Stochastic Simulation Exercises

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Exercise 28.-We are asked to compute the integral :

$$I = \int_0^1 dx \; cos(\frac{\pi x}{2})$$

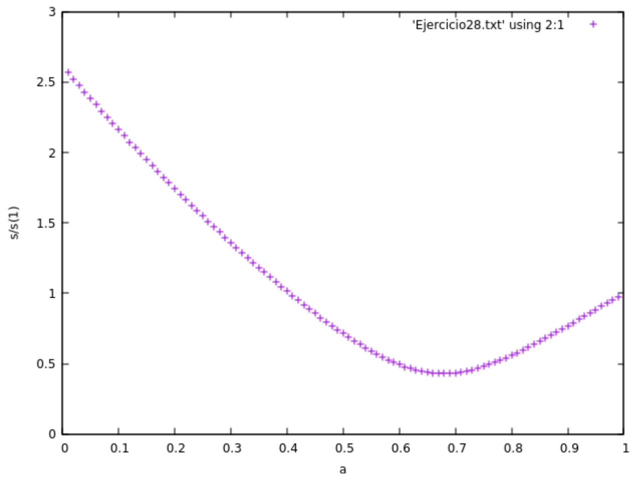
Using the sampling function:

$$f_{\hat{x}}(x) = \frac{6}{3+a}\left(1 + x(a-1) - ax^2\right)$$

As we have to do a whole scan through a in order to obtain the optimal value we go across a hundred values in [0,1] with regular step. The value of the integral below is the average of these hundred values:

$$I = 0.636619 \; \pm \; 1.1 * 10^{-5}$$

Which corresponds to the analytical value , $2/\pi$.
As it was said, we have to find the most optimal value and compare it's efficiency with the case a=1. We first present a plot of the standard deviation compared to the case a=1 vs a :



A more precise value for this minimum would be :

$$a = 0.68 \pm 0.01$$

$$s(a = 0.68) = 0.4294 * s(a = 1)$$

The error in a corresponds to the resolution we have taken.

Exercise 31.- We are asked to compute the integral :

$$\int_0^1 d^n x \; exp(-1/2(\Sigma x_i^2)))cos^2(\Sigma x_i x_{i+1})$$

For the case n =1 we took the expression $exp(-x^2)$. We present a summary of the results:

Dimension	Hit and Miss	Uniform Sampling	Simpson
1	0.85560 +/− 0.00011	0.855618+/− 0.000038	0.855624
2	0.67560+/− 0.00030	0.676173+/− 0.000065	
5	0.19443+/− 0.00044	0.194537+/− 0.000065	
10	0.082624+/− 0.000090	0.082621+/− 0.000020	

As we can see, all the values fit well for the methods that have been done.

Exercise 48.-We are asked to design a random number generator of the function:

$$f_{\hat{x}}(x) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} \qquad x \geqslant 0, \qquad 0 < \alpha < 1$$

With a rejection method. To do that, we need to split our distribution in two parts: g(x), which is a distribution proportional to the former one, and h(x), which is a quantity between 0 and 1 in the domain of the distribution.
This function has not a trivial h(x) in the whole domain, since $x^{\alpha-1}$ diverges as x goes to zero. The solution is to split the whole domain of x in two parts :

- Domain one, for x ∋ (0,1)
- Domain two, for x ∋ (1,∞)

In domain one, h(x) is the e^{-x}. In the second one, h(x) is $x^{\alpha-1}$ so that both quantities have an upper bound of 1. So , our proportional distribution is :

$$Ax^{\alpha-1} \qquad x \ni (0,1)$$

$$Ae^{-x} \qquad x \ni (1,\infty)$$

Our distribution has to be normalized so :

$$A\left(\int_0^1 x^{\alpha-1}\,dx + \int_1^\infty e^{-x}\,dx\right) = 1$$

This leads to :

$$A\left(\frac{1}{\alpha} + \frac{1}{e}\right) = 1 \qquad\qquad A = \frac{\alpha e}{\alpha + e}$$

The rejection method works as follows:

- We generate a random number between $[0,1]$ and we compare it to the probability of getting a lesser value than 1(F(x=1)). This value is , if we compute the first integral above:

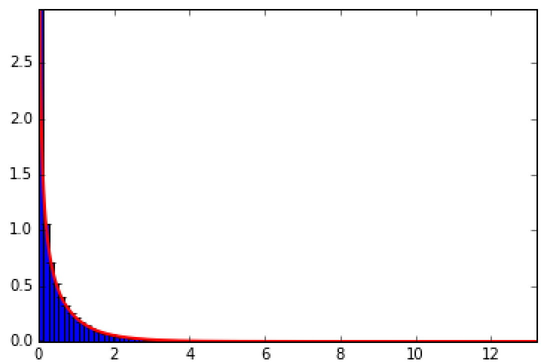
  $$\frac{A}{\alpha} = \frac{e}{e+\alpha} = \beta$$

  This is the probability of getting a value less than one. Our criteria to chose where the random number has to go is :

  $$If \qquad p = \frac{random\ \ number}{\beta} < 1 \qquad \to Domain\ \ 1$$

  $$If \qquad p = \frac{random\ \ number}{\beta} > 1 \qquad \to Domain\ \ 2$$

If we go to the domain one , as our random number is now between 0 and 1, all behaves normal. And if we go to the domain two we do the rescaling in order to compare it with another random uniform number between 0 and 1. We plot the results in a histogram for α=0.5 :



We can see that they fit correctly the functional form (the red line), so we have generated the distribution. Other results can be extracted from the program. The errors of the histogram are also computed as a binomial process with p=probability of acceptance for M trials :

$$\sigma = \sqrt{\frac{p(1-p)}{M}}$$

And this is used for this and for the next histograms.

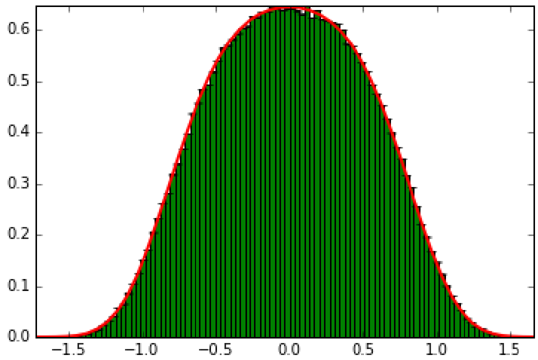
Exercise 50.-We are asked to generate by the rejection method the distribution:

$$f(x) = Ce^{-x^2/2-x^4}$$

This distribution is easy to split because we can get :

$$g(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad\qquad h(x) = e^{-x^4}$$

So we do the rejection method with a gaussian random number. We obtain results and plot them in a histogram with the theoretical curve (the line) :



We are asked also to compute the constant through the efficiency. To do so :

$$\epsilon = \int_{-\infty}^{\infty} dx \;\; h(x)g(x) = \frac{1}{\sqrt{2\pi}C} \int_{-\infty}^{\infty} dx \;\; f(x) = \frac{1}{\sqrt{2\pi}C}$$

$$C = \frac{1}{\sqrt{2\pi}\epsilon}$$

So we can compute the constant through the acceptance probability. The error in the constant is , by error propagation:

$$s(C) = \frac{1}{\sqrt{2\pi}p^2}\sqrt{\frac{p(1-p)}{M}}$$

Where we have take into account that being accepted or rejected corresponds to a binomial distribution that depends on the total number of trials M with probability p. The result obtained is :

$$C = 0.64395 \;\pm\; 5.0 * 10^{-4}$$

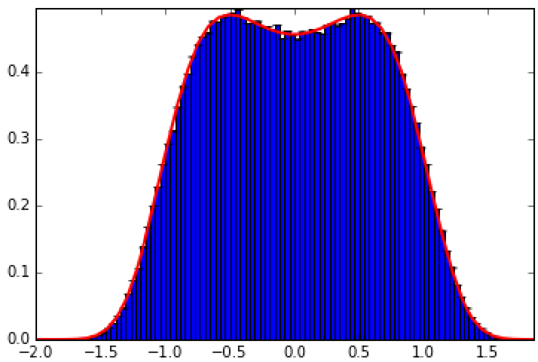
Exercise 51.- We have to sample :

$$f(x) = Ce^{x^2/2-x^4}$$

So we do the rejection method with the following functions :

$$g(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad\qquad h(x) = e^{x^2-x^4-1/4}$$

Where the 1/4 at h(x) corresponds to dividing by the maximum of the original h(x) (without the 1/4) so it fits the condition of being in [0,1] in the whole range. We do the histogram :



We are also asked to compute the constant. If we repeat the procedure of the former exercise we get:

$$C = \frac{e^{1/4}}{\sqrt{2\pi\epsilon}}$$

And for the error :

$$s(C) = \frac{e^{1/4}}{\sqrt{2\pi}p^2}\sqrt{\frac{p(1-p)}{M}}$$

We obtain the following result : distribution. The result obtained is :

$$C = 0.45715 \ \pm \ 5.2 * 10^{-4}$$