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Spectra of general hypergraphs

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Here, we show a method to reconstruct connectivity hypermatrices of a general hypergraph (without any self loop or multiple edge) using tensor. We also study the different spectral properties of these hypermatrices and find that these properties are similar for graphs and uniform hypergraphs. The representation of a connectivity hypermatrix that is proposed here can be very useful for the further development in spectral hypergraph theory.

1. Introduction

Spectral graph theory has a long history behind its development. In spectral graph theory, we analyze the eigenvalues of a connectivity matrix which is uniquely defined on

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a graph. Many researchers have had a great interest to study the eigenvalues of different connectivity matrices, such as, adjacency matrix, Laplacian matrix, signless Laplacian matrix, normalized Laplacian matrix, etc. Now, a recent trend has been developed to explore spectral hypergraph theory. Unlike in a graph, an edge of a hypergraph can be constructed with more than two vertices, i.e., the edge set of a hypergraph is the subset of the power set of the vertex set of that hypergraph [22]. Now, one of the main challenges is to uniquely represent a hypergraph by a connectivity hypermatrix or by a tensor, and vice versa. It is not trivial for a non-uniform hypergraph, where the cardinalities of the edges are not the same. Recently, the study of the spectrum of uniform hypergraph becomes popular. In a (m-)uniform hypergraph, each edge contains the same, (m), number of vertices. Thus an m-uniform hypergraph of order n can be easily represented by an m order n dimensional connectivity hypermatrix (or tensor). In [7], the results on the spectrum of adjacency matrix of a graph are extended for uniform hypergraphs by using characteristic polynomial. Spectral properties of adjacency uniform hypermatrix are deduced from matroids in [16]. In 1993, Fan Chung defined Laplacian of a uniform hypergraph by considering various homological aspects of hypergraphs and studied the eigenvalues of the same [5]. In [9–11,18,19], different spectral properties of Laplacian and signless Laplacian of a uniform hypergraph, defined by using tensor, have been studied. In 2015, Hu and Qi introduced the normalized Laplacian of a uniform hypergraph and analyzed its spectral properties [8]. The important tool that has been used in spectral hypergraph theory is tensor. In 2005, Liqun Qi introduced the different eigenvalues of a real supersymmetric tensor [17]. The various properties of the eigenvalues of a tensor have been studied in [3,4,13,14,20,21,23,24].

But, still the challenge remains to come up with a mathematical framework to construct a connectivity hypermatrix for a non-uniform hypergraph, such that, based on this connectivity hypermatrix the spectral graph theory for a general hypergraph can be developed. Here, we propose a unique representation of a general hypergraph (without any self loop or multiple edge) by connectivity hypermatrices, such as, adjacency hypermatrix, Laplacian hypermatrix, signless Laplacian hypermatrix, normalized Laplacian hypermatrix and analyze the different spectral properties of these matrices. These properties are very similar with the same for graphs and uniform hypergraphs. Studying the spectrum of a uniform hypergraphs could be considered as a special case of the spectral graph theory of general hypergraphs.

2. Preliminary

Let $\mathbb{R}$ be the set of real numbers. We consider an m order n dimensional hypermatrix A having n^m elements from $\mathbb{R}$, where

$$A = (a_{i_1,i_2,\ldots,i_m}), a_{i_1,i_2,\ldots,i_m} \in \mathbb{R} \text{ and } 1 \leq i_1, i_2, \ldots, i_m \leq n.$$

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. If we write x^m as an m order n dimension hypermatrix with $(i_1, i_2, \dots, i_m)$-th entry $x_{i_1} x_{i_2} \dots x_{i_m}$, then Ax^{m-1}, where the multiplication is taken as tensor contraction over all indices, is an n tuple whose i-th component is

$$\sum_{i_2, i_3, \dots, i_m = 1}^{n} a_{i i_2 i_3 \dots i_m} x_{i_2} x_{i_3} \dots x_{i_m}.$$

Definition 2.1. Let A be a nonzero hypermatrix. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called eigenvalue and eigenvector (or simply an eigenpair) if they satisfy the following equation

$$Ax^{m-1} = \lambda x^{[m-1]}.$$

Here, $x^{[m]}$ is a vector with i-th entry x_i^m. We call (λ, x) an H-eigenpair (i.e., λ and x are called H-eigenvalue and H-eigenvector, respectively) if they are both real. An H-eigenvalue λ is called $H^+(H^{++})$-eigenvalue if the corresponding eigenvector $x \in \mathbb{R}^n_+$ $(\mathbb{R}^n_{++})$.

Definition 2.2. Let A be a nonzero hypermatrix. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an E-eigenpair (where λ and x are called E-eigenvalue and E-eigenvector, respectively) if they satisfy the following equations

$$Ax^{m-1} = \lambda x,$$
$$\sum_{i=1}^{n} x_i^2 = 1.$$

We call (λ, x) a Z-eigenpair if both of them are real.

From the above definitions it is clear that, a constant multiplication of an eigenvector is also an eigenvector corresponding to an H-eigenvalue, but, this is not always true for E-eigenvalue and Z-eigenvalue. Now, we recall some results that are used in the next section.

Theorem 2.1 *([17]). The eigenvalues of A lie in the union of n disks in $\mathbb{C}$. These n disks have the diagonal elements of the supersymmetric tensor as their centers, and the sums of the absolute values of the off-diagonal elements as their radii.*

The above theorem helps us to bound the eigenvalues of a tensor.

Lemma 2.1. *Let A be an m order and n dimensional tensor and $D = diag(d_1, \dots, d_n)$ be a positive diagonal matrix. Define a new tensor*

$$B = A.D^{-(m-1)}.\overbrace{D \dots D}^{m-1}$$

with the entries

$$B_{i_1 i_2 \ldots i_m} = A_{i_1 i_2 \ldots i_m} d_{i_1}^{-(m-1)} d_{i_2} \ldots d_{i_m}.$$

Then A and B have the same H-eigenvalues.

Proof. From the remarks of lemma (3.2) in [23]. $\square$

Some results of spectral graph theory[1] also hold for general hypergraphs. If λ is any eigenvalue of an adjacency matrix of a graph G with the maximal degree Δ, then $\lambda \leqslant \Delta$. For a k-regular graph k is the maximum eigenvalue with a constant eigenvector of the adjacency matrix of that graph. If λ and μ are the eigenvalues of the adjacency matrices, represent the graphs G and H, respectively, then $\lambda + \mu$ is also an eigenvalue of the same for $G \times H$, the Cartesian product of G and H. All the eigenvalues of a Laplacian matrix of a graph are nonnegative and a very rough upper bound of these eigenvalues is 2Δ, whereas, any eigenvalue of a normalized Laplacian matrix of a graph lies in the interval $[0, 2]$. Zero is always an eigenvalue for both, Laplacian and normalized Laplacian matrices, of a graph, with a constant eigenvector. If $\mathbb{A}$ and $\mathcal{L}$ are the normalized adjacency matrix and normalized Laplacian matrix, respectively, of a graph (such that $\mathcal{L} = I - \mathbb{A}$) then the spectrum of $\mathbb{A}$, $\sigma(\mathbb{A}) = 1 - \sigma(\mathcal{L})$. If M is a connectivity matrix of a graph with r connected components then $\sigma(M) = \sigma(M_1) \cup \sigma(M_2) \cdots \cup \sigma(M_r)$, where M_i is the same connectivity matrix corresponding to the component i.

3. Spectral properties of general hypergraphs

Definition 3.1. A (general) hypergraph G is a pair $G = (V, E)$ where V is a set of elements called vertices, and E is a set of non-empty subsets of V called edges. Therefore, E is a subset of $\mathcal{P}(V) \setminus \{\emptyset\}$, where $\mathcal{P}(V)$ is the power set of V.

Example 3.1. Let $G = (V, E)$, where $V = \{1, 2, 3, 4, 5\}$ and $E = \big\{\{1\}, \{2, 3\}, \{1, 4, 5\}\big\}$. Here, G is a hypergraph of 5 vertices and 3 edges.

3.1. Adjacency hypermatrix and eigenvalues

Definition 3.2. Let $G = (V, E)$ be the hypergraph where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_k\}$. Let $m = max\{|e_i| : e_i \in E\}$ be the maximum cardinality of edges, $m.c.e(G)$, of G. Define the adjacency hypermatrix of G as

$$\mathcal{A}_G = (a_{i_1 i_2 \ldots i_m}),\ 1 \le i_1, i_2, \ldots, i_m \le n.$$

For all edges $e = \{v_{l_1}, v_{l_2}, \ldots, v_{l_s}\} \in E$ of cardinality $s \le m$,

[1] For different spectral properties of a graph see [2,6].

$$a_{p_1 p_2 \dots p_m} = \frac{s}{\alpha}, \text{ where } \alpha = \sum_{k_1, k_2, \dots, k_s \geq 1, \sum k_i = m} \frac{m!}{k_1! k_2! \dots k_s!},$$

and $p_1, p_2, \dots, p_m$ chosen in all possible way from $\{l_1, l_2, \dots, l_s\}$ with at least once for each element of the set. The other positions of the hypermatrix are zero.[2]

Example 3.2. Let $G = (V, E)$ be a hypergraph in Example 3.1. Here, the maximum cardinality of edges is 3. The adjacency hypermatrix of G is $\mathcal{A}_G = (a_{i_1 i_2 i_3})$, where $1 \leq i_1, i_2, i_3 \leq 5$. Here, $a_{111} = 1, a_{233} = a_{232} = a_{223} = a_{323} = a_{332} = a_{322} = \frac{1}{3}, a_{145} = a_{154} = a_{451} = a_{415} = a_{541} = a_{514} = \frac{1}{2}$, and the other elements of $\mathcal{A}_G$ are zero.

Definition 3.3. Let $G = (V, E)$ be a hypergraph. The degree, $d(v)$, of a vertex $v \in V$ is the number of edges consist of v.

Let $G = (V, E)$ be a hypergraph, where $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_k\}$. Then, the degree of a vertex v_i is given by

$$d(v_i) = \sum_{i_2, i_3, \dots, i_m = 1}^{n} a_{i i_2 i_3 \dots i_m}.$$

Definition 3.4. A hypergraph is called k-*regular* if every vertex has the same degree k.

Now, we discuss some spectral properties of $\mathcal{A}_G$ of a hypergraph G. Some of these properties are very similar as in general graph (i.e. for a 2-uniform hypergraph).

Theorem 3.1. *Let μ be an H-eigenvalue of $\mathcal{A}_G$. Then $|\mu| \leq \Delta$, where Δ is the maximum degree of G.*

Proof. Let G be a hypergraph with n vertices and $m.c.e(G) = m$. Let μ be an H-eigenvalue of $\mathcal{A}_G = (a_{i_1 i_2 \dots i_m})$ with an eigenvector $x = (x_1, x_2, \dots, x_n)$. Let $x_p = max\{|x_1|, |x_2|, \dots, |x_n|\}$. Without loss of any generality we can assume that $x_p = 1$. Now,

$$|\mu| = |\mu x_p^{m-1}| = \left| \sum_{i_2, i_3, \dots, i_m = 1}^{n} a_{p i_2 i_3 \dots i_m} x_{i_2} x_{i_3} \dots x_{i_m} \right|$$

$$\leq \sum_{i_2, i_3, \dots, i_m = 1}^{n} |a_{p i_2 i_3 \dots i_m}| |x_p|^{m-1} = d(v_p) \leq \Delta. \quad \square$$

Thus, for a k-regular hypergraph the theorem (3.1) implies $|\mu| \leq k$.

[2] For a similar construction on uniform multi-hypergraph see [15].

Theorem 3.2. *Let $G = (V, E)$ be a k-regular hypergraph with n vertices. Then, $\mathcal{A}_G = (a_{i_1 i_2 \dots i_m})$ has an H-eigenvalue k.*

Proof. Since, G is k-regular, then $d(v_i) = k$ for all $v_i \in V$, $i \in \{1, 2, 3, \dots, n\}$. Now, for a vector $x = (1, 1, 1, \dots, 1) \in \mathbb{R}^n$, we have

$$\mathcal{A}_G x^{m-1} = \sum_{i_2, i_3, \dots i_m = 1}^{n} a_{i i_2 i_3 \dots i_m} = k.$$

Thus the proof. □

Theorem 3.3. *Let $G = (V, E)$ be a k-regular hypergraph with n vertices. Then, $\mathcal{A}_G = (a_{i_1 i_2 \dots i_m})$ has a Z-eigenvalue $k(\frac{1}{\sqrt{n}})^{m-2}$.*

Proof. The vector $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}) \in \mathbb{R}^n$ satisfies the Z-eigenvalue equations for $\lambda = k(\frac{1}{\sqrt{n}})^{m-2}$. □

Theorem 3.4. *Let G be a hypergraph with n vertices and maximum degree Δ. Let $x = (x_1, x_2, \dots, x_n)$ be a Z-eigenvector of $\mathcal{A}_G = (a_{i_1 i_2 \dots i_m})$ corresponding to an eigenvalue μ. If $x_p = max\{|x_1|, |x_2|, \dots, |x_n|\}$, then $|\mu| \leq \frac{\Delta}{x_p}$.*

Proof. The Z-eigenvalue equations of $\mathcal{A}_G$ for μ and x are $Ax^{m-1} = \mu x$, and $\sum x_i^2 = 1$. Therefore, $|x_i| \leq 1$, for all $i = 1, 2, 3, \dots, n$. Now,

$$|\mu||x_j| = \left| \sum_{i_2, i_3, \dots, i_m = 1}^{n} a_{j i_2 i_3 \dots i_m} x_{i_2} x_{i_3} \dots x_{i_m} \right|,$$

which implies $|\mu||x_j| \leq d(j) \leq \Delta, \forall j = 1, 2, 3, \dots, n$. Therefore, $|\mu| \leq \frac{\Delta}{x_p}$. □

Definition 3.5. A hypergraph $H = (V_1, E_1)$ is said to be a spanning subhypergraph of a hypergraph $G = (V, E)$, if $V = V_1$ and $E_1 \subseteq E$.

Theorem 3.5. *Let $G = (V, E)$ be hypergraph. Let $H = (V', E')$ be a subhypergraph of G, such that, $m.c.e(G) = m.c.e(H)$ be even. Then, $\mu_{max}(H) \leq \mu_{max}(G)$, where μ_{max} is the highest Z-eigenvalue of the corresponding adjacency hypermatrix.*

Proof. Let $|V| = n$, $|V'| = n'$ $(\leq n)$ and $m.c.e(G) = m.c.e(H) = m$. Now,

$$\mu_{max}(H) = max_{||x||=1} x^t \mathcal{A}_H x^{m-1} \text{ (by using lemma (3.1) in [13])}$$
$$= max_{||x||=1} \left(\sum_{i_1, i_2, \dots, i_m = 1}^{n'} a^H_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m} \right)$$

$$= max_{||x||=1}\Bigg(\sum_{i_1,i_2,\ldots,i_m=1}^{n} a^H_{i_1 i_2\ldots i_m} x_{i_1}x_{i_2}\ldots x_{i_m}\Bigg),$$

$$\text{where } a^H_{i_1..i_m} = x_{i_r} = 0 \text{ when } i_r > n'$$

$$\leq \Bigg(\sum_{i_1,i_2,\ldots,i_m=1}^{n} a^G_{i_1 i_2\ldots i_m} x_{i_1}x_{i_2}\ldots x_{i_m}\Bigg)$$

$$\leq \mu_{max}(G),$$

since each component of x is nonnegative (by Perron–Frobenious theorem [3]) and the number of edges of G is greater than or equal to the number of edges of H. Hence the proof. □

Definition 3.6. Let $G = (V,E)$ be a hypergraph with $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_k\}$, and $m.c.e(G) = m$. Let $x = (x_1, x_2, \ldots, x_n)$ be a vector in $\mathbb{R}^n$ and $p \geq s-1$ be an integer. For an edge $e = \{v_{l_1}, v_{l_2}, \ldots, v_{l_s}\}$ and a vertex v_{l_i}, we define

$$x_p^{e/v_{l_i}} := \sum x_{r_1}x_{r_2}\ldots x_{r_p},$$

where the sum is over $r_1, r_2, \ldots, r_p$ chosen in all possible way from $\{l_1, l_2, \ldots, l_s\}$, such that, all $l_j (j \neq i)$ occur at least once. Whereas,

$$x_p^e := \sum x_{r_1}x_{r_2}\ldots x_{r_p},$$

where the sum is over $r_1, r_2, \ldots, r_p$ chosen in all possible way from $\{l_1, l_2, \ldots, l_s\}$ with at least once for each element of the set.

The symmetric (adjacency) hypermatrix $\mathcal{A}_G$ of order m and dimension n uniquely defines a homogeneous polynomial of degree m and in n variables by

$$F_{\mathcal{A}_G}(x) = \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1 i_2\ldots i_m} x_{i_1}x_{i_2}\ldots x_{i_m}.$$

We rewrite the above polynomial as:

$$F_{\mathcal{A}_G}(x) = \sum_{e\in E} a_e^G x_m^e,$$

where $a_e^G = \frac{s}{\alpha}$, $\alpha = \sum_{k_1,k_2,\ldots,k_s\geq 1, \sum k_i = m} \frac{m!}{k_1!k_2!\ldots k_s!}$, and s is the cardinality of the edge e.

Definition 3.7. Let G and H be two hypergraphs. The Cartesian product, $G \times H$, of G and H is defined by the vertex set $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H) = \big\{\{v\} \times e : v \in V(G), e \in E(H)\big\} \bigcup \big\{e \times \{v\} : e \in E(G), v \in V(H)\big\}$.

Definition 3.8. Let G be a hypergraph with the vertex set $V = \{v_1, v_2, \dots, v_n\}$ and $m.c.e(G) = m$. For an edge $e = \{v_{l_1}, v_{l_2}, \dots, v_{l_s}\}$ and an integer $r \geq m$, the arrangement $(v_{p_1} v_{p_2} \dots v_{p_r})$ (where $p_1, p_2, \dots, p_r$ are chosen in all possible way from $\{l_1, l_2, \dots, l_s\}$ with at least once for each element of the set) represents the edge e in order r.

Example 3.3. Let $G = (V, E)$ where $V = \{1, 2, 3, 4, 5\}$ and $E = \big\{\{1, 2, 3\}, \{2, 3, 5\}, \{1, 3, 4, 5\}\big\}$, then the arrangement (12233) represents the edge $\{1, 2, 3\}$ in order 5. (12123) is also a representation of the edge $\{1, 2, 3\}$ in order five, whereas, (111123) represents the edge $\{1, 2, 3\}$ in 6 order.

Let $G = (V, E)$ be a hypergraph with $m.c.e(G) = m$ and $E_i = \{e \in E : v_i \in e\}$. Now, the H-eigenvalue equation for $\mathcal{A}_G$ becomes

$$\sum_{e \in E_i} a_e^G x_{m-1}^{e/v_i} = \lambda x_i^{(m-1)}, \text{ for all } i.$$

Theorem 3.6. *Let G and H be two hypergraphs with $m.c.e(G) = m.c.e(H)$. If λ and μ are H-eigenvalue for G and H, respectively, then $\lambda + \mu$ is an H-eigenvalue for $G \times H$.*

Proof. Let n_1 and n_2 be the number of vertices in G and H, respectively, and $m.c.e(G) = m.c.e(H) = m$. Let $(\lambda, \mathbf{u})$ and $(\mu, \mathbf{v})$ be H-eigenpairs of $\mathcal{A}_G$ and $\mathcal{A}_H$, respectively. Let $\mathbf{w} \in \mathbb{C}^{n_1 n_2}$ be a vector with the entries indexed by the pairs $(a, b) \in [n_1] \times [n_2]$, such that, $w(a, b) = u(a)v(b)$. Now, we show that $(\lambda + \mu, \mathbf{w})$ is an H-eigenpair of $\mathcal{A}_{G \times H}$.

$$\begin{aligned}
\sum_{e \in E_{(a,b)}} a_e^{G \times H} w_{m-1}^{e/(a,b)} &= \sum_{\substack{\{a\} \times e \in E_{(a,b)} \\ \text{with } e \in E_b}} a_e^{G \times H} w_{m-1}^{\{a\} \times e/(a,b)} + \sum_{\substack{e \times \{b\} \in E_{(a,b)} \\ \text{with } e \in E_a}} a_e^{G \times H} w_{m-1}^{e \times \{b\}/(a,b)} \\
&= \sum_{e \in H_b} a_e^{G \times H} u^{m-1}(a) v_{m-1}^{e/b} + \sum_{e \in G_a} a_e^{G \times H} u_{m-1}^{e/a} v^{m-1}(b) \\
&= u^{m-1}(a) \sum_{e \in H_b} a_e^H v_{m-1}^{e/b} + v^{m-1}(b) \sum_{e \in E_a} a_e^G u_{m-1}^{e/a} \\
&= u^{m-1}(a) \mu v^{m-1}(b) + v^{m-1}(b) \lambda u^{m-1}(a) \\
&= (\lambda + \mu) w^{m-1}(a, b).
\end{aligned}$$

Hence the proof.[3] □

Lemma 3.1. *Let A and B be two symmetric hypermatrices of order m and dimension n, where m is even. Then $\lambda_{max}(A + B) \leq \lambda_{max}(A) + \lambda_{max}(B)$, where $\lambda_{max}(A)$ denotes the largest Z-eigenvalue of A.*

[3] For similar proof on uniform hypergraph see [7].

Proof.

$$\begin{aligned}\lambda_{max}(A+B) &= max_{||x||=1}x^t(A+B)x^{m-1} \text{ (by using lemma (3.1) in [13])}\\ &\leq max_{||x||=1}x^tAx^{m-1} + max_{||x||=1}x^tBx^{m-1}\\ &= \lambda_{max}(A) + \lambda_{max}(B). \quad \square\end{aligned}$$

Let $G = (V, E)$ be a hypergraph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and $m = max\{|e_i| : e_i \in E\}$. We partition the edge set E as, $E = E_1 \cup E_2 \cup \cdots \cup E_m$, where E_i contains all the edges of the cardinality i and construct a hypergraph $G_i = (V, E_i)$, for a nonempty E_i.

Definition 3.9. Define the adjacency hypermatrix of G_i in $m(> i)$-order by an n dimensional m order hypermatrix

$$\mathcal{A}^m_{G_i} = \left((a^m_{G_i})_{p_1p_2\ldots p_m}\right),\ 1 \leq p_1, p_2, \ldots, p_m \leq n,$$

such that, for any $e = \{v_{l_1}, v_{l_2}, \ldots, v_{l_i}\} \in E_i$,

$$(a^m_{G_i})_{p_1p_2\ldots p_m} = \frac{i}{\alpha},\text{ where } \alpha = \sum_{k_1,k_2,\ldots,k_i\geq 1, \sum k_j=m} \frac{m!}{k_1!k_2!\ldots k_i!}$$

and $p_1, p_2, \ldots, p_m$ are chosen in all possible way from $\{l_1, l_2, \ldots, l_i\}$ with at least once for each element of the set. The other positions of $\mathcal{A}^m_{G_i}$ are zero.

Thus, we can represent a hypergraph G, with $m.c.e(G) = s$, in higher order $m > s$ by the hypermatrix $\mathcal{A}^m_G$. Clearly, all the eigenvalue equations show that the eigenvalues of $\mathcal{A}^{m_1}_G$ and $\mathcal{A}^{m_2}_G$ are not equal for $m_1 \neq m_2$.

Theorem 3.7. *Let $G = (V, E)$ be a hypergraph and $m.c.e(G) = m$ be even. Then $\lambda_{max}(\mathcal{A}_G) \leq \sum_{i=1}^m \lambda_{max}(\mathcal{A}^m_{G_i})$, where $\lambda_{max}(A)$ is the largest Z-eigenvalue of A.*

Proof. Since $\mathcal{A}_G = \sum_{i=1}^m \mathcal{A}^m_{G_i}$, the proof follows from the lemma (3.1). $\square$

Moreover, the theorem (3.7) implies $\lambda_{max}(\mathcal{A}_G) \leq \sum_{i=1}^m n_i\lambda_{max}(\mathcal{A}^m_i)$, where n_i is the number of edges of cardinality i and $\mathcal{A}^m_i$ is the adjacency hypermatrix in m-order of a hypergraph contains a single edge of cardinality i.

3.2. Laplacian hypermatrix and eigenvalues

Definition 3.10. Let $G = (V, E)$ be a (general) hypergraph without any isolated vertex where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_k\}$. Let $m.c.e(G) = m$. We define the Laplacian hypermatrix, L_G, of $G = (V, E)$ as $L_G = D_G - \mathcal{A}_G = (l_{i_1i_2\ldots i_m}), 1 \leq$

$i_1, i_2, \ldots, i_m \leq n$, where $D_G = (d_{i_1 i_2 \ldots i_m})$ is the m order n dimensional diagonal hypermatrix with $d_{ii\ldots i} = d(v_i)$ and others are zero. The signless Laplacian of G is defined as $L_G = D_G + \mathcal{A}_G$.

Let $G = (V, E)$ be a hypergraph with $m.c.e(G) = m$. For any edge $e = \{v_{l_1}, v_{l_2}, \ldots, v_{l_s}\}$, we define a homogeneous polynomial of degree m and in n variables by

$$L(e)x^m = \sum_{j=1}^{s} x_{i_j}^m - \frac{s}{\alpha} x_m^e \ (s \leq m).$$

Proposition 3.1. $\sum_{j=1}^{s} x_{i_j}^m \geq \frac{s}{\alpha} x_m^e \ (x_{i_j} \in \mathbb{R}_+)$.

Proof. x_m^e is the sum of all possible terms, $x_{i_1}^{k_1} x_{i_2}^{k_2} \ldots x_{i_s}^{k_s}$ (where $\sum k_i = m$ and $k_i \geq 1$)

$$\text{where } \alpha = \sum_{k_1, k_2, \ldots, k_s \geq 1, \sum k_i = m} \frac{m!}{k_1! k_2! \ldots k_s!},$$

with some natural coefficient. Now, by applying AM-GM inequality on $k_1 x_{i_1}^m, k_2 x_{i_2}^m, \ldots, k_s x_{i_s}^m$ we get

$$\frac{1}{m} \sum_{j=1}^{s} k_j x_{i_j}^m \geq x_{i_1}^{k_1} x_{i_2}^{k_2} \ldots x_{i_s}^{k_s}. \tag{1}$$

If we apply (1) for each term of x_m^e and take the sum, we get

$$\frac{\alpha}{s} \sum_{j=1}^{s} x_{i_j}^m \geq x_m^e. \quad \square$$

Many properties of Laplacian and signless Laplacian tensors are discussed in [18]. Now we show that some of the results in general graph are also true for non-uniform (general) hypergraph. Note that, here, L is co-positive tensor since, $Lx^m = \sum_{e \in E} L(e)x^m \geq 0$ for all $x \in \mathbb{R}_+^n$.

Theorem 3.8. *Let $G = (V, E)$ be a general hypergraph. Let $L = (l_{i_1 i_2 \ldots i_m})$ where $1 \leq i_1, i_2, \ldots, i_m \leq n$, be the Laplacian hypermatrix of G. Then $0 \leq \lambda \leq 2\Delta$, where λ is an H-eigenvalue of L.*

Proof. For a vector $y = (\frac{1}{n^{\frac{1}{m}}}, \frac{1}{n^{\frac{1}{m}}}, \ldots, \frac{1}{n^{\frac{1}{m}}}), Ly^m = 0$. Since L is a co-positive tensor, thus $min\{Lx^m : x \in \mathbb{R}_+^n, \sum_{i=1}^{n} x_i^m = 1\} = 0$. Therefore $\lambda \geq 0$.

Again using theorem $(6(a))$ of [17] we have

$$|\lambda - l_{ii\ldots i}| \leq \sum_{\substack{i_2,i_3,\ldots,i_m=1,\\ \delta_{i,i_2,\ldots,i_m}=0}}^{n} |l_{ii_2i_3\ldots i_m}| = \Delta,$$

i.e., $|\lambda| \leq 2\Delta$. Thus $0 \leq \lambda \leq 2\Delta$. □

Theorem 3.9. *Let $G = (V, E)$ be a general hypergraph with m.c.e$(G) = m \geq 3$. Let L be the Laplacian hypermatrix of G. Then*

(i) L has an H-eigenvalue 0 with eigenvector $(1, 1, \ldots, 1) \in \mathbb{R}^n$ and an Z-eigenvalue 0 with eigenvector $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}) \in \mathbb{R}^n$. Moreover, 0 is the unique H^{++}-eigenvalue of L.
(ii) Δ is the largest H^+-eigenvalue of L.
(iii) $(d(i), e^{(j)})$ is an H-eigenpair, where $e^{(j)} \in \mathbb{R}^n$ and $e_i^{(j)} = 1$ if $i = j$, otherwise 0.
(iv) For a nonzero $x \in \mathbb{R}^n$ $(d(v_i), x)$ is an eigenpair if $\sum_{e\in E_i} a_G^e x_{m-1}^{e/i} = 0$.

Proof. (i) It is easy to check that 0 is an H-eigenvalue with the eigenvector $(1, 1, 1, \ldots, 1) \in \mathbb{R}^n$ and 0 is an Z-eigenvalue with the eigenvector $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}) \in \mathbb{R}^n$. Let x is an H^{++}-eigenvector of L with eigenvalue λ. By theorem (3.8), $\lambda \geq 0$. Suppose $x_j = \min_i\{x_i\}$. Therefore x_j is positive. Now,

$$\lambda x_j^{m-1} = d(v_j)x_j^{m-1} - \sum_{e\in E, j\in e, |e|=s} \frac{s}{\alpha} \sum_{\substack{e\equiv\{i,i_2,\ldots,i_m\}\\ \text{as set, } i,i_2,\ldots,i_m=1}}^{n} x_{i_2}x_{i_3}\ldots x_{i_m},$$

which implies that

$$\lambda = d(v_j) - \sum_{e\in E, j\in e, |e|=s} \frac{s}{\alpha} \sum_{\substack{e\equiv\{i,i_2,\ldots,i_m\}\\ \text{as set, } i,i_2,\ldots,i_m=1}}^{n} \frac{x_{i_2}}{x_j}\frac{x_{i_3}}{x_j}\ldots\frac{x_{i_m}}{x_j}.$$

Thus, $\lambda \leq d(v_j) - d(v_j) = 0$. Hence $\lambda = 0$.

(ii) Suppose λ is an H^+-eigenvalue with non-negative H^+-eigenvector, x of L. Assume that $x_j > 0$. Now, we have

$$\lambda x_j^{m-1} = d(v_j)x_j^{m-1} - \sum_{e\in E, j\in e, |e|=s} \frac{s}{\alpha} \sum_{\substack{e\equiv\{i,i_2,\ldots,i_m\}\\ \text{as set, } i,i_2,\ldots,i_m=1}}^{n} x_{i_2}x_{i_3}\ldots x_{i_m} \leq d(v_j)x_j^{m-1}.$$

Therefore $\lambda \leq d(v_j) \leq \Delta$. Thus, Δ is the largest H^+-eigenvalue of L.

(iii) Proof is obvious.

(iv) It is clear from the eigenvalue equation. □

Let $G = (V, E)$ be a general hypergraph and $m.c.e(G) = m$. The *analytic connectivity*, $\alpha(G)$, of G is defined as $\alpha(G) = \underset{j=1,\dots,n}{min}\ min\{Lx^m | x \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^m = 1, x_j = 0\}$.

Theorem 3.10. *The general hypergraph $G = (V, E)$ with $m.c.e(G) \geq 3$ is connected if and only if $\alpha(G) > 0$.*

Proof. Suppose $G = (V, E)$ is not connected. Let $G_1 = (V_1, E_1)$ be a component of G. Then there exists $j \in V \setminus V_1$. Let $x = \frac{1}{|V_1|^{\frac{1}{m}}} \sum_{i \in V_1} e^{(i)}$. Then x is a feasible point. Therefore $min\{Lx^m | x \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^m = 1, x_j = 0\} = 0$, which implies $\alpha(G) = 0$.

Let $\alpha(G) = 0$. Thus there exists j such that $min\{Lx^m | x \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^m = 1, x_j = 0\} = 0$. Suppose that y is a minimizer of this minimization problem. Therefore $y_j = 0$, $Ly^m = 0$. By optimization theory, there exists a Lagrange multiplier μ such that for $i = 1, 2, \dots, n$ and $i \neq j$, either, $y_i = 0$ and

$$\frac{\partial}{\partial y_i}(Ly^m) \geq \mu \frac{\partial}{\partial y_i}(\sum_{i=1}^n y_i^m - 1) \tag{2}$$

or, $y_i > 0$ and

$$\frac{\partial}{\partial y_i}(Ly^m) = \mu \frac{\partial}{\partial y_i}(\sum_{i=1}^n y_i^m - 1). \tag{3}$$

In (2) and (3) $y \in \mathbb{R}^n_+, \sum_{i=1}^n y_i^m = 1, y_j = 0$. Now, multiplying (2) and (3) by y_i and summing them for $i = 1, \dots, n$, we have $Ly^m = \mu(\sum_{i=1}^n y_i^m)$. Thus $Ly^m = \mu$. Hence $\mu = 0$. Therefore, for $i = 1, 2, \dots, n$ and $i \neq j$, either $y_i = 0$ or $\frac{\partial}{\partial y_i}(Ly^m) = 0$. Hence, either $y_i = 0$ or $d_i(y_i)^{m-1} - \sum_{i_2,i_3,\dots,i_m=1}^n a_{ii_2i_3\dots i_m} y_{i_2} y_{i_3} \dots y_{i_m} = 0$. Let $y_k = max\{y_i : i = 1, 2, \dots, n\}$. Hence, we have

$$d_k - \sum_{i_2,i_3,\dots,i_m=1}^n a_{ii_2i_3\dots i_m} \frac{y_{i_2}}{y_k} \frac{y_{i_3}}{y_k} \dots \frac{y_{i_m}}{y_k} = 0.$$

Again, we know that

$$d(v_k) = \sum_{i_2,i_3,\dots,i_m=1}^n a_{ii_2i_3\dots i_m}.$$

Therefore, $x_i = x_k$ as long as i and k belong to same edge. Thus, $x_i = x_k$ as long as i and k are in same component of G. Since $y_j = 0$, we have, j and k are in different components of G. Hence, G is not connected. This proves the theorem. □

3.3. Normalized Laplacian hypermatrix and eigenvalues

Now, we define normalized Laplacian hypermatrix for a general hypergraph. For any graph, there are two ways to construct normalized Laplacian matrix (see [1] and [6] for details).[4] Motivated by these two similar constructions, here, we also define the normalized Laplacian hypermatrix in two different ways and show that they are cospectral. The first definition is similar to the normalized Laplacian matrix defined in [1].

Definition 3.11. Let $G = (V, E)$ be a general hypergraph without any isolated vertex where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_k\}$. Let $m.c.e(G) = m$. The normalized Laplacian hypermatrix $\mathcal{L} = (l_{i_1 i_2 \ldots i_m})$, which is an n-dimensional m-th order hypermatrix, is defined as: for any edge $e = \{v_{l_1}, v_{l_2}, \ldots, v_{l_s}\} \in E$ of cardinality $s \leq m$,

$$l_{p_1 p_2 \ldots p_m} = -\frac{s/\alpha}{d(v_{p_1})}, \text{ where } \alpha = \sum_{\substack{k_1, k_2, \ldots, k_s \geq 1, \\ \sum k_i = m}} \frac{m!}{k_1! k_2! \ldots k_s!}$$

and $p_1, p_2, \ldots, p_m$ are chosen in all possible way from $\{l_1, l_2, \ldots, l_s\}$, such that, all l_j occur at least once. All the diagonal entries are 1 and the rest are zero.

Clearly, the hypermatrix $A = \mathcal{I} - \mathcal{L}$, which is known as normalized adjacency hypermatrix, is a stochastic tensor, that is, A is non-negative and $\sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \ldots i_m} = 1$, where $a_{i_1 i_2 \ldots i_m}$ is the $(i_1, i_2, \ldots, i_m)$-th entry of A. The different properties of a stochastic tensor are discussed in [24] and which can be used to study the hypermatrices A and $\mathcal{L}$. Now, we define the normalized Laplacian hypermatrix of a general hypergraph as it is defined for a graph in [6].

Definition 3.12. Let $G = (V, E)$ be a general hypergraph without any isolated vertex, where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_k\}$. Let $m.c.e(G) = m$. The normalized Laplacian hypermatrix $\mathfrak{L} = (l_{i_1 i_2 \ldots i_m})$, which is an n-dimension m-th order symmetric hypermatrix, is defined as: for any edge $e = \{v_{l_1}, v_{l_2}, \ldots, v_{l_s}\} \in E$ of cardinality $s \leq m$,

$$l_{p_1 p_2 \ldots p_m} = -\frac{s}{\alpha} \prod_{j=1}^{m} \frac{1}{\sqrt[m]{d(v_{p_j})}}, \text{ where } \alpha = \sum_{k_1, k_2, \ldots, k_s \geq 1, \sum k_i = m} \frac{m!}{k_1! k_2! \ldots k_s!}$$

and $p_1, p_2, \ldots, p_m$ chosen in all possible way from $\{l_1, l_2, \ldots, l_s\}$ with at least once for each element of the set. The diagonal entries of $\mathfrak{L}$ are 1 and the rest of the positions are zero.

Theorem 3.11. *$\mathcal{L}$ and $\mathfrak{L}$ are co-spectral.*

[4] These two matrices are similar, i.e., they have same eigenvalues.

Proof. In the lemma (2.1) choose a diagonal matrix $D = (d_{ij})_{n\times n}$ where $d_{ii} = (d(v_i))^{1/m}$. □

Theorem 3.12. *Let $G = (V, E)$ be a general hypergraph. Let $\mathcal{L}$, A be the normalized Laplacian and normalized adjacency hypermatrices of G, respectively. If G has at least one edge, then $\lambda \in \sigma(A)$ if and only if $(1-\lambda) \in \sigma(\mathcal{L})$, otherwise, $\sigma(A) = \sigma(\mathcal{L}) = 0$, where $\sigma(\mathcal{L})$ denotes the spectrum of $\mathcal{L}$.*

Proof. Since, $\mathcal{L} = \mathcal{I} - A$ and λ is the eigenvalue of A *iff* $\det(A - \lambda\mathcal{I}) = 0$, thus, $\det(\mathcal{L} - (1-\lambda)\mathcal{I}) = 0$ implies $(1-\lambda) \in \sigma(\mathcal{L})$. □

Theorem 3.13. *Let $G = (V, E)$ be a general hypergraph. Let $\mathcal{L} = (l_{i_1 i_2 \dots i_m})$ where $1 \le i_1, i_2, \dots, i_m \le n$, and A be the normalized Laplacian and normalized adjacency hypermatrices of G, respectively, then*

(i) $\rho(A) = 1$.
(ii) $0 \le \lambda(\mathcal{L}) \le 2$.
(iii) 1 is the largest H^+-eigenvalue of $\mathcal{L}$.
(iv) 0 *is an eigenvalue of $\mathcal{L}$ with the eigenvector* $(1, 1, \dots, 1)$ *and* 0 *is an Z-eigenvalue with eigenvector* $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}) \in \mathbb{R}^n$.
(v) 0 *is the unique H^{++}-eigenvalue of $\mathcal{L}$.*

Proof. (i) Since A is stochastic tensor, it is obvious that the spectral radius of A is 1. Moreover, $(1, 1, \dots, 1)$ is an eigenvector with eigenvalue 1.

(ii) We know that spectral radius of A is 1 and $\mathcal{L} = \mathcal{I} - A$. By theorem (3.12) $\lambda \in \sigma(A)$ if and only if $(1-\lambda) \in \sigma(\mathcal{L})$. Since 1 is an eigenvalue of A, thus, $\lambda \ge 0$. Again, using theorem $(6(a))$ of [17] we have

$$|\lambda(\mathcal{L}) - 1| \le \sum_{\substack{i_2, i_3, \dots, i_m = 1, \\ \delta_{i, i_2, \dots, i_m} = 0}}^{n} |l_{i i_2 i_3 \dots i_m}| = 1.$$

This implies $|\lambda(\mathcal{L})| \le 2$. Thus, we have $0 \le \lambda(\mathcal{L}) \le 2$.

(iii) Suppose that λ is an H^+-eigenvalue with non-negative H^+-eigenvector, x of $\mathcal{L}$. Assume that $x_j > 0$. Now, we have

$$\lambda x_j^{m-1} = x_j^{m-1} - \frac{1}{d(v_j)} \sum_{e \in E, j \in e, |e| = s} \frac{s}{\alpha} \sum_{\substack{e \equiv \{i, i_2, \dots, i_m\} \\ \text{as set, } i, i_2, \dots, i_m = 1}}^{n} x_{i_2} x_{i_3} \dots x_{i_m}.$$

Hence, $\lambda x_j^{m-1} \le x_j^{m-1}$ implies $\lambda \le 1$. Thus, 1 is the largest H^+-eigenvalue of $\mathcal{L}$.

(iv) It is easy to check that 0 is an H-eigenvalue corresponding an eigenvector $(1,1,\dots,1) \in \mathbb{R}^n$ and 0 is an Z-eigenvalue with the eigenvector $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}) \in \mathbb{R}^n$.

(v) Let x is an H^{++}-eigenvector of $\mathcal{L}$ with eigenvalue λ. From the part (ii) of this theorem we have $\lambda \geq 0$. Suppose $x_j = \min_i\{x_i\}$. Now,

$$\lambda x_j^{m-1} = x_j^{m-1} - \frac{1}{d(v_j)} \sum_{e \in E, j \in e, |e|=s} \frac{s}{\alpha} \sum_{\substack{e \equiv \{i, i_2, \dots, i_m\} \\ \text{as set}, i, i_2, \dots, i_m = 1}}^{n} x_{i_2} x_{i_3} \dots x_{i_m},$$

which implies that

$$\lambda = 1 - \frac{1}{d(v_j)} \sum_{e \in E, j \in e, |e|=s} \frac{s}{\alpha} \sum_{\substack{e \equiv \{i, i_2, \dots, i_m\} \\ \text{as set, } i, i_2, \dots, i_m = 1}}^{n} \frac{x_{i_2}}{x_j} \frac{x_{i_3}}{x_j} \dots \frac{x_{i_m}}{x_j}.$$

Thus $\lambda \leq 1 - 1 = 0$. Hence $\lambda = 0$. □

Theorem 3.14. *Let $G = (V, E)$ be a general hypergraph and $m.c.e(G) = m$. Let $\mathcal{L}$ be the normalized Laplacian hypermatrix of G of order m and dimension n. Let $m(\lambda)$ be the algebraic multiplicity of $\lambda \in \sigma(\mathcal{L})$, then $\sum_{\lambda \in \sigma(\mathcal{L})} m(\lambda)\lambda = n(m-1)^{n-1}$.*

Proof. Since, for any tensor $\mathcal{T} = (t_{i_1 i_2 \dots i_m}), t_{i_1 i_2 \dots i_m} \in \mathbb{C},\ 1 \leq i_1, i_2, \dots, i_m \leq n$,

$$\sum_{\lambda \in \sigma(\mathcal{T})} m(\lambda)\lambda = (m-1)^{(n-1)} \sum_{i=1}^{n} t_{ii\dots i} \text{ (see [12])}.$$

Hence, we have $\sum_{\lambda \in \sigma(\mathcal{L})} m(\lambda)\lambda = n(m-1)^{n-1}$. □

Theorem 3.15. *Let $G = (V, E)$ be a general hypergraph and A be any connectivity hypermatrix of G. If G has $r \geq 1$ connected components, $G_1, G_2, \dots, G_r$, such that, $|V(G_i)| = n_i > 1$ and $m.c.e(G_i) = m.c.e(G)$ for each $i \in \{1, 2, \dots, r\}$. Then, as sets, $\sigma(A) = \sigma(A_1) \cup \sigma(A_2) \cup \dots \cup \sigma(A_r)$, where A_i is the connectivity hypermatrix of G_i.*

Proof. Using corollary (4.2) of [21] we get

$$\phi_A(\lambda) = \prod_{i=1}^{r} (\phi_A(\lambda))^{(m-1)^{n-n_i}},$$

where $\phi_A(\lambda)$ is the characteristic polynomial of the tensor A. Therefore, $\sigma(A) = \sigma(A_1) \cup \sigma(A_2) \cup \dots \cup \sigma(A_r)$. □

4. Discussion and conclusion

Here, we propose a mathematical framework to construct connectivity matrices for a general hypergraph and also study the eigenvalues of adjacency hypermatrix, Laplacian hypermatrix, normalized Laplacian hypermatrix. This connectivity hypermatrix reconstruction can be used for further development of spectral hypergraph theory in many aspects, but, this may not be quite useful to study dynamics on hypergraphs.

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References

[1] A. Banerjee, J. Jost, On the spectrum of the normalized graph Laplacian, Linear Algebra Appl. 428 (2008) 3015–3022.
[2] R.B. Bapat, Graphs and Matrices, Springer, 2010.
[3] K.C. Chang, K. Pearson, T. Zhang, Perron–Frobenius theorem for nonnegative tensors, Commun. Math. Sci. 6 (2) (2008) 507–520.
[4] K.C. Chang, K. Pearson, T. Zhang, On eigenvalue problems of real symmetric tensors, J. Math. Anal. Appl. 350 (2009) 416–422.
[5] F.R. Chung, The Laplacian of a hypergraph, Discrete Math. Theor. Comput. Sci. (1993).
[6] F.R. Chung, Spectral Graph Theory, American Mathematical Society, 1997.
[7] J. Cooper, A. Dutle, Spectra of uniform hypergraphs, Linear Algebra Appl. 436 (2012) 3268–3292.
[8] S. Hu, L. Qi, The Laplacian of a uniform hypergraph, J. Comb. Optim. 29 (2) (2015) 331–366.
[9] S. Hu, L. Qi, J-Y. Shao, Cored hypergraphs, power hypergraphs and their Laplacian H-eigenvalues, Linear Algebra Appl. 439 (2013) 2980–2998.
[10] S. Hu, L. Qi, The eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform hypergraph, Discrete Appl. Math. 169 (2014) 140–151.
[11] S. Hu, L. Qi, J. Xie, The largest Laplacian and signless Laplacian H-eigenvalues of a uniform hypergraph, Linear Algebra Appl. 469 (2015) 1–27.
[12] S. Hu, Z. Huang, C. Ling, L. Qi, On determinants and eigenvalue theory of tensors, J. Symbolic Comput. 50 (2013) 508–531.
[13] G. Li, L. Qi, G. Yu, The Z-eigenvalues of a symmetric tensor and its application to spectral hypergraph theory, Numer. Linear Algebra Appl. 20 (2013) 1001–1029.
[14] M. Ng, L. Qi, G. Zhou, Finding the largest eigenvalue of a nonnegative tensor, SIAM J. Matrix Anal. Appl. 31 (3) (2009) 1090–1099.
[15] K.J. Pearson, T. Zhang, On spectral hypergraph theory of the adjacency tensor, Graphs Combin. 30 (2014) 1233–1248.
[16] K.J. Pearson, Spectral hypergraph theory of the adjacency hypermatrix and matroids, Linear Algebra Appl. 465 (2015) 176–187.
[17] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symbolic Comput. 40 (2005) 1302–1324.
[18] L. Qi, H^+ eigenvalues of Laplacian and signless Laplacian tensor, Commun. Math. Sci. 12 (6) (2014) 1045–1064.
[19] L. Qi, J. Shao, Q. Wang, Regular uniform hypergraphs, s-cycles, s-paths and their largest Laplacian H-eigenvalues, Linear Algebra Appl. 443 (2014) 215–227.
[20] J. Shao, A general product of tensors with applications, Linear Algebra Appl. 439 (2013) 2350–2366.
[21] J. Shao, H. Shan, L. Zhang, On some properties of the determinants of tensors, Linear Algebra Appl. 439 (2013) 3057–3069.

[22] V.I. Voloshin, Introduction to Graph and Hypergraph Theory, Nova Science Publishers Inc., 2012.
[23] Y. Yang, Q. Yang, Further results for Perron–Frobenious theorem for nonnegative tensors, SIAM J. Matrix Anal. Appl. 31 (5) (2010) 2517–2530.
[24] Y. Yang, Q. Yang, Further results for Perron–Frobenious theorem for nonnegative tensors II, SIAM J. Matrix Anal. Appl. 32 (4) (2011) 1236–1250.