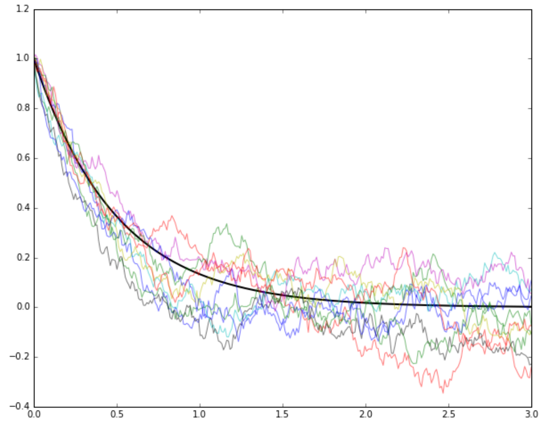
Numerical Methods for Stochastic Differential Equations

Juan Ozaita Corral

Exercise 1 1.1.-Firstly, we show the plot of the ten stochastic trajectories along with the deterministic one.



As we can see, at least, for the first 0.5 seconds the deterministic regime dominates, and beyond that value, the stochastic equations start to fluctuate around the deterministic value.

1.2.-We show the plots for $< x >$ and $< x^2 >$ averaging over ten, a hundred and a thousand trajectories:

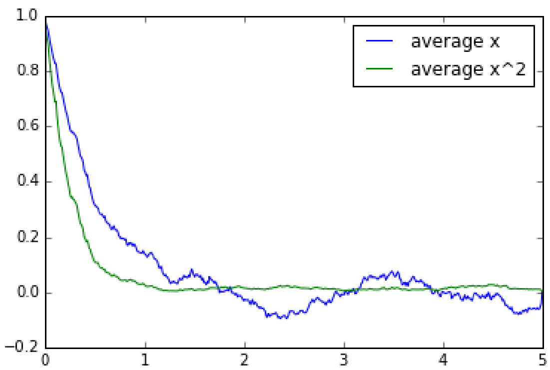


Figure 1: Averages over 10 trajectories

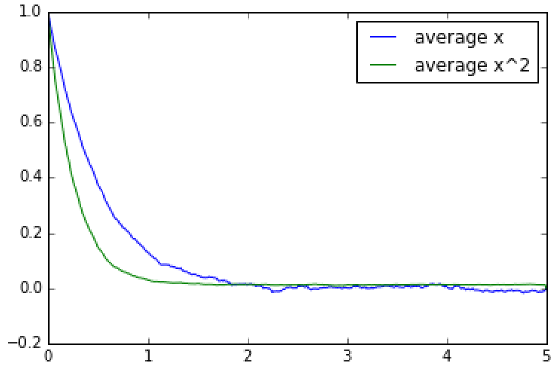


Figure 2: Averages over 100 trajectories

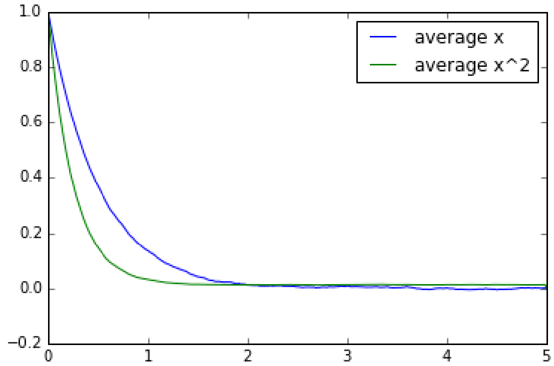


Figure 3: Averages over 1000 trajectories

As we can see, making the average and the variance among more trajectories improves the continuous functional shape of these ones, approximating it to the deterministic case.
1.3-We present a table with the results for $< x(1) >$ and $< x(1)^2 >$ for different timesteps:

h	$< x(1) >$	$< x(1)^2 >$
0.5	-0.0025 $\pm$ 0.0050	0.0245 $\pm$ 0.0011
0.1	0.1103$\pm$ 0.0038	0.02520 $\pm$ 0.00099
0.01	0.1368$\pm$ 0.0034	0.0308 $\pm$ 0.0011
0.001	0.1312$\pm$ 0.0034	0.0299$\pm$ 0.0011

We can see that as h goes to zero, our mean also approaches the deterministic case ($exp(-2) = 0.1353$). We can not say this about $< x(1)^2 >$, because ($exp(-4) = 0.0183$) and our stochastic data almost double it.
1.4.- Firstly, we plot the correlation function $C(s)$ vs s , in the required range.

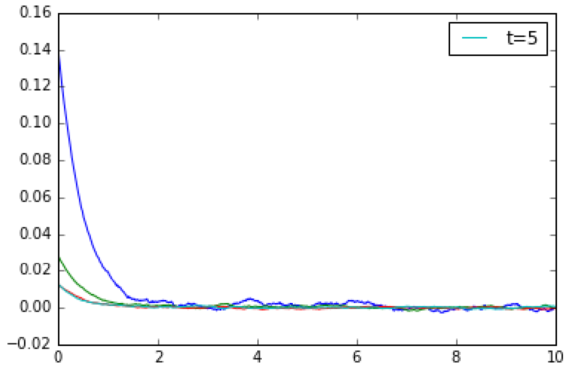
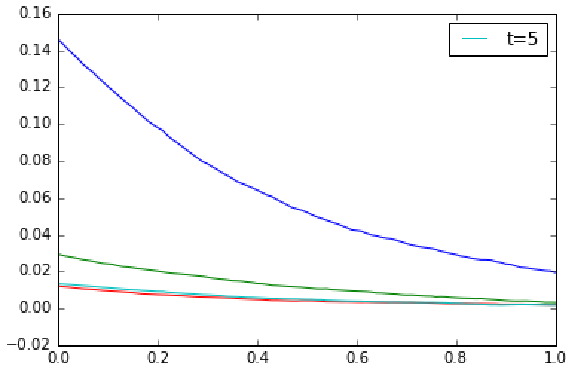


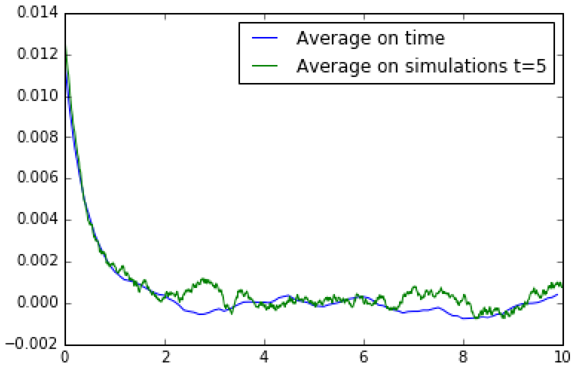
Figure 4: C(s) vs s

As fas as we know, the correlation has to decrease with the time, because at the beginning of the time, the process is dominated by the deterministic regime, with strong correlation. If we amplify the previous picture :



We can see that there is no difference between the lower correlation functions, those are the one for t=2 and t=5. So we can use this figure to say that t=2 approxmately separates the transient and the stationary regime. Taking this function as the stationary correlation function, we can evaluate $corr(s = 0)/e$ and calculate the correlation time, which is 0,5 seconds.

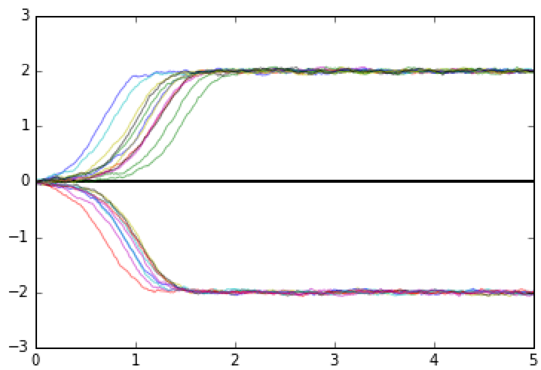
1.5.-As we have calculated the correlation time in the previous step (0,5 seconds) we generate a long trajectory of 500 seconds. We compute all the correlations for $s \ni [0, 10]$ as before, reaching the following plot. We compare it with the case t=5 (stationary regime) of 1.4:



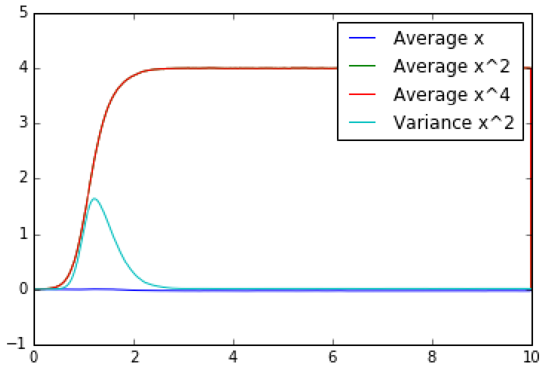
Because of the resolution that we have used, the average in time has less points than the previous, but we can see that they fit correctly and reproduce the stationary regime. That's because of the 10 small trajectories that we can extract from a long one (except the first one , the others are all in the stationary regime).

Exercise 2

2.1.-We plot the trajectories and we find :

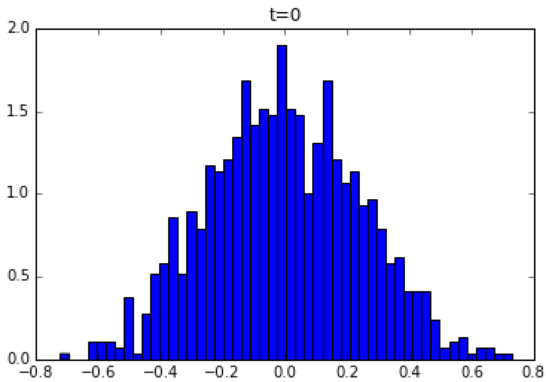


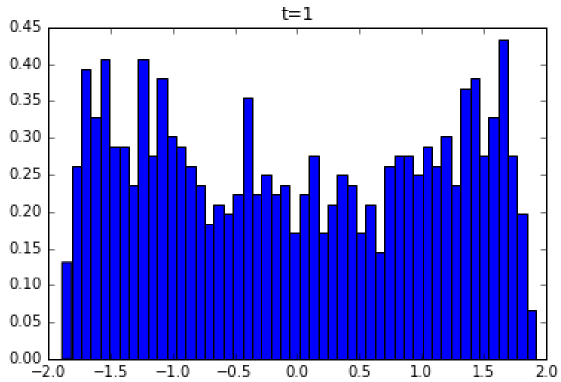
The black thick line is the deterministic trajectory. It happens that the real initial condition depends on the noise and it behaves as a perturbation that takes our system to one of the fixed points : $\sqrt{a/b}$ or $-\sqrt{a/b}$.
2.2.-We present the plot:

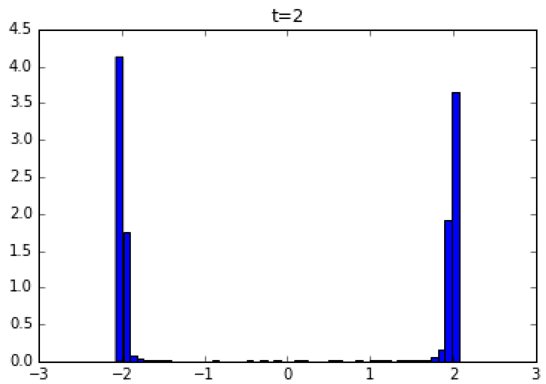


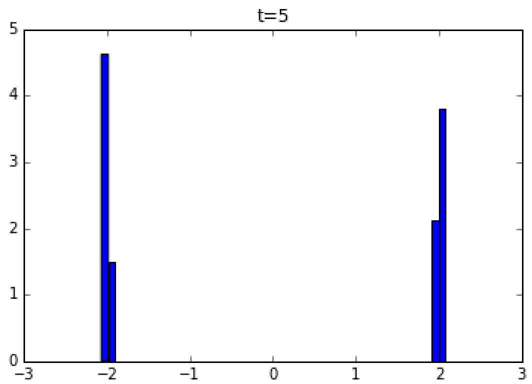
As the distribution is symettric (half of the trajectories go to either fixed point), the average is zero. The average of the squared ($< x^2 >$) tells us the squared evolution of the trajectories. The variance of this quantity tell us the differences between the different stochastic trajectories. It reaches a maximum in an intermediate time (1.22 seconds), and we could clasify this regime (before they all reach the fixed points) as a transient regime.

2.3.-We plot the different normalized histograms for the times : 0.5,1,2,5:



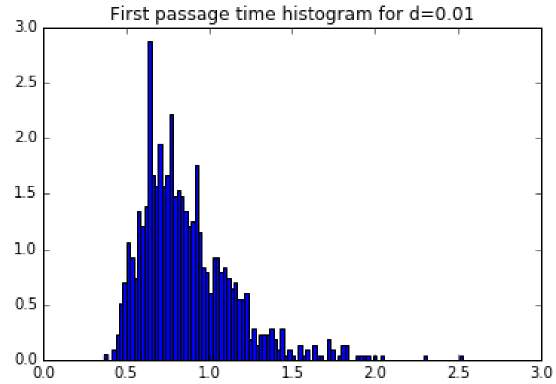




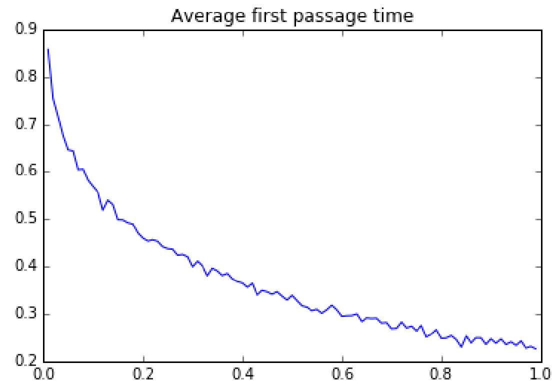


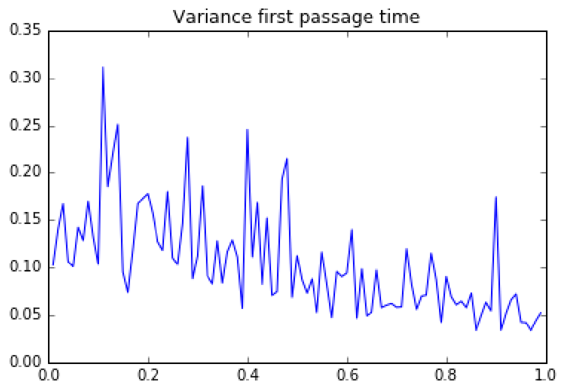
Changes in shape tell us the evolution of trajectories. In a first moment, the equation is dominated by the gaussian white noise. As it evolves, our dynamic equation acts and it pushes our particles to one of the two final stationary states.

2.4.-We plot the histogram for the distribution of the first passage time for d=0.01 obtaining:



As we can see, it is not symmetrical, in fact, it has a similar shape to a poissonian but we can only say this analyzing the average and the variance so we proceed to plot both quantities as a function of d:





As we can see, it is not a poissonian because the functional form of the average is different from the variance. We can see that the average first passenger time decreases as the noise becomes more powerful (which is logical) but the variance just decreases in a noisy way.