We will use the same notation as in the article. So, given a prime number p, we will denote by $G_p = (\mathbb{Z}/p\mathbb{Z})^*$, by $\mathrm{Syl}_\ell(G_p)$ the ℓ-pylow subgroup of G_p. Given a prime $\ell \in \mathcal{P}$ we denote by $\pi_\ell : G_p \to \mathrm{Syl}_\ell(G_p)$ the natural projection given by the Sylow decomposition of G_p. Also, denote by $S \subseteq \mathcal{P}$ the set of prime number defined as follows: $\ell \in S$ if and only if $\ell \mid \#G_p$ and $\ell^{-1} \notin R$, where R is the smallest ring in which the equality Conjecture 0.1, Conjecture 2.11, or Conjecture 2.11 is well-defined (depending on which conjecture one is studying).

Recall that the following function is an isomorphism

$$\begin{aligned} \phi : Q_1(R,G) &\to \bigoplus_{\ell \in S} \mathrm{Syl}_\ell(G) \\ \sum_{g\in G} r_g[g] + I(R,G)^2 &\mapsto \left(\prod_{g\in G} \pi_\ell(g)^{r_g} \right)_{\ell \in S}, \end{aligned}$$

where $(-)_{\ell\in S}$ denotes a tuple in $\bigoplus_{\ell\in S} \mathrm{Syl}_\ell(G)$ indexed by S.

Now, write $S = \{p_1, ..., p_n\}$ the primes that divide G_p. So,

$$G_p = \mathrm{Syl}_{p_1}(G_p) \times \cdots \mathrm{Syl}_{p_n}(G_p)$$

Fix a prime $p_i \in S$. Given a $g \in G_p$ we could try to write

$$g = g_{p_1} \cdots g_{p_n} \tag{1}$$

where $g_{p_i} \in \mathrm{Syl}_{p_i}(G_p)$ and the projection π_{p_i} of g would be $\pi_{p_i}(g) = g_{p_i}$. However, this is a computational intensive task. Another way of getting a projection $\pi_{p_i} : G_p \to \mathrm{Syl}_{p_i}(G_p)$ is raising g to a sufficient large number. More specifically, consider

$$m = \prod_{p_j \in S\setminus p_i} \#\mathrm{Syl}_{p_j}(G_p), \tag{2}$$

then the map $g \mapsto g^m$ is a surjective homomorphism from G_p to Syl_{p_i}. Indeed, because G_p is an abelian group, then by equation (1) we have that

$$g^m = g_{p_1}^m \cdots g_{p_i}^m \cdots g_{p_n}^m = e \cdots g_{p_i}^m \cdots e$$

because we are raising $g_{p_j} \in \mathrm{Syl}_{p_j}(G_p)$ and we are raising g_{p_j} to a multiple of $\mathrm{Syl}_{p_j}(G_p)$ so $g_{p_i}^m = e$ (Lagrange's Theorem). Because $(m, \#\mathrm{Syl}_{p_i}(G_p)) = 1$, then the map $g \mapsto g^m$ is an automorphism of $\mathrm{Syl}_{p_i}(G_p)$. Denote by $\rho : \mathrm{Syl}_{p_i}(G_p) \to \mathrm{Syl}_{p_i}(G_p)$; $g \mapsto g^m$.

So, we have that

$$a \equiv b \text{ in } \mathrm{Syl}_{p_i}(G_p) \Leftrightarrow \rho(a) \equiv \rho(b) \text{ in } \mathrm{Syl}_{p_i}(G_p)$$

Thus, if we consider the equation given by Conjecture 0.1

$$\begin{aligned} \prod_{a\in G_p} \pi_{p_i}(a)^{\lambda(a,p)} \equiv \pi_{p_i}(\tilde{q}_p)^{\frac{\lambda(0,1)}{2\mathrm{ord}_p(q_p)}} &\Leftrightarrow \rho\left(\prod_{a\in G_p} \pi_{p_i}(a)^{\lambda(a,p)} \right) \equiv \rho\left(\pi_{p_i}(\tilde{q}_p)^{\frac{\lambda(0,1)}{2\mathrm{ord}_p(q_p)}} \right) \\ &\Leftrightarrow \prod_{a\in G_p} \rho\left(\pi_{p_i}(a)\right)^{\lambda(a,p)} \equiv \rho\left(\pi_{p_i}(\tilde{q}_p)\right)^{\frac{\lambda(0,1)}{2\mathrm{ord}_p(q_p)}} \\ &\Leftrightarrow \prod_{a\in G_p} \pi_{p_i}(\rho(a))^{\lambda(a,p)} \equiv \pi_{p_i}(\rho(\tilde{q}_p))^{\frac{\lambda(0,1)}{2\mathrm{ord}_p(q_p)}} \end{aligned}$$

Thus, to check Conjecure 0.1, Conjecture 2.11, and Conjecture 2.12, we raise the elements in G before projecting to the corresponding ℓ-Sylow subgroup and then the projection becomes the identity. Which is the reason to why we do not include the left hand-side and right hand-side of the equations in "Table conjecture 0 1.pdf" and "Table conjecture 2 12.pdf".

Remark 0.1. We want to remark that instead of calculating m, as in equation (2), we use the fact that $R = \mathbb{Z}\left[\frac{1}{N}\right]$, and $m \mid N^c$ for $c >> 0$. In our case because the highest conductor is around $90,000$ considering $c = 17$ is sufficient, because $2^{17} > 100,000$. So for any prime $\ell \mid \#G_p$, we have that the image of the map $g \mapsto g^{\ell^{17}}$ is $\mathrm{Syl}_\ell(G_p)$[1].

Finally, note that the map $(\mathbb{Z}/p\mathbb{Z})^* \to (\mathbb{Z}/p\mathbb{Z})^*\ ; g \mapsto g^2$ is precisely $\langle -1 \rangle$, because $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic and thus the only subgroup of order 2 is $\langle -1 \rangle$.

[1]Instead of considering the map $g \mapsto g^{\ell^{17}}$ we compose the map $g \mapsto g^{\ell}$ with itself 17 times, each time considering it modulo p. Thus the calculations do not become so computationally intensive.