $G=(V, E)$, in which one vertex is denoted by s and one by t and each edge (v_i, v_j) has length 1.

Initially all vertices are unlabelled.

Output: A shortest path $s \rightarrow t$ in $G=(V, E)$.

1. Label s with 0.
2. Set $v_i=0$.
3. Find all unlabelled vertices **adjacent to a vertex labelled** v_i.
4. Label the vertices just found with $v_{i+1.}$
5. If vertex t is labelled, then "**back tracking**" gives the shortest path. If k is label of t(i.e., $t=v_k$), then

Output: $v_k, v_{k-1}, \ldots, v_1, 0$.

Else increase i by 1. Go to Step 3.

End Moore.

Remark 2.17. There could be several shortest paths from s to t.

EXAMPLE 2.85

Use BFS algorithm to find shortest path from s to t in the connected graph G given in Figure 2.197.

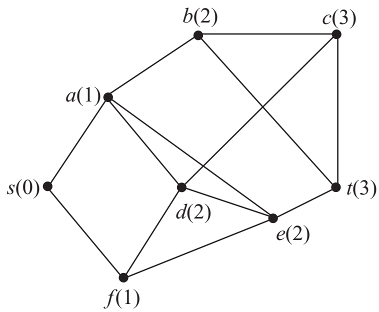


Figure 2.197

Solution. Label s with 0 and then label the adjacent vertices with 1. Thus two vertices have been labelled by 1. Now label the adjacent vertices of all vertices labelled by 1 with label 2. Thus three vertices have been labelled with 2. Label the vertices adjacent to these vertices (labelled by 2) with 3. Thus two vertices have been labelled with 3. We have reached t. Now backtracking yields the following three shortest paths:

(i) $t(3), e(2), f(1), s(0)$, that is, $s\,f\,e\,t$,
(ii) $t(3), b(2), a(1), s(0)$, that is $s\,a\,b\,t$,
(iii) $t(3), e(2), a(1), s(0)$, that is, $s\,a\,e\,t$.

Thus there are three possible shortest paths of length 3.

EXAMPLE 2.86

Find a shortest path $s \to t$ in the graph given in Figure 2.198 (using Moore's BFS algorithm).

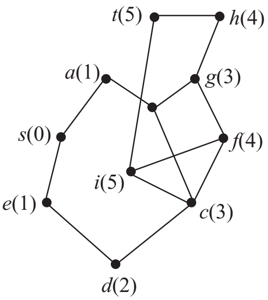


Figure 2.198

Solution. Label s with 0 and then label the adjacent vertices with 1. Thus two vertices have been marked as 1. Label the adjacent vertices of al vertices labelled as 1 with 2. Thus two vertices have been marked as 2. Now label the adjacent vertices of all vertices labelled as 2 with 3. Thus two vertices have been marked as 3. Label the adjacent vertices of all vertices labelled as 3 with 4. Thus three vertices have been marked as 4. Now there is only one vertex t to be labelled. Label it with 5. We have reached at t.

Now backtracking gives the following shortest paths:

$t\,h\,g\,b\,a\,s$	or	$s\,a\,b\,g\,h\,t$	with length 5,
$t\,i\,c\,d\,e\,s$	or	$s\,e\,b\,c\,i\,t$	of length 5,
$t\,i\,c\,b\,a\,s$	or	$s\,a\,b\,c\,i\,t$	of length 5.

Thus, there are three possible shortest paths of length 5.

2.17 MINIMAL SPANNING TREE

Definition 2.113. Let G be a weighted graph. A spanning tree of G with minimum weight is called **minimal spanning tree of** G.

We discuss two algorithms to find a minimal spanning tree for a weighted graph G.

2.17.1 Prim Algorithm

Prim algorithm builds a minimal spanning tree T by expanding outward in connected links from some vertex. In this algorithm, one edge and one vertex are added at each step. The edge added is the one of least weight that connects the vertices already in T with those not in T.

Input: A connected weighted graph G with n vertices.
Output: The set of edges E in a minimal spanning tree.

1. Choose a vertex v_1 of G. Let $V=\{v_1\}$ and $E=\{\ \}$.
2. Choose a nearest neighbour v_i of V that is adjacent to v_j, $v_j \in V$ and for which the edge (v_i, v_j) does not form a cycle with members of E. Add v_i to V and add (v_i, v_j) to E.
3. Repeat Step 2 till number of edges in T is $n-1$. Then V contains all n vertices of G and E contains the edges of a minimal spanning tree for G.

Definition 2.114. A **greedy algorithm** is an algorithm that optimizes the choice at each iteration without regard to previous choices.

For example, **Prim algorithm is a greedy algorithm.**

EXAMPLE 2.87

Find a minimal spanning tree for the graph shown in Figure 2.199.

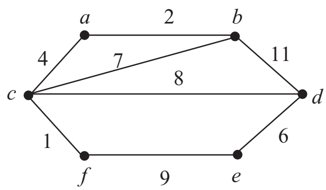


Figure 2.199

Solution. We shall use **Prim algorithm to find the required minimal spanning tree.** We note that number of vertices in this connected weighted graph is 6. Therefore the tree will have five edges.

We start with any vertex, say c. The nearest neighbour of c is f and (c, f) does not form a cycle. Therefore (c, f) is the first edge selected.

Now we consider the set of vertices $V=\{c, f\}$. The vertex a is nearest neighbour to $V=\{c, f\}$ and the edge (c, a) does not form a cycle with the member of set of edges selected so far. Thus,

$$E=\{(c, f), (c, a)\} \quad \text{and} \quad V=\{c, f, a\}.$$

The vertex b is now nearest neighbour to $V=\{c, f, a\}$ and the edge (a, b) do not form a cycle with the member of $E=\{(c, f), (c, a)\}$. Thus,

$$E=\{(c, f), (c, a), (a, b)\} \quad \text{and} \quad V=\{c, f, a, b\}.$$

Now the edge (b, c) cannot be selected because it forms a cycle with the members of E. We note that d is the nearest point to $V=\{c, f, a, b\}$ and (c, d) is the edge which does not form a cycle with members of $E=\{(c, f), (c, a), (a, b)\}$. Thus we get

$$E=\{(c, f), (c, a), (a, b)\}, (c, d)\},$$
$$V=\{c, f, a, b, d\}.$$

The nearest vertex to V is now e and (d, e) is the corresponding edge. Thus,

$$E=\{(c, f), (c, a), (a, b), (c, d), (d, e)\},$$
$$V=\{c, f, a, b, d, e\}$$

Since number of edges in the Prim tree is 5, the process is complete. The minimal spanning tree is shown in Figure 2.200:

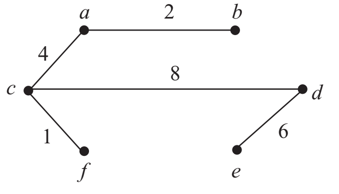


Figure 2.200

The length of the tree is $1+4+2+8+6=21$

EXAMPLE 2.88

Build the lowest-cost road system that will connect the cities in the graph given in Figure 2.201.

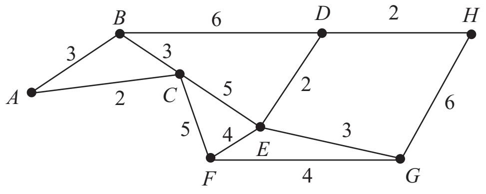


Figure 2.201

Solution. We shall use Prim algorithm to build this system. Here the number of vertices (Cities) is 8. So, the minimal spanning tree would have seven edges.

We start from any vertex, say A and have at the initial stage:

$$V=\{A\}, \quad E=\{\ \}.$$

The nearest neighbour of $V=\{A\}$ is C and the corresponding edge is (A, C). So we have

$$E=\{(A, C)\}, \quad V=\{A, C\}.$$

Now B is the nearest neighbour of $V=\{A, C\}$ and edge (C, B) does not form cycle with the edges in $E=\{(A, C)\}$. Here the edge (A, B) can also be chosen. Thus, choosing arbitrarily one out of (A, B) and (C, B), we have

$$E=\{(A, C), (C, B)\}, \quad V=\{A, C, B\}.$$

Now, E and F are both nearest neighbour to V and (C, E) or (C, F) do not form cycle with the members of $E=\{(A, C), (C, B)\}$. So we may choose any of E or F. Let us choose F and the corresponding edge (C, F). Then,

$$E=\{(A, C), (C, B), (C, F)\}, \quad V=\{A, C, B, F\}.$$

Now E and G are nearest neighbours of $V=\{A, C, B, F\}$ and both (F, E) and (F, G) do not form cycle with the members of $E=\{(A, C), (C, B), (C, F)\}$. Therefore we may pick up any of these vertices. We pick arbitrarily E.

Then,

$$E=\{(A, C), (C, B), (C, F), (F, E)\},$$
$$V=\{A, C, B, F, E\}.$$

Now D is the nearest neighbour of $V=\{A, C, B, F, E\}$ and (E, D) is the corresponding edge to be added to the tree. So, we have

$$E=\{(A, C), (C, B), (C, F), (F, E), (E, D)\},$$
$$V=\{A, C, B, F, E, D\}.$$

Now H is the nearest neighbour and (D, H) is the corresponding edge and we have

$$E=\{(A, C), (C, B), (C, F), (F, E), (E, D), (D, H)\},$$
$$V=\{A, C, B, F, E, D, H\}.$$

Now G is the nearest neighbour and (E, G) is the corresponding edge. So, we have

$$E=\{(A, C), (C, B), (C, F), (F, E), (E, D), (D, H), (E, G)\},$$
$$V=\{A, C, B, F, E, D, H, G\}$$

Since the number of edges in the Prim tree is 7, the process is complete. The minimal spanning tree is shown in Figure 2.202:

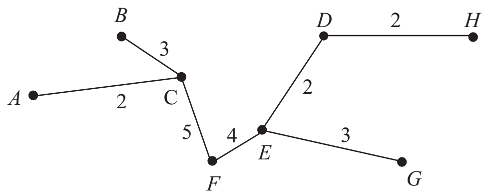


Figure 2.202

The total length of the roads is

$$2+3+5+4+2+2+3=21.$$

EXAMPLE 2.89

Using Prim algorithm, find the minimal spanning tree of the following graph (Figure 2.203):

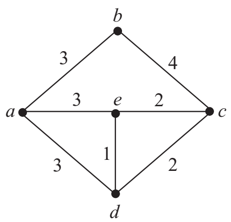


Figure 2.203

Solution. Pick up the vertex a. Then

$$E=\{\ \} \quad \text{and} \quad V=\{a\}.$$

The nearest neighbour of V is b, d or e and the corresponding edges are (a, b) or (a, d) or (a, e). We choose arbitrarily (a, b) and have

$$E=\{(a, b)\}, \quad V=\{a, b\}.$$

Now d is the nearest neighbour of $V=\{a, b\}$ and the corresponding edge (a, d) does not form cycle with (a, b). Thus we get

$$E=\{(a, b), (a, d)\}, \quad V=\{a, b, d\}.$$

Now e is the nearest neighbour of $\{a, b, d\}$ and (d, e) does not form cycle with $\{(a, b), (a, d)\}$.

Hence,

$$E=\{(a, b), (a, d), (d, e)\}, \quad V=\{a, b, d, e\}.$$

Now c is the nearest neighbour of $V=\{a, b, d, e\}$ and the corresponding edges are (e, c), (d, c). Thus we have, choosing (e, c),

$$E=\{(a, b), (a, d), (d, e), (e, c)\}, \quad V=\{a, b, d, e, c\}$$

Total weight $=3+3+1+2=9$.

(If we choose (d, c), then total weight is $3+3+1+2=9$).

The minimal tree is shown in Figure 2.204.

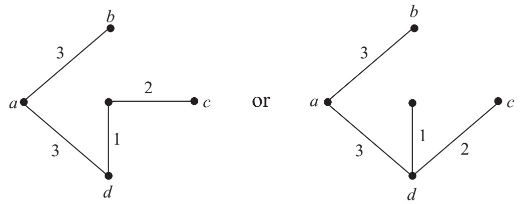


Figure 2.204

2.17.2 Kruskal's Algorithm

In Kruskal's algorithm, the edges of a weighted graph are examined one by one in order of increasing weight. At each stage, an edge with least weight out of edge-set remaining at that stage is added provided this additional edge does not create a circuit with the members of existing edge set at that stage. After $n-1$ edges have been added, these edges together with the n vertices of the connected weighted graph form a minimal tree.

Algorithm

Input: A connected weighted graph G with n vertices and the set $E=\{e_1, e_2, \ldots, e_k\}$ of weighted edges of G.

Output: The set of edges in a minimal spanning tree T for G.

Step 1. Initialize T to have all vertices of G and no edges.

Step 2. Choose an edge e_1 in E of least weight. Let

$$E^* = \{e_1\}, \quad E = E - \{e_1\}.$$

Step 3. Select an edge e_i in E of least weight that does not form circuit with members of E^*. Replace

$$E^* \text{ by } E^* \cup \{e_i\} \quad \text{and} \quad E \text{ with } E - \{e_i\}.$$

Step 4. Repeat Step 3 until number of edges in E^* is equal to $n-1$.

EXAMPLE 2.90

Use Kruskal's algorithm to determine a minimal spanning tree for the connected weighted graph G shown in Figure 2.205.

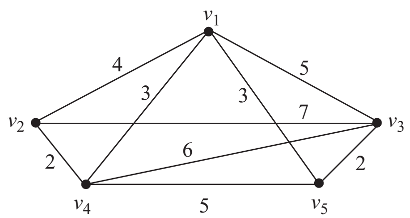


Figure 2.205

Solution. The given weighted graph has five vertices. The minimal spanning tree would, therefore, have four edges.

Let

$$E = \{(v_1, v_2), (v_1, v_4), (v_1, v_5), (v_2, v_3), (v_1, v_3), (v_2, v_4), (v_4, v_5), (v_5, v_3), (v_3, v_4)\}.$$

The edges (v_2, v_4) and (v_3, v_5) have minimum weight. We choose arbitrarily one of these, say (v_2, v_4).

Thus

$$E^* = \{(v_2, v_4)\},$$
$$E = E - \{(v_2, v_4)\}.$$

The edge (v_3, v_5) has minimum weight, so we pick it up. We have thus

$$E^* = \{(v_2, v_4), (v_3, v_5)\},$$
$$E = E - \{(v_2, v_4), (v_3, v_5)\}.$$

The edges (v_1, v_4) and (v_1, v_5) have minimum weight in the remaining edge set. We pick (v_1, v_4) say, as it does not form a cycle with E^*. Thus

$$E^* = \{(v_2, v_4), (v_3, v_5), (v_1, v_4),$$
$$E = E - \{(v_2, v_4), (v_3, v_5), (v_1, v_4)\}.$$

Now the edge (v_1, v_5) has minimum weight in $E - \{(v_1, v_4), (v_3, v_5), (v_1, v_4)\}$ and it does not form a cycle with E^*. So, we have

$$E^* = \{(v_2, v_4), (v_3, v_5), (v_1, v_4), (v_1, v_5)\}$$

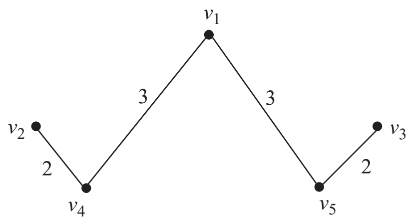
and

$$E = E - \{(v_2, v_4), (v_3, v_5), (v_1, v_4), (v_1, v_5)\}.$$

Thus all the four edges have been selected. The minimal tree has the edges.

$$(v_2, v_4), (v_3, v_5), (v_1, v_4), (v_1, v_5)$$

and is shown in the Figure 2.206.



(Minimal spanning tree)

Figure 2.206

EXAMPLE 2.91

Using Kruskal's algorithm, find the minimal spanning tree for the graph given in the Figure 2.207.

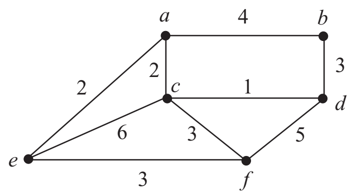


Figure 2.207

Solution. The given graph has six vertices, therefore the minimal spanning tree would have $6-1=5$ edges

Let

$$E=\{(a, b), (a, c), (b, d), (c, f), (d, f), (c, d), (a, e), (e, f), (e, c)\}.$$

Choose the edge (c, d) first because it has minimum weight. Therefore,

$$E^*=\{(c, d)\}, \quad E=E-\{(c, d)\}.$$

Now choose the edge (a, c) because it has minimum weight. So,

$$E^*=\{(c, d), (a, c), \quad E=E-\{(c, d), (a, c)\}.$$

Now choose the edge (a, e), since it has also minimum weight in $E-\{(c, d),(a, c)\}$ and it does not form cycle with E^*. So

$$E^*=\{(c, d), (a, c), (a, e)\}, \quad E=E-\{(c, d), (a, c), (a, e)\}.$$

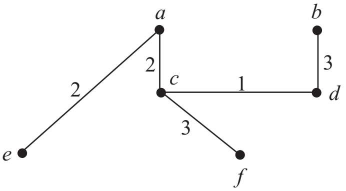
Now choose (b, d) since it has minimum weight and it does not form circuit with E^*. Thus

$$E^*=\{(c, d), (a, c), (a, e), (b, d)\},$$
$$E=E-\{(c, d), (a, c), (a, e), (b, d)\}.$$

Now choose (c, f) or (e, f) because they have equal and minimal weight. Let us choose (c, f). Then

$$E^*=\{(c, d), (a, c), (a, e), (b, d), (c, f)\},$$
$$E=E-\{(c, d), (a, c), (a, e), (b, d), (c, f)\}.$$

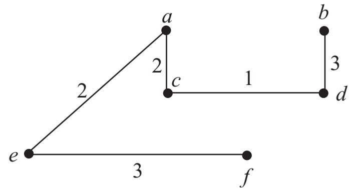
We have thus obtained all the five required edges. The minimal spanning tree is shown in the Figure 2.208.



(Minimal spanning tree)

Figure 2.208

Remark 2.18. In the above example, if we had chosen (e, f) in place of (c, f) in the last step, then the minimal spanning tree would have been as shown in the Figure 2.209.



(Minimal spanning tree)

Figure 2.209

EXAMPLE 2.92

Use Kruskal's algorithm to find a shortest spanning tree for the graph G given in the Figure 2.210.

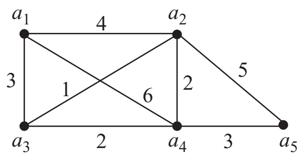


Figure 2.210

Solution. The given graph has five vertices. So the required spanning tree would have $5-1=4$ edges. Let

$$E=\{(a_1, a_2), (a_1, a_3), (a_3, a_4), (a_2, a_4), (a_2, a_5), (a_4, a_5), (a_2, a_3), (a_1, a_4)\}.$$

Choose the edge (a_2, a_3) since it has minimum weight. Then

$$E^*=\{(a_2, a_3)\},$$
$$E=E-\{a_2, a_3\}$$
$$= \{(a_1, a_2), (a_1, a_3), (a_3, a_4), (a_2, a_4), (a_2, a_5), (a_4, a_5), (a_1, a_4)\}.$$

Now choose the edge (a_2, a_4) or (a_3, a_4) having minimum weight in E. Let us take (a_2, a_4), say. Then

$$E^*=\{(a_2, a_3), (a_2, a_4)\}$$

and

$$E=E-\{(a_2, a_3), (a_2, a_4)\}$$
$$= \{(a_1, a_2), (a_1, a_3), (a_3, a_4), (a_2, a_5), (a_4, a_5), (a_1, a_4)\}.$$

We cannot choose (a_3, a_4) now because it will form circuit with already chosen edges (a_2, a_3), (a_2, a_4). Therefore we can choose (a_1, a_3) or (a_4, a_5). Suppose we choose (a_1, a_3). Then

$$E^* = \{(a_2, a_3), (a_2, a_4), (a_1, a_3)\},$$
$$E = E - \{(a_2, a_3), (a_2, a_4), (a_1, a_3)\}$$
$$= \{(a_1, a_2), (a_3, a_4), (a_2, a_5), (a_4, a_5), (a_1, a_4)\}.$$

Now choose (a_4, a_5) as it has minimal weight in E. Then

$$E^* = \{(a_2, a_3), (a_2, a_4), (a_1, a_3), (a_4, a_5)\}$$

and

$$E = E - \{(a_2, a_3), (a_2, a_4), (a_1, a_3), (a_4, a_5)\}$$
$$= \{(a_1, a_2), (a_3, a_4), (a_2, a_5), (a_1, a_4)\}.$$

Thus we have obtained four edges and therefore the process is complete.

The Kruskal minimal spanning tree is shown in the Figure 2.211.

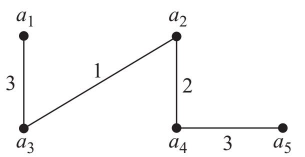


Figure 2.211

Its length is $3+1+2+3=9$.

2.18 CUT SETS

Let G be a connected graph. We know that the distance between two vertices v_1 and v_2, denoted by $d(v_1, v_2)$, is the **length of the shortest path**.

Definition 2.115. The **diameter** of a connected graph G, denoted by diam (G), is the maximum distance between any two vertices in G.

For example, in graph G shown in the Figure 2.212, we have

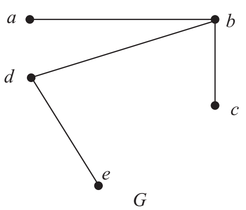


Figure 2.212

$d(a, e)=3$, $d(a, c)=2$, $d(b, e)=2$ and diam $(G)=3$.

Definition 2.116. A vertex v in a connected graph G is called a **cut point** if $G-v$ is disconnected, where $G-v$ is the graph obtained from G by deleting v and all edges containing v.

For example, in the above graph, d is a cut point.

Definition 2.117. An edge e of a connected graph G is called a **bridge** (or **cut edge**) if $G-e$ is disconnected, where $G-e$ is the graph obtained by deleting the edge e.

For example, consider the graph G shown in the Figure 2.213.

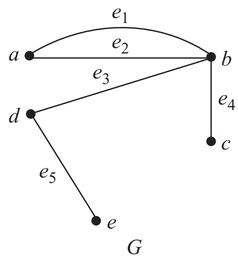


Figure 2.213

We observe that $G-e_3$ is disconnected. Hence the edge e_3 is a bridge.

Definition 2.118. A minimal set C of edges in a connected graph G is said to be a **cut set (**or **minimal edge-cut)** if the subgraph $G-C$ has more connected components than G has.

For example, in the above graph, if we delete the edge $(b, d)=e_3$, the resulting subgraph $G-e_3$ has two connected components shown in the Figure 2.214.

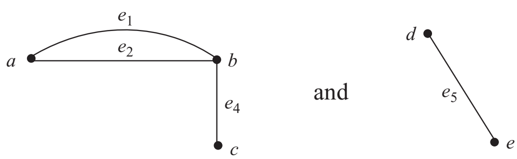


Figure 2.214

So, in this example, the cut set consists of single edge $(b, d)=e_3$, which is called cut edgeor bridge.

EXAMPLE 2.93

Find a cut set for the graph given below in the Figure 2.215.

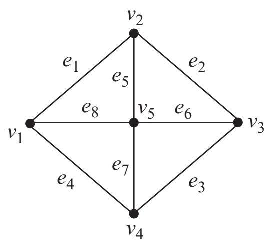


Figure 2.215

Solution. The given graph is connected. It is sufficient to reduce the graph into two connected components. To do so we have to remove the edges e_1, e_4, e_5, e_6, e_7. The two connected components are shown in the Figure 2.216.

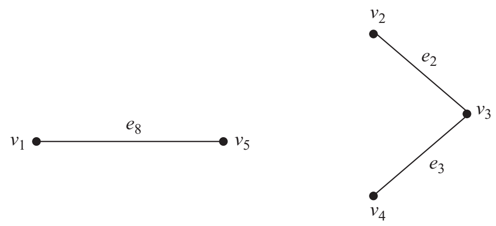


Figure 2.216

But, if we remove any proper subset of $\{e_1, e_4, e_5, e_6, e_7\}$, then there is no increase in connected components of G. Hence $\{e_1, e_4, e_5, e_6, e_7\}$ is a cut set.

EXAMPLE 2.94

Find a cut set for the graph given in the Figure 2.217.

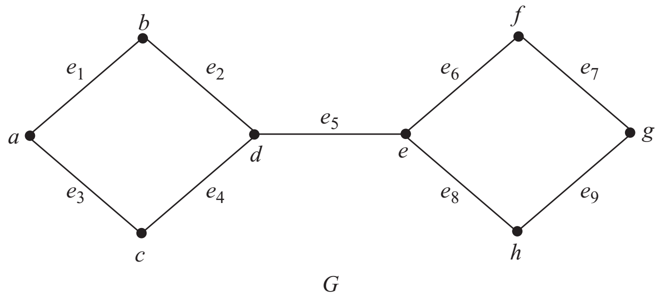


Figure 2.217

The given graph G is connected. Clearly $G-e_5$ has two connected components give in the Figure 2.218.

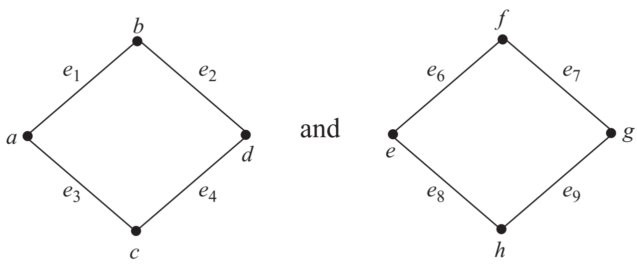


Figure 2.218

Hence a cut set for this graph is $\{e_5\}$. Some other cut sets for G are $\{e_1, e_3\}$, $\{e_3, e_2\}$, $\{e_6, e_9\}$, $\{e_6, e_8\}$.

EXAMPLE 2.95

Find a cut set for the graph represented by the Figure 2.219.

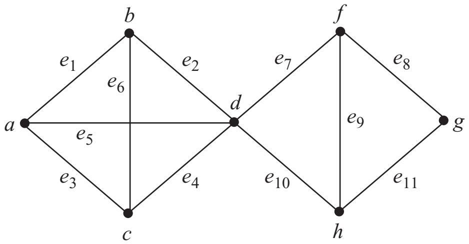


Figure 2.219

Solution. The given graph is a connected graph. We note that removal of the edges e_7 and e_{10} creates two connected components of G shown in Figure 2.220.

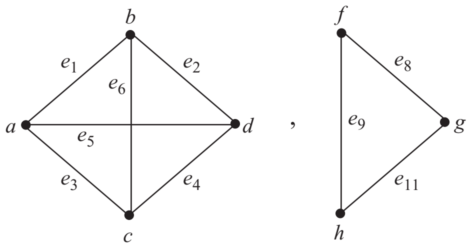


Figure 2.220

Hence the set $\{e_7, e_{10}\}$ is a cut set for the given graph G.

EXAMPLE 2.96

Find a cut set (minimal edge cut) for the following graph (Figure 2.221).

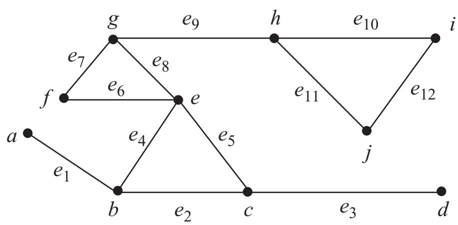
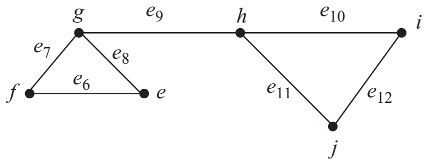


Figure 2.221

Solution. We note that $\{e_4, e_5\}$ is a cut set (minimal edge cut) for the given graph because removal of e_4 and e_5 gives rise to two components shown in the Figure 2.222.



and

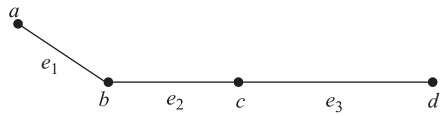


Figure 2.222

Similarly, $\{e_9\}$ forms a cut set (bridge) for this graph.

Theorem 2.30. Let G be a connected graph with n vertices. Then G is a tree if and only if every edge of G is a bridge (cut edge).

(This theorem asserts that every edge in a tree is a bridge).

Proof: Let G be a tree. Then it is connected and has $n-1$ edges (proved already). Let e be an arbitrary edge of G. Since $G-e$ has $n-2$ edges, and also we know that a graph G with n vertices has at least $n-C(G)$ edges, it follows that $n-2 \geq n-C(G-e)$. Thus, $G-e$ has at least two components. Thus removal of the edge e created more components than in the graph G. Hence e is a cut edge. This proves that every edge in a tree is a bridge.

Conversely, suppose that G is connected and every edge of G is a bridge. We have to show that G is a tree. To prove it, we have only to show that G is circuit-free. Suppose on the contrary that there exists a cycle between two points x and y in G (Figure 2.223). Then any edge on this cycle is not a cut edge which contradicts the fact that every edge of G is a cut edge. Hence G has no cycle. Thus G is connected and acyclic and so is a tree.

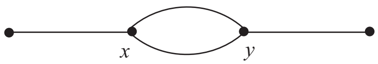


Figure 2.223

2.18.1 Relation Between Spanning Trees, Circuits and Cut Sets

A spanning tree contains a unique path between any two vertices in the graph. Therefore, addition of a chord to the spanning tree yields a subgraph that contains exactly one circuit. For example, consider the graph G shown below in the Figure 2.224.

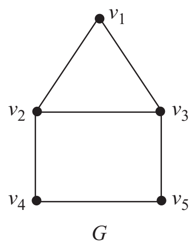


Figure 2.224

For this graph, the Figure 2.225 is a spanning tree.

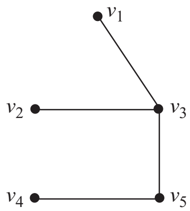


Figure 2.225 (Spanning tree)

The chords of this tree are (v_1, v_2) and (v_2, v_4). If we add (v_1, v_2) to this spanning tree, we get a circuit $v_1\ v_2\ v_3\ v_1$. Similarly addition of (v_2, v_4) gives one more circuit $v_2\ v_3\ v_5\ v_4\ v_2$. If there are v vertices and e edges in a graph, then there are $e-v+1$ chords in a spanning tree. Therefore, if we add all the chords to the spanning tree, there will be $e-v+1$ circuits in the graph.

Definition 2.119. Let v be the number of vertices and e be the number of edges in a graph G. Then the set of $e-v+1$ circuits obtained by adding $e-v+1$ chords to a spanning tree of G is called the **fundamental system of circuits relative to the spanning tree**.

A circuit in the fundamental system is called a **fundamental circuit**.

For example, $\{v_1, v_2, v_3, v_1\}$ is the fundamental circuit corresponding to the chord (v_1, v_2).

On the other hand, since each branch of a tree is cut edge, removal of any branch from a spanning tree breaks the spanning tree into two trees. For example, if we remove (v_1, v_3) from the above figured spanning tree, the resulting components are shown in the Figure 2.226.

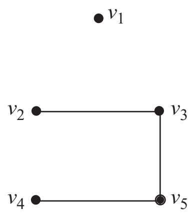


Figure 2.226

Thus, **to every branch in a spanning tree, there is a corresponding cut set.** But, in a spanning tree, there are $v-1$ branches. Therefore, **there are $v-1$ cut sets corresponding to $v-1$ branches.**

Definition 2.120. The set of $v-1$ cut sets corresponding to $v-1$ branches in a spanning tree of a graph with v vertices is called the **fundamental system of cut sets relative to the spanning tree.**

A cut set in the fundamental system of cut sets is called a **fundamental cut set.**

For example, the fundamental cut sets in the spanning tree (figured above) are

$$\{(v_1, v_2), (v_1, v_3)\}, \{(v_1, v_3), (v_2, v_3), (v_3, v_4)\},$$
$$\{(v_3, v_5), (v_4, v_5)\}, \{v_2, v_4), (v_4, v_5)\}.$$

Theorem 2.31. A circuit and the complement of any spanning tree must have at least one edge in common.

Proof: We recall that the set of all chords of a tree is called the complement of the tree. Suppose on the contrary that a circuit has no common edge with the complement of a spanning tree. This means the circuit is wholly contained in the spanning tree. This contradicts the fact that a tree is acyclic (circuit- free). Hence a circuit has at least one edge in common with complement of a spanning tree.

Theorem 2.32. A cut set and any spanning tree must have at least one edge in common.

Proof: Suppose on the contrary that there is a cut set which does not have a common edge with a spanning tree. Then removal of cut set has no effect on the tree, that is, the cut set will not separate the graph into two components. But this contradicts the definition of a cut set. Hence the result.

Theorem 2.33. Every circuit has an even number of edges in common with every cut set.

Proof: We know that a cut set divides the vertices of the graph into two subsets each being set of vertices in one of the two components. Therefore a path connecting two vertices in one subset must traverse the edges in the cut set an even number of times. Since a circuit is a path from some vertex to itself, it has an even number of edges in common with every cut set.

2.19 TREE SEARCHING

Let T be a binary tree of height $h \geq 1$ and root v. Since $h \geq 1$, v has at least one child : v_L and/or v_R. Now v_L and v_R are the roots of the left and right subtrees of v called T_L and T_R, respectively.

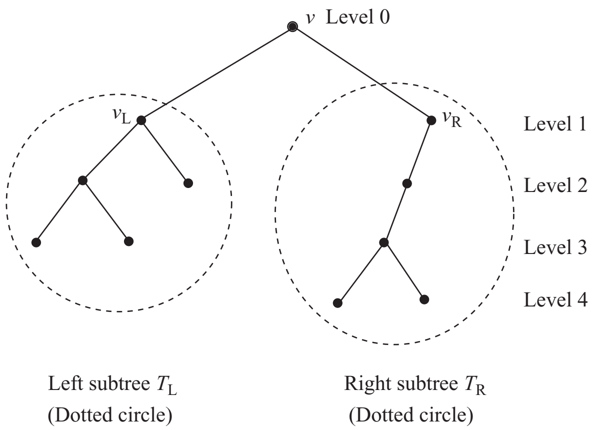


Figure 2.227

Definition 2.121. Performing appropriate tasks at a vertex is called **visiting the vertex**.

Definition 2.122. The process of visiting each vertex of a tree in some specified order is called **searching the tree** or **walking** or **traversing the tree**.

We now discuss methods of searching a tree.

1. Pre-order Search Method

Input: The root v of a binary tree.
Output: Vertices of a binary tree using preorder traversal

1. **Visit** v
2. If v_L (left child of v) exists, then apply the algorithm to $(T(v_L), v_L)$
3. If v_R (right child of v) exists, then apply this algorithm to $(T(v_R), v_R)$

End of algorithm pre-order.

In other words, pre-order search of a tree consists of the following steps:

Step 1. Visit the root.
Step 2. Search the left subtree if it exists.
Step 3. Search the right subtree if it exists.

EXAMPLE 2.97

Find the order of the vertices of the following tree T processed using pre-order traversal.

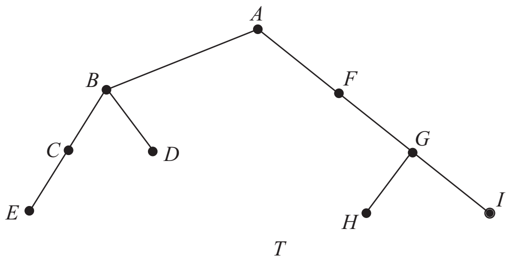


Figure 2.228

Solution. The root of the given tree is A. It has two children B and F. The left subtree and right subtree are shown in Figure 2.229.

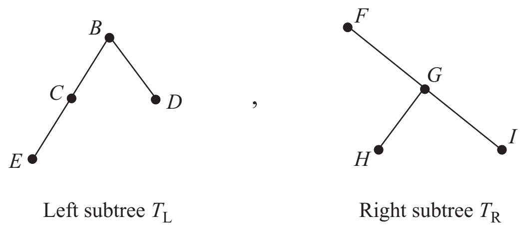


Figure 2.229

If we apply pre-order algorithm to the given tree T, we visit the root and print A. Using pre-order to left subtree, we visit the root B and print B and then search T_L printing C and E and then D. Up to this point, the search has yielded the string $A\ B\ C\ E\ D$. Thus the search of T_L is complete. We now go to subtree T_R. Using the same procedure, we get the string $F\ G\ H\ I$. Thus the complete search of the tree T is

A B C E D F G H I.

EXAMPLE 2.98

Find binary tree representation of the expression

$$(a-b)\times(c+(d\div e))$$

and represent the expression in string form using pre-order traversal.

Solution. In the given expression, $\times$ is the central operator and therefore shall be the root of the binary tree. Then the operator $-$ acts as v_L and the operator $+$ acts as v_R. Thus the tree representation of the given expression is as shown in the Figure 2.230.

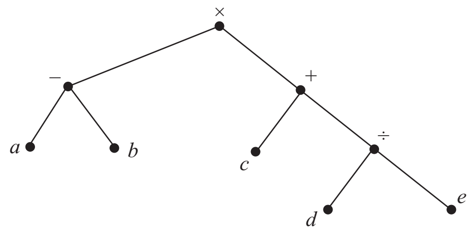


Figure 2.230

The result of the pre-order traversal to this binary tree is the string

$$\times - a\ b + c \div d\ e.$$

This form of the expression is called **prefix form** or **polish form** of the expression

$$(a-b)\times(c+(d\div e)).$$

In a polish form, the variables a, b, c,…are called **operands** and $-,+,\times,\div$ are called **operators**. We observe that, **in polish form, the operands follow the operator**.

EXAMPLE 2.99

Represent the expression $(A+B) * (C-D)$ as a binary tree and write the prefix form of the expression.

Solution. Here * is the central operator. Further + and – operators are v_L and v_R. Hence the binary tree for the given experssion is as shown in the Figure 2.231.

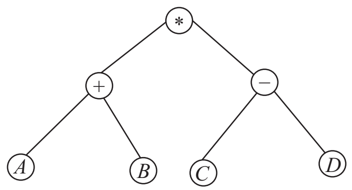


Figure 2.231

Using pre-order traversal, the prefix expression for it is

$$* + A\ B - C\ D.$$

2.19.1 Procedure to Evaluate an Expression Given in Polish Form

To find the value of a polish form, we proceed as follows:

Move from left to right until we find a string of the form K x y, where K is operator and x, y are operands.

Evaluate x K y and substitute the answer for the string K x y. Continue this procedure until only one number remains.

EXAMPLE 2.100

Find parenthesized form of the polish expression

$$-+A\ B\ C.$$

Solution. The parenthesized form of the given polish expression is derived as follows:

$$-(A+B)\ C,$$
$$(A+B)-C.$$

The corresponding binary tree is shown in the Figure 2.232.

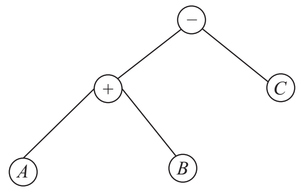


Figure 2.232

EXAMPLE 2.101

Evaluate the polish form

$$\times - 64 + 5 \div 22$$

Solution. We have the following steps in this regard:

1. $\times(6-4)+5\div 22$
2. $\times 2+5\div 22$
3. $\times 2+5\ (2\div 2)$
4. $\times 2+51$
5. $\times 2\ (5+1)$
6. $\times 26$
7. 2×6
8. 12, which is the required value of the expression.

2. Post-order Search Method

Algorithm

Step 1. Search the left subtree if it exists
Step 2. Search the right subtree if it exists
Step 3. Visit the root

End of algorithm

EXAMPLE 2.102

Represent the expression

$$(A+B) * (C-D)$$

as a binary tree and write the result of post-order search for that tree.

Solution. The binary tree expression (as shown earlier) of the given algebraic expression is given in Figure 2.233.

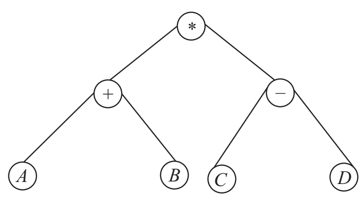


Figure 2.233

The result of post-order search of this tree is

$$A\ B + C\ D - *$$

This form of the expression is called **postfix form** of the expression or **reverse polish form** of the expression.

In postfix form, the operator follows its operands.

EXAMPLE 2.103

Find the parenthesized form of the postfix form

$$A\ B\ C * * C\ D\ E + / -.$$

Solution. We have

1. $A\ B\ C * * C\ D\ E + / -$
2. $A\ (B * C) * C\ (D+E)\ / -$
3. $(A * (B * C))\ (C\ /\ (D+E)) -$
4. $(A * (B * C)) - (C\ /\ (D+E))$.

The corresponding binary tree is shown in the Figure 2.234.

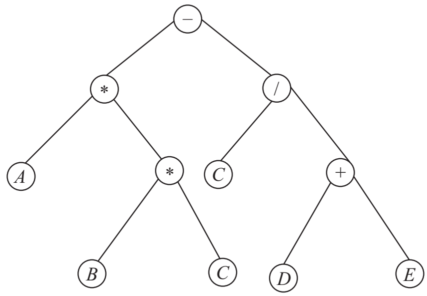


Figure 2.234

EXAMPLE 2.104

Evaluate the postfix form

$$2 1 - 3 4 2 \div + \times.$$

Solution. We have

$$\begin{aligned} 2 1 - 3 4 2 \div + \times &= (2-1)\ 3 4 2 \div + \times \\ &= 1 3\ (4 \div 2) + \times \\ &= 1 3 2 + \times \\ &= 1\ (3+2) \times \\ &= 1 5 \times \\ &= 1 \times 5 = 5. \end{aligned}$$

2.20 TRANSPORT NETWORKS

An important application of weighted directed graph is to model transport networks like oil pipeline, water supply pipeline, communication network, electric power distribution system and highway system. The purpose of a network is to implement the flow of oil, water, electricity, messages, traffic, etc and we desire that the flow through the network should be largest possible value. Such a flow will be called maximum flow.

Definition 2.123. A ***transport network*** or simply a ***network*** is a simple weighted directed graph with the following properties:

(i) There is a designated vertex (node), called the ***source,*** that has no incoming edge.

(ii) There is a designated vertex, called the ***sink,*** that has no outgoing edge.

(iii) There is a non-negative weight c_{ij} on the directed edge (i, j), called the ***capacity*** of the edge (i, j).

Since the graph is simple, it follows that if the edge (i, j) is in the network, then (j, i) is not there.

Definition 2.124. A ***flow*** in a network is a function that assigns to each (i, j) of the network a non-negative number f_{ij} such that

(i) $0 \le f_{ij} \le c_{ij}$, where c_{ij} is the capacity of the edge (i, j)

(ii) For each vertex j, which is neither the source nor the sink,

$$\sum_i f_{ij} = \sum_k f_{jk}.$$

The non-negative number f_{ij} is called the ***flow*** in the edge (i, j). For any vertex (distinct from source and sink) j, the sum $\sum_i f_{ij}$ is called the ***flow into*** j and the sum $\sum_k f_{jk}$ is called the ***flow out of*** j. Thus the condition (ii) implies that material cannot accumulate, be created, dissipate or lost at any vertex other than the source or the sink. Hence the equation

$\sum_i f_{ij} = \sum_k f_{jk}$ is called ***conservation of flow***. As a consequence of conservation, it follows that the ***sum of flows leaving the source must be equal to the sum of the flows entering the sink.*** This sum is called the ***value of the flow*** and is denoted by value (F). A flow F in a network is represented by labelling each edge (i, j) with the pair (c_{ij}, f_{ij}).

For example consider the oil pipeline network shown in Figure 2.235.

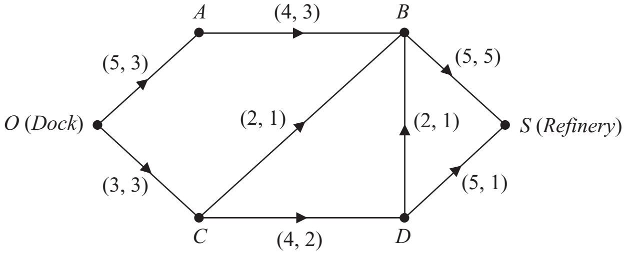


Figure 2.235

In this network, the crude oil unloaded at the dock O (source) is being pumped to the refinery S (sink). The capacities of the flow are

$$c_{OA} = 5, \quad c_{AB} = 4, \quad c_{BS} = 5, \quad c_{OC} = 3,$$
$$c_{CB} = 2, \; c_{CD} = 4, \; c_{DB} = 2, \quad c_{DS} = 5.$$

The flows in the edges are

$$f_{OA} = 3, \quad f_{AB} = 3, \quad f_{BS} = 5, \quad f_{OC} = 3,$$
$$f_{CB} = 1, \quad f_{CD} = 2, \quad f_{DB} = 1, \quad C_{DS} = 1.$$

We note that

Flow into the vertex $B = 3 + 1 + 1 = 5$

Flow out of the vertex $B = 5$.

Therefore, ***conservation of flow*** is satisfied. Further, value $(F) = 3 + 3 = 5 + 1 = 6$.

EXAMPLE 2.105

Consider the water pipeline network shown in Figure 2.236 for two towns A and B in which water is supplied from four borewells w_1, w_2, w_3, w_4 and a, b, c, d represent intermediate pumping stations. Form an equivalent network with unique source and unique sink and determine the value of the flow.

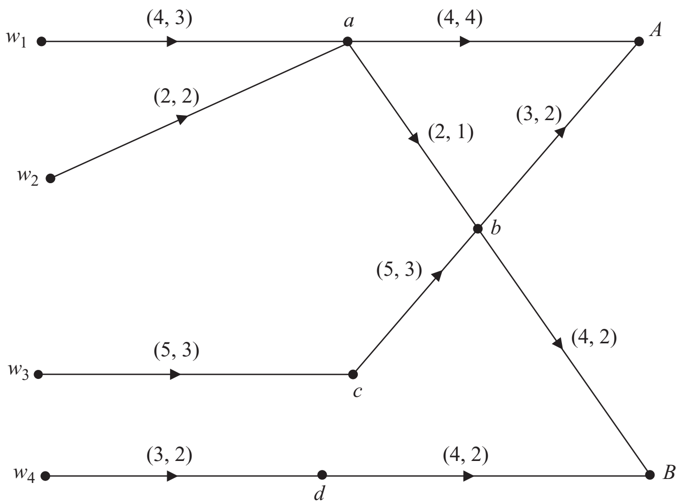


Figure 2.236

Solution. To obtain the required equivalent network with designated source and designated sink, we tie together the given sources w_1, w_2, w_3 and w_4 into a ***super source*** O and the towns A and B into a ***super sink*** S as shown in Figure 2.237.

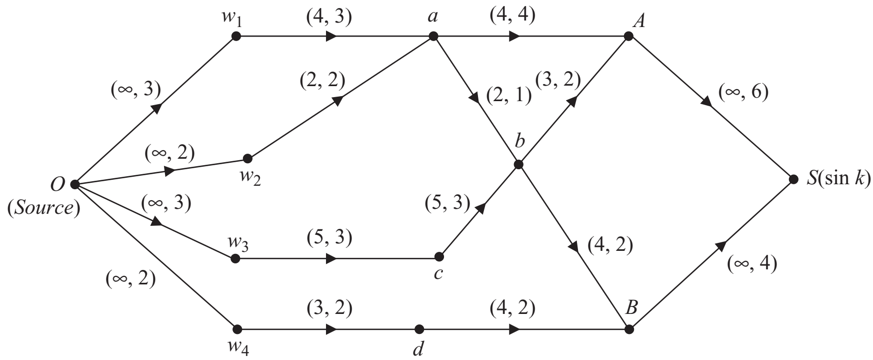


Figure 2.237

The value of the flow is

$$\text{Value }(F) = 3+2+3+2 = 6+4 = 10.$$

Definition 2.125. A ***maximal flow*** in a network N is a flow with maximum value.

In general, there will be several flows having the same maximum value. For example, consider the network shown in Figure 2.238.

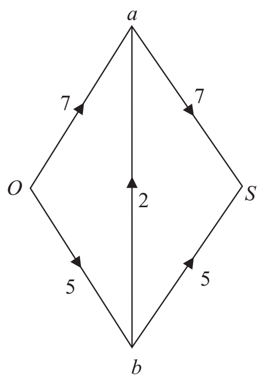


Figure 2.238

Consider the following two flows for this network:

The value of the flow in Figure 2.239(a) is 10 and three of the five edges in this network attain their maximum capacity. But for the same network, the value of the flow in Figure 2.239(b) is 12 and four out of five edges are carrying maximum capacity. Therefore, the flow function in the second case is better than the first one.

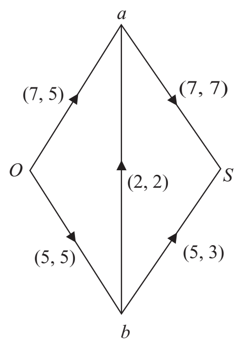


Figure 2.239(a)

and

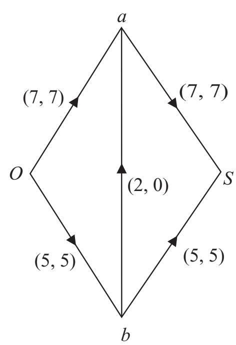


Figure 2.239(b)

EXAMPLE 2.106

Find the maximum possible increase in the following network (Figure 2.240).

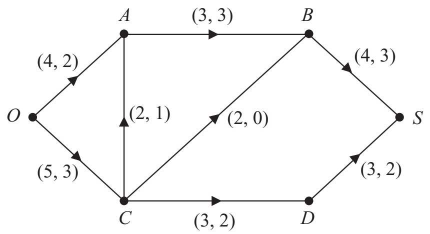


Figure 2.240

Solution. We cannot increase the flow in the path *OABS* because the capacity and the flow of the edge *AB* are equal. But if increase flow in the path *OCDS* by 1, we get the network shown in Figure 2.241.

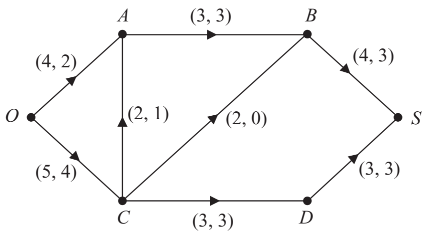


Figure 2.241

Further, if we increase the flow in the path *OCBS* by 1, we get the network shown in Figure 2.242.

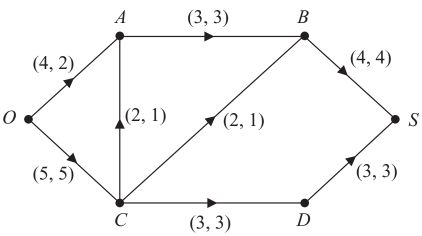


Figure 2.242

No further increase in flow is possible since the capacities of all the edges leading to the sink have been exhausted. The value of the flow is 7.

EXAMPLE 2.107

Find a maximal flow in the network shown in Figure 2.243.

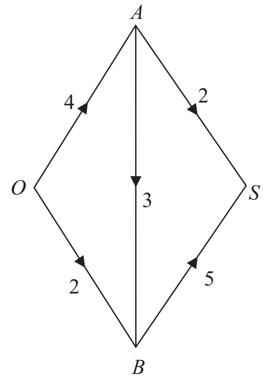


Figure 2.243

Solution. The capacity of each edge is shown in Figure 2.243. Let the initial flow be 0 in each edge. Thus we have the network as shown in Figure 2.244(a).

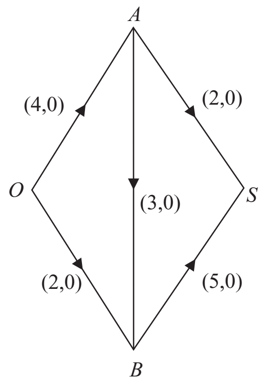


Figure 2.244(a)

Increase the flow by 2 in the path *OAS* to get the network shown in the Figure 2.244(b).

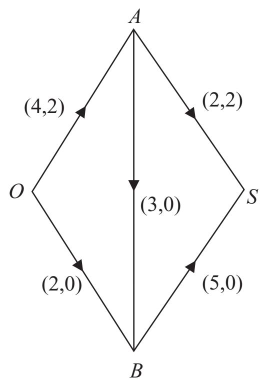


Figure 2.244(b)

Now increase the flow by 2 in the path *OBS* to get the network shown in Figure 2.244(c).

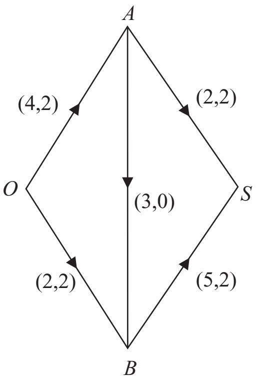


Figure 2.244(c)

Now increase the flow by 2 in the path *OABS* to get the network shown in Figure 2.244(d).

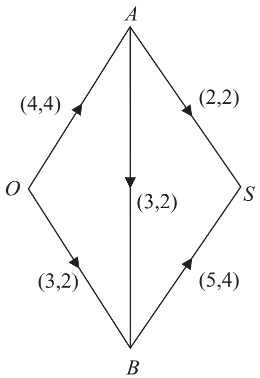


Figure 2.244(d)

Any further increase in the flow is not possible. Thus maximal flow in the network has been reached. Thus, value (F)$=2+4=6$.

EXAMPLE 2.108

Find a maximal flow in the network shown in Figure 2.245.

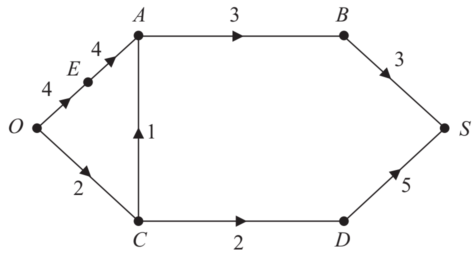


Figure 2.245

Solution. The initial labelling yields the network shown in Figure 2.246(a).

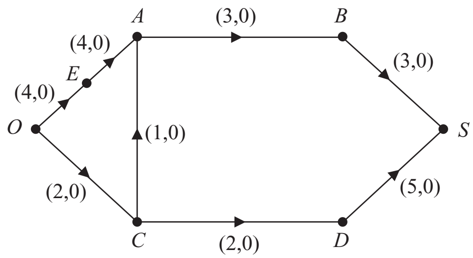


Figure 2.246(a)

Increase the flow by 2 in the path *OCDS* by to get the network shown in Figure 2.246(b).

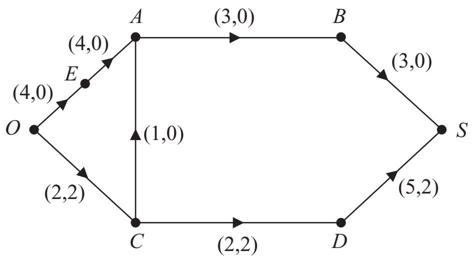


Figure 2.246(b)

Now increase the flow by 3 in the path *OEABS* to get the network shown in Figure 2.246(c).

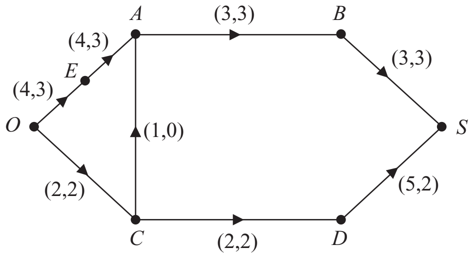


Figure 2.246(c)

No further increase in flow in possible since the capacities in the edges (A, B) and (C, D) has been exhausted. Hence value (F)$=5$.

Definition 2.126. Let N be a network. A ***cut*** in N is a partition $(P, \bar{P})$ of the set of vertices in N such that the source $O \in P$ and the sink $S \in \bar{P}$.

Thus, a cut $(P, \bar{P})$ in a network N is a set K of edges $(v, w), v \epsilon P, w \epsilon \bar{P}$ such that every path from source to sink contains at least one edge from K. In fact, a cut does cut a directed graph into two pieces, one containing the source and the other containing the sink. ***Nothing can flow from source to sink if edges of a cut are removed.*** We indicate a cut by drawing a dashed line to partition the vertex set. As an illustration, consider the network of EXAMPLE 2.106. The dashed line divides the vertex set into the sets $P = \{O, A\}$ and $\bar{P} = \{C, B, D, S\}$.

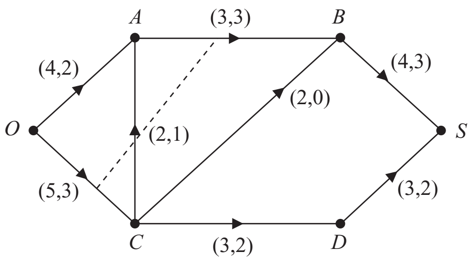


Figure 2.247

Definition 2.127. The ***capacity of a cut*** $K = (P, \bar{P})$ is the sum of the capacities of all edges in K and is denoted by $C(K)$ or $C(P, \bar{P})$.

For example, the capacity of the cut in Figure 2.247 is

$$C(P, \bar{P}) = C_{OC} + C_{AC} + C_{AB} = 5 + 2 + 3 = 10.$$

EXAMPLE 2.109

Find the capacity of the cut, shown by dashed line, in the network shown in Figure 2.248.

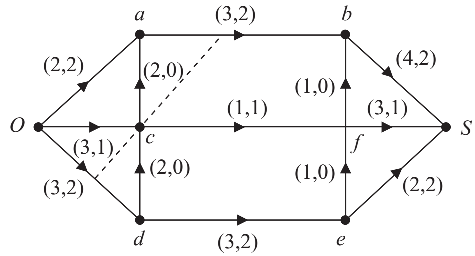


Figure 2.248

Solution. The cut $(P, \bar{P})$ is given by

$$P = \{o, a, c\}, \quad \bar{P} = \{d, e, f, b, S\}.$$

Therefore the capacity of the given cut is

$$C(P, \bar{P}) = c_{Od} + c_{cf} + C_{ab} = 3 + 1 + 3 = 7.$$

Definition 2.128. A ***minimal cut*** is a cut having minimum capacity.

Theorem 2.34. (The Max Flow Min Cut Theorem) A maximum flow F in a network has value equal to the capacity of a minimum cut of the network.

Proof: Let F be any flow and K be any cut in a network. Since all parts of F must pass through the edges of the cut K and since $C(K)$ is the maximum amount that can pass through the edges of K, it follows that ***value***$(F) \le C(K)$. The equality will hold if the flow F uses the full capacity of all edges in the cut K. Thus equality holds if the flow is maximum. Further, K must be a minimum capacity cut since every cut must have capacity at least equal to ***value*** (F). This completes the proof of the theorem.

EXAMPLE 2.110

In the network shown in Figure 2.249, find a maximum flow, give its value and prove that it is maximum by appealing to the max flow min cut theorem.

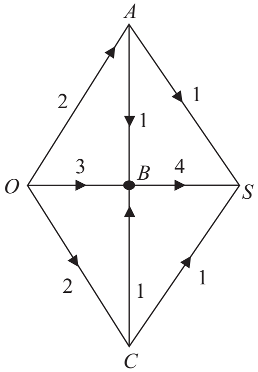


Figure 2.249

Solution. The initial labelling yields the following network shown in Figure 2.250(a):

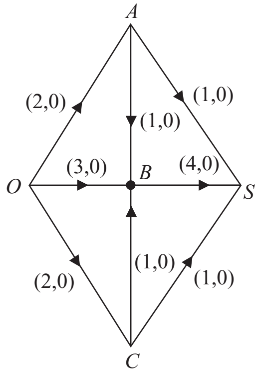


Figure 2.250(a)

Increase the flow by 1 in the path *OAS* to get the network shown in Figure 2.250(b).

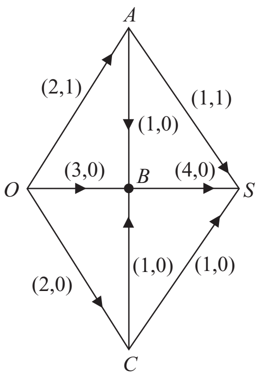


Figure 2.250(b)

Now increase the flow by 1 in the path *OCS* to get the network shown in Figure 2.250(c).

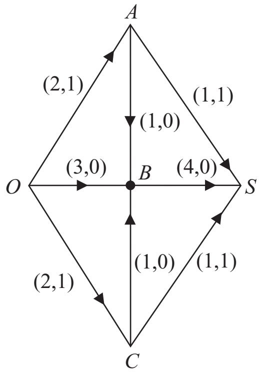


Figure 2.250(c)

Further, increase in flow by 1 in the path *OABS* to get the network of Figure 2.250(d).

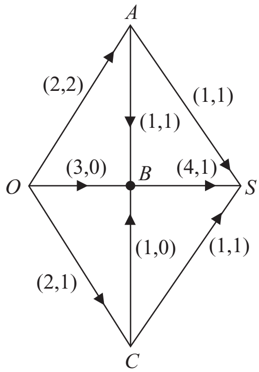


Figure 2.250(d)

Now increase the flow by 1 in the path *OCBS* and get the network shown in Figure 2.250(e).

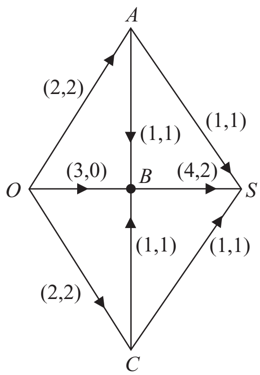


Figure 2.250(e)

Lastly, we increase the flow by 2 in the path *OBS* and get the network with maximum flow as shown in Figure 2.250(f). In fact the capacities in the edges leading to the sink have been completely exhausted.

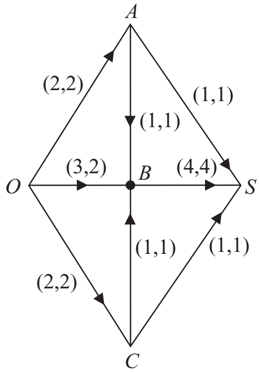


Figure 2.250(f)

We note that value (F=2+2+2=1+4+1=6). To check whether it is maximum flow, we appeal to max flow min cut theorem. We consider the following cuts:

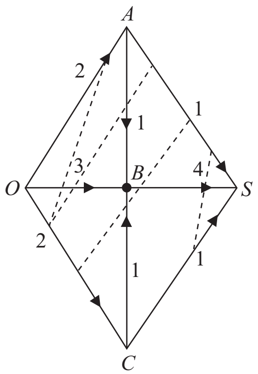


Figure 2.250(g)

(i) $K_1 = \{OC, OB, OA\}$
with cap
$(K_1) = C_{OC} + C_{OB} + C_{OA} = 2 + 3 + 2 = 7,$

(ii) $K_2 = \{OC, BC, BS, AS\}$
with cap
$(K_2) = C_{OC} + C_{BC} + C_{BS} + C_{AS}$
$= 2 + 1 + 4 + 1 = 8,$

(iii) $K_3 = \{CS, BS, AS\}$
with cap$(K_3) = C_{CS} + C_{BS} + C_{AS} = 1 + 4 + 1 = 6,$

(iv) $K_4 = \{OC, OB, AS, AB\}$
with cap $(K_4) = C_{OC} + C_{OB} + C_{AS} + C_{AB}$
$= 2 + 3 + 1 + 1 = 7.$

We note that capacity of the minimum cut in 6. Hence, the maximum flow is 6.

EXERCISES

1. Is there a non-empty simple graph with twice as many edges as vertices?

   Ans. Yes. In fact if n = number of vertices $2n$ = number of edges, then using the formula for number of edges in a complete graph: $\frac{n(n-1)}{2}$, we have $2n = \frac{n(n-1)}{2}$ which yields $n = 5$. But in a complete graph K_5 the number of edges is 10. Thus the complete graph K_5, which is simple by definition, has five vertices and ten edges.
2. Is the graph given below a bipartite graph?

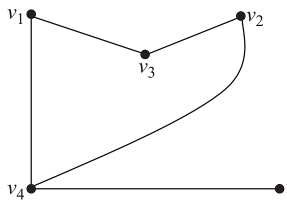


Figure 2.251

   Ans. Yes, it is a bipartite graph, where the set of vertices V has been partitioned into two subsets, $V_1 = \{v_3, v_4\}$, $V_2 = \{v_1, v_2, v_5\}$. But it is not a complete bipartite graph.
3. Find a formula for the number of edges in complete bipartite graph K_{mn}.

   Ans. The $m + n$ vertices having been divided into two subsets V_1 and V_2 containing m vertices and n vertices, respectively. Every verticex in V_1 is joined to all the n vertices in V_2. Thus the task of joining each vertex of V_1 to each vertex in V_2 can be performed in mn ways. Hence the number of edges in K_{mn} is mn.
4. A graph G has vertices of degrees 1, 4, 3, 7, 3 and 2. Find the number of edges in the graph.

Ans. Total degree of G is $1 + 4 + 3 + 7 + 3 + 2 = 20$ which should be twice the number of edges. Hence number of edges in G is 10.

5. How many edges are there in an n-cube?

   Ans. Degree of each vertex in an n-cube in n. The number of vertices in an n cube in 2^n. Therefore total degree of n-cube in $n2^n$. Hence the number of edges is an n-cube is

$$\frac{n \cdot 2^n}{2} = n \cdot 2^{n-1}.$$

6. When does the complete graph K_n possess an Euler circuit?

   Ans. Complete graph is a connected graph. So it will have an Euler cycle if degree of each vertex is even.

7. Does the graph given below have an Euler circuit?

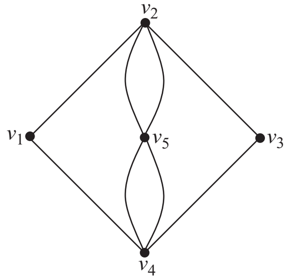


Figure 2.252

   Ans. Yes. The graph is connected and degree of each vertex is even. Hence the graph possesses an Euler circuit. For example, $v_1\ v_2\ v_5\ v_4\ v_5\ v_2\ v_3\ v_4\ v_1$ is an Euler cycle.

8. When does the complete bipartite graph K_{mn} contain Euler cycle?

   Ans. When both m and n are even. In such a case the degree of each vertex will be even.

9. In seven bridges problem, was it possible for a citizen of Konigsberg to make a tour of the city and cross each bridge exactly twice? Give reasons.

   Ans. Yes, because in such a case degree of each vertex of the graph of seven bridges problem shall be even.

10. Show that the graph given below does not contain a Hamiltonian cycle.

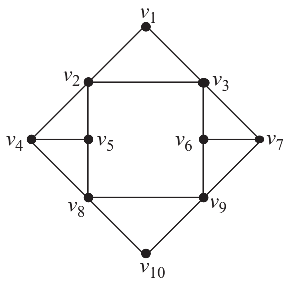
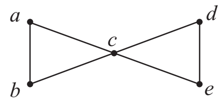


Figure 2.253

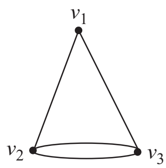
11. Give example of a graph that has Euler circuit but not Hamiltonian circuit.

    Ans.



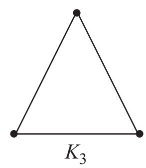
12. Give example of a graph that has an Hamiltonian circuit but not an Euler circuit.

    Ans.



13. Give example of a graph that has both an Euler circuit and an Hamiltonian circuit.

    Ans.



14. Use Dijkstra's shortest path algorithm to find the shortest path from a to f in the graph given below:

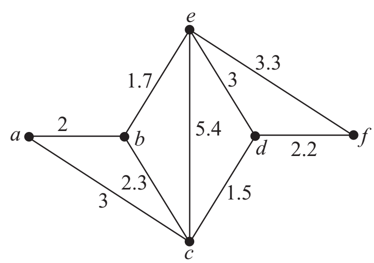


Figure 2.254

Ans. 6.7

15. Find the adjacency matrix of the graph.

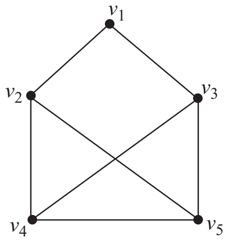


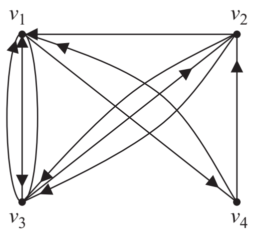
Figure 2.255

Ans. $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$

16. Find directed graph that have the following adjacency matrix.

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Ans.



17. A graph G has the adjacency matrix given below. Verify whether it is connected.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Ans. The graph G is connected because $B = A + A^2 + A^3 + A^4$ has non-zero entries off the main diagonal.

18. Defining an isomorphism. Show that the graphs shown below are isomorphic.

(Clearly, the graph G_2 can be drawn as)

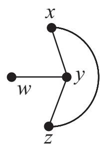


Figure 2.256

Ans. Vertex set of G_1 is $V(G_1) = \{a, b, c, d\}$
vertex set of G_2 is $V(G_2) = \{x, y, z, w\}$
Define $f : V(G_1) \to V(G_2)$ by
$f(a) = y$, $f(b) = w$, $f(c) = x$, $f(d) = z$.
Then f is bijective.

19. Show that the graphs shown below are not isomorphic. Give reasons.

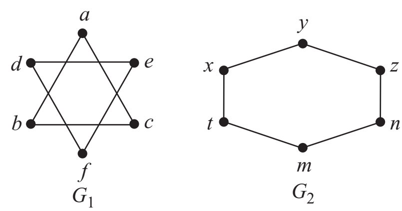


Figure 2.257

Ans. Graph G_2 is connected whereas G_1 is not connected. The connected components of G_1 are

20. Show that the graphs shown below are not isomorphic.

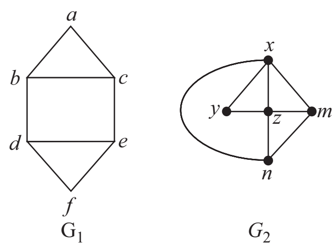


Figure 2.258

Ans. The graphs do not have same number of vertices and so are not isomorphic.

21. Find the complement of the graph shown below.

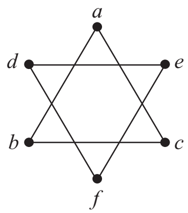
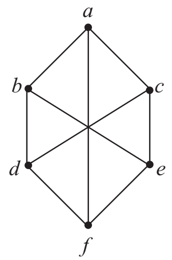


Figure 2.259

Ans. The complement of the given graph is



22. Let G be a graph and V be its vertex set. Then

(i) Eccentricity of $v \in V$ is defined as
$$e(v) = \max \{d(u, v): u \in V, u \neq v\}.$$

(ii) The radius of the graph is defined by
$$\text{rad}(G) = \min \{e(v): v \in V\}.$$

(iii) The diameter of G is defined by
$$\text{diam }(G) = \max \{e(v): v \in V\}.$$
$$= \max [d(u, v): u \in V, v \in V, u \neq v\}.$$

(iv) v is a central point if
$$e(v) = \text{rad}(G).$$

(v) The centre of G is the set of all central points.

Find the centre of the graph given below:

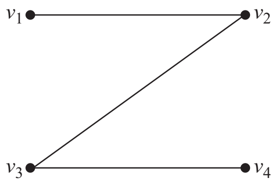


Figure 2.260

Ans. We note that

(i) $d(v_1, v_2) = 1$, $d(v_1, v_3) = 2$, $d(v_1, v_4) = 3$ and so $e(v_1) = 3$

(ii) $d(v_2, v_1) = 1$, $d(v_2, v_3) = 1$, $d(v_2, v_4) = 2$ and so $e(v_2) = 2$

(iii) $d(v_3, v_1) = 2$, $d(v_3, v_2) = 1$, $d(v_3, v_4) = 1$ and so $e(v_3) = 2$

(iv) $d(v_4, v_1) = 3$, $d(v_4, v_2) = 2$, $d(v_4, v_3) = 1$ and so $e(v_4) = 3$

Hence rad $(G) = \min \{3, 2, 2, 3\} = 2$. The centre is $\{v_2, v_3\}$.

23. Show that any graph having n vertices, $n = 1, 2, 3, 4$ is planar.

24. Show that the colouring of map given below requires at least three colours

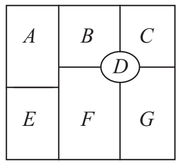


Figure 2.261

Ans. Give one colour to A and D, give second colour to B, E and G, and third colour to C and F. Thus all the adjoining boundaries have different colours.

25. How many colours are required to paint the graph given below?

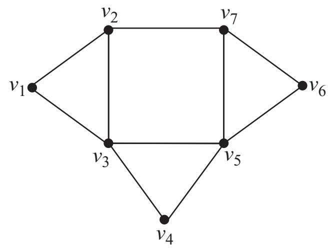


Figure 2.262

Ans. Three. We can paint v_1, v_2, v_3 with colours C_1, C_2, C_3. Then paint v_4 by C_1, v_5 by C_2, v_6 by C_3 and v_7 by C_1.

26. For which values of m and n is the complete bipartite graph K_{mn} a tree?

Ans. Number of edges in $K_{mn} = mn$. Total number of vertices in $K_{mn} = m+n$. If k_{mn} is a tree, then number of edges in K_{mn} should be $(m+n)-1$. Hence, to be a tree k_{mn} satisfies the condition

$$mn = (m+1)-1 \quad \text{or} \quad m+n = mn+1,$$

which is satisfied if either $m=1$ or $n=1$.

27. Let G be a graph with n vertices and $n-2$ or fewer edges. Show that G is not connected.

Ans. Suppose G is connected. The graph itself will be a tree or it can be made a tree by eliminating some edges. In either case it must have $n-1$ edges. But in the questions, number of edges is less than or equal to $n-2$. Hence G cannot be connected.

28. Represent the weighted graph given below in the matrix form.

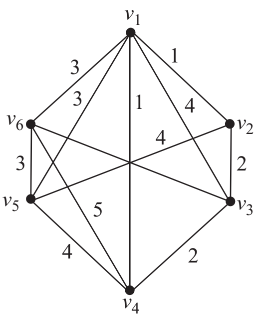


Figure 2.263

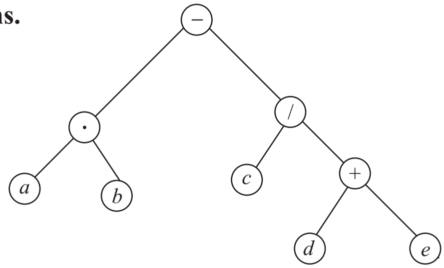
Ans.
$$\begin{bmatrix} \infty & 1 & 4 & 1 & 3 & 3 \\ 1 & \infty & 2 & \infty & 4 & \infty \\ 4 & 2 & \infty & 2 & \infty & \infty \\ 1 & \infty & 2 & \infty & 4 & 5 \\ 3 & 4 & \infty & 4 & \infty & 3 \\ 3 & \infty & \infty & 5 & 3 & \infty \end{bmatrix}$$

The diagonal is $\infty\ \infty \ldots \infty$ and the matrix is symmetric with respect to the diagonal.

29. Represent the expression given below by a binary tree and obtain the prefix form and postfix form of the expression.

$$a \cdot b - (c/(d+e)).$$

Ans.

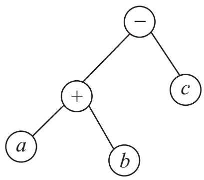


Prefix form: $- \bullet ab - c + de$
Postfix form: $ab \bullet de + c/-$

30. Convert the following postfix form of a binary tree into prefix form and parenthesized infix form and usual infix form

$$ab+c-.$$

Ans.



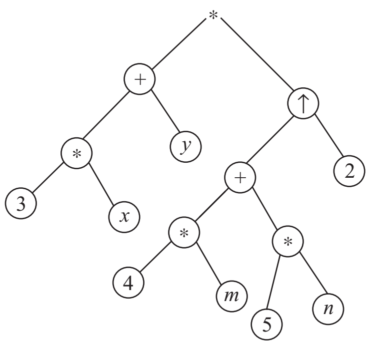
Prefix form: $-+abc$
Parenthesized form: $(a+b)-c$
Usual infix form: $a+b-c$

31. Draw a tree for the algebraic expression

$$(3x+y)\,(4m+5n)^2$$

and find its prefix polish form.

Ans. Using $\uparrow$ for exponentiation and * for multiplication, the binary tree representing the algebraic expression is



The prefix form is

$$* + * 3xy \uparrow + * 4m * 5n$$

$$\begin{aligned} 232*\uparrow + 233 - 24 &= * \uparrow (2+3)\ 3\ (2-4) \\ &= * (2+3)^3 (2-4) \\ &= (2+3)^3 * (2-4) \\ &= 5^3 * (-2) \\ &= -250 \end{aligned}$$

32. If $\uparrow$ denotes exponentiation, evaluate the polish form

$$*\uparrow + 233 - 24.$$

33. Find all cut sets for the graph G given below.

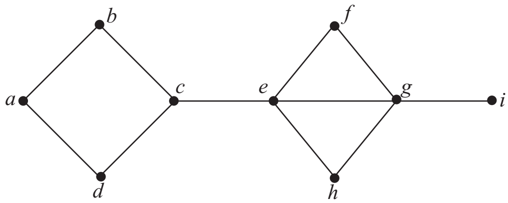


Figure 2.264

Ans. $\{c, e\}$; $\{\{a, b\}, \{a, d\}\}$; $\{\{a, d\}, \{b, c\}\}$; $\{\{e, f\}, \{e, g\}\}$ $\{h, g\}\}$; $\{g, i\}$; $\{\{e, h\}, \{h, g\}\}$. Try for other cut edges.

34. Find a minimal spanning tree for the graph shown below:

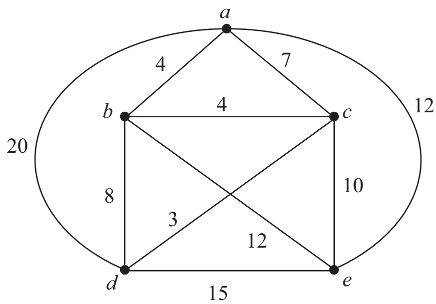
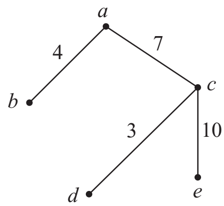


Figure 2.265

Ans.



35. Find a minimal tree for the graph of Exercise 34 by deleting one by one those costliest edges whose deletion does not disconnect the graph.

Ans. First remove $\{a, d\}$, then $\{d, c\}$, $\{a, e\}$, $\{b, e\}$. We cannot remove $\{c, e\}$ now because the graph will be disconnected by doing so. But we can delete $\{b, d\}$. Now delete any of $\{a, c\}$ or $\{b, c\}$. We will get two minimal trees.

36. Find all spanning trees of the graph shown below:

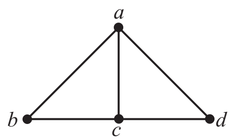


Figure 2.266

Ans. Each spanning tree must have $n - 1 = 4 - 1 = 3$ edges. There are eight such trees.

37. Find the value of the maximum flow in the network of EXAMPLE 2.109 and verify the value obtained by max flow min cut theorem.

Ans. Value (F) for max flow $= 7$, minimum cut (k) is shown in Figure 8.248.

38. Find the value of the maximum flow in the network shown in Figure 2.267 and verify your answer using max flow min cut theorem.

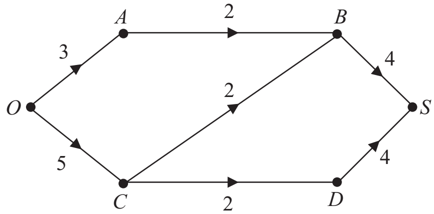


Figure 2.267

Ans. Max flow $= 6$

3 Improper Integrals

3.1 IMPROPER INTEGRAL

While evaluating Riemann integrals, the integrand was assumed to be bounded and the range (interval) of integration was assumed to be finite. The aim of this chapter is to consider the integrals in which the above conditions are not satisfied. This leads us to a concept called the *improper integral.*

Definition 3.1. If the integrand f is unbounded or the limits a or b or both are infinite, then the integral $\int_a^b f(x)dx$ is called an *improper integral*, *generalized integral or an infinite integral.*

For example, the integrals

(i) $\int_0^1 \frac{dx}{x^3}$,

(ii) $\int_1^2 \frac{dx}{(1-x)(2-x)}$ and

(iii) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

are improper integrals. We note that in integral (i), the integrand becomes infinite (unbounded) at the lower limit 0. In such a case, we say that $x = 0$ is a point of infinite discontinuity (singular point) of the integrand f. In integral (ii), both lower and upper limits are singular points of f, whereas in integral (iii), the range of integration is infinite.

An integral that is not improper is called *proper integral.* In view of Definition 3.1, we shall consider the following two cases:

(a) When the integral is unbounded.
(b) When the range of integration is infinite.

3.2 CONVERGENCE OF IMPROPER INTEGRAL WITH UNBOUNDED INTEGRAND

In what follows, we assume that the number of singularities of the integrand in any interval of integration is finite.

Definition 3.2. If the lower limit is the only singularity of the integrand f, then the improper integral $\int_a^b f(x)dx$ is said to converge at a if

$$\lim_{\mu\to 0+} \int_{a+\mu}^{b} f(x)dx$$

exists and is finite.

EXAMPLE 3.1

Examine the convergence of $\displaystyle\int_0^2 \frac{dx}{x^2}$.

Solution. We note that the lower limit 0 is the only singularity of the integrand $\frac{1}{x^2}$ in the interval $[0,\ 2]$. Then

$$\int_0^2 \frac{dx}{x^2} = \lim_{\mu\to 0+} \int_{\mu}^{2} \frac{dx}{x^2}, \qquad 0 < \mu < 2,$$

$$= \lim_{\mu\to 0+} \left[\frac{1}{\mu} - \frac{1}{2}\right] = \infty.$$

Since the limit is not finite, the given improper integral is divergent.

Definition 3.3. If the upper limit is the only singularity of the integrand f, then the improper integral $\int_a^b f(x)dx$ is said to converge at b if

$$\lim_{\mu\to 0+} \int_{a}^{b-\mu} f(x)dx$$

exists and is finite.

EXAMPLE 3.2

Examine the convergence of $\displaystyle\int_0^1 \frac{dx}{(1-x)^2}$.

Solution. Since the upper limit 1 is the only singularity of the integrand, we have

$$\int_0^1 \frac{dx}{(1-x)^2} = \lim_{\mu\to 0+} \int_0^{1-\mu} \frac{dx}{(1-x)^2}, \quad 0<\mu<1$$
$$= \lim_{\mu\to 0+} \left[\frac{1}{1-x}\right]_0^{1-\mu}$$
$$= \lim_{\mu\to +} \left[\frac{1}{\mu}-1\right] = \infty.$$

Hence, the given improper integral is divergent.

Definition 3.4. If both upper and lower limits are the only singularities of the integrand, and c is any point within the interval of integration $[a,b]$, then the improper integral $\int_a^b f(x)dx$ converges if both improper integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge. Moreover,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

EXAMPLE 3.3

Examine the convergence of $\int_0^2 \frac{dx}{x(2-x)}$.

Solution. Both 0 and 2 are singularities of the integrand. Therefore,

$$\int_0^2 \frac{dx}{x(2-x)} = \int_0^1 \frac{dx}{x(2-x)} + \int_1^2 \frac{dx}{x(2-x)}$$
$$= \frac{1}{2}\int_0^1 \left[\frac{1}{x} - \frac{1}{2-x}\right]dx + \frac{1}{2}\int_1^2 \left[\frac{1}{x} - \frac{1}{2-x}\right]dx$$
$$= \frac{1}{2}\lim_{\mu\to 0+}\left[\log\frac{x}{2-x}\right]_\mu^1 + \frac{1}{2}\lim_{\mu\to 0+}\left[\log\frac{x}{2-x}\right]_1^{2-\mu}$$
$$= 0 - \frac{1}{2}\lim_{\mu\to 0+}\left[\log\frac{\mu}{2-\mu}\right]$$
$$+\frac{1}{2}\lim_{\mu\to 0+}\left[\log\frac{2-\mu}{\mu}\right] + 0 = \infty.$$

Hence, the given improper integral diverges.

3.2.1 COMPARISON TESTS

(A) Let f and ϕ be two functions such that $f(x)\geq 0$, $\phi(x)\geq 0$ and $f(x)\leq\phi(x)$ for all $x\in[a,b]$. Then

(i) $\int_a^b f(x)dx$ converges if $\int_a^b \phi(x)dx$ converges.

(ii) $\int_a^b \phi(x)dx$ diverges if $\int_a^b f(x)dx$ diverges.

Proof. Assume that f and ϕ are both bounded and integrable in the interval $[a+\mu\,,\,b]$, $0<\mu\leq(b-a)$ and a is the only singularity. Then, by the given hypothesis,

$$\int_{a+\mu}^b f(x)dx \leq \int_{a+\mu}^b \phi(x)dx, \quad \mu\ \varepsilon\ (0\,,\,b-a). \tag{1}$$

To prove (i), let $\int_a^b \phi(x)dx$ converge so that there exists a number M such that

$$\int_{a+\mu}^b \phi(x)dx < M \text{ for all } \mu \in (0,\ b-a).$$

Hence,

$$\int_{a+\mu}^b f(x)dx < M \text{ for all } \mu \in (0,\ b-a)$$

and so $\int_a^b f(x)dx$ converges at a.

To prove (ii), we note that if $\int_a^b f(x)dx$ does not converge at a, then $\int_{a+\mu}^b f(x)dx$ is not bounded above. Consequently, (1) implies that $\int_{a+\mu}^b \phi(x)dx$ is also not bounded above. Hence, $\int_a^b \phi(x)dx$ does not converge.

(B) If f and g are two positive functions such that $\lim_{x\to a+}\frac{f(x)}{g(x)} = l$ and l is neither 0 nor ∞, then the two integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge or diverge together.

Proof. Under the given hypothesis, $\frac{f(x)}{g(x)}$ is positive for all x and so l cannot be negative. Since $\lim\limits_{x\to a+}\frac{f(x)}{g(x)}=l$, there exists a neighbourhood $[a,\ c]$, $a<c<b$ such that for all $x\in[a,c]$,

$$\left|\frac{f(x)}{g(x)}-l\right|<\varepsilon$$

or

$$l-\varepsilon<\frac{f(x)}{g(x)}<l+\varepsilon$$

or

$$(l-\varepsilon)g(x)<f(x)<(l+\varepsilon)g(x)$$

or

$$(l-\varepsilon)g(x)<f(x) \tag{2}$$

and

$$f(x)<(l+\varepsilon)g(x) \tag{3}$$

Now, due to relation (2),

$$\text{Convergence of } \int_a^b f(x)dx$$

$$\Rightarrow \int_a^c f(x)dx \text{ converges}$$

$$\Rightarrow (l-\varepsilon)\int_a^c g(x)dx \text{ converges (using Test (A))}$$

$$\Rightarrow \int_a^c g(x)dx \text{ converges}$$

$$\Rightarrow \int_a^b g(x)dx \text{ converges at } a.$$

Using (3), it may similarly be shown that

$$\int_a^b g(x)dx \text{ does not converge}$$

$$\Rightarrow \int_a^b f(x)dx \text{ does not converge.}$$

Also from (3), it follows that

(i) Convergence of $\int_a^b g(x)dx \Rightarrow$ convergence of $\int_a^b f(x)dx$.

(ii) Divergence of $\int_a^b f(x)dx \Rightarrow$ divergence of $\int_a^b g(x)dx$.

Remark 3.1.

(i) If $\lim\limits_{x\to a+}\frac{f(x)}{g(x)}=0$ and $\int_a^b g(x)$ converges, then $\int_a^b f(x)dx$ also converges.

(ii) If $\lim\limits_{x\to a+}\frac{f(x)}{g(x)}=\infty$ and $\int_a^b g(x)dx$ diverge, then $\int_a^b f(x)dx$ also diverges.

EXAMPLE 3.4

Examine the convergence of

$$\int_a^b \frac{dx}{(x-a)^n} \quad (\textit{Comparison Integral})$$

Solution. For $n\neq 1$, we have

$$\begin{aligned}
\int_a^b \frac{dx}{(x-a)^n} &= \lim_{\mu\to 0+}\int_{a+\mu}^b \frac{dx}{(x-a)^n}\\
&= \lim_{\mu\to 0+}\left[\frac{1}{(1-n)(x-a)^{n-1}}\right]_{a+\mu}^b\\
&= \frac{1}{1-n}\lim_{\mu\to a+}\left[\frac{1}{(b-a)^{n-1}}-\frac{1}{\mu^{n-1}}\right]\\
&= \begin{cases}\frac{1}{1-n}(b-a)^{1-n} & \text{if } n<1\\ \infty & \text{if } n>1.\end{cases}
\end{aligned}$$

If $n=1$, then we have

$$\begin{aligned}
\int_a^b \frac{dx}{x-a} &= \lim_{\mu\to 0+}\int_{a+\mu}^b \frac{dx}{x-a}\\
&= \lim_{\mu\to 0+}[\log(x-a)]_{a+\mu}^b\\
&= \lim_{\mu\to 0+}[\log(b-a)-\log\mu)=\infty.
\end{aligned}$$

Hence, the given integral converges if $n < 1$ and diverges if $n \geq 1$.

EXAMPLE 3.5

Examine the convergence of $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$.

Solution. Let $f(x) = \frac{1}{x^{1/3}(1+x^2)}$ and $g(x) = \frac{1}{x^{1/3}}$. Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{1}{1+x^2} = 1 \text{ (finite)}.$$

Therefore $\int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ converge or diverge together. But $\int_0^1 f(x)dx = \int_0^1 \frac{dx}{x^{1/3}}$ is convergent since $\frac{1}{3} < 1$ (see Example 3.4). Hence, $\int_0^1 f(x)dx = \int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ also converges.

EXAMPLE 3.6

Examine the convergence of $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}}$.

Solution. The integrand is unbounded at both limits 0 and 1. Therefore, we may write

$$\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}} = \int_0^{1/2} \frac{dx}{x^{1/2}(1-x)^{1/2}} + \int_{1/2}^1 \frac{dx}{x^{1/2}(1-x)^{1/2}} \quad (4)$$

For the first integral on the right side of (4), let

$$f(x) = \frac{1}{x^{1/2}(1-x)^{1/2}} \text{ and } g(x) = \frac{1}{x^{1/2}}.$$

Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = 1 \text{ (finite)}$$

and so $\int_0^{1/2} f(x)dx$ and $\int_0^{1/2} g(x)dx$ behave alike. But the integral $\int_0^{1/2} g(x)dx$ converges since $\frac{1}{2} < 1$.

Hence, the integral $\int_0^{1/2} \frac{dx}{x^{1/2}(1-x)^{1/2}}$ also converges.

For the second integral on the right side of (4), let

$$f(x) = \frac{1}{x^{1/2}(1-x)^{1/2}} \text{ and } \phi(x)\frac{1}{(1-x)^{1/2}}.$$

Then

$$\lim_{x\to 1}\frac{f(x)}{g(x)} = 1 \text{ (finite)}$$

Therefore, $\int_{1/2}^1 f(x)dx$ and $\int_{1/2}^1 \phi(x)dx$ behave alike. But the integral $\int_{1/2}^1 \frac{dx}{(1-x)^{1/2}}$ converges since $\frac{1}{2} < 1$.

Hence, $\int_{1/2}^1 \frac{dx}{x^{1/2}(1-x)^{1/2}}$ also converges. Since both integrals on the right-hand side of (4) converge, it follows that the given integral is also convergent.

EXAMPLE 3.7

Examine the convergence of

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \ (\textit{Beta Function})$$

(For a detailed study on Beta Function, refer Chapter 4.)

Solution.

Case I. If m and n both are greater than or equal to 1, the integrand is finite for all values of x from 0 to 1. Hence, in that case, the given integral is convergent.

Case II. If m and n are less than 1, then the integrand has 0 and 1 as the points of infinite discontinuity. In that case, we have

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^{1/2} x^{m-1}(1-x)^{n-1}dx + \int_{1/2}^1 x^{m-1}(1-x)^{n-1}dx. \quad (5)$$

For the first integral on the right side of (5), we take

$$f(x) = x^{m-1}(1-x)^{n-1} = \frac{(1-x)^{n-1}}{x^{1-m}} \text{ and } g(x) = \frac{1}{x^{1-m}}.$$

Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}(1-x)^{n-1} = 1 \text{ (finite)}.$$

Therefore, $\int\limits_0^{1/2} f(x)dx$ and $\int\limits_0^{1/2} g(x)dx$ converge or diverge together. But $\int\limits_0^{1/2} g(x)dx = \int\limits_0^{1/2} \frac{1}{x^{1-m}}dx$ converges if and only if $1-m<1$ if and only if $m>0$. Hence, $\int\limits_0^{1/2} f(x)dx$ converges at 0 if and only if m is positive.

To test the convergence at 1 of the second integral on the right side of (5), we take

$$f(x) = x^{m-1}(1-x)^{n-1} = \frac{x^{m-1}}{(1-x)^{1-n}} \quad \text{and}$$

$$\phi(x) = \frac{1}{(1-x)^{1-n}}.$$

Then

$$\lim_{x\to 1}\frac{f(x)}{\phi(x)} = \lim_{x\to 1} x^{m-1} = 1 \text{ (finite)}$$

Therefore, $\int\limits_{1/2}^{1} f(x)dx$ and $\int\limits_{1/2}^{1} \phi(x)dx$ converge or diverge together. But the integral $\int\limits_{1/2}^{1} \phi(x)dx = \int\limits_{1/2}^{1} \frac{1}{(1-x)^{1-n}}dx$ converges if and only if $1-n<1$ if and only if n is positive. Thus, it follows from (5) that $\int\limits_0^1 x^{m-1}(1-x)^{n-1}dx$ exists (converges) for all positive values of m and n.

3.3 CONVERGENCE OF IMPROPER INTEGRAL WITH INFINITE LIMITS

Definition 3.5. Let f be bounded and integrable in $[a,b]$, where b is any number greater than or equal to a. If $\lim\limits_{b\to\infty}\int\limits_a^b f(x)dx$ is finite, then the improper integral $\int\limits_a^\infty f(x)dx$ is said to converge at ∞.

Thus, by definition,

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_0^b f(x)dx,$$

provided that the limit exists and is finite.

EXAMPLE 3.8

Evaluate $\int\limits_0^\infty \frac{dx}{1+x^2}$.

Solution. By definition, we have

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty}[\tan^{-1}x]_0^b$$
$$= \lim_{b\to\infty}\tan^{-1}b = \frac{\pi}{2}.$$

Definition 3.6. Let f be bounded and integrable in $[t,b]$, where $t \le b$. If $\lim\limits_{t\to-\infty}\int\limits_t^b f(x)dx$ is finite, then the improper integral $\int\limits_{-\infty}^b f(x)dx$ is said to converge at $-\infty$.

Thus, by definition,

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx,$$

provided that the limit exists and is finite.

EXAMPLE 3.9

Evaluate $\int\limits_{-\infty}^0 \frac{dx}{1+x^2}$.

Solution. We have

$$\int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{t\to-\infty}\int_t^0 \frac{dx}{1+x^2}$$
$$= \lim_{t\to-\infty}[\tan^{-1}x]_t^0$$
$$= \frac{\pi}{2}.$$

Definition 3.7. Let f be bounded and integrable in $[a,\ b]$, where $b \ge a$. If t is any number and both $\int\limits_{-\infty}^t f(x)dx$ and $\int\limits_t^\infty f(x)dx$ converge at $-\infty$ and ∞, respectively, then the improper integral $\int\limits_{-\infty}^\infty f(x)dx$ is said to converge at both limits $-\infty$ and ∞.

Thus, by definition, for any number t

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{t} f(x)dx + \int_{-\infty}^{\infty} f(x)dx,$$

provided that the integrals on the right exist.

EXAMPLE 3.10

Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Solution. We have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi \text{ by Examples 3.8 and 3.9.}$$

EXAMPLE 3.11

Show that

$$\int_{a}^{\infty} \frac{dx}{x^n}, \quad a > 0 \ (\textit{Comparison Integral})$$

converges if and only if $n > 1$.

Solution. When $n \neq 1$, we have

$$\int_{a}^{b} \frac{dx}{x^n} = \frac{1}{1-n}\left[\frac{1}{b^{n-1}} - \frac{1}{a^{n-1}}\right].$$

Therefore,

$$\lim_{b\to\infty} \int_{a}^{b} \frac{dx}{x^n} = \lim_{b\to\infty} \frac{1}{1-n}\left[\frac{1}{b^{n-1}} - \frac{1}{a^{n-1}}\right]$$

$$= \begin{cases} \frac{1}{(n-1)a^{n-1}}, & n > 1 \\ \infty, & n < 1 \end{cases}$$

When $n = 1$, we have

$$\int_{a}^{b} \frac{dx}{x} = [\log x]_a^b = \log\frac{b}{a} \to \infty \quad \text{as } b \to \infty.$$

Thus, the given integral converges if $n > 1$ and diverges if $n \le 1$.

3.3.1 COMPARISON TESTS

(A) If $0 \le f(x) \le g(x)$ for all x in $[a, x]$, then

(i) $\int_{a}^{\infty} f(x)dx$ converges if $\int_{a}^{\infty} g(x)dx$ converges.

(ii) $\int_{a}^{\infty} g(x)dx$ diverges if $\int_{a}^{\infty} f(x)dx$ diverges.

(B) If f and g are positive in $[a, x]$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$, where l is a non-zero finite number, then the integrals $\int_{a}^{\infty} f(x)dx$ and $\int_{a}^{\infty} g(x)dx$ converge or diverge together. Further, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ and $\int_{a}^{\infty} g(x)dx$ converges, then $\int_{a}^{\infty} f(x)dx$ converges and if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ and $\int_{a}^{\infty} g(x)dx$ diverges, then $\int_{a}^{\infty} f(x)dx$ diverges.

(The test (B) is called the Quotient Test).

(C) ***p*-test:** If $\lim_{x\to\infty} x^p f(x) = l$, then

(i) $\int_{a}^{\infty} f(x)dx$ converges if $p > 1$ and l is finite.

(ii) $\int_{a}^{\infty} f(x)dx$ diverges if $p \le 1$ and l is non-zero.

EXAMPLE 3.12

Examine the convergence of

$$\int_{1}^{\infty} e^{-x^2} dx \ (\textit{Euler–Poisson Integral}).$$

Solution. Since $e^{x^2} > x^2$ for all real x, we have $e^{-x^2} < \frac{1}{x^2}$. Since $\int_{1}^{\infty} \frac{1}{x^2} dx$ is convergent, by the comparison test, the given Euler–Poisson integral is also convergent.

EXAMPLE 3.13

Examine the convergence of $\int_{1}^{\infty} \frac{dx}{x^{1/3}(1+x)^{1/2}}$.

Solution. Let

$$f(x)\frac{1}{x^{1/3}(1+x)^{1/2}} \text{ and } g(x)\frac{1}{x^{1/3}x^{1/2}} = \frac{1}{x^{5/6}}.$$

Then $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = 1$. Hence, $\int_1^\infty f(x)dx$ and $\int_1^\infty g(x)dx$ behave identically. But $\int_1^\infty \frac{dx}{x^{5/6}}$ diverges since $\frac{5}{6} < 1$. Hence, $\int_1^\infty \frac{dx}{x^{1/3}(1+x)^{1/2}}$ is also divergent.

Definition 3.8. The improper integral $\int_a^\infty f(x)dx$ is called *absolutely convergent* if $\int_a^\infty |f(x)|dx$ converges.

If $\int_a^\infty f(x)dx$ converges but $\int_a^\infty |f(x)|dx$ diverges, then $\int_a^\infty f(x)dx$ is called *conditionally convergent.*

EXAMPLE 3.14

Show that $\int_0^\infty \frac{\cos x}{x^2+1}dx$ is absolutely convergent.

Solution. Since $\left|\dfrac{\cos x}{x^2+1}\right| \le \dfrac{1}{1+x^2}$ and

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{a\to\infty}\int_0^a \frac{dx}{1+x^2} = \lim_{a\to\infty}[\tan^{-1}x]_0^a = \frac{\pi}{2},$$

the integral $\int_0^\infty \frac{dx}{1+x^2}$ is convergent. Therefore, by the comparison test, the integral $\int_0^\infty \left|\frac{\cos x}{1+x^2}\right|dx$ is convergent. Hence, the result.

EXAMPLE 3.15

Examine the convergence of

$$\int_0^\infty e^{-x}x^{p-1}dx \ (\textit{Gamma Function})$$

(For a detailed study on Gamma Function, refer Chapter 4.)

Solution. Let

$$f(x) = x^{p-1}e^{-x}.$$

If $p < 1$, then 0 is a point of infinite discontinuity of f. Then

$$\int_0^\infty f(x)dx = \int_0^1 f(x)dx + \int_1^\infty f(x)dx. \qquad (6)$$

For the integral $\int_0^1 f(x)dx$, we examine the convergence at 0. So, let $g(x) = x^{p-1} = \frac{1}{x^{1-p}}$. Then

$$\lim_{x\to0}\frac{f(x)}{g(x)} = \lim_{x\to0} e^{-x} = 1$$

and $\int_0^1 g(x)dx = \int_0^1 \frac{1}{x^{1-p}}$ converges if $1-p<1$ or if $p>0$. Therefore $\int_0^1 f(x)dx$ converges if $p > 0$.

For the integral $\int_1^\infty f(x)dx$, we examine the convergence at ∞. So, let $g(x) = \frac{1}{x^2}$ so that

$$\lim_{x\to0}\frac{f(x)}{g(x)} = \lim\frac{x^{p+1}}{e^x} = 0 \text{ for all } p.$$

But $\int_1^\infty \frac{dx}{x^2}$ converges. Therefore, $\int_1^\infty f(x)dx$ converges for all p.

Therefore, from (6), it follows that the Gamma Function converges for all positive values of p.

EXERCISES

1. Examine the convergence of

(i) $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$

Ans. Convergent

(ii) $\int_0^{\pi/2} \log\sin x\, dx$

Ans. Convergent

(iii) $\int_0^1 e^{-mx}x^n dx$

Ans. Converges for $n > -1$

2. Examine the convergence of

(i) $\int_1^\infty \frac{\log x}{x^2}dx$

Ans. Convergent

(ii) $\int_0^\infty \frac{\sin^2 x}{x^2}dx$

Ans. Convergent

(iii) $\int_0^\infty \frac{x^{p-1}}{1+x}dx$

Ans. Converges for $0 < p < 1$

(iv) $\int_0^\infty \frac{\sin x^m}{x^p}dx$

Ans. Converges if $1-m < p < 1+m$

3. Show that $\int_1^\infty \frac{\sin x}{x^n}$ is absolutely convergent.

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4 Beta and Gamma Functions

The beta and gamma functions, also called Euler's Integrals, are the improper integrals, which are extremely useful in the evaluation of integrals.

4.1 BETA FUNCTION

The integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, which converges for $m > 0$ and $n > 0$ is called the *beta function* and is denoted by β (m, n). Thus,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, m > 0, n > 0.$$

Beta function is also known as *Eulerian Integral of First Kind*.

As an illustration, consider the integral $\int_0^1 x^{\frac{1}{2}}(1-x)^4 dx$. We can write this integral as

$$\int_0^1 x^{\frac{3}{2}-1}(1-x)^{5-1}dx,$$

which is a beta function, denoted by $\beta\left(\frac{3}{2}, 5\right)$. But, on the other hand, the integral $\int_0^1 x^{\frac{1}{2}}(1-x)^{-3}dx$ is *not* a beta function as $n-1=-3$ implies $n=-2$ (negative).

4.2 PROPERTIES OF BETA FUNCTION

1. We have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= \int_0^1 (1-x)^{m-1}(1-(1-x))^{n-1}dx, \text{ since}$$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx,$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx$$

$$= \beta(n,m), m > 0, \; n > 0.$$

Thus,

$$\beta(m,n) = \beta(n,m), \; m > 0, n > 0.$$

2. We have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx. \tag{1}$$

Putting $x = \sin^2\theta$, we get $dx = 2\sin\theta\cos\theta\, d\theta$ and therefore, (1) becomes

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}$$

$$.2\sin\theta\cos\theta\; d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\cos^{2n-2}\theta\sin\theta\cos\theta\; d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\; d\theta.$$

Thus,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\; d\theta.$$

3. Let m and n be positive integers. By definition,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m > 0,\ n > 0.$$

Integration by parts yields

$$\beta(m,n) = \left[x^{m-1}\frac{(1-x)^n}{n(-1)}\right]_0^1 - \int_0^1 (m-n)x^{m-2}\left[\frac{(1-x)^n}{n(-1)}\right]dx$$

$$= \frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^n dx$$

$$= \frac{m-1}{n}\beta(m-1,n+1).$$

Similarly,

$$\beta(m-1,n+1) = \frac{m-2}{n+1}\beta(m-2,n+2)$$

$$\beta(2,m+n-2) = \frac{1}{m+n-2}\beta(1,\ m+n-1).$$

Multiplying the preceding equations, we get

$$\beta(m,n) = \frac{(m-1)(m-2).....(2)(1)}{n(n+1)(n+2).....(m+n-2)} \times \beta(1,m+n-1)$$

$$= \frac{(m-1)!\,[1.2.3....(n-1)]}{1.2....(n-1)n(n+1)(n+2)...(m+n-2)} \times \int_0^1 x^{1-1}(1-x)^{m+n-2}dx$$

$$= \frac{(m-1)!\,(n-1)!}{(m+n-2)!}\int_0^1 (1-x)^{m+n-2}dx$$

$$= \frac{(m-1)!\,(n-1)!}{(m+n-2)!}\left[\frac{(1-x)^{m+n-1}}{(m+n-1)(-1)}\right]_0^1$$

$$= \frac{(m-1)!\,(n-1)!}{(m+n-2)!}\cdot\frac{1}{m+n-1}$$

$$= \frac{(m-1)!\,(n-1)!}{(m+n-1)!}.$$

Hence, if m and n are positive integers, then

$$\beta(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}.$$

4. Put $x = \frac{t}{1+t}$ so that $dx = \frac{1}{(1+t)^2}dt$. Then the expression for beta function reduces to

$$\beta(m,n) = \int_0^\infty \left(\frac{t}{1+t}\right)^{m-1}\left(1-\frac{t}{1+t}\right)^{n-1}\cdot\frac{1}{(1+t)^2}dt$$

$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m-1}}\cdot\frac{1}{(1+t)^{n-1}}\cdot\frac{1}{(1+t)^2}dt$$

$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx.$$

Hence,

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx.$$

Since $\beta(m,\ n) = \beta(n,\ m)$, we have

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx.$$

2. From the property (4), we have

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$= I_1 + I_2,\ \text{say.} \tag{2}$$

In I_2, put $x = \frac{1}{t}$ so that $dx = -\frac{1}{t^2}dt$ and so,

$$I_2 = \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(-\frac{1}{t^2}\right)dt = \int_0^1 \frac{t^{m+n}}{t^{m-1}(t+1)^{m+n}}\cdot\frac{1}{t^2}dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx.$$

Hence, (2) reduces to

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx,\ m > 0,\ n > 0.$$

EXAMPLE 4.1

Show that

$$\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx = 2\beta(m,n).$$

Solution. We know that

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx. \qquad (3)$$

Since $\beta(m, n) = \beta(n, m)$, we have

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx. \qquad (4)$$

Adding (3) and (4), we get

$$2\beta(m,n) = \int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx.$$

EXAMPLE 4.2

Show that

$$\int_0^a x^{m-1}(a-x)^{n-1}dx = a^{m+n-1}\beta(m,n).$$

Solution. Putting $x = ay$, we get

$$\int_0^a x^{m-1}(a-x)^{n-1}dx$$

$$= \int_0^1 (ay)^{m-1}(a-ay)^{n-1}.a\ dy$$

$$= \int_0^1 (ay)^{m-1}a^{n-1}(1-y)^{n-1}.a\ dy$$

$$= \int_0^1 a^{m-1+n-1+1}y^{m-1}(1-y)^{n-1}dy$$

$$= a^{m+n-1}\int_0^1 y^{m-1}(1-y)^{n-1}dy$$

$$= a^{m+n-1}\beta(m,n).$$

EXAMPLE 4.3

Show that

$$\frac{\beta(m+1,n)}{\beta(m,n)} = \frac{m}{m+n}.$$

Solution. We have

$$\beta(m+1,n) = \int_0^1 x^m(1-x)^{n-1}dx$$

$$= \int_0^1 (1-x)^m x^{n-1}dx, \text{ since } \beta(m+1,n) = \beta(n,m+1)$$

$$= \left[(1-x)^m\frac{x^n}{n}\right]_0^1 - \int_0^1 m(1-x)^{m-1}(-1).\frac{x^n}{n}dx$$

$$= \frac{m}{n}\int_0^1 x^{n-1}.x(1-x)^{m-1}dx$$

$$= \frac{m}{n}\int_0^1 x^{n-1}[1-(1-x)](1-x)^{m-1}dx$$

$$= \frac{m}{n}\left[\int_0^1 x^{n-1}(1-x)^{m-1}dx - \int_0^1 x^{n-1}(1-x)^m dx\right]$$

$$= \frac{m}{n}[\beta(n,m)-\beta(n,m+1)]$$

$$= \frac{m}{n}\beta(m,n) - \frac{m}{n}\beta(m+1,n).$$

Thus,

$$\left(1+\frac{m}{n}\right)\beta(m+1,n) = \frac{m}{n}\beta(m,n)$$

or

$$(n+m)\beta(m+1,n) = m\beta(m,n)$$

or

$$\frac{\beta(m+1,n)}{\beta(m,n)} = \frac{m}{m+n}.$$

EXAMPLE 4.4

Prove that

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right),$$

$$m > -1 \text{ and } n > -1.$$

Solution. We have

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{m-1}\theta\cos^{n-1}\theta.\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{m-1}\theta(1-\sin^2\theta)^{\frac{n-1}{2}}\sin\theta\cos\theta\, d\theta.$$

Putting $\sin^2\theta = x$ so that $2\sin\theta\cos\theta\, d\theta = dx$, we get

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta$$

$$= \frac{1}{2}\int_0^1 x^{\frac{m-1}{2}}(1-x)^{\frac{n-1}{2}}dx = \frac{1}{2}\int_0^1 x^{\frac{m+1}{2}-1}(1-x)^{\frac{n+1}{2}-1}dx$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right),\ m > -1 \text{ and } n > -1.$$

EXAMPLE 4.5

Show that

$$\beta(m,n) = \beta(m+1,n) + \beta(m,n+1).$$

Solution. By definition,

$$\beta(m+1,n) + \beta(m,n+1)$$

$$= \int_0^1 x^m(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^n dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}[x + (1-x)]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}dx = \beta(m,n).$$

EXAMPLE 4.6

Express $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}dx,\ m,\ n,\ a,\ b > 0$ in terms of beta function.

Solution. Put $bx = ay$ so that $dx = \frac{a}{b}dy$ in the given integral. This gives

$$\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}dx = \int_0^\infty \frac{\left(\frac{ay}{b}\right)^{m-1}}{(a+ay)^{m+n}}\cdot\frac{a}{b}dy$$

$$= \frac{1}{a^n b^m}\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy = \frac{1}{a^n b^m}\beta(m,n),$$

using property (4) of beta function.

EXAMPLE 4.7

Show that $\beta\left(m,\frac{1}{2}\right) = 2^{2m-1}\beta(m,m)$.

Solution. We know [see property (2)] that

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\, d\theta. \tag{5}$$

Putting $n = \frac{1}{2}$, we get

$$\beta\left(m,\frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\, d\theta. \tag{6}$$

Now, putting $n = m$ in (5), we have

$$\beta(m,m) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2m-1}\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2m-1}d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin 2\theta\right)^{2m-1}d\theta$$

$$= \frac{1}{2^{2m-1}}.2\int_0^{\frac{\pi}{2}} \sin^{2m-1}2\theta\, d\theta$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\, d\phi,\ \phi = 2\theta$$

$$= \frac{1}{2^{2m-1}} . 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \phi \, d\phi$$

$$= \frac{1}{2^{2m-1}} \beta\left(m, \frac{1}{2}\right), \text{ using (6)},$$

and so,

$$\beta\left(m, \frac{1}{2}\right) = 2^{2m-1} \beta(m, m).$$

4.3 GAMMA FUNCTION

The *gamma function* is defined as the definite integral

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx, \; n > 0.$$

The gamma function is also known as *Euler's Integral of Second Kind*.

4.4 PROPERTIES OF GAMMA FUNCTION

1. We have

$$\Gamma(n+1) = \int_0^{\infty} e^{-x} x^n dx = [-x^n e^{-x}]_0^{\infty} + n \int_0^{\infty} x^{n-1} e^{-x} dx$$

$$= n \int_0^{\infty} e^{-x} x^{n-1} dx = n\Gamma(n).$$

Hence,

$$\Gamma(n+1) = n\Gamma(n),$$

which is called the *recurrence formula* for $\Gamma(n)$.

2. Let n be a positive integer. By property (1), we have

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1) \\ &= n(n-1)(n-2)\Gamma(n-2) \\ &= n(n-1)(n-2)...3.2.1 \; \Gamma(1) \\ &= n!\Gamma(1).\end{aligned}$$

But, by definition,

$$\Gamma(1) = \int_0^{\infty} e^{-x} dx = [-e^{-x}]_0^{\infty} = 1.$$

Hence,

$$\Gamma(n+1) = n!, \text{ when } n \text{ is a positive integer.}$$

If we take $n = 0$, then

$$0! = \Gamma(1) = 1,$$

and so, gamma function defines 0!

Further, from the relation $\Gamma(n+1) = n\Gamma(n)$, we deduce that

$$\begin{aligned}\Gamma(2) &= 1.\Gamma(1) = 1!, \\ \Gamma(3) &= 2.\Gamma(2) = 2.1 = 2!, \\ \Gamma(4) &= 3.\Gamma(3) = 3.2.1 = 3!, \text{ and so on.}\end{aligned}$$

Moreover, $\Gamma(0) = \infty$ and $\Gamma(-n) = -\infty$ if $n > 0$.

Also,

$$\begin{aligned}\Gamma(n) &= \frac{\Gamma(n+1)}{n}, n \neq 0 = \frac{(n+1)\Gamma(n+1)}{n(n+1)} \\ &= \frac{\Gamma(n+2)}{n(n+1)}, \; n \neq 0 \text{ and } n \neq -1 \\ &= \frac{(n+2)\Gamma(n+2)}{n(n+1)(n+2)} \\ &= \frac{\Gamma(n+3)}{n(n+1)(n+2)}, \; n \neq 0, \; n \neq -1, \text{ and } n \neq -2 \\ &= \frac{\Gamma(n+k+1)}{n(n+1)(n+2)...(n+k)}, \; n \neq 0, \; n \neq -1, \\ &\quad n \neq -2, \text{ and } n \neq -k.\end{aligned}$$

Thus, $\Gamma(n)$ for $n < 0$ is defined, where k is a least-positive integer such that $n + k + 1 > 0$.

4.5 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

We know that

$$\Gamma(m) = \int_0^{\infty} e^{-t} t^{m-1} dt.$$

Putting $t = x^2$ so that $dt = 2xdx$, we get

$$\Gamma(m) = 2 \int_0^{\infty} e^{-x^2} x^{2m-1} dx. \tag{7}$$

Similarly, we can have

$$\Gamma(n) = 2 \int_0^{\infty} e^{-y^2} y^{2n-1} dy.$$

Therefore,

$$\Gamma(m)\Gamma(n) = 4 \int_0^{\infty} e^{-x^2} x^{2m-1} dx . \int_0^{\infty} e^{-y^2} y^{2n-1} dy$$

$$= 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx \, dy.$$

Taking $x = r\cos\theta$ and $y = r\sin\theta$, we have $dx\,dy = r d\theta\,dr$.
Therefore,

$$\Gamma(m)\Gamma(n) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2} r^{2(m+n)-1}\cos^{2m-1}\theta \times \sin^{2n-1}\theta\, dr d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\left[2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr\right] \times \cos^{2m-1}\theta\sin^{2n-1}\theta\, d\theta$$

$$= \Gamma(m+n)\left[2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \times \sin^{2n-1}\theta\, d\theta\right],\ \text{using}(1)$$

$$= \Gamma(m+n)\beta(m,n) \text{ using property (7) of beta function.}$$

Hence,

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

EXAMPLE 4.8

Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Solution. We know that

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Putting $m = n = \frac{1}{2}$, we get

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2.$$

Thus,

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}dx$$

$$= \int_0^1 \frac{dx}{\sqrt{x}\sqrt{1-x}} = \int_0^1 \frac{dx}{\sqrt{x-x^2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}} = \left[\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$= \sin^{-1}(1) - \sin^{-1}(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Hence,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Second Method

We know that (see Example 4.4)

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}.$$

Putting $m = n = 0$, we get

$$\int_0^{\frac{\pi}{2}} d\theta = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{2\Gamma(1)} = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{2}.$$

Thus,

$$\frac{\pi}{2} = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{2}.$$

Hence,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

EXAMPLE 4.9

Express the integrals $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta$ and $\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta$ in terms of gamma function.

Solution. We have

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{1}{2}}\theta}{\cos^{\frac{1}{2}}\theta} d\theta = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta\, d\theta$$

$$= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)}$$

$$= \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right).$$

Similarly, we can show that

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right).$$

EXAMPLE 4.10

Show that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$, $0 < n < 1$.

(Euler's Reflection Formula)

Solution. We know that

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx.$$

Also,

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \; m > 0 \text{ and } n > 0.$$

Therefore,

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx.$$

Putting $m = 1-n$ so that $m > 0$ implies $n < 1$, we get

$$\frac{\Gamma(n)\Gamma(1-n)}{\Gamma(1)} = \int_0^\infty \frac{x^{n-1}}{(1+x)}dx$$

or

$$\Gamma(n)\Gamma(1-n) = \int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}, \; 0 < n < 1.$$

EXAMPLE 4.11

Show that

$$\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}.$$

Hence, evaluate $\int_0^{\frac{\pi}{2}} \sin^p\theta \, d\theta$ and $\int_0^{\frac{\pi}{2}} \cos^p\theta \, d\theta$.

Solution. We know that

$$2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \beta(m,n)$$

or

$$\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}.\beta(m,n).$$

But, $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. Therefore,

$$\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}.$$

If we put $2m-1 = p$ and $2n-1 = q$, then this result reduces to

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}. \tag{8}$$

Putting $q = 0$ in (8), we get

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)} = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}\frac{\sqrt{\pi}}{2}.$$

Similarly, taking $p = 0$, we get

$$\int_0^{\frac{\pi}{2}} \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{(q+1)}{2}\right)}{\Gamma\left(\frac{q+2}{2}\right)}\frac{\sqrt{\pi}}{2}.$$

EXAMPLE 4.12

Show that

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

(Duplication Formula).

Solution. In Example 4.7, we have shown that

$$\beta\left(m,\frac{1}{2}\right) = 2^{2m-1}\beta(m,m).$$

Converting into gamma function, we get

$$\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} = 2^{2m-1}\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)}.$$

Since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we get

$$\frac{\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} = 2^{2m-1}\frac{\Gamma(m)}{\Gamma(2m)}$$

or

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m).$$

EXAMPLE 4.13

Show that

$$\int_0^\infty e^{-ax}x^{n-1}dx = \frac{\Gamma(n)}{a^n},$$

where a and n are positive. Deduce that

(i) $\int_0^\infty e^{-ax}x^{n-1}\cos bx \, dx = \frac{\Gamma(n)}{r^n}\cos n\theta$

(ii) $\int_0^\infty e^{-ax}x^{n-1}\sin bx \, dx = \frac{\Gamma(n)}{r^n}\sin n\theta$,

where $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\frac{b}{a}$. Also evaluate $\int_0^\infty e^{-ax}\cos bx \, dx$ and $\int_0^\infty e^{-ax}\sin bx \, dx$.

Solution. Put $ax = z$, so that $adx = dz$, to get

$$\int_0^\infty e^{-ax}x^{n-1}dx = \int_0^\infty e^{-z}\left(\frac{z}{a}\right)^{n-1}\cdot\frac{dz}{a}$$

$$= \frac{1}{a^n}\int_0^\infty e^{-z}z^{n-1}dz = \frac{\Gamma(n)}{a^n}. \quad (9)$$

Replacing a by $a + ib$ in (9), we get

$$\int_0^\infty e^{-(a+ib)x}x^{n-1}dx = \frac{\Gamma(n)}{(a+ib)^n}. \quad (10)$$

But as,

$$e^{-(a+ib)x} = e^{-ax}.e^{-ibx} = e^{-ax}(\cos bx - i\sin bx)$$

and taking $a = r\cos\theta$ and $b = r\sin\theta$, De-Moivre's Theorem implies

$$(a+ib)^n = (r\cos\theta + ir\sin\theta)^n$$
$$= r^n(\cos\theta + i\sin\theta)^n$$
$$= r^n(\cos n\theta + i\sin n\theta).$$

Therefore, (10) reduces to

$$\int_0^\infty [e^{-ax}(\cos bx - i\sin bx)]x^{n-1}dx$$

$$= \frac{\Gamma(n)}{r^n(\cos n\theta + i\sin n\theta)} = \frac{\Gamma(n)}{r^n}(\cos n\theta + i\sin n\theta)^{-1}$$

$$= \frac{\Gamma(n)}{r^n}(\cos n\theta - i\sin n\theta).$$

Equating real- and imaginary parts on both sides, we get

$$\int_0^\infty e^{-ax}x^{n-1}\cos bx\ dx = \frac{\Gamma(n)}{r^n}\cos n\theta$$

and

$$\int_0^\infty e^{-ax}x^{n-1}\sin bx\ dx = \frac{\Gamma(n)}{r^n}\sin n\theta.$$

If we put $n = 1$, then

$$\int_0^{\frac{\pi}{2}} e^{-ax}\cos bx\ dx = \frac{\Gamma(1)}{r}\cos\theta = \frac{r\cos\theta}{r^2} = \frac{a}{a^2+b^2}$$

and

$$\int_0^{\frac{\pi}{2}} e^{-ax}\sin bx\ dx = \frac{\Gamma(1)}{r}\sin\theta = \frac{r\sin\theta}{r^2} = \frac{b}{a^2+b^2}.$$

EXAMPLE 4.14

Show that

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\Gamma(n+1)}.$$

Hence, deduce that $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$.

Solution. We know that

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\ d\theta.$$

Therefore,

$$\beta\left(n+\frac{1}{2}, n+\frac{1}{2}\right)$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2n}\theta\cos^{2n}\theta\ d\theta = 2\int_0^{\frac{\pi}{2}}\left(\frac{\sin 2\theta}{2}\right)^{2n} d\theta$$

$$= \frac{1}{2^{2n-1}}\int_0^{\frac{\pi}{2}}\sin^{2n}2\theta\ d\theta = \frac{1}{2^{2n}}.\int_0^{\pi}\sin^{2n}\phi\ d\phi,\ \phi = 2\theta$$

$$= \frac{1}{2^{2n}}.2\int_0^{\frac{\pi}{2}}\sin^{2n}\phi\ d\phi = \frac{1}{2^{2n-1}}\left[\frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma\left(\frac{2n+2}{2}\right)}\,\frac{\sqrt{\pi}}{2}\right]$$

(see Example 2.11)

$$= \frac{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{2^{2n}\,\Gamma(n+1)}. \quad (11)$$

Also,

$$\beta\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)}$$

$$= \frac{\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2}{\Gamma(2n+1)}. \quad (12)$$

From (11) and (12), we have

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n}}.\frac{\Gamma(2n+1)}{\Gamma(n+1)}.$$

Further, putting $n = \frac{1}{4}$, we have

$$\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{\pi}}{\sqrt{2}}.\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} = \sqrt{\frac{\pi}{2}}\,\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}.$$

Hence,

$$\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \pi\sqrt{2}.$$

EXAMPLE 4.15

Show that

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sqrt{2}}.$$

Solution. In Example 4.9, we have proved that

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right).$$

But,

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2} \text{ (Example 2.14.)}$$

Hence,

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{\pi}{\sqrt{2}}.$$

EXAMPLE 4.16

Show that

$$\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p},\ p > 0,\ q > 0.$$

Solution. Putting $\log\frac{1}{y} = x$, we have $\frac{1}{y} = e^x$ or $y = e^{-x}$ and so, $dy = -e^{-x}\, dx$. Therefore,

$$\begin{aligned}
\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy
&= \int_\infty^0 e^{-(q-1)x} x^{p-1}(-e^{-x})dx \\
&= \int_0^\infty e^{-qx} x^{p-1} dx \\
&= \int_0^\infty e^{-t}\left(\frac{t}{q}\right)^{p-1} . \frac{dt}{q},\ \text{putting } qx = t \\
&= \frac{1}{q^p}\int_0^\infty e^{-t} t^{p-1} dt = \frac{\Gamma(p)}{q^p}.
\end{aligned}$$

EXAMPLE 4.17

Show that

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta . \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} = \pi.$$

Solution. We have

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta . \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} \\
&= \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\, d\theta . \int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\theta\, d\theta. \\
&= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+2}{2}\right)} . \frac{\sqrt{\pi}}{2} . \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+2}{2}\right)} . \frac{\sqrt{\pi}}{2} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)} . \frac{\pi}{4} \\
&\text{(see Example 5.11)} \\
&= \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} . \frac{\pi}{4} = \frac{\Gamma\left(\frac{1}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} . \frac{\pi}{4} = \pi.
\end{aligned}$$

EXAMPLE 4.18

Prove that

$$\int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy = \Gamma(n),\ n > 0.$$

Solution. Putting $\log\frac{1}{y} = x$, that is, $\frac{1}{y} = e^x$ or $y = e^{-x}$, we have $dy = -e^{-x}dx$. Hence,

$$\begin{aligned}
\int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy &= -\int_\infty^0 x^{n-1}e^{-x}dx \\
&= \int_0^\infty e^{-x}x^{n-1}dx \\
&= \Gamma(n),\ n > 0.
\end{aligned}$$

EXAMPLE 4.19

Evaluate $\int_0^1 x^4\left(\log\frac{1}{x}\right)^3 dx$.

Solution. Putting $\log\frac{1}{x} = y$, that is, $\frac{1}{x} = e^y$ or $x = e^{-y}$, we have $dx = -e^y\, dy$. Therefore,

$$\int_0^1 x^4\left(\log\frac{1}{x}\right)^3 dx = -\int_\infty^0 e^{-4y}.y^3.e^{-y}dy = \int_0^\infty e^{-5y}.y^3dy$$

$$= \int_0^\infty e^{-t}.\frac{t^3}{125}.\frac{dt}{5}, \text{ putting } 5y = \text{t}$$

$$= \frac{1}{625}\int_0^\infty e^{-t}.t^3dt = \frac{1}{625}\Gamma(3) = \frac{6}{625}.$$

EXAMPLE 4.20

Show that

$$\int_0^1 x^m(\log x)^n dx = \frac{(-1)^n n!}{(m+n)^{n+1}}.$$

Solution. Putting $\log x = -y$, we have $x = e^{-y}$ and so, $dx = -e^{-y}$. When $x = 1$, $y = 0$ and when $x \to 0$, $y \to \infty$. Therefore,

$$\int_0^1 x^m(\log x)^n dx$$

$$= \int_\infty^0 e^{-my}(-y)^n(-e^{-y})dy = (-1)^n\int_0^\infty e^{-(m+1)y}.y^n dy$$

$$= (-1)^n\int_0^\infty e^{-t}.\left(\frac{t}{m+1}\right)^n\frac{dt}{m+1}, \text{ putting } (m+1)y = t$$

$$= \frac{(-1)^n}{(m+1)^{n+1}}\int_0^\infty e^{-t}t^n dt = \frac{(-1)^n}{(m+1)^{n+1}}\Gamma(n+1)$$

$$= \frac{(-1)^n n!}{(m+1)^{n+1}}.$$

EXAMPLE 4.21

Prove that

$$\int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right).$$

Solution. Putting $x^2 = \tan\theta$, we have $2xdx = \sec^2\theta\, d\theta$. Therefore,

$$\int_0^1 \frac{dx}{\sqrt{1+x^4}} = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sqrt{2\tan\theta}}\cdot\frac{1}{\sqrt{1+\tan^2\theta}}d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sqrt{\tan\theta}}.\frac{1}{\sec\theta}d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sin^{\frac{1}{2}}\theta\cos^{\frac{1}{2}}\theta}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{(\sin\theta\cos\theta)^{\frac{1}{2}}} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\left(\frac{\sin 2\theta}{2}\right)^{\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\sin 2\theta}} = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{2\sqrt{\sin\phi}}, \quad \phi = 2\theta$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\phi\cos^0\phi\, d\phi$$

$$= \frac{2}{4\sqrt{2}}\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\phi\cos^0\phi\, d\phi$$

$$= \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right),$$

since $\beta(m,n) = 2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\phi\cos^{2n-1}\phi\, d\phi$.

EXAMPLE 4.22

Show that

$$\int_0^{\frac{\pi}{2}}\left(\sqrt{\tan\theta}+\sqrt{\sec\theta}\right)d\theta$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[\Gamma\left(\frac{3}{4}\right)+\frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}\right].$$

Solution. We have

$$\int_0^{\frac{\pi}{2}}\left(\sqrt{\tan\theta}+\sqrt{\sec\theta}\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}}\sqrt{\tan\theta}\,d\theta + \int_0^{\frac{\pi}{2}}\sqrt{\sec\theta\, d\theta}$$

$$= \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta \cos^{-\frac{1}{2}}\theta \, d\theta + \int_0^{\frac{\pi}{2}} \sin^0\theta \cos^{-\frac{1}{2}}\theta \, d\theta$$

$$= \frac{2}{2}\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta \cos^{-\frac{1}{2}}\theta \, d\theta + \frac{2}{2}\int_0^{\frac{\pi}{2}} \sin^0\theta \cos^{-\frac{1}{2}}\theta \, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right) + \frac{1}{2}\beta\left(\frac{1}{2},\frac{1}{4}\right)$$

$$= \frac{1}{2}\left[\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[\Gamma\left(\frac{3}{4}\right) + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[\Gamma\left(\frac{3}{4}\right) + \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}\right].$$

4.6 DIRICHLET'S AND LIOUVILLE'S THEOREMS

The following theorems of Dirichlet and Liouville are useful in evaluating multiple integrals.

Theorem 4.1 (Dirichlet). If V is the region, where $x \geq 0, y \geq 0, z \geq 0,$ *and* $x + y + z \leq 1$, then

$$\iiint_V x^{p-1}y^{q-1}z^{r-1}dx\,dy\,dz = \frac{\Gamma(p)\Gamma(q)\Gamma(r)}{\Gamma(p+q+r+1)}.$$

(The Dirichlet's Theorem can be extended to a finite number of variables).

Proof: Since $x + y + z \leq 1$, we have $y + z \leq 1 - x = a$. Therefore,

$$\iiint_V x^{p-1}y^{q-1}z^{r-1}\,dx\,dy\,dz$$

$$= \int_0^1\int_x^{1-x}\int_0^{1-x-y} x^{p-1}y^{q-1}z^{r-1}\,dx\,dy\,dz$$

$$= \int_0^1 x^{p-1}\left[\int_0^a\int_0^{a-y} y^{q-1}z^{r-1}dz\,dy\right]dx. \qquad (1)$$

Let

$$I = \int_0^a\int_0^{a-y} y^{q-1}z^{r-1}dz\,dy.$$

Putting $y = aY$ and $z = aZ$, this integral reduces to

$$I = \int_D (aY)^{q-1}(aZ)^{r-1}.a^2dZ\,dY,$$

where D is the domain where $X \geq 0$, $Y \geq 0$, and $Y + Z \geq 1$. Thus,

$$I = a^{q+r}\int_0^1\int_0^{1-Y} Y^{q-1}Z^{r-1}dZ\,dY$$

$$= a^{q+r}\int_0^1 Y^{q-1}\left[\frac{Z^r}{r}\right]_0^{1-Y}dY$$

$$= \frac{a^{q+r}}{r}\int_0^1 Y^{q-1}(1-Y)^r dY$$

$$= \frac{a^{q+r}}{r}\beta(q,\ r+1) = \frac{a^{q+r}}{r}\,\frac{\Gamma(q)\Gamma(r+1)}{\Gamma(q+r+1)}$$

$$= a^{q+r}\frac{\Gamma(q)\Gamma(r)}{\Gamma(q+r+1)}.$$

Hence, (1) yields

$$\iiint_V x^{p-1}y^{q-1}z^{r-1}dx\,dy\,dz$$

$$= \int_0^1 \frac{\Gamma(q)\Gamma(r)}{\Gamma(q+r+1)}x^{p-1}a^{q+r}dx$$

$$= \frac{\Gamma(q)\Gamma(r)}{\Gamma(q+r+1)}\int x^{p-1}(1-x)^{q+r}dx, \text{ since } a = 1-x$$

$$= \frac{\Gamma(q)\Gamma(r)}{\Gamma(q+r+1)}\beta(p, q+r+1)$$

$$= \frac{\Gamma(q)\Gamma(r)}{\Gamma(q+r+1)}.\frac{\Gamma(p)\Gamma(q+r+1)}{\Gamma(p+q+r+1)} = \frac{\Gamma(p)\Gamma(q)\Gamma(r)}{\Gamma(p+q+r+1)}.$$

Remark 4.1. If $x + y + z \leq h$, then by putting $\frac{x}{h} = X$, $\frac{y}{h} = Y$, and $\frac{z}{h} = Z$, we have $X + Y + Z \leq \frac{h}{h} = 1$ and so, the Dirichlet's Theorem takes the form

$$\iiint_V x^{p-1}y^{q-1}z^{r-1}dx\,dy\,dz$$

$$= \frac{\Gamma(p)\Gamma(q)\Gamma(r)}{\Gamma(q+r+1)}.h^{p+q+r}.$$

Theorem 4.2 (Liouville). If x, y, and z are all positive such that $h_1 < x + y + z < h_2$, then

$$\iiint f(x+y+z)x^{p-1}y^{q-1}z^{r-1}dx\,dy\,dz = \frac{\Gamma(p)\Gamma(q)\Gamma(r)}{\Gamma(p+q+r)}\int_{h_1}^{h_2} f(h)h^{p+q+r-1}dh.$$

(Proof, not provided here, is a slight modification of the proof of Dirichlet's Theorem).

EXAMPLE 4.23

Evaluate $\iiint x\ y\ z\ dx\ dy\ dz$ taken throughout the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1.$$

Solution. Put $\frac{x^2}{a^2} = X$, $\frac{y^2}{b^2} = Y$, and $\frac{z^2}{c^2} = Z$ to get

$$x = aX^{\frac{1}{2}},\ y = bY^{\frac{1}{2}},\ \text{and } z = cZ^{\frac{1}{2}}$$

and

$$xdx = \frac{a^2}{2}dX,\ ydy = \frac{b^2}{2}dY,\ \text{and } zdz = \frac{c^2}{2}dZ.$$

The condition, under this substitution, becomes

$$X + Y + Z \le 1.$$

Therefore, for the first quadrant,

$$\begin{aligned}
&\iiint xyz\ dx\ dy\ dz \\
&= \iiint (xdx)(ydy)(zdz) \\
&= \iiint \left(\frac{a^2}{2}dX\right)\left(\frac{b^2}{2}dY\right)\left(\frac{c^2}{2}dZ\right) \\
&= \frac{a^2b^2c^2}{8}\iiint X^{1-1}Y^{1-1}Z^{1-1}dX\ dY\ dZ \\
&= \frac{a^2b^2c^2}{8}\ \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1+1)},
\end{aligned}$$

by Dirichlet's Theorem

$$= \frac{a^2b^2c^2}{8}\cdot\frac{1}{\Gamma(4)} = \frac{a^2b^2c^2}{8.3!} = \frac{a^2b^2c^2}{48}.$$

Therefore, value of the integral for the whole of the ellipsoid is $8\left(\frac{a^2b^2c^2}{48}\right) = \frac{a^2b^2c^2}{6}$.

EXAMPLE 4.24

The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$ meets the axes in A, B, and C. Find the volume of the tetrahedron OABC.

Solution. We wish to evaluate

$$\iiint dx\ dy\ dz$$

under the condition $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$. Putting $\frac{x}{a} = X$, $\frac{y}{b} = Y$, and $\frac{z}{c} = Z$, we get $X + Y + Z = 1$. Also $dx = adX$, $dy = bdY$, and $dz = cdZ$. Therefore, using Dirichlet's Theorem, the required volume of the tetrahedron is

$$\begin{aligned}
V &= \iiint dx\ dy\ dz \\
&= \iiint abc\ dX\ dY\ dZ \\
&= abc\iiint X^{1-1}Y^{1-1}Z^{1-1}dX\ dY\ dZ \\
&= abc\frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1+1)} \\
&= \frac{abc}{\Gamma(4)} = \frac{abc}{3!} = \frac{abc}{6}.
\end{aligned}$$

EXAMPLE 4.25

Evaluate $\iiint x^{l-1}\ y^{m-1}\ z^{n-1}\ dx\ dy\ dz$, where $x > 0$, $y > 0$, and $z > 0$ under the condition $\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r \le 1$.

Solution. Put $\left(\frac{x}{a}\right)^p = X$, $\left(\frac{y}{b}\right)^q = Y$, and $\left(\frac{z}{c}\right)^r = Z$ so that

$$\begin{aligned}
dx &= \frac{a}{p}X^{\frac{1}{p}-1}dX, \\
dy &= \frac{b}{q}Y^{\frac{1}{q}-1}dY,\ \text{and} \\
dz &= \frac{c}{r}Z^{\frac{1}{r}-1}dZ.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\iiint x^{l-1}y^{m-1}z^{n-1}dx\ dy\ dz \\
&= \iiint \left(aX^{\frac{1}{p}}\right)^{l-1}\left(bY^{\frac{1}{q}}\right)^{m-1}\left(cZ^{\frac{1}{r}}\right)^{n-1} \\
&\quad \times \frac{abc}{pqr}X^{\frac{1}{p}-1}Y^{\frac{1}{q}-1}Z^{\frac{1}{r}-1}dX\ dY\ dZ \\
&= \frac{a^lb^mc^n}{pqr}\iiint X^{\frac{l}{p}-1}Y^{\frac{m}{q}-1}Z^{\frac{n}{r}-1}dX\ dY\ dZ \\
&= \frac{a^lb^mc^n}{pqr}\cdot\frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{r}\right)}{\Gamma\left(1+\frac{l}{p}+\frac{m}{q}+\frac{n}{r}\right)}.
\end{aligned}$$

EXAMPLE 4.26

Show that $\iint x^{m-1}\ y^{n-1}\ dx\ dy$ over the positive octant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^m b^n}{2n}\beta\left(\frac{m}{2},\frac{n}{2}+1\right)$.

Solution. Putting $\frac{x^2}{a^2}=X$ and $\frac{y^2}{b^2}=Y$, we get $x=a\sqrt{X}$ *and* $y=b\sqrt{Y}$ and $dx=\frac{a}{2}X^{-\frac{1}{2}}dX$ and $dy=\frac{b}{2}Y^{-\frac{1}{2}}dY$. Therefore,

$$\begin{aligned}
&\iint x^{m-1}y^{n-1}dxdy\\
&=\iint a^{m-1}X^{\frac{m-1}{2}}b^{n-1}Y^{\frac{n-1}{2}}\frac{a}{2}X^{-\frac{1}{2}}\frac{b}{2}Y^{-\frac{1}{2}}dXdY\\
&=\frac{a^m b^n}{4}\int\int X^{\frac{m}{2}-1}Y^{\frac{n}{2}-1}dX\ dY\\
&=\frac{a^m b^n}{4}\cdot\frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(1+\frac{m}{2}+\frac{n}{2}\right)}=\frac{a^m b^n}{\frac{4n}{2}}\cdot\frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)\frac{n}{2}}{\Gamma\left(\frac{m+n}{2}+1\right)}\\
&=\frac{a^m b^n}{2n}\cdot\frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{m+n}{2}+1\right)}=\frac{a^m b^n}{2n}\beta\left(\frac{m}{2},\frac{n}{2}+1\right).
\end{aligned}$$

4.7. MISCELLANEOUS EXAMPLES

EXAMPLE 4.27

Evaluate $\int_0^\infty \frac{x\,dx}{1+x^6}$ using Beta and Gamma functions.

Solution. Putting $x^6=t$, we have $x=t^{\frac{1}{6}}$ and so $dx=\frac{1}{6}t^{-\frac{5}{6}}$. When $x=0$, $t=0$ and when $x=\infty$, $t=\infty$. Therefore

$$\begin{aligned}
\int_0^\infty \frac{x\,dx}{1+x^6}&=\frac{1}{6}\int_0^\infty \frac{t^{\frac{1}{6}}}{1+t}t^{-\frac{5}{6}}dt\\
&=\frac{1}{6}\int_0^\infty \frac{t^{-\frac{2}{3}}}{1+t}dt=\frac{1}{6}\int_0^\infty \frac{t^{\frac{1}{3}-1}}{(1+t)^{\frac{1}{3}+\frac{2}{3}}}dt\\
&=\frac{1}{6}\beta\left(\frac{1}{3},\frac{2}{3}\right),\quad \text{since}\quad \beta(m,n)\\
&\qquad\qquad=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx\\
&=\frac{1}{6}\,\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)}=\frac{1}{6}\,\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)}\\
&=\frac{1}{6}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\\
&=\frac{1}{6}\left(\frac{\pi}{\sin\frac{\pi}{3}}\right),\quad \text{using } \Gamma(n)\,\Gamma\left(1-\frac{2}{n}\right)\\
&\qquad\qquad=\frac{\pi}{\sin n\pi}(0<n).
\end{aligned}$$

EXAMPLE 4.28

Evaluate $\int_0^a x^4\sqrt{a^2-x^2}dx$ using Gamma function.

Solution. Putting $x=a\sin\ \theta$, we get $dx=a\cos\ \theta\ d\theta$ and $\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}=a\ \cos\ \theta$. When $x=0$, $\theta=0$ and when $x=a$, $\theta=\frac{\pi}{2}$. Therefore

$$\begin{aligned}
I&=\int_0^a x^4\sqrt{a^2-x^2}dx\\
&=\int_0^{\frac{\pi}{2}} a^4\sin^4\theta(a\cos\theta)(a\cos\theta)d\theta\\
&=a^6\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^2\theta\ d\theta.
\end{aligned}$$

Since $\int_0^{\frac{\pi}{2}}\sin^m\theta\ \cos^n\theta\,d\theta=\frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}$, we have

$$\begin{aligned}
I&=a^6\left[\frac{\Gamma\left(\frac{5}{2}\right)\ \Gamma\left(\frac{3}{2}\right)}{2\ \Gamma(4)}\right]\\
&=\frac{a^6}{2}\left[\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}.\frac{1}{2}\sqrt{\pi}}{3!}\right]=\frac{\pi a^6}{32}.
\end{aligned}$$

EXAMPLE 4.29

Show that $\int_0^1\frac{x^n}{\sqrt{1-x^2}}dx=\frac{2.4.\dots(n-1)}{1.3.5.\dots n}$, where n is odd integer.

Solution. Putting $x=\sin\theta$, we have $dx=\cos\theta\ d\theta$. Therefore

$$\begin{aligned}
\int_0^1\frac{x^n}{\sqrt{1-x^2}}dx&=\int_0^{\frac{\pi}{2}}\frac{\sin^n\theta}{\cos\theta}\cos\theta\,d\theta\\
&=\int_0^{\frac{\pi}{2}}\sin^n\theta\,d\theta\\
&=\frac{\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{n+2}{2}\right)}\sqrt{\pi}\quad\text{(see Example 4.11).}
\end{aligned}$$

Since n is odd, we take $n = 2m+1$. Therefore

$$\int_0^1 \frac{x^n}{\sqrt{1-x^2}}dx$$

$$= \frac{\Gamma(m+1)}{2\,\Gamma\left(m+\frac{3}{2}\right)}\frac{\sqrt{\pi}}{2}$$

$$= \frac{m(m-1)(m-2)\ldots 3.2.1}{(m+\frac{1}{2})(m-\frac{1}{2})(m-\frac{3}{2})\ldots\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}\frac{\sqrt{\pi}}{2}$$

$$= \frac{2m(2m-2)(2m-4)\ldots 6.4.2}{(2m+1)(2m-1)(2m-3)\ldots 3.1}$$

$$= \frac{(n-1)(n-3)(n-5)\ldots 6.4.2}{n(n-2)\ldots 5.3.1}, \quad (n \text{ odd})$$

which was to be established.

EXAMPLE 4.30

Show that $\int_0^\infty x^m e^{-a^2x^2}dx = \frac{1}{2a^{m+1}}\,\Gamma\left(\frac{m+1}{2}\right)$.

Solution. Putting $ax = \sqrt{z}$, we have $a\,dx = \frac{1}{2}z^{-\frac{1}{2}}dz$. Therefore

$$\int_0^\infty x^m e^{-a^2x^2}dx = \frac{1}{2a}\int_0^\infty e^{-z}\left(\frac{\sqrt{z}}{a}\right)^m z^{-\frac{1}{2}}dz$$

$$= \frac{1}{2a^{m+1}}\int_0^\infty e^{-z}z^{\frac{m-1}{2}}dz$$

$$= \frac{1}{2a^{m+1}}\int_0^\infty e^{-z}z^{\frac{m+1}{2}-1}dz$$

$$= \frac{1}{2a^{m+1}}\,\Gamma\left(\frac{m+1}{2}\right).$$

EXERCISES

1. Show that

(i) $\beta(2.5, 1.5) = \frac{\pi}{16}$.

(ii) $\beta\left(\frac{9}{2}, \frac{7}{2}\right) = \frac{5\pi}{2048}$.

(iii) $\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \pi\sqrt{2}$.

2. Show that

$$\int_0^\infty \sqrt{x}e^{-x^3}dx = \frac{\sqrt{\pi}}{3}.$$

3. Show that

$$\int_0^\infty x^{2n-1}e^{-ax^2}dx = \frac{\Gamma(n)}{2a^n}.$$

4. Show that

$$\int_0^\infty \frac{x^a}{a^x}dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}, \text{ if a} > 1.$$

5. Show that

$$\int_0^{\frac{\pi}{2}} \sin^3 x\cos^{\frac{5}{2}} x dx = \frac{8}{77}.$$

6. Show that

$$\int_a^b (x-a)^m(b-x)^n dx = (b-a)^{m+n+1}\beta(m+1, n+1).$$

7. Prove that

$$\int_0^\infty x^n e^{-a^2x^2}dx = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right), \quad n > -1.$$

8. Show that

$$\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}}\cdot\int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}.$$

9. Prove that

$$\int_0^1 \frac{xdx}{\sqrt{1-x^5}} = \frac{1}{5}\beta\left(\frac{2}{5}, \frac{1}{2}\right).$$

10. Show that

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}.$$

11. Express the integrals in terms of gamma function:

(i) $\int_0^\infty x^{p-1}e^{-kx}dx, k>0$ **Ans.** (i) $\frac{\Gamma(n)}{k^n}$.

(ii) $\int_0^1 \left(\log\frac{1}{x}\right)^{n-1}dx$ (ii) $\int_0^\infty e^{-y}y^{n-1}dy = \Gamma(n)$.

(iii) $\int_0^\infty e^{-x^2}dx$ (iii) $\frac{1}{2}\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$.

12. Show that $\int_0^\infty x^3 e^{-x^2}dx = \frac{1}{9}\Gamma\left(\frac{1}{3}\right)$.

13. Show that $\int_0^\infty x^{2n-1}e^{-ax^2}dx = \frac{\Gamma(n)}{2a^n}$.

14. Show that $y\beta(x+1, y) = x\beta(x, y+1)$.

15. Show that $\int_0^\infty \sqrt{x}e^{-x^2}dx. \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}dx = \frac{\pi}{2\sqrt{2}}$.

16. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B, and C, respectively. Find the mass of the tetrahedron OABC if the density at any point is $\rho = \mu\, xyz$.

Hint: Mass $= \iiint \rho\, dx\, dy\, dz, 0 \le \frac{x}{a}+\frac{y}{b}$
$+\frac{z}{c} \le 1$
$= \iiint \mu xyz\, dx\, dy\, dz$

Put $\frac{x}{a}=X, \frac{y}{b}=Y,$ and $\frac{z}{c}=Z$ and proceed.

Ans. $\frac{\mu a^2b^2c^2}{720}$.

17. Show that the volume of the solid bounded by the coordinate planes and the surface $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}=1$ is $\frac{abc}{90}$.

18. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Hint: $V = 8\int\int\int dxdydz.$

Put $\frac{x^2}{a^2}=X, \frac{y^2}{b^2}=Y,$ and $\frac{z^2}{c^2}=Z,$ and use Dirichlet's Theorem to get $V = \frac{4\pi}{3}abc.$

19. Show that the entire volume of the solid $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$ is $\frac{4}{35}\pi\, abc.$

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5 Laplace Transform

The study of Laplace transform is essential for engineers and scientists because these transforms provide easy and powerful means of solving differential and integral equations. The Laplace transforms directly provides the solution of differential equations with given boundary values without finding the general solution first. A Laplace transform is an extension of the continuous-time Fourier transform motivated by the fact that this transform can be used to a wider class of signals than the Fourier transform can. In fact, Fourier transform does not converge for many signals whereas the Laplace transform does. Fourier transform is not applicable to initial-value problems whereas Laplace transform is applicable. Also some functions like sinusoidal functions and polynomials do not have Fourier transform in the usual sense without the introduction of generalized functions. Causal functions (which assume zero value for $t < 0$) are best handled by Laplace transforms.

5.1 DEFINITION AND EXAMPLES OF LAPLACE TRANSFORM

Definition 5.1. Let $f(t)$ be a function of t defined for $t > 0$. Then the *one-sided Laplace transform* (or merely *Laplace transform*) of $f(t)$, denoted by $L\{f(t)\}$ or $F(s)$, is defined by

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt, \quad s \in \mathbb{R} \text{ or } \mathbb{C},$$

provided that the integral converges for some value of s.

The defining integral is called the *Laplace integral*.

Definition 5.2. The *two-sided Laplace transform* of a function $f(t)$ is defined by

$$L\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t)dt$$

for all values of s, real or complex for which the integral converges.

The defining integral in this case is called *two-sided Laplace integral*.

The symbol L, which transforms $f(t)$ into $F(s)$ is called *Laplace operator*. Thus the Laplace transform of a function exists only if the Laplace integral converges. The following theorem provides sufficient conditions for the existence of Laplace transform.

Theorem 5.1. Let $f(t)$ be piecewise continuous in every finite interval $0 \le t \le N$ and be of exponential order γ for $t > N$. Then Laplace transform of $f(t)$ exists for all $s > \gamma$.

Proof: Since $f(t)$ is piecewise continuous on every finite interval [0, N] and e^{-st} is also piecewise continuous on [0, N] for $N > 0$, it follows that $e^{-st} f(t)$ is integrable on [0, N]. For any positive number N, we have

$$\int_0^{\infty} e^{-st} f(t)\, dt = \int_0^{N} e^{-st} f(t)dt + \int_N^{\infty} e^{-st} f(t)dt.$$

By the above arguments, the first integral on the right exists. Further, since $f(t)$ is of exponential order γ for $t > N$, there exists constant M such that $|f(t)| \le M\, e^{\gamma t}$ for $t \ge 0$ and so

$$\left|\int_N^{\infty} e^{-st} f(t)dt\right| \le \int_N^{\infty} |e^{-st} f(t)|dt \le \int_0^{\infty} |e^{-st} f(t)|dt$$
$$\le \int_0^{\infty} e^{-st} M\, e^{\gamma t}\, dt = \frac{M}{s-\gamma}.$$

Thus the Laplace transform $L\{f(t)\}$ exists for $s > \gamma$.

Remark 5.1. The conditions of the Theorem 5.1 are *sufficient but not necessary* for the existence of Laplace transform of a function. Thus Laplace transform may exist even if these conditions are not satisfied. For example, $f(t) = t^{-1/2}$ does not satisfy these conditions but its Laplace transform does exist (see Example 5.8).

EXAMPLE 5.1

Find Laplace transform of *unit step function f* defined by $f(t) = 1$, $t \geq 0$.

Solution. By definition of Laplace transform, we have

$$L\{f(t)\} = \int_0^\infty e^{-st}\,dt = \lim_{T\to\infty}\int_0^T e^{-st}\,dt$$

$$= \lim_{T\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^T = \lim_{T\to\infty}\frac{1-e^{-st}}{s} = \frac{1}{s} \text{ if } s > 0.$$

EXAMPLE 5.2

Find the Laplace transform of the *unit ramp function f* defined by $f(t) = t$, $t \geq 0$.

Solution. Using integration by parts, we get

$$L\{f(t)\} = \int_0^\infty t e^{-st}\,dt = \lim_{T\to 0}\int_0^T t e^{-st}\,dt$$

$$= \lim_{T\to\infty}\left\{\left[t\left(\frac{e^{-st}}{-s}\right)\right]_0^T - \left[\frac{e^{-st}}{s^2}\right]_0^T\right\}$$

$$= \lim_{T\to\infty}\left(\frac{1}{s^2} - \frac{e^{-sT}}{s^2} - \frac{Te^{-sT}}{s}\right) = \frac{1}{s^2} \text{ if } s > 0.$$

EXAMPLE 5.3

Find $L\{f(t)\}$, where $f(t) = [t]$, $t > 0$.

Solution. We have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}(0)\,dt + \int_1^2 e^{-st}\,dt + \int_2^3 e^{-st}\,2dt + \ldots$$

$$= \left|\frac{e^{-st}}{-s}\right|_1^2 + 2\left|\frac{e^{-st}}{-s}\right|_2^3 + 3\left|\frac{e^{-st}}{-s}\right|_3^4 + \ldots$$

$$= \frac{e^{-s}}{s}(1-e^{-s}) + 2\frac{e^{-2s}}{s}(1-e^{-s}) + 3\frac{e^{-3s}}{s}(1-e^{-s}) + \ldots$$

$$= \frac{e^{-s}}{s}(1-e^{-s})[1 + 2e^{-s} + 3e^{-2s} + \ldots]$$

$$= \frac{e^{-s}}{s}(1-e^{-s})\frac{1}{(1-e^{-s})^2}$$

$$= \frac{e^{-s}}{s(1-e^{-s})} = \frac{1}{s(e^s-1)}.$$

EXAMPLE 5.4

Find Laplace transform of $f(t) = e^{at}$, $t \geq 0$.

Solution. By the definition of Laplace transform, we have

$$L\{f(t)\} = \int_0^\infty e^{-st}e^{at}\,dt = \lim_{T\to\infty}\int_0^T e^{-(s-a)t}\,dt$$

$$= \lim_{T\to\infty}\left[\frac{e^{-(s-a)}}{-(s-a)}\right]_0^T = \lim_{T\to\infty}\frac{1-e^{-(s-a)T}}{s-a}$$

$$= \frac{1}{s-a}, \text{ if } s > a.$$

The result of this example holds for complex numbers also.

EXAMPLE 5.5

Find Laplace transforms of $f(t) = \sin at$ and $g(t) = \cos at$.

Solution. Since

$$\int e^{at}\sin bt\,dt = \frac{e^{at}(a\sin bt - b\cos bt)}{a^2+b^2},$$

and

$$\int e^{at}\cos bt\,dt = \frac{e^{at}(a\cos bt + b\sin bt)}{a^2+b^2},$$

we have

$$L\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt$$

$$= \lim_{T\to\infty}\int_0^T e^{-st}\sin at\,dt$$

$$= \lim_{T\to\infty}\left[\frac{e^{-st}(-s\sin at - a\cos at)}{s^2+a^2}\right]_0^T$$

$$= \lim_{T\to\infty}\left[\frac{a}{s^2+a^2} - \frac{e^{-st}(s\sin aT + a\cos aT)}{s^2+a^2}\right]$$
$$= \frac{a}{s^2+a^2} \; if \; s>0,$$

and

$$L\{\cos at\} = \int_0^\infty e^{-st}\cos at\, dt$$
$$= \lim_{T\to\infty}\int_0^T e^{-st}\cos at\, dt$$
$$= \lim_{T\to\infty}\left[\frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2}\right]_0^T$$
$$= \lim_{T\to\infty}\left[\frac{s}{s^2+a^2} - \frac{e^{-st}(s\cos aT - a\sin aT)}{s^2+a^2}\right]$$
$$= \frac{s}{s^2+a^2} \; if \; s>0.$$

EXAMPLE 5.6

Find the Laplace transforms of $f(t) = \sinh at$ and $g(t) = \cosh at$.

Solution. Since $\sinh at = \frac{e^{at}-e^{-at}}{2}$, we have

$$\mathrm{L}\{\sinh at\} = \mathrm{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt$$
$$= \frac{1}{2}\int_0^\infty e^{-st}e^{at}dt - \frac{1}{2}\int_0^\infty e^{-st}e^{-at}dt$$
$$= \frac{1}{2}\mathrm{L}\{e^{at}\} - \frac{1}{2}\mathrm{L}\{e^{-at}\}$$
$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2},\; s>|a|.$$

Again, since $\cosh at = \frac{e^{at}+e^{-at}}{2}$, proceeding as above, we have

$$\mathrm{L}\{\cosh at\} = \frac{1}{2}\mathrm{L}\{e^{at}\} + \frac{1}{2}\mathrm{L}\{e^{-at}\}$$
$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$
$$= \frac{s}{s^2-a^2},\; s>|a|.$$

EXAMPLE 5.7

Find Laplace transform of $f(t) = t^n$, where n is a positive integer.

Solution. Putting $st = u$, we have

$$\mathrm{L}\{f(t)\} = \int_0^\infty e^{-st}t^n\, dt = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^n . \frac{du}{s}$$
$$= \frac{1}{s^{n+1}}\int e^{-u}u^{(n+1)-1}\,du = \frac{\Gamma(n+1)}{s^{n+1}}$$
$$\text{for } s>0 \text{ and } n+1\geq 0,$$

by the definition of gamma function. Since n is positive integer, $\Gamma(n+1) = n!$ and so

$$\mathrm{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Remark 5.2. Integrating the defining formula for Laplace transform of t^n by parts, we have

$$\mathrm{L}\{t^n\} = \int_0^\infty t^n e^{-st}\, dt$$
$$= \left[\frac{-t^n e^{-st}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}\, e^{-st}\, dt.$$

The integral on the right exists and the lower limit can be used in the first term if $n \geq 1$. Since $s>0$, the exponent in the first term goes to zero as t tends to infinity. Thus, we obtain a general *recurrence formula*,

$$\mathrm{L}\{t^n\} = \frac{n}{s}\mathrm{L}\{t^{n-1}\}, \quad n\geq 1.$$

Hence, by induction, we get the sequence

$$\mathrm{L}\{t^0\} = \mathrm{L}\{1\} = \frac{1}{s} \quad \text{(by Example 5.1)}$$
$$\mathrm{L}\{t\} = \frac{1}{s}L\{t^0\} = \frac{1}{s^2}$$
$$\mathrm{L}\{t\} = \frac{2}{s^3}$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$\mathrm{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

EXAMPLE 5.8

Find the Laplace transform of $f(t) = t^{-1/2}$.

Solution. The condition $n + 1 > 0$ of Example 5.7 is satisfied and so

$$L\{t^{-1/2}\} = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}} = \sqrt{\frac{\pi}{s}}.$$

It may be mentioned here that the function $f(t) = t^{-1/2}$ does not satisfy the conditions of Theorem 5.1, even then the Laplace transform of this function exists. Thus the conditions of Theorem 5.1 are sufficient but not necessary for the existence of Laplace transform of a given function.

EXAMPLE 5.9

Find the Laplace transform of the function *f* defined by

$$f(t) = \begin{cases} t & \text{for } 0 \le t \le 3 \\ 0 & \text{for } t > 3. \end{cases}$$

Solution. The graph of the function *f* is shown in the Figure 5.1.

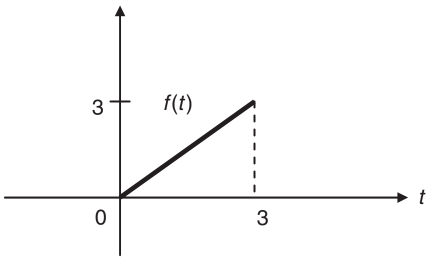

Figure 5.1

Using integration by parts, we have

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st} f(t)\,dt = \int_0^3 e^{-st}t\,dt \\ &= \left[t.\frac{e^{-st}}{-s}\right]_0^3 - \int_0^3 \frac{e^{-st}}{-s}\,dt \\ &= -\frac{3}{s}e^{-3s} + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^3 \\ &= -\frac{3}{s}e^{-3s} - \frac{1}{s^2}[e^{-3s} - 1] \\ &= \frac{1}{s^2}[1 - e^{-3s}] - \frac{3}{s}e^{-3s} \quad \text{for } s > 0. \end{aligned}$$

EXAMPLE 5.10

Find the Laplace transform of the function *f* defined by $f(t) = \sqrt{t^n}$, $n \ge 1$ and odd integer.

Solution. Integration by parts yields

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty \sqrt{t^n}\, e^{-st} dt \\ &= -\left[\frac{\sqrt{t^n}\, e^{-st}}{s}\right]_0^\infty + \frac{n}{2s}\int_0^\infty \sqrt{t^{n-2}}\, e^{-st}\, dt. \end{aligned}$$

If $n \ge 1$, the lower limit can be used in the first term on the right and thus the integral exists. Thus

$$L\{\sqrt{t^n}\} = \frac{n}{2s} L\{\sqrt{t^{n-2}}\}, \; n \ge 1 \text{ and odd.}$$

Thus we obtain a sequence of formulas given below:

$$L\left\{\frac{1}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{\sqrt{s}} \qquad \text{(Example 5.8)}$$

$$L\{\sqrt{t}\} = \frac{\sqrt{\pi}}{2\sqrt{s^3}}$$

$$L\{\sqrt{t^3}\} = \frac{3\sqrt{\pi}}{4\sqrt{s^5}}$$

......

......

$$L\{\sqrt{t^n}\} = \frac{(n+1)!\,\sqrt{\pi}}{2^{(n+1)}\left(\frac{n+1}{2}\right)!\,\sqrt{s^{(n+2)}}}.$$

EXAMPLE 5.11

Find the Laplace transform of erf($\sqrt{z}$) and erf(z).

Solution. Recall that the *error function* is defined by the integral

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt,$$

where the variable z may be real or complex.

The graph of erf(z), where z is real is shown in the Figure 5.2.

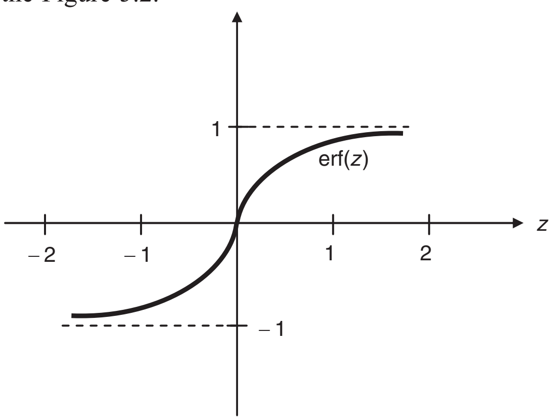

Figure 5.2 The Error Function

Let us find Laplace transform of erf $(\sqrt{z})$. Using series expansion of e^{-t^2}, we have

$$L\{\text{erf}(\sqrt{z})\}$$

$$= L\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left[1-t^2+\frac{t^4}{2!}-\frac{t^6}{3!}+\ldots\right]dt\right\}$$

$$= \frac{2}{\sqrt{\pi}}L\left\{t^{1/2}-\frac{t^{3/2}}{3}+\frac{t^{5/2}}{5.2!}-\frac{t^{7/2}}{7.3!}+\ldots\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left\{\frac{\Gamma(3/2)}{s^{3/2}}-\frac{\Gamma(5/2)}{3s^{5/2}}+\frac{\Gamma(7/2)}{5.2!\,s^{7/2}}-\frac{\Gamma(9/2)}{7.3!\,s^{9/2}}+\ldots\right\}$$

$$= \frac{1}{s^{3/2}}-\frac{1}{2s^{5/2}}+\frac{1.3}{2.4s^{7/2}}-\frac{1.3.5}{2.4.6\,s^{9/2}}+\ldots$$

$$= \frac{1}{s^{3/2}}\left[1-\frac{1}{2s}+\frac{1.3}{2.4}\frac{1}{s^2}-\frac{1.3.5}{2.4.6}\frac{1}{s^3}+\ldots\right]$$

$$= \frac{1}{s^{3/2}}\left(1+\frac{1}{s}\right)^{-1/2}$$

$$= \frac{1}{s\sqrt{s+1}}, \quad s > -1.$$

We now find the Laplace transform of erf (z). We have

$$L\{\text{erf}(z)\} = \int_0^\infty e^{-st}\text{erf}(z)\,dz$$

$$= \int_0^\infty e^{-st}\frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}dx\,dz$$

Changing the order of integration (Figure 5.3), we get

$$L\{\text{erf}(z)\} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\int_x^\infty e^{-st}\,dt\,dx = \frac{2}{s\sqrt{\pi}}\int_0^\infty e^{-(x^2+sx)}dx$$

$$= \frac{2}{s\sqrt{\pi}}e^{s^2/4}\int_0^\infty e^{-\left(x+\frac{s}{2}\right)^2}dx,$$

because $x^2+sx = \left(x+\frac{s}{2}\right)^2-\frac{s^2}{4}$.

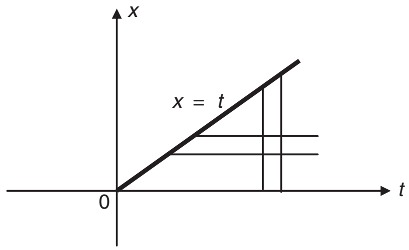

Figure 5.3

Taking $u = x+\frac{s}{2}$, we have

$$L\{\text{erf}(z)\} = \frac{2}{s\sqrt{\pi}}e^{s^2/4}\int_{s/2}^\infty e^{-u^2}du$$

and so

$$L\{\text{erf}(z)\} = \frac{2}{s\sqrt{\pi}}e^{s^2/4}\text{erf }c\left(\frac{s}{2}\right), \quad s > 0.$$

EXAMPLE 5.12

Find the Laplace transform of $f(t) = \sin\sqrt{t}$.

Solution. The series expansion for sin sin $\sqrt{t}$ is

$$\sin\sqrt{t} = t^{1/2}-\frac{t^{3/2}}{3!}+\frac{t^{5/2}}{5!}-\frac{t^{7/2}}{7!}+\ldots$$

$$= \sum_{n=0}^\infty\frac{(-1)^n t^{n+\frac{1}{2}}}{(2n+1)!}$$

Therefore

$$L\{\sin\sqrt{t}\} = \sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)!}L\{t^{n+\frac{1}{2}}\}$$

$$= \sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)!}\frac{\Gamma\left(n+\frac{3}{2}\right)}{s^{n+\frac{3}{2}}}$$

$$= \sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)!}\cdot\frac{(2n+2)!}{2^{2n+2}(n+1)!}\sqrt{\pi}.\frac{1}{s^{n+\frac{3}{2}}}$$

$$= \sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)!}\cdot\frac{(2n+1)!}{2^{2n+1}n!}\cdot\frac{\sqrt{\pi}}{s^{n+\frac{3}{2}}}$$

$$= \frac{1}{2s}\sqrt{\frac{\pi}{s}}\sum_{n=0}^\infty\frac{(-1)^n}{n!}\left(\frac{1}{4s}\right)^n$$

$$= \frac{1}{2s}\sqrt{\frac{\pi}{s}}\,e^{-\frac{1}{4s}}, \quad s > 0.$$

EXAMPLE 5.13

Find the Laplace transform of the *pulse of unit height and duration* T.

Solution. The pulse of unit height and duration T is defined by

$$f(t) = \begin{cases} 1 & \text{for } 0 < t < T \\ 0 & \text{for } T < t. \end{cases}$$

Therefore,

$$L\{f(t)\} = \int_0^T e^{-st}\, dt = \frac{1 - e^{-sT}}{s}.$$

EXAMPLE 5.14

Find the Laplace transform of sinusoidal (sine) pulse.

Solution. Sinusoidal pulse is defined by

$$f(t) = \begin{cases} \sin at & \text{for } 0 < t < \pi/a \\ 0 & \text{for } \pi/a < t. \end{cases}$$

Therefore, by the definition of Laplace transform we have

$$L\{f(t)\} = \int_0^\infty \sin at\; e^{-st} dt = \frac{a(1 + e^{-s\pi/a})}{s^2 + a^2}.$$

The denominator of $L\{f(t)\}$ here is zero at $s = \pm\, ia$. But, since $e^{\pm ia} = -1$, the numerator also becomes zero. Thus $L\{f(t)\}$ have no poles and is an entire function.

EXAMPLE 5.15

Find the Laplace transform of *triangular pulse of duration T.*

Solution. The triangular pulse of duration T is defined by

$$f(t) = \begin{cases} \frac{2}{T}t & \text{for } 0 < t < \mathrm{T}/2 \\ 2 - \frac{2}{T}t & \text{for } \mathrm{T}/2 < t < \mathrm{T} \\ 0 & \text{for } \mathrm{T} < t. \end{cases}$$

By the definition of Laplace transform, we have

$$L\{f(t)\} = \frac{2}{T}\int_0^{T/2} t\, e^{-st} dt + \int_{T/2}^{T} \left(2 - \frac{2}{T}t\right) e^{-st}\, dt$$

$$= \frac{2}{T}\left(\frac{1 - 2e^{-sT/2} + e^{-sT}}{s^2}\right).$$

EXAMPLE 5.16

Find the Laplace transform of a function defined by

$$f(t) = \begin{cases} \frac{t}{a} & \text{for } 0 < t < a \\ 1 & \text{for } t > a. \end{cases}$$

Solution. Integrating by parts,

$$L\{f(t)\} = \int_0^a \frac{t}{a}\, e^{-st} dt + \int_a^\infty e^{-st}\, dt$$

$$= \frac{-e^{-sa}}{s} - \frac{e^{-sa} - 1}{as^2} + \frac{e^{-sa}}{s} = \frac{1 - e^{-sa}}{as^2}.$$

EXAMPLE 5.17

Find Laplace transform of *f* defined by

$$f(t) = \begin{cases} e^t & \text{for } 0 < t < 1 \\ 0 & \text{for } t > 1. \end{cases}$$

Solution. By definition of Laplace transform, we have

$$L\{f(t)\} = \int_0^1 e^t . s^{-st}\, dt = \int_0^1 e^{(-s+1)t}\; dt = \frac{e^{1-s} - 1}{1 - s}.$$

EXAMPLE 5.18

Find the Laplace transform of a function *f* defined by

$$f(t) = \begin{cases} \sin t & \text{for } 0 < t < \pi \\ 0 & \text{for } t > \pi. \end{cases}$$

Solution. The integration by parts yields

$$L\{f(t)\} = \int_0^\pi e^{-st} \sin t\; dt = 0 + \frac{1}{s}\int_0^\pi e^{-st} \cos t dt$$

$$= \frac{1}{s}\left[\frac{e^{-st}}{-s} \cos t\right]_0^\pi + \frac{1}{s}\int_0^\pi \frac{e^{-st}}{-s} \sin t\; dt$$

$$= \frac{1}{s^2}[1 + e^{-s\pi}] - \frac{1}{s^2}\int_0^\pi e^{-st} \sin t\; dt.$$

Thus,

$$\left(\frac{s^2+1}{s^2}\right) L\{f(t)\} = \frac{1}{s^2}[1 + e^{-s\pi}],$$

which yields

$$L\{f(t)\} = \frac{1 + e^{-s\pi}}{s^2 + 1}.$$

EXAMPLE 5.19

Find the Laplace transform of the function f_ε defined by

$$f_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon} & \text{for } 0 \le t \le \varepsilon \\ 0 & \text{for } t > \varepsilon, \end{cases}$$

where $\varepsilon > 0$. Deduce the Laplace transform of the *Dirac delta function*.

Solution. The graph of the function f_ε is shown in the Figure 5.4.

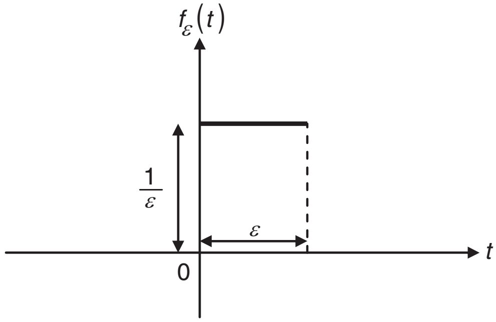

Figure 5.4

We observe that as $\varepsilon \to 0$, the height of the rectangle increases indefinitely and the width decreases in such a way that its area is always equals to 1.

The Laplace transform of f_ε is given by

$$L\{f_\varepsilon(t)\} = \int_0^\infty e^{-st} f_\varepsilon(t)\,dt = \int_0^\varepsilon e^{-st} f_\varepsilon(t)\,dt$$
$$+ \int_\varepsilon^\delta e^{-st} f_\varepsilon(t)\,dt$$
$$= \frac{1}{\varepsilon}\int_0^\varepsilon e^{-st}\,dt = \frac{1-e^{-s\varepsilon}}{s\varepsilon}.$$

Further we note that

$$\lim_{\varepsilon\to 0} L\{f_\varepsilon(t)\} = \lim_{\varepsilon\to 0}\frac{1-e^{-s\varepsilon}}{s\varepsilon}$$
$$= \lim_{\varepsilon\to 0}\frac{1-(1-s\varepsilon+\frac{s^2\varepsilon^2}{2!}-\ldots)}{s\varepsilon}$$
$$= \lim_{\varepsilon\to 0}\left(1-\frac{s\varepsilon}{2}+\ldots\right) = 1.$$

Also, we observe from the definition of f_ε that $\lim\limits_{\varepsilon\to 0} f_\varepsilon(t)$ does not exist and so L $\left\{\lim\limits_{\varepsilon\to 0} f_\varepsilon(t)\right\}$ is not defined. Even then it is useful to define a function δ as $\delta(t) = \left\{\lim\limits_{\varepsilon\to 0} f_\varepsilon(t)\right\}$ such that

$$\mathrm{L}\{\delta(t)\} = \lim_{\varepsilon\leftarrow 0}\{f_\varepsilon(t)\} = 1.$$

The function $\delta(t)$ is called the *Dirac delta function* or *unit impulse function* having the properties

$$\delta(\mathrm{t}) = 0, t \ne 0, \text{ and } \int_0^\infty \delta(t)\,dt = 1.$$

EXAMPLE 5.20

Find the Laplace transform of *Heaviside's unit step function* defined by

$$\mathrm{H}(t-a) = \begin{cases} 1 & \text{for } t > a \\ 0 & \text{for } t < a. \end{cases}$$

Solution. The Heaviside's unit step function is also known as *delayed unit step function* and occurs in the electrical systems. It delays the output until $t = a$ and then assumes a constant value of one unit. Its graph is shown in the Figure 5.5.

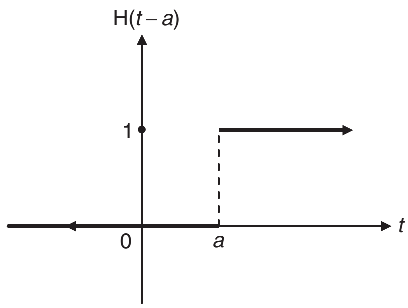

Figure 5.5

The Laplace transform of Heaviside's unit function is given by

$$\mathrm{L}\{\mathrm{H}(t-a)\} = \int_0^\infty e^{-st}\mathrm{H}(t-a)\,dt = \int_a^\infty e^{-st}\,dt$$
$$= \lim_{T\to\infty}\int_a^T e^{-st}\,dt = \lim_{T\to\infty}\left[\frac{e^{-st}}{-s}\right]_a^T = \frac{e^{-sa}}{s}.$$

EXAMPLE 5.21

Find the Laplace transform of *rectangle function* defined by

$$g(t) = \begin{cases} 1 & \text{for } a < t < b \\ 0 & \text{otherwise.} \end{cases}$$

Solution. The graph of this function is shown in the Figure 5.6. Clearly, this function can be expressed in terms of Heaviside's unit function as

$$g(t) = \mathrm{H}(t-a) - \mathrm{H}(t-b).$$

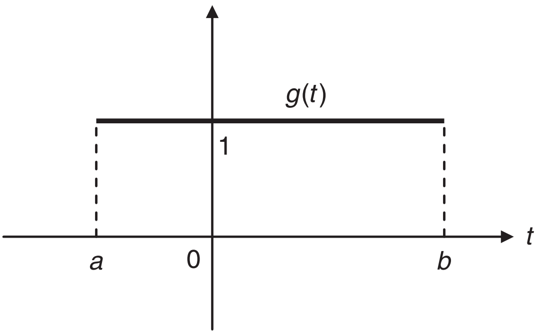

Figure 5.6

Further if $a = 0$, then g becomes pulse of unit height and duration b (Example 5.13). The Laplace transform of rectangle function g is given by

$$\begin{aligned} \mathrm{L}\{g(t)\} &= \mathrm{L}\{\mathrm{H}(t-a\} - \mathrm{L}\{\mathrm{H}(t-b)\} \\ &= \frac{e^{-sa}}{s} - \frac{e^{-sb}}{s} \\ &= \frac{e^{-sa} - e^{-sb}}{s}. \end{aligned}$$

5.2 PROPERTIES OF LAPLACE TRANSFORMS

While studying the following properties of Laplace transforms, we assume that the Laplace transforms of the given functions exist.

Theorem 5.2. (Linearity of the Laplace Transform). If c_1 and c_2 are arbitrary constants (real or complex) and $f_1(t)$ and $f_2(t)$ are functions with Laplace transforms $F_1(s)$ and $F_2(s)$, respectively, then

$$\begin{aligned} L\{c_1 f_1(t) + c_2 f_2(t)\} &= c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\} \\ &= c_1 F_1(s) + c_2 F_2(s). \end{aligned}$$

Thus L is a linear operator.

Proof: Using the definition of Laplace transform and the linearity property of integral, we have

$$\begin{aligned} &\mathrm{L}\{c_1 f_1(t) + c_2 f_2(t)\} \\ &= \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]\,dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t)\,dt + c_2 \int_0^\infty e^{-st} f_2(t)\,dt \\ &= c_1 F_1(s) + c_2 F_2(s). \end{aligned}$$

EXAMPLE 5.22

Find the Laplace transform of $f(t) = \sin^2 3t$.

Solution. Since

$$\sin^2 3t = \frac{1-\cos 6t}{2} = \frac{1}{2} - \frac{1}{2}\cos 6t,$$

we have

$$\begin{aligned} \mathrm{L}\{\sin^2 3t\} &= \mathrm{L}\left\{\frac{1}{2} - \frac{1}{2}\cos 6t\right\} = \frac{1}{2}L\{1\} - \frac{1}{2}L\{\cos 6t\} \\ &= \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s}{s^2+6^2}\right),\ s > 0 \\ &= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right] = \frac{18}{s(s^2+36)},\ s > 0. \end{aligned}$$

EXAMPLE 5.23

Find the Laplace transform of $f(t) = e^{4t} + e^{2t} + t^3 + \sin^2 t$.

Solution. Since $\sin^2 t = \frac{1-\cos 2t}{2}$, by Theorem 5.2, we have

$$\begin{aligned} \mathrm{L}\{f(t)\} &= \mathrm{L}\{e^{4t} + e^{2t} + t^3 + \sin^2 t\} \\ &= L\{e^{4t}\} + L\{e^{2t}\} + L\{t^3\} + \frac{1}{2}L\{1\} \\ &\quad - \frac{1}{2}L\{\cos 2t\} \\ &= \frac{1}{s-4} + \frac{1}{s-2} + \frac{6}{s^4} + \frac{1}{2s} - \frac{s}{2(s^2+4)} \\ &= \frac{1}{s-4} + \frac{1}{s-2} + \frac{6}{s^4} + \frac{2}{s(s^2+4)},\ s > 0. \end{aligned}$$

EXAMPLE 5.24

Find Laplace transform of $f(t) = \sin^3 2t$.

Solution. Since $\sin 3t = 3\sin t - 4\sin^3 t$, we have

$$\sin^3 t = \frac{3}{4}\sin t + \frac{1}{4}\sin 3t$$

and so

$$\sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t.$$

Hence, by linearity of L, we get

$$\begin{aligned}
L\{f(t)\} &= \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\} \\
&= \frac{3}{4}\left[\frac{2}{s^2+4}\right] - \frac{1}{4}\left[\frac{6}{s^2+36}\right], \; s > 0 \\
&= \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right] \\
&= \frac{48}{(s^2+4)\,(s^2+36)}.
\end{aligned}$$

EXAMPLE 5.25

Find the Laplace transform of $f(t) = \sin at \sin bt$.

Solution. We have

$$\begin{aligned}
f(t) &= \frac{1}{2}(2\sin at \sin bt) \\
&= \frac{1}{2}[\cos(at - bt) - \cos(at + bt)] \\
&= \frac{1}{2}\cos(a-b)t - \frac{1}{2}\cos(a+b)t.
\end{aligned}$$

Therefore, using linearity, we have

$$\begin{aligned}
L\{f(t)\} &= \frac{1}{2}L\{\cos(a-b)t\} - \frac{1}{2}L\{\cos(a+b)t\} \\
&= \frac{1}{2}\left[\frac{s}{s^2+(a-b)^2}\right] - \frac{1}{2}\left[\frac{s}{s^2+(a+b)^2}\right], s > 0 \\
&= \frac{2abs}{(s^2+(a-b)^2)\,(s^2+(a+b)^2)}, \; s > 0.
\end{aligned}$$

EXAMPLE 5.26

Find the Laplace transform of $f(t) = \sin(\omega t + \phi)$, $t \geq 0$.

Solution. Since

$$\sin(\omega t + \phi) = \sin \omega t \cos \phi + \cos \omega t \sin \phi,$$

we have by linearity of the operator L,

$$\begin{aligned}
L\{f(t)\} &= \cos\phi L\{\sin\omega t\} + \sin\phi L\{\cos\omega t\} \\
&= \cos\phi\left(\frac{\omega}{s^2+\omega^2}\right) + \sin\phi\left(\frac{s}{s^2+\omega^2}\right), \; s > 0 \\
&= \frac{1}{s^2+\omega^2}(\omega\cos\phi + s\sin\phi), \; s > 0.
\end{aligned}$$

EXAMPLE 5.27

Determine Laplace transform of the *square wave function* f defined by

$$\begin{aligned}
f(t) = H(t) - 2H(t-a) + 2H(t-2a) \\
- 2H(t-3a) + \ldots
\end{aligned}$$

Solution. We note that

$$f(t) = H(t) - 2H(t-a) = 1 - 2(0) = 1, \; 0 < t < a$$

$$\begin{aligned}
f(t) &= H(t) - 2H(t-a) + 2H(t-2a) \\
&= 1 - 2(1) + 2(0) = -1, \quad 0 < a < t < 2a
\end{aligned}$$

and so on. Thus the graph of the function is as shown in the Figure 5.7.

By linearity of Laplace operator, we have

$$\begin{aligned}
L\{f(t)\} &= L\{H(t) - 2L\{H(t-a)\} \\
&\quad + 2L\{H(t-2a)\} - 2L\{H(t-3a)\} + \ldots \\
&= \frac{1}{s} - 2\frac{e^{-sa}}{s} + 2\frac{e^{-2sa}}{s} - 2\frac{e^{-3sa}}{s} + \ldots
\end{aligned}$$

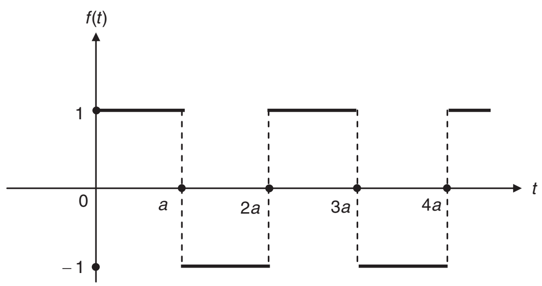

Figure 5.7

$$= \frac{1}{s}[1 - 2e^{-sa}\{1 - e^{-sa} + e^{-2sa} - \ldots\}]$$
$$= \frac{1}{s}\left[1 - 2e^{-sa}\left(\frac{1}{1+e^{-sa}}\right)\right]$$
$$= \frac{1}{s}\left(\frac{1-e^{-sa}}{1+e^{-sa}}\right) = \frac{1}{s}\left(\frac{e^{\frac{sa}{2}} - e^{-\frac{sa}{2}}}{e^{\frac{sa}{2}} + e^{-\frac{sa}{2}}}\right)$$
$$= \frac{1}{s}\tanh\left(\frac{sa}{2}\right).$$

EXAMPLE 5.28

Find the Laplace transform of $f(t) = (\sin t - \cos t)^2$, $t \geq 0$.

Solution. Since
$$(\sin t - \cos t)^2 = \sin^2 t + \cos^2 t - 2\sin t\cos t$$
$$= 1 - \sin 2t,$$
we have
$$L\{f(t)\} = L\{1\} - L\{\sin 2t\} = \frac{1}{s} - \frac{2}{s^2+4}$$
$$= \frac{s^2 - 2s + 4}{s(s^2+4)}, \; s > 0.$$

EXAMPLE 5.29

Find Laplace transform of $f(t) = 2 + \sqrt{t} + \frac{1}{\sqrt{t}}, t > 0$.

Solution. By linearity of L, we have
$$L\{f(t)\} = 2L\{1\} + L\{\sqrt{t}\} + L\left\{\frac{1}{\sqrt{t}}\right\}$$
$$= \frac{2}{s} + \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} + \frac{\sqrt{\pi}}{s}$$
$$= \frac{2}{s} + \frac{\sqrt{\pi}}{2s^{3/2}} + \sqrt{\frac{\pi}{s}}, \; s > 0.$$

EXAMPLE 5.30

Find Laplace transform of $e^{at}\cos bt$ and $e^{at}\sin bt$, where a and b are real.

Solution. Let $f(t) = e^{(a+ib)t}$. Then (see Example 5.4)
$$L\{f(t)\} = \frac{1}{s-(a+ib)} = \frac{1}{s-a-ib}$$
$$= \frac{1}{(s-a)-ib}\cdot\frac{(s-a)+ib}{(s-a)+ib}$$
$$= \frac{(s-a)+ib}{(s-a)^2+b^2} \qquad (1)$$

Also
$$e^{(a+ib)t} = e^{at}[\cos bt + i\sin bt]$$
$$= e^{at}\cos bt + i\,e^{at}\sin bt$$

Hence
$$L\{f(t)\} = L\{e^{at}\cos bt + i\,e^{at}\sin bt\}$$
$$= L\{e^{at}\cos bt\} + i\,L\{e^{at}\sin bt\}$$
$$\text{(by linearity of L)} \qquad (2)$$

Thus, by (1) and (2), we have
$$L\{e^{at}\cos bt\} + i\,L\{e^{at}\sin bt\} = \frac{(s-a)+ib}{(s-a)^2+b^2}.$$

Comparing real and imaginary parts, we have
$$L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$$
and
$$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}.$$

EXAMPLE 5.31

Find the Laplace transform of $f(t) = \int_0^t \frac{\sin u}{u}\,du$.

Solution. Using series expansion of sin u, we have
$$\int_0^t \frac{\sin u}{u}\,du = \int_0^t \frac{1}{u}\left[u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \ldots\right]du$$
$$= \int_0^t \left[1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \frac{u^6}{7!} + \ldots\right]du$$
$$= t - \frac{t^3}{3.3!} + \frac{t^5}{5.5!} - \frac{t^7}{7.7!} + \ldots$$

Therefore,
$$L\{f(t)\} = L\left\{t - \frac{t^3}{3.3!} + \frac{t^5}{5.5!} - \frac{t^7}{7.7!} + \ldots\right\}$$
$$= \frac{1}{s^2} - \frac{1}{3.3!}\cdot\frac{3!}{s^4} + \frac{1}{5.5!}\cdot\frac{5!}{s^6} - \frac{1}{7.7!}\cdot\frac{7!}{s^8}$$
$$+ \ldots \text{(by linearity of L)}$$
$$= \frac{1}{s^2} - \frac{1}{3s^4} + \frac{1}{5s^6} - \frac{1}{7s^8} + \ldots$$
$$= \frac{1}{s}\left[\frac{1}{s} - \frac{(1/s)^3}{3} + \frac{(1/s)^5}{5} - \frac{(1/s)^7}{7} + \ldots\right]$$
$$= \frac{1}{s}\tan^{-1}\frac{1}{s}.$$

EXAMPLE 5.32

Find Laplace transform $f(t) = \cosh at - \cos at$.

Solution. By linearity of the Laplace operator, we have

$$\begin{aligned}
\mathrm{L}\{f(t)\} &= \mathrm{L}\{\cosh at - \cos at\} \\
&= \mathrm{L}\{\cosh at\} - \mathrm{L}\{\cos at\} \\
&= \frac{s}{s^2 - a^2} - \frac{s}{s^2 + a^2}, \quad s > |a| \\
&= \frac{s^3 + a^2 s - s^3 + a^2 s}{s^4 - a^4} = \frac{2a^2 s}{s^4 - a^4}.
\end{aligned}$$

EXAMPLE 5.33

Find the Laplace transform of *Bessel's function of order zero*.

Solution. Recall that Bessel's function of order zero is defined by

$$\mathrm{J}_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \ldots$$

Therefore,

$$\begin{aligned}
\mathrm{L}\{J_0(t)\} &= \mathrm{L}\left\{1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \ldots\right\} \\
&= \mathrm{L}\{1\} - \frac{1}{2^2}\mathrm{L}\{t^2\} + \frac{1}{2^2.4^2}\mathrm{L}\{t^4\} \\
&\qquad - \frac{1}{2^2.4^2.6^2}\mathrm{L}\{t^6\} + \ldots \\
&= \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2 4^2}\frac{4!}{s^5} - \frac{1}{2^2.4^2.6^2}\frac{6!}{s^7} + \ldots \\
&= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{s^4}\right)\right. \\
&\qquad \left. - \frac{1.3.5}{2.4.6}\left(\frac{1}{s^6}\right) + \ldots\right] \\
&= \frac{1}{s}\left[\left(1 + \frac{1}{s^2}\right)^{-1/2}\right] \text{(using binomial theorem)} \\
&= \frac{1}{\sqrt{s^2+1}}.
\end{aligned}$$

Theorem 5.3. [First Shifting (Translation) Property]. If $f(t)$ is a function of t for $t > 0$ and $L\{f(t)\} = F(s)$, then

$$\mathrm{L}\{e^{at} f(t)\} = F(s - a).$$

Proof: We are given that

$$\mathrm{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

By the definition of Laplace transform, we have

$$\begin{aligned}
\mathrm{L}\{e^{at} f(t)\} &= \int_0^\infty e^{-st}(e^{at} f(t))\, dt = \int_0^\infty e^{-(s-a)} f(t)\, dt \\
&= \mathrm{F}(s - a).
\end{aligned}$$

EXAMPLE 5.34

Find the Laplace transform of $g(t) = e^{-t} \sin^2 t$.

Solution. We have (see Example 5.23)

$$\mathrm{L}\{\sin^2 t\} = \mathrm{F}(s) = \frac{2}{s(s^2 + 4)}.$$

Therefore, using first shifting property, we get

$$\begin{aligned}
\mathrm{L}\{g(t)\} &= \mathrm{F}(s - a) \\
&= \frac{2}{(s+1)\left[(s+1)^2 + 4\right]}, \quad \text{since } a = -1. \\
&= \frac{2}{(s+1)(s^2 + 2s + 5)}.
\end{aligned}$$

EXAMPLE 5.35

Find Laplace transform of $g(t) = t^3 e^{-3t}$.

Solution. Since

$$\mathrm{L}\{t^3\} = \mathrm{F}(s) = \frac{3!}{s^4} = \frac{6}{s^4},$$

we have by shifting property,

$$\mathrm{L}\{g(t)\} = \mathrm{L}\{e^{-3t}.t^3\} = \mathrm{F}(s - a) = \frac{6}{(s+3)^4},$$

since $a = -3$.

EXAMPLE 5.36

Using first-shifting property, find Laplace transforms of $t \sin at$ and $t \cos at$.

Solution. Since $\mathrm{L}\{t\} = \frac{1}{s^2}$, we have

$$\begin{aligned}
\mathrm{L}\{t e^{iat}\} &= \mathrm{L}\{t \cos at\} + i\mathrm{L}\{t \sin at\} \\
&= \mathrm{F}(s - a) = \frac{1}{(s - ia)^2} = \frac{(s + ia)^2}{\left[(s - ia)(s + ia)\right]^2} \\
&= \frac{(s^2 - a^2) + i(2as)}{(s^2 + a^2)^2}.
\end{aligned}$$

Equating real and imaginary parts, we have

$$L\{t\cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

and

$$L\{t\sin at\} = \frac{2as}{(s^2 + a^2)^2}.$$

EXAMPLE 5.37

Find the Laplace transform of $f(t) = e^{at}\cosh bt$.

Solution. Since

$$L\{\cosh bt\} = \frac{s}{s^2 - b^2}, \quad s > |b|,$$

the shifting property yields

$$L\{e^{at}\cosh bt\} = F(s-a)$$
$$= \frac{s-a}{(s-a)^2 - b^2}, \quad s > |b| + a.$$

EXAMPLE 5.38

Find Laplace transform of $f(t) = e^{-3t}$ (2cos $5t$ + 3 sin $5t$).

Solution. Since

$$L\{2\cos 5t - 3\sin 5t\}$$
$$= 2L\{\cos 5t\} - 3L\{\sin 5t\}$$
$$= \frac{2s}{s^2+25} - \frac{3\times 5}{s^2+25} = \frac{2s-15}{s^2+25} = F(s),$$

therefore, shifting property yields

$$L\{f(t)\} = F(s-a) \text{ with } a = -3$$
$$= \frac{2(s+3) - 15}{(s+3)^2 + 25} = \frac{2s-9}{s^2+6s+34}.$$

EXAMPLE 5.39

Find Laplace transform of $f(t) = \sinh 3t \cos^2 t$.

Solution. We know that $\cos^2 t = \frac{1+\cos 2t}{2}$. Therefore,

$$L\{\cos^2 t\} = \frac{1}{2}L\{1\} + \frac{1}{2}L\{\cos 2t\}$$
$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] = \frac{s^2+2}{s(s^2+4)}, \quad s > 0.$$

Therefore, by first shifting theorem, we have

$$L\{f(t)\} = L\{\sinh 3t\cos^2 t\} = L\left\{\frac{e^{3t} - e^{-3t}}{2}\cos^2 t\right\}$$
$$= \frac{1}{2}L\{e^{3t}\cos^2 t\} - \frac{1}{2}L\{e^{-3t}\cos^2 t\}$$

(by linearity of L)

$$= \frac{1}{2}\left[\frac{(s-3)^2+2}{(s-3)[(s-3)^2+4]}\right]$$
$$- \frac{1}{2}\left[\frac{(s+3)^2+2}{(s+3)\,[(s+3)^2+4]}\right]$$
$$= \frac{1}{2}\left[\frac{s^2-6s+11}{(s-3)\,(s^2-6s+13)}\right.$$
$$\left. - \frac{s^2+6s+11}{(s+3)\,(s^2+6x+13)}\right].$$

EXAMPLE 5.40

Find the Laplace transform of cosh at sin bt.

Solution. Let F(s) be Laplace transform of $f(t)$, $t > 0$ and let

$$g(t) = f(t)\cosh at.$$

Then

$$L\{g(t)\} = L[f(t)\cosh at] = L\left[\frac{e^{at} + e^{-at}}{2}f(t)\right]$$
$$= \frac{1}{2}L(e^{at}f(t)) + \frac{1}{2}L(e^{-at}f(t))$$
$$= \frac{1}{2}[F(s-a) + F(s+a)]$$

(use of first shifting theorem).

We take $f(t)$ = sin bt. Then $F(s) = \frac{b}{s^2+b^2}$ and, therefore, using above result, we have

$$L\{(\cosh at)\sin bt\}$$
$$= \frac{1}{2}\left[\frac{b}{(s-a)^2+b^2} + \frac{b}{(s+a)^2+b^2}.\right]$$

EXAMPLE 5.41

Find the Laplace transform of $f(t) = \cosh 4t \sin 6t$.

Solution. Taking $a = 4$, $b = 6$ in Example 5.40, we get

$$L\{\cosh 4t\sin 6t\} = \frac{6(s^2+52)}{s^4 + 40s^2 + 2704}.$$

Theorem 5.4. (Second Shifting Property). Let $F(s)$ be the Laplace transform of $f(t)$, $t > 0$ and let g be a function defined by

$$g(t) = \begin{cases} f(t-a) & \text{for } t > a \\ 0 & \text{for } t < a. \end{cases}$$

Then
$$L\{g(t)\} = e^{-as}F(s).$$

Proof: Using the substitution $t - a = u$, we have

$$\begin{aligned} L\{g(t)\} &= \int_0^\infty e^{-st} g(t)\, dt \\ &= \int_0^a e^{-st} g(t) dt + \int_a^\infty e^{-st} g(t)\, dt \\ &= 0 + \int_a^\infty e^{-st} f(t-a)\, dt \\ &= \int_0^\infty e^{-s(u+a)} f(u)\, du \\ &= e^{-as} \int_0^\infty e^{-su} f(u)\, du = e^{-as}F(s). \end{aligned}$$

EXAMPLE 5.42

Find the Laplace transform of the function f defined by

$$f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & \text{for } t > \frac{2\pi}{3} \\ 0 & \text{for } t < \frac{2\pi}{3}. \end{cases}$$

Solution. We know that $L\{\cos t\} = \frac{s}{s^2+1}$, $s > 0$. Therefore, by second shifting property,

$$L\{f(t)\} \quad = e^{-\frac{2\pi s}{3}} L\{\cos t\} = \frac{s e^{-\frac{2\pi s}{3}}}{s^2+1},\ s > 0.$$

EXAMPLE 5.43

Find the Laplace transform of the *sine function switched on at time* $t = 3$.

Solution. The given function is defined by

$$f(t) = \begin{cases} \sin t & \text{for } t \geq 3 \\ 0 & \text{for } t < 3\,. \end{cases}$$

Using Heaviside's unit step function H, this function can be expressed as

$$f(t) = H(t-3)\sin t.$$

To use second-shift theorem, we first write $\sin t$ as

$$\begin{aligned} \sin t &= \sin(t - 3 + 3) \\ &= \sin(t-3)\cos 3 + \cos(t-3)\sin 3. \end{aligned}$$

Then

$$\begin{aligned} L\{f(t)\} &= L\{H(t-3)\sin(t-3)\cos 3\} \\ &\quad + L\{H(t-3)\cos(t-3)\sin 3\} \\ &= \cos 3 e^{-3s} L\{\sin t\} + \sin 3 e^{-3s} L\{\cos t\} \\ &= \cos 3\, e^{-3s} \frac{1}{s^2+1} + \sin 3 e^{-3s} \frac{s}{s^2+1} \\ &= \frac{e^{-3s}}{s^2+1}(\cos 3 + s \sin 3). \end{aligned}$$

EXAMPLE 5.44

Find the Laplace transform of the function f defined by

$$f(t) = \begin{cases} (t-1)^2 & \text{for } t \geq 1 \\ 0 & \text{for } 0 \leq t < 1. \end{cases}$$

Solution. This function is just the function $g(t) = t^2$ delayed by 1 unit of time and its graph is shown in Figure 5.8.

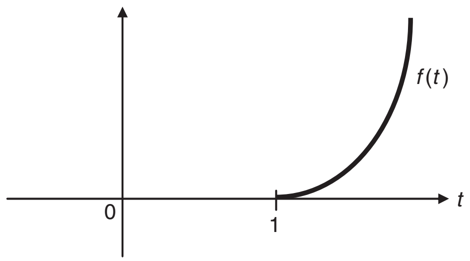

Figure 5.8

Therefore, by second shift property, we have

$$L\{f(t)\} = e^{-s} L\{t^2\} = \frac{2e^{-s}}{s^3}, \quad \text{Re}(s) > 0.$$

EXAMPLE 5.45

Find Laplace transform of the function f defined by

$$f(t) = \begin{cases} (t-4)^5 & \text{for } t > 4 \\ 0 & \text{for } t < 4. \end{cases}$$

Solution. Using second shift property, we have

$$\mathrm{L}\{f(t)\} = e^{-4s}[\mathrm{L}\{t^5\}] = e^{-4s}.\frac{5!}{s^6} = 120\frac{e^{-4s}}{s^6}.$$

Theorem 5.5. (Change of Scale Property). If $F(s)$ is the Laplace transform of $f(t)$ for $t > 0$, then for any positive constant a,

$$\mathrm{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right).$$

Proof: We are given that

$$\mathrm{F}(s) = \mathrm{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt.$$

Taking $u = at$, we have

$$\mathrm{L}\{f(at)\} = \int_0^\infty e^{-st}f(at)dt = \int_0^\infty e^{-su/a}f(u)\frac{du}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-su/a}f(u)\,du = \frac{1}{a}F\left(\frac{s}{a}\right).$$

EXAMPLE 5.46

Find the Laplace of $f(t) = \cos 6t$.

Solution. Since $\mathrm{L}\{\cos t\} = \frac{s}{s^2+1}$, the change of scale property implies

$$\mathrm{L}\{\cos 6t\} = \frac{1}{6}\left(\frac{s/6}{(s/6)^2+1}\right)$$

$$= \frac{1}{6}\left(\frac{s}{6[(s^2/36)+1]}\right) = \frac{s}{s^2+36}.$$

EXAMPLE 5.47

Using change of scale property, find the Laplace transform $J_0(at)$.

Solution. Let $J_0(t)$ be Bessel's function of order zero. By Example 5.33, $\mathrm{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$. Therefore, by change of scale property,

$$\mathrm{L}\{J_0(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right) = \frac{1}{a}.\frac{1}{\sqrt{(s/a)^2+1}} = \frac{1}{\sqrt{s^2+a^2}}.$$

Theorem 5.6. (Laplace Transform of Derivatives). Let f be a function such that

(a) f is continuous for all t, $0 \le t \le N$

(b) f is of exponential order γ for $t > N$

(c) f' is sectionally continuous for $0 \le t \le N$.

Then the Laplace transform of f' exists and is given by

$$L\{f'(t)\} = sF(s) - f(0),$$

where $F(s)$ is the Laplace transform of f.

Proof: The existence of the Laplace transform is established by Theorem 5.1. Further, integrating by parts, we have

$$\mathrm{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt = \lim_{T\to\infty}\int_0^T e^{-st}f'(t)\,dt$$

$$= \lim_{T\to\infty}\left\{[e^{-st}f(t)]_0^T + s\int_0^T e^{-st}f(t)\,dt\right\}$$

$$= \lim_{T\to\infty}\left\{[e^{-sT}f(T) - f(0)] + s\int_0^T e^{-st}f(t)dt\right\}$$

$$= s\int_0^\infty e^{-st}f(t)\,dt - f(0)$$

$$= s\mathrm{F}(s) - f(0),$$

the last but one step being the consequence of the fact that f is of exponential order and so $\lim_{T\to\infty} e^{-sT}f(T) = 0$ for $s > \gamma$.

EXAMPLE 5.48

Find Laplace transform of $g(t) = \sin at \cos at$.

Solution. Let $f(t) = \sin^2 at$. Then

$$f'(t) = 2a\sin at\cos at.$$

Since

$$\mathrm{L}\{f'(t)\} = s\mathrm{F}(s) - f(0),$$

we have

$$\mathrm{L}\{2a\sin at\cos at\}$$

$$= s\mathrm{L}\{\sin^2 at\} - 0 = s\mathrm{L}\{\sin^2 at\}$$

$$= \frac{2a^2}{s(s^2+4a^2)} \text{ (see Example 5.22).}$$

Hence

$$\mathrm{L}\{\sin at\cos at\} = \frac{a}{(s^2+4a^2)}.$$

EXAMPLE 5.49

Using Laplace transform of cos bt, find the Laplace transform of sin bt.

Solution. We want to find L{sin bt} from L{cos bt}. So, let $f(t)$ = cos bt. Then $f'(t) = -b$ sin bt and so

$$\begin{aligned} \mathrm{L}\{f'(t)\} &= s\mathrm{F}(s) - f(0) = s\mathrm{L}\{\cos bt\} - 1 \\ &= s\left(\frac{s}{s^2+b^2}\right) - 1 = \frac{s^2}{s^2+b^2} - 1 \\ &= \frac{-b^2}{s^2+b^2}. \end{aligned}$$

Thus

$$\mathrm{L}\{-b\sin bt\} = \frac{-b^2}{s^2+b^2}.$$

Hence

$$\mathrm{L}\{\sin bt\} = \frac{b}{s^2+b^2}.$$

EXAMPLE 5.50

Find Laplace transform of *Bessel's function of order 1*.

Solution. Let $J_1(t)$ be Bessel's function of order 1. We know that

$$\frac{d}{dt}\{t^n \mathrm{J}_n(t)\} = t^n\, \mathrm{J}_{n-1}(t).$$

If n = 0, we have

$$J_0'(t) = J_{-1}(t) = -J_1(t).$$

Hence

$$\begin{aligned} \mathrm{L}\{J_1(t)\} &= -L\{J_0'(t)\} = -[s\mathrm{L}\{J_0(t) - \mathrm{J}_0(0)]\} \\ &= -\left[\frac{s}{\sqrt{s^2+1}} - 1\right] \\ &= 1 - \frac{s}{\sqrt{s^2+1}} \text{(see Example 5.33)} \\ &= \frac{\sqrt{s^2+1} - s}{\sqrt{s^2+1}}. \end{aligned}$$

Theorem 5.7. If $L\{f(t)\} = F(s)$, then

$$\mathrm{L}\{f''(t) = s^2 F(s) - sf(0) - f'(0)$$

if $f(t)$ and $f'(t)$ are continuous for $0 \le t \le N$ and of exponential order for $t > N$ whereas $f''(t)$ is sectionally continuous for $0 \le t \le N$.

Proof: By Theorem 5.6, we have

$$\mathrm{L}\{g'(t)\} = s\ \mathrm{G}(s) - g(0).$$

Taking $g(t) = f'(t)$, we have

$$\begin{aligned} \mathrm{L}\{f''(t)\} &= s\mathrm{L}\{f'(t)\} - f'(0) \\ &= s[s\mathrm{F}(s) - f(0)] - f'(0) \\ &= s^2\mathrm{F}(s) - sf(0) - f'(0). \end{aligned}$$

EXAMPLE 5.51

Using Theorem 5.7, find L{sin at}, $t \ge 0$.

Solution. Let $f(t)$ = sin at. Then

$$f'(t) = a\cos at, \quad f''(t) = -a^2 \sin at.$$

By Theorem 5.7,

$$\mathrm{L}\{f''(t)\} = s^2\mathrm{F}(s) - sf(0) - f'(0)$$

and so

$$\mathrm{L}\{-a^2\sin at\} = s^2\mathrm{L}\{\sin at\} - a$$

which yields

$$(s^2+a^2)\mathrm{L}\{\sin at\} = a$$

and hence

$$\mathrm{L}\{\sin at\} = \frac{a}{s^2+a^2}, \quad s > 0.$$

EXAMPLE 5.52

Using Laplace transform of derivatives, find L{t cos at}.

Solution. Let $f(t) = t$ cos at. Then

$$\begin{aligned} f'(t) &= \cos at - at\sin at \\ f''(t) &= -2a\sin at - a^2 t\cos at. \end{aligned}$$

But

$$\begin{aligned} \mathrm{L}\{f''(t)\} &= s^2\mathrm{L}\{f(t)\} - sf(0) - f'(0) \\ &= s^2\mathrm{L}\{f(t)\} - 1 \end{aligned}$$

and so

$$\mathrm{L}\{-2a\sin at - a^2 t\cos t\} = s^2\mathrm{L}\{t\cos at\} - 1,$$

that is,

$$\begin{aligned} (s^2+a^2)\mathrm{L}\{t\cos at\} &= 2a\mathrm{L}\{\sin at\} + 1 \\ &= -2a\left(\frac{a}{s^2+a^2}\right) + 1 \\ &= \frac{s^2-a^2}{s^2+a^2}, \end{aligned}$$

and so

$$\mathrm{L}\{t\cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}.$$

Theorem 5.7. can be generalized to higher order derivatives in the form of the following result:

Theorem 5.8. Let $L\{f(t)\} = F(s)$. Then

$$L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0),$$

if $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous for $0 \le t \le N$ and of exponential order for $t > N$ whereas $f^{(n)}(t)$ is piecewise continuous for $0 \le t \le N$.

Proof: We shall prove our result using mathematical induction. By Theorems 5.6 and 5.7, we have

$$\mathrm{L}\{f'(t)\} = s\mathrm{F}(s) - f(0),$$
$$\mathrm{L}\{f''(t)\} = s^2\mathrm{F}(s) - sf(0) - f'(0).$$

Thus the theorem is true for $f'(t)$ and $f''(t)$. Suppose that the result is true for $f^{(n)}(t)$. Then

$$\mathrm{L}\{f^{(n)}(t) = s^n\mathrm{F}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$$

Then application of Theorem 5.6 yields

$$\begin{aligned}\mathrm{L}\{f^{(n+1)}(t)\} &= s[s^n F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)] - f^{(n)}(0)\\ &= s^{n+1}F(s) - s^n f(0) - \ldots - sf^{(n-1)}(0) + f^{(n)}(0),\end{aligned}$$

which shows that the result holds for $(n + 1)^{\text{th}}$ derivative also. Hence by mathematical induction, the result holds.

EXAMPLE 5.53

Using Theorem 5.8, find $\mathrm{L}\{t^n\}$.

Solution. We have $f(t) = t^n$. Therefore,

$$f'(t) = nt^{n-1}, f''(t) = n(n-1)t^{n-2}, \ldots, f^{(n)}(t) = n!$$

Now use of Theorem 5.8 yields

$$\mathrm{L}\{f^{(n)}(t)\} = L\{n!\} = s^n L\{t^n\} - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$$

But

$$f(0) = f'(0) = f''(0) = \ldots = f^{(n-1)}(0) = 0.$$

Therefore

$$\mathrm{L}\{n!\} = s^n L\{t^n\},$$

which gives

$$\mathrm{L}\{t^n\} = \frac{L\{n!\}}{s^n} = \frac{n!\,\mathrm{L}\{1\}}{s^n} = \frac{n!}{s^{n+1}}.$$

Theorem 5.9. (Multiplication by t^n). If $L\{f(t)\} = F(s)$, then

$$\mathrm{L}\{tf(t)\} = -\frac{d}{ds}F(s),$$

and in general

$$\mathrm{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s).$$

Proof: By definition of Laplace transform,

$$\mathrm{F}(s) = \int_0^\infty e^{-st}f(t)\,dt.$$

Then, by Leibnitz-rule for differentiating under the integral sign, we have

$$\begin{aligned}\frac{dF}{ds} &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{d}{ds}(e^{-st}f(t)\,dt)\\ &= \int_0^\infty -t\,e^{-st}f(t)\,dt = -\int_0^\infty e^{-st}(tf(t))\,dt\\ &= -\mathrm{L}\{tf(t)\}\end{aligned}$$

and so

$$\mathrm{L}\{tf(t)\} = -\frac{d}{ds}\mathrm{F}(s).$$

Thus the theorem is true for $n = 1$. To obtain the general form, we use mathematical induction. So, assume that the result is true for $n = m$. Thus

$$\mathrm{L}\{t^m f(t)\} = (-1)^m \frac{d^m}{ds^m}\mathrm{F}(s) = (-1)^m\mathrm{F}^{(m)}(s).$$

Therefore,

$$\frac{d}{ds}[\mathrm{L}(t^m f(t)] = (-1)^m F^{(m+1)}(s),$$

that is,

$$\frac{d}{ds}\int_0^\infty e^{-st}t^m f(t)\,dt = (-1)^m\mathrm{F}^{(m+1)}(s),$$

which, on using Leibnitz rule, yields

$$-\int_0^\infty e^{-st}t^{m+1}f(t)\,dt = (-1)^m F^{(m+1)}(s),$$

and so

$$L\{t^{m+1}f(t)\} = (-1)^{m+1}F^{(m+1)}(s).$$

Hence, the result follows by mathematical induction.

EXAMPLE 5.54

Find Laplace transform of $f(t) = t\sin^2 t$.

Solution. Let $f(t) = t\sin^2 t$. We know that $L\{\sin^2 t\} = \frac{2}{s(s^2+4)}$. Therefore

$$L\{t\sin^2 t\} = -\frac{d}{ds}\left\{\frac{2}{s(s^2+4)}\right\} = 2\left[\frac{3s^2+4}{s^2(s^2+4)^2}\right].$$

EXAMPLE 5.55

Find Laplace transform of $f(t) = te^{-t}\cosh t$.

Solution. We know that

$$L\{\cosh t\} = \frac{s}{s^2-1}.$$

Therefore,

$$L\{t\cosh t\} = -\frac{d}{ds}\left(\frac{s}{s^2-1}\right) = \frac{s^2+1}{(s^2-1)^2}.$$

Then, by Theorem 5.3, we have

$$L\{e^{-t}t\cosh t\} = \frac{(s+1)^2+1}{((s+1)^2-1)^2} = \frac{s^2+2s+2}{(s^2+2s)^2}.$$

EXAMPLE 5.56

Using Theorem 5.9, evaluate $\int_0^\infty t\,e^{-2t}\sin t\,dt$ and $\int_0^\infty te^{-2t}\cos t\,dt$.

Solution. We know that $L\{\sin t\} = \frac{1}{s^2+1}$. Therefore, by Theorem 5.9, we have

$$L\{t\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}.$$

But $\int_0^\infty e^{-2t}(t\sin t)dt$ is the Laplace transform of $t\sin t$ with $s = 2$. Hence

$$\int_0^\infty e^{-2t}(t\sin t)dt = L\{t\sin t\} = \left[\frac{2s}{(s^2+1)^2}\right]_{s=2} = \frac{4}{25}.$$

In a similar way, we can show that

$$\int_0^\infty t\,e^{-2t}\cos t\,dt = L\{t\cos t\} \text{ with } s = 2$$

$$= \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s=2} = \frac{3}{25}.$$

EXAMPLE 5.57

Evaluate the integral $I = \int_0^\infty e^{-x^2}dx$.

Solution. Putting $t = x^2$, we get

$$I = \frac{1}{2}\int_0^\infty e^{-t}t^{-1/2}\,dt = \frac{1}{2}L\{t^{-1/2}\} \text{ with } s = 1$$

$$= \frac{1}{2}\cdot\frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{1/2}} \text{ with } s = 1$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

EXAMPLE 5.58

Find Laplace transform of $f(t) = e^{-2t}\,t\cos t$.

Solution. We know that $L\{\cos t\} = \frac{s}{s^2+1}$. Therefore, by Theorem 5.9, we have

$$L\{t\cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2}.$$

Now using first shifting property, we have

$$L\{e^{-2t}t\cos t\} = \frac{(s+2)^2-1}{((s+2)^2+1)^2} = \frac{s^2+4s+3}{(s^2+4s+5)^2}.$$

EXAMPLE 5.59

Find the Laplace transform of $f(t) = t^2 e^{-2t}\cos t$.

Solution. As in Example 5.58, $L\{t\cos t\} = \frac{s^2-1}{(s^2+1)^2}$. Therefore,

$$L\{t^2\cos t\} = -\frac{d}{ds}\left(\frac{s^2-1}{(s^2+1)^2}\right).$$

Then using first-shifting property, we have

$$L\{t^2e^{-2t}\cos t\} = 2\left(\frac{s^3+10s^2+25s+22}{(s^2+4s+5)^3}\right).$$

EXAMPLE 5.60

Find Laplace transform of $f(t) = t^n e^{-at}$.

Solution. Since $L\{e^{-at}\} = \frac{1}{s+a}$, we have

$$L\{t^n e^{-at}\} = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s+a}\right) = (-1)^{2n} \frac{n!}{(s+a)^{n+1}}.$$

Theorem 5.10. (Division by t). If $L\{f(t)\} = F(s)$, then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du,$$

provided $\lim_{t\to 0} \frac{f(t)}{t}$ exists.

Proof: Put $g(t) = \frac{f(t)}{t}$. So, $f(t) = t\,g(t)$ and

$$\begin{aligned} L\{f(t)\} &= L\{tg(t)\} \\ &= -\frac{d}{ds} L\{g(t)\}, \text{ by Theorem 5.9} \\ &= -\frac{dG}{ds}. \end{aligned}$$

Then integration yields

$$G(s) = -\int_\infty^s F(u)\,du = \int_s^\infty F(u)\,du,$$

that is,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du.$$

Remark 5.3. By Theorem 5.10, we have

$$L\left\{\frac{f(t)}{t}\right\} = \int_0^\infty e^{-st} \frac{f(t)}{t}\,dt = \int_s^\infty F(u)\,du.$$

Letting $s \to 0+$ and assuming that the integral converges, it follows that

$$\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty F(u)\,du.$$

For example, if $f(t) = \sin t$, then $F(s) = \frac{1}{s^2+1}$ and so

$$\int_0^\infty \frac{\sin t}{t}\,dt = \int_0^\infty \frac{du}{u^2+1} = [\tan^{-1} u]_0^\infty = \frac{\pi}{2}.$$

EXAMPLE 5.61

Find the Laplace transform of $f(t) = \frac{\cos 2t - \cos 3t}{t}$.

Solution. By linearity of L, we have

$$\begin{aligned} L\{\cos 2t - \cos 3t\} &= L\{\cos 2t\} - L\{\cos 3t\} \\ &= \frac{s}{s^2+4} - \frac{s}{s^2+9}. \end{aligned}$$

Therefore, by Theorem 5.10, we get

$$\begin{aligned} &L\left\{\frac{\cos 2t - \cos 3t}{t}\right\} \\ &= \int_s^\infty \frac{u}{u^2+4}\,du - \int_s^\infty \frac{u}{u^2+9}\,du \\ &= \frac{1}{2}[\log(u^2+4)]_s^\infty - \frac{1}{2}[\log(u^2+9)]_s^\infty \\ &= \frac{1}{2}\left[\log \frac{u^2+4}{u^2+9}\right]_s^\infty \\ &= \frac{1}{2}\lim_{u\to\infty}\left(\log\frac{u^2+4}{u^2+9}\right) - \frac{1}{2}\log\frac{s^2+4}{s^2+9} \\ &= \frac{1}{2}\log\left(\lim_{u\to\infty}\frac{1+(4/u^2)}{1+(9/u^2)}\right) - \frac{1}{2}\log\frac{s^2+4}{s^2+9} \\ &= 0 + \frac{1}{2}\log\frac{s^2+9}{s^2+4} = \frac{1}{2}\log\frac{s^2+9}{s^2+4}. \end{aligned}$$

EXAMPLE 5.62

Find the Laplace transform of $f(t) = \frac{e^{-at} - e^{-bt}}{t}$.

Solution. We have

$$\begin{aligned} L\{e^{-at} - e^{-bt}\} &= L\{e^{-at}\} - L\{e^{-bt}\} \\ &= \frac{1}{s+a} - \frac{1}{s+b}. \end{aligned}$$

Therefore, proceeding as in Example 5.61, we have

$$\begin{aligned} L\{f(t)\} &= \int_s^\infty \left[\frac{1}{u+a} - \frac{1}{u+b}\right] du = \left[\log\frac{u+a}{u+b}\right]_s^\infty \\ &= \lim_{u\to\infty} \log\frac{u+a}{u+b} - \log\frac{s+a}{s+b} \\ &= 0 - \log\frac{s+a}{s+b} = \log\frac{s+b}{s+a}. \end{aligned}$$

EXAMPLE 5.63

Find the Laplace transfrom of $f(t) = \frac{1-\cos 2t}{t}$.

Solution. We have

$$L\{1 - \cos 2t\} = L\{1\} - L\{\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}.$$

Therefore, by Theorem 5.10, we get

$$\begin{aligned}
\mathrm{L}\left\{\frac{1-\cos 2t}{t}\right\} &= \int_s^\infty \left[\frac{1}{u} - \frac{u}{u^2+4}\right] du \\
&= \left[\log u - \left(\frac{1}{2}\log(u^2+4)\right)\right]_s^\infty \\
&= \left[\frac{1}{2}\log u^2 - \frac{1}{2}\log(u^2+4)\right]_s^\infty \\
&= \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right).
\end{aligned}$$

EXAMPLE 5.64

Using Remark 5.3, evaluate the integral

$$\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\, dt.$$

Solution. By Remark 5.3, we have

$$\begin{aligned}
&\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\, dt \\
&= \int_0^\infty L\{e^{-t}-e^{-3t}\}\, ds = \int_0^\infty \left[\frac{1}{u+1} - \frac{1}{u+3}\right] du \\
&= [\log(u+1) - \log(u+3)]_0^\infty = \left[\log\frac{u+1}{u+3}\right]_0^\infty \\
&= \log\left[\lim_{u\to\infty}\frac{u+1}{u+3}\right] - \log\frac{1}{3} \\
&= \log\left[\lim_{u\to\infty}\frac{1+(1/u)}{1+(3/u)}\right] - \log\frac{1}{3} \\
&= \log 1 - \log\frac{1}{3} = \log 3.
\end{aligned}$$

EXAMPLE 5.65

Find the Laplace transform of $f(t) = \frac{1-e^t}{t}$.

Solution. Since $\mathrm{L}\{1-e^t\} = \mathrm{L}\{1\} - \mathrm{L}\{e^t\} = \frac{1}{s} - \frac{1}{s-1}$, we have

$$\begin{aligned}
\mathrm{L}\{f(t)\} &= \int_s^\infty \left(\frac{1}{u} - \frac{1}{u-1}\right) du \\
&= [\log u - \log(u-1)]_s^\infty \\
&= \left[\log\frac{u}{u-1}\right]_s^\infty = -\log\left[\frac{1}{1-(1/s)}\right] \\
&= \log\left(\frac{s-1}{s}\right).
\end{aligned}$$

EXAMPLE 5.66

Find Laplace transform of $f(t) = \frac{\sin at}{t}$.

Solution. We know that

$$\mathrm{L}\{\sin at\} = \frac{a}{s^2+a^2}.$$

Therefore, by Theorem 5.10,

$$\mathrm{L}\left\{\frac{\sin at}{t}\right\} = a\int_s^\infty \frac{du}{u^2+a^2} = \tan^{-1}\left(\frac{a}{s}\right).$$

Theorem 5.11. (Laplace Transform of Integrals). If $L\{f(t)\} = F(s)$, then

$$L\left\{\int_0^t f(u)\, du\right\} = \frac{F(s)}{s}.$$

Proof: The function $f(t)$ should be integrable in such a way that

$$g(t) = \int_0^t f(u)\, du$$

is of exponential order. Then $g(0) = 0$ and $g'(t) = f(t)$. Therefore,

$$\mathrm{L}\{g'(t)\} = s\,\mathrm{L}\{g(t)\} - g(0) = s\,\mathrm{L}\{g(t)\}$$

and so

$$\begin{aligned}
\mathrm{L}\left\{\int_0^t f(u)du\right\} &= \mathrm{L}\{g(t)\} = \frac{\mathrm{L}\{g'(t)\}}{s} \\
&= \frac{\mathrm{L}\{f(t)\}}{s} = \frac{\mathrm{F}(s)}{s}.
\end{aligned}$$

EXAMPLE 5.67

Find Laplace transform of $\int_0^t \frac{\sin u}{u}\, du$.

Solution. From Example 5.66, we have

$$\mathrm{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right).$$

Therefore, by Theorem 5.11, we have

$$\mathrm{L}\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right).$$

(The function $\int_0^t \frac{\sin u}{u}\, du$ is called *sine integral function*, denoted by $\mathrm{S}i(t)$, which is used in optics).

EXAMPLE 5.68

Find the Laplace transform of $\int_0^t \cos 2u\, du$.

Solution. We know that $L\{\cos 2t\} = \frac{s}{s^2+4}$ and so

$$L\left\{\int_0^t \cos 2u\, du\right\} = \frac{1}{s}\cdot\frac{s}{s^2+4} = \frac{1}{s^2+4}.$$

EXAMPLE 5.69

Find Laplace transform of *cosine integral function*

$$Ci(t) = \int_\infty^t \frac{\cos u}{u}\, du,\ t > 0$$

Solution. We know that $L\{\cos t\} = \frac{s}{s^2+1}$. Therefore, by Theorem 5.10

$$L\left\{\frac{\cos t}{t}\right\} = \int_s^\infty \frac{u}{u^2+1}\, du = \frac{1}{2}\int_s^\infty \frac{2u}{u^2+1}\, du$$

$$= \frac{1}{2}\log(1+s^2).$$

Now Theorem 5.11 yields the Laplace transform of cosine integral function $Ci(t)$ as given below:

$$L\{Ci(t)\} = L\left\{\int_\infty^t \frac{\cos u}{u}\, du\right\} = L\left\{-\int_t^\infty \frac{\cos u}{u}\, du\right\}$$

$$= -L\left\{\int_t^\infty \frac{\cos u}{u}\, du\right\}$$

$$= -\left[\frac{1}{s}\left(\frac{1}{2}\log(1+s^2)\right)\right]$$

$$= -\frac{1}{2s}\log(1+s^2).$$

EXAMPLE 5.70

Find the Laplace transform of the *exponential integral* defined by

$$Ei(t) = \int_t^\infty \frac{e^{-u}}{u}\, du, \quad t > 0.$$

Solution. We know that $L\{e^{-u}\} = \frac{1}{s+1}$. Therefore, by Theorem 5.10, we have

$$L\left\{\frac{e^{-u}}{u}\right\} = \int_s^\infty \frac{1}{u+1}\, du = \log(1+s).$$

Now application of Theorem 5.11 yields

$$L\{Ei(t)\} = \frac{1}{s}\log(1+s).$$

EXAMPLE 5.71

Find Laplace transform of $\int_0^t e^t \frac{\sin t}{t}\, dt$.

Solution. We know that $L\{\sin t\} = \frac{1}{s^2+1}$. Therefore,

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1}\, du = [\tan^{-1} u]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s.$$

Therefore, by first-shifting property, we have

$$L\left\{e^t \frac{\sin t}{t}\right\} = \cot^{-1}(s-1).$$

Hence

$$L\left\{\int_0^t e^t \frac{\sin t}{t}\, dt\right\} = \frac{1}{s}\cot^{-1}(s-1).$$

Theorem 5.12. (Laplace Transform of a Periodic Function). Let f be a periodic function with period T so that $f(t) = f(t+T)$. Then

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt.$$

Proof: We begin with the definition of Laplace transform and evaluate the integral using periodicity of $f(t)$. Thus we have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^T e^{-st} f(t)\, dt + \int_T^{2T} e^{-st} f(t)\, dt$$

$$+ \int_{2T}^{3T} e^{-st} f(t)\, dt + \ldots$$

$$+ \int_{(n-1)T}^{nT} e^{-st} f(t)\, dt + \ldots$$

provided that the series on the right-hand side converges. This is true since the function f satisfies the condition for the existence of its Laplace transform. Consider the integral

$$\int_{(n-1)T}^{nT} e^{-st} f(t)\, dt$$

and substitute $u = t-(n-1)$T. Since f has period T, we have

$$\int_{(n-1)T}^{nT} e^{-st} f(t) dt = e^{-s(n-1)T} \int_0^T e^{-su} f(u)\, du,$$

$$n = 1, 2, \ldots$$

Thus, we have

$$L\{f(t)\} = (1 + e^{-sT} + e^{-2sT} + \ldots) \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt \text{ (summing the G.P.)},$$

since $1 + r + r^2 + \text{r}^3 + \ldots = \frac{1}{1-r}$, $|r| < 1$.

EXAMPLE 5.72

Find Laplace transform of the *half-wave rectified sinusoidal*

$$f(t) = \begin{cases} \sin t & \text{for } 2n\pi < t < (2n+1)\pi \\ 0 & \text{for } (2n+1)\pi < t < (2n+2)\pi, \end{cases}$$

for $n = 0, 1, 2, \ldots$.

Solution. The graph of the half-wave rectified sinusoidal function f is shown in the Figure 5.9

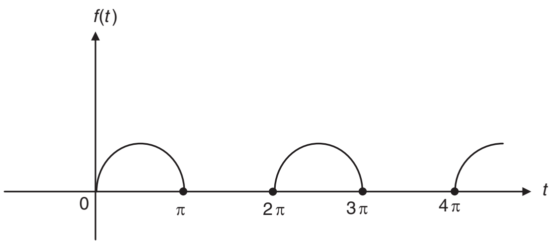

Figure 5.9

The function is of period 2π. So, Theorem 5.12 yields

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}(-s \sin t - \cos t)}{s^2+1} \right]_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}} \left(\frac{1+e^{-\pi s}}{s^2+1} \right)$$

$$= \frac{1}{(1-e^{-\pi s})(s^2+1)}.$$

EXAMPLE 5.73

Find the Laplace transform of *full rectified sine wave* defined by the expression

$$f(t) = \begin{cases} \sin t & \text{for } 0 < t < \pi \\ -\sin t & \text{for } \pi < t < 2\pi \end{cases}$$

$$f(t) = f(t+\pi).$$

Solution. The graph of the rectified sine wave is shown in the Figure 5.10.

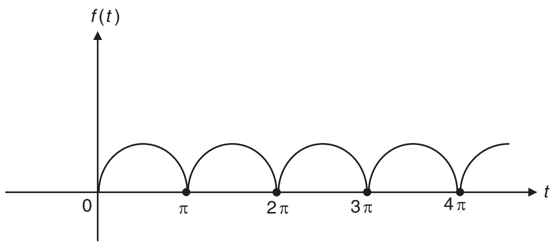

Figure 5.10

The function f has period π and, therefore, application of Theorem 5.12 yields

$$\begin{aligned}
L\{f(t)\} &= \frac{1}{1-e^{-\pi s}}\int_0^{\pi} e^{-st}\sin t\,dt \\
&= \frac{1}{1-e^{-\pi s}}\left[\frac{1+e^{-\pi s}}{s^2+1}\right] \\
&= \frac{1+e^{-\pi s}}{(1-e^{-\pi s})(s^2+1)}.
\end{aligned}$$

EXAMPLE 5.74

Find Laplace transform of the *triangular wave function* defined by

$$f(t) = \begin{cases} t & \text{for } 0 \le t < a \\ 2a - t & \text{for } 0 \le t < 2a, \end{cases}$$

$$f(2a+t) = f(t).$$

Solution. The graph of triangular wave function is shown in the Figure 5.11.

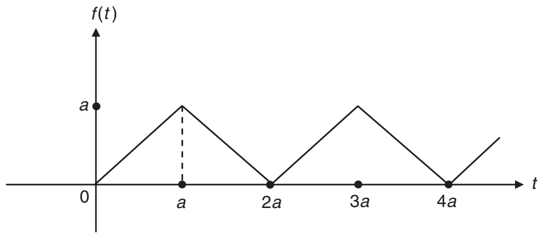

Figure 5.11

Using the formula for the Laplace transform of a periodic function, we have

$$\begin{aligned}
L\{f(t)\} &= \frac{1}{1-e^{2as}}\int_0^{2a} e^{-st}f(t)\,dt \\
&= \frac{1}{1-e^{2as}}\left[\int_0^{a} t e^{-st}dt + \int_a^{2a}(2a-t)e^{-st}\,dt\right].
\end{aligned}$$

Now integration by parts gives

$$\begin{aligned}
L\{f(s)\} &= \frac{1}{1-e^{2as}}(e^{-as}-1)^2 . \frac{1}{s^2} \\
&= \frac{1}{s^2}\left(\frac{1-e^{-as}}{1+e^{as}}\right) = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right).
\end{aligned}$$

EXAMPLE 5.75

Find the Laplace transform of the *saw tooth wave function*, whose graph is shown in the Figure 5.12.

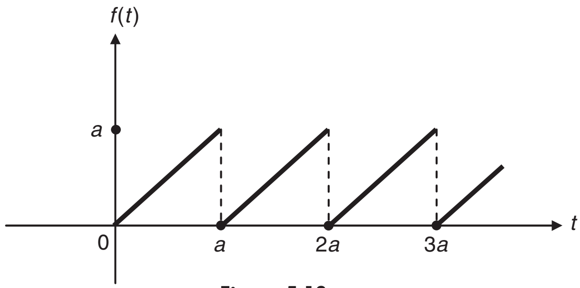

Figure 5.12

Solution. It is a periodic function with period a. Therefore,

$$\begin{aligned}
L\{f(t)\} &= \frac{1}{1-e^{-as}}\int_0^{a} e^{-st}f(t)\,dt = \frac{1}{1-e^{-as}}\int_0^{a} t e^{-st}dt \\
&= \frac{1}{1-e^{-as}}\left\{\left[\frac{te^{-st}}{-s}\right]_0^a - \int_0^a \frac{e^{-st}}{-s}dt\right\} \\
&= \frac{1}{1-e^{-as}}\left\{\left[t\frac{e^{-st}}{-s}\right]_0^t - \left[\frac{e^{-st}}{s^2}\right]_0^a\right\} \\
&= \frac{1}{1-e^{-as}}\left[t\frac{e^{st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^a \\
&= \frac{1}{1-e^{-as}}\left[\frac{ae^{-sa}}{-s} - \frac{e^{-sa}}{s^2} + \frac{1}{s^2}\right] \\
&= \frac{1}{1-e^{-as}}\left[-\frac{ae^{-sa}}{s} + \frac{1-e^{-sa}}{s^2}\right] \\
&= \frac{1}{s^2} - \frac{ae^{-sa}}{s(1-e^{-as})}.
\end{aligned}$$

5.3 LIMITING THEOREMS

Theorem 5.13. (Initial Value). Let $f(t)$ and its derivative $f'(t)$ be piecewise continuous and of exponential order, then

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s),$$

provided the limits exist.

Proof: We know that

$$L\{f'(t)\} = sF(s) - f(0). \qquad (3)$$

Further, since $f'(t)$ obeys the criteria for the existence of Laplace transform, we have

$$\left|\int_0^\infty e^{-st} f'(t)dt\right| \le \int_0^\infty |e^{-st} f'(t)|\,dt$$

$$\le \int_0^\infty e^{-st} e^{Mt}\,dt$$

$$= -\frac{1}{M-s} \to 0 \text{ as } s \to \infty.$$

Hence (3) implies $s\,F(s) - f(0) \to 0$ as $s \to \infty$, and so

$$\lim_{s\to\infty} s\,F(s) = f(0) = \lim_{t\to 0} f(t).$$

Theorem 5.14 (Final Value). If the limits indicated exist, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s).$$

Proof: Using Laplace transform of the derivative, we have

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0) \qquad (4)$$

Then

$$\lim_{s\to 0}\int_0^\infty e^{-st} f'(t)\,dt = \lim_{s\to 0}\lim_{T\to\infty}\int_0^T e^{-st} f'(t)\,dt$$

$$= \lim_{s\to 0}\lim_{T\to\infty}\{e^{-sT} f(T) - f(0)\}$$

$$= \lim_{T\to\infty} f(T) - f(0)$$

$$= \lim_{t\to\infty} f(t) - f(0).$$

Hence relation (4) yields

$$\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0)$$

and so

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

EXAMPLE 5.76

Verify the validity of limiting theorems by considering $f(t) = e^t$.

Solution. We observe that

$$L\{e^t\} = F(s) = \frac{1}{s+1}.$$

$$\lim_{t\to 0} f(t) = f(0) = 1,$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\frac{s}{s+1} = 1.$$

Thus, initial value theorem is verified in this case.

On the other hand,

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} e^{-t} = 0,$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\frac{s}{s+1} = 0,$$

and so the final value theorem also holds for e^{-t}.

EXAMPLE 5.77

Suppose Laplace transform of a function f is given by

$$F(s) = \frac{18}{s(s^2+36)}.$$

Find $f(0)$ and $f'(0)$.

Solution. By initial value theorem, we have

$$f(0) = \lim_{s\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$$

$$= \lim_{s\to\infty}\frac{18\,s}{s(s^2+36)} = 0,$$

and

$$f'(0) = \lim_{s\to\infty}[s^2\,F(s) - s\,f(0)]$$

$$= \lim_{s\to\infty}\left[\frac{18s^2}{s(s^2+36)} - 0\right] = 0.$$

EXAMPLE 5.78

Let $F(s) = \frac{1}{s-1}$. Can we find $f(\infty)$ using final value theorem?

Solution. In this case, the final value theorem does not apply because

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} e^t \to \infty \text{ as } t \to \infty.$$

5.4 MISCELLANEOUS EXAMPLES

EXAMPLE 5.79

Find the Laplace Transforms of

(i) $e^{4t}\sin 2t\cos t$,
(ii) $\sinh\,t\cos^2 t$ and
(iii) $e^{-t}\cos^2 t$.

Solution. (i) We have

$$\sin 2t\ \cos t = \frac{1}{2}[\sin 3t + \sin t].$$

Therefore

$$\begin{aligned} L\{\sin 2t\ \cos t\} &= \frac{1}{2}L\{\sin 3t\} + \frac{1}{2}L\{\sin t\} \\ &= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right]. \end{aligned}$$

Now using shifting property, we get

$$\begin{aligned} &L\{e^{4t}\sin 2t\ \cos t\} \\ &= \frac{1}{2}\left[\frac{3}{(s-4)^2+9} + \frac{1}{(s-4)^2+1}\right] \\ &= \frac{1}{2}\left[\frac{3}{s^2-8s+25} + \frac{1}{s^2-8s+17}\right]. \end{aligned}$$

(ii) Since $\cos^2 t = \frac{1+\cos 2t}{2}$. Therefore

$$\begin{aligned} L\{\cos^2 t\} &= \frac{1}{2}L\{1\} + \frac{1}{2}L\{\cos 2t\} \\ &= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] = \frac{s^2+2}{s(s^2+4)}\ ,\ s > 0. \end{aligned}$$

Therefore, by shifting property,

$$\begin{aligned} L\{\sinh t\ \cos^2 t\} &= L\left\{\frac{e^t - e^{-t}}{2}\cos^2 t\right\} \\ &= \frac{1}{2}L\{e^t\cos^2 t\} - \frac{1}{2}L\{e^{-t}\cos^2 t\} \\ &= \frac{1}{2}\left\{\frac{(s-1)^2+2}{(s-1)[(s-1)^2+4]}\right\} \\ &\quad - \frac{1}{2}\left\{\frac{(s+1)^2+2}{(s+1)[(s+1)^2+4]}\right\} \\ &= \frac{1}{2}\left[\frac{s^2-2s+3}{(s-1)(s^2-2s+5)}\right. \\ &\quad \left. - \frac{s^2+2s+3}{(s+1)(s^2+2s+5)}\right]. \end{aligned}$$

(iii) As in part (ii),

$$L\{\cos^2 t\} = \frac{s^2+2}{s(s^2+4)}, s > 0.$$

Therefore, by shifting property,

$$\begin{aligned} L\{e^{-t}\cos^2 t\} &= \frac{(s+1)^2+2}{(s+1)[(s+1)^2+4]} \\ &= \frac{s^2+2s+3}{(s+1)(s^2+2s+5)}. \end{aligned}$$

EXAMPLE 5.80

Find the Laplace transforms of the following:

(i) $e^{-3t}(2\cos 5t - 3\sin 5t)$ and

(ii) $\frac{1}{t}$ $(\cos at - \cos bt)$.

Solution. (i) Using linearty and shifting properties, we have

$$\begin{aligned} &L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} \\ &= 2L\{e^{-3t}\cos 5t\} - 3L\{e^{-3t}\sin 5t\} \\ &= 2\frac{s+3}{(s+3)^2+25} - 3\frac{5}{(s+3)^2+25} \\ &= \frac{2s-9}{(s+3)^2+25} \\ &= \frac{2s-9}{s^2+6s+34}. \end{aligned}$$

(ii) As in Example 5.64, we have

$$\begin{aligned} &\int_0^\infty \frac{\cos at - \cos bt}{t}dt \\ &= \int_0^\infty L\{\cos at - \cos bt\}ds \\ &= \int_0^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]ds \\ &= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_0^\infty \\ &= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_0^\infty \\ &= \frac{1}{2}\left[\log\left\{\lim_{s\to\infty}\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right\}\right] - \frac{1}{2}\log\frac{a^2}{b^2} \\ &= -\frac{1}{2}\log\frac{a^2}{b^2} = \frac{1}{2}\log\frac{b^2}{a^2}. \end{aligned}$$

EXAMPLE 5.81

Find the Laplace transform of cosh at sin at.

Solution. Let

$$g(t) = \cosh at \ \sin at.$$

Then

$$L\{g(t)\} = L\left[\frac{e^{at}+e^{-at}}{2}\sin at\right]$$
$$= \frac{1}{2}L\{e^{at}\sin at + e^{-at}\sin at\}$$
$$= \frac{1}{2}L\{e^{at}\sin at\} + \frac{1}{2}L\{e^{-at}\sin at\}.$$

Now

$$L\{\sin at\} = \frac{a}{s^2+a^2}.$$

Therefore, by shifting property,

$$L\{e^{at}\sin\ at\} = \frac{a}{(s-a)^2+a^2},$$
$$L\{e^{-at}\sin\ at\} = \frac{a}{(s+a)^2+a^2}.$$

Hence

$$L\{g(t)\} = \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right]$$

EXAMPLE 5.82

Find the Laplace transform of $f(t) = t^2\ \sin^2 at$.

Solution. Let $f(t) = t^2\ \sin^2\ at$. We know that

$$L\left\{\sin^2 at\right\} = \frac{2a^2}{s\left[s^2+(2a)^2\right]}.$$

Therefore

$$L\left\{t^2\sin^2\ at\right\} = (-1)^2\frac{d^2}{ds^2}\left[\frac{2a^2}{s(s^2+4a^2)}\right]$$
$$= 2a^2\frac{d^2}{ds^2}\left[\frac{1}{s(s^2+4a^2)}\right]$$
$$= 8a^2\left[\frac{3s^4+6a^2s^2-8a^4}{s^2(s^2+4a^2)^3}\right].$$

EXAMPLE 5.83

Evaluate:

$$L\left\{te^{-2t}\sin\ 4t\right\}$$

Solution. We know that

$$L\{\sin 4t\} = \frac{1}{s^2+16}.$$

Now, by shifting property,

$$L\{e^{-2t}\sin 4t\} = \frac{1}{(s+2)^2+16}.$$

Further,

$$L\{e^{-2t}\sin 4t\} = -\frac{d}{ds}\left[\frac{1}{(s+2)^2+16}\right]$$
$$= \frac{2s+4}{(s^2+4s+20)^2}.$$

EXAMPLE 5.84

Express $f(t) = \left\{\begin{array}{lll} 1 & if & 0 < t \le 1 \\ t & if & 1 < t \le 2 \\ t^2 & if & t > 2 \end{array}\right\}$ in terms of unit step function and hence find $Lf(t)$.

Solution. In terms of Heavyside s unit step function, we have

$$f(t) = H(t) + tH(t-1) + t^2H(t-2).$$

Therefore

$$L\{f(t)\} = L\{H(t)\} + L\{tH(t-1)\} + L\left\{t^2H(t-2)\right\}$$
$$= \frac{1}{s} - \frac{d}{ds}\left\{\frac{e^{-s}}{s}\right\} + (-1)^2\frac{d}{ds}\left\{\frac{e^{-2s}}{s}\right\}$$
$$= \frac{1}{s} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s}$$
$$= \frac{1}{s}[1-e^{-s}-2e^{-2s}] - \frac{1}{s^2}[e^{-s}+e^{-2s}].$$

EXAMPLE 5.85

Find the Laplace Transform of $e^{3t}[t\cos 2t]$ and $\frac{1-e^{-2t}}{t}$.

Solution. We know that

$$F(s) = L\{t\cos 2t\} = \frac{s^2-4}{(s^2+4)^2}.$$

Therefore, by shifting property,

$$L\{e^{3t}(t\cos 2t)\} = F(s-3)$$
$$= \frac{(s-3)^2-4}{((s-3)^2+4)^2} = \frac{s^2-6s+5}{(s^2-6s+13)^2}.$$

Also, As in Example 5.65,

$$L\{1-e^{-2t}\} = L\{1\} - L\{e^{-2t}\} = \frac{1}{s} - \frac{1}{s+2}.$$

Therefore,

$$L\{f(t)\} = \int_s^\infty \left(\frac{1}{u} - \frac{1}{u+2}\right) du$$
$$= [\log u - \log(u+2)]_s^\infty$$
$$= \log\left[\frac{u}{u+2}\right]_s^\infty = -\log\left[\frac{1}{1+\frac{2}{s}}\right]$$
$$= \log\left[\frac{s+2}{s}\right].$$

EXAMPLE 5.86

Evaluate:

$$L\left\{\frac{1-\cos\ at}{t}\right\}$$

Solution. See Example 5.63. We have

$$L\{1-\cos\ at\} = L\{1\} - L\{\cos\ at\} = \frac{1}{s} - \frac{s}{s^2+a^2}.$$

Therefore

$$L\left\{\frac{1-\cos\ at}{t}\right\} = \int_s^\infty \left[\frac{1}{u} - \frac{u}{u^2+a^2}\right] du$$
$$= \left[\log u - \left(\frac{1}{2}\log(u^2+a^2)\right)\right]_s^\infty$$
$$= \left[\frac{1}{2}\log u^2 - \frac{1}{2}\log(u^2+a^2)\right]_s^\infty$$
$$= \frac{1}{2}\log\frac{s^2+a^2}{s^2}.$$

EXAMPLE 5.87

Using Laplace transform evaluate $\int_0^\infty \frac{e^{-at}\sin^2 t}{t}\, dt.$

Solution. We have

$$L\{\sin^2 t\} = L\left\{\frac{1}{2}(1-\cos 2t)\right\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right].$$

Therefore

$$L\{e^{-at}\sin^2 t\} = \frac{1}{2}\left[\frac{1}{s+a} - \frac{s+a}{(s+a)^2+4}\right].$$

Further,

$$\int_0^\infty \frac{e^{-at}\sin^2 t}{t} dt = \int_0^\infty L\{e^{-at}\sin^2 t\} ds$$
$$= \frac{1}{2}\left[\int_0^\infty \frac{ds}{s+a} - \int_0^\infty \frac{s+a}{(s+a)^2+4} ds\right]$$
$$= \frac{1}{2}\left[\log\frac{s+a}{\sqrt{(s+a)^2+4}}\right]_0^\infty$$
$$= \frac{1}{2}\left[0 - \frac{1}{2}\log\frac{a}{\sqrt{a^2+4}}\right]$$
$$= \frac{1}{2}\log\frac{\sqrt{a^2+4}}{a}.$$

EXERCISES

1. Find the Laplace transforms of
   (a) $4t + 6e^{4t}$ **Ans.** $\left(\frac{4}{s^2} + \frac{6}{s-4}\right)$
   (b) $e^{-4t}\sin 5t$ **Ans.** $\left(\frac{5}{s^2+8s+41}\right)$
   (c) $\frac{1}{t}(\sin at - at\cos at)$ **Ans.** $\tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2+a^2}$
   (d) $\frac{t}{2a}\sin at$ **Ans.** $\left(\frac{s}{(s^2+a^2)^2}\right)$
   (e) $\frac{t^{x-1}e^{at}}{\Gamma(x)}$ **Ans.** $\frac{1}{(s-a)^x}$, $x > 0$
   (f) $f(t) = \begin{cases} t+1 & \text{for } 0 \le t \le 2 \\ 3 & \text{for } t > 2 \end{cases}$
   Ans. $\frac{1}{s} + \frac{1}{s^2}(e^{-2s} - 1)$
   (g) $f(t) = t\,\mathrm{H}(t-a)$ **Ans.** $\frac{(1+as)\,e^{-as}}{s^2}$
   (h) Null function defined by $\int_0^t n(t)\,dt = 0$ for all t. **Ans.** 0
2. Show that the Laplace transforms of the following functions do not exist:
   (a) e^{t^2} **Hint:** Not of exponential order
   (b) $e^{1/t}$ **Hint:** Not defined at $t = 0$
   (c) $f(t) = \begin{cases} 1 & \text{for even } t \\ 0 & \text{for odd } t \end{cases}$

Hint: has infinite number of finite jumps and so condition of piecewise continuity is not satisfied

(d) t^{-n}, n is positive integer.

3. Find $L\left\{\frac{\sinh \omega t}{t}\right\}$ **Ans.** $\frac{1}{2}\log\frac{s+\omega}{s-\omega}$, $s > |\omega|$

4. Find the Laplace transform of a function whose graph is shown in the Figure 5.13.

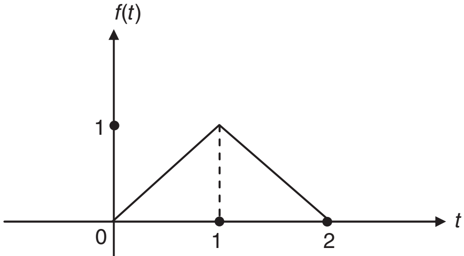

Figure 5.13

Hint: The function is defined by

$$f(t) = \begin{cases} t & \text{for } 0 \le t < 1 \\ 2-t & \text{for } 1 \le t < 2. \end{cases}$$

Therefore, $L\{f(t)\} = \int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt = \frac{1}{s^2}(1-e^{-s})^2$.

5. Find the Laplace transform of step function f defined by $f(t) = n, n \le t < n+1, n = 0, 1, 2, \ldots$
Hint: The graph of step function is shown in the Figure 5.14.

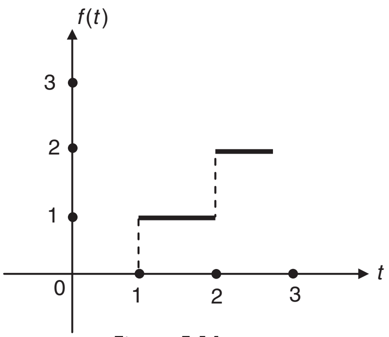

Figure 5.14

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_1^2 e^{-st}\, dt + 2\int_2^3 e^{-st}\, dt + 3\int_3^4 e^{-st}\, dt + \ldots$$

$$= \left[\frac{e^{-st}}{-s}\right]_1^2 + 2\left[\frac{e^{-st}}{-s}\right]_2^3 + 3\left[\frac{e^{-st}}{-s}\right]_3^4 + \ldots$$

$$= \frac{e^{-s}}{s}(1 + e^{-s} + e^{-2s} + \ldots)$$

$$= \frac{e^{-s}}{s(e^s - 1)}.$$

6. Find the Laplace transforms of
(a) $t\, e^{-2t} \sin 2t$ **Ans.** $\frac{4(s-2)}{(s^2-4s+8)^2}$
(b) $t^2 \cos at$ **Ans.** $\frac{2s^3-6a^2s}{(s^2+a^2)^3}$
(c) $t \sin 3t \cos 2t$.

Hint: $\sin 3t \cos 2t = \frac{1}{2}(2\sin 3t \cos 2t)$

$$= \frac{1}{2}(\sin(3t+2t) + \sin(3t-2t)) = \frac{1}{2}\sin 5t + \frac{1}{2}\sin t.$$

Therefore,

$$L\{\sin 3t \cos 2t\} = \frac{3s^2+15}{(s^2+1)(s^2+25)},\ s > 0 \text{ and so}$$

$$L\{s \sin 3t \cos 2t) = -\frac{d}{ds}\frac{(3s^2+15)}{(s^2+1)(s^2+25)}$$

$$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}.$$

7. Find Laplace transforms of
(a) $\frac{e^{-t}\sin t}{t}$ **Hint:** First find $L\{e^{-t}\sin t\}$ and then use

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds.$$ **Ans.** $\cot^{-1}(s+1)$.

(b) $\frac{e^{at}-\cos 6t}{t}$ **Ans.** $\log\frac{s^2-36}{s-a}$

8. Evaluate $I = \int_0^\infty \frac{e^{-t}\sin^2 t}{t}\, dt$.

Hint: $I = \int_0^\infty L\{f(t)\}\, du$, where $f(t) = e^{-t}\sin^2 t$. But $L\{e^{-t}\sin^2 t\} = \frac{2}{(s+1)(s^2+2s+5)}$.
Therefore,

$$I = \int_0^\infty \frac{2}{(s+1)(s^2+2s+5)}\, ds$$ **Ans.** $\frac{1}{2}\log 5$

9. Show that $L\{tf'(t)\} = -\left\{s\frac{d}{ds}F(s) + F(s)\right\}$.

10. Show that $L\left\{\frac{1}{t}(\sin at - at\cos at)\right\} = \tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2+a^2}$.

11. Show that $L\{t^2 f''(t)\} = s^2\frac{d^2}{ds^2}F(s) + 4s\frac{d}{ds}F(s) + 2F(s)$.

12. Use initial value theorem to find $f(0)$ and $f'(0)$ for the function f for which $F(s) = \frac{s}{s^2-5s+12}$

Ans. $f(0) = 1, f'(0) = 5$

13. Find the Laplace transform of the square wave function with graph shown in the Figure 5.15.

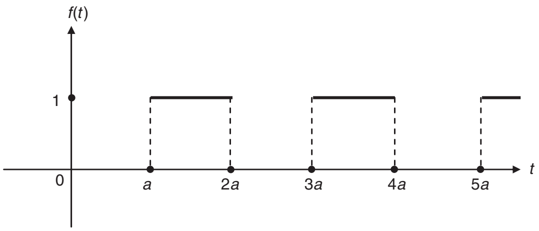

Figure 5.15

Hint: The function is of period $2a$. Therefore,

$$L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_a^{2a} e^{-st}\,dt = \frac{e^{-as}}{s(1+e^{-as})}$$

14. Express the function $f(t)$ in Exercise 13 in terms of Heaviside's unit step function and then find its Laplace transform.

Hint: $f(t) = H(t-a) - H(t-2a) + H(t+3a) - H(t-4a) + \ldots$

15. Find the Laplace transform the of square wave function with graph given in the Figure 5.16.

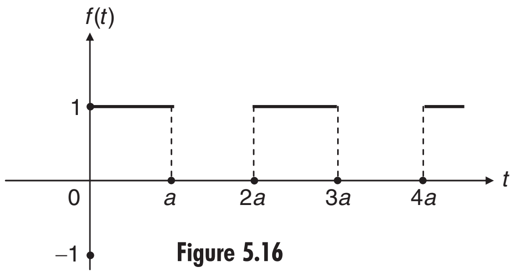

Figure 5.16

Hint: Here the function is of period $2a$. Therefore,

$$L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{a} e^{-st}\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{e^{-st}}{-s}\right]_0^a = \frac{1}{s(1+e^{-as})}.$$

16. Solve Example 5.27 using periodicity

Hint: $L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2as}}$ $\times \left[\int_0^a e^{-st}dt + \int_a^{2a} -e^{-st}dt\right]$ which on using Examples 5.13 and 5.15 yields the transform as

$$\frac{1}{s}\tanh\frac{as}{2}.$$

17. Find Laplace transform of the half-wave rectified sine function f defined by

$$f(t) = \begin{cases} \sin\omega t & \text{for } \frac{2n\pi}{\omega} < t < \frac{(2n+1)\pi}{\omega} \\ 0 & \text{for } \frac{(2n+1)\pi}{\omega} < t < \frac{(2n+2)\pi}{\omega} \end{cases}$$

Hint: The function is periodic with period $\frac{2\pi}{\omega}$. Therefore,

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} \sin\omega t\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin\omega t\,dt$$

$$= \frac{\omega}{s^2+\omega^2}(1+e^{-\pi s/\omega}).$$

18. Establish relation between Laplace transform and Fourier transform of a function.

Hint: Let ϕ be a function defined by

$$\phi(t) = \begin{cases} e^{-xt}f(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0. \end{cases}$$

Then the Fourier transform of ϕ is

$$F(y) = F\{\phi(t)\} = \int_0^\infty e^{-iyt}.e^{-xt}f(t)dt$$

$$= \int_0^\infty e^{-(x+iy)t}f(t)dt$$

$$= \int_0^\infty e^{-st}f(t)dt = L\{f(t)\}.$$

Thus Laplace transform of $f(t)$ is equal to the Fourier transform of $e^{-xt}f(t)$.

6 Inverse Laplace Transform

Like the operations of addition, multiplication, and differentiation, the Laplace transform has also its inverse. During the process of solving physical problems like differential equations, it is necessary to invoke the inverse transform of the Laplace transform. Thus given a Laplace transform F(s) of a function f, we would like to know what f is. Hence, we are concerned with the solution of the integral equation,

$$\int_0^\infty e^{-st} f(t)\, dt = \mathrm{F}(s).$$

6.1 DEFINITION AND EXAMPLES OF INVERSE LAPLACE TRANSFORM

Definition 6.1. Let f have Laplace transform F(s), that is, L{ $f(t)$} = F(s), then $f(t)$ is called an *inverse Laplace transform* of F(s) and we write

$$L^{-1}\{\mathrm{F}(s)\} = f(t), t \geq 0.$$

The transformation L^{-1} is called *inverse Laplace operator* and it maps the Laplace transform of a function back to the original function.

We know that Laplace transform F(s) of a function $f(t)$ is uniquely determined due to the properties of integrals. However, this is not true for the inverse transform. For example, if $f(t)$ and $g(t)$ are two functions that are identical except for a finite number of points, they have the same transform F(s) since their integrals are identified. Therefore, either $f(t)$ or $g(t)$ is the inverse transform of F(s). Thus inverse transform of a given function F(s) is uniquely determined only upto an additive null function [a function $n(t)$ for which $\int_0^t n(u)\, du = 0$ for all t]. The following examples show that L^{-1} {F(s)} can be more than one function.

(a) Let $f(t) = \sin \omega t$, $t \geq 0$. Then $\mathrm{L}\{f(t)\} = \frac{\omega}{s^2+\omega^2}$. Thus

$$L^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin \omega t.$$

Now let

$$g(t) = \begin{cases} \sin \omega t & \text{for } t > 0 \\ 1 & \text{for } t = 0. \end{cases}$$

Then

$$\mathrm{L}\{g(t)\} = \frac{\omega}{s^2+\omega^2},$$

and so

$$L^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = g(t).$$

Hence there are two inverse transforms of $\frac{\omega}{s^2+\omega^2}$.
(b) Let $f(t) = e^{-3t}$ and

$$g(t) = \begin{cases} 0 & \text{for } t = 1 \\ e^{-3t} & \text{otherwise.} \end{cases}$$

Then both $f(t)$ and $g(t)$ have same Laplace transform $\frac{1}{s+3}$. Thus $\frac{1}{s+3}$ has two inverse Laplace transforms $f(t)$ and $g(t)$.

But the following theorem shows that the Laplace transform is one-one mapping.

Theorem 6.1. (Lerch's Theorem). Distinct continuous functions on $[0, \infty)$ have distinct Laplace transforms.

Thus, if we restrict ourselves to continuous functions on $[0, \infty)$, then the inverse transform $L^{-1}\{\mathrm{F}(s)\} = f(t)$ is uniquely defined. Since many of the functions, we generally deal with, are solutions to the differential equations and hence continuous, the assumption of the theorem is satisfied.

EXAMPLE 6.1

Find inverse Laplace transform of $\frac{a}{s^2+a^2}, \frac{s}{s^2+a^2}, \frac{1}{s-a}, \frac{e^{-sa}}{s}, \frac{s}{s^2-a^2}$ and $\frac{n!}{s^{n+1}}$.

Solution. We know that

$$\mathrm{L}\{\sin at\} = \frac{a}{s^2+a^2},\ \mathrm{L}\{\cos at\} = \frac{s}{s^2+a^2},$$

$$\mathrm{L}\{e^{at}\} = \frac{1}{s-a}, \mathrm{L}\{h(t-a)\} = \frac{e^{-sa}}{s},$$

$$\mathrm{L}\{\cosh at\} = \frac{s}{s^2-a^2}, \text{ and}$$

$$\mathrm{L}\{t^n\} = \frac{n!}{s^{n+1}}, n \text{ being non-negative integer.}$$

Therefore,

$$L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at,$$

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at,$$

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at},$$

$$L^{-1}\left\{\frac{e^{-sa}}{s}\right\} = \mathrm{H}(t-a), \text{Heavyside's unit step function.}$$

$$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at,$$

and

$$L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, n \text{ being non-negative integer,}$$

EXAMPLE 6.2

Find $L^{-1}\left(\frac{1}{\sqrt{s}}\right)$.

Solution. Since $\mathrm{L}\left\{\frac{1}{t^{1/2}}\right\} = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$, it follows that

$$L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}.$$

6.2 PROPERTIES OF INVERSE LAPLACE TRANSFORM

The operational properties used in finding the Laplace transform of a function are also used in constructing the inverse transform. We, thus, have the following properties of inverse transform.

Theorem 6.2. (Linearity Property). If $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$, respectively, and a_1 and a_2 are arbitrary constants, then

$$\begin{aligned} &L^{-1}\{a_1F_1(s) + a_2F_2(s)\} \\ &= a_1\, L^{-1}\{F_1(s)\} + a_2L^{-1}\{F_2(s)\} \\ &= a_1f_1(t) + a_2f_2(t). \end{aligned}$$

Proof: Since

$$\begin{aligned} L\{a_1f_1(t) + a_2f_2(t)\} &= a_1\mathrm{L}\{f_1(t)\} + a_2\mathrm{L}\{f_2(t)\} \\ &= a_1\mathrm{F}_1(s) + a_2\mathrm{F}_2(s), \end{aligned}$$

we have

$$\begin{aligned} &L^{-1}\{a_1\mathrm{F}_1(s) + a_2\mathrm{F}_2(s)\} \\ &= a_1f_1(t) + a_2f_2(t) \\ &= a_1\, L^{-1}\{\mathrm{F}_1(s)\} + a_2\, L^{-1}\{\mathrm{F}_2(s)\}. \end{aligned}$$

EXAMPLE 6.3

Find the inverse Laplace transform of

$$\frac{1}{2s} + \frac{4}{3(s-a)} + \frac{s}{s^2+16}.$$

Solution. By linearity of inverse Laplace transform, we have

$$\begin{aligned} &L^{-1}\left\{\frac{1}{2s} + \frac{4}{3(s-a)} + \frac{s}{s^2+16}\right\} \\ &= \frac{1}{2}L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{3}L^{-1}\left\{\frac{1}{s-a}\right\} \\ &+ L^{-1}\left\{\frac{s}{s^2+16}\right\} = \frac{1}{2} + \frac{4}{3}e^{at} + \cos 4t. \end{aligned}$$

EXAMPLE 6.4

Find inverse Laplace transform of

$$\frac{5}{s-3} + \frac{s}{s^2+4} + \frac{3}{s-7}.$$

Solution. By linearity, we have

$$\begin{aligned} &L^{-1}\left\{\frac{5}{s-3} + \frac{s}{s^2+4} + \frac{3}{s-7}\right\} \\ &= 5L^{-1}\left\{\frac{1}{s-3}\right\} + L^{-1}\left\{\frac{s}{s^2+4}\right\} \\ &+ 3L^{-1}\left\{\frac{1}{s-7}\right\} = 5e^{3t} + \cos 2t + 3\, e^{7t}. \end{aligned}$$

EXAMPLE 6.5

Find inverse Laplace transform of

$$-\frac{3}{2s}+\frac{4}{3(s-1)}+\frac{1}{6(s+2)}.$$

Solution. By linearity of inverse Laplace transform, we have

$$\begin{aligned}
L^{-1}&\left\{-\frac{3}{2s}+\frac{4}{3(s-1)}+\frac{1}{6(s+2)}\right\}\\
&=-\frac{3}{2}L^{-1}\left\{\frac{1}{s}\right\}+\frac{4}{3}L^{-1}\left\{\frac{1}{s-1}\right\}+\frac{1}{6}L^{-1}\left\{\frac{1}{s+2}\right\}\\
&=-\frac{3}{2}+\frac{4}{3}e^{t}+\frac{1}{6}e^{-2t}.
\end{aligned}$$

EXAMPLE 6.6

Find $L^{-1}\left\{\frac{1}{\sqrt{s}+a}\right\}$.

Solution. Since

$$\frac{1}{\sqrt{s}+a}=\frac{1}{\sqrt{s}}-\frac{a}{\sqrt{s}(\sqrt{s}+a)}=\frac{1}{\sqrt{s}}-\frac{a(\sqrt{s}-a)}{\sqrt{s}(s-a^2)},$$

therefore,

$$\begin{aligned}
L^{-1}\left\{\frac{1}{\sqrt{s}+a}\right\}&=L^{-1}\left\{\frac{1}{\sqrt{s}}\right\}-aL^{-1}\left\{\frac{\sqrt{s}-a}{\sqrt{s}(s-a^2)}\right\}\\
&=L^{-1}\left\{\frac{1}{\sqrt{s}}\right\}-aL^{-1}\left\{\frac{1}{s-a^2}\right\}\\
&\quad+a^2L^{-1}\left\{\frac{1}{\sqrt{s}(s-a^2)}\right\}\\
&=\frac{1}{\sqrt{\pi t}}-ae^{a^2t}+ae^{a^2t}\mathit{erf}(a\sqrt{t})\\
&=\frac{1}{\sqrt{\pi t}}-ae^{a^2t}(1-\mathit{erf}(a\sqrt{t}))\\
&=\frac{1}{\sqrt{\pi t}}-ae^{a^2t}\mathit{erfc}\ (a\sqrt{t})
\end{aligned}$$

Theorem 6.3. (First Shifting Property). If $F(s)$ is Laplace transform of $f(t)$, then

$$L^{-1}\ \{\mathrm{F}(s-a)\}=e^{at}f(t).$$

Proof: We know that

$$\mathrm{L}\{e^{at}f(t)\}\quad=\mathrm{F}(s-a).$$

Therefore,

$$L^{-1}\{\mathrm{F}(s-a)\}=e^{at}f(t).$$

EXAMPLE 6.7

Find inverse Laplace transform of $\frac{s-5}{s^2+6s+13}$.

Solution. Since

$$\frac{s-5}{s^2+6s+13}=\frac{(s+3)-8}{(s+3)^2+4},$$

we have

$$\begin{aligned}
L^{-1}\left\{\frac{s-5}{s^2+6s+13}\right\}&=L^{-1}\left\{\frac{s+3}{(s+3)^2+4}\right\}\\
&\quad-4\ L^{-1}\left\{\frac{2}{(s+3)^2+4}\right\}\\
&=e^{-3t}\cos 2t-4e^{-3t}\sin 2t\\
&=e^{-3t}(\cos 2t-4\sin 2t).
\end{aligned}$$

EXAMPLE 6.8

Find inverse Laplace transform of $\frac{2s-3}{s^2+4s+13}$.

Solution. Since

$$\frac{2s-3}{s^2+4s+3}=\frac{2s+4-7}{(s+2)^2+9}=\frac{2(s+2)-7}{(s+2)^2+9},$$

we have

$$\begin{aligned}
L^{-1}\left\{\frac{2s-3}{s^2+4s+13}\right\}&=L^{-1}\left\{\frac{2(s+2)}{(s+2)^2+9}\right\}\\
&\quad-7\ L^{-1}\left\{\frac{1}{(s+2)^2+9}\right\}\\
&=2e^{-2t}\cos 3t-\frac{7}{3}e^{-2t}\sin 3t.
\end{aligned}$$

EXAMPLE 6.9

Find inverse Laplace transform of $\frac{s}{(s+1)^2}$.

Solution. We note that

$$\frac{s}{(s+1)^2}=\frac{s+1-1}{(s+1)^2}=\frac{1}{s+1}-\frac{1}{(s+1)^2}.$$

Therefore,

$$\begin{aligned}
L^{-1}\left\{\frac{s}{(s+1)^2}\right\}&=L^{-1}\left\{\frac{1}{s+1}\right\}-L^{-1}\left\{\frac{1}{(s+1)^2}\right\}\\
&=e^{-t}.1-e^{-t}.t=e^{-t}(1-t).
\end{aligned}$$

EXAMPLE 6.10

Find inverse Laplace transform of

$$\frac{1}{s^2+4s+13}-\frac{s+4}{s^2+8s+97}+\frac{s+2}{s^2-4s+29}.$$

Solution. We have

$$\begin{aligned}F(s)&=\frac{1}{s^2+4s+13}-\frac{s+4}{s^2+8s+97}+\frac{s+2}{s^2-4s+29}\\&=\frac{1}{(s+2)^2+9}\\&\quad-\frac{s+4}{(s+4)^2+81}+\frac{s-2+4}{(s-2)^2+25}.\end{aligned}$$

Therefore,

$$\begin{aligned}L^{-1}\{F(s)\}&=\frac{1}{3}L^{-1}\left\{\frac{3}{(s+2)^2+9}\right\}\\&\quad-L^{-1}\left\{\frac{s+4}{(s+4)^2+81}\right\}\\&\quad+L^{-1}\left\{\frac{s-2}{(s-2)^2+25}\right\}\\&\quad+\frac{4}{5}L^{-1}\left\{\frac{5}{(s-2)^2+25}\right\}\\&=\frac{1}{3}e^{-2t}\sin 3t-e^{-4t}\cos 9t\\&\quad+e^{2t}\cos 5t+\frac{4}{5}e^{2t}\sin 5t.\end{aligned}$$

EXAMPLE 6.11

Find the inverse Laplace transform of $\frac{s+2}{s^2-4s+13}$.

Solution. We have

$$\begin{aligned}\frac{s+2}{s^2-4s+13}&=\frac{(s-2)+4}{(s-2)^2+9}\\&=\frac{(s-2)}{(s-2)^2+9}+\frac{4}{(s-2)^2+9}\\&=\frac{s-2}{(s-2)^2+9}+\frac{4}{3}\cdot\frac{3}{(s-2)^2+9}\end{aligned}$$

Therefore, by linearity of inverse Laplace transform and shifting property, we get

$$\begin{aligned}L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\}&=L^{-1}\left\{\frac{s-2}{(s-2)^2+9}\right\}\\&\quad+\frac{4}{3}L^{-1}\left\{\frac{3}{(s-2)^2+9}\right\}\\&=e^{2t}\cos 3t+\frac{4}{3}e^{2t}\sin 3t\\&=e^{2t}(\cos 3t+\frac{4}{3}\sin 3t).\end{aligned}$$

Theorem 6.4. (Second Shifting Property). If $L^{-1}\{F(s)\}=f(t)$, then $L^{-1}\{e^{-sa}F(s)\}=g(t)$, where

$$g(t)=\begin{cases}f(t-a) & \text{for } t>a\\ 0 & \text{for } t<a.\end{cases}$$

Proof: Since $L\{g(t)\}=e^{-sa}F(s)$, it follows that

$$L^{-1}\{e^{-sa}F(s)\}=g(t).$$

Second Proof. By definition of Laplace transform, we have

$$F(s)=\int_0^\infty e^{-st}f(t)\,dt.$$

Therefore,

$$\begin{aligned}e^{-sa}F(s)&=\int_0^\infty e^{-sa}e^{-st}f(t)\,dt=\int_0^\infty e^{-s(t+a)}f(t)\,dt\\&=\int_a^\infty e^{-su}f(u-a)\,du,\quad t+a=u.\\&=\int_0^a e^{-su}(0)\,du+\int_a^\infty e^{-su}f(u-a)\,du\\&=L\{g(t)\}.\end{aligned}$$

Hence

$$L^{-1}\{e^{-sa}F(s)\}=g(t).$$

EXAMPLE 6.12

Find inverse Laplace transform of $-\frac{e^{-\pi s/2}}{s^2+1}$.

Solution. We have

$$\begin{aligned}-\frac{e^{-\pi s/2}}{s^2+1}&=e^{-\pi s/2}\left(-\frac{1}{s^2+1}\right)\\&=e^{-\pi s/2}F(s),\text{ where }F(s)=-\frac{1}{s^2+1}.\end{aligned}$$

But

$$L^{-1}\{F(s)\} = L^{-1}\left\{-\frac{1}{s^2+1}\right\} = -L^{-1}\left\{\frac{1}{s^2+1}\right\}.$$

Therefore, by second-shifting property,

$$L^{-1}\left\{-\frac{e^{-\pi s/2}}{s^2+1}\right\} = g(t),$$

where

$$g(t) = \begin{cases} -\sin(t-\pi/2) & \text{for } t > \pi/2 \\ 0 & \text{for } t < \pi/2 \end{cases}$$

$$= -\sin\left(t-\frac{\pi}{2}\right)\left[H\left(t-\frac{\pi}{2}\right)\right]$$

$$= \cos t\left[H\left(t-\frac{\pi}{2}\right)\right],$$

where $H(t)$ denotes Heavyside's unit step function.

EXAMPLE 6.13

Find inverse Laplace transform of $\frac{\omega e^{-sa}}{s^2+\omega^2}$.

Solution. We have

$$L^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin\omega t.$$

Therefore, by second-shifting property,

$$L^{-1}\left\{e^{-sa}\frac{\omega}{s^2+\omega^2}\right\} = g(t),$$

where

$$g(t) = \begin{cases} \sin\omega(t-a) & \text{for } t > a \\ 0 & \text{for } t < a. \end{cases}$$

EXAMPLE 6.14

Find inverse transform of $\frac{2e^{-s}}{s^3}$, $\mathrm{Re}(s) > 0$.

Solution. We have

$$\frac{2e^{-s}}{s^3} = 2e^{-s}.\frac{1}{s^3}.$$

Since $L^{-1}\left\{\frac{1}{s^3}\right\} = t^2$, therefore, by second-shifting property,

$$L^{-1}\left\{\frac{2e^{-s}}{s^3}\right\} = g(t),$$

where

$$g(t) = \begin{cases} 2(t-1)^2 & \text{for } t \ge 1 \\ 0 & \text{for } 0 \le t < 1. \end{cases}$$

EXAMPLE 6.15

Find $L^{-1}\left\{\frac{A}{s} - \frac{s}{s^2+1}\ e^{-sa}\right\}$, where A is a constant.

Solution. We have

$$L^{-1}\left\{\frac{1}{s}\right\} = 1 \quad \text{and} \quad L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t.$$

Therefore, using linearity property and second-shifting property, we have

$$L^{-1}\left\{\frac{A}{s} - \frac{s}{s^2+1}\ e^{-sa}\right\}$$

$$= AL^{-1}\left\{\frac{A}{s}\right\} - L\left\{e^{-sa}\frac{s}{s^2+1}\right\} = A - g(t),$$

where

$$g(t) = \begin{cases} \cos(t-a) & \text{for } t > a \\ 0 & \text{for } t < a. \end{cases}$$

$$= H(t-a)\cos(t-a).$$

Hence

$$L^{-1}\left\{\frac{A}{s} - \frac{s}{s^2+1}\ e^{-sa}\right\}$$

$$= A - H\ (t-a)\cos(t-a).$$

EXAMPLE 6.16

Find the inverse Laplace transform of $\frac{e^{-7s}}{(s-3)^3}$.

Solution. Since

$$L^{-1}\left(\frac{1}{(s-3)^3}\right) = \frac{1}{2}t^2e^{3t},$$

by second-shifting property, we have

$$L^{-1}\left\{e^{-7s}\frac{1}{(s-3)^3}\right\} = g(t),$$

where

$$g(t) = \begin{cases} \frac{1}{2}(t-7)^2e^{3(t-7)} & \text{for } t > 7 \\ 0 & \text{for } 0 \le t \le 7 \end{cases}$$

$$= \frac{1}{2}H(t-7)(t-7)^2e^{3(t-7)}.$$

Theorem 6.5. (Change of Scale Property). If $L^{-1}\ \{F(s)\} = f(t)$, then

$$L^{-1}\{F(as)\} = \frac{1}{a}f\ \left(\frac{t}{a}\right).$$

Proof: By the definition of Laplace transform,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Therefore,

$$F(as) = \int_0^\infty e^{-ast} f(t)\, dt$$

$$= \frac{1}{a}\int_0^\infty e^{-su} f\left(\frac{u}{a}\right) du, \quad u = at$$

$$= \frac{1}{a}\, L\left\{f\left(\frac{t}{a}\right)\right\}.$$

Hence

$$L^{-1}\{F(as)\} = \frac{1}{a}\, f\left(\frac{t}{a}\right).$$

Remark 6.1. It follows from Theorem 6.5 that if $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{F\left(\frac{s}{a}\right)\right\} = af(at)$ for $a > 0$.

EXAMPLE 6.17

Find the inverse transform of $\frac{s}{(s/2)^2+4}$.

Solution. Since

$$L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t,$$

we have

$$L^{-1}\left\{\frac{s}{(s/2)^2+4}\right\} = 2L^{-1}\left\{\frac{s/2}{(s/2)^2+4}\right\}$$

$$= 2.2\cos(2(2t)) = 4\cos 4t.$$

Theorem 6.6. (Inverse Laplace Transform of Derivatives). If $L^{-1}\{F(s)\} = f(t)$, then

$$L^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t).$$

Proof: Since

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s),$$

we have

$$L^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

EXAMPLE 6.18

Find $L^{-1}\left\{\frac{1-s^2}{(s^2+1)^2}\right\}$.

Solution. We know that

$$L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t.$$

Further

$$\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{1-s^2}{(s^2+1)^2}.$$

Therefore, by Theorem 6.6, we have

$$L^{-1}\left\{\frac{1-s^2}{(s^2+1)^2}\right\} = -t\cos t.$$

EXAMPLE 6.19

Find $L^{-1}\left\{\log\frac{s+a}{s+b}\right\}$.

Solution. We note that

$$\frac{d}{ds}\log\frac{s+a}{s+b} = \frac{d}{ds}[\log(s+a) - \log(s+b)]$$

$$= \frac{1}{s+a} - \frac{1}{s+b}.$$

Therefore, the use of Theorem 6.6 yields

$$L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\} = -t\, f(t)$$

and so

$$L^{-1}\left\{\frac{1}{s+a}\right\} - L^{-1}\left\{\frac{1}{s+b}\right\} = -t\, f(t),$$

that is,

$$e^{-at} - e^{-bt} = -t\, f(t).$$

Hence

$$f(t) = \frac{1}{t}\left(e^{-bt} - e^{-at}\right).$$

EXAMPLE 6.20

Find $L^{-1}\left\{\log\frac{s^2+a^2}{s^2+b^2}\right\}$.

Solution. Since

$$\frac{d}{ds}\log\frac{s^2+a^2}{s^2+b^2} = \frac{d}{ds}[\log(s^2+a^2) - \log(s^2+b^2)]$$

$$= \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2},$$

Theorem 6.6 yields

$$L^{-1}\left\{\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right\} = -t\, f(t),$$

or

$$2\cos at - 2\cos bt = -t\, f(t),$$

or

$$f(t) = \frac{2}{t}(\cos bt - \cos at).$$

EXAMPLE 6.21

Find $L^{-1}\{\log \frac{1+s}{s}\}$.

Solution. Since

$$\frac{d}{ds}\log\frac{1+s}{s} = \frac{d}{ds}[\log(1+s) - \log s]$$
$$= \frac{1}{s+1} - \frac{1}{s},$$

Therefore,

$$L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s}\right\} = -tf(t)$$

or

$$e^{-1} - 1 = -tf(t)$$

or

$$f(t) = \frac{1-e^{-t}}{t}.$$

EXAMPLE 6.22

Find $L^{-1}\{\log(1+\frac{1}{s^2})\}$.

Solution. Since

$$\frac{d}{ds}\left\{\log\frac{s^2+1}{s^2}\right\} = \frac{d}{ds}[\log(s^2+1) - \log s^2]$$
$$= \frac{2s}{s^2+1} - \frac{2s}{s^2} = 2\left[\frac{s}{s^2+1} - \frac{1}{s}\right],$$

we have

$$L^{-1}\left\{2\left(\frac{s}{s^2+1} - \frac{1}{s}\right)\right\} = -tf(t)$$

and so

$$2\cos t - 2 = -t\,f(t)$$

or

$$f(t) = \frac{2(1-\cos t)}{t}.$$

EXAMPLE 6.23

Find $L^{-1}\{\tan^{-1}\frac{1}{s}\}$'$s > 0$,

Solution. Since

$$\frac{d}{ds}\left(\tan^{-1}\frac{1}{s}\right) = \frac{1}{1+(1/s)^2}\left(-\frac{1}{s^2}\right)$$
$$= \frac{-1}{1+(1/s^2)}\left(\frac{1}{s^2}\right) = -\frac{1}{s^2+1},$$

it follows that

$$L^{-1}\left\{-\frac{1}{s^2+1}\right\} = -tf(t),$$

that is,

$$-1(\sin t) = -t\,f(t).$$

Hence

$$f(t) = \frac{\sin t}{t}.$$

EXAMPLE 6.24

Find $L^{-1}\left\{\log\frac{s^2+1}{(s-1)^2}\right\}$.

Solution. Since

$$\frac{d}{ds}\left(\log\frac{s^2+1}{(s-1)^2}\right) = \frac{d}{ds}[\log(s^2+1) - 2\log(s-1)]$$
$$= \frac{2s}{s^2+1} - \frac{2}{s-1},$$

we have

$$L^{-1}\left\{\frac{2s}{s^2+1} - \frac{2}{s-1}\right\} = -tf(t),$$

which yields

$$2[\cos t - e^t] = -t\,f(t)$$

or

$$f(t) = \frac{2(e^t - \cos t)}{t}.$$

EXAMPLE 6.25

Find $L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}$.

Solution. We have

$$\frac{s+2}{(s^2+4s+5)^2} = \frac{s+2}{((s+2)^2+1)^2}$$

Therefore,

$$L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = L^{-1}\left\{\frac{s+2}{((s+2)^2+1)^2}\right\}$$
$$= e^{-2t}.\mathrm{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}.$$

We now find $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$. We note that

$$\frac{d}{ds}\left(\frac{1}{(s^2+1)}\right) = -\frac{2s}{(s^2+1)^2}.$$

Therefore, by Theorem 6.6

$$L^{-1}\left\{-\frac{2s}{(s^2+1)^2}\right\} = -\text{t } L^{-1}\left\{\frac{1}{s^2+1}\right\} = -t\sin t$$

and so

$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t.$$

Hence

$$L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = \frac{1}{2}\ t\ e^{-2t}\sin t.$$

EXAMPLE 6.26

Find $L^{-1}\left\{\frac{s+3}{(s^2+6s+13)^2}\right\}$.

Solution. We have

$$\frac{s+3}{(s^2+6s+13)^2} = \frac{s+3}{((s+3)^2+4)^2}.$$

Therefore,

$$L^{-1}\left\{\frac{s+3}{(s^2+6s+13)^2}\right\} = L^{-1}\left\{\frac{s+3}{((s+3)^2+4)^2}\right\}$$

$$= e^{-3t}L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}.$$

To find $L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$, we note that

$$\frac{d}{ds}\left(\frac{1}{s^2+4}\right) = \frac{-2s}{(s^2+4)^2}.$$

Therefore,

$$L^{-1}\left\{-\frac{2s}{(s^2+4)^2}\right\} = -tL^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= -\frac{t}{2}\sin 2t$$

and so

$$L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}t\sin 2t.$$

Consequently, we get

$$L^{-1}\left\{\frac{s+3}{(s^2+6s+13)^2}\right\} = \frac{1}{4}t\ e^{-3t}\sin 2t.$$

Theorem 6.7. (Inverse Laplace Transform of Integrals). If $L^{-1}\{F(s)\} = f(t)$, then

$$L^{-1}\left\{\int_s^\infty F(u)\ du\right\} = \frac{f(t)}{t}.$$

Proof: Since,

$$\text{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\ du,$$

we have

$$L^{-1}\left\{\int_s^\infty F(u)\ du\right\} = \frac{f(t)}{t}.$$

EXAMPLE 6.27

Find

$$L^{-1}\left\{\int_s^\infty \left(\frac{1}{u-1} - \frac{1}{u+1}\right)\ du\right\}.$$

Solution. By Theorem 6.7, we have

$$L^{-1}\left\{\int_s^\infty \left(\frac{1}{u-1} - \frac{1}{u+1}\right)\ du\right\} = \frac{L^{-1}\left\{\frac{1}{u-1} - \frac{1}{u+1}\right\}}{t}.$$

But

$$L^{-1}\left\{\frac{1}{u-1} - \frac{1}{u+1}\right\} = e^t - e^{-t}.$$

Therefore,

$$L^{-1}\left\{\int_s^\infty \left(\frac{1}{u-1} - \frac{1}{u+1}\right)\ du\right\} = \frac{e^t - e^{-t}}{t}.$$

Theorem 6.8. (Multiplication by s^n). If $L^{-1}\{F(s)\} = f(t)$ and $f(0) = 0$, then

$$L^{-1}\{sF(s)\} = f'(t).$$

Proof: We know that

$$\text{L}\{f'(t)\} = s\ \text{F}(s) - f(0)$$

and so

$$L^{-1}\{sF(s) - f(0)\} = f'(t),$$

that is,

$$L^{-1}\{sF(s)\} = f'(t).$$

Remark 6.2. If $f(0) \neq 0$, then

$$L^{-1}\{sF(s) - f(0)\} = f'(t)$$

and so

$$L^{-1}\{sF(s)\} = f'(t) + f(0)\delta(t),$$

where $\delta(t)$ is the Dirac delta function.

EXAMPLE 6.28

Find $L^{-1}\left\{\frac{s^2}{s^2+1}\right\}$.

Solution. We know that

$$L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t = f(t)$$

and $f(0) = 1 \neq 0$ for $t = 0$. Therefore,

$$L^{-1}\left\{\frac{s^2}{s^2+1}\right\} = L^{-1}\{sF(s)\}$$
$$= f'(t) + f(0)\delta(t) = \delta(t) - \sin t.$$

EXAMPLE 6.29

Find $L^{-1}\left\{\frac{s^3}{s^2+1}\right\}$.

Solution. From the Example 6.28,

$$L^{-1}\left\{\frac{s^2}{s^2+1}\right\} = \delta(t) - \sin t = f(t).$$

Since $f(0) = 0$, we have

$$L^{-1}\left\{\frac{s^3}{s^2+1}\right\} = L^{-1}\{sF(s)\} = f'(t)$$
$$= \frac{d}{dt}\{\delta(t) - \sin t\} = \delta'(t) - \cos t.$$

Theorem 6.9. (Division by s). If $L^{-1}\{F(s)\} = f(t)$, then

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\, du.$$

Proof: We know that

$$\mathrm{L}\left\{\int_0^t f(u)\, du\right\} = \frac{F(s)}{s}.$$

Therefore,

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)du.$$

Remark 6.3. Consider

$$g(t) = \int_0^t \int_0^v f(u)\, du\, dv.$$

Then

$$g'(t) = \int_0^t f(u)\, du \quad \text{and} \quad g''(t) = f(t).$$

Also $g(0) = g'(0) = 0$. Thus

$$\mathrm{L}\{g''(t)\} = s^2\mathrm{L}\{g(t)\} - s\, g(0) - g'(0)$$
$$= s^2\mathrm{L}\{g(t)\},$$

that is,

$$\mathrm{L}\{f(t)\} = s^2\,\mathrm{L}\{g(t)\}$$

or

$$\mathrm{F}(s) = s^2\,\mathrm{L}\{g(t)\}$$

or

$$\mathrm{L}\{g(t)\} = \frac{F(s)}{s^2}$$

Hence

$$\mathrm{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = g(t) = \int_0^t \int_0^v f(u)\, du\, dv$$
$$= \int_0^t \int_0^t f(t)\, dt^2.$$

In general, we have

$$\mathrm{L}^{-1}\left\{\frac{F(s)}{s^n}\right\} = \underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{n \text{ times}} f(t)dt^n.$$

EXAMPLE 6.30

Find $L^{-1}\left\{\frac{1}{s(s+1)}\right\}$.

Solution. Since $L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$, we have

$$L^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t e^{-u}\, du = [-e^{-u}]_0^t$$
$$= -[e^{-t} - 1] = 1 - e^{-t}.$$

EXAMPLE 6.31

Find $L^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$.

Solution. From Example 6.30, we have

$$L^{-1}\left\{\frac{1}{s(s+1)}\right\} = 1 - e^{-t}$$

Therefore, by Theorem 6.9, we have

$$L^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = \int_0^t (1 - e^{-u})du = [u + e^u]_0^t$$
$$= t + e^t - 1.$$

EXAMPLE 6.32

Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ and deduce $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$ from it.

Solution. Since

$$\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = -\frac{2as}{(s^2+a^2)^2},$$

we have

$$L^{-1}\left\{-\frac{2as}{(s^2+a^2)^2}\right\} = -tf(t),$$

where $f(t) = L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$ and so

$$-2a\,L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -t\sin at$$

Hence

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t\sin at}{2a}$$

Now Theorem 6.9 yields

$$\begin{aligned}
L^{-1}&\left\{\frac{1}{(s^2+a^2)^2}\right\}\\
&= \frac{1}{2a}\int_0^t u\sin au\, du\\
&= \frac{1}{2a}\left\{\left[u\frac{-\cos au}{a}\right]_0^t - \int_0^t -\frac{\cos au}{a}\,du\right\}\\
&= \frac{1}{2a}\left[-\frac{t\cos at}{a} + \frac{\sin at}{a^2}\right]\\
&= \frac{1}{2a^3}(\sin at - at\cos at).
\end{aligned}$$

6.3 PARTIAL FRACTIONS METHOD TO FIND INVERSE LAPLACE TRANSFORM

While working with physical problems, we come across some functions that are not immediately recognizable as the Laplace transform of some elementary functions. In such cases, we decompose the given function into partial fractions and then write down the inverse of each fraction. This method is used when inverse transform of a rational function is required.

We can also use the concept of *simple poles* of the rational function F(s) to find inverse Laplace transform. Let

$$\mathrm{F}(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{(s-\alpha_1)(s-\alpha_2)\ldots(s-\alpha_n)},$$
$$\alpha_i \neq \alpha_j,$$

where P(s) is a polynomial of degree less than n. In terms of complex analysis, we call $\alpha_1, \alpha_2, \ldots, \alpha_n$ the simple poles of F(s). The partial fraction decomposition shall be

$$\mathrm{F}(s) = \frac{A_1}{s-\alpha_1} + \frac{A_2}{s-\alpha_2} + \ldots + \frac{A_n}{s-\alpha_n}.$$

Multiplying both sides of this equation by $s-\alpha_i$ and letting $s \to \alpha_i$, we have

$$\mathrm{A_i} = \lim_{s\to\alpha_i}(s-\alpha_i)\mathrm{F}(s).$$

Again, in terms of complex analysis, A_i is called the *residue of F(s) at the poles* α_i.
Now, let L{$f(t)$} = F(s). Then

$$\begin{aligned}
f(t) = \mathrm{L}^{-1}\{\mathrm{F}(s)\} &= \sum_{i=1}^{n} L^{-1}\left(\frac{A_i}{s-\alpha_i}\right) = \sum_{i=1}^{n} A_i e^{\alpha_i t}\\
&= \sum_{i=1}^{n}\lim_{s\to a_i}(s-\alpha_i)F(s)e^{\alpha_i t}.
\end{aligned}$$

EXAMPLE 6.33

Find $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$.

Solution. Write

$$\frac{1}{s^3(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+1},$$

which yields

$$\begin{aligned}
1 = \mathrm{A}s^2(s^2+1) &+ \mathrm{B}s(s^2+1)\\
&+ \mathrm{C}(s^2+1) + (\mathrm{D}s+\mathrm{E})s^3.
\end{aligned}$$

Putting $s=0$, we get C = 1. Comparing coefficients of s on both sides, we get B = 0. Comparing coefficients of s^2 on both sides, we get A + C = 0 which yields A = −C = −1. Comparing coefficients of s^3, we get B + E = 0 and so E = −B = 0. Comparing coefficients of x^4, we have A + D = 0 and so D = −A = 1. Thus

$$\frac{1}{s^3(s^2+1)} = -\frac{1}{s} + \frac{1}{s^3} + \frac{s}{s^2+1}.$$

Hence, by Example 6.1,

$$\mathrm{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \mathrm{L}^{-1}\left\{-\frac{1}{s}+\frac{1}{s^3}+\frac{s}{s^2+1}\right\}$$
$$= -1+\frac{1}{2}t^2+\cos t$$
$$= \frac{1}{2}(t^2+2\cos t-2).$$

EXAMPLE 6.34

Find $\mathrm{L}^{-1}\left\{\frac{2}{(s+1)\ (s^2+1)}\right\}$.

Solution. Write

$$\frac{2}{(s+1)\ (s^2+1)} = \frac{A}{s+1}+\frac{Bs+C}{s^2+1}.$$

Clearing fractions, we have

$$2 = \mathrm{A}(s^2+1)+(\mathrm{B}s+\mathrm{C})(s+1).$$

Putting $s=-1$, we have $\mathrm{A}=1$. Comparing the coefficient of s^2 and that of s^0, we get

$$0=\mathrm{A}+\mathrm{B}, \text{ which yields } \mathrm{B}=-1$$

and

$$2=\mathrm{A}+\mathrm{C}, \text{ which yields } \mathrm{C}=1.$$

Hence

$$\frac{2}{(s+1)(s^2+1)} = \frac{1}{s+1}+\frac{-s+1}{s^2+1}$$
$$= \frac{1}{s+1}-\frac{s}{s^2+1}+\frac{1}{s^2+1}.$$

Therefore,

$$\mathrm{L}^{-1}\left\{\frac{2}{(s+1)\ (s^2+1)}\right\} = e^{-t}-\cos t+\sin t.$$

EXAMPLE 6.35

Find $\mathrm{L}^{-1}\left\{\frac{s^2}{(s-1)\ (s+2)\ (s-3)}\right\}$.

Solution. The simple poles of the rational function F(s) are $s=1$, -2, and 3. Therefore,

$$f(t)=\mathrm{L}^{-1}\{\mathrm{F}(s)\} = \lim_{s\to 1}(s-1)\,F(s)e^t$$
$$+\lim_{s\to -2}(s+2)\,F(s)\,e^{-2t} \quad +\lim_{s\to 3}(s-3)\,F(s)\,e^{3t}$$
$$= -\frac{1}{6}e^t+\frac{4}{(-3)\ (-5)}e^{-2t}+\frac{9}{(2)\ (5)}e^{3t}$$
$$= -\frac{1}{6}e^t+\frac{4}{15}e^{-2t}+\frac{9}{10}e^{3t}.$$

EXAMPLE 6.36

Find $\mathrm{L}^{-1}\left\{\frac{4s+5}{(s-1)^2\ (s+2)}\right\}$.

Solution. Write

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s+2},$$

which gives

$$4\mathrm{s}+5=\mathrm{A}(s+2)(s-1)+\mathrm{B}(s+2)+\mathrm{C}(s-1)^2.$$

Putting $s=1$, we get $9=3\mathrm{B}$ and so $\mathrm{B}=3$. Equating the coefficients of s and s^2, we get $\mathrm{A}+\mathrm{B}-2\mathrm{C}=4$ and $\mathrm{A}+\mathrm{C}=0$. These equations give $\mathrm{C}=-\frac{1}{3}$ and $\mathrm{A}=\frac{1}{3}$. Thus

$$\frac{4x+5}{(s-1)^2(s+2)} = \frac{1}{3(s-1)}+\frac{3}{(s-1)^2}-\frac{1}{3(s+2)}.$$

Hence

$$\mathrm{L}^{-1}\left\{\frac{4s+5}{(s-1)^2\ (s+2)}\right\} = \frac{1}{3}e^t+3t\ e^t-\frac{1}{3}e^{-2t}.$$

EXAMPLE 6.37

Find $\mathrm{L}^{-1}\left\{\frac{s^2}{s^3+6s^2+11s+6}\right\}$.

Solution. We have

$$\mathrm{F}(s) = \frac{s^2}{s^3+6s^2+11s+6}.$$

Clearly $s=-1$, -2, and -3 are roots of the polynomial $s^3+6s^2+11s+10$. Hence the simple poles of F(s) are -1, -2, and -3. Therefore,

$$f(t)=\mathrm{L}^{-1}\{\mathrm{F}(s)\} = \lim_{s\to -1}(s+1)\,F(s)e^{-t}$$
$$+\lim_{s\to -2}(s+2)\,F(s)e^{-2t}+\lim_{s\to -3}(s+3)\,F(s)e^{-3t}$$
$$=\frac{1}{2}e^{-t}-4e^{-2t}+\frac{9}{2}e^{-3t}.$$

EXAMPLE 6.38

Find $\mathrm{L}^{-1}\left\{\frac{s^3}{s^4-a^4}\right\}$.

Solution. We have

$$\frac{s^3}{s^4-a^4} = \frac{s^3}{(s-a)\ (s+a)\ (s^2+a^2)}$$
$$= \frac{A}{s-a}+\frac{B}{s+a}+\frac{Cs+D}{s^2+a^2},$$

which yields

$$s^3 = A(s+a)(s^2+a^2) + B(s-a)(s^2+a^2) + (Cs+D)(s^2-a^2)$$
$$= A(s^3+as^2+a^2s+a^3) + B(s^3-as^2+a^2s-a^3) + C(s^3-sa^2) + D(s^2-a^2).$$

Putting $s = a, -a$, we get $A = B = \frac{1}{4}$. Comparing coefficients of s, and s^2, we get $C = \frac{1}{2}$, $D = 0$. Thus

$$\frac{s^3}{s^4-a^4} = \frac{1}{4(s-a)} + \frac{1}{4(s+a)} + \frac{s}{2(s^2+a^2)}.$$

Hence

$$L^{-1}\left\{\frac{s^3}{s^4-a^4}\right\} = \frac{1}{4}e^{at} + \frac{1}{4}e^{-at} + \frac{1}{2}\cos at$$
$$= \frac{1}{2}\cos at + \frac{1}{2}\left(\frac{e^{at}+e^{-at}}{2}\right)$$
$$= \frac{1}{2}[\cos at + \cosh at].$$

EXAMPLE 6.39

Find $L^{-1}\left\{\frac{s^2}{(s^2-a^2)\,(s^2-b^2)\,(s^2-c^2)}\right\}$.

Solution. Let

$$F(s) = \frac{s^2}{(s^2-a^2)\,(s^2-b^2)\,(s^2-c^2)} = \frac{s^2}{(s-a)\,(s+a)\,(s-b)\,(s+b)(s-c)\,(s+c)}.$$

The simple poles of $F(s)$ are $a, -a, b, -b, c$, and $-c$. Therefore,

$$L^{-1}\{F(s)\} = \lim_{s\to a}(s-a)F(s)e^{at} + \lim_{s\to -a}(s+a)F(s)\,e^{-at} + \lim_{s\to b}(s-b)F(s)e^{bt} + \lim_{s\to -b}(s+b)F(s)e^{-bt} + \lim_{s\to c}(s-c)F(s)e^{ct} + \lim_{s\to -c}(s+c)F(s)e^{-ct}$$

$$= \frac{a^2}{2a(a^2-b^2)(a^2-c^2)}e^{at} - \frac{a^2}{2a(a^2-b^2)\,(a^2-c^2)}e^{-at} + \frac{b^2}{2b(b^2-a^2)(b^2-c^2)}e^{bt} - \frac{b^2}{2b(b^2-a^2)\,(b^2-c^2)}e^{-bt} + \frac{c^2}{2c(c^2-a^2)\,(c^2-b^2)}e^{ct} - \frac{c^2}{2c(c^2-a^2)(c^2-b^2)}e^{-ct}$$

$$= \frac{a\,(e^{at}-e^{-at})}{2(a^2-b^2)\,(a^2-c^2)} + \frac{b\,(e^{bt}-e^{-bt})}{2(b^2-a^2)\,(b^2-c^2)} + \frac{c(e^{ct}-e^{-ct})}{2(c^2-a^2)\,(c^2-b^2)} = \frac{a\sinh at}{(a^2-b^2)\,(a^2-c^2)} + \frac{b\sinh bt}{(b^2-a^2)\,(b^2-c^2)} + \frac{c\sinh ct}{(c^2-a^2)\,(c^2-b^2)}.$$

EXAMPLE 6.40

Find $L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$.

Solution. Let

$$F(s) = \frac{2s^2-6s+5}{s^3-6s^2+11s-6}.$$

We observe that $s = 1$, $s = 2$, and $s = 3$ are the roots of the equation $s^3 - 6s^2 + 11s - 6 = 0$. Thus

$$F(s) = \frac{2s^2-6s+5}{(s-1)\,(s-2)\,(s-3)}.$$

Therefore,

$$L^{-1}\{F(s)\} = \lim_{s\to 1}(s-1)\,F(s)\,e^t + \lim_{s\to 2}(s-2)\,F(s)e^{2t} + \lim_{s\to 3}(s-3)\,F(s)\,e^{3t} = \frac{1}{2}\,e^t - e^{2t} + \frac{5}{2}e^{3t}.$$

EXAMPLE 6.41

Find $L^{-1}\left\{\frac{1}{s^3-a^3}\right\}$.

Solution. We have

$$F(s) = \frac{1}{s^3-a^3} = \frac{1}{(s-a)\,(s^2+as+a^2)} = \frac{A}{s-a} + \frac{Bs+C}{s^2+as+a^2}.$$

Therefore,

$$1 = A(s^2+as+a^2) + (Bs+C)(s-a).$$

Putting $s = a$, we have $A = \frac{1}{3a^2}$. Comparing the coefficients of constant terms and of s^2, we get respectively, $a^2A - aC = 1$ and $A + B = 0$ which in turn imply $B = -A = -\frac{1}{3a^2}$ and $C = -\frac{2}{3a}$. Thus

$$F(s) = \frac{1}{3a^2(s-a)} - \frac{1}{3a^2}\left(\frac{s+2a}{s^2+as+a^2}\right).$$

Hence

$$L^{-1}\{F(s)\}$$

$$=\frac{1}{3a^2}L^{-1}\left\{\frac{1}{s-a}\right\}-\frac{1}{3a^2}L^{-1}\left(\frac{\left(s+\frac{a}{2}\right)+\frac{3a}{2}}{\left(s+\frac{a}{2}\right)^2+\frac{3a^2}{4}}\right)$$

$$=\frac{1}{3a^2}e^{at}-\frac{1}{3a^2}\left[L^{-1}\left(\frac{s+\frac{a}{2}}{\left(s+\frac{a}{2}\right)^2+\frac{3a^2}{4}}\right)\right.$$

$$\left.+L^{-1}\left(\frac{\frac{3a}{2}}{\left(s+\frac{a}{2}\right)^2+\frac{3a^2}{4}}\right)\right]$$

$$=\frac{1}{3a^2}e^{at}-\frac{1}{3a^2}\left[e^{-\frac{at}{2}}\left(\cos\frac{\sqrt{3}\,a}{2}t+\sqrt{3}\sin\frac{\sqrt{3}\,a}{2}\,t\right)\right]$$

$$=\frac{1}{3a^2}\left[e^{at}-e^{-\frac{at}{2}}\left(\cos\frac{\sqrt{3}\,a}{2}\,t+\sqrt{3}\sin\frac{\sqrt{3}\,a}{2}\,t\right)\right].$$

EXAMPLE 6.42

Find $L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$.

Solution. Write

$$F(s)=\frac{4s+5}{(s-1)^2\,(s+2)}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s+2},$$

and so

$$4s+5=A(s-1)(s+2)+B(s+2)+C(s-1)^2.$$

Taking $s=1$, we get $B=3$. Similarly, taking $s=-2$, we get $C=-\frac{1}{3}$. Now equating coefficients of s^2, we have $A+C=0$ and so $A=-C=\frac{1}{3}$. Hence

$$F(s)=\frac{1}{3(s-1)}+\frac{3}{(s-1)^2}-\frac{1}{3(s+2)}.$$

Hence

$$L^{-1}\{F(s)\}=\frac{1}{3}e^t+3t\,e^t-\frac{1}{3}e^{-2t}.$$

EXAMPLE 6.43

Find $L^{-1}\left\{\frac{2s+1}{(s+2)^2(s-1)^2}\right\}$.

Solution. Write

$$F(s)=\frac{2s+1}{(s+2)^2\,(s-1)^2}$$

$$=\frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{C}{s-1}+\frac{D}{(s-1)^2}$$

and so

$$2s+1=A(s+2)(s-1)^2+B(s-1)^2$$
$$+C(s-1)(s+2)^2+D(s+2)^2.$$

Putting $s=1$ yields $D=\frac{1}{3}$. Putting $s=-2$, we have $B=-\frac{1}{3}$. Comparing coefficients of s^3, we have $A+C=0$ and comparing coefficient of s^2, we have $B+D-3C=0$. Thus $C=A=0$ and so

$$F(s)=\frac{2s-1}{(s+2)^2(s-1)^2}=-\frac{1}{3(s+2)^2}+\frac{1}{3(s-1)^2}.$$

Hence

$$L^{-1}\{F(s)\}=-\frac{1}{3}t\,e^{-2t}+\frac{1}{3}t\,e^t=\frac{1}{3}t(e^t-e^{-2t}).$$

6.4 HEAVISIDE'S EXPANSION THEOREM

Let $F(s)=\frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ have no common factors and the degree of $Q(s)$ is greater than the degree of $P(s)$. If $Q(s)$ has distinct zeros, then the following theorem gives us the inverse Laplace transform of $F(s)$.

Theorem 6.10. (Heaviside's Expansion Formula). Let $P(s)$ and $Q(s)$ be polynomials in s, where degree of $Q(s)$ is greater than that of $P(s)$. If $Q(s)$ has n distinct zeros $a_1, a_2, \ldots, a_n$, then

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\}=\sum_{i=1}^{n}\frac{P(a_i)}{Q'(a_i)}\;e^{a_i t}.$$

Proof: Since $Q(s)$ has n distinct zeros, it factorizes into n linear factors and so by partial fractions, we have

$$\frac{P(s)}{Q(s)}=\frac{A_1}{s-a_1}+\frac{A_2}{s-a_2}+\ldots+\frac{A_i}{s-a_i}+\ldots+\frac{A_n}{s-a_n}.$$

Multiplying throughout by $s-a_i$ and letting $s\to a_i$, we obtain

$$A_i=\lim_{s\to a_i}\frac{P(s)}{Q(s)}\;(s-a_i)=\lim_{s\to a_i}P(s)\left(\frac{s-a_i}{Q(s)}\right)$$

$$=\lim_{s\to a_i}P(s)\lim_{s\to a_i}\left(\frac{s-a_i}{Q(s)}\right)$$

$$=P(a_i)\lim_{s\to a_i}\frac{1}{Q'(s)}\quad\text{(using L'Hospital's rule)}$$

$$=\frac{P(a_i)}{Q'(a_i)}.$$

Hence

$$\frac{P(s)}{Q(s)}=\sum_{i=1}^{n}\frac{P(Q_i)}{Q'(a_i)}\frac{1}{(s-a_i)}$$

and so

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{i=1}^{n} \frac{P(a_i)}{Q'(a_i)} L^{-1}\left\{\frac{1}{s-a_i}\right\}$$
$$= \sum_{i=1}^{n} \frac{P(a_i)}{Q'(a_i)}\, e^{a_i t}.$$

EXAMPLE 6.44

Find $L^{-1}\left\{\frac{s}{(s-3)\,(s^2+4)}\right\}$ using Heaviside's expansion formula.

Solution. Let

$$F(s) = \frac{P(s)}{Q(s)} = \frac{s}{(s-3)\,(s^2+4)}.$$

Thus

$$P(s) = s, \quad Q(s) = (s-3)(s^2+4)$$
$$= s^3 - 3s^2 + 4s - 12$$

and

$$Q'(s) = 3s^2 - 6s + 4.$$

Also roots of $Q(s) = 0$ are 3, $2i$, and $-2i$. Therefore, by Heaviside's expansion formula

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{i=1}^{3} \frac{P(a_i)}{Q'(a_i)}\, e^{a_i t} = \frac{P(3)}{Q'(3)} e^{3t}$$
$$+ \frac{P(2i)}{Q'(2i)} e^{2it} + \frac{P(-2i)}{Q'(-2i)} e^{-2it}$$
$$= \frac{3}{13} e^{3t} + \frac{2i}{-8-12i} e^{2it} - \frac{2i}{-8+12i} e^{-2it}$$
$$= \frac{3}{13} e^{3t} - \frac{(3+2i)}{26}(\cos 2t + i\sin 2t)$$
$$+ \frac{2i-3}{26}(\cos 2t - i\sin 2t)$$
$$= \frac{3}{13} e^{3t} - \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t.$$

EXAMPLE 6.45

Find $L^{-1}\left\{\frac{3s+7}{(s-3)\,(s+1)}\right\}$ using Heaviside's expansion formula.

Solution. We have

$$F(s) = \frac{P(s)}{Q(s)} = \frac{3s+7}{(s-3)\,(s+1)} = \frac{3s+7}{s^2-2s-3}.$$

Here $P(s) = 3s + 7$, $Q(s) = s^2 - 2s - 3$, $Q'(s) = 2s - 2$, and zeros of $Q(s)$ are 3 and -1. Hence, by Heaviside's expansion formula,

$$L^{-1}\{F(s)\} = \sum_{i=1}^{2} \frac{P(a_i)}{Q'(a_i)}\, e^{a_i t} = \frac{P(3)}{Q'(3)} e^{3t} + \frac{P(-1)}{Q'(-1)} e^{-t}$$
$$= \frac{16}{4} e^{3t} + \frac{4}{-4} e^{-t} = 4e^{3t} - e^{-t}.$$

6.5 SERIES METHOD TO DETERMINE INVERSE LAPLACE TRANSFORM

Let $L\{f(t)\} = F(s)$. In certain situation, we observed that Laplace transform of $f(t)$ can be obtained by expressing $f(t)$ in terms of power series and then by taking transform term-by-term. The same technique proves useful in finding inverse Laplace transforms. Thus, if

$$F(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3} + \ldots,$$

then

$$f(t) = a_0 + a_1 + a_2 \frac{t^2}{2!} + \ldots,$$

where the series on the right-hand side may be summable to a known function.

EXAMPLE 6.46

Find $L^{-1}\left\{\frac{e^{-1/s}}{s}\right\}$.

Solution. Expansion in series yields

$$\frac{1}{s} e^{-1/s} = \frac{1}{s}\left[1 - \frac{1}{s} + \frac{1}{2!\, s^2} - \frac{1}{3!\, s^3} + \frac{1}{4!\, s^4} - \ldots\right]$$
$$= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{2!\, s^3} - \frac{1}{3! s^4} + \ldots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, s^{n+1}}.$$

Inversion of the series term-by-term yields

$$L^{-1}\{F(s)\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L^{-1}\left\{\frac{1}{s^{n+1}}\right\}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(n!)^2} = J_0(2\sqrt{t}),$$

where $J_0(t)$ is the Bessel function of order zero.

EXAMPLE 6.47

Find $L^{-1}\left\{\frac{1}{s}\sin\frac{1}{s}\right\}$.

Solution. We have

$$\frac{1}{s}\sin\frac{1}{s} = \frac{1}{s}\left[\frac{1}{s} - \frac{(1/s)^3}{3!} - \frac{(1/s)^5}{5!} - \frac{(1/s)^7}{7!} + \ldots\right]$$
$$= \frac{1}{s^2} - \frac{1}{s^4 .3!} + \frac{1}{s^6 .5!} - \frac{1}{s^8 7!} + \ldots$$

Therefore,

$$L^{-1}\left\{\frac{1}{s}\sin\frac{1}{s}\right\} = L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{3!}L^{-1}\left\{\frac{1}{s^4}\right\} + \frac{1}{5!}L^{-1}\left\{\frac{1}{s^6}\right\} - \frac{1}{7!}L^{-1}\left\{\frac{1}{s^8}\right\} + \ldots$$

$$= t - \frac{1}{(3!)^2}t^3 - \frac{1}{(5!)^2}t^5 - \frac{1}{(7!)^2}t^7 + \ldots$$

EXAMPLE 6.48

Find $L^{-1}\{\tan^{-1}\frac{1}{s}\}$.

Solution. We have

$$\tan^{-1}\left(\frac{1}{s}\right) = \frac{1/s}{1} - \frac{(1/s)^3}{3} + \frac{(1/s)^5}{5} - \frac{(1/s)^7}{7} + \ldots$$

$$= \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \ldots$$

Therefore, inversion of the series term-by-term yields

$$L^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\} = 1 - \frac{1}{3}.\frac{t^2}{2!} + \frac{1}{5}.\frac{t^4}{4!} - \frac{1}{7}.\frac{t^6}{6!} + \ldots$$

$$= 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \ldots$$

$$= \frac{1}{t}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots\right) = \frac{\sin t}{t}.$$

EXAMPLE 6.49

Find $L^{-1}\left\{\frac{1}{\sqrt{s+a}}\right\}$.

Solution. Let

$$F(s) = \frac{1}{\sqrt{s+a}} = \frac{1}{\sqrt{s}}\left(1 + \frac{a}{s}\right)^{-1/2}$$

$$= \frac{1}{\sqrt{s}}\left[1 - \frac{1}{2}\left(\frac{a}{s}\right) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(\frac{a}{s}\right)^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\left(\frac{a}{s}\right)^3 + \ldots\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n 1.3.5\ldots(2n-1)a^n}{2^n\, n!\, s^{n+(1/2)}} |s| > |a|.$$

Hence

$$L^{-1}\{F(s)\} = \sum_{n=0}^{\infty}\frac{(-1)^n 1.3.5\ldots(2n-1)a^n t^{n-(1/2)}}{2^n\, n!\Gamma(n+1/2)}$$

$$= \frac{1}{\sqrt{t}}\sum_{n=0}^{\infty}\frac{(-1)^n 1.3.5\ldots.(2n-1)a^n t^n}{2^n\, n!\, \Gamma(n+1/2)}.$$

But, using $\Gamma(v+1) = v\Gamma(v)$, we have

$$\Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\left(\frac{1.3.5\ldots.(2n-1)}{2^n}\right)$$

$$= \sqrt{\pi}\left(\frac{1.3.5\ldots.(2n-1)}{2^n}\right).$$

Hence

$$L^{-1}\{F(s)\} = \frac{1}{\sqrt{t}}\sum_{n=0}^{\infty}\frac{(-1)^n a^n t^n}{\sqrt{\pi}\, n!} = \frac{1}{\sqrt{\pi t}}\, e^{-at}.$$

6.6 CONVOLUTION THEOREM

The convolution of two functions plays an important role in a number of physical applications. It is generally expedient to resolve a Laplace transform into the product of two transforms when the inverse transform of both transforms are known.

Definition 6.2. The *convolution* (or *Faltung*) of two given functions $f(t)$ and $g(t)$, $t > 0$ is defined by the integral

$$(f*g)(t) = \int_0^t f(u)g(t-u)\, du,$$

which, of course, exists if f and g are piecewise continuous.

The integral $\int_0^t f(u)\, g(t-u)\, du$ represents a *superposition* of effects of magnitude $f(u)$ occurring at time $t = u$ for which $g(t-u)$ is the *influence function* or *response* of a system to a unit impulse defined at time $t = u$.

If we substitute $v = t - u$, then

$$(f*g)(t) = \int_0^t g(v)\, f(t-v)dv = (g*f)(t).$$

Hence, the convolution is *commutative*. Also, it follows from the definition that

$a(f*g) = af*g = f*ag$, a constant

$f*(g+h) = (f*g) + (f*h)$ (distributive property).

Further, we note that

$$[f*(g*h)](t)$$

$$= \int_0^t f(u)(g*h)(t-u)du$$

$$= \int_0^t f(u)\left(\int_0^{t-u} g(v)h(t-u-v)\, dv\right) du$$

$$= \int_0^t f(u) \int_u^t g(z-u)\ h(t-z)dz)du, \quad v = z-u$$

$$= \int_0^t \left(\int_0^z f(u)g(z-u)\ du \right) h(t-z)\ dz$$

$$= [(f*g)*h](t).$$

Hence

$$f*(g*h) = (f*g)*h,$$

and so *convolution is associative*.

EXAMPLE 6.50

Evaluate $\cos t * \sin t$.

Solution. By definition,

$$\cos t * \sin t = \int_0^t \cos u \sin(t-u)du$$

$$= \int_0^t \frac{1}{2}[\sin t + \sin(t-2u)]\ du$$

$$= \frac{1}{2}\sin t[u]_0^t + \frac{1}{4}[\cos(t-2u)]_0^t$$

$$= \frac{1}{2}t\sin t + \frac{1}{4}[\cos(-t) - \cos t] = \frac{1}{2}\ t\sin t.$$

EXAMPLE 6.51

Evaluate $e^t * t$.

Solution. By definition of convolution,

$$e^t * t = \int_0^t e^u(t-u)\ du$$

$$= [t\ e^u]_0^t - [(ue^u - e^u)]_0^t = e^t - t - 1.$$

Convolution is very significant in the sense that the Laplace transform of the convolution of two functions is the product of their respective Laplace transforms. We prove this assertion in the form of the following theorem.

Theorem 6.11. (Convolution Theorem). If f and g are piecewise continuous on $[0, \infty)$ and of exponential order γ, then

$$L\{(f*g)(t)\} = L\{f(t)\}.L\{g(t)\} = F(s)G(s),$$

or equivalently,

$$L^{-1}\{F(s)G(s)\} = (f*g)(t).$$

Proof: Using the definition of the Laplace transform, we have

$$\mathrm{L}\{(f*g)(t)\} = \int_0^\infty e^{-st} \int_0^t f(u)\ g(t-u)\ du\ dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(u)\ g(t-u)du\ dt.$$

The domain of this double integral is an infinite wedge bounded by $t = 0$, $t = \infty$, $u = 0$, and $u = t$ as displayed in Figure 6.1. Due to hypotheses on f and g, the double integral on the right hand side converges absolutely and so we can perform change of order of integration.

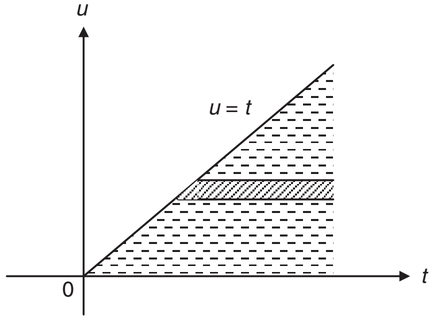

Figure 6.1 Region of Integration

We take a strip parallel to the axis of t. Then t varies from u to ∞ and u varies from 0 to ∞. Hence

$$\mathrm{L}\{f(t) *‘g(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u)g(t-u)\,dt\,du$$

$$= \int_0^\infty \int_u^\infty e^{-su}.e^{-s(t-u)} f(u)\,g(t-u)\,dt\,du$$

$$= \int_0^\infty e^{-su} f(u) \int_u^\infty e^{-s(t-u)} g(t-u)dt\,du.$$

Implementing the change of variable $t - u = v$ in the inner integral, we get

$$\mathrm{L}\{f(t)*g(t)\} = \int_0^\infty e^{-su} f(u) \int_0^\infty e^{-sv} g(v)dv\ du$$

$$= \int_0^\infty e^{-su} f(u)G(s)du$$

$$= \mathrm{G}(s) \int_0^\infty e^{-su} f(u)\ du$$

$$= \mathrm{G}(s)\mathrm{F}(s) = \mathrm{F}(s)\mathrm{G}(s).$$

Consequently,

$$L^{-1}\{F(s)G(s)\} = f(t) * g(t).$$

EXAMPLE 6.52

Find $L\{e^{at} * e^{bt}\}$.

Solution. By Convolution Theorem,

$$L\{e^{at} * e^{bt}\} = L\{e^{at}\}.L\{e^{bt}\} = \frac{1}{(s-a)\ (s-b)}.$$

EXAMPLE 6.53

Find $L^{-1}\left\{\frac{1}{(s-a)\ (s-b)}\right\}$.

Solution. By Convolution Theorem

$$L^{-1}\left\{\frac{1}{(s-a)\ (s-b)}\right\} = e^{at} * e^{bt}$$

$$\int_0^t e^{ut}e^{b(t-u)}du = \frac{e^{at}-e^{bt}}{a-b}, a \neq b.$$

EXAMPLE 6.54

Using Convolution Theorem, find $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$.

Solution. Let $F(s) = \frac{1}{s^3}$, $G(s) = \frac{1}{s^2+1}$ and so $f(t) = \frac{1}{2}t^2$ and $g(t) = \sin t$.

By Convolution Theorem and integration by parts, we have

$$L^{-1}\{F(s)G(s)\}$$

$$= f * g = g * f = \frac{1}{2}\int_0^t \sin u(t-u)^2 du$$

$$= \frac{1}{2}\left\{[-(t-u)^2\cos u]_0^t - \int_0^t 2(t-u)\cos u\ du\right\}$$

$$= \frac{1}{2}\left[t^2 - 2\left\{[(t-u)\sin u]_0^t + \int_0^t \sin u\ du\right\}\right]$$

$$= \frac{1}{2}[t^2 + 2\cos t - 2].$$

EXAMPLE 6.55

Using Convolution Theorem, find $L^{-1}\left\{\frac{1}{s^2(s-1)}\right\}$.

Solution. Let $F(s) = \frac{1}{s^2}$ and $G(s) = \frac{1}{s-1}$. Then, $f(t) = t$ and $g(t) = e^t$. Therefore, by Convolution Theorem,

$$L^{-1}\{F(s)G(s)\} = f * g = t * e^t$$

$$= \int_0^t e^u(t-u)\ du = e^t - t - 1.$$

EXAMPLE 6.56

Use Convolution Theorem to find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$. (see also Example 6.32).

Solution. Let

$$\frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2} = F(s)G(s).$$

Then

$$f(t) = L^{-1}\{F(s)\} = \cos at \quad \text{and}$$

$$g(t) = L^{-1}\{G(s)\} = \frac{\sin at}{a}.$$

By Convolution Theorem,

$$L^{-1}\{F(s)G(s)\} = f * g = \int_0^t \cos au \frac{\sin a(t-u)}{a}\ du$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin a(t-2u)]\ du$$

$$= \frac{1}{2a}\sin at[u]_0^t + \frac{1}{4a^2}[\cos a(t-2u)]_0^t$$

$$= \frac{1}{2a}t\sin at + \frac{1}{4a^2}[\cos a(-t) - \cos at]$$

$$= \frac{1}{2a}t\sin at.$$

EXAMPLE 6.57

Using Convolution Theorem, find $L - 1\left\{\frac{1}{s^2(s^2+a^2)}\right\}$.

Solution. Let

$$\frac{1}{s^2(s^2+a^2)} = \frac{1}{s^2}\cdot\frac{1}{(s^2+a^2)} = F(s)G(s).$$

Then

$$f(t) = L^{-1}\{F(s)\} = t \quad \text{and}$$

$$g(t) = L^{-1}\{G(s)\} = \frac{\sin at}{a}.$$

By Convolution Theorem,

$$L^{-1}\{F(s)G(s)\}$$

$$= \frac{1}{a}\sin at * t = \frac{1}{a}\int_0^t (t-u)\sin au\, du$$

$$= \frac{1}{a}\left\{-\left[\frac{1}{a}(t-u)\cos au\right]_0^t - \int_0^t \frac{\cos au}{a}\, du\right\}$$

$$= \frac{1}{a}\left\{\frac{t}{a} - \left[\frac{\sin au}{a^2}\right]_0^t\right\} = \frac{1}{a}\left[\frac{t}{a} - \frac{\sin at}{a^2}\right]$$

$$= \frac{1}{a^2}\left[t - \frac{1}{a}\sin at\right].$$

EXAMPLE 6.58

Using Convolution Theorem, find $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$.

Solution. Let

$$\frac{1}{s^2(s+1)^2} = \frac{1}{s^2}\cdot\frac{1}{(s+1)^2} = F(s)G(s).$$

Then

$$f(t) = L^{-1}\{F(s)\} = t \quad \text{and}$$

$$g(t) = L^{-1}\{G(s)\} = t\, e^{-t}$$

So, by Convolution Theorem, we have

$$L^{-1}\{F(s)G(s)\} = g * f$$

$$= \int_0^t (ue^{-u})\,(t-u)\, du = \int_0^t (tu-u^2)\, e^{-u}\, du$$

$$= -[(tu-u^2)e^{-u}]_0^t - \int_0^t -(t-2u)e^{-u}du$$

$$= \int_0^t (t-2u)\, e^{-u}du = -[(t-2u)e^{-u}]_0^t - 2\int_0^t e^{-u}du$$

$$= te^{-t} + t - 2\left[\frac{e^{-u}}{-1}\right]_0^t = te^{-t} + t + 2e^{-t} - 2$$

$$= (t+2)e^{-t} + t - 2.$$

EXAMPLE 6.59

Using Convolution Theorem, find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$.

Solution. Let

$$\frac{1}{(s^2+a^2)^2} = \frac{1}{(s^2+a^2)}\cdot\frac{1}{(s^2+a^2)} = F(s)G(s).$$

Then

$$f(t) = g(t) = \frac{\sin at}{a}.$$

Therefore, by Convolution Theorem,

$$L^{-1}\{F(s)G(s)\}$$

$$= f * g = \frac{1}{a^2}\int_0^t \sin au \sin a(t-u)du$$

$$= \frac{1}{a^2}\int_0^t \frac{1}{2}[\cos a(2u-t) - \cos at]\, du$$

$$= \frac{1}{2a^2}\left[\int_0^t \cos a(2u-t)du - \int_0^t \cos atdu\right]$$

$$= \frac{1}{2a^2}\left\{\left[\frac{\sin a(2u-t)}{2a}\right]_0^t - \cos at[u]_0^t\right\}$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} - \frac{\sin a(-t)}{2a} - t\cos at\right]$$

$$= \frac{1}{2a^3}[\sin at - at\cos at].$$

EXAMPLE 6.60

Use Convolution Theorem to find $L^{-1}\left\{\frac{a^2}{s(s+a)^2}\right\}$.

Solution. Let

$$\frac{a^2}{s(s+a)^2} = \frac{1}{s}\cdot\frac{a^2}{(s+a)^2} = F(s)G(s).$$

Then

$$f(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1.$$

To find $g(t) = L^{-1}\left\{\frac{a^2}{s(s+a)^2}\right\}$, we note that

$$\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{-1}{(s+a)^2}.$$

Therefore,

$$L^{-1}\left\{\frac{-1}{(s+a)^2}\right\} = -t\, L^{-1}\left\{\frac{1}{s+a}\right\} = -t\, e^{-at}$$

and so

$$L^{-1}\left\{\frac{a^2}{(s+a)^2}\right\} = a^2\ t\ e^{-at}.$$

Application of Convolution Theorem yields

$$L^{-1}\{F(s)G(s)\} = \int_0^t a^2 u e^{-au} du = a^2 \int_0^t u\, e^{-au} du$$

$$= a^2 \left\{ \left[\frac{u\, e^{-au}}{-a} \right]_0^t - \int_0^t \frac{e^{-au}}{-a}\, du \right\}$$

$$= a^2 \left\{ \frac{te^{-at}}{-a} - \left[\frac{e^{-au}}{a^2} \right]_0^t \right\}$$

$$= -at\, e^{-at} - e^{-at} + 1$$

$$= 1 - e^{-at}(at + 1).$$

EXAMPLE 6.61

Find $L^{-1}\left\{\frac{1}{\sqrt{s}(s-1)}\right\}$ using Convolution Theorem.

Solution. Let

$$\frac{1}{\sqrt{s}(s-1)} = \frac{1}{\sqrt{s}} \cdot \frac{1}{s-1} = F(s)G(s).$$

Then

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{\sqrt{\pi t}} \quad \text{and}$$

$$g(t) = L^{-1}\{G(s)\} = e^t.$$

Hence Convolution Theorem yields

$$L^{-1}\left\{\frac{1}{\sqrt{s}(s+1)}\right\} = L^{-1}\{F(s)G(s)\} = f * g$$

$$= \int_0^t \frac{1}{\sqrt{\pi u}} e^{(t-u)} du = \frac{e^t}{\sqrt{\pi}} \int_0^t \frac{e^{-u}}{\sqrt{u}} du$$

$$= \frac{2e^t}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-v^2} dv, u = v^2$$

$$= e^t \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-v^2} dv = e^t erf \sqrt{t},$$

where erf$\sqrt{t}$ is the *error function*.

EXAMPLE 6.62

Using Convolution Theorem, establish Euler's formula for Beta function:

$$\beta(a, b) = \frac{\Gamma(a)\ \Gamma(b)}{\Gamma(a+b)}.$$

Solution. We know that Beta function is defined by

$$\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\ dx.$$

Let

$$f(t) = t^{a-1}, g(t) = t^{b-1}, a, b > 0.$$

Then

$$(f * g)(t) = \int_0^t u^{a-1}(t-u)^{b-1} du.$$

Putting $u = vt$, we get

$$(f * g)(t) = t^{a+b-1} \int_0^1 v^{a-1}(1-v)^{b-1}\ dv$$

$$= t^{a+b-1}\beta(a, b).$$

But by Convolution Theorem,

$$L\{f * g\} = L\{f(t)\}L\{g(t)\}.$$

Therefore,

$$L\{t^{a+b-1}\beta(a, b)\} = L\{t^{a-1}\}\ L\{t^{b-1}\}$$

$$= \frac{\Gamma(a)}{s^a} \cdot \frac{\Gamma(b)}{s^b} = \frac{\Gamma(a)\ \Gamma(b)}{s^{a+b}}$$

and so

$$t^{a+b-1}\beta(a, b) = L^{-1}\left\{\frac{\Gamma(a)\ \Gamma(b)}{s^{a+b}}\right\}$$

$$= \Gamma(a)\ \Gamma(b) L^{-1}\left\{\frac{1}{s^{a+b}}\right\}$$

$$= \Gamma(a)\ \Gamma(b) \frac{t^{a+b-1}}{\Gamma(a+b)},$$

which implies

$$\beta(a, b) = \frac{\Gamma(a)\ \Gamma(b)}{\Gamma(a+b)},$$

the required Euler's Formula for Beta function.

Remark 6.4. It also follows from Example 6.62 that

$$L\left\{\int_0^t u^{a-1}(t-u)^{b-1} du\right\} = \frac{\Gamma(a)\ \Gamma(b)}{s^{a+b}}.$$

EXAMPLE 6.63

Using Convolution Theorem, show that

$$J_0(t) = \frac{2}{\pi} \int_0^1 \frac{\cos tx}{\sqrt{1-x^2}}\ dx,$$

where J_0 represents Bessel's function of order zero.

Solution. We know that

$$\mathrm{L}\{\mathrm{J}_0(t)\} = \frac{1}{\sqrt{s^2+1}} = \frac{1}{\sqrt{s+i}}\cdot\frac{1}{\sqrt{s-i}}$$

and so

$$J_0(t) = L^{-1}\left\{\frac{1}{\sqrt{s+i}}\cdot\frac{1}{\sqrt{s-i}}\right\} = L^{-1}\{F(s)G(s)\},$$

where

$$\mathrm{F}(s) = \frac{1}{\sqrt{s+i}},\ \mathrm{G}(s) = \frac{1}{\sqrt{s-i}}.$$

Since

$$L^{-1}\left\{\frac{1}{\sqrt{s+a}}\right\} = \frac{1}{\sqrt{\pi t}}e^{-at},$$

we have

$$f(t) = \frac{1}{\sqrt{\pi t}}e^{-it} \quad \text{and} \quad g(t) = \frac{1}{\sqrt{\pi t}}\ e^{it}.$$

Hence, by Convolution Theorem

$$J_0(t) = \mathrm{L}^{-1}\{\mathrm{F}(s)\mathrm{G}(s)\} = f * g$$

$$= \int_0^t \frac{1}{\sqrt{\pi u}}\ e^{-iu}\left(\frac{1}{\sqrt{\pi}(t-u)^{1/2}}\ e^{i(t-u)}\right) du$$

$$= \frac{1}{\pi}\int_0^t \frac{e^{i(t-2u)}}{\sqrt{u(t-u)}}\ du.$$

Putting $u = tv$, we get

$$J_0(t) = \frac{1}{\pi}\int_0^1 \frac{e^{it}(1-2v)}{\sqrt{v(1-v)}}\ dv.$$

Now putting $x = 1 - 2v$, we have

$$J_0(t) = \frac{1}{\pi}\int_{-1}^1 \frac{e^{itx}}{\sqrt{1-x^2}}\ dx$$

$$= \frac{1}{\pi}\int_{-1}^1 \frac{\cos tx}{\sqrt{1-x^2}}dx + \frac{i}{\pi}\int_{-1}^1 \frac{\sin tx}{\sqrt{1-x^2}}\ dx$$

$$= \frac{2}{\pi}\int_0^1 \frac{\cos tx}{\sqrt{1-x^2}}\ dx,$$

the second integral vanishes because sin x is odd.

6.7 COMPLEX INVERSION FORMULA

The complex inversion formula is a technique for computing directly the inverse of a Laplace transform. For this technique, we require the parameter s to be a complex variable. The contour integration is the main tool applied in this technique.

Let f be a continuous function possessing a Laplace transform and defined in $(-\infty, \infty)$ such that $f(t) = 0$ for $t < 0$. Let $s = x + iy$ be a complex variable. Then

$$\mathrm{L}\{f(t)\} = \mathrm{F}(s) = \int_0^\infty e^{-st}f(t)\ dt$$

$$= \int_{-\infty}^\infty e^{-(x+iy)t}f(t)\ dt$$

$$= \int_{-\infty}^\infty e^{-iyt}(e^{-xt}f(t))\ dt.$$

The integral on the right is the Fourier transform of the function $e^{-xt}f(t)$. Thus the *Laplace transform of $f(t)$ is the Fourier transform of the function $e^{-xt}f(t)$.*

Suppose that f and its derivative f' are piecewise continuous on $(-\infty, \infty)$, that is, both f and f' are continuous in any finite interval except possibly for a finite number of jump discontinuities. Suppose, further, that f is absolutely integrable, that is, $\int_{-\infty}^{\infty} |f(t)|dt < \infty$. Then Fourier integral theorem asserts that

$$\frac{f(t+)+f(t-)}{2} = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(u)\ e^{is(t-u)}du\ ds,$$

where the integration with respect to s is in the Cauchy principal value sense.

If f were continuous on $(-\infty, \infty)$, then $\frac{f(t+)+f(t-)}{2}$ converts to $f(t)$ in the above expression. Further, in terms of Fourier transform of $f(t)$, the above expression becomes

$$\frac{f(t+)+f(t-)}{2} = \frac{1}{2\pi}\int_{-\infty}^\infty F(s)e^{-ist}ds,$$

where $\mathrm{F}(s) = \int_{-\infty}^{\infty} e^{-isu}f(u)du$ is Fourier transform of $f(t)$.

Theorem 6.12. (Complex Inversion Formula). Let f be continuous on $[0, \infty)$, $f(t) = 0$ for $t < 0$, f be of exponential order γ and f' be piecewise continuous on $[0, \infty)$. If $L\{f(t)\} = F(s)$ and the real number γ is so chosen that all the singularities of $F(s)$ lie in the left of the line Re $(s) > \gamma$, then

$$f(t) = \mathrm{L}^{-1}\{\mathrm{F}(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds, \quad t > 0.$$

Proof: By definition of Laplace transform,

$$\mathrm{F}(s) = \int_0^\infty e^{-su}f(u)\,du.$$

We have then

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)ds = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT}\int_0^\infty e^{st}e^{-su}f(u)\,du\,ds.$$

Let $s = \gamma + iy$ so that $ds = idy$. Therefore,

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)ds = \lim_{T\to\infty}\frac{1}{2\pi}e^{\gamma t}\int_{-T}^{T} e^{iyt}dy\int_0^\infty e^{-iyu}e^{-\gamma u}f(u)du$$

$$= \frac{1}{2\pi}e^{\gamma t}\int_{-\infty}^{\infty}\int_0^\infty e^{iy(t-u)}(e^{-\gamma u}f(u))\,du\,dy.$$

The given hypothesis and Theorem 6.1 imply that $L\{f(t)\}$ converges absolutely for Re $(s) > \gamma$, that is,

$$\int_0^\infty |e^{-st}f(t)|dt = \int_{-\infty}^{\infty} e^{-xt}|f(t)|dt < \infty, x > \gamma.$$

Thus $e^{-xt}f(t)$ is absolutely integrable and so Fourier integral theorem asserts that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{-\gamma u}f(u))e^{i\,y(t-u)}du\,dy$$

$$= e^{-\gamma t}f(t) \text{ for } t > 0.$$

Hence

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)ds = \frac{1}{2\pi}.2\pi e^{-\gamma t}f(t)\,e^{\gamma t}$$

$$= f(t), t > 0,$$

and so

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds, \quad t > 0.$$

Remark 6.5. The expression

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds$$

is called *complex inversion formula, Bromwich integral formula*, *Fourier–Mellin inversion formula*, or *fundamental theorem of Laplace transform*. In practice, the integral in the above expression is evaluated by considering the contour integration along the contour Γ_R, ABCDEA, known as *Bromwich contour* shown in the Figure 6.2.

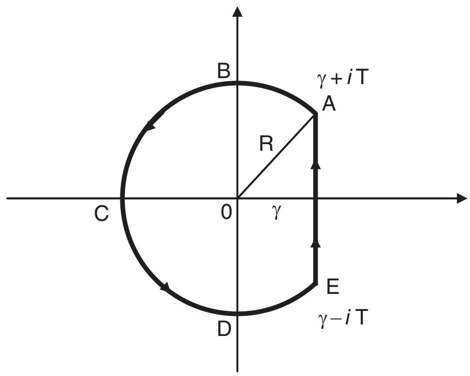

Figure 6.2 Bromwich Contour

The vertical line at γ is known as *Bromwich line*. Thus the contour Γ_R consists of arc C_R (ABCDE) of radius R and centre at the origin and the Bromwich line EA. Thus,

$$\frac{1}{2\pi i}\int_{\Gamma_R} e^{st}F(s)\,ds = \frac{1}{2\pi i}\int_{C_R} e^{st}F(s)\,ds$$

$$+\frac{1}{2\pi i}\int_{EA} e^{st}F(s)\,ds.$$

Since F(s) is analytic for Re(s) $= x > \gamma$, all singularities of F(s) must lie to the left of the Bromwich line. Thus, by Cauchy residue theorem, we have

$$\frac{1}{2\pi i}\int_{\Gamma_R} e^{st}F(s)\,ds = \sum_{k=1}^{n}\text{Res }(z_k),$$

where Res(z_k) is the residue of the function at the pole $s = z_k$. Since $e^{st} \neq 0$, multiplying F(s) by e^{st} does not affect the status of the poles z_k of F(s).

If we can show that

$$\lim_{R\to\infty}\int_{C_R} e^{st}F(s)\,ds = 0,$$

then letting R → ∞, we get

$$f(t) = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{ts}F(s)\,ds = \sum_{k=1}^{n}\text{Res}(z_k),$$

and so inverse function f can be determined.

The following theorem shows that *Laplace transform is one-to-one*.

Theorem 6.13. Let $f(t)$ and $g(t)$ be two piecewise smooth functions of exponential order and let $F(s)$ and $G(s)$ be the Laplace transforms of $f(t)$ and $g(t)$ respectively. If $F(s) = G(s)$ in a half-place $Re(s) > \gamma$, then $f(t) = g(t)$ at all points where f and g are continuous.

Proof: Suppose f and g are continuous at $t \in$ R. By complex inversion formula, we have

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds,$$

$$g(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}G(s)\,ds.$$

Since F(s) = G(s), it follows that $f(t) = g(t)$. Thus L$\{f(t)\}$ = L$\{g(t)\}$ implies that $f(t) = g(t)$ and so Laplace operator is one-to-one.

Generally, we see that most of the Laplace transforms satisfy the growth restriction

$$|\text{F}(s)| \le \frac{M}{|s|^p}$$

for all sufficiently large values of $|s|$ and some $p > 0$. Obviously, F(s) → 0 as $|s| \to \infty$. Therefore, the following result (stated without proof) is helpful.

Theorem 6.14. Let for s on C_R, $F(s)$ satisfies the growth restriction

$$|F(s)| \le \frac{M}{|s|^p} \text{ for } p > 0, \text{all } R > R_0.$$

Then

$$\lim_{R\to\infty}\int_{C_R} e^{st}F(s)ds = 0, t > 0.$$

EXAMPLE 6.64

Show that F(s) = $\frac{s}{s^2-a^2}$ satisfies growth restriction condition.

Solution. We have

$$\text{F}(s) = \frac{s}{s^2-a^2},$$

and so

$$|\text{F}(s)| \le \frac{|s|}{|s^2-a^2|} \le \frac{|s|}{|s|^2-|a|^2}.$$

If $|s| \ge 2|a|$, then $|a|^2 \le \frac{|s|^2}{4}$ and so $|s|^2 - |a|^2 \ge \frac{3}{4}|s|^2$ and we have

$$|\text{F}(s)| \le \frac{4/3}{|s|}.$$

EXAMPLE 6.65

Find the Laplace transform of $f(t)$ = cosh at and verify the inversion formula.

Solution. By Example 6.6, we have F(s) = $\frac{s}{s^2-a^2}$. The function F(s) is analytic except at poles $s = a$ and $s = -a$. The Bromwich contour is shown in the Figure 6.3.

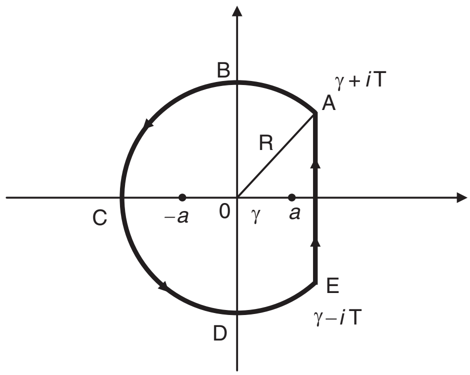

Figure 6.3

By inversion formula, we have

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\left(\frac{s}{s^2-a^2}\right)ds$$

$$= \frac{1}{2\pi i}\int_{\Gamma_R}\frac{se^{st}}{s^2-a^2}\,ds.$$

Further, F(s) satisfies growth restriction condition. Therefore, integral over contour C_R (arc ABCDE) tends to zero as R $\to \infty$. Now

$$\text{Res}(a) = \lim_{s\to a}(s-a)\,e^{st}F(s) = \lim_{s\to a}\frac{se^{st}}{s+a} = \frac{ae^{at}}{2a},$$

$$\text{Res}(-a) = \lim_{s\to(-a)}(s+a)e^{st}F(s) = \lim_{s\to(-a)}\frac{se^{st}}{s-a} = \frac{e^{-at}}{-2a}(-a).$$

Hence

$$\text{L}^{-1}\{\text{F}(s)\} = \frac{e^{at}+e^{-at}}{2} = \cosh at,$$

and so inversion formula is verified.

EXAMPLE 6.66

Find $L^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\}$, $s > 0$ using inversion formula.

Solution. We have

$$\text{F}(s) = \frac{\omega}{s^2+\omega^2},\, s > 0$$

The function F(s) has two simple poles at $s = \pm\, i\omega$. By inversion formula

$$\text{L}^{-1}\{\text{F}(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\left(\frac{\omega}{s^2+\omega^2}\right)ds$$

$$= \frac{\omega}{2\pi i}\int_{\Gamma_R}\frac{e^{st}}{s^2+\omega^2}\,ds,$$

where Γ_R is the Bromwich contour shown in the Figure 6.4:

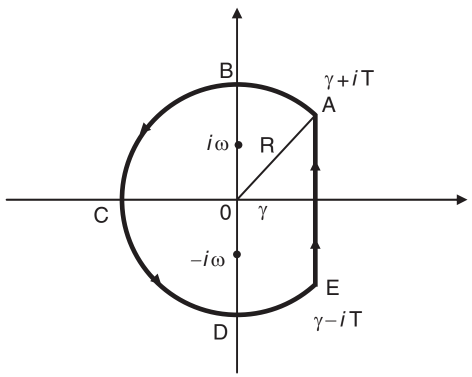

Figure 6.4

We have

$$\text{Res}(i\omega) = \lim_{s\to i\omega}(s-i\omega)e^{st}F(s) = \lim_{s\to i\omega}\frac{e^{st}}{s+i\omega} = \frac{e^{i\omega t}}{2i\omega},$$

$$\text{Res}(-i\omega) = \lim_{s\to -i\omega}(s+i\omega)e^{st}F(s) = \lim_{s\to -i\omega}\frac{e^{st}}{s-i\omega} = \frac{e^{-i\omega t}}{-2i\omega}.$$

Further, let $s = \gamma + \text{R}\; e^{i\theta}$, $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$. Then the integral over the contour C_R yields

$$\left|\int_{\pi/2}^{3\pi/2}\frac{e^{\gamma t}e^{t\,R(\cos\theta+i\sin\theta)}R\;e^{i\,\theta}}{(\gamma+\text{R}e^{i\,\theta})^2+\omega^2}\right| \le \frac{\pi e^{\gamma\,t}R}{R^2-\gamma^2-\omega^2}$$

$\to 0$ as R $\to \infty$.

Hence

$$\text{L}^{-1}\{\text{F}(s)\} = \frac{\omega}{2\pi i}2\pi i(\text{sum of residue at} \pm i\omega)$$

$$= \frac{e^{i\omega t}-e^{-i\omega t}}{2i} = \sin\omega t.$$

EXAMPLE 6.67

Find $L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\}$ using inversion formula.

Solution. We have

$$\text{F}(s) = \frac{1}{s(s^2+a^2)^2} = \frac{1}{s(s-ia)^2(s+ia)^2}.$$

Thus F(s) has a simple pole at $s = 0$ and a pole of order 2 at $s = \pm\, ia$. Further, F(s) satisfies growth restriction condition. Therefore, integral over the contour C_R goes to zero as R $\to \infty$. Further,

$$\text{Res}(0) = \lim_{s\to 0}se^{st}F(s) = \lim_{s\to 0}\frac{e^{st}}{(s^2+a^2)^2} = \frac{1}{a^4}$$

$$\text{Res}(ia) = \lim_{s\to ai}\frac{d}{ds}(s-ia)^2e^{ts}F(s)$$

$$= \lim_{s\to ai}\frac{d}{ds}\left(\frac{e^{ts}}{s(s+ia)^2}\right) = \frac{it}{4a^3}e^{iat} - \frac{e^{iat}}{2a^4},$$

$$\text{Res}(-ia) = \lim_{s\to -ia}\frac{d}{ds}(s+ia)^2e^{ts}F(s)$$

$$= \lim_{s\to -ia}\frac{d}{ds}\left(\frac{e^{ts}}{s(s-ia)^2}\right) = \frac{-it}{4a^3}e^{-iat} - \frac{e^{-iat}}{2a^4}.$$

Hence

$$f(t) = \frac{1}{a^4} + \frac{it}{4a^3}(e^{iat}-e^{-iat}) - \frac{1}{2a^4}(e^{iat}+e^{-iat})$$

$$= \frac{1}{a^4}\left(1 - \frac{a}{2}t\sin at - \cos at\right).$$

EXAMPLE 6.68

Find $L^{-1}\left\{\frac{s}{(s+1)^3\,(s-1)^2}\right\}$ using inversion formula.

Solution. We have

$$\text{F}(s) = \frac{s}{(s+1)^3\,(s-1)^2}.$$

The function F(s) has poles of multiplicity 2 at $s = 1$ and poles of multiplicity 3 at $s = -1$. Residue at $s = 1$ is given by

$$\begin{aligned}\text{Res}(1) &= \lim_{s\to 1}\frac{d}{ds}\left[\frac{(s-1)^2 se^{st}}{(s+1)^3(s-1)^2}\right]\\ &= \lim_{s\to 1}\frac{d}{ds}\left(\frac{se^{st}}{(s+1)^3}\right)\\ &= \frac{1}{16}\ e^t(2t-1),\end{aligned}$$

$$\begin{aligned}\text{Res}(-1) &= \lim_{s\to -1}\frac{1}{2!}\frac{d^2}{ds^2}\left[\frac{(s+1)^3 se^{st}}{(s+1)^3(s-1)^2}\right]\\ &= \lim_{s\to -1}\frac{1}{2}\frac{d^2}{ds^2}\left(\frac{se^{st}}{(s-1)^2}\right)\\ &= \frac{1}{16}e^{-t}(1-2t^2).\end{aligned}$$

The value of the integral over the contour C_R tends to zero as $R \to \infty$. Hence

$$\begin{aligned}f(t) &= \text{sum of residues at the poles}\\ &= \frac{1}{16}e^{-t}(1-2t^2) + \frac{1}{16}e^t(2t-1).\end{aligned}$$

EXAMPLE 6.69

Derive Heaviside's expansion formula using complex inversion formula.

Solution. Let F(s) $= \frac{P(s)}{Q(s)}$, where P(s) and Q(s) are polynomials having no common factors (roots) and degree of Q(s) is greater than the degree of P(s). Suppose Q(s) has simple zeros at $z_1, z_2, \ldots, z_m$. If degree of P(s) and Q(s) are n and m, respectively, then for $a_0 \neq 0$, $b_m \neq 0$,

$$\begin{aligned}\frac{P(s)}{Q(s)} &= \frac{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}\\ &= \frac{a_n + \frac{a_{n+1}}{s} + \ldots + \frac{a_0}{s^n}}{s^{m-n}\left(b_m + \frac{b_{m-1}}{s} + \ldots + \frac{b_0}{s^m}\right)}.\end{aligned}$$

For sufficiently large $|s|$, we have

$$|a_n + \frac{a_{n-1}}{s} + \ldots + \frac{a_0}{s^n}| \le |a_n| + |a_{n-1}| + \ldots + |a_0| = C_1 \text{ say},$$

$$|b_m + \frac{a_{m-1}}{s} + \ldots + \frac{b_0}{s^n}| \ge |b_m| - \frac{|b_{m-1}|}{|s|} - \ldots - \frac{|b_0|}{|s|^m} \ge \frac{|b_m|}{2} = C_2 \text{ say},$$

and so

$$|F(s)| = \left|\frac{P(s)}{Q(s)}\right| \le \frac{C_1/C_2}{|s|^{m-n}}.$$

Thus, F(s) satisfies growth restriction condition. Further,

$$\begin{aligned}\text{Res}(z_n) &= \lim_{s\to z_n}(s-z_n)e^{st}F(s)\\ &= \lim_{s\to z_n}\frac{e^{st}P(s)}{\frac{Q(s)-Q(z_n)}{s-z_n}}, \quad \text{since } Q(Z_n) = 0\\ &= \frac{e^{z_n t}P(z_n)}{Q'(z_n)}.\end{aligned}$$

Hence, by inversion formula, we have

$$f(z) = \sum_{n=1}^{m}\frac{P(z_n)}{Q'(z_n)}\ e^{tz_n},$$

which is the required Heaviside's expansion formula.

EXAMPLE 6.70

Find $L^{-1}\left(\frac{1}{s(1+e^{as})}\right)$, using complex inversion formula.

Solution. Let F(s) $= \frac{1}{s(1+e^{as})}$. Then F(s) has a simple pole at $s = 0$. Further, $1 + e^{as} = 0$ yields

$$e^{as} = -1 = e^{(2n-1)\pi i}, \quad n = 0, \pm 1, \pm 2, \ldots$$

and so $s_n = \left(\frac{2n-1}{a}\right)\pi i$, $n = 0, \pm 1, \pm 2, \ldots$ are also poles of F(s). Also, $\frac{d}{ds}(1+e^{as})_{s=s_n} = -a \neq 0$. Therefore, s_n are simple poles. Now

$$\text{Res}(0) = \lim_{s\to 0} s\,e^{st}F(s) = \frac{1}{2},$$

$$\begin{aligned}\text{Res}\left[\left(\frac{2n-1}{a}\right)\pi i\right] &= \lim_{s\to s_n}(s-s_n)e^{st}F(s)\\ &= \lim_{s\to s_n}\frac{(s-s_n)e^{st}}{s(1+e^{as})}\left(\frac{0}{0}\text{ form}\right)\\ &= \lim_{s\to s_n}\frac{e^{st}+t\,e^{st}(s-s_n)}{a\,s\,e^{as}+1+e^{as}}, \text{by L'Hospital rule}\\ &= \frac{e^{tsn}}{a\,s_n e^{asn}} = -\frac{e^{t\left(\frac{2n-1}{a}\right)\pi i}}{(2n-1)\pi i}.\end{aligned}$$

Also, it can be shown that F(s) satisfies growth restriction condition. Hence, by inversion formula, at the points of continuity of f, we have

$$f(t) = \text{sum of residues at the poles}$$

$$= \frac{1}{2} - \sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)\pi\, i}\; e^{t\left(\frac{2n-1}{a}\right)\pi i}$$

$$= \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin\left(\frac{2n-1}{a}\right)\pi\, t.$$

EXAMPLE 6.71

Find $L^{-1}\{e^{-a\sqrt{s}}\}$, $a > 0$.

Solution. We know (Exercise 12) that

$$L^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = erfc\left(\frac{a}{2\sqrt{t}}\right),$$

that is,

$$\mathrm{L}\left\{erfc\left(\frac{a}{2\sqrt{t}}\right)\right\} = e^{-\frac{a\sqrt{s}}{s}}.$$

Therefore,

$$\mathrm{L}\left\{\frac{d}{dt} erfc\left(\frac{a}{2\sqrt{t}}\right)\right\} = \mathrm{s}\ \mathrm{F}(s) - f(0)$$

$$= s\frac{e^{-a\sqrt{s}}}{s}$$

because $erfc\left(\frac{a}{2\sqrt{t}}\right) \to 0$ as $t \to 0$.

Thus

$$\mathrm{L}\left\{\frac{a}{2\sqrt{\pi t^3}}\; e^{-a^2/(4t)}\right\} = e^{-a\sqrt{s}}.$$

Hence

$$L^{-1}\{e^{-a\sqrt{s}}\} = \frac{a}{2\sqrt{\pi t^3}}\; e^{-a^2/(4t)}.$$

6.8 MISCELLANEOUS EXAMPLES

EXAMPLE 6.72. Find $L^{-1}\left[\frac{1}{(s-3)^2}\right]$.

Solution. Since $L\{e^{at}t\} = \frac{1}{(s-a)^2}$ by shifting property, we have

$$L^{-1}\left[\frac{1}{(s-3)^2}\right] = te^{3t}$$

EXAMPLE 6.73. Find the inverse Laplace transforms of: (i) $\frac{3s+1}{s^2(s^2+4)}\, e^{-3s}$ and (ii) $\tan^{-1}\left(\frac{2}{s^2}\right)$.

Solution. (i) We first find partial fraction of $\frac{3s+1}{s^2(s^2+4)}$. We have

$$\frac{3s+1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{cs+D}{s^2+4}$$

or

$$3s + 1 = As(s^2+4) + B(s^2+4) + s(cs+D).$$

Putting $s = 0$, we get

$$1 = 4B \text{ which yields } B = \frac{1}{4}.$$

Taking $s = -4$, we get

$$-11 = 16C - 4D \qquad (1)$$

Comparing coefficients of s, we get

$$3 = 4A + D \qquad (2)$$

Comparing coefficients of s^2, we get

$$0 = B + C \text{ which yields } C = -B = -\frac{1}{4}.$$

Then (1) yields $D = \frac{16c+11}{4} = \frac{7}{4}$. Now (2) gives

$$A = \frac{3-D}{4} = \frac{3-\frac{7}{4}}{4} = \frac{5}{16}.$$

Hence

$$\frac{3(s+1)}{s^2(s^2+4)} = \frac{5}{16s} + \frac{1}{4s^2} - \frac{s}{4(s^2+4)} + \frac{7}{4(s^2+4)}.$$

Therefore

$$L^{-1}\left\{\frac{3s+1}{s^2(s^2+4)}\right\} = \frac{5}{16} + \frac{1}{4}t - \frac{1}{4}\cos 2t + \frac{7}{8}.\sin 2t.$$

Using second shifting property, we have

$$L^{-1}\left\{\frac{3s+1}{s^2(s^2+4)}e^{-3s}\right\} = \frac{5}{16} + \frac{1}{4}(t-3) - \frac{1}{4}\cos 2(t-3) + \frac{7}{8}\sin 2(t-3)$$

for $t > 3$.

(ii). We note that

$$\frac{d}{ds}\left(\tan^{-1}\frac{2}{s^2}\right) = \frac{1}{1+\frac{4}{s^4}}\left(\frac{-4}{s^3}\right) = \frac{-4s}{s^4+4}$$

$$= -\frac{4s}{(s^2+2)^2-(2s)^2}$$

$$= -\frac{4s}{(s^2+2+2s)(s^2+2-2s)}$$

$$= -\frac{(s^2+2s+2)-(s^2-2s+2)}{(s^2+2s+2)(s^2-2s+2)}$$

$$= -\frac{1}{s^2-2s+2}+\frac{1}{s^2+2s+2}$$

$$= -\frac{1}{(s-1)^2+1}+\frac{1}{(s+1)^2+1}.$$

Therefore

$$L^{-1}\left\{-\frac{1}{(s-1)^2+1}+\frac{1}{(s+1)^2+1}\right\} = -t f(t),$$

that is,

$$-e^t \sin t + e^{-t}\sin t = -t f(t).$$

Hence

$$f(t) = \frac{\sin t(e^t - e^{-t})}{t} = \frac{2\sin t\ \sinh t}{t}.$$

EXAMPLE 6.74. Find the inverse Laplace transform of $(5s+3)/(s-1)(s^2+2s+5)$.

Solution. Using partial fractions, we have

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1}+\frac{-s+2}{s^2+2s+5}$$

$$= \frac{1}{s-1}-\frac{(s+1)-3}{(s+1)^2+4}$$

$$= \frac{1}{s-1}-\frac{s+1}{(s+1)^2+4} + \frac{3}{(s+1)^2+4}$$

Therefore

$$L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$$

$$= e^t - e^{-t}\cos 2t + 3e^{-t}\sin 2t.$$

EXAMPLE 6.75. Find inverse Laplace transform of $(se^{-s/2}+\pi e^{-s})/(s^2+\pi^2)$.

Solution. Using linearity and shifting properties, we have

$$L^{-1}\left\{\frac{se^{-\frac{s}{2}}+\pi e^{-s}}{s^2+\pi^2}\right\}$$

$$= L^{-1}\left\{\frac{se^{-\frac{s}{2}}}{s^2+\pi^2}\right\} + L^{-1}\left\{\frac{\pi e^{-s}}{s^2+\pi^2}\right\}$$

$$= \cos\pi\left(t-\frac{1}{2}\right)H\left(t-\frac{1}{2}\right) + \sin\pi(t-1)H(t-1)$$

$$= \sin\pi t\left[H\left(t-\frac{1}{2}\right)\right] - \sin\pi t[H(t-1)]$$

$$= \sin\pi t\left[H\left(t-\frac{1}{2}\right) - H(t-1)\right],$$

where $H(t)$ denotes Heavyside's unit step function.

EXAMPLE 6.76. Find the inverse Laplace transform of

$$\frac{1}{s^2(s^2+a^2)}.$$

Solution. We know that

$$L\{\sin at\} = \frac{a}{s^2+a^2}.$$

Therefore

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at.$$

Then

$$L^{-1}\left\{\frac{a}{s(s^2+a^2)}\right\} = \frac{1}{a}\int_0^t \sin at\,dt = \frac{1}{a}\left(-\frac{\cos at}{a}\right)_0^t$$

$$= -\frac{1}{a^2}[\cos at - 1]$$

and

$$L^{-1}\left\{\frac{a}{s^2(s^2+a^2)}\right\} = -\frac{1}{a^2}\int_0^t (\cos at - 1)dt$$

$$= -\frac{1}{a^2}\left[\frac{\sin at}{a} - t\right]_0^t$$

$$= -\frac{1}{a^3}\sin at + \frac{t}{a^2}$$

$$= \frac{1}{a^2}\left[t - \frac{1}{a}\sin at\right].$$

Note: This question can also be solved using Convolution Theorem (see Example 6.57)

EXAMPLE 6.77. Find $L^{-1}\left[\cot^{-1}\left(\frac{2}{s+1}\right)\right]$.

Solution. Since

$$\frac{d}{ds}\left[\cot^{-1}\left(\frac{2}{s+1}\right)\right] = \frac{-1}{1+\left(\frac{2}{s+1}\right)^2}\left(-\frac{2}{(s+1)^2}\right)$$
$$= \frac{2}{s^2+2s+5},$$

we have

$$L^{-1}\left\{\frac{2}{s^2+2s+5}\right\} = -tf(t)$$

or

$$L^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} = -tf(t)$$

or

$$e^{-t}\sin\ 2t\ = -tf(t).$$

Hence

$$f(t) = -\frac{1}{t}e^{-t}\ \sin\ 2t.$$

EXAMPLE 6.78. Find:

$$L^{-1}\left\{\log\frac{s^2+1}{s(s+1)}\right\}.$$

Solution. Similar to Example 6.22. In fact,

$$\frac{d}{ds}\left\{\log\frac{s^2+1}{s(s+1)}\right\}$$
$$= \frac{d}{ds}[\log(s^2+1) - \log s - \log(s+1)]$$
$$= \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}.$$

Therefore

$$L^{-1}\left\{\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right\} = -tf(t)$$

or

$$2\cos t - 1 - e^{-t} = -tf(t).$$

Hence

$$f(t) = \frac{1}{t}[e^{-t} + 1 - 2\cos t].$$

EXAMPLE 6.79. Find the inverse Laplace transform of $\frac{s^2}{(s^2+4)^2}$.

Solution. Let $F(s) = \frac{s}{s^2+4} = G(s)$. Then

$$f(t) = L^{-1}\{F(s)\} = \cos\ 2t \text{ and}$$
$$g(t) = L^{-1}\{G(s)\} = \cos\ 2t.$$

Therefore, by Convolution Theorem

$$L^{-1}\{F(s)G(s)\} = f*g = \int_0^t \cos\ 2u\cos\ 2(t-u)\ du$$
$$= \frac{1}{2}\int_0^t [\cos\ 2t + \cos\ (4u-2t)]\ du$$
$$= \frac{1}{2}\left[u\ \cos\ 2t + \frac{\sin\ (4u-2t)}{4}\right]_0^t$$
$$= \frac{1}{2}\left[t\ \cos\ 2t + \frac{\sin\ 2t}{4}\right]$$
$$= \frac{1}{4}[2t\cos\ 2t\ + \sin\ 2t].$$

EXAMPLE 6.80. Using Convolution Theorem find the Laplace inverse of

$$(s+2)/(s^2+4s+13)^2.$$

Solution. We have

$$\frac{s+2}{(s^2+4s+13)^2} = \frac{s+2}{((s+2)^2+3^2)^2}.$$

Therefore

$$L^{-1}\left\{\frac{s+2}{(s^2+4s+13)^2}\right\} = e^{-2t}L^{-1}\left\{\frac{s}{(s^2+3^2)^2}\right\}.$$

Let

$$\frac{s}{(s+3^2)^2} = \frac{s}{s^2+3^2}\cdot\frac{1}{s^2+3^2} = F(s)\ g(s).$$

$$\text{Then } f(t) = L^{-1}\{F(s)\} = \cos 3t \text{ and}$$
$$g(t) = L^{-1}\{G(s)\} = \frac{\sin 3t}{3}.$$

By Convolution Theorem

$$L^{-1}\{F(s)\,G(s)\} = f*g = \int_0^t \cos 3u\frac{\sin 3(t-u)}{3}du$$
$$= \frac{1}{3}\cdot\frac{1}{2}\int_0^t [\sin 3t + \sin 3(t-2u)]du$$
$$= \frac{1}{6}\sin 3t|u|_0^t + \frac{1}{12}[\cos 3(t-2u)]_0^t$$

$$= \frac{1}{6} t \sin 3t + \frac{1}{12}[\cos 3(-t) - \cos 3t]$$
$$= \frac{1}{6} t \sin 3t.$$

Hence

$$L^{-1}\left\{\frac{s+2}{(s^2+4s+13)^2}\right\} = e^{-2t}\left[\frac{1}{6} t \sin 3t\right]$$
$$= \frac{1}{6} t\, e^{-2t} \sin 3t.$$

EXAMPLE 6.81. Apply Convolution Theorem to evaluate

$$L^{-1}\left(\frac{1}{s^2+a^2}\right)^2.$$

Solution. Proceeding as in Example 6.56, we have

$$F(s) = G(s) = \frac{1}{s^2+a^2}.$$

Then

$$f(t) = L^{-1}\{F(s)\} = \frac{\sin at}{a}$$
$$g(t) = L^{-1}\{G(s)\} = \frac{\sin at}{a}.$$

Therefore, by Convolution Theorem,

$$L^{-1}\{F(s)G(s)\}$$
$$= f * g = \int_0^t \frac{\sin au}{a} \cdot \frac{\sin a(t-u)}{a} du$$
$$= \frac{1}{a^2}\int_0^t \frac{1}{2}[\cos a\ (2u-t) - \cos at]\ du$$
$$= \frac{1}{2a^2}\left[\int_0^t \cos a\ (2u-t) du - \int_0^t \cos at\, du\right]$$
$$= \frac{1}{2a^2}\left\{\left[\frac{\sin a(2u-t)}{2a}\right]_0^t - t \cos at\right\}$$
$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} + \frac{\sin at}{2a} - t \cos at\right]$$
$$= \frac{1}{2a^2}\left[\frac{\sin at}{a} - t \cos at\right]$$
$$= \frac{1}{2a^3}[\sin at - at \cos at].$$

EXAMPLE 6.82. Find the inverse Laplace transform of $\frac{s}{(s^2+1)(s^2+4)}$ using Convolution Theorem.

Solution. Let

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{1}{(s^2+1)} \cdot \frac{s}{s^2+4} = F(s)G(S).$$

Then

$$f(t) = L^{-1}\{F(s)\} = \sin t$$
$$g(t) = L^{-1}\{G(s)\} = \cos 2t.$$

By Convolution Theorem,

$$L^{-1}\{F(s)G(s)\}$$
$$= \sin t * \cos 2t$$
$$= \int_0^t \sin u \cos 2(t-u) du$$
$$= \frac{1}{2}\int_0^t [\sin(2t-u) - \sin(2t-3u)] du$$
$$= \frac{1}{2}\left[\frac{-\cos(2t-u)}{-1} + \frac{\cos(2t-3u)}{-3}\right]_0^t$$
$$= \frac{1}{2}\left[\cos t - \frac{1}{3}\cos t - \cos 2t + \frac{1}{3}\cos 2t\right]$$
$$= \frac{1}{3}\cos t - \frac{1}{3}\cos 2t = \frac{1}{3}(\cos t - \cos 2t).$$

EXAMPLE 6.83. Using Convolution Theorem, find inverse Laplace transform of (a) $\frac{s}{(s^2+a^2)^3}$ and (b) $\frac{1}{s(s+1)(s+2)}$

Solution. (a) Let

$$F(s) = \frac{s}{s^2+a^2} \text{ and}$$
$$G(s) = \frac{1}{(s^2+a^2)^2}$$

Then (see Example 6.59)

$$f(t) = \cos at \text{ and } g(t) = \frac{1}{2a^3}[\sin at - at \cos at].$$

By Convolution Theorem, we have

$$L^{-1}\{F(s)G(s)\}$$
$$= f * g = \int_0^t \frac{1}{2a^3}\cos au[\sin a(t-u)$$
$$- a(t-u)\cos a(t-u)] du$$
$$= \frac{1}{2a^3}\int_0^t [\cos au \sin a(t-u)$$
$$- a \cos au(t-u)\cos a(t-u)] du$$

$$= \frac{1}{2a^3}\left[\int_0^t \cos au \, \sin a(t-u)du\right.$$
$$\left. -a\int_0^2 (t-u)\cos au \, \cos a(t-u)du\right]$$
$$= \frac{1}{2a^3}\int_0^t \cos au \sin a(t-u)du$$
$$-\frac{1}{2a^2}\left[\int_0^t (t-u)\frac{1}{2}\{\cos at + \cos a(t-2u)\}\right]du$$
$$= \frac{1}{4a^3}t\sin at - \frac{t^2}{8a^2}\cos at - \frac{1}{8a^3}t\sin at$$
$$= \frac{t}{8a^3}[\sin at - at\cos at].$$

(b) Let

$$F(s) = \frac{1}{s(s+1)} \text{ and } G(s) = \frac{1}{s+2}.$$

Then

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = 1 - e^{-t}$$

and

$$g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}.$$

Therefore, by Convolution Theorem,

$$L^{-1}\{F(s)G(s)\} = f * g = \int_0^t (1-e^{-u})e^{-2(t-u)}du$$
$$= e^{-2t}\int_0^t (e^{2u} - e^u)du$$
$$= e^{-2t}\left[\frac{2u}{2} - e^u\right]_0^t$$
$$= e^{-2t}\left[\frac{e^{2t}}{2} - e^t + \frac{1}{2}\right]$$
$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$

EXERCISES

1. Find inverse Laplace transform of
   (a) $\frac{2s+6}{(s^2+6s+10)^2}$ **Ans.** $te^{-3t}\sin t$
   (b) $\frac{s}{(s^2+4)^2}$ **Ans.** $\frac{1}{4}t\sin 2t$
   (c) $\log\frac{s+1}{s-1}$ **Ans.** $\frac{2\sinh t}{t}$
   (d) $\cot^{-1}\frac{s}{\pi}$ **Hint:** $\frac{d}{ds}\left\{\cot^{-1}\frac{s}{\pi}\right\} = -\frac{\pi}{s^2+\pi^2}$ implies
   $L^{-1}\left\{\frac{-\pi}{s^2+\pi^2}\right\} = -tf(t)$ implies
   $\pi\left[\frac{1}{\pi}\sin\pi t\right] = tf(t)$ **Ans.** $f(t) = \frac{\sin\pi t}{t}$
   (e) $\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)$ **Hint:** Use Example 6.22 and Theorem 6.9
   Ans. $2\int_0^t \frac{1-\cos u}{u}\,du$
   (f) $\frac{2s-3}{s^2+4s+13}$ **Ans.** $2e^{-2t}\cos 3t - \frac{7}{3}e^{-2t}\sin 3t$
   (g) $\frac{s}{s^4+4a^4}$ **Hint:** Use partial fraction method
   Ans. $\frac{1}{2a^2}\sin at\sinh at$
   (h) $\frac{a(s^2-2a^2)}{s^4+4a^4}$ **Ans.** $\cos at \sinh at$
   (i) $\frac{s^2+6}{(s^2+1)\,(s^2+4)}$ **Ans.** $f(t) = \frac{1}{3}(5\sin t - \sin 2t)$
   (j) $\frac{s}{s^4+s^2+1}$ (partial fraction method)
   Ans. $\frac{2}{\sqrt{3}}\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}\,t$
   (k) $\frac{2s^2+5s-4}{s^3+s^2-2s}$ **Hint:** Has simple poles, so use residue method
   Ans. $2 + e^t - e^{-2t}$
   (l) $\frac{2s^2-4}{(s+1)\,(s-2)\,(s-3)}$ **Ans.** $-\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$
2. Use first shift property to find $L^{-1}\left\{\frac{1}{\sqrt{s+a}}\right\}$.
   Hint: $L^{-1}\left\{\frac{1}{\sqrt{s+a}}\right\} = e^{-at}L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = e^{-at}\cdot\frac{t^{\frac{1}{2}-1}}{\Gamma(1/2)}$
   Ans. $\frac{e^{-at}}{\sqrt{\pi t}}$
3. Solve Exercise 1(k) using Heaviside's expansion formula.
4. Use Heaviside's expansion formula to find $L^{-1}\left\{\frac{27-12s}{(s+4)\,(s^2+9)}\right\}$
   Ans. $3e^{-4t} - 3\cos 3t$
5. Use series method to find $L^{-1}\{e^{-\sqrt{s}}\}$.
   Ans. $\frac{1}{2\sqrt{\pi}\,t^{3/2}}\;e^{-\frac{1}{4t}}$
6. Show that
   $L^{-1}\left\{\frac{1}{s}\cos\frac{1}{s}\right\} = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \ldots$

7. Evaluate $\sin t * t^2$. **Ans.** $t^2 + 2\cos t - 2$

8. Find $L\{\sin t * t^2\}$. **Ans.** $\frac{2}{(s^2+1)\, s^3}$

9. Use Convolution Theorem to find the inverse Laplace transforms of the following :

   (a) $\frac{1}{s(s-a)}$ **Ans.** $\frac{e^{at}-1}{a}$.

   (b) $\frac{a^2}{(s^2+a^2)^2}$ **Ans.** $\frac{1}{2a}(\sin at - at\cos at)$

   (c) $\frac{4}{s^3+s^2+s+1}$ **Ans.** $2(e^{-t} - \cos t + \sin t)$

   (d) $\frac{s+2}{(s^2+4s+5)^2}$ **Ans.** $\frac{1}{2}te^{-2t}\sin t$

10. Verify complex inversion formula for $F(s) = \frac{1}{s(s-a)}$.

    Hint: Simple poles at 0 and a, satisfies growth restriction condition, $\text{Res}(0) = -1/a$, $\text{Res}(a) = e^{at}/a$

    Ans. $f(t) = \frac{1}{a}(e^{at} - 1)$

11. Using complex inversion formula, find the inverse Laplace transform of the following:

    (a) $\frac{s}{s^2+a^2}$ **Ans.** $\cos at$

    (b) $\frac{1}{(s+1)\,(s-2)^2}$ **Ans.** $\frac{1}{9}e^{-t} + \frac{1}{3}te^{2t} - \frac{1}{9}e^{2t}$

    (c) $\frac{1}{(s^2+1)^2}$ **Ans.** $\frac{1}{2}(\sin t - t\cos t)$

12. Find $L^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\}$, $a > 0$. **Ans.** $1 - \text{erf}\left(\frac{a}{2\sqrt{t}}\right)$ or $\text{erfc}\left(\frac{a}{2\sqrt{t}}\right)$.

7 Applications of Laplace Transform

Laplace transform is utilized as a tool for solving linear differential equations, integral equations, and partial differential equations. It is also used to evaluate the integrals. The aim of this chapter is to discuss these applications.

7.1 ORDINARY DIFFERENTIAL EQUATIONS

Recall that a *differential equation* is an equation where the unknown is in the form of a derivative. The *order* of an ordinary differential equation is the highest derivative attained by the unknown. Thus the equation

$$\frac{d^2y}{dt^2} + \alpha \frac{dy}{dy} + \beta y = f(t)$$

is of *second order*, whereas the equation.

$$\left(\frac{dy}{dx}\right)^3 + y = sinx$$

is a *first order* differential equation.

Theorem 6.8, opens up the possibility of using Laplace transform as a tool for solving *ordinary differential equations*. Laplace transforms, being linear, are useful only for solving linear differential equations. Differential equations containing powers of the unknown or expression such as tan x, e^x cannot be solved using Laplace transforms.

The results

$$\mathrm{L}\{f'(t)\} = s\mathrm{F}(s) - f(0)$$

and

$$\mathrm{L}\{f''(t) = s^2\mathrm{F}(s) - sf(0) - f'(0)$$

will be used frequently for solving ordinary differential equations. To solve linear ordinary differential equation by the Laplace transform method, we first convert the equation in the unknown function $f(t)$ into an equation in $\mathrm{F}(s)$ and find $\mathrm{F}(s)$. The inversion of $\mathrm{F}(s)$ then yields $f(t)$.

Since $f(0)$, $f'(0)$, and $f''(0)$ appear in Laplace transform of derivatives of f, the Laplace transform method is best suited to *initial value problems* (where auxiliary conditions are *all* imposed at $t = 0$). The solution by Laplace method with initial conditions automatically built into it. We need not add particular integral to complementary function and then apply the auxiliary conditions.

(a) Ordinary Differential Equations with Constant Solution

In case of an ordinary differential equation with constant coefficients, the transformed equation for $\mathrm{F}(s)$ turns out to be an algebraic one and, therefore, the Laplace transform method is powerful tool for solving this type of ordinary differential equations. If

$$a_n \frac{d^ny}{dt^n} + a_{n-1} \frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0 y = f(t)$$

with $y(0) = y_0, y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}$, then $f(t)$ is called *input*, *excitation*, or *forcing function* and $y(t)$ is called the *output* or *response*. Further, the following results suggests that if $f(t)$ is continuous and of exponential order, then $y(t)$ is also continuous and of exponential order.

Theorem 7.1. If $a_n y^{(n)} + a_{n-1}\, y^{(n-1)} + \ldots + a_0 y = f(t)$ *is* n^{th} order linear non-homogeneous equation with constant coefficients and f is continuous on $[0, \infty)$ and of exponential order, then $y(t)$ is also continuous and of exponential order.

EXAMPLE 7.1

Find the general solution of the differential equation

$$y''(t) + k^2 y(t) = 0.$$

Solution. Assume that the value of the unknown function at $t = 0$ be denoted by the constant A, and the value of its first derivative at $t = 0$ by the constant B. Thus

$$y(0) = \text{A and } y'(0) = \text{B}.$$

Taking Laplace transform of both sides of the given differential equations, we have

$$\text{L}\{y''(t)\} + k^2\text{L}\{y(t)\} = 0$$

But

$$\begin{aligned} \text{L}\{y''(t)\} &= s^2\text{Y}(s) - sy(0) - y'(0) \\ &= s^2\text{Y}(s) - \text{A}s - \text{B}. \end{aligned}$$

Therefore,

$$s^2\text{Y}(s) - \text{A}s - \text{B} + k^2\text{Y}(s) = 0.$$

The solution of this algebraic equation in Y(s) is

$$\text{Y}(s) = A\frac{s}{s^2 + k^2} + \frac{B}{k}\cdot\frac{k}{s^2 + k^2}.$$

Taking inverse Laplace transform, we get

$$y(t) = \text{A}\cos kt + \frac{B}{k}\sin kt,$$

where A and B are constants since the initial conditions were not given.

EXAMPLE 7.2

Solve

$$\frac{d^2x}{dt^2} + x = \text{A}\sin t,\ \ x(0) = x_0,\ \ x'(0) = v_0.$$

Show that the phenomenon of *resonance* occurs in this case.

Solution. Taking Laplace transform, we get

$$s^2\text{X}(s) - sx(0) - x'(0) + \text{X}(s) = \frac{A}{s^2 + 1}$$

or

$$(s^2 + 1)\,\text{X}(s) = \frac{A}{s^2 + 1} + sx_0 + v_0$$

or

$$\text{X}(s) = \frac{A}{(s^2 + 1)^2} + \frac{s}{s^2 + 1}x_0 + \frac{v_0}{s^2 + 1}.$$

Taking inverse Laplace transform, we have

$$x(t) = \frac{A}{2}(\sin t - t\cos t) + x_0\cos t + v_0\sin t.$$

We note that $x(t) \to \infty$ as $t \to \infty$ due to the term $t \cos t$. This term is called a *secular term*. *The presence of secular term causes resonance*, because the solution becomes unbounded.

Remark 7.1. If we consider the equation $\frac{d^2x}{dt^2} + k^2x = \text{A} \sin t$, $k \neq 1$, then there will be no secular term in the solution and so the system will be *purely oscillatory*.

EXAMPLE 7.3

Solve the initial value problem

$$y'(t) + 3y(t) = 0, y(0) = 1.$$

Solution. Taking Laplace transform, we get

$$\text{L}\{y'(t)\} + 3\text{L}\{y(t)\} = 0,$$

which yields

$$s\text{Y}(s) - y(0) + 3\text{Y}(s) = 0.$$

Since $y(0) = 1$, we have

$$s\text{Y}(s) - 3\text{Y}(s) = 1,$$

an algebraic equation whose solution is

$$\text{Y}(s) = \frac{1}{s + 3}.$$

Taking inverse Laplace transform leads to

$$y(t) = e^{-3t}.$$

EXAMPLE 7.4

Solve the initial value problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 8y = 0, y(0) = 3, \quad y'(0) = 6.$$

Solution. The given equation is

$$y''(t) - 2y'(t) - 8y = 0,\ \ y(0) = 3,\ \ y'(0) = 6.$$

Laplace transform leads to

$$\text{L}\{y''(t\} - 2\text{L}\{y'(t)\} - 8\text{L}\{y\} = 0,$$

that is,

$$s^2\text{Y}(s) - sy(0) - y'(0) - 2\{s\text{Y}(s) - y(0)\} - 8\text{Y}(s) = 0$$

and so using initial conditions, we have

$$(s^2 - 2s - 8)\text{Y}(s) - 3s = 0.$$

Hence

$$Y(s) = \frac{3s}{s^2 - 2s - 8} = 3\left[\frac{s-1+1}{(s-1)^2 - 9}\right]$$
$$= 3\left[\frac{s-1}{(s-1)^2-9} + \frac{1}{(s-1)^2-9}\right].$$

Taking inverse Laplace transform, we get

$$y(t) = 3L^{-1}\left\{\frac{s-1}{(s-1)^2-9}\right\} + 3L^{-1}\left\{\frac{1}{(s-1)^2-9}\right\}$$
$$= 3e^t \cosh 3t + e^t \sinh 3t.$$

EXAMPLE 7.5

Solve the initial value problem

$$y''' + y'' = e^t + t + 1, \; y(0) = y'(0) = y''(0) = 0.$$

Solution. Taking Laplace transform of both sides of the given equation, we have

$$L\{y'''(t) + L\{y''(t)\} = L\{e^t\} + L\{t\} + L\{1\},$$

that is,

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$
$$+ s^2Y(s) - sy(0) - y'(0) = \frac{1}{s-1} + \frac{1}{s^2} + \frac{1}{s}$$

Since $y(0) = y'(0) = y''(0) = 0$, we have

$$s^3Y(s) + s^2Y(s) = \frac{1}{s-1} + \frac{1}{s^2} + \frac{1}{s},$$

and so

$$Y(s) = \frac{2s^2 - 1}{s^4(s-1)\,(s+1)}.$$

Using partial fraction decomposition, we have

$$Y(s) = -\frac{1}{s^2} + \frac{1}{s^4} - \frac{1}{2(s+1)} - \frac{1}{2(s-1)}.$$

Taking inverse transform yields

$$y(t) = L^{-1}\left\{-\frac{1}{s^2} + \frac{1}{s^4} - \frac{1}{2(s+1)} + \frac{1}{2(s-1)}\right\}$$
$$= -t + \frac{1}{6}t^3 - \frac{1}{2}e^{-t} + \frac{1}{2}e^t.$$

Verification: We have

$$y' = -1 + \frac{1}{2}t^2 + \frac{1}{2}e^{-t} + \frac{1}{2}e^t,$$
$$y'' = t - \frac{1}{2}e^{-t} + \frac{1}{2}e^t,$$
$$y''' = 1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^t.$$

Adding y'' and y''', we get

$$y'' + y''' = t + e^t + 1 \text{ (the given equation)}.$$

EXAMPLE 7.6

Solve

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t, \; y(0) = y'(0) = 0.$$

Solution. Taking Laplace transform of both sides of the given equation, we take

$$L\{y'''(t)\} + 2L\{y'(t)\} - 3L\{y(t)\} = L\{\sin t\},$$

which yields

$$s^2Y(s) - sy(0) - y'(0) + 2\{sY(s) - y(0)\} - 3Y(s)$$
$$= \frac{1}{s^2+1}.$$

Using the given initial conditions, we have

$$s^2Y(s) + 2sY(s) - 3Y(s) = \frac{1}{s^2+1}$$

and so

$$Y(s) = \frac{1}{(s^2+1)(s^2+2s-3)} = \frac{s-1}{2(s^2+1)} - \frac{s+1}{2(s^2+2s-3)}$$
$$= \frac{s}{2(s^2+1)} - \frac{1}{2(s^2+1)} - \frac{1}{2}\left[\frac{s+1}{(s+1)^2-4}\right].$$

Taking inverse Laplace transform, we have

$$y(t) = \frac{1}{2}\cos t - \frac{1}{2}\sin t - \frac{1}{2}e^{-t}\sinh 2t.$$

EXAMPLE 7.7

Solve

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = t^2e^{3t}, \; y(0) = 2, \; y'(0) = 6.$$

Solution. Taking Laplace transform, we get

$$s^2Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 9Y(s) = \frac{2}{(s-3)^3}.$$

Using initial conditions, we have

$$s^2Y(s) - 2s - 6 - 6sY(s) + 12 + 9Y(s) = \frac{2}{(s-3)^3}$$

or

$$(s^2 - 6s + 9)Y(s) = 2(s-3) + \frac{2}{(s-3)^3}$$

or

$$Y(s) = \frac{2}{s-3} + \frac{2}{(s-3)^5}.$$

Taking inverse Laplace transform yields

$$y(t) = 2e^{3t} + \frac{1}{12}t^4e^{-3t}.$$

EXAMPLE 7.8

Solve

$$y'' - 3y' + 2y = t, \quad y(0) = 0 \text{ and } y'(0) = 0.$$

Solution. Taking Laplace transform yields

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s^2}.$$

Making use of initial value conditions, we have

$$(s^2 - 3s + 2)Y(s) = \frac{1}{s^2}$$

and so

$$Y(s) = \frac{1}{s^2(s^2 - 3s + 2)} = \frac{1}{4(s-2)} - \frac{1}{(s-1)} + \frac{3}{4s} + \frac{1}{2s^2}.$$

Taking inverse Laplace transform, we get

$$y(t) = \frac{1}{4}e^{2t} - e^t + \frac{3}{4} + \frac{t}{2}.$$

EXAMPLE 7.9

Solve

$$y' + 2y = 1 - H(t-1), \quad y(0) = 2,$$

where $H(t)$ is Heaviside's unit step function.

Solution. Taking Laplace transform leads to

$$sY(s) + 2 + 2Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

or

$$(s+2)Y(s) = \frac{1}{s} - \frac{e^{-s}}{s} - 2$$

or

$$Y(s) = \frac{1}{s(s+2)} - \frac{e^{-s}}{s(s+2)} - \frac{2}{s+2}.$$

But, by partial fraction, we have

$$\frac{1}{s(s+2)} = \frac{1}{2s} - \frac{1}{2(s+2)}.$$

Therefore,

$$Y(s) = \frac{1}{2s} - \frac{1}{2(s+2)} - \left(\frac{e^{-s}}{2s} - \frac{e^{-s}}{2(s+2)}\right) - \frac{2}{s+2}.$$

Taking inverse transform, we get

$$y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} - \frac{1}{2}H(t-1) + \frac{1}{2}e^{-2(t-1)}H(t-1) + 2e^{-2t}$$
$$= \begin{cases} \frac{1}{2} + \frac{3}{2}e^{-2t} & \text{for } 0 \le t < 1 \\ \frac{3}{2}e^{-2t} + \frac{1}{2}e^{-2(t-1)} & \text{for } t \ge 1. \end{cases}$$

EXAMPLE 7.10

Solve

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution. Taking Laplace transform yields

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 5Y(s) = \frac{1}{(s+1)^2 + 1}.$$

Using initial conditions, we get

$$s^2Y(s) + 2sY(s) + 5Y(s) = 1 + \frac{1}{(s+1)^2 + 1}$$

or

$$(s^2 + 2s + 5)Y(s) = 1 + \frac{1}{(s+1)^2 + 1}$$

or

$$Y(s) = \frac{1}{s^2 + 2s + 5} + \frac{1}{(s^2 + 2s + 5)(s^2 + 2s + 2)} = \frac{s^2 + 2s + 3}{(s^2 + 2s + 5)\,(s^2 + 2s + 2)}.$$

Using partial fractions, we have

$$\begin{aligned} Y(s) &= \frac{2}{3(s^2+2s+5)} + \frac{1}{3(s^2+2s+2)} \\ &= \frac{2}{3((s+1)^2+1)}. \end{aligned}$$

Taking inverse transform, we get

$$\begin{aligned} y(t) &= \frac{2}{3}\cdot\frac{1}{2}e^{-t}\sin 2t + \frac{1}{3}e^{-t}\sin t \\ &= \frac{1}{3}e^{-t}(\sin t + \sin 2t). \end{aligned}$$

EXAMPLE 7.11

Solve the equation of motion

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} = mv_0\delta(t), \quad x(0) = x'(0) = 0,$$

which represents the motion of a pellet of mass m fired into a viscous gas from a gun at time $t = 0$ with a muzzle velocity v_0 and where $\delta(t)$ is Dirac delta function, $x(t)$ is displacement at time $t \geq 0$ and $k > 0$ is a constant.

Solution. The condition $x'(0) = 0$ implies that the pellet is initially at rest for $t < 0$. Taking the Laplace transform of both sides, we have

$$m[s^2X(s) - sx(0) - x'(0)] + k[s\,X(s) - x(0)] = 1.mv_0.$$

Using the given conditions, this expression reduces to

$$(ms^2 + ks)\ X(s) = mv_0$$

or

$$X(s) = \frac{mv_0}{ms^2+ks} = \frac{v_0}{s\left(s+\frac{k}{m}\right)}$$

Use of partial fractions yields

$$X(s) = \frac{v_0}{s\left(s+\frac{k}{m}\right)} = \frac{A}{s} + \frac{B}{s+\frac{k}{m}}$$

or

$$v_0 = A\left(s + \frac{k}{m}\right) + Bs.$$

Comparing coefficients, we get

$$v_0 = A\frac{k}{m}, \quad \text{which yields } A = \frac{mv_0}{k}$$

and

$$0 = A + B, \text{ which gives } B = -\frac{mv_0}{k}.$$

Hence

$$X(s) = \frac{mv_0}{ks} - \frac{mv_0}{k(s+\frac{k}{m})}.$$

Then, application of inverse Laplace transform yields

$$x(t) = \frac{mv_0}{k}(1 - e^{-\frac{k}{m}t}).$$

The graph of $x(t)$ is shown in the Figure 7.1

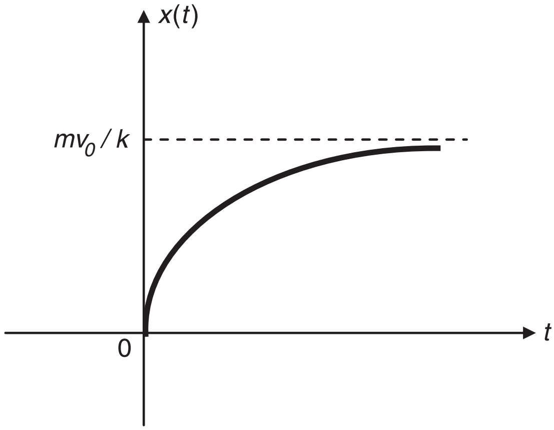

Figure 7.1

The velocity is given by

$$\frac{dx}{dt} = x'(t) = v_0e^{-\frac{k}{m}t}.$$

We observe that $\lim_{t\to 0+} x'(t) = v_0$ and $\lim_{t\to 0-} x'(t) = 0$. This indicates instantaneous jump in velocity at $t = 0$ from a rest state to the value v_0. The graph of $x'(t)$ is shown in the Figure 7.2.

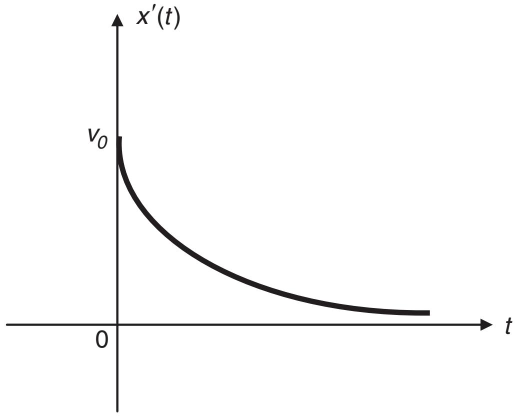

Figure 7.2

EXAMPLE 7.12

Solve boundary value problem

$$\frac{d^2y}{dt^2}+9y=\cos 2t,\quad y(0)=1,\quad y\left(\frac{\pi}{2}\right)=-1.$$

Solution. Suppose that $y'(0) = A$. Then taking Laplace transform, we have

$$s^2Y(s)-sy(0)-y'(0)+9\,Y(s)=\frac{s}{s^2+4}$$

or

$$(s^2+9)Y(s)=s+A+\frac{s}{s^2+4},$$

and so

$$\begin{aligned}Y(s)&=\frac{s+A}{s^2+9}+\frac{s}{(s^2+9)\,(s^2+4)}\\&=\frac{4s}{5(s^2+9)}+\frac{A}{s^2+9}+\frac{s}{5(s^2+4)}\\&\qquad\text{(partial fractions).}\end{aligned}$$

Taking inverse Laplace transform yields

$$y(t)=\frac{4}{5}\cos 3t+\frac{A}{3}\sin 3t+\frac{1}{5}\cos 2t.$$

Since $y\left(\frac{\pi}{2}\right)=1$, putting $t=\frac{\pi}{2}$, we get $A=\frac{12}{5}$. Hence

$$y(t)=\frac{4}{5}\cos 3t+\frac{4}{5}\sin 3t+\frac{1}{5}\cos 2t.$$

EXAMPLE 7.13

Solve

$$\frac{d^2x}{dt^2}+6\frac{dx}{dt}+9x=\sin t\ (t\geq 0)$$

subject to the conditions $x(0)=x'(0)=0$.

Solution. Taking Laplace transform of both sides of the given equations yields

$$s^2X(s)-sx(0)-x'(0)+6(sX(s)-x(0))+9X(s)=\frac{1}{s^2+1}.$$

Using the initial conditions, we have

$$(s^2+6s+9)X(s)=\frac{1}{s^2+1}$$

or

$$\begin{aligned}X(s)&=\frac{1}{(s^2+1)\,(s+3)^2}\\&=\frac{A}{s+3}+\frac{B}{(s+3)^2}+\frac{Cs+D}{s^2+1}.\end{aligned}$$

Comparing coefficients of the powers of s, we get

$$A=\frac{3}{50},\qquad B=\frac{1}{10},\qquad C=-\frac{3}{50},\qquad D=\frac{2}{25}.$$

Hence

$$X(s)=\frac{3}{50(s+3)}+\frac{1}{10(s+3)^2}-\frac{3s}{50(s^2+1)}+\frac{2}{25(s^2+1)}.$$

Application of inverse Laplace transform gives

$$\begin{aligned}x(t)&=\frac{3}{50}e^{-3t}+\frac{e^{-3t}t}{10}-\frac{3}{50}\cos t+\frac{2}{25}\sin t\\&=\frac{e^{-3t}}{50}(5t+3)-\frac{3}{50}\cos t+\frac{2}{25}\sin t.\end{aligned}$$

The term $\frac{e^{-3t}}{50}(5t+3)$ is the particular solution, called the *transient response* since it dies away for large time, whereas the terms $-\frac{3}{50}\cos t+\frac{2}{25}\sin t$ is called the complementary function (sometimes called *steady state response* by engineers since it persists). However, there is nothing steady about it.

(b) Problems Related to Electrical Circuits

Consider the RCL circuit, shown in the Figure 7.3, consisting of *resistance*, *capacitor*, and *inductor* connected to a battery.

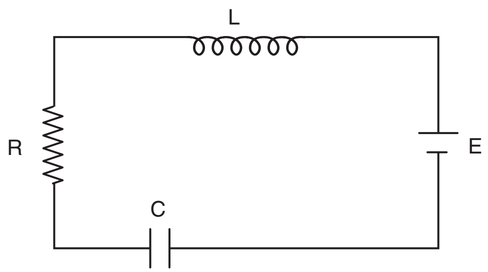

Figure 7.3

We know that resistance R is measured in *ohms*, capacitance C is measured in *farads*, and inductance is measured in *henrys*.

Let I denote the current flowing through the circuit and Q denote the charge. Then current I is related to Q by the relation $I = \frac{dQ}{dt}$. Also

(a) By Ohm's law, $\frac{V}{I} = R$ (resistance). Therefore, the voltage drop V across a resistor R is RI.

(b) The voltage drop across the inductor L is $L\frac{dI}{dt}$.

(c) The voltage drop across a capacitor is $\frac{Q}{C}$.

Thus, if E is the voltage (potential difference) of the battery, then by Kirchhoff's law, we have

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t),$$

where L, C, and R are constants. In terms of current, this equation becomes

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(u)\, du = E(t),$$

because $I = \frac{dQ}{dt}$ implies $Q = \int_0^t I(u)\, du$.

In terms of charge, this differential equation takes the form

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t),$$

which is a differential equation of second order with constant coefficients L, R, and 1/C. The forcing function (input function) E(t) is supplied by the battery (voltage source). The system described by the above differential equation is known as *harmonic oscillator*.

EXAMPLE 7.14

Given that $I = Q = 0$ at $t = 0$, find I in the LR circuit (Figure 7.4) for $t > 0$.

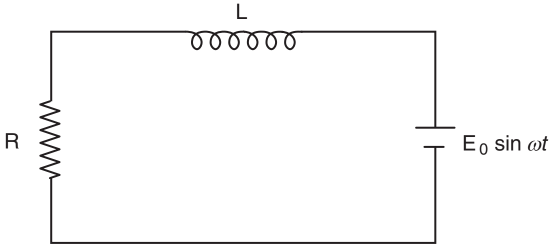

Figure 7.4

Solution. By Kirchhoff's law, the differential equation governing the given circuit is

$$L\frac{dI}{dt} + RI = E_0 \sin \omega t, \qquad I(0) = 0,$$

where L, R, E_0, and ω are constants. Taking Laplace transform of both sides, we have

$$L[sF(s) - I(0)] + RF(s) = \frac{E_0\omega}{s^2 + \omega^2},$$

where F(s) denotes the Laplace transform of I. Using the given initial condition, we have

$$(Ls + R)F(s) = \frac{E_0\omega}{s^2 + \omega^2}$$

which yields

$$F(s) = \frac{E_0\omega}{(Ls + R)\,(s^2 + \omega^2)} = \frac{E_0\frac{\omega}{L}}{\left(s + \frac{R}{L}\right)(s^2 + \omega^2)}$$
$$= \frac{A}{\left(s + \frac{R}{L}\right)} + \frac{Bs + C}{s^2 + \omega^2}.$$

Comparison of coefficients of different powers of s yields

$$A = \frac{E_0L\omega}{L^2\omega^2 + R^2},\ B = \frac{-E_0L\omega}{L^2\omega^2 + R^2},\ C = \frac{E_0R\omega}{L^2\omega^2 + R^2}.$$

Hence

$$F(s) = \frac{E_0L\omega}{\left(s+\frac{R}{L}\right)(L^2\omega^2 + R^2)} - \frac{sE_0L\omega}{(s^2+\omega^2)(L^2\omega^2+R^2)} + \frac{E_0R\omega}{(s^2+\omega^2)(L^2\omega^2+R^2)}.$$

Taking inverse Laplace transform yields

$$I(t) = \frac{E_0L\omega}{L^2\omega^2 + R^2}e^{-\frac{R}{L}t} - \frac{E_0L\omega}{L^2\omega^2 + R^2}\cos \omega t + \frac{E_0R}{L^2\omega^2 + R^2}\sin \omega t.$$

EXAMPLE 7.15

Given that $I = Q = 0$ at $t = 0$, find charge Q and current I in the following circuit (Figure 7.5) for $t > 0$.

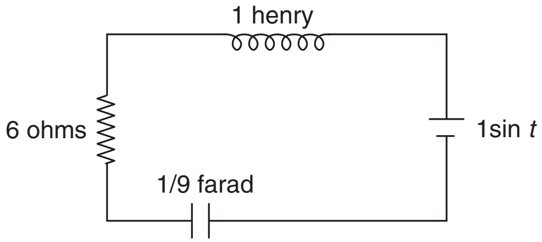

Figure 7.5

Solution. By Kirchhoff's law, the differential equation for the given circuit is

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=E(t).$$

Here L = 1, R = 6, C = $\frac{1}{9}$, E(t) = sin t. Thus we have

$$\frac{d^2Q}{dt^2}+6\frac{dQ}{dt}+9Q=\sin t(t>0),$$

subject to Q(0) = 0, Q′(0) = I(0) = 0. By Example 7.13, the solution of this equation is

$$Q(t)=\frac{e^{-3t}}{50}(5t+3)-\frac{3}{50}\cos t+\frac{2}{25}\sin t.$$

Then

$$I(t)=\frac{dQ}{dt}=\frac{5e^{-3t}}{50}-\frac{3}{50}(5t+3)e^{-3t}+\frac{3}{50}\sin t+\frac{2}{25}\cos t$$
$$=-\frac{e^{-3t}}{50}(15t+4)+\frac{3}{50}\sin t+\frac{2}{25}\cos t.$$

EXAMPLE 7.16

Solve

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=\delta(t)\text{ (Dirac delta function)}$$

under conditions $q(0)=q'(0)=0$.

Solution. Applying Laplace transform to both sides of the given equation, we find

$$\left(Ls^2+Rs+\frac{1}{C}\right)Q(s)=1$$

or

$$Q(s)=\frac{1}{Ls^2+Rs+\frac{1}{C}}=\frac{1}{L\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)}.$$

Suppose the roots of $s^2+\frac{R}{L}s+\frac{1}{LC}$ are s_1 and s_2. Then

$$s_1=\frac{-R+\sqrt{R^2-(4L/C)}}{2L}\text{ and }s_2=\frac{-R-\sqrt{R^2-(4L/C)}}{2L}.$$

Let us suppose R > 0. Then three cases arise:

(a) If $R^2-\frac{4L}{C}<0$, then s_1 and s_2 are complex and $s_1=\bar{s}_2$.

(b) If $R^2-\frac{4L}{C}=0$, then s_1 and s_2 are real and $s_1=s_2$.

(c) If $R^2-\frac{4L}{C}>0$, then s_1 and s_2 are real and $s_1\neq s_2$.

Case (a). Using partial fractions, we have

$$Q(s)=\frac{1}{L(s-s_1)(s-s_2)}=\frac{1}{L(s_1-s_2)}\left[\frac{1}{s-s_1}-\frac{1}{s-s_2}\right].$$

Taking inverse Laplace transform yields

$$q(t)=\frac{1}{L(s_1-s_2)[e^{s_1t}-e^{s_2t}]}.$$

If we put

$$\omega_0=\frac{1}{2L}\sqrt{\frac{4L}{C}-R^2}\quad\text{and}\quad\sigma=-\frac{R}{L},$$

then $s_1=\bar{s}_2=\sigma+i\omega_0$ and so $s_1-s_2=2i\omega_0$. Therefore,

$$q(t)=\frac{1}{2Li\omega_0}\left(e^{(\sigma+i\omega_0)t}-e^{(\sigma-i\omega_0)t}\right)$$
$$=\frac{1}{L\omega_0}e^{\sigma t}\frac{e^{i\omega_0t}-e^{-i\omega_0t}}{2i}$$
$$=\frac{1}{L\omega_0}e^{\sigma t}\sin\omega_0t,\quad\sigma<0.$$

Thus, the impulse response $q(t)$ is a *damped sinusoidal* with frequency ω_0. That is why, this case is called *damped vibration* or *undercritical damping* (Figure 7.6).

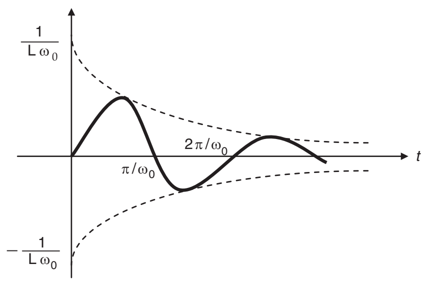

Figure 7.6

Case (b) In this case $s_1 = s_2 = -\frac{R}{2L}$ and so

$$Q(s) = \frac{1}{L(s-\sigma)^2}.$$

Taking inverse transform, we get

$$q(t) = \frac{te^{\sigma t}}{L}, \quad \sigma < 0.$$

This case is called *critical damping* (Figure 7.7)

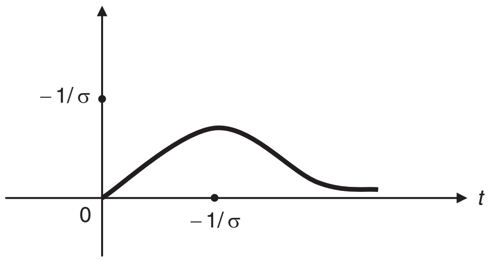

Figure 7.7

Case (c) As in case (a), we have

$$q(t) = \frac{1}{L(s_1 - s_2)} \; (e^{s_1 t} - e^{s_2 t}).$$

Since L > 0 and C > 0, we have R > $\sqrt{R^2 - \frac{4L}{C}}$ and so $s_2 < s_1 < 0$. Thus $q(t)$ is the sum of two exponentially damped functions. Put

$$\omega_0 = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}} \quad \text{and } \sigma = -\frac{R}{2L}.$$

Then, we have

$$s_1 = \sigma + \omega_0, \quad s_2 = \sigma - \omega_0.$$

Therefore,

$$s_1 - s_2 = 2\omega_0$$

and

$$q(t) = \frac{1}{2L\omega_0}(e^{(\sigma+\omega_0)t} - e^{(\sigma-\omega_0)t}) = \frac{1}{L\omega_0}\sinh \omega_0 t$$
$$= \frac{1}{2L\omega_o}e^{(\sigma+\omega_0)t}(1 - e^{-2\omega_0 t}), \quad \sigma < 0.$$

Since $\sigma + \omega_0 < 0$, the impulse response $q(t)$ is damped hyperbolic sine. This case is called *overdamped* or *overcritical damping* (Figure 7.8).

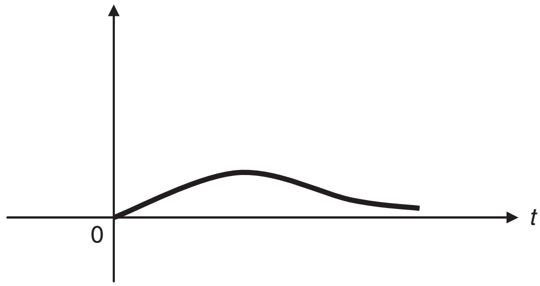

Figure 7.8

(c) Mechanical System (Mass-Spring System)

Let m be the mass suspended on a spring that is rigidly supported from one end (Figure 7.9). The rest position is denoted by $x = 0$, downward displacement by $x > 0$, and upward displacement is represented by $x < 0$. Let

(i) $k > 0$ be the *spring constant* (or *stiffness*) and $a > 0$ be the *damping constant*.

(ii) $a\,\frac{dx}{dt}$ be the damping force due to medium (air, etc.). Thus, damping force is proportional to the velocity.

(iii) $f(t)$ represents all external impressed forces on m. It is also called *forcing* or *excitation*.

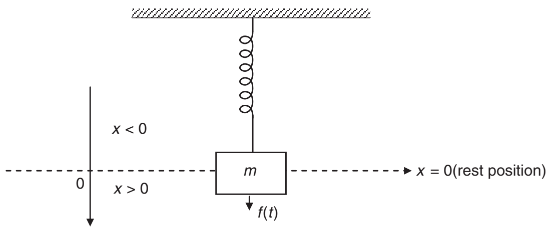

Figure 7.9

By Newton's second law of motion, the sum of forces acting on m equals $m\frac{d^2x}{dt^2}$ and so

$$m\frac{d^2x}{dt^2} = -kx - a\frac{dx}{dt} + f(t).$$

Thus the equation of motion is

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = f(t) \tag{1}$$

This is exactly the same differential equation which occurs in harmonic oscillator.

If $a = 0$, the motion is called *undamped* whereas if $a \neq 0$, the motion is called *damped*. Moreover, if $f(t) = 0$, that is, if there is no impressed forces, then the motion is called *forced*.

The equation (1) can be written as

$$\frac{d^2x}{dt^2} + \frac{a}{m}\frac{dx}{dt} + \frac{k}{m} = f(t)/m, \qquad (2)$$

where $f(t)/m$ is now the external impressed force (or excitation force) per unit mass.

EXAMPLE 7.17

Solve the equation of motion

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \lambda^2 x = \delta(t), x(0) = x'(0) = 0$$

for $0 < b < \lambda$.

[Clearly this is equation (2) with $\frac{a}{m} = 2b$, $\frac{k}{m} = \lambda^2$]

Solution. We want to find the response of the given mechanical system to a unit impulse. Taking Laplace transform, we get

$$\{s^2X(s) - sx(0) - x'(0)\} + 2b\{sX(s) - x(0)\} + \lambda^2 X(s) = 1.$$

Taking note of the given conditions, we have

$$(s^2 + 2bs + \lambda^2)\, X(s) = 1$$

or

$$X(s) = \frac{1}{s^2 + 2bs + \lambda^2} = \frac{1}{(s+b)^2 + (\lambda^2 - b^2)}.$$

Taking inverse Laplace transform yields

$$x(t) = e^{-bt}\left(\frac{1}{\sqrt{\lambda^2 - b^2}} \sin\sqrt{\lambda^2 - b^2}\; t\right),$$

which is clearly a case of damped oscillation (Figure 7.10).

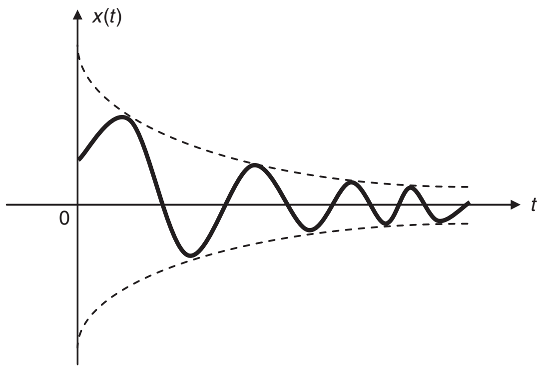

Figure 7.10

Also we note that

$$X(s) = \frac{1}{s^2 + 2bs + \lambda^2}\; L\{\delta(t)\}.$$

Thus we conclude that

Response = Transfer function × Input.

(d) Ordinary Differential Equations with Polynomial (Variable) Coefficients

We know that

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s),$$

where $F(s) = L\{f(t)\}$. Thus for $n = 1$, we have

$$L\{tf(t) = -F'(s).$$

Hence, if $f'(t)$ satisfies the sufficient condition for the existence of Laplace transform, then

$$L\{tf'(t) = -\frac{d}{ds}L\{f'(t)\} = -\frac{d}{ds}(sF(s) - f(0))$$
$$= -sF'(s) - F(s).$$

Similarly for $f''(t)$,

$$L\{tf''(t)\} = -\frac{d}{ds}L\{f''(t)\} = -\frac{d}{ds}\{s^2F(s) - sf(0) - f'(0)\}$$
$$= -s^2F'(s) - 2sF(s) + f(0).$$

The above-mentioned derivations are used to solve linear differential equations whose coefficients are first degree polynomials.

EXAMPLE 7.18

Solve

$$ty'' + y' + ty = 0,\; y(0) = 1, \quad y'(0) = 0.$$

Solution. Taking Laplace transform, we have

$$L\{ty''\} + L\{y'\} + L\{ty\} = 0$$

or

$$-\frac{d}{ds}L\{y''(t)\} + \{sY(s) - y(0)\} - \frac{d}{ds}\{Y(s)\} = 0$$

or

$$-\frac{d}{ds}\{s^2Y(s) - sy(0) - y'(0)\} + \{sY(s) - y(0)\}$$
$$-\frac{d}{ds}Y(s) = 0$$

which on using initial conditions yields

$$-\left[s^2\frac{dY(s)}{ds} + 2sY(s)\right] + sY(s) - \frac{dY(s)}{ds} = 0$$

or

$$(s^2+1)\frac{dY(s)}{ds}+sY(s)=0$$

or

$$\frac{dY(s)}{Y(s)}+\frac{s\,ds}{s^2+1}=0.$$

Integrating, we have

$$\log Y(s)+\frac{1}{2}\log(s^2+1)=A \text{ (constant)}$$

and so

$$Y(s)=\frac{A}{\sqrt{s^2+1}}.$$

Taking inverse Laplace transform, we get

$$y(t)=A\,J_0(t),$$

where $J_0(t)$ is Bessel function of order zero.

Putting $t=0$ and using initial condition $y(0)=1$, we have

$$1=AJ_0(0)=A.$$

Hence the required solution is

$$y(t)=J_0(t).$$

EXAMPLE 7.19

Solve

$$y''+ty'-2y=4,\ y(0)=-1,\ y'(0)=0.$$

Solution. Taking Laplace transform yields

$$L\{y''(t)\}+L\{ty'(t)\}-2L\{y(t)\}=4L\{1\}$$

or

$$s^2Y(s)-sy(0)-y'(0)-\frac{d}{ds}L\{y'(t)\}-2Y(s)=\frac{4}{s}$$

or

$$s^2Y(s)-sy(0)-y'(0)-\frac{d}{ds}(sY(s)-y(0))$$
$$-2Y(s)=\frac{4}{s}.$$

On using the initial values, we have

$$s^2Y(s)+s-\left(s\frac{dY(s)}{ds}+Y(s)\right)-2Y(s)=\frac{4}{s}$$

or

$$\frac{s\,dY(s)}{ds}-(s^2-3)Y(s)=-\frac{4}{s}+s$$

or

$$\frac{dY(s)}{ds}+\left(\frac{3}{s}-s\right)Y(s)=-\frac{4}{s^2}+1.$$

The integrating factor is

$$e^{\int\left(\frac{3}{s}-s\right)ds}=s^3e^{-\frac{s^2}{2}}.$$

Therefore,

$$\frac{d}{ds}[Y(s).s^3e^{-\frac{s^2}{2}}]=-\frac{4}{s^2}s^3e^{-\frac{s^2}{2}}+s^3e^{-\frac{s^2}{2}},$$

and so integration yields

$$Y(s)s^3e^{-\frac{s^2}{2}}=-4\int se^{-\frac{s^2}{2}}ds+\int s^3e^{-\frac{s^2}{2}}ds.$$

Putting $u=-\frac{s^2}{2}$, we get

$$\begin{aligned}Y(s)s^3\ e^{-\frac{s^2}{2}}&=4\int e^u du+2\int u\,e^u du\\&=4e^{-\frac{s^2}{2}}+2\left(\frac{-s^2}{2}e^{-\frac{s^2}{2}}-e^{-\frac{s^2}{2}}\right)+A\\&=2e^{-\frac{s^2}{2}}-s^2\ e^{-\frac{s^2}{2}}+C.\end{aligned}$$

Thus,

$$Y(s)=\frac{2}{s^3}-\frac{1}{s}+\frac{C}{s^3}\ e^{\frac{s^2}{2}}.$$

Since $Y(s)\to 0$ as $s\to\infty$, we must have $C=0$ and so

$$Y(s)=\frac{2}{s^3}-\frac{1}{s}.$$

Taking inverse Laplace transform, we get

$$y(t)=t^2-1.$$

EXAMPLE 7.20

Solve

$$ty''+2y'+ty=0,\qquad y(0)=1,\qquad y(\pi)=0.$$

Solution. Let $y'(0)=A$ (constant). Taking Laplace transform of both sides, we obtain

$$-\frac{d}{ds}\{s^2Y(s)-sy(0)-y'(0)\}+2\{sY(s)-y(0)\}$$
$$-\frac{d}{ds}\{Y(s)\}=0$$

and so

$$-s^2Y'(s)-2sY(s)+y(0)+2sY(s)-2y(0)$$
$$-Y'(s)=0.$$

Using boundary conditions, we get

$$-(s^2+1)Y'(s)-1=0$$

or

$$Y'(s)=\frac{-1}{s^2+1}.$$

Integration yields

$$Y(s)=-\tan^{-1}s+\text{ B (constant)}.$$

Since $Y(s)$ tends to zero as $s\to\infty$, we must have $B=\pi/2$. Hence,

$$Y(s)=\frac{\pi}{2}-\tan^{-1}s=\tan^{-1}\left(\frac{1}{s}\right).$$

Taking inverse Laplace transform, we have (see Example 21.48).

$$y(t)=L^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\}=\frac{\sin t}{t}.$$

This solution clearly satisfies $y(\pi)=0$.

EXAMPLE 7.21

Solve

$$ty''+y'+2y=0,\quad y(0)=1.$$

Solution. Taking Laplace transform gives

$$-\frac{d}{ds}(s^2Y(s)-sy(0)-y'(0))+(sY(s)-y(0))$$
$$+2Y(s)=0$$

or

$$-s^2Y'(s)-2sY(s)+y(0)+sY(s)-y(0)+2Y(s)=0$$

or

$$-s^2Y'(s)-sY(s)+2Y(s)=0$$

or

$$Y'(s)+\left(\frac{1}{s}-\frac{2}{s^2}\right)Y(s)=0.$$

The integrating factor is

$$e^{\int\left(\frac{1}{s}-\frac{2}{s^2}\right)ds}=e^{\log s+\frac{2}{s}}=se^{\frac{2}{s}}.$$

Therefore,

$$\frac{d}{ds}\{Y(s)se^{2/s}\}=0.$$

Integrating, we have

$$Y(s)\,se^{2/s}=A\text{ (constant)}$$

or

$$Y(s)=\frac{A\,e^{-\frac{2}{s}}}{s}.$$

Since $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, taking $x=-\frac{2}{s}$, we have

$$Y(s)=A\sum_{n=0}^{\infty}\frac{(-1)^n2^n}{n!\,s^{n+1}}.$$

Taking inverse Laplace transform,we get

$$y(t)=A\sum_{n=0}^{\infty}\frac{(-1)^n\,2^nt^n}{(n!)^2}.$$

The condition $y(0)=1$ now yields $A=1$. Hence

$$y(t)=\sum_{n=0}^{\infty}\frac{(-1)^n\,2^nt^n}{(n!)^2}=J_0(2\sqrt{2t}),$$

where J_0 is Bessel's function of order zero.

EXAMPLE 7.22

Solve

$$ty''-y'=-1,\quad y(0)=0.$$

Solution. Taking Laplace transform of both sides of the given equation,

$$-\frac{d}{ds}\{s^2Y'(s)-sy(0)-y'(0)\}-\{sY(s)-y(0)\}=-\frac{1}{s}$$

or

$$-s^2Y'(s)-2sY(s)+y(0)-sY(s)+y(0)=-\frac{1}{s}$$

or

$$-s^2Y'(s)-3sY(s)=-\frac{1}{s}$$

or

$$Y'(s)+\frac{3}{s}Y(s)=\frac{1}{s^3}.$$

The integrating factor is

$$e^{\int\frac{3}{s}ds}=e^{3\log s}=s^3.$$

Therefore,

$$\frac{d}{ds}(Y(s)s^3)=\frac{1}{s^3}s^3=1.$$

Integrating

$$Y(s)s^3=s+A(\text{constant})$$

and so

$$Y(s)=\frac{1}{s^2}+\frac{A}{s^3}.$$

Taking inverse Laplace transform, we get

$$y(t) = t + Bt^2,$$

where B is constant. Obviously, the solution satisfies $y(0) = 0$.

EXAMPLE 7.23

Solve

$$ty'' + (t+1)y' + 2y = e^{-t},\ y(0) = 0.$$

Solution. Taking Laplace transform of both sides gives

$$-\frac{d}{ds}\{s^2Y(s) - sy(0) - y'(0)\} - \frac{d}{ds}\{sY(s) - y(0)\}$$
$$+\{sY(s) - y(0)\} + 2Y(s) = \frac{1}{s+1}$$

or

$$-s^2Y'(s) - 2sY(s) - \{sY'(s) + Y(s)\} + sY(s)$$
$$-y(0) + 2Y(s) = \frac{1}{s+1}$$

or

$$-s^2Y'(s) - sY'(s) - 2sY(s) + sY(s) + Y(s) = \frac{1}{s+1}$$

or

$$-(s^2+s)Y'(s) - (s-1)Y(s) = \frac{1}{s+1}$$

or

$$Y'(s) + \frac{s-1}{s^2+s} = \frac{-1}{s(s+1)^2}.$$

The integration factor is

$$e^{\int \frac{s-1}{s^2+s}\,ds} = e^{\int \left(-\frac{1}{s}+\frac{2}{s+1}\right)ds} = \frac{(s+1)^2}{s}.$$

Therefore,

$$\frac{d}{ds}\left(Y(s)\frac{(s+1)^2}{s}\right) = \frac{-1}{s(s+1)^2}\cdot\frac{(s+1)^2}{s} = -\frac{1}{s^2}.$$

Integrating, we get

$$Y(s)\,\frac{(s+1)^2}{s} = \int -\frac{1}{s^2}\,ds = \frac{1}{s} + C$$

and so

$$Y(s) = \frac{1}{(s+1)^2} + \frac{Cs}{(s+1)^2}.$$

By initial value theorem $y(0) = \lim_{s\to 0} sY(s) = 0$ and so $C = 0$. Hence

$$Y(s) = \frac{1}{(s+1)^2}.$$

Taking inverse Laplace transform, we get

$$y(t) = te^{-t}.$$

7.2 SIMULTANEOUS DIFFERENTIAL EQUATIONS

The Laplace transforms convert a pair of differential equations into simultaneous algebraic equations in parameters. After that we solve these equations for Laplace transforms of the variables and then apply inverse Laplace operators to get the required solution.

EXAMPLE 7.24

Solve the simultaneous differential equations

$$3x' + y' + 2x = 1, \quad x' + 4y' + 3y = 0$$

subject to the conditions $x(0) = 0$, $y(0) = 0$.

Solution. Taking Laplace transform, we get

$$3\{sX(x) - x(0)\} + \{sY(s) - y(0)\} + 2X(x) = \frac{1}{s}$$

and

$$sX(s) - x(0) + 4\{sY(s) - y(0)\} + 3Y(s) = 0.$$

Using the initial conditions, these equations reduce to

$$(3s+2)X(s) + sY(s) = \frac{1}{s} \qquad (3)$$

and

$$sX(s) + (4s+3)Y(s) = 0. \qquad (4)$$

Multiplying (3) and (4) by s and $(3s+2)$ respectively and then subtracting, we get

$$Y(s) = -\frac{1}{11s^2+17s+6} = \frac{-1}{(11s+6)(s+1)},$$

and then using (4), we have

$$X(s) = \frac{4s+3}{s(11s+6)(s+1)}.$$

We deal with X(s) first. Using partial fraction, we have

$$X(s) = \frac{1}{2s} - \frac{3}{10}\left(\frac{1}{s+(6/11)}\right) - \frac{1}{5(s+1)}.$$

Taking inverse transform, we have

$$\mathrm{x}(t) = \frac{1}{2} - \frac{3}{10}e^{-\frac{6}{11}t} - \frac{1}{5}e^{-t}$$
$$= \frac{1}{10}(5 - 3e^{-\frac{6}{\pi}t} - 2e^{-t}).$$

Further, poles of Y(s) are $-\frac{6}{11}$ and –1. Hence

$$y(t) = \lim_{s\to\frac{-6}{11}}\left(s + \frac{6}{11}\right)\mathrm{Y}(s)\,e^{-\frac{6}{11}t}$$
$$+ \lim_{s\to-1}(s+1)\,\mathrm{Y}(s)e^{-t} = \frac{1}{5}\,(e^{-t} - e^{-\frac{6}{11}t}).$$

EXAMPLE 7.25

Solve the simultaneous differential equations

$$\frac{dx}{dt} = 2x - 3y, \quad \frac{dy}{dt} = y - 2x$$

subject to the conditions $x(0) = 8$, $y(0) = 3$.

Solution. Taking Laplace transform and using the given conditions, we have

$$s\mathrm{X}(s) = 2\mathrm{X}(s) - 3\mathrm{Y}(s) + 8$$

and

$$s\mathrm{Y}(s) = \mathrm{Y}(s) - 2\mathrm{X}(s) + 3.$$

Thus

$$(s-2)\mathrm{X}(s) + 3\mathrm{Y}(s) = 8,$$

and

$$2\mathrm{X}(s) + (s-1)\mathrm{Y}(s) = 3.$$

Solving these algebraic equations, we get

$$\mathrm{X}(s) = \frac{8s - 17}{s^2 - 3s - 4} = \frac{8s-17}{(s+1)\,(s-4)},$$

and

$$\mathrm{Y}(s) = \frac{3s-22}{s^2 - 3s - 4} = \frac{3s-22}{(s+1)\,(s-4)}.$$

Using partial fractions, these yields

$$\mathrm{X}(s) = \frac{5}{s+1} + \frac{3}{s-4}, \mathrm{Y}(s) = \frac{5}{s+1} - \frac{2}{s-4}.$$

Hence taking inverse Laplace transform, we get

$$x(t) = 5e^{-t} + 3e^{4t}, \quad y(t) = 5e^{-t} - 2e^{4t}.$$

EXAMPLE 7.26

Solve

$$\frac{dx}{dt} - \mathrm{y} = e^t, \quad \frac{dy}{dt} + x = \sin t$$

subject to the conditions $x(0) = 1$, $y(0) = 0$.

Solution. Taking Laplace transform and using the given conditions, we have

$$s\mathrm{X}(s) - \mathrm{Y}(s) = \frac{1}{s-1} + 1 = \frac{s}{s-1},$$
$$s\mathrm{Y}(s) + \mathrm{X}(s) = \frac{1}{s^2+1}.$$

Solving these equations, we get

$$\mathrm{X}(s) = \frac{s^4 + s^2 + s - 1}{(s-1)\,(s^2+1)^2},$$
$$\mathrm{Y}(s) = \frac{-s^4 + s^3 - 2s^2}{s(s-1)\,(s^2+1)^2} = \frac{-s^3 + s^2 - 2s}{(s-1)\,(s^2+1)^2}.$$

Now

$$\mathrm{X}(s) = \frac{s^4 + s^2 + s - 1}{(s-1)\,(s^2+1)^2}$$
$$= \frac{A}{s-1} + \frac{Bs + C}{(s^2+1)} + \frac{Ds+E}{(s^2+1)^2}.$$

Comparison of coefficients yields

$$\mathrm{A} = \mathrm{B} = \mathrm{C} = \frac{1}{2} \quad \text{and} \quad \mathrm{E} = 1.$$

Thus

$$\mathrm{X}(s) = \frac{1}{2(s-1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)} + \frac{1}{(s^2+1)^2}.$$

Hence

$$x(t) = \frac{1}{2}e^t + \frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}(\sin t - t\cos t)$$
$$= \frac{1}{2}[e^t + \cos t + 2\sin t - t\cos t].$$

Now consider Y(s). We have

$$\mathrm{Y}(s) = \frac{-s^3 + s^2 - 2s}{(s-1)\,(s^2+1)^2}$$
$$= \frac{A}{s-1} + \frac{Bs + C}{(s^2+1)} + \frac{Ds+E}{(s^2+1)^2}.$$

Comparing coefficients, we get

$$\mathrm{A} = -\frac{1}{2}, \mathrm{B} = \frac{1}{2}, \mathrm{C} = -\frac{1}{2}, \mathrm{D} = 2, \mathrm{E} = 0,$$

and so

$$\mathrm{Y}(s) = \frac{-1}{2(s-1)} + \frac{s}{2(s^2+1)} - \frac{1}{2(s^2+1)} + \frac{2s}{(s^2+1)^2}.$$

Hence

$$y(t) = -\frac{1}{2}e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t + t\sin t.$$

EXAMPLE 7.27

The co-ordinates (x, y) of a particle moving along a plane curve at any time t are given by

$$\frac{dy}{dt} + 2x = \sin 2t, \quad \frac{dx}{dt} - 2y = \cos 2t, \quad t > 0.$$

If at $t = 0$, $x = 1$ and $y = 0$, show by using transforms, that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.

Solution. Using Laplace transform, we get

$$sY(s) - y(0) + 2X(s) = \frac{2}{s^2+4}$$

and

$$sX(s) - x(0) - 2Y(s) = \frac{s}{s^2+4}.$$

Using the given conditions, we have

$$sY(s) + 2X(s) = \frac{2}{s^2+4}$$

and

$$sX(s) - 2Y(s) = 1 + \frac{s}{s^2+4} = \frac{s^2+s+4}{s^2+4}.$$

Solving for X(s) and Y(s), and using partial fractions, we have

$$X(s) = \frac{s^3+s^2+4s+4}{(s^2+4)^2} = \frac{s}{s^2+4} + \frac{1}{s^2+4}$$
$$= \frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4},$$
$$Y(s) = \frac{-2s^2-8}{(s^2+4)^2} = -\frac{2}{s^2+4}.$$

Hence taking inverse transform, we get

$$x(\mathrm{t}) = \cos 2\mathrm{t} + \frac{1}{2}\sin 2t,$$
$$y(\mathrm{t}) = -\sin 2\mathrm{t}.$$

We observe that

$$4x^2 + 4xy + 5y^2 = 4(\cos^2 2t + \sin^2 2t) = 4,$$

and hence the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.

EXAMPLE 7.28

Solve the following system of equations:

$$x(t) - y''(t) + y(t) = e^{-t} - 1,$$
$$x'(t) + y'(t) - y(t) = -3e^{-t} + t,$$

subject to $x(0) = 0$, $y(0) = 1$, $y'(0) = -2$.

Solution. Taking Laplace transform yields

$$X(s) - \{s^2Y(s) - sy(0) - y'(0)\} + Y(s) = \frac{1}{s+1} - \frac{1}{s}$$

and

$$sX(s) - x(0) + sY(s) - y(0) - Y(s) = \frac{-3}{s+1} + \frac{1}{s^2}.$$

Using the given conditions, we have

$$X(s) - s^2Y(s) + s - 2 + Y(s) = \frac{-1}{s(s+1)}$$

and

$$sX(s) + sY(s) - 1 - Y(s) = \frac{-3s^2+s+1}{(s+1)s^2}$$

or

$$X(s) - (s^2-1)Y(s) = 2 - s - \frac{1}{s(s+1)}$$
$$= \frac{-s^3+s^2+2s-1}{s(s+1)}$$

and

$$sX(s) + (s-1)Y(s) = 1 - \frac{3s^2-s-1}{(s+1)s^2}$$
$$= \frac{s^3-2s^2+s+1}{(s+1)s^2}.$$

Solving for X(s) and Y(s), we have

$$X(s) = \frac{1}{s^2(s+1)} = \frac{1}{s^2} + \frac{1}{s+1} - \frac{1}{s},$$

$$Y(s) = \frac{s^2-s-1}{s^2(s+1)} = \frac{1}{s+1} - \frac{1}{s^2}.$$

Hence, taking inverse Laplace transform, we get

$$x(t) = t + e^{-t} - 1, y(t) = e^{-t} - t.$$

EXAMPLE 7.29

Given that I(0) = 0, find the current I in RL-network shown in the Figure 7.11.

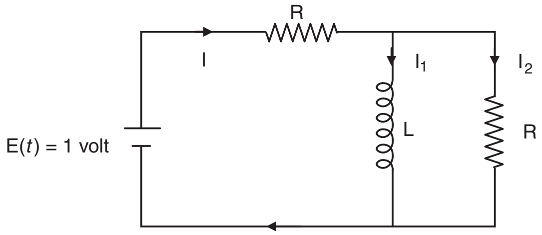

Figure 7.11

Solution. We note that $I = I_1 + I_2$ and so $RI = RI_1 + RI_2$, or equivalently, $RI_2 = RI - RI_1$. By Kirchhoff's law, we have

(a) In the closed loop containing R and L,

$$RI + L\frac{dI_1}{dt} = E = 1 \tag{5}$$

(b) In the closed loop containing two resistances R,

$$RI + RI_2 = E = 1$$

or

$$RI + RI - RI_1 = 1$$

or

$$2RI - RI_1 = 1. \tag{6}$$

We want to solve (5) and (6) under the conditions $I(0) = I_1(0) = 0$. Taking Laplace transform yields

$$RF(s) + L\{sG(s)I_1(0)\} = \frac{1}{s}$$

and

$$2R\ F(s) - RG(s) = \frac{1}{s}.$$

Using $I_1(0) = 0$, we have

$$RF(s) + LsG(s) = \frac{1}{s} \tag{7}$$

and

$$2RF(s) - RG(s) = \frac{1}{s}. \tag{8}$$

Multiplying (7) by R and (8) by Ls and adding, we get

$$(R^2 + 2RLs)F(s) = \frac{R}{s} + L = \frac{R + Ls}{s}$$

or

$$F(s) = \frac{R+Ls}{Rs(R+2Ls)} = \frac{1}{R}\left(\frac{R+Ls}{s(R+2Ls)}\right).$$

Using partial fractions, we get

$$F(s) = \frac{1}{R}\left(\frac{1}{s} - \frac{1}{2(s+(R/2L))}\right).$$

Taking inverse Laplace transform yields

$$I(t) = \frac{1}{R}\left[1 - \frac{1}{2}e^{-\frac{R}{2L}t}\right] = \frac{1}{2R}\left[2 - e^{-\frac{R}{2L}t}\right].$$

7.3 DIFFERENCE EQUATIONS

A relationship between the values of a function $y(t)$ and the values of the function at different arguments $y(t + h)$, h constant, is called a *difference equation*. For example,

$$y(n+2) - y(n+1) + y(n) = 2$$

and

$$y(n+2) - 2y(n) + y(n-1) = 1$$

are difference equations.

A relation between the terms of a sequence $\{x_n\}$ is also a difference equation. For example,

$$x_{n+1} + 2x_n = 8$$

is a difference equation.

Difference equations (also called *recurrence relations*) are closely related to differential equations and their theory is basically the same as that of differential equations.

Order of a difference equation is the difference between the largest and smallest arguments occurring in the difference equation divided by the unit of increment.

For example, the order of the difference equation $a_{n+2} - 3a_{n+1} + 2a_n = 5^n$ is $\frac{n+2-n}{1} = 2.$

Solution of a difference equation is an expression for y_n which satisfies the given difference equation.

The aim of this section is to solve difference equations using Laplace transform.

We first make the following *observations*:

(A) Let $f(t) = a^{[t]}$, where $[t]$ is the greatest integer less than or equal to t and $a > 0$. Then $f(t)$ is of exponential order and by definition,

$$L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt = \int_0^\infty e^{-st}a^{[t]}dt$$

$$= \int_0^1 e^{-st}a^0dt + \int_1^2 e^{-st}a^1dt + \int_2^3 e^{-st}a^2dt + \ldots$$

$$= \frac{1-e^{-s}}{s} + \frac{a(e^{-s}-e^{-2s})}{s} + \frac{a^2(e^{-2s}-e^{-3s})}{s} + \ldots$$
$$= \frac{1-e^{-s}}{s}[1+ae^{-s}+a^2e^{-s}+\ldots]$$
$$= \frac{1-e^{-s}}{s(1-ae^{-s})} \quad (\mathrm{Re}(s) > \max(0, \log a)).$$

(B) If $\mathrm{L}^{-1}\mathrm{F}\{(s)\} = f(t)$, then we know that

$$\mathrm{L}^{-1}\{e^{-s}\mathrm{F}(s)\} = \begin{cases} f(t-1) & \text{for } t > 1 \\ 0 & \text{for } t < 1. \end{cases}$$

Also, by observation (1) above, we have

$$\mathrm{L}^{-1}\left\{\frac{1-e^{-s}}{s(1-ae^{-s})}\right\} = a^n \text{ for } n = 0,\ 1,\ 2,\ \ldots,$$
$$n \le t < n+1.$$

Therefore,

$$\mathrm{L}^{-1}\left\{\frac{(1-e^{-s})e^{-s}}{s(1-ae^{-s})}\right\}$$
$$= \mathrm{f(t-1)}$$
$$= a^n \quad \text{for } n \le t-1 < n+1, n = 0, 1, 2,$$
$$= a^n \quad \text{for } n \le t < n+1, n = 1, 2, 3, \ldots$$

(C) If $f(t) = na^{n-1}$ for $n \le t < n+1$, $n = 0, 1, 2, \ldots$, then

$$\mathrm{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$
$$= \int_1^2 e^{-st}dt + 2a\int_2^3 e^{-st}dt + 4a\int_3^4 e^{-st}dt + \ldots$$
$$= \frac{e^{-s}-e^{-2s}}{s} + 2a\left[\frac{e^{-2s}-e^{-3s}}{s}\right] + 4a^2\left[\frac{e^{-3s}-e^{4s}}{s}\right] + \ldots$$
$$= \frac{e^{-s}(1-e^{-s})}{s}[1+2ae^{-s}+4a^2e^{-2s}+\ldots]$$
$$= \frac{e^{-s}(1-e^{-s})}{s} \cdot \frac{1}{(1-ae^{-s})^2} = \frac{e^{-s}(1-e^{-s})}{s(1-ae^{-s})^2}.$$

Hence

$$\mathrm{L}^{-1}\left\{\frac{e^{-s}(1+e^{-s})}{s(1-ae^{-s})^2}\right\} = f(t) = na^{n-1},\ \mathrm{n} = 0, 1, 2, \ldots$$

EXAMPLE 7.30

Solve

$$a_{n+2} - 4a_{n+1} + 3a_n = 0, \quad a_0 = 0, a_1 = 1.$$

Solution. Let us define

$$y(t) = a_n, \quad n \le t < n+1, \quad n = 0, 1, 2, \ldots$$

Then the given difference equation reduces to

$$y(t+2) - 4y(t+1) + 3y(t) = 0.$$

Thus

$$\mathrm{L}\{y(t+2)\} - 4\mathrm{L}\{y(t+1)\} + 3\mathrm{L}\{y(t)\} = 0 \quad (9)$$

Now

$$L\{zy(t+2)\} = \int_0^\infty e^{-st}y(t+2)\,dt$$
$$= \int_2^\infty e^{-s(u-2)}y(u)\,du,\ u = t+2$$
$$= e^{2s}\int_0^\infty e^{-su}y(u)\,du - e^{2s}\int_0^2 e^{-su}y(u)\,du.$$
$$= e^{2s}\mathrm{L}\{y(t)\} - e^{2s}\int_0^1 e^{-su}a_0\,du - e^{2s}\int_1^2 e^{-su}a_1\,du$$
$$= e^{2s}\mathrm{L}\{y(t)\} - e^{2s}\left(\frac{e^{-s}-e^{-2s}}{s}\right) \text{ since } a_0 = 0,\ a_1 = 1$$
$$= \mathrm{e}^{2s}L\{y(t)\} - \frac{e^s}{s}(1-e^{-s}),$$

$$\mathrm{L}\{y(t+1)\} = \int_0^\infty e^{-st}y(t+1)\,dt$$
$$= \int_1^\infty e^{-s(u-1)}y(u)\,du,\ u = t+1$$
$$= \int_0^\infty e^{-s(u-1)}y(u)\,du - \int_0^1 e^{-s(u-1)}y(u)\,du$$
$$= e^s\int_0^\infty e^{-su}y(u)\,du - e^s\int_0^1 e^{-su}y(u)\,du$$
$$= e^sL\{y(t)\} - e^s\int_0^1 e^{-su}a_0\,du$$
$$= e^s\mathrm{L}\{y(t)\} \text{ since } a_0 = 0.$$

Hence (9) becomes

$$e^{2s}\,\mathrm{L}\{y(t)\} - \frac{e^s}{s}(1-e^{-s}) - 4e^s\mathrm{L}\{\mathrm{y(t)}\}$$
$$+ 3L\{\mathrm{y(t)}\} = 0,$$

which yields

$$\begin{aligned}
L\{y(t)\} &= \frac{e^s(1-e^{-s})}{s(e^{2s}-4e^s+3)} \\
&= \frac{e^s(1-e^{-s})}{2s}\left(\frac{1}{e^s-3}-\frac{1}{e^s-1}\right) \\
&= \frac{1-e^{-s}}{2s}\left(\frac{1}{1-3e^{-s}}-\frac{1}{1-e^{-s}}\right) \\
&= \frac{1-e^{-s}}{2s(1-3e^{-s})}-\frac{1-e^{-s}}{2s(1-e^{-s})} \\
&= \frac{1}{2}L\{3^{[t]}\}-\frac{1}{2}L\{1\}, \text{ by observation (1).}
\end{aligned}$$

Hence inversion yields

$$a_n = \frac{1}{2}[3^n - 1], \quad n = 0, 1, 2, \ldots$$

EXAMPLE 7.31

Solve the difference equation

$$y(t+1) - y(t) = 1,\ y(t) = 0, t < 1.$$

Solution. Taking Laplace transformation of both sides, we get

$$L\{y(t+1)\} - L\{y(t)\} = L\{1\}.$$

But, as in Example 7.30, we have

and so
$$\begin{aligned}
L\{y(t+1)\} &= e^s L\{y(t)\}, \\
e^s L\{y(t)\} - L\{y(t)\} &= \frac{1}{s}
\end{aligned}$$

or

$$L\{y(t)\} = \frac{1}{s(e^s-1)}.$$

Taking inverse Laplace transform, we have

$$\begin{aligned}
y(t) &= L^{-1}\left\{\frac{1}{s(e^s-1)}\right\} \\
&= [t], t > 0 \text{(see Example 20.3)}
\end{aligned}$$

EXAMPLE 7.32

Solve

$$a_{n+2} - 4a_{n+1} + 3a_n = 5^n, \quad a_0 = 0, \quad a_1 = 1$$

Solution. We define

$$y(t) = a_n, \quad n \le t < n+1, \quad n = 0, 1, 2, \ldots$$

Then the difference equation becomes

$$y(t+2) - 4y(t+1) + 3y(t) = 5^n. \qquad (10)$$

By observation (B) and Example 7.30, we have

$$\begin{aligned}
L\{5^n\} &= \frac{1-e^{-s}}{s(1-5e^{-s})}, \\
L\{y(t+2)\} &= e^{2s}L\{y(t)\} - \frac{e^s}{s}(1-e^{-s}), \\
L\{y(t+1)\} &= e^s L\{y(t)\}.
\end{aligned}$$

Taking Laplace transform of both sides of (10), we have

$$L\{y(t+2)\} - 4L\{y(t+1)\} + 3L\{y(t)\} = L\{5^{[t]}\}$$

or

$$e^{2s}L\{y(t)\} - \frac{e^s}{s}(1-e^{-s}) - 4e^s L\{y(t)\} + 3L\{y(t)\} = L\{5[t]\}$$

or

$$\{e^{2s} - 4e^s + 3\}L\{y(t)\} = \frac{e^s(1-e^{-s})}{s} + L\{5^{[t]}\}.$$

Hence

$$\begin{aligned}
L\{y(t)\} &= \frac{e^s(1-e^{-s})}{s(e^{2s}-4e^s+3)} + \frac{L\{5^{[t]}\}}{e^{2s}-4e^s+3} \\
&= \frac{1}{2}L\{3^{[t]}\} - \frac{1}{2}L\{1\} + \frac{L\{5^{[t]}\}}{e^{2s}-4e^s+3}.
\end{aligned}$$

But

$$\begin{aligned}
\frac{L\{5^{[t]}\}}{e^{2s}-4e^s+3} &= \frac{1-e^{-s}}{s(1-5e^{-s})}\cdot\frac{1}{e^{2s}-4e^s+3} \\
&= \frac{e^s-1}{s(e^s-5)(e^s-3)(e^s-1)} \\
&= \frac{e^s-1}{s}\left(\frac{1/8}{e^s-5}-\frac{1/4}{e^s-3}+\frac{1/8}{e^s-1}\right) \\
&= \frac{1-e^{-s}}{s}\left(\frac{1/8}{1-5e^{-s}}-\frac{1/4}{1-3e^{-s}}+\frac{1/8}{1-e^{-s}}\right) \\
&= \frac{1}{8}L\{1\} + \frac{1}{8}L\{5^{[t]}\} - \frac{1}{4}L\{3^{[t]}\}.
\end{aligned}$$

Hence

$$L\{y(t)\} = \frac{-3}{8}L\{1\} + \frac{1}{4}L\{3^{[t]}\} + \frac{1}{8}L\{5^{[t]}\}$$

and so

$$a_n = \frac{-3}{8} + \frac{1}{4}3^n + \frac{1}{8}5^n.$$

EXAMPLE 7.33

Solve

$$a_{n+2} - 3a_{n+1} + 2a_n = 2^n, \quad a_0 = 0, \quad a_1 = 1.$$

Solution. We define

$$y(t) = a_n,\ n \le t < n+1.$$

Then the given equation reduces to

$$y(t+2) - 3y(t+1) + 2y(t) = 2^{[t]}.$$

Taking Laplace transform of both sides, we get

$$\mathrm{L}\{y(t+2)\} - 3\mathrm{L}\{y(t+1)\} + 2\mathrm{L}\{y(t)\} = \mathrm{L}\{2^{[t]}\}.$$

But, as in the previous examples,

$$\mathrm{L}\{y(t+2)\} = e^{2s}\mathrm{L}\{y(t)\} - \frac{e^s}{s}(1-e^{-s}),$$

$$\mathrm{L}\{y(t+1)\} = e^s L\{y(t)\}.$$

Therefore,

$$(e^{2s} - 3e^s + 2)\mathrm{L}\{y(t)\} = \frac{e^s}{s}(1-e^{-s}) + \mathrm{L}\{2^{[t]}\},$$

which gives

$$\begin{aligned}
\mathrm{L}\{y(t)\} &= \frac{e^s}{s(e^{2s}-3e^s+2)}(1-e^{-s}) + \frac{\mathrm{L}\{2^{[t]}\}}{e^{2s}-3e^s+2} \\
&= \frac{e^s(1-e^{-s})}{s}\left(\frac{1}{e^s-2} - \frac{1}{e^s-1}\right) + \frac{\mathrm{L}\{2^{[t]}\}}{e^{2s}-3e^s+2} \\
&= \frac{1-e^{-s}}{s}\left(\frac{1}{1-2e^{-s}} - \frac{1}{1-e^{-s}}\right) + \frac{\mathrm{L}\{2^{[t]}\}}{e^{2s}-3e^s+2} \\
&= \frac{1-e^{-s}}{s(1-e^{-s})} - \frac{1}{s} + \frac{\mathrm{L}\{2^{[t]}\}}{e^{2s}-3e^s+2} \\
&= \mathrm{L}\{2^{[t]}\} - \mathrm{L}\{1\} + \frac{\mathrm{L}\{2^{[t]}\}}{e^{2s}-3e^s+2}.
\end{aligned}$$

But

$$\begin{aligned}
\frac{\mathrm{L}\{2^{[t]}\}}{e^{2s}-3e^s+2} &= \frac{1-e^{-s}}{s(1-2e^{-s})}\cdot\frac{1}{e^{2s}-3e^s+2} \\
&= \frac{e^s-1}{s(e^s-2)(e^s-2)(e^s-1)} \\
&= \frac{e^s-1}{s}\left[\frac{-1}{e^s-2} + \frac{1}{(e^s-2)^2} + \frac{1}{e^s-1}\right] \\
&= \frac{1-e^{-s}}{s}\left[\frac{-1}{1-2e^{-s}} + \frac{e^{-s}}{(1-2e^{-s})^2} + \frac{1}{1-e^{-s}}\right] \\
&= -\mathrm{L}\{2^{[t]}\} + \mathrm{L}\{n2^{n-1}\} + \mathrm{L}\{1\}.
\end{aligned}$$

Therefore,

$$\mathrm{L}\{y(t)\} \quad = \mathrm{L}\{n2^{n-1}\}.$$

Hence

$$a_n = n2^{n-1}, \quad n = 0, 1, 2, \ldots$$

Verification. We note that

$$\begin{aligned}
a_{n+1} &= (\mathrm{n}+1)2^n, a_{n+2} = (n+2)2^{n+2-1} \\
&= (n+2)2^{n+1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_{n+2} - 3a_{n+1} + 2a_n &= (n+2)2^{n+1} - (3n+3)2^n + 2n2^{n-1} \\
&= 2^n[2n+4-3n-3+n] = 2^n.
\end{aligned}$$

EXAMPLE 7.34

Solve

$$y(t) - y(t-\pi) = \sin t, \quad y(t) = 0, \quad t \le 0.$$

Solution. Taking Laplace transform, we have

$$\mathrm{L}\{y(t)\} - \mathrm{L}\{y(t-\pi)\} = \mathrm{L}\{\sin t\}. \tag{11}$$

But

$$\begin{aligned}
\mathrm{L}\{\mathrm{y}(t-\pi)\} &= \int_0^\infty e^{-st}y(t-\pi)\,dt \\
&= \int_{-\pi}^\infty e^{-s(u+\pi)}y(u)\,du,\ u = t-\pi \\
&= e^{-s\pi}\int_0^\infty e^{-su}y(u)\,du,\ y(u) = 0,\ u < 0 \\
&= e^{-s\pi}\mathrm{L}\{y(t)\}.
\end{aligned}$$

Hence (11) reduces to

$$\mathrm{L}\{y(t)\} - e^{-s\pi}\mathrm{L}\{y(t)\} = \frac{1}{s^2+1}$$

and so

$$\mathrm{L}\{y(t)\} = \frac{1}{(s^2+1)(1-e^{-s\pi})}.$$

Taking inverse transform, we get (see Exapmle 5.72)

$$\begin{aligned}
y(t) &= \begin{cases} \sin t & \text{for } 0 < t < \pi \\ 0 & \text{for } \pi < t < 2\pi \end{cases} \\
&= \begin{cases} \sin t & \text{for } 2n\pi < t < (2n+1)\pi \\ 0 & \text{for } (2n+1)\pi < t < (2n+2)\pi \end{cases}
\end{aligned}$$

(due to periodicity), for $n = 0, 1, 2, \ldots$

This is half-wave rectified sinusoidal function.

EXAMPLE 7.35

Find explicit formula (solution) for *Fibonacci sequence*:

$$a_{n+2} = a_{n+1} + a_n, \quad a_0 = 0, \quad a_1 = 1.$$

Solution. Define

$$y(t) = a_n, \quad n \le t < n+1, \quad n = 0, 1, 2, \ldots$$

Then the given difference equation reduces to

$$y(t+2) - y(t+1) - y(t) = 0.$$

Taking Laplace transform, we have

$$\mathrm{L}\{y(t+2)\} - \mathrm{L}\{y(t+1)\} - \mathrm{L}\{y(t)\} = 0.$$

But

$$\mathrm{L}\{y(t+2)\} = e^{2s}\mathrm{L}\{y(t)\} - \frac{e^s(1-e^{-s})}{s}$$

$$\mathrm{L}\{y(t+1)\} = e^s\mathrm{L}\{y(t)\}.$$

Therefore, we get

$$(e^{2s} - e^s - 1)\mathrm{L}\{y(t)\} = \frac{e^s(1-e^{-s})}{s}$$

or

$$\begin{aligned}\mathrm{L}\{y(t)\} &= \frac{e^s(1-e^{-s})}{s(e^{2s}-e^s-1)} \\ &= \frac{e^s(1-e^{-s})}{s}\left[\frac{\frac{1}{\sqrt{5}}}{e^s - \frac{1+\sqrt{5}}{2}} - \frac{\frac{1}{\sqrt{5}}}{e^s - \frac{1-\sqrt{5}}{2}}\right] \\ &= \frac{1-e^{-s}}{s}\left[\frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{1+\sqrt{5}}{2}e^{-s}} - \frac{1}{1-\frac{1-\sqrt{5}}{2}e^{-s}}\right)\right] \\ &= \frac{1}{\sqrt{5}}\left[\mathrm{L}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{[t]}\right\} - \mathrm{L}\left\{\left(\frac{1-\sqrt{5}}{2}\right)^{[t]}\right\}\right].\end{aligned}$$

Hence

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], n \ge 0.$$

EXAMPLE 7.36

Solve the difference equation

$$y(t) + y(t-1) = e^t, \quad y(t) = 0, \quad t \le 0.$$

Solution. Taking Laplace transform of both sides of the given equation, we get

$$\mathrm{L}\{y(t)\} + \mathrm{L}\{y(t-1)\} = \mathrm{L}\{e^t\}.$$

But

$$\begin{aligned}\mathrm{L}\{\mathrm{y(t-1)}\} &= \int_0^\infty e^{-st}y(t-1)\,dt \\ &= \int_{-1}^\infty e^{s(u+1)}y(u)\,du,\ u=t-1 \\ &= e^{-s}\int_{-1}^0 e^{-su}y(u)\,du + e^{-s}\int_0^\infty e^{-su}y(u)\,du \\ &= e^{-s}\int_0^\infty e^{-su}y(u)\,du \text{ since y(t)=0 for t}\ge 0 \\ &= e^{-s}L\{y(t)\}.\end{aligned}$$

Therefore, we have

$$L\{y(t)\} + e^{-s}L\{y(t)\} = \frac{1}{s-1}$$

or

$$\begin{aligned}\mathrm{L}\{y(t)\} &= \frac{1}{(s-1)(1+e^{-s})} \\ &= \frac{1}{(s-1)}[1 - e^{-s} + e^{-2s} - e^{-3s} + \ldots] \\ &= \sum_{n=0}^\infty \frac{(-1)^n e^{-ns}}{s-1}.\end{aligned}$$

Hence

$$y(t) = \sum_{n=0}^{[t]} (-1)^n e^{t-n}.$$

EXAMPLE 7.37

Solve the *differential-difference equation*

$$y'(t) - y(t-1) = t, \quad y(t) = 0, \quad t \le 0.$$

Solution. Taking Laplace transform of both sides, we have

$$\mathrm{L}\{y'(t)\} - \mathrm{L}\{y(t-1)\} = \mathrm{L}\{t\}.$$

Now

$$\mathrm{L}\{y'(t)\} = s\,\mathrm{L}\{y(t)\} - y(0) = s\,\mathrm{L}\{y(t)\}$$

and

$$\mathrm{L}\{y(t-1)\} = e^{-s}\mathrm{L}\{y(t)\}.$$

Therefore,

$$\{e^{-s} + s\}\mathrm{L}\{y(s)\} = \frac{1}{s^2}$$

or

$$\begin{aligned} L\{y(t)\} &= \frac{1}{s^2(e^{-s}+s)} = \frac{1}{s^3\left(1+\frac{e^{-s}}{s}\right)} \\ &= \frac{1}{s^3}\left(1-\frac{e^{-s}}{s}+\frac{e^{-2s}}{s^2}-\frac{e^{-3s}}{s^3}+\dots\right) \\ &= \frac{1}{s^3}-\frac{e^{-s}}{s^4}+\frac{e^{-2s}}{s^5}-\frac{e^{-3s}}{s^6}+\dots \\ &= \sum_{n=0}^{\infty}\frac{e^{-ns}}{s^{n+3}}. \end{aligned}$$

But

$$L^{-1}\left\{\frac{e^{-ns}}{s^{n+3}}\right\} = \begin{cases} \frac{(t-n)^{n+2}}{(n+2)!} & \text{for } t \geq n \\ 0 & \text{otherwise}. \end{cases}$$

Therefore, if $[t]$ denotes the greatest integer less than or equal to t, then

$$y(t) = \sum_{n=0}^{[t]}\frac{(t-n)^{n+2}}{(n+2)!}.$$

EXAMPLE 7.38

Solve the *differential-difference equation*

$$y''(t)-y(t-1) = f(t), \quad y(t)=0, \quad y'(t)=0 \text{ for } t\leq 0,$$

$$f(t)=\begin{cases} 0 & \text{for } t\leq 0 \\ 2t & \text{for } t>0. \end{cases}$$

Solution. Taking Laplace transform of both sides, we get

$$L\{y''(t)\} - L\{y(t-1)\} = L\{f(t)\}$$

or

$$s^2L\{y(t)\}-sy(0)-y(0) - e^{-s}L\{y(t)\} = \frac{2}{s^2}$$

or

$$(s^2-e^{-s})L\{y(t)\} = \frac{2}{s^2}$$

or

$$\begin{aligned} L\{y(t)\} &= \frac{2}{s^2(s^2-e^{-s})} = \frac{2}{s^4\left(1-\frac{e^{-s}}{s^2}\right)} \\ &= \frac{2}{s^4}\left(1+\frac{e^{-s}}{s^2}+\frac{e^{-2s}}{s^4}+\frac{e^{-3s}}{s^6}+\dots\right) \\ &= 2\left(\frac{1}{s^4}+\frac{e^{-s}}{s^6}+\frac{e^{-2s}}{s^8}+\frac{e^{-3s}}{s^{10}}+\dots\right) \\ &= 2\sum_{n=0}^{\infty}\frac{e^{-ns}}{s^{2n+4}}. \end{aligned}$$

But

$$L^{-1}\left\{\frac{e^{-ns}}{s^{n+4}}\right\} = \frac{(t-n)^{2n+3}}{(2n+3)!}.$$

Hence

$$y(t) = 2\sum_{n=0}^{[t]}\frac{(t-n)^{2n+3}}{(2n+3)!}.$$

7.4 INTEGRAL EQUATIONS

Equations of the form

$$f(t) = g(t) + \int_a^b K(t, u)f(u)\,du$$

and

$$g(t) = \int_a^b K(t, u)f(u)\,du,$$

where the function $f(t)$ to be determined appears under the integral sign are called *integral equations*.

In an integral equation, $K(t, u)$ is called the *kernel*. If a and b are constants, the equation is called a *Fredholm integral equation*. If a is a constant and $b = t$, then the equation is called a *Volterra integral equation*.

If the kernel $K(t, u)$ is of the form $K(t - u)$, then the integral $\int_0^t K(t-u)f(u)\,du$ represents convolution. Thus, we have

$$f(t)=g(t)+\int_0^t K(t-u)f(u)\,du=g(t)+K(t)*f(t).$$

Such equations are called *convolution-type integral equations*. Taking Laplace transform of convolution-type integral equation, we have

$$\begin{aligned} L\{f(t)\} &= L\{g(t)\} + L\{K(t) * f(t)\} \\ &= L\{g(t)\} + L\{K(t)\}\,L\{f(t)\}, \end{aligned}$$

by using Convolution theorem. Hence

$$(1 - L\{K(t)\}\,L\{f(t)\}) = L\{g(t)\},$$

which implies

$$L\{f(t)\} = \frac{L\{g(t)\}}{1-L\{K(t)\}}.$$

Taking inverse Laplace transform yields the solution $f(t)$.

EXAMPLE 7.39

Solve the integral equation

$$f(t) = e^{-t} + \int_0^t \sin(t-u) f(u)\, du.$$

Solution. Taking Laplace transform of both sides of the given equation, we get

$$\mathrm{L}\{f(t)\} = \mathrm{L}\{e^{-t}\} + \mathrm{L}\{\sin t\}\,\mathrm{L}\{f(t)\},$$

which yields

$$\mathrm{L}\{f(t)\} = \frac{\mathrm{L}\{e^{-t}\}}{1-\mathrm{L}\{\sin t\}} = \frac{s^2+1}{s^2(s+1)}.$$

Using partial fractions, we obtain

$$\mathrm{L}\{f(t)\} = \frac{2}{s+1} + \frac{1}{s^2} - \frac{1}{s}.$$

Taking inverse Laplace transform yields

$$f(t) = 2e^{-t} + t - 1.$$

EXAMPLE 7.40

Solve the integral equation

$$f(t) = 1 + \int_0^t \sin(t-u) f(u)\, du.$$

Solution. We have

$$f(t) = 1 + \sin t * f(t).$$

Taking Laplace transform yields

$$\mathrm{L}\{f(t)\} = \mathrm{L}\{1\} + \mathrm{L}\{f(t)\}\mathrm{L}\{\sin t\}$$

or

$$\mathrm{L}\{f(t)\} = \frac{L\{1\}}{1-L\{\sin t\}} = \frac{1}{s\left(1-\frac{1}{s^2+1}\right)}$$

$$= \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3}.$$

Taking inverse Laplace transform, we get

$$f(t) = 1 + \frac{t^2}{2}.$$

EXAMPLE 7.41

Solve

$$\int_0^t f(u) f(t-u)\, du = 16 \sin 4t.$$

Solution. The given equation in convolution form is

$$f(t) * f(t) = 16 \sin 4t.$$

Taking Laplace transform, we get

$$\mathrm{L}\{f(t) * f(t)\} = 16\,\mathrm{L}\{\sin 4t\}$$

or

$$\mathrm{L}\{f(t)\}\mathrm{L}\{f(t)\} = 16\,\mathrm{L}\{\sin 4t\}$$

(using convolution theorem).

or

$$[\mathrm{L}\{f(t)\}]^2 = \frac{16(4)}{s^2+16} = \frac{64}{s^2+16}.$$

or

$$\mathrm{L}\{f(t)\} = \frac{\pm 8}{\sqrt{s^2+16}}.$$

Taking inverse Laplace transform yields

$$f(t) = \pm 8 J_0(4t),$$

where J_0 is Bessel?s function of order zero.

EXAMPLE 7.42

Solve the integral equation

$$g(x) = f(x) - \int_0^t e^{t-u} f(u)\, du.$$

Solution. The given equation, in convolution form, is

$$g(t) = f(t) - e^t * f(t).$$

Taking Laplace transform of both sides, we get

$$\mathrm{L}\{g(t)\} = \mathrm{L}\{f(t)\} - \mathrm{L}\{e^t\}\,\mathrm{L}\{f(t)\}$$

or

$$\mathrm{L}\{f(t)\} = \frac{\mathrm{L}\{g(t)\}}{1-\mathrm{L}\{e^t\}} = \frac{(s-1)\,\mathrm{L}\{g(t)\}}{(s-2)}$$

$$= \mathrm{L}\{g(t)\} + \frac{\mathrm{L}\{g(t)\}}{s-2}$$

$$= \mathrm{L}\{g(t)\} + \mathrm{L}\{g(t)\,\mathrm{L}\{e^{2t}\}\}.$$

Taking inverse Laplace transform yields

$$f(t) = g(t) + g(t) * e^{2t}$$

$$= g(t) + \int_0^t g(u)\, e^{2(t-u)} du.$$

Definition 7.1. The convolution-type integral equation of the form

$$\int_0^t \frac{f(u)}{(t-u)^n}\,du = g(t),\ 0 < n < 1$$

is called *Abel?s integral equation*.

We consider below examples of this type of integral equations.

EXAMPLE 7.43

Solve the integral equation

$$1 + 2t - t^2 = \int_0^t f(u)\frac{1}{\sqrt{t-u}}du.$$

Solution. The given equation is a special case of Abel?s integral equation. The convolution form of this equation is

$$1 + 2t - t^2 = f(t) * \frac{1}{\sqrt{t}}.$$

Taking Laplace transform yields

$$\mathrm{L}\{f(t)\}\,\mathrm{L}\left\{\frac{1}{\sqrt{t}}\right\} = \mathrm{L}\{1\} + 2\mathrm{L}\{t\} - \mathrm{L}\{t^2\}$$

or

$$L\{f(t)\}\sqrt{\frac{\pi}{s}} = \frac{1}{s} + \frac{2}{s^2} - \frac{2}{s^3}$$

or

$$L\{f(t)\} = \frac{1}{\sqrt{\pi}}\left[\frac{1}{s^{1/2}} + \frac{2}{s^{3/2}} - \frac{2}{s^{5/2}}\right].$$

Taking inverse transform, we get

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{\pi}}\left[\frac{t^{-1/2}}{\Gamma(1/2)} + \frac{2t^{1/2}}{\Gamma(3/2)} - \frac{2t^{3/2}}{\Gamma(5/2)}\right] \\ &= \frac{1}{\sqrt{\pi}}\left[\frac{t^{-1/2}}{\sqrt{\pi}} + \frac{2t^{1/2}}{(1/2)\sqrt{\pi}} - \frac{2t^{3/2}}{(3/2)(1/2)\sqrt{\pi}}\right] \\ &= \frac{1}{\pi}[t^{-1/2} + 4t^{1/2} - 8t^{3/2}]. \end{aligned}$$

EXAMPLE 7.44

Solve the integral equation

$$\int_0^t f(u)\,\frac{1}{\sqrt{t-u}} = 1 + t + t^2.$$

Solution. Proceeding as in Example 7.43 above, we have

$$L\{f(t)\} = \frac{1}{\sqrt{\pi}}\left[\frac{1}{s^{1/2}} + \frac{1}{s^{3/2}} + \frac{2}{s^{5/2}}\right],$$

which on inversion yields

$$f(t) = \frac{1}{\pi}[t^{-1/2} + 2t^{1/2} + \frac{8}{3}\,t^{3/2}].$$

EXAMPLE 7.45 (Tautochrone Curve)

A particle (bead) of mass m is to slide down a frictionless curve such that the duration T_0 of descent due to gravity is independent of the starting point. Find the shape of such curve (known as *Tautochrone curve*).

Solution. Velocity of the bead at the starting point is zero since it starts from rest at that point, say P with co-ordinates (x, y). Let Q = (x, u) be some intermediate point during the motion. Let the origin O be the lowest point of the curve (Figure 7.12). Let the length of the arc OQ be s.

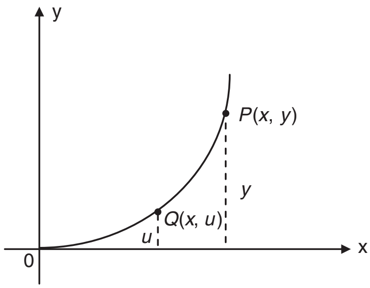

Figure 7.12

By law of conservation of energy, potential energy at P + kinetic energy at P = potential energy at Q + Kinetic energy at Q, that is,

$$mgy + 0 = mgu + \frac{1}{2}m\left(\frac{ds}{dt}\right)^2,$$

where $\frac{ds}{dt}$ is the instantaneous velocity of the particle at Q.

Thus

$$\left(\frac{ds}{dt}\right)^2 = 2g(y-u)$$

and so

$$\frac{ds}{dt} = -\sqrt{2g(y-u)},$$

negative sign since s decreases with time. The total time T_0 taken by the particle to go from P to Q is

$$T_0 = \int_0^{T_0} dt = \int_y^0 \frac{-ds}{\sqrt{2g(y-u)}} = \int_0^y \frac{ds}{\sqrt{2g(y-u)}}.$$

If $\frac{ds}{du} = f(u)$, then $ds = f(u)\,du$ and so

$$T_0 = \frac{1}{\sqrt{2g}} \int_0^y \frac{f(u)}{\sqrt{y-u}} du$$

The convolution form of this integral equation is

$$T_0 = \frac{1}{\sqrt{2g}} f(y) * \frac{1}{\sqrt{y}}.$$

Taking Laplace transform of both sides and using Convolution theorem, we have

$$L\{T_0\} = \frac{1}{\sqrt{2g}} L\{f(y)\}\, L\left\{\frac{1}{\sqrt{y}}\right\}$$

or

$$L\{f(y)\} = \frac{\sqrt{2g}\,T_0/s}{\sqrt{T/s}} = \frac{\sqrt{2g/\pi}}{s^{1/2}}\,T_0 = \frac{C_0}{s^{1/2}},$$

where C_0 is a constant. Inverse Laplace transform then yields

$$f(y) = \frac{C}{\sqrt{y}}.$$

Since $f(y) = \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$, we get

$$1 + \left(\frac{dx}{dy}\right)^2 = \frac{C^2}{y}$$

or

$$\left(\frac{dx}{dy}\right)^2 = \frac{C^2}{y} - 1 = \frac{C^2 - y}{y}$$

or

$$\frac{dx}{dy} = \sqrt{\frac{C^2 - y}{y}}$$

or

$$x = \int \sqrt{\frac{C^2 - y}{y}}\, dy.$$

Putting $y = C^2 \sin^2 \frac{\theta}{2}$, we get

$$x = \frac{C^2}{2}(\theta + \sin\theta), \quad y = \frac{C^2}{2}(1 - \cos\theta),$$

which are the parametric equations of a *cycloid*.

7.5 INTEGRO-DIFFERENTIAL EQUATIONS

An integral equation in which various derivatives of the unknown function $f(t)$ are also present is called an *integro-differential equation*. These types of equations can also be solved by the method of Laplace transform.

EXAMPLE 7.46

Solve the following integro-differential equation:

$$y'(t) = \int_0^t y(u)\cos(t-u)\,du, \quad y(0) = 1.$$

Solution. We write the given equation in convolution form as

$$y'(t) = y(t) * \cos t.$$

Taking Laplace transform and using Convolution theorem yields

$$L\{y'(t)\} = L\{y(t)\}\, L\{\cos t\}$$

or

$$sL\{y(t)\} - y(0) = L\{y(t)\}\frac{s}{s^2+1}$$

or

$$\left(s - \frac{s}{s^2+1}\right) L\{y(t)\} = 1, \quad \text{since } y(0) = 1$$

or

$$L\{y(t)\} = \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3}.$$

Taking inverse Laplace transform, we get

$$y(t) = 1 + \frac{1}{2}t^2.$$

EXAMPLE 7.47

Solve

$$y'(t) + 5\int_0^t y(u)\cos 2(t-u)\,dy = 10, \; y(0) = 2.$$

Solution. Convolution form of the equation is

$$y'(t) + 5\cos t * y(t) = 10.$$

Taking Laplace transform and using Convolution theorem, we have

$$sL\{y(t)\} - y(0) + \frac{5sL\{y(t)\}}{s^2+4} = \frac{10}{s}$$

or

$$L\{y(t)\} = \frac{2s^3 + 10s^2 + 8s + 40}{s^2(s^2+9)}$$
$$= \frac{1}{9}\left\{\frac{8}{s} + \frac{40}{s^2} + \frac{10s}{s^2+9} + \frac{50}{s^2+9}\right\}.$$

Hence

$$y(t) = \frac{1}{9}\left(8 + 40t + 10\cos t + \frac{50}{3}\sin 3t\right).$$

7.6 SOLUTION OF PARTIAL DIFFERENTIAL EQUATION

Consider the function $u = u(x, t)$, where $t \geq 0$ is a time variable. Suppose that $u(x, y)$, when regarded as a function of t, satisfies the sufficient conditions for the existence of its Laplace transform.

Denoting the Laplace transform of $u(x, t)$ with respect to t by $U(x, s)$, we see that

$$U(x, s) = L\{u(x, t)\} = \int_0^\infty e^{-st} u(x, t)\, dt.$$

The variable x is the untransformed variable. For example,

$$L\{e^{a(x-t)}\} = e^{ax} L\{e^{-at}\} = e^{ax}\frac{1}{s+a}.$$

Theorem 7.2. Let $u(x, t)$ be defined for $t \geq 0$. Then

(a) $L\left\{\frac{\partial u}{\partial x}\right\} = \frac{d}{dx}(U(x, s))$

(b) $L\left\{\frac{\partial u}{\partial t}\right\} = s\,U(x, s) - u(x, 0)$

(c) $L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2}{dx^2}(U(x, s))$

(d) $L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2\,U(x, s) - s\,u(x, 0) - \frac{\partial u}{\partial t}(x, 0).$

Proof: (a) We have, by Leibnitz?s rule for differentiating under the integration,

$$L\left\{\frac{\partial u}{\partial x}\right\} = \int_0^\infty e^{-st}\frac{\partial u}{\partial x}\, dt = \frac{d}{dx}\int_0^\infty e^{-st} u(x, t)\, dt$$
$$= \frac{d}{dx}(U(x, s)).$$

(b) Integrating by parts, we get

$$L\left\{\frac{\partial u}{\partial t}\right\} = \int_0^\infty e^{-st}\frac{\partial u(x,t)}{\partial t}\, dt$$
$$= \lim_{T\to 0}\int_0^T e^{-st}\frac{\partial u(x,t)}{\partial t}\, dt$$
$$= \lim_{T\to 0}\left\{\left[e^{-st}u(x,t)\right]_0^T + s\int_0^T e^{-st}u(x,t)dt\right\}$$
$$= s\int_0^T e^{-st}u(x,t)dt - u(x,0)$$
$$= sU(x, s) - u(x, 0).$$

(c) Taking $V = \frac{\partial u}{\partial x}$, we have by(a),

$$L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = L\left\{\frac{\partial V}{\partial x}\right\} = \frac{d}{dx}(V(x, s))$$
$$= \frac{d}{dx}\left(\frac{d}{dx}(U(x,s))\right) = \frac{d^2}{dx^2}(U(x, s)).$$

(d) Let $v = \frac{\partial u}{\partial t}$. Then

$$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = L\left\{\frac{\partial v}{\partial t}\right\} = sV(x, s) - v(x, 0)$$
$$= s[sU(x, s) - u(x, 0)] - \frac{\partial u}{\partial t}(x, 0)$$
$$= s^2U(x, s) - u(x, 0) - \frac{\partial u}{\partial t}(x, 0).$$

Theorem 7.2 suggest that if we apply Laplace transform to both sides of the given partial differential equation, we shall get an ordinary differential equation in U as a function of single variable x. This ordinary differential equation is then solved by the usual methods.

EXAMPLE 7.48

Solve

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}, \quad u(x, 0) = x, \quad u(0, t) = t,$$

Solution. Taking Laplace transform, we get

$$L\left\{\frac{\partial u}{\partial x}\right\} = L\left\{\frac{\partial u}{\partial t}\right\}.$$

Using Theorem 7.2, we get

$$\frac{d}{dx}[U(x, s)] = s\,U(x, s) - u(x, 0) = s\,U(x, s) - x.$$

Thus, we have first order differential equation

$$\frac{d}{dx}[\mathrm{U}(x, s)] - s\,\mathrm{U}(x, s) = -x$$

The integrating factor is

$$e^{\int -s\,dx} = e^{-sx}.$$

Therefore,

$$U(x, s)\,e^{-sx} = \int -xe^{-sx}dx$$

$$= -\left[x\frac{e^{-sx}}{-s} - \int \frac{e^{-sx}}{-s}\,dx\right] + \mathrm{C}$$

$$= \frac{x\,e^{-sx}}{s} + \frac{e^{-sx}}{s^2} + \mathrm{C}$$

(constant of integration).

This yields

$$\mathrm{U}(x, s) = \frac{x}{s} + \frac{1}{s^2} + \mathrm{C}\,e^{sx}. \quad (12)$$

Now the boundary condition $u(0, t)$ is a function of t. Taking Laplace transform of this function, we have

$$\mathrm{U}(0, s) = \mathrm{L}\{u(0, t)\} = \mathrm{L}\{t\} = \frac{1}{s^2}.$$

Then taking $x = 0$ in (12), we have

$$\frac{1}{s^2} = \frac{1}{s^2} + \mathrm{C}$$

and so C = 0. Thus, we have

$$\mathrm{U}(x, s) = \frac{x}{s} + \frac{1}{s^2}.$$

Taking inverse Laplace transform, we have

$$u(x, t) = x + t.$$

EXAMPLE 7.49

Solve the partial differential equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = x, \quad x > 0, \quad t > 0$$

with the initial and boundary conditions $u(x, 0) = 0$, $x > 0$ and $u(0, t) = 0$ for $t > 0$.

Solution. Taking Laplace transform with respect to t, we get

$$\mathrm{L}\left\{\frac{\partial u}{\partial t}\right\} + \mathrm{L}\left\{x\frac{\partial u}{\partial x}\right\} = \mathrm{L}\{x\}$$

which yields

$$s\mathrm{U}(x, s) - u(x, 0) + x\frac{d}{dx}\mathrm{U}(x, s) = \frac{x}{s}.$$

Since $u(x, 0) = 0$, this reduces to

$$\frac{d}{dx}\mathrm{U}(x, s) + \frac{s}{x}\mathrm{U}(x, s) = \frac{1}{s}. \quad (13)$$

The integrating factor is

$$e^{\int \frac{s}{x}\,dx} = e^{s\log x} = x^s.$$

Therefore solution of (13) is

$$\mathrm{U}(x,s)x^s = \frac{1}{s}\int x^s dx + \mathrm{C} = \frac{1}{s}\frac{x^{s+1}}{s+1} + \mathrm{C} = \frac{x^{s+1}}{s(s+1)} + C$$

and so

$\mathrm{U}(x, s) = \frac{x}{s(s+1)} + \mathrm{C}$ (constant of integration). (14)

Now since $\mathrm{U}(0, t) = 0$, its Laplace transform is 0, that is, $\mathrm{U}(0, s) = 0$. Therefore, (14) implies C = 0. Hence

$$\mathrm{U}(x, s) = \frac{x}{s(s+1)} = x\left(\frac{1}{s} - \frac{1}{s+1}\right).$$

Taking inverse Laplace transform, we get the solution as

$$u(x, t) = x(1 - e^{-t}).$$

EXAMPLE 7.50

Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0$$

under the conditions

$$u(x, 0) = 1,\ u(0, t) = 0 \text{ and } \lim_{x\to\infty} u(x, t) = 1.$$

Solution. The given equation is heat conduction equation in a solid, where $u(x, t)$ is the temperature at position x at any time t and diffusivity is 1. The boundary condition $u(0, t) = 0$ indicates that temperature at $x = 0$ is 0 and $\lim_{x\to\infty} u(x, t) = 1$ indicates that the temperature for large values of x is 1 whereas

$u(x, 0) = 1$ represents the initial temperature 1 in the semi-infinite medium ($x > 0$) (Figure 7.13).

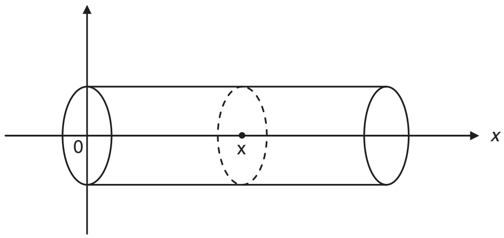

Figure 7.13

Taking Laplace transform, yields

$$sU(x, s) - u(x, 0) = \frac{d^2}{dx^2}U(x, s).$$

Since $u(x, 0) = 1$, we have

$$\frac{d^2}{dx^2}U(x, s) - s\,U(x, s) = -1.$$

The general solution of this equation is

$$\begin{aligned} U(x, s) &= \text{C.F.} + \text{P.I.} \\ &= [c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}] + \frac{1}{s}. \end{aligned} \tag{15}$$

The conditions $u(0, t) = 0$ yields

$$U(0, s) = L\{u(0, t)\} = 0, \tag{16}$$

whereas $\lim_{x\to\infty} u(x, t) = 1$ yields

$$\begin{aligned} \lim_{x\to\infty} U(x, s) &= \lim_{x\to\infty} L\{u(x, t)\} = L\{\lim_{x\to\infty} u(x, t)\} \\ &= L\{1\} = \frac{1}{s}. \end{aligned} \tag{17}$$

Now (15) and (17) imply $c_1 = 0$. Then (16) implies $c_2 = -\frac{1}{s}$. Hence

$$U(x, s) = \frac{1}{s} - \frac{e^{-\sqrt{s}x}}{s}.$$

Taking inverse Laplace transform, we get

$$\begin{aligned} u(x, t) &= 1 - L^{-1}\left\{\frac{e^{-\sqrt{s}x}}{s}\right\} \\ &= 1 - \left(1 - \text{erf}\left(\frac{x}{2\sqrt{t}}\right)\right) = erf\left(\frac{x}{2\sqrt{t}}\right). \end{aligned}$$

EXAMPLE 7.51

Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ x > 0, \ t > 0$$

subject to the conditions

$$\begin{aligned} &u(x, 0) = 0, \ x > 0, \\ &u(0, t) = t, \ t > 0 \text{ and } \lim_{x\to\infty} u(x, t) = 0. \end{aligned}$$

Solution. Taking Laplace transform, we have

$$\frac{d^2}{dx^2}U(x, s) - s\,U(x, s) = 0.$$

The solution of this equation is

$$U(x, s) = c_1\, e^{\sqrt{s}\,x} + c_2\, e^{-\sqrt{s}\,x} \tag{18}$$

Since $\lim_{x\to\infty} u(x, t) = 0$, we have

$$\begin{aligned} \lim_{x\to\infty} U(x, s) &= \lim_{x\to\infty} L\{u(x, t)\} \\ &= L\{\lim_{x\to\infty} u(x, t)\} = L\{0\} = 0(\text{finite}). \end{aligned}$$

Therefore, $c_1 = 0$ and (18) reduces to

$$U(x, s) = c_2 e^{-\sqrt{s}\,x}. \tag{19}$$

Also, since $u(0, t) = t$, we have

$$U(0, s) = \frac{1}{s^2}.$$

Hence, (19) yields $c_2 = \frac{1}{s^2}$. Thus

$$U(x, s) = \frac{1}{s^2} e^{-\sqrt{s}\,x}.$$

Since $L^{-1}\{e^{-\sqrt{s}\,x}\} = \frac{x}{2\sqrt{\pi t^3}}\, e^{-\frac{x^2}{4t}}$, by Convolution theorem, we have

$$u(x, t) = \int_0^t (t - u)\, \frac{x}{2\sqrt{\pi u^3}}\, e^{-\frac{x^2}{4u}}\, du.$$

Putting $\lambda = \frac{x^2}{4u}$, we get

$$u(x, t) = \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{t}}^{\infty} e^{-\lambda^2}\left(t - \frac{x^2}{4\lambda^2}\right) d\lambda.$$

EXAMPLE 7.52

Solve

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$$

subject to the conditions $u(0, t) = 0$, $u(5, t) = 0$, $u(x, 0) = \sin \pi x$.

Solution. Taking Laplace transform and using $u(x, 0) = \sin \pi x$, we get

$$\frac{d^2}{dx^2}\mathrm{U}(x,s) - \frac{s}{2}\mathrm{U}(x,s) = -\frac{1}{2}\sin \pi x.$$

Complementary function for this equation is $c_1 e^{\sqrt{\frac{s}{2}}x} + c_2 e^{-\sqrt{\frac{s}{2}}x}$ and particular integral is $\frac{1}{2(\pi^2+(s/2))}\sin \pi x$. Thus the complete solution is

$$\mathrm{U}(x,s) = c_1 e^{\sqrt{\frac{s}{2}}x} + c_2 e^{-\sqrt{\frac{s}{2}}x} + \frac{1}{2(\pi^2+(s/2))}\sin \pi x. \quad (20)$$

Since $u(0, t) = 0$, we have $\mathrm{U}(0, t) = 0$ and since $u(5, t) = 0$, $\mathrm{U}(5, t) = 0$. Therefore, (20) gives

$$c_1 + c_2 = 0 \quad \text{and} \quad c_1 e^{5\sqrt{s/2}} + c_2 e^{5(-\sqrt{s}/2)} = 0.$$

These relations imply $c_1 = c_2 = 0$. Hence

$$\mathrm{U}(x,s) = \frac{1}{2(\pi^2 + s/2)}\sin \pi x = \frac{1}{s + 2\pi^2}\sin \pi x.$$

Taking inverse Laplace transform, we get
$\mathrm{U}(x,t) = e^{-2\pi^2 t}\sin \pi s.$

EXAMPLE 7.53

Solve one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}, \qquad x > 0, \qquad t > 0$$

subject to the condition $y(x, 0) = 0, x > 0$; $y_t(x, 0) = 0, x > 0$, $y(0, t) = \sin \omega t$ and $\lim_{x\to\infty} y(x, t) = 0$.

Solution. The displacement is only in the vertical direction and is given by $y(x, t)$ at position x and time t. For a vibrating string, the constant a equals $\sqrt{\frac{T}{\rho}}$, where T is tension in the string and ρ is mass per unit length of the vibrating string (Figure 7.14).

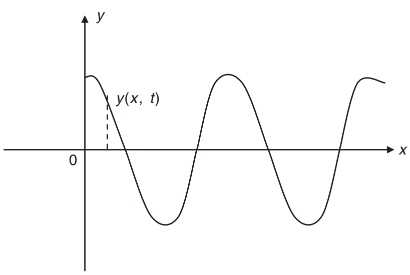

Figure 7.14

Taking Laplace transform, we get
$s^2\mathrm{Y}(x,s) - sy(x,0) - y_t(x,0) - a^2\frac{d^2}{dx^2}\mathrm{Y}(x,s) = 0$
or

$$\frac{d^2}{ds^2}\mathrm{Y}(x,s) - \frac{s^2}{a^2} = 0 \quad (21)$$

The general solution of (21) is

$$\mathrm{Y}(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x} \quad (22)$$

The condition $\lim_{x\to\infty} y(x, t)$ implies $c_1 = 0$.
Since $y(0, t) = \sin \omega t$, we have

$$\mathrm{Y}(0,s) = \{y(0,t)\} = \frac{\omega}{s^2+\omega^2}.$$

Therefore, (22) implies $c_2 = \frac{\omega}{s^2+\omega^2}$ and so

$$\mathrm{Y}(x,s) = \frac{\omega}{s^2+\omega^2}e^{-\frac{s}{a}x}.$$

Taking inverse Laplace transform, we have

$$y(x,t) = \begin{cases} \sin\omega\left(t - \frac{x}{a}\right) & \text{for } t > \frac{x}{a} \\ 0 & \text{for } t < \frac{x}{a}. \end{cases}$$
$$= \sin\omega\left(t - \frac{x}{a}\right)\mathrm{H}\left(t - \frac{x}{a}\right).$$

EXAMPLE 7.54

Solve

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad \text{for } 0 < x < 1, \quad t > 0$$

subject to $y(x, 0) = 0$, $0 < x < 1$; $y(0, t) = 0$, $t > 0$, $y(1, t) = 0$, $t > 0$ and $y_t(x, 0) = x$, $0 < x < 1$.

Solution. Taking Laplace transform and using $y(x, 0) = 0$ and $y_t(x, 0) = x$, we get

$$\frac{d^2\mathrm{Y}(x,s)}{dt^2} - s^2\mathrm{Y}(x,s) = x,$$

whose solution is given by

$$\mathrm{Y}(x,s) = c_1\cosh sx + c_2\sinh sx - \frac{x}{s^2}.$$

Now $y(0, t) = 0$ implies that $\mathrm{Y}(0, s) = 0$ and so $c_1 = 0$. Similarly, $y(1, t) = 0$ implies $\mathrm{Y}(1, s) = 0$ and so $c_2 \sinh s - \frac{1}{s^2} = 0$. Thus $c_2 = \frac{1}{s^2\sinh x}$. Hence

$$\mathrm{Y}(x,s) = \frac{1}{s^2\sinh x}.\sinh sx - \frac{x}{s^2}.$$

This function has simple poles at $n\pi i$, $n = \pm 1, \pm 2, \ldots$, and a pole of order 2 at $s = 0$. Now

$$\begin{aligned}
\text{Res}(n\pi i) &= \lim_{s\to n\pi i}(s - n\pi i)e^{ts}.\frac{\sinh sx}{s^2\sinh s} \\
&= \lim_{s\to n\pi i}\frac{(s-n\pi i)}{\sinh s}\lim_{s\to n\pi i} e^{ts}\frac{\sinh sx}{s^2} \\
&= \frac{1}{\cosh n\pi i}.\frac{e^{n\pi i t}\sinh n\pi i x}{-n^2\pi^2} \\
&= \frac{(-1)^{n+1}}{n^2\pi^2}e^{n\pi i t}\sin n\pi x, \\
\text{Res}(0) &= xt.
\end{aligned}$$

Hence, by Complex inversion formula,

$$\begin{aligned}
y(x,t) &= xt + \sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2}e^{n\pi i t}\sin n\pi x - xt \\
&= \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\sin n\pi x\sin n\pi t.
\end{aligned}$$

7.7 EVALUATION OF INTEGRALS

Laplace transforms can be used to evaluate certain integrals. In some cases the given integral is a special case of a Laplace transform for a particular value of the transform variable s. To evaluate an integral containing a free parameter, we first take Laplace transform of the integrand with respect to the free parameter. The resulting integral is then easily evaluated. Then we apply inverse Laplace transform to get the value of the given integral. In some cases, Theorem 5.9, regarding Laplace transform is used to evaluate the given integral.

EXAMPLE 7.55

Evaluate the integral

$$\text{I} = \int_0^\infty e^{-t}\frac{\sin t}{t}\,dt,$$

and show that $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$.

Solution. We know that

$$\text{L}\left\{\frac{\sin t}{t}\right\} = \int_0^\infty e^{-st}\frac{\sin}{t}\,dt = \tan^{-1}\frac{1}{s}. \qquad (23)$$

Setting $s = 1$, we get

$$\int_0^\infty e^{-t}\frac{\sin t}{t}\,dt = \tan^{-1}1 = \frac{\pi}{4}.$$

Further, letting $s \to 0$ in (23), we get

$$\int_0^\infty \frac{\sin t}{t}\,dt = \tan^{-1}\infty = \frac{\pi}{2}.$$

EXAMPLE 7.56

Evaluate the integral

$$\int_0^\infty \frac{\sin tx}{x(1+x^2)}\,dx.$$

Solution. Let

$f(t) = \int_0^\infty \frac{\sin tx}{x(1+x^2)}\,dx.$

Taking Laplace transform with respect to t, we have

$$\begin{aligned}
\text{F}(s) &= \int_0^\infty \frac{1}{x(1+x^2)}\,dx\int_0^\infty e^{-st}\sin tx\,dx \\
&= \int_0^\infty \frac{dx}{x(1+x^2)\,(s^2+x^2)} \\
&= \frac{1}{s^2-1}\int_0^\infty\left(\frac{1}{1+x^2} - \frac{1}{s^2+x^2}\right)dx \\
&= \frac{1}{s^2-1}\left(\frac{\pi}{2} - \frac{\pi}{2s}\right) = \frac{\pi}{2}\frac{1}{s^2-1}\left(\frac{s-1}{s}\right) \\
&= \frac{\pi}{2}\left(\frac{1}{s(s+1)}\right) = \frac{\pi}{2}\left(\frac{1}{s} - \frac{1}{s+1}\right).
\end{aligned}$$

Taking inverse Laplace transforms, we get

$f(t) = \frac{\pi}{2}(1 - e^{-t}).$

EXAMPLE 7.57

Evaluate

$$\int_0^\infty \frac{\sin^2 tx}{x^2}dx.$$

Solution. We have

$$f(t) = \int_0^\infty \frac{\sin^2 tx}{x^2}\,dx = \int_0^\infty \frac{1-\cos(2tx)}{2x^2}dx.$$

Taking Laplace transform with respect to t, we have

$$F(s)=\frac{1}{2}\int_0^\infty \frac{1}{x^2}\left(\frac{1}{s}-\frac{s}{4x^2+s^2}\right)dx=\frac{2}{s}\int_0^\infty \frac{dx}{4x^2+s^2}$$

$$=\frac{1}{s}\int_0^\infty \frac{dy}{y^2+s^2}=\frac{1}{s^2}\left[\tan^{-1}\frac{y}{s}\right]_0^\infty=\pm\frac{\pi}{2s^2}.$$

Thus, taking inverse Laplace transformation, we get

$$f(t)\quad=\pm\frac{\pi t}{2}=\frac{\pi t}{2}\,\mathrm{sgn}\, t.$$

EXAMPLE 7.58

Evaluate

$$\int_{-\infty}^\infty \sin x^2\,dx.$$

Solution. Let

$$f(t)=\int_0^\infty \sin t x^2\,dx$$

Taking Laplace transform, we get

$$L\{f(t)\}=\int_0^\infty e^{-st}\,dt\int_0^\infty \sin t x^2\,dx$$

$$=\int_0^\infty dx\int_0^\infty e^{-st}\sin tx^2 dt$$

$$=\int_0^\infty L\{\sin tx^2\}\,dx=\int_0^\infty \frac{x^2}{s^2+x^4}\,dx.$$

Put $x^2=s\tan\theta$, that is, $x=\sqrt{s}\sqrt{\tan\theta}$. Then

$$dx=\sqrt{s}.\frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta\,d\theta.$$

Therefore,

$$L\{f(t)\}=\frac{1}{2}\int_0^{\pi/2}\frac{s^{3/2}\tan\theta(\tan\theta)^{-1/2}\sec^2\theta d\theta}{s^2(1+\tan^2\theta)}$$

$$=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}\sqrt{\tan\theta}d\theta=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}d\theta.$$

But we know that

$$\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta=\frac{1}{2}\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}.$$

Taking $2m-1=\frac{1}{2}$ and $2n-1=-\frac{1}{2}$, we get $m=\frac{3}{4}$ and $n=\frac{1}{4}$. Hence using the relation $\Gamma(p)\,\Gamma(1-p)=\frac{\pi}{\sin p\pi}$, $0<p<1$, we have

$$L\{f(t)\}=\frac{1}{4\sqrt{s}}\frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma(1)}=\frac{1}{4\sqrt{s}}\pi\sqrt{2}=\frac{\pi\sqrt{2}}{4\sqrt{s}}.$$

Taking inverse Laplace transform yields

$$f(t)=\frac{\pi\sqrt{2}}{4}\left(\frac{t^{-1/2}}{\sqrt{\pi}}\right)=\frac{\sqrt{2\pi}}{4}\,t^{-1/2}.$$

Putting $t=1$, we get

$$\int_0^\infty \sin x^2\,dx=\frac{1}{2}\sqrt{\frac{\pi}{2}}$$

and so

$$\int_{-\infty}^\infty \sin x^2\,dx=\sqrt{\frac{\pi}{2}}.$$

EXAMPLE 7.59

Evaluate the integral

$$\int_0^\infty \frac{\cos tx}{x^2+1}\,dx,\quad t>0.$$

Solution. Let

$$f(t)=\int_0^\infty \frac{\cos tx}{x^2+1}\,dx.$$

Taking Laplace transform with respect to t, we get

$$L\{f(t)\}=L\left\{\int_0^\infty \frac{\cos tx}{x^2+1}\,dx\right\}=\int_0^\infty L\left\{\frac{\cos tx}{x^2+1}\right\}dx$$

$$=\int_0^\infty \frac{s}{(x^2+1)(s^2+x^2)}\,dx$$

$$=s\int_0^\infty \frac{dx}{(x^2+1)(s^2+x^2)}$$

$$=\frac{s}{s^2-1}\int_0^\infty\left(\frac{1}{x^2+1}-\frac{1}{s^2+x^2}\right)dx$$

$$=\frac{s}{s^2-1}\left[\tan^{-1}x-\tan^{-1}\frac{x}{s}\right]_0^\infty$$

$$=\frac{s}{s^2-1}\left(\frac{\pi}{2}-\frac{\pi}{2s}\right)=\frac{\pi/2}{s+1}.$$

Now, taking inverse Laplace transform, we have

$$f(t) = \frac{\pi}{2} e^{-t}, \quad t > 0.$$

EXAMPLE 7.60

Evaluate

$$\int_0^\infty t\,J_0(t)\,dt,$$

where J_0 is Bessel?s function of order zero.

Solution. We know that

$$L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}.$$

Therefore,

$$L\{tJ_0(t)\} = -\frac{d}{ds}\{L\{J_0(t)\}\} = -\frac{d}{ds}\left\{\frac{1}{\sqrt{s^2+1}}\right\}$$
$$= -\frac{2s}{(s^2+1)^{3/2}}.$$

But, by definition

$$L\{tJ_0(t)\} = \int_0^\infty e^{-st} t\,J_0(t)\,dt.$$

Hence

$$\int_0^\infty e^{-st}\,t\,J_0(t)\,dt = -\frac{2s}{(s^2+1)^{3/2}}.$$

Taking $s = 0$, we get

$$\int_0^\infty t\,J_0(t)\,dt = 0.$$

EXAMPLE 7.61

Evaluate

$$\int_0^\infty e^{-2t}\operatorname{erf}\sqrt{t}dt.$$

Solution. We have

$$L\{\operatorname{erf}\sqrt{t}\} = \int_0^\infty e^{-st}\operatorname{erf}\sqrt{t}dt = \frac{1}{s\sqrt{s+1}}.$$

Taking $s = 2$, we get

$$\int_0^\infty e^{-2t}\operatorname{erf}\sqrt{t}dt = \frac{1}{2\sqrt{3}}.$$

EXAMPLE 7.62

Evaluate

$$\int_0^t \operatorname{erf}\sqrt{u}\operatorname{erf}\sqrt{(t-u)}\,du.$$

Solution. Let

$$f(t) = \int_0^t \operatorname{erf}\sqrt{u}\operatorname{erf}\sqrt{t-u}\,du.$$

Then, by Convolution theorem, we have

$$F(t) = L\{\operatorname{erf}\sqrt{t}\}L\{\operatorname{erf}\sqrt{t}\}$$
$$= \frac{1}{s\sqrt{s+1}}\cdot\frac{1}{s\sqrt{s+1}} = \frac{1}{s^2(s+1)}$$
$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}.$$

Taking inverse transform, we get

$$f(t) = -1 + t + e^{-t}.$$

7.8 MISCELLANEOUS EXAMPLES

EXAMPLE 7.63

Solve the differential equation, using Laplace transform $y'' + 4y' + 4y = e^{-t}$ given that $y(0) = 0$ and $y'(0) = 0$.

Solution. Taking Laplace transform, we have

$$L\{y''(t)\} + 4L\{y'(t)\} + 4L\{y(t)\} = L\{e^{-t}\}$$

or

$$s^2Y(s) - sy(0) - y'(0) + 4\{sY(s) - y(0)\} + 4Y(s)$$
$$= \frac{1}{s+1}.$$

Using the initial conditions, we have

$$s^2Y(s) + 4sY(s) + 4Y(s) = \frac{1}{s+1}$$

or

$$Y(s) = \frac{1}{(s+1)(s^2+4s+4)} = \frac{1}{(s+1)(s+2)^2}$$
$$= \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}.$$

Taking inverse Laplace transform, we get

$$y(t) = e^{-t} - e^{-2t} - te^{-2t}.$$

EXAMPLE 7.64

Using Laplace transform, solve

$$(D^2+5D-6)y = x^2e^{-x}, \quad y(0)=a, \quad y'(0)=b.$$

Solution. Applying Laplace transform to the given equation, we get

$$s^2L\{y\} - sy(0) - y(0) + 5[sL\{y\} - y(0)] - 6L\{y\}$$
$$= \frac{(-1)^4 2\,!}{(s+1)^3} = \frac{2}{(s+1)^3}$$

or

$$s^2L\{y\} - as - b + 5[sL\{y\} - a] - 6L\{y\} = \frac{2}{(s+1)^3}$$

or

$$L\{y\}[s^2+5s-6] = \frac{2}{(s+1)^3} + as + b + 5a$$

or

$$L\{y\} = \frac{as+5a+b}{s^2+5s-6} + \frac{2}{(S+1)^3(s^2+5s-6)}$$
$$= -\frac{1}{5}\frac{1}{(s+1)^3} - \frac{3}{50}\frac{1}{(s+1)^2} - \frac{19}{500}\frac{1}{(s+1)}$$
$$+ \frac{1}{28}\frac{1}{(s-1)} + \frac{2}{875}\frac{1}{(s+6)}$$

(by partial fractions).

Taking inverse Laplace transform, we have

$$y(x) = \frac{a-b}{7}e^{-6x} + \frac{6a+b}{7}e^{x} - \frac{1}{50}x^2e^{-x}$$
$$- \left(\frac{19}{500}e^{-x} + \frac{1}{28}e^{x} + \frac{2}{875}e^{-6x}\right).$$

EXAMPLE 7.65

Solve the differential equation $\frac{d^2x}{dt^2} + x = t\cos 2t$ under the conditions: $x(0) = x'(0) = 0$.

Solution. The given differential equation is

$$\frac{d^2x}{dt^2} + x = t\;\cos 2t, \quad x(0) = x'(0) = 0.$$

Taking Laplace transform of both sides of the given equation, we have

$$L\{x''(t)\} + L\{x(t)\} = L\{t\cos 2t\}$$

or

$$s^2X(s) - sx(0) - x'(0) + X(s) = \frac{s^2-4}{(s^2+4)^2}.$$

Using initial conditions $x(0) = x'(0) = 0$, we get

$$X(s)(s^2+1) = \frac{s^2-4}{(s^2+4)^2}$$

or

$$X(s) = \frac{s^2-4}{(s^2-1)(s^2+4)^2}$$
$$= \frac{5}{9(s^2+4)} + \frac{3}{3(s^2+4)^2} - \frac{5}{9(s^2+1)}.$$

Taking inverse Laplace transform, we have

$$x(t) = \frac{5}{9}\frac{\sin 2t}{2} + \frac{8}{3}[\sin 2t - 2t\cos 2t] - \frac{5}{9}\sin t.$$

EXAMPLE 7.66

Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{2x}, \; y(0) = 2, \; y'(0) = -1$$

by using Laplace transforms.

Solution. Taking Laplace transform, we have

$$s^2Y(s) - sy(0) - y'(0) - 2\{sY(s) - y(0)\} + Y(s)$$
$$= \frac{1}{s-2}.$$

Since $y(0) = 2$, $y'(0) = -1$, we have

$$s^2Y(s) - 2s + 1 - 2sY(s) + 4 + Y(s) = \frac{1}{s-2}$$

or

$$s^2Y(s) - 2sY(s) + Y(s) = 2s - 5 + \frac{1}{s-2}$$

or

$$(s^2-2s+1)Y(s) = \frac{2s^2-7s+11}{s-2}$$

or

$$Y(s) = \frac{2s^2-9s+11}{(s-2)(s-1)^2}$$
$$= \frac{1}{s-2} + \frac{1}{s-1} - \frac{4}{(s-1)^2}.$$

Taking inverse Laplace transform, we get

$$y(x) = e^{2x} + e^{x} - 4\,L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$
$$= e^{2x} + e^{x} - 4e^{x}(x) = e^{x}(1-4x) + e^{2x}.$$

EXAMPLE 7.67

Solve the following differential equation using Laplace transforms:

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2e^t \quad \text{where}$$

$$y(0) = 1, \left(\frac{dy}{dt}\right)_{t=0} = 0, \left(\frac{d^2y}{dt^2}\right)_{t=0} = -2.$$

Solution. The given differential equation is

$$y''' - 3y'' + 3y' - y = t^2e^t$$

with conditions

$$y(0) = 1 \ , \ y'(0) = 0 \ \ , \ y''(0) = 2.$$

Taking Laplace transform, we get

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - 3[s^2y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] - Y(s) = \frac{2}{(s-1)^3}.$$

Using the given initial conditions, we have

$$s^3Y(s) - s^2 + 2 - 3[s^2Y(s) - s] + 3[sY(S) - 1] - Y(s) = \frac{2}{(s-1)^3}$$

or

$$[s^3 - 3s^2 + 3s - 1]Y(s) - s^2 + 3s - 1 = \frac{2}{(s-1)^3}$$

or

$$(s-1)^3Y(s) = \frac{2}{(s-1)^3} + s^2 - 3s + 1 = \frac{2}{(s-1)^3} + (s-1)^2 - s$$

Thus

$$Y(s) = \frac{2}{(s-1)^6} + \frac{(s-1)^2}{(s-1)^3} - \frac{s}{(s-1)^3} = \frac{2}{(s-1)^6} + \frac{1}{s-1} - \left[\frac{(s-1)+1}{(s-1)^3}\right] = \frac{2}{(s-1)^6} + \frac{1}{s-1} - \frac{1}{(s-1)^3} - \frac{1}{(s-1)^3}$$

Taking inverse Laplace transform, we get

$$y = \frac{t^5e^t}{60} + e^t - te^t - \frac{t^2}{2}e^t.$$

EXAMPLE 7.68

Solve the following simultaneous differential equations:

(a) $3\frac{dx}{dt} - y = 2t$, $\frac{dx}{dt} + \frac{dy}{dt} - y = 0$ with the condition $x(0) = y(0) = 0$.

(b) $\frac{dx}{dt} + 4\frac{dy}{dt} - y = 0$, $\frac{dx}{dt} + 2y = e^{-t}$ with the condition $x(0) = y(0) = 0$.

Solution. (a) The given simultaneous differential equations are

$$3x' - y = 2t$$
$$x' + y' - y = 0$$

with $x(0) = y(0) = 0$.Taking Laplace transforms, we get

$$3\{sX(s) - x(0)\} - Y(s) = \frac{2}{s^2}$$

and

$$sX(s) - x(0) + sY(s) - y(0) - Y(s) = 0.$$

Using the initial conditions, the above equations reduce to

$$3sX(s) - Y(s) = \frac{2}{s^2}$$

and

$$sX(s) + sY(s) - y(s) = 0$$

or

$$3sX(s) - Y(s) = \frac{2}{s^2} \tag{24}$$

and

$$sX(s) + (s-1)Y(s) = 0 \tag{25}$$

Multiplying the equation (25) by 3 and then subtracting (24) from it, we get

$$(3s-2)Y(s) = -\frac{2}{s^2}$$

and so

$$Y(s) = -\frac{2}{s^2(3s-2)} = \frac{1}{s^2} + \frac{3}{2s} - \frac{3}{2\left(s - \frac{2}{3}\right)} \tag{26}$$

(by partial fractions)

Taking inverse Laplace transform, we have

$$y = t + \frac{3}{2} - \frac{3}{2}e^{\frac{2t}{3}}. \tag{27}$$

Substituting the value of $Y(s)$ from (26) in (24), we get

$$3sX(s) = \frac{2}{s^2} + \frac{1}{s^2} + \frac{3}{2s} - \frac{3}{2\left(s - \frac{2}{3}\right)}$$

so that

$$X(s) = \frac{1}{s^3} + \frac{1}{2s^2} - \frac{1}{2s\left(s - \frac{2}{3}\right)}$$
$$= \frac{1}{s^3} + \frac{1}{2s^2} - \frac{3}{4}\left[\frac{1}{s - \frac{2}{3}} - \frac{1}{s}\right]$$

Taking inverse Laplace transform, we have

$$x = \frac{t^2}{2} + \frac{t}{2} + \frac{3}{4} - \frac{3}{4}e^{\frac{2t}{3}} \tag{28}$$

Thus (27) and (28) provides the solution to the given system.

(b) Taking Laplace transform, we have

$$sX(s) - x(0) + 4(sY(s) - y(0) - Y(s) = 0$$

and

$$sX(s) - x(0) + 2Y(s) = \frac{1}{s+1}.$$

Using the given initial conditions, we get

$$sX(s) + (4s - 1)Y(s) = 0 \tag{29}$$

and

$$SX(s) + 2Y(s) = 0 \tag{30}$$

Subtracting (30) from (29), we get

$$(4s - 3)Y(s) = -\frac{1}{s+1}$$

or

$$Y(s) = \frac{1}{(s+1)(4s-3)}$$
$$= \frac{1}{7}\left[\frac{1}{s+1} - \frac{1}{s - \frac{3}{4}}\right]. \tag{31}$$

Taking inverse Laplace transform, we get

$$y = \frac{1}{7}[e^{-t} - e^{\frac{3t}{4}}].$$

Putting the value of $Y(s)$ from (31) in (30), we get

$$sX(s) = \frac{5}{7(s+1)} + \frac{2}{7\left(s - \frac{3}{4}\right)}$$

or

$$X(s) = \frac{5}{7s(s+1)} + \frac{2}{7s\left(s - \frac{3}{4}\right)}$$
$$= \frac{5}{7}\left(\frac{1}{s} - \frac{1}{s+1}\right) + \frac{8}{21}\left[\frac{1}{s - \frac{3}{4}} - \frac{1}{s}\right].$$

Taking inverse transform, we get

$$x = \frac{1}{3} - \frac{5}{7}e^{-t} + \frac{8}{21}e^{\frac{3t}{4}}.$$

EXAMPLE 7.69

Solve the simultaneous differential equations using Laplace transforms:

$$x'(t) + y'(t) + x(t) = -e^{-t},$$
$$x'(t) + 2y'(t) + 2x(t) + 2y(t) = 0,$$

where $x(0) = -1,\ y(0) = 1.$

Solution. We want to solve

$$x'(t) + y'(t) + x(t) = -e^{t},$$
$$x'(t) + 2y'(t) + 2x(t) + 2y(t) = 0,$$

subject to the conditions $x(0) = -1$, $y(0) = 1$. Taking Laplace transform of the given equations, we have

$$sX(s) - x(0) + sY(s) - y(0) + X(s) = -\frac{1}{s+1}$$

and

$$sX(s) - x(0) + 2[sY(s) - y(0)] + 2X(s) + 2Y(s)$$
$$= 0.$$

Using the given initial conditions, we get

$$sX(s) + X(s) + sY(s) = -\frac{1}{s+1}$$

and

$$sX(s) + 1 + 2sY(s) - 2 + 2X(s) + 2Y(s) = 0$$

that is,

$$(s+1)X(s) + sY(s) = -\frac{1}{s+1} \tag{32}$$

and

$$(s+2)X(s) + 2(s+1)Y(s) = 1 \tag{33}$$

Multiplying (32) by $2(s+1)$ and (33) by s and then subtracting, we get

$$[2(s+1)^2 - s(s+2)]X(s) = -2 - s$$

or

$$(s^2+2s+2)X(s) = -(s+2)$$

or

$$X(s) = -\frac{s+2}{s^2+2s+2} = -\frac{s+2}{(s+1)^2+1}$$
$$= -\left[\frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}\right]. \quad (34)$$

Taking inverse Laplace transform, we get

$$x(t) = -e^{-t}\cos t - e^{-t}\sin t.$$

Further, putting the value of $X(s)$ in (33) ,we get

$$2(s+1)Y(s) = 1 + \frac{(s+2)(s+2)}{s^2+2s+2}$$
$$= \frac{s^2+2s+2+s^2+4s+4}{s^2+2s+2}$$
$$= \frac{2s^2+6s+6}{s^2+2s+2}$$

or

$$Y(s) = \frac{s^2+3s+3}{(s+1)(s^2+2s+2)}$$
$$= \frac{1}{s+1} + \frac{1}{s^2+2s+2}$$
$$= \frac{1}{s+1} + \frac{1}{(s+1)^2+1}.$$

Taking inverse Laplace transform, we have

$$y(t) = e^{-t} - e^{-t}\sin t.$$

EXAMPLE 7.70

Use Laplace transform method to solve the simultaneous equations: $D^2x - Dy = \cos t$; $Dx + D^2y = -\sin\ t$, given that $x = 0$, $Dx = 0$, $y = 0$ $Dy = 1$, when $t = 0$.

Solution. Taking Laplace transforms of the given equations, we have

$$s^2X(s) - sx(0) - x'\ (0) - sy\ (s) - y(0) = \frac{s}{s^2+1}$$

and

$$s\ X(s) - x(0)\ + s^2y(s) - sy(0) - y'(0)$$
$$= -\frac{1}{s^2+1}.$$

Using the given condition, the above equations transform to

$$s^2X(s) - sY(s) = \frac{s}{s^2+1} \quad (35)$$

and

$$sX(s) + s^2Y(s) = 1 - \frac{1}{s^2+1} = \frac{s^2}{s^2+1}. \quad (36)$$

Multiplying (35) by s and adding to (36), we get

$$s^3X(s) + sX(s) = \frac{s^2}{s^2+1} + \frac{s^2}{s^2+1} = \frac{2s^2}{s^2+1}.$$

Thus

$$X(s) = \frac{2s^2}{(s^2+1)(s^3+s)} = \frac{2s}{(s^2+1)^2}.$$

Taking inverse Laplace transform, we have

$$x(t) = t\sin t.$$

Putting the value of $X(s)$ in (35), we get

$$\frac{2s^3}{(s^2+1)^2} - sy\ (s) = \frac{s}{s^2+1}$$

or

$$y(s) = \frac{s^2-1}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}.$$

Taking inverse Laplace transform, we get

$$y(t) = \sin\ t - (\sin\ t - t\ \cos\ t) = t\ \cos\ t.$$

Hence the required solution is

$$x(t) = t\sin t \quad \text{and} \quad y(t) = t\cos t.$$

EXAMPLE 7.71

Using Laplace transform, solve the integral equation

$$y(t) = 1 - e^{-t} + \int_0^t y(t-u)\sin u\, du.$$

Solution. Taking Laplace transform of both sides of the given integral equation, we have

$$L\{y\} = L\{1\} - L\{e^{-t}\} + L\{y\}.\ L\{\sin u\}$$
$$= \frac{1}{s} - \frac{1}{s+1} + L\{y\}.\frac{1}{s^2+1}.$$

This relation yields

$$L\{y\}\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s} - \frac{1}{s+1}$$

or

$$L\{y\} = \frac{s^2+1}{s^2}\left(\frac{1}{s} - \frac{1}{s+1}\right) = \frac{s^2+1}{s^3(s+1)}$$

$$= \frac{2}{s} - \frac{2}{s+1} - \frac{1}{s^2} + \frac{1}{s^3}$$

(by partial fractions).

Taking inverse Laplace transform, we get

$$y(t) = 2 - 2\,e^{-t} - t + \frac{1}{2}t^2.$$

EXERCISES

1. Solve the following initial value problems:

   (a) $y'(t) + 3y(t) = 0$, $x(1) = 1$.

   Ans. $y(t) = e^{3(1-t)}$

   (b) $\frac{d^2y}{dt^2} + y = 1$, $y(0) = y(0) = 0$.

   Ans. $1 - \cos t$

   (c) $y'' + y = e^{-t}$, $y(0) = A$, $y'(0) = B$.

   Ans. $y(t) = \frac{1}{2}e^{-t} + \left(A - \frac{1}{2}\right)\cos t + \left(B + \frac{1}{2}\right)\sin t$

   (d) $\frac{d^2y}{dt^2} + y = 0$, $y(0) = 1$, $y'(0) = 0$.

   Ans. $y(t) = \cos t$.

   (e) $\frac{d^2y}{dt^2} + a^2y = f(t)$, $y(0) = 1$, $y'(0) = -2$.

   Hint: $Y(s) = \frac{s-2}{s^2+a^2} + \frac{F(s)}{s^2+a^2}$. But by Convolution theorem

   $$L^{-1}\left\{\frac{F(s)}{s^2+a^2}\right\} = f(t) * \frac{\sin at}{a} \text{ and so}$$

   $$y(t) = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} - L^{-1}\left\{\frac{2}{s^2+a^2}\right\} + f(t) * \frac{\sin at}{a}$$

   Ans. $\cos at - \frac{2\sin at}{a} + \frac{1}{a}\int_0^t f(u)\sin a(t-u)\,du$

   (f) $\frac{d^2y}{dt^2} + y = 3\sin 2t$, $y(0) = 3$, $y'(0) = 1$.

   Ans. $-\sin 2t + 3\cos t + 3\sin t$

   (g) $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t}$ $(t \geq 0)$, $x(0) = 1$ and $x'(0) = 0$

   Ans. $e^{-t} + e^{-2t} - e^{-3t}$, $t \geq 0$

   (h) $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$, $x(0) = x(0) = 0$.

   Ans. $x(t) = 0$

   (i) $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt^2} + \lambda^2x = 0$, $x(0) = x'(0) = 0$.

   Ans. $x(t) = e^{-bt}(c_1 \sin\sqrt{\lambda^2 - b^2}t + c_2\cos\sqrt{\lambda^2 - b^2}t$

2. Solve $y' - 2ty = 0$, $y(0) = 1$ and show that its solution does not have Laplace transform.

   Ans. $y(t) = e^{t^2}$ (not of exponential order)

3. Solve $ty'' + y = 0$, $y(0) = 0$.

   Hint: Proceed as in Example 7.20

   Ans. $C\sum_{n=0}^{\infty}\frac{(-1)^n t^{n+1}}{(n+1)!\,n!}$.

4. Given that I = Q = 0 at $t = 0$, find current I in the LC circuit given for $t > 0$ in Figure 7.15.

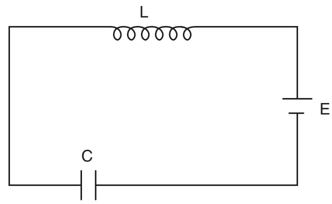

Figure 7.15

Hint: The differential equation governing the circuit is $L\frac{dI}{dt} + \frac{1}{C}\int_0^t I(u)\,du = E$. Application of Laplace transform yields

$Ls\,F(s) + \frac{F(s)}{Cs} = \frac{E}{s}$, that is, $F(s) = \frac{EC}{LCs^2+1} = \frac{E}{L\left(s^2+\frac{1}{LC}\right)}$

Ans. $I(t) = E\sqrt{\frac{C}{L}}\sin\frac{1}{\sqrt{LC}}t$.

5. Given that I = Q = 0 at $t = 0$, find charge and current in the circuit shown in Figure 7.16.

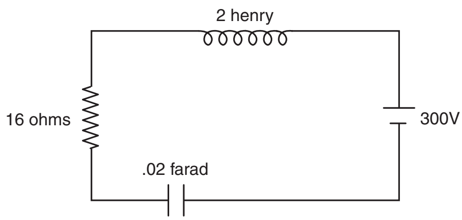

Figure 7.16

Hint: The governing equation is

$\frac{d^2Q}{dt^2} + 8\frac{dQ}{dt} + 25Q = 150$, $F(s) = \frac{150}{s(s^2+8s+25)}$ and so inversion gives $Q(t) = 6 - 6e^{-4t}\cos 3t - 8e^{-4t}\sin 3t$. Then $I(t) = 50\,e^{-4t}\sin 3t$.

6. Solve the following systems of differential equations:

(a) $\frac{dx}{dy} + x - y = 1 + \sin t$, $\frac{dy}{dt} - \frac{dx}{dt} + y = t - \sin t$, with $x(0) = 0$, $y(0) = 1$

Ans. $x(t) = t + \sin t$, $y(t) = t + \cos t$

(b) $\frac{dy}{dt} = -z$, $\frac{dz}{dt} = y$ with $y(0) = 1$, $z(0) = 0$.

Ans. $y(t) = \cos t$, $z(t) = \sin t$

7. Solve $y'' + 4y = 4 \cos 2t$, $y(0) = y'(0) = 0$. Does resonance occur in this case? **Hint:** $Y(s) = \frac{4s}{(s^2+4)^2}$ and so $y(t) = 4\left[\frac{t}{4} \sin 2t\right] = t \sin 2t$.

Note that $y(t)$? $\to \infty$? as $t \to \infty$??. Hence, there shall be resonance.

8. Solve the following difference equations:

(a) $3y(t) - 4y(t-1) + y(t-2) = t$, $y(t) = 0$ for $t <$ 0.

Ans. $y(t) = \frac{t}{3} + \frac{1}{2} \sum_{n=1}^{[t]} \left(1 - \frac{1}{3^n}\right)(t-n)$

(b) $a_{n+2} - 2a_{n+1} + a_n = 0$, $a_0 = 0$, $a_1 = 1$.

Ans. $a_n = n$

(c) $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 1$, $a_1 = 8$.

Ans. $a_n = 3(2^n) - 2(-1)^n$, $n \geq 0$

(d) $a_n = 2a_{n-1} - a_{n-2}$, $a_1 = 1.5$, $a_2 = 3$

Ans. 1.5 n

(e) $y(t) - y(t-1) = t^2$

Ans. $y(t) = 2 \sum_{n=0}^{[t]} \frac{(t-n)^{n+3}}{(n+3)!}$

(f) $y''(t) - y(t-1) = \delta(t)$, $y(t) = y(t) = 0$, $t \leq 0$.

Hint: $s^2 L\{y(t)\} - e^{-s} L\{y(t)\} = L\{\ (t)\}$ and so $L\{y(t)\} = \frac{1}{s^2\left(1 - \frac{e^{-s}}{s^2}\right)}$. But

$$L^{-1}\left\{\frac{e^{-ns}}{s^{2n+2}}\right\} = \begin{cases} \frac{(t-n)^{2n+1}}{(2n+1)!} & \text{for } t \geq n \\ 0 & \text{otherwise.} \end{cases}$$

Hence $y(t) = \sum_{n=0}^{[t]} \frac{(t-n)^{2n+1}}{(2n+1)!}$

9. Solve the integral equations:

(a) $f(t) = 1 + \int_0^t \cos(t-u) f(u)\, du$

Ans. $f(t) = 1 + \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t\right) e^{t/2}$

(b) $y(t) = \sin t + 2 \int_0^t y(u) \cos(t-u)\, du$

Ans. $y(t) = te^t$

(c) $y(t) = t + \frac{1}{6} \int_0^t y(u)(t-u)^3\, du$

Ans. $y(t) = \frac{1}{2}(\sinh t + \sin t)$

(d) $f(t) = \int_0^t \sin u(t-u)\, du$

Ans. 0

(e) $y'(t) + 3y(t) + 2 \int_0^t y(u) du = t$, $y(0) = 1$.

Hint: $L\{y(t)\} = \frac{s^2+1}{s(s^2+3s+2)}$. Use partial fractions and then use inversion to give $y(t) = \frac{1}{2} - 2e^{-t} + \frac{5}{2} e^{-2t}$

10. Solve the partial differential equation $x \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = x$, $x > 0$, $t > 0$ subject to the conditions $u(x, 0) = 0$ for $x > 0$ and $u(0, t) = 0$ for $t > 0$.

Hint: Using Laplace transform, $\frac{d}{dx} U(x, s) + x\, sU(x, s) = \frac{x}{s}$, integrating factor is $e^{\frac{1}{2}x^2}$ and so $U(x, s) = \frac{1}{s^2}\left[1 - e^{-\frac{1}{2}sx^2}\right]$. Inversion yields $u(x, t) = \begin{cases} t & \text{for } t < x^2/2 \\ x^2/2 & \text{for } 2t > x^2 \end{cases}$.

11. Find the bounded solution of $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x > 0$, $t > 0$ for $u(0, t) = 1$, $u(x, 0) = 0$.

Hint: Application of Laplace transform yields $U(x, s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$ for bounded $u(x, t)$, $U(x, s)$ must be bounded and so $c_1 = 0$. Further $U(0, s) = L\{u(0, t)\} = L\{1\} = 1/s$. Therefore, $1/s = c_2$ and so $U(x, s) = \text{erfc}\,(x/2\sqrt{t})$.

12. Solve $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$, $x > 0$, $t > 0$ for $u(x, 0) = 0$, $y_t(x, 0) = 0$, $x > 0$, $y(0, t) = f(t)$ with $f(0) = 0$ and $\lim_{x \to \infty} y(x, t) = 0$.

Hint: see Example 7.54

Ans. $y(x, t) = f\left(t - \frac{x}{a}\right) H\left(t - \frac{x}{a}\right)$

13. Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $x>0$, $t>0$ for $u(0, t) = 10 \sin 2t$, $u(x, 0) = 0$, $u_t(x, 0) = 0$ and $\lim_{x \to \infty} u(x, t) = 0$

Ans. $u(x, t) = \begin{cases} 10 \sin 2(t-x) & \text{for } t > x \\ 0 & \text{for } t < x. \end{cases}$

14. Evaluate the integrals:

(a) $\int_0^\infty J_0(t)\, dt$

Ans. 1

(b) $\int_0^\infty \frac{e^{-t} - e^{-2t}}{t} dt$

Hint: $L\{e^{-t} - e^{-2t}\} = \frac{1}{s+1} - \frac{1}{s+2}$ and so

$L\left\{\frac{e^{-t}-e^{-2t}}{t}\right\} = \int_s^\infty \left(\frac{1}{t+1} - \frac{1}{t+2}\right) dt = \log\left(\frac{s+2}{s+1}\right)$,

that is,

$\int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-2t}}{t}\right) dt = \log\left(\frac{s+2}{s+1}\right)$. Taking $s = 0$, we get the value of the given integral equal to $\log 2$

(c) $\int_0^\infty \cos x^2 dx$ (Proceed as in Example 7.59)

Ans. $\frac{1}{2}\sqrt{\frac{\pi}{2}}$

(d) $\int_0^\infty \frac{x \sin at}{x^2+a^2} dx$; a, $t > 0$.

Ans. $f(t) = \frac{\pi}{2} e^{-t}$

ENGINEERING MATHEMATICS—YEAR 2007
SEMESTER—2

GROUP—A

(Multiple Choice Type Questions)

1. Choose the correct alternatives for any *ten* of the following questions: **$10 \times 1 = 10$**

(i) Laplace transform of the function cos (at) is

(a) $\frac{s}{s^2 - a^2}$ (b) $\frac{a}{s^2 + a^2}$

(c) $\frac{s}{s^2 + a^2}$ (d) $\frac{1}{s^2 - a^2}$

Ans. (c) $\frac{s}{s^2 + a^2}$

(ii) The rank of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ is

(a) 0 (b) 1

(c) 3 (d) 2

Ans. (d) 2

(iii) The order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = y$ are

(a) 2, 2 (b) 2, 1

(c) 1, 2 (d) 1, 1

Ans. (a) 2, 2

(iv) Value of the determinant $\begin{vmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{vmatrix}$ is

(a) 0 (b) abc

(c) $-abc$ (d) 2 abc

Ans. (a) 0

(v) Integrating factor of $\frac{dy}{dx} + y = 1$ is

(a) ex (b) x^2

(c) x (d) 2

Ans. (a) e^x **Hint:** $e^{\int dx} = e^x$

(vi) The operator equivalent to shift operator E is

(a) $I + \Delta$ (b) $(I + \Delta)^{-1}$

(c) $I - \Delta$ (d) $I - \Delta^2$

Ans. (a) $I + \Delta$

(vii) The number of significant digits in 3·0044 is

(a) 5 (b) 2

(c) 3 (d) 4

Ans. (a) 5

(viii) If α, β are the roots of the equation $x^2 - 3x + 2 = 0$, then $\begin{vmatrix} 0 & \alpha & \beta \\ \beta & 0 & 0 \\ 1 & -\alpha & \alpha \end{vmatrix}$ is

(a) 6 (b) $\frac{3}{2}$

(c) –6 (d) 3

Ans. (a) 5 **Hint:** $(x-1)(x-2) = 0, \alpha = 1, \beta = 1.$

(ix) The sum of the eigen values of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ is

(a) 5 (b) 2

(c) 1 (d) 6

Ans. (a) 5

(x) $(\Delta - \nabla)x^2$ is equal to

(a) h^2 (b) $-2h^2$

(c) $2h^2$ (d) None of these.

Ans. (c) $2h^2$

Hint: $(\Delta - \nabla)x^2 = \Delta x^2 - \nabla x^2 = (x+h)^2 - x^2 - \{x^2 - (x-h)^2\} = 2h^2$

(xi) If a linear transformation $T : IR^2 \to IR^2$ be defined by $T(x_1, x_2) = (x_1 + x_2, 0)$ then ker (T) is

(a) $\{(1,-1)\}$ (b) $\{(1,0)\}$

(c) $\{(0,0)\}$ (d) $\{(1,0),(0,1)\}$

Ans. (a) $\{(1,-1)\}$

(xii) The value of $\begin{vmatrix} 2000 & 2001 & 2002 \\ 2003 & 2004 & 2005 \\ 2006 & 2007 & 2008 \end{vmatrix}$ is

(a) 2000 (b) 0

(c) 45 (d) None of these.

Ans. (b)

Hint: $\xrightarrow[C_3 - C_2]{C_2 - C_1} \begin{vmatrix} 2000 & 1 & 1 \\ 2003 & 1 & 1 \\ 2004 & 1 & 1 \end{vmatrix} = 0$

(xiii) The value of K for which the vectors (1, 2, 1), (K, 1, 1) and (0, 1, 1) are linearly dependent

(a) 1 (b) 2

(c) 0 (d) 3.

Ans. (c)

Hint: $\begin{vmatrix} 1 & 2 & 1 \\ k & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$ or $k = 0$

GROUP—B
(Short Answer Type Questions)
Answer any *three* of the following

2. Apply convolution theorem to find the inverse of $\dfrac{s}{\left(s^2+9\right)^2}$

Ans. Same as 5(a) of 2005.

3. Prove that: $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$

Ans. We have, $\Delta = \begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$

$$= \begin{vmatrix} 0 & -2c^2 & -2b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} R_1 = R_1 - \left(R_2 + R_3\right)$$

$$= 2 \times b^2 \times c^2 \begin{vmatrix} 0 & -c^2 & -b^2 \\ 1 & \dfrac{c^2+a^2}{b^2} & 1 \\ 1 & 1 & \dfrac{a^2+b^2}{c^2} \end{vmatrix}$$

$$= 2b^2c^2 \begin{vmatrix} 0 & -c^2 & -b^2 \\ 0 & \dfrac{c^2+a^2}{b^2} & 1-\dfrac{a^2+b^2}{c^2} \\ 1 & 1 & \dfrac{a^2+b^2}{c^2} \end{vmatrix} R'_2 = R_2 - R_3$$

$$= -2b^2c^2 \left\{-b^2\left(\frac{c^2+a^2}{b^2} - 1\right) + c^2\left(1 - \frac{a^2+b^2}{c^2}\right)\right\}$$

$$= -2b^2c^2 \left\{-\left(c^2+a^2-b^2\right) + \left(c^2-a^2-b^2\right)\right\}$$

$$= 4a^2b^2c^2$$

4. Solve the differential equation by Laplace transform : $\dfrac{d^2x}{dt^2} + 4x = \sin 3t,\ x(0) = 0, x'(0) = 0$

Ans. We have $\dfrac{d^2x}{dt^2}+4x=\sin 3t$

Taking Laplace transform on both side

$$s^2X(s)-sx(0)-x(0)+4X(s)=\frac{3}{s^2+3^2}$$

or, $\quad s^2X(s)-s.0-0+4X(0)=\dfrac{3}{s^2+9}$

or, $\quad (s^2+4)X(s)=\dfrac{3}{s^2+9}$

$\therefore \quad X(s)=\dfrac{3}{(s^2+9)(s^2+4)}=\dfrac{3}{5}\left(\dfrac{1}{s^2+4}-\dfrac{1}{s^2+9}\right)$

$\therefore \quad x(t)=\mathrm{L}^{-1}\{X(s)\}$

$$=\frac{3}{5\cdot 2}\sin 2t-\frac{3}{5}\cdot\frac{\sin 3t}{3}$$

$$=\frac{3}{10}\sin 2t-\frac{1}{5}\sin 3t$$

5. Solve $(x^2+y^2+2x)\,dx+xydy=0$

Ans. We have $(x^2+y^2+2x)\,dx+xydy=0$

here $\quad M=x^2+y^2+2x$

$\quad N=xy$

$\therefore \quad \dfrac{\partial M}{\partial y}=2y\dfrac{\partial N}{\partial x}=y$

$\therefore \quad \dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$

$\therefore \quad$ Given equation is not exact.

Now, $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\dfrac{2y-y}{xy}=\dfrac{1}{x}$

So, $\quad$ I. F. $=e^{\int\frac{1}{x}dx}=e^{\log x}=x$

Multiplying both side of given equation by x

$$(x^2+xy^2+2x^2)\,dx+x^2ydy=0$$

or, $\quad (x^3+2x^2)\,dx+xy^2dx+x^2ydy=0$

or, $\quad \left(x^3+2x^2\right)dx+\dfrac{1}{2}d\left(x^2y^2\right)=0$

Integrating both side

$$\frac{x^4}{4}+\frac{2x^3}{3}+\frac{1}{2}x^2y^2=c$$

c — integrating constant.

6. Show that $W = \{x, y, z) \in R^3, 2x - y + 3z = 0\}$ is a subspace of R^3. Find basic of W. What is dimension?

Ans. Same as 13(b) of 2008.

7. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then verify that A satisfies its own characteristic equation. Hence find A^{-1}.

Ans. Same as 8(b) of 2008.

GROUP—C
(Long-Answer Questions)
Answer any *three* questions

8. (a) Evaluate : $\left(\dfrac{\Delta^2}{E}\right)x^3$

Ans. $\left(\dfrac{\Delta^2}{E}\right)x^3 = \dfrac{(E-1)^2}{E}x^3$

$$= \left(\frac{E^2 - 2E + 1}{E}\right)x^3$$

$$= (E - 2 + E - 1)x^3$$

$$= (x+h)^3 - 2x^3(x-h)^3$$

$$= x^3 + 3x^2h + 3xh^2 + h^3 - 2x^3 + x^3 - 3x^2h$$

$$+3xh^2 - h^3 = 6xh^2$$

(b) Find the missing data in the following table

x	–2	–1	0	1	2
$f(x)$	6	0	–	0	6

Ans. As y has four given values

$$\Delta^4 y_0 = 0$$

$$\therefore (E-1)^4 y_0 = 0$$

or, $\quad (E^4 - 4E^3 + 6E^2 - 4E + 1)y_0 = 0$

or, $\quad y_4 - 3y_3 + 6y_2 - 4y_1 - y_0 = 0$

$$\therefore y_2 = \frac{-y_4 + 3y_3 + 4y_1 - y_0}{6}$$

$$= \frac{-6 + 3.0 + 4.0 - 6}{6} = -2$$

(c) Show that, (3, 1, –2), (2, 1, 4) and (1, –1, 2) form a basis of R^3.

Ans. Consult 8(a) of 2005.

9. (a) Prove that $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$, by Laplace transform.

Ans. See 11(b) of 2005

(b) Apply convolution theorem to prove that

$$\int_0^t \sin u \cos(t-u) du = \frac{t}{2}\sin t$$

Ans. Let, $f(t) = \int_0^t \sin u \cos(t-u) du$

Using convolution theorem,

$$\begin{aligned} L\{f(t)\} &= F_1(s)\cdot F_2(s) \\ &= \alpha\{\sin t\}\cdot \alpha\{\cos t\} \\ &= \frac{1}{s^2+1}\cdot\frac{s}{s^2+1} \\ &= \frac{s}{s^2+1} = \frac{1}{2}\cdot\frac{2s}{s^2+1} \\ &= \frac{1}{2}(-1)\cdot\frac{d}{ds}\left(\frac{1}{s^2+1}\right) \end{aligned}$$

$\therefore\ f(t) = \frac{1}{2}\cdot t^1 \cdot \sin t = \frac{1}{2}\sin t$ (Proved)

(c) Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

Ans. We have $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

or, $\cos y\frac{dy}{dx} - \cos y\frac{\tan y}{1+x} = (1+x)e^x$

or, $\cos y\frac{dy}{dx} - \cos y\frac{\tan y}{1+x} = (1+x)e^x$

Let, $\sin y = z$

$\therefore \cos y\frac{dy}{dx} = \frac{dz}{dx}$

or, $\frac{dz}{dx} - \frac{1}{1+x}z = (1+x)e^x$ ....(i)

So, I. F. $= e^{\int\left(\frac{1}{1+x}\right)dx}$

$= \frac{1}{1+x}$

Multiplying both sides of (i) by $\frac{1}{1+x}$

$$\frac{dz}{dx}\cdot\frac{1}{1+x}-\frac{1}{(1+x)^2}\cdot z=e^x$$

$$\therefore \quad d\left(\frac{z}{1+x}\right)=e^x dx$$

Integrating both sides

$$\frac{z}{1+x}=e^x+c, \quad c \text{ — integrating constant}$$

or, $\sin y=\left(e^x+c\right)(1+x)$ [as $z=\sin y$]

10. (a) Solve by Cramer's rule : $x+y+z=7$
$x+2y+3z=10$
$x-y+z=3$

Ans. Same as 14(b) of 2008.

(b) Find general solution of $p=\cos(y-px)$, where $p=\dfrac{dy}{dx}$

Ans. We have, $p=\cos(y-px)$

or, $y-px=\cos^{-1}p$

Differentiating both side w.r.t. x

$$\frac{dy}{dx}-p.1-x\frac{dp}{dx}=\frac{-1}{\sqrt{1-p^2}}\cdot\frac{dp}{dx}$$

or, $p-p-x\dfrac{dp}{dx}=-\dfrac{1}{\sqrt{1-p^2}}\dfrac{dp}{dx}$

or, $\left(x-\dfrac{1}{\sqrt{1-p^2}}\right)\dfrac{dp}{dx}=0$

when, $x-\dfrac{1}{\sqrt{1-p^2}}=0$

or, $x=\dfrac{1}{\sqrt{1-p^2}}$

or, $x^2=\dfrac{1}{1-p^2}$

$\therefore p^2=\dfrac{x^2-1}{x^2}$

$\therefore p=\dfrac{\sqrt{x^2-1}}{x}$

$\therefore$ Singular solution

$$y=\sqrt{x^2-1}+\cos^{-1}\frac{\sqrt{x^2-1}}{x}$$

again when $\frac{dp}{dx}=0$ or, $p=c, c\rightarrow$ constant, general solution is $y=e^{x}+\cos^{-1}c$

(c) Solve: $\frac{dy}{dx}+\frac{y\log y}{x}=\frac{y(\log y)^{2}}{x^{2}}$

Ans. We have $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^{2}}(\log y)^{2}$ dividing both sides by $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^{2}}(\log y)^{2}$

$$\frac{1}{y(\log y)^{2}}\frac{dy}{dx}+\frac{1}{x}\frac{1}{\log y}=\frac{1}{x^{2}}\text{..........(i)}$$

Assume, $z=\frac{1}{\log y}$

$$\therefore \frac{dz}{dx}=-\frac{1}{y(\log y)^{2}}\frac{dy}{dx}$$

equation (i) become

$$-\frac{dz}{dx}+\frac{z}{x}=\frac{1}{x^{2}}$$

or, $\frac{dz}{dx}-\frac{z}{x}=-\frac{1}{x^{2}}$............(ii)

$\therefore$ I. F. $=e^{-\int\frac{dx}{x}}=\frac{1}{x}$

multiplying both side of (ii) by $\frac{1}{x}$

$$\frac{1}{x}\frac{dz}{dx}-\frac{z}{x^{2}}=-\frac{1}{x^{3}}$$

or, $\frac{d}{dx}\left(\frac{z}{x}\right)=-\frac{1}{x^{3}}$ or, $d\left(\frac{z}{x}\right)=-\frac{1}{x^{3}}dx$

Integrating, $\frac{z}{x}=+\frac{1}{2x^{2}}+c,\ c\rightarrow$ Constant

or, $\frac{1}{x\log y}=+\frac{1}{2x^{2}}+c$

or, $\frac{1}{\log y}=\frac{1}{2x}+cx=\frac{1+2cx^{2}}{2x}$

$\therefore \log y=\frac{2x}{1+2cx^{2}}$

or, $y=e^{\frac{2x}{1+2cx^{2}}}$

11. (a) Solve: $(D^{2}-2D)y=e^{x}\sin x$, where $D\equiv\frac{d}{dx}$

Ans. We have $(D^2 - 2D)\ \ y = e^x \sin x$

The auxiliary equation is $m^2 - 2m = 0$

or, $\quad m(m-2) = 0$

$\therefore \quad m = 0, 2$

$\therefore \quad$ C.F. $= (A + Bx)e2x,\ A$ and B constants

$$\text{Again, P. I.} = \frac{1}{D^2 - 2D}(e^x \sin x)$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1)} \cdot \sin x$$

$$= e^x \cdot \frac{1}{D^2 + 2D + 1 - 2D - 1} \cdot \sin x = e^x \cdot \frac{1}{D^2 - 1}(\sin x)$$

$$= e^x \cdot \frac{1}{-1-1} \sin x = -\frac{1}{2} e^x \sin x$$

$\therefore$ Total solution is

$$y = (A + Bx)e^{2x} - \frac{1}{2} e^x \sin x$$

(b) Solve $\dfrac{dx}{dt} - 7x + y = 0$ and $\dfrac{dy}{dt} - 2x - 5y = 0$

Ans. We have $\dfrac{dx}{dt} - 7x + y = 0$

$$\frac{dy}{dt} - 2x - 5y = 0$$

Let, $D \equiv \dfrac{d}{dt}$

$\therefore (D - 7)\ x + y = 0$.........(i)

$(D - 5)\ y - 2x = 0$...........(ii)

(i) × 2 + (ii) × $(D - 7)$

$$2(D-7)\ x + 2y = 0$$

$$\underline{+(D-7)\ (D-5)y - 2x(D-7) = 0}$$

$$(D-7)\ (D-5)y + 2y = 0$$

or, $\quad (D^2 - 12D + 35)y + 2y = 0$

or, $\quad (D^2 - 12D + 37)y = 0$

Auxiliary equation is $m^2 - 12m + 37 = 0$

$$\therefore m = \frac{12 \pm \sqrt{(12)^2 - 4.1.37}}{2} = 6 \pm i$$

$\therefore$ Solution is $y = e^{6i}(A\cos t + B\sin t)$ A and B constant from (ii)

$$x=\frac{1}{2}(D-5)y=\frac{1}{2}(Dy-5y)$$

or, $x=\frac{1}{2}\left\{e^{6t}\left[A(-\sin t)+B\cos t\right]+6e^{6t}(A\cos t+B\sin t)-5e^{et}(A\cos t+B\sin t)\right\}$

$$=\frac{1}{2}\left\{e^{6t}\left[-A+6B-5B\right]\sin t+e^{6t}\left[B+6A-5A\right]\cos t\right\}$$

$$=\frac{1}{2}e^{6t}\left\{(B-A)\sin t+(A+B)\cos t\right\}$$

$$=e^{6t}\left\{C\cos t+D\sin t\right\}$$

where, $C=\dfrac{A+B}{2}$; $\qquad D=\dfrac{B-A}{2}$

$\therefore$ Required solution is

$$x=e^{6t}\left\{C\cos t+D\sin t\right\}$$

$$y=e^{6t}\left\{A\cos t+B\sin t\right\}$$

(c) Solve by Gauss elimination method

$$2x+2y+z=12$$

$$3x+2y+2z=8$$

$$5x+10y-8z=10$$

Ans. We have,

$$2x+2y+z=12\text{..........(i)}$$

$$3x+2y+2z=8\text{..........(ii)}$$

$$5x+10y-8z=10\text{........(iii)}$$

$\text{(i)}\times\left(-\dfrac{3}{5}\right)+\text{(ii)}$ gives

$$-4y+\frac{34}{5}z=2\text{..........(iv)}$$

$\text{(i)}\times\left(-\dfrac{2}{5}\right)+\text{(iii)}$ gives

$$-2y+\frac{21}{5}z=8\text{...........(v)}$$

Again $\text{(iv)}\times\left(-\dfrac{1}{2}\right)+\text{(v)}$ gives

$$\frac{4}{5}z=7$$

$$\therefore z=\frac{35}{4}$$

from (iv) $-4y+\dfrac{34}{5}\cdot\dfrac{35}{4}=2$

or, $\quad y=\dfrac{115}{8}$

from (i) $5x+10\cdot\dfrac{115}{8}-8\cdot\dfrac{35}{4}=10$

or, $\quad 5x+\dfrac{575-280}{4}=10$

$\therefore\ 5x+\dfrac{295}{4}=10$

or, $\quad 5x+10-\dfrac{295}{4}=\dfrac{40-295}{4}$

$\therefore\ x=\dfrac{40-295}{20}=-\dfrac{51}{4}$

12. (a) Use Lagrange's interpolation formula to find $f(x)$, where $f(0)=-18$, $f(1)=0$, $f(3)=0$, $f(5)=-248$, $f(6)=0$ and $f(9)=13104$

Ans. We have,

$x:$	0	1	3	5	6	9
$f(x)$	–18	0	0	–248	0	13104

Now using Lagrange's interpolation,

$$f(x)=\frac{(x-1)(x-3)(x-5)(x-6)(x-9)}{(0-1)(0-3)(0-5)(0-6)(0-9)}\times(-18)$$

$$+\frac{(x-0)(x-1)(x-3)(x-6)(x-9)}{(5-0)(5-1)(5-3)(5-6)(5-9)}\times(-248)$$

$$+\frac{(x-0)(x-1)(x-3)(x-5)(x-6)}{(9-0)(9-1)(9-3)(9-6)(9-9)}\times(13104)$$

$$=(x-1)(x-3)(x-6)\left\{\frac{(x-5)(x-9)}{45}-\frac{x(x-9)\times 31}{20}+\frac{x(x-5)91}{36}\right\}$$

$$=\left(x^3-10x^2+27x-18\right)\left(\frac{180x^2+180x+180}{180}\right)$$

$$=x^5-9x^4+18x^3-x^2+9x-18$$

(b) Apply appropriate interporlation formula to calculate $f(2.1)$. Correct upto two significant figures from the following data.

x	0	2	4	6	8	10
$f(x)$	1	5	17	37	45	51

Ans. From the given data we can design the following difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	1					
2	5	4	8	0	–20	–10
4	17	12	8	–20	–30	
6	37	20	–12	+10		
8	45	8	–2			
10	51	6				

Ans. Here, $x_0 = 0, h = 2, d = \frac{x - x_0}{2} = \frac{2.1-0}{2} = 1\cdot 05$

$\therefore$ Using Newton's forward interpolation

$$f(x) = f(x_0) + d\ \Delta f(x_0) + \frac{d(d-1)}{2!}\Delta^2 f(x_0)$$

$$+\frac{d(d-1)(d-2)}{3!}\Delta^3 f(x_0) + \frac{d(d-1)(d-2)(d-3)}{4!}\Delta^4 f(x_0)$$

$$= 1 + (1\cdot 05)\cdot 4 + \frac{1\cdot 05(1\cdot 05-1)}{2}\cdot 8$$

$$+\frac{1\cdot 05(1\cdot 05-1)(1\cdot 05-2)}{6}(0)$$

$$+\frac{1\cdot 05(1\cdot 05-1)(1\cdot 05-2)(1\cdot 05-3)}{24}(-20)$$

$$+\frac{1\times 05(1\times 05\text{-}1)(1\times 05\text{-}2)(1\times 05\text{-}3)(1\times 05\text{-}4)}{120}(50)$$

$$= 5\cdot 21.$$

13. (a) Apply the method variation of parameters to solve the equation $\frac{d^2y}{dx^2} + y = \sec^3 x.\tan x$

Ans. We have $\frac{d^2y}{dx^2} + y = \sec^3 x.\tan x$

Auxiliary equation is $m^2 + 1 = 0 \therefore m = \pm i$

$\therefore$ P. F. $= A\ \cos x + B\ \sin x$

$$\therefore W = \begin{vmatrix} \cos x & \sin x \\ -\tan x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$\therefore \text{P. I.} = \sin x \int \cos x \sec^3 x \tan x\, dx - \cos x \int \sin x \sec^3 x \tan x\, dx$$

$$= \sin x \int \sec^3 x \tan x\, dx - \cos x \int \sec^2 x \tan^2 x \tan x\, dx$$

$$= \sin x \int \tan x \sec^2 x dx - \cos x \int \tan^2 x \sec^2 x\, dx$$

$$= \sin x \int \tan x \, d(\tan x) - \cos x \int \tan^2 x \, d(\tan x)$$

$$= \sin x \frac{(\tan x)^2}{2} - \cos x \frac{(\tan x)^3}{3} = \frac{1}{2} \sin x \tan^2 x - \frac{1}{3} \cos x \tan^3 x$$

$\therefore$ Required solution is

$$y = A\cos x + B\sin x + \frac{1}{2}\sin x \cdot \tan^2 x - \frac{1}{3}\cos x \tan^3 x$$

(b) Expand by Laplace's method

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (af - be + cd)^2$$

Ans. We have

$$\Delta = \begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$$

Expanding using Laplace method

$$\Delta = (-1)^{1+2+1+2} \begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} \begin{vmatrix} 0 & f \\ -f & 0 \end{vmatrix} + (-1)^{1+2+1+3} \begin{vmatrix} 0 & b \\ -a & d \end{vmatrix} \begin{vmatrix} -d & f \\ -c & 0 \end{vmatrix}$$

$$+ (-1)^{1+2+1+4} \begin{vmatrix} 0 & c \\ -a & 0 \end{vmatrix} \begin{vmatrix} -d & 0 \\ -e & -f \end{vmatrix} + (-1)^{1+2+2+3} \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} \begin{vmatrix} -b & f \\ -c & 0 \end{vmatrix}$$

$$+ (-1)^{1+2+2+4} \begin{vmatrix} a & c \\ 0 & e \end{vmatrix} \begin{vmatrix} -b & 0 \\ -c & -f \end{vmatrix} + (-1)^{1+2+3+4} \begin{vmatrix} b & c \\ d & e \end{vmatrix} \begin{vmatrix} -b & f \\ -c & -e \end{vmatrix}$$

$$= (0 + a^2)\ (0 + f^2) - (0 + ab)\ (0 + ef) + (0 + ca)\ (df - 0)$$

$$+ (ad - 0) + \ (0 + cf) + (be - cd)(be - cd)$$

$$= a^2 f^2 - 2af(be - cd) + (be - cd)^2 = (af - be + cd)^2$$

(c) Given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}$, find $\left\{\frac{\sin at}{t}\right\}$

Ans. According to change of scale property, we know if $L\{f(t)\} = F(s)$, then $L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

$$\therefore \quad L\left[\frac{\sin at}{t}\right] = a.L\left[\frac{\sin at}{at}\right] = a \cdot \frac{1}{a} F\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{1}{\frac{s}{a}}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

14. (a) Assuring the orthogonal properties of Legendre function prove that, $\int_{-1}^{1} [p_n(x)]^2 \, dx = \frac{2}{2n+1}$

Ans. The Legendre's equation can be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n(n+1)y=0$$

As, $p_n(x)$ is the solution,

$$\frac{d}{dx}\left\{(1-x^2)\frac{dp_n}{dx}\right\}+n(n+1)p_n=0$$

Again if $p_m(x)$ is a solution of

$$\frac{d}{dx}\left\{(1-x^2)\frac{dp_m}{dx}\right\}+n(n+1)p_m=0$$

Doing $(\text{i})\times p_m-(\text{ii})\times p_n$

$$p_m\frac{d}{dx}\left\{(1-x^2)\frac{dp_n}{dx}\right\}-p_n\frac{d}{dx}\left\{(1-x^2)\frac{dp_m}{dx}\right\}$$
$$+p_mp_n\{n(n+1)-m(m+1)\}=0$$

Integrating both side w.r.t x from –1 to 1, we get

$$\left[p_m(1-x^2)\frac{dp_n}{dx}\right]_{-1}^{1}-\int_{-1}^{1}\frac{dp_n}{dx}(1-x^2)\frac{dp_n}{dx}dx$$
$$\left[p_n(1-x^2)\frac{dp_m}{dx}\right]_{-1}^{1}-\int_{-1}^{1}\frac{dp_n}{dx}(1-x^2)\frac{dp_m}{dx}dx$$
$$+(n-m)(n+m+1)\int_{-1}^{1}p_mp_ndx=0$$

$$\therefore\int_{-1}^{1}p_m(x)p_ndx=0$$

But $(1-2xh+h^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}h^n p_n(x)$

or, $\quad (1-2xh+h^2)^{-1}=\sum_{n=0}^{\infty}h^{2n}\left[p_n(x)\right]^2$

$$+2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h^{m+n}p_m(x)p_n(x)\quad\text{[squaring both side]}$$

Integrating both side w.r.t. x and taking limit –1 to 1, we get

$$-\frac{1}{2h}\left[\log(1-2xh+h^2)\right]_{-1}^{1}=\sum_{n=0}^{\infty}h^{2n}\int_{-1}^{1}\left[p_n(x)\right]^2dx$$

or, $\quad 2\left[1+\frac{h^2}{3}+\frac{h^4}{5}+\ldots+\frac{h^{2n}}{2n+1}+\ldots\right]=\sum_{n=0}^{\infty}h^{2n}\int_{-1}^{1}\left[p_n(x)\right]^2dx$

equating the coefficient of h^{2n}

$$\int_{-1}^{1} [p_n(x)]^2\,dx = \frac{2}{2x+1} \text{ (Proved)}$$

14. (b) Show that $J_{\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}} \cdot \sin x$

Ans. As $J_n(x) = \left(\dfrac{x}{2}\right)^n \left\{ \dfrac{1}{\Gamma(x+1)} - \dfrac{1}{1!\Gamma(x+2)} \left(x/2\right)^2 \right.$

$$\left. + \frac{1}{2!\Gamma(x+3)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\Gamma(x+4)}\left(\frac{x}{2}\right)^6 + \ldots \right\}$$

Putting $n = \dfrac{1}{2}$

$$J_{\frac{1}{2}}(x) = \left(\frac{x}{2}\right)^{\frac{1}{2}} \left\{ \frac{1}{\Gamma\left(1+\frac{1}{2}\right)} - \frac{1}{1!\Gamma\left(2+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\Gamma\left(3+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!}\frac{1}{\Gamma\left(\frac{x}{2}\right)}\cdot\left(\frac{x}{2}\right)^6 + \ldots \right\}$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}} \left\{ \frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \frac{1}{1!\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\cdot\left(\frac{x}{2}\right)^6 + \ldots \right\}$$

$$= \sqrt{\frac{x}{2}} \cdot \frac{1}{\Gamma\left(\frac{1}{2}\right)} \left\{ \frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!} - \frac{2x^6}{7!} + \ldots \right\} = \sqrt{\frac{2}{x}} \cdot \frac{1}{\sqrt{\pi}} \left\{ \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \right\} = \sqrt{\frac{2}{\pi x}} \sin x.$$

14. (c) State Cayley–Hamilton theorem and so that matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} \text{ satisfies the above theorem :}$$

Ans. Cayley–Hamilton theorem states that every square matrix satisfies its own characteristic equation.

We have $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$

Characteristic equation of A

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 3 & 1-\lambda & 0 \\ -2 & 1 & 4-\lambda \end{vmatrix} = 0$$

or, $-\lambda\{(1-\lambda)(4-\lambda)-0\}0+1\{3+2(1-\lambda)\}=0$

or, $\lambda^3-5\lambda^2+6\lambda-5=0$

Now, $A=\begin{bmatrix}0&0&1\\3&1&0\\-2&1&4\end{bmatrix}$, $A^2=A.A=\begin{bmatrix}0&0&1\\3&1&0\\-2&1&4\end{bmatrix}\begin{bmatrix}0&0&1\\3&1&0\\-2&1&4\end{bmatrix}=\begin{bmatrix}-2&1&4\\3&1&3\\-5&5&14\end{bmatrix}$

$$A^3=A^2\cdot A=\begin{bmatrix}-2&1&4\\3&1&3\\-5&5&14\end{bmatrix}\begin{bmatrix}0&0&1\\3&1&0\\-2&1&4\end{bmatrix}=\begin{bmatrix}-5&-5&14\\-3&4&15\\-13&9&51\end{bmatrix}$$

$$A^3-5A^2+6A-5I$$

$$=\begin{pmatrix}-5&5&14\\-3&4&15\\-13&19&51\end{pmatrix}-5\begin{pmatrix}-2&1&4\\3&1&3\\-5&5&14\end{pmatrix}+6\begin{pmatrix}0&0&1\\3&1&0\\-2&1&4\end{pmatrix}-5\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}=0$$

$\therefore$ A satisfies its characteristic equation.

So, A satisfies the Cayley–Hamilton theorem.

ENGINEERING MATHEMATICS—YEAR 2008

GROUP–A

(Multiple Choice Type Questions)

1. Choose the correct alternatives for any *ten* of the following:

(i) $\frac{1}{D-1}x^2$ is equal to

(a) x^2+2x+2 (b) $-(x^2+2x+2)$

(c) $2x-x^2$ (d) $-(2x-x^2)$

Ans. $-(x^2+2x+2)$

Hint: $\frac{1}{D-1}(x^2)=-(1-D)^{-1}x^2=-(1+D+D^2+D^3+...)x^2=-(x^2+2x+2+0+...)$

(ii) An integrating factor for the differential equation $\frac{dy}{dt}+y=1$ is

(a) e^t (b) $\frac{e}{t}$

(c) et (d) $\frac{t}{e}$

Ans. (a) e^t

Hint: $\frac{dy}{dt}+y=1$

$I.F.=e^{\int dt}=e^t$

(iii) The general solution of the differential equation $D^2y+9y=0$ is

(a) $Ae^{3x}+Be^{-3x}$ (b) $(A+Bx)e^{3x}$

(c) $A\cos 3x+B\sin 3x$ (d) $(A+Bx)\sin 3x$

Ans. (c) $A\cos 3x+B\sin 3x$

Hint: $(D^2+9)y=0.$

$\therefore\ m^2+9=0$

$m=\pm 3i$

$\text{C.F.}=e^{0.x}\{A\cos 3x+B\sin 3x\}$

$\therefore\ y=A\cos 3x+B\sin 3x$

(iv) If $L\{f(t)\} = \tan^{-1}\left(\frac{1}{P}\right)$, then $L\{tf(t)\}$ is

(a) $\tan^{-1}\left(\frac{1}{P^2}\right)$ (b) $\frac{1}{1+P^2}$

(c) $\frac{1}{1+P}$ (d) $\tan^{-1}\left(\frac{2}{\pi P}\right)$

Ans. (b) $\frac{1}{1+P^2}$

$L\{f(t)\} = \tan^{-1}\frac{1}{p}\cot^{-1} p$

Now $L\{tf(t)\} = -\frac{d}{dp}(\cot^{-1} p) = \frac{1}{1+p^2}$

(v) In Simpson's 1/3 rule of finding $\int_a^b f(x)dx, f(x)$ is approximated by

(a) line segment (b) parabola
(c) circular sector (d) parts of ellipse

Ans. (b) parabola.

(vi) The system of equations $x+2y=5, 2x+4y=7$ has

(a) unique solution (b) no solution

(c) infinite number of solutions (d) none of these

Ans. (b) no solution.

$x+2y=5$

$2x+4y=7$

The equations can be written as $AX = B$

Augmented Matrix $A_b = \begin{pmatrix} 1 & 2 & : & 5 \\ 2 & 4 & : & 7 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & : & 5 \\ 0 & 0 & : & -3 \end{pmatrix} R'_2 \to R_2 \to 2R_1$$

Rank of augmented matrix is 2 Rank of coefficient matrix is 1 which are not equal.
Hence equations have no solution.

(vii) The four vectors (1, 1, 0, 0), (1, 0, 0, 1), (1, 0, a, 0), (0, 1, a, b) are linearly independent if

(a) $a \neq 0, b \neq 2$ (b) $a \neq 2, b \neq 0$

(c) $a \neq 0, b \neq -2$ (d) $a \neq -2, b \neq 0$

Ans. (c) $a \neq 0, b \neq -2$

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & a & 0 \\ 0 & 1 & a & b \end{vmatrix}$$

$$= -a-(ab+a)$$

$$= -2a-ab$$

$$= -a(2+b)$$

$\therefore$ $|A| \neq 0$ i.e. $a \neq 0$ and $b \neq -2$.

(viii) The rank of a matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ is

(a) 1 (b) 2

(c) 3 (d) none of these.

Ans. (b) 2.

det $A = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$

$\therefore$ Rank is 2.

(ix) The matrix $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

(a) an orthogonal (b) a symmetric matrix

(c) an idempotent matrix (d) a null matrix

Ans. (a) an orthogonal

$$A^T A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

(x) $T: R^2 \to R^2$ is defined by $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$. Then kernel of T is

(a) $\{(1,2)\}$ (b) $\{(1,-1)\}$
(c) $\{(0,0)\}$ (d) $\{(1,2)\},(1,-1)$

Ans. (c) $\{(0,0)\}$.

We know $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2), (x_1, x_2) \in R^2$

Let, $(x_1, x_2) \in \ker T$

Then, $T(x_1, x_2) = (0,0)$

$\Rightarrow 2x_1 - x_2 = 0$

and $x_1 + x_2 = 0$

$\Rightarrow x_1 = 0, x_2 = 0$

(xi) $\Delta^2 e^x$ is equal to (where $h = 1$)

(a) $(e-1)^2 e^x$ (b) $(e-1)e^x$
(c) $e^{2x}(e-1)$ (d) e^{2x+1}

Ans. (a)

$$\Delta^2 e^x = \Delta(\Delta^2 e^x)$$
$$= \Delta\left\{e^{x+1} - \Delta e^x\right\}\{\because \Delta f(x) = f(x+h) - f(x)\}$$
$$= \Delta e^{x+1} - \Delta e^x, \text{as } h = 1$$
$$= \left(e^{x+2} - e^{x+1}\right) - \left(e^{x+1} - e^x\right)$$
$$= e^x\left(e^2 - 2e + 1\right)$$
$$= e^x(e-1)^2$$

(xii) The eigen value of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ is

(a) 0, 0, 1 (b) 1, 2, 3
(c) 2, 3, 6 (d) none of these.

Hint: Let λ be the eigen value of the matrix A

∴ the characteristic equation is

or, $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$

or, $(1-\lambda)(2-\lambda)(3-\lambda) = 0$

or, $\lambda = 1, 2, 3$

(xiii) The mapping $f: R \to R$ where $f(x) = \sin s, x \in R$ is

(a) one-one (b) onto
(c) neither one-one nor onto (d) both one-one and onto

Ans. (c) neither one-one nor onto.

GROUP—B

2. Solve the differential equation by Laplace transformation. $\frac{d^2y}{dt^2}+9y=1\ y(0)=1, y\left(\frac{\pi}{2}\right)=-1.$

Ans. We have $\frac{d^2y}{dt^2}+9y=1$

Taking Laplace transform on both side

$$s^2Y(s)-sy(0)-y'(0)+9Y(s)=\frac{1}{s}$$

or, $\quad x^2y(s)-s.1-a+9Y(s)=\frac{1}{s}$ [Let $y'(0)=a$]

or, $\quad (s^2+9)\ Y(s)=a+s+\frac{1}{s}$

$$\therefore Y(s)=\frac{a}{s^2+9}+\frac{s}{s^2+9}+\frac{1}{s(s^2+9)}$$

$$=\frac{a}{s^2+9}+\frac{s}{s^2+9}+\frac{1}{9}\left(\frac{1}{s}-\frac{s}{s^2+9}\right)$$

$$\therefore y(t)=L^{-1}\{Y(s)\}$$

$$=\frac{a}{3}\sin 3t+\cos 3t+\frac{1}{9}(1-\cos 3t)$$

$$=\frac{1}{9}+\frac{a}{3}\sin 3t+\frac{8}{9}\cos 3t$$

$$\therefore y\left(\frac{\pi}{2}\right)=\frac{1}{9}+\frac{a}{3}(-1)+0=-1$$

or, $\quad a=\frac{10}{3}$

$\therefore$ required solution is

$$y(t)=\frac{1}{9}+\frac{10}{9}\sin 3t+\frac{8}{9}\cos 3t$$

3. Solve by the method of variation of parameters $\frac{d^2y}{dx^2}+9y=\sec 3x.$

Ans. Refer to 3. (a) of 2005.

4. Examine the consistancy of the following system of equations and solve if possible.

$$x+y+z=1$$
$$2x+y+2z=2$$
$$3x+2y+3z=5$$

Ans. Refer to Q. 3. (ii) a of 2006.

5. If $W=\{(x,y,z)\in R_3; x+y+z=0\}$, show that W is a subspace of R^3, and find a basis of W.

Ans. Same as Q. 3. (b) of 2005.

6. Examine whether the mappint $T : R^2 \to R$ defined by $T(x, y) = (2x - y, x)$ is linear.

Ans. Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be two (???) R^2

Now, $T(\alpha) = T(\alpha_1, \alpha_2) = (2\alpha_1 - \alpha_2, \alpha_1)$

$T(\beta) = T(\beta_1, \beta_2) = (2\beta_1 - \beta_2, \beta_1)$

Now,
$$\begin{aligned} T(a\alpha + b\beta) &= T\{a(\alpha_1, \alpha_2) + b(\beta_1, \beta_2)\} \\ &= T\{a\alpha_1 + b\beta_1,\ a\alpha_2 + b\beta_2\} \\ &= \{2(a\alpha_1 + b\beta_1) - a\alpha_2 - b\beta_2, a\alpha_1 + b\beta_1\} \\ &= a(2\alpha_1 - \alpha_2, \alpha_1) + b(2\beta_1 - \beta_2, \beta_1) \\ &= aT(\alpha) + bT(\beta) \end{aligned}$$

$\therefore$ Given T is a linear transformation.

7. Evaluate $\int_0^1 \frac{dx}{1+x}$ using Trapezoidal rule, taking four equial sub-intervals.

Ans. We have $f(x) = \frac{1}{1+x}; a = 0, b = 1, x = 4$

So, $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25$

x	0	0.25	.5	.75	1
$f(x)$	1	0.8	0.6666	0.5714	0.5

from, Trapezoidal rule.

$$\begin{aligned} I &= \int_0^1 \frac{dx}{1+x} = \frac{h}{2}[y_0 + 2(y_1 + y_3 + y_3) + y_4] \\ &= \frac{0.25}{2}[1 + 2(0.8 + 0.6666 + 0.5714) + 0.5] \\ &= 0.695 \text{ Ans.} \end{aligned}$$

GROUP—C

8. (a) Find Laplace transform of $f(t) = \sin t, 0 < t < \pi = 0,\ t > \pi$

Ans. Same as Q. 8. (b) of 2004.

(b) It $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then verify that A satisfies its own characteristic equation. Hence find A^{-1}.

Ans. Same as Q. 4. (a) of 2002.

(c) Show that (3, 1, –2), (2, 1, 4), (1, –1, 2) form a basis of R^3.

Ans. As $\begin{vmatrix} 3 & 1 & -2 \\ 2 & 1 & 4 \\ 1 & -1 & 2 \end{vmatrix}$

$$= 3\{2+4\} - 1\{4-4\} - 2\{-2-1\}$$
$$= 18 - 0 + 6$$
$$= 24 \neq 0$$

Given three vectors are linearly independent so the vectors belong to R^3.

$\therefore$ The vectors form a basis.

9. (a) Solve: $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2 \log x$

Ans. Let, $\log x = z$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{1}{x} \cdot \frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}$$

$$= -\frac{1}{x}\frac{dy}{dz} + \frac{1}{x^2} \cdot \frac{d^2y}{dz^2}$$

Naturally $x\dfrac{dy}{dx} = \dfrac{dy}{dz}$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

Putting these values in given equation.

$$\frac{d^2y}{dz^2} - 2 \cdot \frac{dy}{dz} - 3y = ze^{2z}$$

The auxileary equation is

$$m^2 - 2m - 3 = 0$$

or, $\quad (m-3)(m+1) = 0$

$\therefore$ C.F. $= A_{11}e^{3z} + A_{22}e^{-z}$, A_{11} and A_{22} constants

$$\text{P.I.} = \frac{1}{D^2 - 2D - 3}(ze^{2z})$$

$$= e^{2z} \cdot \frac{1}{(D+2)^2 - 2(D+2) - 3}(z)$$

$$= e^{2z} \cdot \frac{1}{D^2 + 4D + 4 - 2D - 4 - 3}(z)$$

$$= e^{2z} \cdot \frac{1}{D^2 + 2D - 3}(z)$$

$$= -\frac{1}{3} \cdot e^{2z}\left(1 - \frac{D^2 + 2D}{3}\right)^{-1} z$$

$$= -\frac{1}{3}\cdot e^{2z}\left\{1+\frac{D^2+2D}{3}+\left(\frac{D^2+2D}{3}\right)^2+\ldots\right\}z$$

$$= -\frac{1}{3}\cdot 3e^{2t}(3z+2) = -\frac{1}{9}e^{2t}(3z+2)$$

$\therefore$ Total solution is

$$= A_{11}e^{3z} + A_{22}e^{-z} - \frac{1}{9}e^{2z}(3z+2)$$

$$= A_{11}x^3 + A_{22}.x^{-1} - \frac{1}{9}x^2(3\log x+2) \quad [\text{ as } z = \log x]$$

9. (b) Solve: $(D^2+4)y = x\sin^2 x$, which $D = \dfrac{d}{dx}$

Ans. Auxiliary equation 0 is

$m^2+4=0$

$\therefore \quad m^2 = -4 = (2i)^2$

$\therefore \quad m = \pm 2i$

$\therefore \quad$ C.F. $= A_1\cos 2x + A_2\sin 2x$

$$\text{P.I.} = \frac{1}{D^2+4}x\sin^2 x$$

$$= \frac{1}{2}\frac{1}{D^2+4}x(1-\cos 2x)$$

$$= \frac{1}{2}\frac{1}{D^2+4}(x) - \frac{1}{2}\frac{1}{D^2+4}\cdot x\cos x$$

$$= \frac{1}{2}(D^2+4)^{-1}(x) - \frac{1}{2}\cdot\left(x - \frac{1}{D^2+4}\cdot 2D\right)\frac{1}{D^2+4}\cos 2x$$

$$= \frac{1}{2.4}\left(1+\frac{D^2}{4}\right)^1(x) - \frac{1}{2}\left(x - \frac{2D}{D^2+4}\right)x\frac{1}{2D}\cos 2x$$

$$= \frac{x}{8} - \frac{1}{4}\left(x - \frac{2D}{D^2+4}\right)x\frac{1}{2}\sin 2x$$

$$= \frac{x}{8} - \frac{1}{8}\left(x - \frac{2D}{D^2+4}\right)x\sin 2x$$

$$= \frac{x}{8} - \frac{x^2\sin 2x}{8} - \frac{1}{4}\frac{1}{D^2+4}\sin 2x - \frac{1}{2} - \frac{1}{2}\frac{1}{D^2+4}(x\cos 2x)$$

If $\quad I = \dfrac{1}{D^2+4}(x\cos 2x)$

$$= \frac{x^2\sin 2x}{4} + \frac{1}{2}\cdot x\cdot\frac{1}{2}\sin 2x - 1$$

$$\therefore 2I = \frac{x^2\sin 2x}{4} + \frac{x\cos 2x}{8}$$

$$\therefore I = \frac{1}{8}x^2\sin 2x + \frac{x\cos 2x}{16}$$

$\therefore$ P.I. $= \frac{1}{16}x^2 \sin 2x + \frac{1}{32} x \cos 2x$

$\therefore$ general solution : C.F. + P.I.

$$= A_1 \cos 2x + A_2 \sin 2x - \frac{1}{16}x^2 \sin 2x - \frac{1}{32} x \cos 2x.$$

9. (c) Solve, $x\frac{dy}{dx} + y = y^2 \log x$

Ans. $x\frac{dy}{dx} + y = y^2 \log x$

or, $\frac{x}{y^2}\frac{dy}{dx} + \frac{1}{y} = \log x$

Let, $\frac{1}{y} = z$

$\therefore -\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore -x\frac{dz}{dx} + z = \log x$

or, $\frac{dz}{dx} - \frac{z}{x} = -\frac{\log x}{x}$ ....(i)

As it is linear equation of 2

$$\therefore \text{I.F.} = e^{-\int \frac{1}{x}dx} = e^{-\log|x|} = \frac{1}{x}$$

Multiplying both sides of (i) by I.F. $= \frac{1}{x}$

$$\frac{1}{x}.\frac{dz}{dx} - \frac{1}{x^2}.z = \frac{\log x}{x^2}$$

or, $\frac{d}{dx}\left(\frac{z}{x}\right) = -\frac{\log x}{x^2}$

$\therefore \frac{z}{x} = -\int \frac{\log x}{x^2} dx$

$= -\int pe^{-p} dp$ Let, $\log x = P$

$= (P+1)e^{-P} + c.\ c =$ constant $\therefore \frac{1}{x}dx = dp$

$= \frac{1}{x}(\log x + 1) + c$ $\therefore x = e^P$

[as $P = \log x$]

10. (a) Compute $f(0.5)$ and $f(0.9)$ from the following table

x	0	1	2	3
$f(x)$	1	2	11	34

Ans. Required difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		1		
1	2			
			8	
		9		6
2	11		14	
		23		
3	34			

to evaluate $f(0\cdot 5)$ using Newton's Farward interpolation formula where

$x = 0\cdot 5, x_0 = 0, h = 1, d = \dfrac{x - x_0}{h} = \dfrac{0\cdot 5 - 0}{1} = 0\cdot 5$

$$\therefore f(x) = f(x_0) + d\Delta f(x_0) + \frac{d(d-1)}{2!}\Delta^2 f(x_0) + \frac{d(d-1)(d-2)}{3!}\Delta^2 f(x_0)$$

$$= 1 + 0\cdot 5.1 + \frac{0\cdot 5(0\cdot 5 - 1)}{2}.8 + \frac{0\cdot 5(0\cdot 5-1)(0\cdot 5-2)}{6}.6$$

$$= 0\cdot 875$$

Again for $f(0\cdot 9), x_0 = 0\cdot 9, x_0 = 0, h = 1$

$$d = \frac{x - x_0}{h} = \frac{0\cdot 9 - 0}{1} = 0\cdot 9$$

$$\therefore f(0\cdot 9) = 1 + 0\cdot 9.1 + \frac{0\cdot 9(0\cdot 9 - 1)}{2!}.8 + \frac{0\cdot 9(0\cdot 9-1)(0\cdot 9-2)}{6}.6$$

(b) Apply convolution theorem to evaluate $L^{-1}\left\{\dfrac{1}{\left(s^2+2s+5\right)^2}\right\}$

Ans. We know $L^{-1}\left\{\dfrac{1}{s^2+2^2}\right\} = \dfrac{1}{2}\sin 2t$

$$\therefore L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\} = \frac{1}{2}e^{-1}\sin 2t$$

$$\therefore L^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = \frac{1}{2}e^{-1}\sin 2t$$

$\therefore$ Using convoluation theorem

$$L^{-1}\left\{\frac{1}{\left(s^2+2s+5\right)^2}\right\} = \int_0^1 e^{-u}\,\frac{\sin 2u}{2}\,e^{-(t-u)}\,\frac{\sin^{2(t-u)}}{2}\,du$$

$$= \frac{e^{-t}}{8} \int_0^1 \left[\cos(4u - 2t) - \cos 2t\right] du$$

$$= \frac{e^{-t}}{8} \int_0^1 \left[\frac{\sin(4u - 2t)}{4} - u \cos 2t\right]_0^t$$

$$= \frac{e^{-t}}{8} \left(\frac{\sin 2t}{2} - t \cos 2t\right)$$

10. (c) Prove that $\begin{vmatrix} (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (y-a)^2 & (y-b)^2 & (y-c)^2 \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix}$

$= 2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$

Ans. Let, $\Delta = \begin{vmatrix} (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (y-a)^2 & (y-b)^2 & (y-c)^2 \\ (z-a)^2 & (z-b)^2 & (z-c)^2 \end{vmatrix}$

$$= \begin{vmatrix} x^2 & -2x & 1 \\ y^2 & -2y & 1 \\ z^2 & -2z & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = -2 \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= -2\, \Delta_1\, \Delta_2$$

Now, $\Delta_1 = \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z^2 & 1 \end{vmatrix} = \begin{vmatrix} x^2 & x & 1 \\ y^2 - x^2 & y - x & 0 \\ z^2 - x^2 & z - x & 0 \end{vmatrix} \begin{bmatrix} R_2^1 = R_2 - R_1 \\ R_3 = R_2 - R_1 \end{bmatrix}$

$$= (y-x)(z-x) \begin{vmatrix} x^2 & x & 1 \\ y^2 - x^2 & y - x & 0 \\ z^2 - x^2 & z - x & 0 \end{vmatrix}$$

$$= (y-x)(z-x)\{y - x - z + x\} = -(x-y)(y-z)(z-x)$$

Similarly $\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

$\therefore \Delta = \Delta_1\, \Delta_2 = 2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$ (Proved)

11. (a) Prove that $\Delta.\nabla = \Delta - \nabla$, where symbols have their usual meaning.

$$\nabla\left[f(x)\right] = f(x) - f(x-h)$$

$$\therefore \Delta.\nabla\left[f(x)\right] = \Delta f(x) - \Delta f(x-h) = \Delta f(x) - \left[f(x) - f(x-h)\right\}$$

$$= \Delta f(x) - \nabla f(x) = (\Delta - \nabla) f(x)$$

$$\therefore \Delta.\nabla = \Delta - \nabla$$

(b) Estimate missing term in the following table:

x	0	1	2	3	4
$f(x)$	1	3	9	–	31

Ans. Here y has four given values. Naturally $\Delta^4 y_0 = 0$

or, $(E-1)^4\ y_0 = 0$

or, $\left(E^4 - 4E^3 + 6E^2 - 4E + 1\right) y_0 = 0$

or, $y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

$$\therefore y_3 = \frac{y_4 + 6y_2 - 4y_1 + y_0}{4} = \frac{81 + 6.9 - 4.3 + 1}{4}$$

$$= 31$$

11. (c) Solve: $\left(x^2 y - 2xy^2\right) dx + \left(3x^2 y - x^3\right) dy = 0$

Ans. We have $\left(x^2 y - 2ky^2\right) dx + \left(3x^2 y - x^3\right) dy = 0$........(i)

$\therefore M = x^2 y - 2xy^2 \qquad N = 3x^2 y - x^3$

$$\therefore \frac{\partial M}{\partial y} = x^2 - 4xy \quad \frac{\partial N}{\partial x} = 6xy - 3x^2$$

$$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\left(10xy - 4x^2\right) \neq 0$$

$\therefore$ It is not a exact equation.

Now $Mx + Ny = \left(x^2 y - 2xy^2\right) x + \left(3x^2 y - x^3\right) y = x^2 y^2$

So the integrating factor $\dfrac{1}{x^2 y^2}$

Multiplying both sides of (i) by $\dfrac{1}{x^2 y^2}$

$$\frac{x^2 y - 2xy}{x^2 y^2} dx + \frac{3x^2 y - yx^3}{x^2 y^2} dy = 0$$

or, $$\frac{ydx - xdy}{y^2} - \frac{2}{x} dx + \frac{3}{y} dy = 0$$

Inlogration both sides we get $\frac{x}{y}-2\log x+3\log y=c, c\rightarrow$ constant

or, $\quad \frac{x}{y}+\log\frac{y^3}{x^3}=c$ **Ans.**

12. (a) Show that $\frac{d}{dx}\left\{x^n J_n(x)\right\}=x^n J_{n-1}(x)$

Ans. We know,

$$J_n(x)=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!\,\Gamma(n+r+1)}$$

$$\therefore\ x^n J_n(x)=\sum_{r=0}^{\infty}(-1)^r\frac{x^{2x+2r}}{2^{x+2r}\,r!\,\Gamma(x+r+1)}$$

Differentiating both side w.r.t. x

$$\frac{d}{dx}\left[x^n J_n(x)\right]=\sum_{r=0}^{\alpha}(-1)^r\cdot\frac{2x+2r.x^{2x+2r-1}}{2^{x+2r}\,r!\,\Gamma(n+r+1)}$$

$$=\sum_{r=0}^{\alpha}(-1)^r\cdot\left(\frac{x}{2}\right)_{(n+r)}^{n+2r-1}\frac{1.(x+r)}{r!\,\Gamma(x+r)!}.x^n$$

$$=x^n\sum_{r=0}^{\alpha}(-1)^r\cdot\left(\frac{x}{2}\right)^{n+2r-1}\frac{1}{\Gamma(x+r)}$$

$$=x_n J_{n-1}(x)\ \text{(Proved)}$$

12. (b) Obtain a function whose 1st difference is $9x^2+11x+5$, where $h=1$

Ans. Let the required function is $f(x)$

$\therefore\quad \Delta f(x)=9x^2+11x+5$

$$f(x)=\frac{1}{\Delta}\left[9x^2+11x+5\right]$$

$$=\frac{1}{\Delta}\left[9(x^{(2)}+x^{(1)}+11x^{(1)}+5\right]$$

$$\therefore f(x)=\frac{1}{\Delta}\left[9x^{(2)}+20x^{(1)}+5\right]$$

$$=9.\frac{x^{(3)}}{3}+20.\frac{x^{(2)}}{2}+5x^{(1)}+k\qquad[k=\text{constant}]$$

$$=\left(\frac{9}{3}\right)x(x-1)(x-2)+\left(\frac{20}{2}\right)x(x-1)+5x+k$$

$$=3x(x-1)(x-2)+10x(x-1)+5x+k$$

$$=3x(x-1)(x-2)+5x\left\{2x-2+1\right\}+k$$

$$=3x(x-1)(x-2)+5x(2x-1)+k$$

$$=3x^3+x^2+x+k$$

12. (e) Solve the differential equation by Laplace transform—

$$\frac{d^2y}{dx^2}+\frac{2dy}{dx}+5y=e^t \sin t;\ y(0)=0\quad y'(0)=1$$

Ans. We have

$$\frac{d^2y}{dx^2}+2.\frac{dy}{dx}+5y=e^t \sin t$$

Taking Laplace transform on both side

$$s^2Y(s)-sy(0)-y'(0)+2\left[sY(s)-y(0)\right]+5Y(s)=\frac{1}{(s-1)^2+1}$$

Putting $y(0)=0$ and $y'(0)=1$

$$s^2Y(s)-0-1+2sY(s)-0+5Y(s)=1$$

$$s^2Y(s)-0-1+2sY(s)-0+5Y(s)=\frac{1}{(s-1)^2+1}$$

or, $\left(s^2+2s+5\right)Y(s)=1+\dfrac{1}{(s-1)^2+1}$

$$Y(s)=\frac{1}{s^2+2s+5}+\frac{1}{[(s-1)^2+1][s^2+2x+5]}$$

$$=\frac{1}{(s+1)^2+4}+\frac{1}{[(s-1)^2+1][(s+1)^2+4]}$$

$$=\frac{1}{65}\left[\frac{4(s+1)}{(s+1)^2+2^2}+\frac{66}{(s+1)^2+2^2}-\frac{4(s-1)}{(s-1)^2+1}+\frac{7}{(s-1)^2+1}\right]$$

$$\therefore\ y(f)=L^{-1}\{Y(s)\}$$

$$=\frac{1}{65}\left\{4e^{-t}\cos 2t+33e^{-t}\sin 2t-4e^t\cos t+7e^t\sin t\right\}.$$

13. (a) What is meant by linear independence of a set of n-vectors?

Show that the vectors $(1,2,1),(-1,1,0)$ and $(5,-1,2)$ linearly independent.

Ans. Assume V be a vector over a field F.

A finite subset $\{x_1,x_2,...,x_n\}$ of vectors of v is said to be linearly independent if every relation satisfies

$$a_1x_1+a_2x_2+...+a_nx_n=0, ai\in F$$

$1\le i\le x\Rightarrow ai-0$ for each $1\le i\le n.$

$$\begin{vmatrix}1 & 2 & 1\\ -1 & 1 & 10\\ 5 & -1 & 2\end{vmatrix}=1\{2-0\}-2\{-2-0\}+1\{1-5\}=2\neq 0$$

$\therefore$ Vectors are linearly independent.

(b) Show that $W = \{(x, y, z) \in R^3 : 2x - y + 3z = 0\}$ is a subspace of $\boldsymbol{R}^3$. Find a basis W. What is its dimension?

Ans. Same as Q. 3, (a) of 2002.

(c) Prove that
$$\Delta^2 = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$
$$= 2abc(a+b+c)^3$$

Ans. Same as Q.2(a) of 2004.

14. (a) Find the rank of the matrix.
$$\begin{vmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{vmatrix}$$

Ans. Same as Q.2. (b) of 2002.

14. (b) Solve by Cramer's Rule
$$2x - y = 3$$
$$3y - 2x = 5$$
$$-2z + x = 4$$

Ans. Here, $\Delta = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & 0 & -2 \end{bmatrix} = -10$

$$\Delta_1 = \begin{bmatrix} 3 & -1 & 0 \\ 5 & 3 & -2 \\ 4 & 0 & -2 \end{bmatrix} = -20$$

$$\Delta_2 = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 5 & -2 \\ 1 & 4 & -2 \end{bmatrix} = -10$$

$$\Delta_3 = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & 5 \\ 1 & 0 & 4 \end{bmatrix} = 10$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{-20}{-10} = 2$$
$$y = \frac{\Delta_2}{\Delta} = \frac{-10}{-10} = 1$$
$$z = \frac{\Delta_3}{\Delta} = \frac{10}{-10} = -1$$

$$\therefore \text{ Required solution is } \left.\begin{matrix} x=2 \\ y=1 \\ z=1 \end{matrix}\right\} \textbf{ Ans.}$$

14. (c) Find the eigen values and corresponding eigen vectors of the matrix $A=\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}$

Ans. Characteristic equation of given matrix is

$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ 2 & -2-\lambda & 4 \\ 3 & -3 & 6-\lambda \end{vmatrix} = 0$$

or, $(1-\lambda)\{(6-\lambda)-(-2-\lambda)\}+1.\{2(6-\lambda)-4.3\}+2\{-6+(2+\lambda).3\}=0$

or, $\lambda^2(5-\lambda)=0$

$\therefore \lambda = 0, 0, 5$

for $\lambda = 0$ let there be a non-zero eigen vector X.

$$\therefore \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow[R_3^1 \to R_3 - 3R_1]{R_2^1 \to R_2 - 2R_1} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore\ x_1 - x_2 + 2x_3 = 0$

$\therefore$ Required region vectors for $(\lambda = 0)$ are $A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $B\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

A and B non-zero real number.

$$\text{Again for } \lambda = 5 \begin{pmatrix} -4 & -1 & 2 \\ 2 & -7 & 4 \\ 3 & -3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow{R_1^1 = R_1 + R_3} \begin{pmatrix} -1 & -4 & 3 \\ 2 & -7 & 4 \\ 3 & -3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow[R_3^1 \to R_3 - 3R_1]{R_2^1 \to R_2 - 2R_1} \begin{pmatrix} -1 & -4 & 3 \\ 0 & -15 & 10 \\ 0 & -15 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow{R_3^1 = R_3 + R_2} \begin{pmatrix} -1 & -4 & 3 \\ 0 & -15 & 10 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \; -x_1 - 4x_2 + 3x_3 = 0$$
$$-15x_2 + 10x_3 = 0$$

From these two equaliers $\dfrac{x_2}{2} = \dfrac{x_3}{3} = C,\ C \to$ constant non-zero real no.

Hence required legen veecler is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = C \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

ENGINEERING & MANAGEMENT EXAM—YEAR JUNE, 2009
SEMESTER—2

GROUP—A

Multiple Choice Type Questions

1. Choose the correct alternatives for any *ten* of the following: **$10 \times 1 = 10$**

(i) If $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, then A^{100} is

(a) $\begin{bmatrix} 1 & 0 \\ -150 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ -50 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ -100 & 1 \end{bmatrix}$ (d) None of these

Ans. $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ -1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow A^2 - 2A + I_2 = 0$$

$\therefore\ A^2 = 2A - I_2$

$\therefore\ A^4 = A^2.A^2 = (2A - I_2)(2A - I_2) = 4A^2 - 4A + I_2$

$= 8A - 4I_2 - 4A + I_2 = 4A - 3I_2$

$A^3 = A^2(A) = (2A - I_2)A \qquad = 2A^2 - A$

$= 4A - 2I_2 - A = 3A - 2I_2$

$\therefore\ A^{100} = 100A - 99I_2 = \begin{pmatrix} 100 & 0 \\ -100 & 100 \end{pmatrix} - \begin{pmatrix} 99 & 0 \\ 0 & 99 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -100 & 1 \end{pmatrix}$

$\therefore$ **Ans.** (c)

(ii) The set of vectors {(2, 1, 1), (1, 2, 2), (1, 1, 1)} in R^3 is—

(a) linearly dependent (b) linearly independent

(c) basis of R^3 (d) none of these

Ans. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \underset{R_1 \longleftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix} \underset{R_3 - 2R_1}{\overset{R_2 - R_1}{\sim}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \underset{R_3 + R_1}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

— Raw echelon form

rank = no of non zero rows

= 2 < no of unknowns = 3

$\therefore$ the set of vectors are linearly dependent.

Ans. (a)

(iii) The matrix $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is

(a) an orthogonal matrix (b) a symmetric matrix
(c) an idempotent matrix (d) a null matrix

Ans. $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}; \; A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$\therefore A.T^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\therefore A$ is orthogonal matrix

Ans. (a)

(iv) The value of the determinant $\begin{vmatrix} 1 & 4 & 16 \\ 1^2 & 2^2 & 4^2 \\ 0 & 1 & 6 \end{vmatrix}$ is

(a) 0 (b) 1
(c) 4 (d) 22

Ans. $\begin{vmatrix} 1 & 4 & 16 \\ 1^2 & 2^2 & 4^2 \\ 0 & 1 & 6 \end{vmatrix} = 0$ as R_1 and R_2 are identical

Ans. (a)

(v) The solution of a system of n linear equations with n unknowns is unique if and only if

(a) $\det A = 0$ (b) $\det A > 0$
(c) $\det A < 0$ (d) $\det A \neq 0$.

where A is the matrix of the coefficients of the unknowns in the linear equations.

Ans. unique solution iff $\det A \neq 0$

$\therefore$ **Ans.** (d).

(vi) The eigen values of the matrix $\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$ are

(a) –5, –3 (b) –5, 3
(c) 3, –5 (d) 5, 3

Ans.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 16 = 0 \Rightarrow \lambda^2 - 2\lambda - 15 = 0 \Rightarrow \boxed{\lambda = 5, -3}$$

(vii) The general solution of $p = \log(px - y)$ where $p = \dfrac{dy}{dx}$ is

(a) $y = cx - c$ (b) $y = cx - e^c$

(c) $y = c^2x - e^{-c}$ (d) none of these

Ans. (b)

$p = \log(px - y) \Rightarrow px - y = e^p \Rightarrow y = px - e^p$

$$\Rightarrow p = p.1 + x\frac{dp}{dx} - e^p\frac{dp}{dx} \Rightarrow (x - e^p)\frac{dp}{dx} = 0 \Rightarrow p = c \ or\ x = e^p$$

$\therefore$ general solution $y = cx - e^c$

(viii) Which of the following is not true (the notations have their usual meanings)?

(a) $\Delta = E - 1$ (b) $\Delta.\nabla = \Delta - \nabla$

(c) $\dfrac{\Delta}{\nabla} = \Delta + \nabla$ (d) $\nabla = 1 - E^{-1}$

Ans. (c)

$$\Delta + \nabla = (E-1) + (1 - E^{-1}) = (E-1) + \left(\frac{E-1}{E}\right)$$

$$= (E-1)\left(1 + \frac{1}{E}\right) = (E-1)\left(\frac{E+1}{E}\right) = \left(\frac{E^2-1}{E}\right)$$

(ix) $\Delta^2 e^x$ is equal to $(h = 1)$

(a) $(e-1)^2 e^x$ (b) $(e-1)e^x$

(c) $e^{2x}(e-1)$ (d) e^{2x}

Ans. (a)

$$\Delta^2 e^x = \Delta(\Delta(e^x)) = \Delta(e^{x+1} - e^x) = \Delta(e^{x+1}) - \Delta e^x$$

$$= (e^{x+2}) - e^{x+1}) - (e^{x+1} - e^x) = e^{x+2} - 2e^{x+1} + e^x$$

$$= e^x(e^2 - 2^e + 1) = (e-1)^2 e^x$$

(x) The value of $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt$ is equal to

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{6}$

(c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{2}$

Ans. (d)

Since $L\left(\dfrac{\sin t}{t}\right) = \tan^{-1}\left(\dfrac{1}{s}\right) \Rightarrow \displaystyle\int_0^\infty e^{-st}\frac{\sin t}{t}\,dt = \tan^{-1}\left(\frac{1}{s}\right);$

Putting $s = 0$ we have $\int_0^\infty \frac{\sin t}{t} dt = \tan^{-1}(\infty) = \frac{\pi}{2}$

(xi) If S and T are two subspaces of a vector space V, then which one of the following is a subspace of V also?

(a) $S \cup T$ (b) $S \cap T$

(c) $S - T$ (d) $T - S$

Ans. Since $S \cap T = \{v : v \in s \; and \; v \in T\}$ and S, T are subspace of V

$v_1, v_2 \in \; S \cap T \Rightarrow v_1, v_2 \in S$ and $v_1, v_2 \in T$

$\Rightarrow C_1V_1 + C_2V_2 \in S$ and $C_1V_1 + C_2V_2 \in T$

$\Rightarrow C_1V_1 + C_2V_2 \in S \cap T$

$\Rightarrow S \cap T$ is a subspace of V

Ans. (b).

(xii) If $\lambda^3 - 6\lambda^2 + 9\lambda - 4$ is the characteristic equation of a square matrix A, then A^{-1} is equal to

(a) $A^2 - 6A + 9I$ (b) $\frac{1}{4}A^2 - \frac{3}{2}A + \frac{9}{4}I$

(c) $\frac{1}{4}A^2 - \frac{3}{2}A + \frac{9}{4}$ (d) $A^2 - 6A + 9$

Ans. Every square matrix satisfies its own characteristic equation

$\Rightarrow A^3 - 6A^2 + 9A - 4I = 0$

$\Rightarrow A(A^2 - 6A + 9I) = 4I$

$\Rightarrow A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$

$= \frac{1}{4}A^2 - \frac{3}{2}A + \frac{9}{4}I$

Ans. (b)

(xiii) Co-factor of –3 in the determinant $\begin{vmatrix} -2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{vmatrix}$ is

(a) 4 (b) –4

(c) 0 (d) none of these

Ans. Co-factor of $(-3) = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} = -4$

Ans. (b)

GROUP—B

(Multiple Choice Type Questions)

Answer any *three* of the following **3 × 5 = 15**

2. If A be a skew symmetric and $(I+A)$ be a non-singular matrix, then show that $B=(I-A)(I+A)^{-1}$ is orthogonal.

Ans. Given $B=(I-A)(I+A)^{-1}$

$$\therefore B^T B=\left\{(I+A)^{-1}\right\}^T (I-A)^T (I-A)(I+A)^{-1}$$
$$=\{(I+A)^T\}^{-1} (I-A)^T (I-A)(I+A)^{-1}$$
$$=\left((I+A)^T\right)^{-1} (I-A)^T (I-A)(I+A)^{-1}$$
$$=\left(I+A^T\right)^{-1} (I-A)(I+A)(I+A)^{-1} [\because A^T=-A]$$
$$=(I-A)^{-1} (I-A)(I+A)(I+A)^{-1}$$
$$=I.I$$
$$=I\ (\because A^{-1}A=1)$$

$\therefore B^T B=I \Rightarrow B$ is orthogonal matrix.

3. Evaluate: $L^{-1}\left\{\dfrac{1}{(s-1)^2 (s-2)^3}\right\}$

Ans. Let $\dfrac{1}{(S-2)^3 (S-1)^2}=\dfrac{A}{S-1}+\dfrac{B}{(S-1)^2}+\dfrac{C}{(S-2)}+\dfrac{D}{(S-2)^2}+\dfrac{E}{(S-2)^3}$

Where A, B, C, D, E are constants.

$\therefore 1=A(S-1)(S-1)^3+B(S-2)^3+C(S-1)^2(S-2)^2+D(S-2)(S-1)^2+E(S-1)^2$

Let, $S=1 \Rightarrow B(-1)^3 \Rightarrow B=-1$

Let, $S=2 \Rightarrow 1=E \Rightarrow E=1$

equating coeff. of S^3 from both sides.

$$0=A+B+(-6C)+D$$
$$\Rightarrow A-6C+D=1 \text{.....(i)}$$

equation coeff. of S^4 and constant term from both sides

$$0=A+C \Rightarrow A=-C$$
$$\text{from}\quad (1)-7C+D=1 \quad \text{...(ii)}$$
$$1=8A-8B+4C-2D+E$$

or, $-8=-4C-2D \Rightarrow 2C+D=4$ ....(iii)

$$\text{(ii)} - \text{(iv)} - 9C = -3 \Rightarrow \boxed{C = \frac{1}{3}}$$

$$\therefore \boxed{A = -\frac{1}{3}} \therefore D = 4 - \frac{2}{3} = \boxed{\frac{10}{3}}$$

$$\therefore \quad \frac{1}{(S-1)^2 (S-2)^3} = (-)\frac{1}{3(S-1)} + (-1)\frac{1}{(S-1)^2} + \frac{1}{3}\cdot\frac{1}{(S-2)} + \frac{10}{3}\frac{1}{(S-2)^2} + \frac{1}{(S-2)^3}$$

$$\therefore \quad L^{-1}\left[\frac{1}{(S-1)^2 (S-2)^3}\right] = -\frac{1}{3}L^{-1}\left(\frac{1}{S-1}\right) - L^{1}\left(\frac{1}{(S-1)^2}\right) + \frac{1}{3}L^{-1}\left(\frac{1}{S-2}\right)$$

$$+\frac{10}{3}L^{-1}\left[\frac{1}{(S-2)^2}\right] + L^{-1}\left[\frac{1}{(S-2)^3}\right]$$

$$= -\frac{1}{3}e^{t} - e^{t}t + \frac{1}{3}e^{2t} + \frac{10}{3}e^{2t}.t + e^{2t}.\frac{t^2}{2}$$

4. Solve the differential equation—

$$\frac{dy}{dx} + y = y^3 (\cos x - \sin x)$$

Ans. $\dfrac{dy}{dx} + y = y^3 (\cos x - \sin x)1$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = (\cos x - \sin x)$$

Let, $\dfrac{1}{y^2} = z \Rightarrow -\dfrac{2}{y^3}\dfrac{dy}{dx} = \dfrac{dz}{dx} \Rightarrow \dfrac{1}{y^3}\dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{dz}{dx}$

$$\therefore -\frac{1}{2}\frac{dz}{dx} + z = (\cos x - \sin x)$$

$$\Rightarrow \frac{dz}{dx} - 2z = -2(\cos x - \sin x) \ \ldots \ (I)$$

$\therefore$ I.F. $= e^{-\int 2dx} = e^{-2x}$

$\therefore (I)$ becomes on multiplication by I.F.

$$\frac{d}{dx}\left(ze^{-2x}\right) = -2e^{-2x} (\cos x - \sin x)$$

$$\Rightarrow d\left(Ze^{-2x}\right) = -2\int e^{-2x} (\cos x - \sin x)\, dx$$

$$\Rightarrow Ze^{-2x} = -2\left[\int e^{-2x} \cos dx - \int e^{-2x} \sin x dx\right]$$

$$= -2\left[\frac{e^{-2x}}{5}(-2\cos x + \sin x) - \frac{e^{-2x}}{5}(-2\sin x - \cos x)\right] + c$$

$$= 2e\frac{-2x}{5}[3\sin x - \cos x] + c$$

$$\therefore Z = -\frac{2}{5}[3\sin x - \cos x] + ce^{2x}$$

$$\therefore \frac{1}{y^2} = -\frac{2}{5}(3\sin x - \cos x) + ce^{2x}$$

where C is an arbitrary const.

5. Evaluate the definite integral $\int_1^4 (x + x^3)\,dx$ by using Trapezoidal rule, taking five ordinates and calculate the error.

Ans. $\int_1^4 (x + x^3)\,dx$ $\quad\quad \therefore f(x) = x + x^3$

$a = 1, b = 4, nh = (b - a) = 3$

gives there are 5 ordinates i.e., there are 4 subintgervals $\Rightarrow n = 4$

$$\therefore h = \frac{3}{4} = \cdot 75$$

$\therefore \quad x$:	1	$1\cdot 75$	$2\cdot 5$	$3\cdot 75$	4
$f(x)$:	2	$7\cdot 109$	18.125	$37\cdot 578$	68

Trapezoidal integral formula:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + y_n + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right]$$

$$\therefore \quad \int_1^4 (x + x^3)\,dx = \frac{75}{2}\left[2 + 68 + 2(7\cdot 109 + 18\cdot 125 + 37\cdot 578)\right] = 73.359 = I_{\text{approx}}$$

$$\text{again } I_{exact} = \int_1^4 (x + x^3)\,dx = \left[\frac{x^2}{2} + \frac{x^4}{4}\right]_1^4 = (8 + 64) - \left(\frac{1}{2} + \frac{1}{4}\right)$$

$$= 72 - \frac{3}{4}$$

$$= 72 - .75$$

$$= 71.25$$

$$\therefore \quad \text{error } = \left|I_{exact} \quad - I_{approx}\right| = 2.109.$$

6. If $A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then show that

$$A(\theta)\,A(\phi) = A(\phi).\,A(\theta) = A(\theta + \phi)$$

Ans. Refer to the question 2. (vi) 2006.

GROUP—C

(Short Answer Type Questions)
Answer any *three* of the following **3 × 15 = 45**

7. (a) If $A=\begin{bmatrix}1 & 2 & 1\\ 1 & -4 & 1\\ 3 & 0 & -3\end{bmatrix}$, $B=\begin{bmatrix}2 & 1 & 1\\ 1 & -1 & 0\\ 2 & 1 & -1\end{bmatrix}$, show that $AB = 6I_3$. Utilize this result to solve the following system of equations:

$2x+y+z=5$

$x-y=0$

$2x+y-z=1$

Ans. $A=\begin{bmatrix}1 & 2 & 1\\ 1 & -4 & 1\\ 3 & 0 & -3\end{bmatrix}$; $B=\begin{bmatrix}2 & 1 & 1\\ 1 & -1 & 0\\ 2 & 1 & -1\end{bmatrix}$

$|\det(B) = 2(1-0)-1(-1-0)+1(1+2)$

$= 2+1+3 = 6 \neq 0 \Rightarrow B^{-1}$ exists $|$

$$AB=\begin{bmatrix}1 & 2 & 1\\ 1 & -4 & 1\\ 3 & 0 & -3\end{bmatrix}\begin{bmatrix}2 & 1 & 1\\ 1 & -1 & 0\\ 2 & 1 & -1\end{bmatrix}=\begin{bmatrix}6 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 6\end{bmatrix}=6\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=6I_3 \quad \text{(Proved).}$$

again $\left.\begin{aligned}2x+y+z&=5\\ x-y&=0\\ 2x+y-z&=0\end{aligned}\right\}$ ....(II) Let us consider $P=\begin{bmatrix}2 & 1 & 1\\ 1 & -1 & 0\\ 2 & 1 & -1\end{bmatrix}$ (i.e. the coefficient matrix) by given problem $P = B$

Let (II) can be written as $PX = Q$

$\Rightarrow P^{-1}(PX) = P^{-1}Q$ (as matrix multiplication is associative)

$\Rightarrow (P^{-1}P)X = P^{-1}Q$

$\Rightarrow I.X = P^{-1}Q$

$\Rightarrow X = P^{-1}Q$ ($I \rightarrow$ Identity matrix)

$= B^{-1}Q$

From (I)

$AB = 6I_3$

$\Rightarrow (AB)B^{-1} = (6I_3).B^{-1}$

$\Rightarrow A(BB^{-1}) = 6(I_3.B^{-1})$

$\Rightarrow AI = 6.B^{-1}$

$$\Rightarrow A = 6B^{-1} \Rightarrow B^{-1} = \frac{1}{6}A = \begin{bmatrix} \frac{1}{6} & \frac{2}{6} & \frac{1}{6} \\ \frac{1}{6} & -\frac{4}{6} & \frac{1}{6} \\ \frac{3}{6} & 0 & -\frac{3}{6} \end{bmatrix}$$

$$\therefore X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{2}{6} & \frac{1}{6} \\ \frac{1}{6} & -\frac{4}{6} & \frac{1}{6} \\ \frac{3}{6} & 0 & -\frac{3}{6} \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$\therefore x = 1, y = 1, z = 2$.

(b) Solve: $(y - px)(p-1) = p$ and obtain the singular solution. Here $p = \frac{dy}{dx}$.

Ans. $(y - px)(p-1) = p$

$$\Rightarrow y - px = \frac{p}{p-1}$$

$$\Rightarrow y - px = \frac{p}{p-1} \rightarrow \text{ clairaut's equation ....(I)}$$

differentiating both sides of (I) we get

$$p = p.1 + x.\frac{dp}{dx} + \frac{(p-1).1 - p}{(p-1)^2} \cdot \frac{dp}{dx}$$

$$\Rightarrow x\frac{dp}{dx} - \frac{1}{(p-1)^2}\frac{dp}{dx} \quad \left[\text{Here } p = \frac{dy}{dx}\right]$$

or, $\frac{dp}{dx}\left[x - \frac{1}{(p-1)^2}\right] = 0$ From (I)

$\therefore \frac{dp}{dx} = 0$ or, $p = c$ .... (II) $\frac{dy}{dx} = c \Rightarrow y = \boxed{cx + \frac{c}{c-1}}$ general solution

or, $x = \frac{1}{(p-1)^2} \Rightarrow (p-1)^2 = \frac{1}{x} \Rightarrow p - 1 = \pm\frac{1}{\sqrt{x}}$

$$\Rightarrow p = 1 \pm \frac{1}{\sqrt{x}} \quad \text{.........(III)}$$

Solving (I) and (III)

$$y = px + \frac{p}{(p-1)} = p\left[x + \frac{1}{(p-1)}\right]$$

$$= \left(1 \pm \frac{1}{\sqrt{x}}\right)\left[x + \frac{1}{\left(1 \pm \frac{1}{\sqrt{x}} - 1\right)}\right]$$

$$= \frac{\left(\sqrt{x} \pm 1\right)}{\sqrt{x}} \times \left(x \pm \sqrt{x}\right)$$

$y = \left(\sqrt{x} \pm 1\right)^2 \rightarrow$ singular solution.

(c) Construct the interpolation polynomial for the function $y = \sin \pi x$, taking the points $x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{1}{2}$

Hence find $f\left(\frac{1}{3}\right)$ where $y = f(x)$.

Ans. $y = \sin \pi x$

$$y_0 = \sin \pi x_0 = \sin 0 = 0$$

$$y_1 = \sin \frac{\pi}{6} = \frac{1}{2}$$

$$y_2 = \sin \frac{\pi}{2} = 1$$

x	$x_0 = 0$	$x_1 = \frac{1}{6}$	$x_2 = \frac{1}{2}$
y	$y_0 = 0$	$y_1 = \frac{1}{2}$	$y_2 = 1$

Since value of x are not equispaced we use Lagrarge's interpolation to find the polynormial for $f(x)$

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

$$= \frac{\left(x - \frac{1}{6}\right)\left(x - \frac{1}{2}\right)}{\left(-\frac{1}{6}\right)\left(-\frac{1}{2}\right)} \times 0 + \frac{(x-0)\left(x - \frac{1}{2}\right)}{\frac{1}{6}\left(-\frac{1}{3}\right)} \times \frac{1}{2} + \frac{(x-0)\left(x - \frac{1}{6}\right)}{\left(\frac{1}{2} - 0\right)\left(\frac{1}{2} - \frac{1}{6}\right)} 1$$

$$= -9x\left(x - \frac{1}{2}\right) + 6(x)\left(x - \frac{1}{6}\right) = -\frac{9}{2}x(2x-1) + x(6x-1)$$

$$= -9x^2 + 9x + 6x^2 - 2$$

$$= -3x^2 + \frac{7x}{2}$$

$$\therefore f\left(\frac{1}{3}\right)=-3\left(\frac{1}{9}\right)+\frac{7}{6}=\frac{7}{6}-\frac{1}{3}=\frac{5}{6}$$

8. (a) Solve the differential equation

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=x^2e^{3x}$$

Ans. $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=x^2e^{3x}$

We can write the above equation in the standard form as $(D^2-5D+6)y$

x^2e^{3x} where $D\equiv\frac{d}{dx}$ ....(I)

Let us consider $y=e^{mx}$ be a trial solution of homogeneous eqn. $(D^2-5D+6)y=0$

$\therefore Dy=me^{mx}$; $D^2y=m^2e^{mx}$

$\therefore$ The auxiliary equation is $(m^2e^{mx}-5me^{mx}+6e^{mx})=0$

$\because e^{mx}\neq 0$ for finite values of x

$\Rightarrow m^2-5m+6=0\Rightarrow(m-3)(m-2)=0$

$\Rightarrow m=3,2$

$\therefore$ Complementary function = C.F. $=C_1e^{3x}+c_2e^{2x}$ where c_1,c_2 are arbitrary constants now

$$\text{P.I.}=\frac{1}{(D^2-5D+6)}(x^2e^{3x})$$

$$=\frac{1}{e^{3x}\,(D+3)^2+5(D+3)+6}(x^2)$$

$$\left[\because\frac{1}{f(D)}(e^{ax}v(x))=e^{ax}\frac{1}{f(D+a)}(v(x))\right]$$

$$=e^{3x}\frac{1}{D^2+D}(x^2)$$

$$=e^{3x}\frac{1}{D(1+D)}(x^2)$$

$$=e^{3x}\frac{1}{D}.(1+D)^{-1}(x^2)$$

$$=e^{3x}\frac{1}{D}.(1-D+D^2-D^3+\ldots\infty)(x^2)$$

$$=e^{3x}\frac{1}{D}(x^2-2x+2)\qquad[\because D^r(x^2)=0\text{ for }r>2$$

$$=e^{3x}\left(\frac{x^3}{3}-x^2+2x\right)\text{ and }D^r(x^2)=2!\text{ for }r=2]$$

$$=e^{3x}\left(\frac{x^3}{3}-x^2+2x\right)\left[\because\frac{1}{D}\equiv\int dx\right]$$

$\therefore$ General solution is $y = C_1e^{3x} + C_2e^{2x} + e^{3x}\left(\dfrac{x^3}{3} - x^2 + 2x\right)$

(b) Apply suitable interpolation formula to calculate $f(3)$ correct <??> two significant figures from the following data:

x :	2	4	6	8	10
$f(x)$:	5	10	17	29	49

Ans. Since values of x are equispaced and we are to find $f(x)$ at x were going to apply Newton's backward interpolation formula for problem.

Newton's backward difference table

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
2	5				
4	10	5			
6	17	7	2		
8	29	12	5	3	
10	49	20	8	3	0

$$f(x) = y_n + s\nabla y_n + \frac{s(s+1)}{\underline{|2}}\nabla^2 y_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 y_n + \ldots$$

Here $x = 9$; $h = 2$ (difference between values of x); $n = 5$ we take $x_n = x_5$, $y_n = y_5$

$$\therefore x_n = 10, y_n = 49;\ s = \frac{x - x_n}{h} = \frac{9-10}{2} = -\frac{1}{2} = -0.5$$

$$\therefore f(9) = 49 + (-0.5)\ 20 + \frac{(-.5)(-.5+1)}{2}8 + \frac{(-.5)(-.5+1)(-.5+2)}{6}\times 3$$

$$= 49 - 10 + 4(.5)(-.5) + \frac{1}{2}(.5)(1.5)(-.5)$$

$$= 39 - 1 - .1875 = 37.8125 \approx 38$$

(c) Determine the conditions under which the system of equations

$$x + y + z = 1$$
$$x + 2y - z = b$$
$$5x + 7y + az = b^2$$

admits of

(i) only one solution

(ii) no solution

(iii) many solutions.

Ans.

$$x + y + z = 1$$
$$x + 2y - z = b$$
$$5x + 7y + az = b^2$$

Here the coefficient matrix is $P=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & -1\\ 5 & 7 & a\end{pmatrix}$ and argumented matrix is $=\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 2 & -1 & b\\ 5 & 7 & a & b^2\end{pmatrix}$

Let us transform Q to Echelon matrix by elementary row operations

$$\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 2 & -1 & b\\ 5 & 7 & a & b^2\end{pmatrix}\xrightarrow[R_3-5R_1]{R_2-R_1}\begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1 & -2 & b-1\\ 0 & 2 & a-5 & b^2-5\end{pmatrix}$$

$$\xrightarrow[R_3-2R_2]{}\begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1 & -2 & b-1\\ 0 & 0 & a-1 & b^2-2b-3\end{pmatrix}=R\text{ (let)}$$

$\therefore\ P\sim$ ecelon matrix $\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & -2\\ 0 & 0 & a-1\end{pmatrix}=L$ (say)

(i) for only one solution Rank of P = Rank of Q = number of unknowns = 3

$\therefore$ Rank of L = Rank of R = 3

$\therefore$ Rank of $L=3\Rightarrow a-1\neq 0\Rightarrow a\neq 1$

$\therefore$ for $a\neq 1$ and for any value of b the gives system has luique solution.

(ii) For no solution Rank of $P\neq$ Rank of $Q\Rightarrow$ Rank of $L\neq$ Rank of R

Let Rank of $L=2$ and Rank of $R=3$

$\therefore\ a-1=0$ and $b^2-2b-3\neq 0$

$\Rightarrow a=1$ and $(b-3)\ (b+1)\neq 0$

$\Rightarrow a=1$ and $b\neq -1,\ 3$

$\therefore$ the system admits no solution for $a=1, b\neq -1, 3$

(iii) For many solutions Rank of P = Rank of $Q<3$ (= no unknowns)

$\Rightarrow$ Rank of L = Rank of $R<3$

Since first two rous of R and L are non zero rows

$\therefore$ Rank of R = Rank of $L<2$

$\therefore$ Rank of L = Rank of $R=2$

$\therefore\ a-1=0$ and $b^2-2b-3=0$

$\Rightarrow a=1$ and $b=-1, 3$

$\therefore$ when $a=1$; $b=-1,\ 3$ the system admits many solutions.

9. (a) Prove that $P^T AP$ is a symmetric or a skew-symmetric matrix according as A is symmetric or skew-symmetric.

Ans. Symmetric q skew-symmetric matrix is always a space matrix

$\therefore A$ must be square matrix

Let A be of order $n\times n$ and order of P be $n\times r$

$\therefore AP$ is of order $n\times r$

but P^T is of order $r \times n$

$\therefore$ order of $P^T AP$ is $r \times r \Rightarrow P^T \; AP$ is a square matrix

[If P is itself a square matrix of order $n \times n$ the result is obvious]

$$\begin{aligned}\therefore (P^T AP)^T &= (P^T (AP))^T = (AP)^T (P^T)^T \,' \\ &= (AP)^T P \quad [\because (AB)^T = B^T A^T] \\ &= P^T A^T P\end{aligned}$$

Now $P^T A^T P = P^T AP$ if $A^T = A$ i.e., A is symmetric

$= -P^T AP$ if $A^T = -A$ i.e., A is skew symmetric

$\therefore (P^T AP)^T = P^T AP$ or $(P^T AP)^T = -P^T AP$ according as A is symmetric or skew symmetric.

Hence the result.

(b) Find the eigen values and the eigenvectors of the matrix $\begin{bmatrix} 4 & 6 \\ 2 & 9 \end{bmatrix}$

Ans. Let $A = \begin{pmatrix} 4 & 6 \\ 2 & 9 \end{pmatrix}$

For eigen values $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 4-\lambda & 6 \\ 2 & 9-\lambda \end{vmatrix} = 0$

$$\Rightarrow (4-\lambda)(9-\lambda) - 12 = 0$$

$$\Rightarrow 36 - 13\lambda + \lambda^2 - 12 = 0$$

$$\Rightarrow \lambda^2 - 13\lambda + 24 = 0$$

$$\therefore \lambda = \frac{13 \pm \sqrt{169-96}}{2} = \frac{13 \pm \sqrt{173}}{2}$$

$\therefore$ eigen values are $\dfrac{13+\sqrt{73}}{2}, \dfrac{13-\sqrt{73}}{2}$;

Let $X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ bet the eigen vector corresponding to $\dfrac{13+\sqrt{73}}{2}$

$$\therefore AX_1 = \left(\frac{13+\sqrt{73}}{2}\right) X_1 \Rightarrow \begin{pmatrix} 4 & 6 \\ 2 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\frac{13+\sqrt{73}}{2}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow 4x_1 + 6x_2 = \frac{13+\sqrt{73}}{2} x_1 \Rightarrow 6x_2 = \left(\frac{\sqrt{73}+5}{2}\right) x_1$$

$$\Rightarrow x_1 = \frac{12}{\left(\sqrt{73}+5\right)} x_2$$

$$= \frac{\left(\sqrt{73}-5\right)}{4} x_2$$

again $2x_1 + 9x_2 = \dfrac{13+\sqrt{73}}{2} x_2$

$$\Rightarrow 2x_1 = \frac{\left(\sqrt{73}-5\right)}{2}x_2 \Rightarrow x_1 = \frac{\sqrt{73}-5}{4}x_2$$

$$\therefore x_1 = \begin{pmatrix} \frac{\sqrt{73}-5}{4}c \\ c \end{pmatrix} = c\begin{pmatrix} \frac{\sqrt{73}-5}{4} \\ 1 \end{pmatrix} \text{ when } c \text{ is a non zero real no.}$$

Let $X_2 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be the eigen vector corresponding to $\frac{13-\sqrt{73}}{2}$

$$\therefore AX_2 = \left(\frac{13-\sqrt{73}}{2}\right)X_2 \Rightarrow \begin{pmatrix} 4 & 6 \\ 2 & 9 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(\frac{13-\sqrt{73}}{2}\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\therefore 4y_1 + 6y_2 = \frac{13-\sqrt{73}}{2}y_1 \Rightarrow \left(\frac{\sqrt{73}-5}{2}\right)y_1 = -6y_2 \Rightarrow y_1 = -\frac{\sqrt{73}+5}{4}y_2$$

$$2y_1 + 9y_2 = \frac{13-\sqrt{73}}{2}y_2 \Rightarrow -2y_1 = \frac{5+\sqrt{73}}{2}y_2 \Rightarrow 1y_1 = -\frac{1}{4}\left(\sqrt{73}+5\right)y_2$$

$$\therefore X_2 = k\begin{bmatrix} -\frac{1}{4}\left(5+\sqrt{3}\right) \\ 1 \end{bmatrix} \text{ where } k \neq 0 \text{ real number.}$$

(c) Solve by Cramer's rule the following system of equations :

$$3x + y + z = 4$$
$$x - y + 2z = 6$$
$$x + 2y - z = -3$$

Ans. $3x + y + z = 4$

$$x - y + 2z = 6$$
$$x + 2y - z = -3$$

$$\text{Let } \Delta = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 3(1-4) - 1(-1-2) + 1(2+1)$$

$$= -9 + 3 + 3 = -3 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 4 & 1 & 1 \\ 6 & -1 & 2 \\ -3 & 2 & -1 \end{vmatrix} = 4(1-4) - 1(-6+6) + 1(12-3)$$

$$= -12 + 9 = -3$$

$$\Delta_2 = \begin{vmatrix} 3 & 4 & 1 \\ 1 & 6 & 2 \\ 1 & -3 & -1 \end{vmatrix} = 3(-6+6) - 4(-1-2) + 1(-3-6)$$

$$= 12 - 9 = 3$$

$$\Delta_3 = \begin{vmatrix} 3 & 1 & 4 \\ 1 & -1 & 6 \\ 1 & 2 & -3 \end{vmatrix} = 3(3-12)-1(-3-6)+4(2+1) = -27+9+12 = 6$$

$$\therefore x = \frac{\Delta_1}{\Delta} = 1;\ y = \frac{\Delta_2}{\Delta} = -1;\ z = \frac{\Delta_3}{\Delta} = 2$$

10. (a) What is meant by linear independence of a set of n-vectors ?

Ans. Refer to the soln of Q. 13. (a)/2008.

(b) Solve by the method of variation of parameters the equation

$$\frac{d^2y}{dx^2} + 9y = \sec 3x.$$

Ans. $\dfrac{d^2y}{dx^2} + 9y = \sec 3x$

We can write the given equation in standard form as $(D^2+9)y = \sec\ 3x$ $\left[D = \dfrac{d}{dx}\right]$.

Let, $y = e^{mx}$ be the trial solution of homogereous eqution $(D^2+9)y = 0$

$\therefore\ Dy = me^{mx}\ ;\ D^2y = m^2e^{mx}$

$\therefore$ auxiliary equation is $(m^2+9)\ me^{mx} \neq 0$

$\because e^{mx} \neq$ for finite values of x

$\therefore m^2+9 = 0 \Rightarrow m = \pm 3i$ where $i = \sqrt{-1}$

$\therefore$ C.F.: $c_1 \cos\ 3x + c_2 \sin 3x$; $(c_1, c_2$ are arbitrary constants)

Let us consider P.I. $=$ $u\ \cos\ 3x + v \sin\ 3x$ where u, v are function of x.........(1)

$\therefore\ y_p = u\ \cos 3x + v\ \sin 3x$

$$Dy_p = -3u\ \sin 3x + 3v\ \cos 3x + \cos 3x\frac{du}{dx} + \sin 3x\frac{dv}{dx}$$

Let $\boxed{\cos 3x\dfrac{du}{dx} + \sin\dfrac{dv}{dx} = 0}$ ....(II)

$$\therefore D^3y_p = -9u\ \cos 3x + (-3\sin 3x)\frac{du}{dx} - 9v\sin 3x + 3\cos 3x\left(\frac{dv}{dx}\right) \text{........(III)}$$

substitution (II) and (III) in (I) we get

$$\therefore \boxed{3\cos 3x\frac{dv}{dx} - 3\sin 3x\frac{du}{dx} = \sec 3x} \text{ .... (IV)}$$

$$\therefore \frac{du}{dx} = \frac{\begin{vmatrix} 0 & \sin 3x \\ \sin 3x & 3\cos 3x \end{vmatrix}}{\begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}} = \frac{-\tan 3x}{3} = -\frac{1}{3}\tan 3x$$

$$u = -\frac{1}{3}\int \tan 3x dx = -\frac{1}{9}\log \sec 3x = \frac{1}{9}\log \cos 3x$$

$$\frac{dv}{dx} = \frac{\begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \sec 3x \end{vmatrix}}{\begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\sec 3x \end{vmatrix}} = \frac{1}{3}$$

$$\therefore \int dv = \int \frac{1}{3} dx = v = \frac{x}{3}$$

$$\therefore \text{ P.I.} = \left(\frac{1}{9}\log \cos 3x\right)\cos 3x + \frac{x}{3}(\sin 3x)$$

$\therefore$ general solution $y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{9}\cos 3x \log \cos 3x + \frac{x}{3}\sin 3x.$

(c) Prove that $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

Ans. Refer to the solution of Q. 2(a) (2nd part)/2004.

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