AN ALPHA MODIFICATION OF NEWMARK'S METHOD

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SUMMARY

This paper discusses the Bossak–Newmark algorithm, which is an extension of the well-known Newmark algorithm[1] for the numerical integration of the equations of discretized structural dynamics problems. The extra parameter introduced here enables the method (when used on the test equation $\ddot{x} = -\omega^2 x$) to be simultaneously second order, unconditionally stable and with positive artificial damping.

Comparisons are made with another modification of Newmark introduced by Hilber, Hughes and Taylor.[2]

In many structural dynamics applications the equations of motion for the discretized system have the form

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{F} \tag{1}$$

where M, C, K are the mass, damping and stiffness matrices, respectively; $\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}$ are the displacement, velocity and acceleration vectors, respectively; and $\mathbf{F}$ is the external force vector.

The well-known Newmark algorithm[1] for the numerical integration of equation (1) is defined by

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t \dot{\mathbf{x}}_n + (\Delta t)^2(\tfrac{1}{2} - \beta_{\mathrm{N}})\ddot{\mathbf{x}}_n + (\Delta t)^2 \beta_{\mathrm{N}} \ddot{\mathbf{x}}_{n+1} \tag{2}$$

$$\dot{\mathbf{x}}_{n+1} = \dot{\mathbf{x}}_n + \Delta t(1 - \gamma_{\mathrm{N}})\ddot{\mathbf{x}}_n + \Delta t \gamma_{\mathrm{N}} \ddot{\mathbf{x}}_{n+1} \tag{3}$$

$$M\ddot{\mathbf{x}}_{n+1} + C\mathbf{x}_{n+1} + K\mathbf{x}_{n+1} = \mathbf{F}_{n+1} \tag{4}$$

where the Newmark parameters are distinguished by the subscript 'N' to avoid confusion with other parameters in this paper.

The idea of introducing an additional parameter for controlling the damping properties of Newmark's algorithm was proposed in 1977 by Hilber, Hughes and Taylor.[2] Hilber, Hughes and Taylor introduce a parameter, called here α_{H} to avoid confusion, which they apply to the equation without natural damping and with which equation (4) is replaced by

$$M\ddot{\mathbf{x}}_{n+1} + (1 + \alpha_{\mathrm{H}})K\mathbf{x}_{n+1} - \alpha_{\mathrm{H}} K\mathbf{x}_n = \mathbf{F}_{n+1} \tag{4'}$$

A particular version of this algorithm has also been used by Aboudi.[3]

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This paper considers an extension of Newmark's method suggested by Bossak and defined by

$$\mathbf{x}_{n+1}=\mathbf{x}_n+\Delta t\dot{\mathbf{x}}_n+(\Delta t)^2(\tfrac{1}{2}-\beta_{\mathrm{B}})\ddot{\mathbf{x}}_n+(\Delta t)^2\beta\ddot{\mathbf{x}}_{n+1} \tag{5}$$

$$\dot{\mathbf{x}}_{n+1}=\dot{\mathbf{x}}_n+\Delta t(1-\gamma_{\mathrm{B}})\ddot{\mathbf{x}}_n+\Delta t\gamma_{\mathrm{B}}\ddot{\mathbf{x}}_{n+1} \tag{6}$$

$$(1-\alpha_{\mathrm{B}})M\ddot{\mathbf{x}}_{n+1}+\alpha_{\mathrm{B}}M\ddot{\mathbf{x}}_n+C\dot{\mathbf{x}}_{n+1}+K\mathbf{x}_{n+1}=\mathbf{F}_{n+1} \tag{7}$$

where the Bossak parameters are distinguished by the subscript 'B'. Thus, when $\alpha_{\mathrm{B}}=0$ we have $\beta_{\mathrm{B}}=\beta_{\mathrm{N}}$, $\gamma_{\mathrm{B}}=\gamma_{\mathrm{N}}$.

The Bossak–Newmark algorithm in equations (5), (6), (7) employs only two time levels and can be used iteratively to cope with nonlinear terms or in an equivalent form for linear problems, just as the Newmark algorithm, but has the additional parameter α_{B}.

We analyse the Bossak–Newmark algorithm in the usual way by looking at its effect on the scalar equation

$$\ddot{x}=-\omega^2x \tag{8}$$

thus avoiding any assumptions about the damping matrix which are necessary in order to make a modal decomposition of equation (1).

The Bossak–Newmark method, when applied to equation (8) and reduced to a difference equation in the displacements, becomes

$$\sum_{j=0}^{3}\alpha_jx_{n+j}=-\omega^2(\Delta t)^2\sum_{j=0}^{3}\beta_jx_{n+j} \tag{9}$$

where

$$\alpha_3=1,\qquad \beta_3=\frac{\beta_{\mathrm{B}}}{1-\alpha_{\mathrm{B}}}$$

$$\alpha_2=\frac{3\alpha_{\mathrm{B}}-2}{1-\alpha_{\mathrm{B}}},\qquad \beta_2=\frac{1-4\beta_{\mathrm{B}}+2\gamma_{\mathrm{B}}}{2(1-\alpha_{\mathrm{B}})}$$

$$\alpha_1=\frac{1-3\alpha_{\mathrm{B}}}{1-\alpha_{\mathrm{B}}},\qquad \beta_1=\frac{1+2\beta_{\mathrm{B}}-2\gamma_{\mathrm{B}}}{2(1-\alpha_{\mathrm{B}})}$$

$$\alpha_0=\frac{\alpha_{\mathrm{B}}}{1-\alpha_{\mathrm{B}}},\qquad \beta_0=0$$

To analyse this method as in Lambert,[4] we define

$$\rho(r)=\sum_{j=0}^{3}\alpha_jr^j,\qquad \sigma(r)=\sum_{j=0}^{3}\beta_jr^j$$

Then the Bossak–Newmark method is consistent because $\rho(1)=0$, $\rho'(1)=0$, $\rho''(1)-2\sigma(1)=0$. If we define

$$C_q=\frac{1}{q!}[\alpha_1+2^q\alpha_2+3^q\alpha_3]-\frac{1}{(q-2)!}[\beta_1+2^{q-2}\beta_2+3^{q-2}\beta_3],\quad q\geqslant 2 \tag{10}$$

then following Lambert[2] we can say that the method has order 1 in general, since $C_3\neq 0$.

Since the Bossak–Newmark method is equivalent to a four time-level scheme in the displacements, equation (9), it is interesting to compare it with the general family of four time-level schemes proposed by Zienkiewicz[5] and analysed by Wood.[6] Because of their construction by Weighted Residual Methods, the Zienkiewicz family have $C_3 = 0$ for any values of the Zienkiewicz parameters, here referred to as α_Z, β_Z, γ_Z.

For the Bossak–Newmark method

$$C_3 = \frac{1 - 2\alpha_B - 2\gamma_B}{2(1 - \alpha_B)} \tag{11}$$

Hence we can make this method comparable with the Zienkiewicz family (i.e. of order 2) if we take

$$\alpha_B = \tfrac{1}{2} - \gamma_B \tag{12}$$

and choose α_Z so that $\beta_0 = 0$ in the Zienkiewicz method:[5]

$$\alpha_Z = 6\beta_Z - 11\gamma_Z + 6 \tag{13}$$

The methods are then equivalent with

and
$$\begin{cases} \gamma_Z = \tfrac{3}{2} + \gamma_B \\ \\ \beta_Z = 2\beta_B + 3\gamma_B + \tfrac{5}{2} \end{cases} \tag{14}$$

each having two free parameters.

The Bossak–Newmark three-parameter method has unconditional stability for

$$\alpha_B \leqslant \tfrac{1}{2} \qquad \beta_B \geqslant \frac{\gamma_B}{2} \geqslant \tfrac{1}{4}, \qquad \alpha_B + \gamma_B \geqslant \tfrac{1}{2} \tag{15}$$

using the Routh–Hurwitz criterion.[7]

With α_B chosen as in equation (12) so that the method is second order, we have unconditional stability if

$$\beta_B \geqslant \frac{\gamma_B}{2} \geqslant \tfrac{1}{4} \text{ only}$$

Of course, if $\alpha_B = 0$, $\gamma_B = \gamma_N = \tfrac{1}{2}$ gives zero artificial damping. The Newmark method can only be second order with zero artificial damping.

For a second-order method with positive artificial damping we require, for unconditional stability,

$$\beta_B > \frac{\gamma_B}{2} > \tfrac{1}{4} \qquad \alpha_B = \tfrac{1}{2} - \gamma_B. \tag{16}$$

For the Bossak–Newmark method with $C_3 = 0$ then

$$C_4 = \frac{1 - 12\beta_B}{6(1 - 12\gamma_B)} \tag{17}$$

Thus $\beta_B = \tfrac{1}{12}$ gives the most accurate following of phase (as with the Newmark $\beta_N = \tfrac{1}{12}$, $\gamma_N = \tfrac{1}{2}$), but this is not consistent with the conditions (16) for unconditional stability.

Figures 1 and 2 illustrate some numerical results. The curve labelled N in each is Newmark with $\beta_N = 0{\cdot}25$, $\gamma_N = 0{\cdot}5$. This is the Newmark method which gives optimum following of phase with unconditional stability but, of course, it gives zero artificial damping.

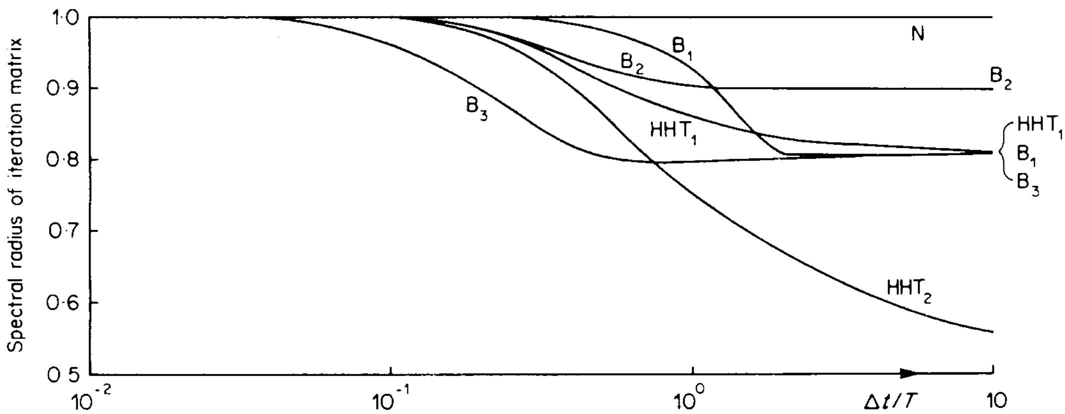

Figure 1

The curves B_1, B_2, B_3 show some results with Bossak–Newmark:

$$B_1: \alpha_B = -0{\cdot}1,\ \beta_B = 0{\cdot}3025, \qquad \gamma_B = 0{\cdot}6$$

$$B_2: \alpha_B = -0{\cdot}1,\ \beta_B = 0{\cdot}5, \qquad \gamma_B = 0{\cdot}6$$

$$B_3: \alpha_B = +0{\cdot}1,\ \beta_B = 0{\cdot}3025, \qquad \gamma_B = 0{\cdot}6$$

In B_1 and B_2 the parameters are chosen to satisfy the condition (12) for second-order accuracy. It is then not possible to have a better phase following than with the Newmark N curve

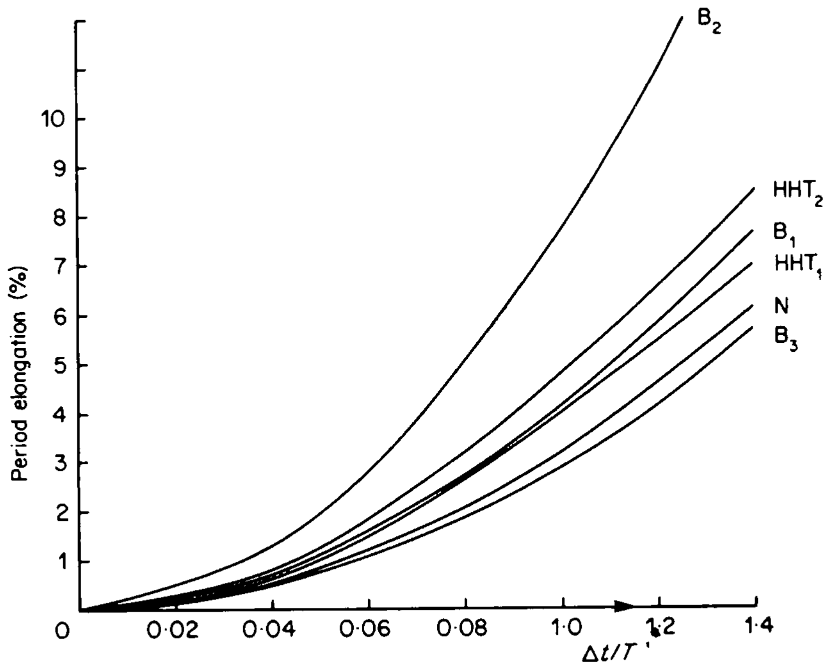

Figure 2

parameters. However, with B_3 (with some cancellation of errors at these frequencies) the phase following is better than with Newmark and there is also the desirable damping of the higher modes.

Results from Hilber, Hughes and Taylor[2] are also shown for comparison:

The curves HHT_1 are for $\alpha_H = -0{\cdot}1$, $\beta = 0{\cdot}3025$, $\gamma = 0{\cdot}6$

and HHT_2 are for $\alpha_H = -0{\cdot}3$, $\beta = 0{\cdot}3025$, $\gamma = 0{\cdot}6$.

The graphs show that B_3 falls away from a spectral radius of unity for a lower value of $\Delta t/T$ than the HHT curves, but its per cent period of elongation is better than the optimum Newmark.

CONCLUSION

The Bossak–Newmark method has the same facility as the original Newmark for use either iteratively with nonlinear problems or in an equivalent form for linear problems, and also uses acceleration, velocity and displacement vectors at only two time levels. With its extra parameter it is possible to make the method when applied to equations of the form $\ddot{x} = -\omega^2 x$, unconditionally stable, with positive artificial damping of the higher modes and with a better phase following in the lower modes than with Newmark.

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