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Information Processing Letters 68 (1998) 55–61

Information Processing Letters

Some remarks on the Kronecker product of graphs ☆

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Received 19 February 1998; received in revised form 2 September 1998
Communicated by L. Boasson

Abstract

This short note is concerned with the Kronecker product of graphs; we give some properties linked to graph minors, planarity, cut vertex and cut edge. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Kronecker product; Cut edge; Cut vertex; Graph minor; Planarity; Combinatorial problems

1. Introduction

The Kronecker product of graphs is one of the usual names of the categorical product of graphs. This product is also called the tensor product or the strong product. The names tensor product or Kronecker product come from the matricial product commonly attributed to Kronecker. This product of graphs was studied by various authors. P.M. Weichsel [8] was interested in the connectedness of the Kronecker product of two connected graphs. D.J. Miller studied the connectivity in [6]. W. Dörfler compared the Kronecker product to the Cartesian product in [3]. D.A. Waller worked in [7] on an extension of the Kronecker product by the complete graph with two vertices (he called it the *double cover*). Some authors studied the planarity, for example Farzan and Waller in [4], and the outerplanarity, for example Jha and Slutzki in [5].

Our work on the Kronecker product of graphs is motivated by the study of local computations on graphs [2], and more particularly by the fact that the Kronecker product of a graph G by K_2 is a covering of G (we recall that a graph G' is a covering of a graph G if there exists a surjective homomorphism γ from G' onto G such that for every vertex v of $V(G')$ the restriction of γ to the neighbours of v is a bijection onto the neighbors of $\gamma(v)$). By this way, we study the following question: *given two connected graphs whose properties are known, what's happening to these properties when we do the Kronecker product of these two graphs*?

We give some properties of the Kronecker product linked to graph minors, planarity, cut vertex, and cut edge. Concerning graph minors, we prove that, for every connected graph G, G is a minor of $G \wedge K_3$, a minor of $G \wedge H$ where H has an odd cycle, and then a minor of $G \wedge G$. Finally we conjecture that P_3 (the simple path with 3 vertices) is the smallest graph verifying the fact that G is a minor of $G \wedge P_3$.

☆ This work has been supported by the EC TMR Network GETGRATS (General Theory of Graph Transformation) through the University of Bordeaux.

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PII: S0020-0190(98)00145-8

2. Basic notions and notation

In this part, we fix notation and recall the basic notions on graphs used through the paper [1].

2.1. Graph

All graphs considered in this paper are finite, undirected and simple (i.e., without multiple edges and self-loops). A graph G, denoted $(V(G), E(G))$, is defined by a finite vertex-set and a finite edge-set. An edge with end-points v and v' is denoted $\{v, v'\}$. An *odd* (respectively *even*) *cycle* is a cycle of odd (respectively even) length. An *homomorphism* between two graphs G and H is a mapping γ from $V(G)$ to $V(H)$ such that if $\{u, v\}$ is an edge of G, then $\{\gamma(u), \gamma(v)\}$ is an edge of H. We say that γ is an *isomorphism* if γ is bijective and γ^{-1} is also an homomorphism. An *automorphism* γ of a graph G is an isomorphism of G onto G. By K_n we denote the complete graph with n vertices. Moreover we denote by C_n the cycle graph with n vertices (i.e., a connected graph with n vertices of degree 2). If it is possible to partition the vertices of a graph G into two subsets V_1 and V_2 such that every edge of G connects a vertex in V_1 to a vertex in V_2 then G is called a *bipartite graph* and is sometimes denoted (V_1, V_2, E) with E as edge-set. If every vertex of V_1 is connected to every vertex of V_2 then G is a *complete bipartite graph* denoted $K_{i,j}$ where $i = |V_1|$ and $j = |V_2|$.

2.2. The Kronecker product of graphs

The Kronecker product was firstly defined on matrices. Here we deal with its extension on graphs as it is presented in [8].

Definition 1. Let G and H be two connected graphs, the Kronecker product of G by H denoted $G \wedge H$ is the graph defined by the couple (V, E) where

$$V = \{(v, w) \mid v \in V(G),\ w \in V(H)\} = V(G) \times V(H),$$

$$E = \{\{(v, w), (v', w')\} \mid \{v, v'\} \in E(G),\ \{w, w'\} \in E(H)\}.$$

This graph product is a commutative and associative operation through isomorphisms. It is easy to prove that the product of a cycle of length n by an edge is a cycle of length $2n$ if n is odd, or a disjoint union of two cycles of length n if n is even. From the definition we get immediately the following properties. Let the graph $K = G \wedge H = (V, E)$, then:

(1) $|V| = |V(G)| * |V(H)|$,
(2) $|E| = 2 * |E(G)| * |E(H)|$,
(3) for every $(v, w) \in V$,

$$\deg((v, w)) = \deg_G(v) * \deg_H(w).$$

We give some simple facts and recall properties.

Fact 1. *Let $G = (V, E)$ be a connected graph, and $H = (V_1, V_2, E')$ be a bipartite connected graph, then $G \wedge H$ is a bipartite graph, the partition of the vertex-set is $(V \times V_1)$ and $(V \times V_2)$.*

An automorphism of a graph is *trivial* if it is the identity. The automorphism group of a graph is *trivial* if it is restricted to the identity automorphism.

Given two connected graphs G and H, it is easy to define an automorphism of $G \wedge H$ proceeding from automorphisms of G and H. Let g (respectively h) be an automorphism of G (respectively H):

$$\gamma : V(G \wedge H) \to V(G \wedge H),\quad (x, y) \mapsto (g(x), h(y)).$$

Then γ is an automorphism of $G \wedge H$. Therefore,

Fact 2. *Let G and H be two connected graphs. If the automorphism group of G or H is nontrivial, then the automorphism group of $G \wedge H$ is nontrivial.*

The last property deals with the relation of subgraph.

Fact 3. *Let G and H be two connected graphs such that G is a subgraph of H, then for every graph K, $G \wedge K$ is a subgraph of $H \wedge K$.*

P.M. Weichsel [8] characterizes the connectivity of the Kronecker product of two connected graphs.

Theorem 1. *Let G and H be connected graphs. The graph $G \wedge H$ is connected if and only if any G or H contains an odd cycle.*

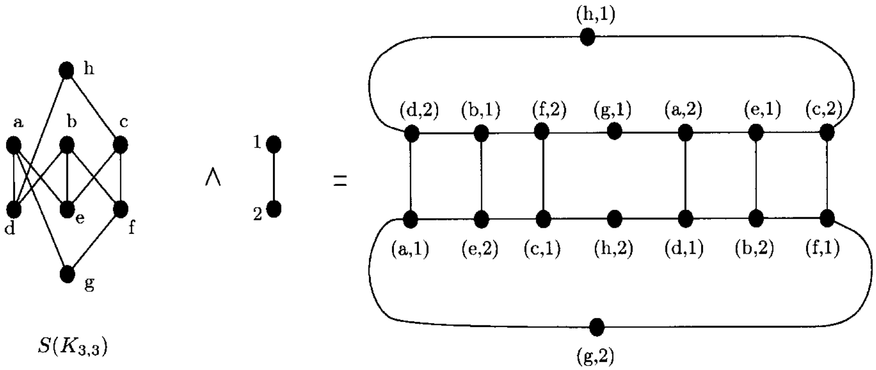

Fig. 1. The product of a subdivision of $K_{3,3}$ by K_2 is planar.

Corollary 2. *If G and H are connected graphs with no odd cycles then $G \wedge H$ has exactly two connected components.*

M. Farzan and D.A. Waller [4] gave a simpler proof of this theorem. As bipartite graphs are exactly the graphs without odd cycles, Weichsel's theorem can be stated in the following way:

Proposition 3. *The Kronecker product of two connected graphs is a non-connected graph if and only if both are bipartite.*

3. Graph properties and Kronecker product

3.1. Graph minors

Let us recall some well-known definitions of operations on edges. *Contracting an edge* linking vertices u and v consists in fusing u and v, deleting the resulting loop and replacing multiple edges, that may arise, by a simple edge. By *deleting an edge* we mean deleting this edge and the isolated vertex created. We say that G is a *minor* of H, denoted $G \lhd H$, if there is a sequence of contractions and of deletions of edges of H which leads to a graph isomorphic to G.

In this part, our interest is to find the minimal graph H such that every graph G is a minor of its Kronecker product by H. Jha and Slutzki prove in [5] that for a certain family of graphs (the almost bipartite graphs), any graph is a minor of its product by K_2. They conjecture in [5] that any graph is a minor of its product by K_2. We give a counter-example to this conjecture in Fig. 1. It is a subdivision of $K_{3,3}$, thus a non-planar graph, whose Kronecker product by K_2 gives a planar graph.

As K_2 is not a solution for our problem (of finding the minimal graph H such that for every graph G, G is a minor of the Kronecker product of G by H), we consider the case of K_3.

Proposition 4. *For every connected graph G, G is a minor of $G \wedge K_3$.*

Proof. Let G be a connected graph, we denote by G_3 the graph $G \wedge K_3$. Let $\{1, 2, 3\}$ be the vertex-set of K_3. In order to prove that G is a minor of G_3 we construct a sequence of deletions and contractions of edges of G_3, which leads to a subgraph isomorphic to G. Let $\{x, y\}$ be an edge of G, then there is in G_3 the cycle:

$$C(x, y) = \big((x, 1), (y, 2), (x, 3), (y, 1), (x, 2), (y, 3)\big).$$

The vertices x and y are *marked* in G as soon as we contract an edge of $C(x, y)$ or we delete an edge incident with a vertex of $C(x, y)$ for some edge $\{x, y\}$.

Initially all the vertices of G are unmarked. Then for each edge $\{x, y\}$ of G such that neither x nor y are marked we do the following sequence of operations on $C(x, y)$:

(1) Deletion of all the edges adjacent to $(x, 2)$ and $(y, 2)$ but which are not edges of $C(x, y)$.
(2) Contraction of the edges $\{(x, 1), (y, 2)\}$, $\{(y, 2), (x, 3)\}$, $\{(y, 1), (x, 2)\}$ and $\{(x, 2), (y, 3)\}$; we obtain two vertices renamed, in a natural way, $(x, 1)$ and $(y, 1)$.

Thus, after the operation, we have:

- the vertices x and y are marked in G;
- the edge $\{(x,1),(y,1)\}$ is in the new graph.

We go on until we cannot find any edge of G with both unmarked end-points.

We denote by R the graph obtained at the end of the procedure. By construction, R is a minor of G_3.

Moreover we have:

Let $\{x, y\}$ be an edge of G, then in G_3 there are the edges $\{(x,1),(y,3)\}$ and $\{(x,3),(y,1)\}$. These edges are not deleted by the operations. If x (respectively y) is marked, $(x,3)$ (respectively $(y,3)$) was renamed $(x,1)$ (respectively $(y,1)$) in R. At the end of the computation, at least one vertex of each edge of G is marked, thus for each edge $\{x,y\}$ of G, there is at least $\{(x,1),(y,1)\}$ in R. Conversely, if $\{(x,1),(y,1)\} \in E(R)$ then by construction $(x,y) \in E(G)$. Finally,

$$\{x,y\} \in E(G) \Leftrightarrow \big\{(x,1),(y,1)\big\} \in E(R).$$

Let H be the subgraph of R induced by the vertex-set

$$V(H) = \big\{(v,1) \in V(R) \mid v \in V(G)\big\}.$$

The graph H is isomorphic to G and is a minor of G_3. □

The graph K_3 is obviously an odd cycle of length 3. The previous result may be generalized:

Proposition 5. *For every connected graph H containing an odd cycle, and for every connected graph G, G is a minor of $G \wedge H$.*

Proof. Let G be a connected graph, let H be a connected graph containing at least one odd cycle. As for Proposition 4, we want to give a sequence of deletions and contractions of edges of $G \wedge H$ in order to obtain a subgraph isomorphic to G.

Let us denote by $(1, 2, \ldots, p)$ an odd cycle of H. Let $\{x, y\}$ be an edge of G, there is in $G \wedge H$ the cycle:

$$C(x,y) = \big((x,1),(y,2),\ldots,(y,p-1),(x,p),\\ (y,1),(x,2),\ldots,(x,p-1),(y,p)\big).$$

We denote by $P(x,y)$ the following set of vertices:

$$P(x,y) = \big\{(y,2),(x,3),\ldots,(y,p-1),\\ (x,2),(y,3),\ldots,(x,p-1)\big\}.$$

The vertices x and y are marked in G as soon as we contract an edge of $C(x,y)$ or we delete an edge incident with a vertex of $P(x,y)$.

At the beginning all the vertices of G are unmarked. Then for each edge $\{x,y\}$ of G such that neither x nor y are marked we do the following sequence of operations on $C(x,y)$ in $G \wedge H$:

(1) For each $(v,j) \in P(x,y)$, deletion of all the edges incident with (v,j) but which are not edges of the cycle $C(x,y)$.

(2) Contraction of the edges $\{(x,1),(y,2)\}, \{(y,2),(x,3)\}, \ldots, \{(y,p-1),(x,p)\}$ and $\{(y,1),(x,2)\}, \{(x,2),(y,3)\}, \ldots, \{(x,p-1),(y,p)\}$, we rename, in a natural way, the resulting vertices $(x,1)$ and $(y,1)$.

Then we have:

- the vertices x and y are now marked in G;
- the edge $\{(x,1),(y,1)\}$ is an edge of the new graph obtained after a sequence of operations on $C(x,y)$.

We go on until we cannot find any edge of G with end-points both unmarked.

We denote by R the graph we have at the end of the procedure. By construction R is a minor of $G \wedge H$. As in Proposition 4, we verify

$$\{x,y\} \in E(G) \Leftrightarrow \big\{(x,1),(y,1)\big\} \in E(R).$$

Let S_R be the subgraph of R induced by the vertex-set $V(S_R) = \{(v,1) \in V(R) \mid v \in V(G)\}$, we have also:

$$E(S_R) = \big\{\{(v,1),(w,1)\} \in E(R) \mid \{v,w\} \in E(G)\big\}.$$

The subgraph S_R is isomorphic to G and is a minor of $G \wedge H$. □

Eventually, from Fact 3, Propositions 5 and 7, proved below, we get:

Proposition 6. *For every graph G, G is a minor of $G \wedge G$.*

Proof. If G has an odd cycle, this is obvious by Proposition 5.

Assume that G has no odd cycle, then by Proposition 7, $G \wedge K_2$ consists of two disjoint connected components isomorphic to G. The graph K_2 is an obvious subgraph of G, thus by Fact 3, $G \wedge K_2$ is a subgraph of $G \wedge G$, and G is a minor of $G \wedge G$. □

4. Properties of the Kronecker product by K_2

We are now interested in the properties of the Kronecker product by the complete graph K_2. We assume that $\{1,2\}$ is the vertex-set of K_2. By taking into account the particularities of the complete graph K_2, we have for any connected graph G:

(1) $|V(G \wedge K_2)| = 2 * |V(G)|$.
(2) $|E(G \wedge K_2)| = 2 * |E(G)|$.
(3) $\forall (x,i) \in V(G \wedge K_2)$, $\deg_{G \wedge K_2}((x,i)) = \deg_G(x)$.
(4) If G is regular of degree d, then $G \wedge K_2$ is regular of degree d.
(5) The Kronecker product of G by K_2 is a bipartite graph with $\{(x,1) \mid x \in V(G)\} \cup \{(x,2) \mid x \in V(G)\}$ as vertex-set.

From [8], $G \wedge K_2$ is a connected graph if and only if G has an odd cycle. More precisely,

Proposition 7. *Let G be a connected graph. If G has no odd cycle, then $G \wedge K_2$ has exactly two connected components isomorphic to G.*

Proof. Let $G = (V, E)$ be a connected graph, let H be $G \wedge K_2$. Let $\{1,2\}$ be the vertex-set of K_2.

We assume that G has no odd cycle. There is no path between $(v,1)$ and $(v,2)$ in H. If not, let $P = ((v_1,1),(v_2,2),\ldots,(v_{n-1},1),(v_n,2))$ be such a path where $v_1 = v_n = v$. It induces in G the odd cycle $(v_1, v_2, \ldots, v_n)$ which is in contradiction with the hypothesis on G.

Now let w be another vertex in G, if the distance between v and w in G is even, then $(w,1)$ (respectively $(w,2)$) is connected with $(v,1)$ (respectively $(v,2)$) in H. Otherwise the distance between v and w is odd and $(w,1)$ (respectively $(w,2)$) is connected with $(v,2)$ (respectively $(v,1)$). Therefore H consists of two connected components: H_1 and H_2.

Let γ_i (for $i = 1,2$) be the map from $V(H_i)$ to $V(G)$ such that for every (v,i) in $V(H_i)$, $\gamma_i((v,i)) = v$. It is easy to see that γ_i is an homomorphism ($i = 0,1$).

Moreover for every v in G, $(v,1)$ is in H_1 iff $(v,2)$ is in H_2. Thus γ_i is an isomorphism. The graph H_1 (respectively H_2) is isomorphic to G via γ_1 (respectively γ_2). □

The graph K_2 has a nontrivial automorphism, thus from Fact 2:

Fact 4. *For any graph G, the automorphism group of $G \wedge K_2$ is nontrivial.*

Aut$(G \wedge K_2)$ contains at least ϕ:

$$\phi : V(G \wedge K_2) \to V(G \wedge K_2)$$
$$(x,1) \mapsto (x,2)$$
$$(x,2) \mapsto (x,1)$$

Cut vertex

A connected graph may contain a vertex such that its deletion disconnects the graph, such a vertex is called a *cut vertex*. An edge of a connected graph such that its removal disconnects the graph is called a *cut edge*.

Thanks to the automorphism ϕ, it is obvious that there is an even number of cut vertices and cut edges in $G \wedge K_2$: the image by ϕ of a cut vertex (respectively cut edge) is a cut vertex (respectively cut edge).

Fact 5. *Let G be a connected graph, x a cut vertex of G. Let $\{G_1, G_2, \ldots, G_k\}$ be the set of the connected components which results from the deletion of x. Let i be an integer of $\{1,\ldots,k\}$, we denote by $N_i(x)$ the neighbourhood of x in G_i:*

$$N_i(x) = \{v \in V(G_i) \mid \{v,x\} \in E(G)\}.$$

Then $G \wedge K_2$ is composed of $(x,1)$ and $(x,2)$ connected with all the components $G_i \wedge K_2$:

$$V(G \wedge K_2) = \{(x,1)\} \cup \{(x,2)\} \cup \left(\bigcup_{i=1}^{k} V(G_i \wedge K_2)\right),$$

$$E(G \wedge K_2) = \bigcup_{i=1}^{k} E(G_i \wedge K_2) \cup \{\{(x,1),(v,2)\} \mid v \in N_i(x)\} \cup \{\{(x,2),(v,1)\} \mid v \in N_i(x)\}.$$

Let x be a vertex of G. If x is not a cut vertex of G then it belongs to a cycle in G. Thus $(x,1)$ and $(x,2)$ are not cut vertices of $(G \wedge K_2)$. Therefore cut vertices of $G \wedge K_2$ may be produced only by cut vertices of G. We have:

Proposition 8. *Let G be a connected graph, x a cut vertex of G. We denote by $\{G_1, G_2, \ldots, G_k\}$ the set of the connected components which results from the deletion of x. We denote for each $i \in \{1, \ldots, k\}$, $N_i(x)$ the neighbourhood of x in G_i. Then $(x, 1)$ and $(x, 2)$ are cut vertices of $G \wedge K_2$ if and only if there exists an $i \in \{1, \ldots, k\}$ such that*

- *G_i contains no odd cycle (i.e., G_i is a bipartite graph: we assume that $G_i = (V_i, V_i', E(G_i)))$, and*
- *$N_i(x) \subseteq V_i$ or $N_i(x) \subseteq V_i'$.*

Proof. Let G be a connected graph, x a cut vertex of G. We denote $\{G_1, G_2, \ldots, G_k\}$ the set of the connected components which results from the deletion of x.

First we suppose that $(x, 1)$ is a cut vertex of $G \wedge K_2$. We want to prove by contradiction that there is i such that G_i has no odd cycle. So we suppose that for all i, G_i has odd cycles. Applying Proposition 7, for all i, $G_i \wedge K_2$ is a connected graph. Moreover by Fact 5, we have: $\forall i$, $\exists (u, 1) \in V(G_i \wedge K_2)$ such that $\{(x, 2), (u, 1)\} \in E(G \wedge K_2)$.

Hence the deletion of $(x, 1)$ does not disconnect the graph $G \wedge K_2$. This leads to a contradiction. Therefore, there is i such that $G_i \wedge K_2$ is disconnected, i.e. (by Proposition 7) G_i has no odd cycle or is bipartite. We denote $V(G_i) = V_i \cup V_i'$ with $\forall u, v \in V_i$ (respectively V_i'), $\{u, v\} \notin E(G_i)$. Now we prove by contradiction that $N_i(x) \subseteq V_i$ or $N_i(x) \subseteq V_i'$. Suppose now that there is u and v in $N_i(x)$ such that $u \in V_i$ and $v \in V_i'$. As G_i is a bipartite and connected component: there is an odd path between u and v in G_i. This implies that $(u, 1)$ and $(v, 2)$ are in the same component in $G_i \wedge K_2$, denoted C_1. As in $G \wedge K_2$ we have the edges $\{(u, 1), (x, 2)\}$ and $\{(v, 2), (x, 1)\}$: the deletion of $(x, 1)$ does not disconnect C_1 from the rest of $G \wedge K_2$ which is a contradiction with the hypotheses.

Conversely, we suppose that there is a bipartite component G_i in G. Let us denote its vertex-set as $V_i \cup V_i'$. We also suppose that $N_i(x) \subseteq V_i$ (it would be the same with V_i'). Then, applying Proposition 7, the component $G_i \wedge K_2$ is disconnected. The vertex $(x, 1)$ is adjacent with the vertices of $(V_i \times \{2\})$ whereas the vertex $(x, 2)$ is adjacent with the vertices of $(V_i \times \{1\})$. Therefore, one of the components of $G_i \wedge K_2$ is only connected with $(x, 1)$ and the other with $(x, 2)$. This implies that $(x, 1)$ (respectively $(x, 2)$) is a cut vertex of $G \wedge K_2$. □

Cut edge

If we look at the structure of a graph with a cut edge, we have:

Fact 6. *Let G be a connected graph with a cut edge $\{x, y\}$, let G_1 and G_2 be the two connected components resulting from the removal of $\{x, y\}$. We assume that $x \in V(G_1)$ and $y \in V(G_2)$. Then, $G \wedge K_2$ is composed of the components $G_1 \wedge K_2$ and $G_2 \wedge K_2$ connected to each other by the edges $\{(x, 1), (y, 2)\}$ and $\{(x, 2), (y, 1)\}$:*

$$V(G \wedge K_2) = V(G_1 \wedge K_2) \cup V(G_2 \wedge K_2),$$

$$E(G \wedge K_2) = E(G_1 \wedge K_2) \cup E(G_2 \wedge K_2) \cup \big\{(x, 1), (y, 2)\big\} \cup \big\{(x, 2), (y, 1)\big\}.$$

A consequence of this fact is that a cut edge in a connected graph G introduces two cut edges into $G \wedge K_2$ under simplified conditions:

Proposition 9. *Let G be a connected graph with a cut edge $\{x, y\}$, let G_1 and G_2 be the two connected components resulting from the removal of $\{x, y\}$. Then the edges $\{(x, 1), (y, 2)\}$ and $\{(x, 2), (y, 1)\}$ are cut edges in $G \wedge K_2$ if and only if G_1 or G_2 has no odd cycle.*

Proof. First, we consider that e_1 and e_2 are cut edges in $G \wedge K_2$. We want to prove by contradiction that G_1 or G_2 has no odd cycle. Suppose that G_1 and G_2 contain odd cycles. Applying Theorem 1, $G_1 \wedge K_2$ and $G_2 \wedge K_2$ are two connected components. Fact 6 shows that e_1 and e_2 connect these two connected components therefore the removal of e_1 (respectively e_2) does not disconnect the graph $G \wedge K_2$ which is in contradiction with the hypothesis.

Conversely, we consider that G_1 or G_2 has no odd cycle, say G_1, for example. Applying Proposition 7, $G_1 \wedge K_2$ contains exactly two connected components:

- one contains the vertex $(x, 1)$ and is connected to $G_2 \wedge K_2$ thanks to the edge e_1;
- the other contains the vertex $(x, 2)$ and is connected to $G_2 \wedge K_2$ thanks to the edge e_2.

So, the removal of e_1 or e_2 disconnects the graph $G \wedge K_2$. □

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