B.7 MISCELLANEOUS

B.7-1 L'Hôpital's Rule

If $\lim f(x)/g(x)$ results in the indeterministic form $0/0$ or ∞/∞, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{\dot{f}(x)}{\dot{g}(x)}$$

B.7-2 The Taylor and Maclaurin Series

$$f(x) = f(a) + \frac{(x-a)}{1!}\dot{f}(a) + \frac{(x-a)^2}{2!}\ddot{f}(a) + \cdots$$

$$f(x) = f(0) + \frac{x}{1!}\dot{f}(0) + \frac{x^2}{2!}\ddot{f}(0) + \cdots$$

B.7-3 Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots \qquad x^2 < \pi^2/4$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \qquad x^2 < \pi^2/4$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \binom{n}{k}x^k + \cdots + x^n$$

$$\approx 1 + nx \qquad |x| \ll 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

B.7-4 Sums

$$\sum_{k=m}^{n} r^k = \frac{r^{n+1} - r^m}{r - 1} \qquad r \neq 1$$

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=0}^{n} k\,r^k = \frac{r + [n\,(r-1) - 1]\,r^{n+1}}{(r-1)^2} \qquad r \neq 1$$

$$\sum_{k=0}^{n} k^2\,r^k = \frac{r[(1+r)(1-r^n) - 2n(1-r)r^n - n^2(1-r)^2 r^n]}{(1-r)^3} \qquad r \neq 1$$

B.7-5 Complex Numbers

$$e^{\pm j\pi/2} = \pm j$$

$$e^{\pm jn\pi} = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$a + jb = re^{j\theta} \qquad r = \sqrt{a^2+b^2},\ \theta = \tan^{-1}\left(\tfrac{b}{a}\right)$$

$$(re^{j\theta})^k = r^k e^{jk\theta}$$

$$(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

B.7-6 Trigonometric Identities

$$e^{\pm jx} = \cos x \pm j\sin x$$

$$\cos x = \tfrac{1}{2}[e^{jx} + e^{-jx}]$$

$$\sin x = \tfrac{1}{2j}[e^{jx} - e^{-jx}]$$

$$\cos\left(x \pm \tfrac{\pi}{2}\right) = \mp\sin x$$

$$\sin\left(x \pm \tfrac{\pi}{2}\right) = \pm\cos x$$

$$2\sin x\cos x = \sin 2x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cos^2 x - \sin^2 x = \cos 2x$$

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

$$\cos^3 x = \tfrac{1}{4}(3\cos x + \cos 3x)$$

$$\sin^3 x = \tfrac{1}{4}(3\sin x - \sin 3x)$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x - y) + \cos(x + y)]$$

$$\sin x \cos y = \tfrac{1}{2}[\sin(x - y) + \sin(x + y)]$$

$$a\cos x + b\sin x = C\cos(x + \theta) \qquad C = \sqrt{a^2 + b^2},\ \theta = \tan^{-1}\left(\tfrac{-b}{a}\right)$$

B.7-7 Indefinite Integrals

$$\int u\,dv = uv - \int v\,du$$

$$\int f(x)\dot{g}(x)\,dx = f(x)g(x) - \int \dot{f}(x)g(x)\,dx$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int x\sin ax\,dx = \frac{1}{a^2}(\sin ax - ax\cos ax)$$

$$\int x\cos ax\,dx = \frac{1}{a^2}(\cos ax + ax\sin ax)$$

$$\int x^2\sin ax\,dx = \frac{1}{a^3}(2ax\sin ax + 2\cos ax - a^2x^2\cos ax)$$

$$\int x^2\cos ax\,dx = \frac{1}{a^3}(2ax\cos ax - 2\sin ax + a^2x^2\sin ax)$$

$$\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$

$$\int \sin ax\cos bx\,dx = -\left[\frac{\cos(a-b)x}{2(a-b)} + \frac{\cos(a+b)x}{2(a+b)}\right] \qquad a^2 \neq b^2$$

$$\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int x^2e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx)$$

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x}{x^2 + a^2}\,dx = \frac{1}{2}\ln(x^2 + a^2)$$

B.7-8 Common Derivative Formulas

$$\frac{d}{dx}f(u) = \frac{d}{du}f(u)\frac{du}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dx^n}{dx} = nx^{n-1}$$

$$\frac{d}{dx}\ln(ax) = \frac{1}{x}$$

$$\frac{d}{dx}\log(ax) = \frac{\log e}{x}$$

$$\frac{d}{dx}e^{bx} = be^{bx}$$

$$\frac{d}{dx}a^{bx} = b(\ln a)a^{bx}$$

$$\frac{d}{dx}\sin ax = a\cos ax$$

$$\frac{d}{dx}\cos ax = -a\sin ax$$

$$\frac{d}{dx}\tan ax = \frac{a}{\cos^2 ax}$$

$$\frac{d}{dx}(\sin^{-1} ax) = \frac{a}{\sqrt{1-a^2x^2}}$$

$$\frac{d}{dx}(\cos^{-1} ax) = \frac{-a}{\sqrt{1-a^2x^2}}$$

$$\frac{d}{dx}(\tan^{-1} ax) = \frac{a}{1+a^2x^2}$$

B.7-9 Some Useful Constants

$\pi \approx 3.1415926535$

$e \approx 2.7182818284$

$\frac{1}{e} \approx 0.3678794411$

$\log_{10} 2 = 0.30103$

$\log_{10} 3 = 0.47712$

B.7-10 Solution of Quadratic and Cubic Equations

Any *quadratic* equation can be reduced to the form

$$ax^2 + bx + c = 0$$

The solution of this equation is provided by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A general *cubic* equation

$$y^3 + py^2 + qy + r = 0$$

may be reduced to the *depressed cubic* form

$$x^3 + ax + b = 0$$

by substituting

$$y = x - \frac{p}{3}$$

This yields

$$a = \tfrac{1}{3}(3q - p^2) \qquad b = \tfrac{1}{27}(2p^3 - 9pq + 27r)$$

Now let

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \qquad B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

The solution of the depressed cubic is

$$x = A + B, \qquad x = -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3}, \qquad x = -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3}$$

and

$$y = x - \frac{p}{3}$$

TABLE 4.1 A Short Table of (Unilateral) Laplace Transforms

No.	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
6	$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
7	$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
8a	$\cos bt\, u(t)$	$\frac{s}{s^2+b^2}$
8b	$\sin bt\, u(t)$	$\frac{b}{s^2+b^2}$
9a	$e^{-at}\cos bt\, u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
9b	$e^{-at}\sin bt\, u(t)$	$\frac{b}{(s+a)^2+b^2}$
10a	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{(r\cos\theta)s + (ar\cos\theta - br\sin\theta)}{s^2+2as+(a^2+b^2)}$
10b	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb} + \frac{0.5re^{-j\theta}}{s+a+jb}$
10c	$re^{-at}\cos(bt+\theta)\, u(t)$ $r = \sqrt{\frac{A^2c + B^2 - 2ABa}{c-a^2}}$ $\theta = \tan^{-1}\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right)$ $b = \sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$
10d	$e^{-at}\left[A\cos bt + \frac{B-Aa}{b}\sin bt\right]u(t)$ $b = \sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$

TABLE 4.2 The Laplace Transform Properties

Operation	$x(t)$	$X(s)$	
Addition	$x_1(t)+x_2(t)$	$X_1(s)+X_2(s)$	
Scalar multiplication	$kx(t)$	$kX(s)$	
Time differentiation	$\frac{dx}{dt}$	$sX(s)-x(0^-)$	
	$\frac{d^2x}{dt^2}$	$s^2X(s)-sx(0^-)-\dot{x}(0^-)$	
	$\frac{d^3x}{dt^3}$	$s^3X(s)-s^2x(0^-)-s\dot{x}(0^-)-\ddot{x}(0^-)$	
	$\frac{d^nx}{dt^n}$	$s^nX(s)-\sum_{k=1}^{n}s^{n-k}x^{(k-1)}(0^-)$	
Time integration	$\int_{0^-}^{t}x(\tau)\,d\tau$	$\frac{1}{s}X(s)$	
	$\int_{-\infty}^{t}x(\tau)\,d\tau$	$\frac{1}{s}X(s)+\frac{1}{s}\int_{-\infty}^{0^-}x(t)\,dt$	
Time shifting	$x(t-t_0)u(t-t_0)$	$X(s)e^{-st_0}$	$t_0\geq 0$
Frequency shifting	$x(t)e^{s_0t}$	$X(s-s_0)$	
Frequency differentiation	$-tx(t)$	$\frac{dX(s)}{ds}$	
Frequency integration	$\frac{x(t)}{t}$	$\int_{s}^{\infty}X(z)\,dz$	
Scaling	$x(at),\ a\geq 0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$	
Time convolution	$x_1(t)*x_2(t)$	$X_1(s)X_2(s)$	
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi j}X_1(s)*X_2(s)$	
Initial value	$x(0^+)$	$\lim_{s\to\infty}sX(s)$	$(n>m)$
Final value	$x(\infty)$	$\lim_{s\to 0}sX(s)$	[poles of $sX(s)$ in LHP]

TABLE 7.1 Fourier Transforms

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\frac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$	$a>0$
5	$t^ne^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
12	$\operatorname{sgn}t$	$\frac{2}{j\omega}$	
13	$\cos\omega_0t\,u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0t\,u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0t\,u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0t\,u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi}\operatorname{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

Table of Z-Transform Pairs

$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$	ROC
$x[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X(z)$	R_x
$x[-n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X(\frac{1}{z})$	$\frac{1}{R_x}$
$x^*[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X^*(z^*)$	R_x
$x^*[-n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X^*(\frac{1}{z^*})$	$\frac{1}{R_x}$
$\mathfrak{Re}\{x[n]\}$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	R_x
$\mathfrak{Im}\{x[n]\}$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	R_x
time shifting $\quad x[n-n_0]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$z^{-n_0}X(z)$	R_x
$a^n x[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X\left(\frac{z}{a}\right)$	$\|a\|R_x$
downsampling by N $\quad x[Nn] \quad N \in \mathbb{N}_0$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{1}{N}\sum_{k=0}^{N-1} X\left(W_N^k z^{\frac{1}{N}}\right) \quad W_N = e^{-\frac{j2\omega}{N}}$	R_x
$ax_1[n]+bx_2[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$aX_1(z)+bX_2(z)$	$R_x \cap R_y$
$x_1[n]x_2[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{1}{2\pi j}\oint X_1(u)X_2\left(\frac{z}{u}\right)u^{-1}du$	$R_x \cap R_y$
$x_1[n] * x_2[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X_1(z)X_2(t)$	$R_x \cap R_y$
$\delta[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	1	$\forall\ z$
$\delta[n-n_0]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	z^{-n_0}	$\forall\ z$
$u[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{z-1}$	$\|z\| > 1$
$-u[-n-1]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{z-1}$	$\|z\| < 1$
$nu[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
$n^2u[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
$n^3u[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\|z\| > 1$
$(-1)^n$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{z+1}$	$\|z\| < 1$
$a^nu[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
$-a^nu[-n-1]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{z-a}$	$\|z\| < \|a\|$
$a^{n-1}u[n-1]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{1}{z-a}$	$\|z\| > \|a\|$
$na^nu[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
$n^2a^nu[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{az(z+a}{(z-a)^3}$	$\|z\| > \|a\|$
$e^{-an}u[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{z-e^{-a}}$	$\|z\| > \|e^{-a}\|$
$\begin{cases} a^n & n=0,\ldots,N-1 \\ 0 & otherwise \end{cases}$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\sin(\omega_0 n)\,u[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z\sin(\omega_0)}{z^2-2\cos(\omega_0)z+1}$	$\|z\| > 1$
$\cos(\omega_0 n)\,u[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z(z-\cos(\omega_0))}{z^2-2\cos(\omega_0)z+1}$	$\|z\| > 1$
$a^n\sin(\omega_0 n)\,u[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{za\sin(\omega_0)}{z^2-2a\cos(\omega_0)z+a^2}$	$\|z\| > a$
$a^n\cos(\omega_0 n)\,u[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z(z-a\cos(\omega_0))}{z^2-2a\cos(\omega_0)z+a^2}$	$\|z\| > a$
$nx[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$-z\frac{d}{dz}X(z)$	R_x
$\frac{x[n]}{n}$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$-\int_0^z \frac{X(z)}{z}dz$	R_x
$\frac{\prod_{i=1}^{m}(n-i+1)}{a^m m!}a^m u[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{(z-a)^{m+1}}$	

Please note : $\frac{z}{z-1} = \frac{z^{-1}}{1-z^{-1}}$

Table 8.1 Denominator Polynomial Coefficients for Normalized Low-Pass Filters of Order $n = 1$ Through $n = 4$

n	b_0	b_1	b_2	b_3
Butterworth				
1	1.0000	—	—	—
2	1.0000	1.4142	—	—
3	1.0000	2.0000	2.0000	—
4	1.0000	2.6131	3.4142	2.6131
0.5-dB Passband Ripple Chebyshev ($\varepsilon^2 = 0.1220$)				
1	2.8628	—	—	—
2	1.5162	1.4256	—	—
3	0.7157	1.5439	1.2529	—
4	0.3791	1.0255	1.7169	1.1974
1.0-dB Passband Ripple Chebyshev ($\varepsilon^2 = 0.2589$)				
1	1.9652	—	—	—
2	1.1025	1.0977	—	—
3	0.4913	1.2384	0.9883	—
4	0.2756	0.7426	1.4539	0.9528
3.0-dB Passband Ripple Chebyshev ($\varepsilon^2 = 1.0000$)				
1	1.0000	—	—	—
2	0.7071	0.6436	—	—
3	0.2500	0.9277	0.5961	—
4	0.1768	0.4039	1.1685	0.5805

Table 7.2 Summary of ideal impulse responses for standard frequency selective filters.

Filter type	***Ideal impulse response, $h_D(n)$*** $h_D(n), n \neq 0$	$h_D(0)$
Lowpass	$2f_c \frac{\sin(n\omega_c)}{n\omega_c}$	$2f_c$
Highpass	$-2f_c \frac{\sin(n\omega_c)}{n\omega_c}$	$1 - 2f_c$
Bandpass	$2f_2 \frac{\sin(n\omega_2)}{n\omega_2} - 2f_1 \frac{\sin(n\omega_1)}{n\omega_1}$	$2(f_2 - f_1)$
Bandstop	$2f_1 \frac{\sin(n\omega_1)}{n\omega_1} - 2f_2 \frac{\sin(n\omega_2)}{n\omega_2}$	$1 - 2(f_2 - f_1)$

f_c, f_1 and f_2 are the normalized passband or stopband edge frequencies; N is the length of filter.

Table 7.3 Summary of important features of common window functions.

Name of window function	*Transition width* (**Hz**) (*normalized*)	*Passband ripple* (**dB**)	*Main lobe relative to side lobe* (**dB**)	*Stopband attenuation* (**dB**) (*maximum*)	*Window function* $w(n), \lvert n \rvert \leq (N-1)/2$
Rectangular	$0.9/N$	0.7416	13	21	1
Hanning	$3.1/N$	0.0546	31	44	$0.5 + 0.5\cos\left(\frac{2\pi n}{N}\right)$
Hamming	$3.3/N$	0.0194	41	53	$0.54 + 0.46\cos\left(\frac{2\pi n}{N}\right)$
Blackman	$5.5/N$	0.0017	57	75	$0.42 + 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right)$
	$2.93/N\ (\beta = 4.54)$	0.0274		50	$\frac{I_0(\beta\{1-[2n/(N-1)]^2\}^{1/2})}{I_0(\beta)}$
Kaiser	$4.32/N\ (\beta = 6.76)$ $5.71/N\ (\beta = 8.96)$	0.002 75 0.000 275		70 90	

$$A_{min} = 20\log\left(\sqrt{1.5} \times 2^B\right)$$

$$|H(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^{2n}}}$$

$$F_s = f_a + f$$

$$q = \frac{V_{fs}}{2^B - 1} \approx \frac{V_{fs}}{2^B}$$

RMS quantization noise:

$$\frac{q}{\sqrt{12}} = \frac{A}{\sqrt{3} \times 2^B}$$

Passband to Stopband signal level:

$$\sqrt{1.5} \times 2^B$$

Aperture time:

$$\tau = aperture\ time$$

$$f_{max} = \frac{1}{\pi 2^{B+1}\tau}$$

Integer bandpass sampling:

$$\frac{2f_H}{n} \le F_s \le \frac{2f_L}{n-1}$$

$$n = \frac{f_H}{B} = \frac{f_L}{B}$$

$$F_{s(\min)} = 2B$$

Non integer bandpass sampling:

$$f_{L,new} = \left(\frac{n-1}{n}\right) f_H$$

$$f_{H,new} = \left(\frac{n}{n-1}\right) f_L$$

$$n = \left\lfloor \frac{f_H}{B} \right\rfloor$$

ADC dynamic range:

$$20\log 2^B$$

Quantization noise power:

$$\sigma_e^2 = \frac{q^2}{12} = \frac{2^{-2(B-1)}}{12}$$

SQNR:

$$6.02B + 1.76\ \text{dB}$$

In-band noise power:

$$P_e = \frac{2f_{max}}{F_s}\sigma_e^2$$

$$\frac{2f_{max}}{F_s} = 2^{-2(B_2 - B_1)}$$

N-th order SDM:

$$Y(z) = X(z) + E(z)(1 - z^{-1})^N$$

SDM transfer function:

$$\left|N\left(e^{j\omega T}\right)\right|^2 = \left|\left(1 - e^{-j\omega T}\right)^n\right|^2$$

DAC sinc attenuation:

$$20\log\left(\frac{\sin x}{x}\right), x = \frac{\omega T}{2}$$

DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, k = 0, .., N-1$$

$$X(N-k) = X(k)^*$$

Fourier transform:

$$F(j\omega) = TX(k)$$

IDFT:

$$x(nT) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}$$

FFT:

$$W_N^k = e^{-j2\pi k/N}$$

$$W_N^{k+N/2} = -W_N^k$$

Power series method IZT:

$$x(0) = \frac{b_0}{a_0}$$

$$x(n) = \left[b_n - \sum_{i=1}^{n} x(n-i)a_i\right] \Big/ a_0, n = 1, ..$$

Residues IZT:

$$Res[F(z), p_k] = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z - p_k)F(z)]_{z=p_k}$$

$$F(z) = z^{n-1}X(z)$$

$$nx(n) \xrightarrow{z} -z\frac{d}{dz}X(z)$$

PF multi-order poles IZT:

$$D_i = \frac{1}{(m-i)!}\frac{d^{m-i}}{dz^{m-i}}\left[(z-p_k)^m\ \frac{X(z)}{z}\right]_{z=p_k}$$

If $n = m, B_0 = \frac{b_n}{a_n}$

Correlation:

$$r_{12}(j) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)x_2(n+j)$$

Cross-correlation coefficient:

$$\rho_{12}(j) = \frac{r_{12}(j)}{\frac{1}{N}\sqrt{\sum_{n=0}^{N-1} x_1(n)^2 . \sum_{n=0}^{N-1} x_2(n)^2}}$$

Convolution:

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k)$$

System identification:

$$h(n) = \frac{y(n) - \sum_{k=0}^{n-1} h(k)x(n-k)}{x(0)}, n \geq 1$$

$$h(0) = \frac{y(0)}{x(0)}$$

Deconvolution:

$$x(n) = \frac{y(n) - \sum_{k=1}^{n} h(k)x(n-k)}{h(0)}, n \geq 1$$

FIR filters:

$$y(n) = \sum_{k=0}^{N-1} h(k)x(n-k)$$

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k}$$

$$A_s = -20\log(\delta_s)$$

$$A_\mathrm{p} = 20\log(1+\delta_p)$$

Linear phase:

$$T_P = \frac{-\theta(\omega)}{\omega}$$

$$T_g = \frac{-d\theta(\omega)}{d\omega}$$

$\theta(\omega) = -\alpha\omega \ \Rightarrow T_P$ and T_g constant

$\therefore$ positive symm., Type 1 and 2

$\theta(\omega) = \beta - \alpha\omega \ \Rightarrow \ T_g$ constant

$\therefore$ negative symm., Type 3 and 4

$$\alpha = \frac{N-1}{2}$$

$$\beta = \pi/2$$

Kaiser window function:

$\beta = 0$	if $A \leq 21\ dB$
$\beta = 0.5842(A-21)^{0.4} + 0.07886(A-21)$	if $21\ dB < A < 50\ dB$
$\beta = 0.1102(A-8.7)$	if $A \geq 50\ dB$

$$N \geq \frac{A - 7.95}{14.36\Delta f}$$

Frequency sampling method, Type 1:

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j2\pi nk/N}$$

For linear phase, pos. symm. and N even

$$h(n) = \frac{1}{N}\left[\sum_{k=1}^{\frac{N}{2}-1} 2|H(k)|\cos[2\pi k(n-\alpha)/N] + H(0)\right]$$

N odd summation over (N-1)/2

IIR filters:

$$y(n) = \sum_{k=0}^{N} b_k x(n-k) - \sum_{k=1}^{M} a_k y(n-k)$$

$$H(z) = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1 + \sum_{k=1}^{M} a_k z^{-k}}$$

$$A_\mathrm{p} = 10\log(1+\varepsilon^2) = -20\log(1-\delta_p)$$

Lowest ripple at $1/\sqrt{1+\varepsilon^2}$

Pole-zero placement:

Angle: $360^o \times f/F_s$

r > 0.9: r = $1 - (bw/F_s)\pi$

Impulse invariant method:

$s = s/\alpha$ $\qquad \alpha = 2\pi f_c/F_s$, T = 1

$$\frac{C}{s-p} \longrightarrow \frac{C}{1-e^{pT}z^{-1}}$$

$$\frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} = \frac{C_1 + C_2 - (C_1 e^{p_2 T} + C_2 e^{p_1 T})z^{-1}}{1-(e^{p_1 T} + e^{p_2 T})z^{-1} + e^{(p_1+p_2)T}z^{-2}}$$

complex conjugate poles:

$$\frac{2C_r - [C_r \cos(p_i T) + C_i \sin(p_i T)]2e^{p_r T}z^{-1}}{1 - 2e^{p_r T}\cos(p_i T)\, z^{-1} + e^{2p_r T}z^{-2}}$$

MZT:

$$s = \frac{s}{\omega_c}$$

$$(s-a) \to (1 - z^{-1}e^{aT})$$

$$H(z) = \frac{1-(e^{z_1 T} + e^{z_2 T})z^{-1} + e^{(z_1+z_2)T}z^{-2}}{1-(e^{p_1 T} + e^{p_2 T})z^{-1} + e^{(p_1+p_2)T}z^{-2}}$$

$$H(z) = \frac{1 - 2e^{z_r T}\cos(z_i T)z^{-1} + e^{z_r T}z^{-2}}{1 - 2e^{p_r T}\cos(p_i T)z^{-1} + e^{p_r T}z^{-2}}$$

$$H(s) = \frac{A_0 + A_1 s + A_2 s^2}{B_0 + B_1 s + B_2 s^2}$$

$$p_{1,2} = -\frac{B_1}{2B_2} \pm \left[\left(\frac{B_1}{2B_2}\right)^2 - \frac{B_0}{B_2}\right]^{1/2}$$

$$z_{1,2} = -\frac{A_1}{2A_2} \pm \left[\left(\frac{A_1}{2A_2}\right)^2 - \frac{A_0}{A_2}\right]^{1/2}$$

BZT:

$$\omega_p' = k\tan\left(\frac{\omega_p T}{2}\right)$$

lp→lp:$s = \frac{s}{\omega_p'}$

lp→hp:$s = \frac{\omega_p'}{s}$

lp→bp:$s = \frac{s^2 + \omega_0^2}{Ws}$

lp→bs:$s = \frac{Ws}{s^2 + \omega_0^2}$

$W = \omega_{p2}' - \omega_{p1}'$

$\omega_0^2 = \omega_{p1}'\omega_{p2}'$

$$s = \frac{z-1}{z+1}$$

Butterworth filter:

$$N \geq \frac{\log\left(\frac{10^{A_s/10}}{10^{A_p/10}} - 1\right)}{2\log\left(\frac{\omega_s^p}{\omega_p^p}\right)}$$

Poles $= s_k = e^{j\pi(2k+N-1)/2N}$

$$= \cos\left[\frac{(2k+N-1)\pi}{2N}\right] + j\sin\left[\frac{(2k+N-1)\pi}{2N}\right]$$

$k = 1, 2, \ldots$

Chebyshev:

$$N \geq \frac{\cosh^{-1}\left(\frac{10^{A_s/10}}{10^{A_p/10}} - 1\right)}{\cosh^{-1}\left(\frac{\omega_s^p}{\omega_p^p}\right)}$$

$$s_k = \sinh(\alpha)\cos(\beta_k) + j\cosh(\alpha)\sin(\beta_k)$$

$$\alpha = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right), \quad \beta_k = \frac{(2k+N-1)\pi}{2N}, k = 1, 2, \ldots$$

BZT Prototype Filters:

Lowpass:

$\omega^p = \frac{\omega_{lp}}{\omega_p'}$ $\qquad \omega_s^p = \frac{\omega_s'}{\omega_p'}$

Highpass:

$\omega^p = -\frac{\omega_p'}{\omega_{hp}}$ $\qquad \omega_s^p = \frac{\omega_p'}{\omega_s'}$

Bandpass:

$\omega^p = \frac{\omega_{bp}^2 - \omega_0^2}{W\omega_{bp}}$ $\qquad \omega_s^p = min\left(\omega_{s1}^p, \left|\omega_{s2}^p\right|\right)$

Bandstop:

$\omega^p = \frac{W\omega_{bs}}{\omega_0^2 - \omega_{bs}^2}$ $\qquad \omega_s^p = min\left(\omega_{s1}^p, \left|\omega_{s2}^p\right|\right)$

Digital filter errors:

$$H_q(\omega) = H(\omega) + E(\omega)$$
$$|E(\omega)| = N2^{-B}$$
$$|E(\omega)| = 2^{-B}\sqrt{N/3}$$
$$|E(\omega)| = 2^{-B}\sqrt{(N\ln N)/3}$$

LMS adaptive algorithm:

$$\hat{n}_k = \sum_{i=0}^{N-1} w_k(i)x_{k-i}$$
$$e_k = y_k - \hat{n}_k$$
$$w_{k+1}(i) = w_k(i) + 2\mu e_k x_{k-i}$$

Bandstop concepts

(1) when $\omega_{bs} = \omega'_{p1}$, $\omega^p = \dfrac{W\omega'_{p1}}{\omega_0'^2 - \omega_{p1}'^2} = \dfrac{(\omega'_{p2} - \omega'_{p1})\omega'_{p1}}{\omega'_{p1}\omega'_{p2} - \omega_{p1}'^2} = 1$

(2) when $\omega_{bs} = \omega'_{s1}$, $\omega^p = \omega_s^{p(1)} = \dfrac{W\omega'_{s1}}{\omega_0^2 - \omega_{s1}^2}$

(3) when $\omega_{bs} = \omega'_{s2}$, $\omega^p = \omega_s^{p(2)} = \dfrac{W\omega'_{s2}}{\omega_0^2 - \omega_{s2}^2}$

(4) when $\omega_{bs} = \omega_0$, $\omega^p = \dfrac{W\omega_0}{\omega_0^2 - \omega_0^2} = \infty$

(5) when $\omega_{bs} = \omega'_{p2}$, $\omega^p = \dfrac{W\omega'_{p2}}{\omega_0^2 - \omega_{p2}'^2} = \dfrac{(\omega'_{p2} - \omega'_{p1})\omega'_{p2}}{\omega'_{p1}\omega'_{p2} - \omega_{p2}'^2} = -1.$

Lowpass concepts

(1) when $\omega_{lp} = 0$, $\omega^p = 0$ (from Equation 8.31)
(2) when $\omega_{lp} = \omega'_p$ (i.e. the passband edge frequency), $\omega^p = \omega'_p/\omega'_p = 1 = \omega_p^p$
(3) when $\omega_{lp} = \omega'_s$, $\omega^p = \omega'_s/\omega'_p = \omega_s^p$.

Highpass concepts

(1) when $\omega_{hp} = 0$, $\omega^p = \infty$ (using Equation 8.32)
(2) when $\omega_{hp} = \omega'_p$ (i.e. the passband edge frequency), $\omega^p = -1$
(3) when $\omega_{hp} = \omega'_s$, $\omega^p = -\dfrac{\omega'_p}{\omega'_s}$
(4) when $\omega_{hp} = -\omega'_p$, $\omega^p = 1$
(5) when $\omega_{hp} = -\omega'_s$, $\omega^p = \dfrac{\omega'_p}{\omega'_s}$.

Bandpass concepts

(1) when $\omega_{bp} = \omega'_{s1}$, $\omega^p = \omega_{s1}'^p = \dfrac{\omega_{s1}'^2 - \omega_0^2}{W\omega'_{s1}}$

(2) when $\omega_{bp} = \omega'_{p1}$, $\omega^p = \dfrac{\omega_{p1}'^2 - \omega_0^2}{W\omega'_{p1}} = \dfrac{\omega_{p1}'^2 - \omega'_{p1}\omega'_{p2}}{(\omega'_{p2} - \omega'_{p1})\omega'_{p1}} = -1$

(3) when $\omega_{bp} = \omega'_{p2}$, $\omega^p = \dfrac{\omega_{p2}'^2 - \omega_0^2}{W\omega'_{p2}} = \dfrac{\omega_{p2}'^2 - \omega'_{p1}\omega'_{p2}}{(\omega'_{p2} - \omega'_{p1})\omega'_{p2}} = 1$

(4) when $\omega_{bp} = \omega'_{s2}$, $\omega^p = \omega_{s2}'^p = \dfrac{\omega_{s2}'^2 - \omega_0^2}{W\omega'_{s2}}$

(5) $\omega_{bp} = \omega_0$, $\omega^p = \dfrac{\omega_0'^2 - \omega_0^2}{W\omega_0^2} = 0$

(6) $\omega_s^p = \min(\omega_{s1}'^p, \omega_{s2}'^p)$.