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# An eigenvalue problem for even order tensors with its applications

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In this paper, we study an eigenvalue problem for even order tensors. Using the matrix unfolding of even order tensors, we can establish the relationship between a tensor eigenvalue problem and a multilevel matrix eigenvalue problem. By considering a higher order singular value decomposition of a tensor, we show that higher order singular values are the square root of the eigenvalues of the product of the tensor and its conjugate transpose. This result is similar to that in matrix case. Also we study an eigenvalue problem for Toeplitz/circulant tensors, and give the lower and upper bounds of eigenvalues of Toeplitz tensors. An application in image restoration is also discussed.

**Keywords:** tensors; eigenvalues; eigenvectors; higher order singular value decomposition; multilevel matrices; Toeplitz tensors; circulant tensors

**AMS Subject Classifications:** 15A18; 15A69

## 1. Introduction

A tensor is a multidimensional array. Let $\mathbb{C}$ be the complex field. An $m$th-order $n$-dimensional tensor $\mathcal{A}$ consisting of $n^m$ entries in $\mathbb{C}$ is denoted by:

$$\mathcal{A} = (a_{i_1,i_2,\ldots,i_m}), \quad a_{i_1,i_2,\ldots,i_m} \in \mathbb{C},\ 1 \le i_k \le n, \quad k = 1, 2, \ldots, m. \tag{1}$$

In the following discussion, we also use $\mathcal{A}(i_1, i_2, \ldots, i_m)$ to denote $(i_1, i_2, \ldots, i_m)$th entry of $\mathcal{A}$.

The tensor eigenpair was introduced by Lim [1] and Qi [2] independently in 2005. At present, the tensor eigenvalue problem becomes a hot topic because of its applications in diffusion tensor imaging, higher order Markov chains and data mining et al., see e.g. [1–13]. Below is the definition of eigenvalues of tensors.[2]

*Definition 1.1* Let $\mathcal{A}$ be a real $m$th-order $n$-dimensional tensor. If $\mathbf{x} \in \mathbb{C}^n \backslash \{\mathbf{0}\}$, $\lambda \in \mathbb{C}$, $\mathbf{x}$ and $\lambda$ satisfy

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$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}, \tag{2}$$

then we call $\lambda$ an eigenvalue of $\mathcal{A}$, and $\mathbf{x}$ its corresponding eigenvector. In particular, if $\mathbf{x}$ is real, then $\lambda$ is also real. Here $\mathbf{x}^T$ is the transpose of $\mathbf{x}$,

$$\mathcal{A}\mathbf{x}^{m-1} := \left( \sum_{i_2,\ldots,i_m=1}^{n} a_{i_1,i_2,\ldots,i_m} x_{i_2} \ldots x_{i_m} \right)_{1 \le i_1 \le n} \quad \text{and } \mathbf{x}^{[m-1]} = \left( x_i^{m-1} \right)_{1 \le i \le n}$$

with $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$.

It is clear when $m = 2$, the above definition is same as that of eigenvalues and eigenvectors of real matrices. Hence, the tensor eigenvalue can be regarded as a generalization of matrix eigenvalues. According to Definition 1.1, we see that a tensor eigenvalue problem is equivalent to solving a set of multivariate polynomials of variables $x_1, x_2, \ldots, x_n$ and an unknown $\lambda$. In general, the tensor eigenvalue problem given by Definition 1.1 is NP-hard.[4]

However, it is interesting to note that there are some other ways to define the tensor eigenvalue. For example, in computational mechanics [14–16] and signal processing[17], the eigenvalue of a fourth-order symmetric tensor $\mathcal{C}$ was introduced as follows when they studied the elasticity of isotropic materials,

$$\mathcal{C} \cdot \mathbf{E} = \left( \sum_{k,l=1}^{n} \mathcal{C}_{ijkl} \mathbf{E}_{kl} \right) = \lambda \mathbf{E}, \tag{3}$$

where $\mathcal{C}$ is a fourth-order $n$-dimension symmetric tensor and $\mathbf{E}$ is an $n \times n$ square matrix. Here, the symmetry means that $\mathcal{C}_{ijkl} = \mathcal{C}_{klij}$ for all $i, j, k, l \in \{1, 2, \ldots, n\}$ holds.

Obviously, the eigenvalue problem given by (3) can be considered as a linear transformation as an fourth-order square tensor on a second-order square tensor (matrix), which generalizes the idea of the matrix case. Recently, Qi [18] extended (3) to a $(2m)$th-order square tensor, i.e.

*Definition 1.2* [17,18] Let $\mathcal{A}$ be a complex $2m$th-order $n$-dimensional tensor. If $\mathcal{X}$ is a nonzero complex $m$th-order $n$-dimensional tensor, $\lambda \in \mathbb{C}$, and $\mathcal{X}$ and $\lambda$ satisfy

$$\mathcal{A} \cdot \mathcal{X} = \lambda \mathcal{X}, \tag{4}$$

where

$$(\mathcal{A} \cdot \mathcal{X})_{i_1,\ldots,i_m} = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_m=1}^{n} a_{i_1,\ldots,i_m,j_1,\ldots,j_m} x_{j_1,\ldots,j_m}, \quad 1 \le i_k \le n, \ 1 \le k \le m. \tag{5}$$

We call $\lambda$ and $\mathcal{X}$ are eigenvalue and eigentensor of $\mathcal{A}$.

It is noted that if $m = 1$, then $\mathcal{A}$ is a square matrix and Definition 1.2 reduces to the matrix eigenvalue. It is also noted that Definition 1.2 is very different from the one in (2). From now on, if no other special illustration, we say the eigenvalue means the eigenvalue given by Definition 1.2.

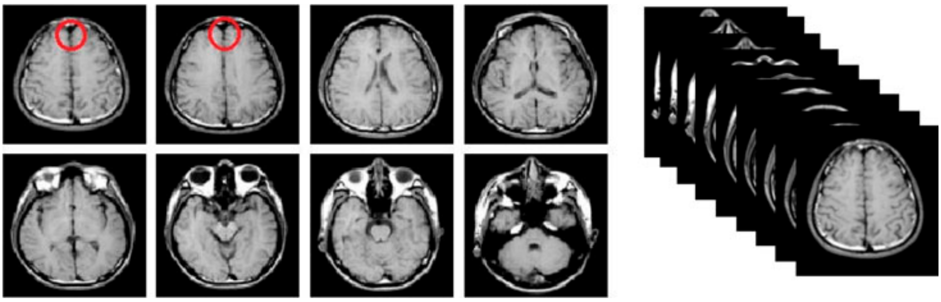

Figure 1. A stack of brain MRI images. The first and second images in the first row are different, but the parts in the red circles in these two images are similar.

A natural question is that why we study the tensor eigenvalue problem? In fact, for high-dimensional problems, the data have inherent tensor structure. Firstly, the tensor structure means that the data is high dimensional. The difference slices of the data may have some relationship. For example, human brain is a three-dimensional structure, any point in the brain can be localized on the $x$, $y$ and $z$ planes. The brain can be cut on any of these planes and are named the coronal plane, the horizontal plane or the sagittal plane. Figure 1 shows a stack of brain MRI images on horizontal plane. The data of the stack of brain MRI images on horizontal plane is high dimensional. But the slices of brain are not independent. As we know, brain is made up of many specialized regions. For example, thinking and voluntary movements are controlled by telencephalon. If we want to reconstruct a part of telencephalon, only one slice of brain MRI images on horizontal is not enough, many slices are needed. These slices contain information of the part of telencephalon. So they have a relationship. If we just process these MRI images slice by slice, we may lose some information of the tensor structure of MRI images. So it is necessary to study this tensor eigenvalue problem.

As we know, the matrix unfolding of a tensor is a useful tool for studying tensor problems. For example, in [19], the authors used the matrix unfolding to solve multilinear systems in quantum mechanical models and high-dimensional PDEs. In this paper, we will apply the matrix unfolding technique to study the eigenvalue problem.

The singular value of a matrix $\mathcal{A}$ is the square root of the eigenvalue of $A^*A$. In multilinear algebra, the higher order singular value decomposition (HOSVD) [20] was a generalization of SVD of a matrix. Naturally, one may ask: what is the relationship between the eigenvalue problem in Definition 1.2 and HOSVD? One contribution of this paper is to establish the relationship between the tensor eigenvalue problem in Definition 1.2 and HOSVD. More precisely, we show that the singular values and the associated singular vectors of $\mathcal{A}$ are just the eigenvalues and the associated eigentensors of $\mathcal{A}^* \star \mathcal{A}$. Here $\mathcal{A}^*$ represents the conjugate transpose of $\mathcal{A}$, and $\star$ refers to the multiplication of two tensors. Their definitions will be given in Section 2.

Another contribution of this paper is to study eigenvalue problems for Toeplitz/circulant tensors, which can be applied to image processing.[21,22] In particular, we construct eigentensors to diagonalize circulant tensors to obtain the eigenvalues. For Toeplitz tensors, we present the lower and upper bounds of the eigenvalues based on generating functions.

The remaining of the paper is organized as follows. In Section 2, we give the basic properties of the tensor eigenvalue problem and demonstrate the relationship between tensor eigenvalue problem and HOSVD. In Section 3, we study Toepltiz and circulant tensors and analyse their eigenvalues, and give an application in image restoration. The concluding remarks are given in Section 4.

## 2. Properties of tensor eigenvalues

### 2.1. *Unfolding operations*

Suppose that $\mathcal{A}$ is a $2m$th-order $n$-dimensional tensor. We can reorder $\mathcal{A}$ as a square matrix using the square matrix unfolding of tensors. In [23], Kofidis et al. employed the square matrix unfolding of tensors, and their aim is to study the problem of the best rank-one approximation of a super-symmetric tensor. Here, the super-symmetric means that $\mathcal{A}_{i_1,i_2,\ldots,i_m} = \mathcal{A}_{i'_1,i'_2\ldots,i'_m}$, where $(i'_1, i'_2 \ldots, i'_m)$ is any permutation of $(i_1, i_2, \ldots, i_m)$, $i_k \in \{1, 2, \ldots, n\}$, $k \in \{1, 2, \ldots, m\}$.

*Definition 2.1* [23] Let $\mathcal{A}$ be a $(2m)$th-order $n$-dimensional tensor. The square matrix unfolding of $\mathcal{A}$ with an ordering $P$ is an $n^m$-by-$n^m$ matrix $\mathbf{A}_P$ where its $(k, h)$th entry is given by

$$\mathbf{A}_P(k, h) = \mathcal{A}(i'_1, i'_2, \ldots, i'_m, j'_1, j'_2, \ldots, j'_m),$$

with

$$k = n^{m-1}(i'_1 - 1) + n^{m-2}(i'_2 - 1) + \cdots + n(i'_{m-1} - 1) + i'_m,\ 1 \le i'_k \le n,\ 1 \le k \le m,$$
$$h = n^{m-1}(j'_1 - 1) + n^{m-2}(j'_2 - 1) + \cdots + n(j'_{m-1} - 1) + j'_m,\ 1 \le j'_k \le n,\ 1 \le k \le m,$$

and $\mathbf{P}$ is the permutation matrix corresponding to the ordering $P$:

$$(i'_1, i'_2, \ldots, i'_m) = (i_1, i_2, \ldots, i_m)\mathbf{P}, \quad (j'_1, j'_2, \ldots, j'_m) = (j_1, j_2, \ldots, j_m)\mathbf{P}.$$

Let us consider a simple example of the square matrix unfolding with the natural ordering $I$. The permutation matrix $\mathbf{P}$ is just the identity matrix $\mathbf{I}$. Suppose that $\mathcal{A} = (a_{i,j,k,l})$ is a 4th-order three-dimensional tensor, the square matrix unfolding of $\mathcal{A}$ with the natural ordering $I$ is a $3^2$-by-$3^2$ matrix given by

$$\mathbf{A}_I = \left(\begin{array}{ccc|ccc|ccc}
a_{1111} & a_{1112} & a_{1113} & a_{1121} & a_{1122} & a_{1123} & a_{1131} & a_{1132} & a_{1133} \\
a_{1211} & a_{1212} & a_{1213} & a_{1221} & a_{1222} & a_{1223} & a_{1231} & a_{1232} & a_{1233} \\
a_{1311} & a_{1312} & a_{1313} & a_{1321} & a_{1322} & a_{1323} & a_{1331} & a_{1332} & a_{1333} \\
\hline
a_{2111} & a_{2112} & a_{2113} & a_{2121} & a_{2122} & a_{2123} & a_{2131} & a_{2132} & a_{2133} \\
a_{2211} & a_{2212} & a_{2213} & a_{2221} & a_{2222} & a_{2223} & a_{2231} & a_{2232} & a_{2233} \\
a_{2311} & a_{2312} & a_{2313} & a_{2321} & a_{2322} & a_{2323} & a_{2331} & a_{2332} & a_{2333} \\
\hline
a_{3111} & a_{3112} & a_{3113} & a_{3121} & a_{3122} & a_{3123} & a_{3131} & a_{3132} & a_{3133} \\
a_{3211} & a_{3212} & a_{3213} & a_{3221} & a_{3222} & a_{3223} & a_{3231} & a_{3232} & a_{3233} \\
a_{3311} & a_{3312} & a_{3313} & a_{3321} & a_{3322} & a_{3323} & a_{3331} & a_{3332} & a_{3333}
\end{array}\right). \tag{6}$$

Since $\mathcal{A}$ is a 4th-order tensor, we have the other square matrix unfolding and the permutation matrix is given as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We note that $(i'_1, i'_2) = (i_1, i_2)\mathbf{P} = (i_2, i_1)$. The corresponding square matrix unfolding of $\mathcal{A}$ is equal to a $3^2$-by-$3^2$ matrix:

$$\mathbf{A}_P = \left(\begin{array}{ccc|ccc|ccc} a_{1111} & a_{1121} & a_{1131} & a_{1112} & a_{1122} & a_{1132} & a_{1113} & a_{1123} & a_{1133} \\ a_{2111} & a_{2121} & a_{2131} & a_{2112} & a_{2122} & a_{2132} & a_{2113} & a_{2123} & a_{2133} \\ a_{3111} & a_{3121} & a_{3131} & a_{3112} & a_{3122} & a_{3132} & a_{3113} & a_{3123} & a_{3133} \\ \hline a_{1211} & a_{1221} & a_{1231} & a_{1212} & a_{1222} & a_{1232} & a_{1213} & a_{1223} & a_{1233} \\ a_{2211} & a_{2221} & a_{2231} & a_{2212} & a_{2222} & a_{2232} & a_{2213} & a_{2223} & a_{2233} \\ a_{3211} & a_{3221} & a_{3231} & a_{3212} & a_{3222} & a_{3232} & a_{3213} & a_{3223} & a_{3233} \\ \hline a_{1311} & a_{1321} & a_{1331} & a_{1312} & a_{1322} & a_{1332} & a_{1313} & a_{1323} & a_{1333} \\ a_{2311} & a_{2321} & a_{2331} & a_{2312} & a_{2322} & a_{2332} & a_{2313} & a_{2323} & a_{2333} \\ a_{3311} & a_{3321} & a_{3331} & a_{3312} & a_{3322} & a_{3332} & a_{3313} & a_{3323} & a_{3333} \end{array}\right). \tag{7}$$

*Remark 1* In Definition 1.1, the tensor is a $(2m)$th-order $n$-dimensional tensor. We may generalize this definition to a $(2m)$th-order $n_1 \times n_2 \times \cdots \times n_m \times n'_1 \times n'_2 \times \cdots \times n'_m$-dimensional tensor, where $n_i = n'_i$, $i = 1, \ldots, m$. In this paper, we study the eigenvalue problem of a $(2m)$th-order $n$-dimensional tensor. But those results can be generalized to a $(2m)$th-order $n_1 \times n_2 \times \cdots \times n_m \times n'_1 \times n'_2 \times \cdots \times n'_m$-dimensional tensor, where $n_i = n'_i$, $i = 1, \ldots, m$, easily.

Given two different orderings $P$ and $P'$, it is interesting to note that $\mathbf{A}_P$ and $\mathbf{A}_{P'}$ are similar via a permutation matrix. The permutation matrix is called a perfect shuffle permutation.[24]

Proposition 2.2 *Suppose $P$ and $P'$ are two different orderings. Then there exist a permutation matrix $\Pi_{P,P'}$ such that*

$$\Pi_{P,P'}\mathbf{A}_P\Pi^T_{P,P'} = \mathbf{A}_{P'}. \tag{8}$$

For example, (6) and (7) are the two square matrix unfoldings of $\mathcal{A}$ with two different orderings. It is easy to check that the perfect shuffle permutation matrix $\Pi_{P,I}$ is given by

$$\mathbf{\Pi}_{P,I} = \left(\begin{array}{ccc|ccc|ccc} 1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ \hline 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ \hline 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \end{array}\right)$$

and $\Pi_{P,I}\mathbf{A}_P\Pi^T_{P,I} = \mathbf{A}_I$.

Without loss of generality, we can assume that the natural ordering $I$ is used in the square matrix unfolding of tensors in the following discussion. For simplicity, we denote $\mathbf{A}_I$ by $\mathbf{A}$.

We remark that the square matrix unfolding of an $(2m)$th-order $n$-dimensional tensor is a multilevel matrix with $m$ levels. More precisely, at the first level, $\mathbf{A}$ is an $n$-by-$n$ block matrix with $n^{m-1}$-by-$n^{m-1}$ blocks:

$$\mathbf{A} = \mathbf{A}^{(1)} = \begin{pmatrix} \mathbf{A}^{(2)}_{(1,1)} & \mathbf{A}^{(2)}_{(1,2)} & \cdots & \mathbf{A}^{(2)}_{(1,n)} \\ \mathbf{A}^{(2)}_{(2,1)} & \mathbf{A}^{(2)}_{(2,2)} & \cdots & \mathbf{A}^{(2)}_{(2,n)} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{A}^{(2)}_{(n,1)} & \mathbf{A}^{(2)}_{(n,2)} & \cdots & \mathbf{A}^{(2)}_{(n,n)} \end{pmatrix}, \tag{9}$$

where the $(i, j)$th-block $\mathbf{A}^{(2)}_{(i,j)}$ is a second-level matrix given by a $n$-by-$n$ block with $n^{m-2}$-by-$n^{m-2}$ block matrix:

$$\mathbf{A}^{(2)}_{(i,j)} = \begin{pmatrix} \mathbf{A}^{(3)}_{(i,1,j,1)} & \mathbf{A}^{(3)}_{(i,1,j,2)} & \cdots & \mathbf{A}^{(3)}_{(i,1,j,n)} \\ \mathbf{A}^{(3)}_{(i,2,j,1)} & \mathbf{A}^{(3)}_{(i,2,j,2)} & \cdots & \mathbf{A}^{(3)}_{(i,2,j,n)} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{A}^{(3)}_{(i,n,j,1)} & \mathbf{A}^{(3)}_{(i,n,j,2)} & \cdots & \mathbf{A}^{(3)}_{(i,n,j,n)} \end{pmatrix}, \quad 1 \leq i, j \leq n.$$

In general, the $\ell$th-level matrix is an $n$-by-$n$ block matrix with $n^{m-\ell}$-by-$n^{m-\ell}$ blocks given by

$$\begin{aligned} &\mathbf{A}^{(\ell)}_{(i_1,\ldots,i_{\ell-1},j_1,\ldots,j_{\ell-1})} \\ &= \begin{pmatrix} \mathbf{A}^{(\ell+1)}_{(i_1,\ldots,i_{\ell-1},1,j_1,\ldots,j_{\ell-1},1)} & \mathbf{A}^{(\ell+1)}_{(i_1,\ldots,i_{\ell-1},1,j_1,\ldots,j_{\ell-1},2)} & \cdots & \mathbf{A}^{(\ell+1)}_{(i_1,\ldots,i_{\ell-1},1,j_1,\ldots,j_{\ell-1},n)} \\ \mathbf{A}^{(\ell+1)}_{(i_1,\ldots,i_{\ell-1},2,j_1,\ldots,j_{\ell-1},1)} & \mathbf{A}^{(\ell+1)}_{(i_1,\ldots,i_{\ell-1},2,j_1,\ldots,j_{\ell-1},2)} & \cdots & \mathbf{A}^{(\ell+1)}_{(i_1,\ldots,i_{\ell-1},2,j_1,\ldots,j_{\ell-1},n)} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{A}^{(\ell+1)}_{(i_1,\ldots,i_{\ell-1},n,j_1,\ldots,j_{\ell-1},1)} & \mathbf{A}^{(\ell+1)}_{(i_1,\ldots,i_{\ell-1},n,j_1,\ldots,j_{\ell-1},2)} & \cdots & \mathbf{A}^{(\ell+1)}_{(i_1,\ldots,i_{\ell-1},n,j_1,\ldots,j_{\ell-1},n)} \end{pmatrix}, \end{aligned}$$

for $1 \leq i_\ell, j_\ell \leq n$ and $2 \leq \ell \leq m$. It is clear that when $\ell = m$, $\mathbf{A}^{(m+1)}_{(i_1,\ldots,i_{m-1},j_1,\ldots,j_{m-1})}$ is an $n$-by-$n$ matrix. We will discuss multilevel Toeplitz and circulant matrices in Section 3.

To change the new tensor eigenvalue problem into a multilevel matrix eigenvalue problem, we also need to change the eigentensor into a column vector.

*Definition 2.3* Let $\mathcal{X}$ be an $m$th-order $n$-dimensional tensor. The vectorization of $\mathcal{X}$ with an ordering $P$ is an $n^m$-vector $\mathbf{x}_P$ where its $i$th entry $\mathbf{x}_P(j)$ is given by

$$\mathbf{x}_P(j) = \mathcal{X}_{i_1,i_2,\ldots,i_m}, \quad 1 \leq i_k \leq n, \ 1 \leq k \leq m,$$

with $j = \sum_{k=1}^{m-1} n^{m-k}(i'_k - 1) + i'_m$ and $\mathbf{P}$ is the permutation matrix corresponding to the ordering $P$: $(i'_1, i'_2, \ldots, i'_m) = (i_1, i_2, \ldots, i_m)\mathbf{P}$.

Using the same ordering $P$ on $\mathcal{A}$ and $\mathcal{X}$, we have the following characterization for the tensor eigenvalue problem.

Proposition 2.4 *The tensor eigenvalue problem in (4) is equivalent to the following the matrix eigenvalue system:*

$$\mathbf{A}_P \mathbf{x}_P = \lambda \mathbf{x}_P.$$

*Remark 2* According to Proposition 2.4, we can calculate the eigenvalues of $\mathcal{A}$ and its associated eigentensor by solving the eigenvalue problem of the corresponding matrix $\mathbf{A}_P$ based on the ordering $P$.

Indeed we see by Proposition 2.2 that both the eigenvalue and eigentensor are unique up to the ordering $P$.

PROPOSITION 2.5 *Let $\mathcal{A}$ be a $(2m)$th-order $n$-dimensional tensor. Suppose $P$ and $P'$ are two different orderings, $\lambda$ is an eigenvalue of $\mathbf{A}_P$ and $\mathbf{y}$ is the associated eigenvector. Then $\lambda$ is also eigenvalue of $\mathbf{A}_{P'}$ and $\Pi_{P,P'}\mathbf{y}$ is an eigenvector of $\mathbf{A}_{P'}$, where $\Pi_{P,P'}$ is perfect shuffle permutation matrix in Proposition 2.2.*

*Proof* We know that $\mathbf{A}_P\mathbf{y} = \lambda\mathbf{y}$ and $\Pi_{P,P'}\mathbf{A}_P\Pi^T_{P,P'} = \mathbf{A}_{P'}$. It implies that $\Pi_{P,P'}\mathbf{A}_P\Pi^T_{P,P'}\Pi_{P,P'}\mathbf{y} = \lambda\Pi_{P,P'}\mathbf{y}$. The result follows. □

It is noted that different from the tensor eigenvalue problem in Definition 1.1, the new tensor eigenvalue problem in Definition 1.2 is solvable and computable.

### 2.2. *Relationship with HOSVD*

In this subsection, we will establish the relationship between the proposed tensor eigenvalue problem and HOSVD [20] of a tensor. In numerical multilinear algebra, there are many applications for HOSVD [20]. The computational procedure of HOSVD involves the calculation of singular value decomposition of matrices with respect to tensor unfolding at different indices. The HOSVD of $\mathcal{A}$ is given by

$$\mathcal{A} = \mathcal{S} \times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2 \times_3 \cdots \times_{2m} \mathbf{U}_{2m}, \tag{10}$$

where $\mathcal{S}$ is a $(2m)$th-order $n$-dimensional all-orthogonal and ordering tensor and $\mathbf{U}_k$ are $n$-by-$n$ unitary matrices. Here, the multiplication $\times_k$ of a tensor $\mathcal{A}$ with a matrix $\mathbf{U}_k$ is a $(2m)$th-order $n$-dimensional tensor given by

$$(\mathcal{A} \times_k \mathbf{U}_k)_{i_1,\ldots,i_{k-1},j_k,i_{k+1},\ldots,i_{2m}} = \sum_{i_k=1}^{n} a_{i_1,\ldots,i_{k-1},i_k,i_{k+1},\ldots,i_{2m}}\mathbf{U}_k(j_k, i_k),$$

for $1 \le i_l, j_k \le n$ and $1 \le l \le 2m$.

Next, we will show that the HOSVD of $\mathcal{A}$ can provide the information for the eigenvalues and their associated eigentensors for the multiplication of $\mathcal{A}$ and its conjugate transpose. Let us first define the conjugate transpose of a tensor.

*Definition 2.6* Let $\mathcal{A}$ be a $(2m)$th-order $n$-dimensional tensor. $\mathcal{A}^*$ is called the conjugate transpose of $\mathcal{A}$ where its entry is given by $a_{j_1,\ldots,j_m,i_1,\ldots,i_m}$ for $1 \le i_k, j_k \le n$ and $1 \le k \le m$. $\mathcal{A}$ is called Hermitian if

$$a_{i_1,\ldots,i_m,j_1,\ldots,j_m} = \overline{a}_{j_1,\ldots,j_m,i_1,\ldots,i_m}, \quad 1 \le i_k, j_k \le n, \ 1 \le k \le m,$$

i.e. $\mathcal{A} = \mathcal{A}^*$.

The contraction product of two $(2m)$th-order $n$-dimensional tensors (the multiplication of two square tensors) can be defined as follows:

$$(\mathcal{A} \star \mathcal{B})_{i_1,\ldots,i_m,j_1,\ldots,j_m} = \sum_{k_1,\ldots,k_m=1}^{n} a_{i_1,\ldots,i_m,k_1,\ldots,k_m} b_{k_1,\ldots,k_m,j_1,\ldots,j_m}, \tag{11}$$

for $1 \le i_l, j_l \le n$ and $1 \le l \le m$. Indeed, the contraction product of two square tensors can be expressed in terms of the multiplication of two multilevel matrices. Also we have

$$(\mathcal{A} \star \mathcal{B})(:, \ldots, :, j_1, \ldots, j_m) = \mathcal{A} \cdot \mathcal{B}(:, \ldots, :, j_1, \ldots, j_m), \quad 1 \le j_k \le n, \ 1 \le k \le m. \tag{12}$$

Proposition 2.7 *Let $\mathcal{A}$ and $\mathcal{B}$ be $(2m)$th-order $n$-dimensional tensors. If their corresponding matrix unfolding are $\mathbf{A}_P$ and $\mathbf{B}_P$ under the ordering $P$, then the multilevel matrix of $\mathcal{A} \star \mathcal{B}$ under the ordering $P$ is equal to $\mathbf{A}_P\mathbf{B}_P$.*

*Proof* Using Proposition 2.2, it is sufficient to consider the natural ordering. The $(i, j)$ entry of $\mathbf{AB}$ is

$$\sum_{k=1}^{n^m} \mathbf{A}(i, k)\mathbf{B}(k, j) = \sum_{k_1,\ldots,k_m=1}^{n} a_{i_1,\ldots,i_m,k_1,\ldots,k_m} b_{k_1,\ldots,k_m,j_1,\ldots,j_m}.$$

Therefore, the multilevel matrix $\mathbf{AB}$ is the square matrix unfolding of $\mathcal{A} \star \mathcal{B}$ under the natural ordering. □

Because $\mathcal{A}$ has a HOSVD given in (10), we can make use of the decomposition to express the multilevel matrix $\mathbf{A}$ corresponding to $\mathcal{A}$ according to the natural ordering:

$$\mathbf{A} = (\mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \cdots \otimes \mathbf{U}_m)\mathbf{S}\left(\mathbf{U}_{m+1}^T \otimes \mathbf{U}_{m+2}^T \otimes \cdots \otimes \mathbf{U}_{2m}^T\right),$$

where $\mathbf{S}$ is the square matrix unfolding of $\mathcal{S}$ under the natural ordering. This implies that

$$\mathbf{A}^*\mathbf{A} = (\bar{\mathbf{U}}_{m+1} \otimes \bar{\mathbf{U}}_{m+2} \otimes \cdots \otimes \bar{\mathbf{U}}_{2m})\mathbf{S}^*\mathbf{S}\left(\mathbf{U}_{m+1}^T \otimes \mathbf{U}_{m+2}^T \otimes \cdots \otimes \mathbf{U}_{2m}^T\right), \tag{13}$$

where $\bar{\mathbf{U}}_k$ is a matrix where its entry is the complex conjugate of the entry of $\mathbf{U}_k$. We make use of the singular value decomposition of $\mathbf{S}$, i.e. $\mathbf{S} = \mathbf{QDW}^*$ where $\mathbf{Q}$ and $\mathbf{W}$ are unitary matrices and $\mathbf{D}$ is a real non-negative diagonal matrix. Therefore, we have

$$\mathbf{S}^*\mathbf{S} = \mathbf{WD}^2\mathbf{W}^*. \tag{14}$$

By substituting (14) into (13), we obtain

$$\mathbf{A}^*\mathbf{A} = (\bar{\mathbf{U}}_{m+1} \otimes \bar{\mathbf{U}}_{m+2} \otimes \cdots \otimes \bar{\mathbf{U}}_{2m})\mathbf{WD}^2\mathbf{W}^*\left(\mathbf{U}_{m+1}^T \otimes \mathbf{U}_{m+2}^T \otimes \cdots \otimes \mathbf{U}_{2m}^T\right). \tag{15}$$

It is clear that the eigenvectors of $\mathbf{A}^*\mathbf{A}$ are given by the unitary matrix:

$$\mathbf{V} = (\bar{\mathbf{U}}_{m+1} \otimes \bar{\mathbf{U}}_{m+2} \otimes \cdots \otimes \bar{\mathbf{U}}_{2m})\mathbf{W}$$

or the eigentensors of $\mathcal{A}^*\mathcal{A}$ are just given by

$$\mathcal{V} = \mathcal{W} \times_1 \bar{\mathbf{U}}_{m+1} \times_2 \bar{\mathbf{U}}_{m+2} \times_3 \cdots \times_m \bar{\mathbf{U}}_{2m},$$

where $\mathcal{V}$ and $\mathcal{W}$ are $(2m)$th-order $n$-dimensional tensors and their square matrix unfoldings are equal to $\mathbf{V}$ and $\mathbf{W}$, respectively. Correspondingly, the eigenvalues of $\mathcal{A}^*\mathcal{A}$ are given by the eigenvalues $\mathbf{D}^2$ of $\mathbf{S}^*\mathbf{S}$ or $\mathcal{S}^* \star \mathcal{S}$, i.e. the square of singular values $\mathbf{D}$ of $\mathbf{S}$. In other words, the square root of the eigenvalue of $\mathcal{A}^*\mathcal{A}$ is just equal to the singular value of $\mathbf{S}$ corresponding to square matrix unfolding of $\mathcal{S}$ which is the core tensor of $\mathcal{A}$. In other words, we obtain the eigen-decomposition of $\mathcal{A}^* \star \mathcal{A}$:

$$(\mathcal{A}^* \star \mathcal{A}) \star \mathcal{V} = \mathcal{V} \star \mathcal{D},$$

where $\mathcal{D}$ is a $(2m)$th-order $n$-dimensional diagonal tensor with its entries given by

$$\mathcal{D}_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = \begin{cases} \mathbf{D}^2_{k,k}, & k = \sum_{l=1}^{n}(i_l - 1)n^{m-l} + i_m,\ i_l = j_l,\ 1 \le l \le m, \\ 0, & \text{otherwise.} \end{cases}$$

*Example 1* Let $\mathcal{A}$ be a 4th-order three-dimensional with

$$\mathcal{A}(1,:,1,:) = \begin{pmatrix} -0.1050 & -0.1447 & 0.1417 \\ 0.0592 & 0.0821 & 0.0888 \\ 0.1414 & 0.1262 & 0.0567 \end{pmatrix},$$

$$\mathcal{A}(1,:,2,:) = \begin{pmatrix} -0.2156 & -0.0306 & 0.2127 \\ 0.2208 & 0.0832 & 0.0224 \\ 0.3945 & 0.2600 & -0.0458 \end{pmatrix},$$

$$\mathcal{A}(1,:,3,:) = \begin{pmatrix} 0.2830 & -0.8334 & -0.0230 \\ -0.6153 & 0.1710 & 0.4950 \\ -0.8596 & -0.3439 & 0.5920 \end{pmatrix},$$

$$\mathcal{A}(2,:,1,:) = \begin{pmatrix} 0.2631 & -0.4137 & -0.4889 \\ 0.8970 & 0.1507 & 0.2317 \\ -0.0113 & 0.3217 & 0.5491 \end{pmatrix},$$

$$\mathcal{A}(2,:,2,:) = \begin{pmatrix} 0.2353 & -0.6650 & -0.7135 \\ 1.3066 & 0.4597 & 0.4064 \\ -0.1215 & 0.5602 & 0.8921 \end{pmatrix},$$

$$\mathcal{A}(2,:,3,:) = \begin{pmatrix} 0.6910 & 0.2683 & -0.0154 \\ 0.0391 & -1.0960 & -0.3063 \\ 0.4814 & -0.4065 & -0.3998 \end{pmatrix},$$

$$\mathcal{A}(3,:,1,:) = \begin{pmatrix} -0.1438 & -0.2205 & 0.0344 \\ -0.0674 & 0.2719 & 0.3131 \\ 0.0330 & 0.2564 & 0.2619 \end{pmatrix},$$

$$\mathcal{A}(3,:,2,:) = \begin{pmatrix} -0.3521 & -0.2077 & 0.0935 \\ -0.0886 & 0.3047 & 0.4961 \\ 0.1673 & 0.4424 & 0.3168 \end{pmatrix},$$

$$\mathcal{A}(3,:,3,:) = \begin{pmatrix} 0.6487 & -0.5308 & -0.1976 \\ -0.0471 & 0.4314 & -0.1700 \\ -0.5455 & -0.3052 & 0.3088 \end{pmatrix}.$$

The unitary matrices in the HOSVD of $\mathcal{A}$ are given by

$$\mathbf{U}_1 = \begin{pmatrix} -0.5875 & 0.4445 & -0.6762 \\ -0.4185 & -0.8821 & -0.2163 \\ -0.6926 & 0.1559 & 0.7043 \end{pmatrix}, \quad \mathbf{U}_2 = \begin{pmatrix} -0.2879 & -0.8442 & 0.4522 \\ -0.9295 & 0.3601 & 0.0804 \\ -0.2307 & -0.3971 & -0.8883 \end{pmatrix},$$

$$\mathbf{U}_3 = \begin{pmatrix} -0.5579 & -0.1280 & -0.8200 \\ -0.7333 & -0.3866 & 0.5593 \\ -0.3885 & 0.9133 & 0.1218 \end{pmatrix}, \quad \mathbf{U}_4 = \begin{pmatrix} -0.5798 & -0.5521 & 0.5992 \\ -0.2881 & 0.8268 & 0.4830 \\ -0.7621 & 0.1074 & -0.6385 \end{pmatrix}.$$

The core tensor of $\mathcal{A}$ is $\mathcal{S}$, and its entries are given by

$$\mathcal{S}(1,:,1,:) = \begin{pmatrix} 0.9172 & -0.0540 & -0.0119 \\ -0.2858 & 0.5308 & 0.3371 \\ 0.7572 & -0.7792 & 0.1622 \end{pmatrix},$$

$$\mathcal{S}(1,:,2,:) = \begin{pmatrix} 0.6020 & -0.2290 & -0.4427 \\ -0.2630 & -0.9133 & 0.1067 \\ 0.6541 & 0.1524 & -0.9619 \end{pmatrix},$$

$$\mathcal{S}(2,:,1,:) = \begin{pmatrix} 0.7537 & 0.9340 & -0.7943 \\ -0.3804 & 0.1299 & -0.3112 \\ 0.5678 & -0.5688 & 0.5285 \end{pmatrix},$$

$$\mathcal{S}(2,:,2,:) = \begin{pmatrix} 0.6892 & -0.8258 & 0.0046 \\ -0.7482 & 0.5383 & 0.7749 \\ 0.4505 & -0.9961 & 0.8173 \end{pmatrix},$$

$$\mathcal{S}(3,:,1,:) = \begin{pmatrix} 0.0759 & 0.4694 & -0.1656 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{S}(3,:,2,:) = \begin{pmatrix} 0.0838 & 0.0782 & -0.8687 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{S}(1,:,3,:) = \mathcal{S}(2,:,3,:) = \mathcal{S}(3,:,3,:) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The square matrix unfolding of $\mathcal{S}$ with the natural ordering is given by

$$\mathbf{S} = \begin{pmatrix} 0.9172 & -0.0540 & -0.0119 & 0.6020 & -0.2290 & -0.4427 & 0 & 0 & 0 \\ -0.2858 & 0.5308 & 0.3371 & -0.2630 & -0.9133 & 0.1067 & 0 & 0 & 0 \\ 0.7572 & -0.7792 & 0.1622 & 0.6541 & 0.1524 & -0.9619 & 0 & 0 & 0 \\ 0.7537 & 0.9340 & -0.7943 & 0.6892 & -0.8258 & 0.0046 & 0 & 0 & 0 \\ -0.3804 & 0.1299 & -0.3112 & -0.7482 & 0.5383 & 0.7749 & 0 & 0 & 0 \\ 0.5678 & -0.5688 & 0.5285 & 0.4505 & -0.9961 & 0.8173 & 0 & 0 & 0 \\ 0.0759 & 0.4694 & -0.1656 & 0.0838 & 0.0782 & -0.8687 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then we obtain

$$\mathbf{A} = (\mathbf{U}_1 \otimes \mathbf{U}_2)\mathbf{S}(\mathbf{U}_3^T \otimes \mathbf{U}_4^T).$$

Next, we will find an unitary matrix $\mathbf{W}$ such that $\mathbf{W}^*\mathbf{S}^*\mathbf{S}\mathbf{W} = \Sigma^2$ is a diagonal matrix, where

$$\mathbf{W} = \begin{pmatrix} -0.6130 & 0.0214 & 0.0793 & -0.4162 & 0.6646 & 0.0505 & 0 & 0 & 0 \\ 0.0598 & 0.3451 & -0.7759 & 0.1160 & 0.1726 & 0.4817 & 0 & 0 & 0 \\ 0.0105 & -0.0074 & 0.5060 & 0.5747 & 0.2650 & 0.5858 & 0 & 0 & 0 \\ -0.5777 & -0.0002 & 0.0490 & -0.1615 & -0.6725 & 0.4307 & 0 & 0 & 0 \\ 0.3679 & -0.6915 & -0.0601 & -0.4354 & 0.0630 & 0.4352 & 0 & 0 & 0 \\ 0.3892 & 0.6343 & 0.3599 & -0.5170 & -0.0447 & 0.2177 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix $\mathbf{V}$ is given by

$$\mathbf{V} = (\mathbf{U}_3 \otimes \mathbf{U}_4)\mathbf{W}$$
$$= \begin{pmatrix} -0.2301 & 0.0182 & -0.4107 & -0.2941 & 0.1375 & 0.0148 & -0.4913 & 0.4527 & 0.4754 \\ -0.2132 & -0.1198 & 0.2202 & -0.2032 & -0.0729 & -0.4156 & -0.3961 & -0.6780 & 0.2362 \\ -0.2901 & 0.0471 & 0.2955 & -0.0312 & 0.2965 & 0.2551 & 0.5235 & -0.0881 & 0.6249 \\ -0.3821 & -0.1425 & -0.5880 & -0.3919 & 0.1091 & 0.0980 & 0.3351 & -0.3088 & -0.3243 \\ -0.4241 & -0.0995 & 0.2655 & -0.1442 & -0.1448 & -0.6207 & 0.2701 & 0.4624 & -0.1611 \\ -0.4318 & 0.1665 & 0.4481 & -0.1298 & 0.2701 & 0.4271 & -0.3571 & 0.0601 & -0.4262 \\ 0.2057 & 0.7764 & -0.0650 & -0.1806 & 0.4249 & -0.3501 & 0.0730 & -0.0672 & -0.0706 \\ 0.5118 & -0.3494 & 0.2637 & -0.7061 & 0.1740 & 0.0523 & 0.0588 & 0.1007 & -0.0351 \\ 0.0298 & -0.4475 & -0.0685 & 0.3857 & 0.7557 & -0.2439 & -0.0778 & 0.0131 & -0.0928 \end{pmatrix},$$

and

$$\Sigma^2 = \mathrm{diag}(6.0484, 3.8184, 3.0304, 1.1585, 0.0968, 0.0240, 0, 0, 0).$$

Therefore, the eigenvalues of $\mathcal{A}^* \star \mathcal{A}$ are

$$6.0484, 3.8184, 3.0304, 1.1585, 0.0968, 0.0240, 0, 0, 0$$

and the eigentensors are $\mathcal{V}$. The entries of the tensor $\mathcal{V}$ can be constructed correspondingly.

## 3. Toeplitz and circulant tensors

In this section, we will study the eigenvalue problem for special structured tensors: Toeplitz and circulant tensors.

*Definition 3.1* A $(2m)$th-order $n$-dimensional tensor $\mathcal{T} = (t_{i_1,\ldots,i_m,j_1,\ldots,j_m})$ is called a Toeplitz tensor if

$$t_{i_1,\ldots,i_m,j_1,\ldots,j_m} = r_{j_1-i_1,\ldots,j_m-i_m}, \quad 1 \le i_k,\, j_k \le n,\ 1 \le k \le m, \tag{16}$$

where $\mathcal{R} = (r_{i_1,\ldots,i_m})$ is an $m$th-order $(2n-1)$-dimensional tensor.

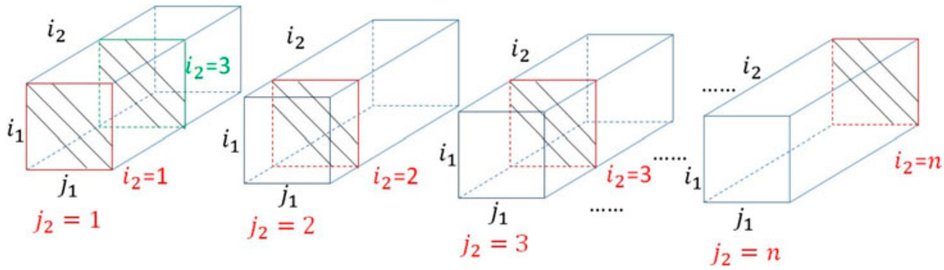


Figure 2. A 4th-order Toeplitz Tensor.

It is clear when $m = 1$, the above definition is the same as that of Toeplitz matrix. Because of its special structure, we only require $m^{2n-1}$ entries to construct a Toeplitz tensor. This Toeplitz tensor has been studied in [17].

In Figure 2, we give an example to show the structure of a 4th-order $n$-dimensional Toeplitz tensor. Let us fix two indices $i_2$ and $j_2$. For example, when $i_2 = 3$ and $j_2 = 1$, $\mathcal{T}(:, 3, :, 1)$ is a Toeplitz matrix (the green one as shown in the figure). Indeed, all front slices of each cuboid, $\mathcal{T}(:, i_2, :, j_2)$, are Toeplitz matrices. Similarly, all the slices $\mathcal{T}(i_1, :, j_1, :)$ are also Toeplitz matrices. It is worth to noting that if $j_2 - i_2$ (or $j_1 - i_1$) are fixed, then the corresponding matrices $\mathcal{T}(:, i_2, :, j_2)$ (or $\mathcal{T}(i_1, :, j_1, :)$) are the same. For example, when $j_2 - i_2 = 0$, these matrices shown in red are the same in different cuboids in Figure 2.

Similar to (9), we can use the square matrix unfolding with natural ordering to present a Toeplitz tensor. It is an $n^m \times n^m$ multilevel Toeplitz matrix $\mathbf{T}$. Indeed, $\mathbf{T}$ can be regarded as the first-level matrix, i.e.

$$\mathbf{T} = \mathbf{T}^{(1)} = \begin{pmatrix} \mathbf{T}^{(2)}_{(0)} & \mathbf{T}^{(2)}_{(1)} & \cdots & \mathbf{T}^{(2)}_{(n-1)} \\ \mathbf{T}^{(2)}_{(-1)} & \mathbf{T}^{(2)}_{(0)} & \cdots & \mathbf{T}^{(2)}_{(n-2)} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{T}^{(2)}_{(1-n)} & \mathbf{T}^{(2)}_{(2-n)} & \cdots & \mathbf{T}^{(2)}_{(0)} \end{pmatrix},$$

where $(i, j)$th-block $\mathbf{T}^{(2)}_{(j-i)}$ is a second-level matrix. $\mathbf{T}^{(2)}_{(k_1)}$ is given by:

$$\mathbf{T}^{(2)}_{(k_1)} = \begin{pmatrix} \mathbf{T}^{(3)}_{(k_1,0)} & \mathbf{T}^{(3)}_{(k_1,1)} & \cdots & \mathbf{T}^{(3)}_{(k_1,n-1)} \\ \mathbf{T}^{(3)}_{(k_1,-1)} & \mathbf{T}^{(3)}_{(k_1,0)} & \cdots & \mathbf{T}^{(3)}_{(k_1,n-2)} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{T}^{(3)}_{(k_1,1-n)} & \mathbf{T}^{(3)}_{(k_1,2-n)} & \cdots & \mathbf{T}^{(3)}_{(k_1,0)} \end{pmatrix}, \quad 1-n \le k_1 \le n-1.$$

In general, the $\ell$th-level matrix is an $n$-by-$n$ block matrix with $n^{m-\ell}$-by-$n^{m-\ell}$ blocks given by

$$\mathbf{T}^{(\ell)}_{(k_1,k_2,\ldots,k_{\ell-1})} = \begin{pmatrix} \mathbf{T}^{(\ell+1)}_{(k_1,k_2,\ldots,k_{\ell-1},0)} & \mathbf{T}^{(\ell+1)}_{(k_1,k_2,\ldots,k_{\ell-1},1)} & \cdots & \mathbf{T}^{(\ell+1)}_{(k_1,k_2,\ldots,k_{\ell-1},n-1)} \\ \mathbf{T}^{(\ell+1)}_{(k_1,k_2,\ldots,k_{\ell-1},-1)} & \mathbf{T}^{(\ell+1)}_{(k_1,k_2,\ldots,k_{\ell-1},0)} & \cdots & \mathbf{T}^{(\ell+1)}_{(k_1,k_2,\ldots,k_{\ell-1},n-2)} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{T}^{(\ell+1)}_{(k_1,k_2,\ldots,k_{\ell-1},1-n)} & \mathbf{T}^{(\ell+1)}_{(k_1,k_2,\ldots,k_{\ell-1},2-n)} & \cdots & \mathbf{T}^{(\ell+1)}_{(k_1,k_2,\ldots,k_{\ell-1},0)} \end{pmatrix},$$

for $1-n \leq k_1, k_2, \ldots, k_{\ell-1} \leq n-1$ and $2 \leq \ell \leq m$. It is clear that when $\ell = m$, $\mathbf{T}^{(m+1)}_{(k_1,k_2,\ldots,k_{m-1})}$ is an $n$-by-$n$ Toeplitz matrix with entries given by

$$\mathbf{T}^{(m+1)}_{(k_1,k_2,\ldots,k_{m-1})}(i,j) = r_{k_1,k_2,\ldots,k_{m-1},j-i}.$$

As an example, the square matrix unfolding with natural ordering of the Toeplitz tensor shown in Figure 2 is a block-Toeplitz with Toeplitz-block matrix $\mathbf{T}$ with

$$\mathbf{T}^{(2)}_{(k_1)} = \begin{pmatrix} r_{k_1,0} & r_{k_1,1} & \cdots & r_{k_1,n-1} \\ r_{k_1,-1} & r_{k_1,0} & \cdots & r_{k_1,n-2} \\ \vdots & \ddots & \ddots & \vdots \\ r_{k_1,1-n} & r_{k_1,2-n} & \cdots & r_{k_1,0} \end{pmatrix}, \quad 1-n \leq k_1 \leq n-1.$$

In [22], it is known that a Toeplitz matrix can be viewed as a principal submatrix of a fixed singly infinite Toeplitz matrix. A matrix is called singly infinite matrix if the indices $i$ and $j$ go from 1 up to infinity. We can associate with the singly infinite Toeplitz matrix a generating function defined on $[-\pi, \pi]$. We note that in practical applications generating functions are usually available in time series, signal processing and image processing. [21,22]

Similarly, we consider a sequence of a $(2m)$th-order Toeplitz tensor $\mathcal{T}_n$. Let

$$f(z_1, z_2, \ldots, z_m) = \sum_{k_1,k_2,\ldots,k_m=1}^{n} r_{k_1,k_2,\ldots,k_m} \exp\left(-i \sum_{\ell=1}^{m} k_\ell z_\ell\right), \tag{17}$$

The function $f$ is called a generating function of the sequence of Toeplitz tensor $\mathcal{T}_n$. The coefficients of $f$ are

$$r_{k_1,k_2,\ldots,k_m} = \frac{1}{2\pi} \int_S f(z_1, z_2, \ldots, z_m) \exp\left(i \sum_{\ell=1}^{m} k_\ell z_\ell\right) \mathrm{d}z_1 \mathrm{d}z_2 \ldots \mathrm{d}z_m, \tag{18}$$

where $S = \underbrace{[-\pi,\pi] \times [-\pi,\pi] \times \cdots \times [-\pi,\pi]}_{m \text{ times}}$. The coefficients can be used to construct a Toeplitz tensor $\mathcal{T}_n$.

There is a close relationship between the spectrum of $\mathcal{T}_n$ and its generating function $f$.

Theorem 3.2 *Let $f$ be a $2\pi$-periodic real continuous function defined on $S$. Then the spectrum $\sigma(\mathcal{T}_n)$ of $\mathcal{T}_n$ satisfies*

$$\sigma(\mathcal{T}_n) \subseteq [f_{\min}, f_{\max}], \quad \forall n \geq 1, \tag{19}$$

*where $f_{\min}$ and $f_{\max}$ denote the minimum and maximum values of $f$.*

*Proof* Because $f$ is real, $r_{i_1,i_2,\ldots,i_m} = \bar{r}_{-i_1,-i_2,\ldots,-i_m}$ for $1 \leq i_k \leq n$ and $1 \leq k \leq m$. This implies that $\mathcal{T}_n$ is Hermitian, i.e.

$$t_{i_1,\ldots,i_m,j_1,\ldots,j_m} = r_{j_1-i_1,\ldots,j_m-i_m} = \bar{r}_{i_1-j_1,\ldots,i_m-j_m} = \bar{t}_{j_1,\ldots,j_m,i_1,\ldots,i_m}.$$

For any $m$th-order $n$-dimensional tensor $\mathcal{X}$, we consider $\mathbf{x}^*\mathbf{T}_n\mathbf{x}$, where $\mathbf{T}_n$ is the matrix unfolding of $\mathcal{T}_n$ under the natural ordering and $\mathbf{x}$ is the vectorization of $\mathcal{X}$ under the natural ordering. Now we know that

$$
\begin{aligned}
&\mathbf{x}^*\mathbf{T}_n\mathbf{x}\\
&\quad= \sum_{i_1,\ldots,i_m=1}^{n} \sum_{j_1,\ldots,j_m=1}^{n} \mathcal{T}_n(i_1,\ldots,i_m,j_1,\ldots,j_m)\mathcal{X}(j_1,\ldots,j_m)\bar{\mathcal{X}}(i_1,\ldots,i_m)\\
&\quad= \sum_{i_1,\ldots,i_m=1}^{n} \sum_{j_1,\ldots,j_m=1}^{n} r_{j_1-i_1,\ldots,j_m-i_m}\mathcal{X}(j_1,\ldots,j_m)\bar{\mathcal{X}}(i_1,\ldots,i_m)\\
&\quad= \frac{1}{2\pi} \sum_{i_1,\ldots,i_m=1}^{n} \sum_{j_1,\ldots,j_m=1}^{n} \int_S f(z_1,z_2,\ldots,z_m)\exp\left(i\sum_{\ell=1}^{m}(j_\ell-i_\ell)z_\ell\right)\mathrm{d}z_1\mathrm{d}z_2\ldots\mathrm{d}z_m\\
&\qquad \mathcal{X}(j_1,\ldots,j_m)\bar{\mathcal{X}}(i_1,\ldots,i_m)\\
&\quad= \frac{1}{2\pi}\int_S \left|\sum_{j_1,\ldots,j_m=1}^{n}\mathcal{X}(j_1,\ldots,j_m)\exp\left(i\sum_{\ell=1}^{m} j_\ell z_\ell\right)\right|^2 f(z_1,\ldots,z_m)\mathrm{d}z_1\ldots\mathrm{d}z_m \qquad (20)
\end{aligned}
$$

where

$$
\begin{aligned}
&\left|\sum_{j_1,\ldots,j_m=1}^{n}\mathcal{X}(j_1,\ldots,j_m)\exp\left(i\sum_{\ell=1}^{m} j_\ell z_\ell\right)\right|^2\\
&=\left[\sum_{j_1,\ldots,j_m=1}^{n}\mathcal{X}(j_1,\ldots,j_m)\exp\left(i\sum_{\ell=1}^{m} j_\ell z_\ell\right)\right]\left[\sum_{j_1,\ldots,j_m=1}^{n}\bar{\mathcal{X}}(j_1,\ldots,j_m)\exp\left(-i\sum_{\ell=1}^{m} j_\ell z_\ell\right)\right].
\end{aligned}
$$

We note that when $\mathbf{x}^*\mathbf{x} = 1$, i.e.

$$
\sum_{i_1,\ldots,i_m=1}^{n}\mathcal{X}(i_1,\ldots,i_m)\bar{\mathcal{X}}(i_1,\ldots,i_m) = 1,
$$

we obtain

$$
\frac{1}{2\pi}\int_S \left|\sum_{j_1,\ldots,j_m=1}^{n}\mathcal{X}(j_1,\ldots,j_m)\exp\left(i\sum_{\ell=1}^{m} j_\ell z_\ell\right)\right|^2 \mathrm{d}z_1\mathrm{d}z_2\ldots\mathrm{d}z_m = 1. \qquad (21)
$$

It follows from (20) that

$$
f_{\min} \le \mathbf{x}^*\mathbf{T}_n\mathbf{x} \le f_{\max},
$$

and therefore the lower bound and the upper bound of eigenvalues of $\mathbf{T}_n$ are given by $f_{\min}$ and $f_{\max}$, respectively. Because the spectrum of $\mathbf{T}_n$ is the same as the one of $\mathcal{T}_n$, this proves the result. □

According to the Toeplitz structure, we can further define a circulant tensor.

*Definition 3.3* A $(2m)$th-order $n$-dimensional tensor $\mathcal{C} = (c_{i_1,\ldots,i_m,j_1,\ldots,j_m})$ is a circulant tensor if it is a Toeplitz tensor and

$$
c_{i_1,\ldots,i_m,j_1,\ldots,j_m} = r_{j_1-i_1(\mathrm{mod}\ n),\ldots,j_m-i_m(\mathrm{mod}\ n)}, \quad 1 \le i_k,\ j_k \le n,\ 1 \le k \le m, \qquad (22)
$$

where $\mathcal{R} = (r_{i_1,\ldots,i_m})$ is an $m$th-order $n$-dimensional tensor.

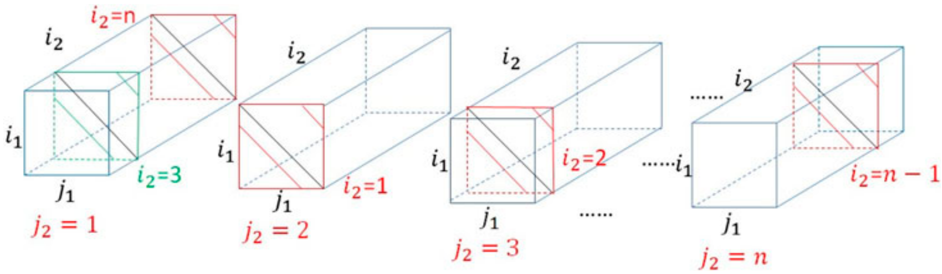


Figure 3. A 4th-order $n$-dimensional circulant tensor.

It is clear when $m = 1$, the above definition is the same as that of a circulant matrix. Because of the circulant structure, we only require $m^n$ entries to construct a circulant tensor. This kind of circulant tensor has been studied in [25]. Any $n \times n$ circulant matrix $C_n$ can always be diagonalized by discrete Fourier matrices $F_n$,[22] where

$$(F_n)_{jk} = \frac{1}{\sqrt{n}} e^{-2\pi i(j-1)(k-1)/n}, \quad 1 \le j, k \le n.$$

Similarly, we show the structure of a 4th-order $n$-dimensional circulant tensor in Figure 3. Again when we fix two indices $i_2$ and $j_2$. For example, when $i_2 = 3$ and $j_2 = 1$, $\mathcal{C}(:, 3, :, 1)$ is a circulant matrix (the green one as shown in the figure). And, all the front slices of each cuboid, $\mathcal{C}(:, i_2, :, j_2)$, are circulant matrices. Similarly, all the slices $\mathcal{C}(i_1, :, j_1, :)$, are also circulant matrices. Notice that if $j_2 - i_2 (\mathrm{mod}\, n)$ (or $j_1 - i_1 (\mathrm{mod}\, n)$) are fixed, then the corresponding matrices $\mathcal{C}(:, i_2, :, j_2)$ (or $\mathcal{C}(i_1, :, j_1, :)$) are the same. For example, when $j_2 - i_2 (\mathrm{mod}\ n) = n - 1$, these matrices shown in red are circulant matrices and they are same in different cuboids in Figure 3.

As an example, $\mathcal{C}$ is a 4th-order $n$-dimensional circulant tensor. Similar to (9), we can use the square matrix unfolding with natural ordering of $\mathcal{C}$ to obtain a multilevel matrix $\mathbf{C}$. It is interesting to note that $\mathbf{C}$ is a block-circulant and circulant-block matrix:

$$\mathbf{C} = \mathbf{C}^{(1)} = \begin{pmatrix} \mathbf{C}^{(2)}_{(0)} & \mathbf{C}^{(2)}_{(1)} & \cdots & \mathbf{C}^{(2)}_{(n-2)} & \mathbf{C}^{(2)}_{(n-1)} \\ \mathbf{C}^{(2)}_{(n-1)} & \mathbf{C}^{(2)}_{(0)} & \cdots & \mathbf{C}^{(3)}_{(n-3)} & \mathbf{C}^{(2)}_{(n-2)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{C}^{(2)}_{(2)} & \mathbf{C}^{(2)}_{(3)} & \cdots & \mathbf{C}^{(2)}_{(0)} & \mathbf{C}^{(2)}_{(1)} \\ \mathbf{C}^{(2)}_{(1)} & \mathbf{C}^{(2)}_{(2)} & \cdots & \mathbf{C}^{(2)}_{(n-1)} & \mathbf{C}^{(2)}_{(0)} \end{pmatrix},$$

where

$$\mathbf{C}^{(2)}_{(k_1)} = \begin{pmatrix} r_{k_1,0} & r_{k_1,1} & \cdots & r_{k_1,n-2} & r_{k_1,n-1} \\ r_{k_1,n-1} & r_{k_1,0} & \cdots & r_{k_1,n-3} & r_{k_1,n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ r_{k_1,2} & r_{k_1,3} & \cdots & r_{k_1,0} & r_{k_1,1} \\ r_{k_1,1} & r_{k_1,2} & \cdots & r_{k_1,n-1} & r_{k_1,0} \end{pmatrix}, \quad 0 \le k_1 \le n-1.$$

It is well known that a block-circulant with circulant-block matrix can be diagonalized by the discrete Fourier matrix. This implies that the eigentensor of a circulant tensor can be obtained similarly.

THEOREM 3.4 *Let $\mathcal{C}$ be a $(2m)$th-order $n$-dimensional circulant tensor given by (22). Then*

$$\mathcal{C} \star \mathcal{F} = \mathcal{F} \star \mathcal{D}$$

*where $\mathcal{D}$ is a $(2m)$th-order $n$-dimensional diagonal tensor with its entries given by*

$$\mathcal{D}_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = \begin{cases} \lambda_{i_1,i_2,\ldots,i_m}, & i_p = j_p,\ 1 \le p \le m, \\ 0, & \text{otherwise.} \end{cases}$$

*where*

$$\lambda_{i_1,i_2,\ldots,i_m} = \sum_{k_1,j_2,\ldots,k_m=1}^{n} r_{k_1-1,k_2-1,\ldots,k_m-1} \exp\left(\frac{-2\pi i}{n}\sum_{\ell=1}^{m}(i_\ell-1)(k_\ell-1)\right),$$

*and $\mathcal{F}$ is a $(2m)$th-order $n$-dimensional tensor with its entries given by*

$$\mathcal{F}_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} = \exp\left(\frac{-2\pi i}{n}\sum_{\ell=1}^{m}(i_\ell-1)(j_\ell-1)\right), \quad 1 \le i_p, j_p \le n,\ 1 \le p \le m.$$

*Proof* For any $1 \le i_\ell, j_\ell \le n$, we have

$$\begin{aligned}
&(\mathcal{C} \star \mathcal{F})(i_1,\ldots,i_m,j_1,\ldots,j_m) \\
&= \sum_{k_1,\ldots,k_m=1}^{n} \mathcal{C}(i_1,\ldots,i_m,k_1,\ldots,k_m)\mathcal{F}(k_1,\ldots,k_m,j_1,\ldots,j_m) \\
&= \sum_{k_1,\ldots,k_m=1}^{n} r_{k_1-i_1 (\text{mod } n),\ldots,k_m-i_m (\text{mod } n)} \exp\left(\frac{-2\pi i}{n}\sum_{\ell=1}^{m}(k_\ell-1)(j_\ell-1)\right).
\end{aligned}$$

Let

$$k'_\ell = k_\ell - i_\ell (\text{mod } n) + 1 = \begin{cases} k_\ell - i_\ell + 1, & k_\ell \ge i_\ell, \\ n + k_\ell - i_\ell + 1, & k_\ell < i_\ell. \end{cases}$$

Because $\exp\left(\frac{-2\pi i}{n} n\right) = 1$, for any $1 \le i_\ell, j_\ell \le n$, we have

$$\begin{aligned}
&(\mathcal{C} \star \mathcal{F})(i_1,\ldots,i_m,j_1,\ldots,j_m) \\
&= \sum_{k'_1,\ldots,k'_m=1}^{n} r_{k'_1-1,\ldots,k'_m-1} \exp\left(\frac{-2\pi i}{n}\sum_{\ell=1}^{m}(k'_\ell+i_\ell-2)(j_\ell-1)\right) \\
&= \sum_{k'_1,\ldots,k'_m=1}^{n} r_{k'_1-1,\ldots,k'_m-1} \exp\left(\frac{-2\pi i}{n}\sum_{\ell=1}^{m}(k'_\ell-1)(j_\ell-1)\right) \cdot \\
&\quad \exp\left(\frac{-2\pi i}{n}\sum_{\ell=1}^{m}(i_\ell-1)(j_\ell-1)\right)
\end{aligned}$$

$$\begin{aligned}
&= \lambda_{j_1,\ldots,j_m} \mathcal{F}_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} \\
&= \mathcal{F}_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m} \mathcal{D}_{j_1,j_2,\ldots,j_m,j_1,j_2,\ldots,j_m} \\
&= \sum_{k_1,\ldots,k_m=1}^{n} \mathcal{F}_{i_1,i_2,\ldots,i_m,k_1,k_2,\ldots,k_m} \mathcal{D}_{k_1,k_2,\ldots,k_m,j_1,j_2,\ldots,j_m} \\
&= (\mathcal{F} \star \mathcal{D})(i_1,\ldots,i_m,j_1,\ldots,j_m).
\end{aligned}$$

The result follows. □

### 3.1. *Applications of image restoration*

In image processing, digital image restoration and reconstruction play an important role in various areas of applied sciences.[26] In most cases of focus-diverse phase retrieval and deconvolution problems, three-dimensional point spread functions $S(x, y, z)$ can be expressed as the three-dimensional Fourier transform expression of focal field distributions and aperture function.[27–29] A third-order tensor $\mathcal{S}$ can be used to describe a three-dimensional point spread function. The three-dimensional model of image formation can be expressed as the convolution of an object and a point spread function associated with the noise [27]:

$$G(x, y, z) = F(x, y, z) * S(x, y, z) + N(x, y, z), \tag{23}$$

where $G(x, y, z)$ is the image, $N(x, y, z)$ is the noise, $F(x, y, z)$ is the object, and $*$ denotes the three-dimensional convolution operator. Using tensors representation for (23), we have

$$\mathcal{G} = \mathcal{T} \cdot \mathcal{F} + \mathcal{N}, \tag{24}$$

where $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{N}$ are third-order tensors for $G$, $F$ and $N$, respectively, $\mathcal{T}$ is the sixth-order Toeplitz tensor (convolution operator) obtained from $\mathcal{S}$, i.e.

$$\mathcal{T}(i_1, i_2, i_3, j_1, j_2, j_3) = \mathcal{S}(j_1 - i_1, j_2 - i_2, j_3 - i_3), \quad 1 \le i_k, j_k \le n,\ 1 \le k \le m,$$

see (16). Figure 4 gave an example of a stack of brain MRI images restoration. The size of each image is $100 \times 100$. The images in the first row of Figure 4 are clean images. We get a stack of blurred images by adding a random noise on each clean image. And the blurred images are arranged at the second row of Figure 4. Our purpose is to recover a stack of images from the stack of blurred images at same time, but not getting them slice by slice. If we know the convolution operator tensor $\mathcal{T}$, we may use the method proposed in [19] to solve the tensor system (24) to find the tensor $\mathcal{F}$ (the original MRI images). And the result are arranged at the third row of Figure 4. Although we can restore each slice in the stack of images by image processing method, the tensor method can recover all images in the stack at same time. This shows that the tensor method is efficient. And the tensor method can use the information between difference slices. For example, each slice is different but the image structure of adjacent slices is similar.

It is well known that restoring an image is an ill-conditioned problem. It is effective to estimate the eigenvalues of a Toeplitz tensor (convolution operator) $\mathcal{T}$ so that a suitable regularization method can be used in the image restoration process.[30] In particular, we make use of Theorem 3.2 to estimate the smallest and largest eigenvalue of $\mathcal{T}$.

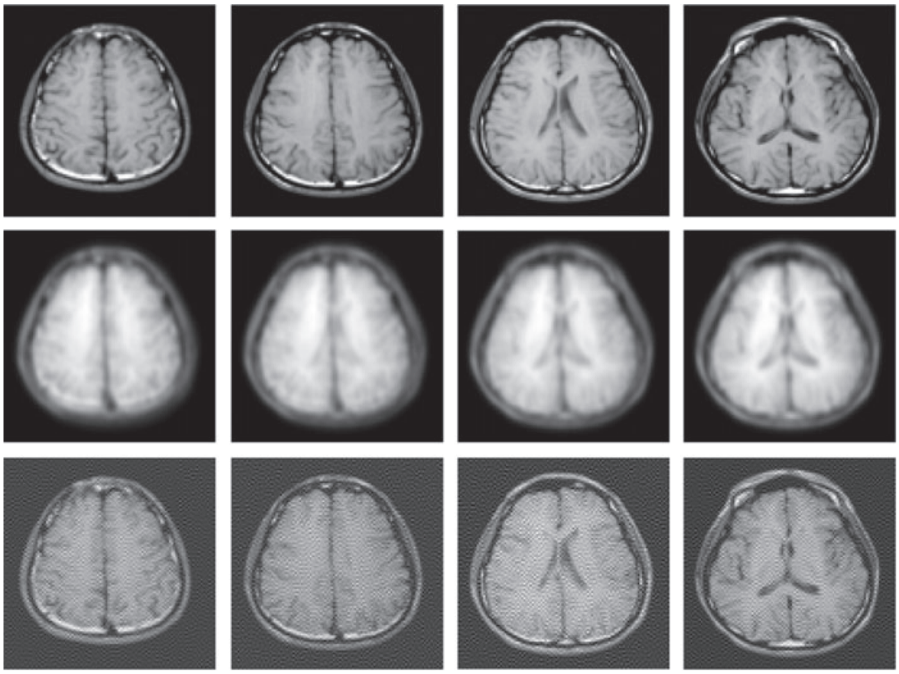

Figure 4. An example of a stack of brain MRI images restoration. The images in the first, second and third row are the clean MRI images, blurred images with random noise and restoration images by tensor method in (24), respectively.

## 4. Concluding remarks

In this article, we have studied eigenvalue problem for even order tensors using the matrix unfolding, and established the HOSVD of a tensor. In addition, the theory of the eigenvalues and the associated eigentensors for special tensors: Toeplitz/circulant tensors are given. An application in image restoration is also discussed. In the future, we may consider to extend our work to odd order tensors, and study their singular value problems. And then, we will explore the low-rank tensor approximation problem [20,23] using the singular value problem.


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## References

[1] Lim L. Singular values and eigenvalues of tensors: a variational approach. In: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP ’05); 2005 Dec 13–15; Puerto Vallarta. p. 129–132.
[2] Qi L. Eigenvalues of a real supersymmetric tensor. J. Symbol. Comput. 2005;40:1302–1324.
[3] Chang K, Pearson K, Zhang T. Perron–Frobenius theorem for nonnegative tensors. Commun. Math. Sci. 2008;6:507–520.
[4] Chang K, Qi L, Zhang T. A survey on the spectral theory of nonnegative tensors. Numer. Linear Algebra Appl. 2013;20:891–912.
[5] Li W, Cui L, Ng M. The perturbation bound for the Perron vector of a transition probability tensor. Numer. Linear Algebra Appl. 2013;20:985–1000.
[6] Li W, Ng M. On the limiting probability distribution of a transition probability tensor. Linear Multilinear Algebra. 2014;62:362–385.
[7] Li W, Ng M. Some bounds for the spectral radius of nonnegative tensors. Numer. Math. 2015;130:315–335.
[8] Li X, Ng M, Ye Y. HAR: hub, authority and relevance scores in multi-relational data for query search. In: The 2012 SIAM International Conference on Data Mining; 2012 Apr 26–28; Anaheim, CA.
[9] Ng M, Qi L, Zhou G. Finding the largest eigenvalue of a nonnegative tensor. SIAM J. Matrix Anal. Appl. 2009;31:1090–1099.
[10] Qi L. Eigenvalues and invariants of tensors. J. Math. Anal. Appl. 2007;325:1363–1377.
[11] Qi L. Symmetric nonnegative tensors and copsitive tensors. Linear Algebra Appl. 2013;439: 228–238.
[12] Qi L, Yu G, Wu EX. Higher order positive semi-definite diffusion tensor imaging. SIAM J. Imaging Sci. 2010;3:416–433.
[13] Yang Q, Yang Y. Further results for Perron–Frobenius theorem for nonnegative tensors II. SIAM J. Matrix Anal. Appl. 2011;32:1236–1250.
[14] Itskov M. On the theory of fourth-order tensors and their applications in computational mechanics. Comput. Methods Appl. Mech. Eng. 2000;189:419–438.
[15] Knowles J. On the representation of the elasticity tensor for isotropic materials. J. Elast. 1995;39:175–180.
[16] Tarantola A. Elements for physics: quantities, qualities, and intrinsic theories. Berlin: Springer; 2005.
[17] Shen S, Berger T. On the distribution formula of the eigenvalue of Toeplitz tensor and its application. Appl. Math. J. Chinese Univ. 1987;2:151–163. Chinese.
[18] Qi L. Theory of tensors (hypermatrices). Hong Kong: Department of Applied Mathematics, The Hong Kong Polytechnic University; 2014.
[19] Brazell M, Li N, Navasca C, Tamon C. Solving multilinear systems via tensor inversion. SIAM J. Matrix Anal. Appl. 2013;34:542–570.
[20] De Lathauwer L, De Moor B, Vandewalle J. A multilinear singular value decomposition. SIAM J. Matrix Anal. Appl. 2000;21:1253–1278.
[21] Chan R, Ng M. Conjugate gradient methods for Toeplitz systems. SIAM Rev. 1996;3:427–482.
[22] Ng M. Iterative methods for Toeplitz systems. Oxford: Oxford University Press; 2004.

[23] Kofidis E, Regalia Ph. On the best rank-1 approximation of higher-order supersymmetric tensors. SIAM J. Matrix Anal. Appl. 2002;23:863–884.
[24] Ragnarsson S, Van Loan C. Block tensors unfoldings. SIAM J. Matrix Anal. Appl. 2012;33: 149–169.
[25] Rezghi M, Elden L. Diagonalization of tensors with circulant structure. Linear Algebra Appl. 2011;435:422–447.
[26] Gonzalez R, Woods R. Digital image processing. New Jersey (NJ): Pearson; 2008.
[27] Liu X, Wang L, Wang J, Meng H. A three-dimensional point spread function for phase retrieval and deconvolution. Opt. Express. 2012;20:15392–15405.
[28] Serre D, Villeneuve E, Carfantan H, et al. Modeling the spatial PSF at the VLT focal plane for MUSE WFM data analysis purpose. In: SPIE, Astronomical Telescopes and Instrumentation: Observational Frontiers of Astronomy for the New Decade; 2010; Bellingham, WA.
[29] Zhao X, Wang F, Huang T, Ng M, Plemmons R. Deblurring and sparse unmixing for hyperspectral images. IEEE Trans. Geosci. Remote Sens. 2013;51:4045–4058.
[30] Hansen P. Rank-deficient and discrete ill-posed problems: numerical aspects of linear inversion. Philadelphia (PA): SIAM; 1998.