Noisy voter model with partial aging and anti-aging

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0.1 Agradecimientos

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Abstract

The noisy voter model is a stochastic binary state model where the agents evolve according to random noise and interactions with its neighbors. The addition of aging, as an individual property that makes the old agents less willing to interact, introduces interesting features. In particular, a second order phase transition appears. In this thesis we have focused on the presence of that phase transition. We characterize an alternative dynamics for the aging (anti-aging) , where old agents are prone to interact, checking that this phase transition is destroyed. We also characterize a system where agents with and without age coexist, checking that theoretically, the second order transition exists for any non-zero quantity of aged agents.

Contents

Chapter 1

Introduction

The study of a two-states system has been widely used to understand many phenomena in very different disciplines. These systems are composed of individuals that are characterized by a state, that can take any of two different values. The dynamics for an individual in the interaction with its neighbors is what characterizes the different models. In our case, the main mechanism of the dynamics will be the herding behavior.
Herding behavior has been studied as an important mechanism in several topics. We present some of them :

- Biology, where the original Kirman's model ([1]) (which we will explain below) appeared originally, as the study of the distributions of food supplying in ant colonies.
- Analysis of financial data ([2]), in order to provide an agent-based hypothesis to the non-Gaussian statistical regularities that it shows (stylized facts).
- Opinion dynamics, to capture statistical properties of the vote distribution in bipartite elections ([3], [4]).

In addition to herding behavior, we will include a feature, often called in the literature "free will", which describes the probability for an individual to change randomly its state. These two elements compose the Kirman's model or, as it is called in opinion dynamics, the noisy voter model ; which has been studied in several topologies ([5]). The noisy voter model depends on one parameter, (a), which is the ratio between the random and the herding process. This model has been widely used in several fields like percolation theory or chemistry. With these two elements, the model has a finite size phase transition where the critical parameter goes to zero in the thermodynamic limit.

Another element that will have a lot of presence in our work is the mechanism of aging, where every individual is labeled with an age. This individual characteristic ranges from zero to infinity in the set of the positive integer numbers. The age measures the resistance of an agent to change its state, modifying the relative weight of the herding mechanism. In the noisy voter model with aging studied in [6], the older an agent is the less important is the herding mechanism for it.

The same concept of aging is also used in several disciplines (non-equilibrium statistical physics, chemistry...) and with several different meanings for it. One example is ecology, where the aging represents literally the age of the population and it is related to the mortality of a species.

Aging introduces important new features ([6]). It transforms the transition of the Kirman's model into a second order one (Ising-like), where the critical parameter is non-zero at the thermodynamic limit. This transition occurs between two different regimes: consensus around one of the states of the system or coexistence of both first order opinions.

There is a social intuition behind the behavior of the aging. The persistence in a state makes more difficult to change. We could make a social analogy considering that the states are two possible opinions of individuals in a population. The mechanism of aging could be translated as the statement: The more you keep an opinion, the more sure you are about it. This will lead to individuals that stay for long time in the same opinion, and convincing the rest to change their mind.

In this work we are going to explore also the opposite mechanism. An agent that stands for a long time with the same opinion is going to be very active interacting with other agents in order to change its state. We will call this modification *anti-aging* and check that the consensus can not be reached as the old agents do not last for enough time in the system.

As we have said, the role of the age in the dynamics is to provide a modification in the importance of the herding mechanism. It's more likely for an old agent to ignore the social interaction, so the old agents will become potential zealots. The strength of this zealots may help us to understand this phase transition. If they are strong enough, the zealots will compete and eventually reach a consensus, but if the random changes in the state are too frequent, zealots can not survive. It is natural to think that this phase transition depends on the number of potential zealots, this is, aged agents.
We are going to check here how this critical point that separates consensus and coexistence varies depending on the density of aged agents in a population. We will call this study on populations with aged and non-aged agents *partial aging*.

The work it's structured as follows: In chapter 2 we will analyze previous works to explain the noisy voter model and the role of the aging, summarizing previous results. In chapter 3 we will study the variations that have been described in the previous lines including a theoretical approximation and simulations. Finally, we will include a chapter of conclusions (4) and an appendix with the calculations that are not exposed during the previous chapters (5).

Chapter 2

State of the art

In this chapter we will present models that have been widely studied in the former literature, like the noisy voter model and the aging variation. We will explain the main conclusions that arise from these models as an introduction to the variations that we will do in the following chapters.

2.1 Noisy voter model

We begin by defining the noisy voter model (or Kirman's model). Let us consider N agents characterized by a binary individual variable: The state. In this work we will use two different notations for the two possible states ($s = \{0 \equiv -, 1 \equiv +\}$).Agents in our system interact pairwise with their neighbors, producing changes in their states. The dynamics can be described in a sequential way. At every time step :

- We choose an agent.
- We choose a mechanism of interaction. The agent can change its state via two different ways :
  - **Free will**, where the agent has a probability , a, of changing at random its state.
  - **Voter-like** interaction, where the agent selects randomly a neighbor and copies its state, hence the probability of copying a state is proportional to the number of neighbors that are in that state.

We present a diagram to summarize the process of the dynamics :

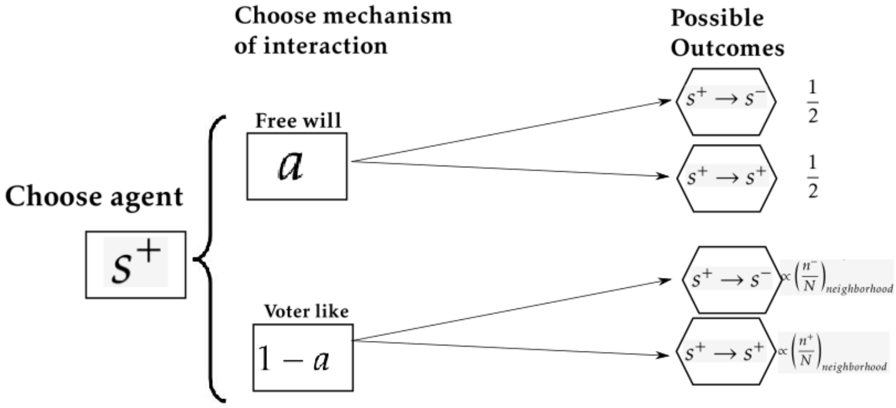


Figure 2.1: Scheme of the possible sources of interaction for an agent with state +

Where $n^{+/-}/N$ are the fraction of agents in each state in the subject's neighborhood. Collecting all the interactions we can build the microscopic transition rates as the probability of changing from one state to the other :

$$\begin{aligned} \omega_1 &\equiv P(s:0\rightarrow 1) = \frac{a}{2} + (1-a)\frac{n}{N}, \\ \omega_2 &\equiv P(s:1\rightarrow 0) = \frac{a}{2} + (1-a)\frac{N-n}{N}. \end{aligned} \tag{2.1}$$

And the global rates:

$$\begin{aligned} \Omega_1 &\equiv P(n\rightarrow n+1) = \frac{N-n}{N}\left(\frac{a}{2} + (1-a)\frac{n}{N}\right), \\ \Omega_2 &\equiv P(n\rightarrow n-1) = \frac{n}{N}\left(\frac{a}{2} + (1-a)\frac{N-n}{N}\right). \end{aligned} \tag{2.2}$$

With these global rates we can arrive to the master equation:

$$\frac{dp(n,t)}{dt} = (E-1)(\Omega_2 p(n,t)) + \left(E^{-1}-1\right)(\Omega_1 p(n,t)). \tag{2.3}$$

At this point, we would like to write the deterministic equation for the averages in this noisy voter model:

$$\frac{d<n>}{dt} =<\Omega_1> - <\Omega_2>. \tag{2.4}$$

It is easy to check that this averaged equation has only one stationary solution: $<n>= N/2$, this is, coexistence between both opinions.

We now introduce a change of variables $x = n/N$ and expand in powers of N^{-1} (the thermodynamic limit). With this procedure (which is standard , it can be checked in [7]) we arrive to the Fokker-Planck equation. Once we have arrived to the equation, we introduce another change of variable to the most natural unit in Ising-like models, the magnetization $m = 2(n/N)-1$. The time is also rescaled to Montecarlo units.The result for the Fokker-Planck and Langevin equations:

$$\begin{aligned} \frac{dp(m,t')}{dt'} &= \frac{\partial}{\partial m}\left[-amp(m,t') + \frac{\partial}{N\partial m}\Big((a+(1-a)(1-m^2))p(m,t')\Big)\right], \\ \frac{dp(m,t')}{dt'} &= -\frac{am}{2} + \sqrt{\frac{a}{2} + (1-a)\frac{1-m^2}{2}}\,\xi(t'). \end{aligned} \tag{2.5}$$

The solution to the stationary distribution of zero flux of the Fokker-Planck equation is :

$$P_{st}(m) = C\left(a + (1-a)(1-m^2)\right)^{\frac{2-a(N+2)}{2(a-1)}}. \tag{2.6}$$

Depending on the sign of the exponent, the probability distribution shows two different behaviors:

- **Coexistence**, where the probability distribution has one unique maximum at $m = 0$.
- **Polarization**, where the probability distribution has two maxima at $m = \pm 1$.

With simple algebra, we can see that the transition occurs at $a_{crit} = 2/(N+2)$. This means that it will disappear in the thermodynamic limit, and that is why we call it a finite size transition.

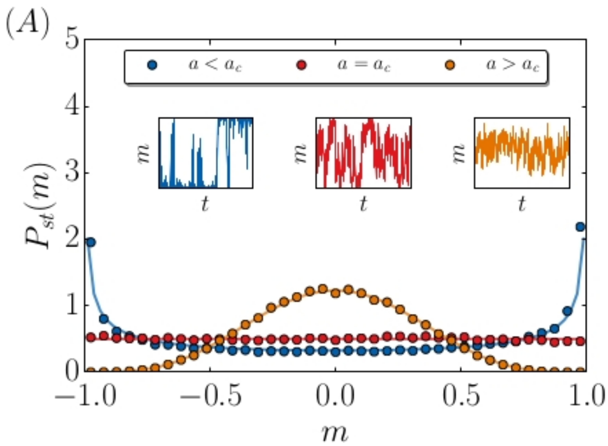
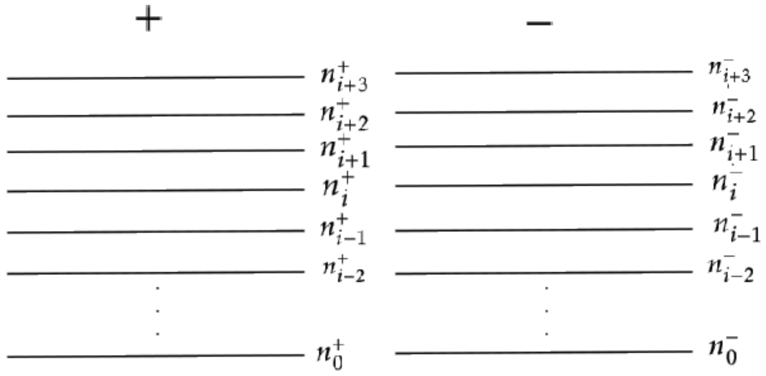


Figure 2.2: Stationary distribution for the three different behaviors in the noisy voter model, [6]

2.2 Aging

At this point we introduce the ”aging” structure. We assign to every agent one new label, called age, that measures the time between changes of states in MonteCarlo time units. This parameter changes between $i = 0$ when the agent has just changed state and grows while the agent sustains its opinion. We define then $s_i^{\pm}$ as an agent with state $\pm$ and age i.
The age also plays a role in the dynamics: When an agent meets another agent, the probability to change opinion via the voter update is $P(i) = 1/(i+2)$, so the longer an agent sustains its opinion the more difficult is for it to copy other opinions.
This means that we can group our whole population in subgroups depending on their age and state, like in a level structure. We call $n_i^{\pm}$ to the number of agents with state $\pm$ and age i.



As a consequence, we can divide our whole population of agents in state 1 (0) in subpopulations using its age. In a population of N agents:

$$n = \sum_{i=0}^{\infty} n_i^+ \qquad , \qquad N - n = \sum_{i=0}^{\infty} n_i^- . \tag{2.7}$$

The dynamics of this model can be explained sequentially. At each timestep:

- We select an agent.
- We select a mechanism of interaction. As in the noisy voter model there are two different paths:
  - **Free will:**Selected with probability a, the particle randomly chooses one of the two available states.
  - **Voter-like interaction:**Selected with probability $(1-a)$, the agent selects randomly one of its neighbors and copies its state. The innovation in this model is that the agent has the chance to ignore the interaction and remain in the same state with probability $P(i) = 1 - 1/(i+2)$. This process is called **aging activation** .
- If the agent remains with the same opinion at the end of the process its age is increased by one. If it changes its state, its age is reseted to zero.

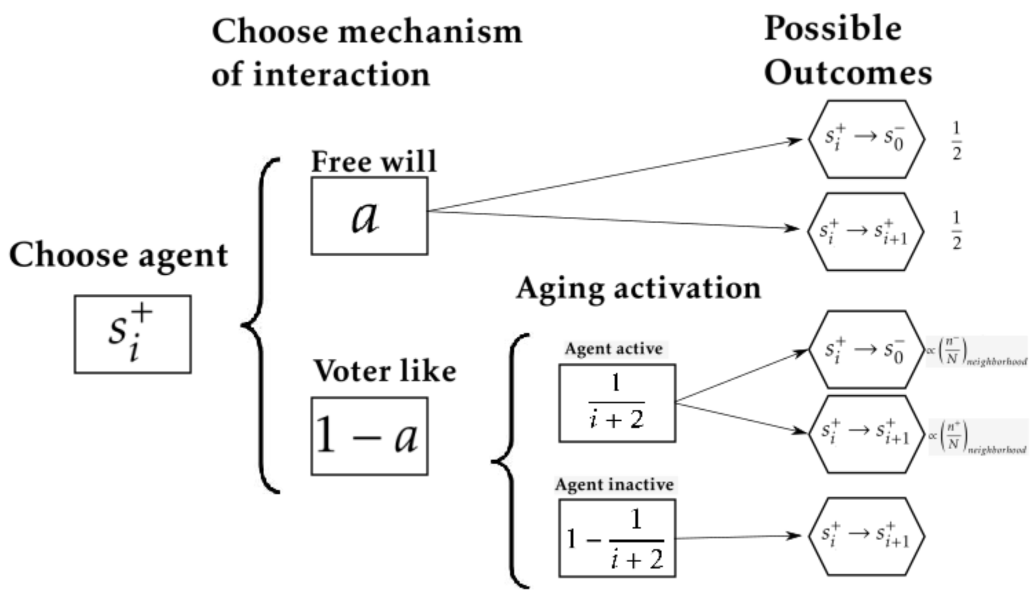


Figure 2.3: Scheme of the possible sources of interaction for the agent s_i^+

As the rates depend on the age, we build the global rates for every subpopulation of agents with age i:

- Rate for the subpopulation with age i in state 1(+) to change to 0(-) and reset one of the element's age.
$$\Omega_1(i) = n_i^+\left(\frac{a}{2} + \frac{1-a}{2+i}\frac{N-n}{N}\right). \tag{2.8}$$
- Rate for the subpopulation with age i in state 0(-) to change to 1(+) and reset one of the element's age.
$$\Omega_2(i) = n_i^-\left(\frac{a}{2} + \frac{1-a}{2+i}\frac{n}{N}\right). \tag{2.9}$$
- Rate for the subpopulation with age i in state 1(+) not to change and increase the age of one its agents by one.
$$\Omega_3(i) = n_i^+\left(\frac{a}{2} + \frac{(1-a)(1+i)}{2+i} + \frac{1-a}{2+i}\frac{n}{N}\right). \tag{2.10}$$
- Rate for the subpopulation with age i in state 0(-) not to change and increase the age of one its agents by one.
$$\Omega_4(i) = n_i^-\left(\frac{a}{2} + \frac{(1-a)(1+i)}{2+i} + \frac{1-a}{2+i}\frac{N-n}{N}\right). \tag{2.11}$$

Again, with these global rates we take into account the elementary processes to arrive to final master equation (see the details in the appendix 5.2, equations 5.11, 5.12 and on):

$$\begin{aligned}
\frac{dp(n_i^+,t)}{dt} &= (E^{-1}-1)(p(n_i^+,t)\Omega_3(i-1)) + (E-1)((p(n_i^+,t)(\Omega_1(i)+\Omega_3(i))),\\
\frac{dp(n_i^-,t)}{dt} &= (E^{-1}-1)(p(n_i^-,t)\Omega_4(i-1)) + (E-1)((p(n_i^-,t)(\Omega_2(i)+\Omega_4(i))),\\
\frac{dp(n_0^+,t)}{dt} &= (E^{-1}-1)\left(p(n_0^+,t)\sum_{i=0}^{\infty}\Omega_2(i)\right) + (E-1)((p(n_0^+,t)(\Omega_1(0)+\Omega_3(0))),\\
\frac{dp(n_0^-,t)}{dt} &= (E^{-1}-1)\left(p(n_0^-,t)\sum_{i=0}^{\infty}\Omega_1(i)\right) + (E-1)((p(n_0^-,t)(\Omega_2(0)+\Omega_4(0))).
\end{aligned} \tag{2.12}$$

With the global rates we build the equation for the averages:

$$\begin{aligned}
\frac{d<n_i^+>}{dt} &=<\Omega_3(i-1)>-<\Omega_3(i)>-<\Omega_1(i)>,\\
\frac{d<n_i^->}{dt} &=<\Omega_4(i-1)>-<\Omega_4(i)>-<\Omega_2(i)>,\\
\frac{d<n_0^+>}{dt} &= \sum_{i=0}^{\infty}<\Omega_2(i)>-<\Omega_3(0)>-<\Omega_1(0)>,\\
\frac{d<n_0^->}{dt} &= \sum_{i=0}^{\infty}<\Omega_1(i)>-<\Omega_4(0)>-<\Omega_2(0)>.
\end{aligned} \tag{2.13}$$

We study the stationary solutions of these equations. After setting the first and second equations in (2.13) equal to zero, we arrive to recursive relations that can be sumed up (see the details on 5.2 in equations 5.17 and following). Defining x as the fraction of agents in state 1(+) we arrive to:

$$\frac{x}{1-x} = \frac{< n_0^+ > \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,x)\right)}{< n_0^- > \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,1-x)\right)}. \tag{2.14}$$

Where:

$$A(j,x) = \left(\frac{a}{2} + \frac{1-a}{1+j}(j+x)\right). \tag{2.15}$$

Once the equilibrium is achieved, it has been proofed that the particles that reset their age are in average the same for both states ($< n_0^+ >=< n_0^- >$). If we apply it in 2.14:

$$\frac{x}{1-x} - \frac{\left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,x)\right)}{\left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,1-x)\right)} = 0 = f_1(a,x). \tag{2.16}$$

The roots for this equation will be the stationary solution of x given a. These roots can be found numerically obtaining:

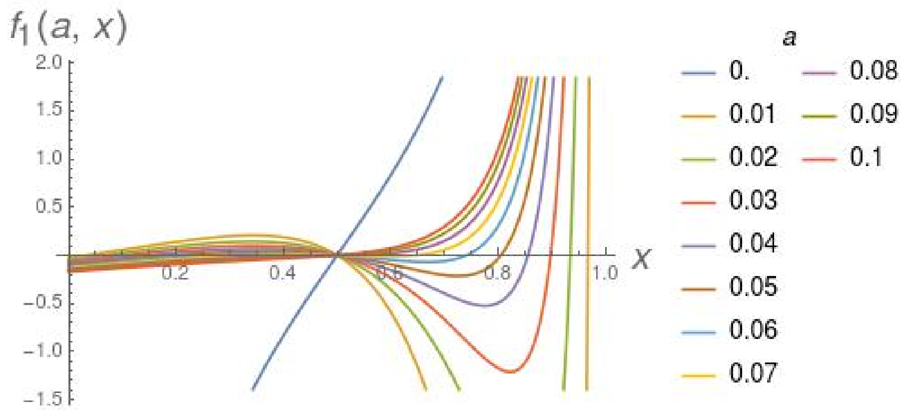


Figure 2.4: Representation of $f_1(a,x) = 0$ in the interval $x \ni \{0,1\}$

It can be seen from figure 2.4 that there is a transition from one stationary solution ($x = 0.5$), where the coexistence is stable, to three stationary solutions ($x < 0.5$,$x = 0.5$,$x > 0.5$), where coexistence becomes unstable and the system tends to polarize. This transition for the stationary distribution has been checked also by numerical simulations:

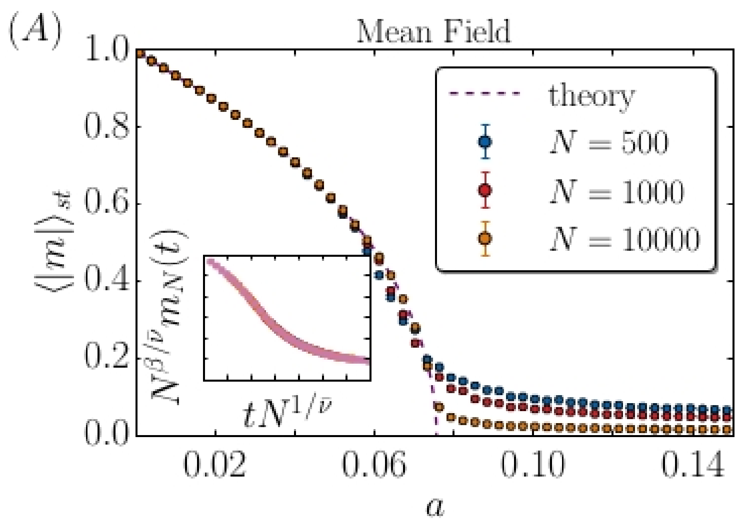


Figure 2.5: Magnetization as a function of the random parameter a, [6]

As a conclusion, the former finite size transition of the noisy voter model becomes a bona fide second order transition. A finite size scaling independent of the transition [6] using some measures like the divergence in the variance of m and the Binder cumulant have shown that this transition belongs to the Ising universality class.

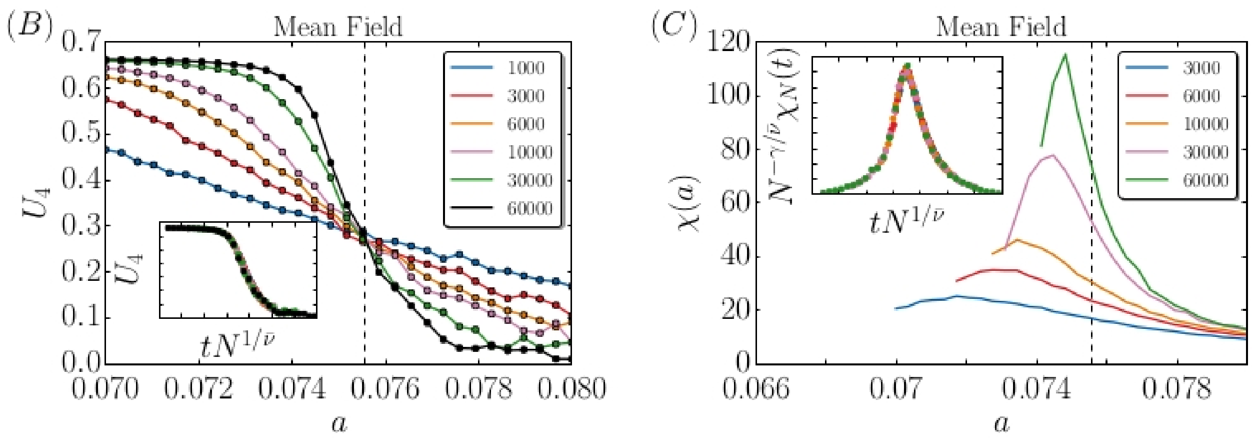


Figure 2.6: Variance and Binder cumulant in the aged noisy voter model, [6]

We define in figure 2.6 the susceptibility as $\chi = N\left(< m^2 > - < m >^2\right)$ and the Binder cumulant as $U_4 = 1 - < m^4 > / \left(3 < m^2 >\right)$.
The crossings of the Binder cumulant for different sizes allows us to obtain the critical parameter a_c and it can be compared, in the all-to-all connection, with a theoretical simile. The idea is that at the critical point of this phase transition, the derivatives in both sides must be equal in the stationary solution of x. This allow us to obtain the critical value for a (a_c) that separates the two phases (as it can be checked in 5.2 (5.22)).
The critical exponent may be extracted expanding in Taylor series the expression (2.16) around the stationary solution and the critical noise, obtaining that the first lower order that survive is $m \sim t^{1/2}$ where $t = 1 - a/a_c$ (as it can be checked in 5.2(5.23)).
Besides the change in the phase transition, the inclusion of aging generates a new order

parameter. We define the mean internal time of a population P:

$$\tau_P = \frac{\sum_i i < n_i^P >}{\sum_i < n_i^P >}. \tag{2.17}$$

Where n_i^P is the number of particles of age i of the population P.
For a realization of the dynamics, the system will reach a stationary value of the magnetization. If $m > 0$, the majority of the agents will be in the state +. If $m < 0$, the majority will be in state −. It has been found that the mean internal time difference between the majority (τ_{maj}) and the minority (τ_{min}) is the mentioned order parameter.

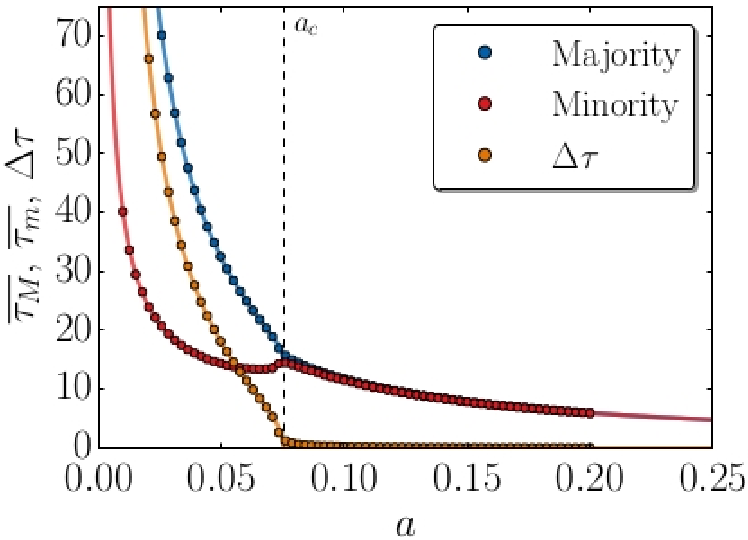


Figure 2.7: Majority and minority internal time as a function of a, [6]

As it can be seen in the figures above, the difference between the majority and the minority ($\Delta\tau$) has a non-zero value in the ferromagnetic phase and a vanishes in the paramagnetic phase.
Coming from this family of models, we are going to study two variations on this aging structure.

Chapter 3

Anti-aging and Partial aging

In this chapter we are going to study two variations of the former age structure. We have seen that in the usual model of aging, the probability of interacting with another agent decays as the age grows. The social simile for this behavior could be the feeling of accommodation, where the more an agent keeps an opinion, its more difficult for it to change.
The first variation that we will include is a modification on this aspect, considering what happens if we make the probability of interacting directly proportional to the age. The intuition behind this modification is to study an opposite social behavior: the willingness to change. This means that the more an agent keeps an opinion, the willing it is to interact with its neighbors.
In the second variation we will study how the second order phase transition appears in a population with two kinds of agents: Individuals without age and aged agents (with the usual way of age dynamics), that we will control using the density of aged agents (ρ) as a second parameter. We could understand this system as a composition of agents prone to accommodation and agents without trends.

3.1 Anti-aging in the noisy voter model

We define the anti-aging in the following way: There are N particles distributed in several levels of age, $i = 0, 1, \ldots \infty$ that evolve during the interaction. The N particles can be in two different states: $+, -$ is the notation that we will use during this text. The dynamics can be described in the following way. For each time step:

- An agent is selected.
- A mechanism of interaction is selected. As in the usual voter model, the particles may interact via:
  - **Free will:** Selected with probability a, the agent chooses at random one of the two states.
  - **Voter-like interaction:**With probability $(1-a)$, the agent will copy one of the states of its neighbors. The process of **aging activation** is also present here, but now the probability to be inactive is $P(i) = 1/(i+2)$ such that old agents have more probability to interact with other agents.

We collect all these behaviors in the following scheme:

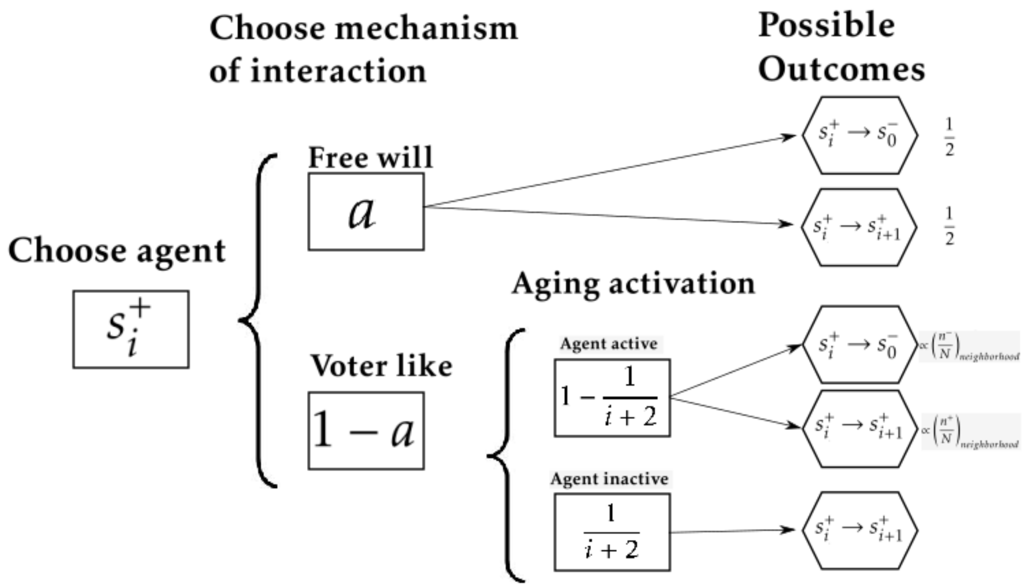


Figure 3.1: Scheme of the possible sources of interaction for an agent s_i^+

As we can see, the dynamics is the same as in the usual voter model in the sense that, either an agent change its opinion and resets its age to zero or it sustains its opinion and increases its age.
Collecting all the behaviors from the former diagram, we can build the local rates. After that, we can translate it to the global rate for an age/sign level multiplying by the number of particles on it.

$$\Omega_1(i) = n_i^+\left(\frac{a}{2} + \frac{(1-a)(1+i)}{2+i}\frac{N-n}{N}\right) \quad , \quad \Omega_2(i) = n_i^-\left(\frac{a}{2} + \frac{(1-a)(1+i)}{2+i}\frac{n}{N}\right),$$
$$\Omega_3(i) = n_i^+\left(\frac{a}{2} + \frac{1-a}{2+i}((1+i)\frac{n}{N}+1)\right) \quad , \quad \Omega_4(i) = n_i^-\left(\frac{a}{2} + \frac{1-a}{2+i}((1+i)\frac{N-n}{N}+1)\right). \tag{3.1}$$

With this global rates for an age/sign level, we analyze the processes that may happen in a differential time, in order to build the master equation. As we are not taking into account the functional form of the global rates, the result is exactly the same as in the usual aging (as it can be seen in 5.3 (5.24,5.25)):

$$\frac{dp(n_i^+)}{dt} = (E^{-1}-1)((p(n_i^+)\Omega_3(i-1)) + (E-1)((p(n_i^+)(\Omega_1(i)+\Omega_3(i))),$$
$$\frac{dp(n_0^+)}{dt} = (E^{-1}-1)((p(n_0^+)\sum_0^\infty \Omega_2(i)) + (E-1)((p(n_0^+)(\Omega_1(0)+\Omega_3(0))),$$
$$\frac{dp(n_i^-)}{dt} = (E^{-1}-1)((p(n_i^-)\Omega_4(i-1)) + (E-1)((p(n_i^-)(\Omega_2(i)+\Omega_4(i))),$$
$$\frac{dp(n_0^-)}{dt} = (E^{-1}-1)((p(n_0^-)\sum_0^\infty \Omega_1(i)) + (E-1)((p(n_0^-)(\Omega_2(0)+\Omega_4(0))). \tag{3.2}$$

As the master equation is the same, again, the averaged equations are also the same as in the usual voter model with aging. The difference in the anti-aging appears when we find the stationary solutions for this averaged equations, as we substitute the functional form.

Making the derivative equal to zero we arrive to:

$$
\begin{aligned}
n_i^+ &= n_{i-1}^+\left(\frac{a}{2}+\frac{1-a}{1+i}\left(1+\frac{in}{N}\right)\right),\\
n_i^- &= n_{i-1}^-\left(\frac{a}{2}+\frac{1-a}{1+i}\left(1+\frac{i(N-n)}{N}\right)\right),\\
n_0^+ &= \sum_{i=0}^{\infty} n_i^-\left(\frac{a}{2}+\frac{(1-a)(1+i)}{2+i}\frac{n}{N}\right),\\
n_0^- &= \sum_{i=0}^{\infty} n_i^+\left(\frac{a}{2}+\frac{(1-a)(1+i)}{2+i}\frac{N-n}{N}\right).
\end{aligned}
\tag{3.3}
$$

As in the usual voter model, from the first two equations in (3.3) we can see a recursive relation, with a different functional form. If we define:

$$n_i^+ = n_{i-1}^+ B(i,x). \tag{3.4}$$

Where B(i,x) is:

$$B(i,x) = \left(\frac{a}{2}+\frac{1-a}{1+i}(1+ix)\right). \tag{3.5}$$

Where again, $x = n/N$ is the fraction of the population in state (+). We can make the finite sum of several levels of age sign:

$$
\begin{aligned}
n_i^+ &= n_0^+\prod_{j=1}^{i} B(j,x),\\
n_i^- &= n_0^-\prod_{j=1}^{i} B(j,1-x).
\end{aligned}
\tag{3.6}
$$

And, as we did in the usual voter model (follow from 5.19 and on), we can sum up to infinity and proceed to solve it with symbolic calculus. In particular, we have to mention that all the symbolic calculus has been solved using the software *Wolfram Mathematica*.

$$n = \sum_{i=0}^{\infty} n_i^+ = n_0^+\left(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i} B(j,x)\right), \tag{3.7a}$$

$$N-n = \sum_{i=0}^{\infty} n_i^- = n_0^-\left(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i} B(j,1-x)\right). \tag{3.7b}$$

$$
\begin{aligned}
&\prod_{j=1}^{i} B(j,x) = \frac{\left(a\left(\frac{1}{2}-x\right)+x\right)^{i}\left(\frac{-2ax+2x+2}{-2xa+a+2x}\right)_i}{\Gamma(i+2)},\\
&\left(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i} B(j,x)\right) = \frac{2^{\frac{-2ax+2x+2}{-2ax+a+2x}}\left(a(2x-1)-2x+2\right)^{-\frac{-4ax+a+4x+2}{-2ax+a+2x}}}{a-2}+\\
&+\frac{\left(-a^2(1-2x)^2+4a\left(2x^2-3x+1\right)+2^{-\frac{2-a}{-2ax+a+2x}}\left(2ax-a-2x+2\right)^{\frac{-4ax+a+4x+2}{-2ax+a+2x}}-4x^2+8x-4\right)}{a-2}.
\end{aligned}
\tag{3.8}
$$

Where $(k)_i$ is the Pochhammer symbol of k. It is defined as :

$$(k)_i = \frac{\Gamma(k+i)}{\Gamma(k)}. \tag{3.9}$$

Where $\Gamma(k)$ is the Gamma function.

Again, we proceed dividing 3.7a by 3.7b and using the condition $< n_0^+ >=< n_0^- >$. Although this condition has not been proofed for the anti-aging during this work, it is reasonable to think that in the equilibrium state the probability to reset its age is the same for each state. We will see that this condition makes the theory match the simulations . We arrive to the expression:

$$\frac{x}{1-x} - \frac{\left(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i} B(j,x)\right)}{\left(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i} B(j,1-x)\right)} = 0 = f_2(a,x)$$

The roots of this expression will be the stationary solutions of the magnetization .
There is another important parameter in the aged noisy voter model, as we stated, the mean age.

$$\begin{aligned}
\overline{\tau^+} &= \frac{\sum_i i < n_i^+ >}{\sum_i < n_i^+ >} = \frac{\left(1+\sum_{i=1}^{\infty} i\prod_{j=1}^{i} B(j,x)\right)}{\left(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i} B(j,x)\right)}, \\
\overline{\tau^-} &= \frac{\sum_i i < n_i^- >}{\sum_i < n_i^- >} = \frac{\left(1+\sum_{i=1}^{\infty} i\prod_{j=1}^{i} B(j,1-x)\right)}{\left(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i} B(j,1-x)\right)}.
\end{aligned} \tag{3.10}$$

In the following we proceed to show the results of this theoretical approximation together with the simulations .

3.1.1 Results

Preliminaries

We present here some specifications for the numerical simulations that will be shown below. We have studied in the next section two different configurations for this dynamics: An all-to-all connection and a 2D lattice. We will compare the all-to-all connection theory (developed before) and the numerics. We will continue with the 2D lattice results. We have checked that $t_{end} = 2000$ is enough time for the system to arrive to a stationary state. We have used one thousand trajectories per point. The initial conditions are set as all particles with age zero, equally splitted in positive and negative.

Results

Coming from the former section, the first result that we are going to compare is the stationary solution of the magnetization. We will compare the roots of $f_2(a,x)$ with its simulated simile.

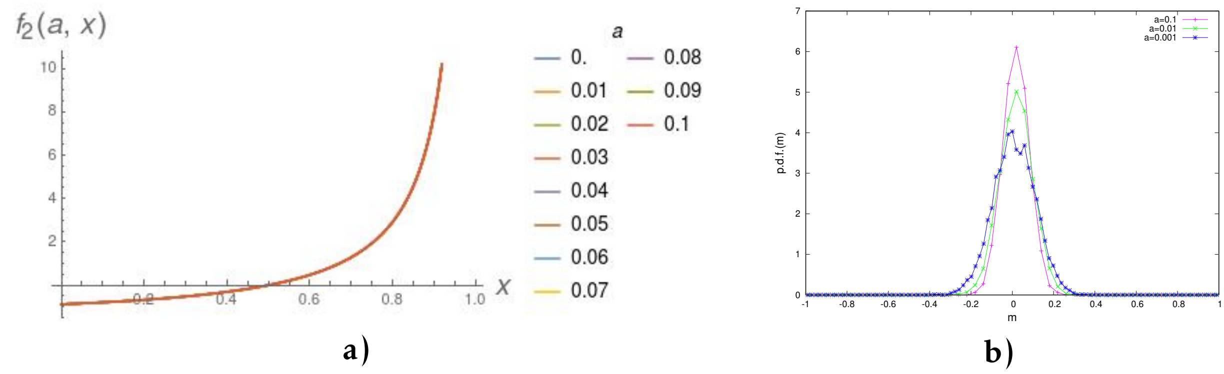


Figure 3.2: **a)** Representation of $f_2(a,x)$ for the interval $x \ni \{0,1\}$, **b)** Histogram of trajectories for $a = 0.1, 0.01, 0.001$

Note that the graph on the left is plotted in the variable $x = n/N$ and the second one in $m = 2x - 1$.

We can conclude that this rule for anti-aging only provides one stationary state: Coexistence between states. As a consequence of the proportional relation between the age and the probability to make a voter-like interaction, an old agent (with an i large) has a great probability to interact with another agent of the network, resetting its age. This is completely different to the aged voter model, where an agent refuses the social interaction as it becomes older, having more probability to sustain its state. To sum up, we could say that as our anti-aged agents become old they are more likely to interact in a system where the noise plays a role towards coexistence of states, independently of the age.

With the former interpretation one could think that, as the old agents only contribute to the social interaction, the initial conditions can be important. But there is an irreversible effect of the free will effect that makes the initial conditions not really relevant.

We can see in the numerical resolution of the equation that independently on the value of a, there is an unique root. This can be checked also by our numerical methods. In the following we present a graph of the magnetization versus the free will parameter a:

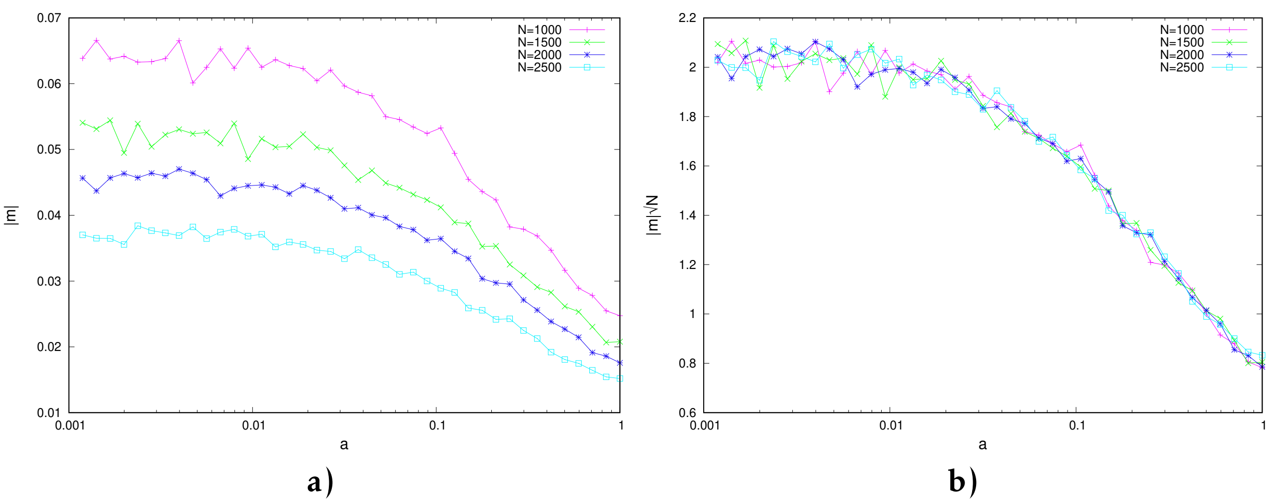


Figure 3.3: **a)**Magnetization in the steady state: all-to-all connection, **b)** Scaling of the magnetization with the numerical fluctuations

This one and some more graphs in this chapter are presented is logarithmic scale in the x axis. That's because we wanted to present the behavior at several scales and equally spaced.

As a conclusion for figure 3.3, we can say that as the number of particles is increased, the stationary magnetization tends to zero. The statistical fluctuations are reduced as the square root of the number of particles. This is the reason why we scale the magnetization.
Although the transition is destroyed, one could ask what is the behavior of the age structure and the mean age (order parameter in the aged voter model) in our problem. As it is expected, the mean age does not show any dramatic change nor a phase transition:

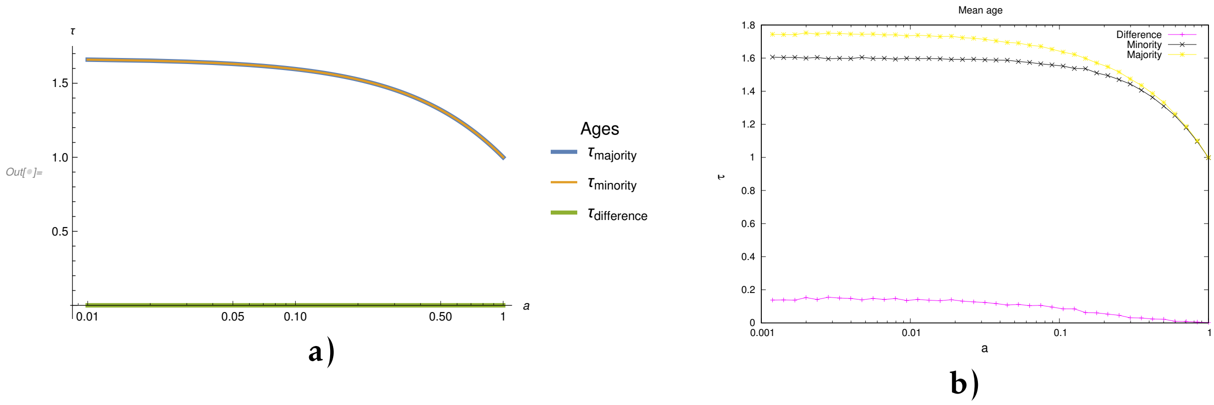


Figure 3.4: **a)** Theoretical dependence on mean age difference, **b)** Numerical dependence on mean age difference

The first interesting thing is that the mean age for both states is the same as the result of the coexistence. As a logical consequence of the destruction of the transition, the difference in the mean age is no longer an order parameter.
The distribution of the age also shows the inexistence of a transition. While in the usual voter model we can check a difference between the phases in the distributions (exponential-like and Poissonian-like) in the anti-aging it we can see that the distribution is independent of a.

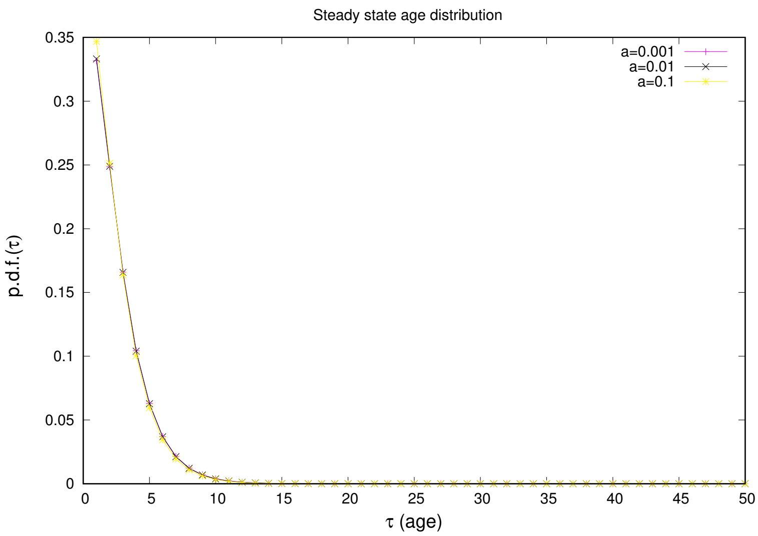
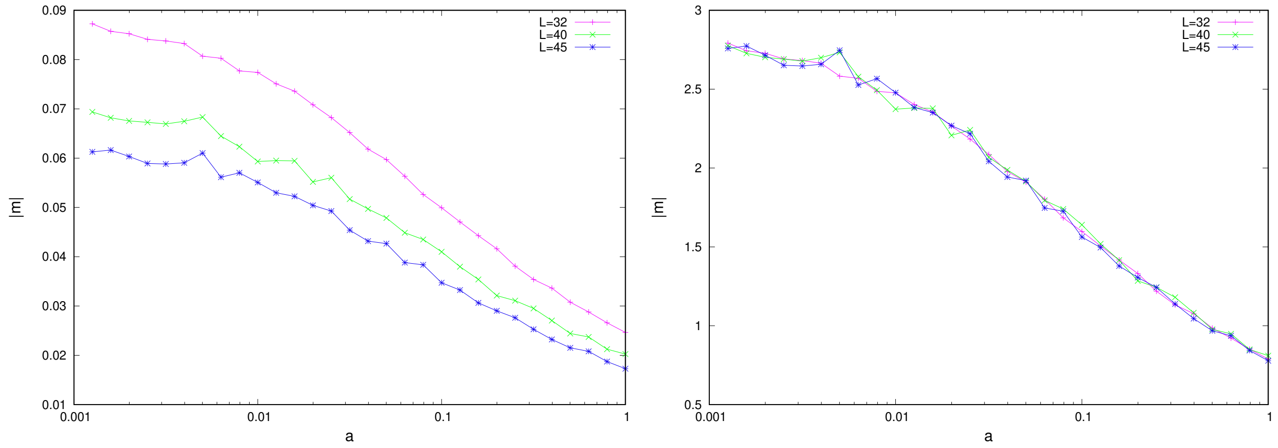


Figure 3.5: Age distribution in the steady state for $a = 0.1, 0.01, 0.001$

We also have studied this dynamics in a 2D lattice. We have considered three lattice sizes: $L = 32, 40, 45$ obtaining for the magnetization in the stationary states the following results:



a) **b)**

Figure 3.6: **a)**Magnetization in the steady state: 2D lattice, **b)** Scaling of the magnetization with the numerical fluctuations

As we can see from figure 3.6 the magnetization tends to zero as the number of particles is increased, reaching the zero in the thermodynamic limit. In order to check this, we have made a scaling with the square root of the number of particles.

Both structures, the all-to-all connection and the 2D lattice, tend to coexistence as the number of particles is increased. Therefore, the disappearance of the phase transition depends exclusively on the dynamics of the anti-aging and not on the particular structure.

3.2 Partial aging in the noisy voter model

The partial aging is defined in the following way: We have a population of N particles that can be in two different states $(1,0)$. Again, we call n the fraction of agents in the 1 state. The peculiarity here is that we have two kinds of agents:

- Agents that are only characterized by their state, without any age effect. They interact as in the usual noisy voter model.
- Agents that are under the effects of normal aging, this is, the probability of social interaction is a function of the age level whether they are. This function is:

$$f(i) = \frac{1}{i+2}.$$

We define $n_i\pm$ as in the former models: the number of particles with age i in the state $\pm$. We will also define $n_x^{\pm}$ as the number of particles without age in state $\pm$.
As we have two kinds of agents, we introduce the parameter ρ, the density of agents with age. Again, the dynamics can be explained sequentially. For each timestep:

- An agent is selected. The probability that an agent with age has been selected is ρ.
- A mechanism of interaction is selected. As in every variation, there are two mechanisms of interaction:

- **Free will:** With probability a, the agent chooses randomly on of the two states.
- **Voter-like interaction:** With probability $1-a$, the agent copies one of its neighbors states. At this important, it's important to know if the agent has age. In that case, there is an **aging activation** such that the probability that the agent ignores the voter interaction is $p(i) = 1 - 1/(i+2)$.

We present a scheme to summarize all possible outcomes:

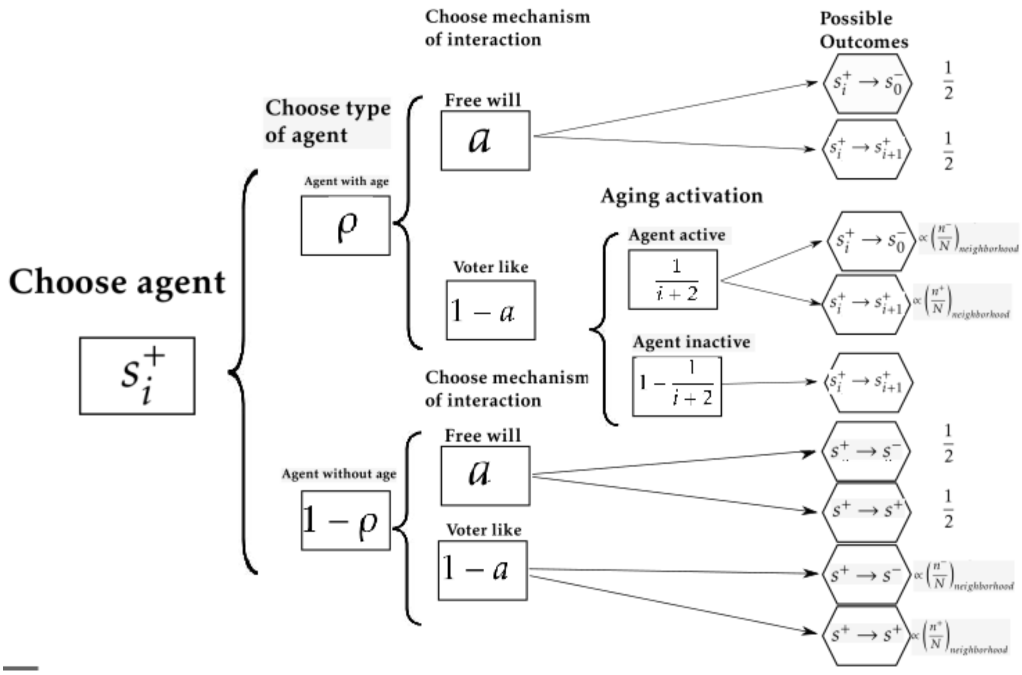


Figure 3.7: Scheme of the possible sources of interaction for an agent

We can see that, as before, an agent can interact sustaining its opinion and increasing its age or it can change opinion resetting its age. Our goal is to build the global rates; as there are two separated types of agents, we will find the same equations as the voter model with aging, multiplied by a factor ρ. There are two more equations for the particles without aging. These equations are the normal noisy voter model rates multiplied by the factor $(1-\rho)$:

$$
\begin{aligned}
\Omega_1(i) &= \rho n_i^+ \left(\frac{a}{2} + \frac{(1-a)}{2+i} \frac{N-n}{N} \right) \quad , \quad \Omega_2(i) = \rho n_i^- \left(\frac{a}{2} + \frac{(1-a)}{2+i} \frac{n}{N} \right), \\
\Omega_3(i) &= \rho n_i^+ \left(\frac{a}{2} + \frac{1-a}{2+i} (\frac{n}{N} + 1 + i) \right) \quad , \quad \Omega_4(i) = \rho n_i^- \left(\frac{a}{2} + \frac{1-a}{2+i} (\frac{N-n}{N} + 1 + i) \right), \\
\Omega_5 &= (1-\rho) n_x^+ \left(\frac{a}{2} + (1-a) \frac{N-n}{N} \right) \quad , \quad \Omega_6 = (1-\rho) n_x^- \left(\frac{a}{2} + (1-a) \frac{n}{N} \right).
\end{aligned}
\tag{3.11}
$$

Again, coming from the global rates we proceed to build the master equations. For this, we have to consider the elementary processes that may occur in a differential time. As our particles are totally splitted between particles with age and particles without age, these fundamental processes are the same that in the former case, and the equations we arrive are functionally the same. This means that the master equation for our population are:

$$\begin{aligned}
\frac{dp(n_i^+,t)}{dt} &= (E^{-1}-1)((p(n_i^+,t)\Omega_3(i-1)) + (E-1)((p(n_i^+,t)(\Omega_1(i)+\Omega_3(i))),\\
\frac{dp(n_i^-,t)}{dt} &= (E^{-1}-1)((p(n_i^-,t)\Omega_4(i-1)) + (E-1)((p(n_i^-,t)(\Omega_2(i)+\Omega_4(i))),\\
\frac{dp(n_0^+,t)}{dt} &= (E^{-1}-1)((p(n_0^+,t)\sum_0^\infty \Omega_2(i)) + (E-1)((p(n_0^+,t)(\Omega_1(0)+\Omega_3(0))),\\
\frac{dp(n_0^-,t)}{dt} &= (E^{-1}-1)((p(n_0^-,t)\sum_0^\infty \Omega_1(i)) + (E-1)((p(n_0^-,t)(\Omega_2(0)+\Omega_4(0))),\\
\frac{dp(n_x^+,t)}{dt} &= (E^{-1}-1)(p(n_x^+,t)\Omega_5) + (E-1)((p(n_i^+,t)\Omega_6),\\
\frac{dp(n_x^-,t)}{dt} &= (E^{-1}-1)(p(n_x^-,t)\Omega_6) + (E-1)((p(n_i^-,t)\Omega_5).
\end{aligned} \tag{3.12}$$

With this master equation we can define the average value equation and find the stationary solution, as we did in former sections. The stationarity condition for n_x^+ is the same as for n_x^- we come out with five stationarity conditions :

$$\begin{aligned}
n_i^+ &= n_{i-1}^+\left(\frac{a}{2} + \frac{1-a}{1+i}\left(i+\frac{n}{N}\right)\right),\\
n_i^- &= n_{i-1}^-\left(\frac{a}{2} + \frac{1-a}{1+i}\left(i+\frac{(N-n)}{N}\right)\right),\\
n_0^+ &= \sum_{i=0}^\infty n_i^-\left(\frac{a}{2} + \frac{1-a}{2+i}\frac{n}{N}\right),\\
n_0^- &= \sum_{i=0}^\infty n_i^+\left(\frac{a}{2} + \frac{1-a}{2+i}\frac{N-n}{N}\right),\\
n_x^+\left(\frac{a}{2} + (1-a)\frac{N-n}{N}\right) &= n_x^-\left(\frac{a}{2} + (1-a)\frac{n}{N}\right).
\end{aligned} \tag{3.13}$$

So we have two equations for the two variables (n_x^+, n_x^-) . If we make the substitutions then we obtain:

$$\begin{aligned}
\frac{n_x^+}{N} &= \left(\frac{a}{2} + (1-a)\frac{n}{N}\right)(1-\rho) = g(x,\rho),\\
\frac{n_x^-}{N} &= \left(\frac{a}{2} + (1-a)\frac{N-n}{N}\right)(1-\rho) = g(1-x,\rho).
\end{aligned}$$

With this information we will procced now with the infinite sum of the age levels. We again have to use the recursive relation, which again we can write as a function depending on (i,x,a):

$$A(i,x,a) = \left(\frac{a}{2} + \frac{1-a}{1+i}(i+x)\right).$$

Such that:

$$n_i^+ = n_0^+\prod_{j=1}^{i} A(j,x,a),$$

$$n_i^- = n_0^-\prod_{j=1}^{i} A(j,1-x,a).$$

And we can write the entire sum of age levels as:

$$\sum_{i=0}^{\infty} n_i^+ = n - n_x^+ = n_0^+ \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,x,a)\right),$$
$$\sum_{i=0}^{\infty} n_i^- = N - n - n_x^- = n_0^- \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,1-x,a)\right). \tag{3.14}$$

If we divide both expressions, assuming that the random effect affects in the same way both states ($< n_0^+ >=< n_0^- >$). We make both equations from 3.14 depend on $x = n/N$. Finally, dividing the first by the second, we arrive to:

$$\frac{x - g(x,\rho,a)}{1 - x - g(1-x,\rho,a)} = \frac{\left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,x,a)\right)}{\left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,1-x,a)\right)}. \tag{3.15}$$

$$\frac{x - g(x,\rho,a)}{1 - x - g(1-x,\rho,a)} - \frac{\left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,x,a)\right)}{\left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,1-x,a)\right)} = 0 = f_3(a,x,\rho). \tag{3.16}$$

So the roots for this equality will be the stationary solutions for the parameter $< x(\rho,a) >$. In the following we will present the results for the theoretical part and the simulations for this dynamics.

3.2.1 Results

Preliminaries

We use these preliminaries to show some specifications about the simulations that are going to be shown below. As we will see, the stationary solution for the system depends on two parameters: The density (ρ) and the free will (a) parameter. We will show 2D density graphs that are produced with the following initial conditions: If we have a density ρ of aged agents, we assign ρN agents equally distributed in the level of age ($i = 0$) and $(1-\rho)N$ non aged agents equally distributed.
We also have to fix the distribution of aged agents. They will be chosen at random from the whole population.
In the partial aging, there is a real process of thermalization, so we wait for it the amount of $3 \cdot 10^6$ interactions which is enough to achieve stability.

Results

We begin studying the all-to-all connection. The first graph that we will present is a density graph of the stationarity solutions for $< x(\rho,a) >$ with 1000 particles. We obtain a theoretical result from the solution of 3.16 and compare it with the simulations.

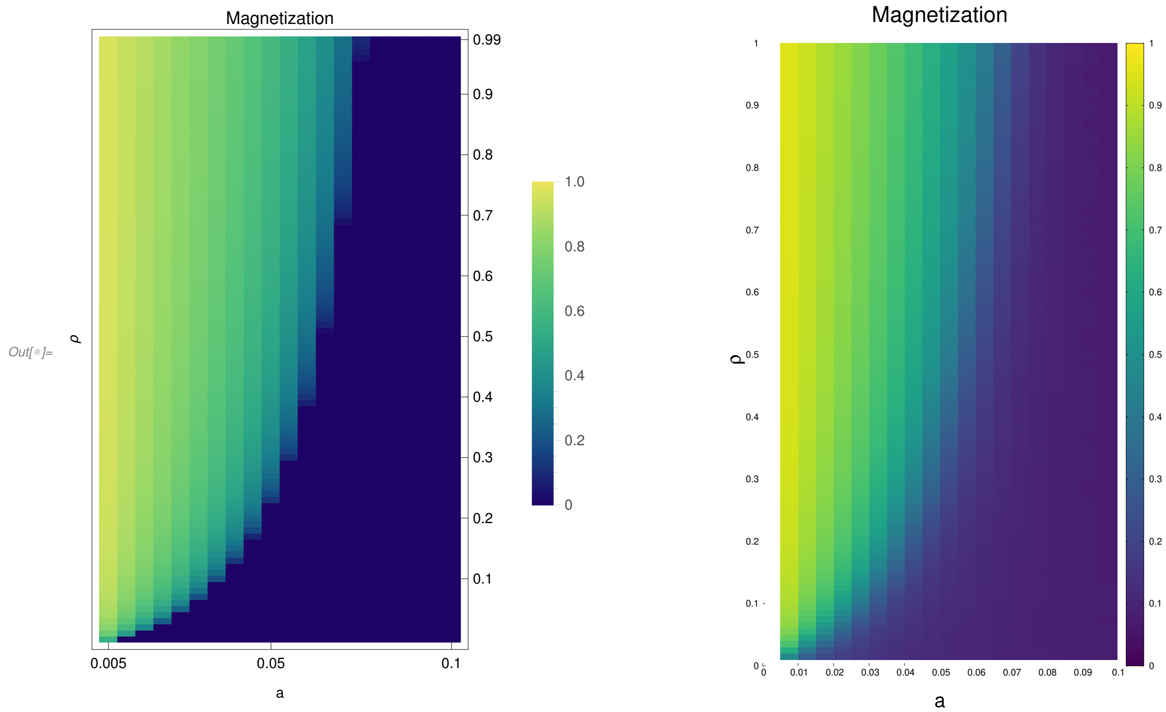


Figure 3.8: Theoretical and simulated density graph of magnetization: All-to-all connection

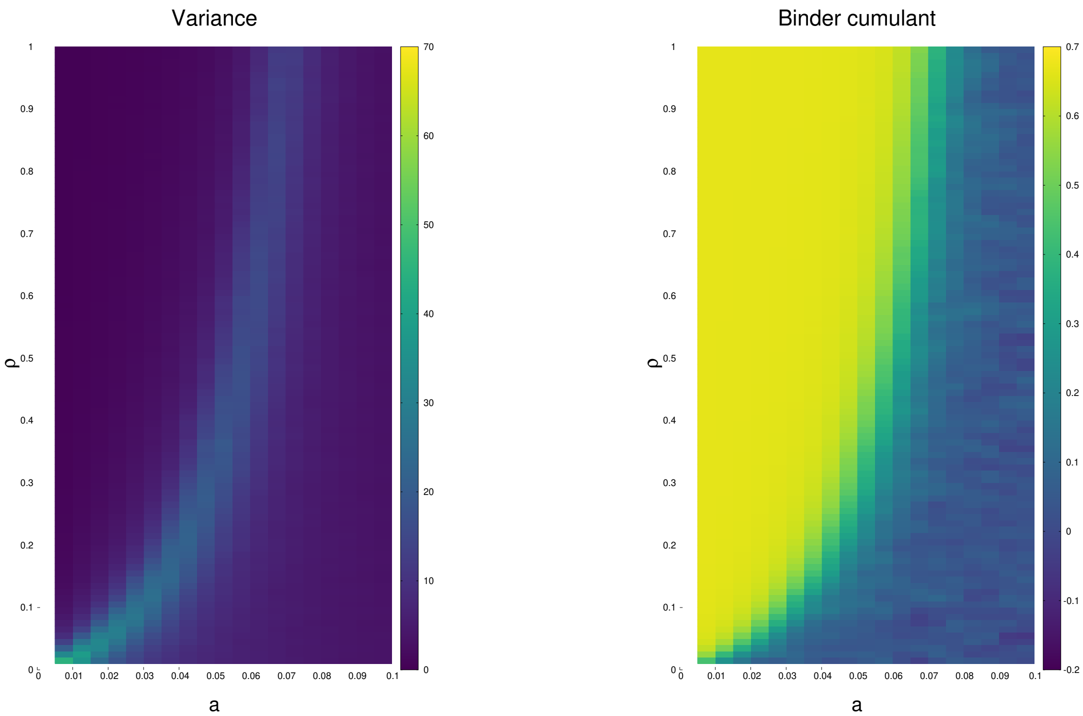


Figure 3.9: Simulated density graph: Variance and Binder cumulant (all-to-all connection)

We can see in figures 3.8 and 3.9 a transition between two regimes: Coexistence of both states ($|m| \simeq 0$) and polarization ($|m| > 0$) of the system towards one of them.
The two regimes of the magnetization, the maximum variance and the decay of the Binder

cumulant tell us where is the critical zone in the (ρ, a) plane. As we can see there is no critical density in order to obtain the aging characteristic phase transition. The critical parameter a_c is a continuous function of the density.

There is another magnitude that is also used as order parameter in the original transition of the voter model with aging, the mean age difference of the population $(\tau_{maj} - \tau_{min})$, that was presented in the former section (2.17). These are the density graphs we obtained for this parameter:

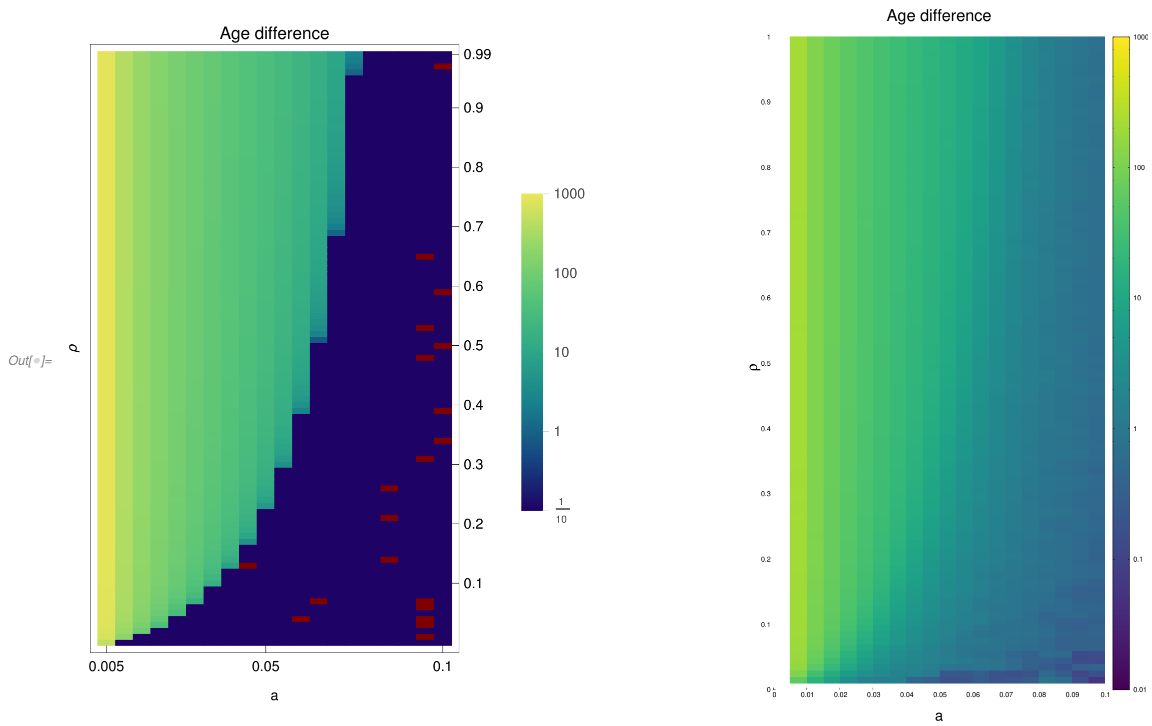


Figure 3.10: Theoretical and simulated density graph of mean age difference

As it was expected, the mean age difference depends strongly on the free will parameter a. But we can see in figure 3.10 that the theoretical and numerical predictions coincide, and conclude that these magnitude (τ_{dif}) can be used to identify the phase in the (ρ, a) plane.

We have also studied these dynamics in a 2D lattice in comparison to the all-to-all connection. Although we can not make the theoretical simile in this case we can check the existence of the phase transition and the existence of a critical region that depends on a. We present density graphs for a $32x32$ square lattice:

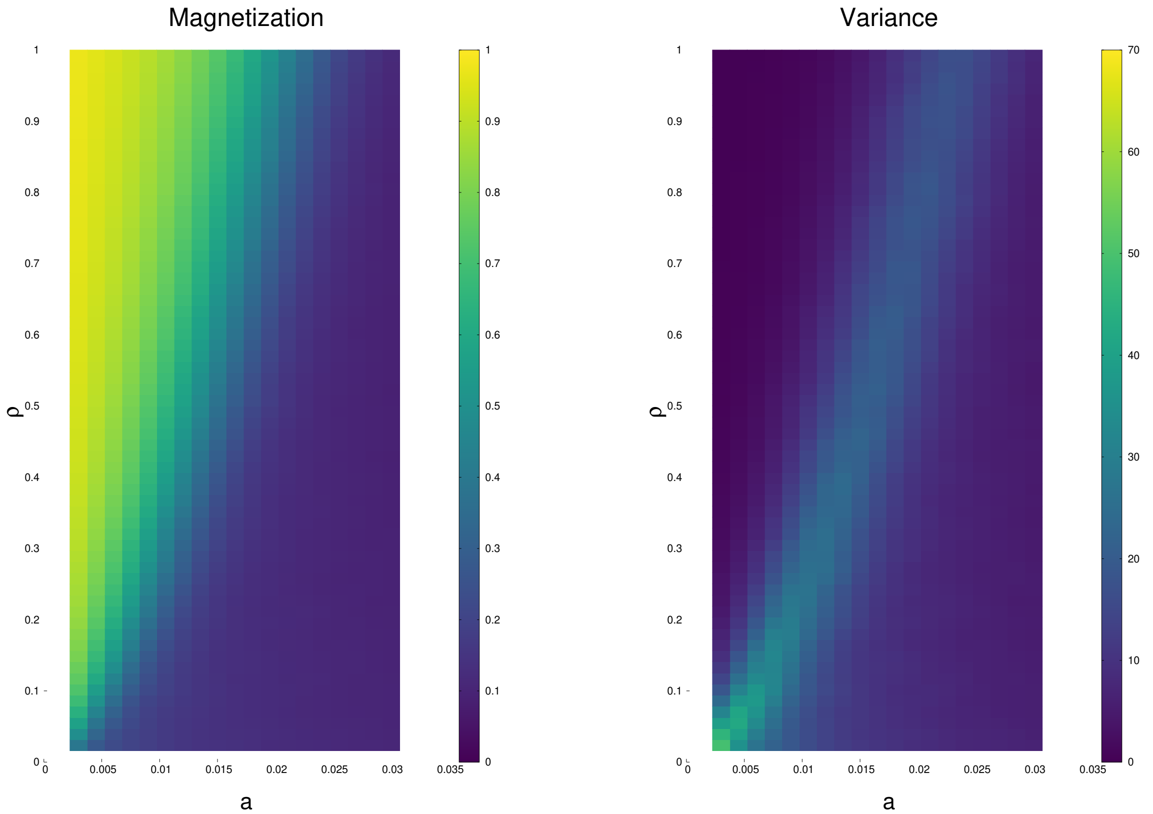


Figure 3.11: Magnetization and variance in a 32x32 lattice

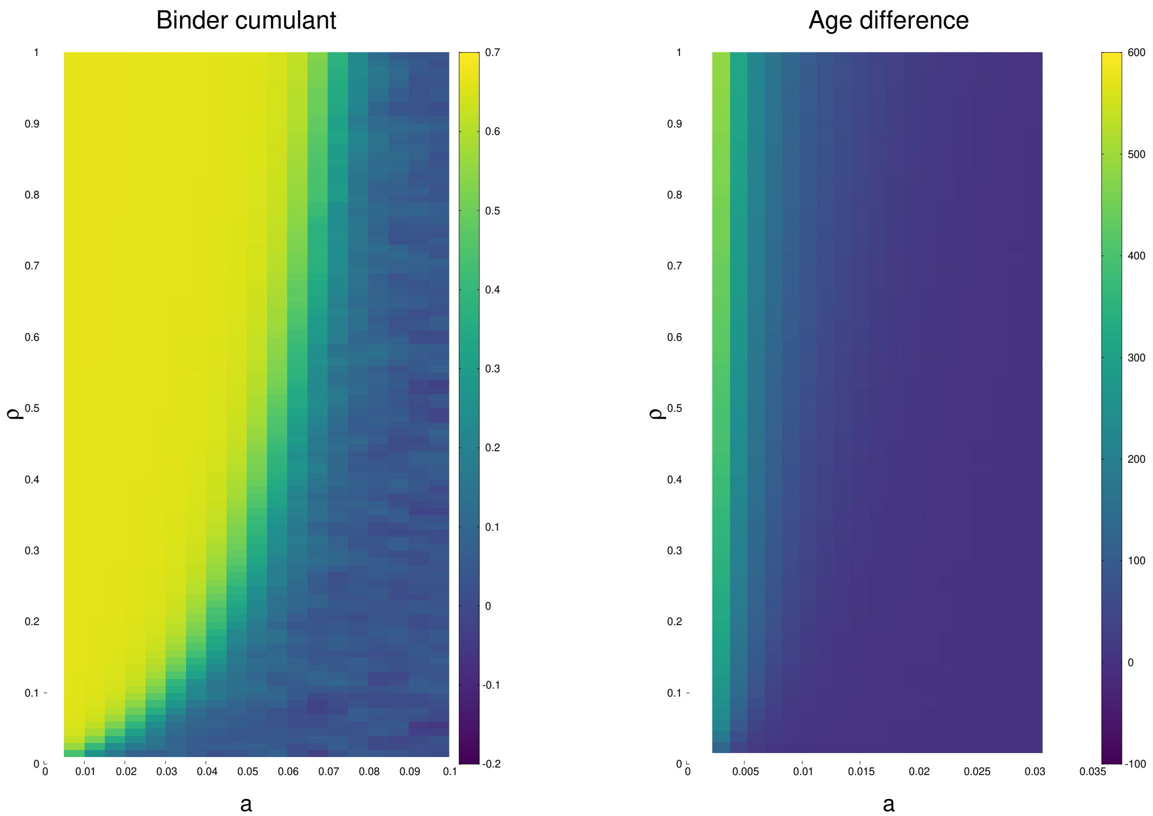


Figure 3.12: Binder cumulant and age difference in a 32x32 lattice

We can see that there is not a qualitative change in the fact of including a lattice. The main difference between the previous case and this one is that, taking magnetization as the order parameter, the functional form that relates the critical parameter (a_c) with the density of aged agents is different. But, overall, the phase transition is sustained as expected, letting us distinguish between coexistence and consensus.

Coming back to the all-to-all connection, we can use the theoretical expression $f_3(a, \rho, x) = 0$ to find $a_c = f(\rho)$. In fact, the derivatives have to be equal at $x = 0.5$ in equation (3.16) . We can solve these conditions with symbolic calculus, obtaining the functional form for $a_c = f(\rho)$. The analytical expression is very long so we just plot it, arriving to the following picture:

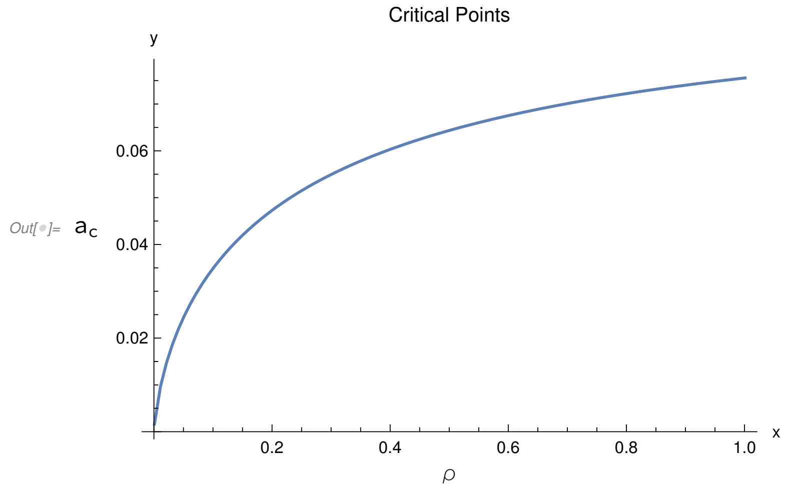


Figure 3.13: Theoretically calculated critical points as function of the ρ and a.

To expose clearly that the phase transition is sustained we are going to present a concrete case, $\rho = 0.2$. With our theoretical background, we can compute that the $a_{crit} = 0.0472754$. We are going to check, using the Binder cumulant, if the simulations match the theory. We present, in the plots below, the magnetization, variance, Binder cumulant and age difference for $1000, 1500, 2000$ and 2500 agents .

It can be seen in the Binder cumulant graph that the trajectories for several sizes cross in the margin of $a_{crit} = 0.0475 \pm 0.0025$, what corresponds to the value we had predicted.
We can conclude that the theoretical approximation is an interesting tool to analyze the displacement of the critical parameter as a function of the density of aged agents.

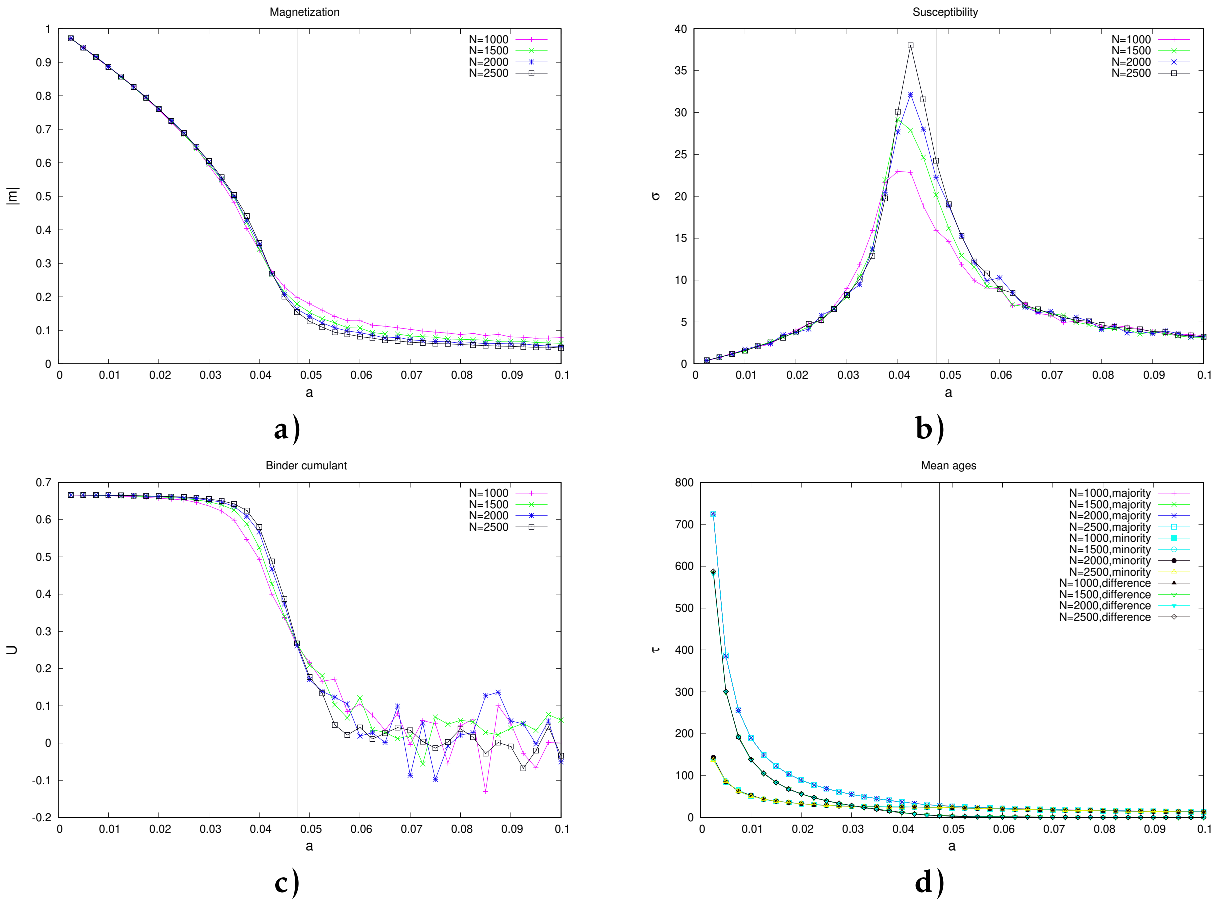


Figure 3.14: All-to-all connection, $\rho = 0.2$: **a)** Magnetization, **b)** Variance, **c)** Binder cumulant, **d)** Mean age for the majority,minority and the difference between them; as a function of a.

Chapter 4

Conclusions

In this work we have proposed two variations on the standard model with aging:

- **Anti-aging:** When the probability of interacting with the population is proportional to the age, the transition that has been found in the usual aging is destroyed. This can be explained because the role of the oldest elements in the system changes completely. In the voter model with aging these agents have more probability to sustain its opinion for a long time , and the competition between these old agents is what breaks the symmetry of the system. In this modification, the older an element is the more it interacts. Favoring the social interaction in a system that is affected by the random noise just biases our system towards coexistence.
  We have also checked that this result is not affected by the all-to-all premise, as the results are practically the same in a 2D square lattice.

- **Partial aging:** We have also studied how are the intermediate phases between the noisy voter model and the noisy voter model with aging introducing a density of aged agents (ρ). We have checked theoretically and with simulations that the aging-induced transition is sustained, and it is continuous. This means that for any density of aged agents it exists an a_{crit} that separates the phases. The existence of the transition can be checked with the two order parameters: The magnetization and the mean age difference between the majority and the minority.

  We have also checked that using a 2D lattice as a structure for our system does not make a qualitative difference. As it has been shown in previous papers ([6]), the a_{crit} is rescaled in the 2D lattice with respect to the all-to-all connection value. In the particular case of the partial aging, we have checked that the main difference between the mean field case and a 2D lattice is the functional form that relates the critical parameter with the density of aged agents.

Chapter 5

Appendix

We will present in this appendix the calculations that lead to the main theoretical results in the noisy voter model, the aging version and the modifications we have studied. At the final part of the appendix it can be found a brief section explaining the numerical methods we have used during this work.

5.1 Noisy voter model

In order to analyze the noisy voter model, we retake the global rates and, collecting all the elementary processes we build the master equation:

$$\begin{aligned} p(n,t+dt) &= p(n,t)(1-\Omega_1 dt)(1-\Omega_2 dt) + p(n+1,t)((\Omega_2 dt) + p(n-1,t)(\Omega_1 dt), \\ \frac{dp(n,t)}{dt} &= (-\Omega_1 - \Omega_2)p(n,t) + \Omega_2 p(n+1,t) + \Omega_1 p(n-1,t), \\ \frac{dp(n,t)}{dt} &= (E-1)(\Omega_2 p(n,t)) + \left(E^{-1}-1\right)(\Omega_1 p(n,t)). \end{aligned} \tag{5.1}$$

With this starting point in the master equation we make a change of variables to the proportion of agents in state $1(+)$ ($x = n/N$).

$$\frac{dp(x,t)}{dt} = (E-1)(\Omega_2 p(x,t)) + \left(E^{-1}-1\right)(\Omega_1 p(x,t)). \tag{5.2}$$

This expressions can be used to extract the Fokker-Planck equation and the Langevin equation (in the Îto interpretation). With this change, we can take the thermodynamic limit ($\lim N \to \infty$), making x a continuous variable. If x is a continuous variable we can expand the step operators:

$$E^l[f(x)] = f(x+l) = f(x) + l\frac{df(x)}{dx} + \frac{l^2}{2!}\frac{d^2f(x)}{dx^2} + \dots \tag{5.3}$$

This can be inserted in the master equation and arrive to :

$$\frac{\partial p(x;t)}{\partial t} = \frac{\partial}{\partial x}\left(F(x)p(x;t) + \frac{1}{2}\frac{\partial}{\partial x}(G(x)p(x;t))\right). \tag{5.4}$$

It is well known ([7]) that respectively, the drift and the noise term of the Fokker-Planck equation can be calculated as:

$$\begin{aligned} F(n) &= \sum_l l\Omega_{n\to n-l}, \\ G(n) &= \sum_l l^2\Omega_{n\to n-l}. \end{aligned} \tag{5.5}$$

The corresponding expressions in our case are the following:

$$\begin{aligned} F(n) &= \Omega_2 - \Omega_1 = \frac{a(2x-1))}{2}, \\ G(n) &= \Omega_2 + \Omega_1 = \frac{a}{2} + 2(1-a)(x(1-x)). \end{aligned} \tag{5.6}$$

The connection between the Fokker-Planck equation and the Langevin equation in the Itô interpretation allows to obtain the following relations:

$$\begin{aligned} \frac{dP(x,t)}{dt} &= \frac{\partial}{N\partial x}\left(\frac{a(2x-1))}{2}p(x,t) + \frac{\partial}{2N\partial x}\left(\frac{a}{2} + 2(1-a)(x(1-x))p(x,t)\right)\right), \\ \frac{dP(x,t)}{dt} &= -\frac{a(2x-1))}{2} + \sqrt{\frac{a}{2} + 2(1-a)(x(1-x))}\xi(t). \end{aligned} \tag{5.7}$$

We translate this equations to the natural magnitude of the Ising-like models, the magnetization ($m = 2x-1$). We also make the change of variables in the Fokker-Planck equation measuring the time in MonteCarlo time units($t' = tN$)

$$\frac{dp(m,t)}{dt'} = \frac{\partial}{\partial m}\left((amp(m,t) + \frac{\partial}{N\partial m}\left((a+(1-a)(1-m^2))p(m,t)\right)\right). \tag{5.8}$$

Once we arrive here we can check the shape of the stationary solution setting the time derivative equal to zero. Applying also the condition of zero flux it is easy to arrive to:

$$P_{st} = C\left((a+(1-a)(1-m^2)\right)^{\frac{2-a(N+2)}{2(a-1)}}. \tag{5.9}$$

The number of maxima and minima is what determines the difference in the behaviour. This change of convexity is produced when the exponent changes sign. This is:

$$a_c = \frac{2}{N+2}. \tag{5.10}$$

Below this critical point, the system shows a bimodal distribution, that jumps from one state to the other. Above it, the distribution is unimodal, exhibiting coexistence between the two states.

5.2 Noisy voter model with aging

Coming from the global rates our goal is to build the master equation. In order to achieve this, we take into account the elementary processes that may occur in our system. If we have $n_i^{\pm}$ particles at time t, they can evolve to time $t+dt$ in three different ways:

- $n_i^{+(-)} \to n_i^{+(-)} + 1$. If there is a particle in a nonzero level of age, the unique option to consider is that it got older from the i-1 level. The probability for this process to occur is: $\Omega_{3(4)}(i-1)dt$
- $n_i^{+(-)} \to n_i^{+(-)} - 1$. Losing a particle may happen because either one particle got older in the i level either one particle changed state. The probability for both processes is the sum of them, as we know: $(\Omega_{3(4)}(i) + \Omega_{1(2)}(i))dt$
- $n_i^{+(-)} \to n_i^{+(-)}$. If none of the former processes happens, we conserve the number of particles at this level. The probability for this is the product of not happening the previous processes: $(1-\Omega_{3(4)}(i)dt)(1-\Omega_{1(2)}(i)dt)(1-\Omega_{3(4)}(i-1)dt)$

If we are talking about the 0 level, we have to consider different situations:

- $n_0^{+(-)} \to n_0^{+(-)} + 1$. If we gain a particle at the zero level, it is because some particle at any level has changed state. The probability for this is: $\sum_0^\infty \Omega_{2(1)}dt$
- $n_0^{+(-)} \to n_0^{+(-)} - 1$. It may happen because some particle got older either changed state. The probability is: $(\Omega_{3(4)}(0) + \Omega_{1(2)}(0))dt$
- $n_i^{+(-)} \to n_i^{+(-)}$. The conservation of particle, as before, is the product of not happening the previous processes.

So we can write down:

$$\begin{aligned}
p(n_i^+, t+dt) &= p(n_i^+, t)(1-\Omega_1(i))(1-\Omega_3(i))(1-\Omega_3(i-1))dt + p(n_i^+ - 1, t)\Omega_3(i-1)dt + \\
&\quad + p(n_i^+ + 1, t)(\Omega_1(i) + \Omega_3(i))dt, \\
\frac{dp(n_i^+)}{dt} &= (-\Omega_1(i) - \Omega_3(i) - \Omega_3(i-1))(p(n_i^+, t) + (\Omega_3(i-1)))p(n_i^+ - 1, t) + p(n_i^+ + 1, t)(\Omega_1(i) + \Omega_3(i)), \\
\frac{dp(n_i^+)}{dt} &= (E^{-1} - 1)((p(n_i^+)\Omega_3(i-1)) + (E-1)((p(n_i^+)(\Omega_1(i) + \Omega_3(i))).
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
p(n_0^+, t+dt) &= p(n_0^+, t)(1-\Omega_3(0)dt)(1-\Omega_1(0)dt)\sum_{i=0}^{\infty}(1-\Omega_2(i)dt) + p(n_0^+ - 1, t)\sum_0^{\infty}\Omega_2(i)dt + \\
&\quad + p(n_0^+ + 1, t)(\Omega_1(0) + \Omega_3(0))dt, \\
\frac{dp(n_0^+)}{dt} &= (-\sum_{i=0}^{\infty}\Omega_2(i) - \Omega_3(0) - \Omega_1(0))(p(n_0^+, t)) + (\sum_{i=0}^{\infty}\Omega_2(i))p(n_0^+ - 1, t) + \\
&\quad + p(n_0^+ + 1, t)(\Omega_1(0) + \Omega_3(0)), \\
\frac{dp(n_0^+)}{dt} &= (E^{-1} - 1)((p(n_0^+)\sum_0^{\infty}\Omega_2(i)) + (E-1)((p(n_0^+)(\Omega_1(0) + \Omega_3(0))).
\end{aligned} \tag{5.12}$$

The other two equations that define the state – can be extracted using the symmetry between the ± states. So if we substitute:

$$\Omega_1 \to \Omega_2 \qquad , \qquad \Omega_3 \to \Omega_4$$

And vice versa we obtain:

$$\frac{dp(n_i^-)}{dt} = (E^{-1}-1)((p(n_i^-)\Omega_4(i-1)) + (E-1)((p(n_i^-)(\Omega_2(i)+\Omega_4(i))). \tag{5.13}$$

$$\frac{dp(n_0^-)}{dt} = (E^{-1}-1)((p(n_0^-)\sum_0^\infty \Omega_1(i)) + (E-1)((p(n_0^-)(\Omega_2(0)+\Omega_4(0))). \tag{5.14}$$

Once we arrive to this point, we are going to find the mean field stationary solution. It is known ([7]) that studying the averages for a master equation with global rates can be written:

$$\frac{d<x>}{dt} = -<l\Omega_{n\to n-l}>. \tag{5.15}$$

Translated to our equation this means:

$$\begin{aligned}
\frac{d<n_i^+>}{dt} &= <\Omega_3(i-1)> - <\Omega_3(i)> - <\Omega_1(i)>,\\
\frac{d<n_i^->}{dt} &= <\Omega_4(i-1)> - <\Omega_4(i)> - <\Omega_2(i)>,\\
\frac{d<n_0^+>}{dt} &= \sum_{i=0}^\infty <\Omega_2(i)> - <\Omega_3(0)> - <\Omega_1(0)>,\\
\frac{d<n_0^->}{dt} &= \sum_{i=0}^\infty <\Omega_1(i)> - <\Omega_4(0)> - <\Omega_2(0)>.
\end{aligned} \tag{5.16}$$

If we set the derivative equal to zero, and substitute the expressions of the rates we arrive to:

$$\begin{aligned}
n_i^+ &= n_{i-1}^+\left(\frac{a}{2}+\frac{1-a}{1+i}\left(i+\frac{n}{N}\right)\right),\\
n_i^- &= n_{i-1}^-\left(\frac{a}{2}+\frac{1-a}{1+i}\left(i+\frac{(N-n)}{N}\right)\right),\\
n_0^+ &= \sum_{i=0}^\infty n_i^-\left(\frac{a}{2}+\frac{1-a}{2+i}\frac{n}{N}\right),\\
n_0^- &= \sum_{i=0}^\infty n_i^+\left(\frac{a}{2}+\frac{1-a}{2+i}\frac{N-n}{N}\right).
\end{aligned} \tag{5.17}$$

As we can see from the first two equations, we have a recursive relation for all the i levels.

$$n_i^+ = n_{i-1}^+ A(i,x).$$

Where $A(i,x)$ is:

$$A(i,x) = \left(\frac{a}{2}+\frac{1-a}{1+i}(i+x)\right).$$

And $x = n/N$. With this recursive relation we can build any level of age as a product until i=0:

$$\begin{aligned}
n_i^+ &= n_0^+\prod_{j=1}^{i} A(j,x),\\
n_i^- &= n_0^-\prod_{j=1}^{i} A(j,1-x).
\end{aligned} \tag{5.18}$$

Futhermore, we can build n as the sum of all the levels 1(+):

$$n = \sum_{i=0}^{\infty} n_i^+ = n_0^+ \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,x) \right). \tag{5.19a}$$

$$N - n = \sum_{i=0}^{\infty} n_i^- = n_0^- \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,1-x) \right). \tag{5.19b}$$

Fortunately, this expressions have a analytic expression which can be obtained using symbolic calculus programs.

$$\prod_{j=1}^{i} A(j,x) = \frac{(1-a/2)^i \left(\frac{-2+2(a-1)x}{a-2} \right)_i}{\Gamma(i+2)},$$
$$\left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,x) \right) = -\frac{2 - 2^{\frac{2(a-1)x-2}{a-2}} a^{\frac{-2ax+a+2x}{a-2}}}{-2ax+a+2x}. \tag{5.20}$$

Where $(z)_i$ is the Pochhammer symbol, defined as the division of gamma functions :

$$(z)_i = \frac{\Gamma(z+i)}{\Gamma(z)}.$$

Arrived at this point we can make the division of 5.19a by 5.19b, introducing in the left part the variable $x =< n > /N$.

$$\frac{x}{1-x} = \frac{< n_0^+ > \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,x) \right)}{< n_0^- > \left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,1-x) \right)}. \tag{5.21}$$

In order to obtain the solution, we use $< n_0^+ > = < n_0^- >$, getting an equation that only depends on x. Again, with symbolic programming we can find the roots of the expression $f_1(a,x)$, which will be the stationary distribution of the magnetization.

$$f_1(a,x) = \frac{x}{1-x} - \frac{\left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,x) \right)}{\left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} A(j,1-x) \right)}.$$

The roots for this equation will be the stationary solution for our system. This solution depends on the parameter (a), this is, there is a critical parameter (a_c), that separate the phases. The derivatives must be the same evaluated at the critical point in the stationary equilibrium. This means that we can use the following condition :

$$\left. \frac{d f_1(a,x)}{dx} \right|_{\substack{x=1/2 \\ a=a_c}} = 0. \tag{5.22}$$

To find the critical parameter. If we numerically solve this equation we can find that $a_c \simeq 0.0755578...$
The critical exponent can be obtained making an expansion around the critical parameter and around the stationary equilibrium ($x = 1/2$).

$$\left. f_1(a,x) \right|_{\substack{x \to 1/2 \\ a \to a_c}} \simeq \left(5.45143(x-0.5)^3 \right) + (a - 0.0755578)(21.1438(x-0.5)) = 0. \tag{5.23}$$

From here it is easy to corroborate that $m = t^{1/2}$ where $t = (1 - a/a_c)$.

5.3 Noisy voter model with anti-aging

Coming from the global rates our goal is to build the master equation. In order to achieve this, we take into account the elementary processes that may ocurr in our system. If we have $n_i^{\pm}$ particles at time t, it can evolve to time $t+dt$ in three different ways :

- $n_i^{+(-)} \to n_i^{+(-)}+1$. If there is a particle in a nonzero level of age, the unique option to consider is that it got older from the $i-1$ level. The probability for this process to occur is : $\Omega_{3(4)}(i-1)dt$
- $n_i^{+(-)} \to n_i^{+(-)}-1$. Losing a particle may happen because either one particle got older in the i level either one particle changed state. The probability for both processes is the sum of them, as we know: $(\Omega_{3(4)}(i)+\Omega_{1(2)}(i))dt$
- $n_i^{+(-)} \to n_i^{+(-)}$. If none of the former processes happens, we conserve the number of particles at this level. The probability for this is the product of not happening the previous processes : $(1-\Omega_{3(4)}(i)dt)(1-\Omega_{1(2)}(i)dt)(1-\Omega_{3(4)}(i)dt)$

If we are talking about the 0 level, we have to consider different processes:

- $n_0^{+(-)} \to n_0^{+(-)}+1$. If we gain a particle at the zero level, it is because some particle at any level has changed state. The probability for this is :$\sum_0^{\infty}\Omega_{2(1)}dt$
- $n_0^{+(-)} \to n_0^{+(-)}-1$. It may happen because some particle got older either changed state. The probability is : $(\Omega_{3(4)}(0)+\Omega_{1(2)}(0))dt$
- $n_i^{+(-)} \to n_i^{+(-)}$. The conservation of particle, as before, is the result of non-happening the former processes.

So we can write down :

$$\begin{aligned}
p(n_i^+,t+dt) &= p(n_i^+,t)(1-\Omega_1(i))(1-\Omega_3(i))(1-\Omega_3(i-1))dt + p(n_i^+-1,t)\Omega_3(i-1)dt+\\
&\quad + p(n_i^+ +1,t)(\Omega_1(i)+\Omega_3(i))dt,\\
\frac{dp(n_i^+)}{dt} &= (-\Omega_1(i)-\Omega_3(i)-\Omega_3(i-1))(p(n_i^+,t)+(\Omega_3(i-1)))p(n_i^+-1,t)+\\
&\quad + p(n_i^+ +1,t)(\Omega_1(i)+\Omega_3(i)),\\
\frac{dp(n_i^+)}{dt} &= (E^{-1}-1)((p(n_i^+)\Omega_3(i-1))+(E-1)((p(n_i^+)(\Omega_1(i)+\Omega_3(i))).
\end{aligned} \tag{5.24}$$

$$\begin{aligned}
p(n_0^+,t+dt) &= p(n_0^+,t)(1-\Omega_3(0)dt)(1-\Omega_1(0)dt)\sum_{i=0}^{\infty}(1-\Omega_2(i)dt)+p(n_0^+-1,t)\sum_0^{\infty}\Omega_2(i)dt+\\
&\quad + p(n_0^+ +1,t)(\Omega_1(0)+\Omega_3(0))dt,\\
\frac{dp(n_0^+)}{dt} &= (-\sum_{i=0}^{\infty}\Omega_2(i)-\Omega_3(0)-\Omega_1(0))(p(n_0^+,t))+(\sum_{i=0}^{\infty}\Omega_2(i))p(n_0^+-1,t)+\\
&\quad + p(n_0^+ +1,t)(\Omega_1(0)+\Omega_3(0)),\\
\frac{dp(n_0^+)}{dt} &= (E^{-1}-1)((p(n_0^+)\sum_0^{\infty}\Omega_2(i))+(E-1)((p(n_0^+)(\Omega_1(0)+\Omega_3(0))).
\end{aligned} \tag{5.25}$$

The other two equations that define the state – can be extracted using that there is no a preferred state. So if we substitute:

$$\Omega_1 \to \Omega_2 \qquad , \qquad \Omega_3 \to \Omega_4$$

And vice versa we obtain :

$$\frac{dp(n_i^-)}{dt} = (E^{-1} - 1)((p(n_i^-)\Omega_4(i-1)) + (E-1)((p(n_i^-)(\Omega_2(i) + \Omega_4(i))). \tag{5.26}$$

$$\frac{dp(n_0^-)}{dt} = (E^{-1} - 1)((p(n_0^-)\sum_0^\infty \Omega_1(i)) + (E-1)((p(n_0^-)(\Omega_2(0) + \Omega_4(0))). \tag{5.27}$$

5.4 Numerical methods

In this section we are going to present the procedures used to simulate a system with N agents with aging that interact in two different structures :

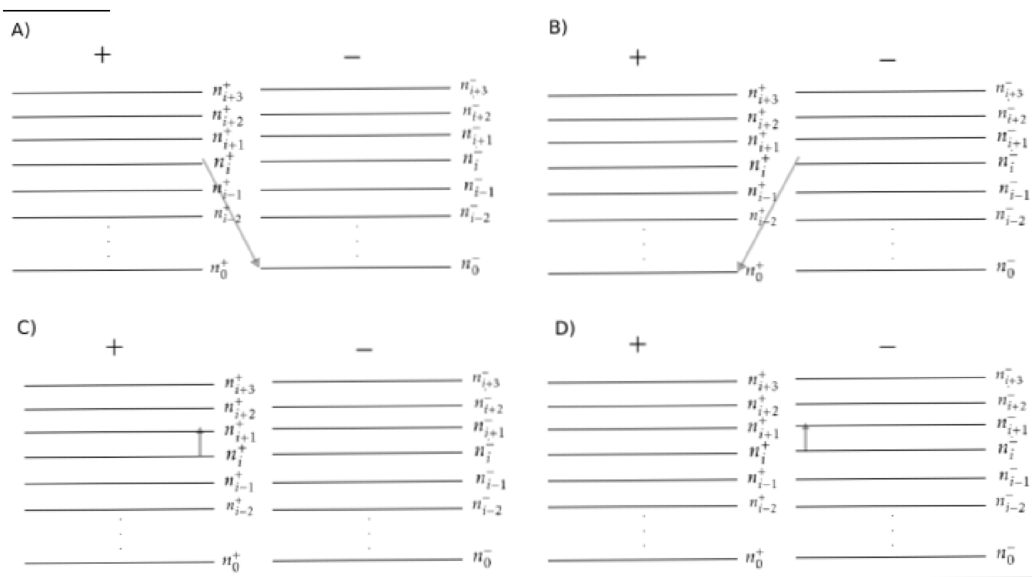
- An all-to-all connection. In this case we have the master equation, so we will use a Gillespie algorithm in order to solve the dynamics.
- A 2D square lattice, where every agent is connected to four neighbors. In this case we haven't studied the master equations, so we will use an agent-based point of view .

We proceed to explain both cases, beginning by the all-to-all connection and the Gillespie algorithm.

5.4.1 Gillespie algorithm

The Gillespie algorithm is a method to simulate master equations using the stochastic process that underlie that master equation. In our case, we will consider systems of N interacting particles in an all-to-all connection that can be in $2(i_{max}+1)$ different states, where i_{max} is an age larger than any observed value.
As all the theoretical discussion has been around the level structure of our system, we will consider the occupation number point of view, where every level is characterized by $n_i^{\pm}$, depending on the sign. A state of the system may be characterized then by a vector of length $2(i_{max}+1)$: $(n_0^+, n_0^-, n_1^+, n_1^- ... n_{i_{max}}^+, n_{i_{max}}^-)$.
The elementary processes that take into account what happens to one particle are the following :

- An agent with age i changes state and its age is resetted to zero (A and B).
- An agent with age i does not change its state and its age is increased by one unity (C and D).



We define W_T as the sum of all the rates in all the age levels, this is, the rate for a process to happen:

$$W_T = \sum_{i=0}^{i_{max}} [\Omega_1(i) + \Omega_2(i) + \Omega_3(i) + \Omega_4(i)]. \tag{5.28}$$

Note that the different $\Omega_j(i)$ are referred to the possible processes that may happen during the dynamics (see 3.1 for example). The probability for a certain process to happen in a certain level of age i is $\Omega_1(i)/W_T, \Omega_2(i)/W_T, \Omega_3(i)/W_T$ and $\Omega_4(i)/W_T$ where W_T is the probability to leave a certain state characterized by a vector $(n_0^+, n_0^-, n_1^+, n_1^- ... n_{i_{max}}^+, n_{i_{max}}^-)$:

The probability for a process to happen in a level of age is :

$$P_H(i) = \frac{\Omega_1(i) + \Omega_2(i) + \Omega_3(i) + \Omega_4(i)}{W_T}. \tag{5.29}$$

At this point we can apply the usual Gillespie technique : We draw a uniform random number(μ) distributed between $(0,1)$ and find the smallest i_H such that $\sum_{i=0}^{i=i_H} P_H(i) > \mu$. This will let us know the age of the agent that is going to interact with the system in this step of the dynamics.
To find out which process is going to happen we rescale the uniform number μ :

$$\mu' = \mu - \sum_{i=0}^{i=i_H-1} P_H(i). \tag{5.30}$$

And know we find out which exact process is happening by finding the smallest j' such that $\sum_{j=1}^{j=j'} \Omega_j(i_H)/W_T > \mu'$. The time lapse for this process can be calculated :

$$t_{i_0 \rightarrow i_1} = -\frac{ln(\mu)}{W_T}. \tag{5.31}$$

And the process is finished. In the new time step, all the probabilities must be computed again for all the age levels.

5.4.2 Agent-based methods

Implementing the Gillespie algorithm is possible in the all-to-all connection because we have extracted the master equation for the process. This is not the case in the 2D lattice . We have a population of N aged agents, where i_{max} is the maximum age allowed. An agent is characterized by its state and age (s,i). Here we will take an individual point of view, where the time is discrete and can be measured in agents interactions. This point of view it's similar to the one has been shown in diagrams like (2.1) or (2.3).
For every time step the dynamics is the following :

- We choose an agent at random among the whole population.
- We draw a uniform random number (μ) to check the mechanism of interaction.
  - If $\mu < a$ the agent will randomly choose one state.

- If $\mu > a$ the agent will make a voter-like interaction with one of its neighbors. In this case we must check the aging activation. We draw another random number (μ') and if $\mu' > p(i)$ the agent will ignore the interaction and remain in the same state. The particular $p(i)$ depends on the case. The ones that have been studied in this work are :

$$p(i) = \frac{1}{i+2} \quad (aging) \qquad\qquad p(i) = 1 - \frac{1}{i+2} \quad (anti-aging) \tag{5.32}$$

At the end of this process, if the agent has changed its state, its age is resetted. In other case, its age is increased by one.

Bibliography

[1] A. Kirman, "Ants, rationality, and recruitment*," *The Quarterly Journal of Economics*, vol. 108, no. 1, pp. 137–156, 1993.

[2] T. Lux and M. Marchesi, "Scaling and criticality in a stochastic multi-agent model of a financial market," *Nature*, vol. 397, no. 6719, p. 498, 1999.

[3] J. Fernández-Gracia, K. Suchecki, J. J. Ramasco, M. San Miguel, and V. M. Eguíluz, "Is the voter model a model for voters?," *Physical review letters*, vol. 112, no. 15, p. 158701, 2014.

[4] A. Carro, R. Toral, and M. San Miguel, "Markets, herding and response to external information," *PloS one*, vol. 10, no. 7, p. e0133287, 2015.

[5] A. Carro, R. Toral, and M. San Miguel, "The noisy voter model on complex networks," *Scientific reports*, vol. 6, p. 24775, 2016.

[6] O. Artime, A. F. Peralta, R. Toral, J. J. Ramasco, and M. San Miguel, "Aging-induced continuous phase transition," *Physical Review E*, vol. 98, no. 3, p. 032104, 2018.

[7] R. Toral and P. Colet, *Stochastic numerical methods: an introduction for students and scientists*. John Wiley & Sons, 2014.